COZ - RELATED AND OTHER SPECIAL QUOTIENTS IN FRAMES

by

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Abstract

We study various quotient maps between frames which are defined by stipulating that they satisfy certain conditions on the cozero parts of their domains and codomains. By way of example, we mention that C-quotient and C^*-quotient maps (as defined by Ball and Walters-Wayland [7]) are typical of the types of homomorphisms we consider in the initial parts of the thesis. To be little more precise, we study uplifting quotient maps, C_1- and C_2-quotient maps and show that these quotient maps possess some properties akin to those of a C-quotient maps. The study also focuses on R^*- and G^*- quotient maps and show, amongst other things, that these quotient maps coincide with the well known C^*- quotient maps in mildly normal frames. We also study quasi-F frames and give a ring-theoretic characterization that L is quasi-F precisely when the ring $\mathcal{R}L$ is quasi-Bézout. We also show that quasi-F frames are preserved and reflected by dense coz-onto R^*-quotient maps. We characterize normality and some of its weaker forms in terms of some of these quotient maps. Normality is characterized in terms of uplifting quotient maps, δ-normally separated frames in terms of C_1-quotient maps and mild normality in terms of R^*- and G^*-quotient maps. Finally we define cozero complemented frames and show that they are preserved and reflected by dense $z^\#$- quotient maps. We end by giving ring-theoretic characterizations of these frames.

Contents

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Chapter 1

Introduction and preliminaries

1.1 History of cozero related quotients in classical and pointfree topology

A subspace S of a topological space X is C-embedded (respectively, C^*-embedded) in X if every function in $C(S)$ (respectively, $C^*(S)$) can be extended to a function in $C(X)$ (respectively, $C^*(X)$). The notions of C- and C^*-embedding have long been known in classical topology. In frames, C- and C^*-quotients were studied by Ball and Walters-Wayland in [7]. In 1978, Ishii and Ohta [40] introduced the notions of C_1-, C_2-, and C_3-embedding which generalize the notion of C-embedding and studied their properties and applications.

Closely related to C^*-embedded subspaces are the notions of R^*-embedded and G^*-embedded subspaces considered by Aull [3]. Prior to that Aull [2] had considered FF-embedded, GG-embedded, FZ-embedded, CG-embedded, and CC-embedded subspaces. The notion of FF-embedded subspaces is what we shall call *uplifting homomorphism* here. The term *uplifting* was introduced in [6] by Ball, Hager and Walters-Wayland.

The notion of z-embedded subspaces was introduced in classical topology in 1963 by Blair (see [17]). In frames, this notion appears to have first been considered by Banaschewski and Gilmour [13] in their study of Oz frames, and subsequently by Ball and Walters-Wayland [7] where the notion is used extensively in their study of C- and C^*-quotients of frames. Furthermore, the theory of coz-onto homomorphisms in frames has been developed extensively in [33] by Dube and Walters-Wayland.

The notions of WN-maps and N-maps were introduced in classical topology by Woods [60] in 1972. These notions were given as modification of the definition of WZ-maps given by Isiwata [41].

The study of δ-normally separated spaces was initiated by Zenor [61], with the descriptor "δ-normally separated space" later coined by Mack [47]. In [45], Kohli and Das introduced four other generalizations of normality, namely, θ-normal spaces, weakly θ-normal spaces, functionally θ-normal and weakly functionally θ-normal spaces. All four of them coincide with normality in the class of θ-regular spaces. However, we will not study these four generalizations of normality in this thesis. Δ-normal spaces were introduced by A. K. Das in [21] and π-normal topological spaces were introduced by L. N. Kalantan in [43]. In this thesis we only extend the notion of Δ-normal spaces to frames and leave out the π-normal topological spaces.

The study of topological concepts from a lattice-theoretic viewpoint was initiated by H. Wallman [57] in 1938. The term *frame* was introduced by C.H. Dowker in 1966 and brought to the fore in the article co-authored with D. Papert [23]. The dual notion *locale* was introduced by J.R. Isbell in 1972 in the pioneering paper, which opened several topics, *Atomless Parts Of Spaces* [39].

1.2 Synopsis of the thesis

Subspaces which are C-embedded, C^*-embedded, z-embedded and other special subspaces play a vital role in the study of topological properties in classical topology. In [1], normal spaces are characterized as those spaces for which every closed subspace is C-embedded. Oz-spaces were introduced in [18] by Blair in 1976.

This thesis, as the title suggests, shows the importance of coz-related and few other special quotients in pointfree topology. In the study, we examine several quotient maps which are "coz-related" in one sense or another. Chapter 1 is essentially introductory. Here we present the relevant definitions pertaining to frames and outline the relevant backgroud for the other chapters. For quick reference and smooth-flowing arguments, we highlighted some of the definitions of Chapter 1 in the body of the thesis.

In Chapter 2, we study four types of quotient maps of frames which are closely related to C- and C^*-quotient maps. We call them C_1-, strong C_1-, C_2- and uplifting quotient maps. We give a characterization of C_1-quotient maps in terms of maximal ideals of cozero parts of their domains and codomains. We show that an onto frame homomorphism is a C-quotient map if and only if it is both a C_1- and a C_2-quotient map. Uplifting quotient maps are used to characterize normal frames as those frames in which every uplifting quotient map out of L is a C^*-quotient map. It also turns out that dense uplifting quotient maps are C^*-quotient maps.

In Chapter 3, we characterize normality and some of its weaker forms in terms of some quotient maps defined in Chapter 2. Normality is also characterized in terms of uplifting quotient maps and δ-normally separated frames are characterized by C_1-quotient maps. We define R^*- and G^*-quotient maps and these turn out to be closely related to C^*-quotient maps. In mildly normal frames, these quotient maps coincide with the C^*-quotient maps. Our study also focuses briefly on Δ-normal frames.

In Chapter 4, we give several characterizations of quasi-F frames. The class of quasi-F frames strictly contains that of F'-frames, which in turn, contains the class of F-frames. Few characterizations of quasi-F frames in terms of the ring of real-valued continuous functions on L are presented. One such characterization is that L is quasi-F precisely when the ring $\mathcal{R}L$ is quasi-Bézout.

In Chapter 5, we study cozero complemented frames. The class of cozero complemented frames contains the class of cozero approximated frames. A *ccc*-frame L (i.e., every collection of pairwise disjoint elements of L is countable) is cozero complemented. A noteworthy observation is that the cozero complemented frames are preserved and reflected by dense $z^{\#}$-quotient maps. We also give a few characterizations of cozero complemented frames in terms of the ring $\mathcal{R}L$ of real-valued continuous functions.

1.3 Frames

In this section we recall some facts about frames that we will need in the sequel. A *frame* is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by 1 and 0 respectively, or by 1_L and 0_L if it is necessary to emphasize the frame in question. A *frame homomorphism* (or a *frame map*) is a map $h : L \to M$ between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. We write **Frm** for the category of frames and frame homomorphisms. By a *subframe* of a frame we mean a subset which is closed under finite meets and all joins.

A typical example of a frame is the lattice $\mathfrak{O}X$ of open sets of a topological space X. If $f : X \to Y$ is a continuous map between topological spaces, then $f^{-1} : \mathfrak{O}Y \to \mathfrak{O}X$ is a frame homomorphism. This establishes a *contravariant functorial* relationship between the category **Top** of topological spaces and continuous maps and the category **Frm** as illustrated below:

$$\mathbf{Top} \xrightarrow{\mathfrak{O}} \mathbf{Frm}$$

$$X \xrightarrow{f} Y \quad \mapsto \quad \mathfrak{O}Y \xrightarrow{f^{-1}=\mathfrak{O}f} \mathfrak{O}X$$

Associated with any frame homomorphism $h : L \to M$ is a map $h_* : M \to L$, known as the *right adjoint* of h, which is not necessarily a frame homomorphism, and is defined by

$$h_*(b) = \bigvee\{a \in L \mid h(a) \leq b\}.$$

The following property holds for every $a \in L$ and every $b \in M$:

$$h(a) \leq b \Leftrightarrow a \leq h_*(b).$$

A frame homomorphism $h : L \to M$ is *dense* if $h(a) = 0$ implies $a = 0$ for every $a \in L$. This holds if and only if $h_*(0) = 0$. A frame homomorphism $h : L \to M$ is onto if and only if $hh_* = id_M$.

By a *quotient* of a frame L, we mean a homomorphic image of L. That is, M is a quotient of L precisely if there is an onto frame homomorphism $h : L \to M$. In such a case h is called a *quotient map*. When we say a quotient $L \to M$ has a property of frames we shall mean that M has that property. Likewise, to say a quotient $L \xrightarrow{h} M$ has a property of homomorphisms means that h has that property.

An *extension* of a frame L is a dense onto homomorphism $h : M \to L$. By abuse of language, we say an extension $h : M \to L$ of L has property Ω of frames if the frame M has the property Ω.

The *pseudocomplement* of an element x of L is the element

$$x^* = \bigvee\{y \in L \mid x \wedge y = 0\}.$$

We note that $x \wedge x^* = 0$. However $x \vee x^* = 1$ does not hold in general.

(i) In the case where $x \vee x^* = 1$, we say x is *complemented.*

(ii) $x \in L$ is dense if $x^* = 0$.

(iii) For every $x \in L,\ x \leq x^{**}$ always holds. If $x = x^{**}$, then x is called a *regular* element.

The *Booleanization* of a frame L is the Boolean frame $\mathcal{B}L$ whose underlying set is $\mathcal{B}L = \{x^{**} \mid x \in L\}$ with meet as in L and join $\bigsqcup S = (\bigvee S)^{**}$ for each $S \subseteq \mathcal{B}L$. The map $L \to \mathcal{B}L$ which sends each $x \in L$ to x^{**} is a dense onto frame homomorphism. We denote it by $\flat$.

Let L be a frame. We call $D \subseteq L$ a *downset* if $x \in D$ and $y \leq x$ implies $y \in D$, and $U \subseteq L$ an *upset* if $u \in U$ and $u \leq v$ implies $v \in U$. For any $a \in L$, we write

$${\downarrow}a = \{x \in L \mid x \leq a\},$$

which is a downset, and

$${\uparrow}a = \{x \in L \mid a \leq x\},$$

which is an upset. We note that ${\downarrow}a$ is a frame whose bottom element is $0 \in L$ and top element a. Similarly, ${\uparrow}a$ has $1 \in L$ as the top element and a as its bottom element. These frames are in fact the quotients of L via the maps $L \to {\uparrow}a$ and $L \to {\downarrow}a$, given respectively by $x \mapsto a \vee x$ and $x \mapsto a \wedge x$. These quotients are known as the *closed quotients* and *open quotients* respectively.

A result often used in frame theory is that every frame homomorphism $h : L \to M$ has a *dense-onto factorization*

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[rr, bend right, "h"'] & {\uparrow}h_*(0) \arrow[r, "\bar{h}"] & M \end{tikzcd}$$

We call $I \subseteq L$ an *ideal* if it satisfies (the following properties):

(i) $0 \in I$.

(ii) $b \in I$ and $a \leq b$ implies $a \in I$ (i.e. I is a downset).

(iii) $a, b \in I$ implies $a \vee b \in I$.

A subset $F \subseteq L$ is called a *filter* if it satisfies the following properties:

(i) $0 \notin F$ and $1 \in F$.

(ii) $a \in F$ and $a \leq b$ implies $b \in F$ (i.e. F is an upset).

(iii) $a, b \in F$ implies $a \wedge b \in F$.

A filter $F \subseteq L$ is called a *prime filter* if it is a filter and satisfies the property that $a \vee b \in F$ implies $a \in F$ or $b \in F$.

A filter $U \subseteq L$ is called an *ultrafilter* if for any filter $F \subseteq L$, whenever $U \subseteq F$, then $U = F$.

We say that a is *rather below* b or a is *well inside* b, written $a \prec b$, if there is a *separating element* $c \in L$ such that $a \wedge c = 0$ and $b \vee c = 1$. We say a frame L is *regular* if every $a \in L$ is expressible as

$$a = \bigvee \{x \in L \mid x \prec a\}.$$

We have the notion of complete regularity, which is defined by means of scales in a frame. By a *scale* in a frame we mean a countable (rational-number) indexed subset

$$\{c_q \mid q \in \mathbb{Q} \cap [0, 1]\} = (c_q)$$

of L such that whenever $p < q$, then $c_p \prec c_q$. We define the completely below relation $\prec\!\!\prec$ on L by: $a \prec\!\!\prec b$ if there is a scale (c_q) such that $a \leq c_0$ and $c_1 \leq b$. We say L is *completely regular* if every $a \in L$ is expressible as

$$a = \bigvee \{x \in L \mid x \prec\!\!\prec a\}.$$

Let L be a frame. We say that a subset $S \subseteq L$ *generates* L if for every element $x \in L$,

$$x = \bigvee \{s \in S \mid s \leq x\}.$$

A *nucleus* on a frame L is a map $j : L \to L$ such that for all $a, b \in L$ the following are satisfied:

(a) $a \leq j(a)$

(b) $j(a \wedge b) = j(a) \wedge j(b)$

(c) $j^2(a) = j(a)$.

The set $\mathrm{Fix}(j) = \{x \in L \mid j(x) = x\}$ is a frame with meet as in L and join $j\left(\bigvee S\right)$ for each $S \subseteq \mathrm{Fix}(j)$. Furthermore, $j : L \to \mathrm{Fix}(j)$ is a quotient map the right adjoint of which is the inclusion $\mathrm{Fix}(j) \to L$.

By a *cover* C of a frame L we mean a subset of L such that $\bigvee C = 1$. We write $\mathrm{Cov}\,(L)$ for the set of all covers of the frame L. The frame L is *compact* if for any $C \in \mathrm{Cov}(L)$, there is a finite $K \subseteq C$ in $\mathrm{Cov}(L)$. The frame L is *Lindelöf* if every cover has a countable subset that is also a cover.

For two covers A and B of L, $A \leq B$ (A *refines* B) means that for each $a \in A$, there exists $b \geq a$ in B. A cover $C \in \mathrm{Cov}(L)$ is said to be *locally finite* if there exists $D \in \mathrm{Cov}(L)$

such that for every $y \in D$, the set

$$\{x \in C \mid x \wedge y \neq 0\}$$

is finite. In this case we say D *finitizes* C. A frame L is said to be *paracompact* if every cover $A \in \mathrm{Cov}(L)$ has a locally finite refinement.

A frame L is *normal* if for any elements a, $b \in L$ such that $a \vee b = 1$, there are elements c, $d \in L$ such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$.

1.4 Cozero part of a frame

An element a of L is a cozero element if there is a sequence (a_n) in L such that $a_n \prec\!\prec a$ for each n and $a = \bigvee a_n$. The *cozero part* of L, denoted by $\mathrm{Coz}\, L$, is the regular sub-σ-frame consisting of all the cozero elements of L.

The *frame of reals* is the frame $\mathcal{L}(\mathbb{R})$ generated by all ordered pairs (p, q) where $p, q \in \mathbb{Q}$, subject to the relations that:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$

(R4) $1_{\mathcal{L}(\mathbb{R})} = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$

Regarding the *frame of reals* $\mathcal{L}(\mathbb{R})$ and the f-ring $\mathcal{R}L$ of *continuous real-valued functions* on L, we refer to [8]. Recall that the *cozero map*, $\mathrm{coz} : \mathcal{R}L \to L$, is given by

$$\mathrm{coz}\, \varphi = \bigvee\{\varphi(p, 0) \vee \varphi(0, q) \mid p, q \in \mathbb{Q}\}.$$

The association $L \mapsto \mathcal{R}L$ is functorial, with $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ taking δ to $h \cdot \delta$, for any $h : L \to M$. Furthermore, $\operatorname{coz}(h \cdot \delta) = h(\operatorname{coz}\delta)$.

The properties of the cozero map which we shall frequently use are the following:

(1) $\operatorname{coz}\gamma = \operatorname{coz}|\gamma| = |\gamma|(\mathbb{R}^+)$, where $\mathbb{R}^+ = (-\infty, 0) \vee (0, \infty)$ in $\mathfrak{L}(\mathbb{R})$,

(2) $\operatorname{coz}\gamma\delta = \operatorname{coz}\gamma \wedge \operatorname{coz}\delta$,

(3) $\operatorname{coz}(\gamma + \delta) \leq \operatorname{coz}\gamma \vee \operatorname{coz}\delta$,

(4) $\varphi \in \mathcal{R}L$ is invertible if and only if $\operatorname{coz}\varphi = 1$,

(5) $\operatorname{coz}\varphi = 0$ if and only if $\varphi = 0$,

(6) $\operatorname{coz}(\gamma + \delta) = \operatorname{coz}\gamma \vee \operatorname{coz}\delta$ if $\gamma, \delta \geq 0$.

There are several ways of realizing the Stone-Čech compactification of a completely regular frame L. We adopt the compactification that is presented in [42]. An ideal J of L is *completely regular* if for each $x \in J$ there exists $y \in J$ such that $x \prec\!\!\prec y$. The Stone-Čech compactification of L is the frame βL consisting of completely regular ideals of L together with the dense onto frame homomorphism $\sigma_L : \beta L \to L$ given by the join. We denote the right adjoint of σ_L by r_L, and recall that $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$ for all $a \in L$. Also, for any c, $d \in \operatorname{Coz} L$, $r_L(c \vee d) = r_L(c) \vee r_L(d)$. The *Stone extension* of a frame homomorphism $h : L \to M$ between completely regular frames is the frame homomorphism $h^\beta : \beta L \to \beta M$ given by

$$h^\beta(I) = \{y \in M \mid y \leq h(x) \text{ for some } x \in I\}$$

for each $I \in \beta L$. It is the unique frame homomorphism that makes the diagram below

$$\begin{array}{ccc} \beta L & \xrightarrow{h^\beta} & \beta M \\ \downarrow{\scriptstyle \sigma_L} & & \downarrow{\scriptstyle \sigma_M} \\ L & \xrightarrow{h} & M \end{array}$$

commute.

1.5 The coreflections λL and υL

Recall that a full subcategory $\mathbf{C}$ of a category $\mathbf{A}$ is said to be a *coreflective* subcategory if for every object A in $\mathbf{A}$, there is an object γA in $\mathbf{C}$ and a morphism $\gamma_A \colon \gamma A \to A$ such that for any morphism $f \colon C \to A$ with domain in $\mathbf{C}$, there is a unique morphism $\bar{f} \colon C \to \gamma A$ satisfying $\gamma_A \cdot \bar{f} = f$. The object γA is called a *coreflection* of A.

Using localic language, Madden and Vermeer [49] have shown that regular Lindelöf locales form a reflective subcategory of the category of locales by actually constructing the reflection, λL, for any completely regular locale L. We recall the construction in frame terms because that is the category of discourse in this thesis.

Let L be a completely regular frame. An ideal of Coz$L$ is a σ-*ideal* if it is closed under countable joins. The regular Lindelöf coreflection of L, denoted by λL, is the frame of σ-ideals of Coz$L$. The join map $\lambda_L \colon \lambda L \to L$ is a dense onto frame homomorphism, and is the attendant coreflection map. This is a special case of a more general result concerning κ-frames (see [48, Proposition 4.4]). We denote by k_L the dense onto frame homomorphism $k_L \colon \beta L \to \lambda L$ defined by $k_L(I) = \langle I \rangle_\sigma$, where $\langle \cdot \rangle_\sigma$ signifies σ-ideal generation in Coz$L$.

Realcompact frames are coreflective in **CRegFrm** (see, for instance, [14] and [50], for details). The realcompact coreflection of L, denoted by υL, is constructed in the following manner. For any $t \in L$, let $[t] = \{x \in \mathrm{Coz}L \mid x \leq t\}$, so that if $c \in \mathrm{Coz}L$, then $[c]$ is the principal ideal of $\mathrm{Coz}L$ generated by c. The map $\ell\colon \lambda L \to \lambda L$ given by

$$\ell(J) = \left[\bigvee J\right] \wedge \bigwedge\{P \in (\lambda L) \mid J \leq P\}$$

is a nucleus. The frame υL is defined to be $\mathrm{Fix}(\ell)$. We denote by ℓ_L the dense onto frame homomorphism $\lambda L \to \upsilon L$ effected by ℓ. The join map $\upsilon_L\colon \upsilon L \to L$ is a dense onto frame homomorphism. For any L we have

$$\mathrm{Coz}(\lambda L) = \mathrm{Coz}(\upsilon L) = \{[c] \mid c \in \mathrm{Coz}L\},$$

a consequence of which is that each of the maps $\lambda_L\colon \lambda L \to L$ and $\upsilon_L\colon \upsilon L \to L$ is a C-quotient map (see [7] for the definition of a C-quotient map).

1.6 C- and C^*-embedding.

The notions of C- and C^*-embedding have elegantly been extended to frames by Ball and Walters-Wayland [7]. Here we recall the definition and some characterizations.

An onto frame homomorphism $h : L \to M$ is said to be a C-quotient map if for every frame homomorphism $\gamma : \mathfrak{O}\mathbb{R} \to M$ there is a frame homomorphism $\delta : \mathfrak{O}\mathbb{R} \to L$ such that $h \circ \delta = \gamma$. Restricting γ to bounded functions defines C^*-quotient maps. As pointed out in [7], these notions are precise extensions to frames of C- and C^*-embeddings of subspaces in the sense that a subspace S of X is C-embedded (resp. C^*-embedded) if and only if the frame homomorphism $\mathfrak{O}X \to \mathfrak{O}S$, induced by the subspace embedding $S \hookrightarrow X$, is a C-quotient (resp. C^*-quotient) map.

A frame homomorphism $h : L \to M$ is said to be

(1) *coz-codense* if the only cozero element it maps to the top element is the top element.

(2) *almost coz-codense* if for each $c \in \operatorname{Coz} L$ such that $h(c) = 1$, there exists $d \in \operatorname{Coz} L$ such that $c \vee d = 1$ and $h(d) = 0$.

(3) *coz-onto* if for every $d \in \operatorname{Coz} M$, there exists $c \in \operatorname{Coz} L$ such that $h(c) = d$.

The following results are taken from [7, Theorem 7.1.1] and [7, Theorem 7.2.7].

Proposition 1.6.1 *The following are equivalent for a quotient map $h : L \to M$:*

(1) h is a C^*-quotient map.

(2) Every binary cozero cover of M is refined by the image of a binary cozero cover of L.

(3) Every binary cozero cover of M is the image of a binary cozero cover of L.

Proposition 1.6.2 *The following are equivalent for a quotient map $h : L \to M$:*

(1) h is a C-quotient map.

(2) h is a C^*-quotient map and almost coz-codense.

(3) h is coz-onto and almost coz-codense.

Chapter 2

On variants of C-embedding

In this chapter, we define C_1- and strong C_1-quotient maps and characterize them in several ways. One such characterization is in terms of contraction of maximal ideals of rings of continuous functions. Incidentally, this characterization leads to a point-sensitive result which has hitherto not been published. In Section 2.2, we define C_2-quotient maps and show that they are coz-onto. The latter part of the chapter consists of what we call uplifting quotient maps. Such maps generalize Aull's [2] notion of FF-embedding. Throughout this chapter our frames are completely regular.

2.1 C_1- and strong C_1-quotient maps

These maps will be defined in terms of complete separation. We therefore start by recalling from [7] what it means to say two quotients are completely separated. Whenever we consider two quotients

$$L \overset{\alpha}{\to} A \quad \text{and} \quad L \overset{\beta}{\to} B$$

of L, we shall frequently display them in a single diagram as follows:

$$A \stackrel{\alpha}{\leftarrow} L \stackrel{\beta}{\rightarrow} B.$$

Two quotients as above are said to be *completely separated* if there are cozero elements c and d of L such that $c \vee d = 1$, $\alpha(c) = 0_A$ and $\beta(d) = 0_B$.

Definition 2.1.1 A quotient map $L \stackrel{h}{\rightarrow} M$ is a C_1-*quotient map* if whenever $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$ are such that $h(c) \vee d = 1$, then the quotients $\uparrow d \stackrel{\varphi}{\leftarrow} L \stackrel{\kappa_c}{\rightarrow} \uparrow c$, where $\varphi = \kappa_d \cdot h$, are completely separated. It is a *strong* C_1-*quotient map* if whenever $a \in L$ and $d \in \operatorname{Coz} M$ are such that $h(a) \in \operatorname{Coz} M$ and $h(a) \vee d = 1$, then the quotients $\uparrow d \stackrel{\varphi}{\leftarrow} L \stackrel{\kappa_c}{\rightarrow} \uparrow c$ are completely separated, where φ is as before, $\kappa_c(x) = x \vee c$ and κ_d is similarly defined.

These definitions are, respectively, adaptations to frames of C_1-embedded subspaces defined by Ishii and Ohta [40] and strongly C_1-embedded subspaces considered in [34]. As in spaces, the former is a weakening of the concept of C-quotients, and it implies almost coz-codensity.

Lemma 2.1.2 *The following statements hold for a quotient map* $L \stackrel{h}{\rightarrow} M$*:*

(a) h *is a* C_1*-quotient map if and only if for every* $c \in \operatorname{Coz} L$ *and* $d \in \operatorname{Coz} M$ *such that* $h(c) \vee d = 1$*, there exists* $u \in \operatorname{Coz} L$ *such that* $u \vee c = 1$ *and* $h(u) \leq d$.

(b) Every C*-quotient map is a* C_1*-quotient map.*

(c) Every C_1*-quotient map is almost coz-codense.*

Proof (a) The claimed elementwise characterization of C_1-quotient map is simply a restatement of the definition.

(b) Suppose h is a C-quotient map. Let $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$ be such that $h(c) \vee d = 1$. Then, in view of h being coz-onto, there exists $u \in \operatorname{Coz} L$ such that $h(u) = d$. Therefore $h(c \vee u) = 1$. Since h is almost coz-codense, there exists $v \in \operatorname{Coz} L$ such that $v \vee c \vee u = 1$ and $h(v) = 0$. Thus, $(v \vee u) \vee c = 1$ and $h(v \vee u) \leq d$. Therefore h is a C_1-quotient map.

(c) Suppose h is a C_1-quotient map. If $c \in \operatorname{Coz} L$ is such that $h(c) = 1$, then $h(c) \vee 0 = 1$. A routine calculation using the charaterization shows that h is almost coz-codense. ∎

In [31] a homomorphism $h : L \to M$ is said to be a *W-map* if $h^{\beta} r_L(c) = r_M h(c)$ for each $c \in \operatorname{Coz} L$. It is shown there that h is a W-map if and only if for each $c \in \operatorname{Coz} L$ and $y \in M$, $y \prec\!\!\prec h(c)$ implies $y \leq h(s)$ for some $s \prec\!\!\prec c$ in L. We show that C_1-quotient maps are precisely the surjective W-maps. We use the preceding lemma.

Corollary 2.1.3 *A quotient map $h : L \to M$ is a C_1-quotient map if and only if it is a W-map.*

Proof ($\Rightarrow$): Suppose $c \in \operatorname{Coz} L$ and $y \in M$ are such that $y \prec\!\!\prec h(c)$. Choose $d \in \operatorname{Coz} M$ such that $y \wedge d = 0$ and $d \vee h(c) = 1$. Since h is a C_1-quotient map, there exists $u \in \operatorname{Coz} L$ such that $u \vee c = 1$ and $h(u) \leq d$. By normality of $\operatorname{Coz} L$, we can find a cozero element $s \prec\!\!\prec c$ such that $s \vee u = 1$. Since $y \wedge h(u) = 0$, $y \leq h(s)$. Therefore h is a W-map.

($\Leftarrow$): Let $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$ be such that $h(c) \vee d = 1$. Since $h(c) \in \operatorname{Coz} M$, the normality of $\operatorname{Coz} M$ yields $y \in M$ such that $y \prec\!\!\prec h(c)$ and $y \vee d = 1$. By hypothesis, $y \leq h(s)$ for some $s \in L$ with $s \prec\!\!\prec c$. Choose $u \in \operatorname{Coz} L$ such that $s \wedge u = 0$ and $u \vee c = 1$. Since $h(u) \wedge h(s) = 0$, $h(u) \wedge y = 0$, and hence $h(u) \leq d$. Therefore h is a C_1-quotient map.
∎

Remark 2.1.4 In [31, Proposition 4.4] it is shown that the Booleanization map is a W-map if and only if L is a *P-frame*, where the latter means $c \vee c^{*} = 1$ for each $c \in \operatorname{Coz} L$. Since the Booleanization map is onto, it follows that it is a C_1-quotient map if and only if L is a P-frame.

In [33, Proposition 4.2], it is shown that an onto homomorphism $h : L \to M$ is a C-quotient map if and only if for each maximal ideal J of $\operatorname{Coz} M$, there is a maximal ideal I of $\operatorname{Coz} L$ such that $h[I] = J$. We characterize C_1-quotient maps similarly. The characterization will also show (what we have already observed) that any C-quotient map is a C_1-quotient map. Following [33], given a frame homomorphism $h : L \to M$ and an ideal J of $\operatorname{Coz} M$, we define the ideal $h^{\#}J$ of $\operatorname{Coz} L$ by

$$h^{\#}J = \{c \in \operatorname{Coz} L \mid h(c) \in J\}.$$

That $h^{\#}J$ is indeed an ideal with $h[h^{\#}J] \subseteq J$ is easy to check. That $h[h^{\#}J] \subseteq J$ follows from the definition of $h^{\#}J$.

Proposition 2.1.5 *The following are equivalent for a quotient map $L \overset{h}{\to} M$.*

(1) h is a C_1-quotient map.

(2) For every maximal ideal J of $\operatorname{Coz} M$, $h^{\#}J$ is a maximal ideal of $\operatorname{Coz} L$.

(3) For every maximal ideal J of $\operatorname{Coz} M$, there is a maximal ideal I of $\operatorname{Coz} L$ such that $h[I] \subseteq J$.

Proof (1) $\Rightarrow$ (2) : Let $u \in \operatorname{Coz} L$ be such that $u \vee c \neq 1$ for all $c \in h^{\#}J$. We must show that $u \in h^{\#}J$, that is, $h(u) \in J$. Suppose, by way of contradiction, that $h(u) \notin J$. Since J is a maximal ideal of $\operatorname{Coz} M$, this implies that there exists $w \in J$ such that $h(u) \vee w = 1$. Since h is a C_1-quotient map, there exists $v \in \operatorname{Coz} L$ such that $u \vee v = 1$ and $h(v) \leq w$. But this implies $h(v) \in J$ and hence $v \in h^{\#}J$, contradicting the nature of u. Therefore $h^{\#}J$ is a maximal ideal of $\operatorname{Coz} L$.

(2) $\Rightarrow$ (3) : This is immediate since $h[h^{\#}J] \subseteq J$ for any ideal J of $\operatorname{Coz} M$.

(3) $\Rightarrow$ (1) : Suppose, on the contrary, that h is not a C_1-quotient map. Then there exists $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$ such that $h(c) \vee d = 1$ but the quotients $\uparrow d \stackrel{\varphi}{\leftarrow} L \stackrel{\kappa_c}{\rightarrow} \uparrow c$, where $\varphi = \kappa_d \cdot h$, are not completely separated. Then, for any $z \in \operatorname{Coz} L$ such that $z \prec\!\prec c$, we cannot have $h(z) \vee d = 1$, for otherwise, if $s \in \operatorname{Coz} L$ is such that $z \wedge s = 0$ and $s \vee c = 1$, then $h(z) \wedge h(s) = 0$, implying $h(s) \leq d$, so that

$$\kappa_c(c) = c = 0_{\uparrow c},\ \varphi(s) = h(s) \vee d = d = 0_{\uparrow d} \text{ and } s \vee c = 1,$$

implying the quotients $\uparrow d \stackrel{\varphi}{\leftarrow} L \stackrel{\kappa_c}{\rightarrow} \uparrow c$ are completely separated. Consequently, the ideal K of $\operatorname{Coz} M$ generated by the set $\{h(x) \vee d \mid x \in \operatorname{Coz} L, x \prec\!\prec c\}$ is proper. Let J be a maximal ideal of $\operatorname{Coz} M$ containing K. The hypothesis yields a maximal ideal I of $\operatorname{Coz} L$ such that $h[I] \subseteq J$. Now $d \in J$, and therefore $c \notin I$ since $h(c) \vee d = 1$. Since I is maximal, there therefore exists $u \in I$ such that $c \vee u = 1$. By the normality of the σ-frame $\operatorname{Coz} L$, there exists $w \in \operatorname{Coz} L$ such that $w \prec\!\prec c$ and $w \vee u = 1$. Then $h(w) \vee h(u) = 1$, a contradiction since both $h(w)$ and $h(u)$ are elements of the proper ideal J. ∎

The equivalence of (1) and (2) in this proposition enables us to present a characterization of C_1-quotient maps in terms of rings of continuous functions. An ideal Q of $\mathcal{R}L$ is a *z-ideal* if, for any $\alpha, \beta \in \mathcal{R}L$, $\operatorname{coz} \alpha = \operatorname{coz} \beta$ and $\alpha \in Q$ together imply $\beta \in Q$. Any maximal ideal of $\mathcal{R}L$ is a z-ideal. One checks routinely that:

1. If I is a maximal ideal of $\operatorname{Coz} L$, then $\operatorname{coz}^{-1}[I]$ is a maximal ideal of $\mathcal{R}L$.
2. If Q is a maximal ideal of $\mathcal{R}L$, then $\operatorname{coz}[Q]$ is a maximal ideal of L.

Lemma 2.1.6 *Let $h : L \rightarrow M$ be a frame homomorphism. Then:*

(1) For any z-ideal Q of $\mathcal{R}M$, $\operatorname{coz}^{-1}[h^{\#}(\operatorname{coz}[Q])] \subseteq (\mathcal{R}h)^{-1}[Q]$.

(2) For any ideal I of $\operatorname{Coz} M$, $\operatorname{coz}^{-1}[h^{\#}I] = (\mathcal{R}h)^{-1}[\operatorname{coz}^{-1}[I]]$.

Proof (1) Let $\alpha \in \operatorname{coz}^{-1}[h^{\#}\operatorname{coz}[Q]]$. Then $\operatorname{coz}\alpha \in h^{\#}(\operatorname{coz}[Q])$, so that

$$h(\operatorname{coz}\alpha) = \operatorname{coz}((\mathcal{R}h)(\alpha)) \in \operatorname{coz}[Q].$$

Since Q is a z-ideal, this implies $(\mathcal{R}h)(\alpha) \in Q$, and hence $\alpha \in (\mathcal{R}h)^{-1}[Q]$. Therefore $\operatorname{coz}^{-1}[h^{\#}(\operatorname{coz}[Q])] \subseteq (\mathcal{R}h)^{-1}[Q]$. In fact, the reverse inclusion also holds even if Q is a mere ideal – but we do not need that.

(2) For any $\alpha \in \mathcal{R}L$, we have

$$\begin{aligned}
\alpha \in (\mathcal{R}h)^{-1}[\operatorname{coz}^{-1}[I]] &\Leftrightarrow (\mathcal{R}h)(\alpha) \in \operatorname{coz}^{-1}[I] \\
&\Leftrightarrow \operatorname{coz}((\mathcal{R}h)(\alpha)) \in I \\
&\Leftrightarrow h(\operatorname{coz}\alpha) \in I \\
&\Leftrightarrow \operatorname{coz}\alpha \in h^{\#}I \\
&\Leftrightarrow \alpha \in \operatorname{coz}^{-1}[h^{\#}I]
\end{aligned}$$

■

Recall that if $f : A \to B$ is a ring homomorphism and I is an ideal of B, then $f^{-1}[I]$ is an ideal of A called the *contraction* of I.

Corollary 2.1.7 *A quotient map $L \xrightarrow{h} M$ is a C_1-quotient map if and only if $\mathcal{R}h$ contracts maximal ideals to maximal ideals.*

Proof $(\Leftarrow)$: Let I be a maximal ideal of $\operatorname{Coz} M$. Then $\operatorname{coz}^{-1}[I]$ is a maximal ideal of $\mathcal{R}M$. By hypothesis, $(\mathcal{R}h)^{-1}[\operatorname{coz}^{-1}[I]]$ is a maximal ideal of $\mathcal{R}L$. That is, by Lemma 2.1.6, $\operatorname{coz}^{-1}[h^{\#}I]$ is a maximal ideal of $\mathcal{R}L$, and hence $\operatorname{coz}[\operatorname{coz}^{-1}[h^{\#}I]]$ is a maximal ideal of $\operatorname{Coz} L$. But clearly,

$\mathrm{coz}\,[\mathrm{coz}^{-1}[h^{\#}I]] \subseteq h^{\#}I$, so $h^{\#}I$ is a maximal ideal of $\mathrm{Coz}\,L$. Therefore h is a C_1-quotient map by Proposition 2.1.5.

$(\Rightarrow)$: Let Q be a maximal ideal of $\mathcal{R}M$. Then $\mathrm{coz}\,[Q]$ is a maximal ideal of $\mathrm{Coz}\,M$. By Proposition 2.1.5, $h^{\#}(\mathrm{coz}\,[Q])$ is a maximal ideal of $\mathrm{Coz}\,L$, and hence $\mathrm{coz}^{-1}[h^{\#}(\mathrm{coz}\,[Q])]$ is a maximal ideal of $\mathcal{R}L$. Since every maximal ideal is a z-ideal, it follows from Lemma 2.1.6 that $\mathrm{coz}^{-1}[h^{\#}(\mathrm{coz}\,[Q])] \subseteq (\mathcal{R}h)^{-1}[Q]$. Therefore $(\mathcal{R}h)^{-1}[Q]$ is a maximal ideal, and we are done. ∎

For any space X, the rings $C(X)$ and $\mathcal{R}(\mathfrak{O}X)$ are isomorphic (see, for instance, [8]). Consequently we have the following:

Corollary 2.1.8 *A subspace S of a Tychonoff space X is C_1-embedded if and only if the ring homomorphism $C(X) \to C(S)$, given by $f \mapsto f|_S$, contracts maximal ideals to maximal ideals.*

Remark 2.1.9 The result in Corollary 2.1.7 can actually be deduced from [31, Proposition 4.4], in light of C_1-quotient maps being W-maps. However, whereas the proof we have presented here does not require knowledge of what maximal ideals of $\mathcal{R}L$ look like, that of [31, Proposition 4.4] makes explicit use of the description of maximal ideals of $\mathcal{R}L$.

We recall from [33] the following definition. A homomorphism $h : L \to M$ is *weakly coz-onto* if $a \wedge b = 0$ in $\mathrm{Coz}\,M$ implies $a = h(c)$ for some $c \in \mathrm{Coz}\,L$ or $b = h(d)$ for some $d \in \mathrm{Coz}\,L$. Coz-onto homomorphisms are obviously weakly coz-onto. It is shown in [33, Proposition 3.13] that if $h : L \to M$ is weakly coz-onto and M is realcompact, then h is coz-onto. The following result shows, among other things, that a C_1-quotient map is coz-onto if and only if it is weakly coz-onto, if and only if it is a C-quotient map. As in [33], we say an ideal I of $\mathrm{Coz}\,L$ is *respected* by a homomorphism $h : L \to M$ in case $h(x) \neq 1$ for each $x \in I$.

Proposition 2.1.10 *The following are equivalent for a C_1-quotient map $L \xrightarrow{h} M$:*

(1) h is a C-quotient map.

(2) h is coz-onto.

(3) h is weakly coz-onto.

(4) For every maximal ideal I of $\operatorname{Coz} L$ respected by h, $h[I]$ is a maximal ideal of $\operatorname{Coz} M$.

Proof The implication (1) $\Rightarrow$ (2) follows from Proposition 1.6.2 and (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) : Since h is almost coz-codense, it suffices to show that it is also a C^*-quotient map by Proposition 1.6.2. We use Proposition 1.6.1. So, let $c \vee d = 1$ in $\operatorname{Coz} M$. Since $\operatorname{Coz} M$ is a normal σ-frame, there exist $u, v \in \operatorname{Coz} M$ such that $u \wedge v = 0$ and $u \vee c = v \vee d = 1$. Since h is weakly coz-onto, at least one of u and v is the image of a cozero element. Say $u = h(s)$ for some $s \in \operatorname{Coz} L$. Then $h(s) \vee c = 1$. Since h is a C_1-quotient map, there exists $t \in \operatorname{Coz} L$ such that $s \vee t = 1$ and $h(t) \leq c$. Therefore the cozero cover $\{c, d\}$ is refined by the image of the cozero cover $\{s, t\}$ since $u \wedge v = 0$ and $v \vee d = 1$ imply $h(s) = u \leq d$.

(2) $\Rightarrow$ (4) : This follows from [33, Proposition 3.7].

(4) $\Rightarrow$ (1) : By [33, Proposition 4.2], it suffices to show that every maximal ideal of $\operatorname{Coz} M$ is the image of a maximal ideal of $\operatorname{Coz} L$ under h. So, let J be a maximal ideal of $\operatorname{Coz} M$. Since h is a C_1-quotient map, there exists a maximal ideal I of $\operatorname{Coz} L$ such that $h[I] \subseteq J$. By (4), $h[I]$ is a maximal ideal of $\operatorname{Coz} M$, and therefore $h[I] = J$, by maximality. ∎

Recall that a frame L is *pseudocompact* if whenever (a_n) is a sequence in L such that $a_n \prec\!\!\prec a_{n+1}$ for each n and $\bigvee a_n = 1$, then $a_k = 1$ for some k. We will use two different characterizations of pseudocompact frames to prove the following corollary. The first is that

L is pseudocompact if and only if every countable cover by cozero elements has a finite subcover (see [12]). The second is that L is pseudocompact if and only if $\beta L \to L$ is coz-codense, as was proved by Walters-Wayland in her doctoral thesis [58].

Recall that an ideal I of a σ-complete (i.e. has all countable joins) lattice is said to be *σ-proper* if for any countable $S \subseteq I$, $\bigvee S \neq 1$. It is a *σ-ideal* if it is closed under countable joins.

Corollary 2.1.11 *The following are equivalent for a completely regular frame L:*

(1) Every homomorphism onto L is a C_1-quotient map.

(2) $\beta L \to L$ is a C_1-quotient map.

(3) L is pseudocompact.

Proof (1) $\Rightarrow$ (2) : Trivial.

(2) $\Rightarrow$ (3) : If $\beta L \to L$ is a C_1-quotient map, then it is almost coz-codense, and hence coz-codense by density. Therefore L is pseudocompact by the characterization cited from [58].

(3) $\Rightarrow$ (1) : Let $h : M \to L$ be a quotient map. Let I be a maximal ideal of $\operatorname{Coz} L$. We aim to show that $h^{\#}I$ is a maximal ideal of $\operatorname{Coz} M$. Observe that I is σ-proper, for if S were a countable subset of I with $\bigvee S = 1$, then pseudocompactness of L would furnish a finite $T \subseteq S$ such that $\bigvee T = 1$, which would imply $1 \in I$, contrary to I being a proper ideal. Thus, by the second part of [33, Lemma 3.6], I is a σ-ideal. For any countable $S \subseteq h^{\#}I$, $h[S]$ is a countable subset of I, and therefore $\bigvee h[S] = h(\bigvee S) \in I$, whence $\bigvee S \in h^{\#}I$. Therefore $h^{\#}I$ is a σ-ideal. Since I is prime (every maximal ideal of $\operatorname{Coz} L$ is prime), a straightforward

calculation shows that $h^{\#}I$ is prime. Consequently, by the first part of [33, Lemma 3.6], $h^{\#}I$ is a maximal ideal of $\operatorname{Coz} M$. Therefore h is a C_1-quotient map by Proposition 2.1.5. ∎

Remark 2.1.12 Following [19], a quotient map $L \xrightarrow{h} M$ is said to *satisfy property* (β) if for every $c, d \in \operatorname{Coz} L$ with $h(c) \vee h(d) = 1$, there exist $u, v \in \operatorname{Coz} L$ such that $u \vee v = 1$, $h(u) \leq h(c)$ and $h(v) \leq h(d)$. A straightforward calculation shows that every coz-onto C_1-quotient map satisfies (β). However, [coz-onto $+(\beta) \not\Rightarrow C_1$]. Indeed, it is shown in [24, Proposition 2.6] that a quotient map is C^* if and only if it is coz-onto and satisfies property (β). Thus, if L is a non-pseudocompact frame, then $\beta L \to L$ is a coz-onto quotient map which satisfies property (β) but, by the preceding corollary, not C_1.

We now turn our attention to strong C_1-quotient maps. In order to characterize them we need the following definition.

Definition 2.1.13 A frame homomorphism $h : L \to M$ is *coz-heavy* if for each $a \in L$ such that $h(a) = 1$, there exists $c \in \operatorname{Coz} L$ such that $c \leq a$ and $h(c) = 1$.

Topologically speaking, recall that a subspace S of X is said to be *normally placed* in case for every open set $U \subseteq X$ which contains S, there is a cozero-set V of X such that $S \subseteq V \subseteq U$. Thus, S is normally placed if and only if the frame homomorphism $\mathfrak{O}X \to \mathfrak{O}S$, sending $O \in \mathfrak{O}X$ to $O \cap S$, is coz-heavy. We will see below that every quotient map onto a Lindelöf frame is coz-heavy.

For brevity, we shall, at times, say a quotient map is C_1 or C or C^*, if it is a C_1-quotient map, etc. In the proof of the following proposition, we use the fact that a quotient map is a C-quotient map if and only if it is C^* and almost coz-codense by Proposition 1.6.2.

Proposition 2.1.14 *A necessary and sufficient condition that a quotient map be strongly C_1 is that it be a coz-heavy C-quotient map.*

Proof Suppose $h : L \to M$ is strongly C_1. We show first that it is C^*. We use Proposition 1.6.1. Consider a, $b \in \operatorname{Coz} M$ such that $a \vee b = 1$. Since h is onto, $h(h_*(a)) \vee b = 1$. Therefore, by hypothesis, the quotients $\uparrow h_*(a) \stackrel{\kappa_{h_*(a)}}{\leftarrow} L \stackrel{g}{\to} \uparrow b$, where $g = \kappa_b \cdot h$, are completely separated. So there exist u, $v \in \operatorname{Coz} L$ such that

$$u \vee v = 1,\ \kappa_{h_*(a)}(u) = 0_{\uparrow h_*(a)} \text{ and } g(v) = 0_{\uparrow b}.$$

The last two equalities reduce to $u \leq h_*(a)$, and $h(v) \leq b$. Therefore h is C^* by the characterization in Proposition 1.6.1.

Next, h is almost coz-codense since a strong C_1-quotient map is obviously C_1, and hence almost coz-codense as observed earlier. That h is also coz-heavy is immediate since $h(a) = 1$ implies $h(a) \vee 0 = 1$, and, of course $0, 1 \in \operatorname{Coz} M$.

Conversely, suppose the condition holds. Let $a \in L$ and $b \in \operatorname{Coz} M$ be such that $h(a) \in \operatorname{Coz} M$ and $h(a) \vee b = 1$. Since h is C^*, there exist u, $v \in \operatorname{Coz} L$ such that

$$u \vee v = 1,\ h(u) = h(a) \text{ and } h(v) = b.$$

Since $\operatorname{Coz} L$ is a normal lattice, there exist c, $d \in \operatorname{Coz} L$ such that

$$c \wedge d = 0 \text{ and } u \vee c = 1 = v \vee d.$$

The element $t = a \vee c$ of L is such that

$$h(t) = h(a) \vee h(c) = h(u) \vee h(c) = 1.$$

Since h is coz-heavy, there is a $w \in \operatorname{Coz} L$ such that $w \leq t$ and $h(w) = 1$. Since h is almost coz-codense, there exists $z \in \operatorname{Coz} L$ such that $w \vee z = 1$ and $h(z) = 0$. Now, $v \vee z$ and $w \wedge d$ are cozero elements of L with

$$(v \vee z) \vee (w \wedge d) = (v \vee z \vee w) \wedge (v \vee z \vee d) = 1.$$

We show that they witness the complete separation of the quotients $\uparrow a \stackrel{\kappa_a}{\leftarrow} L \stackrel{g}{\to} \uparrow b$, where $g = \kappa_b \cdot h$. Since $h(z) = 0$, we have

$$h(v \vee z) = h(v) = b,$$

which implies

$$(\kappa_b \cdot h)(v \vee z) = b \vee b = 0_{\uparrow b}.$$

On the other hand, $w \wedge d \leq d \leq u$, because $u \vee c = 1$ and $d \wedge c = 0$. Consequently,

$$\kappa_a(w \wedge d) = a = 0_{\uparrow a},$$

and hence h is a strong C_1-quotient map. ■

Corollary 2.1.15 *In **RegFrm**, a dense strongly C_1-quotient map is an isomorphism.*

Proof Let $h : L \to M$ be such a homomorphism. Let $a \in L$ be such that $h(a) = 1$. Since h is a strongly C_1-quotient map, it follows, by Proposition 2.1.14, that h is a coz-heavy C-quotient map. By coz-heaviness there exists $c \in \operatorname{Coz} L$ such that $c \leq a$ and $h(c) = 1$. By virtue of h being a dense C-quotient map, it is coz-codense. Thus $c = 1$ and therefore $a = 1$. Thus h is codense, and therefore one-one, and therefore an isomorphism. ■

Since the Booleanization map is dense, Corollary 2.1.15 yields:

Corollary 2.1.16 *In **RegFrm**, the map $\flat : L \to \mathcal{B}L$ is a strongly C_1-quotient map if and only if L is Boolean.*

Now, in the spirit of Corollary 2.1.11, we characterize those frames such that every homomorphism onto them is coz-heavy. They are precisely the Lindelöf frames. From this characterization we shall, en passant, obtain a different proof of [7, Proposition 8.2.14]. We need two lemmas.

Lemma 2.1.17 *In a completely regular frame L, any quotient map onto a Lindelöf frame is coz-heavy.*

Proof Let M be Lindelöf and $h : L \to M$ be a quotient map. Suppose $a \in L$ is such that $h(a) = 1$. By complete regularity, we have

$$\bigvee\{h(c) \mid c \in \operatorname{Coz} L \text{ and } c \leq a\} = 1.$$

Since M is Lindelöf, there are countably many $c_n \leq a$ in $\operatorname{Coz} L$ such that $h(\bigvee c_n) = 1$. Therefore $c = \bigvee c_n$ is a cozero element of L such that $c \leq a$ and $h(c) = 1$. ∎

Lemma 2.1.18 *A homomorphic image of a completely regular Lindelöf frame under a coz-heavy homomorphism is Lindelöf.*

Proof Let $L \xrightarrow{h} M$ be a coz-heavy quotient map with L completely regular and Lindelöf. Let C be a cover of M. For each $c \in C$, take $b_c \in L$ such that $h(b_c) = c$. Put $b = \bigvee\{b_c \mid c \in C\}$. Then $h(b) = 1$, and so, in view of h being coz-heavy, there exists $d \in \operatorname{Coz} L$ such that $d \leq b$ and $h(d) = 1$. Since $d \leq b$, and since cozero elements of any Lindelöf frame are Lindelöf [12], there are countably many elements c_n in C such that $d \leq \bigvee_n b_{c_n}$. Consequently, $1 = h(d) \leq \bigvee h(b_{c_n}) = \bigvee c_n$. Therefore M is Lindelöf. ∎

Recall that a *Lindelöfication* of a frame L is a dense onto homomorphism $M \to L$ with M regular Lindelöf. Recall from Chapter 1 that $\lambda L \to L$ denotes the regular Lindelöf coreflection of L. The following proposition follows from the preceding two lemmas.

Proposition 2.1.19 *The following are equivalent for a completely regular frame L:*

(1) Every homomorphism onto L is coz-heavy.

(2) $\beta L \to L$ is coz-heavy.

(3) L is Lindelöf.

(4) Every compactification $M \to L$ is coz-heavy.

(5) Some compactification $M \to L$ is coz-heavy.

(6) Every Lindelöfication $M \to L$ is coz-heavy.

(7) $\lambda L \to L$ is coz-heavy.

(8) Some Lindelöfication $M \to L$ is coz-heavy.

Ball and Walters-Wayland show in [7, Proposition 8.2.14] that a completely regular frame L is Lindelöf if and only if every dense C-quotient map $M \to L$ is an isomorphism. This result first appeared in [49] stated in localic rather than frame-theoretic terms. It also follows from Lemma 2.1.17 and Proposition 2.1.19.

Corollary 2.1.20 *A completely regular frame L is Lindelöf if and only if every dense C-quotient map $h : M \to L$ is an isomorphism.*

Proof Let L be Lindelöf and $M \to L$ be a dense C-quotient map. Then, by Lemma 2.1.17, $M \to L$ is coz-heavy. Being a C-quotient map, it is almost coz-codense, and being dense, it is therefore coz-codense, and hence codense by coz-heaviness. It is therefore one-one and hence an isomorphism. The converse follows from Proposition 2.1.19 since $\lambda L \to L$ is a dense C-quotient map. ∎

Remark 2.1.21 For a completely regular L, if L is pseudocompact and Lindelöf, then the homomorphism $\beta L \to L$ is codense since it is coz-heavy by Proposition 2.1.19. Therefore it is an isomorphism, and hence L is compact. That a pseudocompact Lindelöf frame is compact was first proved in [12].

We now observe that frames onto which every homomorphism is strongly C_1 are precisely the compact ones.

Corollary 2.1.22 *The following are equivalent for a completely regular frame L:*

(1) Every homomorphism onto L is strongly C_1.

(2) $\beta L \to L$ is strongly C_1.

(3) L is compact.

Proof (1) $\Rightarrow$ (2) : Trivial.

(2) $\Rightarrow$ (3) : If (2) holds, then $\beta L \to L$ is coz-heavy, and therefore L is Lindelöf by Proposition 2.1.19. Also, $\beta L \to L$ is C_1, and therefore L is pseudocompact by Corollary 2.1.11. Therefore L is compact.

(3) $\Rightarrow$ (1) : If L is compact, then it is Lindelöf, and hence every homomorphism onto L is coz-heavy by Proposition 2.1.19. By [25, Proposition 2.8], if L is compact (so that it is almost compact), then every homomorphism onto L is a C-quotient map. Thus, every homomorphism onto L is strongly C_1. ∎

We end the section by examining if the property of being coz-heavy, C_1 or strongly C_1 lifts to the Stone extension of a quotient map. Because we have defined these properties for onto homomorphisms, in order for our investigation to make sense, the Stone extension must be onto. As shown in [24, Proposition 2.1], the Stone extension of a quotient map is onto if and only if the map is a C^*-quotient map. We shall therefore impose that condition. We start with an example of a C^*-quotient map which is not C_1 but the Stone extension of which is C_1.

Example 2.1.23 Let M be a non-pseudocompact frame and let $L \xrightarrow{h} M$ be its Stone-Čech compactification. Then h is a C^*-quotient map, which is not coz-codense since L is not pseudocompact. Consequently, h is not almost coz-codense, and is therefore not a C_1-quotient map. But h^β is an isomorphism since the density of h clearly implies that of h^β, making the latter one-one (and hence an isomorphism) by regularity of its domain and compactness of its codomain.

Proposition 2.1.24 *Let $L \xrightarrow{h} M$ be almost coz-codense and a C_1-quotient map. Then $\beta L \to \beta M$ is a C_1-quotient map. In fact, h^β is a C-quotient map.*

Proof Since βM is Lindelöf, h^β is coz-onto by [33, Proposition 3.2]. To see that it is also almost coz-codense, let $I \in \operatorname{Coz} \beta L$ be such that $h^\beta(I) = 1_{\beta M}$. Since

$$I = \bigvee \{r_L(x) \mid x \in I\},$$

we have

$$1_{\beta M} = h^\beta\left(\bigvee_{x \in I} r_L(x)\right) = \bigvee_{x \in I} h^\beta(r_L(x)).$$

By compactness of βM and the fact that above every element of I is a cozero element belonging to I, there exists a cozero element c in I such that $h^\beta(r_L(c)) = 1_{\beta M}$. Thus

$$h(c) = h\sigma_L(r_L(c)) = \sigma_M h^\beta(r_L(c)) = \sigma_M(1_{\beta M}) = 1.$$

Since h is almost coz-codense, there exists $d \in \operatorname{Coz} L$ such that

$$d \vee c = 1 \text{ and } h(d) = 0.$$

Then $r_L(c) \vee r_L(d) = 1_{\beta L}$, so that $I \vee r_L(d) = 1_{\beta L}$. Since βL is normal, there exists $J \in \operatorname{Coz} \beta L$ such that $J \leq r_L(d)$ and $J \vee I = 1_{\beta L}$. Now, $h^\beta(J) = 0_{\beta M}$. Indeed, if $y \in h^\beta(J)$, then $y \leq h(t)$ for some $t \in J$. Thus $t \leq d$, and so $y \leq h(d) = 0$. Therefore h^β is almost coz-codense, and is therefore a C-quotient map, and hence a C_1-quotient map. ∎

Remark 2.1.25 The reader will have noted that, in the proof of the preceding result, we did not use the fact that I is a cozero element. All we used is that it is mapped to the top element. This might create the impression that h^β satisfies a condition stronger than almost coz-codensity. However, in light of h^β being coz-heavy (since its codomain is Lindelöf), the "stronger" condition is actually equivalent to almost coz-codensity.

Similar to Example 2.1.23, if L is not Lindelöf, then $\beta L \to L$ is a C^*-quotient map which is not coz-heavy, but whose Stone extension is coz-heavy. Of course, if the Stone extension of a quotient map is onto, then it is coz-heavy in view of its codomain being Lindelöf. Also, if L is not compact, then $\beta L \to L$ is a C^*-quotient map which is not strongly C_1 but whose Stone extension is strongly C_1.

Proposition 2.1.26 *Let* $L \xrightarrow{h} M$ *be a strongly* C_1*-quotient map. Then* $\beta L \xrightarrow{h^\beta} \beta M$ *is a strongly* C_1*-quotient map.*

Proof The hypothesis on h makes it almost coz-codense which is C_1. Therefore, by Proposition 2.1.24, h^β is a C-quotient map. It is also coz-heavy since its codomain is Lindelöf. ∎

In analogy with the Stone extension of a homomorphism $h : L \to M$, define the *Lindelöf extension* of h to be the frame homomorphism $h^\lambda : \lambda L \to \lambda M$ given by $I \mapsto \langle h[I] \rangle_\sigma$, where $\langle \cdot \rangle_\sigma$ denotes σ-ideal generation. Then h^λ is the unique "lift" of h to the regular Lindelöf coreflections – just like h^β is the unique lift of h to the Stone-Čech compactifications. It is shown in [24, Lemma 2.3] that h^λ is onto if and only if h is coz-onto. The reader might wonder if the C_1- and strong C_1- properties are inherited by h^λ for coz-onto h. They are. To see this, recall that $\operatorname{Coz} \lambda L = \{[c] | c \in \operatorname{Coz} L\}$, where $[c] = \{u \in \operatorname{Coz} L \mid u \leq c\}$. Furthermore, for each $c \in \operatorname{Coz} L$, $h^\lambda([c]) = [h(c)]$. These observations make it apparent that *if h is a coz-onto quotient map, then h is C_1 if and only if h^λ is C_1.*

On the other hand, if L is not Lindelöf, then $\lambda L \to L$ is a coz-onto quotient map which is not strongly C_1, but whose Lindelöf extension is strongly C_1. Conversely, if $h : L \to M$ is strongly C_1 (so that it is already coz-onto), then h is a C-quotient map, and therefore, by [24, Proposition 2.4], h^λ is a C-quotient map which is coz-heavy, by virtue of its codomain being Lindelöf. Therefore h^λ is strongly C_1.

2.2 C_2-quotient maps

Now we turn to C_2-quotient maps. Recall the definition of a locally finite subset from Chapter 1.

Recall from [40] that a subspace Y of a space X is said to be C_2-*embedded* in X if, for any countable cozero-set cover $\mathcal{U}$ of Y, there exists a locally finite countable collection $\mathcal{V}$ of cozero-sets of X such that $\mathcal{V}$ covers Y and $\mathcal{V} \cap Y$ refines $\mathcal{U}$, where $\mathcal{V} \cap Y$ is the trace of $\mathcal{V}$ on Y, that is $\mathcal{V} \cap Y = \{V \cap Y \mid V \in \mathcal{V}\}$.

Definition 2.2.1 A quotient map $L \xrightarrow{h} M$ is a C_2-*quotient map* if for every countable cozero cover D of M there is a locally finite countable $C \subseteq \operatorname{Coz} L$ such that $\bigvee h[C] = 1$ and $h[C]$ refines D.

As we did in the case of C_1-quotient maps, we show that every C-quotient map is a C_2-quotient map. The argument in this case is not as immediate as in the previous case. A cover A is said to be *normal* if there are covers $A_n,\ n = 1, 2, \ldots$ such that

$$A = A_1 \text{ and } A_n \geq A_{n+1}A_{n+1} \text{ for all } n.$$

In [16], the authors show that every normal cover has a locally finite normal refinement. We shall need to know that if the normal cover is countable, then the locally finite normal refinement can be chosen to be countable as well.

Recall from [25] that if $A = \{a_n \mid n \in \mathbb{N}\}$ is a countable infinite normal cover of a frame L, then there is a countable cozero cover B of L which refines A. The author has shown that if B is finite, then cozero elements can be added to B to form a countably infinite cozero cover B' which refines A. Also if $A = \{a_1, \ldots, a_n\}$ is a finite normal cover of L, then there is a finite cozero cover $B = \{b_1, \ldots, b_k\}$ of L which refines A. If $k < m$, then cozero elements can be added to B to form a cozero cover B' with m elements which refines A. If $k > m$, then grouping together elements of B that are below the same element of A and taking joins, and also adding some cozero elements if necessary, we can form a cozero cover B' with m elements which refines A. Therefore every countable normal cover can be refined by a cozero cover of the same cardinality.

Let A be a countable locally finite normal cover of L. Since A is a countable normal cover, it follows that we can construct a cozero cover B of the same cardinality which refines A by grouping together elements of B that are below the same element of A and taking joins, and also adding some cozero elements if necessary. Furthermore, A is locally finite, so there is a cover W such that each $w \in W$ meets only finitely many elements of A. Since $b_n \leq a_n$ for every $n \in \mathbb{N}$, it follows that each $w \in W$ meets only finitely many elements of B. Thus *every countable locally finite normal cover can be refined by a countable locally finite cozero cover with the same cardinality.*

We observed above that a C_1-quotient map is almost coz-codense. Regarding C_2-quotient maps, we show that they are coz-onto. We use a characterization of coz-onto homomorphisms which is an extension to frames of [40, Lemma 3.5].

Lemma 2.2.2 *A quotient map $h : L \to M$ is coz-onto if and only if for every finite cozero cover D of M there is a countable $C \subseteq \operatorname{Coz} L$ such that $h[C]$ is a cover of M refining D.*

Proof The left-to-right implication is trivial. For the converse, let $d \in \operatorname{Coz} M$ and write $d = \bigvee d_n$, where $d_n \in \operatorname{Coz} M$ for each n and $d_n \prec\!\!\prec d$. For each n, take $s_n \in \operatorname{Coz} M$ such that

$s_n \wedge d_n = 0$ and $s_n \vee d = 1$. Put $U_n = \{s_n, d\}$ and note that U_n is a finite cozero cover of M. By hypothesis, find a countable $V_n \subseteq \operatorname{Coz} L$ satisfying the hypothesized condition. Put $c_n = \bigvee\{x \in V_n \mid h(x) \leq d\}$ and $c = \bigvee c_n$, and notice that, since V_n is countable, $c_n \in \operatorname{Coz} L$ for each n, and hence $c \in \operatorname{Coz} L$. We will show that $h(c) = d$. Clearly, $h(c) \leq d$. Since $\bigvee h[V_n] = 1$, we have that for each n,

$$h(c_n) \vee h\Big(\bigvee\{t \in V_n \mid h(t) \leq s_n\}\Big) = 1,$$

the consequence of which is that $h(c_n) \vee s_n = 1$. Thus, $d_n = (d_n \wedge h(c_n)) \vee (d_n \wedge s_n) = d_n \wedge h(c_n)$. Taking joins over all n, this yields $d = d \wedge \bigvee h(c_n) = d \wedge h(c)$, so that $d \leq h(c)$ and hence equality. Therefore h is coz-onto. ∎

Corollary 2.2.3 *Any C_2-quotient map is coz-onto.*

In light of the fact that a quotient map is a C-quotient map if and only if it is coz-onto and almost coz-codense [7], we have:

Corollary 2.2.4 *A quotient map is a C-quotient map if and only if it is both a C_1- and C_2-quotient map.*

2.3 Uplifting homomorphisms

In this section we consider quotient maps which extend Aull's [2] FF-embedded spaces. These quotient maps bear close resemblance to C^*-quotient maps. There are instances where they coincide.

Definition 2.3.1 A quotient map $L \xrightarrow{h} M$ is *uplifting* if, for every $a, b \in M$, $a \vee b = 1$ implies $h_*(a) \vee h_*(b) = 1$.

The term "uplifting" is borrowed from [6] where it is used to describe a σ-frame homomorphism $f : A \to B$ such that whenever $b_1 \vee b_2 = 1$ in B, there exist $a_1, a_2 \in A$ such that $a_1 \vee a_2 = 1$ and $f(a_i) \leq b_i$. Although our definition is in terms of right adjoints, we observe below that it could have been couched in exactly the same terms as that of [6]. In [51], Martínez considers injective frame homomorphisms with the property stated in the definition and calls them "capping". In [2], a subspace S of a topological space X is said to be FF-embedded if for any pair K_1, K_2 of disjoint closed subsets of S, then there is a pair C_1, C_2 of disjoint closed subsets of X such that $K_i = S \cap C_i$ for $i = 1, 2$. In the observations below, we demonstrate that S is FF-embedded if and only if the quotient map $\mathfrak{O}X \to \mathfrak{O}S$, induced by the subspace inclusion $S \hookrightarrow X$, is uplifting.

Observations 2.3.2 (1) A quotient map $L \xrightarrow{h} M$ is uplifting if and only if for any $x_1, x_2 \in M$ such that $x_1 \vee x_2 = 1$, then there exist $a_1, a_2 \in L$ such that $a_1 \vee a_2 = 1$ and $h(a_i) = x_i$ for $i = 1, 2$. The forward implication holds since $a = hh_*(a)$ for all a as h is onto. The converse holds because $a_i \leq h_*(x_i)$ if $h(a_i) = x_i$. Thus, a subspace is FF-embedded if and only if the induced frame homomorphism is uplifting.

(2) A quotient map $L \xrightarrow{h} M$ is uplifting if and only if for any $b_1, b_2 \in M$ such that $b_1 \vee b_2 = 1$, then there exist $a_1, a_2 \in L$ such that $a_1 \vee a_2 = 1$ and $h(a_i) \leq b_i$ for $i = 1, 2$. Consequently, if L and M are normal, then an onto frame homomorphism is uplifting as a frame homomorphism if and only if it is uplifting as a σ-frame homomorphism.

Examples 2.3.3 (1) Any closed quotient map $L \xrightarrow{h} M$ is uplifting, for if $a \vee b = 1$ in M, then $a \vee hh_*(b) = 1$, and hence by closedness, $1 = h_*(1) = h_*(a) \vee h_*(b)$. In particular, $L \xrightarrow{\kappa_a} \uparrow a$ is uplifting for each $a \in L$.

(2) In his doctoral thesis [20], Chen defines the graph of a homomorphism $h : L \to M$ to be the onto homomorphism $G(h) : L \oplus M \to M$ defined by $x \oplus y \mapsto h(x) \wedge y$. If h is uplifting, then so is its graph. Indeed, let $x \vee y = 1_M$. Then $h_*(x) \vee h_*(y) = 1_L$, and therefore

$(h_*(x) \oplus 1_M) \vee (h_*(y) \oplus 1_M) = 1_{L \oplus M}$. But $G(h)(h_*(x) \oplus 1_M) = x$ and $G(h)(h_*(y) \oplus 1_M) = y$, so $G(h)$ is uplifting. A consequence of this is that the co-diagonal map $\nabla : L \oplus L \to L$, given by $a \oplus b \mapsto a \wedge b$, is uplifting for any frame L since $\nabla = G(id_L)$.

Next we observe that for dense quotient maps, uplifting implies C^*. To prove this, we need the following lemma.

Lemma 2.3.4 *The right adjoint of a dense uplifting homomorphism preserves the rather below and the completely below relations.*

Proof Let $L \xrightarrow{h} M$ be a dense uplifting quotient map, and let $a \prec b$ in M. Pick $s \in M$ such that $a \wedge s = 0$ and $s \vee b = 1$. Then $h_*(a) \wedge h_*(s) = 0$ by denseness, and $h_*(s) \vee h_*(b) = 1$ since h is uplifting. So $h_*(a) \prec h_*(b)$. The other result then follows from this. ∎

Among the many necessary and sufficient conditions that a quotient map $L \xrightarrow{h} M$ be C^* established in [7] is that whenever $c \prec\!\!\prec d$ in M, there exist $a \prec\!\!\prec b$ in L such that $c \leq h(a) \prec\!\!\prec h(b) \leq d$. Consequently, in light of the preceding lemma, we have:

Proposition 2.3.5 *A dense uplifting quotient map $L \xrightarrow{h} M$ is a C^*-quotient map.*

Next, we have the following characterization of extremal disconnectedness in terms of the uplifting property. Recall that a frame L is *extremally disconnected* if $a^* \vee a^{**} = 1$ for all $a \in L$.

Proposition 2.3.6 *A frame L is extremally disconnected if and only if $\flat : L \to \mathcal{B}L$ is uplifting.*

Proof Let us first calculate the right adjoint of $\flat$. For any $a \in \mathcal{B}L$,

$$\flat_*(a) = \bigvee\{x \in L \mid x^{**} \leq a\} = a,$$

since $a^{**} = a$ as $a \in \mathcal{B}L$. Next, notice that pseudocomplementation in $\mathcal{B}L$ is precisely that of L. Assume $\flat$ is uplifting. For any $a \in L$, we have $a^* \sqcup a^{**} = 1$ since $\mathcal{B}L$ is Boolean. Therefore,

$$1 = \flat_*(a^*) \vee \flat_*(a^{**}) = a^* \vee a^{**},$$

implying that L is extremally disconnected. Conversely, let $a \sqcup b = 1$ in $\mathcal{B}L$. Then $(a \vee b)^{**} = 1$, whence $a^{**} \vee b^{**} = 1$, since L is extremally disconnected. Thus, $a \vee b = 1$ since $a = a^{**}$ and $b = b^{**}$. Therefore $\flat_*(a) \vee \flat_*(b) = 1$, and so $\flat$ is uplifting. ■

Of course, that the right adjoint of $\flat$ is simply the inclusion $\mathcal{B}L \to L$ is known since, as a map into L, $\flat$ is a nucleus. Let us now examine if the uplifting property is inherited by the Stone extension.

Proposition 2.3.7 *The Stone extension of an uplifting C^*-quotient map is uplifting.*

Proof Let $L \xrightarrow{h} M$ be an uplifting C^*-quotient map. Let $J \vee K = 1_{\beta M}$. Then $x \vee y = 1$ for some $x \in J$ and $y \in K$. Since J and K are completely regular ideals, there are cozero elements c and d of M such that $c \in J$, $d \in K$ and $c \vee d = 1$. Since h is a C^*-quotient map, there exist $u, v \in \mathrm{Coz}\, L$ such that

$$u \vee v = 1 \text{ and } h(u) = c, \; h(v) = d.$$

Then $r_L(u) \vee r_L(v) = 1_{\beta L}$. We show that $h^\beta(r_L(u)) \subseteq J$. Indeed, if $y \in h^\beta(r_L(u))$, then $y \leq h(s)$ for some $s \prec\!\prec u$. Consequently,

$$y \leq h(s) \leq h(u) = c \in J,$$

which implies that $y \in J$. Similarly, $h^\beta(r_L(v)) \subseteq K$. Therefore h^β is uplifting. ■

We will see in the next chapter that the converse of this result does not hold.

2.4 About meets of quotients

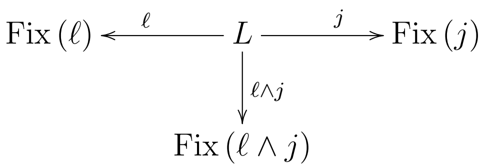
In this section, we explore briefly when the meet (in the assembly) of two quotients is any of the types discussed in the sections 2.1 and 2.3. Recall the definition of a nucleus from Preliminaries. If j and k are nuclei on L, then:

(1) the map $j \wedge k$, defined by $a \mapsto j(a) \wedge k(a)$, is also a nucleus on L

(2) if $j \leq k$ (comparison in the assembly is pointwise), then the mapping $\mathrm{Fix}(j) \to \mathrm{Fix}(k)$, effected by k, is a frame homomorphism.

Note from (2) that if $j \leq k$ and $c \in \mathrm{Coz}\,(\mathrm{Fix}(j))$, then $k(c) \in \mathrm{Coz}\,(\mathrm{Fix}(k))$.

Let ℓ and j be nuclei on L, and consider the diagram



We aim to find reasonable conditions on ℓ and j that ensure that $\ell \wedge j$ is C_1, strongly C_1 or uplifting. In order to avoid ambiguity, we write $\sqcup_\ell$, $\sqcup_j$ and $\sqcup$ to denote the binary joins in $\mathrm{Fix}\,(\ell), \mathrm{Fix}\,(j)$ and $\mathrm{Fix}\,(\ell \wedge j)$, respectively. The symbol $\vee$ is the join in L.

Proposition 2.4.1 *If ℓ and j, as above, are C_1-quotient maps, then $\ell \wedge j$ is a C_1-quotient map. The converse fails.*

Proof Let $c \in \mathrm{Coz}\, L$ and $d \in \mathrm{Coz}\,(\mathrm{Fix}\,(\ell \wedge j))$ be such that $(\ell \wedge j)(c) \sqcup d = 1$. Therefore $(\ell \wedge j)\,[(\ell \wedge j)(c) \vee d] = 1$, which implies $\ell\,(\ell(c) \vee \ell(d)) = 1$, that is $\ell(c) \sqcup_\ell \ell(d) = 1$. But $\ell(d) \in \mathrm{Coz}\,(\mathrm{Fix}\,(\ell))$ as observed above, therefore, in view of ℓ being C_1, there exists $u \in \mathrm{Coz}\, L$

such that $u \vee c = 1$ and $\ell(u) \leq \ell(d)$. Similarly, there exists $v \in \operatorname{Coz} L$ such that $v \vee c = 1$ and $j(v) \leq j(d)$. Thus, $u \wedge v$ is a cozero element of L such that $(u \wedge v) \vee c = 1$ and

$$(\ell \wedge j)(u \wedge v) \leq \ell(u \wedge v) \leq \ell(u) \leq \ell(d).$$

Similarly, $(\ell \wedge j)(u \wedge v) \leq j(d)$, and hence

$$(\ell \wedge j)(u \wedge v) \leq \ell(d) \wedge j(d) = (\ell \wedge j)(d) = d.$$

Therefore $\ell \wedge j$ is a C_1-quotient map.

To see that the converse fails, take $\ell = \mathrm{id}_L$ and $j = r_L \sigma_L$ for any non-pseudocompact L (recall that σ_L is the join map $\beta L \to L$ and r_L its right adjoint). ∎

A corollary of this proposition which is not noted in [40] is that the union of any two C_1-embedded subspaces is C_1-embedded. The following two lemmas are needed to show that, given ℓ and j to be strongly C_1, $\ell \wedge j$ is strongly C_1 precisely when it is coz-onto.

Lemma 2.4.2 *If ℓ and j, as above, are almost coz-codense, then $\ell \wedge j$ is almost coz-codense.*

Proof Let $c \in \operatorname{Coz} L$ be such that $(\ell \wedge j)(c) = 1$. Then $\ell(c) = j(c) = 1$. By hypothesis, there exist $d_1, d_2 \in \operatorname{Coz} L$ such that

$$c \vee d_1 = c \vee d_2 = 1 \text{ and } \ell(d_1) = \ell(0),\ j(d_2) = j(0).$$

Then $d_1 \wedge d_2$ is a cozero element of L such that $c \vee (d_1 \wedge d_2) = 1$ and

$$\begin{aligned} (\ell \wedge j)(d_1 \wedge d_2) &= \ell(d_1 \wedge d_2) \wedge j(d_1 \wedge d_2) \\ &\leq \ell(d_1) \wedge j(d_2) \\ &= \ell(0) \wedge j(0) \\ &= (\ell \wedge j)(0) \\ &= 0_{\mathrm{Fix}\,(\ell \wedge j)} \end{aligned}$$

Therefore $\ell \wedge j$ is almost coz-codense. ∎

Lemma 2.4.3 *If ℓ and j, as above, are coz-heavy, then $\ell \wedge j$ is coz-heavy.*

Proof Let $a \in L$ be such that $(\ell \wedge j)(a) = 1$. Then $\ell(a) = j(a) = 1$. By hypothesis, there exist c, $d \in \operatorname{Coz} L$ such that $c \leq a$, $d \leq a$, $\ell(c) = 1$ and $j(d) = 1$. Then $c \vee d$ is a cozero element of L such that $c \vee d \leq a$ and

$$(\ell \wedge j)(c \vee d) = \ell(c \vee d) \wedge j(c \vee d) \geq \ell(c) \wedge j(d) = 1.$$

Therefore $\ell \wedge j$ is coz-heavy. ■

Proposition 2.4.4 *If ℓ and j, as above, are strongly C_1, then $\ell \wedge j$ is strongly C_1 if and only if it is coz-onto.*

Proof We need only prove the right-to-left implication. In view of Proposition 2.1.14, it suffices to show that $\ell \wedge j$ is a coz-heavy C-quotient map. Since ℓ and j are strongly C_1, they are almost coz-codense, and hence $\ell \wedge j$ is almost coz-codense by Lemma 2.4.2. So, being coz-onto and almost coz-codense, it is a C-quotient map (see Proposition 1.6.2). Also, ℓ and j are coz-heavy, so $\ell \wedge j$ is coz-heavy by Lemma 2.4.3. ■

Although we are not concerned with C- and C^*-quotient maps per se, it is perhaps worth noting that:

Corollary 2.4.5 *If ℓ and j, as above, are almost coz-codense, then the following are equivalent:*

(1) $\ell \wedge j$ is coz-onto.

(2) $\ell \wedge j$ is a C^*-quotient map.

(3) $\ell \wedge j$ is a C-quotient map.

Proof Only the implication (1) $\Rightarrow$ (3) needs verification. By Lemma 2.4.2, $\ell \wedge j$ is almost coz-codense, and hence a C-quotient map if (1) holds. ∎

In preparation for our last result in this chapter, let us observe that if k is a nucleus on L, then $k : L \to \mathrm{Fix}(k)$ is uplifting if and only if for all a, $b \in L$, $k(k(a) \vee k(b)) = 1$ implies $k(a) \vee k(b) = 1$.

Proposition 2.4.6 *Suppose ℓ and j, as above, are uplifting. Denote by $\sqcup$ the join in the frame Fix($\ell \wedge j$). Then $\ell \wedge j$ is uplifting if and only if, for all a, $b \in L$, $(\ell\wedge j)(a) \sqcup (\ell \wedge j)(b) = 1$ implies $\ell(a) \vee j(b) = \ell(b) \vee j(a) = 1$.*

Proof Let us observe first that, for any a, $b \in L$,

$$\begin{aligned}(\ell \wedge j)(a) \vee (\ell \wedge j)(b) &= (\ell(a) \wedge j(a)) \vee (\ell(b) \wedge j(b)) \\ &= ((\ell(a) \wedge j(a)) \vee \ell(b)) \wedge ((\ell(a) \wedge j(a)) \vee j(b)) \\ &= (\ell(a) \vee \ell(b)) \wedge (j(a) \vee \ell(b)) \wedge (\ell(a) \vee j(b)) \wedge (j(a) \vee j(b))\end{aligned}$$

Now let us prove the left-to-right implication. The hypothesis, then, is that ℓ, j and $\ell \wedge j$ are all uplifting. (In fact, that ℓ and j are uplifting is not needed in this case). Suppose that a and b are elements of L such that $(\ell \wedge j)(a) \sqcup (\ell \wedge j)(b) = 1$. Then $(\ell \wedge j)(a)$ and $(\ell \wedge j)(b)$ are elements of the frame Fix $(\ell \wedge j)$ whose join (in this frame) is the top element. Since the homomorphism $\ell \wedge j : L \to \mathrm{Fix}\,(\ell \wedge j)$ is hypothesized to be uplifting, we have that $(\ell \wedge j)(a) \vee (\ell \wedge j)(b) = 1$. Since $\ell \wedge j \leq \ell$ and $\ell \wedge j \leq j$, it follows that

$$\ell(a) \vee \ell(b) = 1 = j(a) \vee j(b).$$

Thus, the calculation at the beginning of the proof yields

$$(\ell(a) \vee j(b)) \wedge (\ell(b) \vee j(a)) = 1,$$

whence the desired result follows.

Conversely, suppose the stated condition holds. Let $u, v \in \operatorname{Fix}(\ell \wedge j)$ be such that $u \sqcup v = 1$. Then $(\ell \wedge j)(u) \sqcup (\ell \wedge j)(v) = 1$. In terms of the join in L, this says that $(\ell \wedge j)(u \vee v) = 1$, which implies that $\ell(u \vee v) = 1$, and hence $\ell(\ell(u) \vee \ell(v)) = 1$. Since ℓ is uplifting, this implies that $\ell(u) \vee \ell(v) = 1$. Similarly, $j(a) \vee j(b) = 1$. Now, this together with the condition and the calculation at the beginning of the proof shows that

$$(\ell \wedge j)(u) \vee (\ell \wedge j)(v) = 1.$$

Therefore $\ell \wedge j$ is uplifting. ■

Chapter 3

Characterizing normality

Our goal in this chapter is to use some of the quotient maps defined in the previous chapter to characterize normality and some of its weaker forms.

3.1 Normality vis-à-vis uplifting maps

What we observed above about uplifting maps bears resemblance to some of the characterizations of C^*-quotient maps (see Proposition 1.6.1). Indeed, if the domain and the codomain of a quotient map are normal, then the two concepts coincide because, in a normal frame, $a \vee b = 1$ implies $c \vee d = 1$ for some cozero elements c and d such that $c \leq a$ and $d \leq b$. In fact, we have the following lemma which we shall use to characterize normal frames in terms of uplifting maps.

Lemma 3.1.1 *Let $L \xrightarrow{h} M$ be a quotient map. Then the following statements hold.*

(a) If M is normal and h is a C^*-quotient map, then h is uplifting.

(b) If L is normal and h is uplifting, then h is a C^*-quotient map.

(c) If L and M are normal, then h is uplifting if and only if it is a C^*-quotient map.

Proof (a) Let $h : L \to M$ be a C^*-quotient map with M normal. Let $a, b \in M$ be such that $a \vee b = 1$. Then by normality of M, there exist $c, d \in \operatorname{Coz} M$ such that $c \leq a$, $d \leq b$ with $c \vee d = 1$. Since h is a C^*-quotient map, there exist $u, v \in \operatorname{Coz} L$ such that $h(u) = c \leq a$ and $h(v) = d \leq b$ with $u \vee v = 1$. Therefore $u \leq h_*(a)$ and $v \leq h_*(b)$. Therefore $1 = u \vee v \leq h_*(a) \vee h_*(b)$. Thus h is uplifting.

(b) Let $a, b \in \operatorname{Coz} M$ be such that $a \vee b = 1$. The map h is uplifting, so $h_*(a) \vee h_*(b) = 1$. Furthermore, L is normal, so there exist $c, d \in \operatorname{Coz} L$ such that $c \leq h_*(a)$, $d \leq h_*(b)$ with $c \vee d = 1$. Therefore $h(c) \leq a$ and $h(d) \leq b$. Thus h is a C^*-quotient map.

(c) This follows immediately from (a) and (b). ∎

In [7], normal frames L are characterized by the property that every closed quotient $L \overset{\kappa_a}{\to} \uparrow a$ is a C^*-quotient map. This enables us to characterize normal frames in terms of uplifting quotient maps.

Proposition 3.1.2 *The following are equivalent for a completely regular frame L:*

(1) L is normal.

(2) Every C^*-quotient map $M \to L$ is uplifting.

(3) $\beta L \to L$ is uplifting.

(4) $\lambda L \to L$ is uplifting.

(5) Every uplifting quotient map $L \xrightarrow{h} M$ is a C^*-quotient map.

Proof By the Lemma 3.1.1, (1) implies (2). Since $\beta L \to L$ is a C^*-quotient map, (2) implies (3); and since $\lambda L \to L$ is a C^*-quotient map, (2) implies (4).

(3) $\Rightarrow$ (1) : Let $a \vee b = 1$ in L. Then $r_L(a) \vee r_L(b) = 1_{\beta L}$, by hypothesis. Therefore, there exist elements $c, d \in L$ such that $c \prec\!\!\prec a$, $d \prec\!\!\prec b$ and $c \vee d = 1$. Therefore L is normal.

(4) $\Rightarrow$ (1) : Let $a \vee b = 1$ in L. Denote the right adjoint of $\lambda L \to L$ by s_L, and notice that, for any $x \in L$, $s_L(x) = \{c \in \operatorname{Coz} L \mid c \leq x\}$. The current hypothesis implies that $s_L(a) \vee s_L(b) = 1_{\lambda L}$. In view of how binary joins are calculated in λL, this implies that there exist $c, d \in \operatorname{Coz} L$ such that $c \prec a$, $d \prec b$ and $c \vee d = 1$. Therefore L is normal.

(1) $\Rightarrow$ (5) : This follows from Lemma 3.1.1 (b).

(5) $\Rightarrow$ (1) : Every closed quotient map $L \to {\uparrow}a$ is uplifting, so the hypothesis makes each such quotient a C^*-quotient map, making L normal by [7, Theorem 8.3.3]. ■

In [5, Proposition 3.7], it is shown that if the right adjoint of the compactification $M \to L$ is a lattice homomorphism, then L is normal and $M \to L$ is (isomorphic to) the Stone-Čech compactification of L. We sharpen this by relaxing the hypothesis somewhat.

Corollary 3.1.3 *If a frame has an uplifting compactification, then the frame is normal and the compactification in question is (isomorphic to) its Stone-Čech compactification.*

Proof Let $M \to L$ be an uplifting compactification of L. Then, in view of M being normal, the homomorphism $M \to L$ is a C^*-quotient map by the implication (1) $\Rightarrow$ (5) of Proposition 3.1.2. Therefore $M \cong \beta L$ by [7, Corollary 8.2.7], and hence L is normal by the implication (3) $\Rightarrow$ (1) in Proposition 3.1.2. ■

The equivalent statements in the following proposition are selected from [25, Proposition 2.8].

Proposition 3.1.4 *The following are equivalent for a completely regular frame L:*

1. *L is almost compact.*
2. *Every onto homomorphism $M \to L$ is a C^*-quotient map.*
3. *Every onto homomorphism $M \to L$ is a C-quotient map.*
4. *L admits only one uniformity.*

The equivalent statements in the following proposition are selected from [44, Theorem 4].

Proposition 3.1.5 *The following are equivalent for a completely regular frame L:*

1. *L admits a unique uniformity.*
2. *L admits only one totally bounded uniform structure.*
3. *L admits a unique strong inclusion.*
4. *L has a unique compactification.*
5. *For $a \prec\prec b \in L$, $\uparrow a^*$ or $\uparrow b$ is compact.*

In [12], a compactification $h : M \to L$ of L is called a *one-point* compactification if there is a maximal element s of M such that h induces an isomorphism $\downarrow s \to L$. Analogous to topological spaces, call a completely regular frame L *almost compact* in case it is compact or $\beta L \to L$ is a one-point compactification. Combining results from Proposition 3.1.4 and Proposition 3.1.5, we have that *a frame is almost compact if and only if it has only one compactification*. This leads to the following result.

Corollary 3.1.6 *The following are equivalent for a completely regular frame L:*

(1) Every homomorphism onto L is uplifting.

(2) L is almost compact and normal.

Proof (1) $\Rightarrow$ (2) : By Corollary 3.1.3, if (1) holds, then L is normal and has only one compactification. Thus, L is also almost compact.

(2) $\Rightarrow$ (1) : If L is almost compact, then, by [25, Proposition 2.8(2)], every quotient map $M \to L$ is a C^*-quotient map. Thus, if L is normal too, then by Proposition 3.1.2, every quotient map $M \to L$ is uplifting. ∎

Based on Lemma 3.1.1, we provide conditions which are equivalent to the Lindelöf extension of a quotient map being uplifting. Let $h : L \to M$ be a coz-onto homomorphism, so that h^λ is onto. Recalling the discussion at the end of the Section 2.1, it is clear, by Proposition 1.6.1, that h is a C^*-quotient map if and only if h^λ is a C^*-quotient map. Since regular Lindelöf frames are normal, Lemma 3.1.1 shows that h^λ is uplifting if and only if it is a C^*-quotient map. Consequently, we have the following result.

Proposition 3.1.7 *The following are equivalent for a quotient map $h : L \to M$:*

(1) h^λ is uplifting.

(2) h^λ is a C^*-quotient map.

(3) h is a C^*-quotient map.

We announced at the end of Section 2.3 of the previous chapter that the converse of Proposition 2.3.7 fails. Here is the verification. For any non-normal completely regular frame L, $\beta L \to L$ is a non-uplifting C^*-quotient map (by Proposition 3.1.2) whose Stone extension is uplifting because it is an isomorphism.

3.2 δ-normally separated frames

The class of δ-normally separated spaces were defined in [47] by Mack. In this section, we define and characterize δ-normally separated frames in terms of C_1-quotient maps.

Definition 3.2.1 A frame L is *δ-normally separated* (respectively, *weakly δ-normally separated*) if for every $a \in L$ (respectively, every regular $a \in L$), the closed quotient map $L \to \uparrow a$ is almost coz-codense.

Remark 3.2.2 The class of δ-normally separated frames was first introduced in [28] by Dube.

Obviously, a normal frame is δ-normally separated. For purposes of computation, we reformulate the definition of δ-normal separation in terms of elements.

Lemma 3.2.3 *A frame L is δ-normally separated (resp. weakly δ-normally separated) if and only if for every $a \in L$ (resp. every regular $a \in L$) and $c \in \operatorname{Coz} L$ such that $a \vee c = 1$, there exists $d \in \operatorname{Coz} L$ such that $d \leq a$ and $c \vee d = 1$.*

Recall that a quotient $h : L \to M$ of L is a C_1-quotient if for every $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$ such that $h(c) \vee d = 1$, then there exists $u \in \operatorname{Coz} L$ such that $u \vee c = 1$ and $h(u) \leq d$.

Frames which are δ-normally separated, and ones weakly so, are characterized as follows:

Proposition 3.2.4 *A frame L is δ-normally separated (resp. weakly δ-normally separated) if and only if every closed (resp. regular-closed) quotient map $L \to \uparrow a$ is C_1-quotient.*

Proof Let L be δ-normally separated and for $a \in L$, consider the closed quotient map $\kappa_a : L \to \uparrow a$. Let $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz}(\uparrow a)$ be such that $\kappa_a(c) \vee d = 1$. Then $(a \vee c) \vee d = c \vee d = 1$, and so, by δ-normal separation, there exists $v \in \operatorname{Coz} L$ such that $v \leq d$ and $c \vee v = 1$. Thus, $\kappa_a(v) = a \vee v \leq a \vee d = d$, and hence κ_a is a C_1-quotient map.

Conversely, let $c \in \operatorname{Coz} L$ and $a \in L$ be such that $c \vee a = 1$. The closed quotient map $\kappa_a : L \to \uparrow a$ is a C_1-quotient map by the current hypothesis, and a is a cozero element of $\operatorname{Coz}(\uparrow a)$ such that $\kappa_a(c) \vee a = 1$. So there exists $v \in \operatorname{Coz} L$ such that $c \vee v = 1$ and $\kappa_a(v) \leq a$. Thus, $a \vee v \leq a$, and therefore $v \leq a$ as desired.

The statement in parenthesis is shown similarly. ∎

The next result shows that in a δ-normally separated frame, a closed C_2-quotient map is C-quotient.

Proposition 3.2.5 *Let $\varphi : L \to \uparrow a$ be a closed quotient of a δ-normally separated frame L. If φ is a C_2-quotient map, then φ is a C-quotient map.*

Proof Since every C_2-quotient map is coz-onto by Corollary 2.2.3 and by definition any closed quotient map of δ-normally separated frames is almost coz-codense, it follows that φ is coz-onto and almost coz-codense. Hence the result follows by [7, Theorem 7.2.3]. ∎

3.3 Mildly normal frames

In this section, we study mildly normal frames and show, amongst other things, that an almost regular Lindelöf frame is mildly normal. We show that mild normality is preserved by dense uplifting quotient maps defined in the previous chapter.

Definition 3.3.1 A frame L is *mildly normal* if for every regular elements a and b in L such that $a \vee b = 1$, there exist $c, d \in L$ such that $c \wedge d = 0$ and $a \vee c = b \vee d = 1$.

Mildly normal frames include extremally disconnected frames. Indeed, if L is extremally disconnected, then $a^* \vee a^{**} = 1$ for every $a \in L$. Let a and b be regular elements such that $a \vee b = 1$. Then $a^* \wedge b^* = 0$ with the property that $a^* \vee a = 1 = b \vee b^*$.

We now present our first characterization of mildly normal frames. The characterizations of mildly normal spaces in [56, Theorem 1] extend to frames. In what follows, we shall denote by L_r, the set of regular elements of L.

Proposition 3.3.2 *The following statements are equivalent for any frame L:*

(1) L is mildly normal.

(2) For all a, $b \in L_r$, if $a \vee b = 1$, then there exists $v \in L$ such that $a \vee v = 1$ and $v \prec b$.

(3) For all a, $b \in L_r$, if $a \vee b = 1$, then there exists a regular element $u \in L$ such that $a \vee u = 1$ and $u \prec b$.

(4) For all a, $b \in L_r$, if $a \vee b = 1$, then there exist u, $v \in L$ such that $a \vee u = 1$, $b \vee v = 1$ and $u^* \vee v^* = 1$.

Proof (1) $\Rightarrow$ (2) : Let L be mildly normal, and let a, $b \in L_r$ be such that $a \vee b = 1$. By mild normality, there exist u, $v \in L$ such that $u \wedge v = 0$ and $a \vee v = b \vee u = 1$. This implies that u is a separating element between v and b, so that $v \prec b$.

(2) $\Rightarrow$ (3) : Since $v \prec b$ implies $v^{**} \prec b$, (3) follows immediately from (2).

(3) $\Rightarrow$ (4) : Let a, $b \in L_r$ be such that $a \vee b = 1$. By (3), find $w \in L_r$ such that $a \vee w = 1$ and $w \prec b$. Then $w^* \vee b = 1$. By applying (3) to $a \vee w = 1$, we find $u \in L_r$ such that $a \vee u = 1$ and $u \prec w$. Then $u^* \vee w = 1$, $w^* \vee b = 1$ and $u^* \vee w^{**} = u^* \vee w = 1$. If we put $v = w^*$, we realize that (4) holds.

(4) $\Rightarrow$ (1) : Let a, $b \in L_r$ be such that $a \vee b = 1$. By (4), there exist u, $v \in L$ such that $a \vee u = 1$, $b \vee v = 1$ and $u^* \vee v^* = 1$. Then $u^{**} \wedge v^{**} = 0$ and $a \vee u^{**} \geq a \vee u = 1$, $b \vee v^{**} \geq b \vee v = 1$. Hence L is mildly normal. ■

In [7], there is a characterization of normal frames in terms of cozero elements. The proof uses properties of the ring $\mathcal{R}L$. A similar characterization of mild normality is valid as we show below. The proof is exactly that of [7], except for the fact that in the case of mild normality, we need to verify that the rather below relation $\prec$ interpolates between regular elements, that is, if $u \prec v$ where u, v are regular, then there exists a regular w such that $u \prec w \prec v$. We shall verify this, and refer to the proof of the analogous result in the case of normality.

Lemma 3.3.3 *Let L be mildly normal. If a, $b \in L_r$ such that $a \prec b$, then there exists $u \in L_r$ such that $a \prec u \prec b$.*

Proof Since $a \prec b$, there is an element $s \in L$ such that $a \wedge s = 0$ and $s \vee b = 1$. Therefore $a \wedge s^{**} = 0$ and $s^{**} \vee b = 1$. Applying the mild normality property of L on s^{**} and b, we find $x, y \in L$ such that $x \wedge y = 0$ and $s^{**} \vee x = 1 = y \vee b$. Hence $x^{**} \wedge y^{**} = 0$ and $s^{**} \vee x^{**} = 1 = y^{**} \vee b$. Thus $a \wedge s^{**} = 0$ and $s^{**} \vee x^{**} = 1$, which implies that $a \prec x^{**}$. Also $x^{**} \wedge y^{**} = 0$ and $y^{**} \vee b = 1$ implies that $x^{**} \prec b$. So $a \prec x^{**} \prec b$ as desired. ∎

Corollary 3.3.4 *If L is mildly normal and $a, b \in L_r$ such that $a \prec b$, then $a \prec\!\prec b$.*

For the following, we mimick the proof of [7, Proposition 8.3.1]. In the following $\mathbb{R}_0 = \mathbb{R} - \{0\}$ and $\mathbb{R}_1 = \mathbb{R} - \{1\}$.

Lemma 3.3.5 *A frame L is mildly normal if and only if whenever $a, b \in L_r$ such that $a \vee b = 1$, there exists $f \in \mathcal{R}L$ such that $f(\mathbb{R}_0) \leq a$ and $f(\mathbb{R}_1) \leq b$.*

Proposition 3.3.6 *A frame L is mildly normal if and only if whenever $a, b \in L_r$ such that $a \vee b = 1$, there exist $c, d \in \operatorname{Coz} L$ such that $c \leq a$, $d \leq b$ and $c \vee d = 1$.*

Proof Put $c = f(\mathbb{R}_0)$ and $d = f(\mathbb{R}_1)$ in Lemma 3.3.5 above. Since each element of $\mathfrak{O}\mathbb{R}$ is a cozero element and frame homomorphisms preserve cozeros, it follows that $c, d \in \operatorname{Coz} L$. Now $c \vee d = f(\mathbb{R}_0) \vee f(\mathbb{R}_1) = f((\mathbb{R} - \{0\}) \cup (\mathbb{R} - \{1\})) = f(\mathbb{R}) = 1$. ∎

Next, recall that in an Oz-frame, every regular element is a cozero element. We show that this class of frames is contained in the class of mildly normal frames.

Proposition 3.3.7 *Every Oz-frame is mildly normal.*

Proof Let $a,\, b \in L_r$ be such that $a \vee b = 1$. Thus, by [11, Proposition 2.2], $a,\, b \in \operatorname{Coz} L$. Since $\operatorname{Coz} L$ is a normal σ-frame, it follows that there exist $u,\, v \in \operatorname{Coz} L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$. ∎

Recall that L is an *F-frame* if every open quotient of a cozero element in L is a C^*-quotient. Furthermore, L is an *F'-frame* if $a^* \vee b^* = 1$ whenever $a,\, b \in \operatorname{Coz} L$ such that $a \wedge b = 0$. It is shown in [7] that L is an F-frame if and only if for all $a,\, b \in \operatorname{Coz} L$ such that $a \wedge b = 0$, there exist $c,\, d \in \operatorname{Coz} L$ such that $c \vee d = 1$ and $c \wedge a = d \wedge b = 0$. It is shown in [33, Corollary 4.7] that a normal F'-frame is an F-frame. In the next proposition we strengthen this result.

Proposition 3.3.8 *A mildly normal F'-frame is an F-frame.*

Proof Let $a,\, b \in \operatorname{Coz} L$ be such that $a \wedge b = 0$. Since L is an F'-frame, $a^* \vee b^* = 1$. Since L is mildly normal and a^*, b^* are regular elements, it follows by Proposition 3.3.6 that there exist $c,\, d \in \operatorname{Coz} L$ such that $c \vee d = 1$, $c \leq a^*$ and $d \leq b^*$. Hence $c \wedge a \leq a^* \wedge a = 0$ and $d \wedge b \leq b^* \wedge b = 0$. Therefore L is an F-frame. ∎

Mild normality is inherited by closed quotients of regular elements as is shown below.

Proposition 3.3.9 *Every regular closed quotient of a mildly normal frame is mildly normal.*

Proof Let L be a mildly normal frame and consider the regular closed quotient $m : L \to {\uparrow}c$, where $c \in L_r$. Let $a,\, b \in {\uparrow}c$ be regular elements such that $a \vee b = 1_{\uparrow c}$. Denote by $(\cdot)^{\#}$ the pseudocomplement in ${\uparrow}c$. Then $a = s^{\#}$ and $b = t^{\#}$ for some $s,\, t \in {\uparrow}c$. From [33, Lemma 4.5], $s^{\#} = (s \wedge c^*)^*$ and $t^{\#} = (t \wedge c^*)^*$, so a and b are regular elements in L such that $a \vee b = 1_L$. Since L is mildly normal, there exist $x, y \in L$ such that $x \wedge y = 0$ and

$a \vee x = 1 = b \vee y$. Now $x \vee c,\ y \vee c \in {\uparrow}c$ and $(x \vee c) \wedge (y \vee c) = c \vee (x \wedge y) = c = 0_{\uparrow c}$, also $a \vee (x \vee c) = (a \vee x) \vee c = 1 \vee c = 1_{\uparrow c}$ and $b \vee (y \vee c) = (b \vee y) \vee c = 1 \vee c = 1_{\uparrow c}$. Therefore ${\uparrow}c$ is mildly normal. ■

In [3], a subspace S of a topological space X is said to be *R^*-embedded (resp. G^*-embedded)* in X if two disjoint regular-closed (resp. closure disjoint open sets) of S are contained in disjoint regular closed sets (resp. extended to closure disjoint open sets) of X. Furthermore, a subset A of S is said to be *extended* to a subset U of X if $U \cap S = A$. If A, B are disjoint regular-closed subsets of S, then $S - A$ and $S - B$ are regular-open subsets of S such that $(S - A) \cup (S - B) = S$. Let U and V be disjoint regular-closed subsets of X such that $A \subseteq U$ and $B \subseteq V$. Then $(X - U) \cap S \subseteq S - A$ and $(X - V) \cap S \subseteq S - B$. The notions of R^*- and G^*-embedded subspaces are captured in frames in the following definition.

Definition 3.3.10 A quotient map $h : L \to M$ is

(i) a G^*-*quotient map* if, for every $a,\ b \in M$ such that $a^* \vee b^* = 1_M$, there exist $c,\ d \in L$ such that $c^* \vee d^* = 1_L,\ h(c) = a$ and $h(d) = b$.

(ii) an R^*-*quotient map* if, for every regular elements $a,\ b \in M$ such that $a \vee b = 1_M$, there exist regular elements $c,\ d \in L$ such that $c \vee d = 1_L,\ h(c) \leq a$ and $h(d) \leq b$.

Proposition 3.3.11 *A quotient map $h : L \to M$ is a G^*-quotient map if and only if for every $u,\ v \in M$ with $u^* \vee v^* = 1_M$, there exist $a,\ b \in L$ such that $a^* \vee b^* = 1_L$ with $u \leq h(a)$ and $v \leq h(b)$.*

Proof $(\Rightarrow)$: This is trivial.

$(\Leftarrow)$: Let $c,\ d \in L,\ c^* \vee d^* = 1_L,\ u \leq h(c)$ and $v \leq h(d)$. Put $a = c \wedge h_*(u)$ and $b =$

$d \wedge h_*(v)$. Now $h(c \wedge h_*(u)) = h(c) \wedge u = u$ and $h(d \wedge h_*(v)) = h(d) \wedge v = v$. Also $(c \wedge h_*(u))^* \vee (d \wedge h_*(v))^* \geq c^* \vee d^* = 1_L$. ∎

Next we show that a G^*-quotient map is an R^*-quotient map.

Proposition 3.3.12 *Let $h : L \to M$ be a quotient of L. If h is a G^*-quotient map, then h is R^*-quotient.*

Proof Let a, $b \in M$ be regular elements such that $a \vee b = 1_M$. Then $a = u^*$ and $b = v^*$ for some u, $v \in M$. Then, by hypothesis, there exist c, $d \in L$ such that $c^* \vee d^* = 1_L$ with $h(c) = u$ and $h(d) = v$. Now $h(c^*) \leq h(c)^* = u^* = a$ and $h(d^*) \leq h(d)^* = v^* = b$. Since pseudocomplements are regular, the result follows. ∎

For the next result we show that if the codomain of a C^*-quotient map is a mildly normal frame, then the map is a G^*-quotient map and hence an R^*-quotient map. If the domain of an R^*-quotient map is mildly normal, then the quotient map is both a G^*-quotient map and a C^*-quotient map.

Theorem 3.3.13 *Let $h : L \to M$ be a quotient map. Then the following hold.*

(a) If M is mildly normal and h is a C^*-quotient map, then h is a G^*-quotient map and hence an R^*-quotient map.

(b) If L is mildly normal and h an R^*-quotient map, then h is a G^*-quotient map and hence a C^*-quotient map.

(c) If L is mildly normal and h is an R^*-quotient map, then M is mildly normal.

Proof (a) $C^* \Rightarrow G^*$: Let $x,\, y \in M$ be such that $x^* \vee y^* = 1_M$. Since M is mildly normal, by Proposition 3.3.6, there exist $c,\, d \in \operatorname{Coz} M$ such that $c \vee d = 1_M$ with $c \leq x^*$ and $d \leq y^*$. Since h is a C^*-quotient map, there exist $s,\, t \in \operatorname{Coz} L$ such that $s \vee t = 1_L,\ h(s) = c$ and $h(t) = d$. Furthermore, $\operatorname{Coz} L$ is a regular σ-frame, so there exist $w,\, z \in \operatorname{Coz} L$ such that $w \prec\!\prec s,\ z \prec\!\prec t$ and $w \vee z = 1_L$. Now $w \prec\!\prec s$ implies $w^* \vee s = 1_L$, and $z \prec\!\prec t$ implies $z^* \vee t = 1_L$. Therefore $h(w^*) \vee h(s) = 1_M$ and $h(z^*) \vee h(t) = 1_M$. Now $h(s) \wedge x = c \wedge x = 0$ and $h(w^*) \vee h(s) = 1_M$ implies $x \leq h(w^*)$. Similarly, $y \leq h(z^*)$. Since $w^{**} \vee z^{**} \geq w \vee z = 1_L$, it follows, by Proposition 3.3.11, that h is a G^*-quotient map and, by Proposition 3.3.12, h is an R^*-quotient map.

(b) $R^* \Rightarrow G^*$: Let $a,\, b \in M$ be such that $a^* \vee b^* = 1_M$. Since h is an R^*-quotient map, there exist regular elements $c,\, d \in L$ such that $c \vee d = 1_L,\ h(c) \leq a^*$ and $h(d) \leq b^*$. Therefore $h(c) \wedge a = 0$ and $h(d) \wedge b = 0$. Furthermore, L is mildly normal, so there exist $u, v \in \operatorname{Coz} L$ such that $u \vee v = 1_L,\ u \leq c$ and $v \leq d$. Since $\operatorname{Coz} L$ is a normal σ-frame, there exist $w,\, z \in \operatorname{Coz} L$ such that $w \prec\!\prec u,\ z \prec\!\prec v$ and $w \vee z = 1_L$. Now $w \prec\!\prec u$ implies $w^* \vee u = 1_L$ and $z \prec\!\prec v$ implies $z^* \vee v = 1_L$. Therefore $h(w^*) \vee h(u) = 1_M$ and $h(z^*) \vee h(v) = 1_M$. Now $h(u) \wedge a \leq h(c) \wedge a = 0$ and $h(w^*) \vee h(u) = 1_M$ implies $a \leq h(w^*)$. Also $h(v) \wedge b \leq h(d) \wedge b = 0$ and $h(z^*) \vee h(v) = 1_M$ implies $b \leq h(z^*)$. Since $w^{**} \vee z^{**} \geq w \vee z = 1_L$, it follows by Proposition 3.3.11, that h is a G^*-quotient map.

$G^* \Rightarrow C^*$: Let $a,\, b \in \operatorname{Coz} M$ be such that $a \vee b = 1_M$. Then there exist $u,\, v \in \operatorname{Coz} M$ such that $u \prec\!\prec a,\, v \prec\!\prec b$ and $u \vee v = 1$. Then $u^{**} \prec\!\prec a,\, v^{**} \prec\!\prec b$ such that $u^{**} \vee v^{**} = 1_M$. Since h is a G^*-quotient map, it follows that there exist $c, d \in L$ such that $c^* \vee d^* = 1_L$ and $h(c) = u^*,\ h(d) = v^*$. Furthermore, L is mildly normal, so there exist $s,\, t \in \operatorname{Coz} L$ such that $s \leq c^*,\ t \leq d^*$ and $s \vee t = 1_L$. Now $h(s) \leq h(c^*) \leq h(c)^* = u^{**} \leq a$ and $h(t) \leq h(d^*) \leq h(d)^* = v^{**} \leq b$. Therefore by Proposition 1.6.1, h is a C^*-quotient map.

(c) Let a and b be regular elements of M such that $a \vee b = 1_M$. Since h is an R^*-quotient map, it follows that there exist regular elements $c,\, d \in L$ such that $c \vee d = 1_L$ with $h(c) \leq a$

and $h(d) \leq b$. Furthermore, L is mildly normal, so there exist $x, y \in L$ such that $x \wedge y = 0$ and $c \vee x = 1 = d \vee y$. Now $h(x) \wedge h(y) = 0$ in M and $a \vee h(x) \geq h(c) \vee h(x) = h(c \vee x) = 1_M$, $b \vee h(y) \geq h(d) \vee h(y) = h(d \vee y) = 1_M$. Thus M is mildly normal. ■

Definition 3.3.14 A frame L is said to be *almost regular* if for every regular element $a \in L$,

$$a = \bigvee \{x \in L \mid x \prec a\}.$$

Call a frame L *nearly compact* if for every cover S of L there is a finite subset K of S such that $(\bigvee K)^* = 0$. In the literature, frames with this property are called almost compact. Since in Chapter 2 we used that name to mean something different, we prefer to use "nearly compact" in order to avoid confusion.

Proposition 3.3.15 *Every almost regular, nearly compact frame is mildly normal.*

Proof Let a and b be regular elements in L such that $a \vee b = 1$. Since L is almost regular, we have that $a = \bigvee \{x \mid x \prec a\}$ and $b = \bigvee \{y \mid y \prec b\}$. Now put $S = \{x \vee y \mid x \prec a,\ y \prec b\}$. Then S is a cover of L. Since L is nearly compact, there exists a finite $K = \{x_1 \vee y_1, \ldots, x_n \vee y_n\} \subseteq S$ such that $(\bigvee K)^* = 0$. Now

$$\begin{aligned} 0 = \left(\bigvee K\right)^* &= \left(\bigvee_{i=1}^{n}(x_i \vee y_i)\right)^* \\ &= \bigwedge_{i=1}^{n}(x_i \vee y_i)^* \\ &= \bigwedge_{i=1}^{n}(x_i^* \wedge y_i^*) \\ &= \left(\bigwedge_{i=1}^{n} x_i^*\right) \wedge \left(\bigwedge_{i=1}^{n} y_i^*\right) \end{aligned}$$

Since each $x_i \prec a$, each $x_i^* \vee a = 1$ and therefore $\bigwedge_{i=1}^{n}(x_i^* \vee a) = a \vee \bigwedge_{i=1}^{n} x_i^* = 1$. Similarly, $\bigwedge_{i=1}^{n}(y_i^* \vee b) = b \vee \bigwedge_{i=1}^{n} y_i^* = 1$. Now put $c = \bigwedge_{i=1}^{n} x_i^*$ and $d = \bigwedge_{i=1}^{n} y_i^*$. Then $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$. Thus, L is mildly normal. ■

In [54, Proposition 10.2], Pultr proves that a regular Lindelöf frame is mildly normal. Mimicking his proof almost verbatim, we obtain the following result.

Proposition 3.3.16 *Every almost regular, Lindelöf frame is mildly normal.*

We end the section by showing in the next proposition that the property of mild normality is inherited by dense uplifting quotients. Let us recall (see, for instance, [7, Lemma 8.2.5]) that if $h : L \to M$ is a dense onto homomorphism with right adjoint r, then $r(b^*) = r(b)^*$ for all $b \in M$.

Proposition 3.3.17 *Let $h : L \to M$ be a dense uplifting quotient map. If L is mildly normal, then so is M.*

Proof Let a and b be regular elements in M such that $a \vee b = 1$. Then $a = a^{**}$ and $b = b^{**}$. Since h is an uplifting quotient map, it follows that $h_*(a^{**}) \vee h_*(b^{**}) = 1$. By the result cited from [7], we have $h_*(a^{**}) = h_*(a)^{**}$ and $h_*(b^{**}) = h_*(b)^{**}$ and therefore $h_*(a^{**})$ and $h_*(b^{**})$ are regular elements in L. By mild normality of L, there exist $c, d \in L$ such that $c \wedge d = 0$ and $c \vee h_*(a^{**}) = 1 = d \vee h_*(b^{**})$. Then $h(c) \wedge h(d) = 0$, $h(c) \vee a^{**} = 1$ and $h(d) \vee b^{**} = 1$. Thus, $h(c) \vee a = 1$ and $h(d) \vee b = 1$, and hence M is mildly normal. ∎

3.4 Δ-normal frames

A subset A of a topological space X is called *regular-closed* if it equals the closure of its interior. On the other hand, it is *regular-open* if it equals the interior of its closure. Clearly, for a subset S of a topological space X, we have that S is regular-open if and only if $X - S$, its complement in X, is regular-closed. Furthermore, for any space X and $U \in \mathfrak{O}X$,

$$U \text{ is regular-open} \quad \Leftrightarrow \quad U = U^{**}.$$

Regular elements of any frame are precisely the pseudocomplements. That is,

$$a \in L \text{ is regular } \Leftrightarrow a = b^* \text{ for some } b \in L.$$

In spaces, there is a variant of normality, called Δ-normality, which is stronger than mild normality. Let us recall the definition.

Let X be a topological space, $A \subseteq X$ and $p \in X$. The point p is said to be a δ-*limit point* of A if every regular-open neighbourhood of p meets A. The δ-*closure* of A, denoted by $\bar{A}^\delta$, is the set

$$\bar{A}^\delta = \{x \in X \mid x \text{ is a } \delta\text{-limit point of } A\}.$$

A set $S \subseteq X$ is δ-*closed* if $S = \bar{S}^\delta$, and δ-*open* if its complement is δ-closed. A topological space X is said to be Δ-*normal* (resp. *weakly* Δ-*normal*) if for any disjoint closed subsets A, B of X, of which one is δ-closed (resp. both δ-closed), there exist disjoint open subsets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

In this section, we extend this notion to frames. We will however not study it in any detail, save to show that it is stronger than mild normality, and coincides with it in the category of regular frames. Since Δ-normality in spaces is defined by means of a condition which makes specific reference to points, our first task will be to cast the definition in the language of open sets only.

Now δ-openness can be phrased in frame terms in the following way: We start by writing $\bar{A}^\delta$ in terms of closed sets with no points mentioned, so that on taking complements we have a set expressed solely in terms of open sets.

Lemma 3.4.1 *For any topological space X and $A \subseteq X$, we have*

$$\bar{A}^\delta = \bigcap\{R \subseteq X \mid R \textit{ is regular-closed and } R \supseteq A\}.$$

Proof Let $p \in \bar{A}^\delta$, and consider any regular-closed set $Q \subseteq X$ with $A \subseteq Q$. We aim to show that $p \in Q$, which will prove that p is in the intersection stated in the Lemma 3.4.1. Suppose, by way of contradiction, that $p \notin Q$. Then $p \in X - Q$. Thus $X - Q$ is a regular-open neighbourhood of p. Since $p \in \bar{A}^\delta$, we must have $(X - Q) \cap A \neq \emptyset$. But this is false since $A \subseteq Q$ and $Q \cap (X - Q) = \emptyset$. Therefore

$$\bar{A}^\delta \subseteq \bigcap \{R \subseteq X \mid R \text{ is regular-closed and } R \supseteq A\}.$$

To show the reverse inclusion, write the collection

$$\{R \subseteq X \mid R \text{ is regular-closed and } R \supseteq A\}$$

as an indexed family $\{R_i\}_{i \in I}$. To show that

$$\bigcap_{i \in I} R_i \subseteq \bar{A}^\delta,$$

it suffices to show that

$$X - \bar{A}^\delta \subseteq X - \bigcap_{i \in I} R_i,$$

that is,

$$X - \bar{A}^\delta \subseteq \bigcup_{i \in I} (X - R_i).$$

Let $z \in X - \bar{A}^\delta$. Then $z \notin \bar{A}^\delta$, and hence z has a regular-open neighbourhood (say P) which misses A. So $A \subseteq X - P$. Consequently, $X - P$ is a regular-closed set containing A, so that, in view of $P = X - (X - P)$, we have that P is one of the sets $X - R_i$. Thus,

$$z \in P \subseteq \bigcup_{i \in I} (X - R_i).$$

This establishes the reverse inclusion, and hence the desired equality. ■

Corollary 3.4.2 *Let X be a topological space and $U \in \mathfrak{O}X$. Then*

$$U \textit{ is } \delta\textit{-open} \quad \Leftrightarrow \quad U = \bigcup \{V \in \mathfrak{O}X \mid V = V^{**} \textit{ and } V \subseteq U\}.$$

In view of this, we say an element of a frame L is a δ-*element* if

$$a = \bigvee\{x \in L \mid x = x^{**} \text{ and } x \leq a\}.$$

Observe that every regular element is a δ-element. Following [21], we formulate the following definition.

Definition 3.4.3 A frame L is said to be

(i) Δ-*normal* if for any a, $b \in L$, with either a or b a δ-element and $a \vee b = 1$, there exist c, $d \in L$ such that $c \wedge d = 0$ and $a \vee c = b \vee d = 1$.

(ii) *weakly* Δ-*normal* if for any a, $b \in L$, with a, b both δ-element and $a \vee b = 1$, there exist c, $d \in L$ such that $c \wedge d = 0$ and $a \vee c = b \vee d = 1$.

Since every regular element is a δ-element, it follows immediately that every Δ-normal frame is mildly normal. Recall that a homomorphism $h\colon L \to M$ is said to be *closed* if for every $a \in L$ and $b \in M$, $h_*(h(a) \vee b) = a \vee h_*(b)$. A frame homomorphism $h\colon L \to M$ is said to be *nearly open* if $h(t^*) = h(t)^*$ for all $t \in L$.

Proposition 3.4.4 *If $h : L \to M$ is closed, one-one and nearly open, then M Δ-normal implies that L is Δ-normal.*

Proof Let a, $b \in L$ with a a δ-element such that $a \vee b = 1$. Then

$$a = \bigvee\{x \in L \mid x = x^{**} \text{ and } x \leq a\}$$

and so

$$\begin{aligned} h(a) &= \bigvee\{h(x) \mid x = x^{**} \text{ and } x \leq a\} \\ &\leq \bigvee\{y \mid y = y^{**} \text{ and } y \leq h(a)\} \\ &\leq h(a), \end{aligned}$$

since $x = x^{**}$ implies $h(x) = h(x)^{**}$ by near openess of h. Therefore $h(a)$ is δ-open and $h(a) \vee h(b) = 1$. Since M is Δ-normal, it follows that there exist c, $d \in M$ such that $c \wedge d = 0$ and $h(a) \vee c = 1 = h(b) \vee d$. Since h is dense (because it is one-one), it follows that $h_*(c) \wedge h_*(d) = 0$. Also since h is closed, $a \vee h_*(c) = 1 = b \vee h_*(d)$. Thus L is Δ-normal. ∎

Chapter 4

Quasi F-frames

4.1 Introduction

In their study of C- and C^*-quotients in pointfree topology, Ball and Walters-Wayland [7] devote a section to disconnectivity in which they define F-frames, F'-frames and quasi-F frames. The class of quasi-F frames contains that of F'-frames, which, in turn, contains that of F-frames. These containments are strict. Furthermore, each of these notions is the exact analogue of its spatial antecedent, by which we mean that a completely regular space X is an F-space, an F'-space or a quasi-F space if and only if the frame of its open subsets is an F-frame, an F'-frame or a quasi-F frame, respectively. Quasi-F spaces were introduced in [22] by Dashiell, et. al. F-frames and F'-frames have been characterized in several ways (see [30], [27] and [33]).

In [7], only one characterization of quasi-F frames is presented, namely:

A completely regular frame L is quasi-F if and only if for all $a, b \in \operatorname{Coz} L$ such *that* $a \wedge b = 0$ *and* $a \vee b$ *is dense, there exist* $c, d \in \operatorname{Coz} L$ *such that* $a \wedge c = b \wedge d = 0$ *and* $c \vee d = 1$.

Our aim in this chapter is to give several characterizations of quasi-F frames. These will include a characterization almost similar to the one cited above but in which we do not require that the cozero elements whose join is dense should also be disjoint.

We shall also present a characterization in terms of rings of continuous functions to the effect that L is quasi-F if and only if $\operatorname{Ann}^2(\alpha) + \operatorname{Ann}^2(\beta) = \mathcal{R}L$ whenever $\alpha + \beta$ is not a zero-divisor. Element-wise characterization includes one stating that a completely regular frame is quasi-F if and only if whenever the join of two cozero elements is dense, their pseudocomplements are completely separated. Another ring-theoretic characterization is that L is quasi-F precisely when the ring $\mathcal{R}L$ is quasi-Bézout.

We call a frame homomorphism "rigid" – a term borrowed from ℓ-groups – if every cozero element in the codomain has a pseudocomplement equal to that of the image of some cozero element. We then show that if a frame is quasi-F, then every dense onto homomorphism out of it is a C^*-quotient map if and only if it is coz-onto, if and only if it is rigid. This leads to the observation that if a Lindelöf frame has a quasi-F compactification, then it is quasi-F, and the compactification in question is the Stone-Čech compactification.

4.2 Characterizations of quasi-F frames

Throughout this chapter, all frames are assumed to be completely regular. Also, by "ring" we mean a commutative ring with identity. We recall from [7] that an onto frame homomorphism $h : L \to M$ is called a C^*-*quotient map* if for each $\alpha \in \mathcal{R}^*M$, there exists $\beta \in \mathcal{R}L$ such that $h \circ \beta = \alpha$. In the same article, a frame L is defined to be *quasi-F* if for every dense $c \in \operatorname{Coz} L$, the open quotient map $L \to {\downarrow}c$ is a C^*-quotient map. A useful characterization given in [7] has already been recited in the introduction. Our first characterization is almost similar to it, but does not require that the cozero elements a and b be disjoint. It is, in fact, a frame analogue of [35, Lemma 2.10].

Proposition 4.2.1 *A completely regular frame L is quasi-F if and only if $a^{**} \vee b^{**} = 1$ for all $a, b \in \operatorname{Coz} L$ with $a \vee b$ dense.*

Proof $(\Rightarrow)$: Let a and b be cozero elements of L such that $a \vee b$ is dense. Put $c = a \vee b$. Then, by hypothesis, the open quotient map $h : L \to {\downarrow}c$ is a C^*-quotient map. Since $h(a) = a$ and $h(b) = b$, it follows that $a, b \in \operatorname{Coz}({\downarrow}c)$ because frame homomorphims preserve cozero elements. Now $a \vee b = 1_{\downarrow c}$, and so, by Proposition 1.6.1, there exist $u, v \in \operatorname{Coz} L$ such that

$$u \vee v = 1_L \text{ and } h(u) = a, \ h(v) = b.$$

Thus,

$$u \wedge (a \vee b) = a \text{ and } v \wedge (a \vee b) = b.$$

Consequently,

$$(u \wedge (a \vee b))^{**} = a^{**},$$

which implies that

$$u^{**} \wedge (a \vee b)^{**} = a^{**},$$

whence $u^{**} = a^{**}$ since $(a \vee b)^{**} = 1$ as $a \vee b$ is dense. Similarly, $v^{**} = b^{**}$. Thus,

$$a^{**} \vee b^{**} = u^{**} \vee v^{**} \geq u \vee v = 1.$$

$(\Leftarrow)$: Let c be a dense cozero element of L. Consider the open quotient map $g : L \to {\downarrow}c$. Note that g is a dense homomorphism, so by [7, Theorem 8.2.6], it suffices to show that for all $a, b \in \operatorname{Coz}({\downarrow}c)$ such that $a \vee b = 1_{\downarrow c}$, $g_*(a) \vee g_*(b) = 1_L$. So take $a, b \in \operatorname{Coz}({\downarrow}c)$ with $a \vee b = 1_{\downarrow c}$. Denote the rather below relation and pseudocomplementation in ${\downarrow}c$ by $\preceq$ and $(\cdot)^{\#}$ respectively. By normality of $\operatorname{Coz}({\downarrow}c)$, there exist $u, v \in \operatorname{Coz}({\downarrow}c)$ such that

$$u \preceq a, \ v \preceq b \text{ and } u \vee v = 1_{\downarrow c}.$$

Then $u^{\#\#} \leq a$ and $v^{\#\#} \leq b$. Since, for any $t \in {\downarrow}c$, $t^{\#} = c \wedge t^*$, it follows that

$$u^{\#\#} = (u^{\#})^{\#} = (c \wedge u^*)^{\#} = c \wedge (c \wedge u^*)^* \geq c \wedge (c^* \vee u^{**}) = c \wedge u^{**} = g(u^{**}).$$

Thus, $g(u^{**}) \leq u^{\#\#} \leq a$, which implies that $u^{**} \leq g_*(a)$. Similarly, $v^{**} \leq g_*(b)$. Since $u \vee v = c$, which is dense, the current hypothesis implies that $u^{**} \vee v^{**} = 1$ since $u, v \in \operatorname{Coz} L$ by [7, Proposition 3.2.10]. Therefore $g_*(a) \vee g_*(b) = 1$, and we are done. ∎

This result enables us to show that if a coproduct of two frames is quasi-F, then each summand is quasi-F. Concerning the coproduct $L \oplus M$, recall that elements of the form $a \oplus b$ generate $L \oplus M$, and that $a \oplus b = 0$ if and only if $a = 0$ or $b = 0$. It is shown in [15] that $(a \oplus b)^{**} = a^{**} \oplus b^{**}$. If i_L and i_M denote the coproduct injections, then, for any $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$, $c \oplus d \in \operatorname{Coz}(L \oplus M)$ since $c \oplus d = i_L(c) \wedge i_M(d)$ – a meet of two cozero elements.

Corollary 4.2.2 *If $L \oplus M$ is quasi-F, then both L and M are quasi-F.*

Proof We show L to be quasi-F. Let $a, b \in \operatorname{Coz} L$ such that $a \vee b$ is dense. We claim that the cozero element $(a \oplus 1) \vee (b \oplus 1)$ is dense. Suppose that

$$(x \oplus y) \wedge ((a \oplus 1) \vee (b \oplus 1)) = 0$$

for some $x \in L$ and $y \in M$. Then

$$((x \oplus y) \wedge (a \oplus 1)) \vee ((x \oplus y) \wedge (b \oplus 1)) = ((x \wedge a) \oplus y) \vee ((x \wedge b) \oplus y) = 0.$$

If $y \neq 0$, then (from above) we must have $x \wedge a = 0$ and $x \wedge b = 0$, which implies that $x \wedge (a \vee b) = 0$, and hence $x = 0$ since $a \vee b$ is dense. Consequently, $x \oplus y = 0$, and hence $(a \oplus 1) \vee (b \oplus 1)$ is dense since the elements $u \oplus v$, $u \in L$, and $v \in M$, generate $L \oplus M$. Now, Proposition 4.2.1 implies

$$(a \oplus 1)^{**} \vee (b \oplus 1)^{**} = 1_{L \oplus M},$$

that is,

$$(a^{**} \oplus 1) \vee (b^{**} \oplus 1) = (a^{**} \vee b^{**}) \oplus 1 = 1_{L \oplus M},$$

which implies that $a^{**} \vee b^{**} = 1$. Therefore L is quasi-F, by the proposition. Similarly, M is quasi-F. ■

Recall that L is an F'-frame if $a^* \vee b^* = 1$ for all $a, b \in \operatorname{Coz} L$ with $a \wedge b = 0$. These frames generalize F-frames which, we recall, are defined by stipulating that for every $c \in \operatorname{Coz} L$, the open quotient map $L \to {\downarrow}c$ be a C^*-quotient map. Every F-frame is an F'-frame, and every mildly normal F'-frame is an F-frame, as we showed in Chapter 3. A P-frame is one in which every cozero element is complemented, and an *almost-P* frame is one such that $a^{**} \in \operatorname{Coz} L$ for each $a \in \operatorname{Coz} L$. We observe that quasi-$F$ frames include all these.

Corollary 4.2.3 *Every almost-P frame and every F'-frame is quasi-F.*

Proof If L is an almost-P frame and $a \vee b$ is dense for some $a, b \in \operatorname{Coz} L$, then, by [26, Proposition 3.3], $a \vee b = 1$, and hence $a^{**} \vee b^{**} = 1$. Therefore L is quasi-F. Now suppose L is an F'-frame and a, $b \in \operatorname{Coz} L$ are such that $a \wedge b = 0$ and $a \vee b$ is dense. Then $a^* \wedge b^* = 0$ by density, and $a^* \vee b^* = 1$ since L is an F'-frame. Thus, a^* and b^* are complemented, and hence are cozero elements satisfying the requirement in the characterization cited from [7].
■

For the following definition, we follow the definition in spaces [38].

Definition 4.2.4 A completely regular frame L is *cozero complemented* if for each $u \in \operatorname{Coz} L$, there is a $v \in \operatorname{Coz} L$ such that $u \wedge v = 0$ and $u \vee v$ is dense in L.

We shall study these types of frames in more detail in the next chapter. Recall that a frame is *basically disconnected* if the pseudocomplement of every cozero element is complemented, that is $c^* \vee c^{**} = 1$ for each cozero element c.

Corollary 4.2.5 *Every basically disconnected frame is quasi-F. On the other hand, every cozero complemented quasi F-frame is basically disconnected.*

Proof Let L be basically disconnected and suppose $a, b \in \operatorname{Coz} L$ such that $a \vee b$ is dense. Then $a^* \wedge b^* = 0$, and therefore $a^* \leq b^{**}$. Since L is basically disconnected, $a^* \vee a^{**} = 1$. Therefore $a^{**} \vee b^{**} = 1$, and hence L is quasi-F by Proposition 4.2.1. This proves the first part.

Now let M be a cozero-complemented quasi-F frame, and let $c \in \operatorname{Coz} M$. Pick $d \in \operatorname{Coz} M$ such that $c \wedge d = 0$ and $c \vee d$ is dense. Then $c^{**} \vee d^{**} = 1$. But now $c^{**} \wedge d^{**} = 0$ since $c \wedge d = 0$, so c^{**} is complemented, that is, $c^* \vee c^{**} = 1$. Therefore M is basically disconnected. ∎

Next, we show that a frame is quasi-F if and only if its Stone-Čech compactification is quasi-F. We need a lemma. Recall that a frame homomorphism $h : L \to M$ is coz-onto if for every $d \in \operatorname{Coz} M$, there exists $c \in \operatorname{Coz} L$ such that $h(c) = d$.

Lemma 4.2.6 *Let $h : L \to M$ be a dense coz-onto frame homomorphism. If L is quasi-F, then so is M.*

Proof Notice that h is onto by complete regularity. Recall that a dense onto frame homomorphism preserves pseudocomplements. Let $u, v \in \operatorname{Coz} M$ such that $u \vee v$ is dense. Since h is coz-onto, there exist $c, d \in \operatorname{Coz} L$ such that $h(c) = u$ and $h(d) = v$. Now,

$$0 = u^* \wedge v^* = h(c)^* \wedge h(d)^* = h(c^*) \wedge h(d^*) = h(c^* \wedge d^*),$$

which implies that $c^* \wedge d^* = 0$ by the density of h. Thus, $c \vee d$ is dense in the quasi-F frame L. Thus, by Proposition 4.2.1, $c^{**} \vee d^{**} = 1$. Consequently,

$$u^{**} \vee v^{**} = h(c)^{**} \vee h(d)^{**} = h(c^{**} \vee d^{**}) = 1,$$

and hence M is quasi-F. ∎

We have an example to show that the converse of this lemma does not hold. That is, if $h : L \to M$ is dense coz-onto and M is a quasi F-frame, it does not follow that L is a quasi F-frame.

Example 4.2.7 Let $L = \mathfrak{O}\mathbb{R}$ and put $a = (-\infty, 0)$, $b = (0, \infty)$. Then $a, b \in \operatorname{Coz} L$ and $a \vee b$ is dense. However

$$a^{**} \vee b^{**} = a \vee b \neq 1_L,$$

so, in view of Proposition 4.2.1, L is not a quasi F-frame. Now consider the Booleanization map $\flat : L \to \mathfrak{B}L$. It is dense and coz-onto, the latter holding since it is onto and $L = \operatorname{Coz} L$. Also $\mathfrak{B}L$ is Boolean, and hence a quasi F-frame. Thus, $\flat$ is a dense coz-onto homomorphism with a quasi-F codomain but not quasi-F domain.

We note, in passing, that this lemma tells us that if L is quasi-F and $c \in \operatorname{Coz} L$ is dense, then $\downarrow c$ is quasi-F because the open quotient map $M \to \downarrow c$ is dense and coz-onto. This can of course be established directly from the definition. In the proof of the following proposition, we will need to recall that the join map $\beta L \to L$ is coz-onto, and that if a frame M is normal, then for any $a, b \in M$ with $a \vee b = 1$, there exist $c, d \in \operatorname{Coz} M$ such that $c \leq a$, $d \leq b$ and $c \vee d = 1$ (see [7, Corollary 8.3.2]).

Proposition 4.2.8 *A completely regular frame is quasi-F if and only if its Stone-Čech compactification is quasi-F.*

Proof If βL is quasi-F, then L is quasi-F by Lemma 4.2.6 since $\beta L \to L$ is dense and coz-onto. Conversely, assume L is quasi-F. We shall use the characterization cited from [7]. Let $I, J \in \operatorname{Coz} \beta L$ such that $I \vee J$ is dense and $I \wedge J = 0_{\beta L}$. Then $\bigvee I \vee \bigvee J$ is a dense element of L and $\bigvee I \wedge \bigvee J = 0$. Since $\bigvee I$ and $\bigvee J$ are cozero elements of L, and since L is quasi-F, by hypothesis, there exist $c, d \in \operatorname{Coz} L$ such that

$$c \wedge \bigvee I = 0,\ d \wedge \bigvee J = 0 \text{ and } c \vee d = 1.$$

Thus, $r(c) \vee r(d) = 1_{\beta L}$. By the normality of βL, there exist $U, V \in \operatorname{Coz} \beta L$ such that

$$U \leq r(c),\ V \leq r(d), \text{ and } U \vee V = 1_{\beta L}.$$

We show that $U \wedge I = V \wedge J = 0_{\beta L}$. Indeed,

$$\bigvee (U \wedge I) = \bigvee U \wedge \bigvee I \leq \bigvee r(c) \wedge \bigvee I = c \wedge \bigvee I = 0,$$

which implies $U \wedge I = 0_{\beta L}$ since $\bigvee : \beta L \to L$ is dense. Similarly, $V \wedge J = 0_{\beta L}$. Therefore βL is quasi-F. ∎

Lemma 4.2.9 *Let $g : L \to M$ be a dense homomorphism. Then for any $a \in L$, $a^* = g_* g(a^*)$.*

Proof Since $t \leq g_* g(t)$, for every $t \in L$, we need only show that $g_* g(a^*) \leq a^*$. Now $g(a \wedge g_* g(a^*)) = g(a) \wedge g(a^*) = 0$, so $a \wedge g_* g(a^*) = 0$, by density. Thus, $g_* g(a^*) \leq a^*$, and hence $a^* = g_* g(a^*)$. ∎

We remind the reader that a dense onto homomorphism commutes with pseudocomplements, i.e., $h(a^*) = h(a)^*$.

As observed in Example 4.2.7, the converse of Lemma 4.2.6 does not hold in general. However, if h is an R^*-quotient map we have the following proposition.

Proposition 4.2.10 *Let $h : L \to M$ be a dense R^*-quotient map which is coz-onto. Then L is quasi-F if and only if M is quasi-F.*

Proof We only show the sufficiency part. Let $a, b \in \operatorname{Coz} L$ such that $a \vee b$ is dense. Since frame homomorphisms preserve cozero elements, it follows that $h(a), h(b) \in \operatorname{Coz} M$. Now, since $a \vee b$ is dense

$$(h(a) \vee h(b))^* = (h(a \vee b))^* = h((a \vee b)^*) = h(0) = 0,$$

and so $h(a) \vee h(b)$ is dense in M. The frame M is quasi-F, so $h(a)^{**} \vee h(b)^{**} = 1_M$. Since h is an R^*-quotient map, it follows that there exist regular elements u, v in L such that $h(u) \leq h(a^{**})$, $h(v) \leq h(b^{**})$ and $u \vee v = 1_L$. Therefore $u \leq h_*h(a^{**})$ and $v \leq h_*h(b^{**})$. Thus, by Lemma 4.2.9,

$$1_L = u \vee v \leq h_*h(a^{**}) \vee h_*h(b^{**}) = a^{**} \vee b^{**}.$$

Hence L is quasi-F. ∎

Let υL and λL denote, respectively, the Hewitt realcompactification and the regular Lindelöf coreflection of L. For $c \in \operatorname{Coz} L$, recall that we write

$$[c] = \{x \in \operatorname{Coz} L \mid x \leq c\},$$

and that

$$\operatorname{Coz} \upsilon L = \operatorname{Coz} \lambda L = \{[c] \mid c \in \operatorname{Coz} L\}.$$

Furthermore, the maps $\upsilon L \to L$ and $\lambda L \to L$, by taking joins, are dense onto. Therefore Lemma 4.2.6 and the characterization from [7] lead us to the following result:

Proposition 4.2.11 *The following are equivalent for L:*

(1) L is quasi-F.

(2) υL is quasi-F.

(3) λL is quasi-F.

We now give a characterization in terms of rings of continuous functions. We remark that the spatial version of this result appears in [53] by putting together certain results therein. It is obtained by regarding $C(X)$ as a Riesz space. Our method of proof is completely different. It uses, among other things, a concept which does not exist in the category of topological spaces; namely, the universal Lindelöfication.

First, some notation from [26]. For each $I \in \beta L$, the ideal $\mathbf{M}^I$ of $\mathcal{R}L$ is defined by

$$\mathbf{M}^I = \{\varphi \in \mathcal{R}L \mid r(\operatorname{coz}\varphi) \subseteq I\}.$$

Keeping in mind that $r(a)^* = r(a^*)$, we have the following lemma, the proof of which can be found in [31]. If $S \subseteq R$, where R is a commutative ring with unity, we let $\operatorname{Ann} S = \{a \in R \mid aS = \{0\}\}$, called the *annihilator* of S.

Lemma 4.2.12 *Let $S \subseteq \mathcal{R}L$ and put $a = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in S\}$. Then $\operatorname{Ann}(S) = \mathbf{M}^{r(a^*)}$. In particular, for any $\gamma \in \mathcal{R}L, \operatorname{Ann}^2(\gamma) = \mathbf{M}^{r(coz\gamma)^{**}}$.*

Note that if an element α of $\mathcal{R}L$ is a zero divisor, then $\operatorname{coz}\alpha$ is not dense as it misses some nonzero cozero element. On the other hand, if a nonzero element of a completely regular frame misses some nonzero element, then it misses some nonzero cozero element. We give a lemma which was proved by Dube in [26]. Here our proof is different and direct.

Lemma 4.2.13 *The homomorphism $\varphi \in \mathcal{R}L$ fails to be a zero divisor if and only if $\operatorname{coz}\varphi$ is dense in L.*

Proof (⇒): Suppose on the contrary that φ is a zero divisor. Let $\delta \in \mathcal{R}L$ with $\operatorname{coz}\delta \neq 0$ be such that $\varphi\delta = 0$. Then $\operatorname{coz}\varphi \wedge \operatorname{coz}\delta = \operatorname{coz}(\varphi\delta) = \operatorname{coz} 0 = 0$. Hence $\operatorname{coz}\varphi$ is not dense in L.

(⇐): Conversely suppose that $\operatorname{coz}\varphi$ is dense in L. Then for any $\delta \in \mathcal{R}L$ with $\operatorname{coz}\delta \neq 0$ in L, we have

$$\begin{aligned} & \operatorname{coz}\varphi \wedge \operatorname{coz}\delta \neq 0 \\ \Rightarrow\ & \operatorname{coz}(\varphi\delta) \neq 0 \\ \Rightarrow\ & \varphi\delta \neq 0 \\ \Rightarrow\ & \varphi \text{ is not a zero divisor.} \end{aligned}$$

■

Recall that an onto frame homomorphism $h : L \to M$ is called a C-quotient map if for every $\alpha \in \mathcal{R}M$, there exists $\beta \in \mathcal{R}L$ such that $h \circ \beta = \alpha$. Thus, h is a C-quotient map precisely when the induced ring homomorphism $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ is onto.

Proposition 4.2.14 *A completely regular frame L is quasi-F if and only if for all $\alpha, \beta \in \mathcal{R}L$, $\operatorname{Ann}^2(\alpha) + \operatorname{Ann}^2(\beta) = \mathcal{R}L$ whenever $\alpha + \beta$ is not a zero divisor.*

Proof (⇐) : Let $c, d \in \operatorname{Coz} L$ such that $c \vee d$ is dense. Let γ and δ be nonnegative elements of $\mathcal{R}L$ such that $c = \operatorname{coz}\gamma$ and $d = \operatorname{coz}\delta$. Now, $\operatorname{coz}(\gamma + \delta) = c \vee d$, which is dense, so $\gamma + \delta$ is not a zero divisor. Thus, by hypothesis, $\operatorname{Ann}^2(\gamma) + \operatorname{Ann}^2(\delta) = \mathcal{R}L$. Take $\alpha \in \operatorname{Ann}^2(\gamma)$ and $\beta \in \operatorname{Ann}^2(\delta)$ such that $\alpha + \beta = 1$. Therefore, by Lemma 4.2.12, $\operatorname{coz}\alpha \leq c^{**}$ and $\operatorname{coz}\beta \leq d^{**}$. Now,

$$1 = \operatorname{coz} 1 = \operatorname{coz}(\alpha + \beta) \leq \operatorname{coz}\alpha \vee \operatorname{coz}\beta,$$

implies $c^{**} \vee d^{**} = 1$. Therefore, by Proposition 4.2.1, L is quasi-F.

(⇒) : Assume, for a moment, that L is normal. Let $\alpha, \beta \in \mathcal{R}L$ such that $\alpha + \beta$ is not a zero divisor. Then $\operatorname{coz}(\alpha + \beta)$ is dense. Since $\operatorname{coz}\alpha \vee \operatorname{coz}\beta \geq \operatorname{coz}(\alpha + \beta)$, $\operatorname{coz}\alpha \vee \operatorname{coz}\beta$

is dense, and so, by Proposition 4.2.1, $(\mathrm{coz}\,\alpha)^{**} \vee (\mathrm{coz}\,\beta)^{**} = 1$. By normality, there exist c, $d \in \mathrm{Coz}\, L$ such that

$$c \leq (\mathrm{coz}\,\alpha)^{**},\ d \leq (\mathrm{coz}\,\beta)^{**} \text{ and } c \vee d = 1.$$

Take nonnegative $\gamma, \delta \in \mathcal{R}L$ such that $c = \mathrm{coz}\,\gamma$ and $d = \mathrm{coz}\,\delta$. Then $r(\mathrm{coz}\,\gamma) \subseteq r(\mathrm{coz}\,\alpha)^{**}$, and hence $\gamma \in \mathbf{M}^{r(\mathrm{coz}\,\alpha)^{**}} = \mathrm{Ann}^2(\alpha)$. Similarly, $\delta \in \mathrm{Ann}^2(\beta)$. Since $\mathrm{coz}\,(\gamma + \delta) = 1, \gamma + \delta$ is invertible, and therefore $\mathrm{Ann}^2(\alpha) + \mathrm{Ann}^2(\beta) = \mathcal{R}L$.

Now, relax the normality assumption. Let us say a ring A *has property* (P) if, for all $a, b \in A$, $\mathrm{Ann}^2(a) + \mathrm{Ann}^2(b) = A$ whenever $a + b$ is not a zero divisor. One checks routinely that property (P) is preserved by ring isomorphisms. We must show that $\mathcal{R}L$ has property (P) under the hypothesis that L is quasi-F. By Proposition 4.2.11, λL is quasi-F. But λL is a regular Lindelöf frame, so it is normal. Thus, $\mathcal{R}(\lambda L)$ has property (P) by what we have shown. Since the map $\lambda L \to L$ is dense, the induced ring homomorphism $\mathcal{R}(\lambda L) \to \mathcal{R}L$ is one-one, by [10, Lemma 2]. Since the map $\lambda L \to L$ is a C-quotient map by [7, Corollary 8.2.13], the induced ring homomorphism is onto. Thus, $\mathcal{R}L$ is isomorphic to $\mathcal{R}(\lambda L)$, and hence has the required property. ■

Since a completely regular space X is a quasi-F space if and only if $\mathfrak{O}X$ is a quasi-F frame, and since the ring $C(X)$ is isomorphic to the ring $\mathcal{R}(\mathfrak{O}X)$, the following corollary is apparent.

Corollary 4.2.15 *A completely regular space X is quasi-F if and only if for all $f, g \in C(X)$, $\mathrm{Ann}^2(f) + \mathrm{Ann}^2(g) = C(X)$ whenever $f + g$ is not a zero divisor.*

Next, we give another characterization which is also a corollary of Proposition 4.2.14. Let us recall that elements a and b of a frame L are said to be *completely separated* if there exist $c, d \in \mathrm{Coz}\, L$ such that $c \vee d = 1$ and $a \wedge c = 0 = b \wedge d$.

Corollary 4.2.16 *A completely regular frame L is quasi-F if and only if c^* and d^* are completely separated whenever $c, d \in \operatorname{Coz} L$ are such that $c \vee d$ is dense.*

Proof ($\Rightarrow$) : Suppose L is quasi-F and that $c \vee d$ is dense for some $c, d \in \operatorname{Coz} L$. Pick nonnegative γ and δ in $\mathcal{R}L$ such that $c = \operatorname{coz}\gamma$ and $d = \operatorname{coz}\delta$. Then $\operatorname{coz}(\gamma+\delta) = \operatorname{coz}\gamma \vee \operatorname{coz}\delta$, which is dense. Therefore $\gamma + \delta$ is not a zero divisor. Thus, by Proposition 4.2.14, $\operatorname{Ann}^2(\gamma) + \operatorname{Ann}^2(\delta) = \mathcal{R}L$. Take $\alpha \in \operatorname{Ann}^2(\gamma)$ and $\beta \in \operatorname{Ann}^2(\delta)$ such that $1 = \alpha + \beta$. Then

$$1 = \operatorname{coz} 1 = \operatorname{coz}(\alpha+\beta) \leq \operatorname{coz}\alpha \vee \operatorname{coz}\beta.$$

Since $\alpha \in \operatorname{Ann}^2(\gamma) = \mathbf{M}^{r(c^{**})}$, it follows that $\operatorname{coz}\alpha \leq c^{**}$. Similarly, $\operatorname{coz}\beta \leq d^{**}$. Thus, $\operatorname{coz}\alpha$ and $\operatorname{coz}\beta$ are cozero elements of L such that

$$\operatorname{coz}\alpha \vee \operatorname{coz}\beta = 1 \text{ and } c^* \wedge \operatorname{coz}\alpha = d^* \wedge \operatorname{coz}\beta = 0.$$

Therefore c^* and d^* are completely separated.

($\Leftarrow$) : Let $a \vee b$ be dense, where $a, b \in \operatorname{Coz} L$. By hypothesis, there exist $u, v \in \operatorname{Coz} L$ such that

$$u \vee v = 1 \text{ and } a^* \wedge u = b^* \wedge v = 0.$$

Thus, $u \leq a^{**}$ and $v \leq b^{**}$, so that $a^{**} \vee b^{**} = 1$. Therefore L is quasi-F by Proposition 4.2.14. ∎

An ideal I of a ring is said to be *regular* if it does not consist entirely of zero divisors. In [4] an ideal I of a reduced commutative ring A is said to be a z^o-*ideal* if for each $a \in I$, $P_a \subseteq I$, where P_a denotes the intersection of all minimal prime ideals containing a. It is then shown that I is a z^o-ideal if and only if $\operatorname{Ann}^2(a) \subseteq I$ for each $a \in I$. Quasi-F frames can be characterized in terms of z^o-ideals as follows. Clearly, every annihilator ideal is a z^o-ideal since $\operatorname{Ann}^3 = \operatorname{Ann}$. It follows, therefore, from Proposition 4.2.14 that:

Corollary 4.2.17 *A frame L is quasi-F if and only if whenever P, Q are z^o-ideals of $\mathcal{R}L$ such that $P + Q$ is regular, then $P + Q = \mathcal{R}L$.*

The next characterization requires a bit of background. First, an f-ring (which is assumed to be reduced and with bounded inversion) is said to be *quasi-Bézout* if every finitely generated ideal which contains a non-zero divisor is principal. In Theorem 5.1 of [22], it is shown that a Tychonoff space X is quasi-F if and only if $C(X)$ is quasi-Bézout. We aim to show that a completely regular frame is quasi-F if and only if $\mathcal{R}L$ is quasi-Bézout. Our argument is similar to that employed by Dube [27] in showing that L is an F-frame if and only if every finitely generated ideal of $\mathcal{R}L$ is principal.

We shall freely use properties of f-rings and characterizations of quasi-Bézout f-rings established in [52]. Clearly, the property of being quasi-Bézout is preserved by ring isomorphisms.

Proposition 4.2.18 *A completely regular frame L is quasi-F if and only if $\mathcal{R}L$ is quasi-Bézout.*

Proof Suppose L is quasi-F. Then βL is quasi-F, so that, by spatiality, $\mathcal{R}(\beta L)$ is quasi-Bézout. This implies that $\mathcal{R}^*L$ is quasi-Bézout since $\mathcal{R}^*L$ is isomorphic to $\mathcal{R}(\beta L)$. We establish from this that $\mathcal{R}L$ is quasi-Bézout. We use the characterization in [52, Theorem 2]. So, let $0 \leq \alpha \leq \beta$ in $\mathcal{R}L$ with β a nonzero divisor. We must show that α is a multiple of β. From $0 \leq \alpha \leq \beta$, it follows that

$$\alpha + \alpha\beta \leq \beta + \alpha\beta,$$

that is,

$$\alpha(1 + \beta) \leq \beta(1 + \alpha). \qquad (\dagger)$$

Since nonnegative elements of $\mathcal{R}L$ are squares ([8, Proposition 11]), if $\gamma \geq 0$ is an invertible element of $\mathcal{R}L$, then $\gamma = \tau^2$ for some $\tau \in \mathcal{R}L$, so that $\gamma^{-1} = (\tau^{-1})^2$, showing that the inverse of γ is nonnegative. Since $\mathcal{R}L$ has bounded inversion, multiplying both sides of (†) with the nonnegative element $(1+\alpha)^{-1}(1+\beta)^{-1}$ yields

$$0 \leq \frac{\alpha}{1+\alpha} \leq \frac{\beta}{1+\beta}. \qquad (\ddagger)$$

But now both $\frac{\alpha}{1+\alpha}$ and $\frac{\beta}{1+\beta}$ are in $\mathcal{R}^*L$ since, for any $\gamma \geq 0$ in $\mathcal{R}L$, $0 \leq \frac{\gamma}{1+\gamma} \leq 1$ by the same kind of reasoning that produced (‡). Notice that $\frac{\beta}{1+\beta}$ is not a zero divisor in $\mathcal{R}^*L$, lest β a zero-divisor. Now from (‡) and the fact that $\mathcal{R}^*L$ is quasi-Bézout, we conclude that there exists $\delta \in \mathcal{R}^*L$ such that

$$\frac{\alpha}{1+\alpha} = \delta\frac{\beta}{1+\beta}$$

whence $\alpha = \gamma\beta$ for some $\gamma \in \mathcal{R}L$.

Conversely, suppose $\mathcal{R}L$ is quasi-Bézout. We show that $\mathcal{R}^*L$ is quasi-Bézout. Let $0 \leq \alpha \leq \beta$ in $\mathcal{R}^*L$ with β not a zero-divisor in $\mathcal{R}^*L$. Then, β is not a zero-divisor in $\mathcal{R}L$ (indeed, $\beta\varphi = 0$ for some nonzero $\varphi \in \mathcal{R}L$, implies $\beta\delta = 0$ for the nonzero element $\delta = \frac{\varphi^2}{1+\varphi^2}$ of $\mathcal{R}^*L$). Since $0 \leq \alpha \leq \beta$ in $\mathcal{R}L$, and since the hypothesis is that $\mathcal{R}L$ is quasi-Bézout, there exists, by [52, Theorem 2], $\gamma \in \mathcal{R}L$ such that $\alpha = \gamma\beta$. Notice that we may assume that $\gamma \geq 0$ since $\alpha = |\alpha| = |\gamma\beta| = |\gamma||\beta| = |\gamma|\beta$. Thus, $0 \leq \gamma \wedge 1 \leq 1$ implies that $\gamma \wedge 1$ is an element of $\mathcal{R}^*L$ with

$$(\gamma \wedge 1)\beta = \gamma\beta \wedge \beta = \alpha \wedge \beta = \alpha.$$

This shows that $\mathcal{R}^*L$ is quasi-Bézout. Consequently, $\mathcal{R}(\beta L)$ is quasi-Bézout, and hence, by spatiality, βL is quasi-F. Therefore L is quasi-F. ∎

Remark 4.2.19 Armed with this result, and taking into account the characterizations of quasi-Bézout rings in [52], one shows easily that most of the characterizations of quasi-F spaces in Theorem 5.1 of [22] extend to frames.

It is shown in [7] that any C^*-quotient map is coz-onto. We show below that any dense onto homomorphism whose source is a quasi-F frame is a C^*-quotient map if and only if it is coz-onto. In fact, these are equivalent to being what we shall call a rigid homomorphism for reasons that shall be apparent. The term "rigid" is usually used to describe certain types of ℓ-subgroups of an ℓ-group.

Definition 4.2.20 A frame homomorphism $h : L \to M$ is *rigid* if for every $d \in \operatorname{Coz} M$, there exists $c \in \operatorname{Coz} L$ such that $h(c)^* = d^*$.

Clearly, any coz-onto frame homomorphism is rigid.

Proposition 4.2.21 *Let L be a quasi-F frame and $h : L \to M$ be a dense onto frame homomorphism. Then the following are equivalent:*

(1) h is a C^*-quotient map.

(2) h is coz-onto.

(3) h is rigid.

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. We need only show that (3) implies (1). We use [7, Theorem 8.2.6]. Let $a, b \in \operatorname{Coz} M$ be such that $a \vee b = 1$. By normality of $\operatorname{Coz} M$, there exist $u, v \in \operatorname{Coz} M$ such that

$$u \prec\!\prec a,\ \ v \prec\!\prec b \text{ and } u \vee v = 1.$$

Since h is rigid, by hypothesis, there exist $c, d \in \operatorname{Coz} L$ such that

$$h(c)^* = u^* \text{ and } h(d)^* = v^*.$$

Now, $u \prec\!\prec a$ implies $u^{**} \leq a$. Similarly, $v^{**} \leq b$. Since $u \vee v = 1$, we have that $u^* \wedge v^* = 0$, and hence, in light of h being dense onto,

$$h(c^* \wedge d^*) = h(c)^* \wedge h(d)^* = 0,$$

implying $c^* \wedge d^* = 0$ by the density of h. Thus, $c \vee d$ is a dense element of a quasi-F frame L. By Proposition 4.2.1, $c^{**} \vee d^{**} = 1$. But now, again using the fact that h is dense onto, $h(c^{**}) = u^{**} \leq a$, which implies that $c^{**} \leq h_*(a)$. Similarly, $d^{**} \leq h_*(b)$. Consequently,

$$h_*(a) \vee h_*(b) \geq c^{**} \vee d^{**} = 1,$$

showing that h is a C^*-quotient map. ∎

In [33, Proposition 4.8], it is shown that a completely regular frame L is an F-frame if and only if every coz-onto quotient map $L \to M$ is a C^*-quotient map. This result has an analogue for quasi-F frames which follows from the foregoing proposition.

Corollary 4.2.22 *A completely regular frame L is quasi-F if and only if every dense coz-onto quotient map $L \to M$ is a C^*-quotient map.*

Proof The left-right implication follows from Proposition 4.2.21. The converse holds because, for any $c \in \operatorname{Coz} L$, the open quotient map $L \to {\downarrow}c$ is coz-onto (see, [7, Corollary 3.2.11]). ∎

We close with the following observation. In [33, Proposition 3.2], it is shown that if L is Lindelöf, then any onto frame homomorphism $M \to L$, with M completely regular, is coz-onto. Therefore we have the following result.

Corollary 4.2.23 *If a Lindelöf frame has a quasi-F compactification, then it is quasi-F, and the compactification in question is its Stone-Čech compactification.*

Proof Let L be a frame and $M \to L$ be a quasi-F compactification of L. By Lemma 4.2.6, L is quasi-F. By Proposition 4.2.21, $M \to L$ is a C^*-quotient map. By [7, Corollary 8.2.7], $M \to L$ is the Stone-Čech compactification of L. ∎

4.3 A few more words on rigidity

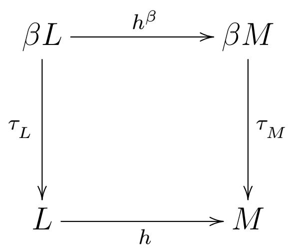
Here we provide the justification alluded to above regarding the term “rigid”. If we keep in mind that an f-subring H of an f-ring G (all viewed as ℓ-groups) is rigid precisely if for every $g \in G$, there exists $h \in H$ such that $\mathrm{Ann}\,(g) = \mathrm{Ann}\,(h)$, where annihilation is computed in G, it makes sense to define a ring homomorphism $\phi : A \to B$ to be *rigid* if for every $b \in B$, there exists $a \in A$ such that $\mathrm{Ann}\,(\phi a) = \mathrm{Ann}\,(b)$. We show that, with this definition of rigidity for ring homomorphisms, a frame homomorphism is rigid if and only if the induced ring homomorphism is rigid. Let us first note that, from Lemma 4.2.12 we have:

Lemma 4.3.1 *Let* $\alpha, \beta \in \mathcal{R}L$. *Then* $\mathrm{Ann}\,(\alpha) = \mathrm{Ann}\,(\beta)$ *if and only if* $(\mathrm{coz}\,\alpha)^* = (\mathrm{coz}\,\beta)^*$.

Recall that if $h : L \to M$ is a frame homomorphism, then the induced ring homomorphism $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ is given by $\mathcal{R}h(\alpha) = h \circ \alpha$. Furthermore, $\mathrm{coz}\,(h \circ \alpha) = h\mathrm{coz}\,\alpha$. Recall also that every frame homomorphism $h : L \to M$ has the *Stone extension* $h^\beta : \beta L \to \beta M$, given by

$$h^\beta(I) = \{y \in M \mid y \leq h(x) \text{ for some } x \in I\},$$

for each $I \in \beta L$. It is the unique frame homomorphism that makes the diagram



commute.

Proposition 4.3.2 *The following are equivalent for a frame homomorphism* $h : L \to M$*:*

(1) h is rigid.

(2) h^β is rigid.

(3) $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ is rigid.

(4) $\mathcal{R}h|_{\mathcal{R}^*L} : \mathcal{R}^*L \to \mathcal{R}^*M$ is rigid.

Proof (1) $\Rightarrow$ (2) : Let $J \in \operatorname{Coz}\beta M$. Then $\bigvee J \in \operatorname{Coz} M$. By (1), there exists $c \in \operatorname{Coz} L$ such that $h(c)^* = (\bigvee J)^*$. Since $\beta L \to L$ is coz-onto, there exists $U \in \operatorname{Coz}\beta L$ such that $\bigvee U = c$. We claim that $h^\beta(U)^* = J^*$. To prove the claim, let $I \in \beta M$ be such that $I \wedge J = 0_{\beta M}$. Take any $i \in I$ and $u \in U$. For any $x \in J$, $i \wedge x = 0$, and so $i \wedge \bigvee J = 0$, so that $i \leq (\bigvee J)^*$. Thus, $i \leq h(c)^*$ and hence $i \wedge h(c) = 0$. Since $c = \bigvee U$ we have that $i \wedge h(u) = 0$. It follows from this that $I \wedge h^\beta(U) = 0_{\beta M}$, and hence $I \leq h^\beta(U)^*$. We have thus shown that if J misses some element of βM, then that element is below $h^\beta(U)^*$. Consequently, $J^* \leq h^\beta(U)^*$.

Next, let $K \in \beta M$ be such that $K \wedge h^\beta(U) = 0_{\beta M}$. For any $k \in K$ and $u \in U$, $k \wedge h(u) = 0$, and therefore $k \wedge h(\bigvee U) = 0$, which implies that

$$k \leq h(\bigvee U)^* = h(c)^* = \left(\bigvee J\right)^*.$$

Thus, $k \wedge \bigvee J = 0$, and hence

$$\bigvee K \wedge \bigvee J = \bigvee (K \wedge J) = 0,$$

whence $K \wedge J = 0_{\beta M}$, implying that $K \leq J^*$. As before, this implies that $h^\beta(U)^* \leq J^*$, and hence equality. Therefore h^β is rigid.

(2) $\Rightarrow$ (1) : Let $c \in \operatorname{Coz} M$. Take $J \in \operatorname{Coz}\beta M$ such that $c = \bigvee J$. By (2), there exists $U \in \operatorname{Coz}\beta L$ such that $h^\beta(U)^* = J^*$. Since $h\tau_L = \tau_M h^\beta$ (recall that the τ-maps are join maps), and each join map commutes with pseudocomplement as it is dense onto, we have

$$\left(h(\bigvee U)\right)^* = \left(\bigvee h^\beta(U)\right)^* = \bigvee h^\beta(U)^* = \bigvee J^* = \left(\bigvee J\right)^* = c^*.$$

Since $\bigvee U \in \operatorname{Coz} L$, (1) follows.

(1) $\Rightarrow$ (3) : Let $\beta \in \mathcal{R}M$. Then $\operatorname{coz}\beta \in \operatorname{Coz} M$, and so, by (1), there exists $\alpha \in \mathcal{R}L$ such that $h(\operatorname{coz}\alpha)^* = (\operatorname{coz}\beta)^*$. Therefore $(\operatorname{coz}(\mathcal{R}h(\alpha)))^* = (\operatorname{coz}\beta)^*$, which, by Lemma 4.3.1, implies that $\operatorname{Ann}(\mathcal{R}h(\alpha)) = \operatorname{Ann}\beta$. Therefore $\mathcal{R}h$ is rigid.

(3) $\Rightarrow$ (1) : Let $d \in \operatorname{Coz} M$ and take $\delta \in \mathcal{R}M$ with $\operatorname{coz}\delta = d$. By (3), there exists $\gamma \in \mathcal{R}L$ such that $\operatorname{Ann}(\mathcal{R}h(\gamma)) = \operatorname{Ann}(\delta)$. By Lemma 4.3.1, this implies that

$$d^* = (\operatorname{coz}\delta)^* = (\operatorname{coz}(h \circ \gamma))^* = h(\operatorname{coz}\gamma)^*,$$

which establishes (1).

(3) $\Rightarrow$ (4) : Suppose (3) holds and let $\beta \in \mathcal{R}^*L$. Then $\beta \in \mathcal{R}L$, and so, by (3), there exists $\alpha \in \mathcal{R}L$ such that $\operatorname{Ann}_{\mathcal{R}M}(h \circ \alpha) = \operatorname{Ann}_{\mathcal{R}M}(\beta)$. Now, $\frac{\alpha^2}{1+\alpha^2} \in \mathcal{R}^*L$ and

$$\operatorname{coz}\left(h \circ \frac{\alpha^2}{1+\alpha^2}\right) = h\left(\operatorname{coz}\frac{\alpha^2}{1+\alpha^2}\right) = h(\operatorname{coz}\alpha).$$

Thus, by Lemma 4.3.1,

$$\operatorname{Ann}_{\mathcal{R}M}\left(h \circ \frac{\alpha^2}{1+\alpha^2}\right) = \operatorname{Ann}_{\mathcal{R}M}(\beta),$$

whence we have

$$\operatorname{Ann}_{\mathcal{R}M}\left(h \circ \frac{\alpha^2}{1+\alpha^2}\right) \cap \mathcal{R}^*M = \operatorname{Ann}_{\mathcal{R}M}(\beta) \cap \mathcal{R}^*M,$$

that is,

$$\operatorname{Ann}_{\mathcal{R}^*M}\left(h \circ \frac{\alpha^2}{1+\alpha^2}\right) = \operatorname{Ann}_{\mathcal{R}^*M}(\beta).$$

This establishes (4).

(4) $\Rightarrow$ (3) : Let $\beta \in \mathcal{R}M$. Then $\frac{\beta^2}{1+\beta^2} \in \mathcal{R}^*M$. By (4), there exists $\alpha \in \mathcal{R}^*L$ such that

$$\operatorname{Ann}_{\mathcal{R}^*M}(h \circ \alpha) = \operatorname{Ann}_{\mathcal{R}^*M}\left(\frac{\beta^2}{1+\beta^2}\right). \qquad (\dagger)$$

We claim that $\mathrm{Ann}_{\mathcal{R}M}(h\circ\alpha) = \mathrm{Ann}_{\mathcal{R}M}(\beta)$. Let $\gamma \in \mathrm{Ann}\,_{\mathcal{R}M}(h\circ\alpha)$. Then $\frac{\gamma^2}{1+\gamma^2}$ is an element of $\mathcal{R}^*M$ such that $\frac{\gamma^2}{1+\gamma^2}\cdot(h\circ\alpha) = 0$. Thus, by (†),

$$\frac{\gamma^2}{1+\gamma^2}\cdot\frac{\beta^2}{1+\beta^2} = 0,$$

which implies $(\gamma\beta)^2 = 0$, and hence $\gamma\beta = 0$ since $\mathcal{R}M$ is reduced. So, $\gamma \in \mathrm{Ann}\,_{\mathcal{R}M}(\beta)$, and therefore $\mathrm{Ann}_{\mathcal{R}M}(h\circ\alpha) \subseteq \mathrm{Ann}\,_{\mathcal{R}M}(\beta)$. Now let $\varphi \in \mathrm{Ann}\,_{\mathcal{R}M}(\beta)$. Then

$$\frac{\varphi^2}{1+\varphi^2}\cdot\frac{\beta^2}{1+\beta^2} = 0,$$

and hence, from (†), $\frac{\varphi^2}{1+\varphi^2} \in \mathrm{Ann}\,_{\mathcal{R}^*M}(h\circ\alpha)$. Thus, $\varphi\cdot(h\circ\alpha) = 0$, and so $\mathrm{Ann}_{\mathcal{R}M}(\beta) \subseteq \mathrm{Ann}\,_{\mathcal{R}M}(h\circ\alpha)$ – hence equality. ∎

Remark 4.3.3 The equivalence of (3) and (4) in Proposition 4.3.2 holds more generally. Namely, if $\phi : A \to B$ is a homomorphism of reduced f-rings with bounded inversion, and if A^* and B^* denote their bounded parts, then ϕ is rigid if and only if the map $\phi|_{A^*} : A^* \to B^*$ is rigid.

Reasoning as in the proof of (2) ⇔ (1) in Proposition 4.3.2 and taking into cognisance the fact that, for any $c \in \mathrm{Coz}\,L, h^\lambda([c]) = [h(c)]$, we obtain:

Proposition 4.3.4 *A frame homomorphism $h : L \to M$ is rigid if and only if $h^\lambda : \lambda L \to \lambda M$ is rigid.*

We recall the following definitions from [37]. A frame L is *weakly Lindelöf* if for every cover C of L there is a countable $S \subseteq C$ such that $(\bigvee S)^* = 0$. An element a of L is weakly Lindelöf if whenever $a = \bigvee T$, then $a^* = (\bigvee S)^*$ for some countable $S \subseteq T$. It is shown in [37, Proposition 7] that a cozero element of a weakly Lindelöf frame is weakly Lindelöf. It is shown in [33, Proposition 3.2] that any frame homomorphism between completely regular frames onto a Lindelöf frame is coz-onto. Weakening the Lindelöf requirement leads to the following result.

Proposition 4.3.5 *Any frame homomorphism between completely regular frames onto a weakly Lindelöf frame is rigid.*

Proof Let M be weakly Lindelöf and $h : L \to M$ be an onto frame homomorphism. For $c \in \operatorname{Coz} M$, take $a \in L$ such that $h(a) = c$. By complete regularity, there is a set $C \subseteq \operatorname{Coz} L$ such that $a = \bigvee C$. Thus, $c = \bigvee h[C]$, and hence, by virtue of c being a cozero element of a weakly Lindelöf frame, there is a countable $S \subseteq C$ such that

$$c^* = (\bigvee\{h(s) \mid s \in S\})^* = h(\bigvee S)^*,$$

by the result cited from [37]. But now $\bigvee S \in \operatorname{Coz} L$, so the result follows. ∎

Chapter 5

Cozero complemented frames

The types of frames to be considered here can be thought of as generalization of P-frames. Their topological antecedents arose from a study by Henriksen and Woods [38] of when the space Min $(C(X))$ of minimal prime ideals of $C(X)$ is compact. Such spaces have since come to be known by the name "cozero complemented spaces". We adopt the same name for frames that generalize them. Our goal in this chapter is to give several characterizations of cozero complemented frames.

5.1 Quotients of cozero-complemented frames

We start by recapitulating the definition of cozero complemented frames, from Definition 4.2.4.

Examples of cozero complemented frames abound. For instance, every basically disconnected frame and every Oz-frame is cozero complemented.

Recall that a point x of a Tychonoff space X is a *P-point* if every zero-set of X containing x is a neighbourhood of x. The space is then a *P-space* if and only if every point in it is a P-point. On the other hand, x is an *almost P-point* of X if every zero-set of X containing x has a non-empty interior. A topological space is an almost P-space if and only if every point in it is an almost P-point. In [46], Levy and Shapiro show that if X has an almost P-point which is not a P-point, then X is not cozero complemented.

This phenomenon actually holds in the category of frames. To justify this assertion, we define the notions of P-point and almost P-point for frames. We use the **O**-ideals and **M**-ideals introduced by Dube [26] as follows: For any $I \in \mathrm{Pt}(\beta L)$, let

$$\mathbf{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\} \text{ and } \mathbf{O}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \prec I\}.$$

Definition 5.1.1 A point I of βL is a *P-point* if $\mathbf{M}^I = \mathbf{O}^I$, and it is an *almost P-point* if for any $\alpha \in \mathbf{M}^I$, $\operatorname{coz}\alpha$ is not dense.

In [29, Proposition 3.9], it is shown that a frame L is a P-frame if and only if $\mathbf{M}^I = \mathbf{O}^I$ for each $I \in \mathrm{Pt}(\beta L)$, that is, if and only if every point of βL is a P-point. Now recall from [7] that L is an *almost P-frame* if the only dense cozero element of L is the top. We claim that L is an almost P-frame if and only if every $I \in \mathrm{Pt}(\beta L)$ is an almost P-point. To see this, assume first that L is an almost P-frame, and let I be a point of βL. For any $\alpha \in \mathbf{M}^I$, $r_L(\operatorname{coz}\alpha) \subseteq I$, and so $\operatorname{coz}\alpha \neq 1$. Therefore $\operatorname{coz}\alpha$ is not dense, and hence I is an almost P-point. Conversely, suppose every point of βL is an almost P-point, and consider any $\alpha \in \mathcal{R}L$ with $\operatorname{coz}\alpha \neq 1$. Then $r_L(\operatorname{coz}\alpha) \neq 1_{\beta L}$, and so since βL is spatial, there is an $I \in \mathrm{Pt}(\beta L)$ with $r_L(\operatorname{coz}\alpha) \subseteq I$. Thus, $\alpha \in \mathbf{M}^I$. Since I is an almost P-point, $\operatorname{coz}\alpha$ is not dense. Therefore L is an almost P-frame.

This shows that the definitions of P-point and almost P-point we have coined are justified.

Example 5.1.2 Let L be a frame with an almost P-point I which is not a P-point. Then L is not cozero complemented. We prove this by demonstrating the existence of a cozero element which violates the requirements in the definition. Since I is not a P-point, there is a positive $\alpha \in \mathbf{M}^I \backslash \mathbf{O}^I$. Then $r_L(\operatorname{coz}\alpha)^* \vee I \neq 1_{\beta L}$. Since $I \in \operatorname{Pt}(\beta L)$ and $I \leq r_L(\operatorname{coz}\alpha)^* \vee I \neq 1_{\beta L}$, it follows that $I = r_L(\operatorname{coz}\alpha)^* \vee I$, so that $r_L(\operatorname{coz}\alpha)^* \subseteq I$. Now suppose, by way of contradiction, that there is a positive $\gamma \in \mathcal{R}L$ such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\gamma = 0$ and $\operatorname{coz}\alpha \vee \operatorname{coz}\gamma$ is dense. Now $\operatorname{coz}\alpha \wedge \operatorname{coz}\gamma = 0$ implies that $r_L(\operatorname{coz}\alpha) \wedge r_L(\operatorname{coz}\gamma) = 0_{\beta L}$, so that $r_L(\operatorname{coz}\gamma) \leq r_L(\operatorname{coz}\alpha)^* \subseteq I$, which in turn implies $\gamma \in \mathbf{M}^I$. Consequently $\alpha + \gamma \in \mathbf{M}^I$ for which $\operatorname{coz}(\alpha+\gamma) = \operatorname{coz}\alpha \vee \operatorname{coz}\gamma$ is dense. This contradicts the fact that I is an almost P-point.

Less obvious examples of cozero complemented frames are given by the following proposition. As in spaces, say a frame has the *countable chain condition* (abbreviated *ccc*) if every collection of pairwise disjoint elements of L is countable.

Proposition 5.1.3 *Every frame with ccc is cozero complemented.*

Proof Let c be a non-dense cozero element of L, where L has ccc, and put

$$\Im = \{S \subseteq {\downarrow}c^* \cap \operatorname{Coz} L \mid \text{any two elements of } S \text{ do not meet and } 0 \notin S\}.$$

The set $\Im \neq \emptyset$ since $\operatorname{Coz} L$ generates L. Partially order $\Im$ by inclusion. Let $\{S_\alpha \mid \alpha \in \Gamma\}$ be a chain in $\Im$. It is easy to show that $\bigcup_{\alpha\in\Gamma} S_\alpha$ is an element of $\Im$, so that every chain in $\Im$ has an upper bound. So, by Zorn's Lemma, $\Im$ has a maximal element, say S. Now S is countable, by the hypothesis on L. Therefore $d = \bigvee S \in \operatorname{Coz} L$. Clearly, $c \wedge d = 0$ since $\bigvee S \leq c^*$. Now it remains to show that $c \vee d$ is dense in L. Since $\operatorname{Coz} L$ generates L, by complete regularity, it suffices to show that $c \vee d$ meets every nonzero cozero element of L. Let $w \in \operatorname{Coz} L$ be such that $w \wedge (c \vee d) = 0$. This implies that $(w \wedge c) \vee (w \wedge d) = 0$. Therefore $w \wedge c = w \wedge d = 0$. Thus $w \leq c^*$, i.e., $w \in {\downarrow}c^* \cap \operatorname{Coz} L$. Since $w \wedge \bigvee S = 0$, it follows that $w \wedge t = 0$ for every $t \in S$. Therefore the set $S \cup \{w\}$ consists of mutually disjoint elements. Since S is maximal with this property, it follows that $w = 0$ (otherwise the maximality of S

is contradicted). So $c \vee d$ is dense. Now if c is dense, then 0 is a cozero element with $c \wedge 0 = 0$ and $c \vee 0$ dense. ■

Remark 5.1.4 Recall that a frame L is *weakly Lindelöf* if every cover of L has a countable subset which is dense. A proof similar to the foregoing one shows that every frame with ccc is weakly Lindelöf.

The following proposition gives an instance of a quotient of a cozero complemented frame being itself cozero complemented. In the proof we will employ the fact that if h is onto, then h_* is one-one.

Proposition 5.1.5 *Let $h : L \to M$ be a dense quotient map where L has ccc. Then M has ccc, and hence is cozero complemented.*

Proof Let $S \subseteq M$ consist of pairwise disjoint elements. The set $h_*[S]$ consists of pairwise disjoint elements because for any distinct $s, t \in S$,

$$h_*(s) \wedge h_*(t) \neq 0 \Rightarrow s \wedge t = hh_*(s) \wedge hh_*(t) \neq 0,$$

which contradicts the nature of S. Since L has ccc, $h_*[S]$ is countable. But h_* is one-one, so the cardinality of S is less than or equal to that of $h_*[S]$, whence S is countable. ■

This result is not a legitimate example of when a quotient inherits the property of being cozero complemented because the frame L satisfies a property stronger than cozero complementedness. Below we provide cases where quotients inherit cozero complementedness from a frame which is assumed only to be cozero complemented. The reader will note that these are extensions of corresponding point-sensitive results such as established by Levy and Shapiro [46] and Henriksen and Woods [38].

For the first of these results we need to collect some facts from the literature. Recall that an element of a frame is dense if it meets every nonzero element of the frame.

Facts 5.1.6 Let L be a frame and $a \in L$.

(1) For any $x \in L$, if x is dense in L, then $a \wedge x$ is dense in ${\downarrow}a$.

(2) For any $x \in {\downarrow}a$, let $x^{\odot}$ denote the pseudocomplement of x in ${\downarrow}a$. Then $x^{\odot} = x^* \wedge a$.

Proof (1) Let $t \in {\downarrow}a$ be nonzero. Then

$$t \wedge (a \wedge x) = (t \wedge a) \wedge x = t \wedge x \neq 0$$

since x is dense in L. So $a \wedge x$ meets every nonzero element of ${\downarrow}a$.

(2) The element $x^* \wedge a \in {\downarrow}a$ and $x \wedge (x^* \wedge a) = 0$. Therefore $x^* \wedge a \leq x^{\odot}$. Now let $z \in {\downarrow}a$ be such that $z \wedge x = 0_{\downarrow a} = 0$. Then $z \leq x^*$. But $z \leq a$, so $z \leq x^* \wedge a$. Since $x^{\odot}$ is an element of ${\downarrow}a$ with $x \wedge x^{\odot} = 0_{\downarrow a} = 0$, it follows that $x^{\odot} \leq x^* \wedge a$. Hence $x^{\odot} = x^* \wedge a$. ∎

Proposition 5.1.7 *Let L be cozero complemented and $a \in \operatorname{Coz} L$. Then ${\downarrow}a$ is cozero complemented.*

Proof Let $c \in \operatorname{Coz}({\downarrow}a)$. Then $c = a \wedge u$ for some $u \in \operatorname{Coz} L$ because, by [7, Proposition 3.2.10], the map $\varphi : L \to {\downarrow}a$ given by $\varphi(x) = a \wedge x$ is coz-onto if $a \in \operatorname{Coz} L$. Since L is cozero complemented, there exists $w \in \operatorname{Coz} L$ such that $u \wedge w = 0$ and $u \vee w$ is dense in L. Now $a \wedge w \in \operatorname{Coz}({\downarrow}a)$ since frame homomorphisms preserve cozero elements. Further,

$$(a \wedge w) \wedge c = (a \wedge c) \wedge w = c \wedge w \leq u \wedge w = 0_{\downarrow a},$$

and

$$c \vee (a \wedge w) = (a \wedge u) \vee (a \wedge w) = a \wedge (u \vee w).$$

But $a \wedge (u \vee w)$ is dense in ${\downarrow}a$ as $u \vee w$ is dense in L, so ${\downarrow}a$ is cozero complemented. ∎

Before presenting the next result – which is an extension of [38, Theorem 2.8(a)] – we give a frame analogue of $z^{\#}$-embedded subspace. As in [38], we say a subspace S of X is *$z^{\#}$-embedded* in X if for each $f \in C(S)$, then there is a $g \in C(X)$ such that

$$\mathrm{cl}_S(\mathrm{int}_S Z(f)) = S \cap \mathrm{cl}_X(\mathrm{int}_X Z(g)).$$

Let us express this notion in terms of the frame homomorphism

$$h : \mathfrak{O}X \to \mathfrak{O}S \quad \text{given by} \quad U \mapsto U \cap S,$$

which will then motivate the frame analogue we seek. We do this for dense subspaces. We recall from [38, Lemma 2.3] that

> *a dense subspace S of X is $z^{\#}$-embedded in X if and only if for every $C \in$ $\mathrm{Coz}\,(\mathfrak{O}S)$, there is a $V \in \mathrm{Coz}\,(\mathfrak{O}X)$ such that $\mathrm{cl}_S C = S \cap \mathrm{cl}_X V$.*

Proposition 5.1.8 *Let S be a dense subspace of X, and let $h : \mathfrak{O}X \to \mathfrak{O}S$ be as above. Then S is $z^{\#}$-embedded in X if and only if for every $U \in \mathrm{Coz}\,(\mathfrak{O}S)$, there is a $V \in \mathrm{Coz}\,(\mathfrak{O}X)$ such that $h(V^*) = U^*$.*

Proof ($\Rightarrow$): Let $U \in \mathrm{Coz}\,(\mathfrak{O}S)$. By the characterization cited from [38], there is a $V \in \mathrm{Coz}\,(\mathfrak{O}X)$ such that $\mathrm{cl}_S U = S \cap \mathrm{cl}_X V$. Denote pseudocomplement in $\mathfrak{O}S$ by $(\cdot)^{\#}$. Then

$$U^{\#} = S \backslash \mathrm{cl}_S U = S \backslash \Big(S \cap \mathrm{cl}_X V \Big) = S \cap (X \backslash \mathrm{cl}_X V) = S \cap V^*.$$

Thus, $U^{\#} = h(V^*)$, as required.

($\Leftarrow$): Suppose that for every $U \in \mathrm{Coz}\,(\mathfrak{O}S)$, there is a $V \in \mathrm{Coz}\,(\mathfrak{O}X)$ such that $h(V^*) = U^{\#}$. That is

$$\begin{aligned} & U^{\#} = h(V^*) = S \cap V^* \\ \Rightarrow \quad & S \backslash \mathrm{cl}_S U = S \cap (X \backslash \mathrm{cl}_X V) \\ \Rightarrow \quad & \mathrm{cl}_S U = S \cap \mathrm{cl}_X V \end{aligned}$$

That is S is $z^{\#}$-embedded in X and we are done. ∎

Based on the above, we formulate the following definition.

Definition 5.1.9 A quotient map $h : L \to M$ is a *$z^{\#}$-quotient map* if for each $v \in \operatorname{Coz} M$, there is a $u \in \operatorname{Coz} L$ such that $h(u^*) = v^*$.

Since dense quotient maps commute with pseudocomplement, it follows that every dense coz-onto quotient map is a $z^{\#}$-quotient map. Dense $z^{\#}$-quotient maps transport cozero complementedness in both directions as the proposition that follows shows.

Proposition 5.1.10 *Let $h : L \to M$ be a dense $z^{\#}$-quotient map. Then L is cozero complemented if and only if M is cozero complemented.*

Proof ($\Rightarrow$): Assume L is cozero complemented, and let $u \in \operatorname{Coz} M$. Since h is a $z^{\#}$-quotient map, there is an $a \in \operatorname{Coz} L$ such that $h(a^*) = u^*$. Since L is cozero complemented, there is a $b \in \operatorname{Coz} L$ with $a \wedge b = 0$ and $a \vee b$ dense. Now $h(b)$ is a cozero element of M such that $h(b) \le h(a^*) = u^*$ since $b \le a^*$. Thus $u \wedge h(b) = 0$. On the other hand,

$$\Big(u \vee h(b)\Big)^* = u^* \wedge h(b)^* = u^* \wedge h(b^*) = h(a^*) \wedge h(b^*) = h((a \vee b)^*) = 0.$$

Therefore $u \vee h(b)$ is dense, and hence M is cozero complemented.

($\Leftarrow$): Let $a \in \operatorname{Coz} L$. Then $h(a) \in \operatorname{Coz} M$. Since M is cozero complemented by the present hypothesis, there is a $u \in \operatorname{Coz} M$ such that $h(a) \wedge u = 0$ and $h(a) \vee u$ is dense. Since h is a $z^{\#}$-quotient map, there is a $b \in \operatorname{Coz} L$ with $h(b^*) = u^*$. Now $u \wedge h(a) = 0$ implies

$$h(a) \le u^* = h(b^*) = h(b)^*,$$

whence we get

$$h(a \wedge b) = h(a) \wedge h(b) = 0,$$

implying $a \wedge b = 0$ since h is dense. On the other hand,

$$0 = \Big(u \vee h(a)\Big)^* = u^* \wedge h(a)^* = h(b^*) \wedge h(a^*) = h((a \vee b)^*).$$

Since h is dense, this implies that $(a \vee b)^* = 0$, so that $a \vee b$ is dense. Therefore L is cozero complemented. ■

Corollary 5.1.11 *If $h : L \to M$ is a dense coz-onto homomorphism between completely regular frames, then L is cozero complemented if and only if M is cozero complemented. Consequently, the following are equivalent for a completely regular frame L:*

(1) L is cozero complemented.

(2) βL is cozero complemented.

(3) λL is cozero complemented.

(4) υL is cozero complemented.

For the next result, the openness of the map f in [38, Lemma 2.4] is relaxed in frames, without somewhat violating the conclusion.

Lemma 5.1.12 *Let $h : L \to M$ be a nearly open quotient map with M weakly Lindelöf. Then h is a $z^{\#}$-quotient map.*

Proof Let $z \in \operatorname{Coz} M$. Since h is onto, it follows that $h(h_*(z)) = z$. Since L is completely regular, there exists $S \subseteq \operatorname{Coz} L$ such that $h_*(z) = \bigvee S$. Therefore

$$z = h\Big(\bigvee S\Big) = \bigvee h[S].$$

Since $z \in \operatorname{Coz} M$ and M is weakly Lindelöf, it follows by [37, Proposition 7] that z is weakly Lindelöf. Therefore there exist countably many elements $(s_n) \subseteq S$ such that

$$z^* = (\bigvee h(s_n))^*.$$

Now put $v = \bigvee s_n$. Then $v \in \operatorname{Coz} L$ and

$$z^* = \Big(\bigvee h(s_n)\Big)^* = \Big(h(\bigvee s_n)\Big)^* = h(v)^* = h(v^*).$$

∎

This lemma enables us to give a strong frame version of Corollary 2.9(a) in [38] as follows:

Corollary 5.1.13 *Let $h : L \to M$ be a nearly open map with M weakly Lindelöf. Then L is cozero complemented if and only if M is cozero complemented.*

Combining Remark 5.1.4 and the preceding corollary, the following corollary is apparent.

Corollary 5.1.14 *Let $h : L \to M$ be a nearly open map with M a ccc-frame. Then L is cozero complemented.*

Next, recall from Chapter 1 that if j is a nucleus on a frame L, $\mathrm{Fix}(j) = \{a \in L \mid j(a) = a\}$. If $\ell \leq j$, then $\mathrm{Fix}(j) \subseteq \mathrm{Fix}\,(\ell)$. It is well known that $\mathrm{Fix}(j)$ is a frame, see for instance, Johnstone [42] and $j : L \to \mathrm{Fix}\,(j)$ is a frame homomorphism. A nucleus is *dense* if it maps only the bottom element to the bottom element. We denote, as usual, by NL the *assembly* of L, that is, the frame of nuclei on L. We recall from [9, p. 32] that if $j, k \in NL$ and $j \leq k$, then the map

$$\mathrm{Fix}(j) \xrightarrow{k} \mathrm{Fix}(k)$$

is a frame homomorphism. Furthermore, if $j \in NL$ and $a \in L$ we have

$$v_a \vee j = v_a \cdot j \quad \text{and} \quad u_a \vee j = j \cdot u_a,$$

where $u_a(x) = a \vee x$ (see, [55]).

Now, in [38], Henriksen and Woods show that:

If $S \cap W$ is a weakly Lindelöf space, where S is a dense subspace and W is an open subspace of a cozero complemented space T, then $S \cap W$ is cozero complemented.

This result is extended to frames as follows:

Proposition 5.1.15 *Let L be a cozero complemented completely regular frame, j be a dense nucleus and $a \in L$. If $\mathrm{Fix}(v_a \vee j)$ is weakly Lindelöf, then it is cozero complemented.*

Proof $\mathrm{Fix}(v_a) \overset{v_a \vee j}{\to} \mathrm{Fix}\,(v_a \vee j)$ is a dense homomorphism. To show denseness, let $s \in \mathrm{Fix}\,(v_a)$ be such that

$$(v_a \vee j)(s) = 0_{\mathrm{Fix}(v_a \vee j)} = v_a(0).$$

That is $v_a(j(s)) = v_a(j(0)) = v_a(0)$. We must show that $s = v_a(0)$. Now $s \in \mathrm{Fix}\,(v_a)$ is such that

$$(v_a \vee j)(s) = v_a(j(s)) = v_a(0);$$

j is dense, hence $s = v_a(s)$ and $s \leq j(s)$. Now

$$s = v_a(s) \leq v_a(j(s)) = v_a(0) \leq v_a(s) = s.$$

Therefore

$$s = v_a(0) = 0_{\mathrm{Fix}(v_a)}.$$

Because $\mathrm{Fix}(v_a)$ is open in L, it follows that v_a is a $z^{\#}$-quotient map by the previous results. Thus, $\mathrm{Fix}(v_a)$ is cozero complemented. But $\mathrm{Fix}(v_a \vee j)$ is dense in $\mathrm{Fix}(v_a)$, so again by the Lemma 5.1.12, $\text{Fix}(v_a \vee j)$ is a $z^{\#}$-quotient of $\text{Fix}(v_a)$. Then, by Proposition 5.1.10, $\text{Fix}(v_a \vee j)$ is cozero complemented. ∎

5.2 Locally cozero-complemented frames

In this section, we introduce the notion of locally cozero complemented frames and, amongst other things, we give the conditions that guarantee a given frame to be cozero complemented whenever it is locally cozero complemented. In particular, we show that a weakly Lindelöf frame is cozero complemented if and only if it is locally cozero complemented. Furthermore, we show that a locally cozero complemented paracompact frame is cozero complemented.

Definition 5.2.1 A frame L is *locally cozero complemented* if for all $a \in L$,

$$a = \bigvee_{\alpha \in I} x_\alpha,$$

for some index set I, where ${\downarrow}x_\alpha$ is cozero complemented for all $\alpha \in I$.

In [38, Proposition 5.1], the authors show that if X is the countable union of cozero complemented cozero sets, then X is cozero complemented. We extend this result to frames. We shall need the following lemma to do that.

Lemma 5.2.2 *Let b, $c \in \operatorname{Coz} L$ be such that ${\downarrow}b$ and ${\downarrow}c$ are cozero complemented. Then ${\downarrow}(b \vee c)$ is cozero complemented.*

Proof Let $z \in \operatorname{Coz}({\downarrow}(b \vee c))$. Then $z \wedge b \in \operatorname{Coz}({\downarrow}b)$ and $z \wedge c \in \operatorname{Coz}({\downarrow}c)$ as they are, respectively, images of the maps ${\downarrow}(b \vee c) \overset{-\wedge b}{\to} {\downarrow}b$ and ${\downarrow}(b \vee c) \overset{-\wedge c}{\to} {\downarrow}c$. Find $u \in \operatorname{Coz}({\downarrow}c)$ and $v \in \operatorname{Coz}({\downarrow}b)$ such that

(i) $(z \wedge c) \wedge u = 0$ and $(z \wedge c) \vee u$ is dense in $\downarrow c$.

(ii) $(z \wedge b) \wedge v = 0$ and $(z \wedge b) \vee v$ is dense in $\downarrow b$.

Now, in view of the fact that $u \wedge c = u$ and $v \wedge b = v$, we have

$$z \wedge (u \vee v) = (z \wedge u) \vee (z \wedge v) = (z \wedge u \wedge c) \vee (z \wedge v \wedge b) = 0.$$

Now if $b \in \mathrm{Coz}\, L$, then $b \in \mathrm{Coz}\,(\downarrow(b \vee c))$. Consider the frame homomorphism $\varphi : L \to \downarrow(b \vee c)$. Then $\varphi(b) = b \wedge (b \vee c) = b \in \mathrm{Coz}\,(\downarrow(b \vee c))$ since frame homomorphisms preserve cozeros. If $u \in \mathrm{Coz}\,(\downarrow b)$, then $u \in \mathrm{Coz}\,(\downarrow(b \vee c))$ by [7, Proposition 3.2.10]. Similarly, $v \in \mathrm{Coz}\,(\downarrow(b \vee c))$. Observe that $u \vee v \in \mathrm{Coz}\,(\downarrow(b \vee c))$ as a join of two cozero elements in the frame $\downarrow(b \vee c)$. Let us show that $z \vee (u \vee v)$ is dense in $\downarrow(b \vee c)$. Take any $p \neq 0$ in $\downarrow(b \vee c)$. Then $p \wedge b \neq 0$ or $p \wedge c \neq 0$ since

$$p = p \wedge (b \vee c) = (p \wedge b) \vee (p \wedge c).$$

If $p \wedge b \neq 0$, then

$$\begin{aligned} p \wedge (z \vee u \vee v) &\geq p \wedge (z \vee v) \\ &\geq (p \wedge b) \wedge (z \vee v) \\ &\geq (p \wedge b) \wedge ((z \vee v) \wedge b) \\ &= (p \wedge b) \wedge ((z \wedge b) \vee v) \\ &\neq 0 \end{aligned}$$

the last step valid since $(z \wedge b) \vee v$ is dense in $\downarrow b$ and $p \wedge b$ is a nonzero element of $\downarrow b$. Similarly, if $p \wedge c \neq 0$, then $p \wedge (z \vee (u \vee v)) \neq 0$. Since $u \vee v \in \mathrm{Coz}\,(\downarrow(b \vee c))$, it follows that $\downarrow(b \vee c)$ is cozero complemented. ∎

Now we extend the result of Henriksen and Woods to frames as follows:

Proposition 5.2.3 *Let L be a completely regular frame. If there is a sequence (a_n) in*

$\operatorname{Coz} L$ *such that* $\bigvee a_n = 1$ *and* $\downarrow a_n$ *is cozero complemented for every* n*, then* L *is cozero complemented.*

Proof Let $u \in \operatorname{Coz} L$. Define the sequence (b_n) in $\operatorname{Coz} L$ by: $b_1 = a_1$ and $b_n = a_1 \vee \ldots \vee a_n$ for $n \geq 2$. Then (b_n) is an increasing sequence with $\bigvee b_n = \bigvee a_n = 1$, and, by Lemma 5.2.2, $\downarrow b_n$ is cozero complemented for each n. The element $u \wedge b_n \in \operatorname{Coz}(\downarrow b_n)$, since $\varphi : L \to \downarrow b_n$ is the map $\varphi(x) = x \wedge b_n$, so that $\varphi(u) = u \wedge b_n$, and frame homomorphisms preserve cozero elements. Since $\downarrow b_n$ is cozero complemented, there exists $w_n \in \operatorname{Coz}(\downarrow b_n)$ such that

$$(u \wedge b_n) \wedge w_n = 0 \text{ and } (u \wedge b_n) \vee w_n \text{ is dense in } \downarrow b_n.$$

Now $w = \bigvee w_n \in \operatorname{Coz} L$. We show that

$$u \wedge w = 0 \text{ and } u \vee w \text{ is dense in } L.$$

To show denseness, let $p \in L$, $p \neq 0$. Then $p \wedge b_n \neq 0$ for some n since $p = \bigvee (p \wedge b_n)$. But $(u \wedge b_n) \vee w_n$ is dense in $(\downarrow b_n)$, so

$$\begin{aligned} p \wedge (u \vee w) &\geq (p \wedge b_n) \wedge (u \vee w) \\ &\geq (p \wedge b_n) \wedge ((u \wedge b_n) \vee w) \\ &\geq (p \wedge b_n) \wedge ((u \wedge b_n) \vee w_n) \\ &\neq 0, \end{aligned}$$

the last step holding since $(u \wedge b_n) \vee w_n$ is dense in $\downarrow b_n$ and $p \wedge b_n$ is a nonzero element in $\downarrow b_n$. Therefore $u \vee w$ is dense in L. On the other hand,

$$\begin{aligned} u \wedge w &= \left(u \wedge \bigvee_n b_n \right) \wedge \left(w \wedge \bigvee_m b_m \right) \\ &= \left(\bigvee_n (u \wedge b_n) \right) \wedge \left(\bigvee_m (w \wedge b_m) \right) \\ &= \bigvee_n \bigvee_m (u \wedge b_n \wedge w \wedge b_m) \\ &= \bigvee_n \bigvee_m (u \wedge b_{\min(m,n)} \wedge w_{\min(m,n)}) \\ &= 0. \end{aligned}$$

This completes the proof. ∎

Recall from [36] that if X is a Tychonoff space, then a cozero-set C of X is called a *complemented cozero-set* if there is a cozero-set D of X such that $C \cap D = \emptyset$ and $C \cup D$ is dense in X.

Definition 5.2.4 Let L be a completely regular frame. Then an element $c \in \operatorname{Coz} L$ is called *pseudo-cozero complemented* if there is a $d \in \operatorname{Coz} L$ such that $c \wedge d = 0$ and $c \vee d$ is dense in L.

The next result shows that for weakly Lindelöf frames, cozero complementedness and local cozero complementedness coincide.

Proposition 5.2.5 *Let L be weakly Lindelöf. Then L is cozero complemented if and only if L is locally cozero complemented.*

Proof ($\Rightarrow$) Since L is completely regular, given $a \in L$, there exist cozero elements x_α such that $a = \bigvee x_\alpha$. Now if $x_\alpha \in \operatorname{Coz} L$, then the quotient map $L \to {\downarrow} x_\alpha$ is a $z^{\#}$-quotient map since it is coz-onto. By [37, Proposition 7], x_α is weakly Lindelöf and so the frame ${\downarrow} x_a$ is weakly Lindelöf. So given L to be cozero complemented, it follows by Corollary 5.1.13 that ${\downarrow} x_\alpha$ is cozero complemented.

($\Leftarrow$) We need to produce an element $d \in L$ such that

(i) d is dense in L

(ii) d is pseudo-cozero complemented.

Given $a \in L$, choose elements $\{x_\alpha^{(a)} \mid \alpha \in A_a\}$ such that

(i) $\downarrow x_\alpha^{(a)}$ is cozero complemented for all α, and

(ii) $a = \bigvee x_\alpha^{(a)}$ – since L is locally cozero complemented.

Now for $\alpha \in A_a$, find (by complete regularity) cozero elements $\{c_\beta^{(a,\alpha)} \mid \beta \in B(a,\alpha)\}$ such that

$$x_\alpha^{(a)} = \bigvee_{\beta \in B(a,\alpha)} c_\beta^{(a,\alpha)}.$$

Then the collection

$$\{c_\beta^{(a,\alpha)} \mid a \in L, \alpha \in A_a, \beta \in B(a,\alpha)\}$$

covers the frame, i.e., has join = top element. But L is weakly Lindelöf, so there exist countably many of these cozero elements whose join is dense. Then the element d is the join of these countably many cozero elements. Hence there is a sequence (c_n) in $\mathrm{Coz}\,(\downarrow d)$ such that $\bigvee c_n = d$ and $\downarrow c_n$ is cozero complemented. Hence, by Proposition 5.2.3, $\downarrow d$ is cozero complemented. Also d is a cozero element as a countable join of cozeros. Since L is weakly Lindelöf, it follows that d is weakly Lindelöf. But d is dense, so the quotient map $h : L \to \downarrow d$ is a $z^{\#}$-quotient map. Hence by Proposition 5.1.10, L is cozero complemented. ■

Next, we show that a hereditarily Lindelöf frame is cozero complemented. We start with a definition.

Definition 5.2.6 A frame L is *hereditarily Lindelöf* if each of its quotients is Lindelöf.

Lemma 5.2.7 *A frame L is hereditarily Lindelöf if and only if $\downarrow a$ is Lindelöf for all $a \in L$.*

Proof ($\Rightarrow$) The left-to-right implication is trivial. Conversely, suppose $\downarrow a$ is Lindelöf for every $a \in L$. Let $h : L \to M$ be a quotient of L, and C a cover of M. For each $x \in C$, pick

$b_x \in L$ such that $h(b_x) = x$. (This is possible, since h is onto). Now

$$1_M = \bigvee C = \bigvee_{x \in C} h(b_x) = h\left(\bigvee_{x \in C} b_x\right).$$

Write $b = \bigvee_{x \in C} b_x$. Then, since $b \in L$, $\downarrow b$ is Lindelöf. Furthermore, $\{b_x\}_{x \in C} \in \mathrm{Cov}(\downarrow b)$. So there exist countably many elements $b_{x_1}, b_{x_2}, \ldots$ such that $b = \bigvee_{i=1}^{\infty} b_{x_i}$. Hence

$$1_M = h(b) = h\left(\bigvee_{i=1}^{\infty} b_{x_i}\right) = \bigvee_{i=1}^{\infty} h(b_{x_i}) = \bigvee_{i=1}^{\infty} x_i.$$

Therefore M is Lindelöf. ∎

Proposition 5.2.8 *If L is hereditarily Lindelöf, then L is ccc.*

Proof Let $A \subseteq L$ be such that its elements are pairwise disjoint. We must show that A is countable. Put $a = \bigvee A$. Then $\downarrow a$ is Lindelöf. Since $A \in \mathrm{Cov}\,(\downarrow a)$, there exists a countable $B \subseteq A$ such that $\bigvee B = a$. If $B \neq A$, let $x \in A - B$. But now $x = x \wedge a = x \wedge \bigvee B = \bigvee\{x \wedge b \mid b \in B\} = 0$. So $A \subseteq B \cup \{0\}$, i.e., A is countable. ∎

From Proposition 5.1.3, the following corollary is apparent.

Corollary 5.2.9 *A hereditarily Lindelöf frame is cozero complemented.*

Next, we show that a locally cozero complemented paracompact frame is cozero complemented. In order to establish that, we shall go through some lemmas.

Lemma 5.2.10 *Let $C = \{c_\alpha\}$ be a chain of cozero complemented elements of L such that* *(i)* $C \subseteq \operatorname{Coz} L$

(ii) for every $D \subseteq C$, $\bigvee D \in \operatorname{Coz} L$.

Then $\downarrow(\bigvee C)$ *is cozero complemented.*

Proof Put $c = \bigvee C = \bigvee c_\alpha$. Let $u \in \operatorname{Coz}(\downarrow c)$. For each α, $u \wedge c_\alpha \in \operatorname{Coz}(\downarrow c_\alpha)$. By cozero complementedness of $\downarrow c_\alpha$, find $d_\alpha \in \operatorname{Coz}(\downarrow c_\alpha)$ such that $u \wedge c_\alpha \wedge d_\alpha = 0$ and $(u \wedge c_\alpha) \vee d_\alpha$ is dense in $\downarrow c_\alpha$. Let $d = \bigvee d_\alpha$. Then d is a cozero element of L and so $d \in \operatorname{Coz}(\downarrow c)$. We need to show that d is a cozero complement of u in $\downarrow c$. That is (i) $u \wedge d = 0$ and (ii) $u \vee d$ is dense in $\downarrow c$. Let $0 \neq p \in \downarrow c$. Then there exist at least one β such that $p \wedge c_\beta \neq 0$. Hence, since $(u \wedge c_\alpha) \vee d_\alpha$ is dense in $\downarrow c_\alpha$, it follows that

$$
\begin{aligned}
& (p \wedge c_\beta) \wedge [(u \wedge c_\alpha) \vee d_\alpha] \neq 0 \\
\Rightarrow\ & (p \wedge c_\beta) \wedge [(u \wedge c_\alpha) \vee d] \neq 0 \\
\Rightarrow\ & [(p \wedge c_\beta) \wedge (u \wedge c_\alpha)] \vee [p \wedge c_\beta \wedge d] \neq 0 \\
\Rightarrow\ & [p \wedge c_\beta \wedge u \wedge c_\alpha] \vee [p \wedge c_\beta \wedge d] \neq 0 \\
\Rightarrow\ & [p \wedge c_{\min(\alpha,\beta)} \wedge u] \vee [p \wedge c_\beta \wedge d] \neq 0 \\
\Rightarrow\ & (p \wedge c_{\min(\alpha,\beta)}) \wedge (u \vee d) \neq 0 \\
\Rightarrow\ & p \wedge (u \vee d) \neq 0
\end{aligned}
$$

That is, $u \vee d$ is dense in $\downarrow c$. Next,

$$
\begin{aligned}
u \wedge d &= (u \wedge c) \wedge (d \wedge c) \\
&= (u \wedge \bigvee c_\alpha) \wedge (d \wedge \bigvee c_\beta) \\
&= \Big(\bigvee (u \wedge c_\alpha)\Big) \wedge \Big(\bigvee (d \wedge c_\beta)\Big) \\
&= \bigvee (u \wedge c_\alpha \wedge c_\beta \wedge d) \\
&= 0.
\end{aligned}
$$

Thus, $\downarrow(\bigvee C)$ is cozero complemented. ∎

Recall the definition of a locally finite subset from Chapter 1.

Lemma 5.2.11 *Let $C \subseteq L$ be a locally finite subset of L consisting of cozero elements, each of which is cozero complemented. Then $\downarrow(\bigvee C)$ is cozero complemented.*

Proof Write $C = \{c_\alpha\}_{\alpha<\nu}$, where α and ν are ordinals. For each $\beta < \nu$, put

$$\bar{c_\beta} = \bigvee\{c_\alpha \mid \alpha \leq \beta\}.$$

If $D \subseteq \mathrm{Coz}\, L$ is locally finite, then $\bigvee D \in \mathrm{Coz}\, L$ by [59, Lemma 1]. Then $\bar{c_\beta} \in \mathrm{Coz}\, L$, for every β, since $\{c_\alpha \mid \alpha \leq \beta\} \subseteq C$ and $c_\alpha \in \mathrm{Coz}\, L$. We claim that $\bar{c_\beta}$ is cozero complemented. Since each c_α is cozero complemented for each $\alpha \leq \beta$, there is a minimal $u_\alpha \in \mathrm{Coz}\, L$ such that u_α misses all c_α and $u_\alpha \vee c_\alpha$ is dense for some $\alpha \leq \beta$. Then u_α misses $\bigvee\{c_\alpha \mid \alpha \leq \beta\}$ and $u_\alpha \vee \bigvee\{c_\alpha \mid \alpha \leq \beta\}$ is obviously dense. This proves the claim. Let $D \subseteq L$ be the set

$$D = \{\bar{c_\beta} \mid \beta < \nu\}.$$

Then D is a chain of cozero complemented cozero elements. That $\bigvee C \leq \bigvee D$ is obvious. Now

$$\begin{aligned} \bar{c_\beta} &= \bigvee_{\alpha\leq\beta} c_\alpha \leq \bigvee C \\ &\Rightarrow \bigvee D \leq \bigvee C \end{aligned}$$

Hence $\bigvee D = \bigvee C$. Hence, by Lemma 5.2.10, $\downarrow(\bigvee C)$ is cozero complemented. ∎

Next, recall the definition of a normal cover in page 31.

The proof of the following lemma can be found in [59]:

Lemma 5.2.12 *Let A be a normal cover of L. Then there exists a locally finite cozero cover C of L such that $C \leq A$.*

Next, recall that a frame L is paracompact if it is regular and if each cover of L has a locally finite refinement. We are now ready to establish the following result.

Theorem 5.2.13 *A locally cozero complemented paracompact frame is cozero complemented.*

Proof Assume L is paracompact and let A be a cover of L. Because L is paracompact, we have that A is a normal cover of L. Then, by Lemma 5.2.12, there exists a locally finite cozero cover B of L such that $B \leq A$. Put $B = \{b_\alpha \mid \alpha < \nu\}$ which is a chain of cozero elements. Because L is locally cozero complemented, we have that each ${\downarrow}b_\alpha$ is cozero complemented for all α. Then, by Lemma 5.2.11, ${\downarrow}(\bigvee B)$ is cozero complemented. Hence L is cozero complemented. ∎

5.3 Characterizations in terms of rings of real-valued continuous functions on L

We now proceed to give ring-theoretic characterizations of cozero complemented frames. We remark that these are extensions of similar characterizations in spaces (see [38]). In this section, we give few characterizations of cozero complemented frames in terms of the ring of a real-valued continuous functions on L, namely $\mathcal{R}L$. Furthermore, denote by $T(R)$, the classical ring of quotients of R. Recall that a ring R is said to be von Neumann regular if for each $a \in R$, there exists $b \in R$ such that $a^2b = a$. In [38], it is shown that X is cozero complemented if and only if the space $\operatorname{Min} C(X)$ of minimal prime ideals of $C(X)$ is compact, if and only if the classical ring of quotients of $C(X)$ is von Neumann regular. In [26], Dube has shown, in the context of frames, that the space $\operatorname{Min}\mathcal{R}L$ is compact if and only if L is cozero complemented.

Let us recall from [31] how annihilators are described in $\mathcal{R}L$. In [27], Dube has shown that, for any $I \in \beta L$, $\operatorname{Ann} \mathbf{O}^I = \operatorname{Ann} \mathbf{M}^I = \mathbf{M}^{r(a^*)}$, where $a = \bigvee I$, whence he deduces that an ideal of $\mathcal{R}L$ is an annihilator ideal if and only if it is of the form $\mathbf{M}^{r(a^*)}$ for some $a \in L$.

In the proof of the following proposition, we shall need certain results from [4] which characterize reduced rings R for which Min$R$ is compact. Recall that a ring R is said to *satisfy a.c.* if for every finitely generated ideal I, there is an $a \in R$ such that $\operatorname{Ann}(I) = \operatorname{Ann}(a)$. We observe that $\mathcal{R}L$ satisfies *a.c.* Indeed, let Q be a finitely generated ideal of $\mathcal{R}L$, generated by the elements $\alpha_1, \ldots, \alpha_n$. Put $\alpha = \alpha_1^2 + \ldots + \alpha_n^2$. Now,

$$\operatorname{Ann}(Q) = \mathbf{M}^{r_L(a^*)} \quad \text{for} \quad a = \operatorname{coz} \alpha_1 \vee \ldots \vee \operatorname{coz} \alpha_n.$$

Since $\operatorname{coz} \alpha = \operatorname{coz} \alpha_1 \vee \ldots \vee \operatorname{coz} \alpha_n$, it follows that $\operatorname{Ann}(Q) = \operatorname{Ann}(\alpha)$. We list as a lemma the results we shall use, less generally in that we will impose the requirement that the ring satisfy *a.c.*

Lemma 5.3.1 *Let R be a reduced ring satisfying a.c., and let $Q(R)$ denote its classical ring of quotients. Then the following statements are equivalent.*

(1) Minimal prime ideals of R are the only prime ideals consisting of zero divisors.

(2) $Q(R)$ is a regular ring.

(3) Min R is compact.

Proposition 5.3.2 *If L is completely regular, then the following are equivalent:*

(1) L is cozero complemented.

(2) For all $\alpha \in \mathcal{R}L$, there exists $\beta \in \mathcal{R}L$ such that $\operatorname{Ann}^2(\alpha) = \operatorname{Ann}(\beta)$.

(3) For all $\alpha \in \mathcal{R}L$, there exists $\beta \in \mathcal{R}L$ such that $(\operatorname{coz} \alpha)^{**} = (\operatorname{coz} \beta)^*$.

(4) *For all* $\alpha \in \mathcal{R}L$*, there exists a nonzero divisor* $d \in \mathcal{R}L$ *such that* $\alpha d = \alpha^2$.

(5) *For all* $\alpha \in \mathcal{R}L$*, there exists* $\beta \in \mathcal{R}L$ *such that* $\alpha\beta = 0$ *and* $|\alpha| + |\beta|$ *is a nonzero divisor.*

(6) *Whenever* $P \subseteq \mathcal{R}L$ *is a prime ideal such that* $P \subseteq Zdv(\mathcal{R}L)$*, where* $Zdv(\mathcal{R}L)$ *stands for the set of zero divisors in* $\mathcal{R}L$*, then* P *is minimal prime.*

(7) *For all* $\alpha \in \mathcal{R}L$*, there exists* $\beta \in \mathcal{R}L$ *such that* $\operatorname{Ann}\alpha = \operatorname{Ann}^2\beta$.

Proof (1) $\Rightarrow$ (2) : Let $\alpha \in \mathcal{R}L$. By cozero complementedness, there is a $\beta \in \mathcal{R}L$ such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = 0$ and $\operatorname{coz}\alpha \vee \operatorname{coz}\beta$ is dense. Now $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = 0$ implies that $\operatorname{coz}\alpha \wedge (\operatorname{coz}\beta)^{**} = 0$, which implies that $(\operatorname{coz}\beta)^{**} \leq (\operatorname{coz}\alpha)^*$. On the other hand, $\operatorname{coz}\alpha \vee \operatorname{coz}\beta$ dense implies that $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = 0$, whence $(\operatorname{coz}\alpha)^* \leq (\operatorname{coz}\beta)^{**}$. Therefore $(\operatorname{coz}\alpha)^* = (\operatorname{coz}\beta)^{**}$, which implies, by Lemma 4.2.12, that $\operatorname{Ann}\alpha = \operatorname{Ann}^2\beta$.

(2) $\Rightarrow$ (3) : $\operatorname{Ann}^2(\alpha) = \operatorname{Ann}(\beta)$ implies $\mathbf{M}^{r_L((\operatorname{coz}\alpha)^{**})} = \mathbf{M}^{r_L((\operatorname{coz}\beta)^{*})}$, which implies $r_L((\operatorname{coz}\alpha)^{**}) = r_L((\operatorname{coz}\beta)^*)$, which implies $(\operatorname{coz}\alpha)^{**} = (\operatorname{coz}\beta)^*$.

(3) $\Rightarrow$ (1) : Let $a \in \operatorname{Coz}L$. Pick $\alpha \in \mathcal{R}L$ such that $\operatorname{coz}\alpha = a$. By (3), there exists $\beta \in \mathcal{R}L$ such that $(\operatorname{coz}\alpha)^{**} = (\operatorname{coz}\beta)^*$. Write $b = \operatorname{coz}\beta$. Now b is a cozero element such that

$$a \wedge b = \operatorname{coz}\alpha \wedge \operatorname{coz}\beta \leq (\operatorname{coz}\alpha)^{**} \wedge \operatorname{coz}\beta = (\operatorname{coz}\beta)^* \wedge \operatorname{coz}\beta = 0,$$

and

$$(a \vee b)^* = a^* \wedge b^* = (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\alpha)^{**} = 0;$$

which implies $a \vee b$ is dense. Therefore L is cozero complemented.

(1) $\Rightarrow$ (4) : Let $\alpha \in \mathcal{R}L$. Then, by hypothesis, there is a $\beta \in \mathcal{R}L$ such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = 0$ and $\operatorname{coz}\alpha \vee \operatorname{coz}\beta$ is dense in L. Now $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \operatorname{coz}(\alpha\beta) = 0$ implies that $\alpha\beta = 0$, and $\operatorname{coz}\alpha \vee \operatorname{coz}\beta = \operatorname{coz}(\alpha^2 + \beta^2)$ is dense in L implies that $\alpha^2 + \beta^2$ is not a zero divisor. We claim that $\alpha + \beta$ is also not a zero divisor. Let $h \in \mathcal{R}L$ such that $h(\alpha + \beta) = 0$. Then $h\alpha + h\beta = 0$

which implies that $h\alpha = -h\beta$. Now $h(\alpha^2+\beta^2) = h\alpha^2+h\beta^2 = h\alpha.\alpha+h\beta.\beta = -h\beta\alpha-h\alpha\beta = 0$ since $\alpha\beta = \beta\alpha = 0$. So, $h = 0$, since $h(\alpha^2 + \beta^2) = 0$ and $\alpha^2 + \beta^2$ is not a zero divisor. Put $d = \alpha + \beta$. Then $\alpha d = \alpha(\alpha + \beta) = \alpha^2 + \alpha\beta = \alpha^2$.

(4) $\Rightarrow$ (5) : Let $\alpha \in \mathcal{R}L$. Then, by hypothesis, there is a nonzero divisor d such that $\alpha d = \alpha^2$. Then $\alpha d-\alpha^2 = 0$, and so $\alpha(d-\alpha) = 0$. Put $\beta = d-\alpha$. Then $\alpha\beta = 0$ and $d = \alpha+\beta$. But $0 \leq |d| = |\alpha + \beta| \leq |\alpha| + |\beta|$ so $\operatorname{coz}(|d|) \leq \operatorname{coz}(|\alpha| + |\beta|)$. But $\operatorname{coz} d = \operatorname{coz}(|d|)$ is dense, so $\operatorname{coz}(|\alpha| + |\beta|)$ is also dense. Thus, $|\alpha| + |\beta|$ is not a zero divisor.

(5) $\Rightarrow$ (6) : Recall first that in a reduced commutative ring with identity, a prime ideal is minimal prime if and only if every element in the ideal is annihilated by an element not in the ideal. Also, observe that in any f-ring A, if $P \subseteq A$ is prime, then for any $a \in A$, we have $a \in P$ if and only if $|a| \in P$ since $a^2 = |a|^2$. Let $\alpha \in P$, with $P \subseteq \mathcal{R}L$ a prime ideal such that $P \subseteq Zdv\,\mathcal{R}L$. By hypothesis, there is a $\beta \in \mathcal{R}L$ such that $\alpha\beta = 0$ and $|\alpha| + |\beta|$ is a nonzero divisor. Hence $|\alpha| + |\beta| \notin P$. Now $\alpha \in P$ and $|\alpha| + |\beta| \notin P$ implies $|\beta| \notin P$, and hence $\beta \notin P$. Since $\alpha\beta = 0$, it follows that P is a minimal prime ideal.

(6) $\Rightarrow$ (1) : Assume (6) holds for $\mathcal{R}L$. By Lemma 5.3.1, $\operatorname{Min}(\mathcal{R}L)$ is compact. By [26], L is cozero complemented.

(3) $\Rightarrow$ (7) : Suppose (3) holds and let $\alpha \in \mathcal{R}L$. Write $a = \operatorname{coz}\alpha$. Pick $b \in \operatorname{Coz} L$ such that $a^* = b^{**}$. Pick $\beta \in \mathcal{R}L$ such that $\operatorname{coz}\beta = b$. Now $\operatorname{Ann}(\operatorname{Ann}\beta) = \operatorname{Ann}\mathbf{M}^{r(b^*)} = \mathbf{M}^{r(b^{**})}$ since $\bigvee r(b^*) = b^*$. Thus, $\operatorname{Ann}(\operatorname{Ann}\beta) = \mathbf{M}^{r(a^*)} = \operatorname{Ann}\alpha$.

(7) $\Rightarrow$ (3) : Conversely, suppose that for every $\alpha \in \mathcal{R}L$, there exists $\beta \in \mathcal{R}L$ such that

$\operatorname{Ann}\alpha = \operatorname{Ann}(\operatorname{Ann}\beta)$. Put $c = \operatorname{coz}\alpha$ and $d = \operatorname{coz}\beta$. So, by Lemma 4.2.12,

$$
\begin{aligned}
& \operatorname{Ann}\alpha = \operatorname{Ann}(\operatorname{Ann}\beta) \\
\Rightarrow\ & \mathbf{M}^{r((\operatorname{coz}\alpha)^*)} = \mathbf{M}^{r((\operatorname{coz}\beta)^{**})} \\
\Rightarrow\ & r((\operatorname{coz}\alpha)^*) = r((\operatorname{coz}\beta)^{**}) \\
\Rightarrow\ & (\operatorname{coz}\alpha)^* = (\operatorname{coz}\beta)^{**}.
\end{aligned}
$$

■

Bibliography

[1] R. A. Alo and H. L. Shapiro, *Normal topological spaces*, Cambridge University Press, (1974).

[2] C. E. Aull, *Embeddings extending various types of disjoint sets*, Rocky Mount. J. Math., **14** (1984), 319-330.

[3] C. E. Aull, *Some embeddings related to C^*-embedding*, J. Austral. Math. Soc. (Series A), **44** (1988), 88-104.

[4] F. Azarpanah, O. A. S. Karamzadeh and A. Rezai Aliabad, *On ideals consisting entirely of zero divisors*, Communications in Algebra, **28** (2) (2000), 1061-1073.

[5] D. Baboolal and B. Banaschewski, *Compactification and local connectedness of frames*, J. Pure Appl. Alg. **70** (1991), 3-16.

[6] R. N. Ball, A. W. Hager and J. Walters-Wayland, *An intrinsic characterization of monomorphisms in regular Lindelöf locales* Appl. Categor. Struct., **15** (2007), 109-118.

[7] R.N. Ball and J. Walters-Wayland, *C- and C^*-quotients in pointfree topology*, Dissertationes Mathematicae (Rozprawy Mat.), Vol. **412** (2002), 62pp.

[8] B. Banaschewski, *The real numbers in pointfree topology*, Textos de Matemática Série B, No. 12 Departamento de Matemática da Universidade de Coimbra, (1997).

[9] B. Banaschewski, *Uniform completion in pointfree topology*, In: Topological and Al-

gebraic Structures in Fuzzy sets, eds S.E. Rodabaugh and E.P. Klement, (2003), 19-56.

[10] B. Banaschewski, *On the function ring functor in pointfree topology*, Appl. Categor. Struct., **13** (2005), 305-328.

[11] B. Banaschewski, T. Dube, C. Gilmour and J. Walters-Wayland, *Oz in pointfree topology*, Quaest. Math., **32** (2009), 215-227.

[12] B. Banaschewski and C. Gilmour, *Pseudocompactness and the cozero part of a frame*, Comment. Math. Univ. Carolinae, **37** (1996), 577-587.

[13] B. Banaschewski and C. Gilmour, *Oz revisited*, Math. Arb. Pap. (2000), 19-23.

[14] B. Banaschewski and C. Gilmour, *Realcompactness and the cozero part of a frame*, Appl. Categor. Struct., **9** (2001), 395-417.

[15] B. Banaschewski and A. Pultr, *Booleanization*, Cah. Top. Geom. Diff. Categ., **37** (1996), 41-60.

[16] B. Banaschewski and A. Pultr, *Paracompactness revisited*, Appl. Categor. Struct., **1** (1993), 181-190.

[17] R. L. Blair, *Filter characterizations of z-, C^*- and C-embeddings*, Fund. Mathematicae XC, (1976), 285-300.

[18] R. L. Blair, *Spaces in which special sets are z-embedded*, Canad. J. Math., **28** (1976), 673-690.

[19] R. L. Blair and A. W. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Z., **136** (1974), 41-52.

[20] X. Chen, *Closed frame homomorphisms*, PhD thesis, McMaster University, (1991).

[21] A. K. Das, Δ-*normal spaces and decompositions of normality*, Applied General Topology, Vol 10. **2** (2009), 197-206.

[22] F. Dashiell, A. Hager and M. Henriksen, *Order-Cauchy completions of rings and vector lattices of continuous functions*, Can. J. Math., **32** (1980), 657-685.

[23] C. H. Dowker and D. Papert, *Quotient frames and subspaces*, Proc. London Math. Soc. **16** (3) (1966), 275-296.

[24] T. Dube, *Some notes on C- and C^*-quotients of frames*, Order **25** (2008), 369-375.

[25] T. Dube, *A little more on coz-unique frames*, Appl. Categor. Struct., **17** (2009), 63-73.

[26] T. Dube, *Some ring-theoretic properties of almost P-frames*, Alg. Univ., **60** (2009), 145-162.

[27] T. Dube, *Some algebraic characterizations of F-frames*, Alg. Univ., **62** (2009), 273-288.

[28] T. Dube, *Remote points and the like in pointfree topology*, Acta Math. Hungar., **123 (3)** (2009), 203-222.

[29] T. Dube, *Concerning P-frames, essential P-frames and strongly zero-dimensional frames*, Alg. Univ., **61** (2009), 115-138.

[30] T. Dube, *Notes on pointfree disconnectivity with a ring-theoretic slant*, Appl. Categor. Struct. **18** (2010), 55-72.

[31] T. Dube, *Contracting the socle in rings of continuous functions*, Rend. Sem. Mat. Univ. Padova **123** (2010), 37-53.

[32] T. Dube and P. Matutu, *A few points on pointfree pseudocompactness*, Quaest. Math., **30** (2007), 451-464.

[33] T. Dube and J. Walters-Wayland, *Coz-onto frame maps and some applications*, Appl. Categor. Struct., **15** (2007), 119-133.

[34] A. García-Máynez, *Generalizations of the concepts of closed functions and C-embedded set*, An. Inst. Univ. Nac. Autónoma. Měxico, **24** (1984), 13-35.

[35] M. Henriksen, J. Vermeer and R.G. Woods, *Quasi-F covers of Tychonoff spaces*, Trans. Amer. Math. Soc., **303** (2) (1987), 779-803.

[36] M. Henriksen, J. Vermeer and R.G. Woods, *Wallman covers of compact spaces*, Disser. Math. (rozprawy Matematyczne) 1989.

[37] M. Henriksen and J. Walters-Wayland, *A pointfree study of bases for spaces of minimal prime ideals*, Quaest. Math., **26** (2002), 333-346.

[38] M. Henriksen and R.G. Woods, *Cozero complemented spaces; when the space of minimal prime ideals of a C(X) is compact*, Topology and its Applications, **141** (2004), 147-170.

[39] J. R. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972), 5-32.

[40] T. Ishii and H. Ohta, *Generalizations of C-embedding and their applications*, Math. Japonica, **23** (4) (1978), 349-368.

[41] T. Isiwata, *Mappings and spaces*, Pacific J. Math. **20** (1967) 455-480; correction, ibid. 23 1967 630-631.

[42] P. T. Johnstone, *Stone spaces*, Cambridge Univ. Press Cambridge, (1982).

[43] L. N. Kalantan, *π-normal Topological Spaces*, Filomat **22** (1) (2008), 173-181.

[44] Y. K. Kim, *Frames with a unique uniformity*, Comm. Korean Math. Soc., **15** (2000), 371-378.

[45] J. K. Kohli and A. K. Das, *New normality axioms and decompositions of normality*, Glasnik Matematički, **37** (57) (2002), 163-173.

[46] R. Levy and J. Shapiro, *Rings of quotients of rings of functions*, Topology and its Applications, **146-147** (2005), 253-265.

[47] J. Mack, *Countable paracompact and weak normality properties*, Trans. Amer. Math. Soc., **148** (1970), 265-272.

[48] J. J. Madden, *κ-frames*, J. Pure Appl. Algebra, **70** (1991), 107-127.

[49] J. Madden and H. Vermeer, *Lindelöf locales and realcompactness*, Math. Proc. Camb. Philos. Soc., **99** (1986), 473-480.

[50] N. Marcus, *Realcompactification of frames*, Comment. Math. Univ. Carolin., **36** (1995), 349-358.

[51] J. Martínez, *Disjointifiable lattice-ordered groups*, Alg. Univ., **59** (2008), 159-178.

[52] J. Martínez and S. Woodward, *Bézout and Prüfer f-rings*, Comm. Algebra, **20** (1992), 2975-2989.

[53] B. de Pagter, *On z-ideals and d-ideals in Riesz spaces III*, Indag. Math., **43** (1981), 409-422.

[54] A. Pultr, *Frames, in*: Handbook of Algebra (ed M. Hazewinkel), Vol 3. Elsevier Science B.V. (2003), 791-857.

[55] H. Simmons, *The Assembly of the frame*, obtainable at www.cs.man.ac.uk/~hsimmons/.

[56] M.K. Singal and A.R. Singal, *Mildly normal spaces*, Kyungpook Math. J., **1** (13) (1973), 27-31.

[57] H. Wallman, *Lattices and topological spaces*, Ann. Math. **39** (1938), 112-126.

[58] J.L. Walters-Wayland, *Completeness and nearly fine uniform frames*, PhD Thesis, University of Catholique de Louvain, (1995).

[59] H.E. Wei and L.U.O. Maokang, *Completely regular paracompact reflection of locales*, Science in China: Series A Math. Vol. 49 No. 6 (2006), 820-826.

[60] R.G. Woods, *Maps that characterize normality properties and pseudocompactness*, J. London Math. Soc., **7** (2) (1974), 453-461.

[61] P. Zenor, *A note on Z-mappings and WZ-mappings*, Proc. Amer. Math. Soc., **23** (1969), 273-275.