Iterated Function Systems in Partial Metric Spaces with Applications

by

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Declaration

I Vuledzani Thomas Makhoshi, declare that this thesis titled **Iterated Function Systems in Partial Metric Spaces with Applications**, and the work presented in it are my own. The work described in this Doctor of Philosophy (PhD) thesis was carried out under the supervision of the Department of Mathematical Sciences, University of South Africa.

This PhD thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Approval

This thesis of Vuledzani Thomas Makhoshi is approved as fulfilling part of the requirements for the award of the degree of Doctor of Philosophy in Mathematics by the University of South Africa.

Contents

Dedication

In memory of my mother.
To my wife Aifheli and our children Muvuledzi, Vhulenda Tshililo, and Phumudzo.

Abstract

The mathematical study of fractals is deeply embedded in iterated function systems (IFS) formulated by Hutchinson in 1981. Since then, the development of iterated function systems in metric space setup, has caught the attention of many researchers.

In the current work, the scope of iterated function systems is extended to more generalized settings such as partial metric spaces, Hausdorff semi-metric spaces, and G-metric spaces. The existence and uniqueness of new attractors and common attractors of generalized iterated function systems in various spaces is proved with the assistance of generalized and generalized cyclic contractive mappings. Well-posedness of attractor based problems of the Hutchinson operators is established. Applications to dynamic programming and nonlinear integral equations are presented.

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Key Terms

- iterated function system
- generalized iterated function system
- fixed point
- common fixed point
- fractal
- attractor
- common attractor
- Hutchinson operator
- contraction mappings
- generalized rational contractions
- generalized contraction
- well-posedness
- collage theorem
- Hausdorff metric
- partial Hausdorff metric
- partial metric space
- complete partial metric space
- cyclic contraction
- functional equations
- semi-metric space
- G-metric space

List of Notation

Below is a list of some important symbols to be used and a brief explanation of their meaning.

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_{[+]}$	The set of non-negative real numbers.
$\mathbb{R}^q$	The set of q-tuples or real numbers.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}_q$	The set of first q natural numbers.

1

Introduction and Preliminaries

1.1. Background

Nowhere has the non-linearity of nature been so well captured than in Mandelbrot's work on fractal geometry [81]. The study of fixed point theory, which has numerous applications in a number of fields of non-linear analysis, is theoretically based on the idea of Banach contraction mapping. For many years now, several researchers have worked extensively to enhance and broaden the discipline of fixed point theory in metric spaces, (see [11, 18, 20, 22, 27, 30, 33, 36, 37, 40, 43, 46, 49, 54, 58, 83, 98, 104]).

Nadler [73], in particular, is credited with pioneering the field of fixed point theory in metric spaces endowed with multi-valued operators, resulting in the Banach fixed point principle being extended to set-valued contraction mappings. In recent years, there has been a substantial surge in interest in the study of metrical fixed point theory, which has resulted in a wide range of applications both within and beyond mathematics (see [5, 26, 24, 40, 62, 57, 69, 94, 95]). Applications to variational inequalities, integral equations, differential equations, optimization, and split feasibility theory are particularly noteworthy [56].

Fixed point theory has numerous helpful and crucial applications in tackling real-world problems. The fixed point theory of Schaefer and Krasnoselski has been shown to be particularly useful in the investigation of the existence of solutions in chemical graph theory [100]. Many scholars have shown that problems in economic theories, neutron transport theory, chemical reactions, epidemiology, steady-state temperature distribution, mathematical psychology, engineering, and applied sciences can be formulated as functional equations whose solutions can be obtained by using fixed point techniques, (see for example in [52, 99]).

Hutchinson's [42] groundbreaking 1981 work, established the concept for constructing fractals called iterated function system (IFS) on a solid mathematical foundation. He accomplished this by demonstrating that the Hutchinson operator, which was developed using a finite collection of contraction mappings on a set of q-tuples of real numbers, denoted by $\mathbb{R}^q$, has as its fixed point a non-void, closed, and bounded subset of $\mathbb{R}^q$ known as an attractor of the iterated function system [25, 53, 76, 77, 78].

Secelean [90, 91] investigated generalized countable iterated function systems on a metric space, however, in our current work, we present some new results on generalized iterated function systems in more generalized settings such as partial metric spaces, semi-metric spaces, and G-metric spaces. The existence and uniqueness of attractors for single valued mappings and like-wise common attractors for multi-valued mappings involving a pair of self-mappings are established with the assistance of finite families of contractive and generalized contractive mappings respectively, defined on partial metric spaces. We confirmed the well-posedness of attractor based problems.

We extended the Banach contraction principle to non-continuous mappings with the aid of cyclic contractive mappings, and obtain valuable results on the existence and uniqueness of attractors. We obtain some results in semi-metric spaces whose definition omits the triangle inequality, followed by non-commutative mappings in G-metric spaces. Applications of iterated function systems in dynamic programming and integral equations are provided.

1.2. Organization of the Thesis

The introduction to this chapter (**Chapter 1**) provides a brief overview of iterated function systems in metric spaces. The remaining chapters in this thesis are organized in the manner described below.

Chapter 2. Iterated Function System of Generalized Contractions in Partial Metric Spaces

In order to create an attractor, this chapter uses a finite family of generalized contraction mappings, each of which belongs to a particular class of mappings defined on a complete partial metric space. As a result, distinct outcomes for iterated function systems satisfying various generalized contractive conditions are established. In order to prove the findings made here, we provide some example. Our present study extends, generalizes, and brings together a number of findings from recent research.

Chapter 3. Generalized iterated function system for common attractors in partial metric spaces

With the help of a finite family of generalized contractive mappings that are members of a distinct class of mappings defined on a complete partial metric space, the purpose of this chapter is to construct new common attractors. As a result, different iterated function system outcomes satisfying various generalized contractive conditions are obtained. To support the findings demonstrated here, we provide an example. These expand upon, generalize, and combine numerous established results found in the literature.

Chapter 4. Iterated Function System of Generalized Cyclic Contractions in Partial Metric Spaces

We generate a fractal using a finite collection of generalized cyclic contraction mappings, belonging to a particular category of mappings defined on a complete partial metric space. As a consequence, different results are obtained for iterated function system that satisfy a different set of generalized cyclic contraction conditions. The chapter will culminate with a brief discussion of applications of cyclic iterated function system to dynamic programing problems and integral equations. With these results, we extend, unify and generalize some common results in recent literature.

Chapter 5. Iterated Function System of Generalized Rational Contractions in Semi-Metric Spaces

Using finite families of generalized contractive mappings from the distinct class of mappings defined on a Hausdorff semi-metric space, we are able to create several new common attractors in this chapter. As a result, different iterated function system outcomes satisfying varied generalized contractive criteria are established. To reinforce the results proved herein, an example is presented. These findings extend, generalize, and consolidate a number of findings from recent literature.

Chapter 6. Common Attractors of Generalized Iterated Function System in G-Metric Spaces

With the aid of a finite family of generalized contractive mappings that are part of a certain class of mappings defined on a G-metric space, we create a new common fractal. As a result, various results are acquired and confirmed by an example for G-iterated function systems that meet a different set of generalized contractive criteria. These findings generalize, integrate, and expand a variety of conclusions seen in recent literature.

Chapter 7. Generalized Iterated Function System of Cyclic Contractions in G-Metric Spaces

This chapter's major goal is to construct fractals using a finite family of generalized cyclic contractions that are members of a particular class of mappings defined on a G-metric space. As a result, different iterated function system outcomes satisfying various generalized cyclic contractive requirements are obtained. An example is provided to support the results demonstrated here. Our findings extend, generalize, and bring together a number of findings from recent literature.

Chapter 8. Conclusion In this chapter, a summary of our work is presented.

2

Iterated Function System of Generalized Contractions in Partial Metric Spaces

2.1. Partial Metric Spaces

This section provides some preliminary definitions and results to serve as a foundation for eventual construction of fractal sets of generalized iterated function systems on complete partial metric spaces.
It is worth noting that the Hutchinson operator, which is defined on a finite family of contractive mappings on a complete partial metric space, is a generalized contractive mapping on a family of compact subsets of a set, say W. A final fractal is generated by successively applying a generalized Hutchinson operator, and following that, a non-trivial example is presented to support the proven result.

We recall the definition of a standard metric space:

Definition 2.1.1. [12, 25] Let W be a (non-void) set, a function $d : W \times W \to \mathbb{R}$ is said to be a *metric* (a *distance* or *dissimilarity* function [31]) on W if for all $\varrho, \varsigma, \varphi \in W$, d satisfies the following properties:

(i) $0 \leq d(\varrho, \varsigma)$ and $0 = d(\varrho, \varsigma)$ if and only if $\varrho = \varsigma$,

(ii) $d(\varrho, \varsigma) = d(\varsigma, \varrho)$,

(iii) $d(\varrho, \varsigma) \leq d(\varrho, \varphi) + d(\varphi, \varsigma)$.

The pair (W, d) consisting of the (non-void) set W and the metric d is called a *metric space*.

We now consider one of the many generalizations of the standard metric spaces, introduced by Matthews [61] in his work on denotational semantics of dataflow networks [102].

Definition 2.1.2. [21, 72] A non-void set W together with a mapping $p_m : W \times W \to \mathbb{R}_{[+]}$ is called a partial metric space denoted by (W, p_m) if for all $\varrho, \varsigma, \varphi \in W$ the following properties hold:

(p_{m_1}) $\varrho = \varsigma$ if and only if $p_m(\varrho, \varrho) = p_m(\varrho, \varsigma) = p_m(\varsigma, \varsigma)$,

(p_{m_2}) $p_m(\varrho, \varrho) \leq p_m(\varrho, \varsigma)$,

(p_{m_3}) $p_m(\varrho, \varsigma) = p_m(\varsigma, \varrho)$,

(p_{m_4}) $p_m(\varrho, \varsigma) \leq p_m(\varrho, \varphi) + p_m(\varphi, \varsigma) - p_m(\varphi, \varphi)$.

Looking at Definition 2.1.2 we observe that, the distance between a point and itself is not necessarly equal to zero as is the case in Definition 2.1.1, and the triangle inequality is expanded by subtracting self distance for the third point under the partial metric. Thus a partial metric space if a generalization of the standard metric space.
Furthermore, from Definition 2.1.2 we note that if $p_m(\varrho, \varsigma) = 0$, then properties (p_{m_1}) and (p_{m_2}) imply that $\varrho = \varsigma$ but the implication is not reversible in general. A partial metric space $(\mathbb{R}_{[+]}, p_m)$, endowed with a partial metric $p_m(\varrho, \varsigma) = \max\{\varrho, \varsigma\}$ is a common elementary example [21].

Example 2.1.1. [21, 61] If $W = \{[\varrho_1, \varrho_2] : \varrho_1, \varrho_2 \in \mathbb{R}, \varrho_1 \leq \varrho_2\}$, then

$$p_m([\varrho_1, \varrho_2], [\varrho_3, \varrho_4]) = \max\{\varrho_2, \varrho_4\} - \min\{\varrho_1, \varrho_3\}$$

defines a partial metric on W.

Following [7, 21, 61], a T_0 topology τ_{p_m} whose base is a class of open p_m-balls $\{B_{p_m}(\varrho, \varepsilon) : \varrho \in W, \varepsilon > 0\}$, such that $B_{p_m}(\varrho, \varepsilon) = \{\varsigma \in W : p_m(\varrho, \varsigma) < p_m(\varrho, \varrho) + \varepsilon\}$, for all $\varrho \in W$ and $\varepsilon > 0$, is generated by each partial metric p_m on W.

In a partial metric space (W, p_m), define $p_m^s : W \times W \to \mathbb{R}_{[+]}$ by $p_m^s(\varrho, \varsigma) = 2p_m(\varrho, \varsigma) - [p_m(\varrho, \varrho) + p_m(\varsigma, \varsigma)]$, for all $\varrho, \varsigma \in W$, then (W, p_m^s) is a metric space [21, 61].

Moreover, the sequence $\{\varrho_a\}$ converges to $\varrho \in W$ if and only if

$$\lim_{a,\eta \to +\infty} p_m(\varrho_a, \varrho_\eta) = \lim_{a \to +\infty} p_m(\varrho_a, \varrho) = p_m(\varrho, \varrho).$$

Definition 2.1.3. [49, 61] In a partial metric space (W, p_m),

(i) $\{\varrho_a\}$ is said to be a Cauchy sequence, provided $\lim_{a,\eta \to +\infty} p_m(\varrho_a, \varrho_\eta)$ exists,

(ii) (W, p_m) is said to be complete, if every Cauchy sequence $\{\varrho_a\}$ in W converges to a point $\varrho \in W$ relative to the topology τ_{p_m} such that $p_m(\varrho, \varrho) = \lim_{a \to +\infty} p(\varrho_a, \varrho)$, and

(iii) a function $h : W \to W$ is continuous at a point $u_0 \in W$ if, for each $\varepsilon > 0$, there exists $\varsigma > 0$ such that $h(B_{p_m}(u_0, \varsigma)) \subseteq B_{p_m}(hu_0, \varepsilon)$.

Lemma 2.1.1. [21] *If (W, p_m) is a partial metric space, then*

(i) *$\{\varrho_a\}$ is a Cauchy sequence in (W, p_m) if and only if it is a Cauchy sequence in (W, p_m^s).*

(ii) *(W, p_m) is a complete partial metric space if and only if (W, p_m^s) is a complete metric space.*

We denote by $\mathcal{CB}^{p_m}(W)$, a family of all non-void closed and bounded subsets of a partial metric space (W, p_m).

Let $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{CB}^{p_m}(W)$ and $\omega \in W$, define

$$p_m(\omega, \mathcal{J}^*) = \inf\{p_m(\omega, \mu) : \mu \in \mathcal{J}^*\}, \quad \delta_{p_m}(\mathcal{J}^*, \mathcal{O}^*) = \sup\{p_m(\mu, \mathcal{O}^*) : \mu \in \mathcal{J}^*\}$$

and

$$\delta_{p_m}(\mathcal{O}^*, \mathcal{J}^*) = \sup\{p_m(\eta, \mathcal{J}^*) : \eta \in \mathcal{O}^*\}.$$

Remark 2.1.1. [21] Let (W, p_m) be a partial metric space and $\mathcal{J}^*$ be any non-void subset of W, then

$$p_m(\mu, \mu) = p_m(\mu, \mathcal{J}^*) \text{ if and only if } \mu \in \overline{\mathcal{J}^*}.$$

Furthermore $\overline{\mathcal{J}^*} = \mathcal{J}^*$ if and only if $\mathcal{J}^*$ is closed in (W, p_m).

Proposition 2.1.1. [21] *Let (W, p_m) be a partial metric space. Then for any $\mathcal{L}^*, \mathcal{J}^*, \mathcal{O}^* \in \mathcal{CB}^{p_m}(W)$, the following statements hold:*

(a) $\delta_{p_m}(\mathcal{L}^*, \mathcal{L}^*) = \sup\{p_m(\ell, \ell) : \ell \in \mathcal{L}^*\}$.

(b) $\delta_{p_m}(\mathcal{L}^*, \mathcal{L}^*) \leq \delta_{p_m}(\mathcal{L}^*, \mathcal{J}^*)$.

(c) $\delta_{p_m}(\mathcal{L}^*, \mathcal{J}^*) = 0$ implies that $\mathcal{L}^* \subseteq \mathcal{J}^*$.

(d) $\delta_{p_m}(\mathcal{L}^*, \mathcal{J}^*) \leq \delta_{p_m}(\mathcal{L}^*, \mathcal{O}^*) + \delta_{p_m}(\mathcal{O}^*, \mathcal{J}^*) - \inf_{\eta \in \mathcal{O}^*} p_m(\eta, \eta)$.

Let (W, p_m) be a partial metric space. Define the mapping $H_{p_m} : \mathcal{CB}^{p_m}(W) \times$

$\mathcal{CB}^{p_m}(W) \to \mathbb{R}_{[+]}$, by

$$H_{p_m}(\mathcal{J}^*, \mathcal{O}^*) = \max\{\delta_{p_m}(\mathcal{J}^*, \mathcal{O}^*), \delta_{p_m}(\mathcal{O}^*, \mathcal{J}^*)\}, \text{ for all } \mathcal{J}^*, \mathcal{O}^* \in \mathcal{CB}^{p_m}(W).$$

Then H_{p_m} is referred to as a partial Hausdorff metric induced by p_m.

Proposition 2.1.2. [21] *Let (W, p_m) be a partial metric space and $\mathcal{L}^*, \mathcal{J}^*, \mathcal{O}^* \in \mathcal{CB}^{p_m}(W)$, then*

(a) $H_{p_m}(\mathcal{L}^*, \mathcal{L}^*) \leq H_{p_m}(\mathcal{L}^*, \mathcal{J}^*)$,

(b) $H_{p_m}(\mathcal{L}^*, \mathcal{J}^*) = H_{p_m}(\mathcal{J}^*, \mathcal{L}^*)$,

(c) $H_{p_m}(\mathcal{L}^*, \mathcal{J}^*) \leq H_{p_m}(\mathcal{L}^*, \mathcal{O}^*) + H_{p_m}(\mathcal{O}^*, \mathcal{J}^*) - \inf_{\eta \in \mathcal{O}^*} p_m(\eta, \eta)$.

Corollary 2.1.1. [21] *If (W, p_m) is a partial metric space, then*

$$H_{p_m}(\mathcal{J}^*, \mathcal{O}^*) = 0 \text{ implies that } \mathcal{J}^* = \mathcal{O}^*$$

for all $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{CB}^{p_m}(W)$.

Next it can be noted, as demonstrated by the example below, that in general, the converse of Corollary 2.1.1 is not true.

Example 2.1.2. [21] Let $W = [0, 1]$ be equipped with the partial metric $p_m : W \times W \to \mathbb{R}_{[+]}$ such that

$$p_m(\varrho, \varsigma) = \max\{\varrho, \varsigma\}.$$

From (a) of Proposition 2.1.1, we get

$$H_{p_m}(W, W) = \delta_{p_m}(W, W) = \sup\{\varrho : 0 \leq \varrho \leq 1\} = 1 \neq 0.$$

Definition 2.1.4. In a partial metric space (W, p_m) let $\mathcal{C}^{p_m} \subseteq W$. Then $\mathcal{C}^{p_m}$ is compact if every sequence $\{v_a\}$ of elements in $\mathcal{C}^{p_m}$ has a subsequence $\{v_{a_i}\}$ which converges to a point in $\mathcal{C}^{p_m}$.

It is crucial to note that closed and bounded subsets of an Euclidean space $\mathbb{R}^q$ are compact. Similarly, every finite subset of $\mathbb{R}^q$ is compact whereas the $(0, 1] \subset \mathbb{R}$ is not compact since $\{1, \frac{1}{2}, \frac{1}{2^2}, ...\} \subset (0, 1]$ does not have any convergent subsequence. Similarly, $\mathbb{Z} \subset \mathbb{R}$ the set of integers, is not compact.

Let (W, p_m) be a partial metric space and let $\mathcal{C}^{p_m}(W)$ denote the collection of all non-void compact subsets of W. If $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$, then

$$H_{p_m}(\mathcal{J}^*, \mathcal{O}^*) = \max\{\sup_{\eta \in \mathcal{O}^*} p_m(\eta, \mathcal{J}^*), \sup_{\mu \in \mathcal{J}^*} p_m(\mu, \mathcal{O}^*)\},$$

where $p_m(\varrho, \mathcal{J}^*) = \inf\{p_m(\varrho, \mu) : \mu \in \mathcal{J}^*\}$ shows how far a point ϱ is from the set $\mathcal{J}^*$. In this case, the mapping H_{p_m} is said to be the Pompeiu-Hausdorff metric induced by the partial metric p_m. If (W, p_m) is a complete partial metric space, then $(\mathcal{C}^p(W), H_{p_m})$ is also a complete partial metric space [76].

Lemma 2.1.2. *In a partial metric space (W, p_m), let $\mathcal{K}^*, \mathcal{L}^*, \mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$, then the following hold:*

(a) *If $\mathcal{L}^* \subseteq \mathcal{J}^*$, then $\sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{J}^*) \leq \sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{L}^*)$.*

(b) $\sup_{\varrho \in \mathcal{K}^* \cup \mathcal{L}^*} p_m(\varrho, \mathcal{J}^*) = \max\{\sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{J}^*), \sup_{\ell \in \mathcal{L}^*} p_m(\ell, \mathcal{J}^*)\}$.

(c) $H_{p_m}(\mathcal{K}^* \cup \mathcal{L}^*, \mathcal{J}^* \cup \mathcal{O}^*) \leq \max\{H_{p_m}(\mathcal{K}^*, \mathcal{J}^*), H_{p_m}(\mathcal{L}^*, \mathcal{O}^*)\}$.

Proof. (a) Since $\mathcal{L}^* \subseteq \mathcal{J}^*$, for all $k \in \mathcal{K}^*$, we have

$$\begin{aligned} p_m(k, \mathcal{J}^*) &= \inf\{p_m(k, \mu) : \mu \in \mathcal{J}^*\} \\ &\leq \inf\{p_m(k, \ell) : \ell \in \mathcal{L}^*\} = p_m(k, \mathcal{L}^*), \end{aligned}$$

which shows that

$$\sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{J}^*) \leq \sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{L}^*).$$

(b)

$$\begin{aligned} \sup_{\varrho \in \mathcal{K}^* \cup \mathcal{L}^*} p_m(\varrho, \mathcal{J}^*) &= \sup\{p_m(\varrho, \mathcal{J}^*) : \varrho \in \mathcal{K}^* \cup \mathcal{L}^*\} \\ &= \max\{\sup\{p_m(\varrho, \mathcal{J}^*) : \varrho \in \mathcal{K}^*\}, \sup\{p_m(\varrho, \mathcal{J}^*) : \varrho \in \mathcal{L}^*\}\} \\ &= \max\{\sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{J}^*), \sup_{\ell \in L^*} p_m(\ell, \mathcal{J}^*)\}. \end{aligned}$$

(c) We observe that

$$\begin{aligned} & \sup_{\varrho \in \mathcal{K}^* \cup \mathcal{L}^*} p_m(\varrho, \mathcal{J}^* \cup \mathcal{O}^*) \\ \leq\ & \max\{\sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{J}^* \cup \mathcal{O}^*), \sup_{\ell \in \mathcal{L}^*} p_m(\ell, \mathcal{L}^* \cup \mathcal{O}^*)\} \text{ (from (b))} \\ \leq\ & \max\{\sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{J}^*), \sup_{\ell \in \mathcal{L}^*} p_m(\ell, \mathcal{O}^*)\} \qquad \text{(from (a))} \\ \leq\ & \max\left\{\max\{\sup_{k \in \mathcal{K}^*} p_m(k, \mathcal{J}^*), \sup_{\mu \in \mathcal{J}^*} p_m(\mu, \mathcal{K}^*)\}, \max\{\sup_{\ell \in \mathcal{L}^*} p_m(\ell, \mathcal{O}^*), \sup_{\eta \in \mathcal{O}^*} p_m(\eta, \mathcal{L}^*)\}\right\} \\ =\ & \max\{H_{p_m}(\mathcal{K}^*, \mathcal{J}^*), H_{p_m}(\mathcal{L}^*, \mathcal{O}^*)\}. \end{aligned}$$

Likewise,

$$\sup_{v\in\mathcal{O}^*\cup\mathcal{J}^*} p_m(v,\mathcal{L}^*\cup\mathcal{K}^*) \leq \max\left\{H_{p_m}\left(\mathcal{K}^*,\mathcal{J}^*\right), H_{p_m}\left(\mathcal{L}^*,\mathcal{O}^*\right)\right\}.$$

Thus, it is evident that

$$\begin{aligned} H_{p_m}(\mathcal{K}^*\cup\mathcal{L}^*,\mathcal{O}^*\cup\mathcal{J}^*) &= \max\left\{\sup_{v\in\mathcal{L}^*\cup\mathcal{O}^*} p_m(v,\mathcal{K}\cup\mathcal{L}^*), \sup_{\varrho\in\mathcal{K}^*\cup\mathcal{L}^*} p_m(\varrho,\mathcal{J}^*\cup\mathcal{O}^*)\right\} \\ &\leq \max\left\{H_{p_m}\left(\mathcal{K}^*,\mathcal{J}^*\right), H_{p_m}\left(\mathcal{L}^*,\mathcal{O}^*\right)\right\}. \end{aligned}$$

□

Theorem 2.1.1. [61] *Let (W,p_m) be a complete partial metric space and $h : W\to W$ be a contraction such that, for any contractive coefficient $\lambda\in[0,1)$,*

$$p_m\left(h\varrho,h\varsigma\right)\leq\lambda p_m(\varrho,\varsigma)$$

is true for all $\varrho,\varsigma\in W$. Then there exists a unique fixed point $\tilde{u}$ of h in W and for every v_0 in W the sequence $\{v_0,hv_0,h^2v_0,...\}$ converges to $\tilde{u}$.

Theorem 2.1.2. [76] *Let (W,p_m) be a partial metric space and $h : W\to W$ a contraction mapping, then the following hold:*

(a) *Elements in $\mathcal{C}^{p_m}(W)$ are mapped to elements in $\mathcal{C}^{p_m}(W)$ by h.*

(b) *If*

$$h(\mathcal{J}^*)=\{h(\varrho_1):\varrho_1\in\mathcal{J}^*\},\ \textit{for any}\ \mathcal{J}^*\in\mathcal{C}^{p_m}(W),$$

then $h:\mathcal{C}^{p_m}(W)\to\mathcal{C}^{p_m}(W)$ is a contraction on $(\mathcal{C}^{p_m}(W),H_{p_m})$.

Proof. (a) It is known that every contraction mapping is continuous. Furthermore, for every continuous mapping $h : W\to W$, a compact subset's image is also compact, which implies that, if

$$\mathcal{J}^*\in\mathcal{C}^{p_m}(W)\text{ then }h(\mathcal{J}^*)\in\mathcal{C}^{p_m}(W).$$

(b) Let $\mathcal{J}^*,\mathcal{O}^*\in\mathcal{C}^{p_m}(W)$. Because $h : W\to W$ is a contraction, we get that

$$p_m\left(h\varrho_1,h\left(\mathcal{O}^*\right)\right)=\inf_{\varrho_2\in\mathcal{O}^*}p_m\left(h\varrho_1,h\varrho_2\right)\leq\lambda\inf_{\varrho_2\in\mathcal{O}^*}p_m\left(\varrho_1,\varrho_2\right)=\lambda p_m\left(\varrho_1,\mathcal{O}^*\right).$$

Also

$$p_m\left(h\varrho_2,h\left(\mathcal{J}^*\right)\right)=\inf_{\varrho_1\in\mathcal{J}^*}p_m\left(h\varrho_2,h\varrho_1\right)<\lambda\inf_{\varrho_1\in\mathcal{J}^*}p_m\left(\varrho_2,\varrho_1\right)=\lambda p_m\left(\varrho_2,\mathcal{J}^*\right).$$

Now

$$\begin{aligned} H_{p_m}\left(h\left(\mathcal{J}^*\right), h\left(\mathcal{O}^*\right)\right) &= \max\{\sup_{\varrho_1\in\mathcal{J}^*} p_m(h\varrho_1, h\left(\mathcal{O}^*\right)), \sup_{\varrho_2\in\mathcal{O}^*} p_m(h\varrho_2, h\left(\mathcal{J}^*\right))\} \\ &\leq \max\{\lambda \sup_{\varrho_1\in\mathcal{J}^*} p_m(\varrho_1, \mathcal{O}^*), \lambda \sup_{\varrho_2\in\mathcal{O}^*} p_m(\varrho_2, \mathcal{J}^*)\} \\ &= \lambda H_{p_m}\left(\mathcal{J}^*, \mathcal{O}^*\right). \end{aligned}$$

As a result, h satisfies.

$$H_{p_m}\left(h\left(\mathcal{J}^*\right), h\left(\mathcal{O}^*\right)\right) \leq \lambda H_{p_m}\left(\mathcal{J}^*, \mathcal{O}^*\right), \text{ for all } \varrho_1, \varrho_2 \in \mathcal{C}^{p_m}(W),$$

and so $h : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$.

□

Theorem 2.1.3. [76] *In a partial metric space (W, p_m), assume $\{h_a : a = 1, 2, ..., q\}$ is a finite collection of contraction mappings on W with contraction constants $\lambda_1, \lambda_2, ..., \lambda_q$, respectively. Let $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ be defined by*

$$\begin{aligned} \Psi(\mathcal{J}^*) &= h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*) \cup \cdots \cup h_q(\mathcal{J}^*) \\ &= \cup_{a=1}^{q} h_a(\mathcal{J}^*), \end{aligned}$$

for each $\mathcal{J}^* \in \mathcal{C}^{p_m}(W)$. Then Ψ is a contraction mapping on $\mathcal{C}^{p_m}(W)$ with contraction constant, $\lambda = \max\{\lambda_1, \lambda_2, ..., \lambda_q\}$.

Proof. We shall demonstrate the claim for $q = 2$. Choose two contractions, $h_1, h_2 : W \to W$ and $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$. From Lemma 2.1.2 (c), we get that

$$\begin{aligned} H_{p_m}(\Psi(\mathcal{J}^*), \Psi(\mathcal{O}^*)) &= H_{p_m}(h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*), h_1(\mathcal{O}^*) \cup h_2(\mathcal{O}^*)) \\ &\leq \max\{H_{p_m}(h_1(\mathcal{J}^*), h_1(\mathcal{O}^*)), H_{p_m}(h_2(\mathcal{J}^*), h_2(\mathcal{O}^*))\} \\ &\leq \max\{\lambda_1 H_{p_m}(\mathcal{J}^*, \mathcal{O}^*)), \lambda_2 H_{p_m}(\mathcal{J}^*, \mathcal{O}^*))\} \\ &\leq \lambda H_{p_m}(\mathcal{J}^*, \mathcal{O}^*), \end{aligned}$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$. □

Theorem 2.1.4. [76] *In a complete partial metric space (W, p_m), let $\{h_a : a = 1, 2, ..., q\}$ be a finite family of contraction mappings on W and*

$$\begin{aligned} \Psi(\mathcal{J}^*) &= h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*) \cup \cdots \cup h_q(\mathcal{J}^*) \\ &= \cup_{a=1}^{q} h_a(\mathcal{J}^*), \end{aligned}$$

for each $\mathcal{J}^* \in \mathcal{C}^{p_m}(W)$*. Then*

(i) $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$,

(ii) Ψ *has a unique fixed point* $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$*, in other words,* $\tilde{U_1} = \Psi(\tilde{U_1}) = \cup_{a=1}^{q} h_a(\tilde{U_1})$,

(iii) *for any choice of an initial set* $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$*, the sequence*

$$\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Psi^2(\mathcal{J}_0^*), ...\}$$

of compact sets converges to $\tilde{U_1}$*.*

Proof. (i) Because each h_a is a contraction, the conclusion follows immediately from the definitions of Ψ and Theorem 2.1.2. (ii) $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ is a contraction as well, by Theorem 2.1.3. Thus if (W, p_m) is a complete partial metric space, so is $(\mathcal{C}^{p_m}(W), H_{p_m})$. As a result, (ii) and (iii) can be deduced from Theorem 2.1.2. □

Definition 2.1.5. A mapping $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ is called a generalised Hutchinson contraction operator in a complete partial metric space (W, p_m) if a constant $\lambda \in [0, 1)$ exists such that for any $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$,

$$H_{p_m}(\Psi(\mathcal{J}^*), \Psi(\mathcal{O}^*)) \leq \lambda \mathcal{S}_\Psi(\mathcal{J}^*, \mathcal{O}^*),$$

where

$$\begin{aligned} \mathcal{S}_\Psi(\mathcal{J}^*, \mathcal{O}^*) \quad = \quad & \max\{H_{p_m}(\mathcal{J}^*, \mathcal{O}^*), H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*)), H_{p_m}(\mathcal{O}^*, \Psi(\mathcal{O}^*)), \\ & \frac{H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{O}^*)) + H_{p_m}(\mathcal{O}^*, \Psi(\mathcal{J}^*))}{2}, H_{p_m}(\Psi^2(\mathcal{J}^*), \Psi(\mathcal{J}^*)), \\ & H_{p_m}(\Psi^2(\mathcal{J}^*), \mathcal{O}^*), H_{p_m}(\Psi^2(\mathcal{J}^*), \Psi(\mathcal{O}^*))\}. \end{aligned}$$

It is important to note that if Ψ (defined in Theorem 2.1.3) is a contraction, then it is a generalised Hutchinson contraction operator but the converse does not hold.

Example 2.1.3. Let $W = [0, 1]$ and $p_m : W \times W \to \mathbb{R}[+]$ be a partial metric space defined as $p_m(w_1, w_2) = \frac{1}{4}|w_1 - w_2| + \frac{1}{2}\max\{w_1, w_2\}$ for all $w_1, w_2 \in W$. Consider $h_1, h_2 : W \to W$ defined as $h_1(w) = \dfrac{w}{3}$ if $w \in [0, 1)$ and $h_1(1) = \dfrac{1}{6}$, $h_2(w) = \dfrac{w}{2}$ if $w \in [0, 1)$ and $h_2(1) = \dfrac{1}{4}$. Let $\mathcal{C}^{p_m}(W^*)$ be the collection of all singleton subsets of W and $\Psi : \mathcal{C}^{p_m}(W^*) \to \mathcal{C}^{p_m}(W^*)$ be a mapping defined as $\Psi(J) = h_1(J) \cup h_2(J)$ for all $J \in \mathcal{C}^{p_m}(W^*)$. Ψ is not a contraction as it is discontinuous at $w = 1$. But it satisfies condition of Definition 2.1.5 for $\lambda = \frac{5}{6}$.

Definition 2.1.6. Let (W, p_m) be a complete partial metric space, then a mapping $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ is called a generalized rational Hutchinson contraction operator if $\lambda_* \in [0, 1)$ exists such that for any $\mathcal{J}^*$, $\mathcal{O}^* \in \mathcal{C}^{p_m}(W)$, the following holds:

$$H_{p_m}\left(\Psi\left(\mathcal{J}^*\right), \Psi\left(\mathcal{O}^*\right)\right) \leq \lambda_* \mathcal{R}_\Psi(\mathcal{J}^*, \mathcal{O}^*),$$

where

$$\begin{aligned} \mathcal{R}_\Psi(\mathcal{J}^*, \mathcal{O}^*) &= \max\left\{ \frac{H_{p_m}(\mathcal{J}^*, \Psi\left(\mathcal{O}^*\right))[1 + H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{2(1 + H_{p_m}\left(\mathcal{J}^*, \mathcal{O}^*\right))}, \right. \\ &\quad \frac{H_{p_m}(\mathcal{O}^*, \Psi\left(\mathcal{O}^*\right))[1 + H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{1 + H_{p_m}\left(\mathcal{J}^*, \mathcal{O}^*\right)}, \\ &\quad \left. \frac{H_{p_m}(\mathcal{O}^*, \Psi\left(\mathcal{J}^*\right))[1 + H_p(\mathcal{J}^*, \Psi\left(\mathcal{J}^*\right))]}{1 + H_{p_m}(\mathcal{J}^*, \mathcal{O}^*)} \right\}. \end{aligned}$$

Definition 2.1.7. Suppose (W, p_m) is a complete partial metric space and let $h_a : W \to W$, $a = 1, 2, ..., q$ be a finite collection of contraction mappings, then $\{W; h_a, a = 1, 2, \cdots, q\}$ is called an iterated function system (IFS).

Definition 2.1.8. [76] If $\mathcal{J}^* \subseteq W$ is a non-void compact set, then $\mathcal{J}^*$ is an attractor of the iterated function system, provided

(i) $\Psi(\mathcal{J}^*) = \mathcal{J}^*$ and

(ii) an open set $V_1 \subseteq W$ exists, such that $\mathcal{J}^* \subseteq V_1$ and $\lim_{a \to +\infty} \Psi^a(\mathcal{O}^*) = \mathcal{J}^*$, for any compact set $\mathcal{O}^* \subseteq V_1$, where the limit is taken relative to the partial Hausdorff metric.

As a result, the maximal open set V_1 satisfying (ii) is known as a basin of attraction.

2.2. Generalised Hutchinson and Generalised Rational Hutchinson Contraction Operators

We now present and prove some theorems regarding the existence and uniqueness of a fixed point of the generalised Hutchinson contraction operator Ψ.

Theorem 2.2.1. *In a complete partial metric space (W, p_m), let $\{W; h_a, a = 1, 2, \cdots, q\}$ be an iterated function system and define a mapping $\Psi : \mathcal{C}^{p_m}(W) \to$*

$\mathcal{C}^{p_m}(W)$ *by*

$$\begin{aligned}\Psi(\mathcal{J}^*) &= h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*) \cup \cdots \cup h_a(\mathcal{J}^*) \\ &= \cup_{a=1}^{q} h_a(\mathcal{J}^*),\end{aligned}$$

for each $\mathcal{J}^* \in \mathcal{C}^{p_m}(W)$. *If* Ψ *is a generalized Hutchinson contraction operator, then it has a unique attractor* $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$, *that is*

$$\tilde{U_1} = \Psi(\tilde{U_1}) = \cup_{a=1}^{q} h_a(\tilde{U_1}).$$

Furthermore, for an arbitrarily chosen initial set $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$, *the sequence*

$$\{\mathcal{J}_0^*, \Psi\left(\mathcal{J}_0^*\right), \Psi^2\left(\mathcal{J}_0^*\right), ...\}$$

of iterates of compact sets converges to the distinct attractor of Ψ.

Proof. Choose $\mathcal{J}_0^*$ randomly in $\mathcal{C}^{p_m}(W)$. If $\mathcal{J}_0^* = \Psi\left(\mathcal{J}_0^*\right)$, then we have the required results. Suppose $\mathcal{J}_0^* \neq \Psi\left(\mathcal{J}_0^*\right)$, and let

$$\mathcal{J}_1^* = \Psi(\mathcal{J}_0^*),\ \mathcal{J}_2^* = \Psi\left(\mathcal{J}_1^*\right), ..., \mathcal{J}_{a+1}^* = \Psi\left(\mathcal{J}_a^*\right)$$

for $a \in \mathbb{N}$.

If $\mathcal{J}_a^* = \mathcal{J}_{a+1}^*$ for some a, then $\mathcal{J}_a^* = \Psi(\mathcal{J}_a^*)$ and the proof is complete. Assume that $\mathcal{J}_a^* \neq \mathcal{J}_{a+1}^*$ for all $a \in \mathbb{N}$, then from Definition 2.1.5, we get

$$\begin{aligned}H_{p_m}(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*) &= H_{p_m}(\Psi\left(\mathcal{J}_a^*\right), \Psi\left(\mathcal{J}_{a+1}^*\right)) \\ &\leq \lambda \mathcal{S}_\Psi\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right),\end{aligned}$$

where

$$
\begin{aligned}
\mathcal{S}_{\Psi}\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right) &= \max\{H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*), \\
&\quad H_{p_m}\left(\mathcal{J}_a^*, \Psi\left(\mathcal{J}_a^*\right)\right), H_{p_m}\left(\mathcal{J}_{a+1}^*, \Psi\left(\mathcal{J}_{a+1}^*\right)\right), \\
&\quad \frac{H_{p_m}\left(\mathcal{J}_a^*, \Psi\left(\mathcal{J}_{a+1}^*\right)\right) + H_{p_m}\left(\mathcal{J}_{a+1}^*, \Psi\left(\mathcal{J}_a^*\right)\right)}{2}, \\
&\quad H_{p_m}(\Psi^2\left(\mathcal{J}_a^*\right), \Psi\left(\mathcal{J}_a^*\right)), H_{p_m}\left(\Psi^2\left(\mathcal{J}_a^*\right), \mathcal{J}_{a+1}^*\right), \\
&\quad H_{p_m}\left(\Psi^2\left(\mathcal{J}_a^*\right), \Psi\left(\mathcal{J}_{a+1}^*\right)\right)\} \\
&= \max\{H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*), H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*), H_{p_m}\left(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*\right), \\
&\quad \frac{H_{p_m}\left(\mathcal{J}_a^*, \mathcal{J}_{a+2}^*\right) + H_{p_m}\left(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+1}^*\right)}{2}, \\
&\quad H_{p_m}(\mathcal{J}_{a+2}^*, \mathcal{J}_{a+1}^*), H_{p_m}(\mathcal{J}_{a+2}^*, \mathcal{J}_{a+1}^*), H_{p_m}\left(\mathcal{J}_{a+2}^*, \mathcal{J}_{a+2}^*\right)\} \\
&\leq \max\{H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*), H_{p_m}\left(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*\right), \\
&\quad \left.\frac{H_{p_m}\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right) + H_{p_m}\left(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*\right)}{2}\right\} \\
&= \max\{H_{p_m}\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right), H_{p_m}\left(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*\right)\}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
H_{p_m}(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*) &\leq \lambda \max\{H_{p_m}\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right), H_{p_m}\left(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*\right)\} \\
&= \lambda H_{p_m}\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right),
\end{aligned}
$$

for all $a \in \mathbb{N}$. Taking $a, n \in \mathbb{N}$ with $n > a$, we have

$$
\begin{aligned}
H_{p_m}\left(\mathcal{J}_a^*, \mathcal{J}_n^*\right) &\leq H_{p_m}\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right) + H_{p_m}\left(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*\right) + ... + H_{p_m}\left(\mathcal{J}_{n-1}^*, \mathcal{J}_n^*\right) \\
&\quad - \inf_{\mu_{a+1} \in \mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}) - \inf_{\mu_{a+2} \in \mathcal{J}_{a+2}^*} p_m(\mu_{a+2}, \mu_{a+2}) - \\
&\quad \cdots - \inf_{\mu_{n-1} \in \mathcal{J}_{n-1}^*} p_m(\mu_{n-1}, \mu_{n-1}) \\
&\leq \sum_{k=1}^{n-a} H_{p_m}\left(\mathcal{J}_{n-k}^*, \mathcal{J}_{n+1-k}^*\right) \\
&= \sum_{k=1}^{n-a} H_{p_m}\left(\Psi^{n-k}(\mathcal{J}_0^*), \Psi^{n-k}(\mathcal{J}_1^*)\right) \\
&\leq \sum_{k=1}^{n-a} \lambda^{n-k} H_{p_m}\left(\mathcal{J}_0^*, \mathcal{J}_1^*\right) \\
&= [\lambda^a + \lambda^{a+1} + ... + \lambda^{n-1}] H_{p_m}\left(\mathcal{J}_0^*, \mathcal{J}_1^*\right) \\
&= \lambda^a [1 + \lambda + \lambda^2 + \cdots + \lambda^{n-a-1}] H_{p_m}(\mathcal{J}_0^*, \mathcal{J}_1^*) \\
&\leq \frac{\lambda^a}{1-\lambda} H_{p_m}\left(\mathcal{J}_0^*, \mathcal{J}_1^*\right),
\end{aligned}
$$

and thus $\lim_{a,n\to+\infty} H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_n^*) = 0$. Hence the sequence $\{\mathcal{J}_a^*\}$ is Cauchy in W. But $(\mathcal{C}^{p_m}(W), H_{p_m})$ is a complete partial metric space, so $\mathcal{J}_a^* \to \tilde{U}_1$ as $a \to +\infty$ for some $\tilde{U}_1 \in \mathcal{C}^{p_m}(W)$, that is,

$$\lim_{a\to+\infty} H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1) = \lim_{a\to+\infty} H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*) = H_{p_m}(\tilde{U}_1, \tilde{U}_1).$$

Now for some $\tilde{U}_1 \in \mathcal{C}^{p_m}(W)$, $\mathcal{J}_a^* \to \tilde{U}_1$ as $a \to +\infty$, that is, $\lim_{a\to\infty} H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1) = 0$.

To show that $\tilde{U}_1$ is the fixed point of Ψ, we assume in the contrary that $H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) > 0$. So

$$\begin{aligned} H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}_{a+1}^*) + H_{p_m}(\mathcal{J}_{a+1}^*, \Psi(\tilde{U}_1)) \\ &\quad - \inf_{\mu_{a+1}\in\mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}) \\ &= H_{p_m}(\tilde{U}_1, \mathcal{J}_{a+1}^*) + H_{p_m}(\Psi(\mathcal{J}_a^*), \Psi(\tilde{U}_1)) \\ &\quad - \inf_{\mu_{a+1}\in\mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}) \\ &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}_{a+1}^*) + \lambda \mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U}_1) - \inf_{\mu_{a+1}\in\mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}) \end{aligned}$$

where

$$\begin{aligned} \mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U}_1) &= \max\{H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1), H_{p_m}(\mathcal{J}_a^*, \Psi(\mathcal{J}_a^*)), H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)), \\ &\quad \frac{H_{p_m}(\mathcal{J}_a^*, \Psi(\tilde{U}_1)) + H_{p_m}(\tilde{U}_1, \Psi(\mathcal{J}_a^*))}{2}, H_{p_m}(\Psi^2(\mathcal{J}_a^*), \Psi(\mathcal{J}_a^*)), \\ &\quad H_{p_m}(\Psi^2(\mathcal{J}_a^*), \tilde{U}_1), H_{p_m}(\Psi^2(\mathcal{J}_a^*), \Psi(\tilde{U}_1))\} \\ &= \max\{H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1), H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*), H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)), \\ &\quad \frac{H_{p_m}(\mathcal{J}_a^*, \Psi(\tilde{U}_1)) + H_{p_m}(\tilde{U}_1, \mathcal{J}_{a+1}^*)}{2}, \\ &\quad H_{p_m}(\mathcal{J}_{a+2}^*, \mathcal{J}_{a+1}^*), H_{p_m}(\mathcal{J}_{a+2}^*, \tilde{U}_1), H_{p_m}(\mathcal{J}_{a+2}^*, \Psi(\tilde{U}_1))\}. \end{aligned}$$

Now we examine the following seven cases:

(1) Suppose $\mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U}_1) = H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1)$, then

$$H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) \leq \lambda H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1)$$

and on taking the limit as $a \to +\infty$, we get

$$H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) \leq \lambda H_{p_m}(\tilde{U}_1, \tilde{U}_1),$$

so $H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) = 0$, and so $\tilde{U}_1 = \Psi(\tilde{U}_1)$.

(2) If $\mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U_1}) = H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)$, then

$$H_{p_m}(\Psi(\tilde{U_1}), \tilde{U_1}) \leq \lambda H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*),$$

and taking the limit as $a \to +\infty$

$$H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq \lambda H_{p_m}(\tilde{U_1}, \tilde{U_1}),$$

which implies that, $\tilde{U_1} = \Psi(\tilde{U_1})$.

(3) In case $\mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U_1}) = H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))$, we get

$$H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq \lambda H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))$$

which gives $\tilde{U_1} = \Psi(\tilde{U_1})$.

(4) Assume $\mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U_1}) = \dfrac{H_p(\mathcal{J}_a^*, \Psi(\tilde{U_1})) + H_{p_m}(\tilde{U_1}, \mathcal{J}_{a+1}^*)}{2}$, then

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \frac{\lambda}{2}[H_{p_m}(\mathcal{J}_a^*, \Psi(\tilde{U_1})) + H_{p_m}(\tilde{U_1}, \mathcal{J}_{a+1}^*)] \\ &\leq \frac{\lambda}{2}[H_{p_m}(\mathcal{J}_a^*, \tilde{U_1}) + H_{p_m}\Big(\tilde{U_1}, \Psi(U_1\Big)) \\ &\quad - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u}) + H_{p_m}(\tilde{U_1}, \mathcal{J}_{a+1}^*)], \end{aligned}$$

and as $a \to +\infty$, we get

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \frac{\lambda}{2}[H_{p_m}(\tilde{U_1}, \tilde{U_1}) + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \\ &\quad - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u}) + H_{p_m}(\tilde{U_1}, \tilde{U_1})] \\ &= \lambda\{H_{p_m}(\tilde{U_1}, \tilde{U_1}) + \frac{1}{2}[H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u})]\}, \end{aligned}$$

that is,

$$H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq \frac{2\lambda}{2 - \lambda}[H_{p_m}(\tilde{U_1}, \tilde{U_1}) - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u})]$$

which gives us $H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) = 0$ and so $\tilde{U_1} = \Psi(\tilde{U_1})$.

(5) For $\mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U_1}) = H_{p_m}(\mathcal{J}_{a+2}^*, \mathcal{J}_{a+1}^*)$, then as $a \to +\infty$, we get

$$H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq \lambda H_{p_m}(\tilde{U_1}, \tilde{U_1}),$$

which gives $\tilde{U_1} = \Psi(\tilde{U_1})$.

(6) Taking $\mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U_1}) = H_{p_m}(\mathcal{J}_{a+2}^*, \tilde{U_1})$, then as $a \to +\infty$, we have

$$H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq \lambda H_{p_m}(\tilde{U_1}, \tilde{U_1}),$$

and so $\tilde{U_1} = \Psi(\tilde{U_1})$.

(7) Lastly if $\mathcal{S}_\Psi(\mathcal{J}_a^*, \tilde{U_1}) = H_{p_m}(\mathcal{J}_{a+2}^*, \Psi(\tilde{U_1}))$, we have

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \lambda H_{p_m}(\mathcal{J}_{a+2}^*, \Psi(\tilde{U_1})) \\ &\leq \lambda[H_{p_m}(\mathcal{J}_{a+2}^*, \tilde{U_1}) + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u})] \end{aligned}$$

and on taking limit as $a \to +\infty$, yields

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \lambda[H_{p_m}(\tilde{U_1}, \tilde{U_1}) + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u})] \\ (1-\lambda) H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \lambda[H_{p_m}(\tilde{U_1}, \tilde{U_1}) - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u})] \end{aligned}$$

which implies that $H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq 0$ and so $\tilde{U_1} = \Psi(\tilde{U_1})$. As a result, in all cases, $\tilde{U_1}$ is the attractor of Ψ. To prove the uniqueness of the attractor, we assume that $\tilde{U_1}$ and $\tilde{U_2}$ are both attractors of Ψ with $H_{p_m}(\tilde{U_1}, \tilde{U_2}) > 0$. From the definition of Ψ, we get

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \tilde{U_2}) &= H_{p_m}(\Psi(\tilde{U_1}), \Psi(\tilde{U_2})). \\ &\leq \lambda \max\{H_{p_m}(\tilde{U_1}, \tilde{U_2}), H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})), H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_2})), \\ &\quad \frac{H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_2})) + H_{p_m}(\tilde{U_2}, \Psi(\tilde{U_1}))}{2}, \\ &\quad H_{p_m}(\Psi^2(\tilde{U_1}), \tilde{U_1}), H_{p_m}(\Psi^2(\tilde{U_1}), \tilde{U_2}), H_{p_m}(\Psi^2(\tilde{U_1}), \Psi(\tilde{U_2}))\} \\ &= \lambda \max\{H_{p_m}(\tilde{U_1}, \tilde{U_2}), H_{p_m}(\tilde{U_1}, \tilde{U_1}), H_{p_m}(\tilde{U_2}, \tilde{U_2}), \\ &\quad \frac{H_{p_m}(\tilde{U_1}, \tilde{U_2}) + H_{p_m}(\tilde{U_2}, \tilde{U_1})}{2}, \\ &\quad H_{p_m}(\tilde{U_1}, \tilde{U_1}), H_{p_m}(\tilde{U_1}, \tilde{U_2}), H_{p_m}(\tilde{U_1}, \tilde{U_2})\} \\ &= \lambda H_{p_m}(\tilde{U_1}, \tilde{U_2}), \end{aligned}$$

which implies that, $(1-\lambda) H_{p_m}(\tilde{U_1}, \tilde{U_2}) \leq 0$, so $H_{p_m}(\tilde{U_1}, \tilde{U_2}) = 0$ and hence $\tilde{U_1} = \tilde{U_2}$. Thus $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$ is the only attractor of Ψ. □

Remark 2.2.1. In Theorem 2.2.1, let $\mathcal{S}^{p_m}(W)$, the collection of all singleton subsets of the space W, then $\mathcal{S}^{p_m}(W) \subseteq \mathcal{C}^{p_m}(W)$. Moreover, taking $h_a = h$ for each $a = 1, 2, \cdots, q$, where $h = h_1$ implies that

$$\Psi(\varrho_1) = h(\varrho_1).$$

As a result, the fixed point result shown below is obtained.

Corollary 2.2.1. *Let* $\{W; h_a, a = 1, 2, \cdots, q\}$ *be a generalized iterated function system defined in a complete partial metric space* (W, p_m)*, and let* $h : W \to W$ *be as in Remark* 2.2.1*. If some* $\lambda \in [0, 1)$ *exists such that for any* $\varrho_1, \varrho_2 \in \mathcal{C}^{p_m}(W)$ *with* $p_m(h\varrho_1, h\varrho_2) \neq 0$*, the following holds:*

$$p_m(h\varrho_1, h\varrho_2) \leq \lambda \mathcal{S}_h(\varrho_1, \varrho_2),$$

where

$$\begin{aligned}\mathcal{S}_h(\varrho_1, \varrho_2) &= \max\{p_m(\varrho_1, \varrho_2), p_m(\varrho_1, h\varrho_1), p_m(\varrho_2, h\varrho_2), \frac{p_m(\varrho_1, h\varrho_2) + p_m(\varrho_2, h\varrho_1)}{2},\\ &\quad p(h^2\varrho_1, \varrho_2), p_m(h^2\varrho_1, h\varrho_1), p_m(h^2\varrho_1, h\varrho_2)\},\end{aligned}$$

then h *has a unique fixed point* $\tilde{u} \in W$*. Furthermore, for any* $v_0 \in W$ *, the sequence* $\{v_0, hv_0, h^2v_0, ...\}$ *has as a limit, a fixed point* $\tilde{u}$ *of* h*.*

Corollary 2.2.2. *Let* $\{W; h_a, a = 1, 2, \cdots, q\}$ *be an iterated function system defined in a complete partial metric space* (W, p_m) *and each* h_a *for* $a = 1, 2, ..., q$ *be a contractive self-mapping on* W*. Then* $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ *defined in Theorem* 2.2.1 *has a unique fixed point in* $\mathcal{C}^{p_m}(W)$*. Furthermore, for any initial set* $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$*, the sequence* $\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Psi^2(\mathcal{J}_0^*), \cdots\}$ *of compact sets converges to a fixed point of* Ψ*.*

Example 2.2.1. [21] Let $W = [0, 10]$ be endowed with the partial metric $p_m : W \times W \to \mathbb{R}_{[+]}$ defined by,

$$p_m(\varrho, \varsigma) = \frac{1}{2}\max\{\varrho, \varsigma\} + \frac{1}{4}|\varrho - \varsigma|$$

for all $\varrho, \varsigma \in W$.

Define $h_1, h_2 : W \to W$ as

$$\begin{aligned}h_1(\varrho) &= \frac{10 - \varrho}{2} \text{ for all } \varrho \in W \text{ and}\\ h_2(\varrho) &= \frac{\varrho + 4}{4} \text{ for all } \varrho \in W.\end{aligned}$$

Now for $\varrho, \varsigma \in W$, we have

$$\begin{aligned} p_m\left(h_1\left(\varrho\right), h_1\left(\varsigma\right)\right) &= \frac{1}{2}\max\left\{\frac{10-\varrho}{2}, \frac{10-\varsigma}{2}\right\} + \frac{1}{4}\left|\frac{10-\varrho}{2} - \frac{10-\varsigma}{2}\right| \\ &= \frac{1}{2}\left[\frac{1}{2}\max\{10-\varrho, 10-\varsigma\} + \frac{1}{4}\left|\varrho-\varsigma\right|\right] \\ &\leq \lambda_1 p_m\left(\varrho, \varsigma\right), \end{aligned}$$

where, $\lambda_1 = \frac{1}{2}$.

Also for $\varrho, \varsigma \in W$, we have

$$\begin{aligned} p_m\left(h_2\left(\varrho\right), h_2\left(\varsigma\right)\right) &= \frac{1}{2}\max\left\{\frac{\varrho+4}{4}, \frac{\varsigma+4}{4}\right\} + \frac{1}{4}\left|\frac{\varrho+4}{4} - \frac{\varsigma+4}{4}\right| \\ &= \frac{1}{4}\left[\frac{1}{2}\max\{\varrho+4, \varsigma+4\} + \frac{1}{4}\left|\varrho-\varsigma\right|\right] \\ &\leq \lambda_2 p_m\left(\varrho, \varsigma\right), \end{aligned}$$

where $\lambda_2 = \frac{1}{4}$.

Let $\{W; h_1, h_2\}$ be an iterated function system and define $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ by

$$\tilde{U} = \Psi(\tilde{U}) = h_1(\tilde{U}) \cup h_2(\tilde{U}) \quad \text{for all } \tilde{U} \in \mathcal{C}^{p_m}(W)$$

then for $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$, we have by Theorem 2.2.1,

$$H_{p_m}\left(\Psi\left(\mathcal{J}^*\right), \Psi\left(\mathcal{O}^*\right)\right) \leq \lambda^* H_{p_m}\left(\mathcal{J}^*, \mathcal{O}^*\right),$$

where, $\lambda^* = \max\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{2}$.

Thus all conditions of Theorem 2.2.1 are satisfied. Moreover, for any initial set $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$, the sequence

$$\{\mathcal{J}_0^*, \Psi\left(\mathcal{J}_0^*\right), \Psi^2\left(\mathcal{J}_0^*\right), ...\}$$

of compact sets is convergent and has for a limit, the attractor of Ψ.

Now we establish the existence and uniqueness of an attractor of the generalized rational Hutchinson contraction operator, Ψ defined in Definition 2.1.6.

Theorem 2.2.2. *In a complete partial metric space (W, p_m), let $\{W; h_a, a = 1, 2, \cdots, q\}$ be an iterated function system. Define $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ as*

$$\begin{aligned} \Psi(\mathcal{J}^*) &= h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*) \cup \cdots \cup h_a(\mathcal{J}^*) \\ &= \cup_{a=1}^q h_a(\mathcal{J}^*), \end{aligned}$$

for each $\mathcal{J}^* \in \mathcal{C}^{p_m}(W)$. *Suppose* Ψ *is a generalized rational Hutchinson contraction operator, then* Ψ *has a unique attractor* $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$, *that is*

$$\tilde{U_1} = \Psi(\tilde{U_1}) = \cup_{a=1}^{q} h_a(\tilde{U_1}).$$

Furthermore, for any arbitrarily chosen initial set $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$, *the sequence of compact sets*

$$\{\mathcal{J}_0^*, \Psi\left(\mathcal{J}_0^*\right), \Psi^2\left(\mathcal{J}_0^*\right), ...\}$$

converges to the attractor of Ψ, *that is* $\tilde{U_1}$.

Proof. Choose an arbitrary element $\mathcal{J}_0^*$ in $\mathcal{C}^{p_m}(W)$. If $\mathcal{J}_0^* = \Psi\left(\mathcal{J}_0^*\right)$, then the proof is complete. Suppose $\mathcal{J}_0^* \neq \Psi\left(\mathcal{J}_0^*\right)$ and define

$$\mathcal{J}_1^* = \Psi(\mathcal{J}_0^*),\ \mathcal{J}_2^* = \Psi\left(\mathcal{J}_1^*\right), ..., \mathcal{J}_{a+1}^* = \Psi\left(\mathcal{J}_a^*\right)$$

for $a \in \mathbb{N}$.

Assume that $\mathcal{J}_a^* \neq \mathcal{J}_{a+1}^*$ for all $a \in \mathbb{N}$, else $\mathcal{J}_a^* = \Psi(\mathcal{J}_a^*)$ for some a and there is nothing further to show. Consider $\mathcal{J}_a^* \neq \mathcal{J}_{a+1}^*$ for all $a \in \mathbb{N}$. Then

$$\begin{aligned} H_{p_m}(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*) &= H_{p_m}(\Psi\left(\mathcal{J}_a^*\right), \Psi\left(\mathcal{J}_{a+1}^*\right)) \\ &\leq \lambda_* \mathcal{R}_\Psi\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right), \end{aligned}$$

where,

$$\begin{aligned}\mathcal{R}_{\Psi}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*) &= \max\left\{\frac{H_{p_m}(\mathcal{J}_a^*,\Psi\left(\mathcal{J}_{a+1}^*\right))[1+H_{p_m}(\mathcal{J}_a^*,\Psi(\mathcal{J}_a^*))]}{2(1+H_{p_m}\left(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*\right))},\right.\\ &\quad \frac{H_{p_m}(\mathcal{J}_{a+1}^*,\Psi\left(\mathcal{J}_{a+1}^*\right))[1+H_{p_m}(\mathcal{J}_a^*,\Psi(\mathcal{J}_a^*))]}{1+H_{p_m}\left(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*\right)},\\ &\quad \left.\frac{H_{p_m}(\mathcal{J}_{a+1}^*,\Psi\left(\mathcal{J}_a^*\right))[1+H_{p_m}(\mathcal{J}_a^*,\Psi\left(\mathcal{J}_a^*\right))]}{1+H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)}\right\}\\ &= \max\left\{\frac{H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+2}^*)[1+H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)]}{2(1+H_{p_m}\left(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*\right))},\right.\\ &\quad \frac{H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*)[1+H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)]}{1+H_{p_m}\left(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*\right)},\\ &\quad \left.\frac{H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+1}^*)[1+H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)]}{1+H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)}\right\}\\ &= \max\left\{\frac{H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+2}^*)}{2},H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*),\right.\\ &\quad H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+1}^*)\}\\ &= \frac{H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+2}^*)}{2}.\end{aligned}$$

Thus, we have

$$\begin{aligned}H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*) &\leq \frac{\lambda_*}{2}[H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)+H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*)\\ &\quad -\inf_{\xi_{a+1}\in\mathcal{J}_{a+1}^*}p_m\left(\xi_{a+1},\xi_{a+1}\right)]\\ &\leq \frac{\lambda_*}{2}[H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)+H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*)],\end{aligned}$$

$$2H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*)-\lambda_*H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*)\leq\lambda_*[H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)],$$

$$H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*)\leq\frac{\lambda_*}{2-\lambda_*}H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*),$$

that is, for $\eta_*=\dfrac{\lambda_*}{2-\lambda_*}<1$, we have

$$H_{p_m}(\mathcal{J}_{a+1}^*,\mathcal{J}_{a+2}^*)\leq\eta_*H_{p_m}(\mathcal{J}_a^*,\mathcal{J}_{a+1}^*)$$

for all $a \in \mathbb{N}$. Thus for $a, n \in \mathbb{N}$ with $a < n$,

$$
\begin{aligned}
H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_n^*) &\leq H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*) + H_{p_m}(\mathcal{J}_{a+1}^*, \mathcal{J}_{a+2}^*) + \cdots + H_{p_m}(\mathcal{J}_{n-1}^*, \mathcal{J}_n^*) \\
&\quad - \inf_{\mu_{a+1} \in \mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}) - \inf_{\mu_{a+2} \in \mathcal{J}_{a+2}^*} p_m(\mu_{a+2}, \mu_{a+2}) - \\
&\quad \cdots - \inf_{\mu_{n-1} \in \mathcal{J}_{n-1}^*} p(\mu_{n-1}, \mu_{n-1}) \\
&\leq \eta_*^a H_{p_m}(\mathcal{J}_0^*, \mathcal{J}_1^*) + \eta_*^{a+1} H_{p_m}(\mathcal{J}_0^*, \mathcal{J}_1^*) + \cdots + \eta_*^{n-1} H_{p_m}(\mathcal{J}_0^*, \mathcal{J}_1^*) \\
&\leq [\eta_*^a + \eta_*^{a+1} + \cdots + \eta_*^{n-1}] H_{p_m}(\mathcal{J}_0^*, \mathcal{J}_1^*) \\
&\leq \eta_*^a [1 + \eta_* + \eta_*^2 + \cdots + \eta_*^{n-a-1}] H_{p_m}(\mathcal{J}_0^*, \mathcal{J}_1^*) \\
&\leq \frac{\eta_*^a}{1 - \eta_*} H_{p_m}(\mathcal{J}_0^*, \mathcal{J}_1^*).
\end{aligned}
$$

This gives us, $H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_n^*) \to 0$ as $a, n \to +\infty$. Therefore $\{\mathcal{J}_a^*\}$ is a Cauchy sequence in W. But $(\mathcal{C}^{p_m}(W), H_{p_m})$ is complete, so $\mathcal{J}_a^* \to \tilde{U}_1$ as $a \to +\infty$ for some $\tilde{U}_1 \in \mathcal{C}^{p_m}(W)$, in other words, $\lim_{a \to +\infty} H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1) = \lim_{a \to +\infty} H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*) = H_{p_m}(\tilde{U}_1, \tilde{U}_1)$.

To prove that $\tilde{U}_1$ is the fixed point of Ψ, we assume in the contrary that $H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) > 0$. This implies that

$$
\begin{aligned}
H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}_{a+1}^*) + H_{p_m}(\mathcal{J}_{a+1}^*, \Psi(\tilde{U}_1)) - \inf_{\mu_{a+1} \in \mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}) \\
&= H_{p_m}(\tilde{U}_1, \mathcal{J}_{a+1}^*) + H_{p_m}(\Psi(\mathcal{J}_a^*), \Psi(\tilde{U}_1)) - \inf_{\mu_{a+1} \in \mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}) \\
&\leq H_{p_m}(\tilde{U}_1, \mathcal{J}_{a+1}^*) + \lambda_* \mathcal{R}_\Psi(\mathcal{J}_a^*, \tilde{U}_1) - \inf_{\mu_{a+1} \in \mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R}_\Psi(\mathcal{J}_a^*, \tilde{U}_1) &= \max \Bigg\{ \frac{H_{p_m}(\mathcal{J}_a^*, \Psi(\tilde{U}_1))[1 + H_{p_m}(\mathcal{J}_a^*, \Psi(\mathcal{J}_a^*))]}{2(1 + H_{p_m}(\mathcal{J}_a^*, U_1))}, \\
&\qquad \frac{H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1))[1 + H_{p_m}(\mathcal{J}_a^*, \Psi(\mathcal{J}_a^*))]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1)}, \\
&\qquad \frac{H_{p_m}(\tilde{U}_1, \Psi(\mathcal{J}_a^*))[1 + H_{p_m}(\mathcal{J}_a^*, \Psi(\mathcal{J}_a^*))]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1)} \Bigg\} \\
&= \max \Bigg\{ \frac{H_{p_m}(\mathcal{J}_a^*, \Psi(\tilde{U}_1))[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{2(1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1))}, \\
&\qquad \frac{H_{p_m}(U_1, \Psi(\tilde{U}_1))[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1)}, \\
&\qquad \frac{H_{p_m}(\tilde{U}_1, \mathcal{J}_{a+1}^*)[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1)} \Bigg\}.
\end{aligned}
$$

Consider the following three cases:

(1) Let

$$\mathcal{R}_{\Psi}(\mathcal{J}_a^*, \tilde{U}_1) = \frac{H_{p_m}(\mathcal{J}_a^*, \Psi(\tilde{U}_1))[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{2(1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1))},$$

then

$$\begin{aligned} H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1) &\leq \frac{\lambda_*[H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1) + H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)}{2(1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1))} \\ &\quad - \frac{\inf_{\tilde{u} \in \tilde{U}_1} p_m(\tilde{u}, \tilde{u})][1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{2(1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1))} \\ &\quad - \inf_{\mu_{a+1} \in \mathcal{J}_{a+1}^*} p_m(\mu_{a+1}, \mu_{a+1}) \\ &\leq \frac{\lambda_*[H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1) + H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)]}{2(1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1))} \\ &\quad \times \frac{[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{2(1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1))}, \end{aligned}$$

and on taking limit as $a \to +\infty$, we get

$$\begin{aligned} H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) &\leq \frac{\lambda_*[H_{p_m}(\tilde{U}_1, \tilde{U}_1) + H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)}{2(1 + H_{p_m}(\tilde{U}_1, \tilde{U}_1))} \\ &\quad - \frac{\inf\limits_{\tilde{u_1} \in \tilde{U}_1} p_m(\tilde{u_1}, \tilde{u_1})][1 + H_{p_m}(\tilde{U}_1, \tilde{U}_1)]}{2(1 + H_{p_m}(\tilde{U}_1, \tilde{U}_1))}. \end{aligned}$$

Which implies that

$$H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) \leq \frac{\lambda_*}{2 - \lambda_*} H_{p_m}(\tilde{U}_1, \tilde{U}_1),$$

where $\dfrac{\lambda_*}{2 - \lambda_*} < 1$ and so $H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)) = 0$.

(2) If $\mathcal{R}_{\Psi}(\mathcal{J}_a^*, \tilde{U}_1) = \dfrac{H_{p_m}(\tilde{U}_1, \Psi\left(\tilde{U}_1\right))[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{1 + H_{p_m}\left(\mathcal{J}_a^*, \tilde{U}_1\right)}$, we have

$$\begin{aligned} H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1) &\leq \lambda_* \left\{ \frac{H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1))[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1)} \right\} \\ &\leq \lambda_* \left\{ \frac{H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1))[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U}_1)} \right\}, \end{aligned}$$

and taking the limit as $a \to +\infty$, yields

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \lambda_* \left\{ \frac{H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))[1 + H_{p_m}(\tilde{U_1}, \tilde{U_1})]}{1 + H_{p_m}(\tilde{U_1}, \tilde{U_1})} \right\} \\ (1 - \lambda_*) H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq 0, \text{a contradiction,} \end{aligned}$$

so $\tilde{U_1} = \Psi(\tilde{U_1})$.

(3) For $\mathcal{R}_\Psi(\mathcal{J}_a^*, \tilde{U_1}) = \dfrac{H_{p_m}(\tilde{U_1}, \mathcal{J}_{a+1}^*)[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U_1})}$, we obtain

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \lambda_* \left\{ \frac{H_{p_m}(\tilde{U_1}, \mathcal{J}_{a+1}^*)[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U_1})} \right\} \\ &\leq \lambda_* \left\{ \frac{H_{p_m}(\tilde{U_1}, \mathcal{J}_{a+1}^*)[1 + H_{p_m}(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*)]}{1 + H_{p_m}(\mathcal{J}_a^*, \tilde{U_1})} \right\}. \end{aligned}$$

Taking the limit as $a \to +\infty$,

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \lambda_* \left\{ \frac{H_{p_m}(\tilde{U_1}, \tilde{U_1})[1 + H_{p_m}(\tilde{U_1}, \tilde{U_1})]}{1 + H_{p_m}(\tilde{U_1}, \tilde{U_1})} \right\} \\ (1 - \lambda_*)\, H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq H_{p_m}(\tilde{U_1}, \tilde{U_1}), \end{aligned}$$

that is $\tilde{U_1} = \Psi(\tilde{U_1})$.

Thus in all three cases it was shown that $\tilde{U_1}$ is an attractor of the mapping Ψ.

For the uniqueness of attractor of Ψ, assume that $\tilde{U_1}$ and $\tilde{U_2}$ are both attractors of Ψ with $H_{p_m}\left(\tilde{U_1}, \tilde{U_2}\right)$ not equal to zero. Since Ψ is a generalized rational contraction, we obtain that

$$
\begin{aligned}
H_{p_m}(\tilde{U_1},\tilde{U_2}) &= H_{p_m}(\Psi(\tilde{U_1}),\Psi(\tilde{U_2})) \\
&\leq \lambda_* \max\left\{\frac{H_{p_m}(\tilde{U_1},\Psi(\tilde{U_2}))[1+H_{p_m}(\tilde{U_1},\Psi(\tilde{U_1}))]}{2(1+H_{p_m}(\tilde{U_1},\tilde{U_2}))},\right. \\
&\quad \frac{H_{p_m}(\tilde{U_2},\Psi\left(\tilde{U_2}\right))[1+H_{p_m}(\tilde{U_1},\Psi(\tilde{U_1}))]}{1+H_{p_m}(\tilde{U_1},\tilde{U_2})}, \\
&\quad \left.\frac{H_{p_m}(\tilde{U_2},\Psi(\tilde{U_1}))[1+H_{p_m}(\tilde{U_1},\Psi\left(\tilde{U_1}\right))]}{1+H_{p_m}(\tilde{U_1},\tilde{U_2})}\right\} \\
&= \lambda_* \max\left\{\frac{H_{p_m}(\tilde{U_1},\tilde{U_2})[1+H_{p_m}(\tilde{U_1},\tilde{U_1})]}{2(1+H_{p_m}(\tilde{U_1},\tilde{U_2}))},\right. \\
&\quad \left.\frac{H_{p_m}(\tilde{U_2},\tilde{U_2})[1+H_{p_m}(\tilde{U_1},\tilde{U_1})]}{1+H_{p_m}(\tilde{U_1},\tilde{U_2})}, \frac{H_{p_m}(\tilde{U_2},\tilde{U_1})[1+H_{p_m}(\tilde{U_1},\tilde{U_1})]}{1+H_{p_m}(\tilde{U_1},\tilde{U_2})}\right\} \\
&\leq \lambda_* H_{p_m}(\tilde{U_1},\tilde{U_2}),
\end{aligned}
$$

and so $(1-\lambda_*)H_{p_m}(\tilde{U_1},\tilde{U_2}) \leq 0$, which implies that $H_{p_m}(\tilde{U_1},\tilde{U_2}) = 0$ and hence $\tilde{U_1} = \tilde{U_2}$. Thus $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$ is a unique attractor of Ψ. □

Corollary 2.2.3. *Let $\{W; h_a, a = 1, 2, \cdots, q\}$ be a generalized iterated function system on a complete partial metric space (W, p_m) and define $h : W \to W$ as in Remark* 2.2.1. *If for any $\varrho_1, \varrho_2 \in \mathcal{C}^{p_m}(W)$ with $p_m(h(\varrho_1), h(\varrho_2)) \neq 0$, there exists some $\lambda_* \in [0, 1)$ satisfying,*

$$
p_m(h\varrho_1, h\varrho_2) \leq \lambda_* \mathcal{R}_h(\varrho_1, \varrho_2),
$$

where

$$
\begin{aligned}
\mathcal{R}_h(\varrho_1,\varrho_2) &= \max\left\{\frac{p_m(\varrho_1,h\varrho_2)[1+p_m(\varrho_1,h\varrho_1)]}{2(1+p_m(\varrho_1,\varrho_2))}, \frac{p_m(\varrho_2,h\varrho_2)[1+p_m(\varrho_1,h\varrho_1)]}{1+p_m(\varrho_1,\varrho_2)},\right. \\
&\quad \left.\frac{p_m(\varrho_2,h\varrho_1)[1+p_m(\varrho_1,h\varrho_1)]}{1+p_m(\varrho_1,\varrho_2)}\right\},
\end{aligned}
$$

then h has a unique fixed point $\tilde{u} \in W$. In addition, for any initial choice of $\tilde{u_0} \in W$, the sequence $\{\tilde{u_0}, h\tilde{u_0}, h^2\tilde{u_0}, ...\}$ converges to $\tilde{u}$.

2.3. Well-posedness of Iterated Function Systems

This section investigates the well-posedness of attractor based problems for generalized Hutchinson contractive operator and generalized rational Hutchinson contractive operator which appear in Definition 2.1.5 and Definition 2.1.6, in a Hausdorff partial metric space setup, respectively. The existence, uniqueness, and stability of solutions to fixed point equations are often connected with well-posedness, which is an important aspect in the construction of fractals. Some significant results on well-posedness of fixed point problems are well presented in [6, 56, 59].

Definition 2.3.1. An attractor based problem of a mapping $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ is said to be well-posed if Ψ has a unique attractor $\Theta^* \in \mathcal{C}^{p_m}(W)$ and for any sequence $\{\Theta_a\}$ in $\mathcal{C}^{p_m}(W)$, $\lim\limits_{a\to+\infty} H_{p_m}(\Psi(\Theta_a), \Theta_a) = 0$ implies that $\lim\limits_{a\to+\infty} H_{p_m}(\Theta_a, \Theta^*) = H_{p_m}(\Theta^*, \Theta^*)$, that is, $\lim\limits_{a\to+\infty} \Theta_a = \Theta^*$.

Theorem 2.3.1. *Let (W, p_m) be a complete partial metric space and define $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ as in Theorem 2.2.1. Then Ψ has a well-posed attractor based problem.*

Proof. According to Theorem 2.2.1, Ψ has a unique attractor $\mathcal{Z}_*$, say. Let $\{\mathcal{Z}_a\}$ be a sequence in $\mathcal{C}^{p_m}(W)$ such that $\lim\limits_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$. We want to show that $\mathcal{Z}_* = \lim\limits_{a\to+\infty} \mathcal{Z}_a$ for every positive integer a. As Ψ is a generalized contractive Hutchinson operator, then

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq H_{p_m}(\Psi(\mathcal{Z}_*), \Psi(\mathcal{Z}_a)) + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p(\beta_a, \beta_a) \\ &\leq \lambda \mathcal{S}_\Psi(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

where

$$\begin{aligned} \mathcal{S}_\Psi(\mathcal{Z}_*, \mathcal{Z}_a) &= \max\{H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a), H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_*)), H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)), \\ &\quad \frac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_*))}{2}, H_{p_m}(\Psi^2(\mathcal{Z}_*), \Psi(\mathcal{Z}_*)), \\ &\quad H_{p_m}(\Psi^2(\mathcal{Z}_*), \mathcal{Z}_a), H_{p_m}(\Psi^2(\mathcal{Z}_*), \Psi(\mathcal{Z}_a))\} \\ &= \max\{H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a), H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)), \\ &\quad \frac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)}{2}, H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a))\}. \end{aligned}$$

The following arise:

(i) For $\mathcal{S}_\Psi(\mathcal{Z}_a, \mathcal{Z}_*) = H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a)$, then

$$H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \lambda H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)$$

$$H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) - \lambda H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*) \leq H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)$$

$$H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \frac{1}{1-\lambda}[H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)],$$

and as $a \to +\infty$ we have

$$\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \frac{1}{1-\lambda}[\lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} \lim_{a\to+\infty} p_m(\beta_a, \beta_a)],$$

thus $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(ii) When $\mathcal{S}_\Psi(\mathcal{Z}_a, \mathcal{Z}_*) = H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))$, then

$$H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \lambda H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} \lim_{a\to+\infty} p_m(\beta_a, \beta_a)],$$

and as $a \to +\infty$ we have,

$$\begin{aligned}\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \quad &\leq \quad \lambda \lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) + \lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) \\ &\quad - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} \lim_{a\to+\infty} p_m(\beta_a, \beta_a),\end{aligned}$$

thus $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(iii) In case $\mathcal{S}_\Psi(\mathcal{Z}_a, \mathcal{Z}_*) = \dfrac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a) + H_p(\mathcal{Z}_a, \mathcal{Z}_*)}{2}$, then

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \quad &\leq \quad \frac{\lambda}{2}[H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)] \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &\leq \quad \frac{\lambda}{2}[H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) \\ &\quad - \inf_{b_a \in \mathcal{Z}_a} p_m(b_a, b_a) + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)] \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) - \lambda H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \quad &\leq \quad \frac{\lambda}{2}[H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) - \inf_{b_a \in \mathcal{Z}_a} p_m(b_a, b_a)] \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \quad &\leq \quad \frac{\lambda}{2(1-\lambda)}[H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) - \inf_{b_a \in \mathcal{Z}_a} p_m(b_a, b_a)] \\ &\quad + \frac{1}{1-\lambda}[H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)], \end{aligned}$$

and as $a \to +\infty$ we have

$$\begin{aligned}\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \frac{\lambda}{2(1-\lambda)}[\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) - \inf_{b_a\in\mathcal{Z}_a} \lim_{a\to+\infty} p_m(b_a, b_a)] \\ &+\frac{1}{1-\lambda}[\lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} \lim_{a\to+\infty} p_m(\beta_a, \beta_a)],\end{aligned}$$

which implies that $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(iv) If $\mathcal{S}_\Psi(\mathcal{Z}_a, \mathcal{Z}_*) = H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a))$, then

$$\begin{aligned}H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \lambda H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a)) + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &\leq \lambda[H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) - \inf_{b_a\in\mathcal{Z}_a} p_m(b_a, b_a)] \\ &\quad +H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a),\end{aligned}$$

$$\begin{aligned}H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) - \lambda H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \lambda[H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) - \inf_{b_a\in\mathcal{Z}_a} p_m(b_a, b_a)] \\ &\quad +H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a),\end{aligned}$$

$$\begin{aligned}H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \frac{\lambda}{1-\lambda}[H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) - \inf_{b_a\in\mathcal{Z}_a} p_m(b_a, b_a)] \\ &+\frac{1}{1-\lambda}[H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)],\end{aligned}$$

and as $a \to +\infty$ we have

$$\begin{aligned}\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \lambda \lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) + \lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) \\ &\quad - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} \lim_{a\to+\infty} p_m(\beta_a, \beta_a),\end{aligned}$$

giving us that $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

□

Theorem 2.3.2. *Consider a complete partial metric space (W, p_m) with $\Psi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ defined as in Theorem 2.2.2. Then Ψ has a well-posed attractor based problem.*

Proof. From Theorem 2.2.2, it follows that the map Ψ has a unique attractor say $\mathcal{Z}_*$. Consider the sequence $\{\mathcal{Z}_a\}$ in $\mathcal{C}^{p_m}(W)$ such that $\lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$. We show that $\mathcal{Z}_* = \lim_{a\to+\infty} \mathcal{Z}_a$ for every $a \in \mathbb{N}$. Since Ψ is a generalized rational contractive Hutchinson operator, then

$$\begin{aligned}
H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*) &\leq H_{p_m}(\Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_*)) + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p(\beta_a, \beta_a) \\
&\leq \lambda_* \mathcal{R}_\Psi(\mathcal{Z}_a, \mathcal{Z}_*) + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_\Psi(\mathcal{Z}_a, \mathcal{Z}_*) = \max\Bigg\{ & \frac{H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_*))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{2(1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*))}, \\
& \frac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_*))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)}, \\
& \frac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)} \Bigg\}.
\end{aligned}$$

We consider the following three cases:

(i) For $\mathcal{R}_\Psi(\mathcal{Z}_a, \mathcal{Z}_*) = \dfrac{H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_*))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{2(1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*))}$, we have

$$\begin{aligned}
H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \lambda_* \frac{H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_*))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{2(1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*))} \\
&\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\
&\leq \lambda_* H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))] \\
&\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a).
\end{aligned}$$

Therefore

$$\begin{aligned}
H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) - \lambda_* H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))] &\leq H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) \\
&\quad - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a),
\end{aligned}$$

thus

$$\begin{aligned}
H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \frac{1}{1 - \lambda_*[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}[H_p(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) \\
&\quad - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)],
\end{aligned}$$

and on taking the limit as $a \to +\infty$, we get

$$\lim_{a \to +\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq 0,$$

which implies that $\lim\limits_{a \to +\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(ii) If $\mathcal{R}_{\Psi}(\mathcal{Z}_a, \mathcal{Z}_*) = \dfrac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_*))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, B_*)}$, then

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \lambda_* \left(\frac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_*))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)} \right) \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &= H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

and applying the limit as $a \to +\infty$, gives

$$\lim_{a \to +\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq 0,$$

which implies that $\lim_{a \to +\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(iii) And if $\mathcal{R}_{\Psi}(\mathcal{Z}_a, \mathcal{Z}_*) = \dfrac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)}$, then

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \lambda_* \frac{H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_a))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)} \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &\leq \lambda_*[H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) - \inf_{\eta_a \in \mathcal{Z}_a} p_m(\eta_a, \eta_a)] \\ &\quad [1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))] + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

so

$$\begin{aligned} & H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) - \lambda_* H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a)[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))] \\ \leq\ & \lambda_*[H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) - \inf_{\eta_a \in \mathcal{Z}_a} p_m(\eta_a, \eta_a)][1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))] \\ & + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

therefore

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \frac{1}{(1 - \lambda_*)\,[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]} [\lambda_*[H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a)) \\ &\quad - \inf_{\eta_a \in \mathcal{Z}_a} p_m(\eta_a, \eta_a)][1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))] + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) \\ &\quad - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

which implies that $\lim_{a \to +\infty} \mathcal{Z}_a = \mathcal{Z}_*$. Hence the required results.

□

3

Generalized iterated function system for common attractors in partial metric spaces

3.1. Introduction

The construction of a common attractor of generalized iterated function system of generalized contractions in the framework of partial metric spaces is the focus of our current discussion. We note that the Hutchinson operator is itself a generalized contractive mapping on a family of compact subsets of W, defined on a finite family of contractive mappings on a complete partial metric space. The final common attractor is generated by using a generalized Hutchinson contraction operator repeatedly, and this is followed by the presentation of a non-trivial example to support the proved result. To conclude the chapter, an application of our findings will be given.

3.2. Generalized Iterated Function System

Some findings on generalized iterated function system for multivalued mappings in metric spaces do appear in [35]. In this section, we define the generalized iterated function system in the context of partial metric spaces.

Definition 3.2.1. [35] Let (W, p_m) be a partial metric space, and let $h, g : W \to W$ be two mappings. Then a pair (h, g) is a generalized contraction provided $\lambda \in [0, 1)$ exists such that

$$p_m(h\varrho, g\varsigma) \leq \lambda p_m(\varrho, \varsigma)$$

for all $\varrho, \varsigma \in W$.

Theorem 3.2.1. *In a partial metric space (W, p_m) let $h, g : W \to W$ be a couple of continuous mappings. If (h, g) is a pair of generalized contractions with $\lambda \in [0, 1)$, then*

(1) *both h and g map elements in $\mathcal{C}^{p_m}(W)$ to elements in $\mathcal{C}^{p_m}(W)$;*

(2) *if for any $\mathcal{J}^* \in \mathcal{C}^{p_m}(W)$, the mappings $h, g : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ are defined as*

$$\begin{aligned} h(\mathcal{J}^*) &= \{h(\varrho) : \varrho \in \mathcal{J}^*\} \text{ and} \\ g(\mathcal{J}^*) &= \{g(\varsigma) : \varsigma \in \mathcal{J}^*\}, \end{aligned}$$

then the pair (h, g) is a generalized contraction on $(\mathcal{C}^{p_m}(W), H_{p_m})$.

Proof. (1) Since h is a continuous mapping and the image of a compact subset under a continuous mapping, $h : W \to W$ is compact, then

$$\mathcal{J}^* \in \mathcal{C}^{p_m}(W) \text{ implies that } h(\mathcal{J}^*) \in \mathcal{C}^{p_m}(W).$$

Similarly, we have

$$\mathcal{J}^* \in \mathcal{C}^{p_m}(W) \text{ implies that } g(\mathcal{J}^*) \in \mathcal{C}^{p_m}(W).$$

(2) Let $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$. Since the pair (h, g) is a generalized contraction, then

$$p_m(h\varrho, g\varsigma) \le \lambda p_m(\varrho, \varsigma) \text{ for all } \varrho, \varsigma \in W,$$

where $\lambda \in [0, 1)$.

Thus, we have

$$\begin{aligned} p_m(h\varrho, g(\mathcal{O}^*)) &= \inf_{\varsigma \in \mathcal{O}^*} p_m(h\varrho, g\varsigma) \\ &\le \inf_{\varsigma \in \mathcal{O}^*} \lambda p_m(\varrho, \varsigma) \\ &= \lambda p_m(\varrho, \mathcal{O}^*). \end{aligned}$$

Also

$$\begin{aligned} p_m(g\varsigma, h(\mathcal{J}^*)) &= \inf_{\varrho \in \mathcal{J}^*} p_m(g\varsigma, h\varrho) \\ &\le \inf_{\varrho \in \mathcal{J}^*} \lambda p_m(\varsigma, \varrho) \\ &= \lambda p_m(\varsigma, \mathcal{J}^*). \end{aligned}$$

Now

$$\begin{aligned}
H_{p_m}\left(h\left(\mathcal{J}^*\right),g\left(\mathcal{O}^*\right)\right) &= \max\{\sup_{\varrho\in\mathcal{J}^*} p_m(h\varrho, g\left(\mathcal{O}^*\right)), \sup_{\varsigma\in\mathcal{O}^*} p_m(g\varsigma, h(\mathcal{J}^*))\} \\
&\leq \max\{\sup_{\varrho\in\mathcal{J}^*} \lambda p_m(\varrho,\mathcal{O}^*), \sup_{\varsigma\in\mathcal{O}^*} \lambda p_m(\varsigma,\mathcal{J}^*)\} \\
&= \max\{\lambda\sup_{\varrho\in\mathcal{J}^*} p_m(\varrho,\mathcal{O}^*), \lambda\sup_{\varsigma\in\mathcal{O}^*} p_m(\varsigma,\mathcal{J}^*)\} \\
&= \lambda\max\{\sup_{\varrho\in\mathcal{J}^*} p_m(\varrho,\mathcal{O}^*), \sup_{\varsigma\in\mathcal{O}^*} p_m(\varsigma,\mathcal{J}^*)\} \\
&= \lambda H_{p_m}\left(\mathcal{J}^*,\mathcal{O}^*\right).
\end{aligned}$$

Consequently,

$$H_{p_m}\left(h\left(\mathcal{J}^*\right),g\left(\mathcal{O}^*\right)\right) \leq \lambda H_{p_m}\left(\mathcal{J}^*,\mathcal{O}^*\right).$$

Thus, (h,g) is a generalized contraction mapping pair on $(\mathcal{C}^{p_m}(W), H_{p_m})$.

□

Proposition 3.2.1. *In a partial metric space (W,p_m). Let $h_a, g_a : W \to W$ for $a = 1,2,\cdots,q$ be a collection of continuous mappings such that*

$$p_m\left(h_a\varrho, g_a\varsigma\right) \leq \lambda_a p_m\left(\varrho,\varsigma\right) \text{ for all } \varrho,\varsigma\in W,$$

where $\lambda_a \in [0,1)$ for each $a \in \{1,2,\cdots,q\}$. Then the mappings $\Psi,\Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ defined as

$$\begin{aligned}
\Psi(\mathcal{J}^*) &= h_1(\mathcal{J}^*)\cup h_2(\mathcal{J}^*)\cup\cdots\cup h_q(\mathcal{J}^*) \\
&= \cup_{a=1}^{q} h_a(\mathcal{J}^*) \text{ for each } \mathcal{J}^*\in\mathcal{C}^{p_m}(W)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\mathcal{J}^*) &= g_1(\mathcal{J}^*)\cup g_2(\mathcal{J}^*)\cup\cdots\cup g_q(\mathcal{J}^*) \\
&= \cup_{a=1}^{q} g_a(\mathcal{J}^*) \text{ for each } \mathcal{J}^*\in\mathcal{C}^{p_m}(W)
\end{aligned}$$

also satisfy

$$H_{p_m}\left(\Psi\mathcal{J}^*,\Phi\mathcal{O}^*\right) \leq \tilde{\lambda} H_{p_m}(\mathcal{J}^*,\mathcal{O}^*) \text{ for all } \mathcal{J}^*,\mathcal{O}^*\in\mathcal{C}^{p_m}(W),$$

where $\tilde{\lambda} = \max\{\lambda_a : a \in \{1,2,...,q\}\}$. Furthermore the pair (Ψ,Φ) is a generalized contraction on $\mathcal{C}^{p_m}(W)$.

Proof. We shall prove the result for $q = 2$. Let $h_1, h_2, g_1, g_2 : W \to W$ be two contractions. For $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$ and using Lemma 2.1.2 (c), we have

$$\begin{aligned}
H_{p_m}(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)) &= H_{p_m}(h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*), g_1(\mathcal{O}^*) \cup g_2(\mathcal{O}^*)) \\
&\leq \max\{H_{p_m}(h_1(\mathcal{J}^*), g_1(\mathcal{O}^*)), H_{p_m}(h_2(\mathcal{J}^*), g_2(\mathcal{O}^*))\} \\
&\leq \max\{\lambda_1 H_{p_m}(\mathcal{J}^*, \mathcal{O}^*), \lambda_2 H_{p_m}(\mathcal{J}^*, \mathcal{O}^*)\} \\
&\leq \widetilde{\lambda} H_{p_m}(\mathcal{J}^*, \mathcal{O}^*).
\end{aligned}$$

□

Definition 3.2.2. Consider a partial metric space (W, p_m) with the mappings $\Psi, \Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$. A pair of mappings (Ψ, Φ) is called

1. a generalized Hutchinson contractive operator if a constant $\lambda \in [0, 1)$ exists such that for any $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$, the following holds:

$$H_{p_m}(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)) \leq \lambda \mathcal{S}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*),$$

   where

$$\begin{aligned}
\mathcal{S}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*) &= \max\{H_{p_m}(\mathcal{J}^*, \mathcal{O}^*), H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*)), H_{p_m}(\mathcal{O}^*, \Phi(\mathcal{O}^*)), \\
&\quad \left.\frac{H_{p_m}(\mathcal{J}^*, \Phi(\mathcal{O}^*)) + H_{p_m}(\mathcal{O}^*, \Psi(\mathcal{J}^*))}{2}\right\},
\end{aligned}$$

2. a generalized rational Hutchinson contractive operator if a constant $\lambda_* \in [0, 1)$ exists such that for any $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$, the following holds:

$$H_{p_m}(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)) \leq \lambda_* \mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*),$$

   where

$$\begin{aligned}
\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*) &= \max\left\{\frac{H_{p_m}(\mathcal{J}^*, \Phi(\mathcal{O}^*))[1 + H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{2(1 + H_{p_m}(\mathcal{J}^*, \mathcal{O}^*))}\right., \\
&\quad \frac{H_{p_m}(\mathcal{O}^*, \Phi(\mathcal{O}^*))[1 + H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{1 + H_{p_m}(\mathcal{J}^*, \mathcal{O}^*)}, \\
&\quad \left.\frac{H_{p_m}(\mathcal{J}^*, \mathcal{O}^*)[1 + H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{1 + H_{p_m}(\mathcal{J}^*, \mathcal{O}^*)}\right\}.
\end{aligned}$$

Note that if the pair (Ψ, Φ) defined as in Proposition 3.2.1 is generalized contraction on $\mathcal{C}^{p_m}(W)$, then the pair (Ψ, Φ) is a generalized Hutchinson contractive operator but not conversely.

Definition 3.2.3. Let (W, p_m) be a complete partial metric space. If

$h_a, g_a : W \to W$, $a = 1, 2, ..., q$ are continuous mappings such that each pair (h_a, g_a) for $a = 1, 2, ..., q$ is a generalized contraction, then $\{W; (h_a, g_a), a = 1, 2, \cdots, q\}$ is called a generalized iterated function system (GIFS).

Definition 3.2.4. Let $\mathcal{J}^* \subseteq W$ be a non-void compact set, then $\mathcal{J}^*$ is a common attractor of the generalized iterated function system if

(i) $\Psi(\mathcal{J}^*) = \Phi(\mathcal{J}^*) = \mathcal{J}^*$ and

(ii) there exists an open set $V_1 \subseteq W$ such that $\mathcal{J}^* \subseteq V_1$ and $\lim_{a\to+\infty} \Psi^a(\mathcal{O}^*) = \lim_{a\to+\infty} \Phi^a(\mathcal{O}^*) = \mathcal{J}^*$ for any compact set $\mathcal{O}^* \subseteq V_1$, where the limit is taken with respect to the partial Hausdorff metric.

As a result, the maximal open set V_1 satisfying (ii) is referred to as a basin of common attraction.

3.3. Generalized common attractors of Hutchinson contractive operators

In the setting of partial metric space, we state and prove some results on the existence and uniqueness of a common attractor of generalized and generalized rational Hutchinson contractive operators, beginning with the following theorem.

Theorem 3.3.1. *Let (W, p_m) be a complete partial metric space and $\{W; (h_a, g_a), a = 1, 2, \cdots, q\}$, a generalized iterated function system. Define $\Psi, \Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ by*

$$\Psi(\mathcal{J}^*) = \cup_{a=1}^{q} h_a(\mathcal{J}^*),$$

and

$$\Phi(\mathcal{O}^*) = \cup_{a=1}^{q} g_a(\mathcal{O}^*)$$

for each $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$. If the pair (Ψ, Φ) is a generalized Hutchinson contractive operator, then Ψ and Φ have a unique common attractor $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$, that is,

$$\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1}).$$

Furthermore, for an arbitrarily chosen initial set $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$, the sequence

$$\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Phi\Psi(\mathcal{J}_0^*), \Psi\Phi\Psi(\mathcal{J}_0^*), ...\}$$

of compact sets converges to the common attractor $\tilde{U_1}$ of Ψ and Φ.

Proof. We choose an arbitrary element $\mathcal{J}_0^*$ in $\mathcal{C}^{p_m}(W)$ and define Ψ and Φ

respectively by

$$\mathcal{J}_1^* = \Psi(\mathcal{J}_0^*),\ \mathcal{J}_3^* = \Psi\left(\mathcal{J}_2^*\right), ..., \mathcal{J}_{2a+1}^* = \Psi\left(\mathcal{J}_{2a}^*\right)$$

and

$$\mathcal{J}_2^* = \Phi(\mathcal{J}_1^*),\ \mathcal{J}_4^* = \Phi\left(\mathcal{J}_3^*\right), ..., \mathcal{J}_{2a+2}^* = \Phi\left(\mathcal{J}_{2a+1}^*\right)$$

for $a \in \{0, 1, 2, ...\}$.

Now, as the pair (Ψ, Φ) is generalized Hutchinson contractive operator, we have

$$\begin{aligned} H_{p_m}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*) &= H_{p_m}(\Psi\left(\mathcal{J}_{2a}^*\right), \Phi\left(\mathcal{J}_{2a+1}^*\right)) \\ &\leq \lambda \mathcal{S}_{\Psi,\Phi}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{S}_{\Psi,\Phi}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right) &= \max\Big\{H_{p_m}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*), H_{p_m}\left(\mathcal{J}_{2a}^*, \Psi\left(\mathcal{J}_{2a}^*\right)\right), \\ &\quad H_{p_m}\left(\mathcal{J}_{2a+1}^*, \Phi\left(\mathcal{J}_{2a+1}^*\right)\right), \\ &\quad \frac{H_{p_m}\left(\mathcal{J}_{2a}^*, \Phi(\mathcal{J}_{2a+1}^*)\right) + H_{p_m}\left(\mathcal{J}_{2a+1}^*, \Psi(\mathcal{J}_{2a}^*)\right)}{2}\Big\} \\ &= \max\Big\{H_{p_m}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*), H_{p_m}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*), \\ &\quad H_{p_m}\left(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*\right), \\ &\quad \frac{H_{p_m}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+2}^*\right) + H_{p_m}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+1}^*)}{2}\Big\} \\ &\leq \max\{H_{p_m}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*), H_{p_m}\left(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*\right), \\ &\quad \frac{H_{p_m}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*) + H_{p_m}\left(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*\right)}{2}\Big\} \\ &= \max\left\{H_{p_m}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right), H_{p_m}\left(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*\right)\right\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} H_{p_m}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*) &\leq \lambda \max\left\{H_{p_m}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right), H_{p_m}\left(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*\right)\right\} \\ &= \lambda H_{p_m}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right). \end{aligned}$$

Also,

$$\begin{aligned} H_{p_m}\left(\mathcal{J}_{2a+2}^*, \mathcal{J}_{2a+3}^*\right) &= H_{p_m}\left(\mathcal{J}_{2a+3}^*, \mathcal{J}_{2a+2}^*\right) \\ &= H_{p_m}(\Psi\left(\mathcal{J}_{2a+2}^*\right), \Phi\left(\mathcal{J}_{2a+1}^*\right)) \\ &\leq \lambda \mathcal{S}_{\Psi,\Phi}\left(\mathcal{J}_{2a+2}^*, \mathcal{J}_{2a+1}^*\right), \end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}_{\Psi,\Phi}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+1}\right) &= \max\{H_{p_m}(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+1}), H_{p_m}\left(\mathcal{J}^*_{2a+2},\Psi\left(\mathcal{J}^*_{2a+2}\right)\right), \\
&\quad H_{p_m}\left(\mathcal{J}^*_{2a+1},\Phi\left(\mathcal{J}^*_{2a+1}\right)\right), \\
&\quad \left.\frac{H_{p_m}\left(\mathcal{J}^*_{2a+2},\Phi\left(\mathcal{J}^*_{2a+1}\right)\right)+H_{p_m}\left(\mathcal{J}^*_{2a+1},\Psi\left(\mathcal{J}^*_{2a+2}\right)\right)}{2}\right\} \\
&= \max\{H_{p_m}(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+1}), H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+3}\right), \\
&\quad H_{p_m}\left(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}\right), \\
&\quad \left.\frac{H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+2}\right)+H_{p_m}\left(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+3}\right)}{2}\right\} \\
&\leq \max\{H_{p_m}(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+1}), H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+3}\right), \\
&\quad \left.\frac{H_{p_m}\left(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}\right)+H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+3}\right)}{2}\right\} \\
&= \max\{H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+1}\right), H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+3}\right)\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
H_{p_m}(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+3}) &\leq \lambda\max\{H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+1}\right), H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+3}\right)\} \\
&= \lambda H_{p_m}\left(\mathcal{J}^*_{2a+2},\mathcal{J}^*_{2a+1}\right).
\end{aligned}$$

Therefore, for all $a \in \{0,1,2,...\}$, we have

$$\begin{aligned}
H_{p_m}(\mathcal{J}^*_{a+1},\mathcal{J}^*_{a+2}) &\leq \lambda H_{p_m}\left(\mathcal{J}^*_{a},\mathcal{J}^*_{a+1}\right) \\
&\leq \lambda^2 H_{p_m}\left(\mathcal{J}^*_{a-1},\mathcal{J}^*_{a}\right) \\
&\leq \cdots \\
&\leq \lambda^{a+1} H_{p_m}\left(\mathcal{J}^*_{0},\mathcal{J}^*_{1}\right).
\end{aligned}$$

Now, we have for $l > a$, with $a, l \in \{0,1,2,...\}$,

$$\begin{aligned}
H_{p_m}\left(\mathcal{J}^*_{a},\mathcal{J}^*_{l}\right) &\leq H_{p_m}\left(\mathcal{J}^*_{a},\mathcal{J}^*_{a+1}\right)+H_{p_m}\left(\mathcal{J}^*_{a+1},\mathcal{J}^*_{a+2}\right)+\cdots+H_{p_m}\left(\mathcal{J}^*_{l-1},\mathcal{J}^*_{l}\right) \\
&\quad -\inf_{m_{a+1}\in\mathcal{J}^*_{a+1}} p_m(m_{a+1},m_{a+1}) - \inf_{m_{a+2}\in\mathcal{J}^*_{a+2}} p_m(m_{a+2},m_{a+2}) - \\
&\quad \cdots - \inf_{m_{a-1}\in\mathcal{J}^*_{a-1}} p_m(m_{a-1},m_{a-1}), \\
&\leq [\lambda^a+\lambda^{a+1}+\cdots+\lambda^{l-1}]H_{p_m}\left(\mathcal{J}^*_{0},\mathcal{J}^*_{1}\right), \\
&= \lambda^a[1+\lambda+\lambda^2+\cdots+\lambda^{l-a-1}]H_{p_m}(\mathcal{J}^*_{0},\mathcal{J}^*_{1})], \\
&\leq \frac{\lambda^a}{1-\lambda}H_{p_m}\left(\mathcal{J}^*_{0},\mathcal{J}^*_{1}\right)
\end{aligned}$$

and so $\lim_{a,l\to+\infty} H_{p_m}(\mathcal{J}^*_{a},\mathcal{J}^*_{l}) = 0$. Thus $\{\mathcal{J}^*_{a}\}$ is a Cauchy sequence in $\mathcal{C}^{p_m}(W)$.

Since $(\mathcal{C}^{p_m}(W), H_{p_m})$ is a complete partial metric space, there exists $\tilde{U}_1 \in \mathcal{C}^{p_m}(W)$ such that $\lim_{a\to+\infty} \mathcal{J}^*_a = \tilde{U}_1$, that is,

$$\lim_{a\to+\infty} H_{p_m}(\mathcal{J}^*_a, \tilde{U}_1) = \lim_{a\to+\infty} H_{p_m}\left(\mathcal{J}^*_a, \mathcal{J}^*_{a+1}\right) = H_{p_m}(\tilde{U}_1, \tilde{U}_1)$$

and so, we have $\lim_{a\to+\infty} H_{p_m}(\mathcal{J}^*_a, \tilde{U}_1) = 0$.

To show that $\Psi(\tilde{U}_1) = \tilde{U}_1$, we consider

$$\begin{aligned} H_{p_m}(\Psi(\tilde{U}_1), \tilde{U}_1) &\leq H_{p_m}(\Psi(\tilde{U}_1), \Phi(\mathcal{J}^*_{2a+1})) + H_{p_m}(\Phi(\mathcal{J}^*_{2a+1}), \tilde{U}_1) \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}), \\ &\leq \lambda \mathcal{S}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U}_1) \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p(m_{2a+1}, m_{2a+1}) \end{aligned}$$

for all $a \in \{0, 1, 2, ...\}$, where

$$\begin{aligned} \mathcal{S}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) &= \max\Big\{ H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}), H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)), \\ &\quad H_{p_m}(\mathcal{J}^*_{2a+1}, \Phi(\mathcal{J}^*_{2a+1})), \\ &\quad \frac{H_{p_m}(\tilde{U}_1, \Phi(\mathcal{J}^*_{2a+1})) + H_{p_m}(\mathcal{J}^*_{2a+1}, \Psi(\tilde{U}_1))}{2} \Big\} \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &= \max\{ H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}), H_{p_m}(\tilde{U}_1, \Psi(\tilde{U}_1)), \\ &\quad H_{p_m}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2}), \\ &\quad \frac{H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+2}) + H_{p_m}(\mathcal{J}^*_{2a+1}, \Psi(\tilde{U}_1))}{2} \Big\} \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}). \end{aligned}$$

Now, we examine the following cases:

(1) If $\mathcal{S}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) = H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1})$, then

$$\begin{aligned} H_{p_m}(\Psi(\tilde{U}_1), \tilde{U}_1) &\leq \lambda H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U}_1) \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq \lambda H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U}_1), \end{aligned}$$

which together with our taking the limit as $a \to +\infty$, gives

$$H_{p_m}(\Psi(\tilde{U}_1), \tilde{U}_1) \leq \lambda H_{p_m}(\tilde{U}_1, \tilde{U}_1) + H_{p_m}(\tilde{U}_1, \tilde{U}_1),$$

and we get $H_{p_m}(\Psi(\tilde{U_1}), \tilde{U_1}) = 0$, that is, $\tilde{U_1} = \Psi(\tilde{U_1})$.

(2) Provided $\mathcal{S}_{\Psi,\Phi}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) = H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))$, then

$$\begin{aligned} H_{p_m}(\Psi(\tilde{U_1}), \tilde{U_1}) &\leq \lambda H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}) \\ &\quad - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq \lambda H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}), \end{aligned}$$

that is,

$$H_{p_m}(\Psi(\tilde{U_1}), \tilde{U_1}) \leq \frac{1}{1-\lambda} H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}),$$

which together with our taking the limit as $a \to +\infty$ implies that $H_{p_m}(\Psi(\tilde{U_1}), \tilde{U_1}) \leq 0$ and so $\tilde{U_1} = \Psi(\tilde{U_1})$.

(3) In the case of $\mathcal{S}_{\Psi,\Phi}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) = H_{p_m}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2})$, we get

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) &\leq \lambda H_{p_m}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2}) + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}) \\ &\quad - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq \lambda H_{p_m}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2}) + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}), \end{aligned}$$

which together with our taking the limit as $a \to +\infty$ implies that $\tilde{U_1} = \Psi(\tilde{U_1})$.

(4) If $\mathcal{S}_{\Psi,\Phi}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) = \dfrac{H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+2}) + H_{p_m}(\mathcal{J}^*_{2a+1}, \Psi(\tilde{U_1}))}{2}$, then

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \frac{\lambda}{2}[H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+2}) + H_{p_m}(\mathcal{J}^*_{2a+1}, \Psi(\tilde{U_1}))] \\ &\quad + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}) - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq \frac{\lambda}{2}[H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+2}) + H_{p_m}(\mathcal{J}^*_{2a+1}, \tilde{U_1}) + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \\ &\quad - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u})] + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}) \\ &\quad - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq \frac{\lambda}{2}[H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+2}) + H_{p_m}(\mathcal{J}^*_{2a+1}, \tilde{U_1}) + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))] \\ &\quad + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}), \end{aligned}$$

which together with our taking the limit as $a \to +\infty$, we get

$$H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq \frac{\lambda}{2} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})),$$

giving us $H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) = 0$, and so $\tilde{U_1} = \Psi(\tilde{U_1})$.

Thus, from the above cases, $\tilde{U}_1$ is the attractor of Ψ.

Similar reasoning, gives

$$\begin{aligned} H_{p_m}(\tilde{U}_1, \Phi(\tilde{U}_1)) &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + H_{p_m}(\mathcal{J}^*_{2a+1}, \Phi(\tilde{U}_1)) \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &= H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + H_{p_m}(\Psi(\mathcal{J}^*_{2a}), \Phi(\tilde{U}_1)) \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + \lambda E_{\Psi,\Phi}(\mathcal{J}^*_{2a}, \tilde{U}_1) \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}), \end{aligned}$$

where

$$\begin{aligned} \mathcal{S}_{\Psi,\Phi}(\mathcal{J}^*_{2a}, \tilde{U}_1) &= \max\{H_{p_m}(\mathcal{J}^*_{2a}, \tilde{U}_1), H_{p_m}(\mathcal{J}^*_{2a}, \Psi(\mathcal{J}^*_{2a})), H_{p_m}(\tilde{U}_1, \Phi(\tilde{U}_1)), \\ &\quad \left.\frac{H_{p_m}(\mathcal{J}^*_{2a}, \Phi(\tilde{U}_1)) + H_{p_m}(\tilde{U}_1, \Psi(\mathcal{J}^*_{2a}))}{2}\right\} \\ &= \max\{H_{p_m}(\mathcal{J}^*_{2a}, \tilde{U}_1), H_{p_m}(\mathcal{J}^*_{2a}, \mathcal{J}^*_{2a+1}), H_{p_m}(U_1, \Phi(\tilde{U}_1)), \\ &\quad \left.\frac{H_{p_m}(\mathcal{J}^*_{2a}, \Phi(\tilde{U}_1)) + H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1})}{2}\right\}. \end{aligned}$$

As a consequence, we observe that:

(1) If $\mathcal{S}_{\Psi,\Phi}(\mathcal{J}^*_{2a}, \tilde{U}_1) = H_{p_m}(\mathcal{J}^*_{2a}, \tilde{U}_1)$, then

$$\begin{aligned} H_{p_m}(\tilde{U}_1, \Phi(\tilde{U}_1)) &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + \lambda H_{p_m}(\mathcal{J}^*_{2a}, \tilde{U}_1) \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + \lambda H_{p_m}(\mathcal{J}^*_{2a}, \tilde{U}_1), \end{aligned}$$

which in combination with our taking the limit as $a \to +\infty$, gives

$$H_{p_m}(\tilde{U}_1, \Phi(\tilde{U}_1)) \leq H_{p_m}(\tilde{U}_1, \tilde{U}_1) + \lambda H_{p_m}(\tilde{U}_1, \tilde{U}_1),$$

and we get $H_{p_m}(\tilde{U}_1, \Phi(\tilde{U}_1)) = 0$, that is, $\tilde{U}_1 = \Phi(\tilde{U}_1)$.

(2) For $\mathcal{S}_{\Psi,\Phi}(\mathcal{J}^*_{2a}, U_1) = H_{p_m}(\mathcal{J}^*_{2a}, \mathcal{J}^*_{2a+1})$, then

$$\begin{aligned} H_{p_m}(\tilde{U}_1, \Phi(\tilde{U}_1)) &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + \lambda H_{p_m}(\mathcal{J}^*_{2a}, \mathcal{J}^*_{2a+1}) \\ &\quad - \inf_{m_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq H_{p_m}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) + \lambda H_{p_m}(\mathcal{J}^*_{2a}, \mathcal{J}^*_{2a+1}), \end{aligned}$$

which together with our taking the limit as $a \to +\infty$, we have

$$H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) \leq H_{p_m}(\tilde{U_1}, \tilde{U_1}) + \lambda H_{p_m}(\tilde{U_1}, \tilde{U_1}),$$

which implies that $H_{p_m}(U_1, \Phi(\tilde{U_1})) \leq 0$ and so $U_1 = \Phi(\tilde{U_1})$.

(3) In the case of $\mathcal{S}_{\Psi,\Phi}(\mathcal{J}^*_{2a}, \tilde{U_1}) = H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1}))$, we get

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) &\leq H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) + \lambda H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) \\ &\quad - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) + \lambda H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})), \end{aligned}$$

that is,

$$H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) \leq \frac{1}{1-\lambda} H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1}),$$

which together with our taking the limit as $a \to +\infty$, we can write $H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) \leq 0$ and so $\tilde{U_1} = \Phi(\tilde{U_1})$.

(4) If $\mathcal{S}_{\Psi,\Phi}(\mathcal{J}^*_{2a}, \tilde{U_1}) = \dfrac{H_{p_m}(\mathcal{J}^*_{2a}, \Phi(\tilde{U_1})) + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})}{2}$, then

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) &\leq H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) + \frac{\lambda}{2}\left[H_{p_m}(\mathcal{J}^*_{2a}, \Phi(\tilde{U_1})) + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})\right] \\ &\quad - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) + \frac{\lambda}{2}[H_{p_m}(\mathcal{J}^*_{2a}, \Phi(\tilde{U_1})) + H_{p_m}(\mathcal{J}^*_{2a}, \tilde{U_1}) \\ &\quad + H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) - \inf_{\tilde{u} \in \tilde{U_1}} p_m(\tilde{u}, \tilde{u}) + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})] \\ &\quad - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) + \frac{\lambda}{2}[H_{p_m}(\mathcal{J}^*_{2a}, \Phi(\tilde{U_1})) + H_{p_m}(\mathcal{J}^*_{2a}, \tilde{U_1}) \\ &\quad + H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})], \end{aligned}$$

which together with our taking the as $a \to +\infty$ implies

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) &\leq H_{p_m}(\tilde{U_1}, \tilde{U_1}) + \frac{\lambda}{2}[H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) + H_{p_m}(\tilde{U_1}, \tilde{U_1}) \\ &\quad + H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) + H_{p_m}(\tilde{U_1}, \tilde{U_1})] \\ &= \lambda H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})), \end{aligned}$$

giving us $H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) = 0$ and so $\tilde{U_1} = \Psi(\tilde{U_1})$.

Thus $\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1})$, which means that $\tilde{U_1}$ is the common attractor of Ψ and Φ.To establish the uniqueness of the common attractor, let $\tilde{U_2}$ be another common attractor of Ψ and Φ. Since the pair (Ψ, Φ) is generalized Hutchinson contractive operator, we have

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \tilde{U_2}) &= H_{p_m}(\Psi(\tilde{U_1}), \Phi(\tilde{U_2})) \\ &\leq \lambda \max\{H_{p_m}(\tilde{U_1}, \tilde{U_2}), H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})), H_{p_m}(\tilde{U_2}, \Phi(\tilde{U_2})), \\ &\quad \left.\frac{H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_2})) + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))}{2}\right\} \\ &= \lambda \max\{H_{p_m}(\tilde{U_1}, \tilde{U_2}), H_{p_m}(\tilde{U_1}, \tilde{U_1}), H_{p_m}(\tilde{U_2}, \tilde{U_2}), \\ &\quad \left.\frac{H_{p_m}(\tilde{U_1}, \tilde{U_2}) + H_{p_m}(\tilde{U_1}, \tilde{U_1})}{2}\right\} \\ &\leq \lambda H_{p_m}(\tilde{U_1}, \tilde{U_2}), \end{aligned}$$

and so $(1-\lambda)H_{p_m}(\tilde{U_1}, \tilde{U_2}) \leq 0$, that is, $H_{p_m}(\tilde{U_1}, \tilde{U_2}) = 0$ and hence $\tilde{U_1} = \tilde{U_2}$. Thus $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$ is a unique common attractor of Ψ and Φ. □

The theorem below shows that, beginning with an arbitrary set, it is possible to find a generalized iterated function system whose common attractor is the given set.

Theorem 3.3.2 (**Generalized Collage**). *Let (W, p_m) be a complete partial metric space. For a given generalized iterated function system $\{W; h_1, h_2..., h_q; g_1, g_{2,}..., g_q\}$ with a common contractive constant $\lambda \in [0, 1)$ and for a given $\varepsilon \geq 0$, if for any $\mathcal{J}^* \in \mathcal{C}^{p_m}(W)$, we have either*

$$H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*)) \leq \varepsilon,$$

or

$$H_{p_m}(\mathcal{J}^*, \Phi(\mathcal{J}^*)) \leq \varepsilon,$$

where $\Psi(\mathcal{J}^*) = \cup_{a=1}^{q} h_a(\mathcal{J}^*)$ and $\Phi(\mathcal{J}^*) = \cup_{a=1}^{q} g_a(\mathcal{J}^*)$, then

$$H_{p_m}(\mathcal{J}^*, \tilde{U_1}) \leq \frac{\varepsilon}{1-\lambda},$$

where $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$ is a common attractor of Ψ and Φ.

Proof. It follows from Proposition 3.2.1 that the pair of the mappings

$$\Psi, \Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$$

satisfies

$$H_{p_m}(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)) \leq \lambda H_{p_m}(\mathcal{J}^*, \mathcal{O}^*) \text{ for all } \mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W).$$

From Theorem 3.3.1, there exists a unique common attractor $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$ of mappings Ψ and Φ, that is, $\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1})$.

In addition, for any $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$, a sequence $\{\mathcal{J}_a^*\}$ defined by $\mathcal{J}_{2a+1}^* = \Psi(\mathcal{J}_{2a}^*)$ and $\mathcal{J}_{2a+2}^* = \Phi\left(\mathcal{J}_{2a+1}^*\right)$ for $a = 0, 1, 2, ...$, we have

$$\lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{J}_{2a}^*), \tilde{U_1}) = \lim_{a\to+\infty} H_{p_m}(\Phi(\mathcal{J}_{2a+1}^*), \tilde{U_1}) = 0.$$

Assume that $H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*)) \leq \varepsilon$ for any $\mathcal{J}^* \in \mathcal{C}^{p_m}(W)$, one can write

$$\begin{aligned} H_{p_m}(\mathcal{J}^*, \tilde{U_1}) &\leq H_{p_m}(\mathcal{J}^*, \Psi(\mathcal{J}^*)) + H_{p_m}(\Psi(\mathcal{J}^*), \Phi(\tilde{U_1})) - \inf_{\alpha\in\Psi(\mathcal{J}^*)} p_m(\alpha, \alpha) \\ &\leq \varepsilon + \lambda H_{p_m}(\mathcal{J}^*, \tilde{U_1}), \end{aligned}$$

which further implies that

$$H_{p_m}(\mathcal{J}^*, \tilde{U_1}) \leq \frac{\varepsilon}{1-\lambda}.$$

Similarly, suppose that $H_{p_m}(\mathcal{J}^*, \Phi(\mathcal{J}^*)) \leq \varepsilon$ for any $\mathcal{J}^* \in \mathcal{C}^{p_m}(W)$. Then,

$$\begin{aligned} H_{p_m}(\mathcal{J}^*, \tilde{U_1}) &\leq H_{p_m}(\mathcal{J}^*, \Phi(\mathcal{J}^*)) + H_{p_m}(\Phi(\mathcal{J}^*), \Psi(\tilde{U_1})) - \inf_{\alpha\in\Phi(\mathcal{J}^*)} p_m(\alpha, \alpha) \\ &\leq \varepsilon + \lambda H_{p_m}(\mathcal{J}^*, \tilde{U_1}), \end{aligned}$$

implies

$$H_{p_m}(\mathcal{J}^*, \tilde{U_1}) \leq \frac{\varepsilon}{1-\lambda}.$$

□

Remark 3.3.1. *If we take in Theorem* 3.3.1, $\mathcal{S}^{p_m}(W)$ *the collection of all singleton subsets of the given space W, then* $\mathcal{S}^{p_m}(W) \subseteq \mathcal{C}^{p_m}(W)$. *Furthermore, if we take a pair of mappings* $(h_a, g_a) = (h, g)$ *for each a, where* $h = h_1$ *and* $g = g_1$ *then the pair of operators* (Ψ, Φ) *becomes*

$$(\Psi(\varrho_1), \Phi(\varrho_2)) = (h(\varrho_1), g(\varrho_2)).$$

Consequently, the following common fixed point result is obtained.

Corollary 3.3.1. *Suppose* $\{W; (h_a, g_a), a = 1, 2, \cdots, q\}$ *is a generalized iterated function system defined in a complete partial metric space* (W, p_m) *and define a* *pair of mappings $h, g : W \to W$ as in Remark 3.3.1. If some $\lambda \in [0,1)$ exists such that for any $\varrho, \varsigma \in W$, the following condition holds:*

$$p_m(h\varrho, g\varsigma) \leq \lambda \mathcal{S}_{h,g}(\varrho, \varsigma),$$

where

$$\mathcal{S}_{h,g}(\varrho, \varsigma) = \max\left\{p_m(\varrho, \varsigma), p_m(\varrho, h\varrho), p_m(\varrho, g\varsigma), \frac{p_m(\varrho, g\varsigma) + p_m(\varsigma, h\varrho_1)}{2}\right\}.$$

Then h and g have a unique common fixed point $\tilde{u} \in W$. Furthermore, for any $\tilde{u}_0 \in W$, the sequence $\{\tilde{u}_0, h\tilde{u}_0, gh\tilde{u}_0, hgh\tilde{u}_0, \cdots\}$ converges to the common fixed point of h and g, that is $\tilde{u}$.

Corollary 3.3.2. *Let $\{W; (h_a, g_a), a = 1, 2, \cdots, q\}$ be a generalized iterated function system defined in a complete partial metric space (W, p_m) and (h_a, g_a) for $a = 1, 2, ..., q$ be a pair of generalized contractive self-mappings on W. Then the pair $(\Psi, \Phi) : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ defined in Theorem 3.3.1 has at most one common attractor in $\mathcal{C}^{p_m}(W)$. Furthermore, for any initial set $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$, the sequence $\{\mathcal{J}_0^*, \Phi\Psi(\mathcal{J}_0^*), \Psi\Phi\Psi(\mathcal{J}_0^*), \cdots\}$ of compact sets has a limit point which is the common attractor of Ψ and Φ.*

With the following example, we establish the validity of Corollary 3.3.1.

Example 3.3.1. Let $W = [0, 10]$ be endowed with the partial metric
$p_m : W \times W \to \mathbb{R}_{[+]}$ defined by,

$$p_m(\varrho, \varsigma) = \frac{1}{2}\max\{\varrho, \varsigma\} + \frac{1}{4}|\varrho - \varsigma| \text{ for all } \varrho, \varsigma \in W.$$

Define $h_1, h_2 : W \to W$ as,

$$\begin{aligned} h_1(\varrho) &= \frac{10 - \varrho}{3} \text{ for all } \varrho \in W, \\ h_2(\varrho) &= \frac{16 - \varrho}{4} \text{ for all } \varrho \in W, \end{aligned}$$

and $g_1, g_2 : W \to W$ as

$$\begin{aligned} g_1(\varsigma) &= \frac{15 - \varsigma}{3} \text{ for all } \varsigma \in W, \\ g_2(\varsigma) &= \frac{\varsigma + 4}{4} \text{ for all } \varsigma \in W. \end{aligned}$$

Now, for $\varrho, \varsigma \in W$, we have

$$\begin{aligned} p_m\left(h_1(\varrho), g_1(\varsigma)\right) &= \frac{1}{2}\max\left\{\frac{10-\varrho}{3}, \frac{15-\varsigma}{3}\right\} + \frac{1}{4}\left|\frac{10-\varrho}{3} - \frac{15-\varsigma}{3}\right| \\ &= \frac{1}{3}\left[\frac{1}{2}\max\{10-\varrho, 15-\varsigma\} + \frac{1}{4}\left|(10-\varrho)-(15+\varsigma)\right|\right] \\ &\leq \lambda_1 p_m(\varrho, \varsigma), \end{aligned}$$

where $\lambda_1 = \dfrac{1}{3}$.

Also, for $\varrho, \varsigma \in W$, we have

$$\begin{aligned} p_m\left(h_2(\varrho), g_2(\varsigma)\right) &= \frac{1}{2}\max\left\{\frac{16-\varrho}{4}, \frac{\varsigma+4}{4}\right\} + \frac{1}{4}\left|\frac{16-\varrho}{4} - \frac{\varsigma+4}{4}\right| \\ &= \frac{1}{4}\left[\frac{1}{2}\max\{16-\varrho, \varsigma+4\} + \frac{1}{4}\left|(16-\varrho)-(\varsigma+4)\right|\right] \\ &\leq \lambda_2 p_m(\varrho, \varsigma), \end{aligned}$$

where $\lambda_2 = \dfrac{1}{4}$.

Consider the generalized iterated function system $\{W; (h_1, g_1), (h_2, g_2)\}$ with the mappings $\Psi, \Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ given as

$$(\Psi, \Phi)(\tilde{U_1}) = (h_1, g_1)(\tilde{U_1}) \cup (h_2, g_2)(\tilde{U_1}) \text{ for all } \tilde{U_1} \in \mathcal{C}^{p_m}(W).$$

Using Proposition 3.2.1, for $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$, we have

$$H_{p_m}\left(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)\right) \leq \lambda^* H_{p_m}\left(\mathcal{J}^*, \mathcal{O}^*\right),$$

where $\lambda^* = \max\left\{\dfrac{1}{3}, \dfrac{1}{4}\right\} = \dfrac{1}{3}$.

Thus, all conditions of Corollary 3.3.1 are satisfied. Moreover, for any initial set $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$, the sequence

$$\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Phi\Psi(\mathcal{J}_0^*), \Psi\Phi\Psi(\mathcal{J}_0^*), \cdots\}$$

of compact sets is convergent and has a limit point which is the common attractor of Ψ and Φ.

The following result shows the existence of unique common attractor of generalized rational Hutchinson contractive operators in partial metric space.

Theorem 3.3.3. *Consider a complete partial metric space (W, p_m) and the generalized iterated function system given as $\{W; (h_a, g_a), a = 1, 2, \cdots, q\}$. Let*

$\Psi, \Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ be defined by

$$\Psi(\mathcal{J}^*) = \cup_{a=1}^{q} h_a(\mathcal{J}^*)$$

and

$$\Phi(\mathcal{O}^*) = \cup_{a=1}^{q} g_a(\mathcal{O}^*),$$

for each $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{p_m}(W)$. If the pair (Ψ, Φ) is generalized rational Hutchinson contractive operator, then Ψ and Φ have a unique common attractor $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$, that is,

$$\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1}).$$

Furthermore, for arbitrarily chosen initial set $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(W)$, the sequence

$$\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Phi\Psi\left(\mathcal{J}_0^*\right), \Psi\Phi\Psi\left(\mathcal{J}_0^*\right), \cdots\}$$

of compact sets converges to a common attractor $\tilde{U_1}$.

Proof. Let $\mathcal{J}_0^*$ be arbitrarily chosen in $\mathcal{C}^{p_m}(W)$. Define

$$\mathcal{J}_1^* = \Psi(\mathcal{J}_0^*),\ \mathcal{J}_3^* = \Psi\left(\mathcal{J}_2^*\right), \cdots, \mathcal{J}_{2a+1}^* = \Psi\left(\mathcal{J}_{2a}^*\right)$$

and

$$\mathcal{J}_2^* = \Phi(\mathcal{J}_1^*),\ \mathcal{J}_4^* = \Phi\left(\mathcal{J}_3^*\right), \cdots, \mathcal{J}_{2a+2}^* = \Phi\left(\mathcal{J}_{2a+1}^*\right)$$

for $a \in \{0, 1, 2, \cdots\}$. Now, since the pair (Ψ, Φ) is a generalized rational Hutchinson contractive operator, we have

$$\begin{aligned} H_{p_m}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*) &= H_{p_m}(\Psi\left(\mathcal{J}_{2a}^*\right), \Phi\left(\mathcal{J}_{2a+1}^*\right)) \\ &\leq \lambda_* \mathcal{R}_{\Psi,\Phi}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right) \end{aligned}$$

for $a \in \{0, 1, 2, \cdots\}$, where

$$\begin{aligned} \mathcal{R}_{\Psi,\Phi}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right) &= \max\left\{ \frac{H_{p_m}(\mathcal{J}_{2a}^*, \Phi\left(\mathcal{J}_{2a+1}^*\right))[1 + H_{p_m}(\mathcal{J}_{2a}^*, \Psi(\mathcal{J}_{2a}^*))]}{2(1 + H_{p_m}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right))}, \right. \\ &\quad \frac{H_{p_m}(\mathcal{J}_{2a+1}^*, \Phi\left(\mathcal{J}_{2a+1}^*\right))[1 + H_{p_m}(\mathcal{J}_{2a}^*, \Psi(\mathcal{J}_{2a}^*))]}{1 + H_{p_m}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right)}, \\ &\quad \left. \frac{H_{p_m}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*)[1 + H_{p_m}(\mathcal{J}_{2a}^*, \Psi\left(\mathcal{J}_{2a}^*\right))]}{1 + H_{p_m}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*)} \right\} \end{aligned}$$

$$\begin{aligned}
&= \max\left\{\frac{H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+2})[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{2(1+H_{p_m}\left(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1}\right))},\right.\\
&\quad \frac{H_{p_m}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2})[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{p_m}\left(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1}\right)},\\
&\quad \left.\frac{H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})}\right\}\\
&= \max\left\{\frac{H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+2})}{2}, H_{p_m}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}),\right.\\
&\quad \left. H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})\right\}\\
&= \max\left\{\frac{H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+2})}{2}, H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})\right\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
H_{p_m}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}) &\leq \frac{\lambda_*}{2}[H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})+H_{p_m}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2})\\
&\quad - \inf_{\alpha_{2a+1}\in\mathcal{J}^*_{2a+1}} p_m\left(\alpha_{2a+1},\alpha_{2a+1}\right)]\\
&\leq \frac{\lambda_*}{2}[H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})+H_{p_m}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2})],
\end{aligned}$$

that is,

$$H_{p_m}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}) \leq \frac{\lambda_*}{2-\lambda_*}H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})$$

and for $\eta_* = \dfrac{\lambda_*}{2-\lambda_*} < 1$, we have

$$H_{p_m}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}) \leq \eta_* H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})$$

for all $a \in \{0,1,2,...\}$. Therefore for $a < l$, with $a, l \in \{0,1,2,...\}$

$$\begin{aligned}
H_{p_m}(\mathcal{J}^*_a,\mathcal{J}^*_l) &\leq H_{p_m}(\mathcal{J}^*_a,\mathcal{J}^*_{a+1})+H_{p_m}(\mathcal{J}^*_{a+1},\mathcal{J}^*_{a+2})+\cdots+H_{p_m}(\mathcal{J}^*_{l-1},\mathcal{J}^*_l)\\
&\quad - \inf_{\alpha_{a+1}\in\mathcal{J}^*_{a+1}} p_m(\alpha_{a+1},\alpha_{a+1}) - \inf_{\alpha_{a+2}\in\mathcal{J}^*_{a+2}} p_m(\alpha_{a+2},\alpha_{a+2}) -\\
&\quad \cdots - \inf_{\alpha_{l-1}\in\mathcal{J}^*_{l-1}} p_m(\alpha_{l-1},\alpha_{l-1}),\\
&\leq \eta^a_* H_{p_m}(\mathcal{J}^*_0,\mathcal{J}^*_1)+\eta^{a+1}_* H_{p_m}(\mathcal{J}^*_0,\mathcal{J}^*_1)+\cdots+\eta^{l-1}_* H_{p_m}(\mathcal{J}^*_0,\mathcal{J}^*_1),\\
&\leq [\eta^a_*+\eta^{a+1}_*+\cdots+\eta^{l-1}_*]H_{p_m}(\mathcal{J}^*_0,\mathcal{J}^*_1),\\
&\leq \eta^a_*[1+\eta_*+\eta^2_*+\cdots+\eta^{l-a-1}_*]H_{p_m}(\mathcal{J}^*_0,\mathcal{J}^*_1),\\
&\leq \frac{\eta^a_*}{1-\eta_*}H_{p_m}(\mathcal{J}^*_0,\mathcal{J}^*_1).
\end{aligned}$$

By convergence towards 0 from the right hand side, we get $H_{p_m}\left(\mathcal{J}^*_a,\mathcal{J}^*_l\right) \to 0$ as $a, l \to +\infty$. Therefore $\{\mathcal{J}^*_a\}$ is a Cauchy sequence in $\mathcal{C}^{p_m}(W)$. But $(\mathcal{C}^{p_m}(W), H_{p_m})$ is complete, so we have $\mathcal{J}^*_a \to \tilde{U}_1$ as $a \to +\infty$ for some $\tilde{U}_1 \in \mathcal{C}^{p_m}(W)$, in other words, $\lim\limits_{a\to+\infty} H_{p_m}(\mathcal{J}_a^*, \tilde{U_1}) = \lim\limits_{a\to+\infty} H_{p_m}\left(\mathcal{J}_a^*, \mathcal{J}_{a+1}^*\right) = H_{p_m}(\tilde{U_1}, \tilde{U_1})$ and we have $\lim\limits_{a\to+\infty} H_{p_m}(\mathcal{J}_a^*, \tilde{U_1}) = 0$.

To prove that $\tilde{U_1}$ is a common attractor of Ψ and Φ, we have

$$\begin{aligned}
H_{p_m}(\Psi(\tilde{U_1}), \tilde{U_1}) &\leq H_{p_m}(\Psi(\tilde{U_1}), \Phi\left(\mathcal{J}_{2a+1}^*\right)) + H_{p_m}(\Phi\left(\mathcal{J}_{2a+1}^*\right), \tilde{U_1}) \\
&\quad - \inf_{\alpha_{2a+1}\in\mathcal{J}_{2a+1}^*} p_m(\alpha_{2a+1}, \alpha_{2a+1}), \\
&\leq \lambda_* \mathcal{R}_{\Psi,\Phi}(\tilde{U_1}, \mathcal{J}_{2a+1}^*) + H_{p_m}(\mathcal{J}_{2a+2}^*, \tilde{U_1}) \\
&\quad - \inf_{\alpha_{2a+1}\in\mathcal{J}_{2a+1}^*} p_m(\alpha_{2a+1}, \alpha_{2a+1}),
\end{aligned}$$

for all $a \in \{0, 1, 2, ...\}$, where

$$\begin{aligned}
\mathcal{R}_{\Psi,\Phi}(\tilde{U_1}, \mathcal{J}_{2a+1}^*) &= \max\left\{ \frac{H_{p_m}(\tilde{U_1}, \Phi(\mathcal{J}_{2a+1}^*))[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{2(1 + H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*))}, \right. \\
&\quad \frac{H_{p_m}(\mathcal{J}_{2a+1}^*, \Phi\left(\mathcal{J}_{2a+1}^*\right))[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*)}, \\
&\quad \left. \frac{H_{p_m}(\mathcal{J}_{2a+1}^*, \tilde{U_1})[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*)} \right\},
\end{aligned}$$

that is,

$$\begin{aligned}
\mathcal{R}_{\Psi,\Phi}(\tilde{U_1}, \mathcal{J}_{2a+1}^*) &= \max\left\{ \frac{H_{p_m}(\mathcal{J}_{2a+2}^*, \tilde{U_1})[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{2(1 + H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*))}, \right. \\
&\quad \frac{H_{p_m}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*)[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*)}, \\
&\quad \left. \frac{H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*)[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*)} \right\}.
\end{aligned}$$

Consider the following three cases:

(1) If $\mathcal{R}_{\Psi,\Phi}(\tilde{U_1}, \mathcal{J}_{2a+1}^*) = \dfrac{H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+2}^*)[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{2(1 + H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*))}$, then we have

$$\begin{aligned}
H_{p_m}(\Psi(\tilde{U_1}), \tilde{U_1}) &\leq \frac{\lambda_* H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+2}^*)[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{2(1 + H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a+1}^*))} + H_{p_m}(\mathcal{J}_{2a+2}^*, \tilde{U_1}) \\
&\quad - \inf_{\alpha_{2a+1}\in\mathcal{J}_{2a+1}^*} p_m(\alpha_{2a+1}, \alpha_{2a+1}),
\end{aligned}$$

which together with our taking the limit as $a \to +\infty$, gives $H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq 0$ and so $\tilde{U_1} = \Psi(\tilde{U_1})$.

(2) If $\mathcal{R}_{\Psi,\Phi}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) = \dfrac{H_{p_m}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2})[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})}$, we have

$$\begin{aligned} H_{p_m}(\Psi(\tilde{U_1}), \tilde{U_1}) &\leq \lambda_* \frac{H_{p_m}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2})[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})} + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}) \\ &\quad - \inf_{\alpha_{2a+1} \in \mathcal{J}^*_{2a+1}} p(\alpha_{2a+1}, \alpha_{2a+1}) \\ &\leq \lambda_* \frac{H_{p_m}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2})[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})} + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1})), \end{aligned}$$

which together with our taking the limit as $a \to +\infty$, yields $H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) \leq 0$ and thus $\tilde{U_1} = \Psi(\tilde{U_1})$.
(3) In case of $\mathcal{R}_{\Psi,\Phi}(U_1, \mathcal{J}^*_{2a+1}) = \dfrac{H_{p_m}(\mathcal{J}^*_{2a+1}, \tilde{U_1})[1 + H_{p_m}(\tilde{U_1}, \Psi(U_1))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})}$, we obtain

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1})) &\leq \lambda_* \frac{H_{p_m}(\mathcal{J}^*_{2a+1}, \tilde{U_1})[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})} + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}) \\ &\quad - \inf_{\alpha_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(\alpha_{2a+1}, \alpha_{2a+1}) \\ &\leq \frac{\lambda_* H_{p_m}(\mathcal{J}^*_{2a+1}, \tilde{U_1})[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1})} + H_{p_m}(\mathcal{J}^*_{2a+2}, \tilde{U_1}), \end{aligned}$$

which together with our taking the limit as $a \to +\infty$, produce

$$H_{p_m}(\tilde{U_1}, \Psi\tilde{U_1})),$$

that is, $\tilde{U_1} = \Psi(\tilde{U_1})$.
In a similar manner, one can obtain

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) &\leq H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) + H_{p_m}(\mathcal{J}^*_{2a+1}, \Phi(\tilde{U_1})) \\ &\quad - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &= H_{p_m}(\tilde{U_1}, \mathcal{J}^*_{2a+1}) + H_{p_m}(\Psi(\mathcal{J}^*_{2a}), \Phi\left(\tilde{U_1}\right)) \\ &\quad - \inf_{m_{2a+1} \in \mathcal{J}^*_{2a+1}} p_m(m_{2a+1}, m_{2a+1}) \\ &\leq H_p(\tilde{U_1}, \mathcal{J}^*_{2a+1}) + \lambda_* \mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*_{2a}, \Phi(\tilde{U_1})), \end{aligned}$$

where

$$\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*_{2a}, \tilde{U_1}) = \max \left\{ \frac{H_{p_m}(\mathcal{J}^*_{2a}, \Phi(\tilde{U_1}))[1 + H_p(\mathcal{J}^*_{2a}, \Psi(\mathcal{J}^*_{2a}))]}{2(1 + H_{p_m}(\mathcal{J}^*_{2a}, \tilde{U_1}))}, \right.$$

$$
\begin{aligned}
&\frac{H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1}))[1+H_{p_m}(\mathcal{J}^*_{2a},\Psi(\mathcal{J}^*_{2a}))]}{1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1})},\\
&\left.\frac{H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1}))[1+H_{p_m}(\mathcal{J}^*_{2a},\Psi(\mathcal{J}^*_{2a}))]}{1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1})}\right\}\\
= \quad & \max\left\{\frac{H_{p_m}(\mathcal{J}^*_{2a},\Phi(\tilde{U_1}))[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{2(1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1}))}\right.,\\
&\frac{H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1}))[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1})},\\
&\left.\frac{H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1})[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1})}\right\}.
\end{aligned}
$$

Again, we have the following three cases:

(1) If $\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*_{2a},\tilde{U_1}) = \dfrac{H_{p_m}(\mathcal{J}^*_{2a},\Phi(\tilde{U_1}))[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{2(1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1}))}$, then

$$
H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1})) \le H_{p_m}(\tilde{U_1},\mathcal{J}^*_{2a+1})+\lambda_*\left\{\frac{H_{p_m}(\mathcal{J}^*_{2a},\Phi(\tilde{U_1}))[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{2(1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1}))}\right\}.
$$

Which together with our taking the limit as $a \to +\infty$, we get

$$
H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1})) \le H_{p_m}(\tilde{U_1},\tilde{U_1}) + \frac{\lambda_*}{2}\left\{\frac{H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1}))[1+H_{p_m}(\tilde{U_1},\tilde{U_1})]}{(1+H_{p_m}(\tilde{U_1},\tilde{U_1}))}\right\},
$$

that is,

$$
\left(1-\frac{\lambda_*}{2}\right)H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1})) \le 0,
$$

thus, $\tilde{U_1} = \Phi(\tilde{U_1})$.

(2) If $R_{\Psi,\Phi}(\mathcal{J}^*_{2a},\tilde{U_1}) = \dfrac{H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1}))[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1})}$, then

$$
H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1})) \le H_{p_m}(\tilde{U_1},\mathcal{J}^*_{2a+1})+\lambda_*\left\{\frac{H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1}))[1+H_{p_m}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{p_m}(\mathcal{J}^*_{2a},\tilde{U_1})}\right\},
$$

which together with our taking the limit as $a \to +\infty$, we get

$$
(1-\lambda_*)H_{p_m}(\tilde{U_1},\Phi(\tilde{U_1})) \le 0,
$$

which implies that $\tilde{U_1} = \Phi(\tilde{U_1})$.

(3) If $\mathcal{R}_{\Psi,\Phi}(\mathcal{J}_{2a}^{*}, \tilde{U_1}) = \dfrac{H_{p_m}(\mathcal{J}_{2a}^{*}, \tilde{U_1})[1 + H_{p_m}(\mathcal{J}_{2a}^{*}, \mathcal{J}_{2a+1}^{*})]}{1 + H_{p_m}(\mathcal{J}_{2a}^{*}, \tilde{U_1})}$ then

$$H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_1})) \leq H_{p_m}(\tilde{U_1}, \mathcal{J}_{2a}^{*}) + \lambda_* \left\{ \frac{H_{p_m}(\mathcal{J}_{2a}^{*}, \tilde{U_1})[1 + H_{p_m}(\mathcal{J}_{2a}^{*}, \mathcal{J}_{2a+1}^{*})]}{1 + H_{p_m}(\mathcal{J}_{2a}^{*}, \tilde{U_1})} \right\},$$

which together with our taking the limit as $a \to +\infty$, we get $H_{p_m}(\tilde{U_1}, \Phi\left(\tilde{U_1}\right)) \leq 0$, which gives $\tilde{U_1} = \Phi(\tilde{U_1})$.

Thus $\tilde{U_1}$ is a common attractor of the mappings Ψ and Φ.

For the uniqueness, assume that $\tilde{U_1}$ and $\tilde{U_2}$ are distinct common attractors of Ψ and Φ. Since the pair (Ψ, Φ) is generalized rational Hutchinson contractive operator, we obtain that

$$\begin{aligned} H_{p_m}(\tilde{U_1}, \tilde{U_2}) &= H_{p_m}(\Psi(\tilde{U_1}), \Phi(\tilde{U_2})) \\ &\leq \lambda_* \max \left\{ \frac{H_{p_m}(\tilde{U_1}, \Phi(\tilde{U_2}))[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{2(1 + H_{p_m}(\tilde{U_1}, \tilde{U_2}))}, \right. \\ &\quad \frac{H_{p_m}(\tilde{U_2}, \Phi(\tilde{U_2}))[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \tilde{U_2})}, \\ &\quad \left. \frac{H_{p_m}(\tilde{U_1}, \tilde{U_2})[1 + H_{p_m}(\tilde{U_1}, \Psi(\tilde{U_1}))]}{1 + H_{p_m}(\tilde{U_1}, \tilde{U_2})} \right\} \\ &= \lambda_* \max \left\{ \frac{H_{p_m}(\tilde{U_1}, \tilde{U_2})[1 + H_{p_m}(\tilde{U_1}, \tilde{U_1})]}{2(1 + H_{p_m}(\tilde{U_1}, \tilde{U_2}))}, \right. \\ &\quad \left. \frac{H_{p_m}(\tilde{U_2}, \tilde{U_2})[1 + H_{p_m}(\tilde{U_1}, \tilde{U_1})]}{1 + H_{p_m}(\tilde{U_1}, \tilde{U_2})}, \frac{H_{p_m}(\tilde{U_1}, \tilde{U_2})[1 + H_{p_m}(\tilde{U_1}, \tilde{U_1})]}{1 + H_{p_m}(\tilde{U_1}, \tilde{U_2})} \right\} \\ &\leq \lambda_* H_{p_m}(\tilde{U_1}, \tilde{U_2}), \end{aligned}$$

and so $(1 - \lambda_*)H_{p_m}(\tilde{U_1}, \tilde{U_2}) \leq 0$, which implies that $H_{p_m}(\tilde{U_1}, \tilde{U_2}) = 0$ and hence $\tilde{U_1} = \tilde{U_2}$. Thus $\tilde{U_1} \in \mathcal{C}^{p_m}(W)$ is a unique common attractor of Ψ and Φ. □

Corollary 3.3.3. *Consider a generalized iterated function system* $\{W; h_a, g_a, a = 1, 2, \cdots, q\}$ *on a complete partial metric space* (W, p_m) *and the mappings* $h, g : W \to W$ *as given in Remark* 3.3.1. *If there exists* $\lambda_* \in [0, 1)$ *such that for any* $\varrho_1, \varrho_2 \in W$*, the following condition holds:*

$$p_m(h\varrho_1, g\varrho_2) \leq \lambda_* \mathcal{R}_{h,g}(\varrho_1, \varrho_2),$$

where

$$\mathcal{R}_{h,g}(\varrho_1,\varrho_2) = \max\left\{\frac{p_m(\varrho_1,g\varrho_2)[1+p_m(\varrho_1,h\varrho_1)]}{2(1+p_m(\varrho_1,\varrho_2))}, \frac{p_m(\varrho_2,g\varrho_2)[1+p_m(\varrho_1,h\varrho_1)]}{1+p_m(\varrho_1,\varrho_2)}, \frac{p_m(\varrho_1,\varrho_2)[1+p_m(\varrho_1,h\varrho_1)]}{1+p_m(\varrho_1,\varrho_2)}\right\}.$$

Then a unique common fixed point for h and g exists. Furthermore, for any initial choice of $v_0 \in W$, the sequence $\{v_0, hv_0, ghv_0, hghv_0, ...\}$ converges to the common fixed point of h and g.

3.4. Well-posedness of common attractor based problems

Now, in the framework of Hausdorff partial metric spaces, we investigate the well-posedness of attractor-based problems of generalized Hutchinson contractive operators pair and generalized rational Hutchinson contractive operators pair given in Definition 3.2.2. [56] contains some useful results on the well-posedness of fixed-point problems.

We begin by defining the well-posedness of the common attractor-based problem.

Definition 3.4.1. For a pair of mappings $\Psi, \Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$, a common attractor-based problem is said to be well-posed if the pair (Ψ, Φ) has a unique common attractor $\Theta_* \in \mathcal{C}^{p_m}(W)$ and for any sequence $\{\Theta_a\}$ in $\mathcal{C}^{p_m}(W)$ such that $\lim_{a\to+\infty} H_{p_m}(\Psi(\Theta_a), \Theta_a) = 0$ and $\lim_{a\to+\infty} H_{p_m}(\Phi(\Theta_a), \Theta_a) = 0$, then $\lim_{a\to+\infty} H_{p_m}(\Theta_a, \Theta_*) = H_{p_m}(\Theta_*, \Theta_*)$, that is, $\lim_{a\to+\infty} \Theta_a = \Theta_*$.

The following result demonstrates the well-posedness of a generalized Hutchinson contractive operators' common attractor-based problem.

Theorem 3.4.1. *Suppose (W, p_m) is a complete partial metric space and define $\Psi, \Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ as in Theorem 3.3.1. The pair (Ψ, Φ) , then has a well-posed common attractor-based problem.*

Proof. According to Theorem 3.3.1, it follows that the mappings Ψ and Φ have a unique common attractor, $\mathcal{Z}_*$.

Let a sequence $\{\mathcal{Z}_a\}$ in $\mathcal{C}^{p_m}(W)$ be such that $\lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$ and $\lim_{a\to+\infty} H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$. We want to show that $\mathcal{Z}_* = \lim_{a\to+\infty} \mathcal{Z}_a$. As the pair

(Ψ, Φ) is generalized Hutchinson contractive operator, so that

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq H_{p_m}(\Psi(\mathcal{Z}_*), \Phi(\mathcal{Z}_a)) + H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &\leq \lambda \mathcal{S}_{\Psi,\Phi}(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

where

$$\begin{aligned} \mathcal{S}_{\Psi,\Phi}(\mathcal{Z}_*, \mathcal{Z}_a) &= \max\{H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a), H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_*)), H_{p_m}(\mathcal{Z}_a, \Phi(\mathcal{Z}_a)), \\ &\quad \frac{H_{p_m}(\mathcal{Z}_*, \Phi(\mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_*))}{2}\}. \end{aligned}$$

Then we have the following cases:

(i) If $\mathcal{S}_{\Psi,\Phi}(\mathcal{Z}_*, \mathcal{Z}_a) = H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a)$, then

$$H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \lambda H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a),$$

which further implies

$$(1-\lambda)\, H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a),$$

that is,

$$H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \frac{1}{1-\lambda}[H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)].$$

As $a \to +\infty$, we have

$$\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \frac{1}{1-\lambda} \lim_{a\to+\infty}[H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)],$$

this implies that $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(ii) In case of $\mathcal{S}_{\Psi,\Phi}(\mathcal{Z}_*, \mathcal{Z}_a) = H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_*))$, we have

$$H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \lambda H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_*)) + H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a).$$

As $a \to +\infty$, we have

$$\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \lambda H_{p_m}(\mathcal{Z}_*, \Psi(\mathcal{Z}_*)) + \lim_{a\to+\infty}[H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)].$$

Thus $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(iii) If $\mathcal{S}_{\Psi,\Phi}(\mathcal{Z}_*, \mathcal{Z}_a) = H_{p_m}(\mathcal{Z}_a, \Phi(\mathcal{Z}_a))$, then

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \lambda H_{p_m}(\mathcal{Z}_a, \Phi(\mathcal{Z}_a)) + H_{p_m}(\mathcal{Z}_a, \Phi(\mathcal{Z}_a)) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &= (\lambda+1) H_{p_m}(\mathcal{Z}_a, \Phi(\mathcal{Z}_a)) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \end{aligned}$$

On taking the limit as $a \to +\infty$, we have that $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(iv) Finally, if $\mathcal{S}_{\Psi,\Phi}(\mathcal{Z}_*, \mathcal{Z}_a) = \dfrac{H_{p_m}(\mathcal{Z}_*, \Phi(\mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_*))}{2}$, then we have

$$\begin{aligned} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) &\leq \frac{\lambda}{2}[H_{p_m}(\mathcal{Z}_*, \Phi(\mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_*))] \\ &\quad + H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &\leq \frac{\lambda}{2}[H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Phi(\mathcal{Z}_a)) + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_*)) \\ &\quad - \inf_{b_a \in \mathcal{Z}_a} p_m(b_a, b_a)] + H_{p_m}(\Phi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Phi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &= \frac{\lambda}{2}[H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) + H_{p_m}(\mathcal{Z}_a, \Phi(\mathcal{Z}_a)) + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*) \\ &\quad - \inf_{b_a \in \mathcal{Z}_a} p_m(b_a, b_a)] + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

which gives

$$H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq \frac{\lambda+2}{2(1-\lambda)}[H_{p_m}(\mathcal{Z}_a, \Phi(\mathcal{Z}_a)) - \inf_{b_a \in \mathcal{Z}_a} p_m(b_a, b_a)] - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a)]$$

and by taking the limit as $a \to +\infty$, we obtain

$$\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a) \leq 0,$$

which implies that $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

□

With the result below, we show the well-posedness of a common attractor-based problem of a generalized rational Hutchinson contractive operators.

Theorem 3.4.2. *Consider a complete partial metric space* (W, p_m) *with* $\Psi, \Phi : \mathcal{C}^{p_m}(W) \to \mathcal{C}^{p_m}(W)$ *defined as in Theorem* 3.3.3. *Then the pair* (Ψ, Φ) *has a well-posed common attractor-based problem.*

Proof. From Theorem 3.3.3, it follows that the mappings Ψ and Φ have a unique common attractor (say) $\mathcal{Z}_*$.

Let a sequence $\{\mathcal{Z}_a\}$ in $\mathcal{C}^{p_m}(W)$ be such that $\lim_{a\to+\infty} H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$ and

$\lim_{a\to+\infty} H_{p_m}(\Phi(\mathcal{Z}_a),\mathcal{Z}_a) = 0$. We want to show that $\mathcal{Z}_* = \lim_{a\to+\infty} \mathcal{Z}_a$. As the pair (Ψ,Φ) is generalized rational Hutchinson contractive operator, so that

$$\begin{aligned} H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*) &\leq H_{p_m}(\Psi(\mathcal{Z}_a),\Psi(\mathcal{Z}_*)) + H_{p_m}(\Psi(\mathcal{Z}_a),\mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a,\beta_a) \\ &\leq \lambda_*\mathcal{R}_{\Psi,\Phi}(\mathcal{Z}_a,\mathcal{Z}_*) + H_{p_m}(\Psi(\mathcal{Z}_a),\mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a,\beta_a), \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_{\Psi,\Phi}(\mathcal{Z}_a,\mathcal{Z}_*) = \max\Bigg\{ &\frac{H_{p_m}(\mathcal{Z}_a,\Phi(\mathcal{Z}_*))[1+H_{p_m}(\mathcal{Z}_a,\Psi(\mathcal{Z}_a))]}{2(1+H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*))}, \\ &\frac{H_{p_m}(\mathcal{Z}_*,\Phi(\mathcal{Z}_*))[1+H_{p_m}(\mathcal{Z}_a,\Psi(\mathcal{Z}_a))]}{1+H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*)}, \\ &\frac{H_{p_m}(\mathcal{Z}_*,\mathcal{Z}_a)[1+H_{p_m}(\mathcal{Z}_a,\Psi(\mathcal{Z}_a))]}{1+H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*)}\Bigg\}. \end{aligned}$$

The following cases arise:

(i) $\mathcal{R}_{\Psi,\Phi}(\mathcal{Z}_a,\mathcal{Z}_*) = \dfrac{H_{p_m}(\mathcal{Z}_a,\Phi(\mathcal{Z}_*))[1+H_{p_m}(\mathcal{Z}_a,\Psi(\mathcal{Z}_a))]}{2(1+H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*))}$, implies that,

$$\begin{aligned} H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*) &\leq \lambda_*\frac{H_{p_m}(\mathcal{Z}_a,\Phi(\mathcal{Z}_*))[1+H_{p_m}(\mathcal{Z}_a,\Psi(\mathcal{Z}_a))]}{2(1+H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*))} \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a),\mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p(\beta_a,\beta_a) \\ &\leq \lambda_* H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*)[1+H_{p_m}(\mathcal{Z}_a,\Psi(\mathcal{Z}_a))] \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a),\mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a,\beta_a), \end{aligned}$$

which leads to,

$$\begin{aligned} &H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*) - \lambda_* H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*)[1+H_{p_m}(\Psi(\mathcal{Z}_a),\mathcal{Z}_a)] \\ &\leq H_{p_m}(\Psi(\mathcal{Z}_a),\mathcal{Z}_a) - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a,\beta_a) \end{aligned}$$

and so

$$\begin{aligned} H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*) \leq\ & \frac{1}{1-\lambda_*[1+H_{p_m}(\Psi(\mathcal{Z}_a),\mathcal{Z}_a)]}[H_{p_m}(\Psi(\mathcal{Z}_a),\mathcal{Z}_a) \\ & - \inf_{\beta_a\in\Psi(\mathcal{Z}_a)} p_m(\beta_a,\beta_a)]. \end{aligned}$$

And taking the limit as $a\to+\infty$ gives,

$$\lim_{a\to+\infty} H_{p_m}(\mathcal{Z}_a,\mathcal{Z}_*) \leq 0,$$

which implies that $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(ii) If $\mathcal{R}_{\Psi,\Phi}(\mathcal{Z}_a, \mathcal{Z}_*) = \dfrac{H_{p_m}(\mathcal{Z}_*, \Phi(\mathcal{Z}_*))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)}$, then

$$\begin{aligned} H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*) &\leq \lambda_* \left(\frac{H_{p_m}(\mathcal{Z}_*, \Phi(\mathcal{Z}_*))[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)} \right) \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a) \\ &= H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a). \end{aligned}$$

Taking the limit as $a \to +\infty$, we have

$$\lim_{a \to +\infty} H_{p_m}(\mathcal{Z}_a, \mathcal{Z}B_*) \leq 0,$$

which implies that $\lim\limits_{a \to +\infty} \mathcal{Z}_a = \mathcal{Z}_*$.

(iii) Finally, assume $\mathcal{R}_{\Psi,\Phi}(\mathcal{Z}_a, \mathcal{Z}_*) = \dfrac{H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a)[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)}$, then

$$\begin{aligned} H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*) &\leq \lambda_* \frac{H_{p_m}(\mathcal{Z}_*, \mathcal{Z}_a)[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)} \\ &\quad + H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a), \end{aligned}$$

that is,

$$\begin{aligned} & H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)[1 - \lambda_* \frac{[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]}{1 + H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)}] \\ \leq\ & H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p(\beta_a, \beta_a), \end{aligned}$$

which further implies

$$\begin{aligned} & H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*)[1 - \lambda_*[1 + H_{p_m}(\mathcal{Z}_a, \Psi(\mathcal{Z}_a))]] \\ \leq\ & H_{p_m}(\Psi(\mathcal{Z}_a), \mathcal{Z}_a) - \inf_{\beta_a \in \Psi(\mathcal{Z}_a)} p_m(\beta_a, \beta_a). \end{aligned}$$

On taking the limit as $a \to +\infty$, gives $\lim\limits_{a \to +\infty} H_{p_m}(\mathcal{Z}_a, \mathcal{Z}_*) \leq 0$ implies that $\lim\limits_{a \to +\infty} \mathcal{Z}_a = \mathcal{Z}_*$. Thus the proof is complete.

□

3.5. Application to dynamic programming problems

In this section, we apply our obtained results to solve functional equations arising in dynamic programming.

Let W_1 and W_2 be two Banach spaces with $\mathcal{J}^* \subseteq W_1$ and $\mathcal{O}^* \subseteq W_2$. Suppose that

$$\kappa \colon \mathcal{J}^* \times \mathcal{O}^* \longrightarrow \mathcal{J}^*, \quad g_1, g_2 \colon \mathcal{J}^* \times \mathcal{O}^* \longrightarrow \mathbb{R}, \quad h_1, h_2 \colon \mathcal{J}^* \times \mathcal{O}^* \times \mathbb{R} \longrightarrow \mathbb{R}.$$

If we consider $\mathcal{J}^*$ and $\mathcal{O}^*$ as the state and decision spaces respectively, then the problem of dynamic programming reduces to the problem of solving the functional equations (see [92]):

$$q_1(\varrho_1) = \sup_{\varsigma \in \mathcal{O}^*} \{g_1(\varrho_1, \varsigma) + h_1(\varrho_1, \varsigma, q_1(\kappa(\varrho_1, \varsigma)))\}, \text{ for } \varrho_1 \in \mathcal{J}^* \tag{3.1}$$

$$q_2(\varrho_1) = \sup_{\varsigma \in \mathcal{O}^*} \{g_1(\varrho_1, \varsigma) + h_2(\varrho_1, \varsigma, q_2(\kappa(\varrho_1, \varsigma)))\}, \text{ for } \varrho_1 \in \mathcal{J}^*. \tag{3.2}$$

Reformulating (3.1) and (3.2), gives

$$q_1(\varrho_1) = \sup_{\varsigma \in \mathcal{O}^*} \{g_2(\varrho_1, \varsigma) + h_1(\varrho_1, \varsigma, q_1(\kappa(\varrho_1, \varsigma)))\} - b, \text{ for } \varrho_1 \in \mathcal{J}^* \tag{3.3}$$

$$q_2(\varrho_1) = \sup_{\varsigma \in \mathcal{O}^*} \{g_2(\varrho_1, \varsigma) + h_2(\varrho_1, \varsigma, q_2(\kappa(\varrho_1, \varsigma)))\} - b, \text{ for } \varrho_1 \in \mathcal{J}^*, \tag{3.4}$$

where $b > 0$.

We study the existence and uniqueness of the bounded solution of the functional equations (3.3) and (3.4) arising in dynamic programming in the setup of partial metric spaces.

Let $\tilde{B}(\mathcal{J}^*)$ denote the set of all bounded real valued functions on $\mathcal{J}^*$. For an arbitrary $\eta^* \in \tilde{B}(\mathcal{J}^*)$, define $\|\eta^*\| = \sup_{t \in \mathcal{J}^*} |\eta^*(t)|$. Then $(\tilde{B}(\mathcal{J}^*), \|\cdot\|)$ is a Banach space. Now consider

$$p_{\tilde{B}}(\eta^*, \xi^*) = \sup_{t \in \mathcal{J}^*} |\eta^*(t) - \xi^*(t)| + b,$$

where $\eta^*, \xi^* \in \tilde{B}(\mathcal{J}^*)$. Then $p_{\tilde{B}}$ is a partial metric on $\tilde{B}(\mathcal{J}^*)$ (see also [4]).

Assume that:

(D_1) : g_1, g_2, h_1 and h_2 are bounded and continuous.

(D_2) : For $\varrho_1 \in \mathcal{J}^*$, $\eta^* \in \tilde{B}(\mathcal{J}^*)$ and $b > 0$, take $\Psi, \Phi : \tilde{B}(\mathcal{J}^*) \to \tilde{B}(\mathcal{J}^*)$ as

$$\Psi\eta^*(\varrho_1) = \sup_{\varsigma\in\mathcal{O}^*}\{g_2(\varrho_1,\varsigma) + h_1(\varrho_1,\varsigma,\eta^*(\kappa(\varrho_1,\varsigma)))\} - b, \text{ for } \varrho_1 \in \mathcal{J}^*, \tag{3.5}$$

$$\Phi\eta^*(\varrho_1) = \sup_{\varsigma\in\mathcal{O}^*}\{g_2(\varrho_1,\varsigma) + h_2(\varrho_1,\varsigma,\eta^*(\kappa(\varrho_1,\varsigma)))\} - b, \text{ for } \varrho_1 \in \mathcal{J}^*. \tag{3.6}$$

Moreover, for every $(\varrho_1, \varsigma) \in \mathcal{J}^* \times \mathcal{O}^*$, $\eta^*, \xi^* \in \tilde{B}(\mathcal{J}^*)$ and $t \in \mathcal{J}^*$ implies

$$|h_1(\varrho_1,\varsigma,\eta^*(t)) - h_2(\varrho_1,\varsigma,\xi^*(t))| \leq \lambda\mathcal{S}_{\Psi,\Phi}(\eta^*(t), \xi^*(t)) - 2b, \tag{3.7}$$

where

$$\begin{aligned}\mathcal{S}_{\Psi,\Phi}(\eta^*(t), \xi^*(t)) \;=\; & \max\Big\{p_{\tilde{B}}(\eta^*(t), \xi^*(t)), p_{\tilde{B}}(\eta^*(t), \Psi\eta^*(t)), p_{\tilde{B}}(\xi^*(t), \Phi\xi^*(t)), \\ & \frac{p_{\tilde{B}}(\eta^*(t), \Phi\xi^*(t)) + p_{\tilde{B}}(\xi^*(t), \Psi\eta^*(t))}{2}\Big\}.\end{aligned}$$

Theorem 3.5.1. *Assume that the conditions (D_1) and (D_2) hold. Then, the functional Equations (3.3) and (3.4) have a unique common and bounded solution in $\tilde{B}(\mathcal{J}^*)$.*

Proof. Note that $(\tilde{B}(\mathcal{J}^*), p_{\tilde{B}})$ is a complete partial metric space. By (D_1), Ψ and Φ are self-mappings of $\tilde{B}(\mathcal{J}^*)$. By (3.5) and (3.6) in (D_2), it follows that for any $\eta^*, \xi^* \in \tilde{B}(\mathcal{J}^*)$ and $b > 0$, choose $\varrho_1 \in \mathcal{J}^*$ and $\varsigma_1, \varsigma_2 \in \mathcal{O}^*$ such that

$$\Psi\eta^* < g_2(\varrho_1,\varsigma_1) + h_1(\varrho_1,\varsigma_1,\eta^*(\kappa(\varrho_1,\varsigma_1))), \tag{3.8}$$

$$\Phi\xi^* < g_2(\varrho_1,\varsigma_2) + h_2(\varrho_1,\varsigma_2,\xi^*(\kappa(\varrho_1,\varsigma_2))), \tag{3.9}$$

which further implies that

$$\Psi\eta^* \geq g_2(\varrho_1,\varsigma_2) + h_1(\varrho_1,\varsigma_2,\eta^*(\kappa(\varrho_1,\varsigma_2))) - b, \tag{3.10}$$

$$\Phi\xi^* \geq g_2(\varrho_1,\varsigma_1) + h_2(\varrho_1,\varsigma_1,\xi^*(\kappa(\varrho_1,\varsigma_1))) - b. \tag{3.11}$$

From (3.8) and (3.11) together with (3.7) implies

$$\begin{aligned}\Psi\eta^*(t) - \Phi\xi^*(t) \;&<\; h_1(\varrho_1,\varsigma_1,\eta^*(\kappa(\varrho_1,\varsigma_1))) - h_2(\varrho_1,\varsigma_1,\xi^*(\kappa(\varrho_1,\varsigma_1))) + b \\ &\leq\; |h_1(\varrho_1,\varsigma_1,\eta^*(\kappa(\varrho_1,\varsigma_1))) - h_2(\varrho_1,\varsigma_1,\xi^*(\kappa(\varrho_1,\varsigma_1)))| + b \\ &\leq\; \lambda\mathcal{S}_{\Psi,\Phi}(\eta^*(t), \xi^*(t)) - b. \end{aligned} \tag{3.12}$$

From (3.9) and (3.10) together with (3.7) implies

$$\begin{aligned}\Phi\xi^{*}(t)-\Psi\eta^{*}(t) &< h_2(\varrho_1,\varsigma_2,\xi^{*}(\kappa(\varrho_1,\varsigma_2)))-h_1(\varrho_1,\varsigma_2,\eta^{*}(\kappa(\varrho_1,\varsigma_2)))+b\\ &\leq |h_1(\varrho_1,\varsigma_2,\eta^{*}(\kappa(\varrho_1,\varsigma_2)))-h_2(\varrho_1,\varsigma_2,\xi^{*}(\kappa(\varrho_1,\varsigma_2)))|+b\\ &\leq \lambda\mathcal{S}_{\Psi,\Phi}(\eta^{*}(t),\xi^{*}(t)-b. \qquad (3.13)\end{aligned}$$

From (3.12) and (3.13), we get

$$|\Psi\eta^{*}(t)-\Phi\xi^{*}(t)|+b\leq\lambda\mathcal{S}_{\Psi,\Phi}(\eta^{*}(t),\xi^{*}(t)). \qquad (3.14)$$

The inequality (3.14) implies that

$$p_{\tilde{B}}(\Psi\eta^{*}(t),\Phi\xi^{*}(t))\leq\lambda\mathcal{S}_{\Psi,\Phi}(\eta^{*}(t),\xi^{*}(t)), \qquad (3.15)$$

where

$$\begin{aligned}\mathcal{S}_{\Psi,\Phi}(\eta^{*}(t),\xi^{*}(t)) = &\max\{p_{\tilde{B}}(\eta^{*}(t),\xi^{*}(t)),p_{\tilde{B}}(\eta^{*}(t),\Psi\eta^{*}(t)),p_{B}(\xi^{*}(t),\Phi\xi^{*}(t)),\\ &\left.\frac{p_{\tilde{B}}(\eta^{*}(t),\Phi\xi^{*}(t))+p_{\tilde{B}}(\xi^{*}(t),\Psi\eta^{*}(t))}{2}\right\}.\end{aligned}$$

Therefore, all conditions of Corollary 3.3.1 hold. Thus, there exists a common fixed point of Ψ and Φ, that is, $\eta^{*}\in\tilde{B}(\mathcal{J}^{*})$, where $\eta^{*}(t)$ is a common solution of functional equations (3.3) and (3.4). □

4

Iterated Function System of Generalized Cyclic Contractions in Partial Metric Spaces

4.1. Introduction

We construct fractal sets of the generalized iterated function systems based on cyclic contractive operators in the framework of partial metric spaces. We notice that the Hutchinson operator, defined on a finite collection of cyclic contraction mappings in a complete partial metric space, is a generalized contractive mapping on a class of compact subsets of a given set W. We apply a generalized Hutchinson operator successively to obtain a final fractal. We conclude this chapter by discussing two applications of our results in Sections 4.5 and 4.6.

We extend the introductory concepts covered in Section 2.1 to the study of iterated function systems of generalized cyclic contractions.

4.2. Cyclic Contractive Mappings

In this section we introduce the notion of cyclic contraction mappings, which need not to be continuous, a key advantage over Banach based contractions [82].

Definition 4.2.1. [55] Consider two non-void subsets $\mathcal{J}^*$ and $\mathcal{O}^*$ of W. $\hbar : W \to W$ is said to be a cyclic mapping if $\hbar(\mathcal{J}^*) \subset \mathcal{O}^*$ and $\hbar(\mathcal{O}^*) \subset \mathcal{J}^*$.

Definition 4.2.2. [8, 50, 51] Let W be a non-void set and $\hbar : W \to W$ a self-map. $W = \cup_{a=1}^{q} W_a$ is a cyclic representation of W relative to $\hbar$ if

- c_1) all the sets W_a, $a = 1, 2, \cdots, q$ are non-void,
- c_2) $\hbar(W_1) \subset W_2, \cdots, \hbar(W_{q-1}) \subset W_q$ and $\hbar(W_q) \subset W_1$.

Definition 4.2.3. Let (W, p_m) be a complete partial metric space and $\{\mathcal{B}_a\}_{a=1}^{q}$ a class of non-void closed subsets of W. A self-map $\hbar : \cup_{a=1}^{q}\mathcal{B}_a \to \cup_{a=1}^{q}\mathcal{B}_a$ is a cyclic contraction on $\{\mathcal{B}_a\}_{a=1}^{q}$ if

(a) $\hbar(\mathcal{B}_a) \subseteq \mathcal{B}_{a+1}$ for $a = 1, 2, \cdots, q$, where $\mathcal{B}_{q+1} = \mathcal{B}_1$,

(b) $p_m(\hbar\mu, \hbar\eta) \leq \lambda p_m(\mu, \eta)$ for all $\mu \in \mathcal{B}_a, \eta \in \mathcal{B}_{a+1}$ with $a = 1, 2, \cdots, q$,

where $\lambda \in [0, 1)$. $\hbar$ is said to be a cyclic function, if condition (a) is satisfied.

Definition 4.2.4. In a complete partial metric space (W, p_m), we say, $\{W;\ \hbar_a, a = 1, 2, \cdots, q\}$ is a cyclic iterated function system if for $a = 1, 2, \cdots, q$, each $\hbar_a : W \to W$ is a cyclic contraction mapping.

Example 4.2.1. Let $W = [0, 2]$ be equipped with a partial metric $p_m : W \times W \to \mathbb{R}_{[+]}$ given by $p_m(\mu, \eta) = \max\{\mu, \eta\}$ for all $\mu, \eta \in W$. Let $\mathcal{B}_1 = [0, 1]$, $\mathcal{B}_2 = [0, 2]$ and define a map $\hbar : \mathcal{B}_1 \cup \mathcal{B}_2 \to \mathcal{B}_1 \cup \mathcal{B}_2$ by

$$\hbar(\mu) = \begin{cases} \frac{\mu}{3} & \text{if } 0 \leq \mu \leq 1 \\ \frac{1}{3} & \text{if } 1 < \mu \leq \frac{3}{2} \\ \frac{1}{5} & \text{if } \frac{3}{2} < \mu \leq 2. \end{cases}$$

Now

$$\begin{aligned} \hbar(\mathcal{B}_1) &= \left[0, \tfrac{1}{3}\right] \subseteq [0, 2] = \mathcal{B}_2 \text{ and} \\ \hbar(\mathcal{B}_2) &= \left[0, \tfrac{1}{5}\right] \subseteq [0, 1] = \mathcal{B}_1. \end{aligned}$$

The map $\hbar$ is not continuous at $\mu = \frac{3}{2}$. Consider,

Case 1: Let $\mu \in \mathcal{B}_1$, $\eta \in \mathcal{B}_2$, then

$\eta \in [0, 1]$, gives

$$\begin{aligned} p_m(\hbar(\mu), \hbar(\eta)) &= p_m(\tfrac{\mu}{3}, \tfrac{\eta}{3}) \\ &= \max\{\tfrac{\mu}{3}, \tfrac{\eta}{3}\} \\ &= \tfrac{1}{3}\max\{\mu, \eta\} \\ &= \tfrac{1}{3}p_m(\mu, \eta) \ \text{ with } \lambda = \tfrac{1}{3}, \end{aligned}$$

$\eta \in (1, \frac{3}{2}]$, gives

$$\begin{aligned} p_m(\hbar(\mu), \hbar(\eta)) &= p_m\left(\tfrac{\mu}{3}, \tfrac{1}{3}\right) \\ &= \max\left\{\tfrac{\mu}{3}, \tfrac{1}{3}\right\} \\ &= \tfrac{1}{3}\max\{\mu, 1\} \\ &\leq \tfrac{1}{3}\max\{\mu, \eta\} \\ &= \tfrac{1}{3}p_m(\mu, \eta) \quad \text{with} \quad \lambda = \tfrac{1}{3}, \end{aligned}$$

and $\eta \in \left(\frac{3}{2}, 2\right]$, gives

$$\begin{aligned} p_m(\hbar(\mu), \hbar(\eta)) &= p_m\left(\tfrac{\mu}{3}, \tfrac{1}{5}\right) \\ &= \max\left\{\tfrac{\mu}{3}, \tfrac{1}{5}\right\} \\ &= \tfrac{1}{3}\max\{\mu, \tfrac{3}{5}\} \\ &\leq \tfrac{1}{3}\max\{\mu, \eta\} \\ &= \tfrac{1}{3}p_m(\mu, \eta) \quad \text{with} \quad \lambda = \tfrac{1}{3}. \end{aligned}$$

Case 2: Let $\mu \in \mathcal{B}_1$, $\eta \in \mathcal{B}_2$, then

$\mu \in [0, 1]$, gives

$$\begin{aligned} p_m(\hbar(\mu), \hbar(\eta)) &= p_m\left(\tfrac{\mu}{3}, \tfrac{\eta}{3}\right) \\ &= \max\left\{\tfrac{\mu}{3}, \tfrac{\eta}{3}\right\} \\ &= \tfrac{1}{3}\max\{\mu, \eta\} \\ &= \tfrac{1}{3}p_m(\mu, \eta) \quad \text{with} \quad \lambda = \tfrac{1}{3}. \end{aligned}$$

$\mu \in \left(1, \frac{3}{2}\right]$, gives

$$\begin{aligned} p_m(\hbar(\mu), \hbar(\eta)) &= p_m\left(\tfrac{\mu}{3}, \tfrac{1}{3}\right) \\ &= \max\left\{\tfrac{\mu}{3}, \tfrac{1}{3}\right\} \\ &= \tfrac{1}{3}\max\{\mu, 1\} \\ &\leq \frac{1}{3}\max\{\mu, \eta\} \\ &= \tfrac{1}{3}p_m(\mu, \eta) \quad \text{with} \quad \lambda = \tfrac{1}{3}. \end{aligned}$$

$\eta \in \left(\frac{3}{2}, 2\right]$, gives

$$\begin{aligned} p_m\left(\hbar\left(\mu\right), \hbar\left(\eta\right)\right) &= p_m\left(\tfrac{1}{5}, \tfrac{\eta}{3}\right) \\ &= \max\left\{\tfrac{1}{5}, \tfrac{\eta}{3}\right\} \\ &= \tfrac{1}{3}\max\{\tfrac{3}{5}, \eta\} \\ &\leq \tfrac{1}{3}\max\{\mu, \eta\} \\ &= \tfrac{1}{3}p_m\left(\mu, \eta\right) \quad \text{with} \quad \lambda = \tfrac{1}{3}. \end{aligned}$$

Thus $\hbar$ is a cyclic contraction on $\mathcal{B}_1 \cup \mathcal{B}_2$ with contraction constant $\lambda = \frac{1}{3}$.

The result stated below confirms that in a complete partial metric space, a cyclic contraction mapping has a fixed point which is unique.

Proposition 4.2.1. [74] *In a complete partial metric space (W, p_m), let $\{\mathcal{B}_a\}_{a=1}^q$ be a class of non-void closed subsets of W. Suppose $\hbar : \{\mathcal{B}_a\}_{a=1}^q \to \{\mathcal{B}_a\}_{a=1}^q$ such that*

$$p_m\left(\hbar\mu, \hbar\eta\right) \leq \lambda \max\left\{p_m\left(\mu, \eta\right), p_m\left(\mu, \hbar\mu\right), p_m\left(\eta, \hbar\eta\right), \frac{p_m\left(\eta, \hbar\mu\right) + p_m\left(\mu, \hbar\eta\right)}{2}\right\}$$

for all $\mu \in \mathcal{B}_a, \eta \in \mathcal{B}_{a+1}$ with $a = 1, 2, \cdots, q$ and $\lambda \in [0, 1)$, then $\hbar$ has a unique fixed point.

Theorem 4.2.1. *In a partial metric space (W, p_m), let $\{\mathcal{B}_a\}_{a=1}^q$, be a class of non-void closed subsets of W, and $\hbar : \cup_{a=1}^q \mathcal{B}_a \to \cup_{a=1}^q \mathcal{B}_a$ be a continuous generalized cyclic contraction map. Then $\hbar : \cup_{a=1}^q \mathcal{C}^{p_m}\left(\mathcal{B}_a\right) \to \cup_{a=1}^q \mathcal{C}^{p_m}\left(\mathcal{B}_a\right)$ is also a generalized cyclic contraction mapping under the Hausdorff partial metric $p_m^{H_*}$ with the contractive constant given by $\lambda \in [0, 1)$.*

Proof. Let $\mathcal{O}^* \in \mathcal{B}_a$ for some $a = 1, 2, \cdots, q$. Using the definition of a cyclic map, we note that $\hbar(\mathcal{O}^*) \subseteq \mathcal{B}_{a+1}$ and since $\hbar$ is continuous, then $\hbar(\mathcal{O}^*)$ is a compact set. Therefore, $\hbar(\mathcal{O}^*) \in \mathcal{C}^{p_m}(\mathcal{B}_{a+1})$ implies that $\hbar(\mathcal{C}^{p_m}(\mathcal{B}_a)) \subseteq \mathcal{C}^{p_m}(\mathcal{B}_{a+1})$ for each $a = 1, 2, \cdots, q$.

Now we take $\mathcal{O}^* \in \mathcal{C}^{p_m}(\mathcal{B}_a)$ and $\mathcal{J}^* \in \mathcal{C}^{p_m}(\mathcal{B}_{a+1})$ for some $a = 1, 2, \cdots, q$. We assume that

$$\sup_{\hbar\mu \in \hbar(\mathcal{O}^*)} p_m\left(\hbar\mu, \hbar(\mathcal{J}^*)\right) \leq \lambda \sup_{\mu \in \mathcal{O}^*} p_m\left(\mu, \mathcal{J}^*\right).$$

But $\hbar$ is a cyclic contraction map, thus we get

$$p_m\left(\hbar\mu, \hbar\eta\right) \leq \lambda p_m\left(\mu, \eta\right) \text{ for all } \mu \in \mathcal{B}_a, \eta \in \mathcal{B}_{a+1} \text{ for } a = 1, 2, \cdots, q,$$

and so

$$\begin{aligned}\sup_{\hbar\mu\in\hbar(\mathcal{O}^*)} p_m\left(\hbar\mu,\hbar(\mathcal{J}^*)\right) &= \sup_{\hbar\mu\in\hbar(\mathcal{O}^*)}\inf_{\hbar\eta\in\hbar(\mathcal{J}^*)} p_m\left(\hbar\mu,\hbar\eta\right)\\ &\le \lambda\sup_{\mu\in\mathcal{O}^*}\inf_{\eta\in\mathcal{J}^*} p_m\left(\mu,\eta\right)\\ &\le \lambda\sup_{\mu\in\mathcal{O}^*} p_m\left(\mu,\mathcal{J}^*\right).\end{aligned}$$

In a similar manner,

$$\sup_{\hbar\eta\in\hbar(\mathcal{J}^*)} p_m\left(\hbar\eta,\hbar(\mathcal{O}^*)\right) \le \lambda\sup_{\eta\in\mathcal{J}^*} p_m\left(\eta,\mathcal{O}^*\right),$$

and so

$$\begin{aligned}p_m^{H_*}\left(\hbar(\mathcal{O}^*),\hbar(\mathcal{J}^*)\right) &= \max\left\{\sup_{\hbar\mu\in\hbar(\mathcal{O}^*)} p_m\left(\hbar\mu,\hbar(\mathcal{J}^*)\right), \sup_{\hbar\eta\in\hbar(\mathcal{J}^*)} p_m\left(\hbar\eta,\hbar(\mathcal{O}^*)\right)\right\}\\ &\le \lambda\max\left\{\sup_{\mu\in\mathcal{O}^*} p_m\left(\mu,\mathcal{J}^*\right), \sup_{\eta\in\mathcal{J}^*} p_m\left(\eta,\mathcal{O}^*\right)\right\}\\ &\le \lambda p_m^{H_*}(\mathcal{O}^*,\mathcal{J}^*).\end{aligned}$$

Hence $\hbar$ is a generalized cyclic contraction mapping on $\mathcal{C}^{p_m}\{\mathcal{B}_a\}_{a=1}^q$. □

Theorem 4.2.2. *Consider a collection* $\{\mathcal{B}_a\}_{a=1}^q$ *of non-void closed subsets of a partial metric space* (W,p_m) *and let* K *be a fixed natural number. If* $\hbar_j : \cup_{a=1}^q\mathcal{B}_a \to \cup_{a=1}^q\mathcal{B}_a$ *for* $j=1,2,\cdots,K$ *are generalized cyclic contractions, then the map* $\Psi : \cup_{a=1}^q\mathcal{C}^{p_m}(\mathcal{B}_a) \to \cup_{a=1}^q\mathcal{C}^{p_m}(\mathcal{B}_a)$ *defined by* $\Psi(\mathcal{O}^*) = \cup_{j=1}^K\hbar_j(\mathcal{O}^*)$ *for every* $\mathcal{O}^*\in\cup_{a=1}^q\mathcal{C}^{p_m}(\mathcal{B}_a)$ *is as well, a generalized cyclic contraction.*

Proof. Let $\mathcal{O}^*\in\mathcal{C}^{p_m}(\mathcal{B}_a)$ for some $a=1,2,\cdots,q$. With the aid of Theorem 4.2.1, for each $j=1,2,\cdots,K$, $\hbar_j$ is a generalized cyclic contraction. Thus $\hbar_j(\mathcal{O}^*)\in\mathcal{C}^p(\mathcal{B}_{a+1})$ for all $j=1,2,\cdots,K$, implying that $\Psi(\mathcal{O}^*)=\cup_{j=1}^K\hbar_j(\mathcal{O}^*)\in\mathcal{C}^{p_m}(\mathcal{B}_{a+1})$ and as a consequence, $\Psi\left(\mathcal{C}^{p_m}(\mathcal{B}_a)\right)\subseteq\mathcal{C}^{p_m}(\mathcal{B}_{a+1})$ for $a=1,2,\cdots,q$. From the cyclic contraction condition of each $\hbar_j$, where $j=1,2,\cdots,K$, we get

$$p_m^{H_*}\left(\hbar_j(\mathcal{O}^*),\hbar_j(\mathcal{J}^*)\right)\le\lambda p_m^{H_*}(\mathcal{O}^*,\mathcal{J}^*),$$

for all $\mathcal{O}^*\in\mathcal{C}^{p_m}(\mathcal{B}_a)$ and $\mathcal{J}^*\in\mathcal{C}^{p_m}(\mathcal{B}_{a+1})$, with $a=1,2,\cdots,q$. Therefore

$$\begin{aligned}p_m^{H_*}\left(\Psi(\mathcal{O}^*),\Psi(\mathcal{J}^*)\right) &= p_m^{H_*}\left(\cup_{j=1}^K\hbar_j(\mathcal{O}^*),\cup_{j=1}^K\hbar_j(\mathcal{J}^*)\right)\\ &\le \max\left\{p_m^{H_*}\left(\hbar_1(\mathcal{O}^*),\hbar_1(\mathcal{J}^*)\right),\cdots,p_m^{H_*}\left(\hbar_K(\mathcal{O}^*),\hbar_K(\mathcal{J}^*)\right)\right\}\\ &\le \lambda p_m^{H_*}(\mathcal{O}^*,\mathcal{J}^*).\end{aligned}$$

□

Definition 4.2.5. Let $\{\mathcal{B}_a\}_{a=1}^{q}$ be a class of non-void closed subsets of W, in a complete partial metric space (W, p_m). We say that $\Psi : \cup_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a) \to \cup_{a=1}^{q}\mathcal{C}^{p_m}(\mathcal{B}_a)$ is a generalized cyclic Hutchinson contractive operator, provided that a contraction factor $\lambda \in [0,1)$ exists with $\mathcal{J}^* \in \mathcal{C}^{p_m}(\mathcal{B}_a)$, $\mathcal{O}^* \in \mathcal{C}^{p_m}(\mathcal{B}_{a+1})$, such that

$$p_m^{H_*}\left(\Psi\left(\mathcal{J}^*\right), \Psi\left(\mathcal{O}^*\right)\right) \leq \lambda \mathcal{S}_\Psi(\mathcal{J}^*, \mathcal{O}^*),$$

where

$$\begin{aligned}\mathcal{S}_\Psi(\mathcal{J}^*, \mathcal{O}^*) \quad = \quad & \max\{p_m^{H_*}(\mathcal{J}^*, \mathcal{O}^*), p_m^{H_*}(\mathcal{J}^*, \Psi\left(\mathcal{J}^*\right)), p_m^{H_*}(\ \mathcal{O}^*, \Psi\left(\mathcal{J}^*\right)), \\ & \frac{p_m^{H_*}(\mathcal{J}^*, \Psi\left(\mathcal{O}^*\right)) + p_m^{H_*}(\mathcal{O}^*, \Psi\left(\mathcal{J}^*\right))}{2}, p_m^{H_*}(\Psi^2\left(\mathcal{J}^*\right), \Psi\left(\mathcal{J}^*\right)), \\ & p_m^{H_*}(\Psi^2\left(\mathcal{J}^*\right), \mathcal{O}^*), p_m^{H_*}(\Psi^2\left(\mathcal{J}^*\right), \Psi\left(\mathcal{O}^*\right))\}.\end{aligned}$$

Definition 4.2.6. Let $\{\mathcal{B}_a\}_{a=1}^{q}$ be a class of non-void closed subsets of W, in a complete partial metric space (W, p_m). We say that $\Psi : \cup_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a) \to \cup_{a=1}^{q}\mathcal{C}^{p_m}(\mathcal{B}_a)$ is a generalized cyclic rational Hutchinson contraction operator, provided that a contractive factor $\lambda \in [0,1)$ exists with $\mathcal{J}^* \in \mathcal{C}^{p_m}(\mathcal{B}_a)$, $\mathcal{O}^* \in \mathcal{C}^{p}(\mathcal{B}_{a+1})$, such that

$$p_m^{H_*}\left(\Psi\left(\mathcal{J}^*\right), \Psi\left(\mathcal{O}^*\right)\right) \leq \lambda_* \mathcal{R}_\Psi(\mathcal{J}^*, \mathcal{O}^*),$$

where

$$\begin{aligned}\mathcal{R}_\Psi(\mathcal{J}^*, \mathcal{O}^*) \quad = \quad & \max\left\{\frac{p_m^{H_*}(\mathcal{J}^*, \Psi\left(\mathcal{J}^*\right))[1 + p_m^{H_*}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{2(1 + p_m^{H_*}\left(\mathcal{J}^*, \mathcal{O}^*\right))},\right. \\ & \frac{p_m^{H_*}(\mathcal{O}^*, \Psi\left(\mathcal{O}^*\right))[1 + p_m^{H_*}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{1 + p_m^{H_*}\left(\mathcal{J}^*, \mathcal{O}^*\right)}, \\ & \left.\frac{p_m^{H_*}(\mathcal{O}^*, \Psi\left(\mathcal{J}^*\right))[1 + p_m^{H_*}(\mathcal{J}^*, \Psi\left(\mathcal{J}^*\right))]}{1 + p_m^{H_*}(\mathcal{J}^*, \mathcal{O}^*)}\right\}.\end{aligned}$$

4.3. Generalized Cyclic Hutchinson Contractive Operator

In this section, we prove that the generalized cyclic Hutchinson contractive operator has a unique attractor.

Theorem 4.3.1. *In a complete partial metric space (W, p_m) with a family $\{\mathcal{B}_a\}_{a=1}^{q}$ of non-void closed subsets of W and $\{W; \hbar_a, a = 1, 2, \cdots, q\}$, a generalized cyclic iterated function system, let the map $\Psi : \cup_{a=1}^{q}\mathcal{C}^{p_m}(\mathcal{B}_a) \to \cup_{a=1}^{q}\mathcal{C}^{p_m}(\mathcal{B}_a)$*

be defined by

$$\Psi(\mathcal{J}^*) = \cup_{a=1}^{q} \hbar_a(\mathcal{J}^*)$$

for each $\mathcal{J}^* \in \cup_{a=1}^{q} \mathcal{C}^{p_m}(\ \mathcal{B}_a)$. *Suppose* Ψ *is a generalized cyclic Hutchinson contraction operator, then* Ψ *has exactly one attractor* $U_1 \in \mathcal{C}^{p_m}(\mathcal{B}_a)$, *that is,*

$$U_1 = \Psi\left(U_1\right) = \cup_{a=1}^{q} \hbar_a(U_1).$$

In addition, for an arbitrary set $\mathcal{J}_0^* \in \cup_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a)$,

$$\left\{\mathcal{J}_0^*, \Psi\left(\mathcal{J}_0^*\right), \Psi^2\left(\mathcal{J}_0^*\right), ...\right\}$$

converges to U_1.

Proof. Let $\mathcal{J}_0^* \in \cup_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a)$ be an arbitrarily chosen set. Then some a_0 exists such that $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(\mathcal{B}_{a_0})$. Similarly, $\Psi\left(\mathcal{C}^{p_m}(\mathcal{B}_{a_0})\right) \subseteq \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+1}})$ implies that $\Psi\left(\mathcal{J}_0^*\right) \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+1}})$. Thus there exists $\mathcal{J}_1^* \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+1}})$, such that $\Psi\left(\mathcal{J}_0^*\right) = \mathcal{J}_1^*$. It follows that $\Psi\left(\mathcal{C}^{p_m}(\mathcal{B}_{a_{0+1}})\right) \subseteq \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+2}})$ which implies that $\mathcal{J}_2^* = \Psi\left(\mathcal{J}_1^*\right) \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+2}})$. The same argument results in the construction of a sequence $\{\mathcal{J}_\delta^*\}$ such that

$$\mathcal{J}_1^* = \Psi(\mathcal{J}_0^*),\ \mathcal{J}_2^* = \Psi\left(\mathcal{J}_1^*\right), ..., \mathcal{J}_{\delta+1}^* = \Psi\left(\mathcal{J}_\delta^*\right)$$

for $\delta \in \mathbb{N} \cup \{0\}$.

Assume that $\mathcal{J}_\delta^* \neq \mathcal{J}_{\delta+1}^*$ for all $\delta \in \mathbb{N} \cup \{0\}$, otherwise, $\mathcal{J}_s^* = \mathcal{J}_{s+1}^*$ for some s, which implies that $\mathcal{J}_s^* = \Psi(\mathcal{J}_s^*)$, and hence the proof. Thus $\mathcal{J}_\delta^* \neq \mathcal{J}_{\delta+1}^*$ for all $\delta \in \mathbb{N} \cup \{0\}$. Definition 4.2.5, with $\mathcal{J}_\delta^* \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{\delta+1}})$ and $\mathcal{J}_{\delta+1}^* = \Psi\left(\mathcal{J}_\delta^*\right) \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{\delta+2}})$, yield

$$\begin{aligned} p_m^{H_*}(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+2}^*) &= p_m^{H_*}(\Psi\left(\mathcal{J}_\delta^*\right), \Psi\left(\mathcal{J}_{\delta+1}^*\right)) \\ &\leq \lambda \mathcal{S}_\Psi\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{S}_\Psi\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right) = \ & \max\{p_m^{H_*}(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*), p_m^{H_*}\left(\mathcal{J}_\delta^*, \Psi\left(\mathcal{J}_\delta^*\right)\right), p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \Psi\left(\mathcal{J}_{\delta+1}^*\right)\right), \\ & \frac{p_m^{H_*}\left(\mathcal{J}_\delta^*, \Psi\left(\mathcal{J}_{\delta+1}^*\right)\right) + p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \Psi\left(\mathcal{J}_\delta^*\right)\right)}{2}, p_m^{H_*}(\Psi^2\left(\mathcal{J}_\delta^*\right), \Psi\left(\mathcal{J}_\delta^*\right)), \\ & p_m^{H_*}\left(\Psi^2\left(\mathcal{J}_\delta^*\right), \mathcal{J}_{\delta+1}^*\right), p_m^{H_*}\left(\Psi^2\left(\mathcal{J}_\delta^*\right), \Psi\left(\mathcal{J}_{\delta+1}^*\right)\right)\} \end{aligned}$$

$$\begin{aligned}
&= \max\{p_m^{H_*}(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*), p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right), p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+2}^*\right), \\
&\quad \frac{p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+2}^*\right) + p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+1}^*\right)}{2}, \\
&\quad p_m^{H_*}(\mathcal{J}_{\delta+2}^*, \mathcal{J}_{\delta+1}^*), p_m^{H_*}\left(\mathcal{J}_{\delta+2}^*, \mathcal{J}_{\delta+1}^*\right), p_m^{H_*}\left(\mathcal{J}_{\delta+2}^*, \mathcal{J}_{\delta+2}^*\right)\} \\
&\leq \max\{p_m^{H_*}(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*), p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+2}^*\right), \\
&\quad \left.\frac{p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right) + p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+2}^*\right)}{2}\right\} \\
&= \max\{p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right), p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+2}^*\right)\}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
p_m^{H_*}(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+2}^*) &\leq \lambda \max\{p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right), p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+2}^*\right)\} \\
&= \lambda p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right),
\end{aligned}$$

for all $\delta \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned}
p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_n^*\right) &\leq p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right) + p_m^{H_*}\left(\mathcal{J}_{\delta+1}^*, \mathcal{J}_{\delta+2}^*\right) + \cdots + p_m^{H_*}\left(\mathcal{J}_{n-1}^*, \mathcal{J}_n^*\right) \\
&\quad - \inf_{\mu_{\delta+1} \in \mathcal{J}_{\delta+1}^*} p_m(\mu_{\delta+1}, \mu_{\delta+1}) - \inf_{\mu_{\delta+2} \in \mathcal{J}_{\delta+2}^*} p_m(\mu_{\delta+2}, \mu_{\delta+2}) - \\
&\quad \cdots - \inf_{\mu_{n-1} \in \mathcal{J}_{n-1}^*} p_m(\mu_{n-1}, \mu_{n-1}) \\
&\leq [\lambda^\delta + \lambda^{\delta+1} + \cdots + \lambda^{n-1}] p_m^{H_*}\left(\mathcal{J}_0^*, \mathcal{J}_1^*\right) \\
&= \lambda^\delta [1 + \lambda + \lambda^2 + \cdots + \lambda^{n-\delta-1}] p_m^{H_*}(\mathcal{J}_0^*, \mathcal{J}_1^*)] \\
&\leq \frac{\lambda^\delta}{1-\lambda} p_m^{H_*}\left(\mathcal{J}_0^*, \mathcal{J}_1^*\right),
\end{aligned}$$

for all $\delta, n \in \mathbb{N} \cup \{0\}$ with $n > \delta$. So $\lim_{\delta, n \to +\infty} p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_n^*\right) = 0$, and so the sequence $\{\mathcal{J}_\delta^*\}$ is Cauchy in W. Since the partial metric space, $(\cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a), p_m^{H_*})$ is complete, then taking the limit as $\delta \to +\infty$ gives $\mathcal{J}_\delta^* \to U_1$ for some $U_1 \in \cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a)$, that is,

$$\lim_{\delta \to +\infty} p_m^{H_*}\left(\mathcal{J}_\delta^*, U_1\right) = \lim_{\delta \to +\infty} p_m^{H_*}\left(\mathcal{J}_\delta^*, \mathcal{J}_{\delta+1}^*\right) = p_m^{H_*}\left(U_1, U_1\right).$$

It turns out that $\{\mathcal{J}_\delta^*\}$ is a sequence with an infinite number of terms in $\mathcal{C}^{p_m}(\mathcal{B}_a)$ for each $a = 1, 2, \cdots, q$. We can therefore construct a convergent subsequence of $\{\mathcal{J}_\delta^*\}$ in each $\mathcal{C}^{p_m}(\mathcal{B}_a)$ for $a = 1, 2, \cdots, q$ which has U_1 as a limit and since each element in $\mathcal{C}^{p_m}(\mathcal{B}_a)$ for $a = 1, 2, \cdots, q$ is closed, we conclude that

$$U_1 \in \cap_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a) \neq \emptyset.$$

Let $V_1 = \cap_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a)$ and $\mathcal{C}^{p_m}(V_1)$ be the collection of all non-void compact subsets of V_1. Then $\Psi|_{\mathcal{C}^{p_m}(V_1)} : \mathcal{C}^{p_m}(V_1) \to \mathcal{C}^{p_m}(V_1)$ is a self-mapping on compact sets and so from Definition 2.1.5 and Theorem 2.2.1, we conclude that $\Psi|_{\mathcal{C}^{p_m}(V_1)}$ has exactly one attractor U_1 in $\mathcal{C}^{p_m}(V_1)$. □

Remark 4.3.1. *If we take $\cup_{a=1}^{q} \mathcal{S}^{p_m}(W_a)$, the union of the family of all singleton subsets of W in Theorem 4.3.1, then $\cup_{a=1}^{q} \mathcal{S}^{p_m}(W_a) \subseteq \cup_{a=1}^{q} \mathcal{C}^{p_m}(W_a)$. In addition, taking $\hbar_a = \hbar$ for each a, with $\hbar = \hbar_1$, we note that the mapping Ψ is expressed as*

$$\Psi(\mu) = \hbar(\mu).$$

As an outcome of Remark 4.3.1, we present the following result.

Corollary 4.3.1. *Suppose (W, p_m) is a complete partial metric space with $\{W; \hbar_a, a = 1, 2, \cdots, q\}$ a generalized cyclic iterated function system and $\hbar : W \to W$ a map defined as in Remark 4.3.1 If*

$$p_m(\hbar\mu, \hbar\eta) \leq \lambda \mathcal{S}_{\hbar}(\mu, \eta),$$

where

$$\begin{aligned} \mathcal{S}_{\hbar}(\mu, \eta) &= \max\{p_m(\mu, \eta), p_m(\mu, \hbar\mu), p(\eta, \hbar\eta), \frac{p_m(\mu, \hbar\eta) + p_m(\eta, \hbar\mu)}{2}, \\ &\quad p(\hbar^2\mu, \eta), p_m(\hbar^2\mu, \hbar\mu), p_m(\hbar^2\mu, \hbar\eta)\}, \end{aligned}$$

for $\mu \in \mathcal{C}^{p_m}(W_a)$, $\eta \in \mathcal{C}^{p_m}(W_{a+1})$ and $\lambda \in [0, 1)$, then $u \in W$ is a unique fixed point for the mapping $\hbar$. In addition, for any $u_0 \in W$, $\{u_0, \hbar u_0, \hbar^2 u_0, ...\}$ converges to u.

Corollary 4.3.2. *In a complete partial metric space (W, p_m), let $\{W; \hbar_a, a = 1, 2, \cdots, q\}$ be a generalized cyclic iterated function system with contraction self-mappings $\hbar_a$ for each $a = 1, 2, ..., q$. Suppose $\{\mathcal{B}_a\}_{a=1}^{q}$ is a class of non-void closed subsets of W. Then $\Psi : \cup_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a) \to \cup_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a)$ defined as in Theorem 4.3.1 has a unique attractor. In addition, for any choice of initial set $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(\mathcal{B}_a)$, $\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Psi^2(\mathcal{J}_0^*), \cdots\}$ is a convergent sequence with the attractor of Ψ as its unique limit.*

Proof. If each $\hbar_a$, is a cyclic contraction mapping on W, for $a = 1, 2, ..., q$, then by Theorem 4.3.1 we have that the mapping $\Psi : \cup_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a) \to \cup_{a=1}^{q} \mathcal{C}^{p_m}(\mathcal{B}_a)$ defined by $\Psi(\mathcal{J}^*) = \cup_{a=1}^{q} \hbar_a(\mathcal{J}^*)$ for all $\mathcal{J}^* \in \mathcal{C}^{p_m}(\mathcal{B}_a)$ is a contraction on $\mathcal{C}^{p_m}(\mathcal{B}_a)$ relative to the Hausdorff partial metric $p_m^{H_*}$, hence the result from Theorem 4.3.1. □

Example 4.3.1. Let (W, p_m) be a complete partial metric space. Suppose $W = [0,2]$ and define $p_m : W \times W \to \mathbb{R}_{[+]}$ by

$$p_m(\mu_1, \eta_1) = \begin{cases} \max\{\mu_1, \eta_1\} & \text{if } \mu_1, \eta_1 \notin [0,1) \\ |\mu_1 - \eta_1| & \text{if } \mu_1, \eta_1 \in [0,1), \end{cases}$$

Suppose $\mathcal{B}_1 = [0,1]$, $\mathcal{B}_2 = [\frac{1}{2}, 2]$, $\mathcal{B}_3 = \mathcal{B}_1$ and $W = \mathcal{B}_1 \cup \mathcal{B}_2 = [0,2]$.
Define $\hbar : W \to W$ by $\hbar(\mu_1) = \frac{1}{2}$ if $\mu_1 \in [0,1)$ and $\hbar(1) = 0$. Note that $\mathcal{B}_1$ and $\mathcal{B}_2$ are closed subsets of (W, p_m). Furthermore $\hbar(\mathcal{B}_a) \subset \mathcal{B}_{a+1}$ for $a = 1,2$ and so $\mathcal{B}_1 \cup \mathcal{B}_2$ is a cyclic representation of W relative to the mapping $\hbar$.

So

$$p_m(\hbar\mu_1, \hbar\eta_1) \leq \lambda \mathcal{S}_\hbar(\mu_1, \eta_1),$$

where

$$\begin{aligned} \mathcal{S}_h(\mu_1, \eta_1) &= \max\{p_m(\mu_1, \eta_1), p_m(\mu_1, \hbar\mu_1), p_m(\eta_1, \hbar\eta_1), \frac{p_m(\mu_1, \hbar\eta_1) + p_m(\eta_1, \hbar\mu_1)}{2}, \\ &\quad p_m(\hbar^2\mu_1, \eta_1), p_m(\hbar^2\mu_1, \hbar\mu_1), p_m(\hbar^2\mu_1, \hbar\eta_1)\}, \end{aligned}$$

holds.

We look at the following:

I. For $\mu_1 \in \mathcal{B}_1, \eta_1 \in \mathcal{B}_2$ with $\mu_1 \in [0, \frac{1}{2}]$ and $\eta_1 \in [\frac{1}{2}, 1)$, we have

$$p_m(\hbar(\mu_1), \hbar(\eta_1)) = p_m(\tfrac{1}{2}, \tfrac{1}{2}) = \left| \frac{1}{2} - \frac{1}{2} \right| = 0,$$

and for $\mu_1 \in [0, \frac{1}{2}]$ and $\eta_1 = 1$,

$$p_m(\hbar(\mu_1), \hbar(\eta_1)) = p_m(\tfrac{1}{2}, 0) = \tfrac{1}{2},$$

and

$$\begin{aligned} \mathcal{S}_h(\mu_1, \eta_1) &= \max\{|\mu_1 - 1|, \left|\mu_1 - \tfrac{1}{2}\right|, |1 - 0|, \frac{|\mu_1 - 1| + \left|1 - \frac{1}{2}\right|}{2}, \\ &\quad \left|\tfrac{1}{2} - 1\right|, \left|\tfrac{1}{2} - \tfrac{1}{2}\right|, \left|\tfrac{1}{2} - 0\right|\}. \end{aligned}$$

Thus

$$p_m(\hbar\mu_1, \hbar\eta_1) \leq \lambda \mathcal{S}_\hbar(\mu_1, \eta_1),$$

with $\lambda = \dfrac{3}{4}$.

II. For $\mu_1 \in \mathcal{B}_2, \eta_1 \in \mathcal{B}_1$ with $\mu_1 \in [\frac{1}{2}, 1]$ and $\eta_1 \in [0, \frac{1}{2})$, we have that

$$p_m(\hbar(\mu_1), \hbar(\eta_1)) = p_m(\tfrac{1}{2}, \tfrac{1}{2}) = 0,$$

and for $\mu_1 = 1$ and $\eta_1 \in [0, \frac{1}{2}]$,

$$p_m(\hbar(\mu_1), \hbar(\eta_1)) = p_m(0, \tfrac{1}{2}) = \tfrac{1}{2}$$

and

$$\begin{aligned} \mathcal{S}_\hbar(\mu_1, \eta_1) &= \max\{|\eta_1 - 1|, \left|\eta_1 - \tfrac{1}{2}\right|, \frac{|\eta_1 - 1| + \left|1 - \frac{1}{2}\right|}{2}, \\ &\quad \left|\tfrac{1}{2} - 1\right|, \left|\tfrac{1}{2} - \tfrac{1}{2}\right|, \left|\tfrac{1}{2} - 0\right|\}. \end{aligned}$$

Thus

$$p_m(\hbar\mu_1, \hbar\eta_1) \leq \lambda \mathcal{S}_\hbar(\mu_1, \eta_1),$$

with $\lambda = \dfrac{3}{4}$.

Therefore Corollary 4.3.1 is verified and, $\frac{1}{2}$ is a distinct fixed point of $\hbar$ and in addition we note that $\hbar$ is not continuous at 1.

4.4. Generalized Cyclic Rational Hutchinson Contraction Operator

In this section, we prove that the generalized cyclic rational Hutchinson contraction operator has a unique attractor.

Theorem 4.4.1. *In a complete partial metric space (W, p_m), let $\{\mathcal{B}_a\}_{a=1}^q$ be a collection of non-void closed subsets of W with $\{W; \hbar_a, a = 1, 2, \cdots, q\}$, a generalized cyclic iterated function system. Suppose $\Psi : \cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a) \to \cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a)$ defined by*

$$\Psi(\mathcal{J}^*) = \cup_{a=1}^q \hbar_a(\mathcal{J}^*)$$

for all $\mathcal{J}^* \in \cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a)$. If Ψ is a generalized cyclic rational Hutchinson contraction operator. Then Ψ has a unique attractor $U_1 \in \cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a)$, which is to say,

$$U_1 = \Psi(U_1) = \cup_{a=1}^q \hbar_a(U_1).$$

In addition, for any initial set $\mathcal{J}_0^* \in \cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a)$, the sequence

$$\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Psi^2(\mathcal{J}_0^*), ...\}$$

of compact sets, converges to the attractor U_1.

Proof. Choose $\mathcal{J}_0^* \in \cup_{a=1}^{q}\mathcal{C}^{p_m}(\mathcal{B}_a)$. Then some a_0 exists such that $\mathcal{J}_0^* \in \mathcal{C}^{p_m}(\mathcal{B}_{a_0})$. Similarly, $\Psi\left(\mathcal{C}^{p_m}(\mathcal{B}_{a_0})\right) \subseteq \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+1}})$ implies that $\Psi\left(\mathcal{J}_0^*\right) \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+1}})$. Thus there exists $\mathcal{J}_1^* \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+1}})$, such that $\Psi\left(\mathcal{J}_0^*\right) = \mathcal{J}_1^*$. It follows that $\Psi\left(\mathcal{C}^{p_m}(\mathcal{B}_{a_{0+1}})\right) \subseteq \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+2}})$ which implies that $\mathcal{J}_2^* = \Psi\left(\mathcal{J}_1^*\right) \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{0+2}})$. Consequently, a sequence $\{\mathcal{J}_b^*\}$ is define by

$$\mathcal{J}_1^* = \Psi(\mathcal{J}_0^*),\ \mathcal{J}_2^* = \Psi\left(\mathcal{J}_1^*\right), ..., \mathcal{J}_{b+1}^* = \Psi\left(\mathcal{J}_b^*\right)$$

for $b \in \mathbb{N} \cup \{0\}$ is obtained.

Assume that $\mathcal{J}_b^* \neq \mathcal{J}_{b+1}^*$ for all $b \in \mathbb{N} \cup \{0\}$. Otherwise, $\mathcal{J}_s^* = \mathcal{J}_{s+1}$ for some s, implies that $\mathcal{J}_s^* = \Psi(\mathcal{J}_s^*)$ and there is nothing further to show. Now take $\mathcal{J}_b^* \neq \mathcal{J}_{b+1}^*$ for all $b \in \mathbb{N} \cup \{0\}$. For $\mathcal{J}_b^* \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{b+1}})$ and $\mathcal{J}_{b+1}^* = \Psi\left(\mathcal{J}_b^*\right) \in \mathcal{C}^{p_m}(\mathcal{B}_{a_{b+2}})$, Definition 4.2.5 gives us that

$$\begin{aligned} p_m^{H_*}(\mathcal{J}_{b+1}^*, \mathcal{J}_{b+2}^*) &= p_m^{H_*}(\Psi\left(\mathcal{J}_b^*\right), \Psi\left(\mathcal{J}_{b+1}^*\right)) \\ &\leq \lambda_* \mathcal{R}_\Psi\left(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_\Psi(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*) &= \max\Bigg\{ \frac{p_m^{H_*}(\mathcal{J}_b^*, \Psi\left(\mathcal{J}_{b+1}^*\right))[1 + p_m^{H_*}(\mathcal{J}_b^*, \Psi(\mathcal{J}_b^*))]}{2(1 + p_m^{H_*}\left(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*\right))}, \\ &\quad \frac{p_m^{H_*}(\mathcal{J}_{b+1}^*, \Psi\left(\mathcal{J}_{b+1}^*\right))[1 + p_m^{H_*}(\mathcal{J}_b^*, \Psi(\mathcal{J}_b^*))]}{1 + p_m^{H_*}\left(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*\right)}, \\ &\quad \frac{p_m^{H_*}(\mathcal{J}_{b+1}^*, \Psi\left(\mathcal{J}_b^*\right))[1 + p_m^{H_*}(\mathcal{J}_b^*, \Psi\left(\mathcal{J}_b^*\right))]}{1 + p_m^{H_*}(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*)} \Bigg\} \\ &= \max\Bigg\{ \frac{p_m^{H_*}(\mathcal{J}_b^*, \mathcal{J}_{b+2}^*)[1 + p_m^{H_*}(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*)]}{2(1 + p_m^{H_*}\left(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*\right))}, \\ &\quad \frac{p_m^{H_*}(\mathcal{J}_{b+1}^*, \mathcal{J}_{b+2}^*)[1 + p_m^{H_*}(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*)]}{1 + p_m^{H_*}\left(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*\right)}, \\ &\quad \frac{p_m^{H_*}(\mathcal{J}_{b+1}^*, \mathcal{J}_{b+1}^*)[1 + p_m^{H_*}(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*)]}{1 + p_m^{H_*}(\mathcal{J}_b^*, \mathcal{J}_{b+1}^*)} \Bigg\} \\ &= \max\left\{ \frac{p_m^{H_*}(\mathcal{J}_b^*, \mathcal{J}_{b+2}^*)}{2}, p_m^{H_*}(\mathcal{J}_{b+1}^*, \mathcal{J}_{b+2}^*), p_m^{H_*}(\mathcal{J}_{b+1}^*, \mathcal{J}_{b+1}^*) \right\} \\ &= \frac{p_m^{H_*}(\mathcal{J}_b^*, \mathcal{J}_{b+2}^*)}{2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} p_m^{H_*}(\mathcal{J}^*_{b+1}, \mathcal{J}^*_{b+2}) &\leq \frac{\lambda_*}{2}[p_m^{H_*}(\mathcal{J}^*_b, \mathcal{J}^*_{b+1}) + p_m^{H_*}(\mathcal{J}^*_{b+1}, \mathcal{J}^*_{b+2}) - \inf_{\xi_{b+1}\in\mathcal{J}^*_{b+1}} p(\xi_{b+1}, \xi_{b+1})] \\ &\leq \frac{\lambda_*}{2}[p_m^{H_*}(\mathcal{J}^*_b, \mathcal{J}^*_{b+1}) + p_m^{H_*}(\mathcal{J}^*_{b+1}, \mathcal{J}^*_{b+2})], \end{aligned}$$

$$2p_m^{H_*}(\mathcal{J}^*_{b+1}, \mathcal{J}^*_{b+2}) - \lambda_* p_m^{H_*}(\mathcal{J}^*_{b+1}, \mathcal{J}^*_{b+2}) \leq \lambda_*[p_m^{H_*}(\mathcal{J}^*_b, \mathcal{J}^*_{b+1})],$$

$$p_m^{H_*}(\mathcal{J}^*_{b+1}, \mathcal{J}^*_{b+2}) \leq \frac{\lambda_*}{2-\lambda_*} p_m^{H_*}(\mathcal{J}^*_b, \mathcal{J}^*_{b+1}),$$

that is, for $\eta_* = \dfrac{\lambda_*}{2-\lambda_*} < 1$, we have

$$p_m^{H_*}(\mathcal{J}^*_{b+1}, \mathcal{J}^*_{b+2}) \leq \eta_* p_m^{H_*}(\mathcal{J}^*_b, \mathcal{J}^*_{b+1})$$

for all $b \in \mathbb{N} \cup \{0\}$. Thus for $b, s \in \mathbb{N} \cup \{0\}$ with $b < s$, we have

$$\begin{aligned} p_m^{H_*}(\mathcal{J}^*_b, \mathcal{J}^*_s) &\leq p_m^{H_*}(\mathcal{J}^*_b, \mathcal{J}^*_{b+1}) + p_m^{H_*}(\mathcal{J}^*_{b+1}, \mathcal{J}^*_{b+2}) + \cdots + p_m^{H_*}(\mathcal{J}^*_{s-1}, \mathcal{J}^*_s) \\ &\quad - \inf_{\mu_{b+1}\in\mathcal{J}^*_{b+1}} p_m(\mu_{b+1}, \mu_{b+1}) - \inf_{\mu_{b+2}\in\mathcal{J}^*_{b+2}} p_m(\mu_{b+2}, \mu_{b+2}) - \\ &\quad \cdots - \inf_{\mu_{s-1}\in\mathcal{J}^*_{s-1}} p_m(\mu_{s-1}, \mu_{s-1}) \\ &\leq \eta_*^b p_m^{H_*}(\mathcal{J}^*_0, \mathcal{J}^*_1) + \eta_*^{b+1} p_m^{H_*}(\mathcal{J}^*_0, \mathcal{J}^*_1) + \cdots + \eta_*^{s-1} p_m^{H_*}(\mathcal{J}^*_0, \mathcal{J}^*_1) \\ &\leq [\eta_*^b + \eta_*^{b+1} + \cdots + \eta_*^{s-1}] p_m^{H_*}(\mathcal{J}^*_0, \mathcal{J}^*_1) \\ &\leq \eta_*^b[1 + \eta_* + \eta_*^2 + \cdots + \eta_*^{s-b-1}] p_m^{H_*}(\mathcal{J}^*_0, \mathcal{J}^*_1) \\ &\leq \frac{\eta_*^b}{1-\eta_*} p_m^{H_*}(\mathcal{J}^*_0, \mathcal{J}^*_1), \end{aligned}$$

and so $\lim_{b,s\to+\infty} p_m^{H_*}(\mathcal{J}^*_b, \mathcal{J}^*_s) = 0$, hence the sequence $\{\mathcal{J}^*_b\}$ is Cauchy in W. Since $(\cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a), p_m^{H_*})$ is a complete partial metric space, $\mathcal{J}^*_b \to U_1$ as $b \to +\infty$ for some $U_1 \in \cup_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a)$, that is, $\lim_{b\to+\infty} p_m^{H_*}(\mathcal{J}^*_b, U_1) = \lim_{b\to+\infty} p_m^{H_*}\left(\mathcal{J}^*_b, \mathcal{J}^*_{b+1}\right) = p_m^{H_*}(U_1, U_1)$.

It can be noted that $\{\mathcal{J}^*_b\}$ has an infinite number of terms in $\mathcal{C}^{p_m}(\mathcal{B}_a)$ for each $a = 1, 2, \cdots, q$. Therefore, a subsequence of $\{\mathcal{J}^*_b\}$ that converges to U_1 can be constructed in each $\mathcal{C}^{p_m}(\mathcal{B}_a)$ with $a = 1, 2, \cdots, q$. Considering that each member in $\mathcal{C}^{p_m}(\mathcal{B}_a)$ for $a = 1, 2, \cdots, q$ is closed, we can conclude that $U_1 \in \cap_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a) \neq \emptyset$.

Now let $V_1 = \cap_{a=1}^q \mathcal{C}^{p_m}(\mathcal{B}_a)$ and set $\mathcal{C}^{p_m}(V_1)$ to be a collection of all nonvoid compact subsets of V_1. Then $\Psi|_{\mathcal{C}^{p_m}(V_1)} : \mathcal{C}^{p_m}(V_1) \to \mathcal{C}^{p_m}(V_1)$ is a self-mapping on compact sets and so using Definition 2.1.6 and adopting the proof of Theorem 2.2.2, we conclude that $\Psi|_{\mathcal{C}^{p_m}(V_1)}$ has a unique attractor in $\mathcal{C}^{p_m}(V_1)$. □

Corollary 4.4.1. *In a complete partial metric space* (W, p_m)*, let* $\{W; \hbar_a, a = 1, 2, \cdots, q\}$ *be a generalized cyclic iterated function system with* $\hbar : W \to W$ *defined as in Remark* 4.3.1*. If for any* $\mu \in \mathcal{C}^{p_m}(W_a)$ *and* $\eta \in \mathcal{C}^{p_m}(W_{a+1})$*, some* $\lambda_* \in [0, 1)$ *exists such that,*

$$p_m(\hbar\mu_1, \hbar\eta_1) \leq \lambda_* \mathcal{R}_\hbar(\mu_1, \eta_1),$$

where

$$\begin{aligned}\mathcal{R}_\hbar(\mu_1, \eta_1) \;=\; \max\Bigg\{ & \frac{p_m(\mu_1, \hbar\eta_1)[1 + p_m(\mu_1, \hbar\mu_1)]}{2(1 + p_m(\mu_1, \eta_1))}, \frac{p_m(\eta_1, \hbar\eta_1)[1 + p_m(\mu_1, \hbar\mu_1)]}{1 + p_m(\mu_1, \eta_1)}, \\ & \frac{p_m(\eta_1, \hbar\mu_1)[1 + p_m(\mu_1, \hbar\mu_1)]}{1 + p_m(\mu_1, \eta_1)} \Bigg\}.\end{aligned}$$

Then $u \in W$ *is a unique fixed point of* $\hbar$*. Additionally, for any choice of* $u_0 \in W$*, the sequence* $\{u_0, \hbar u_0, \hbar^2 u_0, ...\}$ *converges to* u*.*

4.5. Application in dynamic programming

We provide an application of the obtained results in solving functional equations which arise in dynamic programming.

Consider two Banach spaces, $\mathcal{W}_1$ and $\mathcal{W}_2$ with $\mathcal{F}^* \subseteq \mathcal{W}_1$ and $\mathcal{J}^* \subseteq \mathcal{W}_2$. Suppose that

$$\gamma \colon \mathcal{F}^* \times \mathcal{J}^* \longrightarrow \mathcal{F}^*, \quad f_1 \colon \mathcal{F}^* \times \mathcal{J}^* \longrightarrow \mathbb{R}, \quad g_1 \colon \mathcal{F}^* \times \mathcal{J}^* \times \mathbb{R} \longrightarrow \mathbb{R}.$$

If we regard $\mathcal{F}^*$ to be the state space and $\mathcal{J}^*$ the decision space, then the dynamic programming problem may be reduced to that of finding a solution to the functional equation:

$$\rho(m) = \sup_{z \in \mathcal{J}^*} \{f_1(m, z) + g_1(m, z, \rho(\gamma(m, z)))\}, \text{ for } m \in \mathcal{F}^* \tag{4.1}$$

Reformulation of equation (4.1) gives

$$\rho(m) = \sup_{z \in \mathcal{J}^*} \{f_1(m, z) + g_1(m, z, \rho(\gamma(m, z)))\} - \beta, \text{ for } m \in \mathcal{F}^* \tag{4.2}$$

where $\beta > 0$.

We would like to investigate the existence and boundedness of a unique solution of the functional equation (4.2) which arise in dynamic programming in the framework of partial metric spaces.

Let the set of all bounded real-valued functions on $\{ \mathcal{F}_a^*\}_{a=1}^q$ be denoted by $\tilde{B}_1(\mathcal{F}_a^*)$ and choose any $\xi \in \tilde{B}_1(\mathcal{F}_a^*)$, such that $\|\xi\| := \sup_{t_1 \in \mathcal{F}_a^*} |\xi(t_1)|$. Then $(\tilde{B}_1(\mathcal{F}_a^*), \|\cdot\|)$ is a Banach space. Consider

$$p_{B_1}(\xi, \varkappa) = \sup_{t_1 \in \mathcal{M}_a^*} |\xi(t_1) - \varkappa(t_1)| + \beta,$$

where $\xi \in \tilde{B}_1(\mathcal{F}_a^*)$ and $\varkappa \in \tilde{B}_1(\mathcal{F}_{a+1}^*)$. Then p_{B_1} is a partial metric on $\tilde{B}_1(\mathcal{F}_a^*)$ (see [4]).

We assert

(E_1): f_1 and g_1 are bounded and continuous.

(E_2): For $m \in \mathcal{F}_a^*$, $\xi \in \tilde{B}_1(\mathcal{F}_a^*)$ and $\beta > 0$, take $\Psi : \cup_{a=1}^q \tilde{B}(\mathcal{F}_a^*) \to \cup_{a=1}^q \tilde{B}_1(\mathcal{F}_a^*)$ as

$$\Psi\xi(m) = \sup_{z \in \mathcal{J}_a^*} \{f_1(m, z) + g_1(m, z, \xi(\gamma(m, z)))\} - \beta, \text{ for } m \in \mathcal{F}_a^* \tag{4.3}$$

Moreover, for every $(m, z) \in \mathcal{F}_a^* \times \mathcal{J}_a^*$, $\xi \in \tilde{B}_1(\mathcal{F}_a^*)$, $\varkappa \in \tilde{B}_1(\mathcal{F}_{a+1}^*)$ and $t_1 \in \mathcal{F}_a^*$ implies

$$|g_1(m, z, \xi(t_1)) - g_1(m, z, \varkappa(t_1))| \leq \lambda \mathcal{S}_\Psi(\xi(t_1), \varkappa(t_1)) - 2\beta, \tag{4.4}$$

where

$$\begin{aligned} & \mathcal{S}_\Psi(\xi(t_1), \varkappa(t_1)) \\ = \; & \max\{p_{B_1}(\xi(t_1), \varkappa(t_1)), p_{B_1}(\xi(t_1), \Psi(\kappa(t_1))), p_{B_1}(\varkappa(t_1), \Psi(\varkappa(t_1))), \\ & \frac{p_B(\xi(t_1), \Psi(\varkappa(t_1))) + p_{B_1}(\varkappa(t_1), \Psi(\xi(t_1)))}{2}, p_{B_1}(\Psi^2(\xi(t_1)), \Psi(\xi(t_1))), \\ & p_{B_1}(\Psi^2(\xi(t_1)), \varkappa(t_1)), p_{B_1}(\Psi^2\xi(t_1), \Psi(\varkappa(t_1)))\}. \end{aligned}$$

Theorem 4.5.1. *Suppose that* (E_1) *and* (E_2) *are true. Then, there exists a bounded and unique solution to the functional equation* (4.2) *in* $\tilde{B}_1(\mathcal{F}_a^*)$.

Proof. We observe that $(\tilde{B}_1(\mathcal{F}_a^*), p_{B_1})$ is a complete partial metric space. Since Ψ is a self-mapping of $\tilde{B}_1(\mathcal{F}_a^*)$ to itself, using (4.3) in (E_2) we have that for any $\xi, \varkappa \in \tilde{B}_1(\mathcal{F}_a^*)$ and $\beta > 0$, with $m \in \mathcal{F}_a^*$ and $z_1 \in \mathcal{J}_a^*$ such that

$$\Psi\xi < f_1(m, z_1) + g_1(m, z_1, \xi(\gamma(m, z_1))) \tag{4.5}$$

$$\Psi\varkappa < f_1(m, z_1) + g_1(m, z_1, \varkappa(\gamma(m, z_1))), \tag{4.6}$$

implies that

$$\Psi\xi \geq f_1(m, z_1) + g_1(m, z_1, \xi(\gamma(m, z_1))) - \beta \tag{4.7}$$

$$\Psi\varkappa \geq f_1(m, z_1) + g_1(m, z_1, \varkappa(\gamma(m, z_1))) - \beta. \tag{4.8}$$

(4.5) and (4.8) together with (4.4) gives us

$$\begin{aligned}\Psi\xi\left(t_1\right) - \Psi\varkappa\left(t_1\right) &< g_1(m, z_1, \xi(\gamma(m, z_1))) - g_1(m, z_1, \varkappa(\gamma(m, z_1))) + \beta \\ &\leq |g_1(m, z_1, \xi(\gamma(m, z_1))) - g_1(m, z_1, \varkappa(\gamma(m, z_1)))| + \beta \\ &\leq \lambda\mathcal{S}_\Psi(\xi\left(t_1\right), \varkappa\left(t_1\right)) - \beta. \end{aligned} \tag{4.9}$$

(4.6) and (4.7) together with (4.4) implies

$$\begin{aligned}\Psi\varkappa\left(t_1\right) - \Psi\xi\left(t_1\right) &< g_1(m, z_2, \varkappa(\gamma(m, z_1))) - g_1(m, z_1, \xi(\gamma(m, z_1))) + \beta \\ &\leq |g_1(m, z_2, \varkappa(\gamma(m, z_1))) - g_1(m, z_1, \xi(\gamma(m, z_1)))| + \beta \\ &\leq \lambda\mathcal{S}_\Psi(\xi\left(t_1\right), \varkappa\left(t_1\right)) - \beta. \end{aligned} \tag{4.10}$$

From (4.9) and (4.10), we get

$$|\Psi\xi\left(t_1\right) - \Psi\varkappa\left(t_1\right)| + \beta \leq \lambda\mathcal{S}_\Psi(\xi\left(t_1\right), \varkappa\left(t_1\right)). \tag{4.11}$$

Using inequality (4.11) we get

$$p_{B_1}(\Psi\xi\left(t_1\right), \Psi\varkappa\left(t_1\right)) \leq \lambda\mathcal{S}_\Psi(\xi\left(t_1\right), \varkappa\left(t_1\right)), \tag{4.12}$$

where

$$\begin{aligned}& \mathcal{S}_\Psi(\xi\left(t_1\right), \varkappa\left(t_1\right)) \\ = \;& \max\{p_{B_1}(\xi\left(t_1\right), \varkappa\left(t_1\right)), p_{B_1}(\xi\left(t_1\right), \Psi\left(\xi\left(t_1\right)\right)), p_{B_1}(\varkappa\left(t_1\right), \Psi\left(\varkappa\left(t_1\right)\right)), \\ & \frac{p_{B_1}(\xi\left(t_1\right), \Psi\left(\varkappa\left(t_1\right)\right)) + p_{B_1}(\varkappa\left(t_1\right), \Psi\left(\xi\left(t_1\right)\right))}{2}, \\ & p_{B_1}(\Psi^2\left(\xi\left(t_1\right)\right), \Psi\left(\xi\left(t_1\right)\right)), p_{B_1}(\Psi^2\xi\left(t_1\right), \varkappa\left(t_1\right)), \\ & p_{B_1}(\Psi^2\left(\xi\left(t_1\right)\right), \Psi\left(\varkappa\left(t_1\right)\right))\}.\end{aligned}$$

Hence, all conditions of Corollary 4.3.1 are satisfied, thus Ψ has a fixed point, $\xi^* \in \cap_{k=1}^{q} \tilde{B}_1(\mathcal{F}_a^*)$, and so $\xi^*\left(t_1\right)$ is a solution of functional equation (4.2). □

4.6. Application to the solution of Integral Equations

We now look at the existence and uniqueness of solutions to a family of non-linear integral equations shall be established, using Corollary 4.3.1 as a motivation.

Let

$$w(y_1) = \int_0^J K(y_1, t_1)\, h(t_1, w(t_1))dt_1 \text{ for all } y_1 \in [0, J], \tag{4.13}$$

be a nonlinear integral equation, where $J > 0$, $h : [0, J] \times \mathbb{R} \to \mathbb{R}$ and $K : [0, J] \times [0, J] \to \mathbb{R}_{[+]}$ are both continuous mappings (see [75]).

Let $W = C([0, J])$ be the set continuous functions with real values on the interval $[0, J]$ and endow W with a partial metric

$$p(\eta, \varepsilon) = \max_{y_1 \in [0,J]} |\eta(y_1) - \varepsilon(y_1)| + b \text{ for all } \eta, \varepsilon \in W \text{ and some } b > 0.$$

Let $(\alpha_1, \beta_1) \in W \times W$ and $(\alpha_0^*, \beta_0^*) \in \mathbb{R} \times \mathbb{R}$ such that

$$\alpha_0^* \leq \alpha_1(y_1) \leq \beta_1(y_1) \leq \beta_0^* \text{ for all } y_1 \in [0, J]. \tag{4.14}$$

Suppose for all $y_1 \in [0, J]$,

$$\alpha_1(y_1) \leq \int_0^J K(y_1, t_1)\, h(t_1, \beta_1(t_1))dt_1 \tag{4.15}$$

and

$$\beta_1(y_1) \geq \int_0^J K(y_1, t_1)\, h(t_1, \alpha_1(t_1))dt_1. \tag{4.16}$$

Further assume that $h(t_1, \cdot)$ is a decreasing function for all $t_1 \in [0, J]$, that is $r \geq s$ implies that

$$h(y_1, r) \leq h(y_1, s) \text{ for all } r, s \in \mathbb{R}. \tag{4.17}$$

We also suppose that

$$\sup_{y_1 \in [0,T]} \int_0^J K(y_1, t_1)\, dt_1 \leq 1. \tag{4.18}$$

Moreover, for all $t_1 \in [0, J]$, or all $r, s \in \mathbb{R}$ with $(r \leq \beta_0 \text{ and } s \geq \alpha_1)$ or $(r \geq \alpha_0 \text{ and } s \geq \beta_0)$,

$$|h(t_1, r) - h(t_1, s)| \leq \lambda Z_h(r, s) - b \text{ for some } b > 0 \tag{4.19}$$

where

$$Z_h(r,s) = \max\{p(r,s), p(r,hr), p(s,hs), \frac{p(r,hs)+p(r,hs)}{2}, \\ p(h^2r,s), p(h^2r,hr), p(h^2r,hs)\}.$$

Let

$$\mathcal{C} = \{w \in C[0,J] : \alpha_1 \leq w(y_1) \leq \beta_1 \text{ for all } y_1 \in [0,J]\}.$$

Consider the following result.

Theorem 4.6.1. *Suppose that all the conditions (4.14)-(4.17) hold. Then the integral equation (4.13) has at most one solution* $w_* \in \mathcal{C}$.

Proof. Let M_1 and M_2 be closed subsets of W such that

$$M_1 = \{w \in W : w \leq \beta_1\}$$

and

$$M_2 = \{w \in W : w \geq \alpha_1\}.$$

Define the mapping

$$\Psi : W \to W$$

by

$$\Psi w(y_1) = \int_0^J K(y_1,t_1)h(t_1,w(t_1))dt_1 \text{ for all } y_1 \in [0,J].$$

We shall show that

$$\Psi(M_1) \subseteq M_2 \text{ and } \Psi(M_2) \subseteq M_1. \tag{4.20}$$

Let $w_1 \in M_1$, that is,

$$\zeta(t) \leq \beta_1(t) \text{ for all } y_1 \in [0,J].$$

With the aid of condition (4.17), since $K(y_1,t_1) \geq 0$ for all $y_1, t_1 \in [0,J]$, we get that

$$K(y_1,t_1)h(t_1,w(t_1)) \geq K(y_1,t_1)h(t_1,\beta_1(t_1)) \text{ for all } y_1, t_1 \in [0,J].$$

Combining the above inequality with condition (4.15), gives

$$\int_0^J K(y_1,t_1)h(t_1,(\zeta(t_1))dt_1 \geq \int_0^J K(y_1,t_1)h(t_1,\beta_1(t_1))dt_1 \geq \alpha_1(y_1) \text{ for all } y_1 \in [0,J].$$

Thus

$$\Psi w \in M_2.$$

In a similar manner, let $w \in M_2$, so $w(t_1) \geq \alpha_1(t_1)$ for all $y_1 \in [0, J]$. Making use of condition (4.17), since $K(y_1, t_1) \geq 0$ for all $y_1, t_1 \in [0, J]$, we get

$$K(y_1, t_1)h(t_1, w(t_1)) \leq K(y_1, t_1)h(t_1, \alpha_1(t_1)) \text{ for all } y_1, t_1 \in [0, J].$$

Together with condition (4.16) the above inequality implies that

$$\int_0^J K(y_1, t_1)h(t_1, (w(t_1))dt_1 \leq \int_0^J K(y_1, t_1)h(t_1, \alpha_1(t_1))dt_1 \ \leq \beta_1(y_1) \text{ for all } y_1 \in [0, J].$$

Thus we have

$$\Psi w \in M_1.$$

and we conclude that (4.20) holds.

Now, let $(\mu, \eta) \in M_1 \times M_2$, that is for all $y_1 \in [0, J]$,

$$\mu(y_1) \leq \beta_1(y_1), \qquad \eta(y_1) \leq \alpha_1(y_1).$$

Together with (4.14), this implies that

$$\mu(y_1) \leq \beta_0^*, \qquad \eta(y_1) \geq \alpha_0^* \ \text{ for all } y_1 \in [0, J].$$

With the use of conditions (4.18) and (4.19), we have

$$\begin{aligned}
|\Psi\mu(y_1) - \Psi\eta(y_1)| &\leq \int_0^J K(y_1, t_1) \left| h(t_1, \mu(t_1) - h(t_1, \eta(t_1)) \right| dt_1 \\
&\leq \int_0^J K(y_1, t_1) \Big(\lambda \max\{p(\mu, \eta), p(\mu, h\mu), p(\eta, h\eta), \\
&\quad \frac{p(\mu, h\eta) + p(\eta, h\mu)}{2}, p(h^2\mu, \eta), p(h^2\mu, h\mu), p(h^2\mu, h\eta)\} - b \Big) dt_1 \\
&\leq \lambda \max\{p(\mu, \eta), p(\mu, h\mu), p(\eta, h\eta), \frac{p(\mu, h\eta) + p(\eta, h\mu)}{2}, \\
&\quad p(h^2\mu, \eta), p(h^2\mu, h\mu), p(h^2\mu, h\eta)\} \\
&\quad \int_0^J K(y_1, t_1)dt_1 - b\int_0^J K(y_1, t_1)dt_1 \\
&\leq \lambda \max\{p(\mu, \eta), p(\mu, h\mu), p(\eta, h\eta), \frac{p(\mu, h\eta) + p(\eta, h\mu)}{2}, \\
&\quad p(h^2\mu, \eta), p(h^2\mu, h\mu), p(h^2\mu, h\eta)\}.
\end{aligned}$$

Thus

$$p\left(\Psi\mu, \Psi\eta\right) \leq \lambda p\left(\mu, \eta\right)$$

It can be shown, in the same manner, that the above inequality holds for $(\mu, \eta) \in M_2 \times M_{1,}$ so Corollary 4.3.1 is satisfied, and we deduce that Ψ has a unique fixed point $w_* \in M_1 \cap M_2$, and so $w_* \in \mathcal{C}$ is a unique solution of (4.13). $\square$

5

Iterated Function System of Generalized Rational Contractions in Semi-Metric Spaces

5.1. Introduction

In the past decades, metric fixed point theory has proved to be an effective and versatile tool for solving scientific problems. Its vast range of applications, which include among others, iterative methods for solving linear and nonlinear differential, integral and difference equations, split feasibility problems, equilibrium problems and optimization problems attracted several researchers to intensify and extend the scope of fixed point theory in metric spaces, see for example [24, 28, 30, 3, 41, 65, 84, 88]. The notion of metric between two points is important in the definition of the nature of the topology of an underlying space. For example, Frechet [34] defined a metric space on a non-void set W that induces a Hausdorff topology on W. This was followed by several generalizations of the metric function, which includes the notion of a symmetric or semi-metric space giving rise to a non-Hausdorff topology [56].

Some useful results on contractive mappings in semi-metric space were obtained in [1, 2, 19, 38, 39, 44, 45, 48, 66, 93, 101, 105]. Hutchinson [42] introduced iterated function systems in the setting of metric spaces for generating fractals from contractive self-mappings. Since then, numerous researchers have been inspired to acquire a range of iterate function system findings in other spaces [32, 56, 14, 15, 16, 17, 63, 60].

Our primary objective in this chapter is the construction of a fractal set of generalized iterated function system of a generalized rational contraction in semi-metric space. We observe that the Hutchinson operator defined on a finite family of contractive mappings on a complete semi-metric space is itself a generalized contractive mapping on a family of compact subsets of W. By successive application of a generalized Hutchinson operator, a final fractal is obtained without the use of triangle inequality and this shall be followed by a non-trivial example.

For the purposes of our subsequent discussion, we give the following preliminary definitions and results.

Definition 5.1.1. [56] Let W be any non-void set. A mapping $d_s : W \times W \to \mathbb{R}_{[+]}$ is called a Hausdorff semi-metric on W if for all $\varrho, \varsigma \in W$, the following properties hold:

(d_{s_1}) $d_s(\varrho, \varsigma) = 0$ if and only if $\varrho = \varsigma$;

(d_{s_2}) $d_s(\varrho, \varsigma) = d_s(\varsigma, \varrho)$.

A set W equipped with a Hausdorff semi-metric d_s is called a Hausdorff semi-metric space.

Example 5.1.1. [56] Let $W = \mathbb{R}_{[+]}$ and define a semi-metric $d_s : W \times W \to \mathbb{R}_{[+]}$ by

$$\begin{aligned} d_s(2, \varrho) &= d_s(\varrho, 2) = \frac{2+\varrho}{4} \text{ if } \varrho \neq 2, \\ d_s(\varrho, \varsigma) &= |\varrho - \varsigma| \text{ for all } \varrho, \varsigma \in W \backslash \{2\} \text{ and } d_s(2,2) = 0. \end{aligned}$$

We observe that d_s is not a metric on W since the triangle inequality is not satisfied, that is $d_s(0,3) \nleq d_s(0,2) + d_s(2,3)$.

In a Hausdorff semi-metric space (W, d_s), let

$$B_o(\varsigma, r^*) = \{\varrho \in W : d_s(\varsigma, \varrho) < r^*\}$$

define an open ball with center $\varsigma \in W$ and radius any $r^* > 0$. One can represent a topology τ_{d_s} on W by

$$\{\mathcal{U} \in \tau_{d_s} \text{ such that for every } \varsigma \in \mathcal{U},\ B_o(\varsigma, r^*) \subset \mathcal{U} \text{ with } r^* > 0\}.$$

Definition 5.1.2. [103] Let (W, d_s) be a Hausdorff semi-metric space. Then for every $\varrho \in W$ and $\varsigma > 0$, the open ball $B_o(\varrho, r^*)$ is a neighborhood of ϱ with respect to the topology τ_{d_s}. Moreover, $\lim_{a \to +\infty} d_s(\varrho_a, \varrho) = 0$ if and only if the sequence ϱ_a converges to ϱ in the topology τ_{d_s}.

We give some properties each of which serve as a useful partial replacement of the triangle inequalty in a Hausdorff semi-metric space (W, d_s). Let $\{\varrho_a\}, \{\varsigma_a\}$ and $\{\upsilon_a\}$ be sequences in semi-metric space (W, d_s) with $\varrho, \varsigma \in W$. Then [56]

($\mathbf{W}_0$) $\lim_{a\to+\infty} d_s(\varrho_a,\varrho)=0$ and $\lim_{a\to+\infty} d_s(\varrho_a,\varsigma)=0$ implies $\varrho=\varsigma$;

($\mathbf{W}_1$) $\lim_{a\to+\infty} d_s(\varrho_a,\varrho)=0$ and $\lim_{k\to+\infty} d_s(\varrho_a,\varsigma_a)=0$ implies that $\lim_{a\to+\infty} d_s(\varrho,\varsigma_a)=0$;

($\mathbf{W}_2$) $\lim_{a\to+\infty} d_s(\varrho_a,\varsigma_a)=0$ and $\lim_{a\to+\infty} d_s(\varsigma_a,\upsilon_a)=0$ imply that $\lim_{a\to+\infty} d_s(\varrho_a,\upsilon_a)=0$;

($\mathbf{J}$) $\lim_{a\to+\infty} d_s(\varrho_a,\varsigma_a)=0$ and $\lim_{a\to+\infty} d_s(\varsigma_a,\upsilon_a)=0$ imply that $\lim_{a\to+\infty} d_s(\varrho_a,\upsilon_a)\neq+\infty$;

($\mathbf{CC}$) $\lim_{a\to+\infty} d_s(\varrho_a,\varrho)=0$ implies that $\lim_{a\to+\infty} d_s(\varrho_a,\varsigma)=d_s(\varrho,\varsigma)$.

Wilson [103] introduced properties ($\mathbf{W}_0$) and ($\mathbf{W}_1$), Mihet [63] property ($\mathbf{W}_2$), Jachymski et al. [45] property($\mathbf{J}$) and Cho et al. [29] property ($\mathbf{CC}$).

Definition 5.1.3. [39, 45] If $\{\varrho_a\}$ is a sequence in a Hausdorff semi-metric space (W,d_s), then

(a) $\{\varrho_a\}$ is said to be a d_s-Cauchy sequence if, given $\epsilon>0$, there exists a natural number a_ϵ such that, $d_s(\varrho_a,\varrho_k)<\epsilon$, for all $a,k\geq a_\epsilon$.

(b) (W,d_s) is called an S-complete space if for each d_s-Cauchy sequence $\{\varrho_a\}$ in W, an element ϱ in W exists such that $\lim_{a\to+\infty} d_s(\varrho_a,\varrho)=0$.

(c) (W,d_s) is known as a d_s-Cauchy complete Hausdorff semi-metric space if every d_s-Cauchy sequence $\{\varrho_a\}$ in W converges to $\varrho\in W$.

For a non-void set W, we say

$N(W)=\{K : K \text{ is a non-void subset of } W\}$,

$B(W)=\{K : K \text{ is a non-void bounded subset of } W\}$,

$CL(W)=\{K : K \text{ is a non-void closed subset of } W\}$,

$\mathcal{CB}^{d_s}(W)=\{K : K \text{ is a non-void closed and bounded subset of } W\}$,

$\mathcal{C}^{d_s}(W)=\{K : K \text{ is a non-void compact subset of } W\}$.

Definition 5.1.4. [56] Suppose (W,d_s) is a Hausdorff semi-metric space. $\mathcal{V}^*\in N(W)$ is d_s-closed if and only if $\overline{\mathcal{V}^*}=\mathcal{V}^*$, where $\overline{\mathcal{V}^*}=\{v^*\in W : d_s(v^*,\mathcal{V}^*)=0\}$ and $d_s(v^*,W)=\inf\{d_s(v^*,w^*) : w^*\in W\}$.

Let $\mathcal{J}^*,\mathcal{O}^*\in\mathcal{CB}^{d_s}(W)$, define the map $H_{d_s}:\mathcal{CB}^{d_s}(W)\times\mathcal{CB}^{d_s}(W)\to\mathbb{R}_{[+]}$ by

$$H_{d_s}(\mathcal{J}^*,\mathcal{O}^*)=\max\{\sup_{\varsigma\in\mathcal{O}^*} d_s(\varsigma,\mathcal{J}^*),\ \sup_{\varrho\in\mathcal{J}^*} d_s(\varrho,\mathcal{O}^*)\},$$

then we say H_{d_s} is a Pompeiu-Hausdorff semi-metric induced by d_s. If (W,d_s) is a d_s-Cauchy complete semi-metric space, then $(\mathcal{CB}^{d_s}(W),H_{d_s})$ is a d_s-Cauchy complete semi-metric space too.

In the context of a Hausdorff semi-metric space, we state the following Lemmas for later use [73].

Lemma 5.1.1. [56, 73] *Let* (W, d_s) *be a Hausdorff semi-metric space and* $\mathcal{J}^*, \mathcal{O}^* \in CB^{d_s}(W)$. *If* $\varrho \in \mathcal{J}^*$, *then* $d_s(\varrho, \mathcal{O}^*) \leq H_{d_s}(\mathcal{J}^*, \mathcal{O}^*)$.

Lemma 5.1.2. [56, 73] *Let* (W, d_s) *be a Hausdorff semi-metric space and* $\mathcal{J}^*, \mathcal{O}^* \in CB^{d_s}(W)$. *Then for any given* $\epsilon > 0$, *satisfying* $H_{d_s}(\mathcal{J}^*, \mathcal{O}^*) < \epsilon$, *there exist an element* $\varsigma \in \mathcal{O}^*$ *such that* $d_s(\varrho, \varsigma) < \epsilon$ *for every* $\varrho \in \mathcal{J}^*$.

Lemma 5.1.3. [56, 73] *Let* (W, d_s) *be a Hausdorff semi-metric space and* $\mathcal{J}^*, \mathcal{O}^* \in CB^{d_s}(W)$. *Then for any* $\varrho \in \mathcal{J}^*$, *there exists* $\varsigma \in \mathcal{O}^*$ *such that* $d_s(\varrho, \varsigma) \leq \lambda H_{d_s}(\mathcal{J}^*, \mathcal{O}^*)$ *for* $\lambda > 1$.

Lemma 5.1.4. [56, 53, 73] *Let* (W, d_s) *be a Hausdorff semi-metric space. Then for all* $\mathcal{K}^*, \mathcal{L}^*, \mathcal{J}^*, \mathcal{O}^* \in \mathcal{CB}^{d_s}(W)$, *the following conditions hold:*

(a) *If* $\mathcal{L}^* \subseteq \mathcal{J}^*$, *then* $\sup_{k \in \mathcal{K}^*} d_s(k, \mathcal{J}^*) \leq \sup_{k \in \mathcal{K}^*} d_s(k, \mathcal{L}^*)$,

(b) $\sup_{t \in \mathcal{K}^* \cup \mathcal{L}^*} d_s(t, \mathcal{J}^*) = \max\{\sup_{k \in \mathcal{K}^*} d_s(k, \mathcal{J}^*), \sup_{\ell \in \mathcal{L}^*} d_s(\ell, \mathcal{J}^*)\}$,

(c) $H_{d_s}(\mathcal{K}^* \cup \mathcal{L}^*, \mathcal{J}^* \cup \mathcal{O}^*) \leq \max\{H_{d_s}(\mathcal{K}^*, \mathcal{J}^*), H_{d_s}(\mathcal{L}^*, \mathcal{O}^*)\}$.

Theorem 5.1.1. [45, 56] *Let* (W, d_s) *be a* d_s*-Cauchy complete Hausdorff semi-metric space and* $h : W \to W$ *be a contractive mapping such that,*

$$d_s(h\varrho, h\varsigma) \leq \lambda d_s(\varrho, \varsigma)$$

is satisfied for all $\varrho, \varsigma \in W$ *and* $0 \leq \lambda < 1$. *If* (W, d_s) *is a bounded Hausdorff semi-metric space, that is, if there exists some constant* $\mathcal{X}^*$ *such that* $\mathcal{X}^* = \sup\{d_s(\varrho, \varsigma) : \varrho, \varsigma \in W\} < \infty$, *then* h *has a unique fixed point* $\tilde{u}$ *in* W, *in addition for every* $\varrho_0 \in W$, *the sequence* $\{\varrho_0, h\varrho_0, h^2\varrho_0, \cdots\}$ *converges to the unique fixed point* $\tilde{u}$ *of* h.

Theorem 5.1.2. [56, 53] *Let* (W, d_s) *be a Hausdorff semi-metric space and* $h : W \to W$ *be a contraction mapping with* $0 \leq \lambda < 1$, *then*

(a) *elements in* $\mathcal{C}^{d_s}(W)$ *are mapped by* h *to elements in* $\mathcal{C}^{d_s}(W)$.

(b) *If for any* $\mathcal{J}^* \in \mathcal{C}^{d_s}(W)$,

$$h(\mathcal{J}^*) = \{h(\varrho) : \varrho \in \mathcal{J}^*\},$$

then the mapping $h : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ *is a contraction on* $(\mathcal{C}^{d_s}(W), H_{d_s})$.

5.2. Generalized Iterated Function System in Hausdorff Semi-Metric Spaces

Goyal reported some results on generalized iterated function systems for multi-valued mappings in metric spaces in [35]. This section expands on the principles presented in Section 3.2 on generalized iterated function system in partial metric spaces set-up [53] to the context of Hausdorff semi-metric spaces. The definition of a generalized contraction self-map will be followed by some preliminary results.

Definition 5.2.1. Let (W, d_s) be a Hausdorff semi-metric space and $h, g : W \to W$ be two mappings. A couple (h, g) is called a generalized contraction if

$$d_s\left(h\varrho, g\varsigma\right) \leq \lambda d_s\left(\varrho, \varsigma\right)$$

for all $\varrho, \varsigma \in W$, where $0 \leq \lambda < 1$.

Example 5.2.1. Let W be a closed and bounded subset of $\mathbb{R}_{[+]}$, and define a Hausdorff semi-metric d_s on W by $d_s(r, t) = (r - t)^2$ for all $r, t \in W$. Define $h, g : W \to W$ by

$$h(r) = \frac{3r}{5(r+1)} \quad \text{and} \quad g(t) = \frac{3t}{5(t+2)}.$$

Then note that

$$\begin{aligned}
d_s\left(h(r), g(t)\right) &= \left(\frac{3r}{5(r+1)} - \frac{3t}{5(t+2)}\right)^2 \\
&= \frac{9}{25}\left(\frac{r}{r+1} - \frac{t}{t+2}\right)^2 \\
&= \frac{9}{25}\left[\left(\frac{r}{r+1}\right)^2 - 2\left(\frac{r}{r+1}\right)\left(\frac{t}{t+2}\right) + \left(\frac{t}{t+2}\right)^2\right] \\
&\leq \frac{9}{25}\left(r^2 - 2rt + t^2\right) \\
&= \frac{9}{25}(r-t)^2,
\end{aligned}$$

that is, $d_s\left(h(r), g(t)\right) \leq \lambda d_s\left(r, t\right)$ with $\lambda = \dfrac{9}{25}$. Thus the couple (h, g) is a generalized contraction.

Theorem 5.2.1. *Suppose* (W, d_s) *is a Hausdorff semi-metric space and* h, g :

$W \to W$ are two continuous mappings. If the pair (h, g) is a generalized contraction with a common contractive constant λ such that $0 \leq \lambda < 1$, then

(a) *elements in $\mathcal{C}^{d_s}(W)$ are mapped to elements in $\mathcal{C}^{d_s}(W)$ under a pair (h, g),*

(b) *if for any $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W)$, the mappings $h, g : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ are defined as*

$$\begin{aligned} h(\mathcal{J}^*) &= \{h(\varrho) : \varrho \in \mathcal{J}^*\} \text{ and} \\ g(\mathcal{O}^*) &= \{g(\varsigma) : \varsigma \in \mathcal{O}^*\} \end{aligned}$$

then the couple (h, g) is a generalized contraction on $(\mathcal{C}^{d_s}(W), H_{d_s})$.

Proof. (a) Since h is a continuous mapping and the image of a compact subset under a continuous mapping, $h : W \to W$ is compact, then

$$\mathcal{J}^* \in \mathcal{C}^{d_s}(W) \text{ implies that } h(\mathcal{J}^*) \in \mathcal{C}^{d_s}(W).$$

In a similar manner we have

$$\mathcal{O}^* \in \mathcal{C}^{d_s}(W) \text{ implies that } g(\mathcal{O}^*) \in \mathcal{C}^{d_s}(W).$$

(b) Let $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W)$. Since the couple (h, g) is a generalized contraction, then for $0 \leq \lambda < 1)$,

$$d_s(h\varrho, g\varsigma) \leq \lambda d_s(\varrho, \varsigma) \text{ for all } \varrho, \varsigma \in W.$$

Thus we have

$$\begin{aligned} d_s(h\varrho, g(\mathcal{O}^*)) &= \inf_{\varsigma \in \mathcal{O}^*} d_s(h\varrho, g\varsigma) \\ &\leq \inf_{\varsigma \in \mathcal{O}^*} \lambda d_s(\varrho, \varsigma) \\ &= \lambda d_s(\varrho, \mathcal{O}^*). \end{aligned}$$

Also

$$\begin{aligned} d_s(g\varsigma, h(\mathcal{J}^*)) &= \inf_{s \in \mathcal{J}^*} d_s(g\varsigma, h\varrho) \\ &\leq \inf_{\varrho \in \mathcal{J}^*} \lambda d_s(\varsigma, \varrho) \\ &= \lambda d_s(\varrho, \mathcal{J}^*). \end{aligned}$$

Now

$$
\begin{aligned}
H_{d_s}(h(\mathcal{J}^*), g(\mathcal{O}^*)) &= \max\{\sup_{s\in\mathcal{J}^*} d_s(h\varrho, g(\mathcal{O}^*)), \sup_{\varsigma\in\mathcal{O}^*} d_s(g\varsigma, h(\mathcal{J}^*))\} \\
&\leq \max\{\sup_{\varrho\in\mathcal{J}^*} \lambda d_s(\varrho, \mathcal{O}^*), \sup_{\varsigma\in\mathcal{O}^*} \lambda d_s(\varsigma, \mathcal{J}^*)\} \\
&= \lambda \max\{\sup_{\varrho\in\mathcal{J}^*} d_s(\varrho, \mathcal{O}^*), \sup_{\varsigma\in\mathcal{O}^*} d_s(\varsigma, \mathcal{J}^*)\} \\
&= \lambda H_{d_s}(\mathcal{J}^*, \mathcal{O}^*).
\end{aligned}
$$

As a result, the pair (h, g) is a generalized contraction mapping on $(\mathcal{C}^{d_s}(W), H_{d_s})$. □

Proposition 5.2.1. *Let (W, d_s) be a Hausdorff semi-metric space and (h_a, g_a), $a \in \{1, 2, \cdots, q\}$, a finite family of contractive mappings. If*

$$d_s(h_a\varrho, g_a\varsigma) \leq \lambda_a d_s(\varrho, \varsigma) \quad \text{for all} \quad \varrho, \varsigma \in W,$$

for $0 \leq \lambda_a < 1$), $a \in \{1, 2, \cdots, q\}$, then the mappings $\Psi, \Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ such that

$$
\begin{aligned}
\Psi(\mathcal{J}^*) &= h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*) \cup \cdots \cup h_q(\mathcal{J}^*) \\
&= \cup_{a=1}^{q} h_a(\mathcal{J}^*) \text{ for each } \mathcal{J}^* \in \mathcal{C}^{d_s}(W)
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi(\mathcal{O}^*) &= g_1(\mathcal{O}^*) \cup g_2(\mathcal{O}^*) \cup \cdots \cup g_q(\mathcal{O}^*) \\
&= \cup_{a=1}^{q} g_a(\mathcal{O}^*) \text{ for each } \mathcal{O}^* \in \mathcal{C}^{d_s}(W)
\end{aligned}
$$

satisfy

$$H_{d_s}(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)) \leq \tilde{\lambda} H_{d_s}(\mathcal{J}^*, \mathcal{O}^*) \text{ for all } \mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W),$$

where $\widetilde{\lambda} = \max\{\lambda_a : a = 1, 2, \cdots, q\}$ and the pair (Ψ, Φ) is a generalized contraction on $\mathcal{C}^{d_s}(W)$.

Proof. We shall prove the result for $q = 2$. Let $h_1, h_2, g_1, g_2 : W \to W$ be contraction mappings. For $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W)$ and using Lemma 5.1.4 (c), we have

$$\begin{aligned} H_{d_s}(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)) &= H_{d_s}(h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*), g_1(\mathcal{O}^*) \cup g_2(\mathcal{O}^*)) \\ &\leq \max\{H_{d_s}(h_1(\mathcal{J}^*), g_1(\mathcal{O}^*)), H_{d_s}(h_2(\mathcal{J}^*), g_2(\mathcal{O}^*))\} \\ &\leq \max\{\lambda_1 H_d(\mathcal{J}^*, \mathcal{O}^*), \lambda_2 H_d(\mathcal{J}^*, \mathcal{O}^*)\} \\ &\leq \tilde{\lambda} H_{d_s}(\mathcal{J}^*, \mathcal{O}^*) \end{aligned}$$

where $\tilde{\lambda} = \max\{\lambda_1, \lambda_2\}$. □

Theorem 5.2.2. *Suppose (W, d_s) is a d_s-Cauchy complete Hausdorff semi-metric space and $h_a, g_a : W \to W$, $a = 1, 2, \cdots, q$, a finite family of contractive mappings on W with contraction constants λ_a, $a = 1, 2, \cdots, q$, respectively. Define $\Psi, \Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ by*

$$\Psi(\mathcal{J}^*) = \cup_{a=1}^{q} h_a(\mathcal{J}^*),$$

and

$$\Phi(\mathcal{O}^*) = \cup_{a=1}^{q} g_a(\mathcal{O}^*)$$

for each $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W)$. Assume (W, d_s) is a bounded Hausdorff semi-metric space, then the following relations hold:

(a) $\Psi, \Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$.

(b) *The pair (Ψ, Φ) has a unique common fixed point $\tilde{U_1} \in \mathcal{C}^{d_s}(W)$, which implies that, $\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1}) = \cup_{a=1}^{q} h_a(\tilde{U_1}) = \cup_{a=1}^{q} g_a(\tilde{U_1})$.*

(c) *The sequence*

$$\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Phi\Psi(\mathcal{J}_0^*), \Psi\Phi\Psi(\mathcal{J}_0^*), ...\}$$

of compact sets converges to the common fixed point $\tilde{U_1}$ of Ψ and Φ for an arbitrarily chosen initial set $\mathcal{J}_0^* \in \mathcal{C}^{d_s}(W)$.

Proof. (a) Since (h_a, g_a) for $a = 1, 2, \cdots, q$ is a pair of contactive mappings, using the definition of the pair (Ψ, Φ) and Theorem 5.2.1, we get the result. From Proposition 5.2.1, we see that the pair (Ψ, Φ) is a generalized contraction on $\mathcal{C}^{d_s}(W)$. Furthermore, since (W, d_s) is d_s-Cauchy complete then as a concequence $(\mathcal{C}^{d_s}(W), H_{d_s})$ is also complete. As a result, from Theorem 5.1.1 we get (b) and (c). □

Definition 5.2.2. Suppose (W, d_s) is a Hausdorff semi-metric space with $\Psi, \Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$. Then a pair of mappings (Ψ, Φ) is called a generalized rational Hutchinson contractive operator if $\lambda_* \in [0, 1)$ exists, such that for all $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W)$, the following holds:

$$H_{d_s}\left(\Psi\left(\mathcal{J}^*\right), \Phi\left(\mathcal{O}^*\right)\right) \leq \lambda_* \mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*),$$

where

$$\begin{aligned} \mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*) \;=\; & \max\left\{H_{d_s}(\mathcal{J}^*, \mathcal{O}^*), H_{d_s}(\mathcal{J}^*, \Psi\left(\mathcal{J}^*\right)), H_{d_s}(\mathcal{O}^*, \Phi\left(\mathcal{O}^*\right)),\right. \\ & \frac{H_{d_s}(\mathcal{O}^*, \Phi\left(\mathcal{O}^*\right))[1 + H_{d_s}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{1 + H_{d_s}\left(\mathcal{J}^*, \mathcal{O}^*\right)}, \\ & \left.\frac{H_{d_s}(\mathcal{O}^*, \Psi\left(\mathcal{J}^*\right))[1 + H_{d_s}(\mathcal{J}^*, \Psi\left(\mathcal{J}^*\right))]}{1 + H_{d_s}(\mathcal{J}^*, \mathcal{O}^*)}\right\}. \end{aligned}$$

Definition 5.2.3. Let (W, d_s) be a Hausdorff semi-metric space. If $h_a, g_a : W \to W$, are such that each pair (h_a, g_a), $a = 1, 2, ..., q$ is a finite family of generalized contractions, then $\{W; (h_a, g_a), a = 1, 2, \cdots, q\}$ is called a generalized iterated function system.

Definition 5.2.4. Let (W, d_s) be a Hausdorff semi-metric space and $\Psi, \Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ a pair of mappings. Let $\mathcal{Q}^* \subseteq W$ be a non-void closed and bounded set, then $\mathcal{Q}^*$ is a unique common attractor of the generalized iterated function system if

(i) $\Psi(\mathcal{Q}^*) = \Phi(\mathcal{Q}^*) = \mathcal{Q}^*$ and

(ii) there exists an open set $V_1 \subseteq W$ such that $\mathcal{Q}^* \subseteq V_1$ and $\lim\limits_{a\to+\infty} \Psi^a(\mathcal{O}^*) = \lim\limits_{a\to+\infty} \Phi^a(\mathcal{N}^*) = \mathcal{Q}^*$ for any closed and bounded set $\mathcal{N}^* \subseteq V_1$, where the limit is taken relative to the Hausdorff semi-metric.

5.3. Generalized Hutchinson contractive operator in semi-metric spaces

We now turn our attention to some result on the existence and uniqueness of a common attractor of generalized rational Hutchinson contractive operator in a Hausdorff semi-metric space framework.

Theorem 5.3.1. *Let (W, d_s) be a d_s-Cauchy complete Hausdorff semi-metric space and $\{W; (h_a, g_a), a = 1, 2, \cdots, q\}$, the generalized iterated function system. Suppose a pair of mappings $\Psi, \Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ defined by*

$$\Psi(\mathcal{J}^*) = \cup_{a=1}^{q} h_a(\mathcal{J}^*)$$

and

$$\Phi(\mathcal{O}^*) = \cup_{a=1}^{q} g_a(\mathcal{O}^*),$$

for each $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W)$ is a generalized rational Hutchinson contractive operator. If (W, d_s) is bounded, then Ψ and Φ have a unique common attractor $\tilde{U_1} \in \mathcal{C}^{d_s}(W)$, that is,

$$\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1}).$$

Moreover, for the arbitrarily chosen initial set $\mathcal{J}_0^* \in \mathcal{C}^{d_s}(W)$, the sequence

$$\{\mathcal{J}_0^*, \Psi\left(\mathcal{J}_0^*\right), \Phi\Psi\left(\mathcal{J}_0^*\right), \Psi\Phi\Psi\left(\mathcal{J}_0^*\right), \cdots\}$$

of compact sets converges to the unique common attractor of both Ψ and Φ.

Proof. Choose $\mathcal{J}_0^*$ arbitrarily in $\mathcal{C}^{d_s}(W)$ and define the sequences

$$\mathcal{J}_1^* = \Psi(\mathcal{J}_0^*),\ \mathcal{J}_3^* = \Psi\left(\mathcal{J}_2^*\right), \cdots, \mathcal{J}_{2a+1}^* = \Psi\left(\mathcal{J}_{2a}^*\right)$$

and

$$\mathcal{J}_2^* = \Phi(\mathcal{J}_1^*),\ \mathcal{J}_4^* = \Phi\left(\mathcal{J}_3^*\right), \cdots, \mathcal{J}_{2a+2}^* = \Phi\left(\mathcal{J}_{2a+1}^*\right)$$

for $a \in \{0, 1, 2, \cdots\}$.

Now, since the pair (Ψ, Φ) is a generalized rational Hutchinson contractive operator, we have

$$\begin{aligned} H_{d_s}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*) &= H_{d_s}(\Psi\left(\mathcal{J}_{2a}^*\right), \Phi\left(\mathcal{J}_{2a+1}^*\right)) \\ &\leq \lambda_* \mathcal{R}_{\Psi,\Phi}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right) \end{aligned}$$

for $a \in \{0, 1, 2, \cdots,\}$, where

$$\begin{aligned} \mathcal{R}_{\Psi,\Phi}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*) &= \max\Big\{H_{d_s}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*), H_{d_s}(\mathcal{J}_{2a}^*, \Psi\left(\mathcal{J}_{2a}^*\right)), \\ &\quad H_{d_s}(\mathcal{J}_{2a+1}^*, \Phi\left(\mathcal{J}_{2a+1}^*\right)), \\ &\quad \frac{H_{d_s}(\mathcal{J}_{2a+1}^*, \Phi\left(\mathcal{J}_{2a+1}^*\right))[1 + H_{d_s}(\mathcal{J}_{2a}^*, \Psi(\mathcal{J}_{2a}^*))]}{1 + H_{d_s}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right)}, \\ &\quad \frac{H_{d_s}(\mathcal{J}_{2a+1}^*, \Psi\left(\mathcal{J}_{2a}\right))[1 + H_{d_s}(\mathcal{J}_{2a}^*, \Psi\left(\mathcal{J}_{2a}^*\right))]}{1 + H_{d_s}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*)}\Big\}. \\ &= \max\Big\{H_{d_s}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*), H_{d_s}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*), \\ &\quad H_{d_s}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*) \\ &\quad \frac{H_{d_s}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*)[1 + H_{d_s}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*)]}{1 + H_{d_s}\left(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*\right)}, \\ &\quad \frac{H_{d_s}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+1}^*)[1 + H_{d_s}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*)]}{1 + H_{d_s}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*)}\Big\}. \\ &= \max\left\{H_{d_s}(\mathcal{J}_{2a}^*, \mathcal{J}_{2a+1}^*), H_{d_s}(\mathcal{J}_{2a+1}^*, \mathcal{J}_{2a+2}^*)\right\}. \end{aligned}$$

If $\mathcal{R}_{\Psi,\Phi}\left(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1}\right) = H_{d_s}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2})$ then

$$H_{d_s}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}) \leq \lambda_* H_{d_s}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2})$$

which is a contradiction. Thus $\mathcal{R}_{\Psi,\Phi}\left(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1}\right) = H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})$ so that

$$H_{d_s}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}) \leq \lambda_* H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})$$

for $a \in \{0, 1, 2, \cdots\}$. Continuing in this manner, gives

$$\begin{aligned} H_{d_s}(\mathcal{J}^*_{2a+1},\mathcal{J}^*_{2a+2}) &\leq \lambda_* H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1}) \\ &\leq \lambda_*^2 H_{d_s}(\mathcal{J}^*_{2a-1},\mathcal{J}^*_{2a}) \\ &\leq \cdots \\ &\leq \lambda_*^{a+1} H_{d_s}(\mathcal{J}^*_0,\mathcal{J}^*_1). \end{aligned}$$

Furthermore, this implies that

$$\begin{aligned} H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+n}) &\leq \lambda_*^a H_{d_s}(\mathcal{J}^*_0,\mathcal{J}^*_n) \\ &\leq \lambda_*^a \mathcal{X}^* \quad \text{for all } a, n = 0, 1, 2, \cdots . \end{aligned}$$

Thus $\lim\limits_{a\to+\infty} \lambda_*^a \mathcal{X}^* = 0$. Therefore $\{\mathcal{J}^*_a\}$ is a d_s-Cauchy sequence in $\mathcal{C}^{d_s}(W)$. But, $(\mathcal{C}^{d_s}(W), H_{d_s})$ is complete, so we have $\mathcal{J}^*_a \to \tilde{U}_1$ as $a \to +\infty$ for some $\tilde{U}_1 \in \mathcal{C}^{d_s}(W)$, in other words, $\lim\limits_{a\to+\infty} H_{d_s}(\mathcal{J}^*_a,\tilde{U}_1) = \lim\limits_{a\to+\infty} H_{d_s}\left(\mathcal{J}^*_a,\mathcal{J}^*_{a+1}\right) = H_{d_s}(\tilde{U}_1,\tilde{U}_1)$ and so $\lim\limits_{a\to+\infty} H_{d_s}(\mathcal{J}^*_a,\tilde{U}_1) = 0$.

To prove that $\tilde{U}_1$ is a common attractor of both Ψ and Φ, we consider

$$\begin{aligned} H_{d_s}(\mathcal{J}^*_{2a+1},\Phi(\tilde{U}_1)) &= H_{d_s}(\Psi\left(\mathcal{J}^*_{2a}\right),\Phi\left(\tilde{U}_1\right)) \\ &\leq \lambda_* \mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*_{2a},\tilde{U}_1). \end{aligned}$$

for all $a \in \{0, 1, 2, ...\}$, where

$$
\begin{aligned}
\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*_{2a},\tilde{U}_1) &= \max\Big\{H_{d_s}(\mathcal{J}^*_{2a},\tilde{U}_1), H_{d_s}(\mathcal{J}^*_{2a},\Psi(\mathcal{J}^*_{2a})), H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1)),\\
&\quad \frac{H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1))[1+H_{d_s}(\mathcal{J}^*_{2a},\Psi(\mathcal{J}^*_{2a}))]}{1+H_{d_s}(\mathcal{J}_{2a},\tilde{U}_1)},\\
&\quad \frac{H_{d_s}(\tilde{U}_1,\Psi(\mathcal{J}^*_{2a}))[1+H_{d_s}(\mathcal{J}_{2a},\Psi(\mathcal{J}^*_{2a}))]}{1+H_{d_s}(\mathcal{J}^*_{2a},\tilde{U}_1)}\Big\}.\\
&= \max\Big\{H_{d_s}(\mathcal{J}^*_{2a},\tilde{U}_1), H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1}), H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1)),\\
&\quad \frac{H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1))[1+H_{d_s}(\mathcal{J}_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{d}(\mathcal{J}^*_{2a},\tilde{U}_1)},\\
&\quad \frac{H_{d_s}(\tilde{U}_1,\mathcal{J}^*_{2a+1})[1+H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}_{2a+1})]}{1+H_{d_s}(\mathcal{J}^*_{2a},\tilde{U}_1)}\Big\}.
\end{aligned}
$$

We observe that:

(1) In a case where $\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*_{2a},\tilde{U}_1) = H_{d_s}(\mathcal{J}_{2a},\tilde{U}_1)$, we have

$$H_{d_s}(\mathcal{J}^*_{2a+1},\Phi(\tilde{U}_1)) \leq \lambda_* H_{d_s}(\mathcal{J}^*_{2a},\tilde{U}_1),$$

which, on taking the limit as $a \to +\infty$, gives $H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1)) \leq 0$, so $\tilde{U}_1 = \Phi(\tilde{U}_1)$.

(2) Suppose $\mathcal{R}_{\Psi,\Phi}\left(\mathcal{J}^*_{2a},\tilde{U}_1\right) = H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})$, then

$$H_{d_s}(\mathcal{J}^*_{2a+1},\Phi(\tilde{U}_1)) \leq \lambda_* H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})$$

and taking the limit as $k \to +\infty$, yields $H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1)) \leq 0$, thus $\tilde{U}_1 = \Phi(\tilde{U}_1)$.

(3) In case $\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*_{2a},\tilde{U}_1) = H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1))$, we have

$$H_{d_s}(\mathcal{J}^*_{2a+1},\Phi(\tilde{U}_1)) \leq \lambda_* H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1)),$$

which on taking the limit as $a \to +\infty$, implies that $H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1)) \leq 0$ and so $\tilde{U}_1 = \Phi(\tilde{U}_1)$

(4) If $\mathcal{R}_{\Psi,\Phi}\left(\mathcal{J}^*_{2a},\tilde{U}_1\right) = \dfrac{H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1))[1+H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{d_s}\left(\mathcal{J}^*_{2a},\tilde{U}_1\right)}$, we obtain

$$H_{d_s}(\mathcal{J}^*_{2a+1},\Phi(\tilde{U}_1)) \leq \lambda_* \frac{H_{d_s}(\tilde{U}_1,\Phi(\tilde{U}_1))[1+H_{d_s}(\mathcal{J}^*_{2a},\mathcal{J}^*_{2a+1})]}{1+H_{d_s}(\mathcal{J}^*_{2a},\tilde{U}_1)}$$

which, together with our taking the limit as $a \to +\infty$, gives $H_{d_s}(\tilde{U}_1, \Phi(\tilde{U}_1)) \leq \lambda_* H_{d_s}(\tilde{U}_1, \Phi(\tilde{U}_1))$ and so $\tilde{U}_1 = \Phi(\tilde{U}_1)$.

(5) When $\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*_{2a}, \tilde{U}_1) = \dfrac{H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})[1 + H_{d_s}(\mathcal{J}^*_{2a}, \mathcal{J}^*_{2a+1})]}{1 + H_{d_s}(\mathcal{J}^*_{2a}, \tilde{U}_1)}$, we obtain

$$H_{d_s}(\mathcal{J}^*_{2a+1}, \Phi(\tilde{U}_1)) \leq \lambda_* \frac{H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})[1 + H_{d_s}(\mathcal{J}^*_{2a}, \mathcal{J}^*_{2a+1})]}{1 + H_{d_s}(\mathcal{J}^*_{2a}, \tilde{U}_1)}$$

which on taking the limit as $a \to +\infty$ gives, $H_{d_s}(\tilde{U}_1, \Phi(\tilde{U}_1)) \leq 0$, that is $\tilde{U}_1 = \Phi(\tilde{U}_1)$. Thus we conclude that $\tilde{U}_1$ is an attractor of Φ.

Using the same argument, we obtain

$$\begin{aligned} H_{d_s}(\Psi(\tilde{U}_1), \mathcal{J}^*_{2a+2}) &= H_{d_s}(\Psi(\tilde{U}_1), \Phi(\mathcal{J}^*_{2a+1})) \\ &\leq \lambda_* \mathcal{R}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}), \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) &= \max\Big\{ H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1}), H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1)), \\ &\quad H_{d_s}(\mathcal{J}^*_{2a+1}, \Phi(\mathcal{J}^*_{2a+1})), \\ &\quad \frac{H_{d_s}(\mathcal{J}^*_{2a+1}, \Phi(\mathcal{J}^*_{2a+1}))[1 + H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))]}{1 + H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})}, \\ &\quad \frac{H_{d_s}(\mathcal{J}^*_{2a+1}, \Psi(\tilde{U}_1))[1 + H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))]}{1 + H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})} \Big\}. \\ &= \max\Big\{ H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1}), H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1)), \\ &\quad H_{d_s}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2}), \\ &\quad \frac{H_{d_s}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2})[1 + H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))]}{1 + H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})}, \\ &\quad \frac{H_{d_s}(\mathcal{J}^*_{2a+1}, \Psi(\tilde{U}_1))[1 + H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))]}{1 + H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})} \Big\}. \end{aligned}$$

Again, we look at five cases:

(1) Suppose $\mathcal{R}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) = H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})$, then

$$H_{d_s}(\Psi(\tilde{U}_1), \mathcal{J}^*_{2a+2}) \leq \lambda_* H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1}),$$

This, combined with our interpretation of the limit as $a \to +\infty$, leads to $H_{d_s}(\Psi(\tilde{U}_1), \tilde{U}_1) \leq 0$, which is a contradiction, so $\Psi(\tilde{U}_1) = \tilde{U}_1$.

(2) If $\mathcal{R}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) = H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))$, we have

$$H_{d_s}(\Psi(\tilde{U}_1), \mathcal{J}^*_{2a+2}) \leq \lambda_* H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))$$

This, combined with our interpretation of the limit as $a \to +\infty$, implies that $H_{d_s}(\Psi(\tilde{U}_1), \tilde{U}_1) \leq 0$ and thus $\Psi(\tilde{U}_1) = \tilde{U}_1$.

(3) In case, $\mathcal{R}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) = H_{d_s}(\mathcal{J}_{2a+1}, \mathcal{J}_{2a+2})$, we have

$$H_{d_s}(\Psi(\tilde{U}_1), \mathcal{J}^*_{2a+2}) \leq \lambda_* H_{d_s}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2}))$$

This, combined with our interpretation of the limit as $a \to +\infty$, implies that $H_{d_s}(\Psi(\tilde{U}_1), \tilde{U}_1) \leq 0$ and thus $\Psi(\tilde{U}_1) = \tilde{U}_1$.

(4) If $\mathcal{R}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) = \dfrac{H_{d_s}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2})[1 + H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))]}{1 + H_{d_s}\left(\tilde{U}_1, \mathcal{J}^*_{2a+1}\right)}$, we obtain

$$H_{d_s}(\Psi(\tilde{U}_1), \mathcal{J}^*_{2a+2}) \leq \lambda_* \frac{H_{d_s}(\mathcal{J}^*_{2a+1}, \mathcal{J}^*_{2a+2})[1 + H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))]}{1 + H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})}$$

This, combined with our interpretation of the limit as $a \to +\infty$, implies that $H_{d_s}(\Psi(\tilde{U}_1), \tilde{U}_1) \leq 0$ and so $\Psi(\tilde{U}_1) = \tilde{U}_1$.

(5) In case of $\mathcal{R}_{\Psi,\Phi}(\tilde{U}_1, \mathcal{J}^*_{2a+1}) = \dfrac{H_{d_s}(\mathcal{J}^*_{2a+1}, \Psi(\tilde{U}_1))[1 + H_{d_s}(\tilde{U}_1 U_1, \Psi(\tilde{U}_1))]}{1 + H_{d_s}(\tilde{U}_1, \mathcal{J}_{2a+1})}$, we obtain

$$H_{d_s}(\Psi(\tilde{U}_1), \mathcal{J}_{2a+2}) \leq \lambda_* \frac{H_{d_s}(\mathcal{J}_{2a+1}, \Psi(\tilde{U}_1))[1 + H_{d_s}(\tilde{U}_1, \Psi(\tilde{U}_1))]}{1 + H_{d_s}(\tilde{U}_1, \mathcal{J}^*_{2a+1})}$$

This, combined with our interpretation of the limit as $a \to +\infty$, gives $H_{d_s}(\Psi(\tilde{U}_1), \tilde{U}_1) \leq 0$ so that is, $\Psi(\tilde{U}_1) = \tilde{U}_1$.

Thus $\tilde{U}_1$ is an attractor of the mappings Ψ and so we have shown that $\tilde{U}_1$ is a common attractor of both Ψ and Φ.

To prove uniqueness, assume that $\tilde{U}_1$ and $\tilde{U}_2$ are distinct common attractors for both Ψ and Φ. Because the pair (Ψ, Φ) is generalized rational contractive Hutchinson operator, we get

$$\begin{aligned} H_{d_s}(\tilde{U}_1, \tilde{U}_2) &= H_{d_s}(\Psi(\tilde{U}_1), \Phi(\tilde{U}_2)) \\ &\leq \lambda_* \mathcal{R}_{\Psi,\Phi}(\tilde{U}_1, \tilde{U}_2) \end{aligned}$$

where

$$\begin{aligned}\mathcal{R}_{\Psi,\Phi}(\tilde{U_1},\tilde{U_2}) &= \max\Big\{H_{d_s}(\tilde{U_1},\tilde{U_2}), H_{d_s}(\tilde{U_1},\Psi(\tilde{U_1})), H_{d_s}(\tilde{U_2},\Phi(\tilde{U_2}))\ ,\\ &\quad \frac{H_{d_s}(\tilde{U_2},\Phi(\tilde{U_2}))[1+H_{d_s}(\tilde{U_1},\Psi(\tilde{U_1}))]}{1+H_{d_s}(\tilde{U_1},\tilde{U_2})},\\ &\quad \frac{H_{d_s}(\ \Psi(\tilde{U_1}))[1+H_{d_s}(\tilde{U_1},\Psi(\tilde{U_1}))]}{1+H_{d_s}(\tilde{U_1},\tilde{U_2})}\Big\}.\\ &= \max\Big\{H_{d_s}(\tilde{U_1},\tilde{U_2}), H_{d_s}(\tilde{U_1},\tilde{U_1}), H_{d_s}(\tilde{U_2},\tilde{U_2})\ ,\\ &\quad \frac{H_{d_s}(\tilde{U_2},\tilde{U_2})[1+H_{d_s}(\tilde{U_1},\tilde{U_1}))]}{1+H_{d_s}(\tilde{U_1},\tilde{U_2})},\\ &\quad \frac{H_{d_s}(\tilde{U_2},\tilde{U_1})[1+H_{d_s}(\tilde{U_1},\tilde{U_1})]}{1+H_{d_s}(\tilde{U_1},\tilde{U_2})}\Big\}\\ &= \max\left\{H_{d_s}(\tilde{U_1},\tilde{U_2}), \frac{H_{d_s}(\tilde{U_2},\tilde{U_1})}{1+H_{d_s}(\tilde{U_1},\tilde{U_2})}\right\}\\ &= H_{d_s}(\tilde{U_1},\tilde{U_2})\end{aligned}$$

and so $(1-\lambda_*)H_{d_s}(\tilde{U_1},\tilde{U_2}) \leq 0$, which implies that $H_{d_s}(\tilde{U_1},\tilde{U_2}) = 0$ and hence $\tilde{U_1} = \tilde{U_2}$. Thus $\tilde{U_1} \in \mathcal{C}^{d_s}(W)$ is a unique common attractor of Ψ and Φ. □

Remark 5.3.1. *Let $\mathcal{S}^{d_s}(W)$ represent the collection of all singleton subsets of W in Theorem* 5.3.1. *Then $\mathcal{S}^{d_s}(W) \subseteq \mathcal{C}^{d_s}(W)$. Moreover, suppose that a couple of mappings $(h_a, g_a) = (h, g)$ for every a, where $h = h_1$ and $g = g_1$ then, the operator pair $(\Psi,\Phi) : \mathcal{S}^{d_s}(W) \to \mathcal{S}^{d_s}(W)$ becomes*

$$(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)) = (h(\mathcal{J}^*), g(\mathcal{O}^*)) \quad \textit{for all } \mathcal{J}^*, \mathcal{O}^* \in \mathcal{S}^{d_s}(W).$$

As a result, the common fixed point result is as follows.

Corollary 5.3.1. *Suppose (W, d_s) is a generalized d_s-Cauchy complete Hausdorff semi-metric space. Let the mappings $\Psi, \Phi : \mathcal{S}^{d_s}(W) \to \mathcal{S}^{d_s}(W)$ be as in Remark* 5.3.1. *Suppose $0 \leq \lambda_* < 1$ exists such that for all $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W)$, the following condition holds:*

$$H_{d_s}(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)) \leq \lambda_* \mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*),$$

where

$$\begin{aligned}\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*,\mathcal{O}^*) &= \max\{H_{d_s}(\mathcal{J}^*,\mathcal{O}^*), H_{d_s}(\mathcal{J}^*,\Psi(\mathcal{J}^*)), H_{d_s}(\mathcal{O}^*,\Phi(\mathcal{O}^*)),\\ &\quad \frac{H_{d_s}(\mathcal{O}^*,\Phi(\mathcal{O}^*))[1+H_{d_s}(\mathcal{J}^*,\Psi(\mathcal{J}^*))]}{1+H_{d_s}(\mathcal{J}^*,\mathcal{O}^*)},\\ &\quad \left.\frac{H_{d_s}(\mathcal{O}^*,\Psi(\mathcal{J}^*))[1+H_{d_s}(\mathcal{J}^*,\Psi(\mathcal{J}^*))]}{1+H_{d_s}(\mathcal{J}^*,\mathcal{O}^*)}\right\}\end{aligned}$$

and (W,d_s) *is bounded, then* Ψ *and* Φ *have a unique common attractor* $\tilde{U_1} \in \mathcal{C}^{d_s}(W)$*, which means,*

$$\tilde{U_1} = \Psi(\tilde{U_1}) = h(\tilde{U_1}) = g(\tilde{U_1}) = \Phi(\tilde{U_1}).$$

Furthermore, for any singleton set $\mathcal{J}_0^* \in \mathcal{S}^{d_s}(W)$*, the sequence of*

$$\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Phi\Psi(\mathcal{J}_0^*), \Psi\Phi\Psi(\mathcal{J}_0^*), ...\}$$

converges to the unique common attractor of both Ψ *and* Φ*.*

Corollary 5.3.2. *Suppose is* (W,d_s) *a generalized* d_s*-Cauchy complete Hausdorff semi-metric space, and* $\{W; h_a, g_a, a = 1,2,\cdots,q\}$*, a generalized iterated function system. Let a pair of mappings* $\Psi,\Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ *be defined as in Theorem* 5.3.1*. If* (W,d_s) *is bounded, then* Ψ *and* Φ *have a unique common attractor* $\tilde{U_1} \in \mathcal{C}^{d_s}(W)$*. Moreover, for any initial set* $\mathcal{J}_0^* \in \mathcal{C}^{d_s}(W)$*, the sequence* $\{\mathcal{J}_0^*, \Psi(\mathcal{J}_0^*), \Phi\Psi(\mathcal{J}_0^*), \Psi\Phi\Psi(\mathcal{J}_0^*), ...\}$ *converges to the unique common attractor of both* Ψ *and* Φ*.*

Proof. From Proposition 5.2.1, we observe that if every pair of mappings (h_a, g_a), $a = 1,2,\cdots,q$ is a contraction on W, then the pair of mappings $\Psi,\Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ defined by

$$\begin{aligned}\Psi(\mathcal{J}^*) &= h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*) \cup \cdots \cup h_a(\mathcal{J}^*)\\ &= \cup_{a=1}^{q} h_a(\mathcal{J}^*)\end{aligned}$$

and

$$\begin{aligned}\Phi(\mathcal{O}^*) &= g_1(\mathcal{O}^*) \cup g_2(\mathcal{O}^*) \cup \cdots \cup g_a(\mathcal{O}^*)\\ &= \cup_{a=1}^{q} g_a(\mathcal{O}^*),\end{aligned}$$

for each $\mathcal{J}^*,\mathcal{O}^* \in \mathcal{C}^{d_s}(W)$ is a generalized contraction on $\mathcal{C}^{d_s}(W)$ and in reference to Theorem 5.3.1, the result follows. □

Example 5.3.1. Let $W = [0, 1]$ and d_s be a Hausdorff semi-metric on W defined as $d_s(\mathbf{r}, \boldsymbol{t}) = (\boldsymbol{r} - \boldsymbol{t})^2$ for all $\boldsymbol{r}, \mathbf{t} \in W$. Let $h_a, g_a : W \times W \to W$, for $a = 1, 2$ be defined as

$$\begin{aligned} h_1(r,t) &= \left(\frac{2r}{3(r+1)}, \frac{t}{2(t+2)}\right), \\ h_2(r,t) &= \left(\frac{2\sin r}{5(\sin r+1)}, \frac{\sin t}{3(\sin t+3)}\right). \end{aligned}$$

and

$$\begin{aligned} g_1(r,t) &= \left(\frac{2r}{3(r+2)}, \frac{t}{2(t+1)}\right), \\ g_2(r,t) &= \left(\frac{2\sin r}{5(\sin r+2)}, \frac{\sin t}{3(\sin t+2)}\right), \end{aligned}$$

for all $(r, t) \in W$.

Now $\boldsymbol{r} = (r_1, r_2), \boldsymbol{t} = (t_1, t_2) \in W$, gives

$$\begin{aligned} d_s(h_1(\boldsymbol{r}), g_1(\boldsymbol{t})) &= \frac{4}{9}\left(\frac{r_1}{r_1+1} - \frac{t_1}{t_1+2}\right)^2 + \frac{1}{4}\left(\frac{r_2}{r_2+2} - \frac{t_2}{t_2+1}\right)^2 \\ &\quad + \frac{2}{3}\left(\frac{r_1}{r_1+1} - \frac{t_1}{t_1+2}\right)\left(\frac{r_2}{r_2+2} - \frac{t_2}{t_2+1}\right) \\ &\leq \frac{2}{3}\left[(r_1-t_1)^2 + (r_2-t_2)^2 + 2(r_1-t_1)(r_2-t_2)\right] = \lambda_1 d_s(\mathbf{r}, \boldsymbol{t}) \end{aligned}$$

where $\lambda_1 = \frac{2}{3}$. We also have

$$\begin{aligned} d_s(h_2(\boldsymbol{r}), g_2(\boldsymbol{t})) &= \frac{4}{25}\left(\frac{\sin r_1}{\sin r_1+1} - \frac{\sin t_1}{\sin t_1+2}\right)^2 + \frac{1}{9}\left(\frac{\sin r_2}{\sin r_2+2} - \frac{\sin t_2}{\sin t_2+1}\right)^2 \\ &\quad + \frac{1}{3}\left(\frac{\sin r_1}{\sin r_1+1} - \frac{\sin t_1}{\sin t_1+2}\right)\left(\frac{\sin r_2}{\sin r_2+2} - \frac{\sin t_2}{\sin t_2+1}\right) \\ &\leq \frac{4}{5}\left[(r_1-t_1)^2 + (r_2-t_2)^2 + 2(r_1-t_1)(r_2-t_2)\right] = \lambda_2 d_s(\mathbf{r}, \boldsymbol{t}) \end{aligned}$$

where $\lambda_1 = \frac{4}{5}$.

Thus the iterated function system $\{W; h_a, g_a, a = 1, 2\}$ with $\Psi, \Phi : \mathcal{C}^{d_s}(W) \to \mathcal{C}^{d_s}(W)$ defined as

$$\Psi(\mathcal{J}^*) = h_1(\mathcal{J}^*) \cup h_2(\mathcal{J}^*)$$

and

$$\Phi(\mathcal{O}^*) = g_1(\mathcal{O}^*) \cup g_2(\mathcal{O}^*)$$

for all $\mathcal{J}^*, \mathcal{O}^* \in \mathcal{C}^{d_s}(W)$, we have that

$$H_{d_s}\left(\Psi(\mathcal{J}^*), \Phi(\mathcal{O}^*)\right) \leq \lambda \mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*)$$

is satisfied, with $\lambda = \max\{\lambda_1, \lambda_2\} = \dfrac{4}{5}$ and

$$\begin{aligned}\mathcal{R}_{\Psi,\Phi}(\mathcal{J}^*, \mathcal{O}^*) \quad = \quad & \max\left\{H_{d_s}(\mathcal{J}^*, \mathcal{O}^*), H_{d_s}(\mathcal{J}^*, \Psi(\mathcal{J}^*)), H_{d_s}(\mathcal{O}^*, \Phi(\mathcal{O}^*))\right., \\ & \frac{H_{d_s}(\mathcal{O}^*, \Phi\left(\mathcal{O}^*\right))[1 + H_{d_s}(\mathcal{J}^*, \Psi(\mathcal{J}^*))]}{1 + H_{d_s}\left(\mathcal{J}^*, \mathcal{O}^*\right)}, \\ & \left.\frac{H_{d_s}(\mathcal{O}^*, \Psi\left(\mathcal{J}^*\right))[1 + H_{d_s}(\mathcal{J}^*, \Psi\left(\mathcal{J}^*\right))]}{1 + H_{d_s}(\mathcal{J}^*, \mathcal{O}^*)}\right\}.\end{aligned}$$

Thus the pair (Ψ, Φ) satisfies the conditions of generalized rational contractive Hutchinson operator and for an abitrarly chosen initial set $\mathcal{J}_0^* \in \mathcal{C}^{d_s}(W)$, the sequence $\{\mathcal{J}_0^*, \Psi\left(\mathcal{J}_0^*\right), \Phi\Psi\left(\mathcal{J}_0^*\right), \Psi\Phi\Psi\left(\mathcal{J}_0^*\right), ...\}$ converges to the unique common attractor of both Ψ and Φ.

6

Common Attractors of Generalized Iterated Function System in G-Metric Spaces

6.1. Introduction

By introducing the concept of G-metric space, Mustafa and Sims [68] expandeded the generalization of metric spaces. Many authors have since obtained fixed point theorems for mappings satisfying various contractive conditions in G-metric spaces [70, 71, 67, 69, 87, 96]. The study of a common fixed point theory in generalized metric spaces [10] was motivated by Abbas and Rhoades [23, 64, 89].

Several researchers have obtained useful results for iterated function systems in the setting of metric spaces (see [76, 81, 90] and references therein). This chapter deals with the construction of common attractors of generalized iterated function system of generalized contractions in a G-metric space setup. We note that the Hutchinson operator, defined on a finite family of contractive mappings on a complete G-metric space is itself a generalized contractive mapping on a family of compact subsets of W. We apply the generalized Hutchinson operator successively to obtain a final fractal. Our findings do not depend on the concept of continuity nor commutativity of mappings under consideration.
Consistent with Mustafa and Sims [70, 68], we state the following preliminary results.

Definition 6.1.1. [68] Let W be a non-void set. A G-metric on W is a mapping $G : W \times W \times W \to \mathbb{R}_{[+]}$ with the following properties:

(1) $G(\varrho_1, \varrho_2, \varrho_3) = 0$ if $\varrho_1 = \varrho_2 = \varrho_3$ (coincidence),

(2) $0 < G(\varrho_1, \varrho_1, \varrho_2)$ for all $\varrho_1, \varrho_2 \in W$, with $\varrho_1 \neq \varrho_2$,

(3) $G(\varrho_1, \varrho_1, \varrho_2) \leq G(\varrho_1, \varrho_2, \varrho_3)$ for all $\varrho_1, \varrho_2, \varrho_3 \in W$, with $\varrho_2 \neq \varrho_3$,

(4) $G(\varrho_1, \varrho_2, \varrho_3) = G(p\{\varrho_1, \varrho_2, \varrho_3\})$, where p is a permutation of $\varrho_1, \varrho_2, \varrho_3$ (sym-

metry),

(5) $G(\varrho_1, \varrho_2, \varrho_3) \leq G(\varrho_1, b, b) + G(b, \varrho_2, \varrho_3)$ for all $\varrho_1, \varrho_2, \varrho_3, b \in W$.

The pair (W, G) consisting of the non-void set together with the G-metric is called a G-metric space.

If $G(\varrho_1, \varrho_2, \varrho_2) = G(\varrho_2, \varrho_1, \varrho_1)$ for all $\varrho_1, \varrho_2 \in W$, then the G-metric is said to be symmetric.

Example 6.1.1. [9] Consider a usual metric space (W, d) and let the function $G : W \times W \times W \to \mathbb{R}_{[+]}$, be defined by

$$G(\varrho_1, \varrho_2, \varrho_3) = \max\{d(\varrho_1, \varrho_2), d(\varrho_2, \varrho_3), d(\varrho_3, \varrho_1)\}$$

or

$$G(\varrho_1, \varrho_2, \varrho_3) = d(\varrho_1, \varrho_2) + d(\varrho_2, \varrho_3) + d(\varrho_3, \varrho_1)$$

for all $\varrho_1, \varrho_2, \varrho_3 \in W$, then (W, G) is a G-metric space.

Example 6.1.2. [9] Let (W, G) be a G-metric space and define the function $d_G : W \times W \to \mathbb{R}_{[+]}$, by

$$d_G(\varrho_1, \varrho_2) = G(\varrho_1, \varrho_2, \varrho_2) + G(\varrho_2, \varrho_1, \varrho_1) \text{ for all } \varrho_1, \varrho_2 \in W,$$

then (d_G, W) is a usual metric space.

Definition 6.1.2. [9] If $\{z_i\}$ is a sequence in a G-metric space (W, G), then

a) $\{z_i\} \subset W$ is a G-convergent sequence if, for a given $\varepsilon > 0$, there is a point $z \in W$ and a natural number N_0 such that for all $i, j \geq N_0$, $G(z, z_i, z_j) < \varepsilon$;

b) $\{z_i\} \subset W$ is a G-Cauchy sequence if, for any $\varepsilon > 0$, there exist a natural number N_0 such that for all $i, j, k \geq N_0$, $G(z_i, z_j, z_k) < \varepsilon$;

c) (W, G) is G-complete if every G-Cauchy sequence in a space W is convergent in W. $\{z_i\}$ converges to $z \in W$ if and only if $G(z_i, z_j, z) \to 0$ as $i, j \to \infty$ and $\{z_i\}$ is Cauchy if and only if $G(z_i, z_j, z_k) \to 0$ as $i, j, k \to +\infty$.

Definition 6.1.3. [9] Suppose (W, G) and (W', G') are two G-metric spaces. Then the map $h^* : (W, G) \to (W', G')$ is said to be G-continuous at $b \in W$, if and only if, for a given given $\varepsilon > 0$, there exists a $\delta > 0$, such that $\varrho_1, \varrho_2 \in W$ and $G(b, \varrho_1, \varrho_2) < \delta$ implies $G'(h^*(b), h^*(\varrho_1), h^*(\varrho_2)) < \varepsilon$. A map h^* is G-continuous on W if and only if it is G-continuous at every $b \in W$.

Proposition 6.1.1. [67] *Given that (W, G) and (W', G') are G-metric spaces,* *then $h^* : W \to W'$ is continuous at $z \in W$ if and only h^* is G-sequentially continuous at z; in other words, whenever $\{z_i\}$ is G-convergent to z, $\{h^*(z_i)\}$ is G-convergent to $h^*(z)$.*

Proposition 6.1.2. [9] *Let (W, G) be a G-metric space. Then the following claims are true:*

1. *$G(\varrho_1, \varrho_2, \varrho_3)$ is simultaneously continuous in all three of its variables,*
2. *$G(\varrho_1, \varrho_2, \varrho_2) \leq 2G(\varrho_2, \varrho_1, \varrho_1)$.*

Next consider the following families of subsets of a G-metric space (W, G) [56].

$N(W) = \{K : K$ is a non-void subset of $W\}$.

$B(W) = \{K : K$ is a non-void bounded subset of $W\}$.

$CL(W) = \{K : K$ is a non-void closed subset of $W\}$.

$\mathcal{CB}(W) = \{K : K$ is a non-void closed and bounded subset of $W\}$.

$\mathcal{C}^G(W) = \{K : K$ is a non-void compact subset of $W\}$.

Remark 6.1.1. [47] *Let (W, G) be a G-metric space. A mapping $H_G : \mathcal{CB}(W) \times \mathcal{CB}(W) \times \mathcal{CB}(W) \to \mathbb{R}_{[+]}$ defined as*

$$H_G(D, E, F) = \max\{\sup_{\varrho_1 \in D} G(\varrho_1, E, F), \sup_{\varrho_2 \in E} G(\varrho_2, F, D), \sup_{\varrho_3 \in F} G(\varrho_3, D, E)\}$$

for all $D, E, F \in \mathcal{CB}(W)$, where $G(\varrho_1, E, F) = \inf\{G(\varrho_1, \varrho_2, \varrho_3) : \varrho_2 \in E, \varrho_3 \in F\}$ is called a Hausdorff G-metric on $\mathcal{CB}(W)$.

If (W, G) is G-complete, then the pair $(\mathcal{CB}(W), H_G)$ is also an H_G-complete metric space.

Lemma 6.1.1. *Let (W, G) be a G-metric space, then for all $\mathcal{P}^*, \mathcal{Q}^*, \mathcal{R}^*, \mathcal{S}^*, \mathcal{U}^*, \mathcal{V}^* \in \mathcal{C}^G(W)$, the following conditions are true:*

(a) *If $\mathcal{Q}^* \subseteq \mathcal{R}^*$, then $\sup_{k \in \mathcal{P}^*} G(k, \mathcal{R}^*, \mathcal{R}^*) \leq \sup_{k \in \mathcal{P}^*} G(k, \mathcal{Q}^*, \mathcal{Q}^*)$;*

(b) $\sup_{t \in \mathcal{P}^* \cup \mathcal{Q}^*} G(t, \mathcal{R}^*, \mathcal{R}^*) = \max\{\sup_{k \in \mathcal{P}^*} G(k, \mathcal{R}^*, \mathcal{R}^*), \sup_{\ell \in \mathcal{Q}^*} G(\ell, \mathcal{R}^*, \mathcal{R}^*)\}$;

(c) $H_G(\mathcal{P}^* \cup \mathcal{Q}^*, \mathcal{R}^* \cup \mathcal{S}^*, \mathcal{U}^* \cup \mathcal{V}^*) \leq \max\{H_G^*(\mathcal{P}^*, \mathcal{R}^*, \mathcal{U}^*), H_G(\mathcal{Q}^*, \mathcal{S}^*, \mathcal{V}^*)\}$.

Proof. (a) Since $\mathcal{Q}^* \subseteq \mathcal{R}^*$, for all $k \in \mathcal{P}^*$, we have

$$\begin{aligned} G(k, \mathcal{R}^*, \mathcal{R}^*) &= \inf\{G(k, \mu, \mu) : \mu \in \mathcal{R}^*\} \\ &\leq \inf\{G(k, \ell, \ell) : \ell \in \mathcal{Q}^*\} = G(k, \mathcal{Q}^*, \mathcal{Q}^*), \end{aligned}$$

this implies that

$$\sup_{k\in\mathcal{P}^*} G(k,\mathcal{R}^*,\mathcal{R}^*) \leq \sup_{k\in\mathcal{P}^*} G(k,\mathcal{Q}^*,\mathcal{Q}^*).$$

(b) Note that

$$\begin{aligned}\sup_{t\in\mathcal{P}^*\cup\mathcal{Q}^*} G(t,\mathcal{R}^*,\mathcal{R}^*) &= \max\{\sup\{G(t,\mathcal{R}^*,\mathcal{R}^*) : t\in\mathcal{P}^*\},\\ &\quad \sup\{G(t,\mathcal{R}^*,\mathcal{R}^*) : t\in\mathcal{Q}^*\}\}\\ &= \max\{\sup_{k\in\mathcal{P}^*} G(k,\mathcal{R}^*,\mathcal{R}^*), \sup_{\ell\in\mathcal{Q}^*} G(\ell,\mathcal{R}^*,\mathcal{R}^*)\}.\end{aligned}$$

(c) We observe that

$$\begin{aligned}&\sup_{t\in\mathcal{P}^*\cup\mathcal{Q}^*} G(t,\mathcal{R}^*\cup\mathcal{S}^*,\mathcal{U}^*\cup\mathcal{V}^*)\\ \leq\ & \max\{\sup_{k\in\mathcal{P}^*} G(k,\mathcal{R}^*\cup\mathcal{S}^*,\mathcal{U}^*\cup\mathcal{V}^*), \sup_{\ell\in\mathcal{Q}^*} G(\ell,\mathcal{Q}^*\cup\mathcal{S}^*,\mathcal{U}^*\cup\mathcal{V}^*)\}\ \text{(from (b))}\\ \leq\ & \max\{\sup_{k\in\mathcal{P}^*} G(k,\mathcal{R}^*,\mathcal{U}^*), \sup_{\ell\in\mathcal{Q}^*} G(\ell,\mathcal{S}^*,\mathcal{V}^*)\} \qquad \text{(from (a))}\\ \leq\ & \max\Big\{\max\{\sup_{k\in\mathcal{P}^*} G(k,\mathcal{R}^*,\mathcal{U}^*), \sup_{\mu\in\mathcal{R}^*} G(\mu,\mathcal{P}^*,\mathcal{U}^*)\},\\ & \max\{\sup_{\ell\in\mathcal{Q}^*} G(\ell,\mathcal{S}^*,\mathcal{V}^*), \sup_{\eta\in\mathcal{S}^*} G(\eta,\mathcal{Q}^*,\mathcal{V}^*)\}\Big\}\\ =\ & \max\{H_G(\mathcal{P}^*,\mathcal{R}^*,\mathcal{U}^*), H_G(\mathcal{Q}^*,\mathcal{S}^*,\mathcal{V}^*)\}.\end{aligned}$$

Similarly,

$$\sup_{v\in\mathcal{R}^*\cup\mathcal{S}^*} G(v,\mathcal{Q}^*\cup\mathcal{P}^*,\mathcal{U}^*\cup\mathcal{V}^*) \leq \max\{H_G(\mathcal{P}^*,\mathcal{R}^*,\mathcal{U}^*), H_G(\mathcal{Q}^*,\mathcal{S}^*,\mathcal{V}^*)\}.$$

Hence

$$\begin{aligned}H_G(\mathcal{P}^*\cup\mathcal{Q}^*,\mathcal{R}^*\cup\mathcal{S}^*,\mathcal{U}^*\cup\mathcal{V}^*) &= \max\Big\{\sup_{v\in\mathcal{P}^*\cup\mathcal{Q}^*} G(v,\mathcal{R}^*\cup\mathcal{S}^*,\mathcal{U}^*\cup\mathcal{V}^*),\\ &\qquad \sup_{t\in\mathcal{R}^*\cup\mathcal{S}^*} G(t,\mathcal{P}^*\cup\mathcal{Q}^*,\mathcal{U}^*\cup\mathcal{V}^*)\Big\}\\ &\leq \max\{H_G(\mathcal{P}^*,\mathcal{R}^*,\mathcal{U}^*), H_G(\mathcal{Q}^*,\mathcal{S}^*,\mathcal{V}^*)\}.\end{aligned}$$

□

Mustafa et al. [71] obtained the following useful result of a unique fixed point of generalized G-contraction on W in G-metric space (W,G).

Theorem 6.1.1. [71] *In a complete G-metric space* (W,G), *let* $h^*: W\to W$ *be* *a generalized G-contraction on W, that is, for all $\varrho_1, \varrho_2, \varrho_3 \in W$, either*

$$\begin{aligned} G\left(h^*\varrho_1, h^*\varrho_2, h^*\varrho_3\right) &\leq \kappa_1 G\left(\varrho_1, \varrho_2, \varrho_3\right) + \kappa_2 G\left(\varrho_1, h^*\varrho_1, h^*\varrho_1\right) + \kappa_3 G\left(\varrho_2, h^*\varrho_2, h^*\varrho_2\right) \\ &\quad + \kappa_4 G\left(\varrho_3, h^*\varrho_3, h^*\varrho_3\right) \end{aligned}$$

or

$$\begin{aligned} G\left(h^*\varrho_1, h^*\varrho_2, h^*\varrho_3\right) &\leq \kappa_1 G\left(\varrho_1, \varrho_2, \varrho_3\right) + \kappa_2 G\left(\varrho_1, \varrho_1, h^*\varrho_1\right) + \kappa_3 G\left(\varrho_2, \varrho_2, h^*\varrho_2\right) \\ &\quad + \kappa_4 G\left(\varrho_3, \varrho_3, h^*\varrho_3\right), \end{aligned}$$

where $\kappa_j \geq 0$ for $j \in \{1, 2, 3, 4\}$ with $0 \leq \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 < 1$. Then h^* has a unique fixed point, $\tilde{u}$ in W. Moreover, for any choice $v_0 \in W$, the sequence of iterates $\{v_0, h^*v_0, (h^*)^2 v_0, (h^*)^3 v_0, ...\}$ converges to $\tilde{u}$. Furthermore, h^* is G-continuous.

Theorem 6.1.2. *In a G-metric space (W, G) consider a G-contraction, $h^* : W \to W$. Then*

a) h^* maps elements in $\mathcal{C}^G(W)$ to elements in $\mathcal{C}^G(W)$.

b) If for any $\mathcal{R}^* \in \mathcal{C}^G(W)$,

$$h^*(\mathcal{R}^*) = \{h^*(\varrho_1) : \varrho_1 \in \mathcal{R}^*\},$$

then $h^* : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ is a G -contraction on $(\mathcal{C}^G(W), H_G)$.

Proof. (a) We observe that every generalized contraction mapping is continuous. Moreover, under every continuous map $h^* : W \to W$, the image of a compact set is also compact, that is, if

$$\mathcal{R}^* \in \mathcal{C}^G(W), \text{ then } h^*(\mathcal{R}^*) \in \mathcal{C}^G(W).$$

(b) Let $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{S}^* \in \mathcal{C}^G(W)$ and $h^* : W \to W$ be a generalized contraction mapping, then

$$\begin{aligned} G\left(h^*\varrho_1, h^*\left(\mathcal{R}^*\right), h^*\left(\mathcal{S}^*\right)\right) &= \inf\{G\left(h^*\varrho_1, h^*\varrho_2, h^*\varrho_3\right) : \varrho_2 \in \mathcal{R}^*, \varrho_3 \in \mathcal{S}^*\} \\ &\leq \inf\{\kappa G\left(\varrho_1, \varrho_2, \varrho_3\right) : \varrho_2 \in \mathcal{R}^*, \varrho_3 \in \mathcal{S}^*\} \\ &= \kappa \inf\{G\left(\varrho_1, \varrho_2, \varrho_3\right) : \varrho_2 \in \mathcal{R}^*, \varrho_3 \in \mathcal{S}^*\} \\ &= \kappa G\left(\varrho_1, \mathcal{R}^*, \mathcal{S}^*\right), \end{aligned}$$

similarly

$$\begin{aligned} G\left(h^*\varrho_3, h^*\left(\mathcal{R}^*\right), h^*\left(\mathcal{Q}^*\right)\right) &= \inf\{G\left(h^*\varrho_3, h^*\varrho_2, h^*\varrho_1\right) : \varrho_2 \in \mathcal{R}^*, \varrho_1 \in \mathcal{Q}^*\} \\ &\leq \inf\{\kappa G\left(\varrho_3, \varrho_2, \varrho_1\right) : \varrho_2 \in \mathcal{R}^*, \varrho_1 \in \mathcal{Q}^*\} \\ &= \kappa \inf\{G\left(\varrho_3, \varrho_2, \varrho_1\right) : \varrho_2 \in \mathcal{R}^*, \varrho_1 \in \mathcal{Q}^*\} \\ &= \kappa G\left(\varrho_3, \mathcal{R}^*, \mathcal{Q}^*\right), \end{aligned}$$

and

$$\begin{aligned} G\left(h^*\varrho_2, h^*\left(\mathcal{Q}^*\right), h^*\left(\mathcal{S}^*\right)\right) &= \inf\{G\left(h^*\varrho_2, h^*\varrho_1, h^*\varrho_3\right) : \varrho_1 \in \mathcal{Q}^*, \varrho_3 \in \mathcal{S}^*\} \\ &\leq \inf\{\kappa G\left(\varrho_2, \varrho_1, \varrho_3\right) : \varrho_1 \in \mathcal{Q}^*, \varrho_3 \in \mathcal{S}^*\} \\ &= \kappa \inf\{G\left(\varrho_2, \varrho_1, \varrho_3\right) : \varrho_1 \in \mathcal{Q}^*, \varrho_3 \in \mathcal{S}^*\} \\ &= \kappa G\left(\varrho_2, \mathcal{Q}^*, \mathcal{S}^*\right). \end{aligned}$$

Now

$$\begin{aligned} & H_G\left(h^*\left(\mathcal{R}^*\right), h^*\left(\mathcal{S}^*\right), h^*\left(\mathcal{Q}^*\right)\right) \\ = \; & \max\{\sup G\left(h^*\varrho_1, h^*\left(\mathcal{R}^*\right), h^*\left(\mathcal{S}^*\right)\right), \sup G\left(h^*\varrho_3, h^*\left(\mathcal{R}^*\right), h^*\left(\mathcal{Q}^*\right)\right), \\ & \sup G\left(h^*\varrho_2, h^*\left(\mathcal{Q}^*\right), h^*\left(\mathcal{S}^*\right)\right); \varrho_1 \in \mathcal{Q}^*, \varrho_3 \in \mathcal{S}^*, \varrho_2 \in \mathcal{R}^*\} \\ \leq \; & \max\{\sup \kappa G\left(\varrho_1, \mathcal{R}^*, \mathcal{S}^*\right), \sup \kappa G\left(\varrho_3, \mathcal{R}^*, \mathcal{Q}^*\right), \\ & \sup \kappa G\left(\varrho_2, \mathcal{Q}^*, \mathcal{S}^*\right); \varrho_1 \in \mathcal{Q}^*, \varrho_3 \in \mathcal{S}^*, \varrho_2 \in \mathcal{R}^*\} \\ = \; & \kappa \max\{\sup G\left(\varrho_1, \mathcal{R}^*, \mathcal{S}^*\right), \sup G\left(\varrho_3, \mathcal{R}^*, \mathcal{Q}^*\right), \\ & \sup G\left(\varrho_2, \mathcal{Q}^*, \mathcal{S}^*\right); \varrho_1 \in \mathcal{Q}^*, \varrho_3 \in \mathcal{S}^*, \varrho_2 \in \mathcal{R}^*\} \\ = \; & \kappa H_G\left(\mathcal{R}^*, \mathcal{S}^*, \mathcal{Q}^*\right). \end{aligned}$$

Thus $h : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ is a G-contraction. □

Theorem 6.1.3. *Consider a G-metric space (W, G) and let $\{h_a^* : a = 1, 2, ..., q\}$ be a finite family of G-contractions on W with contraction constants $\kappa_1, \kappa_2, ..., \kappa_q$, respectively. Define $\Psi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ by*

$$\begin{aligned} \Psi(\mathcal{R}^*) &= h_1^*(\mathcal{R}^*) \cup h_2^*(\mathcal{R}^*) \cup \cdots \cup h_q^*(\mathcal{R}^*) \\ &= \cup_{a=1}^q h_a^*(\mathcal{R}^*), \end{aligned}$$

for every $\mathcal{R}^* \in \mathcal{C}^G(W)$. Then Ψ is also a G -contractive mapping on $\mathcal{C}^G(W)$ with contraction constant $\kappa = \max\{\kappa_1, \kappa_2, ..., \kappa_q\}$.

Proof. We demonstrate the assertion for $q = 2$. Let $h_1^*, h_2^* : W \to W$ be two contractions. Take $\mathcal{R}^*, \mathcal{S}^*, \mathcal{Q}^* \in \mathcal{C}^G(W)$. From Lemma 6.1.1 (c), we have

$$\begin{aligned} H_G(\Psi(\mathcal{R}^*), \Psi(\mathcal{S}^*), \Psi(\mathcal{Q}^*)) &= H_G(h_1^*(\mathcal{R}^*) \cup h_2^*(\mathcal{R}^*), \\ &\quad h_1^*(\mathcal{S}^*) \cup h_2^*(\mathcal{S}^*), h_1^*(\mathcal{Q}^*) \cup h_2^*(\mathcal{Q}^*)) \\ &\leq \max\{H_G(h_1^*(\mathcal{R}^*), h_1^*(\mathcal{S}^*), h_1^*(\mathcal{Q}^*)), \\ &\quad H_G(h_2^*(\mathcal{R}^*), h_2^*(\mathcal{S}^*), h_2^*(\mathcal{Q}^*))\} \\ &\leq \max\{\kappa_1 H_G(\mathcal{R}^*, \mathcal{S}^*, \mathcal{Q}^*), \kappa_2 H_G(\mathcal{R}^*, \mathcal{S}^*, \mathcal{Q}^*)\} \\ &\leq \kappa H_G(\mathcal{R}^*, \mathcal{S}^*, \mathcal{Q}^*), \end{aligned}$$

where $\kappa = \max\{\kappa_1, \kappa_2\}$. □

Theorem 6.1.4. *In a complete G-metric space (W, G), let $\{h_a^* : a = 1, 2, ..., q\}$ be a finite family of G-contraction mappings on W. Define a mapping Ψ on $\mathcal{C}^G(W)$ by*

$$\begin{aligned} \Psi(\mathcal{R}^*) &= h_1^*(\mathcal{R}^*) \cup h_2^*(\mathcal{R}^*) \cup \cdots \cup h_q^*(\mathcal{R}^*) \\ &= \cup_{a=1}^q h_a^*(\mathcal{R}^*), \end{aligned}$$

for each $\mathcal{R}^* \in \mathcal{C}^G(W)$. Then

(i) $\Psi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$.

(ii) *Ψ has exactly one fixed point $\tilde{U_1} \in \mathcal{C}^G(W)$, that is, $\tilde{U_1} = \Psi(\tilde{U_1}) = \cup_{a=1}^q h_a^*(\tilde{U_1})$.*

(iii) *for any set $\mathcal{R}_0^* \in \mathcal{C}^G(W)$, the sequence*

$$\{\mathcal{R}_0^*, \Psi(\mathcal{R}_0^*), \Psi^2(\mathcal{R}_0^*), ...\}$$

converges to $\tilde{U_1}$.

Proof. (i) Since each h_a^* is a G-contraction mapping, the conclusion follows, from the definition of Ψ and Theorem 6.1.2.

(ii) Using Theorem 6.1.3 we note that $\Psi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ is also a G-contraction mapping. Thus if (W, G) is a complete G-metric space, then $(\mathcal{C}^G(W), H_G)$ is complete. Consequently, we deduce (ii) and (iii) from Theorem 6.1.2. □

Example 6.1.3. Consider $W = [0,1]$ and let

$$G(w_1, w_2, w_3) = \max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\}$$

be a G-metric on W. Define $h_1^*, h_2^*, h_3^* : W \to W$ by

$$h_1^*(w_1) = \begin{cases} \dfrac{w_1}{50} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{48} & \text{if } \frac{1}{2} \leq w_1 \leq 1, \end{cases} \qquad h_2^*(w_1) = \begin{cases} \dfrac{w_1}{46} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{44} & \text{if } \frac{1}{2} \leq w_1 \leq 1, \end{cases}$$

$$h_3^*(w_1) = \begin{cases} \dfrac{w_1}{42} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{40} & \text{if } \frac{1}{2} \leq w_1 \leq 1. \end{cases}$$

Then, clearly $\{h_a^* : a = 1,2,3\}$ is a finite family of G-contraction mappings on W. We define a map $\Psi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ by $\Psi(\mathcal{R}^*) = h_1^*(\mathcal{R}^*) \cup h_2^*(\mathcal{R}^*) \cup h_3^*(\mathcal{R}^*)$ for $\mathcal{R}^* \in \mathcal{C}^G(W)$. Then there exists a unique set $\tilde{U_1} = \{0\}$ in $\mathcal{C}^G(W)$ that satisfies $\Psi\left(\tilde{U_1}\right) = \tilde{U_1}$. Moreover, for any set $\mathcal{R}_0^* \in \mathcal{C}^G(W)$, the sequence $\{\mathcal{R}_0^*, \Psi(\mathcal{R}_0^*), \Psi^2(\mathcal{R}_0^*), ...\}$ converges to $\tilde{U_1}$.

Definition 6.1.4. Let (W, G) be a G-metric space. If $h_a^* : W \to W$, $a = 1,2,...,q$ are G-contraction mappings, then $\{W; h_a^*, a = 1,2,\cdots,q\}$ is a G-iterated function system (G -IFS).

It follows that the G-iterated function system is composed of a G-metric space and a finite family of G-contractions on W.

Definition 6.1.5. Let (W, G) be a G -metric space with $\mathcal{R}^* \in \mathcal{C}^G(W)$, then $\mathcal{R}^*$ is called an attractor of the G-iterated function system if

(i) $\Psi(\mathcal{R}^*) = \mathcal{R}^*$ and

(ii) there exists an open set $V_1^* \subseteq W$ such that $\mathcal{R}^* \subseteq V_1^*$ and $\lim_{a\to\infty} \Psi^a(\mathcal{S}^*) = \mathcal{R}^*$ for any compact set $\mathcal{S}^* \subseteq V_1^*$, where the limit is taken with respect to the G-Hausdorff metric.

The maximal open set V_1^* such that (ii) is satisfied is known as a basin of attraction.

6.2. Generalized Iterated Function System in G-metric Spaces

Some results on generalized iterated function system for multi-valued mapping in a metric space appear in [35]. We discuss a generalized iterated function system in the context of G-metric spaces.

Definition 6.2.1. In a G-metric space (W, G), let $f^*, g^*, h^* : W \to W$ be three self-mappings. (f^*, g^*, h^*) is a triplet of generalized G-contraction mappings if

$$G\left(f^* \varrho_1, g^* \varrho_2, h^* \varrho_3\right) \leq \lambda G\left(\varrho_1, \varrho_2, \varrho_3\right)$$

for all $\varrho_1, \varrho_2, \varrho_3 \in W$, where $\lambda \in [0, 1)$.

Theorem 6.2.1. *Consider a G-metric space (W, G) and let $f^*, g^*, h^* : W \to W$ be continuous mappings. If the triplet (f^*, g^*, h^*) is a generalized G-contraction with $\lambda \in [0, 1)$. Then*

(1) *the elements in $\mathcal{C}^G(W)$ are mapped to elements in $\mathcal{C}^G(W)$ under f^*, g^* and h^*;*

(2) *if for an arbitrary $J^* \in \mathcal{C}^G(W)$, the mappings $f^*, h^*, g^* : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ are defined as*

$$\begin{aligned} f^*(J^*) &= \{f^*(\varrho_1) : \varrho_1 \in J^*\}, \\ g^*(J^*) &= \{g^*(\varrho_2) : \varrho_2 \in J^*\}, \\ h^*(J^*) &= \{h^*(\varrho_3) : \varrho_3 \in J^*\}, \end{aligned}$$

then the triplet (f^*, g^*, h^*) is a generalized G-contraction on $(\mathcal{C}^G(W), H_G)$.

Proof. (1) Since f^* is a continuous mapping and the image of a compact subset under a continuous mapping, $f^* : W \to W$ is compact, then

$$J^* \in \mathcal{C}^G(W) \text{ implies that } f^*(J^*) \in \mathcal{C}^G(W).$$

Similarly,

$$J^* \in \mathcal{C}^G(W) \text{ implies that } g^*(J^*) \in \mathcal{C}^G(W) \text{ and } h^*(J^*) \in \mathcal{C}^G(W).$$

(2) Let $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$. Since the triplet (f^*, g^*, h^*) consists of generalized G-contraction mappings on W, then we have

$$G\left(f^* \varrho_1, g^* \varrho_2, h^* \varrho_3\right) \leq \lambda G\left(\varrho_1, \varrho_2, \varrho_3\right) \text{ for all } \varrho_1, \varrho_2, \varrho_3 \in W,$$

where $\lambda \in [0,1)$.

Now

$$\begin{aligned} G\left(f^*\varrho_1, g^*\left(\mathcal{R}^*\right), h^*\left(\mathcal{N}^*\right)\right) &= \inf\{G\left(f^*\varrho_1, g^*\varrho_2, h^*\varrho_3\right) : \varrho_2 \in \mathcal{R}^*, \varrho_3 \in \mathcal{N}^*\} \\ &\leq \inf\{\lambda G\left(\varrho_1, \varrho_2, \varrho_3\right) : \varrho_2 \in \mathcal{R}^*, \varrho_3 \in \mathcal{N}^*\} \\ &= \lambda G\left(\varrho_1, \mathcal{R}^*, \mathcal{N}^*\right). \end{aligned}$$

In the same manner,

$$\begin{aligned} G\left(g^*\varrho_2, f^*\left(\mathcal{Q}^*\right), h^*\left(\mathcal{N}^*\right)\right) &= \inf\{G\left(g^*\varrho_2, f^*\varrho_1, h^*\varrho_3\right) : \varrho_1 \in \mathcal{Q}^*, \varrho_3 \in \mathcal{N}^*\} \\ &\leq \inf\{\lambda G\left(\varrho_2, \varrho_1, \varrho_3\right) : \varrho_1 \in \mathcal{Q}^*, \varrho_3 \in \mathcal{N}^*\} \\ &= \lambda G\left(\varrho_2, \mathcal{Q}^*, \mathcal{N}^*\right) \end{aligned}$$

and

$$\begin{aligned} G\left(h^*\varrho_3, f^*\left(\mathcal{Q}^*\right), g^*\left(\mathcal{R}^*\right)\right) &= \inf\{G\left(h^*\varrho_3, f^*\varrho_1, g^*\varrho_3\right) : \varrho_1 \in \mathcal{Q}^*, \varrho_2 \in \mathcal{R}^*\} \\ &\leq \inf\{\lambda G\left(\varrho_3, \varrho_1, \varrho_2\right) : \varrho_1 \in \mathcal{Q}^*, \varrho_2 \in \mathcal{R}^*\} \\ &= \lambda G\left(\varrho_3, \mathcal{Q}^*, \mathcal{R}^*\right). \end{aligned}$$

Now

$$\begin{aligned} & H_G\left(f^*\left(\mathcal{Q}^*\right), g^*\left(\mathcal{R}^*\right), h^*\left(\mathcal{N}^*\right)\right) \\ = \;& \max\{\sup_{\varrho_1 \in \mathcal{Q}^*} G(f^*\varrho_1, g^*\left(\mathcal{R}^*\right), h^*\left(\mathcal{N}^*\right)), \\ & \sup_{\varrho_2 \in \mathcal{R}^*} G(g^*\varrho_2, f^*\left(\mathcal{Q}^*\right), h^*\left(\mathcal{N}^*\right)), \sup_{\varrho_3 \in \mathcal{N}^*} G(h^*\varrho_3, f^*\left(\mathcal{Q}^*\right), g^*\left(\mathcal{R}^*\right))\} \\ \leq \;& \max\{\sup_{\varrho_1 \in \mathcal{Q}^*} \lambda G(\varrho_1, \mathcal{R}^*, \mathcal{N}^*), \sup_{\varrho_2 \in \mathcal{R}^*} \lambda G(\varrho_2, \mathcal{Q}^*, \mathcal{N}^*), \sup_{\varrho_3 \in \mathcal{N}^*} \lambda G(\varrho_3, \mathcal{Q}^*, \mathcal{R}^*)\} \\ = \;& \lambda \max\{\sup_{\varrho_1 \in \mathcal{Q}^*} G(\varrho_1, \mathcal{R}^*, \mathcal{N}^*), \sup_{\varrho_2 \in \mathcal{Q}^*} G(\varrho_2, \mathcal{Q}^*, \mathcal{N}^*), \sup_{\varrho_3 \in \mathcal{N}^*} G(\varrho_3, \mathcal{Q}^*, \mathcal{R}^*)\} \\ = \;& \lambda H_G\left(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*\right). \end{aligned}$$

Hence, (f^*, g^*, h^*) is a triplet of generalized G-contraction mappings on $(\mathcal{C}^G(W), H_G)$. □

Proposition 6.2.1. *In a G-metric space (W,G), suppose the mappings $f_a^*, g_a^*, h_a^* : W \to W$ for $a = 1, 2, \cdots, q$ are continuous and satisfy*

$$G\left(f_a^*\varrho_1, g_a^*\varrho_2, h_a^*\varrho_3\right) \leq \lambda_a G\left(\varrho_1, \varrho_2, \varrho_3\right) \text{ for all } \varrho_1, \varrho_2, \varrho_3 \in W,$$

where $\lambda_a \in [0,1)$ for each $a \in \{1, 2, \cdots, q\}$. Then the mappings Υ, Ψ, Φ :

$\mathcal{C}^G(W) \to \mathcal{C}^G(W)$ defined as

$$\begin{aligned}\Upsilon(\mathcal{Q}^*) &= f_1^*(\mathcal{Q}^*) \cup f_2^*(\mathcal{Q}^*) \cup \cdots \cup f_q^*(\mathcal{Q}^*) \\ &= \cup_{a=1}^{q} f_a^*(\mathcal{Q}^*), \text{ for each } \mathcal{Q}^* \in \mathcal{C}^G(W),\end{aligned}$$

$$\begin{aligned}\Psi(\mathcal{R}^*) &= g_1^*(\mathcal{R}^*) \cup g_2^*(\mathcal{R}^*) \cup \cdots \cup g_q(\mathcal{R}^*) \\ &= \cup_{a=1}^{q} g_a^*(\mathcal{R}^*), \text{ for each } \mathcal{R}^* \in \mathcal{C}^G(W)\end{aligned}$$

and

$$\begin{aligned}\Phi(\mathcal{N}^*) &= h_1^*(\mathcal{N}^*) \cup h_2^*(\mathcal{N}^*) \cup \cdots \cup h_q^*(\mathcal{N}^*) \\ &= \cup_{a=1}^{q} h_a^*(\mathcal{N}^*), \text{ for each } \mathcal{N}^* \in \mathcal{C}^G(W)\end{aligned}$$

also satisfy

$$H_G(\Upsilon\mathcal{Q}^*, \Psi\mathcal{R}^*, \Phi\mathcal{N}^*) \leq \lambda_* H_G(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*) \text{ for all } \mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W),$$

where $\lambda_* = \max\{\lambda_a : a = 1, 2, ..., q\}$, that is, the triplet (Υ, Ψ, Φ) is a generalized G-contraction on $\mathcal{C}^G(W)$.

Proof. We give a proof for $q = 2$. Let $f_a^*, g_a^*, h_a^*, : W \to W$, $a \in \{1, 2\}$ be self-mappings such that (f_1^*, g_1^*, h_1^*) and (f_2^*, g_2^*, h_2^*) are triplets of generalized G-contractions. For $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$ and from Lemma 6.1.1 (c),

$$\begin{aligned}H_G(\Upsilon(\mathcal{Q}^*), \Psi(\mathcal{R}^*), \Phi(\mathcal{N}^*)) &= H_G(f_1^*(\mathcal{Q}^*) \cup f_2^*(\mathcal{Q}^*), g_1^*(\mathcal{R}^*) \cup g_2^*(\mathcal{R}^*), \\ &\quad h_1^*(\mathcal{N}^*) \cup h_2^*(\mathcal{N}^*)) \\ &\leq \max\{H_G(f_1^*(\mathcal{Q}^*), g_1^*(\mathcal{R}^*), h_1^*(\mathcal{N}^*)), \\ &\quad H_G(f_2^*(\mathcal{Q}^*), g_2^*(\mathcal{R}^*), h_2^*(\mathcal{N}^*))\} \\ &\leq \max\{\lambda_1 H_G(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*), \lambda_2 H_G(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*)\} \\ &\leq \lambda_* H_G(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*).\end{aligned}$$

□

Definition 6.2.2. In a G-metric space (W, G), let $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$. The mappings (Υ, Ψ, Φ) are called

(I) generalized G-Hutchinson contractive operators (type I) if for any

$\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$,

$$H_G(\Upsilon(\mathcal{Q}^*), \Psi(\mathcal{R}^*), \Phi(\mathcal{N}^*)) \leq A_{\Upsilon,\Psi,\Phi}(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*)$$

holds, where

$$\begin{aligned} A_{\Upsilon,\Psi,\Phi}(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*) &= \alpha H_G(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*) + \beta H_G(\mathcal{Q}^*, \Upsilon(\mathcal{Q}^*), \Upsilon(\mathcal{Q}^*)) \\ &\quad + \gamma H_G(\mathcal{R}^*, \Psi(\mathcal{R}^*), \Psi(\mathcal{R}^*)) + \eta H_G(\mathcal{N}^*, \Phi(\mathcal{N}^*), \Phi(\mathcal{N}^*)), \end{aligned}$$

with $\alpha, \beta, \gamma, \eta \geq 0$ and $\alpha + \beta + \gamma + \eta < 1$.

(II) generalized G-Hutchinson contractive operators (type II) if for any $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$,

$$H_G(\Upsilon(\mathcal{Q}^*), \Psi(\mathcal{R}^*), \Phi(\mathcal{N}^*)) \leq E_{\Upsilon,\Psi,\Phi}(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*)$$

holds, where

$$\begin{aligned} E_{\Upsilon,\Psi,\Phi}(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*) &= \lambda_1 H_G(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*) + \lambda_2[H_G(\mathcal{Q}^*, \mathcal{Q}^*, \Upsilon(\mathcal{Q}^*)) \\ &\quad + H_G(\mathcal{R}^*, \mathcal{R}^*, \Psi(\mathcal{R}^*)) + H_G(\mathcal{N}^*, \mathcal{N}^*, \Phi(\mathcal{N}^*))] \\ &\quad + \lambda_3[H_G(\Upsilon(\mathcal{Q}^*), \mathcal{R}^*, \mathcal{N}^*) + H_G(\mathcal{Q}^*, \Psi(\mathcal{R}^*), \mathcal{N}^*) \\ &\quad + H_G(\mathcal{Q}^*, \mathcal{R}^*, \Phi(\mathcal{N}^*))], \end{aligned}$$

with $\lambda_j \geq 0$ for $j \in \{1, 2, 3\}$ and $\lambda_1 + 3\lambda_2 + 4\lambda_3 < 1$.

Note that if the mappings (Υ, Ψ, Φ) defined as in Proposition 6.2.1 are generalized G-contractions on $\mathcal{C}^G(W)$, then (Υ, Ψ, Φ) is a triplet of generalized G-Hutchinson contractive operators, but the converse is not true.

Definition 6.2.3. In a complete G-metric space (W, G), let $f_a^*, g_a^*, h_a^* : W \to W$, $a = 1, 2, ..., q$ be continuous mappings such that each triplet (f_a^*, g_a^*, h_a^*) for $a = 1, 2, ..., q$ is a generalized G-contraction, then $\{W; (f_a^*, g_a^*, h_a^*), a = 1, 2, \cdots, q\}$ is called the generalized G-iterated function system.

As a consequence, the generalized G-iterated function system consists of a G-metric space and a finite collection of generalized G-contraction mappings on W.

Definition 6.2.4. Let (W, G) be a complete G-metric space and $\tilde{U_1} \subseteq W$ a nonvoid compact set. Then $\tilde{U_1}$ is the common attractor of the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ if

i) $\Upsilon(\tilde{U_1}) = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1}) = \tilde{U_1}$ and

ii) there exists an open set $\mathcal{V}_1^* \subseteq W$ such that $\tilde{U_1} \subseteq \mathcal{V}_1^*$ and $\lim_{a\to+\infty} \Upsilon^a(\mathcal{Q}^*) = \lim_{a\to+\infty} \Psi^a(\mathcal{R}^*) = \lim_{a\to+\infty} \Phi^a(\mathcal{N}^*) = \tilde{U_1}$ for any compact sets $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \subseteq \mathcal{V}_1^*$, where the limit is taken relative to the G-Hausdorff metric.

6.3. Generalized G-Hutchinson contractive operators in G-metric spaces

We state and prove some theorems on the existence and uniqueness of a common attractor of generalized G-Hutchinson contractive operators in the framework of G-metric spaces.

Theorem 6.3.1. *In a complete G-metric space (W, G), let $\{W; (f_a^*, g_a^*, h_a^*), a = 1, 2, \cdots, q\}$ be the generalized G -iterated function system. Define $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ by*

$$\begin{aligned}\Upsilon(\mathcal{Q}^*) &= f_1^*(\mathcal{Q}^*) \cup f_2^*(\mathcal{Q}^*) \cup \cdots \cup f_q^*(\mathcal{Q}^*)\\ &= \cup_{a=1}^q f_a^*(\mathcal{Q}^*),\end{aligned}$$

$$\begin{aligned}\Psi(\mathcal{R}^*) &= g_1^*(\mathcal{R}^*) \cup g_2^*(\mathcal{R}^*) \cup \cdots \cup g_q^*(\mathcal{R}^*)\\ &= \cup_{a=1}^q g_a(\mathcal{R}^*),\end{aligned}$$

and

$$\begin{aligned}\Phi(\mathcal{N}^*) &= h_1^*(\mathcal{N}^*) \cup h_2^*(\mathcal{N}^*) \cup \cdots \cup h_q^*(\mathcal{N}^*)\\ &= \cup_{a=1}^q h_a^*(\mathcal{N}^*)\end{aligned}$$

for $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$. If the mappings (Υ, Ψ, Φ) are a triplet of generalized G-Hutchinson contractive operators (type I), then Υ, Ψ and Φ have a unique common attractor $\tilde{U_1} \in \mathcal{C}^G(W)$, that is,

$$\tilde{U_1} = \Upsilon(\tilde{U_1}) = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1}).$$

Additionally, for any arbitrarily chosen initial set $\mathcal{R}_0^* \in \mathcal{C}^G(W)$, the sequence

$$\{\mathcal{R}_0^*, \Upsilon(\mathcal{R}_0^*), \Psi\Upsilon(\mathcal{R}_0^*), \Phi\Psi\Upsilon(\mathcal{R}_0^*), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0^*), ...\}$$

of compact sets converges to the unique common attractor $\tilde{U_1}$.

Proof. We show that any attractor of Υ is an attractor of Ψ and Φ. To that end, we assume that $\tilde{U_1} \in \mathcal{C}^G(W)$ is such that $\Upsilon(\tilde{U_1}) = \tilde{U_1}$. We need to show that $\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1})$. As the mappings (Υ, Ψ, Φ) are a triplet of generalized G-Hutchinson contractive operators (type I), we get

$$\begin{aligned} H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi(\tilde{U_1})) &= H_G(\Upsilon(\tilde{U_1}), \Psi(\tilde{U_1}), \Phi(\tilde{U_1})) \\ &\leq \alpha H_G(\tilde{U_1}, \tilde{U_1}, \tilde{U_1}) + \beta H_G(\tilde{U_1}, \Upsilon(\tilde{U_1}), \Upsilon(\tilde{U_1})) \\ &\quad + \gamma H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Psi(\tilde{U_1})) + \eta H_G(\tilde{U_1}, \Phi(\tilde{U_1}), \Phi(\tilde{U_1})) \\ &= \gamma H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Psi(\tilde{U_1})) + \eta H_G(\tilde{U_1}, \Phi(\tilde{U_1}), \Phi(\tilde{U_1})) \\ &\leq (\gamma + \eta) H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi(\tilde{U_1})), \end{aligned}$$

thus

$$H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi(\tilde{U_1})) \leq \lambda H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi\left(\tilde{U_1}\right)),$$

where $\lambda = \gamma + \eta < 1$, which implies that $H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi(\tilde{U_1})) = 0$ and so $\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1})$. In an analogous manner, for $\tilde{U_1} = \Phi(\tilde{U_1})$ or for $\tilde{U_1} = \Psi(\tilde{U_1})$, we obtain that $\tilde{U_1}$ is the common attractor of Υ, Ψ and Φ.

We proceed by showing that Υ, Ψ and Φ have a unique common attractor. Let $\mathcal{R}_0^* \in \mathcal{C}^G(W)$ be chosen randomly. Define a sequence $\{\mathcal{R}_a^*\}$ by $\mathcal{R}_{3a+1}^* = \Upsilon(\mathcal{R}_{3a}^*)$, $\mathcal{R}_{3a+2}^* = \Psi(\mathcal{R}_{3a+1}^*)$ and $\mathcal{R}_{3a+3}^* = \Phi(\mathcal{R}_{a+2}^*)$, $a = 0, 1, 2, \cdots$. If $\mathcal{R}_a^* = \mathcal{R}_{a+1}^*$ for some a, with $a = 3n$, then $\tilde{U_1} = \mathcal{R}_{3a}^*$ is an attractor of Υ and from the Proof above, $\tilde{U_1}$ is a common attractor for Υ, Ψ and Φ. The same is true for $a = 3n+1$ or $a = 3n+2$. We assume that $\mathcal{R}_a^* \neq \mathcal{R}_{a+1}^*$ for all $a \in \mathbb{N}$, then

$$\begin{aligned} & H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*) \\ = \; & H_G(\Upsilon(\mathcal{R}_{3a}^*), \Psi\left(\mathcal{R}_{3a+1}^*\right), \Phi\left(\mathcal{R}_{3a+2}^*\right)) \\ \leq \; & \alpha H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + \beta H_G(\mathcal{R}_{3a}^*, \Upsilon(\mathcal{R}_{3a}^*), \Upsilon(\mathcal{R}_{3a}^*)) \\ & + \gamma H_G(\mathcal{R}_{3a+1}^*, \Psi\left(\mathcal{R}_{3a+1}^*\right), \Psi\left(\mathcal{R}_{3a+1}^*\right)) + \eta H_G(\mathcal{R}_{3a+2}^*, \Phi\left(\mathcal{R}_{3a+2}^*\right), \Phi\left(\mathcal{R}_{3a+2}^*\right)) \\ = \; & \alpha H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + \beta H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+1}^*) \\ & + \gamma H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+2}^*) + \eta H_G(\mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*, \mathcal{R}_{3a+3}^*) \\ \leq \; & \alpha H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + \beta H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) \\ & + \gamma H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*) + \eta H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*). \end{aligned}$$

Thus, we have

$$(1 - \gamma - \eta) H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*) \leq (\alpha + \beta) H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*).$$

Hence,

$$H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*) \leq \lambda H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*),$$

where $\lambda = \dfrac{\alpha+\beta}{1-\gamma-\eta}$, with $0<\lambda<1$. Similarly, one can show that

$$H_G(\mathcal{R}^*_{3a+2},\mathcal{R}^*_{3a+3},\mathcal{R}^*_{3a+4}) \leq \lambda H_G(\mathcal{R}^*_{3a+1},\mathcal{R}^*_{3a+2},\mathcal{R}^*_{3a+3})$$

and

$$H_G(\mathcal{R}^*_{3a+3},\mathcal{R}^*_{3a+4},\mathcal{R}^*_{3a+5}) \leq \lambda H_G(\mathcal{R}^*_{3a+2},\mathcal{R}^*_{3a+3},\mathcal{R}^*_{3a+4}).$$

Thus, for all a,

$$\begin{aligned} H_G(\mathcal{R}^*_{a+1},\mathcal{R}^*_{a+2},\mathcal{R}^*_{a+3}) &\leq \lambda H_G\left(\mathcal{R}^*_{a},\mathcal{R}^*_{a+1},\mathcal{R}^*_{a+2}\right) \\ &\leq \cdots \leq \lambda^{a+1} H_G\left(\mathcal{R}^*_{0},\mathcal{R}^*_{1},\mathcal{R}^*_{2}\right). \end{aligned}$$

Now, for l, m, a, with $l > m > a$,

$$\begin{aligned} H_G(\mathcal{R}^*_{a},\mathcal{R}^*_{m},\mathcal{R}^*_{l}) &\leq H_G(\mathcal{R}^*_{a},\mathcal{R}^*_{a+1},\mathcal{R}^*_{a+1}) + H_G(\mathcal{R}^*_{a+1},\mathcal{R}^*_{a+2},\mathcal{R}^*_{a+2}) \\ &\quad + \cdots + H_G(\mathcal{R}^*_{l-1},\mathcal{R}^*_{l-1},\mathcal{R}_{l}) \\ &\leq H_G(\mathcal{R}^*_{a},\mathcal{R}^*_{a+1},\mathcal{R}^*_{a+2}) + H_G(\mathcal{R}^*_{a+1},\mathcal{R}^*_{a+2},\mathcal{R}^*_{a+3}) \\ &\quad + \cdots + H_G(\mathcal{R}^*_{l-2},\mathcal{R}^*_{l-1},\mathcal{R}^*_{l}) \\ &\leq [\lambda^a + \lambda^{a+1} + \cdots + \lambda^{l-2}] H_G\left(\mathcal{R}^*_{0},\mathcal{R}^*_{1},\mathcal{R}^*_{2}\right) \\ &\leq \frac{\lambda^a}{1-\lambda} H_G(\mathcal{R}^*_{0},\mathcal{R}^*_{1},\mathcal{R}^*_{2}). \end{aligned}$$

Note that if $l = m > a$, we get identical results and if $l > m = a$, then

$$H_G(\mathcal{R}^*_{a},\mathcal{R}^*_{m},\mathcal{R}^*_{l}) \leq \frac{\lambda^{a-1}}{1-\lambda} H_G(\mathcal{R}^*_{0},\mathcal{R}^*_{1},\mathcal{R}^*_{2}).$$

and so $\lim\limits_{a,m,l\to+\infty} H_G(\mathcal{R}^*_{a},\mathcal{R}^*_{m},\mathcal{R}^*_{l}) = 0$. Thus $\{\mathcal{R}^*_a\}$ is a G-Cauchy sequence in $\mathcal{C}^G(W)$. Since $(\mathcal{C}^G(W), H_G)$ is a complete G-metric space, there exists $\tilde{U}_1 \in \mathcal{C}^G(W)$ such that $\lim\limits_{a\to+\infty} \mathcal{R}^*_a = \tilde{U}_1$, that is, $\lim\limits_{a\to+\infty} H_G(\mathcal{R}^*_{a},\mathcal{R}^*_{a},\tilde{U}_1) = 0$.

Assume that $\Upsilon(\tilde{U}_1) = \tilde{U}_1$, otherwise, we see that

$$\begin{aligned} & H_G(\Upsilon(\tilde{U}_1),\mathcal{R}^*_{3a+2},\mathcal{R}^*_{3a+3}) \\ = \; & H_G(\Upsilon(\tilde{U}_1),\Psi\left(\mathcal{R}^*_{3a+1}\right),\Phi\left(\mathcal{R}^*_{3a+2}\right)) \\ \leq \; & \alpha H_G(\tilde{U}_1,\mathcal{R}^*_{3a+1},\mathcal{R}^*_{3a+2}) + \beta H_G(\tilde{U}_1,\Upsilon(\tilde{U}_1),\Upsilon(\tilde{U}_1)) \\ & + \gamma H_G(\mathcal{R}^*_{3a+1},\Psi\left(\mathcal{R}^*_{3a+1}\right),\Psi\left(\mathcal{R}^*_{3a+1}\right)) + \eta H_G(\mathcal{R}^*_{3a+2},\Phi\left(\mathcal{R}^*_{3a+2}\right),\Phi\left(\mathcal{R}^*_{3a+2}\right)) \\ = \; & \alpha H_G(\tilde{U}_1,\mathcal{R}^*_{3a+1},\mathcal{R}^*_{3a+2}) + \beta H_G(\Upsilon(\tilde{U}_1),\tilde{U}_1,\mathcal{R}^*_{3a+1}) \\ & + \gamma H_G(\mathcal{R}^*_{3a+1},\mathcal{R}^*_{3a+2},\mathcal{R}^*_{3a+2}) + \eta H_G(\mathcal{R}^*_{3a+2},\mathcal{R}^*_{3a+3},\mathcal{R}^*_{3a+3}). \end{aligned}$$

So $\lim_{a\to+\infty} H_G(\Upsilon(\tilde{U_1}), \mathcal{R}^*_{3a+2}, \mathcal{R}^*_{3a+3}) = H_G(\Upsilon(\tilde{U_1}), \tilde{U_1}, \tilde{U_1})$, that is to say

$$H_G(\Upsilon(\tilde{U_1}), \tilde{U_1}, \tilde{U_1}) \leq \beta H_G(\Upsilon(\tilde{U_1}), \tilde{U_1}, \tilde{U_1}),$$

which is a contradiction as $\beta < 1$. Thus $\Upsilon(\tilde{U_1}) = \tilde{U_1}$. Following the conclusion above, we conclude that $\tilde{U_1}$ is the common attractor of Υ, Ψ and Φ.

For uniqueness, assume that $\tilde{U_2}$ is also a common attractor of Υ, Ψ and Φ. Then

$$\begin{aligned}
H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) &= H_G(\Upsilon(\tilde{U_1}), \Psi(\tilde{U_2}), \Phi(\tilde{U_2})) \\
&\leq \alpha H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) + \beta H_G(\tilde{U_1}, \Upsilon(\tilde{U_1}), \Upsilon(\tilde{U_1})) \\
&\quad +\gamma H_G(\tilde{U_2}, \Psi(\tilde{U_2}), \Psi(\tilde{U_2})) + \eta H_G(\tilde{U_2}, \Phi(\tilde{U_2}), \Phi(\tilde{U_2})) \\
&= \alpha H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) + \beta H_G(\tilde{U_1}, \tilde{U_1}, \tilde{U_1}) \\
&\quad +\gamma H_G(\tilde{U_2}, \tilde{U_2}, \tilde{U_2}) + \eta H_G(\tilde{U_2}, \tilde{U_2}, \tilde{U_2}) \\
&= \alpha H_G(\tilde{U_2}, \tilde{U_2}, \tilde{U_2})
\end{aligned}$$

from which we conclude that $H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) = 0$ and thus $\tilde{U_1} = \tilde{U_2}$. Hence $\tilde{U_1}$ is a unique common attractor of Υ, Ψ and Φ. □

Theorem 6.3.2. *(Generalized Collage I) In a complete G-metric space (W, G), let $\{W; (f^*_a, g^*_a, h^*_a), a = 1, 2, \cdots, q\}$ be the generalized G-iterated function system. Define $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ by*

$$\Upsilon(\mathcal{Q}^*) = \cup_{a=1}^q f^*_a(\mathcal{Q}^*),$$

$$\Psi(\mathcal{R}^*) = \cup_{a=1}^q g^*_a(\mathcal{R}^*),$$

and

$$\Phi(\mathcal{N}^*) = \cup_{a=1}^q h^*_a(\mathcal{N}^*)$$

for $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$. Suppose that the mappings (Υ, Ψ, Φ) are a triplet of generalized G-Hutchinson contractive operators (type I) and $\tilde{U_1} \in \mathcal{C}^p(W)$ is the common attractor for Υ, Ψ and Φ. Then for any given $\varepsilon > 0$ and $\mathcal{R}^* \in \mathcal{C}^G(W)$ the following hold:

(a) $H_G(\mathcal{R}^*, \Upsilon(\mathcal{R}^*), \Upsilon(\mathcal{R}^*)) \leq \varepsilon$, implies that

$$H_G(\mathcal{R}^*, \tilde{U_1}, \tilde{U_1}) \leq \frac{\varepsilon\,(1+\beta)}{1-\alpha}.$$

(b) $H_G(\mathcal{R}^*, \Psi(\mathcal{R}^*), \Psi(\mathcal{R}^*)) \leq \varepsilon$, *implies that*

$$H_G(\mathcal{R}^*, \tilde{U}_1, \tilde{U}_1) \leq \frac{\varepsilon\,(1+\gamma)}{1-\alpha}.$$

(c) $H_G(\mathcal{R}^*, \Phi(\mathcal{R}^*), \Phi(\mathcal{R}^*)) \leq \varepsilon$, *implies that*

$$H_G(\mathcal{R}^*, \tilde{U}_1, \tilde{U}_1) \leq \frac{\varepsilon\,(1+\eta)}{1-\alpha}.$$

Proof. To prove (a): Let $H_G(\mathcal{R}^*, \Upsilon(\mathcal{R}^*), \Upsilon(\mathcal{R}^*)) \leq \varepsilon$ for any $\mathcal{R}^* \in \mathcal{C}^G(W)$, then

$$\begin{aligned}
H_G(\mathcal{R}^*, \tilde{U}_1, \tilde{U}_1) &\leq H_G(\mathcal{R}^*, \Upsilon(\mathcal{R}^*), \Upsilon(\mathcal{R}^*)) + H_G(\Upsilon(\mathcal{R}^*), \tilde{U}_1, \tilde{U}_1) \\
&= H_G(\mathcal{R}^*, \Upsilon(\mathcal{R}^*), \Upsilon(\mathcal{R}^*)) + H_G(\Upsilon(\mathcal{R}^*), \Psi(\tilde{U}_1), \Phi(\tilde{U}_1)) \\
&\leq \varepsilon + \alpha H_G(\mathcal{R}^*, \tilde{U}_1, \tilde{U}_1) + \beta H_G(\mathcal{R}^*, \Upsilon(\mathcal{R}^*), \Upsilon(\mathcal{R}^*)) \\
&\quad +\gamma H_G(\tilde{U}_1, \Psi(\tilde{U}_1), \Psi(\tilde{U}_1)) + \eta H_G(\tilde{U}_1, \Phi(\tilde{U}_1), \Phi(\tilde{U}_1)) \\
&= \varepsilon + \alpha H_G(\mathcal{R}^*, \tilde{U}_1, \tilde{U}_1) + \beta H_G(\mathcal{R}^*, \Upsilon(\mathcal{R}^*), \Upsilon(\mathcal{R}^*)),
\end{aligned}$$

which further implies that

$$H_G(\mathcal{R}^*, \tilde{U}_1, \tilde{U}_1) \leq \frac{\varepsilon\,(1+\beta)}{1-\alpha}.$$

To prove (b): Assume that $H_G(\mathcal{R}^*, \Psi(\mathcal{R}^*), \Psi(\mathcal{R}^*)) \leq \varepsilon$ for any $\mathcal{R}^* \in \mathcal{C}^G(W)$. Then,

$$\begin{aligned}
H_G(\mathcal{R}^*, \tilde{U}_1, \tilde{U}_1) &\leq H_G(\mathcal{R}^*, \Psi(\mathcal{R}^*), \Psi(\mathcal{R}^*)) + H_G(\Psi(\mathcal{R}^*), \tilde{U}_1, \tilde{U}_1) \\
&\leq \varepsilon + H_G(\Upsilon(U_1), \Psi(\mathcal{R}^*), \Phi(\tilde{U}_1)) \\
&\leq \varepsilon + \alpha H_G(\tilde{U}_1, \mathcal{R}^*, \tilde{U}_1) + \beta H_G(\tilde{U}_1, \Upsilon(\tilde{U}_1), \Upsilon(\tilde{U}_1)) \\
&\quad +\gamma H_G(\mathcal{R}^*, \Psi(\mathcal{R}^*), \Psi(\mathcal{R}^*)) + \eta H_G(\tilde{U}_1, \Phi(\tilde{U}_1), \Phi(\tilde{U}_1)) \\
&= \varepsilon + \alpha H_G(\mathcal{R}^*, \tilde{U}_1, \tilde{U}_1) + \gamma H_G(\mathcal{R}^*, \Psi(\mathcal{R}^*), \Psi(\mathcal{R}^*)),
\end{aligned}$$

which further implies that

$$H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U}_1) \leq \frac{\varepsilon\,(1+\gamma)}{1-\alpha}.$$

To prove (c): Assuming that $H_G(\mathcal{R}^*, \Phi(\mathcal{R}^*), \Phi(\mathcal{R}^*)) \leq \varepsilon$ for any $\mathcal{R}^* \in \mathcal{C}^G(W)$, we have

$$
\begin{aligned}
H_G(\mathcal{R}^*, \tilde{U_1}, \tilde{U_1}) &\leq H_G(\mathcal{R}^*, \Phi(\mathcal{R}^*), \Phi(\mathcal{R}^*)) + H_G(\Phi(\mathcal{R}^*), \tilde{U_1}, \tilde{U_1}) \\
&\leq \varepsilon + H_G(\Upsilon(\tilde{U_1}), \Psi(\tilde{U_1}), \Phi(\mathcal{R}^*)) \\
&\leq \varepsilon + \alpha H_G(\tilde{U_1}, \tilde{U_1}, \mathcal{R}^*) + \beta H_G(\tilde{U_1}, \Upsilon(\tilde{U_1}), \Upsilon(\tilde{U_1})) \\
&\quad + \gamma H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Psi(\tilde{U_1})) + \eta H_G(\mathcal{R}^*, \Phi(\mathcal{R}^*), \Phi(\mathcal{R})) \\
&= \varepsilon + \alpha H_G(\mathcal{R}^*, \tilde{U_1}, \tilde{U_1}) + \eta H_G(\mathcal{R}^*, \Phi(\mathcal{R}^*), \Phi(\mathcal{R}^*)),
\end{aligned}
$$

from which we have

$$
H_G(\mathcal{R}^*, \tilde{U_1}, \tilde{U_1}) \leq \frac{\varepsilon\,(1+\eta)}{1-\alpha}.
$$

□

Theorem 6.3.3. ***(Generalized Collage II)*** *In a complete G-metric space (W, G), suppose $\{W; (f_a^*, g_a^*, h_a^*), a = 1, 2, \cdots, q\}$ is a generalized G-iterated function system with contractive constant $\lambda \in [0, 1)$. Given any $\mathcal{R}^* \in \mathcal{C}^G(W)$ and $\varepsilon > 0$ such that either*

$$
H_G(\mathcal{R}^*, \mathcal{R}^*, \Upsilon(\mathcal{R}^*)) \leq \varepsilon
$$

or

$$
H_G(\mathcal{R}^*, \mathcal{R}^*, \Psi(\mathcal{R}^*)) \leq \varepsilon
$$

or

$$
H_G(\mathcal{R}^*, \mathcal{R}^*, \Phi(\mathcal{R}^*)) \leq \varepsilon,
$$

where $\Upsilon(\mathcal{R}^*) = \cup_{a=1}^q f_k^*(\mathcal{R}^*)$, $\Psi(\mathcal{R}^*) = \cup_{a=1}^q g_a^*(\mathcal{R}^*)$ and $\Phi(\mathcal{R}^*) = \cup_{a=1}^q h_a(\mathcal{R}^*)$, there exist a common attractor, $\tilde{U_1} \in \mathcal{C}^p(W)$ for the Hutchinson operators Υ, Ψ and Φ, such that

$$
H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}) \leq \frac{\varepsilon}{1-\lambda}.
$$

Proof. Assume that $H_G(\mathcal{R}^*, \mathcal{R}^*, \Upsilon(\mathcal{R}^*)) \leq \varepsilon$ for any $\mathcal{R}^* \in \mathcal{C}^G(W)$, then

$$
\begin{aligned}
H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}) &\leq H_G(\mathcal{R}^*, \mathcal{R}^*, \Upsilon(\mathcal{R}^*)) + H_G(\Upsilon(\mathcal{R}^*), \Upsilon(\mathcal{R}^*), \tilde{U_1}) \\
&\leq H_G(\mathcal{R}^*, \mathcal{R}^*, \Upsilon(\mathcal{R}^*)) + H_G(\Upsilon(\mathcal{R}^*), \Upsilon(\mathcal{R}^*), \Upsilon(\tilde{U_1})) \\
&\leq \varepsilon + \lambda H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}),
\end{aligned}
$$

which further implies that

$$
H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}) \leq \frac{\varepsilon}{1-\lambda}.
$$

Similarly, if we assume that $H_G(\mathcal{R}^*, \mathcal{R}^*, \Psi(\mathcal{R}^*)) \le \varepsilon$ for any $\mathcal{R}^* \in \mathcal{C}^G(W)$. Then,

$$\begin{aligned} H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}) &\le H_G(\mathcal{R}^*, \mathcal{R}^*, \Psi(\mathcal{R}^*)) + H_G(\Psi(\mathcal{R}^*), \Psi(\mathcal{R}^*), \tilde{U_1}) \\ &\le H_G(\mathcal{R}^*, \mathcal{R}^*, \Psi(\mathcal{R}^*)) + H_G(\Psi(\mathcal{R}^*), \Psi(\mathcal{R}^*), \Psi(\tilde{U_1})) \\ &\le \varepsilon + \lambda H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}), \end{aligned}$$

giving us

$$H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}) \le \frac{\varepsilon}{1-\lambda}.$$

Lastly by assuming that $H_G(\mathcal{R}^*, \mathcal{R}^*, \Phi(\mathcal{R}^*)) \le \varepsilon$ for any $\mathcal{R}^* \in \mathcal{C}^G(W)$, we get

$$\begin{aligned} H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}) &\le H_G(\mathcal{R}^*, \mathcal{R}^*, \Phi(\mathcal{R}^*)) + H_G(\Phi(\mathcal{R}^*), \Phi(\mathcal{R}^*), \tilde{U_1}) \\ &\le H_G(\mathcal{R}^*, \mathcal{R}^*, \Phi(\mathcal{R}^*)) + H_G(\Phi(\mathcal{R}^*), \Phi(\mathcal{R}^*), \Phi\tilde{U_1})) \\ &\le \varepsilon + \lambda H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}), \end{aligned}$$

from which we have

$$H_G(\mathcal{R}^*, \mathcal{R}^*, \tilde{U_1}) \le \frac{\varepsilon}{1-\lambda}.$$

□

Remark 6.3.1. *In Theorem* 6.3.1, *take the collection $\mathcal{S}^G(W)$, of all singleton subsets of the given space W, then $\mathcal{S}^G(W) \subseteq \mathcal{C}^G(W)$. Furthermore, if we take the mappings $(f_a^*, g_a^*, h_a^*) = (f^*, g^*, h^*)$ for each a, where $f^* = f_1^*$, $g^* = g_1^*$ and $h^* = h_1^*$, then the operators (Υ, Ψ, Φ) become*

$$(\Upsilon(\tilde{v_1}), \Psi(\tilde{v_2}), \Phi(\tilde{v_3})) = (f^*(\tilde{v_1}), g^*(\tilde{v_2}), h^*(\tilde{v_3})),$$

for $\tilde{v_1}, \tilde{v_2}, \tilde{v_3} \in W$.

Concequently, the following common fixed point result is established.

Corollary 6.3.1. *Let $\{W; (f_a^*, g_a^*, h_a^*), a = 1, 2, \cdots, q\}$ be a generalized G-iterated function system in a complete G-metric space (W, G) and define the mappings $f^*, g^*, h^* : W \to W$ as in Remark* 6.3.1 *If some $\alpha, \beta, \gamma, \eta \ge 0$ exist with $\alpha + \beta + \gamma + \eta < 1$ such that for any $\tilde{v_1}, \tilde{v_2}, \tilde{v_3} \in W$, the following holds*

$$\begin{aligned} H_G(f^*\tilde{v_1}, g^*\tilde{v_2}, h^*\tilde{v_3}) &\le \alpha H_G(\tilde{v_1}, \tilde{v_2}, \tilde{v_3}) + \beta H_G(\tilde{v_1}, f^*(\tilde{v_1}), f^*(\tilde{v_1})) \\ &\quad + \gamma H_G(\tilde{v_2}, g^*(\tilde{v_2}), g^*(\tilde{v_2})) + \eta H_G(\tilde{v_3}, h^*(\tilde{v_3}), h^*(\tilde{v_3})). \end{aligned}$$

Then f^*, g^* and h^* have a unique common fixed point $\tilde{u_1} \in W$. Additionally, for an arbitrary element $\tilde{u_0} \in W$, the sequence

$\{\tilde{u_0}, f^*\tilde{u_0}, g^*f^*\tilde{u_0}, h^*g^*f^*\tilde{u_0}, f^*h^*g^*f^*\tilde{u_0}, \cdots\}$ *converges to the common fixed point of* f^*, g^* *and* h^*.

Corollary 6.3.2. *Let* $\{W; (f_a^*, g_a^*, h_a^*), a = 1, 2, \cdots, q\}$ *be a generalized G-iterated function system in a complete G-metric space* (W, G) *and define the mappings* $f^*, g^*, h^* : W \to W$ *as in Remark* 6.3.1. *If* (f^*, g^*, h^*) *is a triplet of generalized G-contraction mappings, then* (Υ, Ψ, Φ) *defined on* $\mathcal{C}^G(W)$ *as in Theorem* 6.3.1 *has exactly one common fixed point in* $\mathcal{C}^G(W)$. *Moreover, for any initial set* $\mathcal{R}_0^* \in \mathcal{C}^G(W)$, $\{\mathcal{R}_0^*, \Upsilon(\mathcal{R}_0^*), \Psi\Upsilon(\mathcal{R}_0^*), \Phi\Psi\Upsilon(\mathcal{R}_0^*), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0^*), \cdots\}$ *converges to the common fixed point of* Υ, Ψ *and* Φ.

Example 6.3.1. Let $W = [0, 1]$ and $G(w_1, w_2, w_3) = \max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\}$ be a G-metric on W. Define $f_a^*, g_a^*, h_a^* : W \to W$, $a = 1, 2$ by

$$f_1^*(w_1) = \begin{cases} \dfrac{w_1}{18} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \dfrac{w_1}{16} & \text{if } \frac{1}{2} \leq w_1 \leq 1, \end{cases} \qquad f_2^*(w_1) = \begin{cases} \dfrac{w_1}{14} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \dfrac{w_1}{12} & \text{if } \frac{1}{2} \leq w_1 \leq 1, \end{cases}$$

$$g_1^*(w_1) = \begin{cases} \dfrac{w_1}{10} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \dfrac{w_1}{8} & \text{if } \frac{1}{2} \leq w_1 \leq 1, \end{cases} \qquad g_2^*(w_1) = \begin{cases} \dfrac{w_1}{6} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \dfrac{w_1}{4} & \text{if } \frac{1}{2} \leq w_1 \leq 1, \end{cases}$$

$$h_1^*(w_1) = \begin{cases} \dfrac{w_1}{9} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \dfrac{w_1}{7} & \text{if } \frac{1}{2} \leq w_1 \leq 1, \end{cases} \qquad h_2^*(w_1) = \begin{cases} \dfrac{w_1}{5} & \text{if } 0 \leq w_1 < \frac{1}{2} \\ \dfrac{w_1}{3} & \text{if } \frac{1}{2} \leq w_1 \leq 1. \end{cases}$$

We observe that the maps $f_1^*, f_2^*, g_1^*, g_2^*, h_1^*$ and h_2^* are discontinuous. Moreover,

$$f_1^*g_1^*\left(\tfrac{1}{2}\right)=f_1^*\left(\tfrac{1}{16}\right)=\tfrac{1}{224},\qquad g_1^*f_1^*\left(\tfrac{1}{2}\right)=g_1^*\left(\tfrac{1}{32}\right)=\tfrac{1}{320},$$

$$f_2^*g_2^*\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{8}\right)=\tfrac{1}{112},\qquad g_2^*f_2^*\left(\tfrac{1}{2}\right)=g_2^*\left(\tfrac{1}{24}\right)=\tfrac{1}{48},$$

$$g_1^*h_1^*\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{14}\right)=\tfrac{1}{140},\qquad h_1^*g_1^*\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{16}\right)=\tfrac{1}{144},$$

$$g_2^*h_2^*\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{6}\right)=\tfrac{1}{36},\qquad h_2^*g_2\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{8}\right)=\tfrac{1}{40},$$

$$f_1^*h_1^*\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{14}\right)=\tfrac{1}{252},\qquad h_1^*f_1^*\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{32}\right)=\tfrac{1}{288},$$

$$f_2^*h_2^*\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{6}\right)=\tfrac{1}{84},\qquad h_2^*f_2^*\left(\tfrac{1}{2}\right)=f_2^*\left(\tfrac{1}{24}\right)=\tfrac{1}{120},$$

and so the mappings f_a^*, g_a^* and h_a^* for $a=1,2$ do not commute.

Now, for $w_1, w_2, w_3 \in [0, \frac{1}{2}]$, we have

$$\begin{aligned}
G(w_1,w_2,w_3) &= \max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\},\\
G(w_1,f_1^*w_1,f_1^*w_1) &= \max\{\left|w_1-\tfrac{w_1}{18}\right|,\left|\tfrac{w_1}{18}-\tfrac{w_1}{18}\right|,\left|\tfrac{w_1}{18}-w_1\right|\}=\tfrac{17w_1}{18},\\
G(w_1,f_2^*w_1,f_2^*w_1) &= \max\{\left|w_1-\tfrac{w_1}{14}\right|,\left|\tfrac{w_1}{14}-\tfrac{w_1}{14}\right|,\left|\tfrac{w_1}{14}-w_1\right|\}=\tfrac{13w_1}{14},\\
G(w_2,g_1^*w_2,g_1^*w_2) &= \max\{\left|w_2-\tfrac{w_2}{10}\right|,\left|\tfrac{w_2}{10}-\tfrac{w_2}{10}\right|,\left|\tfrac{w_2}{10}-w_2\right|\}=\tfrac{9w_2}{10},\\
G(w_2,g_2^*w_2,g_2^*w_2) &= \max\{\left|w_2-\tfrac{w_2}{6}\right|,\left|\tfrac{w_2}{6}-\tfrac{w_2}{6}\right|,\left|\tfrac{w_2}{6}-w_2\right|\}=\tfrac{5w_2}{6},\\
G(w_3,h_1^*w_3,h_1^*w_3) &= \max\{\left|w_3-\tfrac{w_3}{9}\right|,\left|\tfrac{w_3}{9}-\tfrac{y_3}{9}\right|,\left|\tfrac{w_3}{9}-w_3\right|\}=\tfrac{8w_3}{9},\\
G(w_3,h_2^*w_3,h_2^*w_3) &= \max\{\left|w_3-\tfrac{w_3}{5}\right|,\left|\tfrac{w_3}{5}-\tfrac{w_3}{5}\right|,\left|\tfrac{w_3}{5}-w_3\right|\}=\tfrac{4w_3}{5}.
\end{aligned}$$

Thus

$$\begin{aligned}
& G(f_1^*w_1,g_1^*w_2,h_1^*w_3)\\
= & \max\{\left|\tfrac{w_1}{18}-\tfrac{w_2}{10}\right|,\left|\tfrac{w_2}{10}-\tfrac{w_3}{9}\right|,\left|\tfrac{w_3}{9}-\tfrac{w_1}{18}\right|\}\\
= & \tfrac{1}{10}\max\{\left|\tfrac{5w_1}{9}-w_2\right|,\left|w_2-\tfrac{10w_3}{9}\right|,\left|\tfrac{10w_3}{9}-\tfrac{5w_1}{9}\right|\}\\
\leq & \tfrac{1}{10}\left[\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+w_1+w_2+w_3\right]\\
= & \tfrac{1}{10}\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+\tfrac{w_1}{10}+\tfrac{w_2}{10}+\tfrac{w_3}{10}\\
= & \tfrac{1}{10}\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+\tfrac{7}{65}\left(\tfrac{13w_1}{14}\right)+\tfrac{1}{9}\left(\tfrac{9w_2}{10}\right)+\tfrac{7}{60}\left(\tfrac{6w_3}{7}\right)\\
= & \alpha_1G(w_1,w_2,w_3)+\beta_1G(w_1,f^*w_1,f^*w_1)+\gamma_1G(w_2,g^*w_2,g^*w_2)+\eta_1G(w_3,h^*w_3,h^*w_3)
\end{aligned}$$

and

$$
\begin{aligned}
& G\left(f_2^* w_1, g_2^* w_2, h_2^* w_3\right) \\
= \;& \max\{\left|\tfrac{w_1}{14}-\tfrac{w_2}{6}\right|,\left|\tfrac{w_2}{6}-\tfrac{w_3}{5}\right|,\left|\tfrac{w_3}{5}-\tfrac{w_1}{14}\right|\} \\
= \;& \tfrac{1}{6}\max\{\left|\tfrac{3w_1}{7}-w_2\right|,\left|w_2-\tfrac{6w_3}{5}\right|,\left|\tfrac{6w_3}{5}-\tfrac{3w_1}{7}\right|\} \\
\leq \;& \tfrac{1}{6}\left[\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+w_1+w_2+w_3\right] \\
= \;& \tfrac{1}{6}\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+\tfrac{w_1}{6}+\tfrac{w_2}{6}+\tfrac{w_3}{6} \\
= \;& \tfrac{1}{6}\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+\tfrac{7}{39}\left(\tfrac{13w_1}{14}\right)+\tfrac{1}{5}\left(\tfrac{5w_2}{6}\right)+\tfrac{5}{24}\left(\tfrac{4w_3}{5}\right) \\
= \;& \alpha_2 G\left(w_1,w_2,w_3\right)+\beta_2 G\left(w_1,f^*w_1,f^*w_1\right)+\gamma_2 G\left(w_2,g^*w_2,g^*w_2\right)+\eta_2 G\left(w_3,h^*w_3,h^*w_3\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
G\left(f_a^* w_1, g_a^* w_2, h_a^* w_3\right) &= \alpha G\left(w_1,w_2,w_3\right)+\beta G\left(w_1,f_a^*w_1,f_a^*w_1\right)+\gamma G\left(w_2,g_k^*w_2,g_k^*w_2\right) \\
&\quad +\eta G\left(w_3,h_a^*w_3,h_a^*w_3\right)
\end{aligned}
$$

for $a=1,2$, where $0<\alpha+\beta+\gamma+\eta=0.755<1$ and

$$
\begin{aligned}
\alpha &= \max\{\alpha_1,\alpha_2\}=\max\{\tfrac{1}{10},\tfrac{1}{6}\}=\tfrac{1}{6}, \\
\beta &= \max\{\beta_1,\beta_2\}=\max\{\tfrac{1}{85},\tfrac{7}{39}\}=\tfrac{7}{39}, \\
\gamma &= \max\{\gamma_1,\gamma_2\}=\max\{\tfrac{1}{9},\tfrac{1}{5}\}=\tfrac{1}{5}, \\
\eta &= \max\{\eta_1,\eta_2\}=\max\{\tfrac{9}{80},\tfrac{5}{24}\}=\tfrac{5}{24}.
\end{aligned}
$$

For $w_1,w_2,w_3\in[\frac{1}{2},1]$,

$$
\begin{aligned}
G\left(w_1,w_2,w_3\right) &= \max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}, \\
G\left(w_1,f_1^*w_1,f_1^*w_1\right) &= \max\{\left|w_1-\tfrac{w_1}{16}\right|,\left|\tfrac{w_1}{16}-\tfrac{w_1}{16}\right|,\left|\tfrac{w_1}{16}-w_1\right|\}=\tfrac{15w_1}{16}, \\
G\left(w_1,f_2^*w_1,f_2^*w_1\right) &= \max\{\left|w_1-\tfrac{w_1}{12}\right|,\left|\tfrac{w_1}{12}-\tfrac{w_1}{12}\right|,\left|\tfrac{w_1}{12}-w_1\right|\}=\tfrac{11w_1}{12}, \\
G\left(w_2,g_1^*w_2,g_1^*w_2\right) &= \max\{\left|w_2-\tfrac{w_2}{8}\right|,\left|\tfrac{w_2}{8}-\tfrac{w_2}{8}\right|,\left|\tfrac{w_2}{8}-w_2\right|\}=\tfrac{7w_2}{8}, \\
G\left(w_2,g_2^*w_2,g_2^*w_2\right) &= \max\{\left|w_2-\tfrac{w_2}{4}\right|,\left|\tfrac{w_2}{4}-\tfrac{w_2}{4}\right|,\left|\tfrac{w_2}{4}-w_2\right|\}=\tfrac{3w_2}{4}, \\
G\left(w_3,h_1^*w_3,h_1^*w_3\right) &= \max\{\left|w_3-\tfrac{w_3}{7}\right|,\left|\tfrac{w_3}{7}-\tfrac{y_3}{7}\right|,\left|\tfrac{w_3}{7}-w_3\right|\}=\tfrac{6w_3}{7}, \\
G\left(w_3,h_2^*w_3,h_2^*w_3\right) &= \max\{\left|w_3-\tfrac{w_3}{3}\right|,\left|\tfrac{w_3}{3}-\tfrac{w_3}{3}\right|,\left|\tfrac{w_3}{3}-w_3\right|\}=\tfrac{2w_3}{3}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& G\left(f_1^* w_1, g_1^* w_2, h_1^* w_3\right) \\
= \;& \max\{\left|\tfrac{w_1}{16} - \tfrac{w_2}{8}\right|, \left|\tfrac{w_2}{8} - \tfrac{w_3}{7}\right|, \left|\tfrac{w_3}{7} - \tfrac{w_1}{16}\right|\} \\
= \;& \tfrac{1}{8}\max\{\left|\tfrac{w_1}{2} - w_2\right|, \left|w_2 - \tfrac{8w_3}{7}\right|, \left|\tfrac{8w_3}{7} - \tfrac{w_1}{2}\right|\} \\
\leq \;& \tfrac{1}{8}\left[\max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\} + w_1 + w_2 + w_3\right] \\
= \;& \tfrac{1}{8}\max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\} + \tfrac{w_1}{8} + \tfrac{w_2}{8} + \tfrac{w_3}{8} \\
= \;& \tfrac{1}{8}\max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\} + \tfrac{2}{15}\left(\tfrac{15w_1}{16}\right) + \tfrac{1}{7}\left(\tfrac{7w_2}{8}\right) + \tfrac{7}{48}\left(\tfrac{6w_3}{7}\right) \\
= \;& \alpha_1 G\left(w_1, w_2, w_3\right) + \beta_1 G\left(w_1, f_1^* w_1, f_1^* w_1\right) + \gamma_1 G\left(w_2, g_1^* w_2, g_1^* w_2\right) + \eta_1 G\left(w_3, h_1^* w_3, h_1^* w_3\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& G\left(f_2^* w_1, g_2^* w_2, h_2^* w_3\right) \\
= \;& \max\{\left|\tfrac{w_1}{12} - \tfrac{w_2}{4}\right|, \left|\tfrac{w_2}{4} - \tfrac{w_3}{3}\right|, \left|\tfrac{w_3}{3} - \tfrac{w_1}{12}\right|\} \\
= \;& \tfrac{1}{12}\max\{|w_1 - 3w_2|, |3w_2 - 4w_3|, |4w_3 - w_1|\} \\
\leq \;& \tfrac{1}{12}\left[\max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\} + w_1 + w_2 + w_3\right] \\
= \;& \tfrac{1}{12}\max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\} + \tfrac{w_1}{12} + \tfrac{w_2}{12} + \tfrac{w_3}{12} \\
= \;& \tfrac{1}{12}\max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\} + \tfrac{1}{11}\left(\tfrac{11}{12}w_1\right) + \tfrac{1}{9}\left(\tfrac{3}{4}w_2\right) + \tfrac{1}{8}\left(\tfrac{2}{3}w_3\right) \\
= \;& \alpha_2 G\left(w_1, w_2, w_3\right) + \beta_2 G\left(w_1, f_2^* w_1, f_2^* w_1\right) + \gamma_2 G\left(w_2, g_2^* w_2, g_2^* w_2\right) + \eta_2 G\left(w_3, h_2^* w_3, h_2^* w_3\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
G\left(f_a^* w_1, g_a^* w_2, h_a^* w_3\right) = \;& \alpha G\left(w_1, w_2, w_3\right) + \beta G\left(w_1, f_a^* w_1, f_a^* w_1\right) + \gamma G\left(w_2, g_a^* w_2, g_a^* w_2\right) \\
& + \eta G\left(w_3, h_k^* w_3, h_k^* w_3\right)
\end{aligned}
$$

for $a = 1, 2$ where $0 < \alpha + \beta + \gamma + \eta = 0.547 < 1$ and

$$
\begin{aligned}
\alpha &= \max\{\alpha_1, \alpha_2\} = \max\{\tfrac{1}{8}, \tfrac{1}{12}\} = \tfrac{1}{8}, \\
\beta &= \max\{\beta_1, \beta_2\} = \max\{\tfrac{2}{15}, \tfrac{1}{11}\} = \tfrac{2}{15}, \\
\gamma &= \max\{\gamma_1, \gamma_2\} = \max\{\tfrac{1}{7}, \tfrac{1}{9}\} = \tfrac{1}{7}, \\
\eta &= \max\{\eta_1, \eta_2\} = \max\{\tfrac{7}{48}, \tfrac{1}{8}\} = \tfrac{7}{48}.
\end{aligned}
$$

For $w_1 \in [0, \frac{1}{2}), w_2, w_3 \in [\frac{1}{2}, 1]$,

$$\begin{aligned}
G\left(w_1, w_2, w_3\right) &= \max\{\left|w_1 - w_2\right|, \left|w_2 - w_3\right|, \left|w_3 - w_1\right|\}, \\
G\left(w_1, f_1^* w_1, f_1^* w_1\right) &= \max\{\left|w_1 - \tfrac{w_1}{18}\right|, \left|\tfrac{w_1}{18} - \tfrac{w_1}{18}\right|, \left|\tfrac{w_1}{18} - w_1\right|\} = \tfrac{17w_1}{18}, \\
G\left(w_1, f_2^* w_1, f_2^* w_1\right) &= \max\{\left|w_1 - \tfrac{w_1}{14}\right|, \left|\tfrac{w_1}{14} - \tfrac{w_1}{14}\right|, \left|\tfrac{w_1}{14} - w_1\right|\} = \tfrac{13w_1}{14}, \\
G\left(w_2, g_1^* w_2, g_1 w_2\right) &= \max\{\left|w_2 - \tfrac{w_2}{8}\right|, \left|\tfrac{w_2}{8} - \tfrac{w_2}{8}\right|, \left|\tfrac{w_2}{8} - w_2\right|\} = \tfrac{7w_2}{8}, \\
G\left(w_2, g_2^* w_2, g_2^* w_2\right) &= \max\{\left|w_2 - \tfrac{w_2}{4}\right|, \left|\tfrac{w_2}{4} - \tfrac{w_2}{4}\right|, \left|\tfrac{w_2}{4} - w_2\right|\} = \tfrac{3w_2}{4}, \\
G\left(w_3, h_1^* w_3, h_1^* w_3\right) &= \max\{\left|w_3 - \tfrac{w_3}{7}\right|, \left|\tfrac{w_3}{7} - \tfrac{w_3}{7}\right|, \left|\tfrac{w_3}{7} - w_3\right|\} = \tfrac{6w_3}{7}, \\
G\left(w_3, h_2^* w_3, h_2^* w_3\right) &= \max\{\left|w_3 - \tfrac{w_3}{3}\right|, \left|\tfrac{w_3}{3} - \tfrac{w_3}{3}\right|, \left|\tfrac{w_3}{3} - w_3\right|\} = \tfrac{2w_3}{3}.
\end{aligned}$$

Thus

$$\begin{aligned}
& G\left(f_1^* w_1, g_1^* w_2, h_1^* w_3\right) \\
= \;& \max\{\left|\tfrac{w_1}{18} - \tfrac{w_2}{8}\right|, \left|\tfrac{w_2}{8} - \tfrac{w_3}{7}\right|, \left|\tfrac{w_3}{7} - \tfrac{w_1}{18}\right|\} \\
= \;& \tfrac{1}{8}\max\{\left|\tfrac{4w_1}{9} - w_2\right|, \left|w_2 - \tfrac{8w_3}{7}\right|, \left|\tfrac{8w_3}{7} - \tfrac{4w_1}{9}\right|\} \\
\leq \;& \tfrac{1}{8}\left[\max\{\left|w_1 - w_2\right|, \left|w_2 - w_3\right|, \left|w_3 - w_1\right|\} + w_1 + w_2 + w_3\right] \\
= \;& \tfrac{1}{8}\max\{\left|w_1 - w_2\right|, \left|w_2 - w_3\right|, \left|w_3 - w_1\right|\} + \tfrac{w_1}{8} + \tfrac{w_2}{8} + \tfrac{w_3}{8} \\
= \;& \tfrac{1}{8}\max\{\left|w_1 - w_2\right|, \left|w_2 - w_3\right|, \left|w_3 - w_1\right|\} + \tfrac{9}{68}\left(\tfrac{17}{18}w_1\right) + \tfrac{1}{7}\left(\tfrac{7}{8}w_2\right) + \tfrac{7}{48}\left(\tfrac{6w_3}{7}\right) \\
= \;& \alpha_1 G\left(w_1, w_2, w_3\right) + \beta_1 G\left(w_1, f_1^* w_1, f_1^* w_1\right) + \gamma_1 G\left(w_2, g_1^* w_2, g_1^* w_2\right) + \eta_1 G\left(w_3, h_1^* w_3, h_1^* w_3\right)
\end{aligned}$$

and

$$\begin{aligned}
& G\left(f_2^* w_1, g_2^* w_2, h_2^* w_3\right) \\
= \;& \max\{\left|\tfrac{w_1}{14} - \tfrac{w_2}{4}\right|, \left|\tfrac{w_2}{4} - \tfrac{w_3}{3}\right|, \left|\tfrac{w_3}{3} - \tfrac{w_1}{14}\right|\} \\
= \;& \tfrac{1}{14}\max\{\left|w_1 - \tfrac{7w_2}{2}\right|, \left|\tfrac{7w_2}{2} - \tfrac{14w_3}{3}\right|, \left|\tfrac{14w_3}{3} - w_1\right|\} \\
\leq \;& \tfrac{1}{14}\left[\max\{\left|w_1 - w_2\right|, \left|w_2 - w_3\right|, \left|w_3 - w_1\right|\} + w_1 + w_2 + w_3\right] \\
= \;& \tfrac{1}{14}\max\{\left|w_1 - w_2\right|, \left|w_2 - w_3\right|, \left|w_3 - w_1\right|\} + \tfrac{w_1}{14} + \tfrac{w_2}{14} + \tfrac{w_3}{14} \\
= \;& \tfrac{1}{14}\max\{\left|w_1 - w_2\right|, \left|w_2 - w_3\right|, \left|w_3 - w_1\right|\} + \tfrac{1}{13}\left(\tfrac{13w_1}{14}\right) + \tfrac{2}{21}\left(\tfrac{3w_2}{4}\right) + \tfrac{3}{28}\left(\tfrac{2w_3}{3}\right) \\
= \;& \alpha_2 G\left(w_1, w_2, w_3\right) + \beta_2 G\left(w_1, f_2^* w_1, f_2^* w_1\right) + \gamma_2 G\left(w_2, g_2^* w_2, g_2^* w_2\right) + \eta_2 G\left(w_3, h_2^* w_3, h_2^* w_3\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
G\left(f_a^* w_1, g_a^* w_2, h_a^* w_3\right) &= \alpha G\left(w_1, w_2, w_3\right) + \beta G\left(w_1, f_a^* w_1, f_a^* w_1\right) + \gamma G\left(w_2, g_a^* w_2, g_a^* w_2\right) \\
&\quad + \eta G\left(w_3, h_a^* w_3, h_a^* w_3\right)
\end{aligned}$$

for $a=1,2$, where $0<\alpha+\beta+\gamma+\eta=0.546<1$ with

$$\begin{aligned}
\alpha &= \max\{\alpha_1,\alpha_2\}=\max\{\tfrac{1}{8},\tfrac{1}{14}\}=\tfrac{1}{8}\\
\beta &= \max\{\beta_1,\beta_2\}=\max\{\tfrac{9}{68},\tfrac{1}{13}\}=\tfrac{9}{68}\\
\gamma &= \max\{\gamma_1,\gamma_2\}=\max\{\tfrac{1}{7},\tfrac{2}{21}\}=\tfrac{1}{7}\\
\eta &= \max\{\eta_1,\eta_2\}=\max\{\tfrac{7}{48},\tfrac{3}{28}\}=\tfrac{7}{48}.
\end{aligned}$$

For $w_1,w_2\in[0,\frac{1}{2})$ and $w_3\in[\frac{1}{2},1]$,

$$\begin{aligned}
G(w_1,w_2,w_3) &= \max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\},\\
G(w_1,f_1^*w_1,f_1^*w_1) &= \max\{\left|w_1-\tfrac{w_1}{18}\right|,\left|\tfrac{w_1}{18}-\tfrac{w_1}{18}\right|,\left|\tfrac{w_1}{18}-w_1\right|\}=\tfrac{17w_1}{18},\\
G(w_1,f_2^*w_1,f_2^*w_1) &= \max\{\left|w_1-\tfrac{w_1}{14}\right|,\left|\tfrac{w_1}{14}-\tfrac{w_1}{14}\right|,\left|\tfrac{w_1}{14}-w_1\right|\}=\tfrac{13w_1}{14},\\
G(w_2,g_1^*w_2,g_1^*w_2) &= \max\{\left|w_2-\tfrac{w_2}{10}\right|,\left|\tfrac{w_2}{10}-\tfrac{w_2}{10}\right|,\left|\tfrac{w_2}{10}-w_2\right|\}=\tfrac{9w_2}{10},\\
G(w_2,g_2^*w_2,g_2^*w_2) &= \max\{\left|w_2-\tfrac{w_2}{6}\right|,\left|\tfrac{w_2}{6}-\tfrac{w_2}{6}\right|,\left|\tfrac{w_2}{6}-w_2\right|\}=\tfrac{5w_2}{6},\\
G(w_3,h_1^*w_3,h_1^*w_3) &= \max\{\left|w_3-\tfrac{w_3}{7}\right|,\left|\tfrac{w_3}{7}-\tfrac{w_3}{7}\right|,\left|\tfrac{w_3}{7}-w_3\right|\}=\tfrac{6w_3}{7},\\
G(w_3,h_2^*w_3,h_2^*w_3) &= \max\{\left|w_3-\tfrac{w_3}{3}\right|,\left|\tfrac{w_3}{3}-\tfrac{w_3}{3}\right|,\left|\tfrac{w_3}{3}-w_3\right|\}=\tfrac{2w_3}{3}.
\end{aligned}$$

Thus

$$\begin{aligned}
& G(f_1^*w_1,g_1^*w_2,h_1^*w_3)\\
=\ & \max\{\left|\tfrac{w_1}{18}-\tfrac{w_2}{10}\right|,\left|\tfrac{w_2}{10}-\tfrac{w_3}{7}\right|,\left|\tfrac{w_3}{7}-\tfrac{w_1}{18}\right|\}\\
=\ & \tfrac{1}{10}\max\{\left|\tfrac{5w_1}{9}-w_2\right|,\left|w_2-\tfrac{10w_3}{7}\right|,\left|\tfrac{10w_3}{7}-\tfrac{5w_1}{9}\right|\}\\
\leq\ & \tfrac{1}{10}\left[\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+w_1+w_2+w_3\right]\\
=\ & \tfrac{1}{10}\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+\tfrac{w_1}{10}+\tfrac{w_2}{10}+\tfrac{w_3}{10}\\
=\ & \tfrac{1}{10}\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+\tfrac{9}{85}\left(\tfrac{17}{18}w_1\right)+\tfrac{1}{9}\left(\tfrac{9w_2}{10}\right)+\tfrac{7}{60}\left(\tfrac{6w_3}{7}\right)\\
=\ & \alpha_1G(w_1,w_2,w_3)+\beta_1G(w_1,f_1^*w_1,f_1^*w_1)+\gamma_1G(w_2,g_1^*w_2,g_1^*w_2)+\eta_1G(w_3,h_1^*w_3,h_1^*w_3)
\end{aligned}$$

and

$$\begin{aligned}
& G(f_2^*w_1,g_2^*w_2,h_2^*w_3)\\
=\ & \max\{\left|\tfrac{w_1}{14}-\tfrac{w_2}{6}\right|,\left|\tfrac{w_2}{6}-\tfrac{w_3}{3}\right|,\left|\tfrac{w_3}{3}-\tfrac{w_1}{14}\right|\}\\
=\ & \tfrac{1}{14}\max\{\left|w_1-\tfrac{7w_2}{3}\right|,\left|\tfrac{7w_2}{3}-\tfrac{14w_3}{3}\right|,\left|\tfrac{14w_3}{3}-w_1\right|\}\\
\leq\ & \tfrac{1}{14}\left[\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+w_1+w_2+w_3\right]\\
=\ & \tfrac{1}{14}\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+\tfrac{w_1}{14}+\tfrac{w_2}{14}+\tfrac{w_3}{14}\\
=\ & \tfrac{1}{14}\max\{|w_1-w_2|,|w_2-w_3|,|w_3-w_1|\}+\tfrac{1}{13}\left(\tfrac{13w_1}{14}\right)+\tfrac{3}{35}\left(\tfrac{5w_2}{6}\right)+\tfrac{3}{28}\left(\tfrac{2w_3}{3}\right)\\
=\ & \alpha_2G(w_1,w_2,w_3)+\beta_2G(w_1,f_2^*w_1,f_2^*w_1)+\gamma_2G(w_2,g_2^*w_2,g_2^*w_2)+\eta_2G(w_3,h_2^*w_3,h_2^*w_3).
\end{aligned}$$

Therefore

$$\begin{aligned} G\left(f_a^* w_1, g_a^* w_2, h_a^* w_3\right) &= \alpha G\left(w_1, w_2, w_3\right) + \beta G\left(w_1, f_a^* w_1, f_a^* w_1\right) + \gamma G\left(w_2, g_a^* w_2, g_a^* w_2\right) \\ &\quad + \eta G\left(w_3, h_a^* w_3, h_a^* w_3\right) \end{aligned}$$

for $a = 1, 2$, where $0 < \alpha + \beta + \gamma + \eta = 0.426 < 1$ with

$$\begin{aligned} \alpha &= \max\{\alpha_1, \alpha_2\} = \max\{\tfrac{1}{10}, \tfrac{1}{14}\} = \tfrac{1}{10}, \\ \beta &= \max\{\beta_1, \beta_2\} = \max\{\tfrac{9}{85}, \tfrac{1}{13}\} = \tfrac{9}{85}, \\ \gamma &= \max\{\gamma_1, \gamma_2\} = \max\{\tfrac{1}{9}, \tfrac{3}{35}\} = \tfrac{1}{9}, \\ \eta &= \max\{\eta_1, \eta_2\} = \max\{\tfrac{7}{60}, \tfrac{3}{28}\} = \tfrac{3}{28}. \end{aligned}$$

We notice that 0 is the only common fixed point of f^*, g^* and h^*.

Let $\{W; (f_1^*, f_2^*, g_1^*, g_2^*, h_1^*, h_2^*)\}$ be the generalized G-iterated function system with the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ defined by

$$\begin{aligned} \Upsilon(\mathcal{Q}^*) &= f_1(\mathcal{Q}^*) \cup f_2^*(\mathcal{Q}^*), \\ \Psi(\mathcal{R}^*) &= g_1(\mathcal{R}^*) \cup g_2(\mathcal{R}^*), \\ \Phi(\mathcal{N}^*) &= h_1(\mathcal{N}^*) \cup h_2(\mathcal{N}^*) \end{aligned}$$

for all $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$. From Proposition 6.2.1, we have that

$$H_G(\Upsilon(\mathcal{Q}^*), \Psi(\mathcal{R}^*), \Phi(\mathcal{N}^*)) \leq \kappa H_G(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*),$$

where $\kappa = \max\{0.755, 0.547, 0.546, 0.426\} = 0.755$. Thus, all of the conditions of Theorem 6.3.1 are met, and additionally, for any initial set $\mathcal{R}_0^* \in \mathcal{C}^G(W)$, the sequence $\{\mathcal{R}_0^*, \Upsilon\left(\mathcal{R}_0^*\right), \Psi\Upsilon\left(\mathcal{R}_0^*\right), \Phi\Psi\Upsilon\left(\mathcal{R}_0^*\right), \Upsilon\Phi\Psi\Upsilon\left(\mathcal{R}_0^*\right), \cdots\}$ of compact sets is convergent and has for a limit, the common attractor of Υ, Ψ and Φ.

Theorem 6.3.4. *Suppose* (W, G) *is a complete* G*-metric space, and let* $\{W; (f_a^*, g_a^*, h_a^*), a = 1, 2, \cdots, q\}$ *be the generalized* G *-iterated function system. Define* $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ *by*

$$\begin{aligned} \Upsilon(\mathcal{Q}^*) &= f_1^*(\mathcal{Q}^*) \cup f_2^*(\mathcal{Q}^*) \cup \cdots \cup f_q^*(\mathcal{Q}^*) \\ &= \cup_{a=1}^{q} f_a^*(\mathcal{Q}^*), \end{aligned}$$

$$\begin{aligned} \Psi(\mathcal{R}^*) &= g_1^*(\mathcal{R}^*) \cup g_2^*(\mathcal{R}^*) \cup \cdots \cup g_q^*(\mathcal{R}^*) \\ &= \cup_{a=1}^{q} g_a^*(\mathcal{R}^*) \end{aligned}$$

and

$$\begin{aligned}\Phi(\mathcal{N}^*) &= h_1^*(\mathcal{N}^*) \cup h_2^*(\mathcal{N}^*) \cup \cdots \cup h_q^*(\mathcal{N}^*) \\ &= \cup_{a=1}^{q} h_a^*(\mathcal{N}^*)\end{aligned}$$

for $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$. If the mappings (Υ, Ψ, Φ) are a triplet of generalized G -Hutchinson contractive operators (type II), then Υ, Ψ and Φ have a unique common attractor $\tilde{U_1} \in \mathcal{C}^G(W)$, that is,

$$\tilde{U_1} = \Upsilon(\tilde{U_1}) = \Psi(\tilde{U_1}) = \Phi(U^*).$$

Moreover, for an arbitrarily chosen initial set $\mathcal{R}_0^* \in \mathcal{C}^G(W)$, the sequence

$$\{\mathcal{R}_0^*, \Upsilon(\mathcal{R}_0^*), \Psi\Upsilon(\mathcal{R}_0^*), \Phi\Psi\Upsilon(\mathcal{R}_0^*), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0^*), ...\}$$

of compact sets converges to the common attractor U^*.

Proof. We show that any attractor of Υ is an attractor of Ψ and Φ. To that end, we assume that $\tilde{U_1} \in \mathcal{C}^G(W)$ is such that $\Upsilon(\tilde{U_1}) = \tilde{U_1}$. We need to show that $\tilde{U_1} = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1})$. As

$$\begin{aligned}
& H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi\tilde{U_1})) \\
= \;& H_G(\Upsilon(\tilde{U_1}), \Psi(\tilde{U_1}), \Phi(U^*)) \\
\leq \;& \lambda_1 H_G(\tilde{U_1}, \tilde{U_1}, \tilde{U_1}) + \lambda_2[H_G(\tilde{U_1}, \tilde{U_1}, \Upsilon(\tilde{U_1}) \\
& + H_G(\tilde{U_1}, \tilde{U_1}, \Psi(\tilde{U_1})) + H_G(\tilde{U_1}, \tilde{U_1}, \Phi(\tilde{U_1}))] \\
& + \lambda_3[H_G(\Upsilon(\tilde{U_1}), \tilde{U_1}, \tilde{U_1}) + H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \tilde{U_1}) \\
& + H_G(\tilde{U_1}, \tilde{U_1}, \Phi(\tilde{U_1}))] \\
= \;& (\lambda_2 + \lambda_3)\left[H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \tilde{U_1}) + H_G(\tilde{U_1}, \tilde{U_1}, \Phi(\tilde{U_1}))\right] \\
\leq \;& (\lambda_2 + \lambda_3)\left[H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi(\tilde{U_1})) + H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi(\tilde{U_1}))\right] \\
= \;& 2(\lambda_2 + \lambda_3) H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi(\tilde{U_1})),
\end{aligned}$$

that is, $(1-2\lambda_2+2\lambda_3))H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi(\tilde{U_1})) \leq 0$ and so $H_G(\tilde{U_1}, \Psi(\tilde{U_1}), \Phi\tilde{U_1})) = 0$ since $2\lambda_2 + 2\lambda_3 < 1$. Thus $\tilde{U_1} = \Upsilon(\tilde{U_1}) = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1})$. Similarly, if we take $\tilde{U_1} = \Phi(\tilde{U_1})$ or $\tilde{U_1} = \Psi(\tilde{U_1})$, we conclude that $\tilde{U_1} = \Upsilon(\tilde{U_1}) = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1})$.

We show that Υ, Ψ, and Φ have a unique common attractor. Let $\mathcal{R}_0^* \in \mathcal{C}^G(W)$ be chosen arbitrarly and define $\{\mathcal{R}_a^*\}$ by $\mathcal{R}_{3a+1}^* = \Upsilon(\mathcal{R}_{3a}^*)$, $\mathcal{R}_{3a+2}^* = \Psi(\mathcal{R}_{3a+1}^*)$, and $\mathcal{R}_{3a+3}^* = \Psi(\mathcal{R}_{3a+2}^*)$, for $a \in \mathbb{N} \cup \{0\}$. If $\mathcal{R}_a^* = \mathcal{R}_{a+1}^*$ for some a, with $a = 3n$, then $\tilde{U_1} = \mathcal{R}_{3a}^*$ is an attractor of Υ and from the proof above, $\tilde{U_1}$ is a common attractor for Υ, Ψ and Φ. The same is true for $a = 3n+1$ or $a = 3n+2$. We assume that $\mathcal{R}_a^* \neq \mathcal{R}_{a+1}^*$ for all $a \in \mathbb{N}$, then

$$
\begin{aligned}
& H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*) \\
= \; & H_G(\Upsilon(\mathcal{R}_{3a}^*), \Psi\left(\mathcal{R}_{3a+1}^*\right), \Phi\left(\mathcal{R}_{3a+2}^*\right)) \\
\leq \; & \lambda_1 H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}) + \lambda_2[H_G(\Upsilon(\mathcal{R}_{3a}^*), \mathcal{R}_{3a}^*, \mathcal{R}_{3a}^*) \\
& + H_G(\mathcal{R}_{3a+1}^*, \Psi(\mathcal{R}_{3a+1}^*), \mathcal{R}_{3a+1}^*) + H_G(\mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+2}^*, \Phi(\mathcal{R}_{3a+2}^*))] \\
& + \lambda_3[H_G(\Upsilon(\mathcal{R}_{3a}^*), \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + H_G(\mathcal{R}_{3a}^*, \Psi\left(\mathcal{R}_{3a+1}^*\right), \mathcal{R}_{3a+2}^*) \\
& + H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \Phi\left(\mathcal{R}_{3a+2}^*\right))] \\
= \; & \lambda_1 H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + \lambda_2[H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a}^*, \mathcal{R}_{3a}^*) \\
& + H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+1}^*) + H_G(\mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*)] \\
& + \lambda_3[H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+2}^*) \\
& + H_G(\mathcal{R}^*{}_{3a}, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+3}^*)] \\
\leq \; & \lambda_1 H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + \lambda_2[H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) \\
& + H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*)] \\
& + \lambda_3[H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) \\
& + \{H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*) + H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*)\}].
\end{aligned}
$$

Thus, we have

$$(1 - \lambda_2 - \lambda_3) H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*) \leq (\lambda_1 + 2\lambda_2 + 3\lambda_3) H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*).$$

Hence,

$$H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*) \leq \lambda H_G(\mathcal{R}_{3a}^*, \mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*),$$

where $\lambda = \dfrac{\lambda_1 + 2\lambda_2 + 3\lambda_3}{1 - \lambda_2 - \lambda_3}$, with $0 < \lambda < 1$. In a similar manner, it can be proved that

$$H_G(\mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*, \mathcal{R}_{3a+4}^*) \leq \lambda H_G(\mathcal{R}_{3a+1}^*, \mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*)$$

and

$$H_G(\mathcal{R}_{3a+3}^*, \mathcal{R}_{3a+4}^*, \mathcal{R}_{3a+5}^*) \leq \lambda H_G(\mathcal{R}_{3a+2}^*, \mathcal{R}_{3a+3}^*, \mathcal{R}_{3a+4}^*).$$

Thus, for all a,

$$
\begin{aligned}
H_G(\mathcal{R}_{a+1}^*, \mathcal{R}_{a+2}^*, \mathcal{R}_{a+3}^*) & \leq \lambda H_G\left(\mathcal{R}_a^*, \mathcal{R}_{a+1}^*, \mathcal{R}_{a+2}^*\right) \\
& \leq \cdots \leq \lambda^{a+1} H_G\left(\mathcal{R}_0^*, \mathcal{R}_1^*, \mathcal{R}_2^*\right).
\end{aligned}
$$

Now, we have for l, m, a, with $l > m > a$,

$$\begin{aligned}
H_G(\mathcal{R}^*_a, \mathcal{R}^*_m, \mathcal{R}^*_l) &\le H_G(\mathcal{R}^*_a, \mathcal{R}^*_{a+1}, \mathcal{R}^*_{a+1}) + H_G(\mathcal{R}^*_{a+1}, \mathcal{R}^*_{a+2}, \mathcal{R}^*_{a+2}) \\
&\quad + \cdots + H_G(\mathcal{R}^*_{l-1}, \mathcal{R}^*{}_{l-1}, \mathcal{R}^*_l) \\
&\le H_G(\mathcal{R}^*_a, \mathcal{R}^*_{a+1}, \mathcal{R}^*_{a+2}) + H_G(\mathcal{R}^*_{a+1}, \mathcal{R}^*_{a+2}, \mathcal{R}^*_{a+3}) \\
&\quad + \cdots + H_G(\mathcal{R}^*_{l-2}, \mathcal{R}^*_{l-1}, \mathcal{R}^*_l) \\
&\le [\lambda^a + \lambda^{a+1} + \cdots + \lambda^{l-2}] H_G(\mathcal{R}^*_0, \mathcal{R}^*_1, \mathcal{R}^*_2) \\
&= \lambda^a [1 + \lambda + \lambda^2 + \cdots + \lambda^{l-a-1}] H_G(\mathcal{R}^*_0, \mathcal{R}^*_1, \mathcal{R}^*_2) \\
&\le \frac{\lambda^a}{1-\lambda} H_G(\mathcal{R}^*_0, \mathcal{R}^*_1, \mathcal{R}^*_2).
\end{aligned}$$

We note that if $l = m > a$, we get similar results and if $l > m = a$, then

$$H_G(\mathcal{R}^*_a, \mathcal{R}^*_m, \mathcal{R}^*_l) \le \frac{\lambda^{a-1}}{1-\lambda} H_G(\mathcal{R}^*_0, \mathcal{R}^*_1, \mathcal{R}^*_2),$$

so $\lim_{a,m,l \to +\infty} H_G(\mathcal{R}^*_a, \mathcal{R}^*_m, \mathcal{R}^*_l) = 0$. Thus $\{\mathcal{R}^*_a\}$ is a G-Cauchy sequence in $\mathcal{C}^G(W)$. Since $(\mathcal{C}^G(W), H_G)$ is a complete G-metric space, there exists $\tilde{U_1} \in \mathcal{C}^G(W)$ such that $\lim_{a \to +\infty} \mathcal{R}^*_a = \tilde{U_1}$, that is, $\lim_{a \to +\infty} H_G(\mathcal{R}^*_a, \mathcal{R}^*_a, \tilde{U_1}) = 0$.

Assume that $\Upsilon(\tilde{U_1}) = \tilde{U_1}$, else

$$\begin{aligned}
& H_G(\Upsilon(\tilde{U_1}), \mathcal{R}_{3a+2}, \mathcal{R}^*_{3a+3}) \\
= {} & H_G(\Upsilon(\tilde{U_1}), \Psi\left(\mathcal{R}^*_{3a+1}\right), \Phi\left(\mathcal{R}^*_{3a+2}\right)) \\
\le {} & \lambda_1 H_G(\tilde{U_1}, \mathcal{R}^*_{3a+1}, \mathcal{R}^*_{3a+2}) + \lambda_2 [H_G(\tilde{U_1}, \tilde{U_1}, \Upsilon(\tilde{U_1}) \\
& + H_G(\mathcal{R}^*_{3a+1}, \mathcal{R}^*_{3a+1}, \Psi\left(\mathcal{R}^*_{3a+1}\right)) + H_G(\mathcal{R}^*_{3a+2}, \mathcal{R}^*_{3a+2}, \Phi(\mathcal{R}^*_{3a+2}))] \\
& + \lambda_3 [H_G(\Upsilon(\tilde{U_1}), \mathcal{R}^*_{3a+1}, \mathcal{R}^*_{3a+2}) + H_G(\tilde{U_1}, \Psi(\mathcal{R}^*_{3a+1}), \mathcal{R}^*_{3a+2}) \\
& + H_G(\tilde{U_1}, \mathcal{R}^*_{3a+1}, \Phi\left(\mathcal{R}^*_{3a+2}\right))] \\
= {} & \lambda_1 H_G(\tilde{U_1}, \mathcal{R}^*_{3a+1}, \mathcal{R}^*_{3a+2}) + \lambda_2 [H_G(\tilde{U_1}, \tilde{U_1}, \Upsilon\tilde{U_1}) \\
& + H_G(\mathcal{R}^*_{3a+1}, \mathcal{R}^*_{3a+1}, \mathcal{R}_{3a+2}) + H_G(\mathcal{R}^*_{3a+2}, \mathcal{R}_{3a+2}, \mathcal{R}_{3a+3})] \\
& + \lambda_3 [H_G(\Upsilon(\tilde{U_1}), \mathcal{R}_{3a+1}, \mathcal{R}^*_{3a+2}) + H_G(\tilde{U_1}, \mathcal{R}^*_{3a+2}, \mathcal{R}^*_{3a+2}) \\
& + H_G(\tilde{U_1}, \mathcal{R}^*_{3a+1}, \mathcal{R}^*_{3a+3})]
\end{aligned}$$

and as $a \to +\infty$, we gives

$$H_G(\Upsilon(\tilde{U_1}), \tilde{U_1}, \tilde{U_1}) \le (\lambda_2 + \lambda_3)\, H_G(\Upsilon(\tilde{U_1}), U_1, \tilde{U_1})$$

which is a contradiction as $(\lambda_2 + \lambda_3) < 1$. Thus $\Upsilon(\tilde{U_1}) = \tilde{U_1}$. Likewise, we can show that $\Psi(\tilde{U_1}) = \tilde{U_1}$ and $\Phi(\tilde{U_1}) = \tilde{U_1}$. Assume that $\tilde{U_2}$ is likewise a common attractor of Υ, Ψ and Φ. to demonstrate uniqueness. Then

$$\begin{aligned}
H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) &= H_G(\Upsilon(\tilde{U_1}), \Psi(\tilde{U_2}), \Phi(\tilde{U_2})) \\
&\leq \lambda_1 H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) + \lambda_2[H_G(\tilde{U_1}, \tilde{U_1}, \Upsilon(\tilde{U_1}) + H_G(\tilde{U_2}, \tilde{U_2}, \Psi(\tilde{U_2})) \\
&\quad + H_G(\tilde{U_2}, \tilde{U_2}, \Phi(V_1))] + \lambda_3[H_G(\Upsilon(\tilde{U_1}), \tilde{U_2}, \tilde{U_2}) \\
&\quad + H_G(\tilde{U_1}, \Psi(\tilde{U_2}), \tilde{U_2}) + H_G(\tilde{\tilde{U_1}}, \tilde{U_2}, \Phi(\tilde{U_2}))] \\
&= \lambda_1 H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) + \lambda_2[H_G(\tilde{U_1}, \tilde{U_1}, \tilde{U_1}) + H_G(\tilde{U_2}, \tilde{U_2}, \tilde{U_2}) \\
&\quad + H_G(\tilde{U_2}, \tilde{U_2}, \tilde{U_2})] + \lambda_3[H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) + H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) \\
&\quad + H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2})] \\
&= (\lambda_1 + 3\lambda_3)\, H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2})
\end{aligned}$$

from which we conclude that $H_G(\tilde{U_1}, \tilde{U_2}, \tilde{U_2}) = 0$ and thus $\tilde{U_1} = \tilde{U_2}$. Hence $\tilde{U_1}$ is a unique common attractor of Υ, Ψ, and Φ. □

Example 6.3.2. Let $W = [0, 1]$ and G be a G-metric on W as defined in Example . Define $f_a^*, g_a^*, h_a^* : W \to W$, $a = 1, 2$ by

$$f_1^*(u_1) = \begin{cases} \dfrac{u_1}{20} & \text{if } 0 \leq u_1 < \frac{1}{2} \\ \dfrac{u_1}{15} & \text{if } \frac{1}{2} \leq u_1 \leq 1, \end{cases} \qquad f_2^*(u_1) = \begin{cases} \dfrac{u_1}{17} & \text{if } 0 \leq u_1 < \frac{1}{2} \\ \dfrac{u_1}{12} & \text{if } \frac{1}{2} \leq u_1 \leq 1, \end{cases}$$

$$g_1^*(u_1) = \begin{cases} \dfrac{u_1}{13} & \text{if } 0 \leq u_1 < \frac{1}{2} \\ \dfrac{u_1}{16} & \text{if } \frac{1}{2} \leq u_1 \leq 1, \end{cases} \qquad g_2^*(u_1) = \begin{cases} \dfrac{u_1}{9} & \text{if } 0 \leq u_1 < \frac{1}{2} \\ \dfrac{u_1}{8} & \text{if } \frac{1}{2} \leq u_1 \leq 1, \end{cases}$$

$$h_1^*(u_1) = \begin{cases} \dfrac{u_1}{10} & \text{if } 0 \leq u_1 < \frac{1}{2} \\ \dfrac{u_1}{14} & \text{if } \frac{1}{2} \leq u_1 \leq 1, \end{cases} \qquad h_2^*(u_1) = \begin{cases} \dfrac{u_1}{12} & \text{if } 0 \leq u_1 < \frac{1}{2} \\ \dfrac{u_1}{13} & \text{if } \frac{1}{2} \leq u_1 \leq 1. \end{cases}$$

Then, clearly the maps $f_1^*, f_2^*, g_1^*, g_2^*, h_1^*$ and h_2^* are discontinuous and satisfying the condition of Theorem Theorem 6.3.4.

Now, by taking the generalized G-iterated function system $\{W; (f_1^*, f_2^*, g_1^*, g_2^*, h_1^*, h_2^*)\}$, we define mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$

by

$$\begin{aligned}\Upsilon(\mathcal{Q}^*) &= f_1(\mathcal{Q}^*) \cup f_2^*(\mathcal{Q}^*),\\ \Psi(\mathcal{R}^*) &= g_1(\mathcal{R}^*) \cup g_2(\mathcal{R}^*),\\ \Phi(\mathcal{N}^*) &= h_1(\mathcal{N}^*) \cup h_2(\mathcal{N}^*)\end{aligned}$$

for all $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$. Then it is easy to verify that mappings (Υ, Ψ, Φ) are a triplet of generalized G-Hutchinson contractive operators (type II), that is, for any $\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^* \in \mathcal{C}^G(W)$,

$$H_G(\Upsilon(\mathcal{Q}^*), \Psi(\mathcal{R}^*), \Phi(\mathcal{N}^*)) \leq E_{\Upsilon,\Psi,\Phi}(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*)$$

holds, where

$$\begin{aligned}E_{\Upsilon,\Psi,\Phi}(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*) &= \lambda_1 H_G(\mathcal{Q}^*, \mathcal{R}^*, \mathcal{N}^*) + \lambda_2[H_G(\mathcal{Q}^*, \mathcal{Q}^*, \Upsilon(\mathcal{Q}^*))\\ &\quad + H_G(\mathcal{R}^*, \mathcal{R}^*, \Psi(\mathcal{R}^*)) + H_G(\mathcal{N}^*, \mathcal{N}^*, \Phi(\mathcal{N}^*))]\\ &\quad + \lambda_3[H_G(\Upsilon(\mathcal{Q}^*), \mathcal{R}^*, \mathcal{N}^*) + H_G(\mathcal{Q}^*, \Psi(\mathcal{R}^*), \mathcal{N}^*)\\ &\quad + H_G(\mathcal{Q}^*, \mathcal{R}^*, \Phi(\mathcal{N}^*))],\end{aligned}$$

with $\lambda_1 = \frac{8}{9}$ and $\lambda_2 = \lambda_3 = \frac{1}{18}$. Clearly, $\lambda_j \geq 0$ for $j \in \{1, 2, 3\}$ and $\lambda_1 + 3\lambda_2 + 4\lambda_3 < 1$. Thus, all of the conditions of Theorem 6.3.4 are met, and additionally, for any initial set $\mathcal{R}_0^* \in \mathcal{C}^G(W)$, the sequence $\{\mathcal{R}_0^*, \Upsilon(\mathcal{R}_0^*), \Psi\Upsilon(\mathcal{R}_0^*), \Phi\Psi\Upsilon(\mathcal{R}_0^*), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0^*), \cdots\}$ of compact sets is convergent and has for a limit, the common attractor of Υ, Ψ and Φ.

Corollary 6.3.3. *In a complete G-metric space (W, G), let $\{W; f_a^*, g_a^*, h_a^*, a = 1, 2, \cdots, q\}$ be a generalized iterated function system and define the mappings $f^*, g^*, h^* : W \to W$ as in Remark* 6.3.1. *If there exist $\lambda_j \geq 0$ for $j \in \{1, 2, 3\}$ with $\lambda_1 + 3\lambda_2 + 4\lambda_3 < 1$ such that for any $w_1, w_2, w_3 \in \mathcal{C}^G(W)$, the following holds:*

$$G(f^* w_1, g^* w_2, h^* w_3) \leq E_{f^*,g^*,h^*}(w_1, w_2, w_3),$$

where

$$\begin{aligned}E_{f^*,g^*,h^*}(w_1, w_2, w_3) &= \lambda_1 H_G(w_1, w_2, w_3) + \lambda_2[H_G(w_1, w_1, f^*(w_1))\\ &\quad + H_G(w_2, w_2, g^*(w_2)) + H_G(w_3, w_3, h^*(w_3))]\\ &\quad + \lambda_3[H_G(f^*(w_1), w_2, w_3) + H_G(w_1, g^*(w_2), w_3)\\ &\quad + H_G(w_1, w_2, h^*(w_3))].\end{aligned}$$

Then f, g and h have a unique common fixed point. In addition, for a ran-

domly chosen $v_0 \in W$, the sequence $\{v_0, f^*v_0, g^*f^*v_0, h^*g^*f^*v_0, f^*h^*g^*f^*v_0, ...\}$ converges to a common fixed point of f^*, g^* and h^*.

6.4. Well-posedness of Attractor based problem in G-Metric Spaces

We extend the discussion in Section 2.3 to the attractor-based problems of generalized Hutchinson contractive operators (type I) and generalized Hutchinson contractive operators (type II) in the framework of Hausdorff G-metric spaces. Some useful results on well-posedness of fixed point problems appear in [8, 56].

Definition 6.4.1. A common attractor-based problem of mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ is said to be well-posed if the triplet (Υ, Ψ, Φ) has a unique common attractor $\Theta_* \in \mathcal{C}^G(W)$ and any sequence $\{\Theta_a\}$ in $\mathcal{C}^G(W)$ is such that $\lim\limits_{a\to+\infty} H_G(\Upsilon(\Theta_a), \Upsilon(\Theta_a), \Theta_a) = 0$, $\lim\limits_{a\to+\infty} H_G(\Psi(\Theta_a), \Psi(\Theta_a), \Theta_a) = 0$, and $\lim\limits_{a\to+\infty} H_G(\Phi(\Theta_a), \Phi(\Theta_a), \Theta_a) = 0$ then $\lim\limits_{a\to+\infty} H_G(\Theta_a, \Theta_a, \Theta_*) = 0$, that is to say, $\lim\limits_{a\to+\infty} \Theta_a = \Theta_*$.

Theorem 6.4.1. *Let (W, G) be a complete G-metric space and $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ be defined as in Theorem 6.3.1. Then the mappings Υ, Ψ, Φ have a well-posed common attractor-based problem.*

Proof. From Theorem 6.3.1, we deduce that the mappings Υ, Ψ and Φ have a unique common attractor $\mathcal{Z}_*$, say. Let a sequence $\{\mathcal{Z}_a\}$ in $\mathcal{C}^G(W)$ be such that $\lim\limits_{a\to+\infty} H_G(\Upsilon(\mathcal{Z}_a), \Upsilon(\mathcal{Z}_a), \mathcal{Z}_a) = 0$, $\lim\limits_{a\to+\infty} H_G(\Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$, and $\lim\limits_{a\to+\infty} H_G(\Phi(\mathcal{Z}_a), \Phi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$.

We show that $\mathcal{Z}_* = \lim\limits_{a\to+\infty} \mathcal{Z}_a$. As the mappings (Υ, Ψ, Φ) are a triplet of generalized G-Hutchinson contractive operators (type I), then

$$\begin{aligned} H_G(\mathcal{Z}_a, \mathcal{Z}_a, \mathcal{Z}_*) &\leq H_G(\mathcal{Z}_a, \Upsilon(\mathcal{Z}_a), \Upsilon(\mathcal{Z}_a)) + H_G(\Upsilon(\mathcal{Z}_a), \Psi(\mathcal{Z}_a), \mathcal{Z}_*)) \\ &= H_G(\Upsilon(\mathcal{Z}_a), \Psi(\mathcal{Z}_a), \Phi(\mathcal{Z}_*)) + H_G(\mathcal{Z}_a, \Upsilon(\mathcal{Z}_a), \Upsilon(\mathcal{Z}_a)) \\ &\leq \alpha H_G(\mathcal{Z}_a, \mathcal{Z}_a, \mathcal{Z}_*) + \beta H_G(\mathcal{Z}_a, \Upsilon(\mathcal{Z}_a), \Upsilon(\mathcal{Z}_a)) \\ &\quad + \gamma H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)) + \eta H_G(\mathcal{Z}_*, \Phi(\mathcal{Z}_*), \Phi(\mathcal{Z}_*)) \\ &\quad + H_G(\mathcal{Z}_a, \Upsilon(\mathcal{Z}_a), \Upsilon(\mathcal{Z}_a)). \end{aligned}$$

Thus

$$H_G(\mathcal{Z}_a, \mathcal{Z}_a, \mathcal{Z}_*) \leq \frac{\beta+1}{1-\alpha} H_G(\mathcal{Z}_a, \Upsilon(\mathcal{Z}_a), \Upsilon(\mathcal{Z}_a)) + \frac{\gamma}{1-\alpha} H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)).$$

Taking limit on both sides as $a \to +\infty$ gives us $\lim_{a\to+\infty} H_G(\mathcal{Z}_a, \mathcal{Z}_a, \mathcal{Z}_*) = 0$ and so $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$. □

Theorem 6.4.2. *Let (W, G) be a complete G-metric space and $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ be defined as in Theorem 6.3.4. Then the mappings Υ, Ψ, Φ have a well-posed common attractor-based problem.*

Proof. From Theorem 6.3.4, it follows that the mappings Υ, Ψ and Φ have a unique common attractor $\mathcal{Z}_*$, say.

Let a sequence $\{\mathcal{Z}_a\}$ in $\mathcal{C}^G(W)$ be such that $\lim_{a\to+\infty} H_G(\Upsilon(\mathcal{Z}_a), \Upsilon(\mathcal{Z}_a), \mathcal{Z}_a) = 0$, $\lim_{a\to+\infty} H_G(\Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$ and $\lim_{a\to+\infty} H_G(\Phi(\mathcal{Z}_a), \Phi(\mathcal{Z}_a), \mathcal{Z}_a) = 0$.
We want to show that $\mathcal{Z}_* = \lim_{a\to+\infty} \mathcal{Z}_a$. As the mappings (Υ, Ψ, Φ) are generalized G-Hutchinson contractive operators (type II), so that

$$
\begin{aligned}
& H_G(\mathcal{Z}_a, \mathcal{Z}_a, \mathcal{Z}_*) \\
\leq\ & H_G(\mathcal{Z}_a, \mathcal{Z}_a, \Psi(\mathcal{Z}_a)) + H_G(\Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a), \mathcal{Z}_*) \\
\leq\ & 2H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)) + H_G(\Upsilon(\mathcal{Z}_*), \Psi(\mathcal{Z}_a), \Phi(\mathcal{Z}_*)) \\
\leq\ & 2H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)) + \lambda_1 H_G(\mathcal{Z}_*, \mathcal{Z}_a, \mathcal{Z}_a) + \lambda_2[H_G(\mathcal{Z}_*, \mathcal{Z}_*, \Upsilon(\mathcal{Z}_*)) \\
& + H_G(\mathcal{Z}_a, \mathcal{Z}_a, \Psi(\mathcal{Z}_a)) + H_G(\mathcal{Z}_a, \mathcal{Z}_a, \Phi(\mathcal{Z}_a))] \\
& + \lambda_3[H_G(\Upsilon(\mathcal{Z}_*), \mathcal{Z}_a, \mathcal{Z}_a) + H_G(\mathcal{Z}_*, \Psi(\mathcal{Z}_a), \mathcal{Z}_a) + H_G(\mathcal{Z}_*, \mathcal{Z}_a, \Phi(\mathcal{Z}_a))] \\
\leq\ & 2H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)) + \lambda_1 H_G(\mathcal{Z}_*, \mathcal{Z}_a, \mathcal{Z}_a) + 2\lambda_2[H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)) \\
& + H_G(\mathcal{Z}_a, \Phi(\mathcal{Z}_a), \Phi(\mathcal{Z}_a))] + \lambda_3[H(\mathcal{Z}_*, \mathcal{Z}_a, \mathcal{Z}_a) + H_G(\mathcal{Z}_*, \mathcal{Z}_a, \mathcal{Z}_a) \\
& + H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \mathcal{Z}_a) + H_G(\mathcal{Z}_*, \mathcal{Z}_a, \mathcal{Z}_a) + H_G(\mathcal{Z}_a, \mathcal{Z}_a, \Phi(\mathcal{Z}_a))] \\
\leq\ & 2H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)) + \lambda_1 H_G(\mathcal{Z}_*, \mathcal{Z}_a, \mathcal{Z}_a) + 2\lambda_2[H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)) \\
& + H_G(\mathcal{Z}_a, \Phi(\mathcal{Z}_a), \Phi(\mathcal{Z}_a))] + \lambda_3[3H(\mathcal{Z}_*, \mathcal{Z}_a, \mathcal{Z}_a) \\
& + 2H_G(\mathcal{Z}_a, \Psi(\mathcal{Z}_a), \Psi(\mathcal{Z}_a)) + 2H_G(\mathcal{Z}_a, \Phi(\mathcal{Z}_a), \Phi(\mathcal{Z}_a))].
\end{aligned}
$$

Thus

$$
\begin{aligned}
H_G(\mathcal{Z}_a, \mathcal{Z}_a, \mathcal{Z}_*) \leq\ & \frac{1}{1 - \lambda_1 - 3\lambda_3}[2(1 + \lambda_2 + \lambda_3) H_G(\mathcal{Z}_a, \Psi(\mathcal{A}_a), \Psi(\mathcal{Z}_a)) \\
& + 2(\lambda_2 + \lambda_3) H_G(\mathcal{Z}_a, \Phi(\mathcal{Z}_a), \Phi(\mathcal{Z}_a))].
\end{aligned}
$$

Taking limit on both side implies that $\lim_{a\to+\infty} \mathcal{Z}_a = \mathcal{Z}_*$. □

7

Generalized Iterated Function System of Cyclic Contractions in G-Metric Spaces

7.1. Introduction

In the previous chapter, we considered the construction of common attractors of generalized iterated function system of generalized contractions in G-metric spaces. We extend our discussion to generalized iterated function system of generalized cyclic contractions in G-metric spaces.

The concept of cyclic contraction mapping was introduced by Rus [85]. Further expansions were made by considering fixed point results for cyclic φ-contractions in the framework of metric spaces [80, 81]. In Chapter 4 we explored iterated function system of generalized cyclic contractions in partial metric spaces. Karapinar et al.[51] obtained some results for cylic contractions on G-metric spaces.

Definition 7.1.1. [80] For a non-void set W, let $h : W \to W$ be a mapping of W to itself. A finite family $\{W_1, W_2, \cdots, W_q\}$ of non-void subsets of W with $W = \cup_{a=1}^{q} W_a$ is said to be a cyclic representation of W with respect to h if $h(W_1) \subset W_2, \cdots, h(W_{q-1}) \subset W_q$, and $h(W_q) \subset W_1$.

Theorem 7.1.1. [51] *In a G -complete G-metric space (W, G), let $\{B\}_{a=1}^{q}$ represent a class of non-void G -closed subsets of W. Let $W = \cup_{a=1}^{q} B_a$ and $\Psi : \cup_{a=1}^{q} B_a \to \cup_{a=1}^{q} B_a$ be a map satisfying $\Psi(B_a) \subseteq W_{a+1}$, $a = 1, 2, \cdots, q$ where $B_{q+1} = B_1$. Suppose there exists $\lambda \in [0, 1)$ such that $G(\Psi u, \Psi v, \Psi w) \leq \lambda G(u, v, w)$ for all $u \in B_a$ and $v, w \in B_{a+1}, a = 1, 2, \cdots, q$, then Ψ has a unique fixed point in $\cap_{a=1}^{q} B_a$.*

7.2. Generalized Iterated Function System of Cyclic Contractions in G-metric spaces

In [35], we find results on generalized iterated function system for multi-valued mappings in a metric spaces. We consider the generalized iterated function system of cyclic contractions in G-metric space setting.

Definition 7.2.1. For a G-metric space, (W, G), let $h_a : W \to W$, $a \in \mathbb{N}_q$ be a finite family of G-contractions, then $\{W; h_a, a \in \mathbb{N}_q\}$ is called a G-iterated function system (G-IFS).

Definition 7.2.2. Let (W, G) be a G-metric space with $\mathcal{J}^* \subseteq W$, a non-void compact set, then $\mathcal{J}^*$ is called an attractor of the G-IFS if

(i) $\Psi(\mathcal{J}^*) = \mathcal{J}^*$ and

(ii) there exists an open set $V_1 \subseteq W$ such that $\mathcal{J}^* \subseteq V_1$ and $\lim_{a\to+\infty} \Psi^a(\mathcal{O}^*) = \mathcal{J}^*$ for any compact set $\mathcal{O}^* \subseteq V_1$, where the limit is taken with respect to the G-Hausdorff metric.

As a necessary consequence, the maximal open set V_1 satisfying (ii) is referred to as a basin of attraction.

Definition 7.2.3. Let $\{B_a\}_{a=1}^q$ be a collection of non-void closed subsets of a G-metric space, (W, G). A self-mapping $h : \cup_{a=1}^q B_a \to \cup_{a=1}^q B_a$ is known as a cyclic G-contraction on $\{B_a\}_{a=1}^q$, provided there exists a $\lambda \in [0, 1)$, such that

(i) $h(B_a) \subseteq B_{a+1}$ for $a \in \mathbb{N}_q$, where $B_{q+1} = B_1$;

(ii) $G(hu, hv, hw) \leq \lambda G(u, v, w)$ for all $u \in B_a$, and $v, w \in B_{a+1}$ for $a \in \mathbb{N}_q$.

If h satisfies condition (i), then h is a cyclic function.

Theorem 7.2.1. [51] *Suppose $\{B_a\}_{a=1}^q$ is a family of non-void G-closed subsets of a G-metric space (W, G). Let $h : \cup_{a=1}^q B_a \to \cup_{a=1}^q B_a$ be a cyclic map satisfying*

$$h(B_a) \subseteq B_{a+1}, a \in \mathbb{N}_q, \text{ where } B_{q+1} = B_1.$$

Suppose there exists $\lambda \in [0, 1)$ such that

$$G(hu, hv, hw) \leq \lambda G(u, v, w)$$

for all $u \in B_a$, and $v, w \in B_{a+1}$ for $a \in \mathbb{N}_q$, then h has a unique fixed point $u \in \cap_{a=1}^q B_a$.

Definition 7.2.4. In a complete G-metric space (W, G), suppose that $f_a, g_a, h_a : W \to W$, $a \in \mathbb{N}_q$ are continuous maps such that each (f_a, g_a, h_a), $a \in \mathbb{N}_q$ is a triplet of generalized G-contractions, then $\{W; (f_a, g_a, h_a), a \in \mathbb{N}_q\}$ is the generalized G-iterated function system.

Definition 7.2.5. Let (W, G) be a complete metric space. A set $\{W; (f_a, g_a, h_a), a \in \mathbb{N}_q\}$ is said to be a generalized cyclic G-iterated function system if each triplet $f_a, g_a, h_a : W \to W$ is a generalized cyclic contraction for $a \in \mathbb{N}_q$.

Theorem 7.2.2. *Let $\{B_a\}_{a=1}^q$ be a family of non-void closed subsets of a G-metric space (W, G) and $f, g, h : \cup_{a=1}^q B_a \to \cup_{a=1}^q B_a$ a triplet of continuous cyclic contractions. Then, the triplet of mappings $f, g, h : \mathcal{C}^G(\cup_{a=1}^q B_a) \to \mathcal{C}^G(\cup_{a=1}^q B_a)$ is also a cyclic contraction relative to the Hausdorff metric H_G sharing a similar contractive constant, λ.*

Proof. Choose $L \in B_a$, for some $a \in \mathbb{N}_q$. From the definition of cyclic map, we obtain that $f(L) \subseteq B_{a+1}$. Also, since f is continuous, then $f(L)$ is a compact set. Therefore, $f(L) \in \mathcal{C}^G(B_{a+1})$ which implies that $f(\mathcal{C}^G(B_a)) \subseteq \mathcal{C}^G(B_{a+1})$ for each $a \in \mathbb{N}_q$.

We take $\mathcal{J}_1^* \in \mathcal{C}^G(B_a)$, $\mathcal{J}_2^* \in \mathcal{C}^G(B_{a+1})$ and $\mathcal{J}_3^* \in \mathcal{C}^G(B_{a+2})$ for some $a \in \mathbb{N}_q$. First, we claim that

$$\sup_{f(m_1) \in f(\mathcal{J}_1^*)} G(f(m_1), g(\mathcal{J}_2^*), h(\mathcal{J}_3^*)) \leq \lambda \sup_{m_1 \in \mathcal{J}_1^*} G(m_1 \mathcal{J}_2^*, \mathcal{J}_3^*).$$

As the triplet (f,g,h) is a cyclic G-contraction, we obtain

$$\begin{aligned} G(f(m_1), g(m_2), h(m_3)) &\leq \lambda G(m_1, m_2, m_3) \text{ for all } m_1 \in B_a,\ m_2 \in B_{a+1} \\ \text{and } m_3 &\in B_{a+2} \text{ where } a \in \mathbb{N}_q. \end{aligned}$$

Thus

$$\begin{aligned} &\sup_{f(m_1) \in f(\mathcal{J}_1^*)} G(f(m_1), g(\mathcal{J}_2^*), h(\mathcal{J}_3^*)) \\ = &\sup_{f(m_1) \in f(\mathcal{J}_1^*)} \left(\inf_{g(m_2) \in g(\mathcal{J}_2^*), h(m_3) \in h(\mathcal{J}_3^*)} G(f(m_1), g(m_2), h(m_3)) \right) \\ \leq &\ \lambda \left(\sup_{m_1 \in \mathcal{J}_1^*} \left(\inf_{m_2 \in \mathcal{J}_2^*, m_3 \in \mathcal{J}_3^*} G(m_1, m_2, m_3) \right) \right) \\ \leq &\ \lambda \left(\sup_{m_1 \in \mathcal{J}_1^*} G(m_1, \mathcal{J}_2^*, \mathcal{J}_3^*) \right). \end{aligned}$$

Similarly, we have

$$\sup_{g(m_2)\in g(\mathcal{J}_2^*)} G(f(\mathcal{J}_1^*), g(m_2), h(\mathcal{J}_3^*)) \leq \lambda\left(\sup_{m_2\in\mathcal{J}_2^*} G(\mathcal{J}_1^*, m_2, \mathcal{J}_3^*)\right)$$

and

$$\sup_{h(m_3)\in h(\mathcal{J}_3^*)} G(f(\mathcal{J}_1^*), g(\mathcal{J}_2^*), h(m_3)) \leq \lambda\left(\sup_{m_3\in\mathcal{J}_3^*} G(\mathcal{J}_1^*, \mathcal{J}_2^*, m_3)\right).$$

So

$$\begin{aligned}
& H_G(f(\mathcal{J}_1^*), g(\mathcal{J}_2^*), \mathcal{J}_3^*)) \\
= \; & \max\{ \sup_{f(m_1)\in f\left(\mathcal{J}_1^*\right)} G(f(m_1), g(\mathcal{J}_2^*), h(\mathcal{J}_3^*)), \sup_{g(m_2)\in g\left(\mathcal{J}_2^*\right)} G(f(\mathcal{J}_1^*), g(m_2), h(\mathcal{J}_3^*)), \\
& \sup_{h(m_3)\in h\left(\mathcal{J}_3^*\right)} G(f(\mathcal{J}_1^*), g(\mathcal{J}_2^*), h(m_3))\} \\
\leq \; & \lambda \max\{ \sup_{m_1\in\mathcal{J}_1^*} G(m_1\mathcal{J}_2^*, \mathcal{J}_3^*), \sup_{m_2\in\mathcal{J}_2^*} G(\mathcal{J}_1^*, m_2, \mathcal{J}_3^*), \sup_{m_3\in\mathcal{J}_3^*} G(\mathcal{J}_1^*, \mathcal{J}_2^*, m_3)\} \\
= \; & \lambda H_G(\mathcal{J}_1^*, \mathcal{J}_2^*, \mathcal{J}_3^*).
\end{aligned}$$

Hence, (*f*,*g*,*h*) is a triplet of cyclic G-contraction mapping on $\{B_a\}_{a=1}^q$. □

Theorem 7.2.3. *Let $\{B_a\}_{a=1}^q$ be the collection of non-void closed subsets of a G-metric space (W, G), and q a fixed natural number. If $f_a, g_a, h_a : \cup_{a=1}^q B_a \to \cup_{a=1}^q B_a$ for all $a \in \mathbb{N}_q$ are generalized cyclic contractions, then the maps $\Upsilon, \Psi, \Phi : \mathcal{C}^G(\cup_{a=1}^q B_a) \to \mathcal{C}^G(\cup_{a=1}^q B_a)$ defined by*

$$\begin{aligned}
\Upsilon(\mathcal{J}_1^*) &= f_1(\mathcal{J}_1^*) \cup f_2(\mathcal{J}_1^*) \cup \cdots \cup f_a(\mathcal{J}_1^*) \\
&= \cup_{a=1}^q f_a(\mathcal{J}_1^*), \text{ for each } \mathcal{J}_1^* \in \mathcal{C}^G(\cup_{a=1}^q B_a),
\end{aligned}$$

$$\begin{aligned}
\Psi(\mathcal{J}_2^*) &= g_1(\mathcal{J}_2^*) \cup g_2(\mathcal{J}_2^*) \cup \cdots \cup g_a(\mathcal{J}_2^*) \\
&= \cup_{a=1}^q g_a(\mathcal{J}_2^*), \text{ for each } \mathcal{J}_2^* \in \mathcal{C}^G(\cup_{a=1}^q B_a)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\mathcal{J}_3^*) &= h_1(\mathcal{J}_3^*) \cup h_2(\mathcal{J}_3^*) \cup \cdots \cup h_a(\mathcal{J}_3^*) \\
&= \cup_{a=1}^q h_a(\mathcal{J}_3^*), \text{ for each } \mathcal{J}_3^* \in \mathcal{C}^G(\cup_{a=1}^q B_a)
\end{aligned}$$

also satisfy

$$H_G\left(\Upsilon(\mathcal{J}_1^*), \Psi(\mathcal{J}_2^*), \Phi(\mathcal{J}_3^*)\right) \leq \lambda_* H_G\left(\mathcal{J}_1^*, \mathcal{J}_2^*, \mathcal{J}_3^*\right) \text{ for all } \mathcal{J}_1^*, \mathcal{J}_2^*, \mathcal{J}_3^* \in \mathcal{C}^G(\cup_{a=1}^q B_a),$$

where $\lambda_* = \max\{\lambda_a : a \in \mathbb{N}_q\}$*, that is, the triplet* (Υ, Ψ, Φ) *is a generalized cyclic contraction map on* $\mathcal{C}^G(W)$*.*

Proof. Let $\mathcal{J}^* \in \mathcal{C}^G(B_a)$ for some $a \in \mathbb{N}_q$. By Theorem 7.2.2, for each $a \in \mathbb{N}_q$, the triplet (f_a, g_a, h_a) is a generalized cyclic contraction. Therefore, $f_a(\mathcal{J}^*) \in \mathcal{C}^G(B_{a+1})$ for all $a \in \mathbb{N}_q$ which implies that $\Upsilon(\mathcal{J}^*) = \cup_{a=1}^q f_a(\mathcal{J}^*) \in \mathcal{C}^G(B_{a+1})$, and consequently, $\Upsilon(\mathcal{C}^G(B_a)) \subseteq \mathcal{C}^G(B_{a+1})$ for $a \in \mathbb{N}_q$. In the same manner we have $\Psi(\mathcal{C}^G(B_a)) \subseteq \mathcal{C}^G(B_{a+1})$ and $\Phi(\mathcal{C}^G(B_a)) \subseteq \mathcal{C}^G(B_{a+1})$ for $a \in \mathbb{N}_q$.

Since the triplet (f_a, g_a, h_a) is generalized cyclic contraction for each $a \in \mathbb{N}_{\mathbf{q}}$, we have

$$H_G(f_a(\mathcal{J}_1^*), g_a(\mathcal{J}_2^*), h_a(\mathcal{J}_3^*)) \leq \lambda H_G \mathcal{J}_1^*, \mathcal{J}_2^*, \mathcal{J}_3^*)$$

for all $\mathcal{J}_1^* \in \mathcal{C}^G(B_a), \mathcal{J}_2^* \in \mathcal{C}^G(B_{a+1})$ and $\mathcal{J}_3^* \in \mathcal{C}^G(B_{a+2})$ for each $a \in \mathbb{N}_q$. If $\mathcal{J}_1^* \in \mathcal{C}^G(B_a)$, $\mathcal{J}_2^* \in \mathcal{C}^G(B_{a+1})$ and $\mathcal{J}_3^* \in \mathcal{C}^G(B_{a+2})$ for some $a \in \mathbb{N}_q$, then we have

$$\begin{aligned} H_G\left(\Upsilon(\mathcal{J}_1^*), \Psi(\mathcal{J}_2^*), \Phi(\mathcal{J}_3^*)\right) &= H_G(\cup_{a=1}^q f_a(\mathcal{J}_1^*), \cup_{a=1}^q g_a(\mathcal{J}_2^*), \cup_{a=1}^q h_a(\mathcal{J}_3^*)) \\ &\leq \max\{H_G(f_1(\mathcal{J}_1^*), g_1(\mathcal{J}_2^*), h_1(\mathcal{J}_3^*)), \\ &\quad \cdots, H_G(f_q(\mathcal{J}_1^*), g_q(\mathcal{J}_2^*), g_q(\mathcal{J}_3^*))\} \\ &\leq \lambda_* H_G(\mathcal{J}_1^*, \mathcal{J}_2^*, \mathcal{J}_3^*). \end{aligned}$$

□

Definition 7.2.6. For a G-metric space (W, G), let $\{B_a\}_{a=1}^q$ be a collection of non-void closed subsets of W. Then $\Upsilon, \Psi, \Phi : \mathcal{C}^G(B_a) \to \mathcal{C}^G(B_a)$ is a triplet of generalized cyclic G-Hutchinson contractive operators (type I) if for any $\mathcal{L} \in \mathcal{C}^G(B_a)$, $\mathcal{M} \in \mathcal{C}^G(B_{a+1})$ and $\mathcal{N} \in \mathcal{C}^G(B_{a+2})$,

$$H_G(\Upsilon(\mathcal{L}), \Psi(\mathcal{M}), \Phi(\mathcal{N})) \leq \mathcal{S}_{\Upsilon,\Psi,\Phi}(\mathcal{L}, \mathcal{M}, \mathcal{N})$$

holds, where

$$\begin{aligned} \mathcal{S}_{\Upsilon,\Psi,\Phi}(\mathcal{L}, \mathcal{M}, \mathcal{N}) &= \alpha H_G(\mathcal{L}, \mathcal{M}, \mathcal{N}) + \beta H_G(\mathcal{L}, \Upsilon(\mathcal{L}), \Upsilon(\mathcal{L})) \\ &\quad + \gamma H_G(\mathcal{M}, \Psi(\mathcal{M}), \Psi(\mathcal{M})) + \eta H_G(\mathcal{N}, \Phi(\mathcal{N}), \Phi(\mathcal{N})), \end{aligned}$$

with $\alpha, \beta, \gamma, \eta \geq 0$ with $\alpha + \beta + \gamma + \eta < 1$.

Definition 7.2.7. In a G-metric space (W, G), let $\{B_a\}_{a=1}^q$ represent a family of non-void closed subsets of W. The triplet $\Upsilon, \Psi, \Phi : \mathcal{C}^G(B_a) \to \mathcal{C}^G(B_a)$ is called a generalized cyclic G-Hutchinson contractive operator (type II) if for any $\mathcal{L} \in \mathcal{C}^G(B_a)$, $\mathcal{M} \in \mathcal{C}^G(B_{a+1})$ and $\mathcal{N} \in \mathcal{C}^G(B_{a+2})$,

$$H_G(\Upsilon(\mathcal{L}), \Psi(\mathcal{M}), \Phi(\mathcal{N})) \leq \mathcal{R}_{\Upsilon,\Psi,\Phi}(\mathcal{L}, \mathcal{M}, \mathcal{N})$$

holds, where

$$\begin{aligned}\mathcal{R}_{\Upsilon,\Psi,\Phi}(\mathcal{L}, \mathcal{M}, \mathcal{N}) &= \lambda_1 H_G(\mathcal{L}, \mathcal{M}, \mathcal{N}) + \lambda_2[H_G(\mathcal{L}, \mathcal{L}, \Upsilon(\mathcal{L})) \\ &\quad + H_G(\mathcal{M}, \mathcal{M}, \Psi(\mathcal{M})) + H_G(\mathcal{N}, \mathcal{N}, \Phi(\mathcal{N}))] \\ &\quad + \lambda_3[H_G(\Upsilon(\mathcal{L}), \mathcal{M}, \mathcal{N}) + H_G(\mathcal{L}, \Psi(\mathcal{M}), \mathcal{N}) \\ &\quad + H_G(\mathcal{L}, \mathcal{M}, \Phi(\mathcal{N}))],\end{aligned}$$

with $\lambda_i \geq 0$ for $i \in \{1, 2, 3\}$ and $\lambda_1 + 3\lambda_2 + 4\lambda_3 < 1$.

Definition 7.2.8. Let (W, G) be a complete G-metric space. If $f_a, g_a, h_a : W \to W$, $a = 1, 2, ..., q$ are continuous mappings such that each triplet (f_a, g_a, h_a) for $a \in \mathbb{N}_q$ is a generalized cyclic G-contraction, then $\{W; (f_a, g_a, h_a), a \in \mathbb{N}_q\}$ is called the generalized cyclic G-iterated function system.

As a result, the generalized cyclic G-iterated function system is made up of a G-metric space and a finite family of generalized cyclic G-contractions on W.

Definition 7.2.9. Let (W, G) be a complete G-metric space and $\tilde{U_1} \subseteq W$ be a non-void compact set. Then $\tilde{U_1}$ is the common attractor of the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(W) \to \mathcal{C}^G(W)$ if

(i) $\Upsilon(\tilde{U_1}) = \Psi(\tilde{U_1}) = \Phi(\tilde{U_1}) = \tilde{U_1}$ and

(ii) there exists an open set $V_1 \subseteq W$ such that $\tilde{U_1} \subseteq V_1$ and $\lim_{a\to\infty} \Upsilon^a(\mathcal{L}) = \lim_{a\to\infty} \Psi^a(\mathcal{M}) = \lim_{a\to\infty} \Phi^a(\mathcal{N}) = \tilde{U_1}$ for any compact sets $\mathcal{L}, \mathcal{M}, \mathcal{N} \subseteq V_1$, where the limit is taken with respect to the G-Hausdorff metric.

7.3. Generalized cyclic G-Hutchinson contractive operators

In the context of G-metric space, we state and prove some results about the existence and uniqueness of a common attractor of generalized cyclic G-Hutchinson contractive operators. To begin, consider the following outcome.

Theorem 7.3.1. *In a complete G-metric space (W, G), suppose $\{B_a\}_{a=1}^q$ is a collection of non-void closed subsets of W and $\{W; (f_a, g_a, h_a), a \in \mathbb{N}_q\}$ is a general-*

ized cyclic G-iterated function system. If the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(\cup_{a=1}^q B_a) \to \mathcal{C}^G(\cup_{a=1}^q B_a)$ *defined by*

$$\begin{aligned}\Upsilon(\mathcal{L}) &= f_1(\mathcal{L}) \cup f_2(\mathcal{L}) \cup \cdots \cup f_q(\mathcal{L}) \\ &= \cup_{a=1}^q f_a(\mathcal{L}) \text{ for } \mathcal{L} \in \mathcal{C}^G(\cup_{a=1}^q B_a)\end{aligned}$$

$$\begin{aligned}\Psi(\mathcal{M}) &= g_1(\mathcal{M}) \cup g_2(\mathcal{M}) \cup \cdots \cup g_q(\mathcal{M}) \\ &= \cup_{a=1}^q g_a(\mathcal{M}) \text{ for } \mathcal{M} \in \mathcal{C}^G(\cup_{a=1}^q B_{a+1})\end{aligned}$$

and

$$\begin{aligned}\Phi(\mathcal{N}) &= h_1(\mathcal{N}) \cup h_2(\mathcal{N}) \cup \cdots \cup h_q(\mathcal{N}) \\ &= \cup_{a=1}^q h_a(\mathcal{N}) \text{ for } \mathcal{N} \in \mathcal{C}^G(\cup_{a=1}^q B_{a+2})\end{aligned}$$

are generalized cyclic G-Hutchinson contractive operators (type I), then Υ, Ψ *and* Φ *have a unique common attractor* $\tilde{U}_1^* \in \mathcal{C}^G(B_a)$*, that is,*

$$\tilde{U}_1^* = \Upsilon(\tilde{U}_1^*) = \Psi(\tilde{U}_1^*) = \Phi(\tilde{U}_1^*) = \cup_{a=1}^q f_a(\tilde{U}_1^*) = \cup_{a=1}^q g_a(\tilde{U}_1^*) = \cup_{a=1}^q h_a(\tilde{U}_1^*).$$

Furthermore, for an arbitrarily chosen initial set $\mathcal{M}_0 \in \mathcal{C}^G(\cup_{a=1}^q B_a)$*, the sequence*

$$\{\mathcal{M}_0, \Upsilon(\mathcal{M}_0), \Psi\Upsilon(\mathcal{M}_0), \Phi\Psi\Upsilon(\mathcal{M}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{M}_0), ...\}$$

of compact sets converges to the common attractor $\tilde{U}_1^*$*.*

Proof. We show that any attractor of Υ is an attractor of Ψ and Φ. To that end, we assume that $\tilde{U}_1^* \in \mathcal{C}^G(W)$ is such that $\Upsilon(\tilde{U}_1^*) = \tilde{U}_1^*$. We need to show that $\tilde{U}_1^* = \Psi(\tilde{U}_1^*) = \Phi(U^*)$, else for $\tilde{U}_1^* \neq \Psi(\tilde{U}_1^*)$ and $\tilde{U}_1^* \neq \Phi(\tilde{U}_1^*)$, we get

$$\begin{aligned}H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) &= H_G(\Upsilon(\tilde{U}_1^*), \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) \\ &\leq \alpha H_G(\tilde{U}_1^*, \tilde{U}_1^*, \tilde{U}_1^*) + \beta H_G(\tilde{U}_1^*, \Upsilon(\tilde{U}_1^*), \Upsilon(\tilde{U}_1^*)) \\ &\quad + \gamma H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Psi(\tilde{U}_1^*)) \\ &\quad + \eta H_G(\tilde{U}_1^*, \Phi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) \\ &= \alpha H_G(\tilde{U}_1^*, \tilde{U}_1^*, \tilde{U}_1^*) + \beta H_G(\tilde{U}_1^*, \tilde{U}_1^*, \tilde{U}_1^*) \\ &\quad + \gamma H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Psi(\tilde{U}_1^*)) \\ &\quad + \eta H_G(\tilde{U}_1^*, \Phi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) \\ &= \gamma H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Psi(\tilde{U}_1^*)) + \eta H_G(\tilde{U}_1^*, \Phi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) \\ &\leq (\gamma + \eta) H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)),\end{aligned}$$

thus

$$H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) \leq \lambda H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)),$$

where $\lambda = \gamma + \eta < 1$, which is a contradiction. In an analogous manner, for $\tilde{U}_1^* \neq \Psi(\tilde{U}_1^*)$ and $U^* = \Phi(U^*)$ or for $\tilde{U}_1^* \neq \Phi(\tilde{U}_1^*)$ and $\tilde{U}_1^* = \Psi(\tilde{U}_1^*)$ similar argument as above yields a contradiction. Hence we conclude that $\tilde{U}_1^* = \Upsilon(\tilde{U}_1^*) = \Psi(\tilde{U}_1^*) = \Phi(\tilde{U}_1^*)$. We also note that the same conclusion holds for $\tilde{U}_1^* = \Psi(\tilde{U}_1^*)$ or $\tilde{U}_1^* = \Phi(\tilde{U}_1^*)$.

Next we show that Υ, Ψ, and Φ have a unique common attractor. Let $\mathcal{M}_0 \in \mathcal{C}^G(W)$ be an arbitrary point. Define a sequence $\{\mathcal{M}_a\}$ by $\mathcal{M}_{3a+1} = \Upsilon(\mathcal{M}_{3a})$, $\mathcal{M}_{3a+2} = \Psi(\mathcal{M}_{3a+1}), \mathcal{M}_{3a+3} = \Phi(\mathcal{M}_{3a+2})$, $a = 0, 1, 2, \cdots$. If $\mathcal{M}_a = \mathcal{M}_{a+1}$ for some n, with $a = 3n$, then $\tilde{U}_1^* = \mathcal{M}_{3a}$ is an attractor of Υ and from the Proof above, $\tilde{U}_1^*$ is a common attractor for Υ, Ψ, and Φ. The same is true for $a = 3n+1$ or $a = 3n + 2$. We assume that $\mathcal{M}_a \neq \mathcal{M}_{a+1}$ for all $a \in \mathbb{N}$, then

$$\begin{aligned}
& H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}) \\
= \ & H_G(\Upsilon(\mathcal{M}_{3a}), \Psi\left(\mathcal{M}_{3a+1}\right), \Phi\left(\mathcal{M}_{3a+2}\right)) \\
\leq \ & \alpha H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + \beta H_G(\mathcal{M}_{3a}, \Upsilon(\mathcal{M}_{3a}), \Upsilon(\mathcal{M}_{3a})) \\
& + \gamma H_G(\mathcal{M}_{3a+1}, \Psi\left(\mathcal{M}_{3a+1}\right), \Psi\left(\mathcal{M}_{3a+1}\right)) + \eta H_G(\mathcal{M}_{3a+2}, \Phi\left(\mathcal{M}_{3a+2}\right), \Phi\left(\mathcal{M}_{3a+2}\right)) \\
= \ & \alpha H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + \beta H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+1}) \\
& + \gamma H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+2}) + \eta H_G(\mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}, \mathcal{M}_{3a+3}) \\
\leq \ & \alpha H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + \beta H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) \\
& + \gamma H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}) + \eta H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}).
\end{aligned}$$

Thus, we have

$$(1 - \gamma - \eta) H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}) \leq (\alpha + \beta) H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}).$$

Hence,

$$H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}) \leq \lambda H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}),$$

where $\lambda = \dfrac{\alpha + \beta}{1 - \gamma - \eta}$, with $0 < \lambda < 1$. In a similar manner, it can be shown that

$$H_G(\mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}, \mathcal{M}_{3a+4}) \leq \lambda H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3})$$

and

$$H_G(\mathcal{M}_{3a+3}, \mathcal{M}_{3a+4}, \mathcal{M}_{3a+5}) \leq \lambda H_G(\mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}, \mathcal{M}_{3a+4}).$$

Thus, for all a,

$$\begin{aligned} H_G(\mathcal{M}_{a+1}, \mathcal{M}_{a+2}, \mathcal{M}_{a+3}) &\leq \lambda H_G\left(\mathcal{M}_a, \mathcal{M}_{a+1}, \mathcal{M}_{a+2}\right) \\ &\leq \cdots \leq \lambda^{a+1} H_G\left(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2\right). \end{aligned}$$

Now, for l, m, a, with $l > m > a$, we have that

$$\begin{aligned} H_G(\mathcal{M}_a, \mathcal{M}_m, \mathcal{M}_l) &\leq H_G(\mathcal{M}_a, \mathcal{M}_{a+1}, \mathcal{M}_{a+1}) + H_G(\mathcal{M}_{a+1}, \mathcal{M}_{a+2}, \mathcal{M}_{a+2}) \\ &\quad + \cdots + H_G(\mathcal{M}_{l-1}, \mathcal{M}_{l-1}, \mathcal{M}_l) \\ &\leq H_G(\mathcal{M}_a, \mathcal{M}_{a+1}, \mathcal{M}_{a+2}) + H_G(\mathcal{M}_{a+1}, \mathcal{M}_{a+2}, \mathcal{M}_{a+3}) \\ &\quad + \cdots + H_G(\mathcal{M}_{l-2}, \mathcal{M}_{l-1}, \mathcal{M}_l) \\ &\leq [\lambda^a + \lambda^{a+1} + \cdots + \lambda^{l-2}] H_G\left(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2\right) \\ &= \lambda^a[1 + \lambda + \lambda^2 + \cdots + \lambda^{l-a-1}] H_G(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2) \\ &\leq \frac{\lambda^a}{1-\lambda} H_G(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2). \end{aligned}$$

We note that if $l = m > a$, we get similar results and if $l > m = a$, then

$$H_G(\mathcal{M}_a, \mathcal{M}_m, \mathcal{M}_l) \leq \frac{\lambda^{a-1}}{1-\lambda} H_G(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2),$$

and so $\lim\limits_{a,m,l \to +\infty} H_G(\mathcal{M}_a, \mathcal{M}_m, \mathcal{M}_l) = 0$. Thus $\{\mathcal{M}_a\}$ is a G-Cauchy sequence in $\mathcal{C}^G(W)$. Since $(\mathcal{C}^G(W), H_G)$ is a complete G-metric space, there exists $\tilde{U}_1^* \in \mathcal{C}^G(W)$ such that $\lim\limits_{a \to +\infty} \mathcal{M}_a = \tilde{U}_1^*$, that is, $\lim\limits_{a \to +\infty} H_G(\mathcal{M}_a, \tilde{U}_1^*) = \lim\limits_{a \to +\infty} H_G(M_a, \mathcal{M}_{a+1}) = H_G(\tilde{U}_1^*, \tilde{U}_1^*)$ and so we have $\lim\limits_{a \to +\infty} H_G(\mathcal{M}_a, \tilde{U}_1^*) = 0$.

To prove that $\Upsilon(\tilde{U}_1^*) = \tilde{U}_1^*$, we claim

$$\begin{aligned} & H_G(\Upsilon(\tilde{U}_1^*), \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}) \\ = \;& H_G(\Upsilon(\tilde{U}_1^*), \Psi(\mathcal{M}_{3a+1}), \Phi(\mathcal{M}_{3a+2})) \\ \leq \;& \alpha H_G(\Upsilon(\tilde{U}_1^*), \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + \beta H_G(\tilde{U}_1^*, \Upsilon(\tilde{U}_1^*), \Upsilon(\tilde{U}_1^*)) \\ & + \gamma H_G(\mathcal{M}_{3a+1}, \Psi(\mathcal{M}_{3a+1}), \Psi\left(\mathcal{M}_{3a+1}\right)) + \eta H_G(\mathcal{M}_{3a+2}, \Phi\left(\mathcal{M}_{3a+2}\right), \Phi\left(\mathcal{M}_{3a+2}\right)) \\ = \;& \alpha H_G(\tilde{U}_1^*, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + \beta H_G(\Upsilon(\tilde{U}_1^*), \tilde{U}_1^*, \mathcal{M}_{3a+1}) \\ & + \gamma H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+2}) + \eta H_G(\mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}, \mathcal{M}_{3a+3}), \end{aligned}$$

where upon taking the limit as $a \to +\infty$, we obtain

$$H_G(\Upsilon(\tilde{U}_1^*), \tilde{U}_1^*, \tilde{U}_1^*) \leq \beta H_G(\tilde{U}_1^*, \tilde{U}_1^*, \tilde{U}_1^*),$$

which is a contradiction. Thus $\Upsilon(\tilde{U}_1^*) = \tilde{U}_1^*$. In a similar manner we can show that $\Psi(\tilde{U}_1^*) = \tilde{U}_1^*$ and $\Phi(\tilde{U}_1^*) = \tilde{U}_1^*$. For uniqueness, we suppose that V_1 is another common attractor of Υ, Ψ, and Φ then

$$\begin{aligned} H_G(\tilde{U}_1^*, V_1, V_1) &= H_G(\Upsilon(\tilde{U}_1^*), \Psi(V_1), \Phi(V_1)) \\ &\leq \alpha H_G(\tilde{U}_1^*, V_1, V_1) + \beta H_G(U^*, \Upsilon(\tilde{U}_1^*), \Upsilon(\tilde{U}_1^*)) \\ &\quad + \gamma H_G(V_1, \Psi(V_1), \Psi(V_1)) + \eta H_G(V_1, \Phi(V_1), \Phi(V_1)) \\ &= \alpha H_G(\tilde{U}_1^*, V_1, V_1) + \beta H_G(\tilde{U}_1^*, \tilde{U}_1^*, \tilde{U}_1^*) \\ &\quad + \gamma H_G(V_1, V_1, V_1) + \eta H_G(V_1, V_1, V_1) \\ &= \alpha H_G(\tilde{U}_1^*, V_1, V_1) \end{aligned}$$

from which we conclude that $H_G(\tilde{U}_1^*, V_1, V_1) = 0$ and thus $\tilde{U}_1^* = V_1$. Hence $\tilde{U}_1^*$ is a unique common attractor of Υ, Ψ, and Φ. □

Example 7.3.1. Let $W = [0, 3]$ be a non-empty set, and $G(w_1, w_2, w_3) = \max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\}$ for all $w_1, w_2, w_3 \in W$ be a complete G-metric. Suppose $\mathcal{Q}_1^* = [0, 1]$, $\mathcal{Q}_2^* = [0, 2]$, and $\mathcal{Q}_3^* = [0, 3]$ are subsets of W. Define $g^* : \cup_{a=1}^3 \mathcal{Q}_a^* \to \cup_{a=1}^3 \mathcal{Q}_a^*$ by

$$g^*(w_1) = \begin{cases} \frac{w_1}{5} & \text{if } 0 \leq w_1 \leq 1 \\ \frac{1}{3} & \text{if } 1 < w_1 \leq 2 \\ \frac{w_1}{3} & \text{if } 2 < w_1 \leq 3. \end{cases}$$

We note that

$$g^*(\mathcal{Q}_1^*) = \left[0, \tfrac{1}{5}\right] \subseteq [0, 2] = \mathcal{Q}_2^*,$$

$$g^*(\mathcal{Q}_2^*) = \left[0, \tfrac{1}{3}\right] \subseteq [0, 3] = \mathcal{Q}_3^*,$$

and

$$g^*(\mathcal{Q}_3^*) = [0, 1] \subseteq [0, 1] = \mathcal{Q}_1^*.$$

Hence, $\mathcal{Q}_1^* \cup \mathcal{Q}_2^* \cup \mathcal{Q}_3^*$ is a cyclic representation of W with respect to g^*. Next, define $f_a^*, g_a^*, g_h^* : W \to W$ by

$$f_1^*(w_1) = \begin{cases} \dfrac{w_1}{24} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{16} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases} \qquad f_2^*(w_1) = \begin{cases} \dfrac{w_1}{18} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{14} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases}$$

$$g_1^*(w_1) = \begin{cases} \dfrac{w_1}{12} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{8} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases} \qquad g_2^*(w_1) = \begin{cases} \dfrac{w_1}{6} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{4} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases}$$

$$h_1^*(w_1) = \begin{cases} \dfrac{w_1}{9} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{7} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases} \qquad h_2^*(w_1) = \begin{cases} \dfrac{w_1}{5} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{3} & \text{if } \ \frac{1}{2} \le w_1 \le 1. \end{cases}$$

Similar arguments as in Example 6.3.2 shows that the results of Theorem 7.3.1 holds.

Remark 7.3.1. Let $W = \cup_{a=1}^{q} W_a$. If we take in Theorem 7.3.1, $\mathcal{S}^G(\cup_{a=1}^{q} W_a)$ the union of all singleton subsets of the given space W, then $\mathcal{S}^G(\cup_{a=1}^{q} W_a) \subseteq \mathcal{C}^G(\cup_{a=1}^{q} W_a)$. Furthermore, if we take the mappings $(f_a, g_a, h_a) = (f, g, h)$ for each a, where $f = f_1, g = g_1$ and $h = h_1$ then the operators (Υ, Ψ, Φ) become

$$(\Upsilon(y_1), \Psi(y_2), \Phi(y_3)) = (f(y_1), g(y_2), h(y_3)).$$

Consequently, obtain the following common fixed point result.

Corollary 7.3.1. *Suppose $\{W; (f_a, g_a, h_a), a \in \mathbb{N}_q\}$ is a generalized cyclic G-iterated function system, defined in a complete G-metric space (W, G) and let the mappings $f, g, h : W \to W$ be defined as in Remark 7.3.1. If some $\lambda \in [0, 1)$ exists such that, for any $y_1 \in \mathcal{C}^G(W_a), y_2 \in \mathcal{C}^G(W_{a+1})$ and $y_3 \in \mathcal{C}^G(W_{a+2})$, the following holds:*

$$\begin{aligned} G(f(y_1), g(y_2), h(y_3)) \le \ & \alpha H_G(y_1, y_2, y_3) + \beta H_G(y_1, \Upsilon(y_1), \Upsilon(y_1)) \\ & + \gamma H_G(y_2, \Psi(y_2), \Psi(y_2)) + \eta H_G(y_3, \Phi(y_3), \Phi(y_3)). \end{aligned}$$

Then f, g and h have a unique common fixed point $u \in W$. In addition, for any initial point $u_0 \in W$, the sequence $\{u_0, fu_0, gfu_0, hgfu_0, fhgfu_0, \cdots\}$ converges to the common fixed point of f, g and h.

Corollary 7.3.2. *Let $\{W; (f_a, g_a, h_a), a \in \mathbb{N}_q\}$ be a generalized cyclic G-iterated* *function system, defined in a complete G-metric space* (W, G)*,and let* (f_a, g_a, h_a) *for* $a \in \mathbb{N}_q$ *be a triplet of generalized cyclic contractive self-mappings on* W*. Suppose* $\{B_a\}_{a=1}^q$ *is a collection of non-void closed subsets of* W*. Then the triple* $(\Upsilon, \Psi, \Phi) : \mathcal{C}^G(\cup_{a=1}^q B_a) \to \mathcal{C}^G(\cup_{a=1}^q B_a)$ *defined in Theorem 7.3.1 has at most one common fixed point. Furthermore, for any initial set* $\mathcal{M}_0 \in \mathcal{C}^G(B_a)$*, the sequence* $\{\mathcal{M}_0, \Upsilon(\mathcal{M}_0), \Psi\Upsilon(\mathcal{M}_0), \Phi\Psi\Upsilon(\mathcal{M}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{M}_0), \cdots\}$ *of compact sets have for a limit, the common fixed point of* Υ, Ψ *and* Φ*.*

Theorem 7.3.2. *For a complete G-metric space* (W, G)*, suppose* $\{B_a\}_{a=1}^q$ *is a family of non-void closed subsets of* W *and* $\{W; (f_a, g_a, h_a), a \in \mathbb{N}_q\}$ *is a generalized cyclic G-iterated function system. If the mappings* $\Upsilon, \Psi, \Phi : \mathcal{C}^G(\cup_{a=1}^q B_a) \to \mathcal{C}^G(\cup_{a=1}^q B_a)$ *defined by*

$$\begin{aligned}\Upsilon(\mathcal{L}) &= f_1(\mathcal{L}) \cup f_2(\mathcal{L}) \cup \cdots \cup f_q(\mathcal{L}) \\ &= \cup_{a=1}^q f_a(\mathcal{L}) \text{ for } \mathcal{L} \in \mathcal{C}^G(\cup_{a=1}^q B_a)\end{aligned}$$

$$\begin{aligned}\Psi(\mathcal{M}) &= g_1(\mathcal{M}) \cup g_2(\mathcal{M}) \cup \cdots \cup g_q(\mathcal{M}) \\ &= \cup_{a=1}^q g_a(\mathcal{M}) \text{ for } \mathcal{M} \in \mathcal{C}^G(\cup_{a=1}^q B_{a+1})\end{aligned}$$

and

$$\begin{aligned}\Phi(\mathcal{N}) &= h_1(\mathcal{N}) \cup h_2(\mathcal{N}) \cup \cdots \cup h_q(\mathcal{N}) \\ &= \cup_{a=1}^q h_a(\mathcal{N}) \text{ for } \mathcal{N} \in \mathcal{C}^G(\cup_{a=1}^q B_{a+2})\end{aligned}$$

are generalized cyclic G-Hutchinson contractive operators (type II), then $\Upsilon, \Psi,$ *and* Φ *have a unique common attractor* $\tilde{U}_1^* \in \mathcal{C}^G(B_a)$*, that is,*

$$\tilde{U}_1^* = \Upsilon(\tilde{U}_1^*) = \Psi(\tilde{U}_1^*) = \Phi(\tilde{U}_1^*) = \cup_{a=1}^q f_a(\tilde{U}_1^*) = \cup_{a=1}^q g_a(\tilde{U}_1^*) = \cup_{a=1}^q h_a(\tilde{U}_1^*).$$

Moreover, for an arbitrarily chosen initial set, $\mathcal{M}_0 \in \mathcal{C}^G(\cup_{a=1}^q B_a)$*, the sequence*

$$\{\mathcal{M}_0, \Upsilon(\mathcal{M}_0), \Psi\Upsilon(\mathcal{M}_0), \Phi\Psi\Upsilon(\mathcal{M}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{M}_0), ...\}$$

of compact sets converges to the common attractor $\tilde{U}_1^*$*.*

Proof. We show that any attractor of Υ is an attractor of Ψ and Φ. To that end, we assume that $\tilde{U}_1^* \in \mathcal{C}^G(W)$ is such that $\Upsilon(\tilde{U}_1^*) = \tilde{U}_1^*$. We need to show that

$\tilde{U}_1^* = \Psi(\tilde{U}_1^*) = \Phi(U^*)$, else for $U^* \neq \Psi(\tilde{U}_1^*)$ and $\tilde{U}_1^* \neq \Phi(\tilde{U}_1^*)$, we get

$$\begin{aligned}
H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) &= H_G(\Upsilon(\tilde{U}_1^*), \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) \\
&\leq \lambda_1 H_G(\tilde{U}_1^*, \tilde{U}_1^*, \tilde{U}_1^*) + \lambda_2[H_G(\tilde{U}_1^*, \tilde{U}_1^*, \Upsilon(\tilde{U}_1^*) \\
&\quad + H_G(\tilde{U}_1^*, \tilde{U}_1^*, \Psi(\tilde{U}_1^*)) + H_G(\tilde{U}_1^*, \tilde{U}_1^*, \Phi(\tilde{U}_1^*))] \\
&\quad + \lambda_3[H_G(\Upsilon(\tilde{U}_1^*), \tilde{U}_1^*, \tilde{U}_1^*) + H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \tilde{U}_1^*) \\
&\quad + H_G(\tilde{U}_1^*, \tilde{U}_1^*, \Phi(\tilde{U}_1^*))] \\
&= (\lambda_2 + \lambda_3)\left[H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \tilde{U}_1^*) + H_G(\tilde{U}_1^*, \tilde{U}_1^*, \Phi(\tilde{U}_1^*))\right] \\
&\leq (\lambda_2 + \lambda_3)\Big[H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)) \\
&\quad + H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*))\Big] \\
&= 2\,(\lambda_2 + \lambda_3)\, H_G(\tilde{U}_1^*, \Psi(\tilde{U}_1^*), \Phi(\tilde{U}_1^*)),
\end{aligned}$$

which is a contradiction. If we take $\tilde{U}_1^* \neq \Psi(\tilde{U}_1^*)$ and $\tilde{U}_1^* = \Phi(\tilde{U}_1^*)$ or $\tilde{U}_1^* \neq \Phi(\tilde{U}_1^*)$ and $\tilde{U}_1^* = \Psi(\tilde{U}_1^*)$, similar argument as above yields a contradiction. Hence we conclude that $\tilde{U}_1^* = \Upsilon(\tilde{U}_1^*) = \Psi(\tilde{U}_1^*) = \Phi(\tilde{U}_1^*)$. We also note that the same conclusion holds for $\tilde{U}_1^* = \Psi(\tilde{U}_1^*)$ or $\tilde{U}_1^* = \Phi(\tilde{U}_1^*)$.

We show that Υ, Ψ, and Φ have a unique common attractor. Let $\mathcal{M}_0 \in \mathcal{C}^G(W)$ be an arbitrary point. Define a sequence $\{\mathcal{M}_a\}$ by $\mathcal{M}_{3a+1} = \Upsilon(\mathcal{M}_{3a})$, $\mathcal{M}_{3a+2} = \Psi(\mathcal{M}_{3a+1}), \mathcal{M}_{3a+3} = \Psi(\mathcal{M}_{3a+2}), a = 0, 1, 2, \cdots$. If $\mathcal{M}_a = \mathcal{M}_{a+1}$ for some a, with $a = 3n$, then $\tilde{U}_1^* = \mathcal{M}_{3a}$ is an attractor of Υ and from the Proof above, $\tilde{U}_1^*$ is a common attractor for Υ, Ψ, and Φ. The same is true for $a = 3n+1$ or $a = 3n+2$. Let us assume that $\mathcal{M}_a \neq \mathcal{M}_{a+1}$ for all $a \in \mathbb{N}$, then

$$\begin{aligned}
& H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}) \\
= \; & H_G(\Upsilon(\mathcal{M}_{3a}), \Psi\left(\mathcal{M}_{3a+1}\right), \Phi\left(\mathcal{M}_{3a+2}\right)) \\
\leq \; & \lambda_1 H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + \lambda_2[H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a}, \Upsilon(\mathcal{M}_{3a}) \\
& + H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+1}, \Psi\left(\mathcal{M}_{3a+1}\right)) + H_G(\mathcal{M}_{3a+2}, \mathcal{M}_{3a+2}, \Phi\left(\mathcal{M}_{3a+2}\right))] \\
& + \lambda_3[H_G(\Upsilon(\mathcal{M}_{3a}), \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + H_G(\mathcal{M}_{3a}, \Psi\left(\mathcal{M}_{3a+1}\right), \mathcal{M}_{3a+2}) \\
& + H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \Phi\left(\mathcal{M}_{3a+2}\right))] \\
= \; & \lambda_1 H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + \lambda_2[H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a}, \mathcal{M}_{3a+1}) \\
& + H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + H_G(\mathcal{M}_{3a+2}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3})] \\
& + \lambda_3[H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) + H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+2}) \\
& + H_G(\mathcal{M}_{3a}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+3})]
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda_1 H_G(\mathcal{M}_{3a},\mathcal{M}_{3a+1},\mathcal{M}_{3a+2}) + \lambda_2[H_G(\mathcal{M}_{3a},\mathcal{M}_{3a+1},\mathcal{M}_{3a+2})\\
&\quad + H_G(\mathcal{M}_{3a},\mathcal{M}_{3a+1},\mathcal{M}_{3a+2}) + H_G(\mathcal{M}_{3a+1},\mathcal{M}_{3a+2},\mathcal{M}_{3a+3})]\\
&\quad + \lambda_3[H_G(\mathcal{M}_{3a},\mathcal{M}_{3a+1},\mathcal{M}_{3a+2}) + H_G(\mathcal{M}_{3a},\mathcal{M}_{3a+1},\mathcal{M}_{3a+2})\\
&\quad + \{H_G(\mathcal{M}_{3a},\mathcal{M}_{3a+1},\mathcal{M}_{3a+2}) + H_G(\mathcal{M}_{3a+1},\mathcal{M}_{3a+2},\mathcal{M}_{3a+3})\}].
\end{aligned}$$

Thus, we have

$$(1-\lambda_2-\lambda_3)H_G(\mathcal{M}_{3a+1},\mathcal{M}_{3a+2},\mathcal{M}_{3a+3}) \leq (\lambda_1+2\lambda_2+3\lambda_3)H_G(\mathcal{M}_{3a},\mathcal{M}_{3a+1},\mathcal{M}_{3a+2}).$$

Hence,

$$H_G(\mathcal{M}_{3a+1},\mathcal{M}_{3a+2},\mathcal{M}_{3a+3}) \leq \lambda H_G(\mathcal{M}_{3a},\mathcal{M}_{3a+1},\mathcal{M}_{3a+2}),$$

where $\lambda = \dfrac{\lambda_1 + 2\lambda_2 + 3\lambda_3}{1-\lambda_2-\lambda_3}$, with $0 < \lambda < 1$. Using the same argument, it can be shown that

$$H_G(\mathcal{M}_{3a+2},\mathcal{M}_{3a+3},\mathcal{M}_{3a+4}) \leq \lambda H_G(\mathcal{M}_{3a+1},\mathcal{M}_{3a+2},\mathcal{M}_{3a+3})$$

and

$$H_G(\mathcal{M}_{3a+3},\mathcal{M}_{3a+4},\mathcal{M}_{3a+5}) \leq \lambda H_G(\mathcal{M}_{3a+2},\mathcal{M}_{3a+3},\mathcal{M}_{3a+4}).$$

Thus, for all a,

$$\begin{aligned}
H_G(\mathcal{M}_{a+1},\mathcal{M}_{a+2},\mathcal{M}_{a+3}) &\leq \lambda H_G\left(\mathcal{M}_{a},\mathcal{M}_{a+1},\mathcal{M}_{a+2}\right)\\
&\leq \cdots \leq \lambda^{a+1} H_G\left(\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2\right).
\end{aligned}$$

Now, we have for l, m, a, with $l > m > a$,

$$\begin{aligned}
H_G(\mathcal{M}_a,\mathcal{M}_m,\mathcal{M}_l) &\leq H_G(\mathcal{M}_a,\mathcal{M}_{a+1},\mathcal{M}_{a+1}) + H_G(\mathcal{M}_{a+1},\mathcal{M}_{a+2},\mathcal{M}_{a+2})\\
&\quad + \cdots + H_G(\mathcal{M}_{l-1},\mathcal{M}_{l-1},\mathcal{M}_l)\\
&\leq H_G(\mathcal{M}_a,\mathcal{M}_{a+1},\mathcal{M}_{ka+2}) + H_G(\mathcal{M}_{ka+1},\mathcal{M}_{ka+2},\mathcal{M}_{ka+3})\\
&\quad + \cdots + H_G(\mathcal{M}_{l-2},\mathcal{M}_{l-1},\mathcal{M}_l)\\
&\leq [\lambda^a + \lambda^{a+1} + \cdots + \lambda^{l-2}]H_G\left(\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2\right)\\
&= \lambda^a[1 + \lambda + \lambda^2 + \cdots + \lambda^{l-a-1}]H_G(\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2)\\
&\leq \frac{\lambda^a}{1-\lambda} H_G(\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2).
\end{aligned}$$

We note that if $l = m > a$, we get similar results and if $l > m = a$, then

$$H_G(\mathcal{M}_a,\mathcal{M}_m,\mathcal{M}_l) \leq \frac{\lambda^{a-1}}{1-\lambda} H_G(\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2).$$

as such, $\lim_{a,m,l\to+\infty} H_G(\mathcal{M}_a, \mathcal{M}_m, \mathcal{M}_l) = 0$. Thus $\{\mathcal{M}_a\}$ is a G-Cauchy sequence in $\mathcal{C}^G(W)$. Since $(\mathcal{C}^G(W), H_G)$ is a complete G-metric space, there exists $\tilde{U}_1^* \in \mathcal{C}^G(W)$ such that $\lim_{a\to\infty} \mathcal{M}_a = \tilde{U}_1^*$, that is, $\lim_{a\to\infty} H_G\left(M_a, \mathcal{M}_{a+1}, \mathcal{M}_{a+2}\right) = H_G(\tilde{U}_1^*, \tilde{U}_1^*, \tilde{U}_1^*)$.

To prove that $\Upsilon(\tilde{U}_1^*) = \tilde{U}_1^*$, we claim in the contrary

$$\begin{aligned}
H_G(\Upsilon(\tilde{U}_1^*), \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}) &= H_G(\Upsilon(\tilde{U}_1^*), \Psi\left(\mathcal{M}_{3a+1}\right), \Phi\left(\mathcal{M}_{3a+2}\right)) \\
&\leq \lambda_1 H_G(\tilde{U}_1^*, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) \\
&\quad +\lambda_2[H_G(\tilde{U}_1^*, \tilde{U}_1^*, \Upsilon(\tilde{U}_1^*) \\
&\quad +H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+1}, \Psi\left(\mathcal{M}_{3a+1}\right)) \\
&\quad +H_G(\mathcal{M}_{3a+2}, \mathcal{M}_{3a+2}, \Phi\left(\mathcal{M}_{3a+2}\right))] \\
&\quad +\lambda_3[H_G(\Upsilon(\tilde{U}_1^*), \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) \\
&\quad +H_G(\tilde{U}_1^*, \Psi\left(\mathcal{M}_{3a+1}\right), \mathcal{M}_{3a+2}) \\
&\quad +H_G(\tilde{U}_1^*, \mathcal{M}_{3a+1}, \Phi\left(\mathcal{M}_{3a+2}\right))],
\end{aligned}$$

that is,

$$\begin{aligned}
H_G(\Upsilon(\tilde{U}_1^*), \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3}) &= \lambda_1 H_G(\tilde{U}_1^*, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) \\
&\quad +\lambda_2[H_G(\tilde{U}_1^*, \tilde{U}_1^*, \Upsilon(U^*) \\
&\quad +H_G(\mathcal{M}_{3a+1}, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+2}) \\
&\quad +H_G(\mathcal{M}_{3a+2}, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+3})] \\
&\quad +\lambda_3[H_G(\Upsilon(\tilde{U}_1^*), \mathcal{M}_{3k+1}, \mathcal{M}_{3a+2}) \\
&\quad +H_G(\tilde{U}_1^*, \mathcal{M}_{3a+2}, \mathcal{M}_{3a+2}) \\
&\quad +H_G(\tilde{U}_1^*, \mathcal{M}_{3a+1}, \mathcal{M}_{3a+3})]
\end{aligned}$$

and taking the limit as $a \to +\infty$, we get

$$H_G(\Upsilon(\tilde{U}_1^*), \tilde{U}_1^*, \tilde{U}_1^*) \leq \left(\lambda_2 + \lambda_3\right) H_G(\Upsilon(\tilde{U}_1^*), \tilde{U}_1^*, \tilde{U}_1^*),$$

which is not possible. Thus $\Upsilon(\tilde{U}_1^*) = \tilde{U}_1^*$. In a similar manner we can show that $\Psi(\tilde{U}_1^*) = \tilde{U}_1^*$ and $\Phi(\tilde{U}_1^*) = \tilde{U}_1^*$. Turning to uniqueness, we suppose that V_1 is another common attractor of Υ, Ψ and Φ, then

$$\begin{aligned} H_G(\tilde{U}_1^*, V_1, V_1) &= H_G(\Upsilon(\tilde{U}_1^*), \Psi(V_1), \Phi(V_1)) \\ &\leq \lambda_1 H_G(\tilde{U}_1^*, V_1, V_1) + \lambda_2[H_G(\tilde{U}_1^*, U^*, \Upsilon(\tilde{U}_1^*)) \\ &\quad + H_G(V_1, V_1, \Psi(V_1)) + H_G(V_1, V_1, \Phi(V_1))] \\ &\quad + \lambda_3[H_G(\Upsilon(\tilde{U}_1^*), V_1, V_1) + H_G(\tilde{U}_1^*, \Psi(V_1), V_1) \\ &\quad + H_G(\tilde{U}_1^*, V_1, \Phi(V_1))] \\ &= \lambda_1 H_G(\tilde{U}_1^*, V_1, V_1) + \lambda_2[H_G(\tilde{U}_1^*, \tilde{U}_1^*, \tilde{U}_1^*) + H_G(V_1, V_1, V_1) \\ &\quad + H_G(V_1, V_1, V_1)] + \lambda_3[H_G(\tilde{U}_1^*, V_1, V_1) + H_G(\tilde{U}_1^*, V_1, V_1) \\ &\quad + H_G(\tilde{U}_1^*, V_1, V_1)] \\ &= (\lambda_1 + 3\lambda_3) H_G(\tilde{U}_1^*, V_1, V_1), \end{aligned}$$

from which we conclude that $H_G(\tilde{U}_1^*, V_1, V_1) = 0$ and thus $\tilde{U}_1^* = V_1$. Hence $\tilde{U}_1^*$ is a unique common attractor of Υ, Ψ, and Φ. □

Example 7.3.2. Let (W, G) be a complete G-metric space having $W = [0, 5]$, and $G(w_1, w_2, w_3) = \max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\}$ for all $w_1, w_2, w_3 \in W$. Let $\mathcal{Q}_1^* = [0, 1]$, $\mathcal{Q}_2^* = [0, 2]$, and $\mathcal{Q}_3^* = [0, 3]$ be subsets of W. Define $g^* : \cup_{a=1}^3 \mathcal{Q}_a^* \to \cup_{a=1}^3 \mathcal{Q}_a^*$, by

$$g^*(w_1) = \begin{cases} \frac{w_1}{7} & \text{if } 0 \leq w_1 \leq 1 \\ \frac{1}{3} & \text{if } 1 < w_1 \leq 2 \\ \frac{w_1}{3} & \text{if } 2 < w_1 \leq 3. \end{cases}$$

Observe that

$$g^*(\mathcal{Q}_1^*) = \left[0, \tfrac{1}{7}\right] \subseteq [0, 2] = \mathcal{Q}_2^*,$$

$$g^*(\mathcal{Q}_2^*) = \left[0, \tfrac{1}{3}\right] \subseteq [0, 3] = \mathcal{Q}_3^*,$$

and

$$g^*(\mathcal{Q}_3^*) = [0, 1] \subseteq [0, 1] = \mathcal{Q}_1^*.$$

Hence, $\mathcal{Q}_1^* \cup \mathcal{Q}_2^* \cup \mathcal{Q}_3^*$ is a cyclic representation of W with respect to g^*. Next, define $f_a^*, g_a^*, h_a^* : W \to W$ by

$$f_1^*(w_1) = \begin{cases} \dfrac{w_1}{54} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{46} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases} \qquad f_2^*(w_1) = \begin{cases} \dfrac{w_1}{38} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{34} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases}$$

$$g_1^*(w_1) = \begin{cases} \dfrac{w_1}{30} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{24} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases} \qquad g_2^*(w_1) = \begin{cases} \dfrac{w_1}{16} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{14} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases}$$

$$h_1^*(w_1) = \begin{cases} \dfrac{w_1}{19} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{13} & \text{if } \ \frac{1}{2} \le w_1 \le 1, \end{cases} \qquad h_2^*(w_1) = \begin{cases} \dfrac{w_1}{11} & \text{if } \ 0 \le w_1 < \frac{1}{2} \\ \\ \dfrac{w_1}{7} & \text{if } \ \frac{1}{2} \le w_1 \le 1. \end{cases}$$

Similar arguments as in Example 6.3.2 confirm the validity of Theorem 7.3.2 holds.

Corollary 7.3.3. *For a generalized cyclic G -iterated function system $\{W; f_a, g_a, h_a, a \in \mathbb{N}_q\}$ on a complete G-metric space (W, G), define the mappings $f, g, h : W \to W$ as in Remark* 7.3.1*. If some $\lambda_* \in [0, 1)$ exists, such that for any $y_1 \in \mathcal{C}^G(W_a), y_2 \in \mathcal{C}^G(W_{a+1})$ and $y_3 \in \mathcal{C}^G(W_{a+2})$, the following holds:*

$$G\left(fy_1, gy_2, hy_3\right) \le \mathcal{R}_{f,g,h}(y_1, y_2, y_3),$$

where

$$\begin{aligned} \mathcal{R}_{f,g,h}(y_1, y_2, y_3) &= \lambda_1 H_G(y_1, y_2, y_3) + \lambda_2[H_G(y_1, y_1, f(y_1)) \\ &\quad + H_G(y_2, y_2, g\left(y_2\right)) + H_G(y_3, y_3, h\left(y_3\right))] \\ &\quad + \lambda_3[H_G(f(y_1), y_2, y_3) + H_G(y_1, g\left(y_2\right), y_3) \\ &\quad + H_G(y_1, y_2, h\left(y_3\right))]. \end{aligned}$$

Then a unique common attractor for $f, g,$ and h exists. Additionaly, for any initial choice of $u_0 \in W$, the sequence $\{u_0, fu_0, gfu_0, hgfu_0, fhgfu_0, \cdots\}$ converges to an attractor of $f, g,$ and h.

8

Conclusion

The results in this thesis expanded the scope of iterated function system to non-standard metric spaces, such as partial metric spaces, semi-metric spaces, and G-metric spaces. The existence and uniqueness of attractors for single valued mappings and like-wise common attractors for multi-valued mappings involving a pair of self-mappings were established with the assistance of finite families of contractive and generalized contractive mappings respectively, defined on a partial metric space. The well-posedness of attractor based problems was confirmed.

With a broader class of cyclic contractive mappings, the Banach contraction principle was extended to include non-continuous mappings, and useful results on the existence and uniqueness of attractors were obtained. This was followed by results in semi-metric space whose definition omits the triangle inequality. The omission was remedied by working in a bounded Hausdorff semi-metric space.

Further investigations yielded some results for non-commutative mappings in G-metric spaces. We culminated our work with a study of generalized iterated function system of cyclic contractions in G-metric spaces.

It was shown that our results not only have applications in the field of dynamic programming where they provide effective tools for solving functional equations, but are very efficient in establishing the existence and uniqueness of solution to integral equations.

Open Problems

There are some open problems for researchers that are working in the filed of pure and applied mathematics. It is envisioned that, current work may be extended by exploring iterated function systems of generalized contractions of integral type on a framework of complete S-metric spaces [79]. One may expand the applications to establish the existence and uniqueness of solution to Volterra integral equation. We believe that characterizing iterated function systems in the setting of parametric metric spaces will be an attractive open challenge. The findings in this paper can be utilized to further research in more general spaces such as quasi metric spaces and controlled metric spaces. The existence of common attractors of a finite set of generalized contractive mappings could be extended to study generalized F-contractions and may also be extended to the problem of Smyth completeness in quasi metric spaces.

It is also very interesting to generate fractals by employing finite family of generalized contractions in the setup of varies generalized metric spaces such as quasi metric spaces, controlled metric spaces and dislocated metric spaces.

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