GREATEST COMMON DWISORS

AND

LEAST COMMON MULTIPLES

OF GRAPHS

by

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submitted in accordance with the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in the subject

MATHEMATICS

at the

UNIVERSITY OF SOUTH AFRICA

PROMOTER: PROFESSOR C M MYNHARDT

NOVEMBER 1991

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ACKNOWLEDGEMENTS

I wish to express my sincere appreciation and deepest gratitude to Professor Gary Chartrand and Professor Christine M Mynhardt for their outstanding mathematical guidance during the writing of this dissertation.

Farrokh Saba

SUMMARY

Chapter I begins with a brief history of the topic of greatest common subgraphs. Then we provide a summaiy of the work done on some variations of greatest common subgraphs. Finally, in this chapter we present results previously obtained on greatest common divisors and least common multiples of graphs.

In Chapter II the concepts of prime graphs, prime divisors of graphs, and prime-connected graphs are presented. We show the existence of prime trees of any odd size and the existence of prime-connected trees that are not prime having any odd composite size. Then the number of prime divisors in a graph is studied. Finally, we present several results involving the existence of graphs whose size satisfies some prescribed condition and which contains a specified number of prime divisors.

Chapter III presents properties of greatest common divisors and least common multiples of graphs. Then graphs with a prescribed number of greatest common divisors or least common multiples are studied.

In Chapter IV we study the sizes of greatest common divisors and least common multiples of specified graphs. We find the sizes of greatest common divisors and least common multiples of stars and that of stripes. Then the size of greatest common divisors and least common multiples of paths and complete graphs are investigated. In particular, the size of least common multiples of paths versus K3 or K4 are determined. Then we present the greatest common divisor index of a graph and we determine this parameter for several classes of graphs.

In Chapter V greatest common divisors and least common multiples of digraphs are introduced. The existence of least common mutliples of two stars is established, and the size of a least common multiple is found for several pairs of stars. Finally, we present the concept of greatest common divisor index of a digraph and determine it for several classes of digraphs.

CHAPTER I

History and Background

Our subject began in 1987 with the study of greatest common subgraphs. In the first section of this chapter, we provide a brief history of this topic. In the second section, we summarize work done on some variations of greatest common subgraphs of graphs. In the third and final section, we summarize results obtained on greatest common divisors and least common multiples of graphs, the main topics of this dissertation. All terms and notation not defined or described in this dissertation may be found in Chartrand and Lesniak [CL].

1.1 Greatest Common Subgraphs

The concept of greatest common subgraphs of graphs was introduced by Chartrand, Saba, and Zou [CSZ1]. A graph G without isolated vertices is called a *greatest common subgraph* of a set *Q* = {Gj, *Q*^ ... , G_n), n > 2, of graphs having the same size if G is a graph of maximum size that is isomorphic to a subgraph of each graph Gi, 1 < i < n. The set of all greatest common subgraphs of Q is denoted by gcs Q or gcs(G_1, G_2> ...» G_n). For example, if Q - {Gj, G2) for the graphs of Figure 1.1, then gcs Q- { H ^ ^ } .

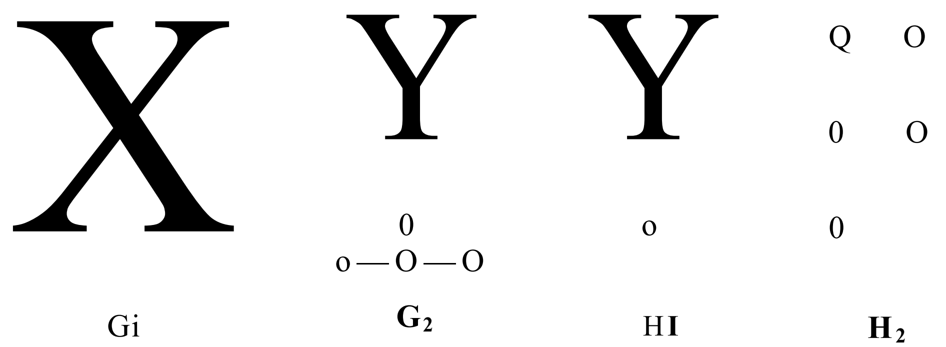


Figure 1.1 Greatest common subgraphs of graphs

Thus, it is clear that a greatest common subgraph'may not be unique. In fact, it is not unusual for a set Q of two or more graphs of equal size to have several greatest common subgraphs. The following result was established in [CSZ2].

Theorem 1A For every pair m, n of positive integers with n > 2, there exist n pairwise nonisomorphic graphs Gj, G 2 , G_n of equal size such that

$$I \; gcs(G_x, G_2, G_n) \; | = m.$$

Another related problem is to find, for a given graph G, two nonisomorphic graphs Gj and G2 of equal size (or a set Q of graphs of equal size) such that G is the *unique* greatest common subgraph of G_x and G2 (respectively, of *Q).* The following result was obtained in [CSZ2], and we state it for future reference.

Theorem IB If G is a graph without isolated vertices, then there exist non-isomorphic graphs Gi and G_2 of equal size such that gcs(Gi, G_2) = {G}.

In the proof of Theorem IB, one of Gi and G_2 is connected while the other graph is disconnected. However, Chartrand, Johnson, and Oellermann [CJO] proved that if G is connected but not complete, then there are nonisomorphic *connected* graphs Gi and G2 of equal size such that gcs(Gi, G2) = {G}. Later, a more general class of problems was investigated.

Let P be a graphical property. For a given graph G without isolated vertices and having property P, do there exist non-isomorphic graphs Gi and G2 of equal size having property P such that gcs(Gi, G2) = {G}? If P is the property of being 2-connected, then the following characterization was given in [COSZ], For a 2-connected graph G, there exist non-isomorphic 2-connected graphs Gj and G2 of equal size such that gcs(Gi, G2) = {G} if and only if G ^ K_n (n > 3) and G ^ K_n - e (n > 4). In the same paper, it was shown that for every n-chromatic graph G (n > 2), there exist non-isomorphic n-chromatic graphs Gi and G2 of the same size such that gcs(Gj, G2) = {G}.

Chartrand and Zou [CZ] characterized trees that are unique greatest common subgraphs of two suitably chosen nonisomorphic trees of equal size. Let D(t) denote a tree consisting of two stars K(1, t) whose central vertices are connected by a path of length 3. If T is a tree, then gcs(Ti, T2) = {T) for some nonisomorphic trees Ti and T2 of equal size if and only if T ^ P_n, n = 2, 4, 5, ... and T £ D(t), t > 2. When the property P is that of being connected outerplanar, connected planar, or unicyclic, then the problem was solved as well, by Kubicki [K],

There are several concepts closely related to greatest common subgraphs that have been studied. Greatest common induced subgraphs have been considered in [CJO], [COSZ], and [CZ], and this concept has proved to be considerably easier than the greatest common subgraph. Also, related problems for digraphs have been considered in [CJO].

Let Q be a set of graphs without isolated vertices, all having the same size. A graph G without isolated vertices is a *least common supergraph* of *Q* if G is a graph of minimum size that is isomorphic to some supergraph of every graph in G. The set of all least common supergraphs of Q is denoted by lcs Q. For the graphs Gx and G_2 of Figure 1.2, lcs(G_{l5} G_2) consists of the three graphs Hj, H_2, and H_3, also shown in Figure 1.2.

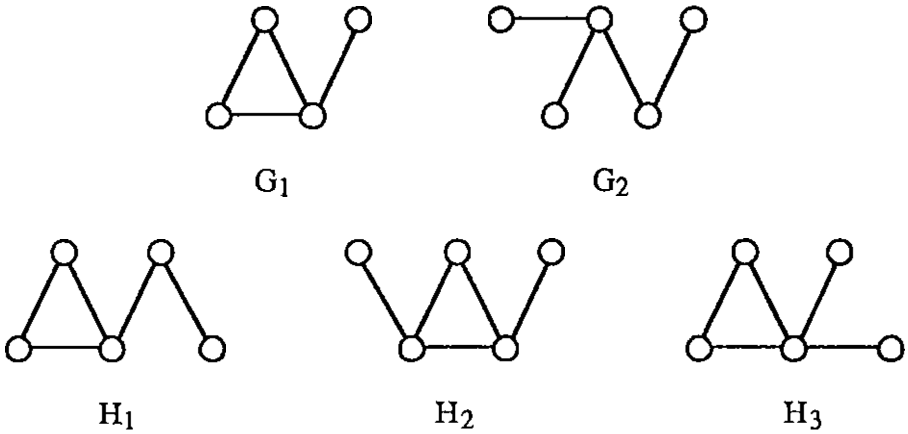


Figure 1.2 Least common supergraphs of graphs

The next result [CHKOSZ] shows a relationship among the size of two given graphs (of equal size) and the sizes of a greatest common subgraph and a least common supergraph of the two given graphs.

Theorem 1C Let Gj and G_2 be graphs without isolated vertices and having size q. If G e gcs *(G_{lt}* G_2) and H e lcs (G_{lf} G_2), then

$$q(G) + q(H) = 2q.$$

In order to present a characterization [CHKOSZ] of graphs that can be least common supergraphs of two graphs, we present a definition. A nonempty graph G is *edge-symmetric* if G - e = G - f for all e , f e E(G).

Theorem ID Let G be a graph without isolated vertices. Then G is a least common supergraph of two nonisomorphic graphs of equal size if and only if G is not edge-symmetric.

The dual nature of greatest common subgraphs and least common supergraphs was described in more detail in [CHKOSZ]. First, some additional notation is useful. For a given graph G, let p be an integer with p>p(G). The graph G(p) is defined by

G^sGuCp-pCO1Kx,

that is, G(p) is obtained by adding p - p(G) isolated vertices to G.

In what follows, least common supergraphs are permitted to have isolated vertices.

Theorem IE Let Q = {G^, G_2,...» G_n} be a family of graphs of equal size and let p = max {p(H) | H e lcs Q and H has no isolated vertices}. Then H e lcs Q if and only if H(p) e gcs (G^p) , G_2(p) > G_n(p)).

The following is a consequence of Theorems **ID** and **IE.**

Theorem IF Let G be a graph of order p without isolated vertices. Then G is a greatest common subgraph of two nonisomorphic graphs of equal size having order p if and only if G is not edge-symmetric.

The final result of this section follows immediately from Theorems 1A and IE.

Theorem 1G For every pair m, n of integers with m > **2** and n > **1,** there exists a set Q of m pairwise nonisomorphic graphs of equal size such that I lcs q\ — n.

1.2 Maximal Common Subgraphs and Absorbing Common Subgraphs

Let Gi and G_2 be nonisomorphic graphs of the same size. The set of all common subgraphs of Gj and G_2 can be considered as a set partially ordered by the relation "is a subgraph o f. Maximal common subgraphs are the maximal elements in this partially ordered set. More formally, a graph H without isolated vertices is a *maximal common subgraph* of Gj and G_2 if H is (isomorphic to) a subgraph of G^ and G_2, and there is no graph F without isolated vertices that is a common subgraph of Gj and G_2 such that H is a proper subgraph of F. The set of all maximal common subgraphs of Gj and G_2 is denoted by mcs(G_{1s} G_2). If Gj = K(3, 3) and G_2 s K(1, 3) u K_4, then mcs(G_{1f} G_2) = {H_2, H_2, H_3}, where H_t s K(1, 3), H_2 = 2K(1, 2), and H_3 = C_4 u K_2 (see Figure 1.3).

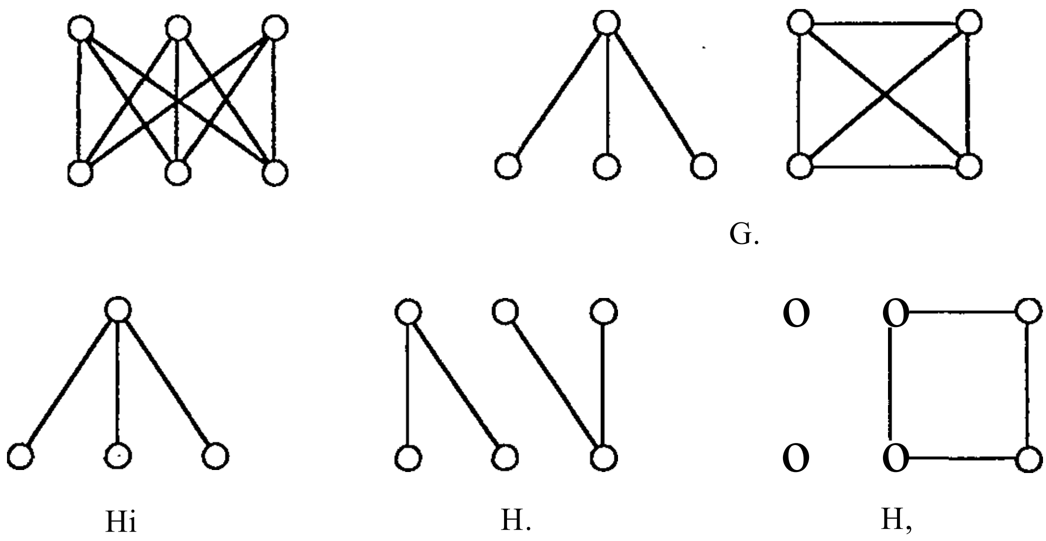


Figure 1.3 Maximal common subgraphs

In this example, the graphs Hj, H_2, and H3 have different sizes (namely 3, 4, and 5, respectively), so H3, having maximum size, is the unique greatest common subgraph of G_x and G_2.

It was shown in [K] that the difference between the sizes of a greatest common subgraph and a maximal common subgraph can be arbitrarily large.

Theorem 1H For every positive integer M, there exist graphs G_x and G_2 of equal size and graphs G e gcs(Gj, G_2) and H e mcs(Gj, G_2) such that q(G)-q(H)>M.

The set of maximal common subgraphs of two graphs can have arbitrarily large cardinality; indeed, a wide range of sizes for maximal common subgraphs is possible [K].

Theorem II For every positive integer N, there exist graphs G_x and G_2 of equal size and N graphs Hj, H_2, H_N with qCHj) * q(Hj) for 1 < i <j < N such that $\{H_1, H_2, \ldots, H_n\}$ **Q** mcs(G_1, G_2).

In [K] graphs are characterized that are maximal common subgraphs of a certain pair of graphs but not greatest common subgraphs of the same pair of graphs.

Theorem 1J Let G be a graph without isolated vertices such that G £ K(l, r) (r = 1, 2). Then there exist nonisomorphic graphs Gj and G_2 of equal size such that G e mcs(Gj, G_2), but G 4 gcs(Gj, G_2).

A graph G without isolated vertices is an *absorbing common subgraph* of two nonisomorphic graphs Gj and G_2 if (1) G is (isomorphic to) a subgraph of G_x and G_2 and (2) whenever a graph H (without isolated vertices) is a common subgraph of G^ and G2, then H is a subgraph of G. Informally, an absorbing common subgraph of Gj and G2 is a common subgraph of Gi and G2 that "absorbs" every other common subgraph of Gi and G2. This concept was introduced by Chartrand, Erdos, and Kubicki [CEK].

Two graphs of equal size need not have an absorbing common subgraph. However, if an absorbing common subgraph of Gj and G2 exists, then it is unique and is denoted by acs (Gj, G2). In fact, when G is the unique maximal common subgraph of graphs Gi and G2, then G is the absorbing common subgraph of Gi and G2, and vice versa, as shown in [CEK].

Theorem IK A graph G is an absorbing common subgraph of two nonisomorphic graphs Gi and G2 of equal size if and only if G is the unique maximal common subgraph of Gj and G2.

From Theorem IK, it then follows that if G is an absorbing common subgraph of two nonisomorphic graphs Gj and G2, then it is the unique greatest common subgraph of Gj and G2.

Thus, the graphs Gj and G2 of Figure 1.3 do not have an absorbing common subgraph since Gj and G2 have more than one maximal common subgraph.

It is not difficult to show that no complete graph of order at least 3 is an absorbing common subgraph. The situation for complete bipartite graphs is presented in [CEK].

Theorem 1L Let $G = K(m, n)$, where $m < n$. Then G is an absorbing common subgraph if and only if $m = 1$, $m = 2$, or $n = m + 1$.

Various other classes of graphs that are (or are not) absorbing common subgraphs are also given in [CEK].

1.3 Greatest Common Divisors and Least Common Multiples

A variation of greatest common subgraphs and least common supergraphs with number-theoretic overtones was introduced in Chartrand, Hansen, Kubicki, and Schultz [CHKS]. A nonempty graph G is said to be *decomposable* into the subgraphs Gi, G_2, G_n of G if no graph Gi, 1 < i < n, has isolated vertices, and the edge set E(G) of G is partitioned into ECGj), $E(G_2)$, $E(G_n)$- If Gi=H for every i (1 < i < n), then G is said to be *H-decomposable.* In fact, this is a generalization of r-factorable graphs, for a positive integer r. If G is H-decomposable into two or more copies of H, and H £ K_2, then we say G is *nontrivially* H-decomposable.

The following observation will prove useful to us later.

Proposition 1.1 If a graph G is decomposable into subgraphs G_{1}} G_2, ... , G_n (n > 2) and each subgraph G^ (1 < i < n) is decomposable into subgraphs F^ Fi_2, ..., Fj_{mi} (mf > 2), then G is decomposable into the subgraphs F_n, F_{12},..., F_{1mi},
F21> F22> » F2m2> - ' Fnl> Fn2> ••• » F_{nm_n}-

This result has the following immediate consequence.

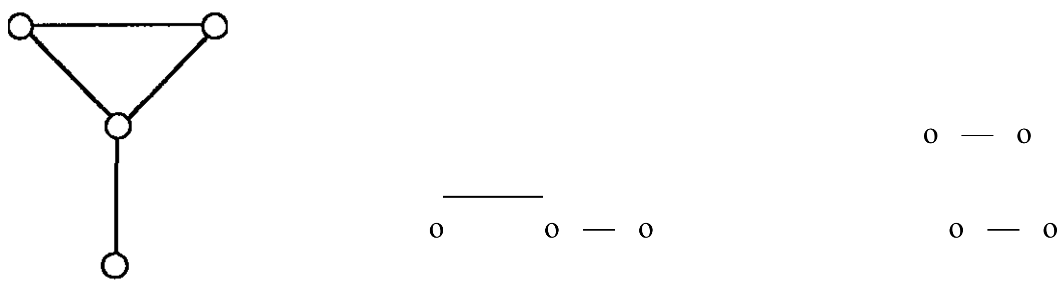
Corollary 1.2 If a graph G is decomposable into subgraphs G_{1s} G_2, ... , G_n (n > 2), each of which is F-decomposable, then G is F-decomposable.

Finally, we have the following result.

Corollary 1.3 If a graph G is F-decomposable and F is H-decomposable, then G is H-decomposable.

If a graph G is H-decomposable, then H is said to *divide* G and is a *divisor* of G. If H divides G, we write H | G. It is clear that for every graph G of size at least 2, we have $K_2$1G and G | G. For such a graph G, the graphs K_2 ajid G are called the *trivial divisors* of G. A graph H is said to be a *proper divisor* of G if G is nontrivially H-decomposable, that is, if H | G and 1 < q(H) < q(G).

If a graph G is H-decomposable, then q(H) | q(G). However, if H is a subgraph of G without isolated vertices such that q(H) | q(G), then G may not be H-decomposable. For example, in Figure 1.4, the graph G is Hj-decomposable but not H_2-decomposable.



G Hi H_2

Figure 1.4 An Hj-decomposable graph G that is not H_2-decomposable

The following theorem, obtained in [CPS], will prove to be useful.

Theorem 1M Every nontrivial connected graph of even size is P3~decomposable.

In [CHKS], a graph G without isolated vertices is defined to be a *greatest common divisor* of two graphs and G_2 if G is a graph of maximum size such that both Gj and G_2 are G-decomposable. We also refer to this as a greatest common divisor of Gj versus G_2. For example, in the graphs of Figure 1.5, Hj is the unique greatest common divisor of Gj and G_2, while H^ and H_2 are the greatest common divisors of G_2 and G3.

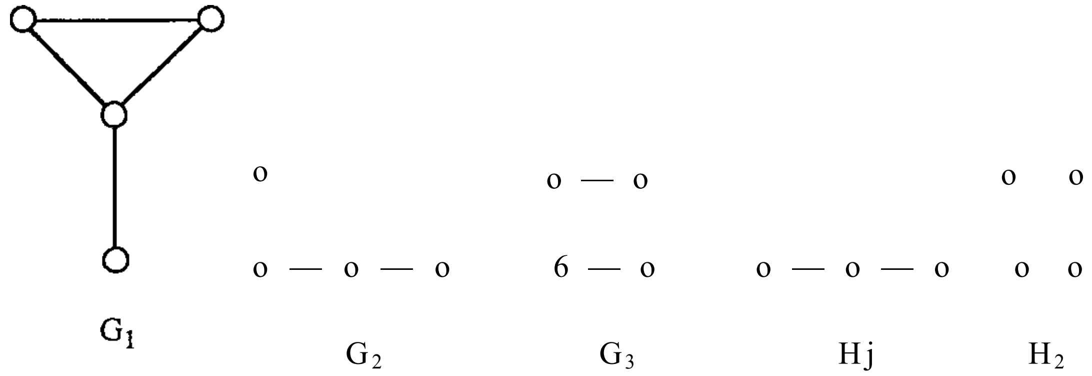


Figure 1.5 Greatest common divisors

A *greatest common divisor* of a set Q = {Gj, G_2, ..., G_n}, n > 2, of nonempty graphs is defined similarly. Since K_2 is a divisor of every graph of Q, there exists a graph of maximum size that is a divisor of every graph of Q. Consequently, every set of two or more nonempty graphs has a greatest common divisor.

The set of all greatest common divisors of a set Q ~ {Gj, G_2,…, G_n), n > 2, of graphs is denoted by GCD Q. In this case, we also write GCD Q = GCD($G_{1\}}$ G_2, ... , G_n). The size of a greatest common divisor of a set (3 = {$G_{1\}}$ G_2,… , G_n), n > 2, of graphs is denoted by gcd Q or gcd($G_{1\}}$ G_2 , G_n) .

In [CHKS], a graph H without isolated vertices is called a *least common multiple* of two nonempty graphs Gj and G_2 if H is a graph of minimum size such that H is both Gj-decomposable and G_2-decomposable. Similarly, a graph H without isolated vertices is called a *least common multiple* of a set Q = {Gj, G_2, G_n}, n > 2, of graphs if H is a graph of minimum size such that H is Gp decomposable for all i (1 < i < n).

The set of least common multiples of a set Q - {Gj, G2, G_n), n > 2, of graphs is denoted by LCM Q or by LCMCG^ G2, G_n). The size of a least common multiple of a set Q - {Gj, G2, G_n}, n > 2 of graphs is denoted by lcm Q or lcm(Gj, G2, G_n). For the graphs Gj and G2 of Figure 1.6, the graphs Hj (1 < i < 5) are the least common multiples of Gj and G2. It is clear that lcm(G|, G2) = 8. This example shows that least common multiples need not be unique.

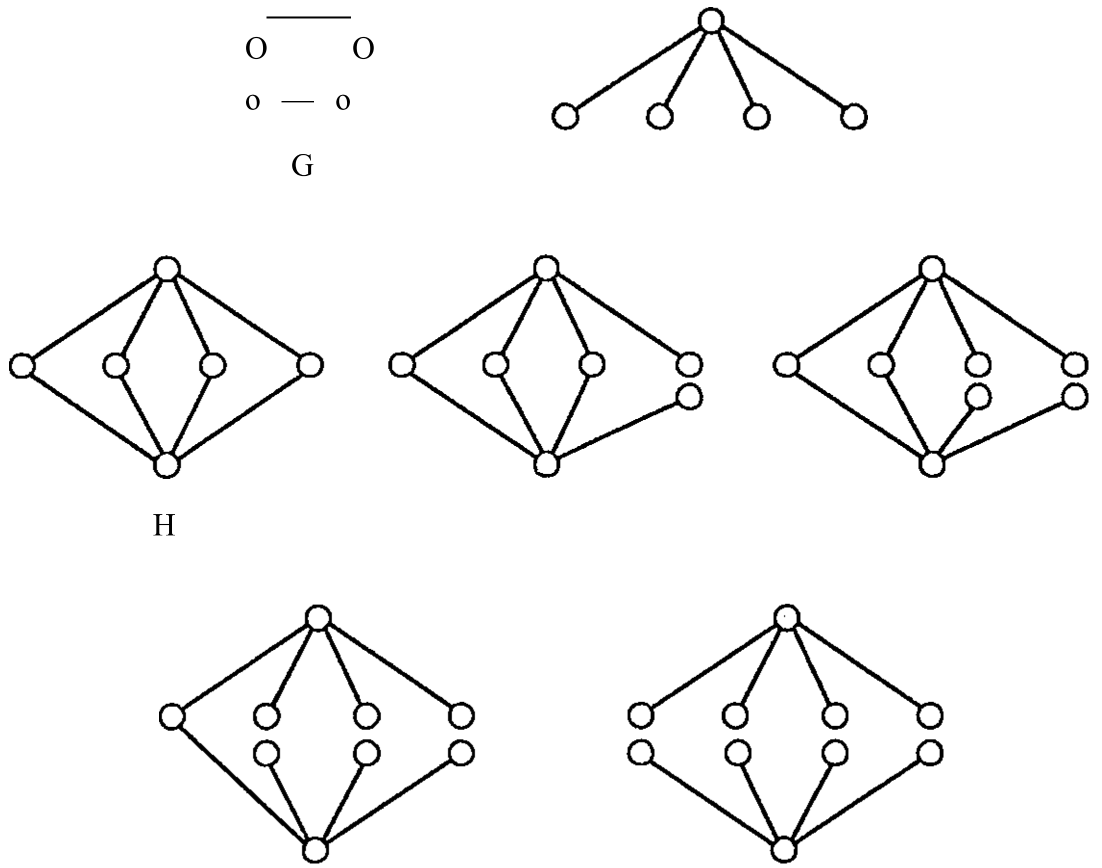


Figure 1.6 Least common multiples

While it is evident that every two (or more) graphs have a greatest common divisor, it is not obvious that they have a least common multiple. It was verified in [CHKS] that every two nonempty graphs do indeed have a least common multiple. The proof of this result made use of the following theorem of Wilson [W].

Theorem IN Let F be a graph of size q (> 1) without isolated vertices. Then $f \mid k_d$ provided n is sufficiently large, $q \mid (_2)$, and d| (n - 1), where d is the greatest common divisor of the degrees of the vertices of F.

With the aid of Theorem IN, we show that every (finite) set of two or more graphs has a least common multiple.

Theorem 1.4 For graphs Gi, G 2 , G_m (m > 2) without isolated vertices, there exists a graph H that is Gi-decomposable for all i (1 < i < m).

Proof Suppose Gj has size qi (1 < i < m), and let d[= gcd {deg v : v e V(Gi)} for all i(1 < i < m). By Theorem IN, for all i (1 < i < m), there exists an integer Ni such that if

(i) n > Ni,

(ii) n(n - 1) = 0 (mod 2qi), and

(iii) (n - 1) = 0 (mod dj),

then K_n is Gi-decomposable.

Then define t = lcm {di> d_2, d_m, q_h q_2, q_m}. Choose k sufficiently large so that 2kt+ 1 > max {Ni, N_2, ..., N_m}, and let n = 2kt + 1. Now di 11 and qi 11 for all i (1 < i < m), and conditions (i) - (iii) of Theorem IN are satisfied and, therefore, H = K_{rt} is Gj-decomposablefor all i(1<i<m). •

Theorem 1.4 has the immediate consequence.

Theorem 1.5 Every set of two or more nonempty graphs has a least common multiple.

Proof Let Q = {G|, G2, G_m), m > 2, be a set of nonempty graphs. Theorem 1.4 shows the existence of a graph H that is Gj-decomposable for all i (1 < i < m). Therefore, H is a common multiple of Q. Consequently, there exists a graph of smallest size that is Gj-decomposable for all i (1 < i < m), implying that a least common multiple of Q exists. •

It is a well-known fact from number theory that for every pair a, b of positive integers, a-b = gcd(a, b>lcm(a, b). It may have been anticipated that there is some relationship between q(Gj)-q(G2) and gcd (Gj, G2)*lcm (Gj, G_2). However, it was shown in [CHKS] that for every positive integer N, there exist graphs Hj and H_2 such that $q(H_1)$-$q(H_2)$ > N-gcd (H_1,.H_2)-lcm (H^ H_2) and graphs Fj and F_2 such that gcd (F_h F_2)-lcm (Fi, F_2) > N-$q(F_1)$-$q(F_2)$.

In [CHKS], lcm (C_n, K(l, m)) was determined when n is even and m is arbitrary and when n = 3 and m is arbitrary. In forthcoming chapters, properties of greatest common divisors, least common multiples, and related concepts are investigated further.

CHAPTER n

PRIME GRAPHS AND PRIME DIVISORS OF GRAPHS

In this chapter we present the concepts of prime graphs, prime divisors of graphs, and prime-connected graphs. We begin by showing the existence of prime trees of any odd size and the existence of prime-connected trees that are not prime and having any odd composite size. Furthermore, prime double stars, prime-connected double stars, and prime-connected caterpillars of diameter 4 or 5 are characterized.

We then investigate the number of prime divisors in a graph. In particular, it is shown that trees and cyclic graphs of every composite size, having a unique prime divisor, exist. Furthermore, this problem is considered for trees and cyclic graphs (of composite size) having exactly two prime divisors. We conclude the chapter by presenting a collection of results involving the existence of graphs (some of which are required to be more highly connected) whose size often satisfies some prescribed condition and which contain a specified number of prime divisors.

2.1 Prime and Prime-Connected Trees

Recall that K_2 and G are the trivial divisors of a nonempty graph G. If G has no isolated vertices and has size at least 2, then G is called a *prime graph*, or simply a *prime*, if it has no nontrivial divisor. If the size of a graph is prime, then the graph is prime. However, the size of a prime graph need not be prime. For example, the graphs K(l, 3) u K_2 and K3 u K_2 are prime graphs of size 4. In fact, for every composite integer q (> 4), there exists a prime graph of size q, namely K_2 u K(l, q- 1). Note, however, that Theorem 1M implies that there are no connected prime graphs of even size at least 4. A *composite* graph is a graph of size 2 or more that is not prime. A divisor that is prime is called a *prime divisor.*

Among all graphs of size 2, 4, or 6, the graphs Gj, 1 < i < 7, in Figure 2.1 are the only prime graphs, as can easily be seen by using Theorem 1M and by checking all remaining cases.

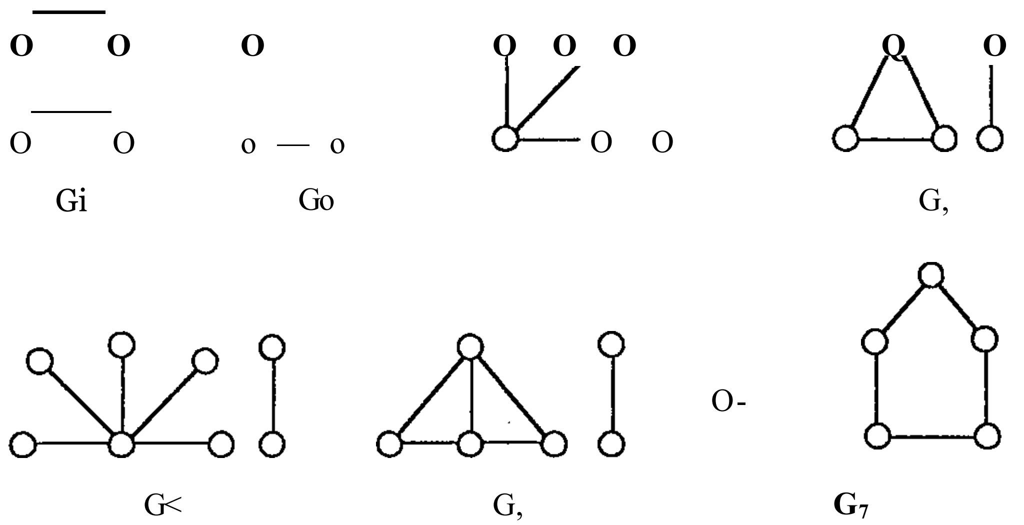


Figure 2.1 Prime graphs of size 2, 4, and 6

A connected graph G of size at least 2 is *prime-connected* if its only *connected* divisors are K_2 and G. By Theorem 1M, every prime-connected graph of size at least 3 must be of odd size. Observe that a connected graph that is prime is also prime-connected. However, the converse is not true in general. The tree T in Figure 2.2 is an example of a prime-connected tree which is not prime, since P3 u K_2 and $3K_2$ are the only nontrivial divisors of T.

o — o — o — o — o — o — o —

T

Figure 2.2 A prime-connected tree that is not prime

We have already indicated that K_2 u K(1, q - 1) is prime for every integer q > 2. Of course, this graph is disconnected. We show that for every odd integer q > 3, there is a connected prime graph of size q. Indeed, we prove that prime trees of size q exist for all odd integers q > 3, and that prime-connected trees of size q that are not prime exist for all odd composite integers q. We begin with a lemma.

Lemma 2.1 A graph of size at least 2 containing an edge adjacent to all other edges lias no disconnected divisor.

Proof Let G be a graph of size at least 2 containing an edge, say e, adjacent to all other edges. Suppose, to the contrary, that H is a disconnected divisor of G. Observe that in any H-decomposition of G, there exists a copy H' of H containing the edge e. Let H_t be a component of H' containing the edge e. Now since any edge of H' other than the edges of Hj is adjacent to e, it follows that no component other than Hj exists. Therefore, H is connected. •

Theorem 2.2 Let q (> 3) be an integer. There exists a prime tree of size q if and only if q is odd.

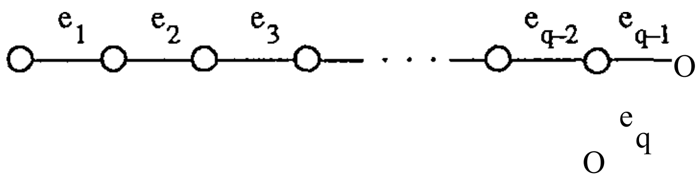
Proof By Theorem 1M, there does not exist a prime tree of even size q (> 4). The result for q odd holds when q is a prime number, since every tree of prime size is a prime tree. Let q (> 9) be an odd composite number, and let T be the tree constructed by joining the central vertices u and v of two copies of K(l,r), where r = (q - l)/2, by the edge e = uv. Observe that diam T = 3. Suppose that T is In-decomposable for some graph H, where 1 < q(H) < q. Since e is adjacent to all other edges of T, Lemma 2.1 implies that H is connected.

If diam H = 3 and H £ T, then e belongs to some copy of H, and T - E(H) is disconnected, implying that diam H < 2 in other copies of H — a contradiction. If diam H < 2, then H = K(l, t), for some t > 1. Suppose that mj and m_2 copies of H have central vertex at u and v, respectively. Without loss of generality, let e be an edge of a copy of H having v as its central vertex. Then m_2t - mjt = 1, that is, (m_2 - mj)t = 1, implying that m_2 - mj = 1 and t = 1. Hence, m_2 = nij + 1 and H s K_2, implying that T is a prime tree. •

We now generalize the tree T depicted in Figure 2.2 to prove the existence of a prime-connected tree of size q that is not prime, where q is any odd composite integer.

Theorem 2.3 For every odd composite integer q, there exists a prime-connected tree of size q that is not prime.

Proof Let q = rs, where r is the smallest prime factor of q. We construct a tree T of size q by identifying an end-vertex of the path P j with the central vertex of the path P_3. We label the edges of T as indicated in Figure 2.3.



T

Figure 2.3 A prime-connected tree that is not prime

We show that T is a prime-connected tree that is not prime. First we show that T is not prime. Observe that T is $(P_r$ u $K_2)$-decomposable into s copies (s > r > 3) of $(P_r$ u $K_2)$ containing edges $e_{(m\_1)(r\_1)+1}$, $e_{(m\_1)(r\_1)+2}$, $t_{\{mr\_m\_1)+r\_v}$ and e_{q-s+m} for m = 1, 2 , s . Therefore, T is not prime.

Next, we show that T is prime-connected. Let Tj be a nontrivial divisor of T. Suppose, to the contrary, that Tj is connected. Since A(T) = 3, it follows that A(Tj) < 3. But Tj is nontrivial, so A(Tj) * 1. Observe that T has exactly one vertex of degree 3, so that A(Tj) * 3. Therefore, A(Tj) = 2 and $T_l = P_k$ for some integer k (> 4), where (k - 1) | q.

Since k - 1 > r (> 3), it follows that e_{q_2} and one of e_{q_j} and e_q belong to the same copy of P_k in any P_k-decomposition of T. Suppose, without loss of generality, that $e_{q,,2}$ and e_{q-1} belong to the same copy of P_k. Therefore, the edge e_q can only belong to a disconnected divisor of T — a contradiction. Hence Tj is disconnected and T is prime-connected but not prime. •

We next consider some specific classes of trees. First, we note that the star K(l, m), m>2, is prime and prime-connected if and only if m is prime. We now turn our attention to double stars. A *double star* is a tree containing exactly two vertices that are not end-vertices. If these two vertices have degrees a and b, we denote this double star by S(a, b) (see Figure 2.4).

1 1

2

a - 1 b - 1

Figure 2.4 The double star S(a, b)

We are now prepared to characterize prime double stars.

Proposition 2.4 For integers a, b (> 2), the double star S(a, b) is prime if and only if gcd(a, b - 1) = gcd(a - 1, b) = 1.

Proof Assume that the double star S(a, b) is a prime graph. We show that gcd(a, b - 1) = gcd(a - 1, b) = 1. If gcd(a, b - 1) = m (> 2), then S(a, b) is K(1, m)-decomposable and, therefore, S(a, b) is not a prime graph — contrary to hypothesis. Similarly, gcd(a - I, b) = 1.

Conversely, assume that gcd(a, b - 1) = gcd(a - 1, b) = 1. Suppose, to the contrary, that S(a, b) is not a prime graph. Then S(a, b) is H-decomposable for some graph H such that H £ K_2 and H ^ S(a, b). Since the edge e = uv, where deg **u** = a and deg v = b, is adjacent to all other edges of S(a, b), Lemma 2.1 implies that H is connected. Therefore, H = K(1, r) for some integer r (> 2). Then (1) r | a and r| (b - 1), or (2) r | (a — 1) and r| b. Therefore, gcd(a, b - 1) * 1 or gcd(a - 1, b) ^ 1 — contrary to hypothesis. •

We now show that a double star is prime if and only if it is prime-connected.

Proposition 2.5 . For integers a, b (> 2), the double star S(a, b) is prime if and only if S(a, b) is prime-connected.

Proof Let the double star S(a, b) be prime for integers a, b (> 2). Clearly, S(a, b) is prime-connected.

Conversely, if S(a, b) is prime-connected, then it has no connected nontrivial divisors, and by Lemma 2.1, also no disconnected divisors. Hence, S(a, b) is prime. •

The following characterization of prime-connected double stars is now obvious.

Proposition 2.6 For integers a, b (> 2), the double star S(a, b) is prime-connected if and only if gcd(a, b - 1) = gcd(a - 1, b) = 1.

Finally, we present a result on double stars having divisors of size 3.

Proposition 2.7 Let T= S(a, b) be a double star of size 3n (> 9). Then T is not decomposable into any graph of size 3 if and only if a = 2(mod 3) and b = 2(mod 3).

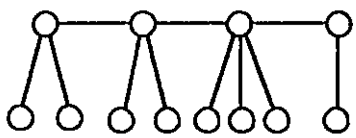
Proof Suppose that T is not decomposable into any graph of size 3. If it is not the case that a = 2(mod 3) and b = 2(mod 3), then either (i) a = 0(mod 2) and b = l(mod 3), or (ii) a ^ l(mod 3) and b = 0(mod 3). Suppose that (i) holds and that u, v e V(T) with deg u = a = 3k and deg v = b = 3k' + 1 for some positive integers k and k'. Then u is the central vertex of k edge-disjoint stars K(l, 3), one of which contains the edge uv, and v is the central vertex of k' edge-disjoint stars K(l, 3) (not containing uv). Thus, T is K(l, 3)-decomposable, contrary to hypothesis.

Case (ii) can be proved similarly to case (i). Therefore, a = 2(mod 3) and b = 2(mod 3).

Suppose next that a = 2(mod 3) and b = 2(mod 3). The only possible subgraphs of size 3 in T are P_4, K(l, 3), and P3 u K_2. We first show that T is not P_4-decomposable. Note that T £ P_4 since T has size at least 9. Suppose, to the contrary, that T is P^decomposable. Then the removal of a copy of P4 from T results in a disconnected graph in which each component has diameter at most 2, implying that the resulting graph is not P4-decomposable — a contradiction. Therefore, T is not P4~decomposable. Next, we show that T is not K(l, 3)-decomposable. Suppose, to the contrary, that T is K(l, 3)-decomposable. Then the central vertex of each copy of K(l, 3) in every K(l, 3)-decomposition of T is either at u or at v, implying that the degree of one of u and v is a multiple of 3, contrary to the hypothesis. That T is not (P3 u K_2)-decomposable follows from Lemma 2.1. Therefore, T is not decomposable into any graph of size 3. •

Next we consider a class of trees that are generalizations of double stars.

Let $a_{1?}$ a $_2$, a $_n$ (n > 2) be integers greater than 1. The *caterpillar* C(ai, a_2, ..., a_n) is the tree obtained from the path P_n: x_{1f} x $_2$, x $_n$ by joining the vertices xj and x_n to aj - 1 and to a_n - 1 new vertices, respectively, and the vertex xj to a^ - 2 new vertices for each i (2 < i < n - 1). Figure 2.5 shows C(3, 4, 5, 2). Note that the double star S(a, b) is isomorphic to the caterpillar C(a, b).



C(3, 4, 5, 2)

Figure 2.5 The caterpillar C(3, 4, 5, 2)

We now characterize prime-connected caterpillars of the type C(a_{ls} a_2, a3).

Proposition 2.8 The caterpillar C(aj, a_2, a_3) is prime-connected if and only if the following conditions hold:

(i) gcd(a_1, a_2-1, a_3-1) = 1,

(ii) gcd(a: - 1, **a2** - 1, **a_3) =** 1,

(iii) gcd(a_1, a_2-2, a_3)= 1,

(iv) gcd(aj - 1 , a^ a_3 - 1) = 1,

(v) a_1 * a_3, or a2 = 2 or an odd integer at least 3,

(vi) a_2 ^ aj + a_3.

Proof Assume that C(aj, a^ a_3) is a prime-connected caterpillar. If gcd($a_{1\}}$ a_2-1, a_3 - 1) = ir^ & 1, then C(aj, a_2, a_3) is K(l, n^)-decomposable — a contradiction. Hence, condition (i) holds. Similarly, conditions (ii), (iii), and (iv) hold. Now suppose, to the contrary, that condition (v) does not hold. Then it follows that aj = a_3 and is an even integer at least 4. In this case, C(a_{is} a_2, a_3) is S(aj, a_2/2)-decomposable, contrary to hypothesis. Finally, suppose, to the contrary, that the condition (vi) does not hold. Then it follows that a_2 = a_l + a_3, implying that C($a_{1?}$ a_2, a_3) is S(aj, a_3)-decomposable, contrary to hypothesis.

Conversely, suppose that conditions (i) - (vi) hold and that C(aj, a_2, a_3) is not prime-connected. Let H be a connected graph such that C(aj, a_2, a_3) is nontrivially H-decomposable.

Case 1 *Assume that the graph* H *is isomorphic to* K(l, m) *for some integer* m (> 2). In this case at least one of the following conditions (a) - (d) must hold:

(a) m|a_x and m|(a_2-1) and m|(a_3-1),

(b) m1faj-1) and m|(a_2-1) and m|a_3,

(c) m | a_x and m | (a_2 - 2) and m | a_3,

(d) m I (a_x - 1) and m^ and m|(a_3-1).

This, in turn, implies that at least one of the conditions (i) - (iv) fails, contrary to hypothesis.

Case 2 *Assume that the graph* H *is isomorphic to* S(a, b) *for some integers* a * 1 *and* b ^ 1. In this case, one of the following conditions holds:

(e) C(aj, a_2, a_3) = C(a, 2b, a) having size aj + a_2 + a_3 - 2 = 2a + 2b - 2,

(f) C(a_{1}} a_2, a_3) = C(a, a + b, b) having size ^ + a2 + a_3 - 2 = 2a + 2b - 2.

But, then

(e') a_x = a_3 (= a) and a2 = 2b (that is, d^ is an even integer at least 4),

(f) ~ ^{a}i $^{+}$ ^{a}3-

Therefore, at least one of the conditions (v) and (vi) fails — contrary to hypothesis. These two cases are exhaustive and the result follows. •

Necessary and sufficient conditions for the caterpillar C(aj, a_2,..., a^) to be prime-connected appear complicated to obtain for large n; however, we do state such a result (without proof) for n = 4.

Proposition 2.9 The caterpillar C(a_1, a_2, a_3, a_4) is prime-connected if and only if the following conditions hold:

(i) gcd(aj, a2 - 1, a_3 - 2, a_4) = 1,

(ii) gcd(aj, a2 - 2, a_3 - 1, a_4) = 1,

(iii) gcd(a_x - 1, a_2, a_3 - 1, a_4 - 1) = 1,

(iv) gcd(a_1-1, a_2-1, a_3, a_4-1) = 1,

(v) aj & a_3 or a_2 * a_4,

(vi) aj * a_4 or a2^a_3.

respectively. Thus, if Gg LCM(W_5, K_5 - e), then q(G) = 72k for some integer k> 1. Let Gj be a copy of W_5 with E ^) = {$e_{1\}}$ e_2,... , e_8} and denote the end-vertices of each ej by r(ej) and sCej). Let G be the graph obtained from G_2 by adding, for each edge ej of Gj, the vertices uj, vj, and wj, joining each of these new vertices to each of r(ej) and s(ej), and Vj to Uj and w_v Then q(G) = 72 and G is \V5-dec0mp0sable into nine copies of W5, namely Gj and ({r(ej), s(ei), u_{-v} v^, ^{w}i)) - ^{e}i for each i (1 < i < 8). Also, G is (K5 - e)-decomposable into eight copies of K_5 - e, namely {{r(ej), s(ei), Uj, v_{-v} wj}) for each i (1 < i < 8). Thus, G e LCM(W5, K5 - e). However, G is not 3-connected since {r(ej), s(ej)} is a cut-set of G for each i.

CHAPTER IV

SIZES OF GREATEST COMMON DIVISORS AND LEAST COMMON MULTIPLES OF SPECIFIED GRAPHS

In this chapter we study the sizes of greatest common divisors and least common multiples for several classes of graphs. In particular, we determine the size of the greatest common divisor and least common multiple of any path and K_3, and of any path and K_4. A lower bound for the size of a least common multiple of a path and a complete graph of any order is established.

The greatest common divisor index is introduced in this chapter. This parameter is determined for any collection of stars and stripes, for paths P_n $(2 < n < 5)$, for all complete graphs, and for the cycle C_4, for example.

In [CHKS] much interest was shown in the sizes of greatest common divisors and least common multiples of graphs. For graphs G_1 and G_2, the size of a greatest common divisor of G_1 and G_2 is denoted by $\mathrm{gcd}(G_1, G_2)$ and the size of a least common multiple by $\mathrm{lcm}(G_1, G_2)$.

4.1 Greatest Common Divisors and Least Common Multiples of Stars and Stripes

The sizes of a greatest common divisor and least common multiple of two matchings (stripes) or of two stars were found in [CHKS].

Theorem 4A For integers **m,** n > **1,**

(1) gcd (mK2, nK2) = gcd (m, n);

(2) 1cm (mK2, nK2) = 1cm (m, n);

(3) gcd (K(l, m), K(l, n)) = gcd (m, n); and

(4) 1cm (K(l, m), K(l, n)) = 1cm (m, n).

These results can be generalized to an arbitrary number of matchings and to an arbitrary number of stars as follows.

Proposition 4.1 For all positive integers m_{1s} m_2 , m_n (n > 2),

(1) gcdCmxK^ $m_2K_2 \ldots m_nK_2$) = gcd(m_{1t} $m_{2>}$ m_n);

(2) lcm(mjK_{2i} $m_2K_2 \ldots m_nK_2$) = lcm(m_{1t} $m_{2>}$ m_n);

(3) gcd(K(l, mj), K(l, m_2), K(l, m_n)) = gcd($m_{1\}}m_2$, ...,m_n);

(4) lcm(K(l, $m_x)_t$ K(l, m_2), K(l, m_n)) = lcm($m_L m_{2i}$ ...$_t$ m_n).

Proof (1) For every i (i = 1, 2, ..., n), the graph m|K_2 is rK_2-decomposable, where r = gcd($m_{1\}}$ m_2, ..., m_n). Therefore,

$$gcd(m_1K_{2t}m_2K2 \ldots m_nK_2) > gcdfmj, m_2,\ldots, m_n).$$

From the definition of the greatest common divisor of graphs it follows that

$$gcd(mjK_{2)}m_2K_2 \ldots m_nK_2) < \mathbf{gcdCmj,}\ m_2,\ldots, \mathbf{m_n).}$$

Therefore, gcd(m_1K_2, $m_2K_{2>}$ m_nK_2) = gcd(m_{ls} m_2 , m_n) .

(2) For i = 1, 2, ..., n, the graph rK_2 is miK_2-decomposable, where r= lcm(mj, m_2, ..., m_n). Therefore,

$$lcm(m_1K_{2>}m_2K_2 \ldots m_nK_2) < lcm(m_{15}\ m_2,\quad m_n).$$

From the definition of the least common multiple of graphs it follows that

$$\operatorname{lcm}(m_1K_2, m_2K_2, \ldots, m_nK_2) \geq \operatorname{lcm}(m_1, m_2, \ldots, m_n).$$

Therefore, $\operatorname{lcm}(m_1K_2, m_2K_2, \ldots, m_nK_2) = \operatorname{lcm}(m_1, m_2, \ldots, m_n)$.

(3) For every i ($i = 1, 2, \ldots, n$), the graph $K(1, m_i)$ is $K(1, r)$-decomposable, where $r = \gcd(m_1, m_2, \ldots, m_n)$. Therefore,

$$\gcd(K(1, m_1), K(1, m_2), \ldots, K(1, m_n)) \geq \gcd(m_1, m_2, \ldots, m_n).$$

From the definition of the greatest common divisor of graphs it follows that

$$\gcd(K(1, m_1), K(1, m_2), \ldots, K(1, m_n)) \leq \gcd(m_1, m_2, \ldots, m_n),$$

and equality follows.

(4) For $i = 1, 2, \ldots, n$, the graph $K(1, r)$ is $K(1, m_i)$-decomposable, where $r = \operatorname{lcm}(m_1, m_2, \ldots, m_n)$. Therefore,

$$\operatorname{lcm}(K(1, m_1), K(1, m_2), \ldots, K(1, m_n)) \leq \operatorname{lcm}(m_1, m_2, \ldots, m_n),$$

and the desired result follows from the definition of the least common multiple of graphs. •

Proposition 3 of [CHKS] gives examples of graphs G_1 and G_2 such that $\gcd(G_1,G_2) = \gcd(q(G_1),q(G_2))$ and $\operatorname{lcm}(G_1, G_2) = \operatorname{lcm}(q(G_1), q(G_2))$. Therefore, Proposition 4.1 can be considered as a generalization of the aforementioned proposition.

For positive integers m and n, we have gcd(m, n)lcm(m, n) = mn. For several classes of graphs and G_2 we have gcd(Gj, G_2)lcm(Gi, G_2) = q(Gi)q(G2>, namely

(1) gcd(mK_2, nK_2)lcm($mK_{2>}$ nK_2) - mn,

(2) gcd(K(l, m), K(l, n))lcm(K(l, m), K(l, n)) = mn.

We now establish some related results.

Proposition 4.2 For positive integers m and n,

(1) gcd (mK_2, K(l, n)) = 1,
(2) lcm(mK_2, K(l, n)) = mn.

Proof (1) The divisors of mK_2 are rK_2, where r | m, and the divisors of K(l, n) are K(l, t), where tin. Therefore, the unique common divisor of mK_2 and K(l, n) is obtained when r = t = 1. Hence, GCD(mK_2, K(l, n)) = {K_2}, and gcd(mK_2, K(l, n)) = 1.

(2) The result is clear when at least one of m or n is 1. Therefore, we may assume that m and n are at least 2. Let G be a graph of smallest size that is both mK_2-decomposable and K(l, n)-decomposable. Suppose that G can be decomposed into r copies of mK_2 and into t copies of K(l, n). Since no copy of K(l, n) can contain more that one edge of mK_2, it is clear that r > n. Furthermore, no copy of mK_2 can contain more than one edge of K(l, n), so t > m. Therefore, G can be decomposed into at least n copies of mK_2 and into at least m copies of K(l, n). Hence, q(G) > mn. Now since mK(l, n) is both mK_2-decomposable and K(l, n)-decomposable, it follows that lcm(mK_2, K(l, n)) - mn. •

Next we generalize the first result of Proposition 4.2 by the next proposition.

Proposition 4.3 For positive integers n, ni, n_2. . . . n_t,

$$\gcd(nK_2, K(1,ni), K(1, n_2), \ldots. K(l, n_t)) = 1.$$

Proof Observe that for every positive integer s, the divisors of K(1, s) are the graphs K(1, r), for every r such that r | s. Furthermore, the divisors of nK_2 are of the form mK_2, where m | n. Therefore, K_2 is the only common divisor of all graphs nK_2, K(1,nj), K(1, n_2), ... K(1, n_t). Hence, gcd(nK_2, Ka**.nj), K(l,** n_2), ... **K(l,n_t))** = 1. •

4.2 Least Common Multiples of Paths and Complete Graphs

We determine gcd (P_n, K_3), 1cm (P_n, K_3), and 1cm (P_n, K_4) for all n > 2.

Proposition 4.4 For all integers n > 2, gcd (P_n, K_3) = 1.

Proof The only divisors of K_3 are K_2 and K_3. The divisors of P_n are P_m where (m - l)I(n - 1). However, K_3 is not a divisor of P_n, implying that K_2 is the only divisor of P_n and K_3 . Therefore, GCD (P_n, K_3) = (K_2) and gcd (P_n, K_3) = 1, for all integers n > 2. •

Next we present a useful proposition that gives a lower bound for the size of a least common multiple of paths versus complete graphs.

Proposition 4.5 For all integers n > 2 and p > 3,

(1) lcm(P_n, K_p) > (P) if n < p and n - 11 (j),

(2) lcm(P_n, K_D) > ML otherwise, where

$L = lcm(n - 1, (\S))$ and $M = max\{[(p - l)(n - 1)/L], [p(n - 1)/2L]\}$.

Proof The result is clear when $n < p$ and $n — 1$ {(JJJ. Now suppose $n > p$ or $n - 1$ does not divide (j).

(i) Let $q = mL$, where $L = lcm(n - 1, (_2))$ and m is an integer such that $m < ["(p - l)(n - 1)/L|$. Suppose G is a connected graph of size q and order at least n such that G is K_p-decomposable. Then G contains a vertex v of degree at least $2(p-l)$ but at most $p - 2$ paths P_n. Hence, G is not P_n-decomposable, for v lies on at least $p - 1$ paths in any decomposition of G into paths.

(ii) Suppose $q = mL$, where m is an integer such that $m <$ |-p(n - 1)/2L]. Let G be a connected graph of size mL that is K_p-decomposable. We show that G has at most $n - 1$ vertices: Note that G contains $s = mL/$^0 edge-disjoint copies of Kp. The maximum number r of vertices of G occur when G consists of s copies Hj, H_2, ... , H_s of Kp where for each $i = 1, 2, ... , s - 1$, Hj has one vertex in common with and with no other Hj, j & i + 1. Hence, $r = p + (s - l)(p - 1) = s(p-l) + 1 = (2mL/p) + 1 < n$. Thus, G has at most $n - 1$ vertices. Therefore, P_n is not a subgraph of G, and therefore, G is not P_n-decomposable. •

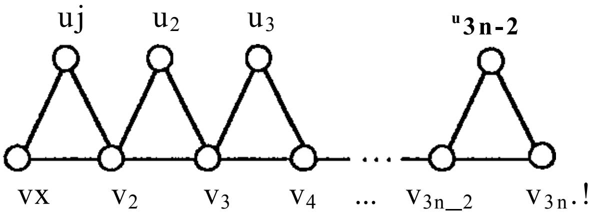
Theorem 4.6 For all integers $m > 2$,

(1) $lcm(P_m, K_3) = 3(m - 1)$ for $m = 0$ or $2(mod3)$,

(2) $lcm(P_m, K_3) - 2(m \quad 1)$ for $m = l(mod 3)$.

Proof (1) Assume that $m = 2(mod 3)$, where $m > 2$. Let $m = 3n - l$ for some integer $n>1$. Since $q(P3_n-i) = 3n - 2$ and the integers $3n - 2$ and 3 are relatively prime, it follows that $lcm(P_{3n\_1}, K_3) > lcm(3n - 2, 3) = 3(3n - 2)$.

Next we show that the graph Gj of Figure 4.1 is both P_{3n-1}-decomposable and K3-decomposable. Observe that Gj is K_3-decomposable into 3n-2 copies of K3 having vertices Vj, Uj, and for all i (1 < i < 3n - 2). Moreover, Gj is $P_{3n\_}$ 1-decomposable into 3 copies of $P_{3n_}$i. Consider the path $v_{1?}$ v_2,... , $v_{3n_}$j. Then let r = [(3n- 1)/2*|. When n is even, consider the paths $v_{1\}}$ u_{1f} v_2, u_2,..., v_r and v_r, u_r, v_{r+1}, u_{r+1},..., $v_{3n_}$i. When n is odd, consider the paths vj, uj, v_2, u_2, ... , u_r and u_r, v_{r+1}, u_{r+1}, ... , v_{3n_1}.

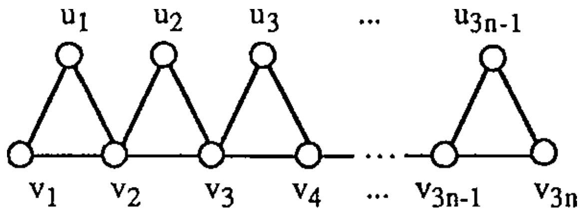


Gi

Figure 4.1 A P_{3n_1}-decomposable and K_3-decomposable graph

Next, assume that m = 0(mod 3), where m > 2. Let m - 3n for some positive integer n. In this case, $q(P3_n)$ = 3n - 1. Then lcm(P_{3n}, K_3) > lcm(3n - 1 , 3) = 3(3n - 1). Now we consider the graph G_2 of Figure 4.2, that is both P_{3n}-decomposable and K_3-decomposable into (3n - 1) copies of K_3 having vertices v^, uj, and vj_{+1} for all i (1 < i < 3n — 1). Moreover, G_2 is P_{3n}-decomposable into 3 copies of P_{3n}. Let r = [3n/2]. When n is even, consider the paths vj, Uj, v_2, u_2, ..., v_r, u_r and u_r, v_{r+1}, u_{r+1}, v_{r+2}, u_{r+2}... , v_{3n} and Vj, v_2, ... , v_{3n}. When n is odd, consider the paths v_2, $u_{1?}$ v_2, u_2,... , $u_{r_}$j, v_r; v_r, u_r, v_{r+1},

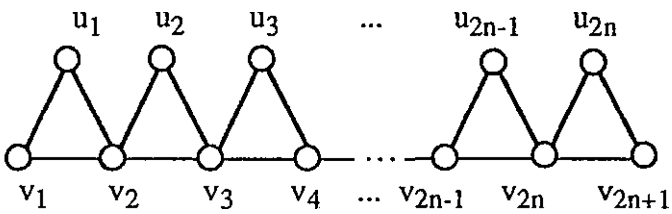
ur+l> - > v3m and vl> v2> - > v3n-



G_2

Figure 4.2 A $P3_n$-decomposable and K_3-decomposable graph

(2) Assume that $m = l(\text{mod } 3)$, where $m > 2$. Let $m = 3n + 1$ for some positive integer n. Since $P3_n+i$ is not K_3-decomposable, it follows that a least common multiple of $P3_{n+}i$ and K_3 has at least $2(3n) = 6n$ edges. Next, we show that the graph G_3 of Figure 4.3 is both P_{3n+1}-decomposable and K_3-decomposable.



g_3

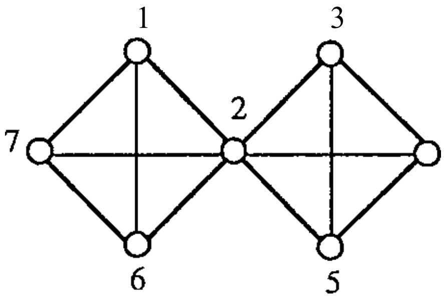
Figure 4.3 A $P_{3n+}j$-decomposable and K_3-decomposable graph

Observe that G_3 is K_3-decomposable into $2n$ copies of K_3 having vertices Vj, Up $Vi_{+}i$ for all i ($1 < i < 2n$). Moreover, G_3 is P_{3n+1}-decomposable into two copies of $P_{3n+}j$, one of which is the path $v_{1\}}$ uj, v_2, u_2,..., v_{n+1}, v_{n+2},... , v_{2n+1} and the other path is obtained from removal of the edges of the aforementioned path.
•

For n = 2, 3, 4, the graph K4 is P_n-decomposable, implying that lcm(P_n, K4) = 6. We now determine lcm(P_n, K_4) for all integers n (> 5) in the following results.

Proposition 4.7 lcm(P_5, K_4) = 12.

Proof Since q(P5) = 4 and q(K_4) = 6, it follows that lcm(P_5, K_4) > 1cm(4, 6) = 12. Furthermore, lcm(P_5, K_4) < 12, since the graph G of Figure 4.4 is K_4-decomposable and P5-decomposable, into the following three p5's in G:
(i) 3-5-2-1-6; (ii) 5-4-2-7-1; (iii) 4-3-2-6-7. •



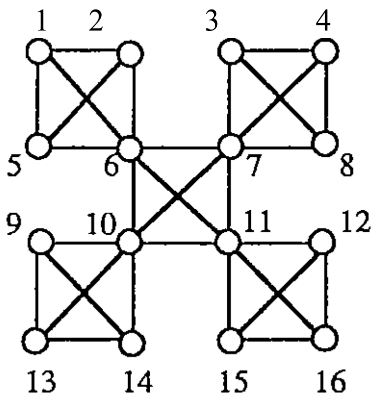
G

Figure 4.4 A P5-decomposable and K^-decomposable graph

We now present an easy proof that lcm(Pg, K4) - 30; a more general result can be found in Theorem 4.17.

Proposition 4.8 lcmtPg, K^) = 30.

Proof Since q(P_6) = 5 and q ^) = 6, it follows that 1cm(P_6,1^) > lcm(5, 6) = 30. Furthermore, lcm(P^, K_4) < 30, since the graph G of Figure 4.5 is both In-decomposable and Pg-decomposable, into the following six P6's in G:



G

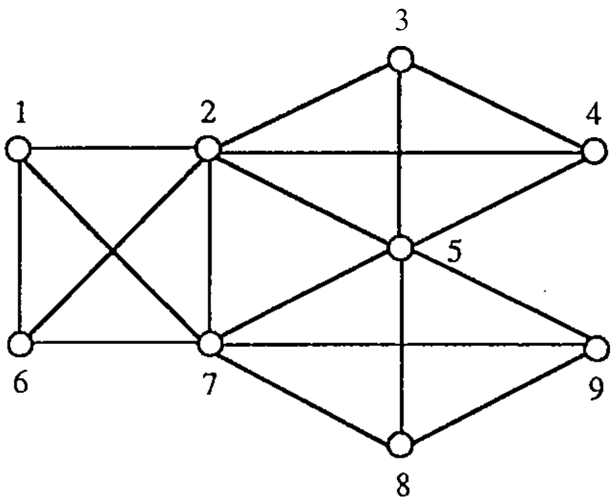
Figure 4.5 A Pg-decomposable and K_4-decomposable graph

2-1-6-7-3-4, 1-5-6-11-16-15, 5-2-6-10-14-13, 8-4-7-10-13-9, 14-9-10-11-12-16, and 3-8-7-11-15-12. •

Observe that the minimum number of paths needed to partition E(G) is equal to half the number of odd vertices of G. This provides a lower bound on the number of paths required, that is, an upper bound on the number of odd vertices graphs which are candidates for $LCM(P_n, K_p)$, for $n > 2$ and $p > 3$, can have.

Proposition 4.9 $lcm(P_7, K_4) = 18$.

Proof Since q(P7) = 6 and $q(K_4) = 6$, it follows from Proposition 4.5 that $lcm(P7, K_4) > 18$. Futhermore, $lcm(P7, K_4) < 18$, since the graph G of Figure 4.6 is both ^-decomposable and P7-decomposable, into the following three P^s in G:



G

Figure 4.6 A p7-decomposable and ^-decomposable graph

6-7-1-2-5-3-4, 3-2-4-5-7-9-8, and 1-6-2-7-8-5-9. •

Proposition 4.10 $lcm(P_8, K_4) = 42$.

Proof Since $q(P_8) = 7$ and $q(K_4) = 6$, it follows that $lcm(P_8, K_4) > lcm(7, 6) =$ 42. Furthermore, lcm(Pg, K_4) < 42, since the graph G of Figure 4.7 is both K_4-decomposable and Pg-decomposable, into the following six Pg's in G:

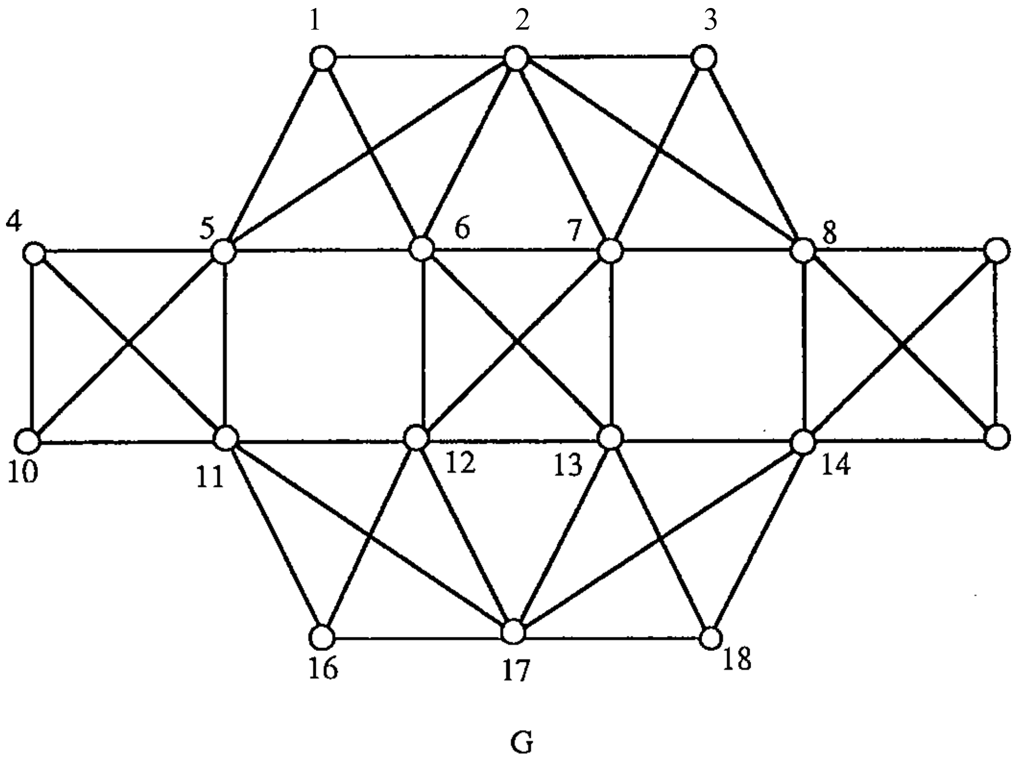


Figure 4.7 A Pg-decomposable and ^-decomposable graph

1-5-2-3-8-9-14-15, 3-7-2-6-5-11-4-10, 2-.8-7-6-13-12-16-17,

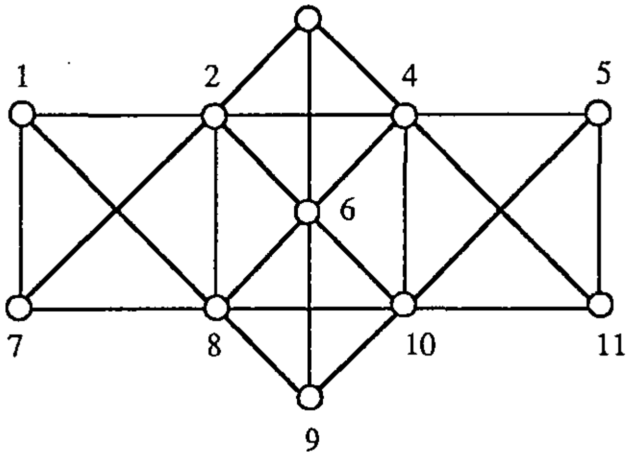
4-5-10-11-12-17-13-18, 9-15-8-14-18-17-11-16, and 2-1-6-12-7-13-14-17. •

(Also see Theorem 4.17 for a more general result.)

Proposition 4.11 lcm(P_9, K_4) = 24.

Proof Since q(P_9) = 8 and q(K_4) = 6, it follows that lcm(P_9, K_4) > lcm(8, 6) = 24. Furthermore, lcm(p9, K_4) < 24, since the graph G of Figure 4.8 is both K_4-decomposable and P_9-decomposable, into the following three copies of P_9 in G:

1 2



G

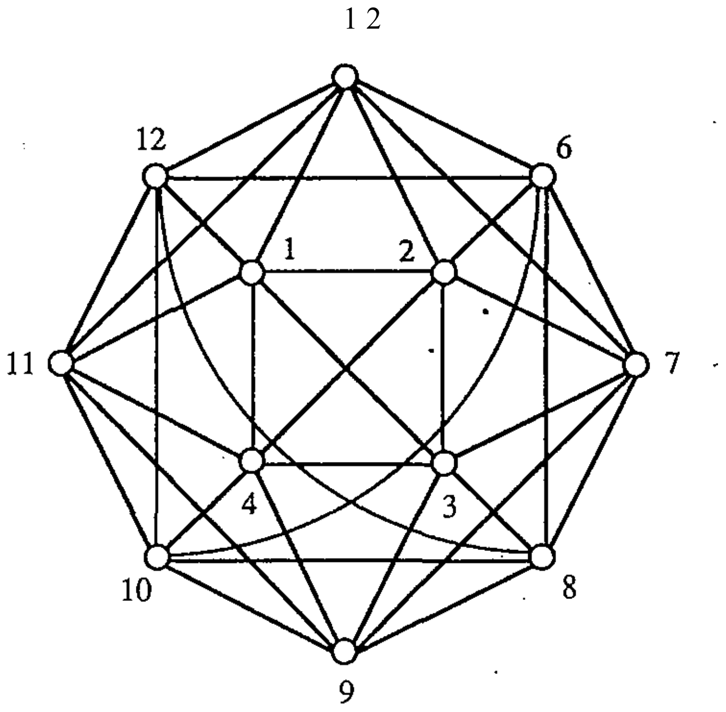
Figure 4.8 A Pp-decomposable and K_4-decomposable graph

1-7-2-8-9-10-6-4-5, 3-4-11-10-8-1-2-6-9, and 7-8-6-3-2-4-10-5-11. •

Proposition 4.12 **lcmCPjQ,** K_4) =36.

Proof Since q(Pio) = 9 and q(K_4). = 6, it follows by Proposition 4.5 that lcm(P_{10}, K_4) > 36.

Observe that lcm(Pio, K_4) < 36, since the graph G of Figure 4.9 is K_4-decomposable and PjQ-decomposable, into the following four copies of Pjq in G:



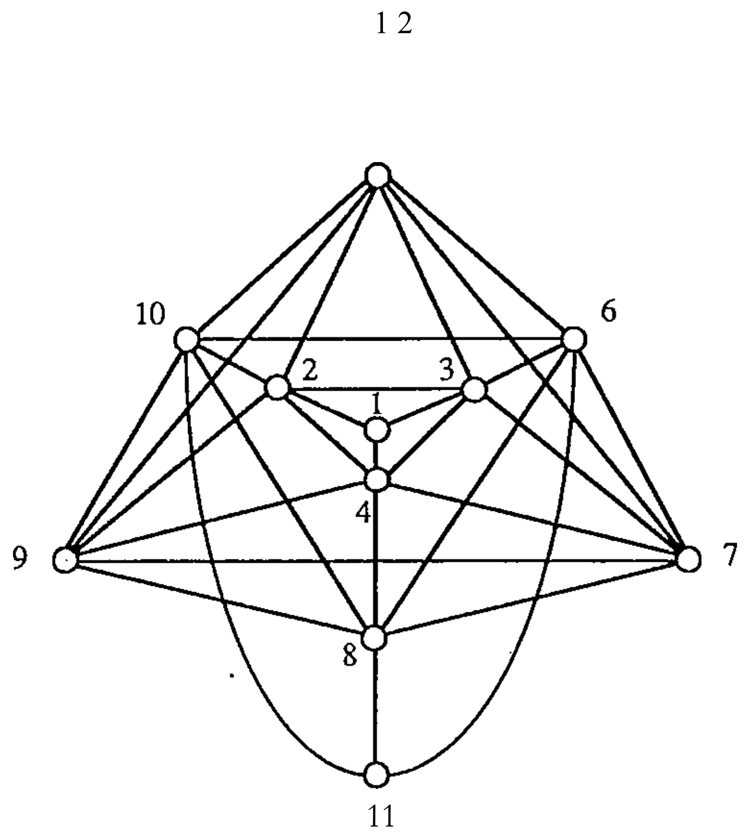
G

Figure 4.9 A Pig-decomposable and ^-decomposable graph

9-7-8-10-12-11-5-1-4-3, 9-10-6-12-5-2-4$_T$1 1-1-3, 5-7-6-8-12-1-2-3-9-11, and 5-6-2-7-3-8-9-4-10-11. •

Proposition 4.13 lcm(P_n, K_4) = 30.

Proof Since q(Pn) = 10 and **q(K_4)** = 6, it follows that lcm(P_n, K_4) > lcm(10, 6) = 30. Furthermore, lcm(P_{11}, K_4) < 30, since the graph of Figure 4.10 is both K_4-decomposable and P^-decomposable, into the following three copies of P_n in G:



G

Figure 4.10 A P^-decomposable and ^-decomposable graph

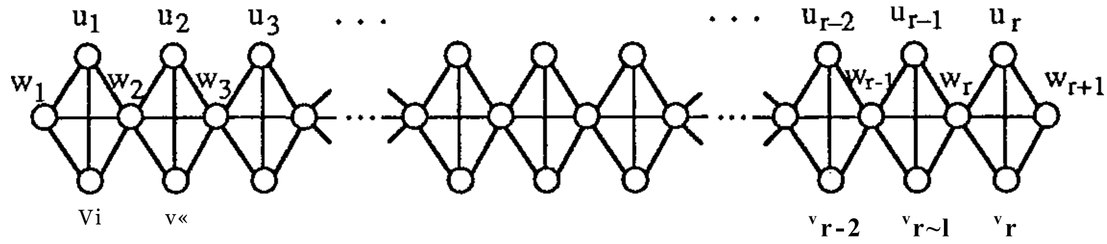
1-3-4-2-10-5-7-9-8-6-11, 1-2-3-7-4-9-5-6-10-8-11, and 1-4-8-7-6-3-5-2-9-10-11.
•

We now obtain lcm(P_{n+}i, K4), where n is an even integer at least 12.

Theorem 4.14 lcm(P_{n+1}, K4) - 3n, where n (> 12) is an even integer.

Proof First, we show that lcm(P_{n+1}, K4) > 3n. Observe that a connected graph that is nontrivially ^-decomposable must have a vertex of degree at least 6. Therefore, a graph that is both P_{n+}j-decomposable and K^-decomposable is decomposable into at least 3 copies of P_{n+}i- This implies that icm(P_{n+}i, K4) > 3n, when n>12.

Next, we show that 1cm(P_{n+1}, K4) < 3n. We construct the graph Gj of Figure 4.11 that is obtained by identifying some of the vertices of n/2 copies of K4 as indicated, where r = n/2.

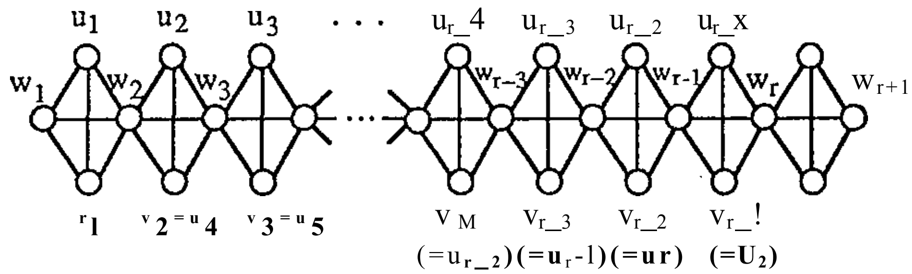


Gi

Figure 4.11 The graph used in the construction of a P_{n+}i-decomposable graph, where r = n/2 is even

Then consider the following cases.

Case 1 *Assume that* r *is even.* We construct the graph G of Figure 4.12 by identifying some of the vertices of Gi as indicated by the vertices of the same labels.



G

Figure 4.12 A P_{n+1}-decomposable and K_4-decomposable graph, where r is even

Observe that G is K_4-decomposable into r copies of K4 having vertices Uj, W|, Vj, and wj_{+1} for each i (1 < i < r). Next, we show that G is P_{n+}}-decomposable into 3 copies of P_{n+1} as indicated in Figures 4.13 to 4.15.

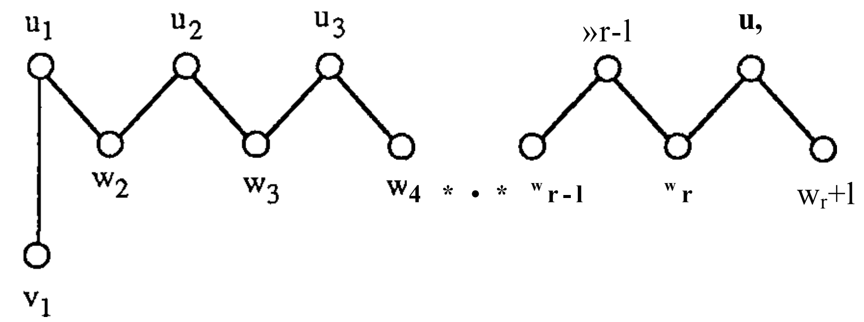


Figure 4.13 The first copy of P_{n+j} in a P_{n+1}-decomposition of G

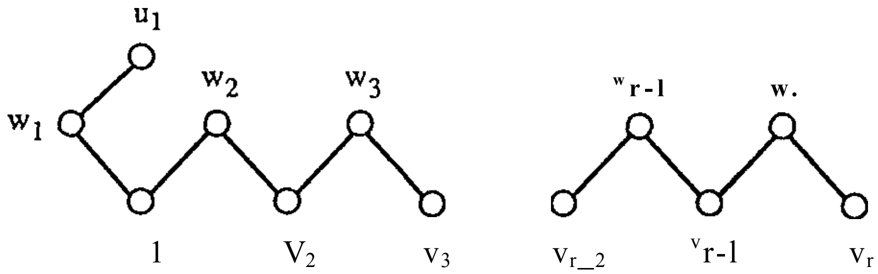


Figure 4.14 The second copy of P_{n+j} in a P_{n+j}-decomposition of G

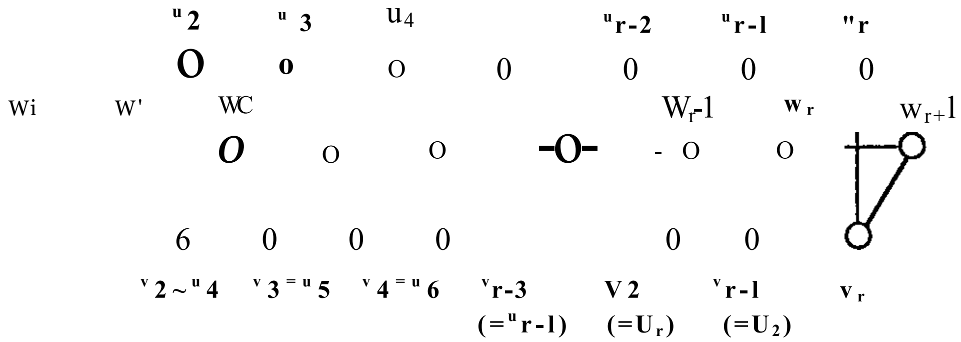
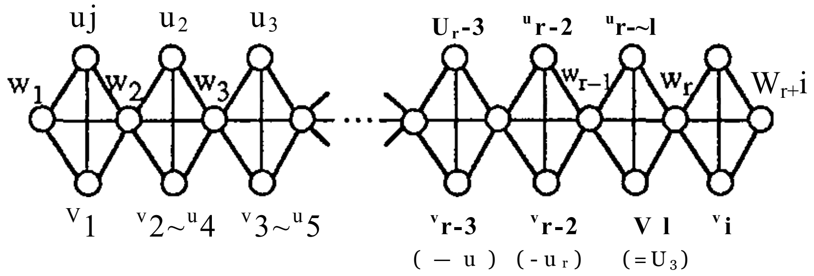


Figure 4.15 The third copy of P_{n+i} in a P_{n+1}-decomposition of G

Therefore, lcm(P_{n+1}, K4) < 3n, where n = 2r (> 12) and r is even.

Case 2 *Assume that* r *is odd.* We construct the graph H of Figure 4.16 by identifying some of the vertices of Gj as indicated by the vertices of the same labels.



H

Figure 4.16 A P_{n+}i-decomposable and ^-decomposable graph, where r is odd

Trivially, H is K^-decomposable into r copies of K4 having vertices Uj, Wj, Vj, and Wi+i for each i (1 < i < r). Finally, we show that H is P_{n+}i-decomposable into 3 copies of P_{n+1} as indicated in Figures 4.17 to 4.19.

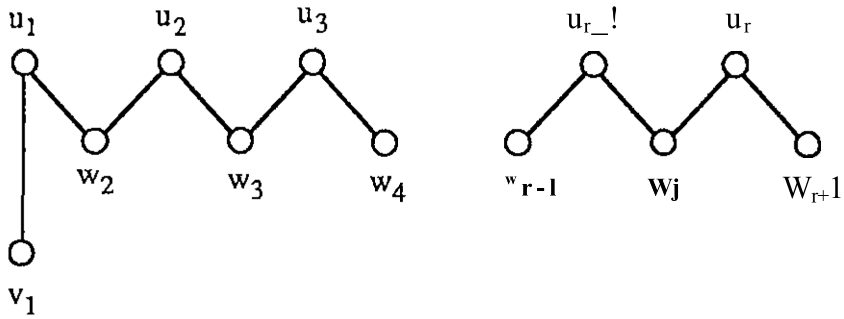


Figure 4.17 The first copy of P_{n+1} in a P_{n+1}-decomposition of H

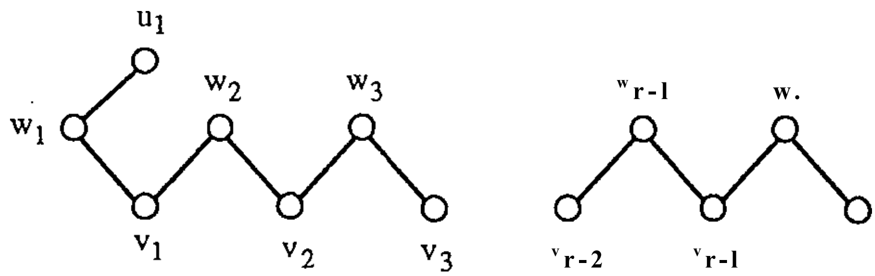


Figure 4.18 The second copy of P_{n+}j in a P_{n+}i-decomposition of H

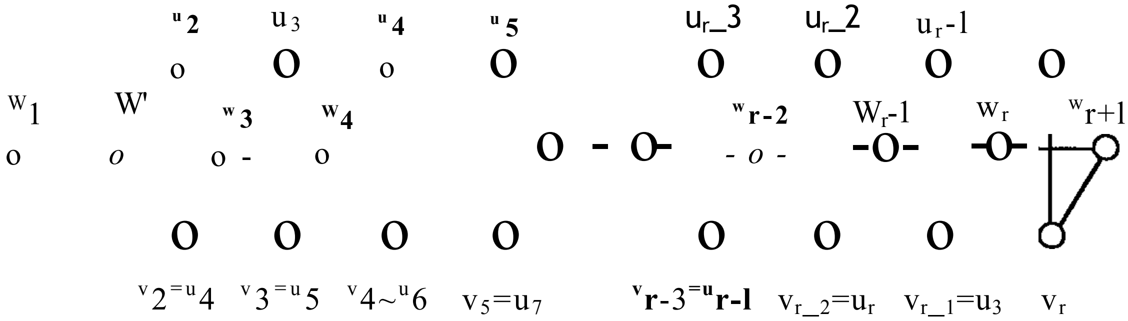


Figure 4.19 The third copy of P_{n+1} in a P_{n+1}-decomposition of H

Therefore, $lcm(P_{n+1}, K_4) < 3n$, where $n = 2r$ (> 12) and r is odd.

Hence, $lcm(P_{n+1}, K4) = 3n$, where n (> 12) is an even integer. •

Next, we obtain $lcm(P_{n+1}, K4)$, where n (> 11) is an odd integer that is not a multiple of 3.

Theorem 4.15 $lcm(P_{n+}i, K4) = 6n$, where n (> 11) is an odd integer that is not a multiple of 3.

Proof Observe that $lcm(P_{n+1}, K4) > lcm(q(P_{n+1}), q(K_4)) = lcm(n, 6) = 6n$. Next, we consider the graph Gi of Figure 4,20 that is obtained by identifying some of the vertices of n copies of K_4 as indicated in this figure, where $r = (n + 1)/2$.

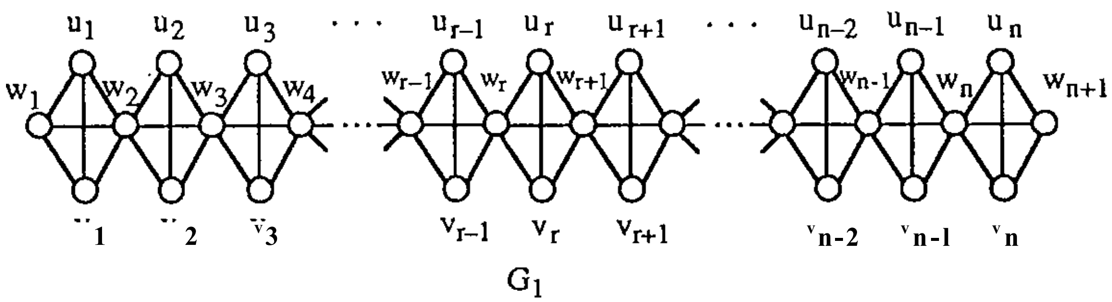


Figure 4.20 The graph used in the construction of a $P_{n+}j$-decomposable graph, where $r = (n + 1)/2$

Then we construct the graph G of Figure 4.21 by identifying some of the vertices of Gj as indicated by the vertices of the same labels.

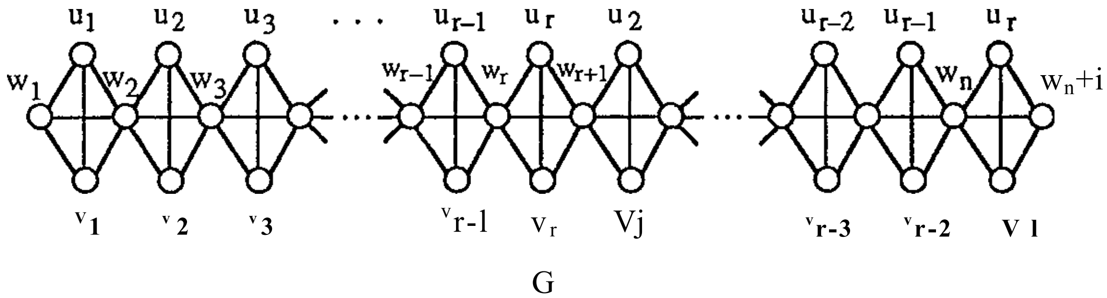


Figure 4.21 A P_{n+}j-decomposable and K^-decomposable graph,

where n is an odd integer that is not a multiple of 3

Note that G is ^-decomposable into n copies of K4. Next, we show that G is P_n+rdecomposable into 6 copies of P_{n+}i- Consider the following 6 copies of P_{n+1} as indicated in Figures 4.22 to 4.27.

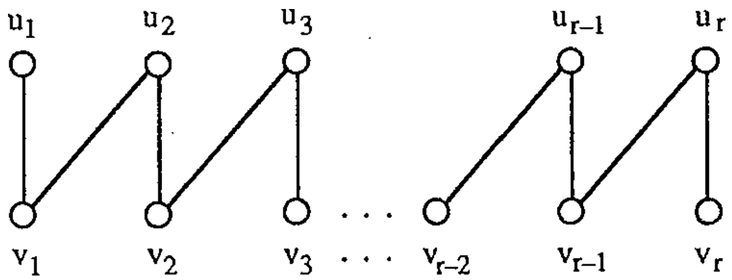


Figure 4.22 The first copy of P_{n+1} in a P_{n+1}-decomposition of G

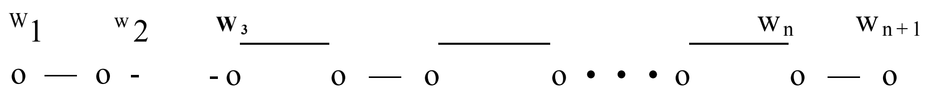


Figure 4.23 The second copy of P_{n+}i in a P_{n+}^decomposition of G

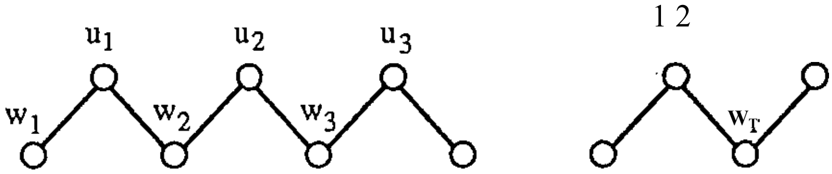


Figure 4.24 The third copy of P_{n+i} in a P_{n+1}-decomposition of G

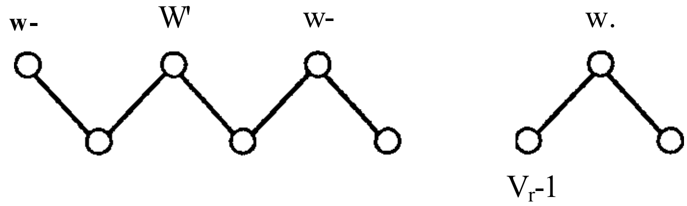


Figure 4.25 The fourth copy of P_{n+j} in a P_{n+j}-decomposition of G

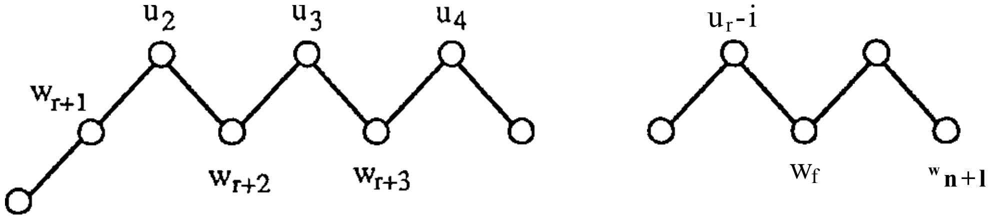


Figure 4.26 The fifth copy of P_{n+1} in a P_{n+1}-decomposition of G

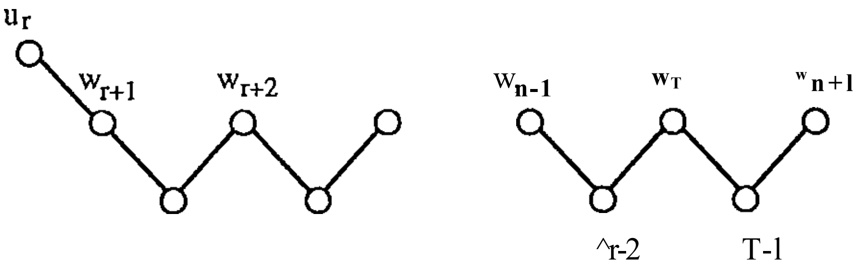


Figure 4.27 The sixth copy of P_{n+i} in a P_{n+1}-decomposition of G

Therefore, $\text{lcm}(P_{n+1}, K4) < 6n$, where n (> 5) is an odd integer that is not a multiple of 3.

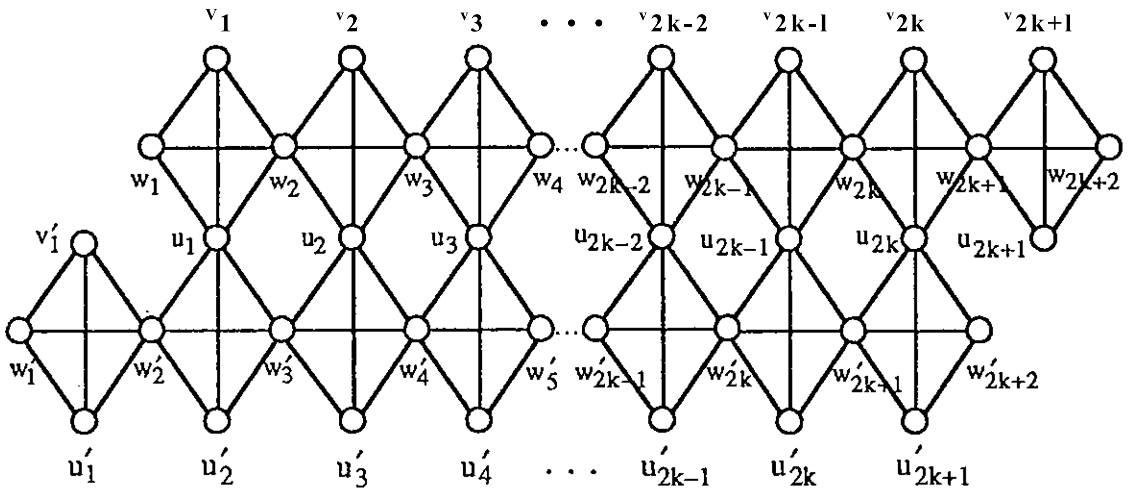
Hence, lcm(P_{n+i}, K_4) = 6n, where n (> 5) is an odd integer that is not a multiple of 3. •

When n is an odd multiple of 3, we have the following result.

Theorem 4.16 lcm(P_{n+i}, K_4) = 4n, where n (> 9) is an odd integer that is a multiple of 3.

Proof Let n = 3(2k +1), where k is a positive integer. Then, by Proposition 4.5, we have lcm(P_{n+1}, K4) > ML, where L = lcm(n, Q)) = lcm(3(2k + 1), 6) - 6(2k + 1). Moreover, M = max{|*3(3)(2k + 1)/6(2k + 1)l, [4(6k + 3)/12(2k + 1)l) = 2. Therefore, lcm(P_{n+1}, K_4) > 12(2k + 1) = 4n.

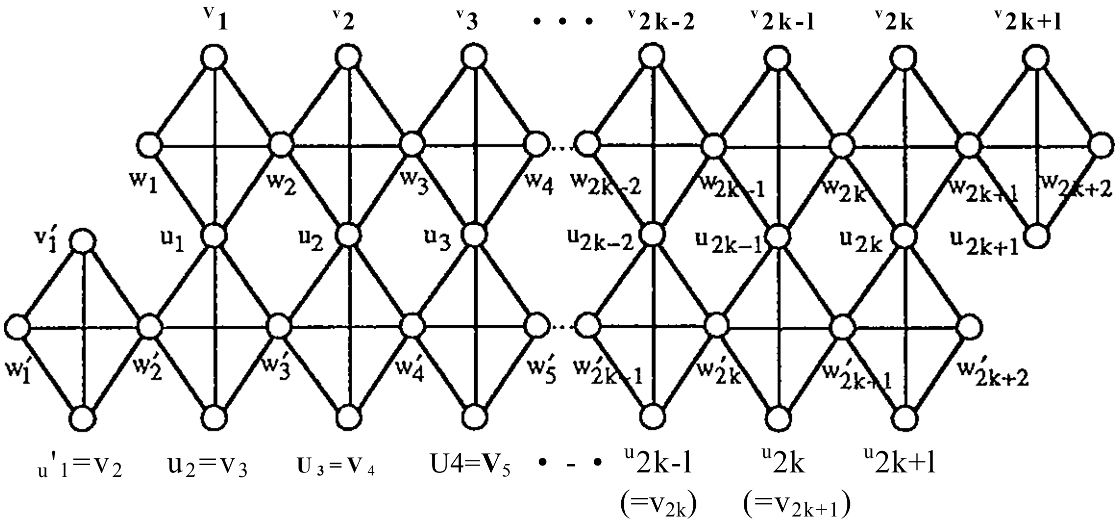
It remains to show that there exists a graph of size 4n that is both P_{n+i}~ decomposable and K_4-decomposable. We consider the graph Gj of Figure 4.28 that is obtained by identifying those pairs of vertices of 2n/3 = 2(2k + 1) copies of K_4 indicated in the figure.



Gl

Figure 4.28 The graph G_2 used in the construction of a P_{n+i}-decomposable graph, where n (> 9) is an odd multiple of 3

Now let G be the graph of Figure 4.29 obtained from Gj by identifying those pairs of vertices of Gj having the same labels.



G

Figure 4.29 A P_{n+}i-decomposable and K^-decomposable graph, where n (> 9) is an odd multiple of 3

Observe that this construction creates no multiple edges, so G is indeed a graph.

We show that G is P_{n+1}-decomposable into four copies of P_{n+}i- Note, firstly, that Gj can be decomposed into the four subgraphs shown in Figures 4.30 - 4.33.

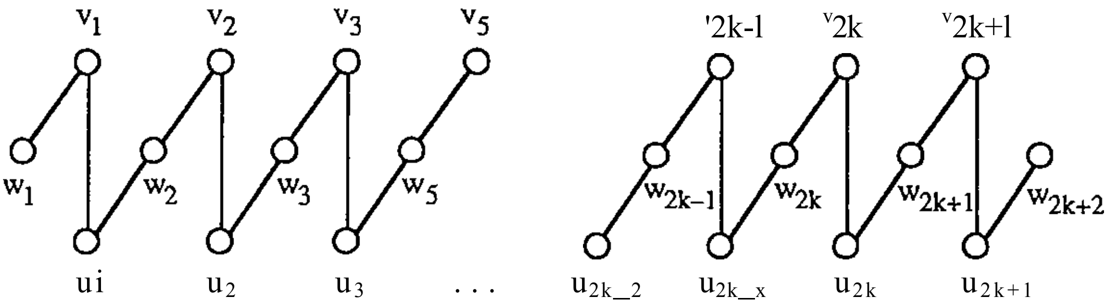


Figure 4.30 The first copy of P_{n+i} in a P_{n+}^-decomposition of G

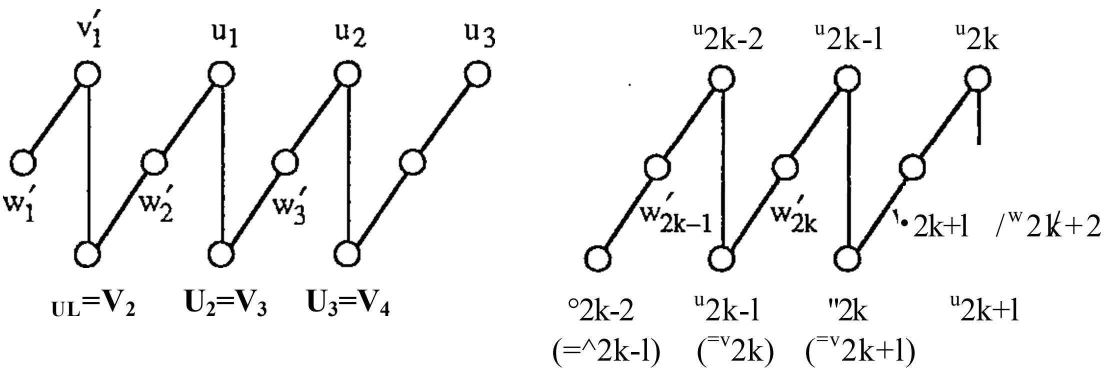


Figure 4.31 The second copy of P_{n+j} in a P_{n+i}-decomposition of G

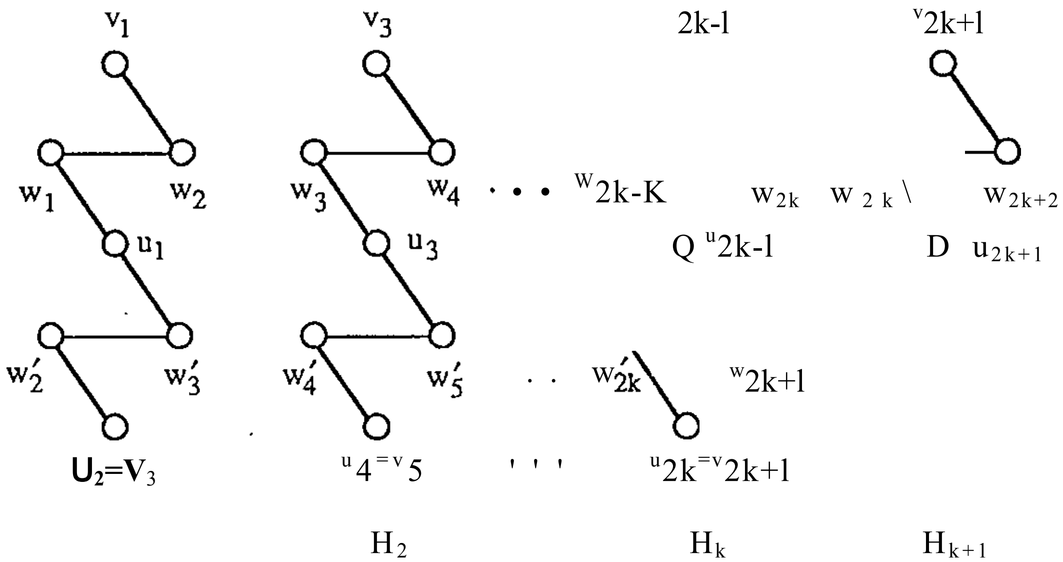


Figure 4.32 The third copy of P_{n+1} in a P_{n+1}-decomposition of G

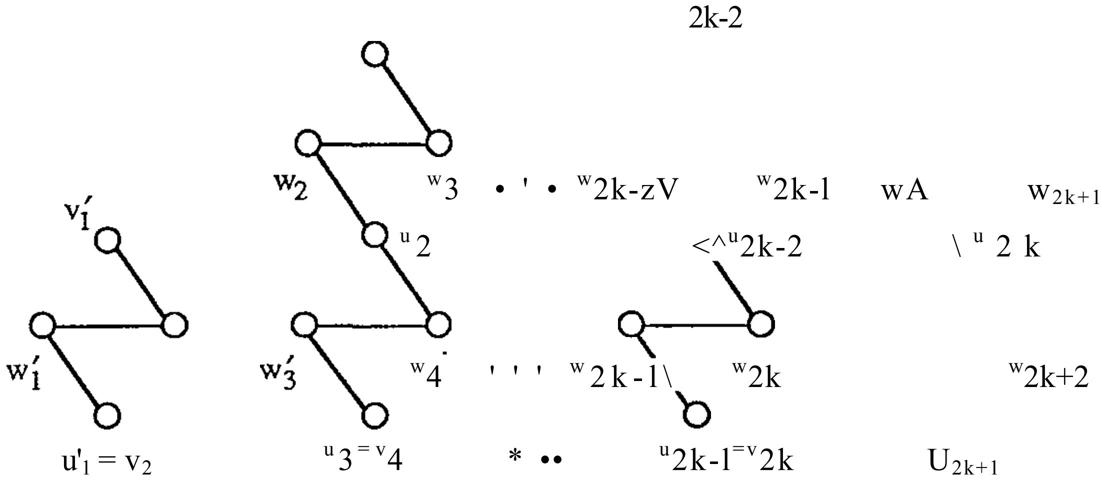


Figure 4.33 The fourth copy of P_{n+1} in a P_{n+1}-decomposition of G

Observe that the two paths shown in Figures 4.30 and 4.31 are paths in G as well as in Gj. Moreover, because of the manner in which the vertices are identified to produce G from G_{1s} the unions of paths shown in Figures 4.32 and 4.33 are, in fact, paths in G. For example, the path of G produced by the union of the paths Hj, H_2, ... , $H_{k+}i$ in Figure 4.32 is obtained by.successively taking the vi ~v_3 path Hi followed by the v_3 - v_5 path H_2, etc., finally concluding with the v_{2k+1} - u_{2k+1} path H_{k+1}.

Since G is obviously K_4-decomposable and since q(G) = 4n < lcm(P_{n+1}, K_4), it follows that G e LCM(P_{n+1}, K_4) and that lcm(P_{n+1}, K_4) = 4n. •

We summarize the previous three results in the next theorem.

Theorem 4.17 For each integer **m** > 2,

(1) lcm(P_m, K_4) = 6 for m = 2, 3, 4

(2) lcm(P_m, K_4) = 3(m - 1) for m = 1, 3, or 5 (mod 6)

(3) lcm(P_m, K_4) = 6(m - 1) for m = 0 or 2 (mod 6)

(4) $\operatorname{lcm}(\mathbf{P_m}, K_4) = 4(m \quad 1)$ for $m = 4 (\bmod 6)$, $m > 10$.

4.3 The Greatest Common Divisor Index of a Graph

For a graph G of size q, define the *greatest common divisor index* i(G) of G, or simply the *index* of G, as the greatest positive integer n for which there exist graphs Gi and G_2, both of size at least nq, such that GCD(Gj, G_2) = {G}. If no such n exists, then we define this index to be

We show that the index of stripes (that is, disjoint copies of K_2) is infinite.

Proposition 4.18 For every integer n (> 1)

$$i(nK_2) = \infty.$$

Proof Let G = niC_2 and suppose, to the contrary, that i(G) = t is finite. Now let m (> t) be an integer and pi and p_2 be distinct primes so that pjn and p_2n are at least m. Then for graphs G_2 = PinK_2 and $G_2 = p_2nK_2$, having size p^n and p_2n respectively, it is clear that GCD(Gj, G_2) = {G}. Therefore, i(G) > m (> t), contrary to the hypothesis. Hence, $i(nK_2) = \infty$. •

Next we show that the index of stars is infinite as well.

Proposition 4.19 For every integer n (> 1),

$$i(K(1, n)) = \infty$$

Proof Let G = K(l, n) and suppose, to the contrary, that i(K(l, n)) = t is finite. Let m(>t) be an integer and pj and p_2 be distinct primes so that p_2n and p_2n .are at least m. Then for graphs Gj = K(l, pjii) and G_2 = K(l, p_2n), having size p_xn and p_2n respectively, it is clear that GCD(G_X, G_2) = {G}. Therefore, i(G) > m (> t)—contrary to the hypothesis. Hence, i(K(l, n)) = 00. •

We combine the previous results in the next proposition.

Proposition 4.20 For all integers a,b (> 1),

$$i(aK_2 \text{ u } K(l, b)) =$$

Proof Let G =• (aK_2 u K(l, b)) and suppose, to the contrary, that i(aK_2 uK(l,b)) = t is finite. Let m (> t) be an integer and pj and p_2 be distinct primes so that p_xa + p^b and p_2a + p_2b are at least m. The graphs G_t = (pjaK_2 u K(l, pjb)) and G_2 = (p_2aK_2 u K(l, p_2b)) provide the desired contradiction. •

We next present a result for the index of an arbitrary number of stars.

Proposition 4.21 For all positive integers nj, n_2, n_m,

$$i(K(l, nj) \text{ u } K(l, n_2) \text{ u } \ldots \text{ u } K(l, n_m)) = -$$

Proof Let G = K(l, nj) u K(l, n_2) u ... K(l, n_m) and suppose, to the contrary, that i(G) = t is finite. Let m (> t) be an integer and $p_{1\}}$ p_2 > m be distinct primes. In this case, the graphs Gi =K(1, Pinj) u K(l, $p_x n_2$) u ... u K(l, pin_m) and G_2s K(l, $p_2$nj) u K(l, $p_2 n_2$) u ... u K(l, $p_2 n_m$) satisfy GCDCGj, G_2) = {G}, again a contradiction. •

We now generalize Propositions 4.20 and 4.21 by finding the index of stars and stripes.

Proposition 4.22 For all positive integers a, m, and n,

$$i(aK_2 \text{ u } K(l, n\{) \text{ u } K(l, n_2) \text{ u } \ldots \text{ u } K(l, n_m)) =$$

Proof The result follows as before by considering the integer m > t, distinct primes Pl and p_2 such that $p_2(a + n_2 + n_2 + ... + n_m)$ and $p_2(a + nj + n_2 + ... + n_m)$ are at least m, and the graphs $Gj = PiaK_2$ u K(l, pjnj) u K(l, p ^) u ... u $K(l, pin_m)$ and G_2 **S** p_2aK_2 **U** $K(l, p_2n_x)$ **U** $K(l, p_2n_2)$ u . . . u K(**1,** p_2n_m). •

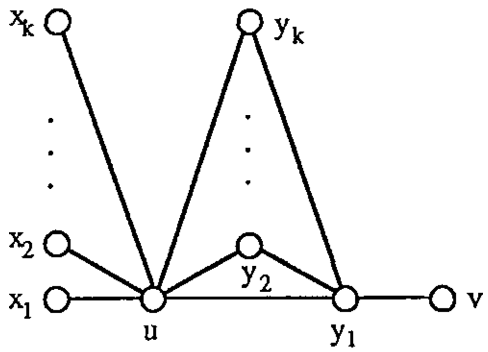
Now we present the index of paths of size 1, 2, 3, and 4.

Proposition 4.23 For n = 2, 3, 4,

$$i(P_n) =$$

Proof Since i(K(l, m)) = <*•, for every integer m (> 1), it follows for m = 1 and m = 2 that $i(P_2)$ = <*> and $i(P_3)$ = respectively.

To show that $i(P_4)$ = <*>, suppose, to the contrary, that $i(P_4) = t$ is finite. Let m (> t) be an integer and pj and p_2 be distinct primes, each of which is at least m. Now for the graphs Gj = p^ and G_2 described in Figure 4.34, having $k = p_2$,



G_2

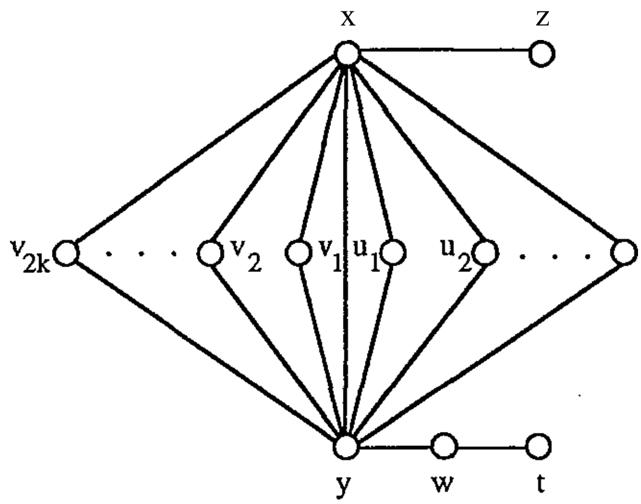
Figure 4.34 The greatest common divisor of Gj and G_2 is P4

we will show that GCD(G_{1t} G_2) — {P4}- We observe that gcd(3p_{1t} 3p_2) = 3, since pi and p_2 are distinct primes. For the graph Gj the divisors of size 3 are P4, P3 u K_2, and $3K_2$. However, by Lemma 2.1, the graph G_2 is not (P3 u K_2)-decomposable, since the edge uy^ is adjacent to all other edges of G_2. Similarly, G_2 is not $3K_2$-decomposable. Observe that G_2 is P4-decomposable into paths $x\text{-}_v$ u, y_{i5} y_l for all i (2 < i < k) and finally the path xj, u, yj, v. Therefore, the path P_4 is the only divisor of size 3 for the graph G_2. Hence, GCD(Gj, *Gq*) = {P_4} implying that i(P_4) > m (> t)—contrary to the hypothesis. Therefore, i(P_4) = •

We determine the index of P5 in the next result.

Proposition 4.24 **i(P5> =**

Proof Suppose, to the contrary, that KP5) = a, where a e N. Let m (> a) be an integer, and let pj and p_2 be distinct primes, where pj > p_2 > m and p_2 = 2k + 1 for some positive integer k. Let G = P1P5 and let G_2 be the graph of Figure 4.35, where the vertices of G_2 are labeled as indicated.



G_2

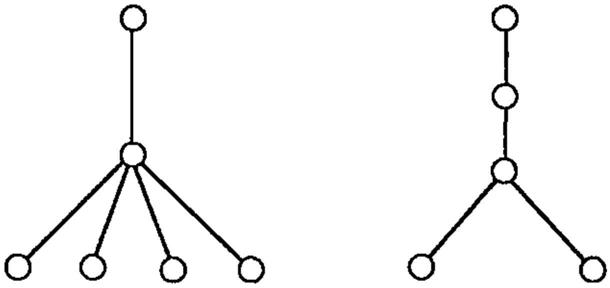
Figure 4.35 The greatest common divisor of Gi and G_2 is P5

We show that $GCD(G_{1t}\ G_2) = (P_5)$: Observe that **gcd(G_{lf}** G_2) < gcd(4pj, $4p_2$) = 4. The graph G_2 is P5-decomposable into p_2 paths, namely UpX, Vj, y, u_{i+1} for i = 1, 2 , 2 k - 1 together with the path u_{2k}, x, v_{2k}, y, m and z, x, y, w, t. The graph Gj is P_5-decomposable. Hence, {P5) c $GCD(G_1, G_2)$. Observe that for the graph Gi the divisors of size 4 are P_5, P_4 u K_2, $2P_3$, P_3 u $2K_2$, and $4K_2$. Every edge of G_2 different from xy and wt is incident with x or y, so Pi(G_2 - wt) = 2. Therefore, G_2 is not G-decomposable, for G *e* {P4 u K_2, P_3 u $2K_2$, $4K_2$}. Also, G_2 is not 2P3-decomposable, for otherwise the edge xy is an edge of P3 in some copy H of 2P3, but no other disjoint copy of P3 in H exists, producing a contradiction. Hence, G_2 is not 2P3~decomposable. Hence, GCD(Gj, G_2) = {P_5} and i(P_5) > m (> a), contrary to hypothesis. •

The index of a path P_n for n > 6 is not known and it appears to be difficult to obtain.

Next, we present another class of graphs and we obtain the index of some special cases of such graphs.

The *broom* B(n, k) for which each of the integers n and k is at least 2 is constructed by identification of the central vertex of the star K(l, n) and an end-vertex of the path P_k. Figure 4.36 shows B(4, 2) and B(2, 3).



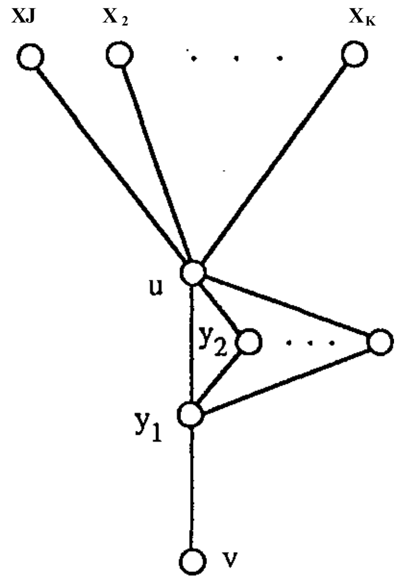
B(4,2) = K(1,5)   B(2, 3)

Figure 4.36 The brooms B(4, 2) and B(2, 3)

Proposition 4.25 i(B(n, 3)) =

Proof We suppose, to the contrary, that i(B(n, 3)) = t is finite. Let m (> t) be an integer and pi, p_2 ^ m be distinct primes. Now for graphs Gj s piB(n, 3) and G_2 described in Figure 4.37, where k = p_2n, we show that GCD(Gx, G_2) = {B(n, 3)}.

Since pi and p_2 are distinct primes, it follows that gcd(pj(n + 2), p_2(n + 2)) = n + 2. Certainly, Gj is B(n, 3)-decomposable. Also, G_2 is B(n, 3)-decomposable, which can be seen by selecting copies of B(n, 3) with vertices xj, x_2, x_n, u, yj, v and p_2 - 1 other copies of B(n, 3) having vertices xJ$_{N+1}$, xI$_{N+2}$, *in+n> u> yi+l» yi. for all i (1 < i < P2 - 1). Thus B(n, 3) e GCD(G!, G_2) and



G_2

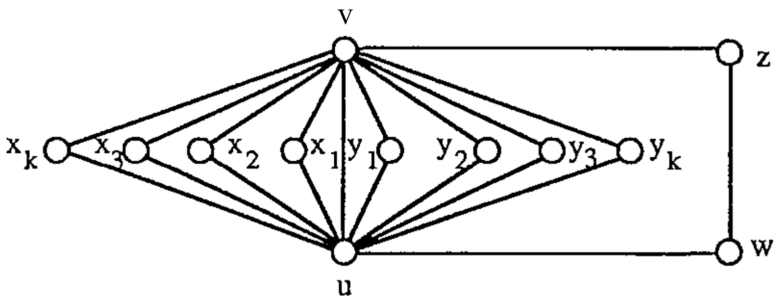
Figure 4.37 The greatest common divisor of Gj and G_2 is B(n, 3)

gcd(Gj, G_2) = n + 2. Let H be any greatest common divisor of Gj and G_2 ; so q(H) = n + 2. The edge uyj in the graph G_2 is adjacent to all other edges, implying, by Lemma 2.1, that H is connected. So H must be a subgraph of each component of Gi- However, each component of Gj is isomorphic to B(n, 3) and so has size n + 2. Thus H= B(n, 3), implying that GCD(G_b G_2) = {B(n, 3)). Therefore, i(B(n, 3)) > m (> t). Hence i(B(n, 3)) = •

Next, we find the index of the cycle C4.

Proposition 4.26 i(C4) = <*>.

Proof Suppose, to the contrary, that i(C4) = t is finite. Let m (> t) be an integer and pi and p_2 primes with p_2 > pj > m. Let Gj s PiC_4 and let G_2 be the graph of Figure 4.38, where the vertices of G_2 are labeled as indicated with k = p_2 - 1. In other words, G_2 is obtained by identifying a vertex of degree 2k in K(2, 2k) with the vertex u of the cycle C: v, u, w, z, v and identifying the other vertex of degree 2k with vertex v of C.



G_2

Figure 4.38 The greatest common divisor of and G_2 is C4

Then gcdCG^ G_2) ^ gcd(4$p_{1>}$ 4p_2) = 4. We show that GCD(G_{15} G_2) = {C4}. For i = 1, 2 , k , define Hj to be the 4-cycle u, Xj, v, y^ u and let H_{k+1} be the 4-cycle u, v, z, w, u. Then we see that G_2 is decomposable into the 4-cycles Hj (1 < i < k + 1). Now since G_x is C_4-decomposable, it follows that C_4 e GCD(Gx, G_2). Observe that graphs C_4, P_4 u K_2, 2P_3, P_3 u 2$K_{2>}$ and 4K_2 are the divisors of size 4 for the graph Gj. Every edge of G_2 different from zw is incident with u or v; so (G_2 - zw) = 2. Notice that the only edge of G_2 not adjacent to uv is zw, that is, uv does not belong to an independent set of three edges. Therefore, G_2 is not G-decomposable, for every G e {4K_2, (P_3 u 2K_2), P4 u K_2}. Since any edge adjacent to zw is also adjacent to uv, there can be no copy of P_3 disjoint from a copy of P_3 containing zw, that is, G_2 is not 2P_3-decomposable. Hence, GCD(Gi, G_2) = {C_4} and i(C_4) > m (> t), contrary to the hypothesis. Therefore, i(C_4) = 00. •

The index of a cycle C_m, for m > 5, is not known.

Every class of graphs we have considered thus far has been shown to have infinite index. This is not always the case, since the complete graph K_n $(n > 3)$ has index equal to 1, a fact which follows directly from Lemma 2.13.

Proposition 4.27 For every integer n (> 3),

$$i(K_n) = 1.$$

In general, the problem of determining the greatest common divisor index of a graph appears to be difficult and it is unknown whether graphs G, with $1 < i(G) <$ exist.

CHAPTER V

ON GREATEST COMMON DIVISORS AND LEAST COMMON MULTIPLES OF DIGRAPHS

In this chapter we introduce the concepts of greatest common divisors and least common multiples for digraphs. It is proved that least common multiples of two directed stars exist. For several pairs of directed stars, the size of a least common multiple is determined. Finally, the greatest common divisor index of a digraph is introduced, and this parameter is found for several classes of digraphs, including directed stars and stripes, directed paths P^ $(2 < n < 5)$, directed cycles C3 and C4, and the complete symmetric digraph Kp, for all integers p (> 3).

5.1 Introduction

A digraph D is said to be *decomposable* into the subdigraphs Di, D 2 , D_n , $n > 1$, of D if no Di $(i = 1,2,\ldots,n)$ has isolated vertices and the arc set E(D) of D is partitioned into E(Di), E(D2), E(D_n). IfDi = H for each i $(1 < i < n)$, then D is said to be *H-decomposable,* and H is said to *divide* D and be a *divisor* of D. If H divides D, we write H | D. Of course, if a digraph D is H-decomposable, then q(H) I q(D). As with graphs, if H is a subdigraph of D without isolated vertices such that q(H) | q(D), then D need not be H-decomposable. For example, in the digraph D of Figure 5.1, Hi, H2 and H3 are all subdigraphs of D such that q(H^) | q(D) for

1 = 1,2,3. While D is Hi-decomposable, D is neither H_2-decomposable nor In-decomposable.

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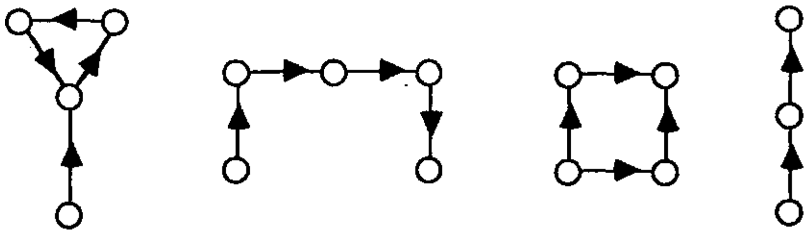
D Hi H_2 H_3

Figure 5.1 Digraphs having q(Hi) | q(D) for i = 1, 2, 3 so that only Hj divides D

For positive integers m and n, let t(m, n) be the digraph whose vertex set can be partitioned into sets Vi and V_2, where | Vj I = m and | $V_2$1 = n so that every vertex of Vj is adjacent to every vertex of V2. Observe that every nonempty digraph is l)-decomposable, where 1t(1, 1) is the unique connected digraph of order 2 and size 1.

A digraph D without isolated vertices is called a *greatest common divisor* of two digraphs Di and D2 if D is a digraph of maximum size such that both Di and D2 are D-decomposable. If Dj and D2 are nonempty digraphs, then they are both Indecomposable. Hence there exists some digraph D of maximum size such that Di and D2 are D-decomposable. Consequently, every two nonempty digraphs have a greatest common divisor. For the digraphs Di and D2 of Figure 5.2, Hi is the unique greatest common divisor of Di and D_2, while Hi and H2 are the greatest common divisors of D2 and D3.



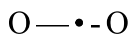
Dl D_2 D_3 HI H_2

Figure 5.2 Digraphs for which GCDCD^ D_2) = {H_x}

and GCD{D_2, D_3} = {H_x, H_2}

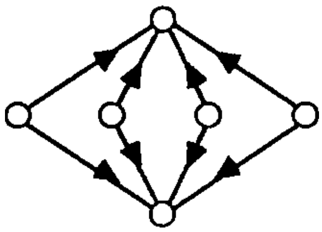
A greatest common divisor of a set D - {D^, D_2,..., D_n}, n > 2 of digraphs is defined similarly, and it follows, as before, that every set of two or more nonempty digraphs has a greatest common divisor.

A digraph H without isolated vertices is called a *least common multiple* of two digraphs Di and D_2 if H is a digraph of minimum size such that it is both Di-decomposable and D_2-decomposable. For the digraphs Dj and D_2 of Figure 5.3, Hi, H_2, H3, H4, and H5 are the least common multiples of Di and D_2.

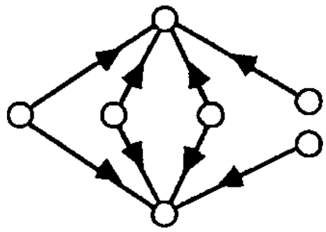


Dl

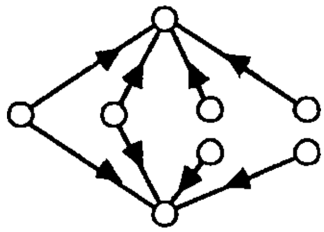
D2



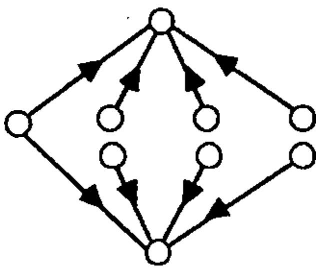
HI



H_2



H_3



H4

H_5

Figure 5.3 The least common multiples of and D_2 are **HJ, H_2,...,** H5

Note that Wilson's result (Theorem IN), which is used to prove the corresponding existence result for graphs, does not hold for digraphs and that no similar result is known for digraphs. Therefore, whether every two nonempty digraphs Di and D2 have a least common multiple is unknown.

For digraphs Di and D2, we denote by gcd (Di, D2) the size of a greatest common divisor of Di and D_2 and by lcm (Di, D_2) the size of a least common multiple of Di and D_2 (if it exists). It is clear that gcd (Di, D_2) < gcd(q(Di), q(p2)) and lcm (Di, D2) ^ lcm (q(Di), q(D2)). There are some digraphs Dj and D2 for which equality holds in both cases. For example, when Dj s (the directed path of length m - 1) and $D_2 = P_n$, we have

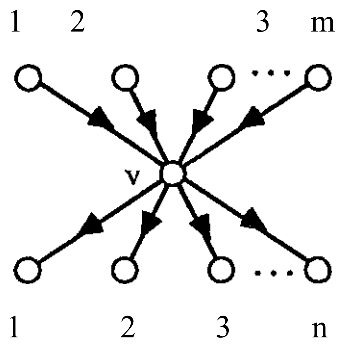
(i) gcd(?$_m$, ?$_n$) = gcd(q(?$_m$), q (?„)) = gcd(m - 1, n - 1) and

(ii) lcrn(?$_m$, t$_n$) = lcm(q(?$_m$), q (?„)) = lcm(m - 1, n - 1).

The set of all greatest common divisors of two digraphs DI and D_2 is denoted by GCD (DI, D_2). Similarly, the set of all least common multiples of two digraphs DI and D_2 is denoted by LCM(DI,D_2). We define GCD (DI, D_2, ..., D_N), LCM (DI, D_2, D_N), gcd *(D$_h$* D_2,..., D_N), and lcm (D_B D_2, D_N), in the expected manner.

5.2 Least Common Multiples of Directed Stars

A *directed star* D(m, n), for nonnegative integers m and n, is a digraph obtained by joining m vertices to a vertex and joining this vertex to n new vertices. A vertex of D(m, n) with indegree r and outdegree s is called an *(r, s) vertex.* Thus, D(m, n) is a digraph having one vertex with indegree m and outdegree n, an (m, n) vertex, m vertices having indegree 0 and outdegree 1, the (0,1) vertices and n vertices having indegree 1 and outdegree 0, the (1, 0) vertices (see Figure 5.4). In Figure 5.4 the vertex v is an (m, n) vertex. Therefore, the digraph 1^(1, 1) is the directed star D(0, 1) or, equivalently, the directed star D(l% 0) with one (0, 1) vertex and one (1,0) vertex.



D(m, n)

Figure 5.4 The digraph D(m, n)

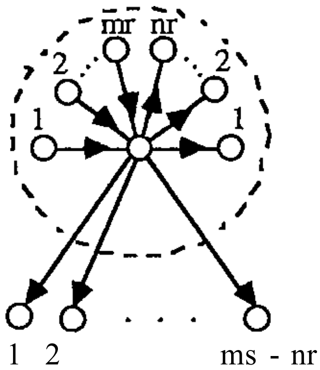
Next, we show that least common multiples of D(m, n) and D(r, s) exist for all positive integers m, n, r, and s.

Theorem 5.1 For all positive integers m, n, r, and s, LCM(D(m, n), D(r, s)) is nonempty.

Proof It suffices to verify the existence of a digraph D that is both D(m, n)-decomposable and D(r, s)-decomposable, implying that LCM(P(m, n), D(r, s)) is nonempty and that lcm(D(m, n), D(r, s)) < I E(D) |. We suppose, without loss of generality, that m > r.

Case 1 *Assume that* ms = nr. Let D = D(mr, ms). Thus, D is decomposable into m copies of D(r, s). By hypothesis D(mr, ms) = D(mr, nr), so that D is decomposable into r copies of D(m, n). Therefore, D is both D(r, s)-decomposable and D(m, n)-decomposable. In this case lcm(D(m, n), D(r, s)) < m(r + s).

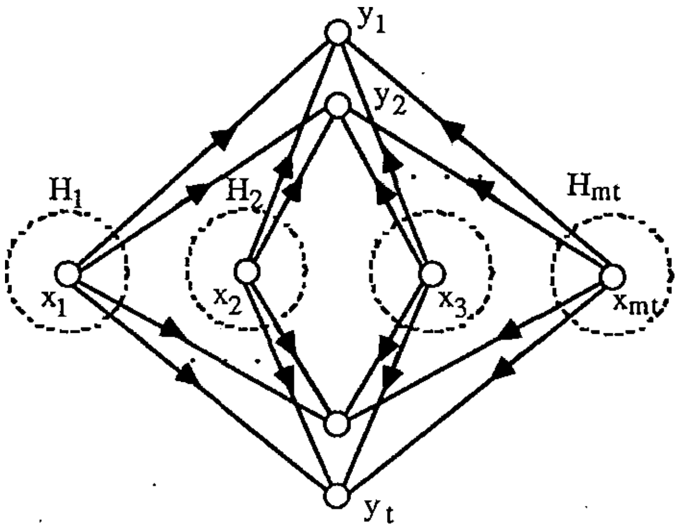
Case 2 *Assume that* ms > nr. First, let H = D(mr, ms) = D(mr, nr + ms - nr). If ms - nr (1,0) vertices and their corresponding arcs are removed from H, then the resulting digraph is D(mr, nr), which is D(m, n)-decomposable into r copies of D(m, n). Therefore, we may present and label the vertices of H as indicated in Figure 5.5, where the encircled part represents a copy of D(mr, nr) whose (mr, nr) vertex is joined to ms - nr vertices outside of the encircled area.



H = D(mr, ms) = D(mr, nr + ms - nr)

Figure 5.5 The digraph H is D(r, s)-decomposable into m copies of D(r, s)

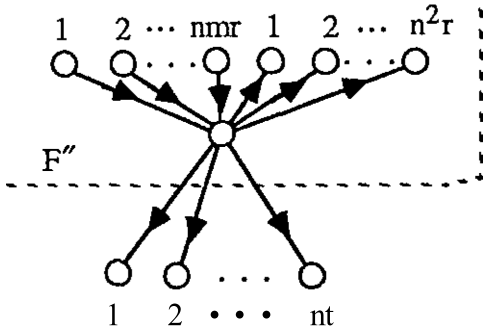
Next, let t = ms - nr and consider mt disjoint digraphs Hj = H for 1 < i < mt. Label the (mr, ms) vertex of Hj by .xj and t of the (1,0) vertices of Hj by Yil> yj2, Yit f°r each i (1 < i < mt). Now we consider the digraph H' of Figure 5.6 obtained by identifying, for every k (1 < k < t), the mt vertices y^, i = 1, 2,..., mt, and denoting the resulting vertex by y^. Observe that y_k is an (mt, 0) vertex for all k (1 < k < t). This completes the construction of H'.



H'

Figure 5.6 Digraph H' used in the construction of D

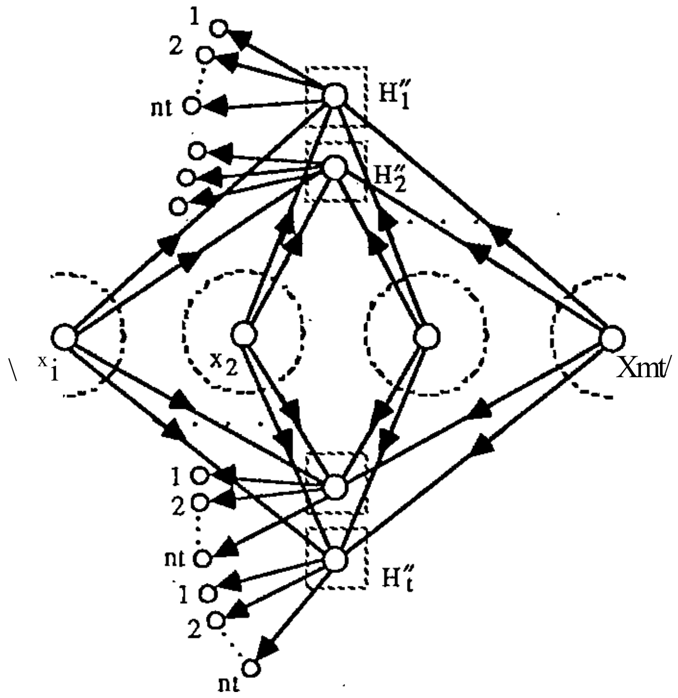
Now consider the digraph $H'' \cong D(nmr, nms)$ of Figure 5.7, and let $H''_1, H''_2, \dots, H''_t$ be t copies of H''. Moreover, let $F'' = D(nmr, n^2r)$ and $F''_k \cong F''$ for every k $(1 \le k \le t)$, where we consider F''_k to be a subdigraph of H''_k for each $1 \le k \le t$. Observe that F'' is D(m, n)-decomposable into nr copies of D(m, n).



$H'' = D(nmr, nms) = D(nmr, n^2r + nt)$

Figure 5.7 A digraph that is D(r, s)-decomposable into nm copies of D(r, s)

Finally, we construct the digraph D (see Figure 5.8) by identifying for each k (1 < k < t) the vertex y_k of H' and the unique vertex of maximum degree of H_k. (The digraph D for the case m = s = 2, n = r= 1 is also illustrated in Figure 5.9.)



D

Figure 5.8 A digraph that is D(m, n)-decomposable and D(r, s)-decomposable

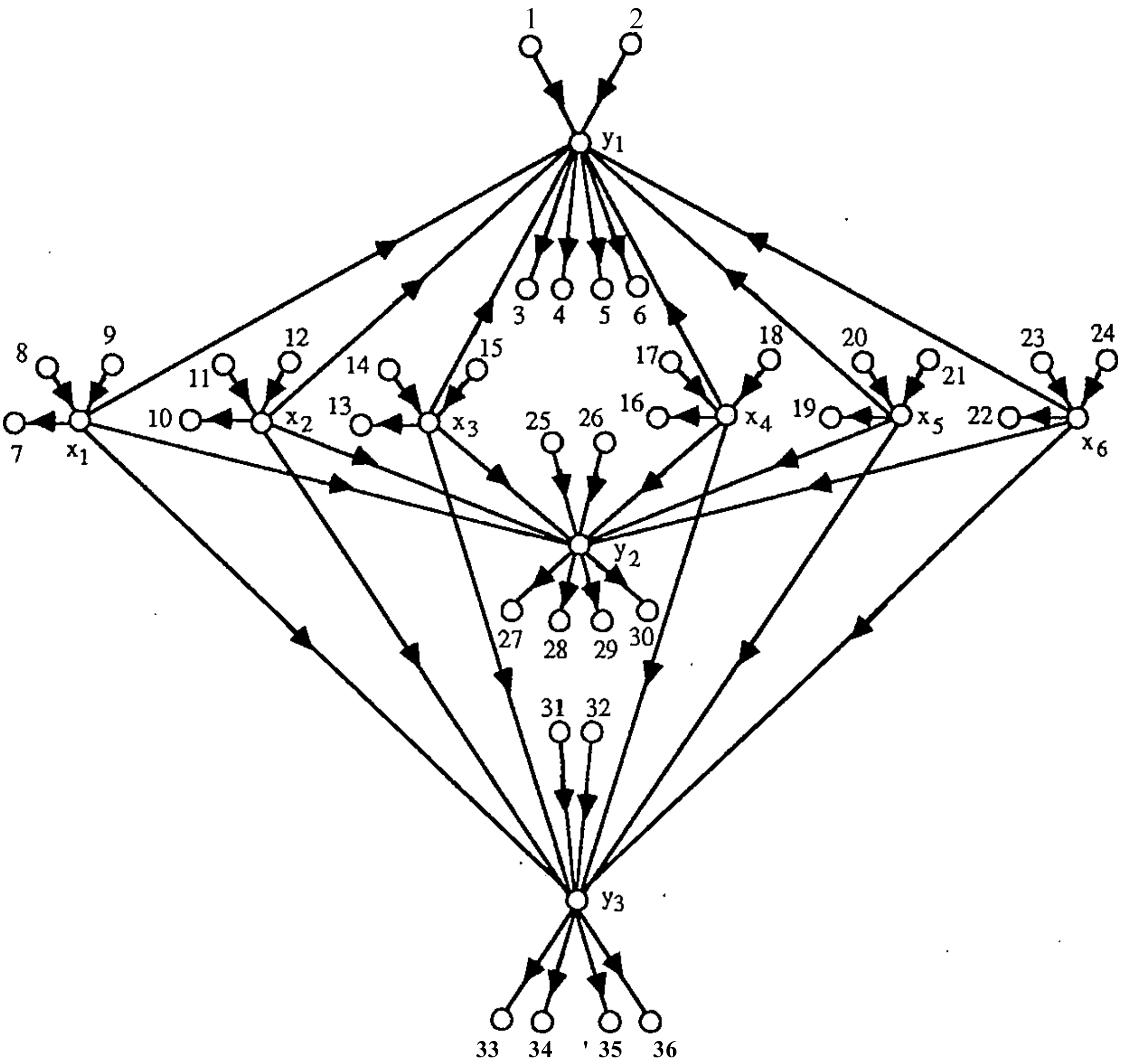
By construction, D is D(m, n)-decomposable with r copies of D(m, n) centered at each vertex Xj (1 < i < mt) and with t + nr copies of D(m, n) centered at each vertex y_k (1 < k < t). D is D(r, s)-decomposable with m copies of D(r, s) centered at each vertex x_i (1 < i < mt) and with mn copies of D(r, s) centered at each vertex y_k (1 < k < t). The size of D is m(m + n)(r + s)(ms - nr), implying that lcm(D(m, n), D(r, s)) < m(m + n)(r + s)(ms - nr).

Case 3 *Assume that* $ms < nr$. Construct a digraph H by identifying the (r, s) vertices of n copies of $D(r, s)$. Therefore, $H = D(nr, ns) = D(nr - ms + ms, ns)$. If $nr - ms$ $(0, 1)$ vertices and the corresponding arcs are removed from H, then the resulting digraph is $D(ms, ns)$ which is $D(m, n)$-decomposable into s copies of $D(m, n)$. Now we follow the technique we used in Case 2 to construct a digraph which is both $D(m, n)$-decomposable and $D(r, s)$-decomposable. •

The above theorem provides an upper bound for the size of a least common multiple of two directed stars. However, the size of a least common multiple of two directed stars can be relatively small.

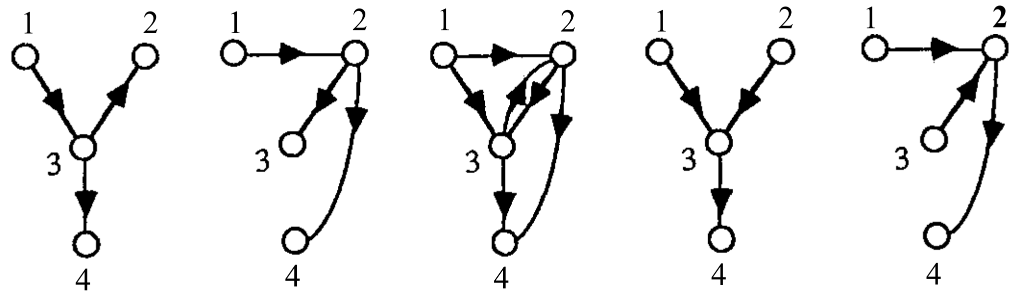
For example, by Theorem 5.1, $\text{lcm}(D(2, 1), D(1, 2)) < 54$. (see Figure 5.9.)

The digraph D' of Figure 5.10 is both $D(2, 1)$-decomposable and $D(1, 2)$-decomposable as indicated. Of course, D' is a digraph of smallest size with this property, implying that $\text{lcm}(D(2, 1), D(1, 2)) = 6$.



D

Figure 5.9 A digraph that is D(2, 1)-decomposable and D(1, 2)-decomposable, having 54 arcs



D(1, 2) D(1, 2) D(2, 1) D(2, 1)

Figure 5.10 A smallest digraph that is D(2, l)-decomposable and D(1, 2)-decomposable

The *converse* D of a digraph D is that digraph with V(D) = V(D) such that (u, v) e E(D) if and only if (v, u) e E(D). In addition to the converse of a digraph, one can also refer to the converse of a concept dealing with digraphs. More specifically, the converse of a concept is the concept that results when the original concept is applied to the converse of a digraph. For example, "adjacent from" is the converse of "adjacent to", "incident from" is the converse of "incident to", and "indegree" is the converse of "outdegree". An elementary, but often useful, observation is the folowing.

Principle of Directional Duality For each theorem concerning digraphs, there is a corresponding theorem obtained by replacing each concept in the theorem by its converse concept.

We illustrate the above ideas with the following result.

Proposition 5.2 For all integers m, n (> 1),

(1) gcd(D(m, 0), D(n, 0)) = gcd(m, n),

(2) gcd(D(0, m), D(0, n)) = gcd(m, n),

(3) lcm(D(m, 0), D(n, 0)) = lcm(m, n),

(4) lcm(D(0, m), D(0, n)) = lcm(m, n).

Proof (1) Observe that a common divisor of D(m, 0) and D(n, 0) is of the form D(k, 0), where k is a common divisor of m and n. Let k* = gcd(m, n). Since D(k*, 0) is a common divisor of D(m, 0) and D(n, 0), it follows that gcd(D(m, 0), D(n, 0)) = k* = gcd(m, n).

(2) This result follows by the Principle of Directional Duality.

(3) Any common multiple of D(m, 0) and D(n, 0) is of the form D(j, 0), where j is a common multiple of m and n. Let j* = lcm (m, n). Since D(j*,0) is a common multiple of D(m, 0) and D(n, 0), we have lcm(D(m, 0), D(n, 0)) = j* = lcm(m, n).

Equality (4) follows by the Principle of Directional Duality. •

The former results can be generalized as follows — the proofs are similar to those above and are omitted.

Proposition 5.3 For all positive integers $m_{1?}$ m2,... , m_n, with n > 2,

(1) gcd(D(mj, 0), D(m_2, 0), ... , D(m_n, 0)) = gcd($m_{1\}}$ m_2,..., m_n),

(2) gcd(D(0, mj), D(0, m_2),..., D(0, m_n)) = gcdCmj, m_2,..., m_n),

(3) lcmCDCm!, 0), D($m_{2>}$ 0),..., D(m_n, 0)) = lcmCmx, m_2,... , m_n),

(4) lcm(D(0, mj), D(0, m_2),..., D(0, m_n)) = IcmCm^ m_2,..., m_n).

Proposition 5.2 considers the stars D(m, *n{)* and D(m_2, n_2) in which mj = m_2 = 0 or nj = n_2 = 0. We now consider those stars for which mj = n_2 = 0 or m_2 = $n_{\}}$ =0.

Proposition 5.4 For all positive integers m and n,

(1) gcd(D(m,0),D(0,n)) = 1,

(2) lcm(D(m, 0), D(0, n)) = mn.

Proof (1) Since only a star is a common divisor of two stars, it follows that, 1) is the only divisor of both D(m, 0) and D(0, n), implying that gcd(D(m, 0), D(0, n)) = 1.

(2) Suppose that lcm(D(m, 0), D(0, n)) = k and that D is a digraph of size k that is both D(m, 0)-decomposable and D(0, n)-decomposable. Let F be a subdigraph isomorphic to D(m, 0) in D. Then every two arcs of F belong to distinct copies of D(n, n). Therefore, k > mn. The digraph n) has size mn and is both D(m, 0)-decomposable and D(0, n)-decomposable, implying that k < mn, and completing the proof. •

We conjecture that these results can be generalized as follows:

Conjecture 5.5 For positive integers mj, m_2,..., m_n and tj, t_2,..., t_k, with n > **2,**

(1) gcd(D(m_{ls} 0), D(m_2, 0), ..., D(m_n, 0), D(0, t^, D(0, t_2),... , D(0, t_k)) = 1,

(2) lcm(D($_{mi}$, 0), D(m_2, 0),..., D(m_n, 0), D(0, tj), D(0, t_2),... , D(0, t_k)) = lcmCmj, m_2,... , m_n)lcm(t_{lf} t_2, ..., t_k).

For (2) consider the digraph ^(r, s), where r = lcm(mi, m_2,..., m_n), and s = lcm(t!, t_2, ... , t_k).

Next, we consider results related to another kind of directed stars.

Theorem 5.6 For every positive integer n,

(1) gcd(D(n,0),D(l, 1)) = 1,

(2) gcd(D(0, n), D(1,l)) = 1,

(3) lcm(D(2n, 0), D(1, 1)) = $4n^2$,

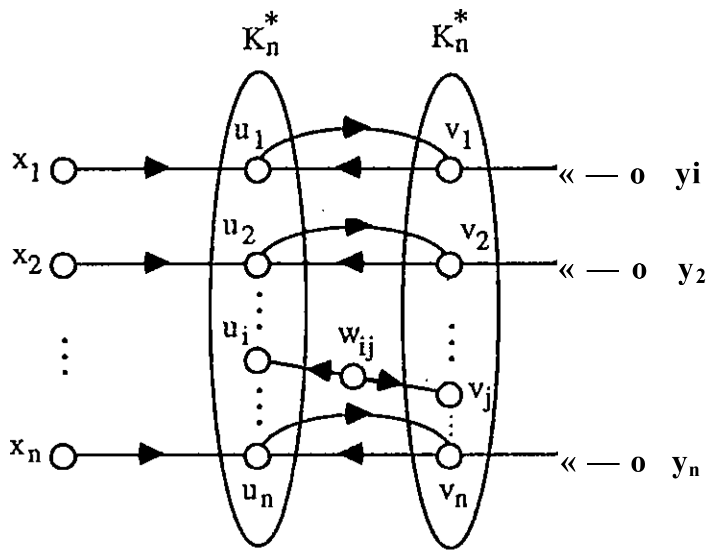
(4) lcm(**D**(2n+l, 0), D(1, 1)) = (2n + 1) (2n + 2).

Proof (1) The only divisors of D(1, 1) are 1) and D(1, 1). But, **D(l,** 1) is not a subgraph of D(n, 0). Therefore, 1) is the only common divisor of D(n, 0) and D(1, 1). Hence, gcd(D(n, 0), D(1, 1)) = 1.

(2) This can be shown similarly.

(3) For a digraph which is both D(2n, 0)-decomposable and D(1,1)-decomposable, eveiy two arcs of a copy of D(2n, 0) belong to two different copies of D(1, 1) and vice versa. Therefore, such a digraph must contain at least 2n copies of D(2n, 0). Hence, lcm(D(2n, 0), D(1, 1)) >$4n^2$

We construct a digraph D having $4n^2$ arcs such that D is both D(2n, 0)-decomposable and D(1, l)-decomposable. Consider two copies of the complete symmetric digraph K* having vertex sets {u^, u_2, ... , u_n} and {vj, v_2, ... ,v_n), respectively. For every i (1 < i < n) join Uj and v^ by a symmetric pair of arcs. We add 2n new vertices x^ x_2,..., x_n and yj, y_2,..., y_n and join x^ to Uj and join yi to Vj for l<i<n. Next, we add new vertices wy (1 <i, j <n, i ^j) and join wy to both Uj and vj. This completes the construction of D, which then has size $4n^2$ (See Figure 5.11).



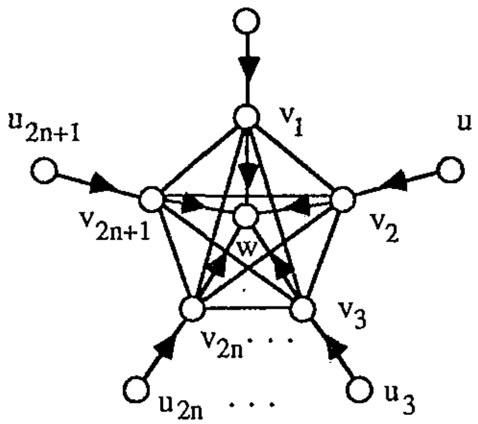
D

Figure 5.11 A digraph that is D(2n, 0)-decomposable and D(1, 1)-decomposable

Observe that for every i (1 < i < n), the vertex uj is adjacent from n - 1 vertices in its copy of K_n and from xi, v*, and wij (i ^ j, and 1 < j < n). Hence, id(uj) = 2n for each i (1 < i < n). Similarly, id(Vi) = 2n for all i (1 < i < n). Thus, D is D(2n, 0)-decomposable such that each (2n, 0) vertex of D(2n, 0) is at vertices u 1, u_2, u_n, vi, v 2 , v n . It remains to show that D is D(1, 1)-decomposable. The paths xj, uj, vi and y_u vj, ui are 2n copies of D(1, 1) and wy, uj, uj (i ^ j) where 1 < i < n , 1 < j < n are n(n - 1) copies of D(1, 1). Finally, wij, vi, vj (i ^ j) with 1 < i <.n, 1 < j < n are n(n - 1) copies of D(1, 1), producing 2n + n(n - 1) + n(n - 1) = $2n^2$ copies of D(1, 1) having a total of $4n^2$ arcs. Therefore, lcm(D(2n, 0), D(1, 1)) < $4n^2$, completing the proof.

(4) In a digraph D which is both D(2n+1, 0)-decomposable and D(1, 1)-decomposable, every two arcs of a copy of D(2n+1, 0) are arcs of two different copies of D(1, 1) and vice versa. Therefore, D must contain at least 2n + 1 copies

of D(2n+1,0). Since D is D(l, l)-decomposable, it must contain an even number of arcs. Hence, D must contain at least 2n + 2 copies of D(2n+1, 0), implying that lcm(D(2n+l, 0), D(l, 1)) > (2n + 1) (2n + 2). We construct a digraph D having (2n+l)(2n + 2) arcs such that D is both D(2n+1, 0)-decomposable and D(l, 1)-decomposable. Consider a copy of the complete symmetric digraph $K2_n^*+i$ having vertices vj, v_2,..., $v_{2n}+i$ and add vertices uj, u_2, $u_{2n}+i$ such that each vertex Uj (1 < i < 2n + 1) is adjacent to the vertex v_i of K_{2n}^*+i. Then add a new vertex w adjacent from all vertices Vj (1 < i < 2n + 1) of $K_{2n+}i$. This completes the construction of D (see Figure 5.12).



D

Figure 5.12 A digraph that is D(2n + 1, 0)-decomposable and D(l, l)-decomposable

Thus, D has (2n + 1) (2n + 2) arcs. It is straight forward to show that D is both D(2n+1, 0)-decomposable and D(l, l)-decomposable. Therefore, lcm((D(2n+l, 0), D(l, 1)) < (2n + 1) (2n + 2) and thus completing the proof. •

The next result follows immediately from the Principle of Directional Duality.

Corollary 5.7 For all positive integers n,

(1) lcm(D(0, 2n), D(l, 1)) = $4n^2$,

(2) lcm(D(0, 2n+l), D(l, 1)) = (2n + 1) (2n + 2).

Based on these results we have the following conjectures.

Conjecture 5.8 For positive integers nj, $n_2, \dots, n_k$, where k > 2,

(1) lcm(D(0, 2nj), D(0, $2n_2$), ... , D(0, $2n_k$), **D**(l, 1)) = r^2,

where r = lcm($2n_1$, $2n_2$, ... , $2n_k$).

(2) lcm(D(0, $2n_x$ + 1), D(0, $2n_2$ + 1), ... , D(0, $2n_k$ + 1),

D(l, 1)) = r(r + 1), where r = lcm($2n_1$ + 1, $2n_2$ + 1,... ,

$2n_k$ + 1).

Now by the Principle of Directional Duality we have:

Conjecture 5.9 For positive integers n^ $n_2, \dots, n_{k>}$ where k > 2,

(1) lcm(D($2n_1$, 0), D($2n_{2>}$ 0), ... , D($2n_k$, 0), D(l, 1)) = r^2,

where r = lcm(2n|, $2n_2$, ... , $2n_k$).

(2) lcm(D($2n_1$ + 1, 0), D($2n_2$ + 1, 0), ... , D($2n_k$ + 1, 0),

D(l, 1)) = r(r + 1), where r = lcm($2n_1$ + 1, $2n_2$ + 1, ... ,

$2n_k$ + 1).

Since D(2, 1) is not a subdigraph of D(n, 0) or D(0, n) for n > 1, we have the following result.

Proposition 5.10 For all positive integers n,

(1) gcd(D(n, 0), D(2, 1)) = 1, .

(2) gcd(D(0, n), D(2, 1)) = 1.

Next we determine lcm(D(m, 0), D(2,1)) for some small values of m.

Proposition 5.11 (1) lcm(D(2, 0), D(2, 1)) = 6,

(2) lcm(D(3, 0), D(2, 1)) = 6,

(3) lcm(D(4, 0), D(2, 1)) = 12,

(4) lcm(D(5, 0), D(2, 1)) = 30.

Proof (1) Since the sizes of D(2, 0) and D(2, 1) are 2 and 3, respectively, lcm(D(2, 0), D(2, 1)) >. 6. The digraph of Figure 5.13, that is both D(2, 0)-decomposable and D(2, l)-decomposable, shows that lcm(D(2, 0), D(2,1)) < 6.

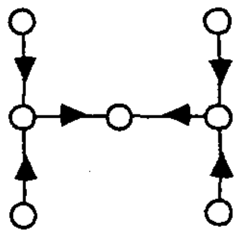
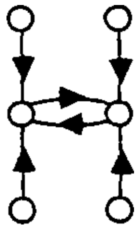


Figure 5.13 A digraph that is D(2,0)-decomposable and D(2, l)-decomposable

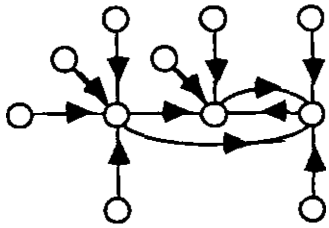
(2) Since | E(D(3, 0)) I = 3 and I E(D(2, 1)) I = 3, it follows that lcm(D(3, 0), D(2, 1)) > lcm(3, 3) = 3. However, D(3, 0) is not D(2, 1)-decomposable, implying lcm(D(3,0), D(2,l)) > 6. The digraph of Figure 5.14 shows that lcrri(D(3, 0), D(2, 1)) < 6. Therefore, lcm(D(3, 0), D(2,1)) = 6.



D_2

Figure 5.14 A digraph that is D(3,0)-decomposable and D(2, l)-decomposable

(3) The digraph D_3 of Figure 5.15 shows that lcm(D(4,0), D(2,1)) = 12.



D_3

Figure 5.15 A digraph of smallest size that is D (4,0)-decomposable andD(2, Indecomposable

(4) It is straight forward to show that the digraph D_4 having size 30 in Figure 5.16 is both D(2, Indecomposable and D(5, 0)-decomposable, so that lcm(D(5, 0), D(2,1)) <30.

D_4

Figure 5.16 A digraph of smallest size that is D(5, 0)-decomposable and D(2, l)-decomposable

We show that lcm(D(5, 0), D(2,1)) * lcm(5, 3) = 15.

Suppose to the contrary, that there is a digraph of size 15 that is both D(5, 0)-decomposable and D(2, l)-decomposable. We show that the vertices of three copies of D(5, 0) cannot be identified in such a way that the resulting digraph D is also D(2, l)-decomposable. Let D_2, and D3 be these three copies of D(5, 0) in any D(5, 0)-decomposition of D. We consider the.following cases.

Case 1 *Assume that the three* (5, 0) *vertices of the copies of* D(5, 0) *are identified.* In this case the (0, 1) vertices of copies of D(5, 0) cannot be identified without causing multiple arcs, and the resulting digraph is not D(2, l)-decomposable.

Case 2 *Assume that two of the* (5, 0) *vertices of two copies of* D(5, 0) *are identified.*. Without loss of generality, let D^ and D_2 be the two copies whose (5, 0) vertices are identified. Suppose firstly that one (2, 1) vertex of a copy of D(2, 1) is at the (5,0) vertex of D^ and two (2, 1) vertices of two copies of D(2, 1) are at the (5, 0) vertex of D_2. Then two (2,1) vertices of two copies of D(2, 1) are at the (5, 0) vertex of D3. This implies that only one arc of D3 is available to be used for three arcs for the three copies of D(2, 1) of and D_2, which is impossible. Therefore, two (2, 1) vertices of two copies of D(2,1) are at the (5, 0) vertex of T>i and two (2,1) vertices of two copies of D(2,1) are at the (5,0) vertex of D_2. Then one (2, 1) vertex of a copy of D(2, 1) is at the (5, 0) vertex of D3. This implies that only three arcs of D3 are available to be used for the four copies of D(2, 1) of D} and D_2, which is impossible.

Case 3 *Assume that none of the* (5, 0) *vertices of copies of* D(5, 0) *are identified.* In this case, at least one arc to the (5, 0) vertex of a copy of D(5, 0) is not an arc from the (2, 1) vertex of any copy of D(2, 1) in any D(2, l)-decomposition of D. Therefore, D is not D(2, l)-decomposable. •

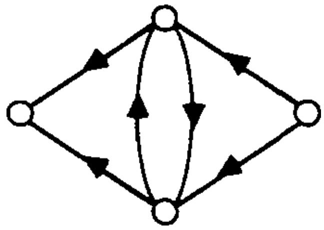
For n > 6, the determination of lcm(D(n, 0), D(2,1)) is still an open problem.

We now turn our attention to lcm(D(0, n), D(2,1)) for n = 2, 3.

Proposition 5.12 **(1)** lcm(D(0, 2), D(2, **1))** = 6,

(2) lcm(D(0, 3), D(2, 1)) = 9.

Proof (1) Since I E(D(0, 2)) | =2 and | E(D(2, 1)) I = 3, it follows that lcm(D(0,2), D(2, 1)) > lcm(2, 3) > 6. The result follows by considering the digraph D of size 6 in Figure 5.17 which is both D(0, 2)-decomposable and D(2,l)-decomposable.

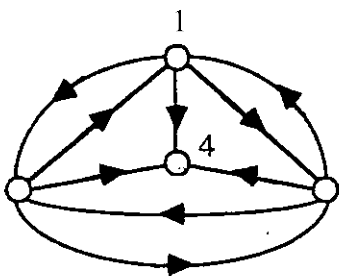


D

Figure 5.17 A digraph of smallest size that is D(0,2)-decomposable and D(2, l)-decomposable

(2) Let H be a digraph that is both D(0, 3)-decomposable and D(2, 1)-decomposable. Since the out-degree of each vertex of D(2, 1) is at most 1 and D(0, 3) has a vertex with out-degree equal to 3, it follows that H contains at least three copies of D(2, 1). Therefore, lcm(D(0, 3), D(2, 1)) > 9.

Consider the digraph H of Figure 5.18, having |E(H)| =9.



H

Figure 5.18 A digraph that is D(0, 3)-decomposable and D(2, l)-decomposable

Observe that H is D(0, 3)-decomposable into three copies of D(0, 3) as described in Figure 5.19.

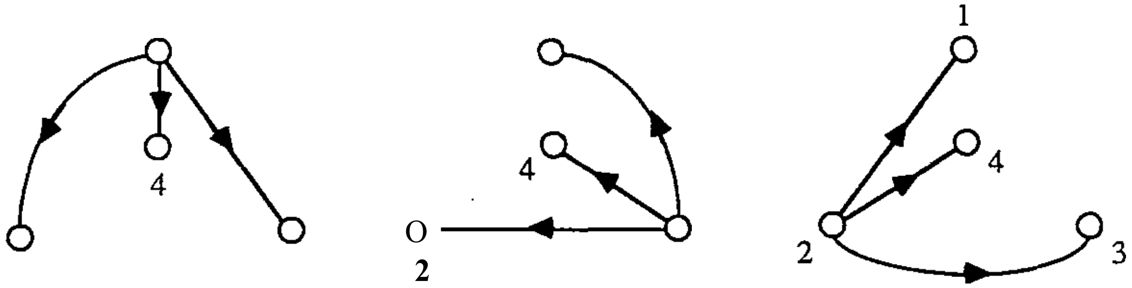


Figure 5.19 A D(0, 3)-decomposition of the digraph H of Figure 5.18

Furthermore, the digraph D is D(2, 1)-decomposable into three copies of D(2, 1) as described in Figure 5.20.

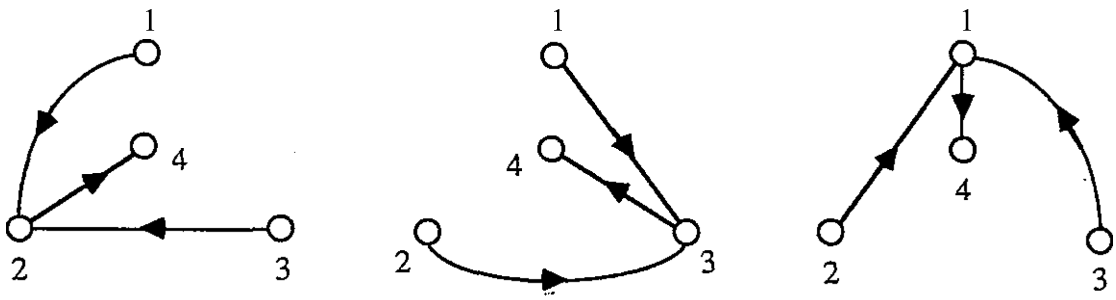


Figure 5.20 A D(2, 1)-decomposition of the digraph H of Figure 5.18

Since the digraph H having | E(D) | = 9 is both D(0, 3)-decomposable and D(2, 1)-decomposable, it follows that lcm(D(0, 3), D(2,1)) < 9, completing the proof.

Next we find the size of a greatest common divisor and a least common multiple of the digraphs D(m, 1) and D(1, 1) for m > 2. In this connection, the following lemma will be helpful.

In the directed star D(m, n), where m, n > 0, an arc from the (m, n) vertex to a (1,0) vertex is called a *central out arc*, while an arc from a (0, 1) vertex to the (m, n) vertex is called a *central in arc*.

Lemma 5.13 Let $D_1, D_2, \ldots, D_k$ be k (> 2) copies of $D(m, 1)$, $m > 2$. Then in any identification of the vertices of Dj with those of Dj (i & j), where distinct vertices of Dj are identified with distinct vertices of Dj, at most two copies of D(l, 1) can be produced that do not use central out arcs of Dj and Dj.

Proof Let C| and c_2 be the central vertices of Dj and Dj (i ^ j), respectively. For t = 1, 2, 3, let H_t be a copy of D(1, 1) having vertices u_t, v_t, and w_t and arcs (u_t, v_t) and (v_t, w_t). Suppose that H_{1s} H_2, and H3 are three edge-disjoint copies of D(1, 1) that do not use central out arcs. Then the arcs of Hj, H_2, and H3 are all central in arcs. This implies that the vertices v_t and w_t are central (distinct) vertices of Dj or Dj. Furthermore, at least two of the vertices v_x, v_2, or v_3 must be the same. .Without loss of generality, let vj and v_2 be the central vertex Cj. Then w^ and w_2 are the central vertex c_2, implying that there are two arcs (vj, Wj) and (v_2, W2) from the vertex cj to the vertex c_2 which is impossible. •

Theorem 5.14 For all positive integers n,

(1) gcd(D(n, 1), D(1, 1)) = 1 (n > 2),

(2) gcd(D(l, n), D(1, 1)) = 1 (n>2),

(3) lcm(D(2n + 1,1), D(1, 1)) = $2(n + 1)^2$,

(4) lcm(D(4n + 2, 1), D(1, 1)) = (2n + 2)(4n + 3),

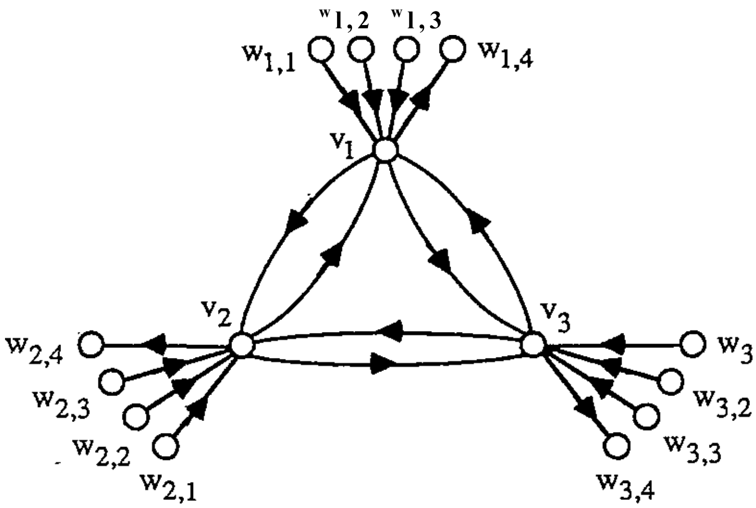
(5) lcm(D(4n, 1), D(1, 1)) = (2n + 2)(4n + 1).

Proof (1) The only divisors of D(l, 1) are S(1, 1) and D(l, 1). However, D(1, 1) is not a divisor of D(n, 1). Therefore, 1) is the only common divisor of D(n, 0) and D(1, 1). Hence, gcd(D(n, 1), D(1, 1)) = 1.

(2) This can be shown similarly.

(3) Let D be a digraph that is both D(2n + 1, l)-decomposable and D(l, 1)-decomposable. We show that I E(D) | > $2(n + 1)^2$. Let D' be a copy of D(2n + 1,1) in a D(2n + 1, l)-decomposition of D. Now consider a D(l, l)-decomposxtion of D. Observe that there is at most one copy F of D(l, 1) in D' having one arc to the (2n +1,1) vertex of D' and one arc from this vertex. Furthermore, by Lemma 5.13, for every copy D" isomorphic to D(2n+1, 1) in D other than D' at most two central in arcs of D' and two central in arcs of D" can be used to produce copies of D(l, 1) distinct from F in D. Since each edge of D' belongs to a copy of D(l, 1) in D, at least n other copies (apart from D') of D(2n + 1,1) in D exist. Hence, I E(D) | > (n + l)(2n + 2) = $2(n + 1)^2$

Now we construct a digraph D having size 2(n + 1) that is both D(2n + 1, l)-decomposable and D(l, l)-decomposable. Define F = K_{n+1}, where V(F) = (vp v_2, ... ,v_{n+1}). We construct the digraph D from F by adding (n + 2)(n + 1) new vertices Wj ^ w_{i2}, , w_{in+2} for i = 1, 2,… , n + 1, together with the arcs (w^, v^, (w_{i2}, Vj), ... , (w_{ijl+1}, and the arc ($v_{i?}$ w_{in+2}). (See Figure 5.21 for case n = 2.)



D

Figure 5.21 A digraph that is D(5, l)-decomposable and D(1, l)-decomposable

The digraph D is D(2n + 1, l)-decomposable into n + 1 copies of D(2n + 1, 1), the i-th copy having vertices . v_{1s} v_2, ... , v_{n+1} and Wj j, w_{i2}, ^{w}i n+2 $^{eac\wedge}$ i = 1, 2, ... , n + 1, together with the arcs (vj, v^), where j * i and 1 <j < n + 1, the arcs (w_u, v^, (w_{i2}, $v_{\{}$),..., (w_{in+1}, vj) and the arc ($v_{i?}$ $w_{i\,n+2}$).

The digraph D is D(l, Indecomposable into n + 1 copies of D(l, 1), the i-th copy having vertices w^ -$_v$ vj, wj $_{n+2}$ together with the arcs ($w_{i?i}$, v^ and (v^ w_{in+2}), and n(n + 1) copies of D(l, 1) having vertices w^., $v_{i?}$ v_k, where k * i, 1 < k < n + 1, and the arcs (wj $_k$, v^ and (v^ v_k) for i = 1, 2, ... , n + 1. Now since I E(D) I = $2(n + l)^2$, it follows that .lcm(D(2n + 1, 1), D(l,- 1)) < $2(n + l)^2$, completing the proof.

(4) Let D be a digraph that is both D(4n + 2, l)-decomposable and D(l, 1)-decomposable. We show that | E(D) | > (2n + 2)(4n + 3).' Let D' be a copy of D(4n + 2, 1) in a D(4n + 2, l)-decomposition of D. Now consider a D(l, 1)-decomposition of D. Observe that there is at most one copy F of D(l, 1) in D'

having one arc to the (4n + 2, 1) vertex of D' and one arc from this vertex. Furthennore, by Lemma 5.13, for every copy D" isomorphic to D(4n + 2, 1) in D other than D' at most two central in arcs of D' and two central in arcs of D" can be used to produce copies of D(l, 1) distinct from F in D. Since each edge of D' belongs to a copy of D(l, 1) in D, at least 2n + 1 other copies of D(4n + 2,1) in D exist. Hence, |E(D) | > (2n + 2)(4n + 3).

Now we construct a digraph D having size (2n + 2)(4n + 3) that is both D(4n + 2, l)-decomposable and D(1, l)-decomposable. Define H ^ K ^ ^, where V(H) = {v_{l3} v_2, ... , v_{2n+2}}. We construct the digraph D from H by adding (2n + 2)(2n + 2) new vertices w^, $w\text{^}_2$, ... , Wi_{2n+2} for i = 1, 2, ..., 2n + 2, together with the arcs (w^j, Vj), (wi_2» vj), ... , (wj 2n+l> ^{v}i) 310

(^{v}i> ^{w}i,2n+2)- (See Figure 5.22 for case n = 2.)

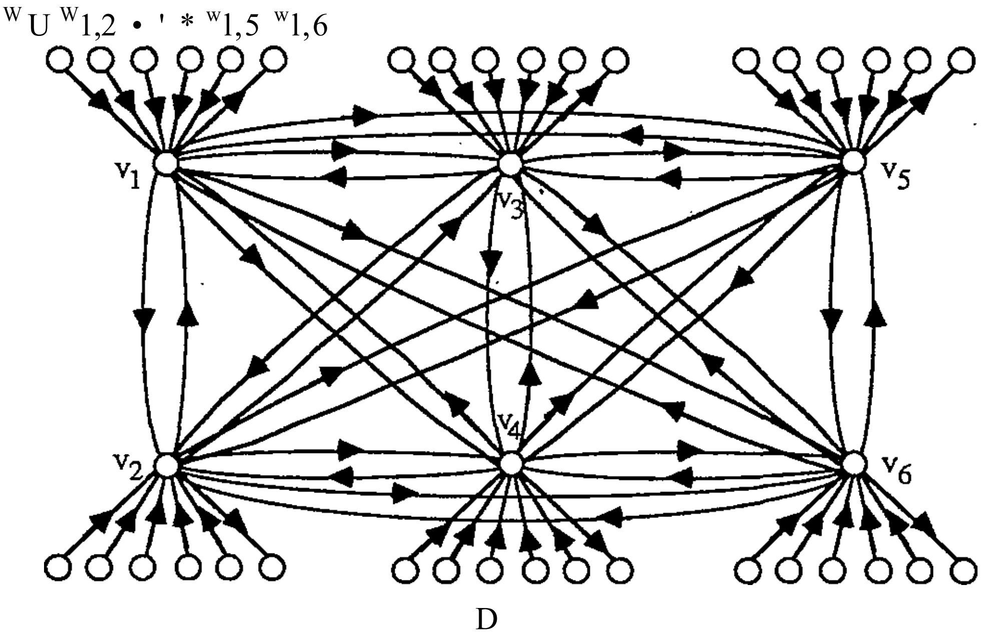
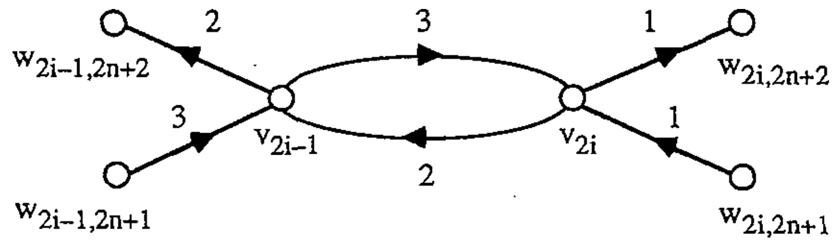


Figure 5.22 A digraph that is D(10, l)-decomposable and D(1, l)-decomposable

The digraph D is D(4n + 2, l)~decomposable into 2n + 2 copies of D(4n + 2, 1), the i-th copy having vertices vj, V2, ... , v_{2n+2}, w^j, w^ 2, ... , ^{w}i,2n+2> ^cs (v_k, Vj), where k ^ i, 1 < k < 2n + 2, and (w^, v^) for each i = 1, 2, ... , 2n + 1, together with the arc (v^, w_{2n+2}). We show that D is D(l, 1)-decomposable into (n + l)(4n + 3) copies of D(1, 1).

Consider the pair of vertices v_2i_i and v_2j for i = 1, 2,... , n + 1, and the subdigraphs Hj of D having symmetric arcs (v_{2i}_j, v_2j) and (v_2j, v_2i_i) together with arcs ($w_{2i\_1)2n+i}$, v_{2i}_i), (v_{2M}, w_{2i_lj2n+2}), (^{w}2i,2n+b ^{v}2i)> and (^{v}2i> $w_{2ij2n+2}$) for i = 1, 2,..., n + 1. It is clear that each subdigraph Hj (1 < i < n + 1) is D(l, 1)-decomposable into three copies of D(l, 1). (See Figure 5.23.)



Hi

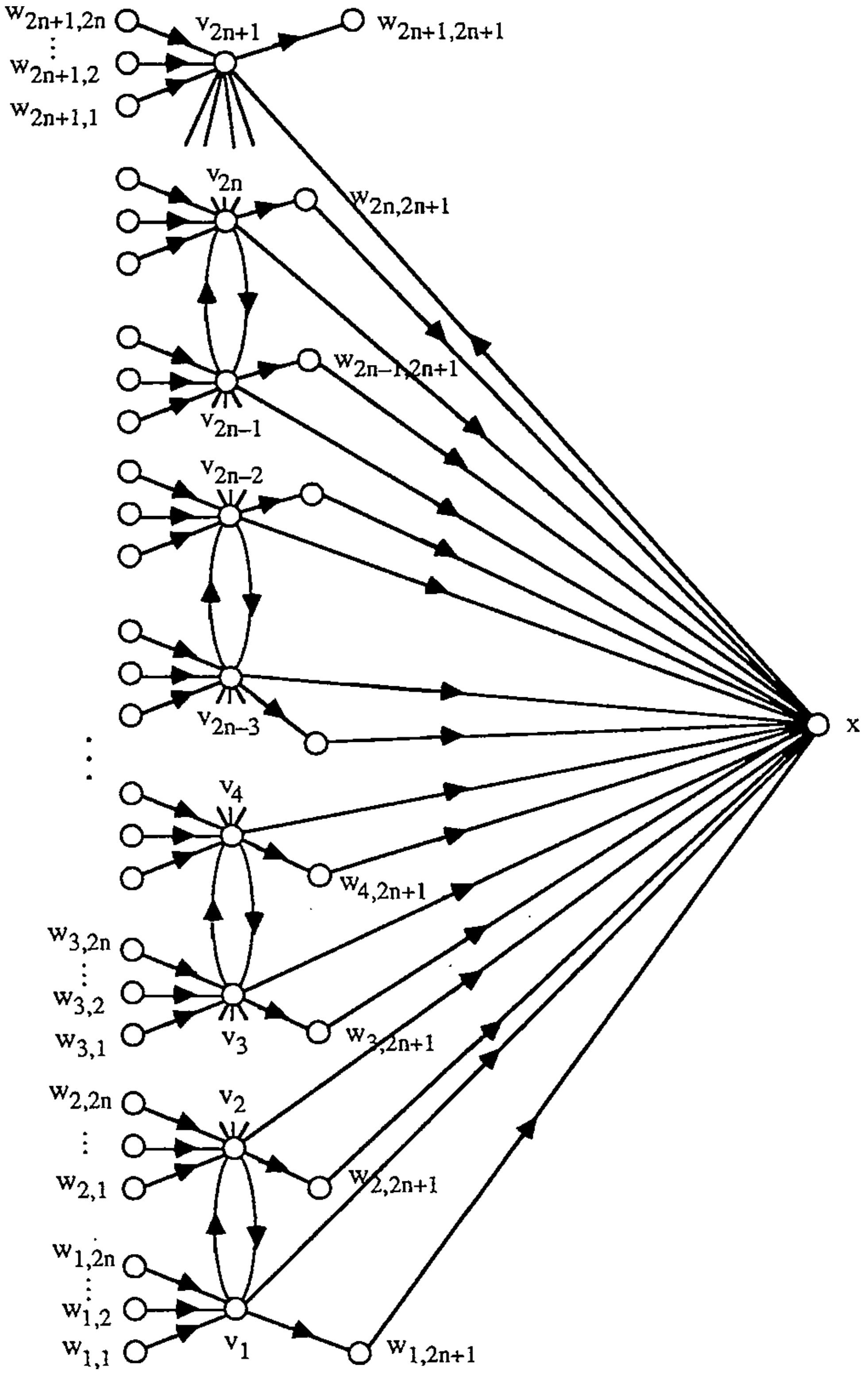
Figure 5.23 The subdigraph Hj corresponding to three copies of D(l, 1)

We remove the arcs of subdigraphs H^, H_2,... , H_{n+1} from D. Then for each t = 1, 2, ... , 2n+2, corresponding to each vertex v_t there are 2n copies of D(1, 1), using the 2n arcs (w_tj, v_t), (w_{t2}, v_t), ... , (w_{t2n}, v_t) and the 2n arcs (v_t, vj), where 1 <j < 2n + 2 and when t is odd, j 4 {t, t + 1} and when t is even (t-1,t).

Therefore, D is D(l, l)-decomposable. Since |H(D)| = (2n + 2)(4n + 3), it follows that lcm(D(4n + 2, 1), D(1, 1)) < (2n + 2)(4n + 3).

(5) Let D be a digraph that is both D(4n, l)-decomposable and D(l, 1)-decomposable. We show that I E(D) I > (2n + 2)(4n +1). Let D' be a copy of D(4n, 1) in a D(4n, l)-decomposition of D. Consider a D(l, l)-decomposition of D. Observe that there is at most one copy F of D(l, 1) in D' having one arc to the (4n, 1) vertex of D' and one arc from this vertex. Furthermore, by Lemma 5.13, for every copy D" isomorphic to D(4n, 1) in D other than D' at most two central in arcs of D' and two central in arcs of D" can be used to produce copies of D(l, 1) distinct from F in D. Since each edge of D' belongs to a copy of D(l, 1) in D, at least 2n other copies of D(4n, 1) in D exist. Now since the size of 2n + 1 copies of D(4n, 1) is odd, it follows that D contains at least 2n + 2 copies of D(4n, 1). Hence, |E(D) | > (2n + 2)(4n + 1).

We construct a digraph D having size (2n + 2)(4n + 1) that is both D(4n, 1)-decomposable and D(l, l)-decomposable. Define F = K2n+i, where V(F) = {vj, v2' — > v2n+l Add 2n + 1 new vertices **WJJ**, Wj 2> — > wi,2n> wi,2n+l for each i = 1, 2, ... , 2n + 1 together with the arcs (wj i, vj), (Wj2, Vj), ... , (Wii2n» vi) arK* (Vi, Wj 2n+i)- Finally, add a new vertex x together with the arcs (vis x) and (wi,2n+l> x) for each i = 1, 2,... , 2n and the arc (x, v2n+i), resulting in the digraph D. (See Figure 5.24.)



D

Figure 5.24 A digraph that is D(4n, 1)-decomposable and D(1, 1)-decomposable

We show that D is both D(4n, l)-decomposable and D(1, l)-decomposable.

We consider the following 2n + 2 copies of D(4n, 1) in D. One copy of D(4n, 1) has x as its (4n, 1) vertex and arcs (vj, x) and (wj^n+b x) f°r 1 ^ i ^ 2n together with the arc (x, v_{2n+1}). The other 2n + 1 copies of D(4n, 1) have their (4n, 1) vertices at Vj (i = 1, 2, ... , 2n + 1) with arcs (Wjj, v^), (wi$_2$, Vj), ... , ($w_{i>2n}$, vj), and (vj, Vj) for all j (1 <j<2n + 1, j * i) and the arc (v_{i9} w_{i2n+1}).

Further, D is D(1, l)-decomposable: One copy of D(1, 1) has vertices x, v_{2n+1}, and $w_{2n+1)2n+1}$ with arcs (x, v_{2n+1}) and (v_{2n+1}, $w_{2n+1)2n+1}$). Another 2n copies of D(1, 1) have vertices v-$_v$ Wi$_{2n+}$i, and x for i = 1, 2, ... , 2n and arcs (^{v}i> $w_{i>2n+}$l) and ($w_{i>2n+1}$, x).

There are 3n copies of D(1, 1) in the n subdigraphs Dj induced by {x, ^{v}**2i**-l» ^{v}**2i>** ^{w}**2i**-l,**2n>** ^{w}**2i,2n)** for i = 2, ... , n. (See Figure 5.25.)

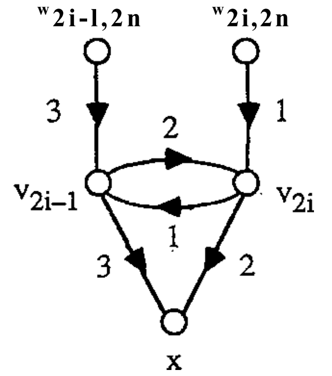


Figure 5.25 Three copies of D(1, 1) in the digraph D of Figure 5.24

Centred at the vertex v_{2n+1} there are 2n copies of D(1, 1) using arcs **(W2n**+l,l» ^{v}**2n**+l)> (w$_2$n+l,**2>** v_{2n+}i). - , (w$_{2n}$+l,**2n>** ^{v}**2n**+l) and arcs (v_{2n+1}, v_x), (^{v}2n+l» ^{v}2)> - > **(**^{v}**2n**+l».^{v}**2n)-** (^{See} Figure 5.26.)

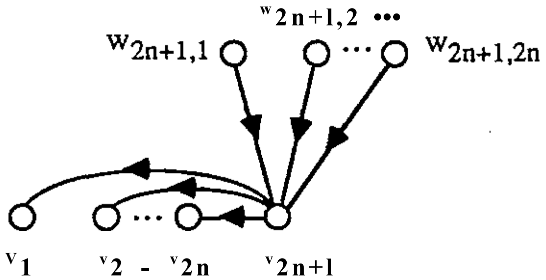


Figure 5.26 2n copies of D(l, 1) in the digraph D of Figure 5.24

Finally, there are 2n(2n - 1) copies of D(l, 1), 2n -1 of which are centred at each of the vertices v_{15} v_2,... , v_{2n}. For each i, the arcs of these copies of D(l, 1) are **(WJJ, VJ), (WJ₂, V}), ... , (WJ** 2n_i, **VJ)** together with the remaining 2n - 1 arcs from the vertex Vj, namely, (vj, vj), where 1 <j < 2n, j & i and when i is odd j ^ i + 1 and when i is even j i — 1. (See Figure 5.27.)

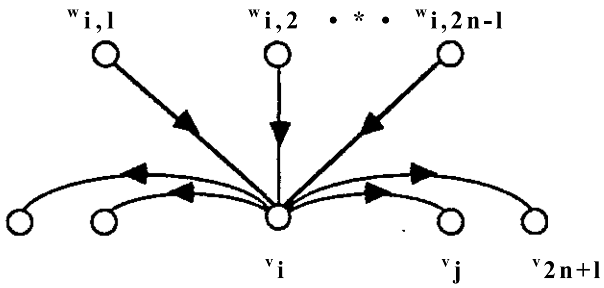


Figure 5.27 2 n - 1 copies of D(l, 1) corresponding to each vertex Vj (i = 1, 2,..., 2n)

Therefore, there are $1 + 2n + 3n + 2n + 2n(2n - 1) = 4n^2 + 5n + 1$ copies of D(l, 1) producing $2(4n^2 + 5n + 1) = (2n + 2)(4n + 1)$ arcs. It follows that lcm(D(4n, 1), D(l, 1)) < (2n + 2)(4n + 1), completing the proof. •

5.3 The Greatest Common Divisor Index of a Digraph

As with graphs, we define, for a given digraph D of size q, the *greatest common divisor index* $i(D)$ as the greatest integer n for which there exist digraphs D_1 and D_2, both of size at least nq, such that $GCD(D_1, D_2) = \{D\}$. If no such n exists, then we define this index to be ∞.

Proposition 5.15 For every positive integer n,

$$i(nD(0, 1)) = \infty.$$

Proof The result is immediate when we follow the technique of the proof of Proposition 4.18. •

Proposition 5.16 For all nonnegative integers m and n, with $(m, n) \neq (0, 0)$,

$$i(D(m, n)) =$$

Proof Suppose, to the contrary, that $i(D(m, n))$ is finite, say $i(D(m, n)) = t$. Let r $(> t)$ be an integer and p_1 and p_2 be distinct primes so that $(m + n)p_1$ and $(m + n)p_2$ are at least r. Define $D_1 = D(p_1 m, p_1 n)$ and $D_2 = D(p_2 m, p_2 n)$. Then $GCD(D_1, D_2) = \{D(m, n)\}$, implying that $i(D(m, n)) \geq r > t$, contrary to hypothesis. Therefore, $i(D(m, n)) = \infty$. •

Now we generalize the former two propositions.

Proposition 5.17 For positive integers a, b, and c

$$i(aD(0, 1) \cup D(b, c)) = \infty.$$

Proof Suppose, to the contrary, that i (a D(0, 1) vj (b, c)) = t, where t € N. Let r (>t) be an integer, and let pj and p_2 be distinct odd primes, where pj(a + b + c) > r for i = 1, 2. Let Dj = p a D(0, 1) u D(p_xb, p_xc) and D_2 a p_2a D(0, 1) u D(p_2b, p_2c). Then GCD(D_X, D_2) = {a D(0, 1) u D(b, c)}. and i (a D(0, 1) u D(b, c)) > r > t, contrary to hypothesis. Therefore, i (a D(0, 1) u D(b, c)) = •

The next propositions are immediate.

Proposition 5.18 For all positive integers bj, b_2, b_n and Cj, c_2, c_n,

$$i\ (D(b_v\ c_x)\ u\ D(b_2,\ c_2)\ u\quad u\ D(b_n,\ c_n)) = «».$$

Proposition 5.19 For all positive integers a, ty, and c^ (1 < i < n), where n > 2,

$$i\ (a\ D(0,1)\ u\ D(b_x,\ c_x)\ u\ D(b_2,\ c_2)\ u\ «\bullet\ u\ D(b_n,\ c_n)\ -$$

The directed path P_n on n vertices is a digraph obtained from assigning direction to the path P_n so that it forms a (directed) path of length n - 1.

Proposition 5.20 For n = 2, 3, 4, 5,

$$i(?_n) =$$

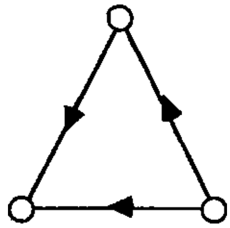
Proof The result follows directly from the corresponding result for graphs. •

For integer n (> 3) we let (* be a directed cycle on n vertices. As with graphs we have the following propositions.

Proposition 5.21 i((*$_3$) = 1.

Proposition 5.22 i(<?$_4$) =

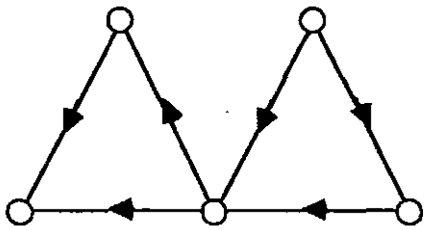
We have shown for graphs $i(K_3) = 1$ and for digraphs $i(C?_3) = 1$, but for the tournaments the result is not similar. For example, we show that $i(T) = {}^{00}$ for the tournament T of Figure 5.28 .



T

Figure 5.28 A transitive tournament T of order 3

Note that a result for digraphs similar to Lemma 2.13 does not hold for tournaments in general, since the digraph D of Figure 5.29 is T-decomposable but not $(\vec{P}_3$ u $\vec{P}_2)$-decomposable.



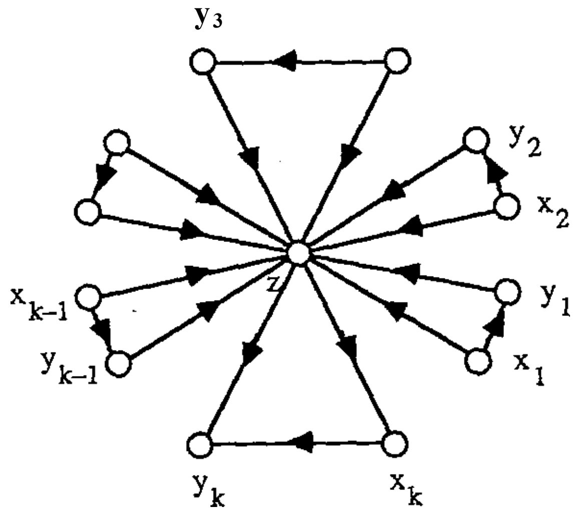
D

Figure 5.29 A digraph that is T-decomposable but not $(\vec{P}_3$ u $\vec{P}_2)$-decomposable

Proposition 5.23 For the tournament T of Figure 5.28,

$$i(T) =$$

Proof Suppose, to the contrary, that $i(T) = t$, for some te N. Let m (> t) be an integer, and let pj and p_2 be primes so that $p_2 > p! > m$. Let $D_x = pjT$ and D_2 be the digraph of Figure 5.30 with $k = p_2$. We show that $GCD(D_{15}\ D_2) = \{T\}$.



D_2

Figure 5.30 The greatest common divisor of Dj and D_2 is T

It is sufficient to show that D_2 is not D-decomposable into k copies of D for any element D of the set <D= {D(2,0) u ? 2 , D(0, 2) u D(1, 1) u ? 2 } .

Suppose, to the contrary, that D_2 is D-decomposable into k copies of D for some D e Since z is a vertex of every copy of D(2, 0), D(0, 2), and D(1, 1), none of the arcs incident to or from z can be an arc of $\vec{P}_2$ in a copy of D. Therefore, all arcs (xj, y^ for 1 < i < k are the arcs of the k copies of $\vec{P}_2$, and the arcs incident from or to z are the arcs of the k copies of one of the digraphs D(2,0), D(0,2), and D(1, 1), implying that z must be a (2k, 0)-vertex, a (0, 2k)-vertex, or a (k, k)-vertex, respectively. However, idz = 2(k - 1) and odz = 2, a contradiction since k > 2. Now since D_x and D_2 are T-decomposable, GCDCD^ D_2) = {T} and i(T) > m > t, contrary to hypothesis. Therefore, i(T) = •

As with graphs, the following lemma, whose proof is omitted, will be useful.

Lemma 5.24 Let p (> 3) be an integer. If D is a nontrivially K*-decomposable digraph, then D is also ((Kp - e) u P^-decomposable, where e is any arc of Kp.

A direct result of this lemma is the next proposition.

Proposition 5.25 For every integer p (> 3),

i(Kp) = 1.

In general, the problem of determining the greatest common divisor index of a digraph appears to be difficult and it is unknown whether digraphs D, with 1 < i(D) < oo, exist.

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