ON QUASI-UNIFORM SPACES VIA STRONG QUASI-UNIFORM COVERS

by

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Abstract

A quasi-uniformity, unlike a uniformity, is not determined by its quasi-uniform covers. It is generally known that covering space theory applies to topological spaces that are connected. In this dissertation, we will define topology induced by strong quasi-uniform cover which will determine the quasi-uniformity of a quasi-uniform space. And then we will expand the discussion on strong quasi-uniform covers, detailing how quasi-uniform spaces via strong quasi-uniform covers give rise to a topological space with some added axioms which will precisely determine the quasi-uniformity of a given quasi-uniform space.. The dissertation will further detail on how uniformizable topological spaces are precisely the completely regular spaces. This erupts from the idea that uniform spaces can be induced by uniform covers, but in this article, we want to ascertain that the same concept can be applied with quasi-uniform spaces via strong quasi-uniform covers.

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Contents

1 CHAPTER 1

1.1 Introduction

A uniform structure on a given set determines a topological structure on the same set, although different uniform structures may determine the same topological structure. However, there exist some topological structures which cannot be obtained from a uniform structure. Metric and topological group structures give rise to uniform structures. As we shall see, a theory which encompasses many of the essentials of both of these important classes of spaces is obviously of considerable interest. But what makes uniform spaces essential, as much as anything, is that every compact Hausdorff space admits a unique uniform structure that is unique.

Uniform spaces are often described as "carriers for the notions of uniform convergence and uniform continuity." Recall, for instance, that any function $f : (X, d) \to (Y, \rho)$ that is between metric spaces is said to be uniformly continuous if it is given that for every $\epsilon > 0$,for any arbitrary two points $a, b \in X$, there is some $\delta > 0$ such that:

$$d(a, b) < \delta \Rightarrow \rho(f(a), f(b)) < \epsilon.$$

Such a notion does not exist for non-metrizable topological spaces. The idea of uniform space corrects this without introducing a metric. Uniform spaces can be introduced via uniform covers. So it goes without saying that a uniform space can actually be completely characterised using uniform covers. But then the same cannot be said about quasi-uniform spaces. We will write a comprehensive account detailing how uniform spaces can be introduced via uniform covers. We will further discuss quasi-uniform spaces and their properties. So we will define a strong quasi-uniform cover which will determine the quasi-uniformity of a given quasi-uniform space.

1.1.1 Motivation

Several studies have been done on quasi-uniform spaces. Others have proven that there exists a topology that can be induced by some quasi-uniformity for every topological space[1]. This consequently lays the foundation of equivalence of topological spaces with quasi-uniform spaces, that is between two topological spaces, if there exists a homeomorphism, continuous map between the spaces which has an continuous inverse. This makes it feasible to study concepts like completeness, Cauchy nets, Cauchy sequences, etc. Even the properties of boundedness may be different from ones in uniform spaces [2]. This necessitates the need for further studies on quasi-uniform spaces.

1.1.2 Research aims and objectives

The aim of the study is to develop a generalised theory that can define a cover that can determine the quasi-uniformity of a given quasi-uniform space. This can be achieved by first digging deeper into properties of quasi-uniform spaces, and then determine why they cannot be formulated in terms of their quasi-uniform covers, even if the quasi-uniform space in question is transitive. From that information, a cover will be defined, and referred to as a strong quasi-uniform cover with some added axioms which will precisely determine the quasi-uniformity of a given quasi-uniform space.

1.1.3 Literature ‘Review

The concept of uniform structure that is on a non-empty set X was introduced and well defined by A. Weil in 1937 in terms of subsets of X[3]. The introduction led to several emerging literature on quasi-uniformities, from L. Nachbin in 1948 who started the study of quasi-uniform structure but used the term "semi-uniformity," with the exclusion of the symmetry axiom, for a structure that satisfies the axioms of uniformity[4]. A quasi-uniformity is not determined by its quasi-uniform covers, unlike uniformity.

Fletcher describes a classical construction which assigns a transitive quasi-uniformity to each family of interior-preserving open covers[5]. He further brought forward the method of constructing compatible quasi-uniformities for any topological space. Tamari (1954) introduced the term "quasi-odoform base", a concept that was later described as "quasi-uniformity" was by A. Császár in 1960 who tried to prove that every topological space admits a compatible quasi-uniformity. Pervin in 1962 tried to back Császár, except that his quasi-uniformity was different in that it had a base that entailed reflexive transitive relations. These uniformities are referred to as non-archimedean uniformities. A.C.M. van Rooy in 1970 made further literature contributions in the idea of uniformities in 1950 as well as H.C. Reichel in 1974, B. Banaschewski in 1955 and going back to A.F. Manna in 1950.[6].

Quasi-uniform spaces are discussed in many advanced Topology textbooks and research work since the early 20th century but we will focus on developing work from some of the authors. In [7], Isbell details how uniform spaces play a key role for uniform continuity as topological spaces for continuity. Naimpally in [8] extends generally known results of function spaces of uniform spaces to quasi-uniform spaces, by additional conditions.

With respect to quasi-uniformity defined in a standard way, Iragi tries to investigate the ideas of completeness of objects and precompactness in [9]. He further shows that every quasi-uniformity on a generally reflective subcategory of some arbitrary category C can be lifted to a coarsest quasi-uniformity on category C for which every reflection morphism is continuous. For a class of uniform spaces referred to as coverable spaces, Berestovskii and Plaut construct a generalized covering space theory in [10] by attempting to assign lifting properties in uniform spaces and uniformly continuous mappings. By detailing its application in obtaining compatible non-transitive and any transitive quasi-uniform structure, Carter in [11] discusses a general method for constructing a compatible quasi-uniform structures.

We will discuss these in detail in this dissertation. What ascertains the importance of uniform space is that a compact Hausdorff space admits a unique uniform structure as will be detailed in subsequent chapters in this paper.

1.2 Quasi-Uniform Space

We need to start by looking into the following definitions before we are able to define quasi-uniform spaces.

Definition 1.2.1. [12] A *filter* $\mathcal{F}$ in any set X is a collection of non-void subsets of X such that

(a) if A belongs to $\mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$ and

(b) for any arbitrary $A, B \in \mathcal{F}, A \cap B \in \mathcal{F}$.

In a topological space X, a filter $\mathcal{F}$ converges to a point x iff each neighbourhood of x belongs to $\mathcal{F}$ (that is, the neighbourhood system of x is a subfamily of $\mathcal{F}$).

Definition 1.2.2. [13] For any set X, we denote the diagonal $\{(x,x)|x \in X\}$ in $X \times X$ by Δ. We specify which set X we are referring to by writting $\Delta(X)$. For $U, V \in X \times X$, then $\mathrm{U} \circ \mathrm{V}$ is the set given by $\{(x,y)|$ for some $z, (x,z) \in U$ and $(z,y) \in V\}$.
In a metric space, we observe that x and y are clustered together, iff the point (x,y) is near the diagonal in $X \times X$. We note that $\circ$ is a natural extension of the idea of composition of functions and that U and V are actually relations on X.

Definition 1.2.3. [13] On a set X, a *diagonal uniformity* is a collection $D(X)$ or just D of subsets of $X \times X$, called *surroundings*, which satisfy:

(a) $\exists E \in \mathcal{D}$ such that $D \in \mathcal{D} \Rightarrow E^{-1} \subset D$

(b) $\exists E \in \mathcal{D}$ such that $D \in \mathcal{D} \Rightarrow E \circ E \subset D$

(c) $D_1, D_2 \in \mathcal{D} \Rightarrow D_1 \cap D_2 \in \mathcal{D}$

(d) $D \in \mathcal{D}, D \subset E \Rightarrow E \in \mathcal{D}$

(e) $D \in \mathcal{D} \Rightarrow \Delta \subset D$

When X has such a structure, we call X a *uniform space*. We refer to uniformity $\mathcal{D}$ as *separating* and the set X is said to be *separated* iff $\bigcap\{D|D \in \mathcal{D}\} = \Delta$.

Definition 1.2.4. [14][6][5][15] On a set X *quasi-uniformity* is a filter $\mathcal{U}$ on $X \times X$ such that:

(a) $\forall U \in \mathcal{U}, \Delta \subseteq U$ and

(b) $\exists V \in \mathcal{U}$ such that if $U \subset \mathcal{U}$, then $V \circ V \subset U$.

A quasi-uniformity is also referred to as a *quasi-uniform structure*. The members of $\mathcal{U}$ are termed *entourage* and the pair $(X,\mathcal{U})$ is referred to as a *quasi-uniform space*. As a result, if $\mathcal{U}$ is a quasi-uniformity on set X, then $\mathcal{U}^{-1} = \{\mathrm{U}^{-1}|\mathrm{U} \in U\}$, the *conjugate* of U, is also a quasi-uniformity on set X. Let $\mathcal{U}, \mathcal{V}$ be quasi-uniformities on X. We describe $\mathcal{U}$ as *coarser* than $\mathcal{V}$ if $\mathcal{U} \subseteq \mathcal{V}$.

Definition 1.2.5. [5] Let $(X, \mathcal{V})$ be a quasi-uniform space. Then the topology *induced* by $\mathcal{V}$ (also trace topology or just the topology $\mathcal{T}(\mathcal{V})$ of $\mathcal{V}$) is the topology defined in Proposition 1.2.7. If $\mathcal{T}(\mathcal{V}) = \mathcal{T}$ (also denoted $\mathcal{T}_{\mathcal{V}} = \mathcal{T}$), then $\mathcal{V}$ is referred to as being *compatible* with $\mathcal{T}$ (and $(X, \mathcal{T})$ is described as *admitting* $\mathcal{V}$). On a set X, two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ are *compatible* if $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{V}}$.

$\mathcal{T}_{\mathcal{U}}$ denotes the topology for X generated by $\mathcal{U}$ where $\mathcal{U}$ is a quasi-uniformity on a set X.

Corollary 1.2.6. *Let $(X, \mathcal{V})$ be a quasi-uniform space. Then $\{V | V \in \mathcal{V}$ and V is $\mathcal{T}(\mathcal{V}^{-1} \times \mathcal{V})$ open$\}$ is a base for $\mathcal{V}$.*

Proposition 1.2.7. *Let $\mathcal{B}$ be a base for a quasi-uniformity $\mathcal{V}$ on X. Let $\mathcal{B}(x) = \{B(x) | B \in \mathcal{B}\}$, for every $x \in X$. Then for every $x \in X$, there is a unique topology on X such that $\mathcal{B}(x)$ is a base for the neighbourhood filter of x in the topology.*

Definition 1.2.8. Let the non-empty collection $\mathcal{B} \subseteq \exp\ (X \times X)$, in a non-empty set X. $\mathcal{B}$ generates, on X, a base for a quasi-uniformity if:

(a) There exists $W \in \mathcal{B}$ such that $W \circ W \subseteq B$ if $B \in \mathcal{B}$.

(b) For each $B \in \mathcal{B}, \Delta \subseteq B$.

Theorem 1.2.9. *[6] Let $\mathcal{B}$ be a collection of relations of X such that for every $B \in \mathcal{B}$, the diagonal $\Delta \subset B$. There is some $A \in \mathcal{B}$ for every $B \in \mathcal{B}$, such that $A \circ A \subset B$. Then for a particularly unique quasi-uniformity, $\mathcal{B}$ is a subbase.*

Definition 1.2.10. [12][13][16] A collection $\mathcal{A}$ is a *cover* of a set B iff $B \subset \bigcup\{A : A \in \mathcal{A}\}$; that is, if and only if every member of B belongs to some member of $\mathcal{A}$. The collection is an *open cover* of B iff every member of $\mathcal{A}$ is an open set. A *subcover* of $\mathcal{A}$ is a subcollection which is also a cover.

Definition 1.2.11. [5] A subfamily $\mathcal{B}$ of a quasi-uniformity $\mathcal{U}$ is a *base* for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. Axiom (b) of Definition 1.2.4 shows that if $\mathcal{B}$ is a base for $\mathcal{U}$ and n is a positive integer, then a countable base $\{B^n | B \in \mathcal{B}\}$ is a base for $\mathcal{U}$. A subfamily U of $\mathcal{U}$ is a *subbase* for $\mathcal{U}$ if the collection of the finite intersection of elements of U is a base for $\mathcal{U}$.

Theorem 1.2.12. *[12] **(Lindelof)** There is a countable subcover of every open cover of a given subset of a space whose topology has a countable base.*

Proof. Suppose that for a set B, $\mathcal{B}$ is an open cover of B, and $\mathcal{A}$ is a countable base for the topology. Because every element of $\mathcal{B}$ is the union of elements of $\mathcal{A}$ there is a subcollection $\mathcal{C}$ of $\mathcal{A}$ which also covers $\mathcal{B}$, such that every element of $\mathcal{C}$ is a subset of some element of $\mathcal{B}$. For all the elements of $\mathcal{C}$ we may select a containing element of $\mathcal{B}$ and so obtain a countable subcollection $\mathcal{D}$ of $\mathcal{B}$. Then $\mathcal{D}$ is also a cover of $\mathcal{A}$ because $\mathcal{C}$ covers $\mathcal{B}$. Hence $\mathcal{B}$ has a countable subcover. □

Definition 1.2.13. [6] A quasi-uniform space $(X, \mathcal{U})$ is *initial with respect to* a family of functions f_a from the quasi-uniform space $(X, \mathcal{U})$ to arbitrary quasi-uniform spaces $(\mathcal{Y}_a, \mathcal{V}_a)$ if the quasi-uniformity $\mathcal{U}$ is the coarsest on X such that all the f_a are continuous quasi-uniformly.

Definition 1.2.14. A set Λ is a *directed set* if and only if there is some relation $\leqslant$ on Λ that satisfies the follow:

(a) for every $\lambda \in \Lambda, \lambda \leqslant \lambda$.

(b) if $\lambda_1 \leqslant \lambda_2$ and $\lambda_2 \leqslant \lambda_3$, then $\lambda_1 \leqslant \lambda_3$.

(c) if $\lambda_1, \lambda_2 \in \Lambda$, then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leqslant \lambda_3, \lambda_2 \leqslant \lambda_3$.

This relation $\leqslant$ is often called a *direction* on Λ, or is referred to as *directing* Λ.

Definition 1.2.15. [13] A *net* is a set X in a function $P : \Lambda \to X$, where Λ is some directed set. The point $P(\lambda)$ is usually denoted x_λ, and we can also refer to it as "the net (x_λ)" or "the net $(x_\lambda)_{\lambda\in\Lambda}$".

Definition 1.2.16. Let X have a diagonal uniformity $\mathcal{D}$. A net $(x_\lambda)_{\lambda\in\Lambda}$ in X is *Cauchy* iff for each $D \subset \mathcal{D}$, there is some $\lambda_0 \in \Lambda$ such that $(x_{\lambda_1}, x_{\lambda_2}) \in D$ whenever $\lambda_1, \lambda_2 \geqslant \lambda_0$. The corresponding covering description is as follows:
$(x_\lambda)_{\lambda\in\Lambda}$ is Cauchy iff for each uniform cover $\mathcal{C}$, there is some $\lambda_0 \in \Lambda$ such that x_{λ_1} and x_{λ_2} lie together in some element $\mathcal{C}$ whenever $\lambda_1, \lambda_2 \geqslant \lambda_0$.

Before we take on the next example, let us consider the following definitions.

Definition 1.2.17. [17] A point x is *adherent* to a set A if each neighbourhood of x meets set A. A set of points adherent to a set A is referred to as the adherence (or the closure) $\overline{A}$ of set A also denoted adh A.

Definition 1.2.18. [18] Let $\mathcal{F}$ be a filter on X, $(X,\mathcal{U})$ be a quasi-uniform space, and $\mathcal{U}(x)$ be the set of all neighbourhoods of $x \in X$. $\mathcal{F}$ is said to be $\mathcal{U}$*-Cauchy* (or just Cauchy) if for each $\mathcal{U}(x) = \{\mathrm{U}(x)|\mathrm{U} \in \mathcal{U}\}$, there is some $x = x(\mathrm{U})$ such that $\mathrm{U}(x) \in \mathcal{F}$. $(X,\mathcal{U})$ is considered *complete* if each $\mathcal{U}$-Cauchy filter has an adherent point and *strongly complete* if each $\mathcal{U}$-Cauchy filter converges.

Now, is there a quasi-uniform space which is complete, but not necessarily strongly complete? This is one mystery that generally arises. We will try to address the question using the following example to show that yes, it does exist, after which we will introduce nets since we will be using them in this dissertation.

Example 1.2.19. *[11] We want to make a T_1-space which does not have a T_1-completion and of a quasi-uniform space which is complete, but not strongly complete. So we let the set X be the set of integers. For every integer n in X, we define $\mathrm{U}_n = \bigcup\{(x,y)$ such that $x \geqslant n, y = 0$ or $1\}$. So for a quasi-uniform structure U on $X \times X$, we have a base $\{\mathrm{U}_n : n \in X\}$ where U generates the discrete topology τ. Now, we let F be a Cauchy filter. So we have $n = \{0, 1, x\}$, for every $x \geqslant n$ and $\mathrm{U}_n(x) = \{x\}$ for every $x < n$. It can be easily concluded that that F would have to be generated by a finite set and also that adh$F \neq \phi$. Let S be the family of all supersets of $\{0, 1\}$. Now lim$S = \phi$, and matter of fact, the U-Cauchy filter S is actually the only one that is non-convergent. Therefore this is enough for us to conclude that the quasi-uniform space (X, U) is complete, but definitely not strong complete.*

We have taken the idea of Cauchy filters to quasi-uniform spaces. So we consider the definitions:

Definition 1.2.20. With topology $\mathcal{T}$, induced by $\mathcal{U}$, let $(X, \mathcal{T})$ be a topological space and let $x \in U$ for some U in $\mathcal{U}$ such that $U(x) \in \mathcal{T}$. A net $(x_\lambda)_{\lambda\in\Lambda} \subset X$ *converges* to $x \in X$ iff for each $U \in \mathcal{U}$, $(x_\lambda)_{\lambda\in\Lambda}$ is in $U(x)$ for sufficiently large λ.

Definition 1.2.21. [8] A net $(x_\lambda)_{\lambda\in\Lambda} \subset X$ is *Cauchy* iff for every $U \in \mathcal{U}, \exists x \in X$ such that $(x_\lambda)_{\lambda\in\Lambda}$ is in $U(x)$ for sufficiently large λ.

We can easily conclude that convergent nets are Cauchy from the above definitions.

Definition 1.2.22. On a set X, let $\{\mathcal{U}_i | i \in I\}$ be a collection of quasi-uniformities. Then the coarsest quasi-uniformity on X which is finer than each $\mathcal{U}_i$ is said to be a *supremum* of $\{\mathcal{U}_i | i \in I\}$. The supremum always exists and is the filter that is coarsest which is finer than each U_i. The *infimum* of $\{\mathcal{U}_i | i \in I\}$ is the finest quasi-uniformity that is coarser than every $\mathcal{U}_i$. On the other hand, in the collection of all quasi-uniformities on X, the supremum that is actually coarser than each $\mathcal{U}_i$ is referred to as the infimum, and it always exists.

The uniformity that has a (Δ) as a base is called the *discrete uniformity*. On a set X, for all quasi-uniformities the supremum is the discrete uniformity is on set X. Every nonempty set X has coarsest quasi-uniformity consisting only of $X \times X$.

Proposition 1.2.23. *[5] Let $\mathcal{B}$ be a base for a quasi-uniformity $\mathcal{U}$ on set X, and for every $x \in X$, let $\mathcal{B}(x) = \{B(x)|B \in \mathcal{B}(x)\}$. Then for every $x \in X$, there is a unique topology on set X such that in this topology, $\mathcal{B}(x)$ is a base for the neighbourhood filter of x.*

If $(X,\mathcal{U})$ is a quasi-uniform space, then the topology $\mathcal{T}(\mathcal{U})$ of $\mathcal{U}$ (or just the topology induced by $\mathcal{U}$) is the topology defined in this Proposition 1.2.23. A subset A of X is an element of $\mathcal{T}(\mathcal{U})$ iff for every $x \in A$, there is an entourage $U \in \mathcal{U}$, such that $U(x) \subset A$.

Definition 1.2.24. On a set X, a *quasi-pseudo-metric* is a function d on $X \times X$ to the non-negative real numbers such that for every $x, y, z \in X, d(x,z) \leqslant d(x,y) + d(x,z)$ and $d(x,x) = 0$. A *quasi-metric* is a quasi-pseudo-metric d such that whenever $d(x,y) = 0$, $x = y$. On a set X, let d be a quasi-pseudo-metric. Then for $n \in \mathbb{N}$, the family of every set of the form $\{(x,y)|d(x,y) < (\frac{1}{2})^n\}$ is a base for a particular quasi-uniformity that is referred to as *the quasi-uniformity generated by* d.

Definition 1.2.25. [13][19][12] A topological space X is a *Hausdorff space* (T_2-space or separated space) iff whenever x and y are distinct points in space X, there exist disjoint neighbourhoods of x and y.

Theorem 1.2.26. *[5] Let $\mathcal{U}$ be a quasi-uniformity and X be a set in a quasi-uniform space $(X,\mathcal{U})$. Then there is a quasi-pseudo-metric d such that $\mathcal{U}$ is a quasi-uniformity generated by d iff $\mathcal{U}$ has a countable base.*

Definition 1.2.27. The *conjugate* $\mathcal{U}^{-1}$ of the quasi-uniformity $\mathcal{U}$, is given by $\mathcal{U}^{-1} = \{U^{-1}(y,x)|U \in \mathcal{U}\}$. If U generates $\mathcal{U}$, then U^{-1} generates $\mathcal{U}^{-1}$. Members of the topology $\mathcal{T}(U^{-1} \times \mathcal{U})$ are called *open* relative to $\mathcal{T}(U^{-1} \times \mathcal{U})$ or $\mathcal{T}(U^{-1} \times \mathcal{U})$-open, or if only one topology is under consideration, simply open sets.

Corollary 1.2.28. *[5] Let $\mathcal{U}$ be the quasi-uniformity for a quasi-uniform space $(X,\mathcal{U})$. Then $\{\mathrm{U}|\mathrm{U} \in \mathcal{U}$ and U is $\mathcal{T}(\mathcal{U}^{-1} \times \mathcal{U})$ open$\}$ is a base for $\mathcal{U}$.*

Definition 1.2.29. [12][13] A topological space is *compact (bicompact)* if and only if every open cover has a finite subcover. A subset A of a topological space is compact iff it is, with relative topology, compact; equivalently A is compact iff every cover of A by open sets in X has a finite subcover.

Definition 1.2.30. [20] A *compact ordered space*, denoted by the pair $(\mathrm{X}, \leqslant)$, is a compact topological space X equipped with a partial order $\leqslant$ on space X, which is closed in the product topology of $X \times X$.

Theorem 1.2.31. *[5] Let $\leqslant$ be a partial order that is closed on X and $(X,\mathcal{T})$ be a compact Hausdorff space. On X, there is precisely one quasi-uniformity U such that $\mathcal{T}(U^*) = \mathcal{T}$ and $\leqslant = \bigcap U$.*

Proof. Suppose that on X, the quasi-uniformity $\mathcal{U}$ is such that the partial order $\leqslant$ is given by $\leqslant = \bigcap U$ and $\mathcal{T}(U^*) = \mathcal{T}$. The quasi-uniformity on X whose set of entourages is the trace of $\mathcal{T} \times \mathcal{T}$ of $\mathcal{T}$, is actually the quasi-uniformity that is induced by the topology $\mathcal{T}$. We prove that $\mathcal{U}$ only has $\mathcal{T} \times \mathcal{T}$ neighbourhoods of $\leqslant$. All elements of $\mathcal{U}$ are $\mathcal{T} \times \mathcal{T}$ neighbourhoods of $\leqslant$, by Corollary 1.2.28. Now, suppose there is a $\mathcal{T} \times \mathcal{T}$ neighbourhood of V or $\leqslant$ which is not an element of $\mathcal{U}$. It can be concluded that $\{\mathrm{U} - \mathrm{V} \mid \mathrm{U} \in \mathcal{U}\}$ is a base for V, a filter on $X \times X$. But then, $(X,\mathcal{T})$ is compact, then V contains (x,y), a $\mathcal{T} \times \mathcal{T}$ cluster point that is not an element of $\leqslant$. But then again, (x,y) is a cluster point of $\mathcal{U}$ from the fact that U is coarser than V. It can be concluded that the intersection of the closures of elements of $\mathcal{U}$ of $\mathcal{T} \times \mathcal{T}$ is $\leqslant$, from the preceding corollary – which is a contradiction.

To conclude the proof, we establishing that U, the collection of all $\mathcal{T} \times \mathcal{T}$ neighbourhoods of $\leqslant$ is a quasi-uniformity on X in a way that $\leqslant = \bigcap U$ and $\mathcal{T}(U^*) = \mathcal{T}$. It follows that $\bigcap U = \leqslant$ and that U is a filter on $X \times X$. Suppose that axiom (b) of Definition 1.2.4 of quasi-uniformity is not satisfied. So, there is some $\mathcal{T} \times \mathcal{T}$ open set $\mathrm{U} \in U$ such that for every $V \in U, V^2 - \mathrm{U} \neq \phi$. Let $V' = \{((x,y),z) \in \mathrm{X}^2 \times X | (x,y) \notin U, (x,z) \in V, (z,y) \in V\}$ for every $V \in U$. Clearly, $B = \{V' | V \in U\}$ is a filter base on $(X \times X - \mathrm{U}) \times X$. But then B has a cluster point $((a,b),c)$ as a consequence of $(X \times X - \mathrm{U}) \times X$ being compact. We ascertain that $(a,c) \in \leqslant$. Now, suppose that as $\leqslant$ is compact, $(a,c) \notin \leqslant$. There are some open disjoint sets H and V such that $(a,c) \in H$ and $\leqslant \subset V$. Set $W = \{((x,y),z) | (x,z) \in H\}$. Consequently, $W \cap V' = \phi$ and $((a,b),c) \in W$; a contradiction. Therefore $(a,c) \in \leqslant$ and it can be concluded as a result that $(c,b) \in \leqslant$. Because $\leqslant$ is transitive, as will be defined in 1.3.1, we can safely say $(a,b) \in \leqslant \subset \mathrm{U}$; turning out to be a contradiction. In conclusion, it is obvious that $\mathcal{T}(U^*) \subset T$ since $\bigcap U$ is a partial order $T(U^*)$, which is a Hausdorff topology. Thus $\mathcal{T}(U^*) = \mathcal{T}$.

□

1.3 Uniform Covers and Transitive Quasi-Uniformities

Definition 1.3.1. [21][9] Let $\mathcal{U}$ be a quasi-uniformity on set X. A base $\mathcal{B}$ for $\mathcal{U}$ is *transitive* if for every $\mathrm{B} \in \mathcal{B}, \mathrm{B} \circ \mathrm{B} = \mathrm{B}$. A *transitive quasi-uniformity* is a quasi-uniformity with a transitive base. Each topological space contains the *fine transitive quasi-uniformity*, which is the finest compatible transitive quasi-uniformity, denoted $\mathcal{FT}$.

Recall that for a topological space $(X, \mathcal{T})$, $\mathcal{U}$ is described as compatible with $(X, \mathcal{T})$ on condition that $\mathcal{T} = \mathcal{T}_U$.

Definition 1.3.2. [16] On the quasi-uniform space $(X, \mathcal{U})$, let $\mathcal{U}$ be a quasi-uniformity. A filter $\mathcal{F}$ on X is a *Cauchy filter on* X on condition for every $\mathrm{U} \in U$ there is some $p \in X$ such that $\mathrm{U}(p) \in \mathcal{F}$. If every Cauchy filter converges in $\mathcal{T}_{\mathcal{U}}$, then the quasi-uniform space $(X, \mathcal{U})$ is *complete*.

Definition 1.3.3. [13] Let $\mathcal{C}$ and $\mathcal{C}'$ be covers of X. $\mathcal{C}'$ is said to refine $\mathcal{C}$ if every C' in $\mathcal{C}'$ is contained in some C in $\mathcal{C}$, that is, $C' \subset \mathcal{C}'$ for some $C \in \mathcal{C}$[22]. Then we refer to $\mathcal{C}'$ as a refinement of $\mathcal{C}$. We describe a *star* of A with respect to $\mathcal{C}$ as the set

$$St(A, \mathcal{C}) = \bigcup \{U \in \mathcal{C} | A \cap U \neq \theta\} \tag{1.1}$$

if $\mathcal{C}$ is a cover of X and $A \subset X$.

We say that $\mathcal{C}$ is a star-refinement of $\mathcal{C}'$, denoted $\mathcal{C}* < \mathcal{C}'$ or that $\mathcal{C}$ *star-refines* $\mathcal{C}'$, if and only if for every $U \in \mathcal{C}$, there exists $V \in \mathcal{C}'$ such that $\mathrm{St}(U, \mathcal{C}) \subset V$. More so, if for $x \in X$, the sets $\mathrm{St}(x, \mathcal{C})$ refine $\mathcal{C}'$, we describe $\mathcal{C}$ as a *barycentric refinement* of $\mathcal{C}'$, denoted $\mathcal{C} \Delta \mathcal{C}'$.

Definition 1.3.4. A quasi-uniform space $(X, \mathcal{U})$ *has the Lebesgue property* provided that for each $\mathcal{T}_U$-open cover $\mathcal{C}$ of X there is some $\mathrm{U} \in U$ such that $\{\mathrm{U}(x) : x \in X\}$ is a refinement of $\mathcal{C}$.

Theorem 1.3.5. *[21] Let $(X, \mathcal{U})$ be a quasi-uniform space. $(X, \mathcal{U})$ is complete provided that it has the Lebesgue property.*

Proof. On X, we let $\mathcal{F}$ be a Cauchy filter and suppose that $\mathcal{F}$ is not convergent. For every $x \in X$, there is some $\mathrm{U}_x \in U$ such that $\mathrm{U}_x(x) \notin \mathcal{F}$. For every $x \in X$, choose one such U_x and $\mathcal{C} = \{\mathrm{int}[\mathrm{U}_x(x)] : x \in X\}$. There is some $\mathrm{U} \in U$ such that $\{\mathrm{U}(x) : x \in X\}$ refines $\mathcal{C}$. Since $\mathcal{F}$ is a Cauchy filter, by Definition 1.2.18, there is some $p \in X$ such that $\mathrm{U}(p) \in \mathcal{F}$. Then $\mathcal{C} \cap \mathcal{F} \neq \phi$. This contradicts the method by which the elements of $\mathcal{C}$ were chosen. □

Definition 1.3.6. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is *orthocompact* provided that if $\mathcal{C}$ is an open cover of X, then there is an open refinement $\mathcal{R}$ of $\mathcal{C}$ such that if $x \in X$ then $\bigcap\{U \in \mathcal{R} : x \in U\} \in \mathcal{T}$. For rotational convenience throughout this dissertation, if $x \in X$ and $\mathcal{R}$ is an open cover of X, we denote $\bigcap\{U \in \mathcal{R} : x \in U\}$ by $U_x^{\mathcal{R}}$.

Theorem 1.3.7. *Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is orthocompact if and only if it is a transitive quasi-uniformity that is compatible with the Lebesgue property.*

Proof. Let $\mathcal{U}$ be a transitive quasi-informity that is compatible with Lebesgue property in the topological space $(X, \mathcal{T})$, and let $\mathcal{C}$ be an open cover of X. There is $U \in \mathcal{U}$, which we may assume, without loss of generality, to be a transitive relation on X, whereby $\{U(x) : x \in X\}$ refines $\mathcal{C}$. Let

$$\mathcal{R} = \{U(x) : x \in X\}. \tag{1.2}$$

We note that $\mathcal{R}$ is an open refinement of $\mathcal{C}$. Let

$$p \in U_z^{\mathcal{R}} = \bigcap\{R \in \mathcal{R} : z \in R\}, \tag{1.3}$$

let $x, z \in X$ and let $q \in U(p)$. Suppose that $z \in U(x)$. Then $p \in U(x)$ so that $q \in U(p) \subset U \circ U(x) = U(x)$. Consequently $q \in U_z^{\mathcal{R}}$ so that $U(p) \subset U_z^{\mathcal{R}}$. It follows that $U_z^{\mathcal{R}} \in \mathcal{T}$.

Now suppose $\mathcal{U}$ is a fine quasi-uniformity that is transitive, whose existence is guaranteed by Definition 1.3.1 (that is, each topological space has a fine transitive quasi-uniformity), let $(X, \mathcal{T})$ be an orthocompact space and let C be an open cover of X. Then for every $x \in X$, there exists an open refinement $\mathcal{R}$ of $\mathcal{C}$ such that, $U_z^{\mathcal{R}} \in \mathcal{T}$. Let $U = \bigcup\{\{x\} \times U_z^{\mathcal{R}} : x \in X\}$. Then $U \in \mathcal{U}$. Clearly:

$$\{U(x) : x \in X\} = \{U_z^{\mathcal{R}} : x \in X\} \tag{1.4}$$

in an open refinement of $\mathcal{C}$. Hence $(X, \mathcal{U})$ has the Lebesgue property. □

Definition 1.3.8. [23] Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is a *transitive base* provided that for $(X, \mathcal{T})$, the fine quasi-uniformity is actually the fine transitive quasi-uniformity $\mathcal{FT}$.

Theorem 1.3.9. *[1] Let R_B be the binary relation $\{(x,y)|x \in B \Rightarrow y \in B\}$ and let $(X, \mathcal{T})$ be a topological space. Let F be a collection of subsets of X. The finite intersection of relation $R_B, B \in F$ form a base B of entourages of a quasi-uniformity $\mathcal{U}$ on X such that their induced topology is exactly the topology $\mathcal{T}$ generated by F on X. This particular quasi-uniformity, denoted $\mathcal{P}$, is said to be a* Pervin quasi-uniformity *of the topology on X. One useful property of $\mathcal{P}$ is that for any topological space, this quasi-uniformity is totally bounded.*

Definition 1.3.10. [6][16] Let $\mathcal{C}$ be a collection of open sets such that $\bigcap\{C \in \mathcal{C} : x \in C\} \in \mathcal{T}$ if $x \in X$, in a topological space $(X, \mathcal{T})$. Then $\mathcal{C}$ is a *Q-collection*. $\mathcal{C}$ is a *Q-cover* provided $\mathcal{C}$ is an open cover of X.

Corollary 1.3.11. *[14] If every compatible quasi-uniformity with the topology $\mathcal{T}$ is a Q-cover, then $(X, \mathcal{T})$ is a unique compatible quasi-proximity as will be defined in 2.3.1 (or just quasi-uniformity) if and only if $\mathcal{T}$ is finite.*

Proposition 1.3.12. *[23] Let $(X, \mathcal{T})$ be a transitive space. Then each Q-cover of X is finite iff $(X, \mathcal{T})$ has only one compatible quasi-uniformity.*

Proof. Let $\mathcal{P}$ be a Pervin quasi-uniformity as described in Theorem 1.3.9. By the same Theorem, $\mathcal{P}$ is totally bounded. If $\mathcal{P} = \mathcal{FT}$, then every Q-cover of X is finite. Consequently, if every Q-cover of X is finite, then $\mathcal{FT}$ is totally bounded. Since $\mathcal{FT}$ is assumed to be the fine quasi-uniformity, the result now follows from Corollary 1.3.11. □

[10] It is well known that traditional covering space theory, including the isomorphism between the fundamental group and the group of deck transformations of the universal cover, applies only to topological spaces that are connected, locally arcwise connected, and semilocally simply connected. Each coverable space is shown to have what we call a uniform universal cover, which is not a cover in the traditional sense (in particular the mapping is not a local homeomorphism in general), but which nonetheless has universal and lifting properties in uniformly continuous functions (as will be defined in this chapter) and uniform spaces categories. One of the main impediments to generalising the classical construction of the universal cover is the traditional definition of covering map, the most important property of which is the ability to lift curves and homotopies. However, this lifting property is traditionally gained at the expense of requiring that a space and its cover be locally homeomorphic in a fairly strong way.

Definition 1.3.13. A cover of a uniform space $(X, \mathcal{D})$ is refined by a cover of a form $\mathcal{C} = \{D[x] | x \in X\}$, where $D[x] = \{y : (x, y) \in D\}$, for some entourage $D \in \mathcal{D}$ if and only if it is a *uniform cover*.

Theorem 1.3.14. *[24][25] The family μ of all uniform covers of a uniform space $(X, \mathcal{D})$ has the properties:*

(a) if $\mathcal{U}_1, \mathcal{U}_2 \in \mu$ then there exists $\mathcal{U}_3 \in \mu, \mathcal{U}_3* < \mathcal{U}_1$ and $\mathcal{U}_3* < \mathcal{U}_2$,

(b) if $\mathcal{U} < \mathcal{U}'$ and $\mathcal{U} \in \mu$, then $\mathcal{U}' \in \mu$.

Conversely, given any family μ of covers of X satisfying (a) and (b), the collection of all sets $D_{\mathcal{U}} = \bigcup\{U \times U | U \in \mathcal{U}, \text{for } \mathcal{U} \in \mu\}$ on X is actually a base for a diagonal uniformity, whose uniform covers are precisely the members of μ.

Proof. (a) It is sufficient to show that any two covers $\mathcal{U}_{D_1}$ and $\mathcal{U}_{D_2}$ have a common barycentric refinement. Recall from Definition 1.3.3 that a barycentric refinement of a barycentric refinement of $\mathcal{U}$ star-refines $\mathcal{U}$. Pick a symmetric $D \in \mathcal{D}$ such that

$$D \circ D \subset D_1 \cap D_2. \tag{1.5}$$

Then for each $x \in X$, $\mathrm{St}(x, \mathcal{U}_D) \subset D_1[x] \cap D_2[x]$ and it follows that $\mathcal{U}_D$ is a common barycentric refinement of $\mathcal{U}_{D_1}$ and $\mathcal{U}_{D_2}$.

(b) is obvious from definition 1.3.13 of uniform cover. □

Thus the uniform covers describe a uniformity as well is its surroundings do. Extended studies about (possibly non-symmetric) uniform structures in the point-free context were conducted in several joint papers by Ferreira and Picado[24]. It is well known that in general, unlike a uniformity, a quasi-uniformity is not determined by its quasi-uniform covers(as will be defined in the next chapter). However, a classical construction, due to P. Fletcher, which assigns a transitive quasi-uniformity to each collection of open covers that are interior-preserving , allows to describe on the topological spaces, in terms of those collections of covers, all transitive quasi-uniformities.

Definition 1.3.15. [12] A topological space X is *regular* iff for every point x and each neighbourhood U of x, there is a closed neighbourhood V of x such that $V \subset U$; that is, the family of closed neighbourhoods of each point is a base for the neighbourhood system of the point.

Definition 1.3.16. A topological space X is referred to as a T_0-*space* if for any distinct points $x, y \in X$ there is an open set $U \subset X$ containing exactly one of these points.

Definition 1.3.17. [19] A topological space X is called a T_1-*space* if for any distinct points $x, y \in X$ the point x has a neighbourhood $U_x \subset X$ such that $y \notin \overline{U}_x$.

Evidently, every T_2-space is T_1.

Definition 1.3.18. [13][19] A topological space X is *completely regular* iff whenever A is a closed set in X and $x \notin A$, there is a continuous function $f : X \to I$ such that $f(x) = 0$ and $f(A) = 1$. A completely regular T_1-space is called a *Tychonoff space*.

Transitive quasi-uniform spaces constitute a fundamental subcategory of the uniformly continuous maps and quasi-uniform spaces categories. They are almost as general and as instrumental and as quasi-uniform spaces in properties of topological spaces. One highly intriguing aspect of transitive quasi-uniformities lies on the fact that they are attainable throgh the Fletcher construction [26] taking into account the open covers that are interior- preserving of their corresponding topological spaces. The challenge of figuring out which topological spaces admit a quasi-uniformity is often swapped under the carpet as hinted by other authors the likes of Hans-Peter, Császár and Fletcher; since it is theorised that every topological space admits a quasi-uniformity[2]. However, this conclusion proved to have strong opposition from other authors, the likes of A. Weil and L. Pontrjagin who ascertained a topological space is completely regular iff it admits a uniformity, however, quasi-uniformities were developed in later years.

Definition 1.3.19. [27] In a non-empty set X, let $\mathcal{U}$ be non-empty family such that $\mathcal{U} \subseteq \exp(X \times X)$ generates a quasi-uniformity also referred to as *generalised quasi-uniformity* on X if:

(a) If $\exists V \in \mathcal{U}$ such that $V \circ V \subseteq U$, then $U \in \mathcal{U}$ where $W \circ W' = \{(x, z) : \exists y \in X$ such that $(x, y) \in W, (y, z) \in W'\}$, for $W, W' \in \mathcal{U}$.

(b) If $U \subseteq V \subseteq X \times X$ and $U \in \mathcal{U}$ then $V \in \mathcal{U}$

(c) $\forall U \in \mathcal{U}$ where $\Delta = \{(x, x) : x \in X\}, \Delta \subseteq U$

Definition 1.3.20. [28] Let X be a non-empty set and $\mu \subseteq \exp(X)$, the set of all finite subsets of X called the *exponential* of X. If $\phi \in \mu$ and $U_\alpha \in \mu(\forall \alpha \in \Lambda)$ results in $\bigcup U_\alpha \in \mu$, then μ is referred to as a *generalised topology*, which is also denoted GT. So (X, μ) is referred to as *generalized topological space* or just a GTS in short.

Definition 1.3.21. Let $\mathcal{U}$ be a quasi uniformity of the quasi-uniform space $(X, \mathcal{U})$. If for each $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \subseteq U$, then $\mathcal{B} \subseteq \mathcal{U}$, which when we we recall is referred to as a base for U.
We refer to as quasi-uniformity generated by $\mathcal{B}$, the quasi-uniformity $\mathcal{U}$ in this particular instance.

Lemma 1.3.22. *Suppose that a strong topological space (X, μ) is as defined in 1.3.24. Now, for $G \in \mu$, $\mathcal{B} = \{B_G : G \in \mu\}$ generates a base for an existing quasi-uniformity on X such that $\mu = \mu(\mathcal{B})$, whereby $B_G = (G \times G) \cup ((X \backslash G) \times X)$.*

A "Pervin g-quasi-uniformity" as described in the above Lemma refers to the induced quasi-uniformity, with reference to simple topological spaces constructed by Pervin in [1].

Theorem 1.3.23. *Let $\mathcal{U}$ be a quasi-uniformity in the quasi-uniform space $(X, \mathcal{U})$. For $U(g) = \{x \in X : (g, x) \in U\}$, the family $\{G \subseteq X : g \in G \Rightarrow \exists U \in \mathcal{U}$ such that $g \in U(g) \subseteq G\}$ yields a strong GT on X.*

Denoted by $\mu(\mathcal{U})$, we refer to this particular GT as a GT induced by $\mathcal{U}$ on X.

Definition 1.3.24. [27] For $X \in \mu$, a *strong GT* is a generalised topology μ on X. If μ is a strong GT, then space (X, μ) is described as a strong generalised topological space (or just a strong topological space).

Definition 1.3.25. Let $\mathcal{U}_X$ and $\mathcal{U}_Y$ be two quasi-uniformities in the quasi-uniform spaces $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ respectively. For every $V \in \mathcal{U}_Y$, there exists $P \in \mathcal{U}_X$ such that $(x_1, x_2) \in P \Rightarrow (f(x_1), f(x_2)) \in V$, the function $f : X \to Y$ is referred to as a *quasi-uniformly continuous function*.

Example 1.3.26. *A Pervin quasi-uniformity is transitive for each given strong topological space.*

2 Chapter 2

2.1 Orthocompact and Preorthocompact Spaces

Definition 2.1.1. [29] For every $x \in X$, if $V(x)$ is a neighborhood of x, then the relation V on X is said to be a *neighbornet* of X.

Definition 2.1.2. [5] If $(X, \mathcal{T})$ is a compact metrizable space, then every metric that induces $\mathcal{T}$ has the following property. For each open cover $\mathcal{C}$ of X there exists a positive real number $\lambda(\mathcal{C})$, called a Lebesgue number of $\mathcal{C}$, such that if $A \subset X$ and $diam(A) \leqslant \lambda(C)$, then there is some $C \in \mathcal{C}$ such that $A \subset C$. Now, a *quasi-uniform cover* of A is a cover $\mathcal{C}$ of a subset A of a quasi-uniform space $(X, \mathcal{U})$ on condition there is some $U \in \mathcal{U}$ such that for every $x \in A$ there is some $C \in \mathcal{C}$ such that $U(x) \subset C$. On a set X, a *Lebesgue quasi-uniformity* is a quasi-uniformity $\mathcal{U}$ whereby each $\mathcal{T}(\mathcal{U})$-open cover is a quasi-uniform cover. An open cover of a topological space $(X, \mathcal{T})$ is a *quasi-normal cover* provided it is just a quasi-uniform cover of $(X, \mathcal{U})$ for some quasi-uniformity $\mathcal{U}$ compatible with $\mathcal{T}$. Thus an open cover $\mathcal{C}$ of X is a quasi-normal cover provided there is a normal neighbornet U of X such that $\{U(x) | x \in X\}$ refines $\mathcal{C}$.

Recall from Definition 1.2.10, the cover $\mathcal{A}$ of a set B as defined by Kelley in [12], as well as its corresponding open cover. A uniform space consists of a set X together with a collection $\mathcal{A}$ of covers of X, and if $(X, \mathcal{U})$ is the corresponding uniform space as given by Weil and Pontryagin, then $\mathcal{A}$ is the collection of all uniform covers.

Proposition 2.1.3. *[5] Let $(X, \mathcal{U})$ be a quasi-uniform space, let K be a compact subspace and let $\mathcal{C}$ be a family of open subsets of X such that $K \subset \bigcup \mathcal{C}$. Then there is an entourage V in $\mathcal{U}$ such that for each $x \in K, \exists C \in \mathcal{C}$ such that $V(x) \subset C$.*

Definition 2.1.4. [13][12][30] A filter $\mathcal{F}$ is an *ultrafilter* iff there is no strictly finer filter $\mathcal{G}$ than $\mathcal{F}$ (that is, iff it is properly contained in no filter in X). Thus the ultrafilters are the maximal filters.

If $\mathcal{F}$ is an ultrafilter in X and the union of two sets is a member of $\mathcal{F}$, then one of the two sets belongs to $\mathcal{F}$.

Definition 2.1.5. [11] Suppose $\mathcal{F}$ is a filter on X on a quasi-uniform space $(X, \mathcal{U})$. We describe $\mathcal{F}$ as a C-filter with respect to the quasi-uniformity $\mathcal{U}$ if $\mathcal{F}$ meets any of the conditions below:

(a) $\lim \mathcal{F} \neq \emptyset$;

(b) $\exists F \in \mathcal{F}$ such that $F \times F \subset U$ where $U \in \mathcal{U}$.

Obviously, when it comes to the case of the uniform space, the idea of C-filter and Cauchy filters are in fact similar. The concepts of C-completion, C-strong complete, C-strong completion and C-complete have definitions in precisely the same apparent manner, incorporating C-filter in Definition 1.2.18.

Theorem 2.1.6. *[13] Every filter $\mathcal{F}$ is contained in some ultrafilter.*

Theorem 2.1.7. *[11] A topological space $(X, \mathcal{T})$ is C-complete with respect to each compatible quasi-uniformity iff it is compact.*

We need to establish the following lemmas before we can prove Theorem 2.1.7.

Lemma 2.1.8. *Each C-filter in a compact quasi-uniform space $(X,\mathcal{U})$ converges.*

Proof. Let us say the C-filter F is non-convergent. x is not a limit point for F, where $x \in X$ and $X \neq \emptyset$. Hence, there exists $U_x \in \mathcal{U}$ such that $U_x(x) \notin F$. Let $V_x \in \mathcal{U}$ be such that $V_x \circ V_x \in U_x$ for every $x \in X$. Now $\{V_x(x) : x \in X\}$ is a neighbourhood covering of X. Thus, for $1 \leqslant i \leqslant k$, there are finitely many points $x_1, ..., x_k$ such that $X = \bigcup\{V_{x_i}(x_i)\}$. For $1 \leqslant i \leqslant k$, let $V = \bigcup\{V_{x_i}\}$. There is some set F$\in F$ such that F$\times$F $\subset V$ as a result of C-filter F being non-convergent. Without loss of generality, we let some p be a member of F. Then for some $1 \leqslant n \leqslant k, p \in V_{x_n}(x_n)$. Then let a be another arbitrary element of F. But then $(x_n, a) \in V_{x_n} \circ V_{x_n} U_{x_n}$ since F$\times$F $\subset V_{x_n}$. Hence F$\subset U_{x_n}(x_n)$ which yields the $U_{x_n}(x_n) \in F$; which is a contradiction.

□

Lemma 2.1.9. *Each ultrafilter is a C-filter for a totally bounded quasi-uniform space $(X,\mathcal{U})$.*

Proof. Let $U \in \mathcal{U}$ and F be an ultrafilter on X. For $\bigcup\{B_i : 1 \leqslant i \leqslant k\} = X$ and $1 \leqslant i \leqslant k$, there are finitely many sets $B_1, ..., B_k \in X$ such that $B_i \times B_i \subset U$ because of the total boundedness of the quasi-uniform space $(X,\mathcal{U})$. Again, we know that F is an ultrafilter, so for some $1 \leqslant j \leqslant k$, B_j is a member of F. □

It has already been mentioned in this chapter that in the uniform space case, the concepts of C-filter and Cauchy filter are precisely the same. It is therefore easy to show that if F is a filter that satisfies the first condition of the definition of C-filter, then adh $F = \lim F$; and therefore, if F is a C-filter such that adh $F \neq \emptyset$, then $\lim F \neq \emptyset$

Lemma 2.1.10. *All quasi-uniform spaces are generated by quasi-uniform structures that are totally bounded.*

Proof. By Theorem 1.3.9, Pervin structures are always totally bounded. □

Lemma 2.1.11. *A totally bounded and C-complete quasi-uniform space $(X,\mathcal{U})$ is compact.*

Proof. In $(X,\mathcal{U})$, suppose F is some ultrafilter. F is consequently a C-filter according to Lemma 2.1.9. F converges as a result of $(X,\mathcal{U})$ being C-complete. Therefore, F has an adherence point and clearly $(X,\mathcal{U})$ is compact.

□

Now we can prove Theorem 2.1.7

Proof. Suppose the topological space $(X, \mathcal{T})$ is compact. $(X,\mathcal{U})$ is C-complete provided that $\mathcal{U}$ is a compatible quasi-uniform structure, according to Lemma 2.1.8. Now we suppose that $(X, \mathcal{T})$ is C-complete with respect to each compatible quasi-uniform structure. Thus $(X, \mathcal{T})$ is compact by Lemma 2.1.11 and 2.1.10. □

Remark 2.1.12. Since all spaces that are finite have unique compatible quasi-uniform structures that are generated by unique sets, then we can conclude that all finite spaces are C-complete.

Definition 2.1.13. A topological space $(X, \mathcal{T})$ is a *transitive space* provided that the fine quasi-uniformity for $(X, \mathcal{T})$ is the fine transitive quasi-uniformity $\mathcal{FT}$.

Definition 2.1.14. $\mathcal{A}$ refines Q or is a Q-refinement iff for each $A \in \mathcal{A}$, there exists $\mathcal{Q} \in Q$ such that $A \subset \mathcal{Q}$

Proposition 2.1.15. *Let $(X, \mathcal{T})$ be an orthocompact space. Then $\mathcal{FT}$ contains every compatible quasi-uniformity for $(X, \mathcal{U})$.*

Proof. Let $\mathcal{U}$ be a compatible quasi-uniformity for the topological space $(X, \mathcal{T})$ and let $U \in \mathcal{U}$. Let W be an open symmetric entourage such that $W \circ W \subset U$, let $\mathcal{C} = \{W(x) : x \in X\}$ and let $\mathcal{R}$ be a Q-refinement of $\mathcal{C}$. Then, for each $x \in X, U_{\mathcal{R}}(x) \subset st(x, \mathcal{C}) \subset U(x)$. Thus $U_{\mathcal{R}} \subset U$ so that $\mathcal{U} \subset \mathcal{FT}$. □

Definition 2.1.16. [31] A family $\mathcal{U}$ of subsets of set X has *subinfinite rank* if whenever $\mathcal{V} \subset \mathcal{U}, \bigcap \mathcal{V} \neq \varnothing$, and $\mathcal{V}$ is infinite, then there are two distinct elements of $\mathcal{V}$, one of which is a subset of the other.

Definition 2.1.17. [32] In a quasi-uniform space $(X, \mathcal{U})$ where $U(x) = \{y \in X : (x, y) \in U\}$, a cover $\mathcal{C}^*$ is called a *strong quasi-uniform cover* for every $x \in X$, if there is some $U \in \mathcal{U}$ such that $U(x) \subseteq \bigcap\{H \in \mathcal{C}^* : x \in H\}$.

Definition 2.1.18. [33][30][5][32] A family $\mathcal{U}$ of open subsets of a space X is said to be *interior-preserving* if for every $\mathcal{U}' \subset \mathcal{U}$, the intersection $\bigcap \mathcal{U}$ is open in X.

Example 2.1.19. *It is obvious that $\{\mathcal{U}\}$ is a strong quasi-uniform cover of X for a quasi-uniform space $(X, \mathcal{U})$.*

The next lemma yields an example of a strong quasi-uniform cover that is non-trivial.

Lemma 2.1.20. *[28] Let B be a transitive member of $\mathcal{T}$. and let $\mathcal{U}$ be a quasi-uniformity of a quasi-uniform space $(X, \mathcal{U})$. It follows that $\{B(x) : x \in X\}$ is a strong quasi-uniform cover of X.*

Proof. Suppose $x \in X$ and that B is a transitive element of the quasi-uniformity $\mathcal{U}$. For some $y, z \in X$, $y \in B(x)$ and let $x \in B(z)$. So, $(z, x), (x, y) \in B$. Now we have $(z, y) \in B$, that is, $y \in B(z)$ since B is transitive. Therefore, $x \in B(z)\}$ and $x \in B(x) \subseteq \bigcap\{B(z) : z \in X\}$. Thus we conclude that $\{B(x) : x \in X\}$ is a strong quasi-uniform cover of X.

□

Definition 2.1.21. [32] For every $\mathcal{C} \in \mathcal{A}$, let $U_{\mathcal{C}} = \{y \in \bigcap\{C \in \mathcal{C} : x \in C\}$ and $(x, y) : x \in X\}$ and let $\mathcal{A}$ be a nonempty collection of covers of a set X. It can be verified that every such $U_{\mathcal{C}}$ is transitive and reflexive relation and that for a quasi-uniformity on X, the family $\{U_{\mathcal{C}} : \mathcal{C} \in \mathcal{A}\}$ is a subbase, which will be denoted $\mathcal{U}_{\mathcal{A}}$.

Theorem 2.1.22. *For a topological space $(X, \mathcal{T})$, these statements below are equivalent:*

(a) the topological space $(X, \mathcal{T})$ is orthocompact.

(b) $\mathcal{FT}$ is a Lebesgue quasi-uniformity.

(c) Every open cover of X has a precise interior-preserving open refinement.

(d) If $\mathcal{C}$ is an open cover of X, there is an open refinement $\mathcal{R}$ of $\mathcal{C}$ such that for each $x \in X$, either $\bigcap \mathcal{R}_X \in \mathcal{T}$ or $\mathcal{R}_X$ is the finite union of monotone collections.

(e) *If $\mathcal{C}$ is an open cover of X, there is an open refinement $\mathcal{R}$ of $\mathcal{C}$ such that for each $x \in X$, either $\bigcap \mathcal{R}_X \in \mathcal{T}$ or $\mathcal{R}_X$ is of subinfinite rank.*

Proof. $(a) \Rightarrow (b)$: Let $\mathcal{R}$ be an open interior-preserving refinement of $\mathcal{C}$ where $\mathcal{C}$ is an open cover of X. It follows that $U_{\mathcal{R}}$ is the required entourage.

$(b) \Rightarrow (c)$: Let $\mathcal{C} = \{C_\alpha | \alpha \in A\}$ be an open cover of X. Then there exists a positive entourage $U \in \mathcal{FT}$ such that $\{U(x)|x \in X\}$ refines $\mathcal{C}$. Define $f : X \to A$ by choosing an arbitrary α in A for every x in X such that $U(x) \subset C_\alpha$. For every $\alpha \in A$, let $R_\alpha = \bigcup\{U(x)|f(x) = \alpha\}$. Then $\mathcal{R} = \{R_\alpha|\alpha \in A\}$ is an open refinement of $\mathcal{C}$. For every $p \in X, U(p) \subset \bigcap \mathcal{R}_p$. Therefore $\mathcal{R}$ is an open interior-preserving refinement of $\mathcal{C}$ such that for every $\alpha \in A, R_\alpha \subset C_\alpha$.

$(c) \Rightarrow (d) \Rightarrow (e)$: Evident.

$(e) \Rightarrow (a)$: Let $\mathcal{C}$ be an open cover of X. Then there exists an open refinement $\mathcal{R}$ of $\mathcal{C}$ such that for each $x \in X$, either $\bigcap \mathcal{R}_X \in \mathcal{T}$ or $\mathcal{R}_X$ is of subinfinite rank. Well order $\mathcal{R}$ and for each $x \in X$ let R_X be the least number of $\mathcal{R}_X$. The family $\mathcal{R}_0 = \{R_X|x \in X\}$ is a subcover of $\mathcal{R}$ with property that each $R \in \mathcal{R}_0$ contains a point of X that belongs to no predecessor of R. Suppose that $\mathcal{R}_0$ is not interior deserving. Then there is an $x \in X$ such that $\mathcal{R}_1 = \{R \in \mathcal{R}_0 | x \in R\}$ is of subinfinite rank and $\bigcap \mathcal{R}_1 \notin \mathcal{T}$. Let R_1 denote the least member of $\mathcal{R}_1$ and let $\mathcal{R}_2 = \{R \in \mathcal{R}_1 | R_1 \text{ is not a subset of } R\}$. Then $\bigcap \mathcal{R}_1 = R_1 \cap (\bigcap \mathcal{R}_2)$. Thus $\bigcap \mathcal{R}_2 \notin \mathcal{T}$ and $\mathcal{R}_2$ is infinite. Inductively, let R_n denote the least member of $\mathcal{R}_n$ and set $\mathcal{R}_{n+1} = \{R \in \mathcal{R}_n | R_n \text{ is not a subset of } R\}$; then $\bigcap \mathcal{R}_n = R_n \cap (\bigcap \mathcal{R}_{n+1})$ and since $\bigcap \mathcal{R}_{n+1} \notin \mathcal{T}, \mathcal{R}_{n+1}$ is infinite. In this way we obtain an infinite subset $\{R_n | n \in N$ of $\mathcal{R}_0$ no two members of which are comparable - a contradiction. Thus $\mathcal{R}_0$ is an interior-preserving open refinement of $\mathcal{C}$. □

2.2 Open-Finite Covers

Our interest in open-finite covers stems from the lemma given below, which enables us to construct many examples of spaces that are not preorthocompact and to establish that a product of metrizable spaces is normal if and only if it is preorthocompact.

Definition 2.2.1. A covering $\{A_\alpha : \alpha \in \Lambda\}$ of a topological space $(X, \mathcal{T})$ is said to be *point-finite* provided there is at most finitely many indices $\alpha \in \Lambda$ such that $x \in A_\alpha$ for every $x \in X$.

Definition 2.2.2. If each open covering has open refinement that is point-finite, the topological space $(X, \mathcal{T})$ is reffered to as *metacompact*.

Definition 2.2.3. An open cover $\mathcal{C}$ of a space is *open finite* provided that no nonempty open set is a subset of infinitely many members of $\mathcal{C}$.

Definition 2.2.4. [34] If there exists a reflexive relation V on X so that, for every $x \in X, V(x)$ is open and $V \circ V(x)$ is a refiner of $\mathcal{C}$ for every open cover $\mathcal{C}$ of X, then the space X is said to be *preorthocompact*.

This is well explained by Fletcher in Corollary 2.2.5 below.

Corollary 2.2.5. *[34] A topological space X is preorthocompact if and only if for each cover $\mathcal{C}$ of X there is a neighbornet V of X such that $\{V^2(x) = V \circ V | x \in X\}$ refines $\mathcal{C}$.*

Definition 2.2.6. [5] If $\mathcal{C}$ and $\mathcal{R}$ are collections of subsets of a topological space X, we say that $\mathcal{R}$ is *cushioned in* $\mathcal{C}$ if one can assign to each $R \in \mathcal{R}$ a $C_R \in \mathcal{C}$ such that for every subcollection $\mathcal{R}'$ of $\mathcal{R}, \overline{\bigcup\{R|R \in \mathcal{R}'\}} \subset \bigcup\{C_R|R \in \mathcal{R}'\}$. A refinement of $\mathcal{C}$ that is cushioned in $\mathcal{C}$ is called a *cushioned refinement* of $\mathcal{C}$. We say that $\mathcal{R}$ is *cocushioned* in $\mathcal{C}$ if one can assign to each $R \in \mathcal{R}$ a $C_R \in \mathcal{C}$ such that for every subcollection $\mathcal{R}'$ of $\mathcal{R}, \bigcap\{R|R \in \mathcal{R}'\} \subset \text{int}\,(\bigcap\{C_R|R \in \mathcal{R}'\})$. A refinement of $\mathcal{C}$ that is cocushioned in $\mathcal{C}$ is called a *cocushioned refinement* of $\mathcal{C}$.

Lemma 2.2.7. *[5] Every open-finite cover of a preorthocompact space has a point-finite open refinement.*

Proof. Let X be a preorthocompact space and let $\mathcal{C}$ be an oGpen-finite cover of X. By Corollary 2.2.5 there is a neighbournet V of X such that $\{V^2(x)|x \in X\}$ refines $\mathcal{C}$. For each $C \in \mathcal{C}$, let $G(C) = int\{x|V(x) \subset C\}$ and let $\mathcal{G} = \{G(C)|C \in \mathcal{C}\}$. Let $p \in X$ and let $C \in \mathcal{C}$ such that $V^2(p) \subset C$. Then $p \in G(C)$ so that $\mathcal{G}$ is a cocushioned open refinement of $\mathcal{C}$. Since $\mathcal{C}$ is open finite, $\mathcal{G}$ is point finite. □

Definition 2.2.8. [6] For every $x \in X$, if $R_{\mathcal{C}}[x]$ is a $\mathcal{T}$-neighbourhood of x, then a cover $\mathcal{C}$ of $(X, \mathcal{T})$, a topological space, is referred to as an SN-cover of $(X, \mathcal{T})$.

Definition 2.2.9. Let $\mathcal{A}$ be a family of SN-covers of the topological space $(X, \mathcal{T})$ such that there are $\mathcal{C}_1, ..., \mathcal{C}_n$ in $\mathcal{A}$ such that $\bigcap_{i=1}^{n}(R_{\mathcal{C}_i}[x]) \subset A$ whenever $x \in A \in \mathcal{T}$. $\mathcal{A}$ is, therefore, said to be an *admissible family of covers*.

Lemma 2.2.10. *[6] $\mathcal{A}$ is an admissible family of Q-covers if it is the family of all open finite ($Q-$, locally finite, point finite) covers of the topological space $(X, \mathcal{T})$. We say $\mathcal{U}_A$ is the open finite (locally finite, point finite, $Q-$) covering quasi-uniformity for $(X, \mathcal{T})$.*

Proposition 2.2.11. *[5] Every closed subspace of a preorthocompact space is preorthocompact, and if each open space of a topological space X is preorthocompact then X is hereditarily preorthocompact.*

Proposition 2.2.12. *[5] Every countably orthocompact separable T_1-space is countably metacompact.*

Proof. Let $\{d_n|n \in \mathbb{N}\}$ be a countable dense subset of X where X is a countably orthocompact separable T_1-space, and let $\{G_n|n \in \mathbb{N}\}$ be a countable open cover of X. For each $n \in \mathbb{N}$, set $R_n = G_n - \{d_k|k \leqslant n$ and $d_k \in \cup_{i<n} G_i\}$. Then $\{R_n|n \in \mathbb{N}$ is an open-finite refinement of $\{G_n|n \in \mathbb{N}\}$, and so by the proof of Lemma 2.2.7 $\{G_n|n \in \mathbb{N}\}$ has a point-finite open refinement. □

Theorem 2.2.13. *[15] Let $(X, \mathcal{T})$ be a symmetric topological space. Now, the finite transitive, point-finite covering, locally finite covering, and the Pervin quasi-uniformities are each locally right symmetric.*

Theorem 2.2.14. *[5] Then the statements below are equivalent in a topological space $(X, \mathcal{T})$:*

(a) the topological space $(X, \mathcal{T})$ is metacompact.

(b) $(X, \mathcal{T})$ is preorthocompact and nearly metacompact.

(c) $(X, \mathcal{T})$ is preorthocompact and semi-metacompact.

Proof. Clearly, it is obvious that $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$ Let $\mathcal{C}$ be an open-finite refinement of V where V is a cover of X and let $(X, \mathcal{T})$ be a semi-metacompact preorthocompact space. By Lemma 2.2.7 there is a point-finite open refinement of $\mathcal{C}$. □

Proposition 2.2.15. *[5] If X is a preorthocompact space, that is the product of non-empty T_1 spaces, then all but countably many factors are countably compact.*

Proof. Suppose that $X = \Pi\{X_i|i \in I\}$ where for uncountably many i, X_i is not countably compact. Then X contains a closed copy of $\mathbb{N}^\omega$, the set of all ω-tuples (ocurrences) of ordered natural numbers, where $\omega \geqslant 0$. As $\mathbb{N}^\omega$ is separable but not normal, it is not metacompact. Since $\mathbb{N}^\omega$ has an open-finite base, it follows from Theorem 2.2.14 that $\mathbb{N}^\omega$ is not preorthocompact - a contradiction to Proposition 2.2.11. □

Corollary 2.2.16. *[5] Let X be a product of metrizable spaces. Then X is normal if and only if it is preorthocompact.*

Proof. It is well-known result of A. H. Stone that a product of metrizable spaces is normal if and only if all but countably many factors are compact. □

Lemma 2.2.17. *Let $\{0,1\}$ have the discrete topology and X be a topological space. Suppose that λ is a cardinal number such that $X \times \{0,1\}$ is preorthocompact; then each open cover of X having cardinality not greater than λ has a point-finite open refinement.*

Proof. Let $\mathcal{C} = \{C_\alpha | \alpha < \lambda\}$ be an open cover of X. For each $\alpha < \lambda$ and $i = 0,1$, set $D(\alpha, i) = C_\alpha \times \{y \in \{0,1\}^\lambda | y(a) = i; a \in \lambda\}$. Let $\mathcal{D} = \{D(\alpha, i) | \alpha < \lambda, i = 0,1\}$. Now $\mathcal{D}$ is an open-finite cover of $X \times \{0,1\}^\lambda$. Since $X \times \{0,1\}^\lambda$ is preorthocompact, from Lemma 2.2.7, it follows that $\mathcal{D}$ has a point-finite open refinement $\mathcal{R}$. Let $y \in \{0,1\}^\lambda$. Then $\{\pi_1(R \cap (X \times \{y\})) | R \in \mathcal{R}\}$ is a point-finite open refinement of $\mathcal{C}$. □

Definition 2.2.18. [35] A space X is *developable* iff there is a sequence $\{G_n\}_{n=1}^\infty$ of open covers of X such that for $x \in X$, $\{\operatorname{st}(x, G_n)\}_{n=1}^\infty$ is a local base at x. This sequence of covers is called *development*.

Definition 2.2.19. Let $\mathcal{B}$ be a base for $\mathcal{T}$ in a topological space $(X, \mathcal{T})$. Then $\mathcal{B}$ is a *uniform base* provided that if $U \in \mathcal{T}$ and $x \in U$ then there are at most finitely many members of $\mathcal{B}$ containing x that are not subsets of U. $(X, \mathcal{T})$ is metacompact and developable (as defined in 2.2.18) iff $\mathcal{T}$ has a uniform base.

Theorem 2.2.20. *[23] Every closed subset of an orthocompact space is orthocompact.*

Theorem 2.2.21. *[5] Let λ be a cardinal number and X be a T_1-space. Suppose that X^λ is preorthocompact.*

(a) If λ is infinite, then X is countably metacompact.

(b) If λ is uncountable, then X is countably compact.

(c) if λ is uncountable and the Lindelöf degree of X does not exceed λ, then X is compact.

Proof. (a) We assume, without loss of generality, that X has two points, which we denote by 0 and 1. Then the subspace $\{0,1\} \times \{0,1\}^{\omega_0}$ of X^λ is closed so that $\omega + 1 = \lambda$ and therefore $X \times \{0,1\}^{\omega_0}$ is preorthocompact. By Lemma 2.2.17 X is countably metacompact.

(b) This assertion is an immediate consequence of Proposition 2.2.15

(c) We have from (b) that X is countably compact so that by the well known theorem of Arens and Dugundji, it suffices to show that X is metacompact. As in the proof of assertion (a) we have that $X \times \{0,1\}^\lambda$ is preorthocompact so that by the previous lemma, each open cover of X having cardinality not greater than λ has a point-finite open refinement. Since the Lindelöf degree of X does not exceed λ, X is metacompact. □

Definition 2.2.22. [36] A topological space X is called *almost 2-fully normal* if the set of neighbourhoods of the diagonal of X is in fact a uniformity, that is, if $\mathcal{C}$ is an open cover of X then there exists an open refinement R of $\mathcal{C}$ such that if R_1 and $R_2 \in \mathcal{R}$ and $R_1 \cap R_2 \neq \emptyset$ then there is set $C \in \mathcal{C}$ such that $R_1 \cup R_2 \subset C$.

Theorem 2.2.23. *[5] Every almost 2-fully normal preorthocompact space is 2-fully normal.*

Proof. Let $(X, \mathcal{T})$ be an almost 2-fully normal preorthocompact space and let $\mathcal{C}$ be an open cover of X. By Corollary 2.2.5 there is a neighbornet R of X such that $\{R^2(x) | x \in X\}$ refines $\mathcal{C}$. Since $(X, \mathcal{T})$ is almost 2-fully normal, the open cover $\{int[R(x)] | x \in X\}$ has a refinement $\mathcal{G}$ with property that if x_1 and $x_2 \in X$ and $st(x_1, \mathcal{G}) \cap st(x_2, \mathcal{G}) \neq \emptyset$ then there exists an $x \in X$ such that $\{x_1, x_2\} \subset int[R(x)]$. For each $x \in X$ let $G(x) \in \mathcal{G}$ such that $x \in G(x)$ and set $H(x) = G(x) \cap int[R(x)]$. We assert that $\mathcal{H} = \{H(x) | x \in X\}$ is an open refinement of $\mathcal{C}$ that satisfies the requirements imposed by the definition of 2-full normality. To establish this assertion let x and y be members of X such that $H(x) \cap H(y) \neq \varnothing$. Since $G(x) \neq \varnothing$, there is some $z \in X$ such that $\{x, y\} \subset int[R(z)]$. Let $C \in \mathcal{C}$ such that $R^2(z) \subset C$. then $H(x) \cup H(y) \subset R(x) \cup R(y) \subset R^2(z) \subset C$.

□

Definition 2.2.24. [36] We call a subset A of a topological space X a *refiner* of a cover $\mathcal{D}$ of X, if A is a subset of some member of $\mathcal{D}$.

Lemma 2.2.25. *[36] A normal topological space X is almost 2-fully normal if and only if for every open cover $\mathcal{C}$ of X there is a locally finite open cover $\mathcal{D}$ of X such that every refiner of $\mathcal{D}$ with at most 2 elements is a refiner of $\mathcal{D}$.*

Proposition 2.2.26. *[34] Every almost 2-fully normal metacompact space is paracompact.*

Proof. Let X be an almost 2-fully normal nearly metacompact space and let $\mathcal{C}$ be an open cover of X. There is an open refinement $\mathcal{R}$ of $\mathcal{C}$ and a dense set D so that $\mathcal{R}$ is point finite on D. There exists a locally finite open cover $\mathcal{G}$ of X such that every two-element refiner of $\mathcal{G}$ is a refiner of $\mathcal{G}$. Let $G \in \mathcal{G}$. There exists $d \in G \bigcap D$. Let $x \in G$. Then there exists $R \in \mathcal{R}$ such that $\{x, d\} \subset R$. It follows that, for each $G \in \mathcal{G}$, there is a finite sub-collection $\mathcal{C}(G)$ of $\mathcal{C}$ so that $G \subset \bigcup \mathcal{C}(G)$. Thus $\{G \bigcap C : C \in \mathcal{C}(G) \text{ and } G \in \mathcal{G}\}$ is a locally finite open refinement of $\mathcal{C}$.

□

Definition 2.2.27. [37] The triple $(X, \mathcal{T}, \leqslant)$, where $(X, \leqslant)$ is linearly ordered (=totally ordered) set, and $\mathcal{T}$ is the order topology by the order $\leqslant$ is called a *linearly ordered topological space* (which is abbreviated as LOTS); that is, $\{(\alpha, +\infty), (-\infty, \alpha) : \alpha \in X\}$ is a subbase for $\mathcal{T}$, here $(\alpha, +\infty) = \{x \in X : x > \alpha\}, (-\infty, \alpha) = \{x \in X : x < \alpha\}$.

A space X is a *generalized ordered space* (abbreviated GO-space) if X is a subspace of a linearly ordered topological space Y, where the order of X is the one induced by the order of Y.

Theorem 2.2.28. *[5] Every GO space is orthocompact.*

Proof. Let $(X, \mathcal{T})$ be a GO space and let $\mathcal{C}$ be a cover of X by possibly degenerate open intervals. Let Y be a subset of X that is maximal with respect to the property that for each $y \in Y, st(y, \mathcal{C}) \cap Y = y$. For each $y \in Y$ set $L(y) = \{(\leftarrow, y) \cap C | C \in \mathcal{C}_y\}$, set $\mathcal{R}(y) = \{(y, \rightarrow) \cap C | C \in \mathcal{C}_y\}$, choose $C_y \in \mathcal{C}$, and set $A(y) = \{C_y\}$. Let $G = \bigcup\{L(y) \cup \mathcal{R}(y) \cup \mathcal{A}(y) | y \in Y\}$. it follows from the maximality of Y that G is an open refinement of $\mathcal{C}$. For each $x \in X$ there is at most one $y \in Y$ such that $y \leqslant x$ and $x \in st(y, \mathcal{C})$ and there is at most one $y \in Y$ such that $y \geqslant x$ and $x \in st(y, \mathcal{C})$. Therefore for each $x \in X, \mathcal{G}_X$ is the union at most four monotone subfamilies and so, by Theorem 2.1.22, X is orthocompact. □

2.3 Quasi-Uniform Covers

In Definition 1.2.4 of chapter 1, we discussed and defined a quasi-uniform space, built up from a quasi-uniformity. We also discussed a transitive set in Definition 1.3.1. In this section, we will start with the following proposition and definitions:

Definition 2.3.1. [38][5] A *quasi-proximity* for a set X is a relation δ in $\mathcal{P}(X)$ such that the following conditions hold:

(a) the set $X \not\delta \varnothing$ and $\varnothing \not\delta X$.

(b) $C\delta A \cup B$ iff $C\delta A$ or $C\delta B$.
$A \cup B\delta C$ if $A\delta C$ or $B\delta C$.

(c) $\{x\}\delta\{x\}$ for every $x \in X$.

(d) If $A \not\delta B$, there is some $C \subset X$ such that $X - C \not\delta B$ and $A \not\delta C$.

δ^{-1} is a quasi-proximity on X if δ is as well. If quasi-proximity $\delta = \delta^{-1}$, then δ is a *proximity*. The pair (X, δ) is referred to as a *quasi-proximity space*.

Proposition 2.3.2. *[5] In $(X, \mathcal{T})$, a topological space, let E be a family of functions that are lower semi-continuous on X such that there is some $f \in E$ such that $f(X - G) = 0$ and $f(x) = 1$ if $G \in \mathcal{T}$ and $x \in G$. Let $U_{(\epsilon,f)} = \{(x,y)|f(x) - f(y) < \epsilon\}$ be a subbase of the uniformity $\mathcal{U}_\delta$ where δ is the quasi-proximity on X and let $S = \{U_{(\epsilon,f)}|f \in E, \epsilon > 0\}$. Therefore, for quasi-uniformity that is compatible with $\mathcal{T}$, S is a subbase.*

Proof. For a quasi-uniformity $\mathcal{U}$ on X, S is a subbase since for $\epsilon > 0$ and every $f \in E, U_{(\frac{\epsilon}{2},f)} \circ (U_{\frac{\epsilon}{2},f)} \subset U_{(\epsilon,f)}$ and $\Delta \subset U_{(\epsilon,f)}$. Let $x \in X, \epsilon > 0$ and $f \in E$. So since f is lower semi-continuous, then $f^{-1}(f(x) - \epsilon, \infty) \in T$ and $U_{(\epsilon,f)}(x) = \{y|f(y) > f(x) - \epsilon\} = f^{-1}(f(x) - \epsilon, \infty)$. Therefore, $T(\mathcal{U}) \subset T$. Let $f \in E$ and $x \in G$ where $G \in T$, such that $f(X - G) = 0$ and $f(x) = 1$. Thus $x \in U_{(1,f)}(x) \subset G$ and hence $\mathcal{U}$ is compatible with T and $T \subset T(\mathcal{U})$.

□

Definition 2.3.3. In a topological space $(X, \mathcal{T})$, a *semi-continuous quasi-uniformity*, denoted SC, is the quasi-uniformity in Proposition 2.3.2 that is generated by making E a set of every lower semi-continuous function.

Lemma 2.3.4. *The statements below hold:*

(a) $A_1 \cap A_2$ is a relation that is transitive on X if A_1 and A_2 are relations that are transitive on a set X.

(b) $(f \times f)^{-1}(B)$ is a relation that is transitive on X if B is a relation that is transitive on Y and $f : X \to Y$ is a function.

Proof. (a) Let's say. Suppose $(x,z) \in (A_1 \cap A_2) \circ (A_1 \cap A_2)$. and let A_1 and A_2 be transitive. So there is some $y \in X$ such that $(x,y) \in (A_1 \cap A_2)$ and $(y,z) \in (A_1 \cap A_2)$. Then $(x,z), (y,z) \in A_1$ and therefore, $(x,z) \in A_1$ by transitivity of A_1. For the same argument, $(x,z) \in A_2$ by the transitivity of A_2. Hence, $(A_1 \cap A_2) \circ (A_1 \cap A_2) \subset (A_1 \cap A_2)$.

(b) Now let's say B is transitive and that $(x,z) \in (f \times f)^{-1}(B) \circ (f \times f)^{-1}(B)$. So there is some $y \in X$ such that $(y,z) \in (f \times f)^{-1}(B)$ and $(x,y) \in (f \times f)^{-1}(B)$. Therefore, $(f(y), f(z)) \in B$ and $(f(x), f(y)) \in B$. Hence, $(f(x), f(z)) \in B$, by the transitivity of B. So, $(x,z) \in (f \times f)^{-1}(B)$.

□

[6] For a quasi-uniformity, if every entourage in subbase $\mathcal{B}$ is transitive, then $\mathcal{B}$ is said to be transitive. A quasi-uniformity has a transitive base if it has a transitive subbase, by (a) above. A transitive quasi-uniformity is a quasi-uniformity with a transitive subbase.

Definition 2.3.5. A cover $\mathcal{C}$ is *directed* (under set inclusion) provided that the union of any two members of $\mathcal{C}$ is a subset of some member of $\mathcal{C}$, and we denote the collection of all finite unions of members of $\mathcal{C}$ as $\mathcal{C}^F$. In particular if $\mathcal{C}$ is a cover, then $\mathcal{C}^F$ is a directed cover and if $\mathcal{C}$ is a directed cover, then $\mathcal{C}^F$ refines $\mathcal{C}$.

Corollary 2.3.6. *[6][5] For every $U \in SC$ in a topological space $(X, \mathcal{T})$, there is some D, a countable subset of X, whereby $U(D)$ induced by the collection D of quasi-pseudometrics, is a quasi-uniformity such that $X = U(D)$.*

Theorem 2.3.7. *[5] For a completely regular topological space $(X, \mathcal{T})$ the following statements are equivalent:*

(a) $(X, \mathcal{T})$ is a Lindelöf space.

(b) *Every directed open cover of the Lebesgue uniformity $(X, C(X))$ is a uniform cover, where $C(X)$ or simply C denote the set of all real-valued continuous functions of X.*

(c) *Every directed open cover of (X, SC) is a quasi-uniform cover.*

Proof. (a) $\Rightarrow$ (b) Suppose that (X, T) is a Lindelöf space. In order to establish condition (b) it suffices to show that every countable directed open cover of X by cozero sets in a uniform cover. Leet $\mathcal{C} = \{C_n | n \in \mathbb{N}\}$ be such a cover and for each $n \in \mathbb{N}$, let $f_n : X \to [0, 1]$ be a continous function such that $c_n = X - f_n^{-1}(0)$. Without loss of generality, we assume that $C_n \subset C_{n+1}$ for each $n \in \mathbb{N}$. Define $g = \frac{1}{\sum_{i=1}^{\infty} \frac{f_i}{2^i}}$. Then g is continous function so that $V = \{(x, y) || g(x) - g(y)| \leqslant 1\} \in CX$. To show that $\mathcal{C}$ is a uniform cover, suppose there is a $p \in X$ such that for each $i \in \mathbb{N}, V(p) - C_i \neq \emptyset$. For each $i \in \mathbb{N}$ there is an $x_i \in V(p)$ such that $f_j(x_i) = 0$ for $1 \leqslant j \leqslant i$. Then for each $n \in \mathbb{N}, g(x_n) > 2^n$ and $g(p) \geqslant g(x_n) - 1$ - a contradiction.

(b) $\Rightarrow$ (c) This implication holds because $C(X) \subset SC$.

(c) $\Rightarrow$ (a) This implication follows from Corollary 2.3.6. □

Definition 2.3.8. Let $\mathcal{C}$ be a family of subsets of a set X and let the family of every finite union of elements of $\mathcal{C}$ be denoted by $\mathcal{C}^F$. A cover $\mathcal{C}$ is *directed* (under inclusion) provided that the union of any two members of $\mathcal{C}$ is a subset of some member of $\mathcal{C}$. In particular, if $\mathcal{C}$ is a cover, then $\mathcal{C}^F$ is a directed cover and if $\mathcal{C}$ is a directed cover, then $\mathcal{C}^F$ refines $\mathcal{C}$.

Corollary 2.3.9. *$\mathcal{SC}$ is a transitive quasi-uniformity in a topological space $(X, \mathcal{T})$.*

Definition 2.3.10. For a cover $\mathcal{C}$ of a set X, we define

$$R_{\mathcal{C}} = \{(x, y) \in X \times X : (\forall C \in \mathcal{C})(x \in C \Rightarrow y \in C)\} \tag{2.1}$$

For a reflexive relation R, the following defines a cover of X:

$$\mathcal{C}_R = \{R[x] : x \in X\} \tag{2.2}$$

Proposition 2.3.11. *[6] Let C_V be a cover of a set X and R be a reflexive relation on X. $R_{\mathcal{C}_V} = V$ provided that V is a transitive reflexive relation on X.*

Proof. Let $x \in X$ Then,

$$\begin{aligned} R_{\mathcal{C}_V} &= \{y : (\forall z \in X)(x \in V[z] \Rightarrow y \in [z])\} \\ &= \{y : (\forall z \in X)((z, x) \in V \Rightarrow (z, y) \in V)\} \\ &= \{y : (x, y) \in V\} = V[x]. \end{aligned} \tag{2.3}$$

□

Definition 2.3.12. If $\mathcal{C}_{R_{\mathcal{C}}} = \mathcal{C}$, then the cover $\mathcal{C}$ is said to be a *reduced cover* of a set X. .

Proposition 2.3.13. *On a set X, between reduced covers and reflexive transitive relations, there exists a one-to-one correspondence.*

Proof. $R_{\mathcal{C}_V} = V$ if V is reflexive and transitive. Therefore, $\mathcal{C}_{R_{\mathcal{C}_V}} = \mathcal{C}_V$ □

Definition 2.3.14. [6] Let a collection of covers of X be $\mathcal{A}$ and define $R_{\mathcal{C}} = \{(x, y) \in X \times X : (\forall C \in \mathcal{C})(x \in C \Rightarrow y \in C)\}$. Take $\mathcal{B} = \{R_{\mathcal{C}} : \mathcal{C} \in \mathcal{A}\}$. So $\mathcal{B}$ is a subbase for a *covering quasi-uniformity* $\mathcal{U}_{\mathcal{A}}$ on X, which is a transitive quasi-uniformity generated by $\mathcal{A}$.

Theorem 2.3.15. *[6] $\mathcal{U}$, a quasi-uniformity on X, is a covering quasi-uniformity iff it is transitive.*

Proof. Suppose $\mathcal{B}$ is a transitive base for B. Then, the collection $\{\mathcal{C}_B : B \in \mathcal{B}\}$ of covers is required, by Proposition 2.3.11. □

Definition 2.3.16. Let $\mathcal{F}$ be a filter on X, in quasi-uniform space $(X, \mathcal{U})$. Then $\mathcal{F}$ is a *weakly Cauchy* filter provided that for each $U \in \mathcal{U}$ there is a $p \in X$ such that for each $F \in \mathcal{F}$, $F \cap U(p) \neq \varnothing$

Proposition 2.3.17. *Each weakly Cauchy filter has a cluster point if and only if in a quasi-uniform space $(X, \mathcal{U})$, each directed open cover is a quasi-uniform cover.*

Proof. Suppose $\mathcal{C}$ is a directed open cover of X and that each weakly Cauchy filter contains a cluster point. We assume, without loss of generality, that $X \notin \mathcal{C}$. Then $\{X - C | C \in \mathcal{C}\}$ is a filter base of closed sets that has no cluster point. Consequently, there is a $U\mathcal{U}$ such that for each $x \in X$ there is a $C_x \in \mathcal{C}$ such that $U(x) \cap (X - C_x) = \varnothing$. Evidently $\{U(x) | x \in X\}$ refines $\mathcal{C}$.

Now suppose a weakly Cauchy filter $\mathcal{F}$ contains no cluster point and that each directed open cover of $(X, \mathcal{U})$ is a quasi-uniform cover. Then $\{X - \overline{F} | F \in \mathcal{F}\}$. There exists $p \in X$ such that for every $F \in \mathcal{F}, U(p) \cap F \neq \varnothing$ and there is some $F_p \in \mathcal{F}$ such that $U(p) \subset X - \overline{F}_p$ - a contradiction.

□

Corollary 2.3.18. *If every directed open cover of a quasi-uniform space $(X, \mathcal{U})$ is a quasi-uniform cover, then $(X, \mathcal{U})$ is complete.*

Theorem 2.3.19. *[5] Each open cover is a quasi-uniform cover iff the locally compact quasi-uniform space is uniformly locally compact.*

Proof. Suppose each directed open cover is a quasi-uniform cover in a locally compact quasi-uniform space $(X, \mathcal{U})$. Let $\mathcal{C} = \{N(x) | x \in X\}$ and $N(x)$ be an open neighbourhood of x which has a compact closure for every $x \in X$. So there is some entourage $U \in \mathcal{U}$ such that $\{U(x) | x \in X\}$ refines $\mathcal{C}^F$. Let $x \in X$. There exists some finite subset $\{y_i | 1 \leqslant i \leqslant n\}$ of X such that $U(x) \subset U_{i=1}^n N(y_i)$. Since $\overline{N(y_i)}$ is compact for each i with $1 \leqslant i \leqslant n, \overline{U(x)}$ is compact.

Now suppose that the quasi-uniform space $(X, \mathcal{U})$ is locally compact uniformly and that $\mathcal{F}$ is a weakly Cauchy filter on X. By Proposition 2.3.17 it suffices to show that $\mathcal{F}$ has a cluster point. Let $V \in \mathcal{U}$ such that for every $x \in X, \overline{V(x)}$ is compact. There is $p \in X$ such that for every $F \in \mathcal{F}, V(p) \cap F \neq \varnothing$. Let $F' = \{\overline{V(p)} \cap F | F \in \mathcal{F}\}$. Then F' is a filter on $\overline{V(p)}$ and $\mathcal{F}'$ contains a cluster point since $\overline{V(p)}$ is compact. Therefore, $\mathcal{F}$ contains a cluster point.

□

In Chapter 1, we defined Q-covers in 1.3.10. This remark follows:

Remark 2.3.20. $R_{\mathcal{C}}[x]$ is open for every $x \in X$ if $R_{\mathcal{C}}[x]$ is a neighbourhood of x for every $x \in X$, by the transitivity of $R_{\mathcal{C}}$. Therefore the Q-covers are simply the open SN-covers.

Theorem 2.3.21. *[6] Let $\mathcal{U}$ be a compatible quasi-uniformity for the topological space $(X, \mathcal{T})$. So, $\mathcal{U}$ is generated by an admissible family of reduced Q-covers of $(X, \mathcal{T})$ if and only if $\mathcal{U}$ is transitive quasi-uniformity.*

Proof. Suppose that $\mathcal{B}$ is a transitive base for $\mathcal{U}$ where for the topological space $(X, \mathcal{T}), \mathcal{U}$ is a compatible transitive quasi-uniformity and $B \in \mathcal{B}$. For every $x \in X$, $R_{\mathcal{C}_B}[x] = B[x]$ which is a neighbourhood of x, according to Proposition 2.3.11. Therefore, for every $B \in \mathcal{B}$, by Remark 2.3.20, $\mathcal{C}_B$ is a (reduced) Q-cover of X. Obviously, there is some $B \in \mathcal{B}$ such that $R_{\mathcal{C}_B}[x] \subset A$, for every $x \in A \in \mathcal{T}$. Hence, for $\mathcal{A} = \{\mathcal{C}_B : B \in \mathcal{B}\}$, $\mathcal{A}$ is an admissible family of covers. $R_{\mathcal{C}_B} = B$ for every $B \in \mathcal{B}$, by Proposition 2.3.11. Therefore $\mathcal{U} = \mathcal{U}_{\mathcal{A}}$.

□

3 Chapter 3

3.1 Completely Regular Quasi-Uniformizable Topological Spaces

In chapter 1, we defined and discussed regular spaces and completely regular spaces. In this section we will look into quasi-uniformizable topological spaces and how they are completely regular spaces and Hausdorff spaces.

Definition 3.1.1. [13] Let A and B be sets, A is said to be *equipotent* with B if and only iff there is a one-one function f from A onto B. Intuitively, equipotent sets have similar cardinal number. So we postulate the existence of sets, called *cardinal numbers*, chosen in a way that each set A is equipotent with exactly one cardinal number, called the *cardinal number of* A and denoted $|A|$.

Definition 3.1.2. The *transitivity degree* of a quasi-metrizable space X is the supremum of all $n \in \mathbb{N}$ such that X admits an effective n-transitive action.

Proposition 3.1.3. *[39] For each infinite cardinal β the exists a quasi-metrizable space X_β such that $tq(X_\beta) > \beta$, where $tq(X)$ denotes the transitivity degree of X.*

Definition 3.1.4. [26] The triple $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a *bitopological space* whereby $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies for set X.

We need to first look at the following definitions before concluding with the Remark 3.1.7 below.

Definition 3.1.5. [40] A filter $\mathcal{V}$ on $X \times X$ is a *semi-uniformity* on a set X if for every $V \in \mathcal{V}$,

(a) $\Delta(X) = \{\langle x, x\rangle : x \in X\} \subseteq V$ and

(b) $V^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in V\} \in \mathcal{V}$.

A base (subbase) for $\mathcal{V}$ is just a base (subbase) for $\mathcal{V}$ considered as a filter provided that on a set X, $\mathcal{V}$ is a semi-uniformity. The members of $\mathcal{V}$ are referred to as *semineighbourhoods of the diagonal in* $X \times X$. The pair $(X, \mathcal{V})$ is referred to as a *semi-uniform space*.

Definition 3.1.6. [40] A subbase $\mathcal{B}$ for a semi-uniformity on a set X is *locally uniform* iff for all $U \in \mathcal{B}$, and for all $x \in X$, there exist $V \in \mathcal{V} : (V \circ V)[x] \subseteq U[x]$. If $\mathcal{V}$ is a locally uniform semi-uniformity, then $\mathcal{V}$ is a *local uniformity*, and $(X, \mathcal{V})$ is a *locally uniform space*.

Remark 3.1.7. We note that for any infinite cardinal m every completely regular topological space that admits a local uniformity with a base of cardinality m also admits a uniformity with a base of cardinality m.

Definition 3.1.8. [41] Let $(X, \mathcal{T}, \mathcal{S})$ be a bitopological space consisting of a (non-empty) set X equipped with the two topologies $\mathcal{T}$ and $\mathcal{S}$. If there exists a quasi-uniformity $\mathcal{U}$ on X such that $\mathcal{T}(\mathcal{U}) = \mathcal{T}$ and $\mathcal{T}(\mathcal{U}^{-1}) = \mathcal{S}$, it said to be *completely regular*. (We can say that $\mathcal{U}$ is *compatible* with the topologies of X). If a completely regular bitopological space's finest compatible totally bounded quasi-uniformity is transitive, it is said to be *strongly zero-dimensional*. It is observed in that the finest compatible quasi-uniformity is transitive on a non-archimedeanly quasi-pseudo-metrizable bitopological space.

Definition 3.1.9. [5][42][18] Let $(X, \mathcal{V})$ and $(X,\mathcal{U})$ be quasi-uniform spaces. A function $f : X \to Y$ is *quasi-uniformly continuous* provided for every $V \in \mathcal{V}$, whenever $(x, y) \in U$, there is some $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$; that is for every $V \in \mathcal{V}$, $f_2^{-1}[V] \in \mathcal{U}$. Now, let $g : Y \to Z$ and $f : X \to Y$. Since $(g \circ f)_2 = g_2 \circ f_2$, the composition of two quasi-uniformly continuous functions is quasi-uniformly continuous. If f and f^{-1} are quasi-uniformly continuous, the bijection $f : X \to Y$ is a *quasi-unimorphism*.

Definition 3.1.10. [5][18] Let $(X,\mathcal{U})$ be a T_1 quasi-uniform space. A complete T_1 quasi-uniform space $(Y, \mathcal{V})$ which possesses a dense subspace quasi-unimorphic (with respect to $\mathcal{U}$ and $\mathcal{V}$) to $(X,\mathcal{U})$ is called a *completion* of $(X,\mathcal{U})$.

Definition 3.1.11. [26] Let $\mathcal{P}$ be the quasi-uniformity on X for which $\{(A \times A) \cup [(X - A) \times X] : A \in \mathcal{T}\}$ is a subbase, in a topological space $(X, \mathcal{T})$. Then $\mathcal{P}$ is the *Pervin quasi-uniformity* for $(X, \mathcal{T})$.

The next construction proves that each quasi-uniform space posseses a simple completion.
For $\beta \notin X$, suppose that $X^* = X \cup \{\beta\}$ in the quasi-uniform space $(X,\mathcal{U})$ and that $S(U) = U \cup \{(\beta, x) : x \in X^*\}$ where $U \in \mathcal{U}$. So for $\mathcal{U}^*$, a quasi-uniform structure, $\mathcal{B} = \{S(U) : U \in \mathcal{U}\}$ forms its base. It should be noted that if $x \in X$, for every $S(U)[x] = U[x]$ and $U \in \mathcal{U}$, $S(U)[\beta] = X^*$. Evidently, on X^* each filter $\mathcal{F}$ converges to β. As a result, $(X^*,\mathcal{U}^*)$ is strongly complete. Furthermore, there is a quasi-uniform isomorphism $i : (X,\mathcal{U}) \to (X^*,\mathcal{U}^*)$ and $\mathcal{U} = \{U^* \cap X \times X : U^* \in \mathcal{U}^*\}$. The quasi-uniform space $(X^*,\mathcal{U}^*)$ is a completion of $(X,\mathcal{U})$, from the fact that X is dense in X^*. Actually, $\mathcal{T}_{\mathcal{U}^*} = \mathcal{T}_{\mathcal{U}} \cup \{X^*\}$ and X^* is compact.
Now suppose some Cauchy filter $\Lambda = \{\hat{\mathcal{F}} : \mathcal{F}$ is nonconvergent on $(X,\mathcal{U})\}$. So we obtain a strong completion of $(X,\mathcal{U})$, using the same construction, which we denote $(\mathring{X},\mathring{\mathcal{U}})$. It is obvious that $(\mathring{X},\mathring{\mathcal{U}})$ is, in general, not the trivial strong completion that was constructed above.

Lemma 3.1.12. *[18] Let $\mathcal{S}$ be a subbase for $\mathcal{U}$ in a quasi-uniform space $(X,\mathcal{U})$. Then we have $S[x] \in \mathcal{F}$ for every $S \in \mathcal{S}$ iff filter $\mathcal{F}$ converges to x.*

For this construction, we let $\mathcal{T}$ generate the Pervin quasi-uniform structure $\mathcal{U}$. Now, let $\mathcal{U} = \{U^* \cap X \times X : U^* \in \mathcal{U}^*\}$, let $(X^*,\mathcal{U}^*)$ be a completion for the quasi-uniform space $(X,\mathcal{U})$ and let $\overline{X} = X^*$. Suppose $\mathcal{F}$ is a filter on X. So there is some $\mathcal{M}$, an ultrafilter on X, such that $\mathcal{M} \supset \mathcal{F}$. $\mathcal{M}$ is a Cauchy filter as a result of $\mathcal{U}$ being pre-compact. Let $\mathcal{M}$ generate the ultrafilter $\widetilde{\mathcal{M}}$ on X^*. $U^* \cap X \times X = U \in \mathcal{U}$ if $U^* \in \mathcal{U}^*$ so that there is some $x \in X$ such that $U[x] \in \mathcal{M}$. Obviously, $U^*[x] \in \widetilde{\mathcal{M}}$ and therefore, $\widetilde{\mathcal{M}}$ is Cauchy. But then $(X^*,\mathcal{U}^*)$ is complete, so $\widetilde{\mathcal{M}}$ converges to x^* for some $x^* \in X^*$.

Suppose two Cauchy ultrafilters $\mathcal{M}_1$ and $\mathcal{M}_2$ on X are equivalent on condition $U[x] \in \mathcal{M}_2$ iff $U[x] \in \mathcal{M}_1$. On the set of every Cauchy ultrafilter on X, this is definitely an equivalence relation. Now, let the equivalence class that has M be denoted by $\hat{\mathcal{M}}$, let $X^* = X \cup \Lambda$ and let $\Lambda = \{\hat{\mathcal{M}} : \mathcal{M}$ is a Cauchy ultrafilter that is convergent on $X\}$. For $V \in \mathcal{B}$, we will denote the set of every mapping δ from Λ to X by $D(V)$ such that $y \in V[\delta(\hat{\mathcal{M}})]\}$, $S(V,\delta) = V \cup \Delta \cup \{(\hat{\mathcal{M}}, y) : \hat{\mathcal{M}} \in \Lambda$ and $V[\delta(\hat{\mathcal{M}})] \in \mathcal{M}$.

Theorem 3.1.13. *[18] $(X^*, \mathcal{U}^*)$ is complete.*

Proof. Suppose $\mathcal{F}$ is a Cauchy filter on X^*. For $\mathcal{M}$ being an ultrafilter, we have $\mathcal{F} \subset \mathcal{M}$. $\mathcal{U}^*$ is complete and $\mathcal{F}$ contains an adherent point whenever $\mathcal{M}$ converges. Let's say $\mathcal{M}$ is not convergent. We want to prove that $x \in \mathcal{M}$. Suppose $S(U,\delta) \in \mathcal{U}^*$. Then, there is some $x^* \in X^*$ such that $S(U,\delta)[x^*] \in \mathcal{M}$ given that $\mathcal{M}$ is Cauchy. If $x^* = \hat{\mathcal{M}}_1 \in \Lambda$, we have

$$S(U,\delta)[x^*] = \{\hat{\mathcal{M}}_1\} \cup U[\delta(\hat{\mathcal{M}}_1)]. \tag{3.1}$$

$\mathcal{M}$ being nonconverge to $\hat{\mathcal{M}}_1$ suggests that there is some $S(V,\delta) \in \mathcal{S}^*$ such that $S(V,\delta)[\hat{\mathcal{M}}_1] \notin \mathcal{M}$. Furthermore, there is some $z^* \neq \hat{\mathcal{M}}_1$ such that $S(V,\gamma)[z^*] \in \mathcal{M}$ given the fact that $\mathcal{M}$ is Cauchy. So, there is nothing to show if $z^* \in X$; so we let $z^* = \hat{\mathcal{M}}_2 \in \Lambda$. Now, we have $\hat{\mathcal{M}}_1 \notin \hat{\mathcal{M}}_2$ and

$$(V[\gamma(\hat{\mathcal{M}}_2)] \cup \{\hat{\mathcal{M}}_2\}) \cap (U[\delta(\hat{\mathcal{M}}_1)] \cup \{\hat{\mathcal{M}}_1\}) \in \mathcal{M}. \tag{3.2}$$

Therefore, $X \in \mathcal{M}$, from the fact that

$$X \supset V[\gamma(\hat{\mathcal{M}}_2)] \cap U[\delta(\hat{\mathcal{M}}_1)] \in \mathcal{M}. \tag{3.3}$$

Let the ultrafilter $\mathcal{M}_0 = \{M \in \mathcal{M} : M \supset X\}$ on X. We prove that $\mathcal{M}_0$ is $\mathcal{U}$-Cauchy. For $V \subset U$, there is some $V \in \mathcal{B}$ provided $U \in \mathcal{U}$. Set $\delta \in D(V)$. So there is some $x^* \in X^*$ such that $S(V,\delta)[x^*] \in \mathcal{M}$ since $S(V,\delta) \in \mathcal{U}^*$. Now, $U[x^*] = \mathcal{M}_0$ since $V[x^*] \in \mathcal{M}$ provided $x^* \in X$. More so, $V[\delta(\underline{\hat{\mathcal{M}}})] \cup \{\underline{\hat{\mathcal{M}}}\} \in \mathcal{M}$ provided $x^* = \underline{\hat{\mathcal{M}}} \in \Lambda$ and the fact that $X \in \mathcal{M}$ means $V[\delta(\underline{\hat{\mathcal{M}}})] \in \mathcal{M}$. As a result, $U[\delta(\underline{\hat{\mathcal{M}}})] \in \mathcal{M}_0$. As a consequence, $\mathcal{M}_0$ is Cauchy on X. There are two cases: $\mathcal{M}_0$ is either convergent or is not, on X.
Case (1) affirmative: $\mathcal{M}_0$ converges to x. Set $S(V,\delta) \in \mathcal{S}^*$ so that $S(V,\delta)[x] = V[x] \in \mathcal{M}_0$. As a result, $S(V,\delta)[x] \in \mathcal{M}$ which yields the convergence of $\mathcal{M}$ to x, a contradiction.
Case (2) negative: $\mathcal{M}_0$ is not convergent on X. Consequently, $\hat{\mathcal{M}}_0 \in \Lambda$ so we prove that $\mathcal{M}$ converges to $\hat{\mathcal{M}}_0$. Now, we have

$$S(V,\delta)[\hat{\mathcal{M}}_0] = V[\delta(\hat{\mathcal{M}}_0)] \cup \{\hat{\mathcal{M}}_0\}, \text{provided} S(V,\delta) \in \mathcal{S}^*. \tag{3.4}$$

So, $V[\delta(\hat{\mathcal{M}})_0] \in \mathcal{M}_0$. Thus $S(V,\delta)[\hat{\mathcal{M}}_0] \in \mathcal{M}$. Hence, $\mathcal{M}$ converges to $\hat{\mathcal{M}}_0$, a contradiction. □

Definition 3.1.14. [2] If quasi-uniformity $\mathcal{U}$ is such that $\mathcal{U} = \mathcal{U}^{-1}$, then then *uniformity* $\mathcal{U}$ is said to be *symmetric*.

Definition 3.1.15. [5] $(X, \delta_{\mathcal{U}})$ is *point-symmetric* in a quasi-uniform space $(X, \mathcal{U})$ provided for every $x \in X$ and $U \in \mathcal{U}$, there is some symmetric $V \in \mathcal{U}$ such that $V(x) \subset U(x)$; equivalently, if for every $x \in X$ and $U \in \mathcal{U}$, there is some $V \in \mathcal{U}$ such that $V^{-1}(x) \subset U(x)$.

Corollary 3.1.16. *Every point-symmetric T_1 quasi-uniform space has a completion.*

Proposition 3.1.17. *Let every quasi-uniformity in the topological space $(X, \mathcal{T})$ compatible with $\mathcal{T}$ have a completion. Then $(X, \mathcal{T})$ is compact.*

Theorem 3.1.18. *[18] $(X^*, \mathcal{U}^*)$ is a completion for the quasi-uniform space $(X, \mathcal{U})$.*

Proof. Clearly, $(X^*, \mathcal{U}^*)$ is complete, by Theorem 3.1.13. Set $U^* \in \mathcal{U}$ and $\hat{\mathcal{M}} \in \Lambda$. So, there is some $S(V_1, \delta_1), ..., S(V_n, \delta_n) \in \mathcal{S}$ such that $\bigcap_{i=1}^{n} S(V_i, \delta_i) \subset U^*$. We have

$$S(V_i, \delta_i)[\hat{\mathcal{M}}] = \{\hat{\mathcal{M}}\} \cup V_i[\delta_i(\hat{\mathcal{M}})] : i = 1, 2, ..., n. \tag{3.5}$$

Now, $\bigcap_{i=1}^{n} V_i[\delta_i(\hat{\mathcal{M}})] \neq \varnothing$, for every $i = 1, 2, ..., n$ from the fact that $V_i[\delta_i(\hat{\mathcal{M}})] \in \mathcal{M}$. So we have

$$\begin{aligned} U^*[\hat{\mathcal{M}}] \cap X &\supset \left(\bigcap_{i=1}^{n} S(V_i, \delta_i) \right) [\hat{\mathcal{M}}] \cap X \\ &= \bigcap_{i=1}^{n} S(V_i, \delta_i)[\hat{\mathcal{M}}] \cap X \\ &= \bigcap_{i=1}^{n} V_i[\delta_i(\hat{\mathcal{M}})] \neq \varnothing. \end{aligned} \tag{3.6}$$

Hence, $\overline{X} = X^*$ and $\hat{\mathcal{M}} \in \overline{X}$. Let the induced quasi-uniform structure of $\mathcal{U}^*$, be denoted by $\mathcal{U}'$, on X. For $V \subset U$, there is some $V \in \mathcal{B}$, provided $U \in \mathcal{U}$. Suppose $\delta \in D(V)$, so that $V = S(V, \delta) \cap X \times X$ and $S(V, \delta) \in \mathcal{U}^*$. Thus $V \in \mathcal{U}'$ and hence $U \in \mathcal{U}'$. Let $U \in \mathcal{U}'$ so that there is some $U^* \in \mathcal{U}^*$ such that $U = U^* \cap X \times X$. There is some $S(V_1, \delta_1), ..., S(V_n, \delta_n) \in \mathcal{S}^*$ such that $U^* \supset \bigcap_{i=1}^{n} S(V_i, \delta_i)$ from the fact that $U^* \in \mathcal{U}^*$. As a result,

$$\left(\bigcap_{i=1}^{n} S(V_i, \delta_i) \right) \cap X \times X \subset U^* \cap X \times X = U. \tag{3.7}$$

Thus, $\bigcap_{i=1}^{n} V_i \subset U$ and therefore $U \in \mathcal{U}$. Hence we have proven that $\mathcal{U} = \mathcal{U}'$. Consequently, it can be concluded that the identity mapping given by $i : (X, \mathcal{U}) \to (X, \mathcal{U}^* \cap X \times X)$ is quasi-uniform isomorphism. □

Corollary 3.1.19. *[5] Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. A quasi-uniformity $\mathcal{W}$ compatible with $\mathcal{T}$ contains a uniformity compatible with $\mathcal{T}$ iff $\mathcal{W}$ is locally symmetric.*

Corollary 3.1.20. *For every $x \in X$ and every $U \in \mathcal{U}$, there is a symmetric $V \in \mathcal{U}$ such that $V \circ V(x) \subset U(x)$.(It is not asserted that $V \circ V \subset U$.)*

Theorem 3.1.21. *[8] Let $(X, \mathcal{U})$ be a quasi-uniform space which satisfies Corollary 3.1.20. Then X is a regular topological space.*

Proof. Let G be any open set that has $x \in X$. So there is some symmetric $V \in \mathcal{U}$ such that $V \circ V(x) \subset G$ by Corollary 3.1.20. Let $z \in \overline{V(x)} - V(x)$. So there is some $y \in X$ such that $y \in V(x) \cap V(z)$. But then V is symmetric, $z \in V(y) \subset V \circ V(x) \subset G$. So $\overline{V(x)} \subset G$ and X is regular. □

Theorem 3.1.22. *[18] The following are properties of $(X^*, \mathcal{U}^*)$ that can be verified easily:*

(a) $(X, \mathcal{U})$ is completely regular provided $(X^*, \mathcal{U}^*)$ is pre-compact and Hausdorff.

(b) the set X is a dense open subset of X^*

(c) $(X, \mathcal{U})$ has property P provided P is an open hereditary property and $(X^*, \mathcal{U}^*)$ has property P.

(d) $(X^*, \mathcal{U}^*)$ is T_0 iff $(X, \mathcal{U})$ is T_0

(e) the subspace topology on Λ is the discrete topology and Λ is closed in X^*.

Apparently the properties hold for $(\mathring{X}, \mathring{\mathcal{U}})$ as well.

Proof. (a) $(X^*, \mathcal{U}^*)$ must be a compact Hausdorff space since it is complete and provided it is Hausdorff and pre-compact. Hence $(X, \mathcal{U})$, the subspace, must be completely regular.

(b) $\overline{X} = X^*$, by Theorem 3.1.18. Let $S(U, \delta) \in \mathcal{U}^*$ and $x \in X$. Then $S(U, \delta)[x] = U[x] \subset X$. Thus X is open in X^*.

(c) Clearly it is a result of (b)

(d) The sufficiency is apparent from the fact that T_0 is a hereditary property. Let $(X, \mathcal{U})$ be T_0, so that given that X is open in X^* it is sufficient to consider the case where $x^*, \hat{\mathcal{M}} \in X^*, x^* \neq \hat{\mathcal{M}} \in \Lambda$. Suppose $S(U, \delta) \in \mathcal{U}^*$ so that $\hat{\mathcal{M}} \notin S(U, \delta)[x^*]$. Therefore X^* is T_0.

(e) By (b), Λ is closed in X^*, and for any $S(U, \delta) \in \mathcal{U}^*$ it follows that $\hat{\mathcal{M}} = \Lambda \cap S(U, \delta)[\hat{\mathcal{M}}]$ for any $\hat{\mathcal{M}} \in \Lambda$. Thus the subspace topology on Λ is discrete.

□

3.2 Completion, completeness and Quasi-Pseudometric Spaces

In this chapter, we defined a completion of a quasi-uniform space and further constructed an example that proved that a quasi-uniform space has a simple completion. In this section, we will further our discussion on how the completion is unique up to isomorphism. We further discussed how quasi-uniform structures generate discrete topologies and what makes them compatible with topologies. Now let us consider the following definition.

Definition 3.2.1. [18] Two quasi-uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are said to be *quasi-uniformly isomorphic* relative to $\mathcal{U}$ and $\mathcal{V}$ if there exists a one-to-one mapping f of X onto Y such that f and f^{-1} are quasi-uniformly continuous.

Definition 3.2.2. [43] A quasi-uniform space $(X, \mathcal{U})$ can be *embedded* in a quasi-uniform space $(Y, \mathcal{V})$ if and only if there exists a quasi-uniform isomorphism from $(X, \mathcal{U})$ onto a subspace of $(Y, \mathcal{V})$.

Definition 3.2.3. [6][5] On a set X, a *quasi-pseudometric* d is a function from $X \times X$ into the set of positive real numbers such that for $x, y, z \in X$:

(a) $d(x, y) \leqq d(x, z) + d(z, y)$ and

(b) $d(x, x) = 0$.

The pair (X, d) is called a *quasi-pseudometric space*.

Remark 3.2.4. [43] Clearly, for a filter $\mathcal{F}$ in a quasi-pseudometric space (X, d) and for every $\epsilon > 0$, there is some $x \in X$ such that $S_\epsilon(x) = \{y \in S : d(x, y) < \epsilon\} \in \mathcal{F}$ iff the filter $\mathcal{F}$ is Cauchy, and that (X, d) is a quasi-uniform space.

Theorem 3.2.5. *[43] Let $\{U_n : n = 0, 1, ...\}$ be sequence of subsets of $X \times X$ such that $U_0 = X \times X, U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ for each n, and each U_n contains the diagonal. Then there is a quasi-pseudometric d on X such that $U_n \subseteq \{(x, y) : d(x, y) < 2^{-n}\} \subseteq U_{n-1}$ for each positive integer n.*

A quasi-uniformity of a set X may be derived from a collection P of quasi-pseudometrics. For p in P, let $U_{p,r} = \{(x, y) : p(x, y) < r\}$. The collection of every set of the form $U_{p,r}$, for $r > 0$ and $p \in P$, is a subbase for a quasi-uniformity $\mathcal{U}$ on X. The quasi-uniformity $\mathcal{U}$ is called the quasi-uniformity generated by P. Each quasi-uniformity for a set X is generated by a family of quasi-pseudometrics. This serves, together with Theorem 3.4.1, as proof for the following theorem.

Theorem 3.2.6. *[43] Each quasi-uniform space $(X, \mathcal{U})$ can be embedded in a product of quasi-pseudometric spaces.*

Definition 3.2.7. [6] A family $\mathcal{D}$ of quasi-pseudometrics *generates* a quasi-uniformity $\mathcal{U}_d$ if the sets $\{(x,y) : d(x,y) < r\}, d \in \mathcal{D}$ and $r > 0$, form a subbase for $\mathcal{U}$.

Proposition 3.2.8. *[22] Let $\{x_n\}$ be a sequence of points in the uniform space X. If $\{x_n\}$ converges in the uniform topology, then $\{x_n\}$ is a Cauchy sequence.*

Proposition 3.2.9. *Let $\mathcal{F}$ be a filter on the uniform space X. If $\mathcal{F}$ converges in the uniform topology, then $\mathcal{F}$ is a Cauchy filter.*

Definition 3.2.10. A uniform space M is called *sequentially complete* if every Cauchy sequence of points in M converges.

Theorem 3.2.11. *A complete uniform space is sequentially complete.*

Theorem 3.2.12. *[44] Every quasi-pseudometric space (X, d) has has a completion (X^*, d^*) which is a quasi-pseudometric space.*

Proof. We may and do assume that $d(x,y) \leqq 1$ for all $x, y \in X$. Let $\overline{X} = \{\overline{x} : \overline{x} = \{x_n : n \in \mathbb{N}\}$ is a d-Cauchy sequence in $X\}$.

Define:

$$\overline{d} : \overline{X} \times \overline{X} \to \mathbb{R} : \begin{cases} \overline{d}(\overline{x}, \overline{y}) = 0 & \text{if } \overline{y} \text{ is a subsequence of } \overline{x}, \\ \overline{d}(\overline{x}, \overline{y}) = \lim_n \inf \lim_m \sup d(x_n, y_m) & \text{otherwise.} \end{cases} \tag{3.8}$$

□

Before we define bicompleteness, lets look at the proposition.

Proposition 3.2.13. *[5] Let the quasi-uniform space $(X, \mathcal{U})$ be locally symmetric. Then in $\mathcal{F}$, a $\mathcal{U}$-Cauchy filter, each cluster point is a limit point of $\mathcal{F}$.*

Proof. Suppose that in $\mathcal{F}$, a $\mathcal{U}$-Cauchy filter, p is a cluster point, and that $U \in \mathcal{U}$. Let V denote a symmetric entourage such that $V^3(p) \subset U(p)$. Then there is some $x \in X$ such that $V(x) \in \mathcal{F}$, from the fact that the filter $\mathcal{F}$ is Cauchy. We have $V(x) \cap V(p) \neq \emptyset$ from the fact that p is a cluster point of $\mathcal{F}$. Therefore $V(x) \subset U(p)$ so that $\mathcal{F}$ converges to p.

□

Definition 3.2.14. [41][5] A quasi-uniform space $(X, \mathcal{U})$ is referred to as *bicomplete* if each $\mathcal{U}^*$-Cauchy filter has a $T(\mathcal{U}^*)$-limit point. By Proposition 3.2.13, a quasi-uniform space $(X, \mathcal{U})$ is called bicomplete if the uniformity $\mathcal{U}^*$ is complete. Therefore $(X, \mathcal{U}^{-1})$ is bicomplete iff $(X, \mathcal{U})$ is bicomplete.

Proposition 3.2.15. *[41] A topological space admits a bicomplete quasi-uniformity if and only if its fine quasi-uniformity is bicomplete.*

Proposition 3.2.16. *The fine quasi-uniformity of any quasi-pseudo-metrizable space is bicomplete.*

Lemma 3.2.17. *Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Let $S \subseteq X$ be dense in X. If Y is complete, then any uniformly continuous map $f : S \to Y$ has a unique continuous extension to X.*

Definition 3.2.18. [42] A quasi-uniform space $(X, \mathcal{U})$ is said to be *half complete* if every $\mathcal{U}^*$-Cauchy filter is $\mathcal{T}(\mathcal{U})$-convergent. A T_1 quasi-uniform space $(Y, \mathcal{V})$ is called a T_1*half completion* of $(X, \mathcal{U})$ if $(Y, \mathcal{V})$ is half complete and $(X, \mathcal{U})$ is quasi-isomorphic to a $\mathcal{T}(\mathcal{V})$-dense subspace of Y. $(X, \mathcal{U})$, quasi-uniform space, is described as being T_1*half completable* if it admits a T_1 half completion.

Remark 3.2.19. The topological space $(X, \mathcal{T}(\mathcal{V}))$ is T_1 provided that $(X,\mathcal{U})$ is a T_1 half completable quasi-uniform space.

Definition 3.2.20. [42] A T_1 quasi-uniform space $(Y, \mathcal{V})$ is referred to as a T_1*-*half completion* of a quasi-uniform space $(X,\mathcal{U})$ provided that $(X,\mathcal{U})$ is quasi-isomorphic to a $\mathcal{T}(\mathcal{V}^*)$-dense subspace of $(Y, \mathcal{V})$ and that $(Y, \mathcal{V})$ is half complete. If a quasi-uniform space has a T_1*-half completion, it is said to be T_1*-half completable.

Proposition 3.2.21. *[42] If a quasi-uniform space $(X,\mathcal{U})$ has a T_1*-half completion $(Y, \mathcal{V})$, then the bicompletion of $(Y, \mathcal{V})$ is quasi-isomorphic to the bicompletion of $(X,\mathcal{U})$.*

Proof. Let $(\widetilde{Y}, \widetilde{\mathcal{V}})$ be the bicompletion of $(Y, \mathcal{V})$. Clearly $(X,\mathcal{U})$ is quasi-isomorphic to a $\mathcal{T}(\widetilde{\mathcal{V}}^*)$-dense subspace of $(\widetilde{Y}, \widetilde{\mathcal{V}})$. Therefore $(\widetilde{Y}, \widetilde{\mathcal{V}})$ is a T_0 bicompletion of $(X,\mathcal{U})$. So $(\widetilde{Y}, \widetilde{\mathcal{V}})$ is quasi-isomorphic to the bicompletion of $(X,\mathcal{U})$. □

Let $(X,\mathcal{U})$ be a T_0 quasi-uniform space and $(\widetilde{X},\widetilde{\mathcal{U}})$ its bicompletion. We will denote by $G(X)$ the set of closed points of $(\widetilde{X}, T(\widetilde{\mathcal{U}}))$. Clearly $G(X) = \widetilde{X}$ whenever $(\widetilde{X},\widetilde{\mathcal{U}})$ is a T_1 quasi-uniform space.

Definition 3.2.22. [5] On $(X,\mathcal{U})$, a quasi-uniform space, a $\mathcal{U}^*$-Cauchy filter is said to be *minimal* if it does not contain any $\mathcal{U}^*$-Cauchy filter other than itself.

Proposition 3.2.23. *[5] On $(X,\mathcal{U})$, a quasi-uniform space, let $\mathcal{F}$ be a $\mathcal{U}^*$-Cauchy filter. There is precisely one minimal $\mathcal{U}^*$-Cauchy filter which is couser than $\mathcal{F}$. More so, $\mathcal{B}_0 = \{U(B)|B \in \mathcal{B}$ and U is a symmetric member of $\mathcal{U}^*\}$ is a base for the minimal $\mathcal{U}^*$-Cauchy filter coarser than $\mathcal{F}$ provided that $\mathcal{B}$ is any base for $\mathcal{F}$.*

Corollary 3.2.24. *For every $x \in X$, let $\eta^*(x)$ denote $\mathcal{T}(\mathcal{U}^*)$-neighbourhood filter of x in a quasi-uniform space $(X,\mathcal{U})$. Then $\eta^*(x)$ is a minimal $\mathcal{U}^*$ filter.*

Theorem 3.2.25. *[5] Let $(X,\mathcal{U})$ be a quasi-uniform space. Then each minimal $\mathcal{U}^*$-Cauchy filter has some base which has $\mathcal{T}(\mathcal{U})^*$-open sets.*

Proof. Suppose $V \in \mathcal{U}$. Then there exists a symmetric entourage $U \in \mathcal{U}^*$ such that $U(x) \in \mathcal{T}(\mathcal{U}^*)$ for every $x \in X$, and that $U \subset V$ by Corollary 1.2.6. $U(A)$ is a $\mathcal{T}(\mathcal{U}^*)$-open subset of $V(A)$, for every subset A of X; therefore, by Proposition 3.2.23, the result follows.

□

[41] Next is a construction of the bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ of a quasi-uniform space $(X,\mathcal{U})$. Let $(X,\mathcal{U})$ be a quasi-uniform space. By $\widetilde{X}$ we denote the set of all minimal $\mathcal{U}^*$-Cauchy filters on X. Moreover, let $\widetilde{\mathcal{U}}$ be the quasi-uniformity on $\widetilde{X}$ that is generated by all sets $\widetilde{U}$ where U belongs to $\mathcal{U}$. Here $\widetilde{U} = \{(\mathcal{F}, \mathcal{G}) \in \widetilde{X} \times \widetilde{X}$: there is some $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $F \times G \subseteq U\}$. Often a $\mathcal{T}(\mathcal{U}^*)$-convergent minimal $\mathcal{U}^*$-Cauchy filter $\eta^*(x) \in \widetilde{X}$ is identified with its limit point in $(X,\mathcal{U}^*)$ and using this identification $(X,\mathcal{U})$ is considered a subspace of $(\widetilde{X},\widetilde{\mathcal{U}})$.

Proposition 3.2.26. *[5] Let $(X,\mathcal{U})$ be a T_0 quasi-uniform space consisting of a bicomplete subspace $(Y, \mathcal{V})$. Then Y is a closed subspace of $(X, \mathcal{T}(\mathcal{U}^*))$.*

For the next Theorem, we will let $(\widetilde{X},\widetilde{\mathcal{U}})$ be a bicompletion of $(X,\mathcal{U})$, where $b(X,\mathcal{U})$ is a T_0 quasi-uniform space.

Theorem 3.2.27. *[42] Let $(X,\mathcal{U})$ be a T_1*-half completable quasi-uniform space. Then any T_1*-half completion of $(X,\mathcal{U})$ is quasi-isomorphic to $(G(X),\widetilde{\mathcal{U}}|G(X))$. Hence it is unique up to quasi-isomorphism. Moreover, $(G(X),\widetilde{\mathcal{U}}|G(X))$ is the uniform completion of $(X,\mathcal{U})$ provided $\mathcal{U}$ is a uniformity.*

Proof. Let $(Y, \mathcal{V})$ be a T_1*-half completion of $(X, \mathcal{U})$. By Proposition 3.2.21, it follows that $(\widetilde{X}, \widetilde{\mathcal{U}})$ is quasi-isomorphic to $(\widetilde{Y}, \widetilde{\mathcal{V}})$, and hence $(G(X), \widetilde{\mathcal{U}}|G(X))$ is quasi-isomorphic to $(G(Y), \widetilde{\mathcal{V}}|G(Y))$ is quasi-isomorphic to $(Y, \mathcal{V})$. More so, $G(X) = \widetilde{(X)}$, so $(G(X), \widetilde{\mathcal{U}}|G(X))$ is the uniform completion of $(X, \mathcal{U})$ provided $\mathcal{U}$ is a uniformity on X. □

Definition 3.2.28. [5] In $(X, \mathcal{U})$, a quasi-uniform space, *bicompletion* $(Y, \mathcal{V})$ is a bicomplete quasi-uniform space that has a $T(V^*)$ dense subspace quasi-unimorphic to $(X, \mathcal{U})$. The main idea towards the bicompletion of a quasi-uniform space construction is the utilisation of minimal $\mathcal{U}^*$-Cauchy filters. On $(X, \mathcal{U})$ the quasi-uniform space, if a $\mathcal{U}$-Cauchy filter does not contains any $\mathcal{U}$-Cauchy filter besides itself, it is called a *minimal*.

Proposition 3.2.29. *[5] Let $\mathcal{D}$ be a dense subset of $(X, \mathcal{T}(\mathcal{U}^*))$ in a quasi-uniform space $(X, \mathcal{U})$. $(X, \mathcal{U})$ is bicomplete provided that each Cauchy filter on $(D, \mathcal{U}^*|D \times D)$ converges in $(X, \mathcal{T}(\mathcal{U}^*))$.*

Proof. It sufficient to prove that on X, each minimal $\mathcal{U}^*$-Cauchy filter $\mathcal{F}$ converges. Each element of $\mathcal{F}$ has a nonempty interior, by Theorem 3.2.25. So, $\mathcal{F}|D$ is a Cauchy filter on $(D, \mathcal{U}^*|D \times D)$ since D is dense in $(X, \mathcal{T}(\mathcal{U}^*))$. Therefore, it is convergent in $(x, \mathcal{T}(\mathcal{U})^*)$. Since $\mathcal{F}$ is coarser than the filter on X, determined by $\mathcal{F}|D$, it can be concluded that $\mathcal{F}$ converges, by Proposition 3.2.13.

□

Definition 3.2.30. [42][5] A *compactification* of a T_1 quasi-uniform space $(X, \mathcal{U})$ is a compact T_1 quasi-uniform space $(Y, \mathcal{V})$ that has $\mathcal{T}(\mathcal{V}^*)$-dense subspace quasi-isomorphic to $(X, \mathcal{U})$. A compactification of a Tychonoff quasi-uniform space $(X, \mathcal{U})$ is a compact Hausdorff quasi uniform space $(Y, \mathcal{V})$ that has a dense subspace quasi-unimorphic to $(X, \mathcal{U})$.

Definition 3.2.31. [42] We say that T_1 quasi-uniform space $(X, \mathcal{U})$ is *compatifiable* if it has a compactification.

Clearly, $(Y, \mathcal{V})$ is a T_1 *-half completion of $(X, \mathcal{U})$ provided that $(X, \mathcal{U})$ has a compactification $(Y, \mathcal{V})$.

Corollary 3.2.32. *[42] If $(X, \mathcal{U})$, a T_1 quasi-uniform space, has a compactification, then any compactification of $(X, \mathcal{U})$ is quasi-isomorphic to $(G(X), \widetilde{\mathcal{U}}|G(X))$. Hence, it is unique up to quasi-isomorphism.*

Proof. Let $(Y, \mathcal{V})$ be a compactification of $(X, \mathcal{U})$. Then $(Y, \mathcal{Y})$ is a T_1*-half completion of $(X, \mathcal{U})$. By Theorem 3.2.27 $(Y, \mathcal{V})$ is quasi-isomorphic to $(G(X), \widetilde{U}|G(X))$, and thus it is unique up to quasi-isomorphism. □

Theorem 3.2.33. *[5] $\mathcal{V}$ consists of a uniformity that is compatible with $\mathcal{T}(\mathcal{V})$ iff $(X, \mathcal{V})$, a Tychonoff quasi-uniform space that is totally bounded contains a compactification.*

3.3 Quasi-Uniform Structures and Symmetric Quasi-Uniformities

In Definition 1.3.10, we introduced quasi-uniform structures, gave an example of their base and noted that they generate discrete topologies.

[15][18] Each quasi-uniform structure generates a topology $\mathcal{T}(\mathcal{U}) = \{O \subset X :$ if $x \in O$, then there exists $U \in \mathcal{U}$ such that $U[x] \subset O\}$. It follows that $\overline{A} = \bigcap\{U^{-1}[A] : U \in \mathcal{U}\}$ and $\text{int}(A) = \{x :$ there exists $U \in \mathcal{U}$ such that $U[x] \subset A\}$. A quasi-uniform structure $\mathcal{U}$ is compatible with a topology $\mathcal{T}$ on X provided $\mathcal{T} = \mathcal{T}(\mathcal{U})$. Suppose $\mathcal{T}_{\mathcal{U}} = \{A \subset X :$ if $a \in A$, there exists $U \in \mathcal{U}$ such that $U[a] \subset A\}$ provided that for a set $X, \mathcal{U}$ is a quasi-uniform structure. Thus, generated by $\mathcal{U}$ on X, $\mathcal{T}_{\mathcal{U}}$ is the *quasi-uniform topology*.

Definition 3.3.1. Let $(X, \mathcal{U})$ be a quasi-uniform space.

(a) $(X, \mathcal{U})$ is called *locally left symmetric* if for each $x \in X$ and $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^{-1}(V[x]) \subset U[x]$.

(b) $(X, \mathcal{U})$ is called *locally right symmetric* if for each $x \in X$ and $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V(V^{-1}[x]) \subset U[x]$.

Lemma 3.3.2. *[15] Let $(X, \mathcal{U})$ be a quasi-uniform space.*

(a) If $(X, \mathcal{U})$ is locally left symmetric, then $\mathcal{T}(\mathcal{U})$ is symmetric.

(b) If $(X, \mathcal{U})$ is locally right symmetric, then $\mathcal{T}(\mathcal{U})$ is symmetric.

Theorem 3.3.3. *Let $(X, \mathcal{T})$ be a symmetric topological space. Then the fine transitive, point-finite covering, locally finite covering, and the Pervin quasi-uniformities are each locally right symmetric.*

Proposition 3.3.4. *[12] A non-void collection $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity for X iff*

(a) each member of $\mathcal{B}$ contains the diagonal Δ;

(b) if $U \in \mathcal{B}$, then U^{-1} contains a member of $\mathcal{B}$

(c) if $U \in \mathcal{B}$, then $V \circ V \subset U$ for some V in $\mathcal{B}$; and

(d) the intersection of two members of $\mathcal{B}$ contains a member.

Then this result follows:

Theorem 3.3.5. *[12] A collection $\mathcal{S}$ of subsets of $X \times X$ is a subbase for some uniformity for X if*

(a) every element of $\mathcal{S}$ has the diagonal Δ,

(b) the set U^{-1} has a member of $\mathcal{S}$ for each $U \in \mathcal{S}$, and

(c) $\exists V \in \mathcal{S}$ such that $V \circ V \subset U$, for every $U \in \mathcal{S}$.

In particular, the union of any families of uniformities for X is the subbase for a uniform for X.

Proof. It must be shown that the collection $\mathcal{B}$ of finite intersections of members of $\mathcal{S}$ satisfies Proposition 3.3.4. This is an apparent consequence of the observation: If $V = \bigcap\{V_i : i = 1, ..., n\}$, if $U = \bigcap\{U_i : i = 1, ..., n\}$ and if $V_1, ..., V_n$ and $U_1, ...U_n$ are subsets of $X \times X$, then for each i, whenever $V_i \subset U_i^{-1}$ (respectively, $V_i \circ V_i \subset U_i$), $V \subset U^{-1}$ (or $V \circ V \subset U$). □

For the uniform space $(X, \mathcal{U})$, the topology $\mathcal{T}$ of the uniformity $\mathcal{U}$, or the uniform topology, is the collection of all subsets T of X such that for each x in T there is U in $\mathcal{U}$ such that $U(x) \subset T$. This is precisely the generalisation of the metric topology, which is the family of all sets which contain a sphere about each point.

Theorem 3.3.6. *[12] The interior of a subset A of X relative to the uniform topology is the set of all points x such that $U[x] \subset A$ for some U in $\mathcal{U}$.*

Proof. It must be shown that the set $B = \{x : U[x] \subset A$ for some U in $\mathcal{U}\}$ is open relative to the uniform topology, for B surely contains every open subset of A and, if B is open, then it must necessarily be the interior of A. If $x \in B$, then there is a member U of $\mathcal{U}$ such that $U[x] \subset A$ and there is V in $\mathcal{U}$ such that $V \circ V \subset U$. If $y \in V[x]$, then $V[y] \subset V \circ V[x] \subset U[x] \subset A$, and hence $y \in B$. Hence $V[x] \subset B$ and B is open. □

Lemma 3.3.7. *Suppose that for a quasi-uniformity $\mathcal{U}$ for E, $\mathcal{S}$ is a subbase. Then by Theorem 3.3.5, $\mathcal{U}$ generates a topology $\mathcal{T}_{\mathcal{U}}$ consisting of all subsets G of E such that there is some $U \in \mathcal{U}$ such that $U[x] \subset G$, for every $x \in G$.*

Definition 3.3.8. A topological space X is said to be *uniformizable* if there is a uniformity $\mathcal{E}$ on X such that the topology of this uniformity coincides with the topology of X.

Definition 3.3.9. [22] A topological space X is called *completely uniformizable* if there exists a uniformity in which X is complete, and which induces the topology of X.

Theorem 3.3.10. *[22] (*Shirota's Theorem*) A topological space X is completely uniformizable iff:*

(a) X is Tychonoff

(b) Every closed discrete subspace of X has nonmeasurable cardinal, and

(c) X is realcompact.

Theorem 3.3.11. *A topological space is completely regular iff it is uniformizable .*

Theorem 3.3.12. *Every topological space is quasi-uniformizable.*

We can prove this theorem using Lemma 3.3.13 below:

Lemma 3.3.13. *[8] Every topological space $(X, \mathcal{T})$ has a quasi-uniformity $\mathcal{U}$ which induces the original topology $\mathcal{T}$; that is, $\mathcal{T}_{\mathcal{U}} = \mathcal{T}$.*

Proof. If $G \in \mathcal{T}$ then for every $x \in G, S_G[x] = G$ and so G will be a $\mathcal{T}_{\mathcal{U}}$-neighbourhood of x; i.e., $G \in \mathcal{T}_{\mathcal{U}}$. Now suppose $G \in \mathcal{T}_{\mathcal{U}}$. Then for every $x \in G, G$ is a $\mathcal{T}_{\mathcal{U}}$-neighbourhood of x and so there exists a $U \in \mathcal{U}$ such that $U[x] \in G$. Now by definition of $\mathcal{U}, U \supset S_{G_1}[x] \cap ... \cap S_{G_n}[x]$ and this intersection certainly contains x and is open since $S_{G_i}[x]$ is either equal to G_i or E and both are open. Thus G is a $\mathcal{T}$-neighbourhood of x and $G \in \mathcal{T}$. □

Definition 3.3.14. [18] A quasi-uniform space $(X,\mathcal{U})$ is said to have property P if each $U \in \mathcal{U}$ is a neighbourhood of Δ in $X \times X$ with respect to the product topology.

Definition 3.3.15. For a quasi-uniform space $(X,\mathcal{U})$, we say that $(X,\mathcal{U})$ has property S if for every $x \in X$, the family $\{V[x] : V \in \mathcal{U}, V \text{ is symmetric}\}$ forms a fundamental system of neighbourhoods for x with respect to the topology generated by $\mathcal{U}$.

Corollary 3.3.16. *[18] For a quasi-uniform space $(X,\mathcal{U}), \mathcal{U}^{-1}$ is compatible with a uniform structure $\mathcal{U}$ on a set X, provided that $\mathcal{U}$ satisfies properties P and S.*

Corollary 3.3.17. *[8] If $(X,\mathcal{U})$ is a regular quasi-uniform space, then there is some symmetric $V \in \mathcal{U}$ such that $V \circ V \circ V(x) \subset U(x)$ for every $x \in X$ and for every $U \in \mathcal{U}$.*

Theorem 3.3.18. *[18] $\mathcal{U}^{-1}$ satisfies property P for a quasi-uniform space $(X,\mathcal{U})$ that satisfies the properties S and P.*

Proof. Let $U^{-1} \in \mathcal{U}^{-1}$. Then there is some $V(x) \in \mathcal{U}$ such that $V(x)[x] \times v(x)[x] \subset U$, for every $x \in X$, from the fact that $\mathcal{U}$ has property P. Therefore, for every $x \in X$, $V(x)[x] \times v(x)[x] \subset U^{-1}$. There is some symmetric $T(x) \in \mathcal{U}$ such that $T(x)[x] :\subset V(x)[x]$, by property S for each $x \in X$. Since $T(x)$ is symmetric, it follows that $T(x) \in \mathcal{U}^{-1}$ for each $x \in X$. Hence $\bigcap\{T(x)[x] \times T(x)[x] : x \in X\} \subset \bigcap\{V(x)[x] \times V(x)[x] : x \in X\} \subset U^{-1}$. Thus U^{-1} is a neighbourhood of Λ with respect to the product conjugate topology and consequently, $\mathcal{U}^{-1}$ has property P. □

Proposition 3.3.19. *[23] Let $(X, \mathcal{T})$ be an orthocompat space. Then $\mathcal{F}\mathcal{T}$ contains every compatible uniformity for $(X, \mathcal{T})$.*

Proof. Let $U \in \mathcal{U}$ where $\mathcal{U}$ is a compatible uniformity for $(X, \mathcal{T})$. Let W be an open symmetric entourage such that $W \circ W \subset U$, let $\mathcal{C} = \{W(x) : x \in X\}$ and let $\mathcal{R}$ be a $\mathcal{Q}$-refinement of $\mathcal{C}$. Then for all $x \in X, U_{\mathcal{R}}(x) \subset \text{st}\,(x, \mathcal{C}) \subset U(x)$. Thus $U_{\mathcal{R}} \subset U$ so that $\mathcal{U} \subset \mathcal{F}\mathcal{T}$.

□

Corollary 3.3.20. *Every open symmetric uniformly regular totally bounded quasi-uniformity is a uniformity.*

Definition 3.3.21. [32] $(X, \mathcal{U})$, a quasi-uniform space, is called a *quiet* quasi-uniform space provided that the next property holds: for any $U \in \mathcal{U}$, there is some $V \in \mathcal{U}$ such that, if $x', x'' \in X$ and $\{x'_\alpha : \alpha \in A\}$ and $\{x''_\beta : \beta \in B\}$ are two nets in X, then from $(x', x'_\alpha) \in V$ for $\alpha \in A$, $(x''_\beta, x'') \in V$ for $\beta \in B$ and $(x''_\beta, x'_\alpha) \to 0$ [i.e. for any $W \in \mathcal{U}$, there is some $\alpha_W \in A$ and $\beta_W \in B$ such that $(x''_\beta, x'_\alpha) \in W$ for $\alpha \geqslant \alpha_W, \beta \geqslant \beta_W$] then $(x', x'') \in U$. When a $V \in \mathcal{U}$ is connected with some $U \in \mathcal{U}$ by the above property, V is said to be *$\mathcal{Q}$-subordinated* to U.

Theorem 3.3.22. *[32] Let $\mathscr{C}$ be the family of all strong quasi-uniform covers of X and $(X, \mathcal{V})$ be a quiet-uniform space. Then the properties below hold:*

(a) If $\mathcal{U} \in \mathscr{C}$ and $\mathcal{U}_1$ is a cover of X such that for each $x \in X$, $\bigcap\{H \in \mathcal{U} : x \in H\} \subseteq \bigcap\{K \in \mathcal{U}_1 : x \in K\}$, then $\mathcal{U}_1 \in \mathcal{C}$.

(b) If $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{C}$, then there is a $\mathcal{U} \in \mathscr{C}$ such that $\bigcap\{H \in \mathcal{U} : x \in H\} \subseteq \bigcap\{K \in \mathcal{U}_1 : x \in K\}$ and $\bigcap\{H \in \mathcal{U} : x \in H\} \subseteq \bigcap\{K \in \mathcal{U}_2 : x \in K\}$, for each $x \in X$.

(c) For $\mathcal{U} \in \mathscr{C}$, if $\{x_\alpha : \alpha \in A\}$ and $y_\beta : \beta \in B$ be two nets in $(X, \mathcal{V})$ such that $x_\alpha \in \bigcap\{H \in \mathcal{U} : x \in H\}$, $y_\beta \in \bigcap\{K \in \mathcal{U} : y \in K\}$ for $\alpha \in A, \beta \in B$ and $(y_\beta, x_\alpha) \to 0$, then $y \in \bigcap\{H \in \mathcal{U} : x \in H\}$.

Conversely, suppose $\mathscr{C}$ is the family of all covers of set X satisfying conditions (a), (b) and (c). Now, on X, there exists $\mathcal{V}$, a quiet quasi-uniformity with respect to $\mathscr{C}$ is exactly the family of all strong quasi-uniform covers of X.

Theorem 3.3.23. *[12] The family of closed symmetric members of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$.*

Proof. If $U \in \mathcal{U}$ and V is a member of $\mathcal{U}$ such that $V \circ V \circ V \subset U$, then $V \circ V \circ V$ contains the closure of V in view of the preceding theorem; hence U contains a closed member of W of $\mathcal{U}$ and $W \cap W^{-1}$ is a closed symmetric member. □

Theorem 3.3.24. *[28] For a quasi-uniform space $(X, \mathcal{T})$, every strong quasi-uniform cover is a $\mu(\mathcal{T})$-open cover.*

Proof. Let $\mathcal{A}$ be a strong quasi-uniform cover of X. So, there exists $T \in \mathcal{T}$ such that for every $x \in X$, $x \in T(x) \subseteq \bigcap \mathcal{A}_x$. Let $x \in A$ and $A \in \mathcal{A}$. So we have $x \in U(x) \subseteq \bigcap \mathcal{A}_x \subseteq A$. Thus A is $\mu(\mathcal{T})$-open, by Theorem 1.3.23. □

Theorem 3.3.25. *[28] With a compatible quasi-uniformity, let (X, μ) be a strong topological space having $\mathcal{B}$, a transitive base . Then $\{B(x) : B \in \mathcal{B}, x \in X\}$ is a base for μ.*

Proof. It follows from Theorem 3.3.24 that $B(x)$ is μ-open for every $x \in X$ and $B \in \mathcal{B}$. So let $x \in G$ and $G \in \mu$. As $\mathcal{B}$ is compatible with μ, $\exists B \in \mathcal{B}$ such that $x \in B(x) \subseteq G$. Thus, $\{B(x) : x \in X, B \in \mathcal{B}\}$ is a base for μ. □

3.4 Applications of Strong Quasi-uniform covers

Theorem 3.4.1. *[28] Let $\mathscr{C}$ be a family of all strong-quasi uniform covers of X in the quasi-uniform space $(X, \mathcal{U})$. Now, $C \in \mathscr{C}$ provided C is a cover of X such that for every $x \in X$, $\bigcap \mathcal{C}_x \subseteq \bigcap C_x$, and that $\mathcal{C} \in \mathscr{C}$.*

Conversely, let $\mathscr{C}$ be a collection of covers of X satisfying the aforesaid condition where X is a non-empty set. So $\{U_\mathcal{C} : \mathcal{C} \in \mathscr{C}\}$ is a transitive base for say $\mathcal{U}_\mathscr{C}$, a quasi-uniformity, on X, with respect to which $\mathscr{C}$ is exactly the family of all strong quasi-uniform covers of X.

Proof. Let $\mathscr{C}$ be the family of all strong quasi-uniform covers of $(X,\mathcal{U})$, a quasi-uniform space. Now, there is some $U \in \mathcal{U}$ such that for every $x \in X, x \in U(x) \subseteq \bigcap \mathcal{C}_x$ holds, since $\mathcal{C} \in \mathscr{C}$. More so, C is a cover of X such that $\bigcap \mathcal{C}_x \subseteq \bigcap C_x$ for every $x \in X$. Therefore, $x \in U(x) \bigcap C_x$ for every $x \in X$. Thus $C \in \mathscr{C}$.

Conversely, le $\mathscr{C}$ be a family of covers of X, such that the aforementioned conditions hold. Then, obviously, for each $\mathcal{C} \in \mathscr{C}$, $\Delta \subseteq U_{\mathcal{C}}$. Now, let $(x,y) \in U_{\mathcal{C}} \circ U_{\mathcal{C}}$ and $\mathcal{C} \in \mathscr{C}$. Then there exists $z \in X$ such that $(z,y),(x,z) \in U_{\mathcal{C}}$. So $y \in \bigcap \mathcal{C}_z$ and $z \in \bigcap \mathcal{C}_x$. Therefore, $(x,y) \in U_{\mathcal{C}}$, that is, $y \in \bigcap \mathcal{C}_x$. Consequently, $U_{\mathcal{C}} \circ U_{\mathcal{C}} = U_{\mathcal{C}}$.

Thus, for all $\mathcal{C} \in \mathscr{C}$, $U_{\mathcal{C}}$ is transitive. Hence, for $\mathcal{U}_{\mathscr{C}}$, a quasi-uniformity on X, $\{U_{\mathcal{C}} : \mathcal{C} \in \mathscr{C}\}$ is a transitive base.

The remainder of the proof serves to show that for the space $(X,\mathcal{U}_{\mathscr{C}})$, the set of strong quasi-uniform covers coincides with $\mathscr{C}$. Then let $C \in \mathscr{C}$. So, for every $x \in X, U_C(x) = D_x$ that is, $x \in U_C(x) \subseteq \bigcap C_x$, for every $x \in X$. Therefore, for $(X,\mathcal{U}_{\mathscr{C}})$, C is a strong quasi-uniform cover. Now, for $(X,\mathcal{U}_{\mathscr{C}})$, let $\mathfrak{C}$ be a strong quasi-uniform cover. Then there is some $\mathcal{C} \in \mathscr{C}$ such that $x \in U_{\mathcal{C}}(x) \subseteq \bigcap \mathfrak{C}_x$, for every $x \in X$ that is, for every $x \in X, \bigcap \mathcal{C}_x \subseteq \bigcap \mathfrak{C}_x$. Thus, $\mathfrak{C} \in \mathscr{C}$ according to the assumed condition. Hence the required outcome.

□

Theorem 3.4.2. *[32] A topological space $(X,\mathcal{T})$ is metacompact if and only if for every open cover $\mathcal{C}$ of $(X,\mathcal{T})$, there is some strong quasi-uniform cover $\mathcal{C}^*$ of $(X,\mathcal{U}_{\mathcal{A}})$ such that $\mathcal{C}^*$ refines $\mathcal{C}$, where $\mathcal{A}$ is the collection of all point-finite open covers of X.*

Proof. Suppose that $\mathcal{C}$ is an open cover of $(X,\mathcal{T})$ where $(X,\mathcal{T})$ is metacompact. Then there is a point-finite open refinement $\mathcal{C}^*$ of $\mathcal{C}$. So, $\mathcal{C}^*$ is a point-finite open cover of $X, U_{\mathcal{C}*} \in \mathcal{U}_{\mathcal{A}}$ where $y \in \cap\{C \in \mathcal{C} : x \in C\}\}$ and $U_{\mathcal{C}*} = \{(x,y) \in X \times X : x \in X$. Then for every $x \in X, U_{\mathcal{C}*}(x) = \cap\{C \in \mathcal{C}^* : x \in C\}$. Thus, for $(X,\mathcal{U}_{\mathcal{A}}), \mathcal{C}^*$ is a strong quasi-uniform cover.

Conversely, suppose $\mathcal{C}$ is an open cover of $(X,\mathcal{T})$. For $(X,\mathcal{U}_{\mathcal{A}})$, there exists a strong quasi-uniform cover $\mathcal{C}^*$ such that $\mathcal{C}^*$ refines $\mathcal{C}$, by the aforesaid condition. Since $\mathcal{C}^*$ is a strong quasi-uniform cover of $(X,\mathcal{U}_{\mathcal{A}})$, there are finitely many $U_{\mathcal{C}_1}, U_{\mathcal{C}_2}, ..., U_{\mathcal{C}_n} \in \mathcal{U}_{\mathcal{A}}$ such that $U(x) \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$, for every $x \in X$, where $U = U_{\mathcal{C}_1}, U_{\mathcal{C}_2}, ..., U_{\mathcal{C}_n}$, i.e., $\bigcap_{i=1}^{n}[\cap\{C \in \mathcal{C}_i : x \in C\}] \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$... (i), i.e.,$\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \cup \mathcal{C}_n : x \in C\} \cap \{C \in \mathcal{C}^* : x \in C\}$.

Let $\mathcal{C}' = \{\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ...\mathcal{C}_n : x \in C\} : x \in X$. As $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n$ are point-finite, $\cap\{\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ...\mathcal{C}_n : x \in C\}$ is a finite intersection for every $x \in X$. Thus every set $\cap\{\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ...\mathcal{C}_n : x \in C\}$ is an open set in $(X,\mathcal{T})$. Now $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n$ being point-finite, C' is as well. By (i) $\mathcal{C}'$ is a refinement of $\mathcal{C}^*$ which in turn is a refinement of $\mathcal{C}$. Thus $\mathcal{C}'$ is an open point-finite refinement of $\mathcal{C}$. Thus $(X,\mathcal{T})$ is consequently metacompact. □

Then, for $(X,\mathcal{U})$, let $\mathscr{C}$ be the family of all strong quasi-uniform covers and for a strong topological space (X,μ), let $\mathcal{U}$ be a compatible quasi-uniformity. So, on X, $\mathscr{U}$ induces a transitive quasi-uniformity $\mathcal{U}_{\mathscr{C}}$, according to Theorem 3.4.1. Now, the problem erupts as to whether or not $\mathcal{U}_{\mathscr{C}}$ and $\mathcal{U}$ are similar. We address the mystery in the theorem below.

Theorem 3.4.3. *[28] Let $\mathcal{U}$ be a quasi-uniformity on X in the strong topological space (X,μ) such that $\mu = \mu(\mathcal{U})$. Let $\mathscr{C}$ be a family of all strong quasi-uniform covers of the quasi-uniform space $(X,\mathcal{U})$. Then the transitive quasi-uniformity $\mathcal{U}_{\mathscr{C}}$, induced on X by $\mathscr{C}$, is a subfamily of $\mathcal{U}$. So $\mathcal{U}_{\mathscr{C}}$ and $\mathcal{U}$ are similar provided $\mathcal{U}$ is transitive. Thus $\mu(\mathcal{U}) = \mu = \mu(\mathcal{U}_{\mathscr{C}})$.*

Proof. Let $\mathcal{C} \in \mathscr{C}$. So, $\exists U \in \mathcal{U}$ such that $\forall x \in X, U(x) \subseteq \bigcap \mathcal{C}_x$. Thus $(x,y) \in U \Rightarrow y \in U(x) \subseteq \bigcap \mathcal{C}_x \Rightarrow (x,y) \in U_{\mathcal{C}}$. Therefore, $U \subseteq U_{\mathcal{C}}$. So $U_{\mathcal{C}} \in \mathcal{U}$. Thus $\mathcal{U}_{\mathscr{C}} \subseteq \mathcal{U}$ that is, $\mathcal{U}_{\mathscr{C}}$ is a subfamily of $\mathcal{U}$. Furthermore, $\mathcal{U}_{\mathscr{C}} \neq \mathcal{U}$ provided $\mathcal{U}$ is not transitie.

So, let $B \in \mathcal{B}$ where $\mathcal{B}$ is a transitive base for transitive quasi-uniformity $\mathcal{U}$. Now, by Lemma 2.1.20, $\mathcal{C} = \{B(x) : x \in X\} \in \mathscr{C}$, so that, $(x,y) \in U_{\mathcal{C}} \Rightarrow y \in \bigcap \mathcal{C}_x \bigcap \{B(z) : x \in B(z), z \in X\} \subseteq B(x)$, as $x \in B(x)$. So $(x,y) \in B$. Thus $U_{\mathcal{C}} \subseteq B$ and hence $\mathcal{U} = \mathcal{U}_{\mathscr{C}}$. Then the result follows.

□

Remark 3.4.4. It can be concluded from the proof above that the family of all strong quasi-uniform covers with respect to a quasi-uniformity $\mathcal{U}$ on a strong topological space X may coincide with that for a strictly smaller quasi-uniformity on X.

Theorem 3.4.5. *[32] A topological space $(X, \mathcal{T})$ is orthocompact if and only if for every open cover $\mathcal{C}$ of $(X, \mathcal{T})$, there is a strong quasi-uniform cover $\mathcal{C}^*$ of $(X, U_{\mathcal{A}})$, the topology induced by $\mathcal{U}_{\mathcal{A}}$, such that $\mathcal{C}^*$ refines $\mathcal{C}$, where $\mathcal{A}$ is the family of all interior preserving open cover of X.*

Proof. Suppose $\mathcal{C}$ is an open cover of $(X, \mathcal{T})$ where $(X, \mathcal{T})$ is orthocompact . Then there exists an interior preserving open refinement $\mathcal{C}^*$ of $\mathcal{C}$. So, $\mathcal{C}^*$ is an interior preserving open cover of $X, U_{\mathcal{C}*} \in \mathcal{U}_{\mathcal{A}}$ where $U_{\mathcal{C}*} = \{(x, y) \in X \times X : x \in X$ and $y \in \cap\{C \in \mathcal{C}^* : x \in C\}\}$. Therefore for every $x \in X, U_{\mathcal{C}*}(x) = \cap\{C \in \mathcal{C}^* : x \in C\}$. Thus $\mathcal{C}^*$ is a strong quasi-uniform cover of $(X,\mathcal{U}_{\mathcal{A}})$.

Conversely, suppose $\mathcal{C}$ is an open cover of $(X, \mathcal{T})$. Then, there exists a strong quasi-uniform cover $\mathcal{C}^*$ of $(X,\mathcal{U}_{\mathcal{A}})$ such that $\mathcal{C}^*$ refines $\mathcal{C}$, according to the aforesaid condition. But then $\mathcal{C}^*$ is a strong quasi-uniform cover of $(X,\mathcal{U}_{\mathcal{A}})$, so there are finitely many $U_{\mathcal{C}_1}, U_{\mathcal{C}_2}, ..., U_{\mathcal{C}_n} \in U_{\mathcal{A}}$ such that $U(x) \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$, for every $x \in X$, where $U = U_{\mathcal{C}_1} \cap U_{\mathcal{C}_2} \cap ... \cap U_{\mathcal{C}_n}$, i.e., $\bigcap_{i=1}^{n}[\cap\{C \in \mathcal{C}_i : x \in C\}] \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}...(i), i.e., \cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \cup \mathcal{C}_n : x \in C\} \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$.

Let $\mathcal{C}' = \{\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ...\mathcal{C}_n : x \in C\} : x \in X\}$. As $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n \in \mathcal{A}, U_{\mathcal{C}_i}(x) \in \mathcal{T}$ for $i = 1, 2, ..., n$ and for all $x \in X$. Hence $\bigcap_{i=1}^{n} U_{\mathcal{C}_i}(x) \in \mathcal{T}$, for all $x \in X$. Thus each set of $\mathcal{C}'$ is open. Also $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n$ being interior preserving, $\mathcal{C}'$ is as well. More so, $\mathcal{C}*$ refines $\mathcal{C}\mathcal{C}'$ also refines $\mathcal{C}*$, by (i). Therefore, there exists an interior preserving open refinement $\mathcal{C}'$ of $\mathcal{C}$ of $(X, \mathcal{T})$. Hence $(X, \mathcal{T})$ is orthocompact. □

4 Conclusion

4.1 Applications

It can be concluded that with respect to a quasi-uniformity $\mathcal{U}$, the family of all strong quasi-uniform covers on a strong topological space X may coincide with that for a strictly smaller quasi-uniformity. Applications include but not limited to the use of the concept of strong quasi-uniform covers to characterize some topological concepts, the likes of Definition **??** of interior-preserving topological spaces. The rest of Chapter 2 further details other applications in topological spaces. These include orthocompact topological spaces whenever every open cover has an interior preserving open refinement. We went on to metacompact topological spaces whenever each open covering has a point-finite open refinement, as well as reflexive and transitive relations of subbases for quasi-uniformities. We gave proof how a topological space is orthocompact if and only if for every open cover of an induced topology, there exists a strong quasi-uniform cover that refines the open cover, in Theorem 3.4.5.

4.2 Summary

In this dissertation, we gave comprehensive details on how uniform spaces can be introduced via uniform covers. We introduced a cover, then defined and gave properties, in a topological spaces, of the concept of strong quasi-uniform cover of X as, for the purpose of this paper, if $\exists U \in \mathcal{U}$ such that for every $x \in X, x \in U(x) \subseteq \bigcap \mathcal{C}^*$. We further characterized quasi-uniformity axiomatically. We then gave account why they cannot be formulated in terms of quasi-uniform covers even if they are transitive. However, by applying the concept of strong quasi-uniform covers by incorporating some axioms, we showed how quasi-uniform spaces give rise to topological space with additional structure that is used to define quasi-uniform properties such as completeness, uniform convergence and uniform continuity. We highlighted that a quasi-uniformizable topological space is precisely the completely regular space and that it is Hausdorff space.

4.3 Key Words

Quasi-uniform space; strong quasi-uniform covers; continuity; quasi-uniformity; space; quasi-uniform structures; quasi-pseudometric spaces; completeness; Hausdorff space; Cauchy filter; transitive; compact.

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