Grade 9 mathematics teachers' strategies to address mathematical proficiency in their teaching of linear equations: A case of selected schools in Gauteng North District.

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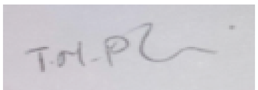

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DEDICATION

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ABSTRACT

The purpose of this research was to look into Grade 9 mathematics teachers' strategies to address mathematical proficiency in their teaching of linear equations. The study was intrigued by the performance of learners in the TIMSS study, and my personal experience as a mathematics teacher, departmental head, Senior Education Specialist for mathematics, and provincial moderator. Grade 9 mathematics teachers were purposively chosen for the study's main subjects. A qualitative method, and case study strategy were used for this research. Videos of lessons conducted by the teachers were viewed to collect data, and interviews were conducted through Microsoft Teams as a follow-up to the lessons. The reason for conducting the research remotely was due to the outbreak of the COVID-19 pandemic; therefore, researchers were not allowed to visit the schools.

The research was conducted in Gauteng North District involving three schools, of which one school was for piloting purposes. Three Grade 9 mathematics teachers were observed on videos; one out of the three was for piloting purposes, and interviews were conducted based on their lessons. The data acquired for this study were analysed using thematic analysis. The study contributes towards a better understanding of teachers' strategies to address learners' mathematical proficiency. Also, some strategies are suggested on how to improve learners' mathematical proficiency.

The research was inspired by Kilpatrick (2001) mathematical proficiency strands which comprise conceptual understanding, procedural fluency, strategic planning, adaptive reasoning, and productive disposition. The conceptual framework and the instruments used were derived from the above-mentioned strands. Each strand consisted of a criterion to use when data were collected.

The analysis shows that teachers were focusing mostly on strategies that addressed learners' procedural fluency when teaching linear equations. The study further shows that lessons were teacher centred. Teachers were providing learners with methods to solve linear equations instead of allowing them to come up with their own initiatives. Thus, the study recommends that teachers should refrain from centralising the lesson around themselves, refresher training on mathematical proficiency to be conducted,

enough time to be allocated for the concepts like linear equations to allow exploration of different methods and representations so that all the mathematical proficiency strands can be addressed.

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CHAPTER ONE: INTRODUCTION TO THE STUDY

1.1 Introduction

The Gauteng Department of Education's vision is that every learner feels valued and inspired in our innovative education system (Department of Basic Education, 2012). What stands out from the vision statement is the innovative education system. This vision statement is in line with what Gulicheva et al. (2017) state, that an innovative education environment enables institutions to provide long-term competitiveness of education services through the elaboration of competitive resources and addressing of innovative approaches, and methods of global education. This view implies that the education system must prepare learners to be equipped to compete with their counterparts from other countries when it comes to education level. The intention is to produce citizens that are competent, and critical thinkers, not only in their usual surroundings but also those that are foreign to them.

The mission statement for the Gauteng Department of Education states that we are committed to providing functional and modern schools that enable quality teaching and learning that protect and promote the right of every learner to quality, equitable, and relevant education (Department of Basic Education, 2012). The highlight of the mission statement is quality teaching, and learning. Nithyanandam (2020) alludes that teaching, and learning is a process in which learners acquire new knowledge, attitudes, behaviours, and abilities to achieve certain agreed objectives. As a result, the product of teaching should benefit learners so that they perform any task according to their acquired skills. Learners should be driven to have a creative, and innovative mindset, and desire to acquire intensive reasoning skills. Furthermore, in his work, Nithyanandam (2020) mentioned the three main components that underpin teaching-and-learning processes, namely (a) teachers, who are the major role players in the educational system, must employ a variety of unique teaching pedagogies to persuade learners to attend class, and master the required skills swiftly, and easily; (b) learners as key participants in the learning process, and (c) a conducive learning environment. The emphasis is that teachers should make learners the centre of the teaching, and learning process, and expose them to different teaching strategies to cater for their various learning needs, and styles. These two aspects, an innovative education system, and quality teaching, and learning, are powerful, and meaningful in producing a highly educated society. At the same time, they will drive the vision of the National Development Plan that by 2030 schools will provide all learners with quality education, especially in Literacy, Mathematics, and Science.

According to the Department of Basic Education (2018), teachers are expected to develop learners' mathematical proficiency by teaching mathematics for understanding. The idea of mathematical proficiency is for teachers to instil a long-lasting understanding of concepts in mathematics. As a result, the Department of Basic Education (DBE) stepped in to ensure that the idea is realised by compiling a document on Mathematics Teaching and Learning Framework: Teaching Mathematics for Understanding (2018). The framework was aligned with Kilpatrick (2001) mathematical proficiency strands. Those strands are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition; however, the DBE'S framework made use of the first four of Kilpatrick (2001) strands, and then replaced productive disposition with a learning-centred classroom. The framework document was compiled after the South African Department of Basic Education had realised that the results from the Trends in Mathematics and Science Study (TIMSS), Annual National Assessment (ANA), Southern and Eastern Consortium for Monitoring Education Quality (SACMEQ) and end of year Examination did not yield the intended outcomes. The sole purpose of the document was to give recommendations to teachers, education planners, and all stakeholders on how to address learners' mathematical proficiency in Mathematics content.

1.2 Problem Statement

1.2.1 Education Sector Background

The South African Department of Basic Education has raised serious concerns about the poor quality of mathematics teaching and learning processes. It was stated in the 2018 mathematics teaching and learning framework that:

The teaching and learning of Mathematics in South African schools are not yielding the intended outcomes of South Africa's education policies, and curricula. This is evident from research from many studies conducted by the Department of Basic Education (DBE), universities and other research agencies in South Africa. The low learner achievement levels revealed by national assessments such as Annual National Assessments (ANA), regional assessments such as Southern and Eastern Consortium for Monitoring Education Quality (SACMEQ) and international assessments such as Trends in Mathematics and Science Study (TIMSS) are indicative, at least in part, of current 'ineffective' teaching and learning practices. (DBE, 2018, p.11)

The Education Sector's numerous interventions such as compiling, and providing high-quality textbooks, DBE workbooks, Sasol Inzalo workbooks, the 1+4 intervention strategy where scripted lesson plans were provided, content workshops for teachers, which advocates professional learning communities (PLC), and video recorded lessons, and online platforms like Siyavula had little impact on the country's mathematics performance.

In 2019, South Africa took part in the research that was conducted by Trends in International Mathematics and Science Study (TIMSS) for Grade 9 mathematics learners through the collaboration of the Department of Basic Education (DBE) and the Human Sciences Research Council (HSRC). Forty-six countries, and entities participated in the study; South Africa was one of the five lowest-performing countries in Africa when compared to those from East Asia (Department of Basic Education, 2020). Despite being in the bottom five, the TIMSS 2019 mathematics achievement score of 389 represents a 17-point gain over the previous TIMSS 2015 cycle. However, Dr Vijay Reddy, Principal Investigator of TIMSS 2019, and Distinguished Research Specialist at the Human Sciences Research Council, cautioned that even though we applaud the improvement in educational achievement, the rate of achievement improvement is decreasing (Department of Basic Education, 2020). She illustrates this by examining two eight-year periods: In the 2003 to 2011 period, the rate of mathematics improvement was 7.4 points a year, and for the 2011 to 2019 period, these figures fell to 4.6 points a year. For South

Africa to meet the TIMSS developmental objectives set in the Medium-Term Strategic Framework (2019–2024), strategically targeted interventions, and additional effort from all education role players are required to accelerate the pace of improvement. The fall from 7.4 to 4.6 points raises concern that needs to be addressed by all stakeholders because it clearly indicates that South Africa still has a lot of work to prepare learners to be globally competitive.

Learners who score above 400 on the TIMSS scale are considered to have mastered the fundamentals of mathematics for that Grade. Greater academic accomplishment indicates that learners can generalise or apply knowledge in both basic, and complicated settings. It is interesting that 3% of mathematics learners, and 1% of learners attained the Advanced Benchmark. 13% of mathematics learners who took the test achieved TIMSS scores greater than 475(the Intermediate Benchmark). The majority of South African learners—41%—had a foundational understanding of mathematics. This suggests that 59% of learners lacked fundamental mathematical understanding. To improve basic math skills, South Africa still has a way to go.

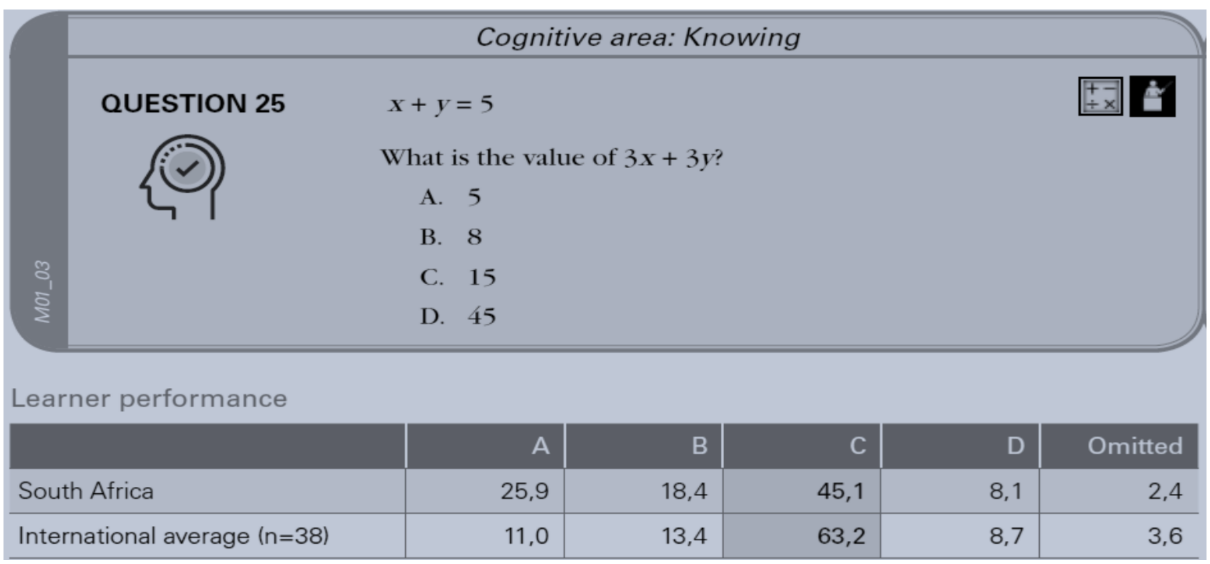
Previously, the results of the 2015 Trends in International Mathematics and Science Study (TIMSS) for Grade 9 in South Africa were also released by the Human Sciences Research Council (HSRC) on 29 November 2016 (HSRC, 2016, Isdale et al., 2017). TIMSS focuses on assessing learners from different countries' mathematics and science levels. TIMSS was established by the International Association for the Evaluation of Educational Achievement (IEA) to allow participating countries to compare learners' educational achievement across borders. The published results have shown that still more work needs to be done when it comes to the teaching of mathematics, particularly algebra. The South African learners performed below the average of their international counterparts in all four content domains, as depicted in Table 1. (DBE, 2016).

Table 1: Content area achievement comparison between South African and international learners (% correct) (all TIMSS items)

	Number	Algebra	Geometry	Data and chance
South Africa	21	20	19	26
International	44	37	37	47

Performance in patterns, functions, and algebra (all included under algebra in TIMSS) was at 20% in South Africa and 37% internationally. This level of learner performance shows a crisis in teaching and learning mathematics in South Africa and internationally. This low level of performance could be attributed to the teaching of mathematics. The comparison in Table 1 regarding algebra and geometry states that these two content areas are difficult for all learners (lowest percentages). For example, Figure 1 shows an item-by-item analysis of released results comparing South African and international learners' their performance in the topic of algebra.

Figure 1: TIMMS item--by-item results analysis


Cognitive area: Knowing

QUESTION 25

M01_03

$x + y = 5$

What is the value of $3x + 3y$?

A. 5

B. 8

C. 15

D. 45

Learner performance

	A	B	C	D	Omitted
South Africa	25,9	18,4	45,1	8,1	2,4
International average (n=38)	11,0	13,4	63,2	8,7	3,6

Forty-five per cent of South African Grade 9 learners answered this question correctly. South African learners, however, lack the basic knowledge required to balance equations. Most learners failed to identify that they were supposed to multiply by three. Twenty-six per cent of learners answered option A, indicating they may have assumed that the second equation would equal the first, and they have not gone beyond that to identify what was required in the question. This kind of performance could be attributed to a lack of good teaching.

Therefore, the 2019 and 2015 results say a lot about the teaching of mathematics in South Africa. Hence, this study sought to explore Grade 9 mathematics teachers' strategies to address mathematical proficiency in their teaching of linear equations

I had the opportunity, and experience to interact with many teachers, and watch how they taught mathematics due to my previous position as a teacher, departmental head, and currently, a senior education specialist. That is why I thought it appropriate to offer my background knowledge.

1.2.2 Personal Experience Background

Throughout my teaching years, linear equations have always been a nightmare to the learners I taught. They have always seen the topic as a concept where we solve for an 'x' that was not easy to find, and they had a negative attitude towards the topic. In the examination, learners would leave out the questions on linear equations without trying to attempt them.

When I was a departmental head, and conducting class visits, algebra was taught only in theory, and not in practical ways by using models where learners could touch and feel to make sense of the concept. Teachers were not creative and taught the topic just on the surface. I also noticed that teachers were teaching learners to memorise concepts and procedures instead of understanding them. The performance in linear equations was very low, presumably caused by learners not being able to recognise, and understand algebraic concepts. Mathematical concepts such as equal signs, and equations were not used appropriately, making it difficult for learners to understand the terminologies used in questions posed in textbooks or examination question papers.

I have experienced that learner taking vernacular as their home language compared to those taking English home language had difficulties translating word problems into algebraic equations. The language barrier contributed to learners' lack of interest in solving word problems. On the other hand, teachers also avoided word problems because they found it difficult to make them accessible for learners to understand. My experience as the subject advisor for mathematics in Grades 7-9 since 2016 allowed me to interact with most teachers in my district. I have been exposed to thoroughly checking the learners’ activity books, moderation of school-based assessment (SBA), and examination at the district, and Provincial levels. Throughout these years, it was evident that learners were struggling with linear equations. Diagnostic analysis of items after moderation indicated that learners’ performance was extremely low, and most did not attempt to answer the questions based on linear equations. Therefore, all this made me realise that it was not only the learners that I taught or those from my District that had challenges with linear equations but even those from other districts in my province (Gauteng). This raised the alarm and made me even more curious.

Furthermore, the literature shows that there are still gaps in terms of addressing mathematical proficiency strands. The mathematical proficiency of Grade 8 learners was investigated in terms of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition by Hlaing and Thein (2020). Their work found that most learners had moderate mathematical proficiency. Furthermore, the results revealed that among the proficiency strands, learners’ procedural fluency was the highest, and strategic competence was the lowest. This implies that the focus was more on the steps to get to the answer, and less on different strategies to reach the same answer.

Arends, a senior research manager in the HSRC’s Inclusive Economic Development research division (2021), investigated two dimensions of instructional quality, cognitive activation, and supportive climate, as indicated in Table 2. This is a report about how teachers applied their instructional practices during lessons. Some cognitive activation and supportive climate indicators are related to the strands of mathematical proficiency indicators. For example, asking learners to explain their answers, encouraging learners to express their ideas in class, and encouraging classroom discussion to relate to adaptive reasoning while asking learners to decide their problem-solving procedure could relate to strategic competency strands. Relating the lesson to learners' daily lives and linking new content to learners' prior knowledge could be related to conceptual understanding.

Table 2: Educators report on applying instructional practices during every or almost every lesson

Criteria: Cognitive activation	%
Ask learners to explain their answers	51
Relate the lesson to learners' daily lives	38
Ask learners to decide their own problem-solving procedures	29
Ask learners to complete challenging exercises	25
Bring interesting materials to class	18
Criteria: Supportive Climate	%
Link new content to learners' prior knowledge	77
Encourage learners to express their ideas in class	58
Encourage classroom discussion	39

The percentage of teachers asking learners to decide their problem-solving procedures was 29%. These findings imply that teachers are not emphasising strategic competence to allow learners to develop their strategies and procedures. Integrating classroom activities into daily life was not encouraged and carried a low percentage. The other most concerning factor about the findings is that teachers were developing learners' productive disposition at a low rate of 39%. The study also showed that learners' participation and discussions were limited. Learners were not exposed to challenging exercises to test their critical thinking and reasoning skills, one skill set that is embedded in mathematical proficiency. These instructional practice tools could be used by teachers in addressing mathematical proficiency.

It was against these backgrounds that my research focused on exploring Grade 9 mathematics teachers' strategies to address mathematical proficiency in their teaching of linear equations

1.3 The Purpose of the Study

The study aims to explore how Grade 9 mathematics teachers' use strategies to address mathematical proficiency in their teaching of linear equations

1.4 Significance of the Study

Learners' poor performance in linear equations may be credited to the reality that learners and teachers, as curriculum implementers, operate at different levels during their interaction in the classroom. Sometimes this happens because teachers are not aware of the level at which their learners operate cognitively. Learners might seem to understand the topic, while the reality is that learners might have memorised the steps towards the solution and reproduced them without the ability to explain in detail how they obtained it. Subsequently, I accept that the discoveries of this study will aid teachers in strengthening mathematics practices within the classroom. The teachers' fortification of their practices will be so that learners not only see mathematics as just another subject but instead as a subject where they create profound conceptual understanding to form a sense of mathematics. It will help learners create confidence and competence to bargain with any mathematical circumstance, coherent thinking, a passion of interest and cherish mathematics.

Teachers are anticipated to educate mathematics for understanding in a learning-centred environment to address the challenges related to educating and learning mathematics. My study was empowered by the vision of the DBE guaranteeing quality education and learning in South African classes. The desire of the DBE, in connection to the framework, is to guarantee that the curriculum is taught by competent and qualified teachers who will motivate learners with competencies for a changing world (DBE, 2018). The system looks to lay a firm establishment for a better approach in which mathematics is taught, and teachers ought to endeavour to (DBE, 2018, p.8):

1. Teach mathematics for conceptual understanding to empower comprehension of mathematical concepts, operations, and relations.
2. Teach so that learners develop procedural fluency, which involves skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
3. Develop learners' strategic competence- the ability to formulate, represent, and decide on appropriate strategies to solve mathematical problems.
4. Provide multiple and varied opportunities for learners to develop their mathematical reasoning skills- the capacity for logical thought, reflection, explanation, and justification; and
5. promote a learning-centred classroom which enables all the above, supported by teachers engaging with learners in a way that foregrounds mathematical learning for all.

The contents of this framework align with my study as it is about exploring how Grade 9 mathematics teachers use strategies to address mathematical proficiency in their teaching of linear equations. The results of my study will enlighten the stakeholders in education on how teachers address mathematical proficiency, and if any further interventions need to occur. The need to do this research also drew inspiration from my encounters with how mathematics was taught and learnt. My quest grew alongside reading the work of Kilpatrick et al. (2001), and the introduction of the Mathematics Teaching and Learning Framework, which emphasises teaching mathematics with understanding. The main driver for this research, which I have observed working with many teachers as a subject advisor, is the lack of good quality teaching of mathematics in schools concerning how the learning of mathematics was influenced by integration with context that draws on learners' everyday experiences, and connections between concepts. This research has also been prompted by my dissatisfaction with existing assumptions that too much teaching and giving many mathematics class-works and home-works create meaningful learning for learners.

Teaching mathematics proficiently and with understanding is one of the required teaching skills teachers need in the 21st century (Kilpatrick et al., 2001; Usiskin, 2012). Mathematical proficiency emphasises learning new concepts and procedures with understanding**.** Learners come to class with existing knowledge that teachers rarely build on. The more in-depth their comprehension of a concept, the more associations learners acquire. This makes it simple for them to put new thoughts into their existing conceptual networks. For example, comprehending the relation among rational numbers plus variables is convenient, because learners must work with rational numbers mostly as coefficients in linear equations. This means that the coefficient of a variable can be anything from negative and positive whole numbers to negative and positive fractions. Without these and numerous other associations, learners will have to learn each new piece of fact they experience as an isolated, irrelevant thought. The issue of the connection of mathematical concepts is also reflected in the South African curriculum. The South African curriculum defines mathematics as:

> *A language that makes use of symbols and notations to describe numerical, geometric, and graphical relationships. It is a human activity that involves observing, representing, and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves. It helps to develop mental processes that enhance logical and critical thinking, accuracy and problem solving which will contribute to decision making. (DBE, 2012, p.8)*

The fact that mathematics is about "describing numerical, geometric and graphical relationships" advocates the learning of mathematics in an integrated manner. This definition, again, suggests that mathematical concepts should not be learnt in isolation from each other; there must be connections "between mathematics objects themselves" and mathematics and its "physical and social phenomenon" (DBE, 2011, p.8).

The component of proficiency in mathematics is additionally inserted within the definition of mathematics in that it makes a difference in creating mental processes that improve consistent and basic considering, exactness and problem-solving. It is critical in such a way that it increments retention and recall. It gets to be simple for learners to recover the information when they have the concept associated with a complete network of thoughts. Reflecting on related thoughts can often lead them to the specified thought inevitably. Mathematical proficiency and understanding of concepts are important because they enhance problem-solving abilities**.** Similarly, Schoenfeld (1992) states that when concepts are well arranged in a rich web of ideas, transferability is essentially improved, and so is problem-solving. This implies that when learners comprehend the relationship between a circumstance and a setting, it gets easy for them to know when to utilise a suitable approach to problem-solving. Once problems increase in difficulty, learners with mathematical proficiency can relate the skills they learnt to find ways to solve new problems. Positive attitude and belief towards mathematics get improved when mathematical proficiency is acquired. Learners tend to attain a positive self-concept and confidence in their ability to learn and understand mathematics when ideas are well understood, and they can make sense of them. The other stakeholders such as policymakers, researchers, material developers, parents, and the nation as a whole, will benefit from the results of the study.

1.5 Objectives of the Study

To attain the intentions of the proposed research, I have set out the following objective:

- Explore how Grade 9 mathematics teachers use strategies to address mathematical proficiency in their teaching of linear equations (conceptual understanding, adaptive reasoning, strategic competence, procedural fluency, and productive disposition) in linear equations.

Purposively, the objectives of this study are:

- To determine how Grade 9 mathematics teachers use strategies to address conceptual understanding in their teaching of linear equations.
- To determine how Grade 9 mathematics teachers use strategies to address procedural fluency in their teaching of linear equations.
- To determine how Grade 9 mathematics teachers use strategies to address strategic competence in their teaching of linear equations.
- To determine how Grade 9 mathematics teachers use strategies to address adaptive reasoning in their teaching of linear equations.
- To determine how Grade 9 mathematics teachers use strategies to address productive disposition in their teaching of linear equations.

Hence, this study intends to answer the research questions set out in the next section.

1.6 Research Questions

1.6.1 Main Research Question.

How do Grade 9 mathematics teachers use strategies to address mathematical proficiency in their teaching of linear equations?

1.6.2 *Secondary Research Questions.*

1. What strategies do teachers use to address conceptual understanding[1] in their teaching of linear equations?
2. What strategies do teachers use to address procedural fluency[2] in their teaching of linear equations?
3. What strategies do teachers use to address strategic competence[3] in their teaching of linear equations?
4. What strategies do teachers use to address adaptive reasoning[4] in their teaching of linear equations?
5. What strategies do teachers use to address productive disposition[5] in their teaching of linear equations?

1.7 Definition of Key Terms

Definitions of key concepts

CONCEPTS	DEFINITIONS
Integration	The action or process of successfully joining or mixing different concepts.
Concepts	A new phenomenon to be learnt.
Assumptions	Something that is accepted as true or certain to happen without proof.
Connection	A link between concepts.
Word problems	Mathematical problems expressed in words.
Procedures	The established way with stepwise nuances of solving mathematical problems and the presentation of the solution.
Strategies	A method or trick to help learners learn mathematics facts.
Algorithms	Algorithms are procedures or descriptions of a set of steps that can be used to solve a mathematical computation.

[1] a comprehension of mathematical concepts, operations, and relations.
[2] the skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
[3] the ability to formulate, represent and solve mathematical problems.
[4] the capacity for logical thought, reflection, explanation, and justification.
[5] a habitual inclination to see mathematics as sensible, useful, and worthwhile.

Mathematical Proficiency	A concept of five intertwined strands, namely, conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.
Algebra	The language for investigating most of mathematics that can be extended to the study of functions and other relationships between variables.
Variables	Letters or other symbols that represent unknown numbers or values.
Integers	A set of positive and negative whole numbers
Coefficients	A number multiplied by a variable
Linear Equations	A first-degree equation in which all variables contain an exponent of 1
Symbols	Mathematical symbols are signs or pictures representing variables or procedures
Notations	Letters, numbers, and other symbols
Relationship	The link between one phenomenon and another.
Prior Knowledge	Previously acquired knowledge that can be linked to new knowledge, that is, learning from known to unknown
SBA	School Based Assessment
Diagnostic Analysis	Locating the root cause of the problem
HSRC	The Human Sciences Research Council
TIMSS	Trends in International Mathematics and Science Study
IEA	International Association for the Evaluation of Educational Achievement
Conceptual Understanding	Comprehending mathematical concepts,

	operations, and relations
Procedural Fluency	The skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
Strategic Competence	The ability to formulate, represent and solve mathematical problems
Adaptive Reasoning	The capacity for logical thought, reflection, explanation, and justification.
Productive Disposition	A habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.
Intertwined	Closely connected in a way that cannot easily be separated
Independent	Two or more events or outcomes are independent if the happening of one has no effect on the other.

1.8 A Snapshot of the Conceptual Framework Used in The Study

I used Kilpatrick et al.'s (2001) model as the conceptual framework in this study. The model emphasises intertwined strands of mathematical proficiency, namely, conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Kilpatrick et al. (2001) used the word "intertwined" since the strands are not free from one another; they express diverse perspectives of a complex entirety. The understanding is that they cannot be addressed in isolation to achieve proficiency. These strands focus mainly on the importance of connecting mathematical concepts, accuracy and effectiveness of procedures used in solving problems, various strategies used to solve problems, reasoning behind procedures and strategies used and making sense of the mathematics.

1.9 A Snapshot of the Methodology Used in the Study

The study used the qualitative approach to gain insight into Grade 9 mathematics teachers' strategies to address mathematical proficiency in their teaching of linear equations. The approach is suitable for the case study adopted for this study because I collected my own data through interviews and observations. The aim was to look into each case and interpret, and describe the realities found in the field. Three teachers were purposively sampled, one for the pilot study and the other two for the main study. A pilot study was conducted to check the credibility of the instruments. I asked permission from the University of South Africa to do this research, and ethical clearance was granted. To obtain access to the schools, I asked permission from Department of Basic Education and was allowed to proceed. Letters explaining my study, permission letters and consent forms were sent to the District Director, School Principals, the Chairperson of the School Governing Body, identified teachers, parents and learners. All stakeholders signed the consent forms as confirmation of giving me permission to conduct my research.

1.10 Organisation of the Dissertation

This dissertation is organised into five chapters. Each chapter has several topics and sub-topics. Chapter 1 focused on introducing the study, Chapter 2 covered the literature review and conceptual framework, Chapter 3 explained the research methodology, Chapter 4 presented and interpreted the data and Chapter 5 presented the conclusions and final discussions. The outline of the dissertation was set out as follows:

Chapter 1: Introduction of the study

Chapter 1 included an overview of the mission and vision of Department of Basic Education. The following were also included: the problem statement, the purpose of the study, the significance of the study, the objectives of the study, the research questions, definition of key concepts, a snapshot of methodology, a snapshot of conceptual framework, and the organisation of the dissertation.

Chapter 2: Literature Review and Conceptual Framework

This chapter aims to introduce the literature review and review existing research focusing on the notion of mathematical proficiency, basic algebra concepts, strategies used by teachers to teach linear equations, teachers' challenges in teaching linear equations and challenges experienced by learners in solving linear equations.

The chapter also includes the conceptual framework inspired by the works of Kilpatrick (2001) on mathematical proficiency strands. The chapter introduces a conceptual framework, a conceptual framework diagram, and a detailed explanation of mathematical proficiency strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition and a chapter summary.

Chapter 3: Research Methodology

This chapter aims to explain in full the research designs, research methods, pilot study, population and sampling, qualitative data analysis and interpretation methods used to ensure dependability, transferability, confirmability and credibility, and a chapter summary.

Chapter 4: Presentation and Interpretation of Data

Chapter 4 represents the study’s core information, focusing on the context of the study, the presentation, and the interpretation of the findings. This chapter describes the events that took place during the lesson observation and the interviews that were conducted in detail and a chapter summary.

Chapter 5: Discussions and Final Reflection

This chapter highlights and discusses Grade 9 mathematics teachers’ strategies to address mathematical proficiency in their teaching of linear equations. The results found in this study are used to make suggestions for further research and recommendations that can be carried out profitably. The study's limitations are also indicated, as they may hinder the results from being generalised, implications of the study, and a final reflection.

The literature from past studies and the conceptual framework relevant to the research problem being examined are reviewed and discussed in the following chapter.

CHAPTER TWO: LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

2.1 Introduction

The literature review explores the existing scholarly work on teaching linear equations. The literature indicates that learners frequently confront challenges in mathematics content, particularly when attempting to make sense of unique concepts such as tackling equations. There are three mutual difficulties that learners come across when trying to solve equations which researchers recognised: insufficient understanding of variables and coefficients in an equation (Kilpatrick & Izsak, 2008; Poon & Leung, 2010), not knowing the purpose of the equal sign, and applying procedures on problems with lack of thorough understanding (Capraro & Joffrion, 2006; Siegler, 2003; Star, 2005). These challenges emanate from how teachers convey their presentation when scaffolding questions for learners to form arithmetic equations and algebraic expressions to algebraic equations. It implies that the teachers do not thoroughly emphasise conceptual understanding of basic concepts of algebra. These findings by researchers influenced my study on teaching linear equations.

The proposed study explores on Grade 9 mathematics teachers' strategies to address mathematical proficiency in their teaching of linear equations. I offered some insight by discussing various topics to trace how mathematical proficiency is addressed in linear equations. These topics include the notion of mathematical proficiency, basic algebra concepts, the strategies teachers use to teach linear equations, the challenges experienced in teaching linear equations and how such challenges are addressed. The information related to this literature review was retrieved from the findings from previous studies and centred around my experience as a subject advisor and provincial moderator.

2.2 The Notion of Mathematical Proficiency

My research revolves around the inclination of mathematical proficiency, emphasising teaching mathematics for understanding. Mathematical proficiency involves logic and imagination, providing mathematical knowledge to learners and laying a premise for future considerations in mathematics and other disciplines. Teachers must acquire a certain level of proficiency for them to be able to instil mathematical proficiency in learners; and they must have an intense understanding of mathematics (Kilpatrick, 2001). Mathematical teaching skills for addressing proficiency are those that the teacher has and uses to facilitate the comprehension, competence, and appreciation of mathematics among learners. Mathematical proficiency for teaching is dynamic and can be detected in a teacher’s activities and the choices made such as interpreting learners’ knowledge of mathematics, creating different illustrations of a mathematical concept, being aware of the level of understanding of the mathematical concept by learners, and using different teaching methods and strategies (Kilpatrick, 2001). Mathematical proficiency comprises five intertwined strands I adopted in this study's conceptual framework: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and a productive disposition. These strands are discussed in detail in Chapter 3. Mathematical proficiency requires elements of mathematical awareness and skills that teachers need themselves and that they strive to cultivate in their learners such as conceptual comprehension and procedural fluency (Kilpatrick, 2001). Learning mathematical skills by learners typically relies on how well-developed the teachers’ skills are. This concept emphasises the notion of mathematical proficiency; the more the teachers’ proficiency has grown in mathematical practice, the better prepared they would be to promote mathematics learning and doing.

2.3 Basic Algebra Concepts

2.3.1 What is Algebra?

Algebra is a thought-provoking content that is introduced and needs vast intellectual thinking and a stimulating skill for learners. It exposes learners beyond simple arithmetic procedures to using variables and mathematical connections. There is a need for a thorough understanding of algebra to comprehend geometry, calculus, and forthcoming mathematics courses. Algebra is the language for examining and communicating most mathematics and can be expanded to study functions and other connections between variables (DBE, 2012, p.10). Algebra serves as a bridge between the other learning areas of mathematics. It is an important building block that prepares learners for mathematical thinking in all areas of mathematics and everyday life (Van de Walle et al., 2014). In connection with my study, it means that the emphasis on addressing mathematical proficiency when teaching linear equations is essential because learners will have to use those acquired skills to relate them to daily experiences. Teaching and learning should not only revolve and end in the classroom walls but must be an extension to real life situations. Algebra is defined as generalised arithmetic (Booth, 1988; Kieran, 1992). As it has different functions, it is defined as a language and thinking tool beyond the calculation with symbols (Booth, 1988; Dede & Argün, 2003; Kieran, 1992; Vance, 1988). It implies that when teaching for proficiency, proper terminologies must be used to understand concepts in algebra better. Learners must be exposed to the relevant use of the correct wording that is expected in algebra. Algebra is covered under the content area patterns, functions, and algebra in the Grade 9 mathematics curriculum (DBE, 2012, p.10). A fundamental element of this content area is for the learners to accomplish effective calculating competencies in the use of algebra, focusing on the following:

- Developing algebraic manipulative skills that recognise the equivalence between different representations of the same relationship.
- Analysing situations in a variety of contexts to make sense of them; and
- Representation and description of situations in algebraic language, formulae, expressions, equations, and graphs.

2.3.2 Linear Equation Concept

Tossavainan et al. (2011) indicated that an equation might be a mathematical articulation presented in symbolic form, indicating that two congruous objects are identical or comparable. For example, the notation A = B is an equation where A and B are compatible. The explanations acknowledge that an equation is mainly characterised by symbols and equal signs. An equation is characterised by the equal sign (=), which indicate the equality between two expressions. An algebraic expression is a group of amounts comprising constants and variables joined by the four crucial operations.

Equations consist of several types such as quadratic equations, exponential equations, radical equations, rational equations, and linear equations. However, this study mainly focuses on linear equations. For example, $5x + 4 = 9$ is called a linear equation, the left and the right-hand side are equal, and the exponent of the variable is 1. Before the introduction of linear equations, arithmetic equations are introduced to learners (Machaba, 2017a), for example, $5 + 4 = 9$. When we write ---- + 4 = 9, it means that an unknown plus 1 should be the same as 9. The line---- represents an unknown value. Variables are then used as unknown values moving from arithmetic equations to linear equations. Instead of a line or a block, in some instances, we use letters such as x, y or any other letter. Therefore, this equation becomes $x + 4 = 9$. This is now called an algebraic linear equation because it has an unknown value expressed as a variable. In more complicated algebra, some equations are more complex, for example, $3x + 5 = 2x + 6$. An equation is like a balance; whatever you do or have on the left-hand side/right-hand side should 'balance' with what is on the left-hand side/right-hand side.

In the South African context, linear equations are taught in term 1 and term 3 for Grade 9 learners (DBE, 2012). Amongst other things, linear equations are characterised by creating equations from situational problems and tackling the equations to get to the solutions. Any effort to scrutinise and expand the practices and knowledge of this topic is significant especially the focus on big ideas, as indicated earlier. In a study, Allsopp et al. (2016) characterised three groupings of big ideas that are relevant to this study: number patterns, equality, and variables. For example, the concept of the equal sign and variables are very important in understanding the concept of linear equations. Linear equations have several representations, namely, flow diagrams, formula/equations, word problems and linear graphs. It will be in the best interest of learners to be exposed to all these representations and different strategies that can be used to strengthen their strategic competence.

2.4 Strategies Used by Teachers to Teach Linear Equations

Understanding how teachers use classroom instruction to impart knowledge and how they adapt their teaching and interaction strategies is critical. Whether they use classroom discussion as a learning tool and how effective their feedback strategies are in enriching the learning environment to improve learners' performance and develop mathematical proficiency in linear equations. Hidayat and Setyawan (2020) argue that the decision of teachers to choose teaching and learning approaches will impact the learning outcome. For learners to be successful in studying mathematics, teachers must carefully select the most appropriate teaching style based on the needs of the learners. The implication is that every teacher's perspective and consideration in establishing the most successful mathematics teaching technique to help their learners understand mathematics may differ based on their expertise. Depending on the environment and the demands of the learners, teachers' considerations about effective teaching practice may remain the same or change. The teachers' awareness of the learning goal and the objectivity of the curriculum may cause them to change their minds regarding the efficiency of teaching methods in achieving the learning goal.

Realistic Mathematics Education (RME) is a theory that can be adopted to investigate problem solving and mathematical communication abilities, as suggested by Oktaviani et al. (2021). RME is a mathematical learning theory that considers mathematics as a human activity that incorporates problem-solving and subject matter structure. RME learning activities stress problems relevant to daily lives, so contextual problems are used as a stepping stone for learning to demonstrate that mathematics is very relevant to learners' daily lives. Learners will become more interactive in improving their knowledge through problems based on real-life scenarios". As a result, mathematics must be cultivated close to learners' lives, linked to everyday life, if possible, with learners' support.

Linear equations revolve around problem-solving, irrespective of the representations that are presented; hence some strategies need to be used. In his book, Van de Walle (2007) indicates that techniques for tackling problems are identifiable strategies for drawing closer to a task that is free of the subject or subject matter. It implies that solving problems can be done using different ways depending on the nature of that problem. The goals of strategies used are to improve learners' ability to analyse an unfamiliar problem, as a result, their strategic competence is reinforced. Selecting strategies assists learners in acquiring relevant problem-solving strategies that are appropriate and improve their ability to validate answers. Therefore, as a teacher, different strategies should be encouraged for finding solutions. Van de Walle (2007) has packaged different approaches to be used when delivering mathematics lessons:

Draw a picture, act it out, and use a model

The focus is on using manipulatives and solving the problem to better understand the situation. This strategy is useful for teachers because it will make their lesson hands-on as learners will be using the models to create the concepts and test emerging ideas. Teachers should ensure that various models are available to assist with important ideas. Manipulatives can present mathematical themes and strengthen conceptual understanding in influential ways. This strategy promotes mathematical proficiency and conceptual understanding because it helps learners to construct mathematical ideas; however, it can help or fail learners in acquiring mathematical proficiency.

Look for a pattern

Many problem-based tasks revolve around pattern searching, especially in procedures to follow and strategies to use in solving linear equations. This strategy will assist teachers in addressing procedural fluency and strategic competence when teaching. Teachers present the lesson in such a way that it will enable learners to pick up the appropriate steps to follow in solving problems; as a result, they generate their approaches depending on their conceptual understanding.

Simplification of the problem

The overall impression is to break down a complex problem into parts that are simple to understand and makes it easy for the problem to be solved. This strategy promotes conceptual understanding to ensure that problems are solved with a deep understanding of the concept; as a result, procedural fluency in systematically solving the problem bit by bit is also stressed. Strategic competence is strengthened to accommodate learners with different cognitive levels. Scaffolding the problem leads to productive disposition because the teacher allows learners to start with problems that most learners can attempt.

Star et al. (2015) also developed three strategies to guide teachers with unique, practical procedures that demonstrated ideas on expanding algebra skills and information. The strategies strive to implement practices that empower learners with a profound understanding of algebra:

Analysing algebraic reasoning and strategies by using solved problems

This strategy engages learners in a conversation to connect the strategies, reasoning, and solution of solved problems. It promotes learners' development of greater comprehension and justification of the rational procedures used in algebra problems; as a result, it promotes their strategic competence and adaptive reasoning. The teacher selects relevant worked-out activities that relate to the aim of the lesson and shares mistakes to present to the class for discussion. With this strategy, learners are also exposed to incomplete and incorrect solved problems, encouraging them to think critically. The limitation of this strategy is that worked-out activities have an undesirable influence on conceptual understanding and procedural fluency because learners only analyse the problems but are not engaged in solving them.

Learners are taught to use the structure of algebraic representations

The strategy aims at promoting mathematical language that learners need to be familiar with when explaining mathematical concepts. When a new topic is introduced, the use of mathematical language encourages learners to define the terms of algebra accurately and appropriately. The teachers avoid using vague and non-mathematical language when explaining concepts; as a result, it helps learners to understand the relationships among concepts better. This strategy also encourages learners to use reflective questioning, which encourages adaptive reasoning, to notice structure as they solve problems. It implies that the teacher encourages learners to ask themselves about the problem at hand, which will lead them to reflect on the structure of the problem and the possible approaches. Learners are instructed that diverse algebraic representations can convey distinctive information about an algebra problem, which addresses a conceptual understanding. This strategy focuses on recognising structures, which help learners to comprehend the characteristics of algebraic equations and problems irrespective of whether they are presented as variables, numbers, words, or graphically, which, in terms of Kilpatrick (2001) strands of mathematical proficiency, is about conceptual understanding. Teaching through reflective questioning, using different algebraic representations, and promoting the use of mathematical language improves mathematical proficiency strands.

Teaching learners to choose relevant strategies when tackling problems

The focus is on teaching learners to identify and create strategies for problem-solving. It implies that this strategy engages in methods for completing a task or problem-solving instead of a set of rules performed in a stereotyped way. Exposing learners to various algebraic strategies allows them to solve algebraic problems with flexibility, apply relevant strategies, and effectively perform diverse solutions when presented with a problem. Using multiple strategies helps learners to get out of their comfort zone of using one approach by letting them spread their knowledge and think conceptually. This strategy encourages learners to provide reasoning and mathematical rationale behind their chosen methods when solving problems. Portraying their thinking makes a difference; learners comprehend their selections and objectives when deciding on a strategy. The strategy also encourages learners to assess and make a comparison between diverse strategies for problem-solving. Concerning my study, teaching alternative algebraic strategies can improve achievement in conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning of algebra.

Cangelosi (1996) defines an algorithm strategy as a multistep procedure for obtaining a result. The algorithm's skills objectives concern learners knowing how to execute the steps in an algorithm. Gaining proficiency with an algorithm usually means that learners must be engaged in learning activities which are more tedious and less interesting than learning activities for other types of objectives such as discovering a relationship. An algorithmic strategy allows for more on procedural fluency and does not promote all the strands of mathematical proficiency.

2.5 Teachers' challenges in teaching mathematics

Goos et al. (2020) argue that mathematics teachers are expected to focus on increasing learners' capacity to apply mathematical knowledge, solve problems, and undertake studies of mathematical phenomena in their world as they continue to seek relevance in mathematics. This includes the ability to analyse and interpret the ever-increasing amount of data and information accessible for active citizenship and decision making in all parts of life, not just work. On the other hand, teachers find this to be a difficult component of their job, even though it may be quite rewarding when done well.

Furthermore, Goos et al. (2020) indicate that the challenge for mathematics education and learning is to assist learners in making sense of mathematics. The implication is that working mathematically to solve the twenty-first century's social, economic, and environmental issues will require the flexibility, depth, and diversity that come from making sense of situations and mathematical abstractions. Teachers are finding it challenging to communicate the idea of sense-making to enhance mathematical understanding and reasoning.

In 2020, we must face the sad reality that most school mathematics teachers have spent their entire lives absorbed in Textbook School Mathematics (TSM) due to the educational establishment’s long-standing malpractice, according to Wu (2020). Furthermore, Wu (2020) states that some challenges that come with Textbook School Mathematics are: a continuous lack of understanding that sound reasoning is impossible to achieve without explicit definitions, deficiency in a broad and cohesive perspective of mathematics, and a lack of understanding of mathematics’ broad hierarchical structure. These challenges mean that one can only rely on previously proven results to progress mathematically and promote mathematical development. As a result, most people lack a thorough understanding of the inner workings of mathematics and a comprehensive picture of the subject. A systemic exposure of mathematics teachers to a detailed exposition of mathematics that honours mathematical integrity to remedy such deficits at both ends of the school mathematics spectrum would be reasonable to suggest.

2.6 Challenges experienced by learners in solving linear equations

2.6.1 Solving Linear Equations

Memories of step-by-step procedures and formulas can be tedious and anxiety-inducing when learning mathematics. It is a prevalent misconception that mathematics is merely a matter of memorisation and methods (Machaba, 2017b). Although mathematics requires a lot of logical reasoning, Kilpatrick et al. (2001) contend that educational mathematics does not always reflect this. According to Kilpatrick et al. (2001), for a long time, the school

system involved complex information, but just on the surface, with little consideration for comprehension, “mathematics learning has often been more a matter of memorizing than of understanding” (Kilpatrick et al., 2001, p.16). Mathematics, without a doubt, requires methods and formulas. However, this is not the essence of mathematics because it requires thinking, reasoning, analysing, and conjecturing; it can be challenging.
According to Mbonambi and Bensilal (2014), most Grade 12 learners lack the basic math skills to solve problems and arrive at the correct solution. This is a clear indicator that teachers did not emphasise competency when teaching algebra, as it is expected that by the end of Grade 12, learners would be competent and able to apply mathematical concepts without difficulty. Furthermore, the researchers suggested that efforts to improve algebraic proficiency should begin in primary school. My study’s conceptual framework focuses on strategies that teachers use to address mathematical proficiency in linear equations, that aligns with what Mbonambi and Bensilal recommend teachers do in the classroom. According to Ally (2011), promoting mathematical proficiency in South African schools is ineffective and unorganised. In terms of my research, it shows that teachers’ concentration on increasing mathematical proficiency is critical for the welfare of the South African country.

Samuel et al. (2016) found in their research that teaching and learning algebraic linear equations was a challenge for teachers. They found that grouping like terms and manipulating algebraic symbols was a challenge. Researchers found that failure to solve algebraic linear equations was due to a lack of necessary pre-requisite knowledge such as simplification of algebraic expressions and equations. They say this could be because of a lack of understanding of complex algebraic equations in general relativity. The transition from algebraic expressions to algebraic linear equations was not done properly by teachers for learners to connect the two concepts. Teachers should be aware of the importance of conceptual understanding of items in algebraic expressions before moving on to linear equations.

In the same study conducted by Samuel et al. (2016), teachers were also requested to explain why learners could not solve algebraic linear equations, mainly because the questions given to them (learners) were based on what was previously learnt. The teachers responded by citing various reasons for learners' deficiencies, including simplifying algebraic expressions, teaching algebra without concrete examples such as manipulative objects like algebra tiles, and learners' difficulty reading and understanding mathematical statements. Learners' failure to recognise and grasp algebraic words such as coefficients, constants, evaluate, simplify, expand, factories, and many more contributed to their low accomplishment in algebraic linear equations. Models for mathematical ideas should be introduced to help learners explore and talk about mathematical ideas so that teachers can establish a strong grasp of linear equations in their learners. As a result, a better understanding of terminology and how to solve linear equations will be achieved. Teachers' replies such as the ones above suggest that while conveying algebraic topics, greater emphasis should also be placed on the language aspect. It is important to emphasise to learners that mathematics has its language, which must be completely understood to master algebraic linear equations in a meaningful way.

2.6.2 Interpretation of the Equal Sign

Many learners did not understand the equal sign, based on my experience as a teacher and head of the department. The equal sign was mentioned by teachers when they used it to distinguish between an algebraic expression and an algebraic equation. For example, $2x - 6$ is an expression because it does not have an equal sign, and $2x - 6 = 4$ is an equation because it originally had an equal sign. The teachers did not detail what the equal sign in an equation signifies. This indicates that those teachers lacked a conceptual knowledge of the equal sign in an equation, making it difficult for them (teachers) to explain it in a way that makes sense to learners.

Machaba (2017a) discovered that Grade 9 learners understood an equal sign as a 'do something' and unidirectional (one-sided) sign rather than as a concept that indicates an equivalence (the idea of maintaining both sides of the equal sign equal) of two quantities in his study. This suggests that most learners did not regard the equal sign as a symbol of identity but rather as a "command" for carrying out the operation shown on its right-hand side, which corresponds to how their teachers taught them. Rather than interpreting the equal sign structurally as a static relationship between two magnitudes, they interpreted it as an operational-computational process with algorithms and actions. As a result, these learners lacked the relational and conceptual grasp of linear equations that were not implanted in them during the teaching and learning process. It is a sign that teachers did not address skills for a deep or relational understanding which takes a lot of time and effort. Relational understanding, which is linked to conceptual understanding, has the critical benefit of making efforts not only worthwhile but also necessary.

According to Van de Walle (2007), the equal sign is one of the most significant symbols in basic arithmetic, algebra, and all mathematics involving numbers and operations. He continues to state that previous and current studies have concluded that the equal sign is a symbol that was poorly understood. According to Van de Walle (2007), learners were taught that the equal sign signifies "is the same as" and that the statements on both sides must be the same. Learners frequently mistake this statement as supposing that the left side must always be worked on to obtain the right-side answer. They connected the equal sign to the button on the calculator that displayed the answer when pressed. Teachers should address a conceptual understanding of the equal sign because it allows learners to recognise the relationships in the number system and use their knowledge and skills when confronted with basic and difficult algebraic equations. When dealing with equal signs, Van de Walle (2007) recommends avoiding imposing relational thinking on learners and proposes instead exposing them to a complex set of true/false and open phrases. For example, learners should be given $6 + 2 = 9 \, and \, 3 + 4 = 7$ to state whether they are true or not to see if they understand the meaning of the equal sign before being exposed to more complex activities. Linear equation examples should be chosen in such a way that their objective is to instil excellent thinking rather than calculation and that they encourage learners to think in relational terms.

Stephens et al. (2020) explain that the equal sign is a relational symbol representing the equivalence or "sameness" of the numbers or expressions on either side of an equation in their study on the effect of balance scales in fostering constructive thinking about equations among diverse learners. They argue that research has shown for years that elementary and middle school learners fail to build a solid understanding of the equal sign as a relational symbol expressing a connection of equivalence. According to Knuth et al. (2008), and McNeil and Alibali (2005), learners who lack a thorough knowledge of the equal sign frequently ascribe an “operational” definition to the symbol, claiming that it signifies providing the solution or total. Before being exposed to the pan balance and balancing scales, Stephens et al. (2020) discovered that many learners first believed that the equations $8 = 8, and\ 5 = 1 + 4$, and $5 = 3 + 2$ were erroneous. These findings suggest that conceptual understanding of the equal sign remains a significant issue. Even in an example where both sides have the same numbers, learners struggled to recognise that both sides are even or level. Effective teachers must support learners in constructing their concepts from pre-existing ideas. How a class is taught, the social climate created in the classroom, and the tools available for learners to participate in classroom mathematics lessons significantly impact what is learnt and how materials such as manipulatives can be used.

Essien and Setati (2006) investigated how learners in Grades 8 and 9 interpret the equal sign. They found that the equal sign is mostly interpreted as a “do- something” or “find-the-answer” symbol by these Grades 8 and 9 learners. Another interpretation of the equal sign that emerged from this research was that it is a unidirectional symbol and a tool for writing the answer. Learners, for example, refused to acknowledge the correctness of the mathematical sentence □ = 2 + 4 because the equal sign was “in the wrong location”. The mathematical sentence was only correct for these learners if it was written in form 2 + 4 = □, with the movement from left to right. In terms of teaching, this means that teachers must expose learners to diverse instances to avoid stereotypes. Teachers should employ the ideas that learners bring to class to construct new concepts and procedures, question their hypotheses, explain their techniques, and solve engaging problems. Essien and Setati (2006) compared their findings to those of research conducted with primary school learners and discovered that they were nearly identical. These findings raise concerns about how learners in Grades 8 and 9 will manipulate basic and complex mathematical equations if they struggle with the idea of the equal sign. Compared to primary school learners, it is expected that learners at this level have a firm foundation and comprehension of the equal sign. This suggests that a proper foundation was not laid in the lower classes. Therefore, teachers in Grades 8 and 9 must repair the harm and not assume that these learners' interpretations of the equal sign are correct.

2.6.3 The Concept of a Variable

When learners are introduced to the content area of algebra, they are also exposed to the concept of a variable. When dealing with algebraic expressions, they are expected to identify the variable, which is the foundation of algebraic equations. Variables are a very strong representation feature that allows generalisations to be expressed. Learners should be able to deal with expressions involving variables without worrying about the precise number or numbers that the letters represent (Van de Walle, 2007). This explanation is referred to as manipulation of opaque formalisms–symbols that can be looked at and worked with instead of looking through or into the symbols to see what they might represent–according to Kaput (1999), as referenced in Van de Walle (2007). Variables, according to these explanations, represent various and distinct numbers.

Van de Walle (2007) identified two ways learners experience variables. These are as follows:

1. Variables used as an unknown value: Variables are originally introduced to learners as symbols that represent an unknown value. For example, the empty box represents the variable in a number sentence, 5+ =8. Similarly, Epp (2011) acknowledges that when algebra is taught, the empty box notation is usually discontinu$\square$, and the focus turns to learning rules for manipulating equations to get a variable, usually 'x', on one side and a number on the other. This means that teachers should introduce variables carefully, noting that 'x' is really a placeholder for the unknown amount, and occasionally use empty-box notation after variables have been introduced.
2. Variables used as quantities that vary: When there are multiple symbols or variables in a single equation, the values of the various variables may differ. For example, in the equation $a + 6 = 10 - b$, one solution is a=3 and b=1, and another solution is for both a and b = 2. Many learners feel that because the two variables are distinct, their values must also differ, which is incorrect. Learners feel that because the two variables are different, their values must be different due to the way they were taught and introduced to the equation. Other teachers utilise variables as items such as bananas and apples, rather than the number of objects, which leads to misunderstandings. Learners tend to write "let a be apples and p be pears" in an equation $a + p = 1$, with two distinct variables, according to Epp (2011). The teachers should emphasise to learners that they should rewrite it as "let a be the number of apples and p be the number of pears". The same is true when applying the formula $d = s \times t$, where 't' does not indicate time but serves as a placeholder for the number of hours travelled. Thus, the number of hours that can be

substituted for the 't' changes. These are the distinctions that mathematics teachers must emphasise to their learners.

Understanding that the value provided by the solution can replace a variable in the original equation is a useful technique that indicates an understanding of the variable nature and the purpose of solving equations. Most learners do not realise that a variable signifies that it can represent any number. Machaba (2017a) discovered that learners could not establish the connection between arithmetic and algebraic equations. They did not realise that an unknown in an arithmetic equation, represented by a box, and an unknown in an algebraic equation, represented by a letter, were the same thing. As a result, learners were unable to understand a variable. When asked to solve n in 7 + n = 6 + 9, some learners just added the numbers in the equation. When asked about how she did it, one said "I have added 6 + 9, and it gave me 15 and I added 7 to get 22n." This implies that when moving from arithmetic equations to algebraic equations, it is important for teachers to indicate that the box is replaced by the letters. Using related examples, the teacher should demonstrate the transition from arithmetic to algebraic equations. For example, when introducing a variable, colours may be used to illustrate the similarities, purposively for the variable:

$5 + \square = 8 \rightarrow 5 + x = 8$

The arrow denotes the transition from arithmetic to the algebraic equation where the letter variable in the algebraic equation substitutes a box in the arithmetic equation.

Kuchemann's approach was used in research by Smith (2011) on mathematics learners' understanding of letters (known in mathematics as variables). Kuchemann's framework integrated letter interpretation with structural complexity to correspond to Piaget's levels of concrete and formal thinking (Kucheman, 1978). In Kuchemann's framework, there were six levels for describing the various ways letters can be used:

Letter evaluated

There are no intermediate steps involving an unknown in this case, therefore the letter can be examined instantly. For example,

$a + 7 = 14$

$a = ?$

Letter ignored

The letter on the equation can be ignored by avoiding using it or solving for it. For example, $\boldsymbol{if\ a + b = 9}$, $\boldsymbol{a + b + c} = ?$ The answer is $\mathbf{9} + \boldsymbol{c}$, but learners might associate $\boldsymbol{c}$ with $\mathbf{3}$ and continue to solve $\mathbf{9 + 3 = 11}$.

Letter as object

At this level, operations can be performed on the letter without having to first evaluate it; however, the letter is not thought of as an unknown number but rather as a name for an object, a shorthand for an object, or both. This concept is frequently conveyed to children when, for example, the simplification of $3a + 2a + 5b\ to\ 5a + 5b$, where "$a$" stands for apples and "b" stands for bananas.

Letter as specific unknown

At this level, the letter is regarded as a distinct, albeit unknown, number that can be used without being analysed. For example, $4n + 6$ can be written as $10n$. This the result of a type of "association," in which the elements in the item are simply combined in the most obvious way, with no regard for what the elements might signify. 4 and 6 add up to 10, and the n is simply appended at the end.

Letter as generalised number

The letter is considered as having the ability to take or express a range of values rather than just one. For example, $c + d = 12$, where $c < d$, therefore, $c = ?$

Letter as variable

Viewing letters as variables entails an awareness that there is some kind of link between the letters, since their value changes in a consistent pattern. For example, blue hats cost 5 Rands each and red hats cost 6 Rands each. I buy some blue and some red hats and altogether it costs me 90 Rands. If b is the number of blue hats bought, and if r is the number of red hats bought, Write down the equation in terms of b and r. The answer here is $5b + 6r = 90$.

Smith (2011) noted the following tendencies committed by learners in the study:

- Rather than standing for numbers, letters were thought of as objects.
- Rather than being viewed as unknowns to be controlled, letters were given numerical values from the start.
- A letter represented a specific number; different letters had to represent different numbers.
- A letter was a specific unknown rather than a generalised number.
- A letter was perceived as standing for just a few possible values, perhaps restricted to discrete positive values, possibly extending to decimals, fractions, or/and negative numbers. Sometimes there was evidence of the student recognising further possibilities as they thought through each set of results and considered the implications.
- Interpretation depended on the perceived context, e.g., a formula as against an equation.
- Letters were associated with roles; often, *x* and *y* were introduced if the student felt two unknowns were needed.
- More than one interpretation might be used in a single question.

These findings show that variable interpretation should be emphasised, because it is a difficult task for learners. It will be difficult to go on to the next step of the problem if learners do not use the variable correctly. Linear equations rely heavily on variables since they make sense in the context of the given statement. The variable determines whether the left side equals the right side or vice versa. If learners do not grasp the concept or interpret the variable correctly, it will significantly affect their procedural fluency. It means that teachers must thoroughly comprehend all linear equation concepts to ensure that learners enhance their mathematical proficiency. To make progress in class, teachers should analyse the tasks they choose while constructing lesson plans based on past lessons. Teachers need to know when to interfere when learners are given a task and how to intervene so that learners are not discouraged from increasing their mathematical proficiency.

2.6.4 Additive and Multiplicative Inverse

In his study, Hall (2002) revealed multiplicative errors that learners committed. For example, learners answered as follows:

$$5 + \frac{x}{2} = 2$$

$5 + x = 4$,

and

$4x = 1$

$x = 1 - 4$.

Learners did not understand the idea of multiplicative inverses. They did not attempt to double-check the answer by substituting it into the original equation for verification. This finding may substantiate Greeno's (1991) findings, which state that it does not always occur to learners that the correct answer to a basic linear equation must be substituted into the original equation so that the left and right sides are equal. This study traces back to the equal sign concept, which states that if teachers do not emphasise conceptual knowledge, learners are more likely to make such errors. Teachers must place a greater emphasis on procedural fluency, as working through the processes of a problem and applying numerous procedures to solve a problem with linear equations enhances conceptual understanding.

The division error was also discovered in learners' answers: It should be noted that this type of error was a combination of the division and rounding off. e.g.,

$5x = 39$

$x = 7.4$

It could be a sign of inadequate subordinate skills such as the ability to divide integers to get a non-integer answer. A critical analysis of a learner's work can occasionally indicate lacking subordinate skills. Many subordinate skills are applied in the solution of linear equations, and their application may have just as much impact on the final success rate as any comprehension of linear equations or their structure.

Hall (2002) also found additive inverse errors from learners' work, for example:

$4x - 2 = x - 1$

$4x + x = -1 - 2$

And

$4 - 3x = 10$

$4 - 3x + 4 = 10 + 4$

$3x = 14$

The learners changed the side but not the sign, which may indicate confusion about the additive and multiplicative inverses. It could also indicate a lack of grasp of the equations' significance. Learners may be familiar with the concept of inverses but are unsure when they should be used. Teachers familiar with errors should be better prepared to anticipate the most typical errors, which might be discussed in class. This may be especially important at the introductory level because it prevents the formation of misconceptions and inaccurate constructions on the part of the learner.

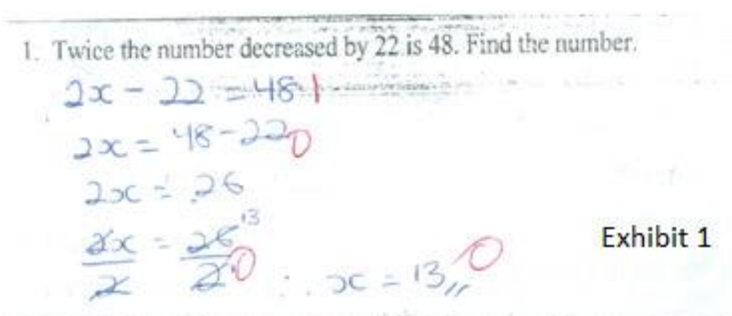
2.6.5 Word problems

The following conclusions were drawn based on the findings of Ibrahim and Yaw's (2019) study on the difficulty learners encounter when completing word problems in linear equations with one variable:

- Learners struggled to solve non-routine word problems.
- They had challenges in recognising patterns in problems before finding solutions.
- They had challenges in analysing and converting word problems into algebraic expressions and, finally into one-variable linear equations.

Teachers must expose learners to several examples of word problems to stimulate their interest in dealing with word problems involving linear equations with one variable. It indicates that teachers should emphasise strategic competency more, in which learners are introduced to various problem-solving strategies and allowed to apply them based on their understanding. In my conceptual framework, strategic competence focused on choosing the best tools to solve problems in linear equations. Teachers should engage learners in a series of non-routine word problems involving linear equations with one variable. Mathematics teachers should engage learners more in topics relating to identifying patterns in word problems. Teachers and learners should analyse and translate word problems into algebraic expressions and linear equations using real-life problems and concrete materials. Teachers should encourage learners to establish a habit of solving several examples of word problems with linear equations with one variable, leading to conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and ultimately productive disposition in word problems.

Many learners did not understand basic terms used in solving linear equation word problems, according to a study conducted by Adu et al. (2015) to examine errors learners make in solving linear equations word problems to expose the nature of these errors and make suggestions for classroom teaching. They also failed to do correct analysis when deducing equations from word problems because they showed no interest in solving linear equation word problems. The learners' incapacity to translate and solve algebraic word problems stems from their inability to break the questions into smaller parts, interpret, and represent words using variables. Mathematics teachers must provide a platform for learners to relate word problems to mathematical ideas and concepts so they can connect or relate them to everyday real-life situations and problems. Strategic competence should be emphasised when learners are introduced to different ways of solving the same problem and coming up with the same number. Teachers should also allow learners to devise their strategies if they are mathematically correct. Several activities administered during Adu et al. (2015) study are reflected below:



The learner was able to represent the word problem in algebraic form in Exhibit 1; however, the notion of additive inverse was difficult to grasp. Even if the answer in Step 3 was erroneous, the learner could appropriately apply the multiplicative inverse skill. A mathematically proficient teacher should be able to use the item analysis of the activity to guide their teaching and the lesson preparations should address the learner's misconception.

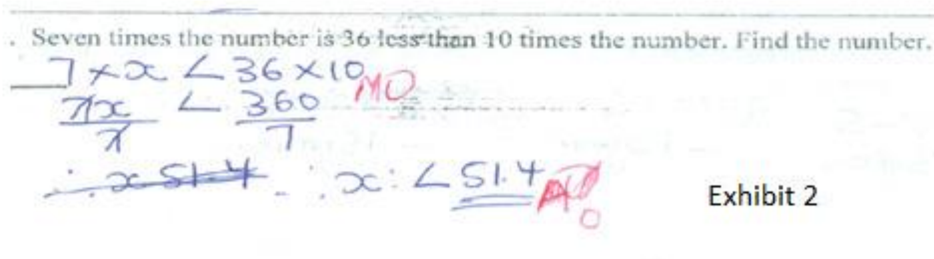


Exhibit 2

In Exhibit 2, the learner could not interpret the word sum to formulate an equation. The proficient teacher must then curb the misconception that the learner might have experienced when answering the question. Usage of mathematical language should be central when working with word sums for learners to understand the terminologies used.

2.7 Conceptual Framework

2.7.1 Introduction

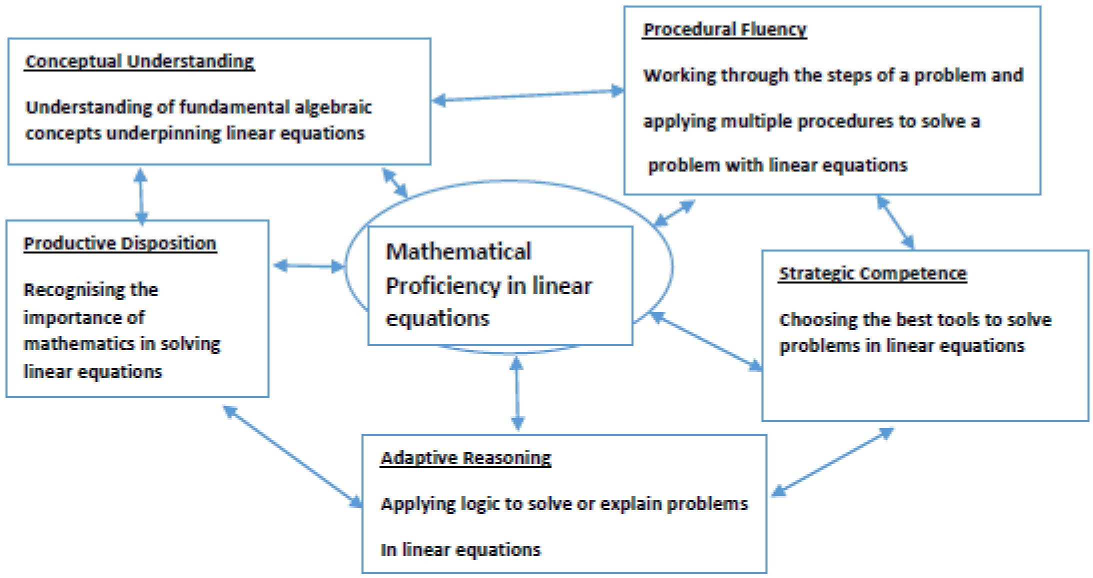
Camp (as cited in Adom et al., 2018) explained a conceptual framework as a structure which can best clarify the common progression of the phenomenon to be examined. Peshkin (1993) accentuated that a conceptual framework relates to the concepts, observational inquiries, and imperative speculations utilised in advancing and systemising the information to be queried. Similarly, Liehr and Smith (1999) concurred that a conceptual framework presents a coordinated way of addressing an issue to be studied. These views are also supported by Luse et al. (2012), who believe that the framework simplifies the concepts within the problem of the study, and it is then easy to define the concepts. In contrast to the definitions mentioned, Grant and Osanloo (2014) accept that a conceptual framework is organised in a coherent structure and using a picture or visual to show how thoughts in a study relate to one another.

In a nutshell, the definitions above have a common understanding that a conceptual framework gives a clear direction of the actions the researcher intended to take to address the problems in the study. Almost all the definitions included a section relating to a logical structure, key concepts, and the relationship among the concepts of the framework. The conceptual framework of research provides numerous benefits. Grant and Osanloo (2014) contended that it helps the researcher recognise and build the worldview on the phenomenon to be examined. Akintoye (2015) and Liehr and Smith (1999) supported that this was the easiest way through which researchers could present their strategies and attest to the issue that has been characterised. Similarly, Evans (2007) argued that it complements why a researcher's theme is worth considering, the presumptions of the researcher, the researchers that concur with the researcher's views and those opposing the views, and how the approach is conceptually grounded. A conceptual framework guided my study towards a constructive, meaningful, and clear picture of the steps to be taken in solving the problems set out in the study.

A conceptual framework utilised in this research was based on Kilpatrick et al.'s (2001) intertwined proficiency model strands. Kilpatrick et al. (2001) used the word "intertwined" since the strands are not free from one another; they expressed diverse perspectives of a complex entirety, meaning that mathematical proficiency can only be achieved by considering all the strands instead of one or two of them. If they are loosely used, there will be no effectiveness and efficiency in the content. These interwoven and interdependent strands of proficiency in mathematics are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. It means that they are locked together, tight, and inseparable. It is a platform for examining the information, aptitudes, capacities, and beliefs that create mathematical capability.

Figure 2 below is an illustration of my conceptual framework. The mathematical proficiency strands do not promote a hierarchy but are all at the same level. They are also indicated by the arrows that point at each other to show they are interdependent and interwoven.

Figure 2: Conceptual framework on strands for mathematical proficiency in linear equations.



Source: Adopted from Kilpatrick's (2001) intertwined strands of proficiency

Figure 2 illustrates the strands used in the study I undertook to explore the strategies that teachers use to address mathematical proficiency in their teaching of linear equations. It guided me on the skills that each learner needed to master to be regarded as mathematically proficient. The figure illustrated skills for teachers to consider when structuring the lesson plans and activities to be administered in class. The solid information and conceptual understanding of content by teachers ought to be shaped so that they can encourage their learners' improvement of mathematical capability. A teacher's mathematical capability can make educational programmes significant, related, pertinent, and beneficial.

2.7.2 Strands of Mathematical Proficiency

According to Barham (2020), the research shows that mathematics teachers need to learn teaching techniques to help their learners learn and apply the strands of mathematical proficiency. Teachers need to learn basic knowledge and specific teaching abilities to assist learners in acquiring conceptual understanding, according to the National Research Council (2001) and the National Council of Teachers of Mathematics.

Furthermore, the councils argued that teachers' pedagogical qualities in employing numerous representations, connecting mathematical ideas, and developing mathematical skills are critical for learners' conceptual understanding.

2.7.2.1 Conceptual understanding

According to Kilpatrick (2001), conceptual understanding is a comprehension of mathematical concepts, operations, and relations. This regularly occurs in learners comprehending associations and likenesses between interrelated actualities (Kilpatrick, 2001). The view was supported by Skemp (1976), who indicated that relational understanding is about not only knowing the method and why it worked, but the relationship between the method and the problem and using it to solve new problems. Skemp (1976) further argued that there is plenty to acquire- the vital part is that once learning has been firmly grasped, the results are everlasting.

Similarly, Hiebert and Lefevre (1986) noted that conceptual knowledge is attained by developing connections between the pre-existing and the new knowledge that has just been learnt. In my view, relational understanding and conceptual knowledge are closely related to conceptual understanding because they all emphasise the connections and the relationship between concepts.

The ability to completely perceive mathematical ideas, convey such concepts in more than one form, link them to appropriate procedures, and infer and judge interrelations reasonably and correctly describes conceptual understanding (Shteiwi et al., 2019). It also entails the precise and rapid use of such symbols in a correct mathematical language, giving the student the flexibility and fluency needed to solve mathematical problems. According to Obeida (2017), conceptual understanding entails correctly processing mathematical concepts in the learners' cognitive structure, all relevant generalisation, and a deep and unambiguous knowledge base. Understanding mathematical concepts, their meanings, qualities, symbols, connected procedures, methods of application in real-life circumstances, and inferring appropriate mathematical generalisations are all signs of conceptual understanding.

Al-Mutawah et al. (2019) argue that while conceptual understanding aids learners in avoiding common errors in problem-solving, procedural knowledge aids them in putting their procedural knowledge to use. Knowing when and how to accomplish tasks effectively and efficiently requires knowing when and how to perform them. Understanding is necessary for learners to tackle challenges they will confront in the future.

Based on the definitions above concerning my study, it is imperative to understand relationships between the concepts of linear equations, mathematics, connections in real-life experiences, and representations to demonstrate conceptual understanding. In this case, it will enable learners to make sense of linear equations.

The relationship between the concepts of linear equations

The relationship between the concepts of linear equations is defined by the symbols used to represent them. Conceptual understanding emphasises a solid understanding of fundamental algebraic concepts, symbols, and keywords. For example, if the teacher considers the problem, $2x + 3(x - 2) = 6$, it is important to understand what each item implies and its importance:

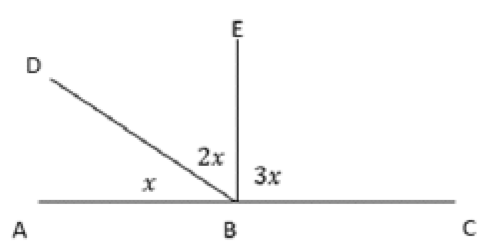
Table 3: Features of the linear equations

$2x + 3(x - 2) = 6$	
2	coefficient
x	variable
$+$	operation
$-$	operation
$=$	equal sign
6	constant

Conceptual understanding in linear equations occurs when an understanding is built within the connections between numbers and variables, the application of operations, and the *big ideas* found in proportional reasoning. A coefficient is a number multiplied by a letter representing an algebraic variable. For example, the number 2 of the term $2x$ in the equation is a coefficient. The word 'coefficient' comes from two Latin words, com, which means 'together' and *efficere,* which means 'work out' or 'accomplish' (My Maths Buddy Dictionary, 2011). A constant is a value that is fixed and does not change. For example, 6 is a constant. Constants are stable data and used to gradually work out the problem and get the solution. The word 'constant' comes from the Latin word *constare,* which means 'to stand firm' (My Maths Buddy Dictionary, 2011). An operation is a mathematical way to get an answer to a problem. The most used operations are addition, subtraction, multiplication, and division.

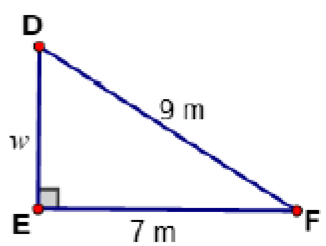
The relationship of linear equations and concepts in mathematics

The relationship between linear equations and other concepts in mathematics is essential for developing deep conceptual understanding. For example, competency development in algebraic equations depends on the capability to find multiples and factors and the prime factorisation of whole numbers. Integers and common and decimal fractions are used in most cases as coefficients in algebraic equations. The ability to apply all four operations with integers and properly use the properties of integers and executing multiple operations using common fractions, decimal fractions, and mixed numbers, illustrates conceptual understanding. The geometry of two-dimensional shapes and of straight lines indicates that solving problems related to them needs the basis of linear equations. For example, the conditions of angles on a straight line must be applied before solving the equation which indicates that angles on a straight line are equal to 180 degrees.

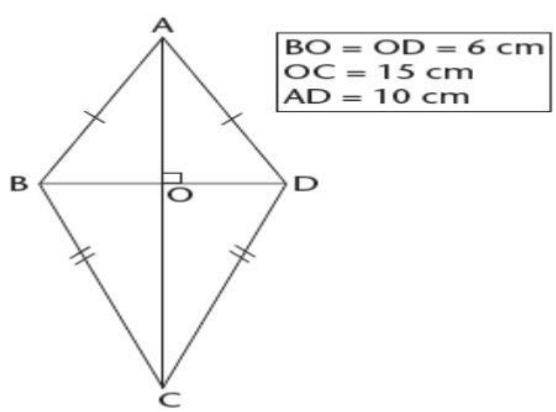


The Pythagoras' theorem, the formulas for calculating the area and perimeter of two-dimensional objects, and the surface area and volume of three-dimensional objects emphasise that the use of formulae provides a context to practise solving equations by inspection or using additive or multiplicative inverses. In the following examples, formulae $r^2 = x^2 + y^2$ $and\ area\ of\ a\ Kite = \frac{1}{2} \times d1 \times\ d2$ $(where\ d1\ and\ d2\ are\ long\ and\ short\ diagonals\ of\ the\ kite)$ are used to do the calculations, which require conceptual and procedural knowledge of linear equations:

1. Determine the length of w:



2. Calculate the area of the following kite:



Different representations of the algebraic concepts in linear equations

Conceptual understanding of linear equations is also demonstrated by the ability to interpret different representations of algebraic concepts such as flow diagrams, formulae,

and word problems. Conceptual understanding encourages a higher level of thinking when problems are solved through investigations and answering questions. Conceptual understanding is evident when learners become fluent in procedures and can use them in situations across their mathematical experiences. Allsopp et al. (2016) and Witzel (2016) strengthen the importance of creating conceptual understanding in learners, since comprehension gives a strong premise for future learning.
Some examples representing linear equations are reflected below:

Flow diagram	Formula/ equation	Word problems
input output $\longrightarrow \boxed{3n+1} \longrightarrow 16$	$3n+1=16$	A number is multiplied with 3 and the answer added to 1 equal to 16.

Different representations enforce a conceptual understanding of linear equations. Numerous teachers and researchers know that the introduction of variable-based mathematics, based on the study of expressions and equations, can pose genuine deterrents in preparing for successful, and important learning (Kieran, 1992). As a result, it is prescribed that teachers should teach so that learners utilise different representations from the start of learning variable-based mathematics (National Council of Teachers of Mathematics [NCTM], 2000). The use of verbal, numerical, graphical, and algebraic representations can make the method of learning variable-based math significant and compelling.

The word-based (verbal) representation is generally used to present a problem, and the final clarification is obtained by representing it in algebra. It accentuates the association between mathematics and other academic and daily living spaces. Learners can recognise the numerical representation at the start of their exposure to algebra. This approach offers a helpful and successful bridge to variable-based mathematics. The graphical representation is compelling in giving a clear picture of a real-valued function of a real variable. Algebraic representation is brief, wide-ranging, and operative in presenting patterns and mathematical models. The importance of working with different representations is up to learners' styles of learning and intellect. In this way, both educational programme designers and teachers ought to be mindful of the consideration of working with multiple representations to solve the problem and its solution in several ways.

2.7.2.2 Procedural fluency

According to Kilpatrick (2001), procedural fluency is the skill of carrying out procedures flexibly, accurately, efficiently, and appropriately. Similarly, Hiebert and Lefevre (1986) assert that procedural knowledge incorporates knowing the formal dialect of the image representation system, calculations and rules for completing projects, and methods and techniques for tackling problems. This implied that procedural fluency is applied with a deep understanding of concepts emphasising the mastering of concepts before procedural skills; therefore, conceptual understanding and procedural fluency are dependent on one another and intertwined.

Selecting an appropriate mathematical method

According to Al-Shammari (2019), procedural fluency is the capacity to select the most appropriate mathematical processes to answer problems skilfully and precisely. Learners with procedural fluency can build processes to solve known situations rather than memorise them to solve familiar problems. The learners' capacity to retain procedures of mathematical operations, perform them quickly, and accurately, and use them appropriately and expertly to relate concepts and relations among operations is also reflected by fluency. For example, procedural fluency focuses on dealing with the steps of a problem and using multiple measures in solving linear equation problems with understanding:

$$2x + 3(x - 2) = 6$$

$$2x + 3x - 6 = 6$$

$$2x + 3x = 6 + 6$$

$$5x = 12$$

$$\frac{1}{5} \times 5x = 12 \times \frac{1}{5}$$

$$x = \frac{12}{5} \; or \; 2{,}4$$

Skills in proficiency include efficiency and accuracy in basic computation, and the knowledge of appropriately and relevantly using procedures and sequencing actions undertaken when solving equations, as seen in the example above. The parenthesis and the implied multiplication in the expression $3(x-2)$, if not explicitly discussed with learners, this might cause further misconception and error in solving this problem. This indicates that knowledge of procedures and algorithms is nested in conceptual understanding.

Procedural fluency executed through memorisation

However, procedural fluency can happen without conceptual understanding, which leads to memorising a set of rules. Memorisation is a less effective strategy than more engaging methods; procedures are performed without understanding. This resonates with Skemp's (1976) idea of instrumental understanding. Skemp (1976) refers to instrumental understanding as just being able to apply multiple steps without knowing why they are being connected in that way or what they imply–rules without reasons. For example, when the thought of inverse operations is obscured by the memorised rule- "when taking a number from the left side/right side it changes the sign or when the number jumps the equal sign it changes the sign", learners will expect to see the numbers jumping the equal sign. This implied that a lack of conceptual understanding of algorithmic methods of solving equations could lead to challenges when attempting different representations of linear equations, using various letters, and having to incorporate real-life situations later when pursuing mathematics for further studies. Perso (1996) communicates concerns that learners who use a set of memorised rules tend to have misguided judgements when solving equations. Perso claimed that baffled teachers educate learners utilising rules rather than empowering conceptual understanding of algebraic forms. Teaching by memorisation needs to be discouraged, as it becomes an obstacle in effectively learning procedures in linear equations. Therefore, for teachers to develop learners' procedural fluency, they need to have a strong procedural fluency in performing essential guideline routines to curb learners' misconceptions. For example, the misconception of an equal sign where learners interpret it as a command to do something,

$7 \pm 3 = - + 6 \rightarrow 7 + 3 = 10 + 6 = 16$, indicates that learners do not conceptualise the concept of the equal sign.

2.7.2.3 Strategic competence

Strategic competence is the ability to formulate, represent, and solve mathematical problems (Kilpatrick, 2001). Similarly, Kieran and Chalouh (1993) alluded that for learners to develop a conceptual understanding of algebra, daily language needs to be used to make it simple to represent it in symbolic form. This view was supported by Esty (1992), Johanning (2000) and Pugalee (2004), who indicated that learners benefit from teaching that incorporates numerous sorts of numerical and verbal communication, writing and understanding word problems, examining solution techniques and ideas, and journaling. For this study, strategic competence is the capacity to characterise word problems in algebraic symbols and solve using more than one way of solving the problem. In this process, learners should not forget that for it to be effective, it must be linked to conceptual understanding and procedural fluency.

Using different strategies to solve the same problem

According to AI-Shammari (2019), strategic competence refers to learners’ capacity to solve mathematical issues, determine important mathematical facts and present them in various ways, find mathematical interrelationships, and elicit different solution approaches that meet problem requirements. As a result, the learner develops the resilience required for mathematical problem-solving processes, which can be presented in various ways. These ways include drawing, mental representation, or writing a formula that reveals interrelationships, using appropriate strategies such as figure drawing, guessing, table construction, logical elicitation, and modelling to present the context of a mathematical problem. Strategic competence, conceptual understanding, and procedural fluency are all linked. To create non-routine solution techniques, the learner must understand implicit knowledge and problem interrelationships, as well as fluency and proficiency in solving problems (NRC, 2004; Qarni & Shalhub, 2019), as indicated in Table 3.

Table 4: Strategic competence example

<table>
<tr><td colspan="2">**Word problem**: On weekends, Chris works in a steakhouse. He earns a basic wage of R105 plus R8 commission for every table he serves. Write an equation to show the amount y, that he can earn and find out how many tables he served on Friday night if he earned R177.</td></tr>
<tr><td colspan="2">**Formulating and representing** – Let the number of tables he served be x:
$y = 8x + 105$</td></tr>
<tr><td colspan="2">Solving the problem using different strategies</td></tr>
<tr><td>Method 1: Trial and error</td><td>Method 2: The balanced approach</td></tr>
<tr><td>Substitute x with positive numbers since tables are dealt with:
For x = 1: 8(1) + 105 = 177
133 ≠ 177
For x = 2: 8(2) + 105 = 177
121 ≠ 177
.
.
For x = 9: 8(9) + 105 = 177
177 = 177
∴ x = 9</td><td>$8x + 105 = 177$
$8x + 105 - 105 = 177 - 105$
$8x = 72$
$\frac{1}{8} \times 8x = 72 \times \frac{1}{8}$
$\therefore x = 9$</td></tr>
</table>

The example above demonstrates strategic competency in exposing learners to multiple ways of solving a mathematical problem. Depending on a fixed method to solve problems is discouraged in strategic fluency.

Selection of own method to solve problems

A better conceptual understanding gives learners the ability to use their own strategy when a new topic is introduced. It encourages creativity and innovation in which strategies that have not been shown before in class are used. Choosing the best tools to solve problems in linear equations strengthens strategic competence. In this case, the teacher solves the problems using different approaches to cater to learners of all cognitive levels;

therefore, learners use specific strategies to solve the equation. It is about allowing learners to make a sensible decision on a suitable strategy depending on their cognitive level. The teacher encourages learners to be innovative and create different strategies from what the teacher has provided during the lesson. They will only be considered if they are correct. This will assist in developing the skill to articulate, characterise and tackle mathematical problems. Using different strategies to solve any mathematical problem indicates a conceptual understanding of the problem. Therefore, a teacher should teach so that learners possess a strategic competence in arranging successful teaching and tackling problems that emerge amid teaching to address them effectively.

2.7.2.4 Adaptive reasoning

Kilpatrick (2001) states that adaptive reasoning is the capacity for logical thought, reflection, explanation, and justification. It is the ability to reason around concepts and conceptual associations. It implies that adaptive reasoning emphasises the justification of mathematical ideas and strategies. It is significant for learners to have the capability to justify their answers to indicate a deeper conceptual understanding of the concept. Reasoning must be enforced to understand the different procedures, proofs, and notions to reach a solution.

Justification of procedures followed in solving problems

Adaptive reasoning remains rooted in metacognition, which emphasises knowing the reason behind the procedures and strategies undertaken. Mathematical reasoning emphasises developing higher-order thinking skills. Prior knowledge is acknowledged to increase reasoning abilities. Encouraging talks about mathematics are crucial in promoting the use of mathematics properly. Conceptual understanding is important as a basis for a better understanding of concepts which leads to comprehensive reasoning. Adaptive reasoning is applying logic to solve or explain problems in linear equations. Throughout solving the problems, educators need to explain why a certain procedure was taken according to the rules and principles of mathematics.

When teachers use their teaching skills in various ways, such as conducting mathematical proofs, estimating, and predicting, finding analogical correspondences that represent powerful reasoning mechanisms, and applying intuitive, inductive, and deductive thinking, adaptive reasoning can succeed (NRC, 2004). The ability to think logically about relationships between mathematical concepts and contexts is known as adaptive reasoning. Because it is derived from contemplation, interpretation, and logical thought, adaptive reasoning is required.

Application of logic to comprehend and justify each step towards the solution

According to Qarni and Shalhub (2019), adaptive reasoning is the ability to think about relationships, concepts, and situations emotionally and use intuition, induction, and guesswork. It is used to understand the problem's various elements thoroughly. It also aids in learning process orientation and identifying the appropriate solution measure. Learners use adaptive reasoning to track their progress by putting their solution idea into action. Reasoning also entails the application of logic to comprehend and justify a problem's solution or to synthesise one.

In linear equations, teachers need to stress the skill that guides them to move from one step to the next in every step of the procedure, as this will build a concrete foundation when solving problems. For example, consider the following:

$2x + 3(x - 2) = 6$

$2x + 3x - 6 = 6$ (BODMAS rule)

$2x + 3x = 6 + 6$ (Additive inverse, grouping like terms, integers, and numbers)

$5x = 12$

$\frac{1}{5} \times 5x = 12 \times \frac{1}{5}$ (Multiplicative inverse)

$x = \frac{12}{5}$ or $2{,}4$ (conversions)

In adaptive reasoning, procedures in linear equations must be backed up by a reason to undertake a specific rule in each step. In adaptive reasoning, the teacher needs to encourage learners to justify their thinking, give learners a chance to explain their procedure and do calculations procedurally with understanding when solving linear equations. Therefore, the justification of responses or mathematical ideas by learners must always be inspired by teachers.

Manipulation of procedures when solving problems

In the study conducted by Monari and Pellegrini (2010), learners were unable to use procedures in mathematics for solving word problems; instead, they used those for physics, which made it difficult for researchers to pronounce how learners used the mathematical proficiency strands. The statement suggested that when learners are given a problem to solve, it is difficult to measure the extent to which they have achieved the skills of mathematical proficiency until they explain it verbally. It implied that learners could manipulate the procedures without understanding, making it challenging to conclude whether they mastered the skills.

2.7.2.5 Productive disposition

Kilpatrick (2001) stated that a productive disposition is a habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's efficacy. When an individual has a positive attitude towards mathematics, it is an indication that they have acquired a productive disposition.

Eliciting courage and enthusiasm to solve problems

Productive disposition advances when learners have a deeper understanding of mathematical concepts, enjoy learning and can do mathematics confidently. This view is supported by Cuoco (1996), who indicates that those with productive disposition could relate to and identify real-life situations with the mathematics done in the classroom setup. They can extend their knowledge from the known to the unknown. It implied that those with productive disposition have the courage and enthusiasm to prosper in mathematics endeavours and learn more by regularly participating in mathematical tasks. Those with productive dispositions are inquisitive about mathematics and are eager to solve problems until the solution is reached, no matter how long it takes. For example, in solving linear equations, learners will see it through to complete the problem at hand until they have a concrete understanding of the procedures. It is an indication of self-belief, courage, and love for mathematics. The more the learners believe in themselves, the more they become courageous to attempt mathematics-related problems irrespective of the difficulty level. I have indicated earlier those linear equations can be represented in different forms; therefore, learners with productive disposition will find pleasure in working out problems in whatever form until it makes sense to them. Productive disposition is about habitual inclination, which encourages the habit of solving problems on own initiative.

Instilling an interest in solving problems

The teachers' mathematical knowledge level, the importance of mathematics and their trust in learners' ability to achieve influence learners' achievement and passion for mathematics. Therefore, it is imperative to demonstrate productive inclination through teaching, learning, and using different practices required for mathematical proficiency. Adolescents' interest in mathematics ranges from obsessive avoidance to obsessive pursuit (Cangelosi, 1996). Teachers need to be mindful of these, because learners will come to class with a wide variety of interests within those two extremes. Most of the learners arrive in class with a distaste for mathematics, especially linear equations. This might create some tension between the teacher, and the learners; however, it is in the teacher's best interest to avoid the tension by motivating these learners. Exposing learners to activities they can relate to is one of the strategies to instil interest in them. Learners are more likely to work eagerly on mathematical tasks that relate to what they consider important than on tasks that they perceive irrelevant to their immediate interest. For example, in linear equations, word problems are closely related to real-life experiences because they include problems involving ages in most instances. Thus, the teacher needs to capitalise on the learners' existing values to design lessons that focus on prior knowledge.

Encouraging effective communication about mathematical aspects

Another aspect that can help teachers to address productive disposition when teaching linear equations is effective communication. Van Horn (1982) indicated that learners feel safer and freer to participate in tasks given when the teachers do not use threatening or descriptive language than when they use judgemental language. It implies that for teachers to help learners develop the love for mathematics, particularly linear equations, they should teach in a way that incorporates all the above strands of mathematical proficiency. For example, when teachers teach conceptually and procedurally, they encourage learners to be innovative and come up with their own strategies and be able to justify their responses, and then learners would acquire a productive disposition. Descriptive language avoids the labelling of learners and the dangerous practice of confounding mathematical and other academic achievements with self-worth (Cangelosi,1996). The implication is that learners must be treated cautiously to avoid denting their image, which might cause them to hate mathematics.

Fostering a positive attitude towards mathematics

Several studies have investigated the role of teacher ability in fostering positive attitudes towards mathematics learning (Fang, 2012; Shriki & Lavy, 2011; Shriki & Patkin, 2016). Engaging learners in mathematics is another aspect that facilitates the acquisition of a productive disposition. When learners are procedurally engaged and interacting with the task and each other, when there is a cognitive challenge within the task, and when they understand that learning mathematics is worthwhile, valuable, and useful both inside and outside the classroom, deep engagement with mathematics occurs.

A three–year longitudinal study on the influences of student engagement yielded the Framework for Engagement with Mathematics (FEM) (Attard, 2014), as indicated in Table 4. It gives mathematics teachers insight into the foundations for learners to engage in Mathematics. The Framework for Engagement with Mathematics emphasises that active learning is influenced by the teachers' pedagogical repertoires, resources, and activities and by the deeper level of pedagogical connections that develop between the learners and teachers–this defines the learning relationships between teachers and learners. If there are none of these links, it is a sign that something is wrong. Regardless of how attractive the activities or duties are, it is then doubtful that engagement will emerge. Table 4 reflects the Framework for Engagement with Mathematics:

Table 5: Framework for Engagement with Mathematics

FRAMEWORK FOR ENGAGEMENT WITH MATHEMATICS

In an engaging mathematics classroom, positive pedagogical relationships exist where:

- Learners' **backgrounds** and pre-existing knowledge are acknowledged and contribute to the learning of others
- The teacher is **aware** of each student's mathematical abilities and learning needs
- **Interaction** among learners and between teacher and learners is continuous
- The teacher models **enthusiasm** and enjoyment of mathematics and has a strong **pedagogical content knowledge**
- **Feedback** to learners is constructive, purposeful, and timely

In an engaging mathematics classroom, engaging pedagogical repertoires mean:

- There is a **substantive conversation** about mathematical concepts and their applications to life
- Tasks are positive, provide an opportunity for all learners to achieve a level of **success** and are **challenging** for all
- Learners are provided with an element of **choice**
- Technology is embedded and used to enhance mathematical understanding through a **student-centred** approach to learning
- The **relevance** of the mathematics curriculum is explicitly linked to learners' lives outside the classroom and empowers learners with the capacity to transform and reform their lives
- Mathematics lessons regularly include a **variety** of tasks that cater to the diverse needs of learners

Learners are engaged with mathematics when:

- Mathematics is a subject they enjoy learning
- They value mathematics learning and see its relevance in their current and future lives
- They see connections between the mathematics learnt at school and those used beyond the classroom.

Adapted from Attard (2014)

When accompanied by serious, hardworking, and competent learners, productive disposition refers to a propensity and feeling for mathematics, therefore seeing its significance and rewards. Furthermore, it instils self-assurance in the learners, causing them to regard the subject as important and deserving of attention. Identifying the rewards of perseverance throughout the mathematics learning process is necessary for developing productive reasoning (Siegfried, 2012).

The conceptual framework assisted my study in observing whether teachers were addressing different mathematical proficiency strands by understanding fundamental algebraic concepts, working through the steps of a problem, and using several procedures towards solving linear equations. In this process, they have to select the relevant strategy to solve problems in linear equations, apply the logic to solve or explain problems in linear equations and recognise the importance of mathematics in solving linear equations in their planning, presentations, interaction with learners, and the assessment they administer. It guided the route taken in ensuring that necessary information was gathered on the addressing mathematical proficiency in teaching linear equations.

In summary, the present study intended to use proficiency strands to explore the strategies used by teachers to address mathematical proficiency in their teaching of linear equations, intending to suggest ways to enhance teachers' practices. The study intended to use the findings to develop a theory that would benefit mathematics curriculum developers and implementers.

2.8 Chapter Summary

The chapter discussed the findings made by previous studies on the topic of linear equations in detail and outlined the strategies and challenges faced by teachers. The other point under discussion was the challenges learners faced when solving linear equation problems. A conceptual framework, which was inspired by the works of Kilpatrick (2001) on mathematical proficiency strands, was under discussion in this chapter. The mathematical proficiency notion emphasises the concepts of intertwined strands. Each strand was discussed in detail to get a clear picture of what it entails.

The research methods and design employed for this study are covered in the following chapter.

CHAPTER THREE: RESEARCH METHODOLOGY

3.1 Introduction

In this section, I will explain the research design, research methodologies, population, sampling, and data collection methods used in my study on how Grade 9 mathematics teachers use the strategies to address mathematical proficiency in their teaching of linear equations. The importance of this section is that it gathered all the methodological elements of the present research project to explain how data were gained

3.2 Research Design

I have followed a qualitative design for this research. Qualitative researchers try to analyse human accomplishment from the insiders' viewpoint, also described as an 'emic' perspective (Babbie & Mouton, 2008). Observation and analysis were the main aspects of this qualitative research and revolved around the participant. Observational methods such as semi-structured interviews and participant lesson observation were employed. A case study approach was also used for this study. This approach enabled me to gather rich qualitative data and better understand how mathematics teachers use strategies to address mathematical proficiency in their teaching of linear equations. The case study offered a greater understanding of the phenomena under study. According to Runeson and Höst (2008), the term case study is utilised similarly with terms like field study and observational study, each concentrating on a specific feature of the research methodology. Leedy and Ormrod (2015) indicate that a case study in some instances is called idiographic research, as a specific person, programme, or occasion is studied profoundly for a stipulated time. Similarly, Van der Walt and Van Rensburg (2006) agree that a case study is a comprehensive study of an individual, a collection of individuals, or an organisation.

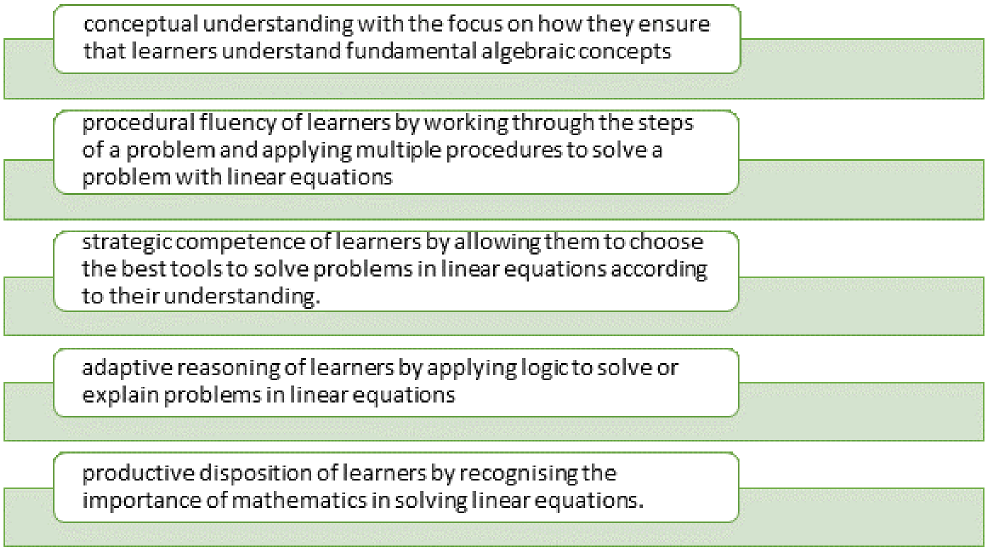
A case study was relevant to my research because I studied individual teachers on addressing mathematical proficiency in their teaching of linear equations. An exploratory approach was also used as it works hand in hand with a case study because I collected my own data through interviews and observations. Babbie (2011) indicates that in-depth interviews would be a useful technique, along with personal observations. This statement informed my research methods; as a result, I used classroom lesson observations and semi-structured interviews for my study. To establish rapport with the teachers I observed and interviewed, a detailed and documented explanation of my research was supplied to the participants. Consent forms were sent to the Principals, SGB, teachers, learners, and parents of participating learners. Because the emphasis of this study was on qualitative research in which meaning was interpreted, an interpretative paradigm was adopted.

3.3 Data Collection Instruments

3.3.1 Classroom Observations

Due to the COVID-19 pandemic, I could not be physically present in the classroom for observation. I relied on the video clips supplied by the teachers. Therefore, a remote classroom observation was conducted in Grade 9 mathematics classes to gather information about my study. A transcript of the lessons was compiled, and an observation checklist (Appendix B) was also used to organise the data obtained from the video clips. The items on the checklist were generated from the conceptual framework that explained the mathematical proficiency strands in detail with a focus on linear equations. When conducting research, my instruments were informed by the strands of mathematical proficiency. Furthermore, the descriptors of the strands assisted me in interpreting the collected data. The instruments entailed how Grade 9 mathematics addresses mathematical proficiency focusing on:

Figure 3: Mathematical proficiency strands



Adapted from the conceptual framework in Chapter 2 of this study

The classroom observations aimed to determine the strategies used by teachers in addressing mathematical proficiency in their teaching of linear equations. The classroom observations gave me an impression of the activities and the interactions that happened in the classroom, and how the teachers presented linear equations to learners.

3.3.2 Semi-structured Interviews

Semi-structured interviews were conducted, which consisted of standard questions (Appendix C) and follow-up questions from the lessons which the teachers conducted. The interviews were conducted using Microsoft Teams due to the COVID-19 regulations. The interviews were recorded after obtaining the teachers' consent.

3.4 Pilot Study

Janghorban et al. (2013) indicated that in-depth interviews, engaging with participants, choosing the right location for the interview, and taking advantage of the opportunities to elicit emergent interview themes are just a few of the unique opportunities that pilot studies can provide to qualitative researchers. This implies that conducting a pilot study before the main study lays a proper foundation for making the participants comfortable when gathering information. In addition to giving researchers a basis for self-evaluation of their readiness and capacity, the pilot study may also enable researchers to engage in qualitative inquiry practice, improving the credibility of their qualitative research (Janghorban et al., 2013). The implication is that it assists in reviewing the content of the instruments, the material to be used and the questions to be asked as to whether they will gather the information that was intended.

A pilot study was conducted with one Grade 9 mathematics teacher, namely Teacher A, to establish if the classroom observation instrument and the interview questions were adequate for this research. The total number of learners who took part in the pilot study in school A was eleven. The pilot study took place from 26-31 August 2021

Table 5 below presents the biographical information of Teacher A.

Table 6: Biographical information of Teacher A

Teacher's Name	Gender	Qualifications	Experience	School's name
Teacher A (Pilot Study)	Male	B.Ed Honours (Mathematics Education)	30 years	School A

The table above indicates that Teacher A had 30 years' experience in teaching mathematics. The acquired qualifications were B. Ed honours in Mathematics Education, which served as a good standard for a mathematics teacher. Teacher A's school was based in a semi-rural location under quintile 2. Quintile 2 means that the school depends on the government for funding and school fees are not payable. The school one of the adopted Mathematics, Science and Technology (MST) grant schools which is supplied with laptops, data projectors, calculators, mathematics kits and answer series booklets for learners.

The advantage of using an experienced and highly qualified teacher for my pilot study was that it served as a good benchmark to ensure that the instruments used, and the questions asked were of a high standard.

Lessons conducted by Teacher A on linear equations were observed through the videoclips sent by the teacher, and semi-structured interviews were conducted afterwards through MSTEAMS. The pilot study's findings showed that the lesson observation instrument used was relevant to the topic that was taught, and the teacher understood every question posed during the interview, as shown by the meaningful answers offered. The aim of having a pilot study was to check if the instruments that I developed were credible to be used in my research. As a result of the findings, no adjustments were made on the lesson observation instrument and the questions for the semi-structured interviews.

3.5 Population and Sampling

I conducted my study in two different secondary schools and a pilot study in one secondary school from Gauteng North District in Gauteng Province, based in South Africa. A purposive sampling technique allowed me to choose the sample based on familiarity with the phenomena being researched. The criteria used was to select participants of different gender, different qualification, teaching same grade, experience in teaching, the type of schools the participants were from and the location of the schools. For the main study, two Grade 9 teachers were observed on video while conducting lessons on linear equations, and they were then interviewed based on the lessons they conducted through MSTEAMS. Pseudonyms for the three teachers and their schools were used to keep their identity and that of the school confidential.

Table 7 presents biographical information of the participants:

Table 7: Biographical Information of the Teacher 1 and Teacher 2

Teacher's Name	Gender	Qualifications	Experience	School's name
Teacher 1	Female	B.Ed Degree in Mathematics & Life Sciences	4 years	School 1
Teacher 2	Male	B.Ed honours in Physical Sciences	3 years	School 2

Table 6 shows that Teacher 1 and Teacher 2 were not at the same level in terms of qualification. The reason for selecting such participants was to have a clear picture of teachers with different qualifications in terms of their mathematical proficiency. Their subject specialisations were different, Teacher 1 specialised in Mathematics and Life Sciences and Teacher 2 specialised in Physical Sciences. Regarding work experience, Teacher 1 and Teacher 2 were not at the same level with a difference of one year.

School 1 is in a semi-rural area and classified as Quintile 2, meaning that school fees are not payable, and the school depends fully on the government for funding. School 2 is in an urban area and is classified as Quintile 4. Quintile 4 means that the school is well developed, has resources and school fees are payable. Therefore, School 1 and School 2 are not at the same level. School 2 is an MST grant school which is supplied with different technology equipment for teachers, and learners. In other words, the school is well-resourced.

The total number of learners who took part in the main study in school A were eleven, and those who took part in school 2 were twelve. The main study took place from 27-30 September 2021.

3.6 Data Analysis Procedures

Transcripts of observed lessons and interviews, a rating scale (evidence of the addressing of mathematical proficiency in the concept) and a bar graph for each teacher were compiled to understand the findings better. The rating scale was developed based on the information from the lesson observation instruments and used to draw a bar graph to give a clear picture of how each teacher addressed mathematical proficiency in their teaching of linear equations. A rating scale with criterion was used to rate the teachers' strategies used in addressing mathematical proficiency in their teaching of linear equations from what was captured on the lesson observation instrument of all the lessons conducted by teachers. The results from the rating scale were then used to draw a bar graph. The scales ranged from 0-4, where the rate represented the following:

0 The teacher shows **no evidence** of addressing mathematical proficiency in teaching linear equations

1 The teacher shows **weak evidence** of addressing mathematical proficiency in teaching linear equations

2 The teacher shows **some evidence** of addressing mathematical proficiency in teaching linear equations

3 The teacher shows **strong evidence** of addressing mathematical proficiency in teaching linear equations

4 The teacher shows **very strong evidence** of addressing mathematical proficiency in teaching linear equations

The following abbreviations were used for mathematical proficiency strands:

CU: Conceptual Understanding

PF: Procedural Fluency

SC: Strategic Competence

AR: Adaptive Reasoning

PD: Productive Reasoning

Table 8 below sets out the rating scale:

Table 8: Rating Scale

Scale	0	1	2	3	4
Criteria	The teacher shows **no evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **weak evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **some evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **strong evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **very strong** of addressing mathematical proficiency in teaching linear equations
Strands					
CU					

Scale	0	1	2	3	4
PF					
SC					
AR					
PD					

The completed rating scale and the bar graphs were done in Chapter 4.

3.7 Qualitative Data Analysis and Interpretation Method

Thematic analysis was used to analyse the data that was gathered for this study. Kiger and Varpio (2020) define thematic analysis as a way of representing data, but it also involves interpretation when choosing codes and creating themes. Kiger and Varpio (2020) divided thematic analysis into six steps, namely,

Step 1: Familiarisation

Learning about data is the first step in thematic data analysis. Before evaluating specific items, it is critical to acquire a comprehensive analysis of the entire information gathered. This could entail listening to the audio several times and transcribing, repeatedly reading text, making some preliminary notes, and generally looking through the data to become familiar with it.

I familiarised myself with data by listening and watching the lessons conducted by Teacher 1 and Teacher 2 on several occasions to clearly understand the content that was taught and how it was presented. I also listened to the recordings of the interviews that were conducted multiple times to make sense of the answers that the teachers provided. This process assisted me in adapting to the information that I gathered.

Step 2: Coding

Coding is the process of identifying specific text passages, usually phrases or sentences, and assigning brief labels or codes to indicate the meaning of those passages. A code needs to be sufficiently well-defined and delineated so it does not conflict with other codes.

After listening several times to both the video clips and the interview recording, I could transcribe the interaction that happened in the classroom between the teachers and the learners and between myself and the teachers during the interview. These transcripts were used to detect the concepts in the text that were similar and somehow aligned with the topic under study. The lesson observation and interview transcripts were looked through a few times to check if anything was missed.

Step 3: Generating themes

At this stage, all themes that have the potential to be significant should be noted by the researcher, whether they are directly relevant to the research topic and regardless of the amount of information that falls within them. Prospective themes that can reveal anything useful about the data should be identified.

The next step was to classify the phrases according to the themes guided by my conceptual framework, which highlighted the mathematical proficiency strands. The themes focused on understanding fundamental concepts underpinning linear equations, working through the steps of a problem, and applying multiple procedures to solve a problem with linear equations. In the process, attention was paid to choosing the best tools to solve problems in linear equations, applying logic to solve or explain problems in linear equations and recognising the importance of mathematics in solving linear equations.

Step 4: Reviewing themes

This step ensures that the themes chosen to serve the purpose of accurately reflecting information. The researcher then examines the coded data presented within each theme. In this case, the researcher determines if specific themes make sense within the collected data and if the thematic map effectively and sufficiently captures the broader information collection.

At this stage, I scrutinised the themes again to check if they were aligned with what my study intended to yield.

Step 5: Defining and naming themes

The process of defining themes entails articulating precisely what we intend from each theme as well as determining how it aids in the comprehension of the facts. The technique of naming themes requires coming up with a brief and clear name for each subject. After scrutinising the themes, I assigned each theme a relevant name and used descriptors to clarify what each theme intended to convey.

Step 6: Writing up

By taking notes, defining themes, and choosing sample data extracts in earlier steps, the writing process has already begun. This final report involves an introduction, research questions, aims and approaches, methodology and conclusion.

3.8 Trustworthiness

According to Pilot & Beck (2014), trustworthiness or thoroughness of a study alludes to the degree of confidence in information, elucidation, and strategies used to guarantee the quality of a study. To ensure trustworthiness in my study, I collected data from teachers through video clips and interviews through MSTEAMS. I ensured that I spend as much time as possible listening to the recording of the lesson observations and interviews so that I could collect authentic data. My aim was to observe and collect data from teachers without influencing their practices and the methods of teaching. To achieve high levels of trustworthiness, I ensured that teachers felt safe and nonthreatened; our relationship was open, relaxed, and trusting. As a result, they felt uninhibited about the amount of information they disclosed to me. I reassured them that all the information revealed to me during the interviews and witnessed during the classroom observations was treated confidentially and with sensitivity.

Trustworthiness is outlined into four criteria, namely, transferability, dependability, confirmability, and credibility.

3.8.1 Transferability

Babbie and Mouton (2008) defines transferability as the degree at which the discoveries can be useful in other settings or with other respondents. To enhance trustworthiness in qualitative research, I was deeply immersed in the study and thoroughly listened to the teachers' videos from lesson presentations and the recordings from the interviews to gain sight on how they used strategies to address mathematical proficiency in teaching linear equations. To ensure transferability, I used purposive sampling of my participants and locations to maximise the range of specific information. I also collected sufficiently detailed information to allow decisions that needs to be taken about my study.

3.8.2 Dependability

Van der Walt & Van Rensburg (2006) describes dependability as a criterion that requires audit, the enquiry auditor takes after the forms and strategies utilized by the researcher within the study and decides whether they are adequate. An inquiry must moreover give its audience prove that in case it was to be rehashed with the same participants within the same setting, its discoveries would be comparative (Babbie & Mouton, 2008). This study used triangulation for the purpose of presenting the research findings that were credible and authentic. I used classroom observation and interviews to collect data. Using these two methods of data collection increased the trustworthiness and credibility of my research.

3.8.3 Confirmability

Van der Walt & Van Rensburg (2006) argue that confirmability ensures that the discoveries, conclusions, and suggestions are backed by the information and that there is correlation between the researcher's clarification and the real prove. I ensured confirmability by repeatedly watching and listening to video clips of the lesson presentations and listening to the recordings of the interviews conducted. I reviewed the themes that were developed, findings and conclusions and a final report.

3.8.4 Credibility

The definition of credibility by Polit & Beck, (2014) emphasises the certainty in the truth of the study and the discoveries. I was immersed in the field for a longer period to ensure deeper penetration in the situation and which allowed me to get authentic data. I conducted my observations and interviews until data saturation occurred. I repeatedly watched and listened to the video clips of the lesson presentations and recordings of the interviews, analysed to get more understanding on the data that I have collected and checked several times if correct interpretation was done.

3.9 Research Ethics

Research ethics according to Babbie (2011) is related with profound quality and both bargain with things of right and wrong. It is adjusting to the guidelines of conduct of a given profession or group. O'Leary (2014) describes research ethics as standards or rules of conduct that act to manage what is worthy or permitted within a profession. Van der Walt & Van Rensburg (2006) indicate that a researcher is accountable for conducting research in a moral way. Van der Walt & Van Rensburg (2006) argues that the researcher must:

- conduct the research competently
- manage the resources honestly
- recognise honestly those who contributed supervision and support
- communicate results precisely
- consider the consequences of the research for the field of study and for society in general.

Following these concerns, I ensured that my research was conducted in a manner that protected participants' human rights, that communication lines were open between me and the participants, and the information gathered was protected so that it does not end up in the wrong hands.

There are four essential moral standards that direct researchers (Van der Walt & Van Rensburg, 2006):

- **Principle of respect for persons** – Individuals have the right to independence, which implies that those who participated in my study were made aware that there is no risk or any penalty that will be imposed on them if they decide to participate or not. In addition, they were made aware that they have the freedom to no longer take part in the study whenever they feel that their privacy is infringed, to withhold information or to request further clarity about the purpose of my study.
- **Principle of beneficence** – It is in the best interest of the researcher to protect the welfare and reputation of the participants and make them comfortable and keep them away from harm. It means that I did not force any teacher to take videos of themselves while teaching, all teachers that recorded the lessons did that freely. During the interviews, if my participants did not want to disclose some information, I did not force them to do so.
- **Principle of justice** – It includes the participants' right to reasonable choice and treatment. I ensured that my sampling of participants represented the population. There

were no favours in selecting the participants in the study. I used purposive sampling technique because it allowed me to choose the sample grounded on the familiarity with the phenomena being considered.

- **Principle of confidentiality** – I complied with ethical issues of confidentiality, anonymity, and privacy by preventing all information collected during the study from being revealed or made accessible to any other person. I kept the partakers' identities a secret by providing each participant with a pseudonym. Pseudonyms were used when deliberating, and after that, the list of real names was kept safe, allowing the participants to determine the extent to which they wanted to share their information. The aim was to gain trust and rapport from the participants. The data gathered was strictly used for the intended research and will eventually be destroyed after it has been used maximally.

The final report was then developed using the information collected in one place. Presentations, interpretations, and discussions were guided by information that was put together according to the relevant themes. I minimised biasness by addressing instrumentation and bias issues with the aid of a well-conducted pilot study. One benefit of performing a pilot study is that it may provide early warning about potential failure areas for the main research project, potential protocol violations, and if suggested procedures or instruments are appropriate or overly complicated. The pilot study allowed my study the opportunity to keep track of the time it takes to complete the interview and decide whether it is reasonable, examine whether each question yields a sufficient range of replies, ascertain those responses can be interpreted in terms of the information needed, ensure that all questions have been answered and ensure that all the items on the observation lesson instrument were relevant.

3.10 Chapter Summary

This chapter explained the research design and methods used to carry out this study. The discussions were also focused on the conducted pilot study, the population, and the sampling of teachers, including their biographic information.

Procedures for conducting the analysis and interpretation of data are discussed in this chapter, which also presents the interpretation of the results.

CHAPTER FOUR: PRESENTATION OF DATA OF THE STUDY

4.1 Introduction

This chapter presents and interprets the study's findings on how do Grade 9 mathematics teachers use strategies to address mathematical proficiency in their teaching of linear equations. The study gathered data through video-recorded lesson observations and interviews.

I watched and listened to each video lesson on several occasions and listened to the recordings of interviews multiple times before I could transcribe data. Transcripts of those lessons and interviews were compiled and analysed, guided by the work of Kiger and Varpio (2020) on thematic analysis of qualitative data – (see Section 3.6 in Chapter 3). The focus was on mathematical proficiency strands, as discussed in Figure 3 of Chapter 3. The descriptors of each strand, as outlined in Figure 4, were used as guidelines to analyse data.

Figure 4 illustrates the descriptions of strands of mathematical proficiency used in this study that assisted in interpreting and analysing the data and making sense of the findings. As I watched the video lessons, I noted issues that were further probed during the interviews. The aim was to get clarity on what transpired during the lessons. All the teachers conducted lessons on linear equations. The concepts they focused on are set out below.

Figure 4: Descriptors of strands of Mathematical proficiency

DESCRIPTORS OF STRANDS OF MATHEMATICAL PROFICIENCY

CONCEPTUAL UNDERSTANDING

(a) Learners' Pre-requisite/ prior knowledge; (b) Explanations of concepts/ terminologies used in linear equations; (c)Correct usage of operations when solving linear equations; (d)Relationship between concepts of linear equations; (e)Relationship of linear equations with concepts within mathematics; (f)Relationship of linear equations with concepts in real-life situation; (g) Different representation of algebraic concepts in linear equations.

PROCEDURAL FLUENCY

(a) Carrying out procedures flexibly; accurately; efficiently; and appropriately; (b) Using multiple measures in solving linear equation problems with understanding; (c) Knowledge of appropriately and relevantly using procedures and sequencing of actions.

STRATEGIC COMPETENCE:

(a) Solving the problem using different strategies; (b) Exposing learners to more than one way of solving mathematical problem.

ADAPTIVE REASONING

(a)Justification of mathematical ideas and strategies; (b) Justification of procedures.

PRODUCTIVE DISPOSITION

(a) Encourage extension of knowledge from the known to the unknown; (b) Promotes courage and enthusiasm; (c) Encourages own initiative.

Figure 5: Lessons conducted by Teacher 1 and Teacher 2

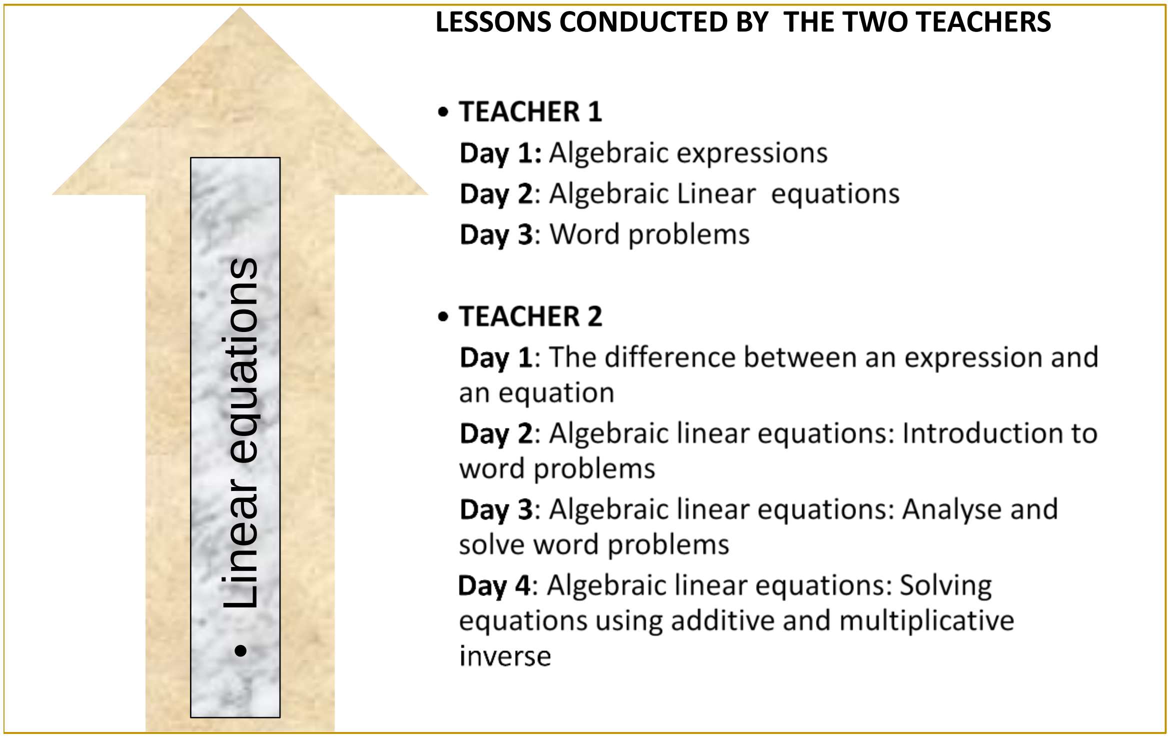


Figure 5 shows that Teacher 1 from School 1 conducted three lessons on three separate days, and Teacher 2 from School 2 conducted four lessons on four separate days. Both teachers started with algebraic expressions, and they used different techniques to address the concept of linear equations.

4.2 Presentations of Findings

In this section, I report on the classroom observations and semi-structured interviews conducted. For the sake of coherence, the findings are reported per participant. Data were presented according to what transpired in each day of the lesson presentation. To present the data, I started by describing what happened during the presentation and supported my description with evidence. In other words, what I have described is backed up by the words of the participants.

4.2.1 Teacher 1: Data presentation From the Lesson Observation and Interviews

Day 1: Algebraic expressions

1hour 45 minutes

On Day 1 of the presentation, Teacher 1 started by introducing the learners to algebraic expressions so that she could swiftly define the different types of expressions. The lesson started as follows:

1. **Teacher 1:** Today, we are going to look at algebraic expressions. The topic that we have done many times. I have made pages for you and on the smartboard so that you can have a better view and I do not have to re-write everything every time. We have done this topic in Grade 8 and even in the recent grade. When we talk about the expressions, we have done it many times. When we talk about expressions, it is divided by [a] positive or negative sign. This positive or negative signs, they divide the expression in either monomial, binomials, trinomials, or polynomials. Anyone tell us what is a monomial? An expression that has how many terms?
2. **Class:** One term!
3. **Teacher 1:** What about a binomial?
4. **Class:** Two terms!
5. **Teacher 1:** Trinomials?
6. **Class:** Three terms!
7. **Teacher 1:** What about polynomials?
8. **Class:** Four!
9. **Teacher 1**: Four or more terms.

To check learners' prior knowledge, Teacher A asked questions to check their understanding of monomials, binomials, trinomials, and polynomials (different types of expressions). Although the learners managed to answer the questions correctly, they responded as a group. Furthermore, Teacher 1 explained the features that made up an algebraic expression and did not confine the variable to 'x' only: The following conversation serves the point in case.

10. **Teacher:** Anyone who can give us the variable.
11. Learner: x
12. **Teacher:** Is the variable always x?
13. Class: No.

14. **Teacher**: The variables differ, it can be 'y' or 'e' or other letters. So, usually we say the variable represents the unknown.
15. Class: Yes

In lines 10-15, Teacher 1 discussed the terminology of a variable and built learners' conceptual understanding that 'x' is not the only variable that can be used. In this way, the teacher removes the stereotype that learners will only solve for x and exposes them to other variables. Using the expression: $12x^2 + 12x + 4$, Teacher 2 continued as follows:

16. **Teacher 1:** Then the coefficient of x^2?
17. Learner A: 12
18. **Teacher 1**: 12, so what is a coefficient? It is a number because someone said 12?
19. **Learner B:** It is the number that is next to the variable
20. **Teacher 1**: It is the number that is multiplied to the variable.
21. **Class**: Yes.
22. **Teacher 1:** The next one, the coefficient of x.
23. **Teacher 1:** Positive 12! So, the coefficient of x is also 12. But this is the 12 that is multiplied to [by] x, where the exponent is one. Then the number of terms in each expression, how many terms are there in each expression? How many terms are there? You said it is a trinomial, so how many terms?
24. **Learner C**: Three terms.

In lines 16-24, the teacher introduced the coefficient concept. A learner indicated that the coefficient was a number next to the variable; however, the teacher clarified that it was a number multiplied by a variable. The teacher addressed learners' conceptual understanding by removing their misconception that a coefficient is a number next to the variable.

Day 2: Algebraic linear equations

3 hours

On Day 2, Teacher 1 introduced learners to an algebraic equation with the emphasis that the equation must have an equal sign:

25. **Teacher 1**: I have tried to add a definition there just to remind you, Algebraic equations are statements that have two equal expressions. Which is true, right?
26. **Class**: Yes.
27. **Teacher 1**: If something is an equation, it needs an equal sign?

28. **Class**: Yes.
29. **Teacher 1**: On each side of an equal sign, there is an expression, we have this one, and we also have this one; they are expressions. This one has two terms; it is a binomial, and this one [is] a monomial. If something is an equation, it must have an equal sign separating the two expressions. So usually, we try to solve for one unknown variable. The way we are going to do [this] we are solving one variable; it can be 'x', 'y' or 'p' any other variable. Then it says we utilise additive and multiplicative inverse to solve for unknown. Even though we do not see these words all the time, but when we solved equations from Grade 8, we have been using them, when we are either adding or subtracting to remove the constant, so we have variables on one side, numbers on one side and constants on one side. Or maybe we are multiplying with those inverses from the fractions to have only the variable by itself. Fine?

In lines 25-29, Teacher 1 emphasised the characteristic of equations. Conceptual understanding was addressed by stressing the constant, variables, and equal sign. The equal sign was mostly deemed as a separator of the two expressions.

Furthermore, Teacher 1 gave learners an example to work on:

30. Teacher 1: These ones are going to be the examples; we will do them here. So, we have the first one here, the easiest of them all. I think everyone can do this one. It says, the example, solve for the unknown:

    $x+1=3$

    I think everyone can solve for x, right?
31. Class: Yes
32. Teacher 1: Can I have someone to solve for x? This one, anyone can do it.

[Learner writing on the board]

33. Teacher 1: So, we have the unknown; what is the unknown? The variable there?
34. Class: x.
35. Teacher 1: We also have a constant on the same side of x, and we do not need that constant, how do we remove it?
36. Class: By subtracting one.
37. Teacher 1: Because we have a positive one, we are subtracting a one. On the other side, we have a three; we also subtract one. We have x+1-1 gives us zero, 3 – 1 gives us 2, x=2. I think everyone can do this, right?
38. Class: Yes.
39. Teacher 1: Anyone who has a question about this? No one! Easy right?
40. Class: Yes.

In lines 30 and 40, the teacher assures the learners that the problem that is being solved was simple, thus addressing their productive disposition. The teacher sets the mood for the learners to approach the problem positively. The learners were also encouraged to solve the problems on the board and to answer as a whole class. Procedural fluency was addressed as learners were taken step-by-step in solving the problem. Learners were introduced to one method of solving the problem, which was additive and multiplicative inverse, and they followed the same method to solve the given problems.

When asked about strategies to teach linear equations during the interview, the conversation went as follows:

> ***Researcher****: During all the lessons that you conducted; I have noticed that you use only one strategy of solving the linear equations. Were there no other methods that could have been used?*
>
> ***Teacher 1****: There are several methods that can be used, but I preferred the additive and multiplicative inverse because it is easy to understand. I did not want to confuse the learners with many strategies, and they end up not knowing which one to use. Again, the issue of time was a problem because to show them all the methods was going to need a lot of time. We are supposed to finish the curriculum on time; therefore, spending time on trying different methods was going to waste time. The methods that I know of are trial-and-error and transposing. Trial-and-error takes time because we have to test several numbers until we get the correct one which wastes a lot of time. The transposing method is quicker, but learners forget to change the sign when using the method.*
>
> ***Researcher (****Follow-up question): Was it not going to be beneficial for learners to be exposed to different ways of solving the same problem?*
>
> ***Teacher 1:*** *It was, but like I said, we always have a challenge with time.*

In the same lesson, adaptive reasoning was not encouraged as neither the teacher nor the learners were justifying the strategy used and the procedures that were followed (See learners' responses-there are no justifications for their answers). When probed during the interview on the justification of procedures when solving linear equations, the conversation went as follows:

Researcher: When solving the linear equations problem there was no indication of justifying the chosen strategy, and the procedures followed. Was there a reason for doing so?

Teacher 1: *I was not aware that I have to give a reason for every step that I was doing. I took for granted that learners memorises what I have said, and they will be able to apply it when I give them problems to solve. What was important for me was that learners could use the method that I gave them and get the correct answers. That is why I gave them one method to use because when we do it over and over again, they will be able to master it.*

Researcher (*Follow-up question): So, what you mean is that the important thing in mathematics problems is to get to the correct answers.*

Teacher 1: Yes*, because when these learners write a test or an examination, we mark the correct answers, therefore, it is important to teach them to get the correct answers.*

Day 3: Algebraic linear equations: Word Problems

1 hour 30 minutes

On the third day, Teacher 1 focused on word problems. The purpose was to ensure that learners could translate word problems into mathematical equations and to see how mathematics can be used in a real-life situation. Teacher 1 engaged the learners as follows:

41. Teacher 1: Alright, so we have word problems which is another topic. So, this one we usually see in the examinations, maybe just a few multiple-choice questions, it does not have many marks. But when we look at them, we see sentences that have equations or expressions in them that you need to solve to find answers. So, it says that when we talk about word problems, these are mathematical exercises, or they are verbal problems written in words that are represented in ordinary language, normally English, that we use in daily basis rather than mathematical expressions or equations. So, we have this equation that is written in words. Then it says to solve the word problems, we need to apply our knowledge to break down these words and write them as an equation.

> So next activity, you are going to break down those activities into equations. So, we have used these words before from Grade 8. We are going to use the same words or the same knowledge to break them down. So, we have a sum; we have talked about the sum, that means we are adding. We have a difference or subtraction. Usually, the word problems, they will use words such as after, then we have dividing or quotient. Dividing means you are sharing something; we are dividing something amongst ourselves. Then we have multiplications. Usually, we use the word times or twice the number, three times a number, multiplication. Then we have a certain number, this one obviously, a certain number is a number that you do not know. If someone speaks about a certain number, it means they are talking about the number, you can try to guess the number, but you are not going to be sure. The unknown number means you can use variables to represent a certain number. You can use x or y or any other variable. Then we have equal to; sometimes they use words such as the total, altogether or combined, is the end results, overall. So, we are going to try and use these words, look for them in the word problems then we solve them, but I will give you examples.

On day 3, the teacher took almost the whole lesson explaining the terminologies used in word problems. Instead of tapping into learners' prior knowledge, the teacher provided learners with information.

As I was observing these lessons, it was evident that Teacher 1 was not tapping into learners' prior knowledge. I commented and asked the question.

> ***Researcher****: I have observed when you were introducing the lessons about word problems, learners were not asked about their previous knowledge about their topic. Was that done on purpose?*
>
> ***Teacher 1****: The reason was that the learners that I taught linear equations did not do it in 2020 because schools were closed for a long time, and learners were only taught fundamentals when they came back. Therefore, it was going to be unfair for me to ask about what is it that they learnt previously, whereas I know that they did not do the topic.*
>
> ***Researcher*** *(Follow-up question): If the learners were taught linear equations previously, were you going to tap on their prior knowledge?*
>
> ***Teacher 1****: I would, but learners do not like to answer questions in front of their classmates, therefore, it means I will have to wait for few minutes for them to answer, and we do not have a lot of time to complete the work.*

The lesson revolved around the teacher, and at the same time, the teacher seemed uncomfortable letting learners participate. Already in the introduction, the teacher told learners that the topic does not carry a lot of marks in the examination, thus making them lose interest and concentration. Productive disposition was not encouraged during the introduction of the lesson. Teacher 1 explained all the concepts used in the word problems and did not allow learners to restate the definition or explain them in their own words or understanding. In trying to assist learners in translating the word problems to mathematical equations, Teacher 1 used the following example:

42. Teacher 1: Example number one, the question says:

Write these word problems as equations. We will just write them as equations first then later we will solve them. The first one it talks about George; it says George has a certain number of apples. If it says a certain number, do you know how many they are?

43. Class: No.
44. Teacher 1: No. So, we can use the variable x or y. Preferably let us use x. So, he has a certain number of apples, x apples. Then they talk about Sarah also; they say Sarah also has apples also but four times, so it is the same x, but this one has four times, times meaning that we are?

In line 44, the teacher channelled the learners which variable to use instead of allowing them to select which variable they prefer as individuals. The teacher was leading the learners, which made them not fully participate and only giving single responses when the teacher was asking questions.

45. Class: Multiplying.
46. Teacher 1: Multiplication. It says she has four times as many apples as George. It says they had a total, the answer, after the equal sign so they have a total of twenty-five apples when they are together, fine?
47. Class: Yes.
48. Teacher 1: Alright. So, to solve them you can start writing. You can just write the answer it doesn't really matter. So, we have a certain number; we took it from the paragraph or a sentence. Then Sarah had 4x, right?
49. Class: Yes.
50. Teacher 1: 4x a certain number, four times a certain number or 4x, then after the equal sign, we have twenty-five. The total was 25. La e bona (Do you see it)?
51. Class: Yes.
52. Teacher 1: So, a certain number ge re e ngwala (when we write it) to write it down we can use x. Because they said this one is four times, meaning it is four multiplied by x because we still do not know how many apples there are and the total is equals to, we include the equal sign, equals to twenty-five. To simplify x+4x=25. Four multiplied by x is equal to 4x. That is what we are going to do; we will read the sentences and write them as equations. For it to be an equation, it must have an equal sign. Fine?

In line 50 and 52, the teacher code-switched, which poses a challenge because mathematics in Grade 9 is done in English, and learners might get confused when they must attempt problems on their own without the teacher's assistance.

During the interview, the teacher was asked the following questions:

> ***Researcher****: During the lesson on word problems, there was a point where you code-switched. How does it help with learners' understanding?*
>
> ***Teacher 1****: Most of the learners do not understand English properly, so code-switching helps me to get their attention using their home language. It also helps me to explain concepts that I find difficult to explain in English.*
>
> ***Researcher:(****Follow-up question): Will the code-switching not be a disadvantage for learners when they have to solve problems in your absence?*
>
> ***Teacher 1:*** *I believe that, as long as they understood what I said in class using their own language, then they will not have a problem in solving the problems on their own.*

The lesson continued as follows:

53. Class: Yes.
54. Teacher 1: Another example, it says there are a certain number, a certain number means it is a variable, an unknown, right?
55. Class: Yes.
56. Teacher 1: So, it is x. A certain number of cupcakes, x cupcakes. It says if the cupcakes are shared, if they are sharing, it means they are?
57. Class: Dividing.
58. Teacher 1: Dividing. So, these cupcakes are shared amongst eight people. They are divided amongst eight people. It says they each get three cupcakes after sharing. It means that after sharing, they each get three cupcakes. So, when we write it down, we have a certain number, x right?
59. Class: Yes.
60. Teacher 1: So, this certain number of cupcakes has been divided by how many people?
61. Class: Eight.
62. Teacher 1: Eight people. Then after dividing them by eight people, how much does each get?
63. Class: Three.
64. Teacher 1: That is all you are going to do. You read a sentence, then you write the word problem, then we just discuss them, and we continue, fine?
65. Class: Yes.

Learners were exposed to solving linear equations in symbolic and word form only as observed from day 1 to day 3. When asked in the interview if the teacher knew of other forms of representing the linear equation, the response was as follows:

> ***Researcher****: Learners were exposed to two representations of linear equations, which were in word and algebraic forms. Are there other ways of representing linear equations that you know of?*

Teacher 1: Yes, linear equations can be represented as flow diagrams, tables, and graphs. I did not want the learners to get overwhelmed by showing them different representations because when they solve problems, they might end up using all the representations instead of one.

Researcher:(Follow-up question): Is it not advisable to expose learners to different representations so that various cognitive levels can be catered for?

Teacher 1: I have realised that learners are comfortable with one or two ways of representing the problem; otherwise, they lose interest in the lesson. It might be important to expose them to different representations, but time is still a challenge because there is still much to cover.

The teacher gave learners examples, but she took a centre stage in solving and explaining how to translate from word to equations. The whole lesson was traditional where the teacher tried to explain each example to the learners without giving them the chance to translate on their own. The focus was mainly on the procedures to be taken in writing the word problems into the algebraic linear equations.

During the interview, the teacher was asked a question:

> ***Researcher:*** *Most of the time in the lessons, you were the one solving the problems and learners were minimally engaged in doing so. The only time they were involved was when they gave short answers. What was the reason?*
>
> ***Teacher 1:*** *As I mentioned, the learners did not do linear equations in grade 8, therefore it was not going to be easy for them to solve most of the problems by themselves. I had to teach them until I am satisfied that they understand how to solve the problem.*

Researcher: *The observations that I made during your lessons was that learners were not given an opportunity to come up with their own strategies or methods to solve the linear equations problems. Is it how you normally conduct the lessons?*

Teacher 1: *Yes, this is how I conduct my lessons because learners hardly know the other methods. They always depend on the teacher for strategies to be used to solve the problems. The other thing is that learners in my class tend to be lazy to think and they prefer either to keep quite or give one answer. Most of the time we do not have enough time to allow space for learners to come up with their strategies. We would need the entire day to ask each learner to give their strategy.*

Furthermore, Teacher 1 was asked questions that impact teaching and learning, although they were not based directly on the observed lessons but may influence what transpires in the classroom.

Researcher*: How often do you meet with other teachers to elaborate goals, investigate learners' thinking and difficulties with linear equations and exploring different representations and strategies?*

Teacher 1: We try to meet when there is a need to plan together as grade 8 and 9 teachers. Sometimes it is a challenge for me because I also have classes for Life Sciences in Grade 10 & 11, and I have to conduct extra classes. To investigate learners' thinking and difficulties with linear equations, we give classwork and informal tests after teaching a topic. We conduct diagnostic analysis, which helps us in identifying their challenges. As teachers, we discuss different representations and strategies to solve linear equations problems, however, most of the time we cannot teach them because of time.

The teacher indicated that meetings were held with other teachers to discuss learners' challenges, conduct diagnostic analysis, and explore different representations and strategies in linear equations. However, the challenge that hinders these collaborations was time constraints because other higher grades need their attention in the form of extra classes.

Researcher*: With your observation, how is the attitude of learners towards linear equations?*

Teacher 1: *Many learners have a challenge dealing with word problems because they first have to understand the words written and translate them to mathematics. Learners have a serious challenge with English; therefore, they have a negative attitude when they are supposed to deal with word problems.*

Researcher (Follow-up question)*: Seeing that learners have challenges with translation because of the barriers they have with English, what is it that you have done to assist them?*

Teacher 1: *We are always advised to implement English Across Curriculum every day, therefore, learners are given few minutes to read the explanations of terminologies used in mathematics. That is how I try to assist my learners every day.*

The teacher indicated that according to the observation of learners, they have a negative attitude towards word problems due to a language barrier. The teacher implemented English across the curriculum, which aims at encouraging learners to read out loud the explanations of terminologies in mathematics, thus capacitating them with mathematical language.

Researcher*: When conducting the lessons on linear equations, are you finding it easy or difficult to present it?*

Teacher 1: *Sometimes it is easy and difficult. It depends on the concept that is presented. Especially when word problems are taught, it is extremely difficult to clarify to learners. I have to use vernacular to try and explain to them so that they understand. I can say linear equations are quite challenging but exciting for me.*

The teacher found it difficult to present word problems, and it was also confirmed during the interview that sometimes code code-switching happens in trying to make learners understand. The teacher also indicated that linear equations are an interesting topic to teach. The teacher was then asked the following:

Researcher*: Is there any comment that you would like to say based on the lessons that you presented or the interview?*

Teacher 1: *Finally! Yes, Mam, let me firstly say that being part of your study was an eye-opener and I believe that from here, I will do things differently in my classroom. There are so many things that we take for granted as teachers, like the* *strategies that we use; we choose them for ourselves but not for the learners. Allowing learners to take part in the lessons I always took it as time-wasting because my goal is to finish the curriculum on time.*

Below is Table 7 which shows the rating scale for Teacher 1 's addressing mathematical proficiency in their teaching linear equations for all the lessons

Table 9: Rating Scale for Teacher 1 's addressing mathematical proficiency in their teaching linear equations

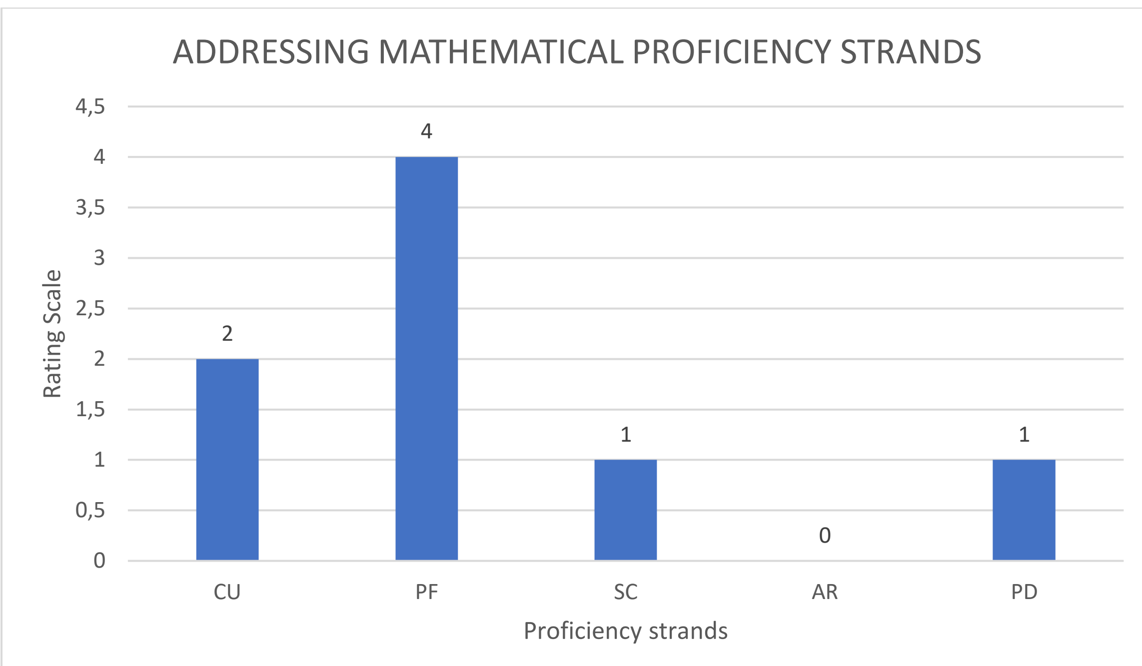
Scale	0	1	2	3	4
Criteria	The teacher shows **no evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **weak evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **some evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **strong evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **very strong** evidence of addressing mathematical proficiency in teaching linear equations
Strands					
CU			2		
PF					4
SC		1			
AR	0				
PD		1			

The lesson observation instrument for Teacher 1 was used to develop a rating scale for each proficiency strand.

Addressing conceptual understanding when teaching linear equations was at Level 2. Procedural fluency was very strongly addressed as the level was at 4. The Teacher showed weak evidence of addressing strategic competence, which was at level 1. There was no evidence of the teacher addressing adaptive reasoning as it stands at 0, meaning that the strategies and procedures were followed without reasoning. Addressing of a productive disposition by the teacher was weak at Level 1, which indicates that learners were minimally allowed to take the initiative.

Figure 6 below shows the bar graph representation of Teacher 1's addressing of mathematical proficiency in teaching linear equations, which was informed by the rating scale in Table 9.

Figure 6: Bar graph for Teacher 1 's addressing of mathematical proficiency in teaching linear equations



The bar graph shows that procedural fluency was allocated a longer bar which implies that the emphasises was on getting the steps done to reach the answer. Conceptual understanding occupied much less, meaning that the strand was not given much attention. Strategic competence and productive disposition carried the same weight, which was less, meaning that fewer strategies were used, and learners' interests were not considered. Adaptive reasoning was at 0, which implies that reasoning of the strategies and steps during the procedures was not done.

When solving a problem, it is expected that all the strands should be equally addressed as they are interconnected. If one strand is addressed more than the other, the teacher did not fully produce mathematically proficient learners.

4.2.2 Teacher 2: Data presentation from the lesson observation and interviews

Day 1: The difference between an expression and an equation

2 hours

On Day 1 of the lesson presentation, Teacher 2 explained to the learners how the lesson would unfold, including looking particularly at the difference between algebraic expressions and equations.

> ***Teacher 2:*** *Today, we are going to look at Algebraic expressions. So, in our textbooks, let us go to page seventy-four. That is where we are going to discuss Algebraic Expression. At the end of the day, we must all understand the difference between algebraic expression and algebraic equation, because these things are not the same. Once you understand algebraic expressions, then we will go to algebraic equations. So, therefore, on algebraic expressions, we start by saying that number one, on this one, we work with variables, coefficient and constant. So, this is like this, for instance let us say we have four x to the power two plus three, let us say this is an expression. So, on that expression, four we refer to it as coefficient, the x we refer to it as the variable and three is a number on its own. So, if it is a number on its own, we refer to it as a constant. So that means on this expression, we are looking at the variable, coefficient, and the constant.*

In Line 1, lesson extracts show that the teacher introduced the topic of the lesson which focused on the difference between the expressions and the equations. The teacher introduced the terminologies that make up the expressions and are also found in the equations. The prior knowledge of learners was not considered as expected to build learners' conceptual understanding. The teacher continued providing learners with explanations of a variable, constant, and coefficient without involving them.

The teacher was asked during the interview in terms of accessing learners' prior knowledge:

> ***Researcher**: During your lessons, I have noticed that you were not connecting with what learners knew and what was taught? Was there a reason?*
>
> ***Teacher 2:** Yes, the reason was that most of the learners did not have a prior knowledge regarding the topic that we were discussing because it seems that the learners had content gap in regard to the knowledge that they have acquired before. With COVID-19, the topic that we were discussing with the learners was never discussed in the previous grade, Grade 8. The learners that I was teaching were in Grade 9, so, therefore, there was never a time where they were taught about the topic that were discussing by that time. That is why there was never prior-knowledge questions.*
>
> ***Researcher** (Follow-up question): So, it means that linear equations was not taught in Grade 8?*
>
> ***Teacher 2:** Yes, it was not there.*

The next extract illustrates how the teacher explained a variable:

> ***Teacher 2:** The letters always represent the variable; the reason why we say it is a variable, it represents the unknown number which can always vary. So that means the variable can always change that is why we refer to it as a variable. Letters are used to represent numbers whose value is unknown, and that value can change. So that is what were refer to as the variable, these are the letters right, and in this case our letters are x.*

Teacher 2 explained the concept of "constant" as follows:

1. **Teacher 2:** Is a constant, and the constant does not change, constant remain like that forever. So, three, this number that is standing alone, we refer to it as a constant. We know that in English we talk about a context; when we talk about a constant, we talk about those things that do not change; they may remain the same. That number is not the same as the letter because the letter can represent an unknown number that can change anytime

In incorporating the understanding of a variable and a constant, Teacher 2 explained the coefficient as follows:

2. **Teacher 2:** The other things are that the coefficient is always found just before the variable. So, all these numbers just before the letter, we refer to them as the coefficient. The coefficient is a constant by which a variable is multiplied in an algebraic expression. In this case [it] is the four, four does not differ much with three, but they differ in such a way that the four multiplies with the variable. Immediately when the number multiplies with the variable, that number is not called a constant but a coefficient. Here is four multiplied by x. If there is no sign of addition or subtraction, it means they multiply, even if it is not indicated that it is a multiple.

The teacher further introduced the Algebraic Equation by first explaining the concept of an equal sign as indicated below:

3. **Teacher 2:** What does equal sign mean? You hear most of the time they talk about equality, right? What does equality mean?

Learner: The equal sign means what is on the left side is equal to what is on the right side.

Teacher 2: It means both sides are equal, right?

It was only in line 5 that the teacher started engaging a learner in terms of asking them questions. In the beginning, the teacher made lengthy explanations of the concepts embedded in algebraic expressions without the learners' involvement.

Teacher 2 provided learners with an example to demonstrate the purpose of an equal sign in an equation. The teacher emphasised that the equal sign should keep the equation equally balanced. The extract below is an indication of what transpired:

4. **Teacher 2:** The other thing is that to solve the equation, we must apply the same principle of expression and keep the two sides of the equation equally balanced. For example, solve

for x and remember you must keep the equation equally balanced. If you are supposed to keep the equation equally balanced, for example:

$3x - 8 = 34$ that means two sides are equal. How do you ensure that the two sides are equal? We must find the value of x that makes the equation to be true, right? So, how are we going to find it?

5. **Class**: We move 8 to the other side.
6. **Teacher 2:** We can move 8 to the other side, right? Or we can look for the additive inverse of 8. If it is negative 8, then that is going to be positive 8. We add the additive inverse of 8 in both sides. What is negative 8 and 8?

In line 8, the teacher indicated that "we can move 8 to the other side". The word "move" was used to explain transferring numbers or variables to the other side of the equation, which is normally regarded as transposing.

The teacher continued with the example for learners to find the value of x that makes the equation to be true:

7. Class: Zero.
8. **Teacher 2:** Zero, right?
9. Class: Yes.
10. **Teacher 2:** Therefore, that will be $3x = 34 + 8$. What is $34 + 8$?
11. Learner: Forty-two.
12. **Teacher 2:** Is 42, right?
13. Class: Yes.
14. **Teacher 2:** And now, since we are looking for x, we must multiply by the one over three. We must multiply both sides by one over three, right? Which is the multiplicative inverse of three. Three divided by three is one, right?
15. Class: Yes.
16. **Teacher 2:** We are left with x. What is 42 divided by 3?
17. Class: Fourteen.

In lines 9-19, Teacher 2 took learners step -by- step in solving for the value of x and introduced them to a single method of solving the problem. The lesson emphasised a bit on conceptual understanding even though the teacher was the one who was explaining the terminologies involved. The method to solve the problem using multiplicative and additive inverse was used. Procedural fluency was effectively done where learners were engaged in solving step -by -step. Adaptive reasoning and productive disposition were not promoted in the lesson. In adaptive reasoning, the expectation is that the teacher should justify the strategies used and the procedures done. There was no attempt of justifying during problem-solving, meaning that adaptive reasoning was not accommodated. This is evident in the learners' responses from lines 6- 19. The teacher was asked during the interview on the justification of steps taken when solving linear equation problems:

> ***Researcher***: *When the learners were responding, you seemed not to ask them why they were giving such responses. Is that what you normally do?*
>
> ***Teacher 2:*** *Yes, because my assumption is that if learners are able to operate and give a proper answer, it means the previous knowledge that they have, they know how to operate the sign with no doubt. Hence, I did not follow up when they give me the answer. Yes, that is how I teach most of the time, especially with the Grade 9s, because the Grade 9s have their own previous knowledge in regard to mathematics so they can be able to operate the multiple, addition, subtraction, and division. So, if I ask them a question to say they must add or divide, my assumption is that they will use their previous knowledge in order to divide, add, subtract, or multiply to get the proper answer.*

There was lack of learner involvement in terms of initiating their strategies to solve the problem at hand. Learners were only involved when responding to the teacher's questions.

Day 2: Algebraic Equations: Introduction to word problems
2 hours

On Day 2, the teacher started the lesson by introducing the day's topic and reminding the learners about the concepts that were dealt with in the previous lesson.

18. **Teacher 2:** Today, we are going to do algebraic equations, word problems. So, remember last time we did algebraic expressions, algebraic equations, and we looked at the difference between algebraic expressions and algebraic equations, where the expression

is made up of variables, numbers and constant and coefficient. So, these are what makes up algebraic expressions. Algebraic equations are made up of the same things, but on algebraic equations, there is an equal sign. That is the difference, right? What does the equal sign mean?

19. **Class:** Both sides are equal.
20. **Teacher 2:** It means that both sides are equal. That means the left-hand side is equals to the right-hand side.

The extracts in lines 20 – 22 show that the teacher was the conveyer of information, instead of asking the learners what they had learnt in the previous lesson. The teacher did not consider the learners' prior knowledge. The teacher spoon-fed learners with information instead of building their conceptual understanding.

The teacher then exposed learners to word problems:

21. **Teacher 2:** Example: A number increased by six equal ten.

So, a certain number that is unknown is increased by six, equals ten. So, we take a number as an x, we say let a number be x. The number is not indicated whether is two, three, four, five or six. But since it is not indicated as exact number, but that number is increased by six. Which sign or operation we can use to indicate an increase?

22. **Learner D**: Multiplication and addition.
23. **Teacher 2:** We can use addition, right?
24. Class: Yes.
25. **Teacher 2:** Therefore, that means x is a number which is increased by six and equal; equal is a sign, right?
26. Class: Yes.
27. **Teacher 2:** Equals to ten.

In lines 23 – 29, the teacher started by giving learners a simple example to translate from word to algebraic equations. The teacher elaborated on words like "increase" and "a certain number" for learners to understand what they mean in mathematical language. There was an interaction between the learners and the teacher; however, the learners answered as a group in most instances. It was only once that an answer came from an individual learner.

The teacher continued with the examples of word problems and still kept them simple, as shown in the extract:

28. **Teacher 2:** Example: Three times a certain number by five equals to thirty.

I want you to try this one on your own. Can you create an equation from that? (Learner writing the answer on the board)

29. **Teacher 2:** Let us do another one. We try to set up the equation.

Example: A certain number increased by five equals to thirty.

30. (Learner writing the answer on the board)
31. (Class clapping hands for the learner)
32. **Teacher 2:** We still set up simple equations, and we start with the easier ones. Example: Three doubled and added to a number equals to fifty. Now, let us talk about the word double; the word double means it was multiplied by two, right?
33. Class: Yes.
34. **Teacher 2**: That means the number that is unknown is multiplied by two and added to a number equals to fifty. How are we going to write this one?
35. (Learner writing on the board. There was an argument between the learners on how a doubled number is written, whether is 3^2 or 3×2)
36. **Teacher:** 2: $3^2 = 3 \times 3 = 9$ and $3 \times 2 = 6$. Therefore, the equation is

$3 \times 2 + x = 50.$

In line 30, Teacher 2 encourages learners to solve the problems on their own and gives individual learners an opportunity to write on the boards, as it is evident in lines 30, 32 and 37. In line 33, learners cheered each other on by clapping their hands for the correct answer. In line 34, Teacher 2 involved learners in discussing doubling, and what it means. It is also evident in line 37 that learners were allowed to deliberate about the meaning of three squared. The teacher alleviated the misconception that learners had by clarifying its meaning.

The teacher further gave an example that involved a real-life situation to expose learners to daily context:

39. **Teacher 2:** Example: Zinzi is a florist she makes a certain number of bouquets and sell all of them at forty Rands each. The money received from the sale of the bouquets is two thousand. How many bouquets did Zinzi make? We know the price of a bouquet; what is the price?
40. **Class:** Forty Rands**.**
41. **Teacher 2:** It is forty Rands. We do not know how many bouquets did she make, right?
42. Class: Yes.
43. **Teacher 2:** We are going to say $x40 = 2000$.
44. **Learner E:** Sir, it is not written like that but is $40x = 2000$
45. **Teacher 2:** It is $x \times 40 = 2000$. What is it that we are going to do to get the number of bouquets?

46. **Learner F:** Divide by forty on both sides.
47. **Teacher 2:** We divide both sides by forty, and x will be how much?
48. **Class**: Fifty.
49. Teacher 2: It is fifty.

In lines 39 – 49, Teacher 2 brought a real-life situation into the classroom by giving learners an example that involved flowers. The teacher allowed learners to give answers as a group and individually. Teacher 2 took learners step- by -step in formulating the algebraic equation from the word problem. Procedural fluency was addressed in lines 23-29 and 39-49, where the teacher translated the word problem to linear equations. Learners were involved in answering the questions posed by the teacher to reach the answer ultimately. The traces of strategic competence were minimal as learners were using the method that was taught by the teacher only. There was no point where they were asked to provide their own methods. When asked during the interview about allowing learners to come up with their own methods, the following transpired:

> ***Researcher****: Seeing that when you were teaching, learners were not given an opportunity to come up with their own strategies or methods to solve the linear equation problems. Was there a purpose in doing so?*
>
> ***Teacher 2:*** *Not really, in particular, I could have asked them to come up with their own method, but the main reason was that they must understand these ones first as a point of departure that if you can understand this one, you can use it to solve the linear equations. The other reason was that we did not have enough time when we were discussing this. Maybe if we have more enough time when the learners are given an ample time to discuss this and now once they grasped the basic ones, then we will give them a chance to come with their own as to say that is not the only way to solve these problems. They can also come up with their own way to solve these problems as long as you understand these ones that we are discussing now.*
>
> ***Researcher*** *(Follow-up question): So, in other words, you are saying should you [be] given another opportunity, and enough time you will give them a chance to come up with their own strategies?*
>
> *Teacher 2: Yes.*

In the same lesson, learners were exposed to the same problem-solving method. In lines 30-38, individual learners were involved in providing the answers on the board, and learners' productive disposition was addressed by being encouraged to take part in translating the word problems into equations according to their understanding. Furthermore, the learners argued about how a "doubled number" is written, whereby the teacher made an example to clarify it for the learners. Productive disposition and conceptual understanding allowed learners to express their understanding of the word doubled, and the teacher addressed the learners' misconceptions. Adaptive reasoning was not addressed, as there were no instances where the teacher or the learners justified the answers that they provided.

Day 3: Algebraic linear equations: Analyse and solve word problems

2 hours

On day 3, Teacher 2 continued with analysing and solving word problems. The teacher started by assuming that all the learners understood how to set up equations from word problems. There were no efforts to check if there were learners who did not grasp the concept of setting up the equations. It was evident in the extract below that the teacher made assumptions:

50. **Teacher 2:** Today, we are going to do algebraic equations but this equation we are going to do word problems. Last time when we did word problems our focus was to set up the equations. So, now we know how to set up the equations because we did last time.

The teacher exposed learners to an example that dealt with real life situation as shown:

51. **Teacher 2:** Example: At honeybee pot, the bulk price for honey is R2,50 per litre with a minimum purchase of 20 litres. If Mary paid R80 for some honey by how many litres did Mary purchase exceed the minimum?
52. **Teacher 2:** The minimum purchase is how much?
53. Class: 20 litres.
54. **Teacher 2:** If Mary paid R80, how may litres do Mary purchase, and it exceeded the minimum with how much?
55. **Class:** Teacher?
56. **Teacher 2:** Mary bought for R80, how many R2,50 is there in R80?

57. Class: Sixteen.
58. **Teacher 2:** Okay, how many R2,50 is there in R10?
59. **Class:** Thirty-two.
60. **Teacher 2:** Thirty-two, therefore that means we are going to start by saying $\frac{R80}{R2,50} = 32\ litres$. We know the minimum litres that the purchase requires is 20, right?
61. Class: Yes.
62. **Teacher 2:** Then is $32 - 20 = 12$, therefore Mary exceeded the minimum purchase with 12 litres.

In lines 51-62, learners showed signs of not understanding the question that the teacher posed, however, the question was rephrased. Procedural fluency was addressed by going through the steps of solving the problem that the teacher did, and learners were giving single answers. The teacher was probed during the interview about solving the problem on his own, and the conversation was as follows:

Researcher: I have noticed that in the lessons, you were much more involved and learners less involved in solving the problems be it on the board or in their books. What was the reason?

Teacher 2: I think the lesson was on introduction level. I needed to introduce the lesson first for the learners to understand, then, later on when we discuss, we start involving them throughout when we are solving the problems. All the lessons on the video were on introductive level; there was nowhere we did in-depth of the topic that it is why it seems like I was doing all the work for the learners.

In the same lesson, the conceptual understanding was not emphasised. One method of solving the problem was used, meaning that strategic competence was not considered. There was no reasoning provided behind the steps involved; thus, adaptive reasoning was not addressed. A productive disposition was not addressed because learners were not encouraged to solve the problem according to their understanding and only followed the teacher’s lead. The lesson revolved around the teacher because all the explanations were from him, and learners gave single answers.

Day 4: Algebraic linear equations: Solving using additive and multiplicative inverse

2 hours

On Day, Teacher 2 started by making an overview of the lessons that have taken place for the past three days. The teacher then indicated that the day's lesson would focus on linear equations. The extract below is the introduction made by the teacher:

63. **Teacher 2:** Today, we are going to deal with algebraic equations. Remember last time we did algebraic expressions, algebraic equations, and under the algebraic equations, we specifically looked at word equations. On the word equations, we were able to analyse the word equation and solve. Now today, we are not going to look at word equations; we are going to look at the equation itself. Under the equations, we know that there are variables. For example, on equations, if we have $2x + 4 = 10$. This is not a word equation but a normal equation where we have a variable. A variable, we know, represents an unknown value, right?

In line 63, prior knowledge was not tapped into; instead, the teacher was providing the learners with information about what they had learnt in the previous lessons. Even though the teacher did not tap into the learners' prior knowledge, a conceptual understanding was addressed.

In the following extracts, Teacher 2 made a snapshot of word problems and introduced another method of solving linear equations, which is trial- and – error.

64. **Teacher 2:** A variable represents a certain number. If it represents a certain number, it says this side is equal to this side. Our duty is to think of a number that is multiplied by 3; when you subtract 8, it gives 34 to make the equation to be true. Think of a number that is standing for a variable. We can use a trial-and-error method, we can put 10. Ten multiplied by 3?
65. **Teacher 2:** Which means x represents 14, because 14 makes the equation to be true. The method that we just used is a trial-and-error method and it is about guessing. Now we do not want to guess, we want to solve properly for us to find the number that makes the equation true. We can say the two sides of an equation will remain balanced if the following are done:
    1 Add or subtract the same value/number from both sides. That is what we should keep in mind that when we add or subtract the same number both sides, the equation remain balanced, that means it does not change. It is only balanced by the equal sign; the equal sign indicates that both sides are equal.

2 Multiply or divide both sides by the same number. When you multiply or divide by the same number, the equation remains balanced.

66. **Teacher 2:** That means when you think of a value that makes the equation to be true, you must think of the value that you can put on the x and make the two sides to be equal.

Strategic competence was addressed in line 64 as the teacher introduced a trial-and-error method to learners in addition to the additive and multiplicative inverse method. During the interview, the teacher was asked about other methods of solving linear equations:

> ***Researcher:*** *I really appreciate the strategies that were used when you were teaching. Were there no other strategies that you could have added or used?*
>
> ***Teacher 2:*** *There were other methods that I could have exposed learners to, but I found the ones that I used as the easier one of the trial-and-error methods as the learners they seemed to understand the trial-and-error method as compared to the one that I was going to use. Through the experience, it taught me that the trial-and-error method is the one that learners can be able to grasp more easier, as compared to the other methods that I can also use when I was teaching these learners. Hence, I opted for this one, as learners it is easier for them to grasp because the other method that can be used, it is going to be difficult for the learners to understand especially when we do additive and multiplicative inverse. That is why I decided for this one.*
>
> **Researcher** *(Follow-up question): If you do not mind, can you share the other methods that you know besides these two?*
>
> ***Teacher 2****: The other method that I can use is the one that we say we transpose instead of using the additive inverse. So, you transpose and change the sign. That is the other method that we usually use, and sometimes it is difficult for the learners where they transpose, and they forget to change the sign to say if it is $2x$ when it ‘jumps’ the equal sign it is going to be negative*

Furthermore, Teacher 2 outlined to learners the conditions of applying the additive and multiplicative inverse method in lines 65 and 66.

The teacher exposed learners to a linear equation example and took them through to solve it:

67. **Teacher 2:** Therefore, $3x = x + 14$. We must also get rid of x to get to the left-hand side. We must group the like terms in one side. The one that has x must be on one side and the ones that are numbers on the other side to get the value of x. If x is on the right-hand side, we must find the inverse of x. If it is positive x that means the inverse of x is what?
68. **Class:** Negative x.
69. **Teacher 2:** Negative x, right?
70. Class: Yes.
71. **Teacher 2:** We are going to subtract negative x on both sides because it balances the equation, then $3x - x = x - x + 14$. What is $3x - x$? If you have 3 oranges and you take one out, how many are you left with?
72. Class: Two.
73. **Teacher 2:** If we have $3x$ and take one out how many are we left with?
74. Class: $\mathbf{2x}$
75. **Teacher 2:** We are left with $2x$. Then x subtract x means we do not have x anymore, right?
76. Class: Yes
77. **Teacher 2:** Therefore, we have $2x = 14$. What is the next thing that we do with 2x? We are looking for the multiplicative inverse of 2. What is the multiplicative inverse of 2?
78. Learner: Seven.
79. **Teacher 2:** Seven? What was the multiplicative inverse of 3?
80. **Class:** One over 3.
81. **Teacher 2:** One over 3. What was the multiplicative inverse of one over 3?
82. Class: Three.
83. **Teacher 2:** What is the multiplicative inverse of 2?
84. **Class:** One over 2
85. **Teacher 2:** One over two, right?
86. Class: Yes.
87. **Teacher 2:** Therefore, $\frac{1}{2} \times 2x = 14 \times \frac{1}{2}$. What is 1 x 2?
88. Class: Two.
89. **Teacher 2:** What is 2 divided by 2?
90. Class: One.
91. **Teacher 2:** What is 14 multiplied by 1?
92. **Class:** Fourteen.
93. **Teacher 2:** What is 14 divided by 2?
94. Class: Seven.

Procedural fluency was addressed from lines 67-94 as the learners were taken step -by-step to solve the problem. Even though learners were involved, the problem solving was teacher-centred because the teacher was solving the problem until the end. Learners were not allowed to attempt the problem on their own. Adaptive reasoning and productive disposition were not addressed because no reasoning was given for the steps taken, and the learners were not encouraged to be innovative. Conceptual understanding was minimally addressed as learners were exposed to two ways of representing linear equations: algebraic and word forms. The teacher was asked about his knowledge of other ways of representing linear equations during the interview, and the response was as follows:

> ***Researcher****: I have noticed during your presentation that you have represented linear equations in word form and algebraic form. Don't you think that there are any other ways of representing linear equations?*
>
> ***Teacher 2:*** *We can represent linear equations on a graph and on tables.*
>
> ***Researcher*** *(Follow-up question): But you have shown learners word form and algebraic form, or is that what they are supposed to do at that level?*
>
> ***Teacher 2:*** *No, they are supposed to know in word form, in equation, in a graph as well as in tables. The time was a challenge as well on this case where I could be able to show them all the other parts of linear equations.*

In addition, questions that could impact teaching and learning were posed to Teacher 2, even though they were not necessarily related to the observed classes.

> ***Researcher****: How often do you meet with other teachers to elaborate goals, investigate learners' thinking and difficulties with linear equations and exploring different representations and strategies?*
>
> ***Teacher 2:*** *Usually, we meet at the beginning of the topic at my school. What we would do is to see how best we can teach this topic so that we can be able to establish the strength and the weaknesses of each other so that we can be able to teach this topic. At the beginning of each topic, we usually meet as Grade 9 educators, and we discuss the issues pertaining to the topic and how best we can be able to teach this topic. The other thing that we also discuss is the strength on how best we understand the topic. So, if there is an educator that has a challenge regarding certain area or concept, we discuss that to help the educator on how best they can teach this topic to the learners.*
>
> ***Researcher*** *(Follow-up question): Then, in terms of the learners' thinking and difficulties about linear equations, what are the discussions around that?*
>
> ***Teacher 2:*** *On these one we depend much on how learners respond when we do active learning. Each teacher will use own discretion when they teach the topic to figure out their thinking in regard to active learning where the oral questions will be*

asked, and the learner must respond immediately. Therefore, the teacher will be able to say this one is not responding positively, maybe she did not grasp the topic according to how it was discussed initially. So, we depend much on active learning than to say that we give them a classwork and they write on their own. But with the active learning we can be able to determine whether these learners are grasping the topic or not and their thinking.

***Researcher** (Follow-up question): So, the only time you meet is when you discuss what you are going to teach, but there is no point where you come back again and do a reflection?*

Teacher 2: No.

The teacher indicated that there was collaboration happening at the school for Grade 9 teachers to plan together for the upcoming topic. They also assist each other in terms of the topic or concept on which they have challenges. In terms of reflecting on the lesson after it has been conducted, there was no collaboration; each teacher used their own discretion to reflect on the learners' understanding. Reflecting on the lesson is essential because it assists the teacher in refining their teaching method, the tasks and the material used. Individual learners can be identified, and remedial work can be done according to their areas of challenge.

***Researcher**: What is the attitude of your learners towards the mathematics, specifically linear equations? Do they like it or see sense in it?*

***Teacher 2:** I can say the learners like the linear equations for the mere fact that it involves a lot of 'x'. They do like the topic, and it seems like the learners are more comfortable with solving the linear equation than drawing on the graph. Learners have a challenge in identifying the x and y axis; they only focus on solving the linear equations*

***Researcher** (Follow-up question): Have you checked with them what is really difficult about it? What were the root cause of them having difficulty in identifying the vertices?*

***Teacher 2:** Yes, they do not take graphs as vertices of x and y; to them is something else, but if we let them practise more, especially with the DBE books, because now it has more graphs where they need to draw on their own. Then they* *start to respond positively but now, if you draw it and want them to draw in their classwork books is a big challenge. They do not know where to put x or y. With the DBE workbooks, it becomes easy because it specifies where x and y are, and it becomes more easier for them to draw. The other thing is the previous knowledge with the previous grade, it seems like they were not taught enough about understanding the graphs.*

The teacher indicated that the learners were not comfortable drawing the graph, but they were never exposed to drawing graphs in all the lessons I observed and according to the initial conversation during the interview.

***Researcher**: What are your challenges when teaching linear equations, and what are the challenges that learners have in learning about linear equations?*

***Teacher 2:** Not having enough models or teaching aids to represent these linear equations. I think if we can have enough models or teaching aids with regard to the topic, I think it is going to be better. The learner's challenge is the language, and I think we need to go easy when we are teaching this topic. Linear is a bombastic word to the learners so we must be able to describe these words in simple English to them so that they can be able to understand what is happening. When we solve, what I have noticed is that when they write a method in words [it] is easier than when you tell them what to do without noting it down. When you give them homework, then when they reach home, they forgot what is it that they must do. But when they write down for each step of what they are going to do, and they note it down, and when you give them homework, they are able to refer that [to] the first step [of] what they did. So, they solve the problem and write the method of what they did. The challenge that I have with linear equations is in regard with mathematical language that we teach these learners. It seems like when you speak to them with the mathematics language; the learners cannot really understand what you are talking about.*

The teacher indicated that the lack of manipulatives to teach linear equations poses a challenge. There are different types of learners in each classroom; amongst them, there are those that learn effectively by visualising and touching; therefore, using manipulatives will be to their advantage. The teacher indicated that learners had a challenge with language in linear equations.

> ***Researcher***: *Is there any comment that you would like to say based on the lessons that you presented or the interview?*
>
> ***Teacher 2:*** *With the lessons that I presented; I think it has helped a lot in understanding more on this topic. Because during the presentation, I can feel that maybe before, I was not doing proper when I was teaching this topic, but now I can be able to move with the topic step- by -step and able to feel if the learners are understanding or not. With the interviews, I think it provoked my thoughts, especially with the topic I was teaching the learners because we are busy now with the topic of linear equations and these questions, I was not expecting them, and they really provoked my thoughts. I thought it was going to be easy questions, but they were not. I think this grows me professionally, these questions because it requires me to always look at and reflect on how we do things every day at schools.*

Table 10: Rating Scale for Teacher 2 's addressing mathematical proficiency in teaching linear equations

Scale	0	1	2	3	4
Criteria	The teacher shows **no evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **weak evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **some evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **strong evidence** of addressing mathematical proficiency in teaching linear equations	The teacher shows **very strong** addressing mathematical proficiency in teaching linear equations
Strands					
CU			2		

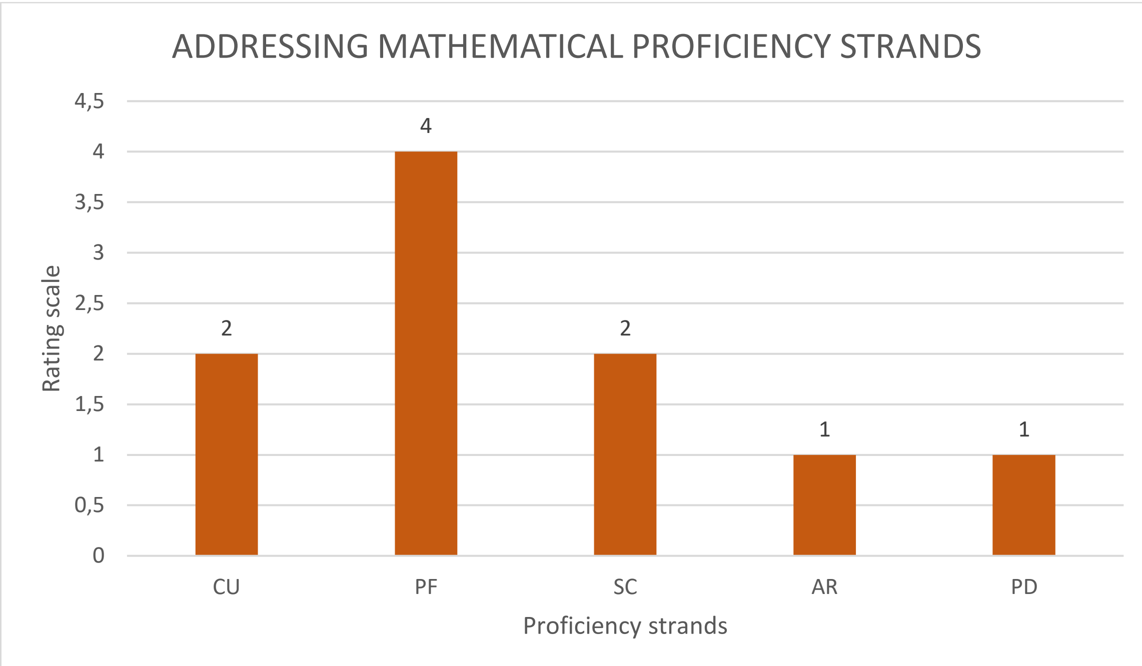
Scale	0	1	2	3	4
PF					4
SC			2		
AR		1			
PD		1			

The Rating Scale table for Teacher 2 shows that the teacher's lesson and activities revolved more around completing procedures accurately. Less attention was paid to Adaptive reasoning and Productive Disposition. Conceptual Understanding and Strategic Competence were slightly encouraged during the lesson.

The table shows that the teacher was somehow using the traditional way of teaching, whereby the important thing was to get to the correct answer. Learners' ideas and views were not considered to strengthen their conceptual understanding. Strengthening learners' conceptual understanding promotes their ability to come up with their own strategies, justifying the chosen strategies and ultimately acquiring productive dispositions. Mathematical proficiency strands are intertwined so that if one is left unattended, its addressing is incomplete.

Figure 7 represents a Bar graph drawn from the levels presented on the rating scale for Teacher 2. The graph gives a picture of how do Grade 9 mathematics teachers use strategies to address mathematical proficiency when teaching linear equations. It divided the proficiency strands into clear bars on how they were incorporated all four lessons that were conducted.

Figure 7: Bar graph for Teacher 2 ’s addressing of mathematical proficiency in teaching linear equations



The bar graph clearly shows that Teacher 2 focused on procedural fluency. The teacher ensured that the steps were done accurately and efficiently. The procedures used to solve the linear equation problems at level 4.

Conceptual Understanding and Strategic Competence were at level 2, which indicates that the teacher attempted to address mathematical proficiency in teaching linear equations. This was followed by Strategic Competence, meaning that the teacher tried exposing learners to more than one way of solving linear equations problems.

Adaptive reasoning and Productive Disposition were at level 1 each, which was a worrying factor. It simply indicated that the teacher was minimally justifying the strategies and procedures used, and learners were not engaged fully in a way that they could come up with their strategies and their justification.

4.3 Chapter Summary

This chapter presented and interpreted the findings of this study on how teachers address mathematical proficiency in teaching linear equations. The extracts of the transcripts from lesson observations and interview responses were used to assist in presenting the findings of my research. Rating scales and bar graphs were presented to give a clear picture on how mathematical proficiency strands were addressed during the lessons that the teachers conducted.

The study's discussions of results and conclusions are presented in the next chapter.

CHAPTER FIVE: DISCUSSIONS AND CONCLUSION

5.1 Introduction

Chapter 5 focuses on the discussions of the findings from - Teacher 1 and Teacher 2’s lessons observations and interviews conducted. Data were also presented and interpreted using rating scales and bar graph. Furthermore, the chapter covers the study's limitations, the recommendations for the Department of Basic Education, ideas for future research and the implication of the study. This qualitative case study aimed to observe how Grade 9 mathematics teachers’ use strategies to address mathematical proficiency in their teaching of linear equations. The following were the research questions:

Main Research Question:

How do Grade 9 mathematics teachers use strategies to address mathematical proficiency in their teaching of linear equations?

Secondary Research Questions:

1. What strategies do teachers use to address conceptual understanding in their teaching of linear equations?
2. What strategies do teachers use to address procedural fluency in their teaching of linear equations?
3. What strategies do teachers use to address strategic competence in their teaching linear equations?
4. What strategies do teachers use to address adaptive reasoning in their teaching linear equations?
5. What strategies do teachers use to address productive disposition in their teaching linear equations?

5.2 Discussions of the Findings

The discussions are based on the secondary questions of the study. The aim was to answer the main question using the analysis and findings presented in Chapter 4. The secondary questions were investigated using the relevant mathematical proficiency strands and their descriptors.

5.2.1 What Strategies do Teachers Use to Address Conceptual Understanding in teaching Linear Equations?

The question mainly focused on how do Grade 9 mathematics teachers use strategies to address conceptual understanding in their teaching of linear equations. The conceptual understanding in this study focuses particularly on pre-requisite/prior knowledge, explanations of concepts/terminologies used in linear equations, correct usage of operations when solving linear equations, the relationship between concepts of linear equations, relationship of linear equations with concepts within mathematics, the relationship of linear equations with concepts in real-life situation, and different representations of algebraic concepts in linear equations.

5.2.1.1 Pre-requisite/ prior knowledge

Neither of the teachers acknowledged the learners' prior knowledge. It was evident that the teachers were the source of information, as they provided learners with the concepts they had previously learned. Listening to the video clips of the lessons, reading through the transcriptions and lesson observation instruments, it showed that learners were not afforded an opportunity to indicate what they already know about the topic. This contradicts Ningsih and Retnowati's (2019) understanding that it is important to recognise specific objects by activating previous knowledge in long-term memory. As a result, it is critical to connect long-term memory knowledge with the information that must be presented. In other words, it is profound to protect the information that has already been stored in learners' memories by letting them share their own experiences. Most importantly, recognising how long-term memory works can help teachers design lessons so that learners can gain and apply their existing information to build new knowledge.

5.2.1.2 Explanations of concepts/ terminologies used in linear equations

Teachers 1 and 2 explained the terminologies used in linear equations to the learners. Each terminology was thoroughly explained; however, that was done by the teachers in all the conducted lessons. Learners were not allowed to explain the terminologies presented. The lessons revolved around the teachers, especially when they were teaching word problems and translating words into equations. Teachers' practice of not allowing learners to express their understanding in their own words might impact their ability to acquire mathematical language skills (Walkington et al., 2019). In support of this view, Mohamed et al. (2020) agree that allowing learners to use words to learn to talk mathematically and construct their own meanings is critical. The implication is that providing learners with meaningful ongoing interactions should be prompted by questions and discussions that will improve their understanding of mathematical language.

5.2.1.3 Correct usage of operations when solving linear equations

Operations involved in linear equations include addition, subtraction, multiplication, and division. The execution of these operations was efficiently done by both Teacher 1 and Teacher 2. Most of the activities solved during the lesson were teacher-centred, and the teachers ensured that the operations were used correctly and sequentially to obtain the desired answer. Learners were given little opportunity to solve the problems; therefore, addressing of conceptual understanding in terms of using the operations was at low level.

Learners must know words like ‘plus’, ‘add’, and ‘sum’ to do the mathematical operations of addition and ‘minus’, ‘subtract’, ‘take away’, ‘difference’ and other similar terms to perform the mathematical operation of subtraction. This means that if learners have not grasped the ideas of addition and subtraction, and they are unlikely to understand the operations’ application, especially when addressing word problems. The terms ‘product’, ‘times’ and ‘multiplied by’ are all used in the multiplication vocabulary. To add to this understanding Booker et al. (2014) indicated that learners should understand division concepts such as ‘divide’, ‘quotient’, and ‘share’. Similarly, learners must also understand the ideas of multiplication and division to comprehend the application of these operations.

5.2.1.4 Relationship between features of linear equations

Teacher 1 and Teacher 2 were able to explain the relationship between the variable, constant, coefficient, equal sign, and the operations that make up the linear equation. The teachers explained each aspect for learners to see how they depend on one another. The aspect that was not thoroughly explained was that of the equal sign. The teachers simply regarded the equal sign as the separator of the two expressions. At some point, Teacher 2 further indicated that the equal sign is meant to keep the equation balanced. To clarify Teacher 2’s explanation, Kieran et al. (2016) emphasised that the expressions on either side of the equal sign in an equation have the same value, and learners need to understand that this equality should always be upheld when solving equations. This suggests that learners should understand the equal sign as a feature of the equation that keeps it balanced, not as a sign to do something.

5.2.1.5 Relationship of linear equations with concepts within mathematics

The relationship of linear equations with concepts within mathematics in the context of this study refers to teachers' awareness of how linear equations that are used in many topics within mathematics like the geometry of straight lines, the Theorem of Pythagoras, the area, and perimeter of two-dimensional shapes and surface area and volume of three-dimensional objects. The relationship that was evident between linear equations, and concepts within mathematics was at the elementary level in the sense that teachers were only able to identify the kinds of numbers that were in an equation, for example, Teacher 2 used a fraction as a coefficient in one of the examples he provided. Teacher 1 did not try to show or ask learners about any relationship within mathematics. In other words, the teachers did not show how linear equations are applied or used in other areas of mathematics. Flagging the importance of emphasising the relationship within mathematical concepts, Adler et al. (2000) argue that relational learning will replace the existing creation of inert knowledge in schools with knowledge that is meaningful, transferrable, and flexible. This insinuates that connectivity should occur at several levels: connectivity of the different aspects of mathematics, connectivity of mathematics and everyday real-world knowledge, and, where appropriate, connectivity across subjects.

5.2.1.6 Relationship of linear equations with concepts in real-life situations

Teachers 1 and 2 exposed learners to examples of real-life situations when they were solving word problems. On this aspect, teachers tried to show learners that mathematics does not only revolve in the classroom but can also be applied to things they use daily. Showing the application of mathematical concepts in real life expands learners' conceptual understanding. This view is supported by Premadasa and Bhatia (2013), who found that, given a set of word problems covering a wide range of application areas, learners prefer problems that intrigue them or to which they can easily relate. They further indicated that teachers employ word problems to introduce the real-world applications to their learners. The implication is that, in most instances, teachers expose learners to real-life situations in linear equations when they deal with word problems.

5.2.1.7 Different representation of algebraic concepts in linear equations

Learners were exposed to word problems and algebraic equations in both teachers' lessons. The expectation was for the teachers to expose learners to different representations of the problem at hand to make it meaningful. The different representations included graphics, tables, flow diagrams and balance scales. Therefore, some objectives of mathematics education include translating between various representations, and associating multiple representations of mathematical concepts, conditions, and operations. For example, translating from verbal statements to equations and then to graphics and tables. The advantages of using numerous representations while solving linear equations problems lead to relevant and high-quality learning. This view is supported by Ainsworth (1999) who used three topics to summarise the advantages of various representations: they support different ideas and processes, constrain interpretations, and promote a deeper understanding of the domain. The implication is that when teachers expose learners to different representations, they instil conceptual understanding and accommodate learners with different cognitive levels.

During the interviews, Teachers 1 and 2 indicated that they did not ask learners about prior knowledge of linear equations because it was not done in 2020 by the Grade 8 learners who were doing Grade 9 in 2021 when the research was conducted. However, the teachers were then asked further if the situation was different, whether they would require learners to indicate prior knowledge. Teacher 1 indicated that it would be a waste of time as learners do not like to answer questions among their classmates, and secondly, teachers still had to cover the curriculum on time. The statement was disturbing because it is expected that teaching has to be guided by what the learners bring into the classroom. Misconceptions should be addressed early so that learners do not find it difficult to understand the concept.

5.2.2 What Strategies Do Teachers Use to Address Procedural Fluency in teaching Linear Equations?

This question focused on teachers' addressing of procedural fluency when teaching about linear equations. Procedural fluency was divided into three descriptors: (*a) Carrying out procedures flexibly; accurately; efficiently; and appropriately, (b) Using multiple measures in solving linear equation problems with understanding, and (c) Knowledge of appropriately and relevantly using procedures and sequencing of actions.* These descriptors were used to answer the question on addressing of procedural fluency.

5.2.2.1 Carrying out procedures flexibly, accurately, efficiently, and appropriately

Although Teacher 1 and Teacher 2 did not justify the steps taken when solving the linear equation problems, they ensured that they were accurate, efficient, and appropriate in all the calculations. They emphasised solving the problems and getting to the correct answer. Teacher 1, at some point, was not happy when the learner got the step incorrect, which showed that, according to her, what was important was to get everything right; that is the other reason why the lessons were viewed as teacher centred. Teacher 2 also ensured that every procedure was correct by doing the calculations himself. It is clear that even today, teachers are focusing more on procedures than learners having to acquire conceptual understanding and being allowed to get wrong answers so that they can learn from their misconceptions. A broader perspective has been adopted by Inayah et al. (2020), who argue that learners' comprehension of a mathematical topic and their ability to solve mathematical problems correlate with procedural fluency. Learners can determine how well they understand a mathematical idea by how smoothly the mathematical methods are applied. Learners can also effectively solve mathematical problems by acquiring their procedural fluency in mathematics. Learners need to perfect this skill since it is crucial. As a result, the instillation of procedural fluency skills in learners is indicated by their mastery of general mathematical methods. For other mathematical skills to function, procedural fluency is a prerequisite.

5.2.2.2 Using multiple measures in solving linear equation problems with understanding

Both teachers did not make much effort to expose learners to multiple ways of solving linear equations. Teacher 1 showed learners only one way, and Teacher 2 exposed learners to only two methods of solving the problems in all the lessons they conducted. Learners were not allowed to suggest other ways of solving the linear equations problems. Even when the learners were solving the examples, they followed the teacher's methods. Learners were subjected to teacher-centred lessons throughout. In other words, both teachers applied instrumental understanding, leading learners to memorisation instead of emphasising rational understanding and solving problems with understanding. Even though it might take longer to address procedural fluency, it is still essential to let learners acquire it. Similarly, Bay-Williams (2020) argued that it takes a long time to become procedurally fluent, which includes linking procedures to concepts. For learners to acquire specific, effective techniques and subsequently learn to choose the strategy that best fits a given situation, it is necessary to employ visual representation and scenarios. The argument implies that teachers need to use manipulatives to address procedural fluency. Manipulatives assist in making abstract concepts to be concrete.

5.2.2.3 Knowledge of appropriately and relevantly using procedures and sequencing actions.

Teacher 1 and Teacher 2 did an excellent job of conducting the problems. They knew when to apply a certain step and the relevancy of those steps. All the procedures conducted were done perfectly by the teachers. Procedural fluency was done appropriately, and it was the core of solving linear equations during all the lessons. Out of all the mathematical proficiency strands, procedural fluency was the one where the teachers did exceptionally well in it. The only concern was that teachers did most of the work, and learners were minimally involved. The lessons were centralised more on the teachers ensuring that the procedures were done correctly to obtain the correct answers. There was no room for wrong answers, especially from the learners’ side. The rating scale and the bar graph clearly indicate that both teachers concentrated on procedural fluency.

5.2.3 What Strategies do Teachers Use to Address Strategic Competence in Teaching Linear Equations?

The descriptors for the strand were used to answer the question on strategic competence by (a) Solving the problem using different strategies, and (b) Exposing learners to more than one way of solving a mathematical problem.

5.2.2.1 Solving the problem using different strategies

Teacher 1 exposed learners to only one way of solving linear equations: using additive and multiplicative inverse procedures. All the lessons conducted were stereotyped in that they were teacher centred in addition to only using one strategy. The teacher was solving the problems, focusing on getting the correct answers. Teacher 2's lessons used two strategies to solve the linear equations: the trial-and-error method and additive and multiplicative inverse. The lessons were mostly teacher centred. The teacher was the only source of information, and learners were receivers by following what the teacher presented. Algani (2019) found that teachers believe that using various innovative strategies to teach mathematics is essential and effective. My view is that these findings encourage teachers to use various proven effective strategies, technological innovation, and creativity when teaching mathematics. Algani (2019) further indicates that it is conspicuous that how the content is conveyed and clarified affects learners' understanding and interest in mathematics. The implication is that teachers need to be creative and know the vast majority of strategies to use in solving linear equations in mathematics. Their teaching should not be stagnant to disadvantage the learners in the process.

5.2.2.2 Exposing learners to more than one way of solving a mathematical problem

Teacher 1 restricted learners from coming up with their own methods for solving linear equations. The teacher was the primary problem-solver in the lesson and largely applied their own knowledge to resolve all the problems using the same method. Teacher 2 tried to expose learners to two strategies to solve the problems but did not give the learners a chance to develop their own strategies. The lessons conducted by Teacher 2 were also teacher-centred, and the teacher initiated the two methods. The study findings conducted by Sari et al. (2019) demonstrate how various methods of imparting knowledge might influence how effective problem-solving skills are developed. Exposing learners to different methods enhances their problem-solving skills, thus developing strategic competence. The learners will then be aware that a problem may be solved in various ways while still yielding the same results. Similarly, Tachie (2019) agrees that the degree to which learners succeed in problem solving is determined by the accessibility and appropriateness of applying teaching techniques, skills, knowledge, and strategies to solve a problem. The implication is that learners' academic achievement, especially when solving linear equations problems, depends on acquiring metacognitive abilities and the methods used. Therefore, during lessons, it is important for teachers to expose learners and also give them an opportunity to develop different strategies that cater to various cognitive levels. Furthermore, it is the learner who will be faced with problems when writing an examination or a test, and they should be able to apply the methods that they are comfortable with and can relate to. The more strategies and approaches learners are exposed to, the deeper their conceptual understanding of the topic.

Both Teacher 1 and Teacher 2 confirmed during the interview that the strategies that they chose were the ones they thought were easy to understand. This implies that learners could not apply the strategies according to their cognitive levels. They had to follow what the teachers taught them. Teacher 1 further indicated that it was also a waste of time to try all the strategies as that would confuse the learners. In this case, it was evident that teachers acted as barriers to learning because they did not allow learners to explore different ways of solving linear equations problems.

5.2.3 What Strategies do Teachers Use to Address Adaptive Reasoning in Teaching Linear Equations?

This question focused on teachers' addressing of adaptive reasoning when teaching about linear equations. The adaptive reasoning in this study was divided into two descriptors, namely (a) Justification of mathematical ideas and strategies, and (b) Justification of procedures.

These descriptors were used to answer the question using the findings concerning Teacher 1 and Teacher 2.

5.2.3.1 Justification of mathematical ideas and strategies

During the lessons conducted by Teacher 1, it was clear that there was no reasoning behind the chosen strategy and the mathematical ideas. The teacher never explained why the problems were done in a certain way or provided step-by-step teaching. Neither the learners nor the teacher explained how they arrived at the answer, meaning adaptive reasoning was not considered. Teacher 2 tried, to a certain extent, to explain the aim of moving from one step to the other by mentioning the BODMAS rule. Answers were substituted into the equations to verify whether they were correct. Thorough justification was not accommodated, so adaptive reasoning was not addressed accordingly. Justification of mathematical ideas is important because to solve word problems; learners must use interpretations, reasons, and representations, all of which are crucial for mathematical communication. Similarly, Bieda et al. (2013) argue that making sense of and comprehending mathematical ideas and concepts embedded in procedures is the process of reasoning in mathematics. The implication is that there must be a reason behind every procedure followed in mathematics. Procedures are not done randomly without logic attached to them.

5.2.3.2 Justification of procedures

There was no justification for the steps used in the process of solving linear equations with lessons that were conducted by Teacher 1. Everything went on as if learners understood what was happening at a deeper level. Follow-up questions were not asked by the teacher to establish the reasoning behind the answers that the learners gave.

No procedures were justified during the calculations by Teacher 2 to support an understanding of the processes. The connections between the concepts employed to tackle the problem were not thoroughly justified. The teacher did not place a strong emphasis on higher-order thinking. To disprove what transpired in the lessons observed, there is an emphasis from NRC (2004) study, that the ability to engage in the processes of mathematical thinking, which entails reasoning, problem-solving, integrating mathematical ideas, drawing conclusions from evidence, and conveying mathematics to others, is necessary for a thorough comprehension of mathematics. Tanudjaya and

Doorman (2020) agree that in our modern, technologically advanced culture, it is crucial to develop higher-order thinking abilities in mathematics. As a result, both the teachers and the learners must understand the steps taken to solve problems and the sense behind them.

When the teachers were asked during the interview why they did not justify each step of the procedure when solving the linear equations, Teacher 1 responded that she was unaware that it was necessary for her to write down the reasoning because it was said verbally, and learners should remember. This response implies that learners have to quickly write and listen to what the teacher is saying or remember at a later stage. Most learners in Grade 9 are not yet at the stage of capturing everything that the teacher has said, which means the expectation by the teacher was impossible. Learners need to be guided by the teacher in all the steps so that their adaptive reasoning is addressed, and they can solve linear equations independently. On the other hand, Teacher 2 indicated that for him not to justify the steps or not to ask learners to justify it was because as long as learners are giving correct answers, there is no need for them to provide reasons; it simply meant that they understand. Both teachers do not find it important to justify the steps that were taken when solving problems. For them, it is either a waste of time or learners understand the concepts.

5.2.4 What Strategies do Teachers Use to Address Productive Disposition in Teaching Linear Equations?

This question focused on the addressing of a productive disposition by teachers when teaching learners in linear equations. The descriptors for productive disposition were: (a) Encourage extension of knowledge from the known to the unknown, (b) Promote courage and enthusiasm, and c) Encourage own initiatives.

5.2.4.1 Encourage the extension of knowledge from the known to the unknown

Neither Teacher 1 nor Teacher 2 acknowledged the information the learners acquired in the lower grades. Learners were spoon-fed throughout the lessons that were conducted.

There was no effective learning because learners were treated as though they had nothing to offer or show what they possessed previously.

The teachers did not try to move learners from known to unknown so that they can connect between what they had learnt and what they were about to learn. Teachers need to make a connection between the concepts in mathematics so that learners can make sense of what they are learning. Similarly, Hasbi et al. (2019) explain the term “mathematical connection” as the capacity to comprehend, use, and identify how mathematical concepts interact with one another. It also refers to the capacity to recognise and apply mathematics in contexts unrelated to mathematics. As a result, concepts in mathematics will not be treated in isolation; at the same time, learners will be able to derive mathematics in complex situations.

5.2.4.2 Promote courage and enthusiasm

During the lessons conducted by Teacher 1, there was an instance where learners were allowed to solve problems on the board; however, the teacher kept on interrupting them and not allowing them to make mistakes so that they could learn from them. The teacher focused on getting the correct answers and using only one method to solve problems. Learners did not explain the reasoning behind the steps taken. The promotion of courage and enthusiasm was not supported, thus productive disposition was not addressed. On the other hand, Teacher 2 also allowed the learners to solve problems on the board, but they were also not allowed a chance to explain how they came to the answer they obtained. The learners represented what the teacher demonstrated through the examples. Productive disposition was not effectively addressed. When learners are actively and mentally participating in the lesson, they learn effectively. Choosing for them will not improve them; instead, it will impede their ability to think critically and prolong their mental activity. In that way, learners will be discouraged and demoralised. To discourage the teachers’ practices, Awofala et al. (2020) indicated that learners who see mathematics as worthwhile, practical, and useful may genuinely love the subject and possess the persistent spirit needed to solve mathematics problems. Therefore, it is the responsibility of mathematics teachers to support learners in developing a regular inclination for exploring the significance underpinning mathematical ideas, making connections in mathematics, and fostering mathematics' real value.

5.2.4.3 Encourage own initiative

In the lessons that Teacher 1 and Teacher 2 conducted, there was no point where learners were required to suggest their own methods of solving the problems. The learners were passive in that respect because the teachers were the only ones who imparted the methods to solve the linear equations problems. It is important as teachers to allow learners some space to make mistakes and learn from them, which means effective teaching and learning take place. Learners cannot come to class and absorb everything the teacher tells them for the whole lesson and leave without putting forward their understanding. It is a very powerful learning experience to have learners creating their own methods and then share them with the class. This view is supported by Kopel et al. (2021), who indicated that by encouraging innovative teaching methods, learners are taught to view setbacks or errors as chances for growth. The implication is that learners must be allowed to create their own way of solving problems according to their understanding of whether those methods are wrong or right, and they will adjust them accordingly. Letting them develop their own initiatives is a good chance to evaluate the misconceptions they might have and their level of conceptual understanding.

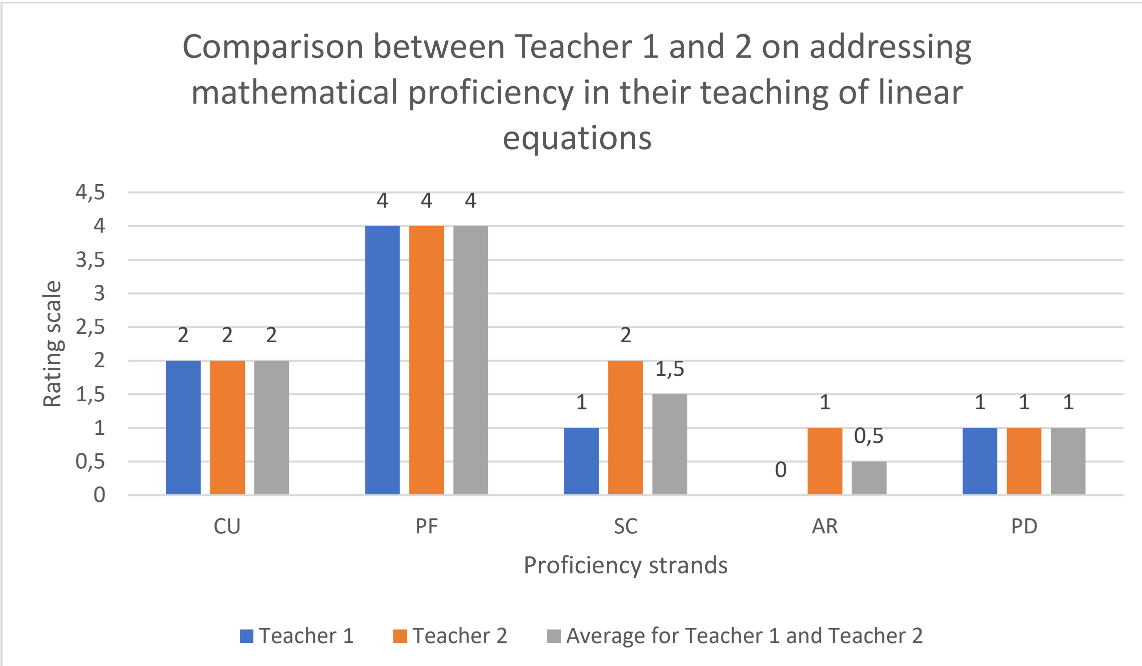
Teacher 1 and Teacher 2 were asked during the interview about not actively involving the learners during their lessons. Teacher 1 responded that learners do not know any other methods and tend to be lazy to answer questions posed to them. The assumption made by the teacher was alarming because learners need to be given a chance to showcase what they know instead of being suppressed. Teacher 2 indicated that it is important for him to teach them his method first until they understand; however, due to time constraints, allowing learners to share the methods they might know is always a challenge. Addressing of productive disposition was not considered in these lessons. According to the teachers' responses, they hardly engage learners in terms of being innovative.

Engaging learners in the learning process stimulates learners' interest in and enjoyment of the lesson. Fostering learners' initiative means being prepared for 21st century skills, emphasising creativity, critical thinking, collaboration, and communication. Learners should be allowed to be creative in their own right and to think critically about the problem presented. They must be allowed to come together as a group to analyse their ideas and converse on how they reached the solution before deciding on the correct way to approach a situation without the teacher's help. In their study, Kurnuiawati et al. (2022) concluded that teachers who nurture learner creativity at a high-level exhibit a certain kind of teaching methodology that enables learners to grow into creative individuals. In other words, it takes a special kind of teacher to allow learners to be innovative in the classroom and generate as many ideas as possible towards solving the problems. In that way, learners' involvement will be considered, and their productive disposition will be instilled. Therefore, to answer the research question: How do Grade 9 mathematics teachers use strategies to address mathematical proficiency in their teaching of linear equations, teachers' emphasis was mainly on procedural fluency. A conceptual understanding was not instilled in learners as they were exposed to two ways of representing linear equations, and their prior knowledge was not considered. For strategic competence, teachers used one or two methods to solve problems. Adaptive reasoning and productive disposition were minimally emphasised.

5.3 A closer look into the bar graphs for Teacher 1 and Teacher 2

An insight into the teachers' addressing the mathematical proficiency in teaching linear equations using bar graphs was done individually which has shown how they did it. The figure below shows a comparison between the two teachers on how they addressed the mathematical proficiency in their teaching of linear equations guided by the rating scales.

Figure 8: Comparison between Teacher 1 and 2 on addressing mathematical proficiency in their teaching of linear equations



The rating scale and the bar graph for both Teacher 1 and Teacher 2 show that in terms of addressing conceptual understanding they were at level 2, which was very low. These findings show that conceptual understanding was not a priority in their teaching. The learners were taught without an actual understanding of concepts within linear equations.

Both Teacher 1 and 2 were at level 4 which was the maximum scale with an average of level 4. It indicates that the emphasis on procedural fluency was done very well; however, most of the work was done by the teachers. The focus was more on problem solving and getting the correct answers.

Looking at the rating scale and bar graph for strategic competence, Teacher 1 was at level 1, and Teacher 2 was at level 2. Both teachers were at an average of level 1,5 in terms of exposing learners to different methods of solving linear equations. This outcome indicates that these teachers still lack the knowledge to teach problem solving in different ways, making the learners think that linear equations can be solved in one or two particular ways only.

Analysing the scores on the rating scale and bar graph in terms of adaptive reasoning, Teacher 1 was at level 0, Teacher 2 was at level 1, and both at an average of 0,5. These results indicate a minimal attempt to justify the procedures and strategies used to solve the linear equation problems. This shows that the teachers went through the calculations with no interest in clarifying the reasons behind each step taken. More work still needs to be done in terms of addressing the importance of justifying each step when solving problems.

Addressing productive disposition was insufficient when looking at the bar graph presented. Both Teacher 1 and Teacher 2 were at level 1 in addressing productive disposition which gave an average of level 1. I agree that a productive disposition cannot be obtained in one or two lessons, but teachers should incorporate strategies that work towards building the strand when conducting lessons and solving problems. Ultimately the learners' interest will grow steadily, resulting in them acquiring a productive disposition.

5.4 Limitations of the Study

My main objective with this study was to learn more about how do Grade 9 mathematics teachers use strategies to address mathematical proficiency in their teaching of linear equations. As previously said, teachers were considered for the research because they influence their learners' proficiency.

However, the findings of this study cannot be generalised due to the following limitations: Firstly, in terms of content, the focus was only on linear equations, whereas many other mathematics concepts are part of the curriculum. Mathematics is a broad subject in such a way that it can be found in many subjects outside the mathematical discipline. There might be a possibility that other teachers can instill mathematical proficiency in other concepts in mathematics besides linear equations.

Secondly, the focus of the study was only on Grade 9 mathematics teachers. Mathematics is taught from preschool to Grade 12 in the South African schooling system. As for linear equations, the concept is taught from Grades 7 to 12. Therefore, having to focus only on Grade 9 and leaving out five other grades that deal with the concept of linear equations was a disadvantage because the findings were restricted.

Thirdly, only two teachers participated in this study, whereas there are many teachers in Gauteng North District teaching mathematics in Grade 9. Therefore, generalising the findings will be an oversight because the whole population was not used, but the results were based on sampled teachers.

Fourth, the research was based on one District in Gauteng. There are 15 Districts in Gauteng Province and many schools in each District. In those schools, there are many grades 9 mathematics teachers. As a result, the findings of this study cannot be generalised.

Fifth, the teachers may have intentionally or unintentionally told me what I wanted to hear during the interviews. Due to their perception that their performance would reflect on them as teachers, they were both obviously invested in performing well. For example, both teachers chose the strategies that they thought were easy to execute and got the correct answers. Although I thought the teachers were honest and upfront in their answers during the interviews, there might have been an expected impact. In other words, what they thought I expected of them as teachers may have impacted what the teachers stated. For example, on the question of teachers collaborating to elaborate goals, investigate learners' thinking and difficulties with linear equations and explore different representations and strategies, they might have indicated that such happens just to give the impression of being proactive teachers.

Sixth, the study only explored teachers' strategies to address mathematical proficiency, whereas quantitative research method could have determined learners' level of development in terms of mathematical proficiency.

5.5 Recommendations

Based on the findings stated in this paper, despite the limitations that were indicated, the following recommendations are suggested:

Training is required on mathematical proficiency strands. Teachers must be capacitated on all the strands incorporated in mathematical proficiency. All the strands need to be unpacked to make it easy for teachers to have a clear understanding of each and one so that they can implement them in their classrooms. The training must further elaborate on the mathematically proficient teacher and the importance of developing learners to also be proficient in mathematics.

Content workshops held for teachers should also be aligned with the mathematical proficiency strands. In other words, when the content workshops are done, there should be a clear demonstration of how teachers can teach specific content to instill mathematical proficiency. Most content workshops focus on procedural fluency, which makes the teachers impart the knowledge as done in the workshops. Therefore, the trainers must bear in mind that workshops should be conducted to empower the teachers with the necessary tools and strategies to effectively address mathematical proficiency.
In one of the interviews conducted with the teachers, it was mentioned that there was not enough time to allow learners to develop their own strategies according to their understanding of solving linear equations. It is then recommended that the allocation of teaching time should be extended and documented in the policy for mathematics teachers to use as many strategies as possible and to allow learners to be innovative in coming up with their own way of solving problems. The focus should not only be on the completion of the curriculum, but the emphasis should also be on teaching and learning with understanding. There are also different approaches that teachers can use, for example, brainstorming, learning pathways, inquiry, jigsaw, snowballing, and many others, which require adequate time to implement. These approaches can also come in handy in terms of addressing mathematical proficiency.

The Annual Teaching Plan (ATP) that teachers follow in terms of which topics to teach should also indicate the suggested approaches teachers can implement for a specific concept. Seeing that in the lessons that I observed, both teachers were using a teacher-centred approach in all the lessons that they conducted. Indicating the relevant approach per concept might assist in taking the teachers out of the stereotyped approach and exploring different approaches. As a result, learners will also be exposed to various ways of learning.

The use of manipulatives is also recommended because learners are unique and have different ways of learning. Exposing learners to manipulatives will also instill learners' interest and assist them in concentration. Teachers can use either visual or concrete manipulatives depending on the concept taught. As a result, learners will acquire conceptual understanding.

Some teachers have been teaching for a while, and they might still be stuck in the traditional ways of teaching. It is recommended that specific training should be organised for teachers who have been in the field for years to instill new and innovative ways of teaching.

The education system depends entirely on teachers to produce proficient learners; therefore, it is important to invest all the resources in capacitating the teachers to become fluent in everything they do in the classroom. They must be provided with all the necessary tool to make it possible for them to execute their duties without any obstacles.

5.6 Suggestions for Future Research

More research is required to ascertain whether these findings even apply to most, if not all teachers. More research needs to be done on mathematical proficiency because it covers all the aspects needed to develop effective lessons and develop fluent learners in mathematics.

The future study needs to be extended to other grades. The involvement of more grades will also allow us to see the progression of the impact of what learners have learnt in the previous grades. Future research can also focus on the same cohort of learners and teachers where possible to make follow-ups of the same learners and teachers as they move up with the grades every year. This approach will allow the researcher to monitor the conceptual growth of both the teacher and the learners.

Mathematics has a variety of topics, therefore, focusing on one topic was not sufficient. For future research, it is advised to expand to other topics to discover how mathematical proficiency is addressed. Every topic in mathematics is important because the concepts within mathematics are connected. Therefore, if one concept is poorly presented, it might affect successive concepts.

For future research, more teachers should be involved in the study to get a clear picture of what is really taking place during teaching and learning. The kind of interactions, methods, approaches, and materials used to address mathematical proficiency should be determined. Everything happening in the classroom is important as it yields different kinds of learners depending on the processes that transpired. Therefore, observation of as many teachers as possible is critical to find out the loopholes in terms of teaching.
The benefits of increasing the number of districts and provinces for future research will be great so that all the stakeholders will get an informed report on how the teachers and learners in their specific areas are doing. It will be more advantageous to get comprehensive information from different places to make an informed conclusion about the teaching and learning of mathematics from different demographics.

5.7 Implications of the Study

The significance of this study was to determine how teachers used strategies to address mathematical proficiency in their teaching of linear equations. Mathematical proficiency encompasses skills that are needed to be confident in solving mathematical problems with understanding and effectiveness.

The result of my study implies that one of the main aims of the teachers in the presented lessons was to address procedural skills by first teaching the approaches and then having learners practise them using similar constructed examples and methods. Such mathematical experiences are common, which is troubling because procedural skills and mathematical comprehension should be addressed simultaneously. Mathematical proficiency strands are intertwined, interwoven and interconnected; therefore, all the strands should be addressed simultaneously to produce a mathematically proficient learner.

Teachers oversee creating and maintaining the mathematics learning environment to which learners like those in this study are exposed. Teachers choose the rules, procedures, and domain-specific approaches in mathematical practices (Goldin, 2018) that are implemented in the learning environment depending on their views and talents in teaching and learning mathematics. Based on these, teachers must develop strategies for teaching algebra to minimise difficulties in learning algebra. They must focus on understanding of concepts in algebra. It is easy to memorise something and even do well in a test without knowing what is going on. As a result, when learners solve problems that are not the same but like the lessons they have learned, they find it hard to obtain the right answer and are challenged with what steps to undertake. Hence, understanding mathematics must be the highest goal. This can be achieved by providing learners with varied teaching and learning strategies.

Rich mathematical tasks that are given in various contexts that can be completed in multiple ways or for which there is no immediately clear approach have a stronger positive correlation with learners' conceptual understanding than procedural tasks (Blazar, 2015). Teachers should emphasise conceptual knowledge more than the steps involved in doing calculations. This can be accomplished by switching from traditional teaching to a more engaging and learner-centric approach like contextual learning (Jazuli et al., 2017). Learners will comprehend topics more fully if they are introduced using a variety of representations, such as manipulatives, pictures, and symbolic representations, to accommodate diverse learning styles.

When it comes to the teaching and learning of mathematics, a representation is any notion or concept that enables us to understand, share, debate, and/or manipulate the idea or concept with others (Goldin, 2014; Kilpatrick et al., 2001; Tripathi, 2008). As a result, learners can exchange ideas according to their understanding based on how the problem is represented. They will be able to conclude what works best for each individual learner. The various representations will assist learners in getting out of their comfort zones, exploring various ways of approaching a problem, and getting to the same answer. Examples of visible representations that encompass mathematical concepts include diagrams, graphs, number lines, physical objects, or mathematical formulae, expressions, and equations (Goldin, 2014; National Council of Teachers of Mathematics, 2014). Since they can help with concept formation, shaping, and mediation in math classrooms, using representations and tools is crucial to learning (Goldin, 2004; National Council of Teachers of Mathematics, 2014; Pape & Tchoshanov,2001). The emphasis is that different representations can make it easy for learners to understand difficult concepts by engaging in varied approaches to learning. Ultimately, conceptual understanding will be instilled in learners with different cognitive levels.

Learners' understanding of mathematics disciplines and their capacity to research and discuss real-world problems depend on their ability to use various representational techniques (Goldin, 2004; Pape & Tchoshanov,2001). This suggests that learners should not only do mathematical operations that are completed within the four walls of the classroom but also connect those operations to their daily experiences. Learners should be able to locate mathematical ideas in contexts where others can't make a connection between those contexts and mathematics. But they can only achieve that if they have significant strategic competence. Learners' understanding of the concept increases and they create new or richer mental images when they use a variety of representations for the same concept (Kilpatrick et al., 2001; Pape & Tchoshanov,2001; Swan, 2005). As a result, it will be advantageous for learners to be exposed to different representations because they intensify their scope of understanding which becomes wider. The wider their understanding, the more they will be able to generate various ways of solving problems and become more affluent.

Given the importance of mathematics in daily life, introducing a topic to learners through real-world challenges where teachers may explain both the concepts and facts to the learners is critical. Teachers can ask learners to describe how a concept works and how to use it to solve issues to other learners to offer them a chance to express their ideas. In my study, teachers presented examples of problems before giving practise questions that were more routine and did not require any reasoning. The teaching approach remained conventional. Traditional mathematics teaching and learning methods that emphasise direct presentation allow learners to mould their capacity to think rationally while limiting their ability to generate information by interacting with their surroundings. As a result, teachers must allow learners to justify their procedures.

Mathematical teaching that involves learners in the learning process by providing opportunities for them to construct mathematical concepts or procedures is effective for mathematics teachers. Learners' knowledge and memory will benefit from placing them in situations where they must build their knowledge under the teacher's guidance. Learners who develop their mathematics knowledge rather than receiving it from their teachers retain more of what they learn. Hidayat and Setyawan (2020) indicate that teachers can determine how far their pupils' understanding of mathematics has progressed through construction. This implies that different perspectives and interpretations of mathematical knowledge result from constructing mathematical knowledge among learners. Mathematical knowledge is built based on whatever information is available and appropriate to the problem, and it reflects the learners' mathematical comprehension.

Teachers need to teach for understanding as it lays a solid foundation for developing high cognitive level for learners. Teachers' level of mathematical proficiency is vital in the growth of mathematically proficient learners. Proficiency levels describe what a teacher can do in mathematics in terms of procedures, i.e., problem-solving, cognitive, and demonstrating procedures, reflecting, choosing tools, and computational approaches, relating, signifying, and collaborating in real-world situations in a spontaneous and non-rehearsed context. Teachers need to develop lessons carefully for conceptual understanding. Activities planned by teachers should purposefully encourage learners to develop from tangible, narrow, to broader concepts. Learners must be able to use appropriate manipulatives when solving problems to show understanding.

Proficiency in mathematics teaching is a question of knowing how to prepare and present the lesson to learners, how to interact with the learners during the lesson and how to conduct activities (Kilpatrick, 2001). This implies that knowing how to prepare and present the lesson focuses on the nature of information teachers must have to develop, lead, and inspire creative teaching successfully. Each prepared lesson needs to aid productive and effective results and the development of learners towards proficiency. Addressing of mathematical proficiency depends on strong content knowledge and conceptual understanding of resources by teachers. Scaffolding examples to cater to learners with different cognitive levels is an important activity in teaching mathematics. The relevancy of chosen examples is important to avoid confusion, and at the same time, a high standard for the level of cognitive demand for learners must be maintained.

Cuoco (2001) states that mathematical proficiency prepares a teacher to build a curriculum that links mathematical concepts and builds a disposition within learners. Mathematically proficient teachers understand that there are multiple strategies for approaching mathematics. Teachers should also acknowledge that there is some form of information that learners bring into the class that should be considered. Some of the information they bring is correct, while some are not; if not acknowledged, it can cause misconceptions that will make it difficult for learners to understand. Interacting with learners in class and outside assist teachers in picking up the knowledge that learners have and their understanding towards mathematics.

In the 21st century, the emphasis is on collaboration, communication, creativity, and critical thinking. The expectation is that between teachers and learners, learners and learners, there should be sharing of thoughts, working together, innovatively assessing activities, integrating subjects and disciplines, and attempting new strategies. Another important aspect of a mathematically proficient teacher is reflection. Through reflection, teachers can improve their teaching practices, the opportunity to observe and correct common learners' errors, and the effectiveness of a lesson. Teachers can use different ways of reflecting on their work to improve their approaches such as administering tasks to gather whether the learners understood, examining video records, working with co-workers, teachers from neighbouring schools, teachers outside the cluster, district, province, and national level, and abroad.

Progressing teachers' mathematical knowledge and their capability to use it to educate is pivotal in creating learners' mathematical proficiency. All these qualities can effectively shape a learner's proficiency towards understanding linear equations and applying the skills in any given situation.

5.8 Final Reflection

Finally, improving the learners' level of knowledge should be the goal of teaching mathematics, according to Cadorna et al. (2021). This implies that teaching mathematics, particularly algebra, should focus not only on procedural skills but also on conceptual understanding, strategic competence, adaptive reasoning, and productive reasoning. Addressing of all these strands is needed to enable learners to solve difficult problems. There is little benefit in learners recalling a formula or procedure to prepare for an assessment tomorrow, only to forget the core concepts by next week. Teachers must ensure that the learners understand the material and not just memorise the procedures. Using strategies to address mathematical proficiency is a powerful approach, and if done properly, it will yield innovative and creative mathematicians.

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Appendices

6.1 Appendix A: Post-Graduate Acceptance Letter

UNISA university of south africa

0919 M1RST

MOTHUDI T N MISS
1330 PABLO STREET
KARENPARK
0182

STUDENT NUMBER : 3646-012-5

ENQUIRIES NAME : MADISENG LS
ENQUIRIES TEL : 0124415702

DATE : 2022-02-11

Dear Student

I wish to inform you that your registration has been accepted for the academic year indicated below. Kindly activate your Unisa mylife (https://myunisa.ac.za/portal) account for future communication purposes and access to research resources.

DEGREE :	MED MATHS EDUC	(90061)
TITLE : proficiency	Grade 9 mathematics teachers' development of learner mathematical in Linear equations: A case of selected schools in Gauteng North district.	
SUPERVISOR :	Mrs SM KODISANG (kodissm@unisa.ac.za)	
CO-SUPERVISOR :	Prof MF MACHABA (EMACHAMF@unisa.ac.za)	
ACADEMIC YEAR :	2022	
TYPE:	DISSERTATION	
SUBJECTS REGISTERED:	DFMED96 MEd - Mathematics Education (Dissertation)	

A statement of account will be sent to you shortly.

You must re-register online and pay every academic year until such time that you can submit your dissertation/thesis for examination.

If you intend submitting your dissertation/thesis for examination you have to submit an Intention to submit form (available on the website www.unisa.ac.za) at least two months before the date of submission. If submission takes place after 15 November, but before the end of January of the following year, you do need not to re-register and pay registration fees for the next academic year. Should you submit after the end of January, you must formally reregister online and pay the full fees.

Please access the information with regard to your personal librarian on the following link: https://bit.ly/3hxNqVr

You are reminded to claim the myLife email address since all communication from the University will be sent to this address and you will also be required to contact the University using the myLife email address.

Yours faithfully,

Prof M S Mothata
Registrar

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6.2 Appendix B: Observation Schedule

Name of School		**Content Area**	Patterns, functions & algebra
Name of Teacher		**Topic**	Algebraic Equations
District	Gauteng North	**Sub-topic**	Linear Equations
Cluster		**Date**	
Grade	9	**Duration**	

1. Teachers' development of learners' mathematical proficiency in linear equations

STRANDS	INDICATORS	FINDINGS/ OBSERVATIONS
Conceptual Understanding	Explanations of concepts/ terminologies used in linear equations	
	Correct usage of operations when solving linear equations	
	Relationship between concepts of linear equations	
	Relationship of linear equations with concepts within mathematics	
	Relationship of linear equations with concepts in real-life situation	
	Different representation of algebraic concepts in linear equations	
Procedural fluency	Carrying out procedures flexibly	
	Carrying out procedures accurately	
	Carrying out procedures efficiently	
	Carrying out procedures appropriately	
	Using multiple measures in solving linear equation problems with understanding	

	Knowledge of appropriately and relevantly using procedures and sequencing of actions	
STRANDS	**INDICATORS**	**FINDINGS/ OBSERVATIONS**
Strategic Competence	Solving the problem using different strategies	
	Exposing learners to more than one way of solving mathematical problem.	
Adaptive Reasoning	Justification of mathematical ideas and strategies	
	Justification of procedures	
Productive disposition	Encourage extension of knowledge from the known to the unknown	
	Promotes courage and enthusiasm	
	Encourages own initiative	

6.3 Appendix C: Interview Questions, Teacher

1. I have observed that when you were introducing the lessons, learners were not asked about their previous knowledge of the topic. Was that done on purpose?
2. During all the lessons that you conducted, I have noticed that you used only one strategy of solving the linear equations. Were there no other methods that could have been used?
3. When solving the linear equation problems, there was no indication of justifying the chosen strategy and the procedures followed, was there a reason for doing so?
4. The observation that I made during your lessons was that learners were not given an opportunity to come up with their own strategies or methods to solve the linear equations problems. Is it how you normally conduct your lessons?
5. Most of the time in the lessons, you were the one solving the problems and learners were minimally engaged in doing so. The only time they were involved was when they gave short answers. What was the reason?
6. During the lesson on word problems, there was a point where you quote switched, why did you do that?
7. Learners were exposed to two representations of linear equations, which were in word form and algebraic form. Are there other ways of representing linear equations that you know of? If yes, what are those representations?
8. How often do you meet with other teachers to elaborate goals, investigate learners' thinking and difficulties with linear equations and exploring different representations and strategies?
9. With your observation, how is the attitude of learners towards linear equations?

10. When conducting the lessons on linear equations, are you finding it easy or difficult to present it?
11. Is there any comment that you would like to say based on the lessons that you presented or the interview?

6.4 Appendix D: Letter to the District Director

Request for permission to conduct research at Gauteng North District in Gauteng Province

Date: 09/09/2021

To Gauteng North District Director: Mr. Zukile Kosi

Dear Sir

Subject: Request to conduct research in schools from Gauteng North District

My name is Teresa Ntsae Mothudi and I am a student at the University of South Africa. I am presently enrolled for the MEd with a specialization in mathematics education. To complete the requirements for the degree, I need to conduct virtual research that is related to the area of my specialization. My research project is entitled: Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District. The purpose of the study is to explore how teachers develop learners' mathematical proficiency (conceptual understanding; adaptive reasoning; strategic competence, procedural fluency, and productive disposition) in linear equations.

My supervisor is Ms Sophy Kodisang, a lecturer in the Department of Mathematics Education. Her office telephone number is 012 429 3964.

Four teachers who are teaching grade 9 mathematics at Gauteng North District will be selected to participate in the study. The respondents' lessons will be observed and thereafter they will be interviewed. All data collected will be analysed and a report regarding the study will be written.

Due to the Covid-19 pandemic where social distancing is emphasised as one of the regulations, the lesson observations and interviews will be conducted remotely. Teachers who give a consent to participate will be requested to capture their lesson presentation on a video camera using their cell phones and send the recordings to the researcher or alternatively use google classroom platform. The contents of the video will always be kept confidential, and no one will be invited in the google classroom besides the researcher. After analysing the lessons maximally, the video recordings will be deleted permanently. The interviews will be conducted telephonically, through Zoom or Microsoft Teams. The participant and the researcher will be the only people having access to any of the platforms. The interviews will be recorded, with the permission of the participant. The recordings of the interview will be deleted permanently after being maximally used.

I believe that the work I am doing could be relevant to mathematics education in Gauteng North District. The information collected will be treated with confidence and I hereby give undertaking that:

- All participation is voluntary
- The school's name, as well as teachers participating, will not be revealed in the findings of the research
- All discussions with participants will be treated with confidentiality
- The schools can withdraw from the research study at any time
- If the school is willing to participate, it will be requested to sign consent forms that will be provided

I therefore request permission to conduct the research in the District.

Thank you for your assistance in this matter.

Sincerely yours,

Teresa Ntsae Mothudi

District Director's Name in print	District Director's signature	Date

6.5 Appendix E: Letter to the Principal

Request to principals for permission to conduct research in schools

Date_______________________________

The principal of ________________________________ school

Gauteng North District

Dear Sir/Madam

Subject: Request to conduct research in your school

My name is Teresa Ntsae Mothudi and I am student at the University of South Africa. I am presently enrolled for the MEd with a specialization in mathematics education. To complete the requirements for the degree, I need to conduct a research that is related to the area of my specialization. My research project is entitled: Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District. The purpose of the study is to explore how teachers develop learners' mathematical proficiency (conceptual understanding; adaptive reasoning; strategic competence, procedural fluency, and productive disposition) in linear equations. My supervisor is Ms Sophy Kodisang, a lecturer in the Department of Mathematics Education. Her office telephone number is 012 429 3964.

I will be an observer for two lessons in linear equations that will be presented by grade 9 mathematics teacher. A semi-structured interview will be conducted with the participant after the second lesson observation at a time and place that is convenient for both the participant and researcher. The interview will be audio recorded for a verbatim transcription. The participants will therefore be requested to sign a consent form provided. All data collected will be analysed and a report regarding the study will be written. This research study will not only benefit the institution involved but will contribute to Mathematics teaching in the Gauteng North District. Please note that if you allow one teacher and his/her grade 9 mathematics class to participate in the research study, the following ethical values will apply:

- the teacher and students' participation is voluntary
- all information will be treated with confidentiality and anonymity to ensure that no harm or bad effect will be caused to participants by the research study
- participants will be granted the right to withdraw when they so wish, they may also refrain from answering questions when they see it necessary.

Due to the Covid-19 pandemic where social distancing is emphasised as one of the regulations, the lesson observations and interviews will be conducted remotely. Teachers who give a consent to participate will be requested to capture their lesson presentation on a video camera using their cell phones and send the recordings to the researcher or use google classroom platform. The contents of the video will always be kept confidential, and no one will be invited in the google classroom besides the researcher. After analysing the lessons maximally, the video recordings will be deleted permanently. The interviews will be conducted telephonically, through Zoom or Microsoft Teams. The participant and the researcher will be the only people having access to any of the platforms. The interviews will be recorded, with the permission of the participant. The recordings will be deleted permanently after being maximally used.

I will avail to you the summary of the study results at the time of completion if you would wish to have the summary.

Thank you in advance for your support.

Sincerely yours,

Teresa Ntsae Mothudi

Please complete the consent form below

6.6 Appendix F: Consent Form for Principal

3

I.....................................the principal of.. understand the context of the research study and I grant permission that the research study (title: **Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District** may be conducted at the school. I am aware that the teacher and learners' participation is voluntary; all information will be treated with confidentiality and anonymity to ensure that no harm or bad effect will be caused to participants by the research study; all observation videos and interview recordings will be destroyed at the end of the study; participants will be granted the right to withdraw when they so wish, they may also refrain from answering questions when they see it necessary.

________________________ ____________________

Principal Date

School Stamp

6.7 Appendix G: Letter to the School Governing Body (SGB)

Request to the school governing body for permission to conduct research in schools

Date____________________________

The SGB of ___________________________ school

Gauteng North District

Dear Sir/Madam

Subject: Request to conduct research in your school

My name is Teresa Ntsae Mothudi and I am student at the University of South Africa. I am presently enrolled for the MEd with a specialization in mathematics education. To complete the requirements for the degree, I need to conduct a research that is related to the area of my specialization. My research project is entitled: Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District. The purpose of the study is to explore how teachers develop learners' mathematical proficiency (conceptual understanding; adaptive reasoning; strategic competence, procedural fluency, and productive disposition) in linear equations. My supervisor is Ms Sophy Kodisang, a lecturer in the Department of Mathematics Education. Her office telephone number is 012 429 3964.

I will be an observer for two lessons in linear equations that will be presented by grade 9 mathematics teacher. A semi-structured interview will be conducted with the participant after the second lesson observation at a time and place that is convenient for both the participant and researcher. The interview will be audio recorded for a verbatim transcription. The participants will therefore be requested to sign a consent form provided. All data collected will be analysed and a report regarding the study will be written. This research study will not only benefit the institution involved but will contribute to Mathematics teaching in the Gauteng North District. Please note that if you allow one teacher and his/her grade 9 mathematics class to participate in the research study, the following ethical values will apply:

- the teacher and students' participation is voluntary
- all information will be treated with confidentiality and anonymity to ensure that no harm or bad effect will be caused to participants by the research study
- participants will be granted the right to withdraw when they so wish, they may also refrain from answering questions when they see it necessary.

Due to the Covid-19 pandemic where social distancing is emphasised as one of the regulations, the lesson observations and interviews will be conducted remotely. Teachers who give a consent to participate will be requested to capture their lesson presentation on a video camera using their cell phones and send the recordings to the researcher or use google classroom platform. The contents of the video will always be kept confidential, and no one will be invited in the google classroom besides the researcher. After analysing the lessons maximally, the video recordings will be deleted permanently. The interviews will be conducted telephonically, through Zoom or Microsoft Teams. The participant and the researcher will be the only people having access to any of the platforms. The interviews will be recorded, with the permission of the participant. The recordings will be deleted permanently after being maximally used.

6.8 Appendix H: Consent Form for SGB

I……………………………… the SGB member of ………………………………… understand the context of the research study and I grant permission that the research study (title: **Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District** may be conducted at the school. I am aware that the teacher and learners' participation is voluntary; all information will be treated with confidentiality and anonymity to ensure that no harm or bad effect will be caused to participants by the research study; all observation videos and interview recordings will be destroyed at the end of the study; participants will be granted the right to withdraw when they so wish, they may also refrain from answering questions when they see it necessary.

________________________ ____________________

SGB Member Date

School Stamp

6.9 Appendix I: Letter to the Teacher

Request to teachers for permission to conduct research in schools

Date

The grade 9 Mathematics Teacher

______________________________ school

Gauteng North District

Dear Sir/Madam

My name is Teresa Ntsae Mothudi and I am student at the University of South Africa. I am presently enrolled for the MEd with a specialization in mathematics education. To complete the requirements for the degree, I need to conduct a research that is related to the area of my specialization. My research project is entitled: Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District. The purpose of the study is to explore how teachers develop learners' mathematical proficiency (conceptual understanding; adaptive reasoning; strategic competence, procedural fluency, and productive disposition) in linear equations.

My supervisor is Ms Sophy Kodisang, a lecturer in the Department of Mathematics Education.

I would like to be a non-participative observer for two lessons in linear equations that will be presented by you. I hope to then conduct a semi-structured interview with you after the second lesson observation at a time and place that is convenient for both of us. The interview will be audio recorded for a verbatim transcription. The participants will therefore be requested to sign a consent form provided. All data collected will be analysed and a report regarding the study will be written.

I believe that the work I am doing could be relevant to mathematics education in Gauteng North District. The information collected will be treated with confidence and I hereby give undertaking that:

- Your participation is voluntary.
- You may discontinue participation at any time if you so wish.
- You may also refrain from answering some interview questions when there is a need to do so.

Due to the Covid-19 pandemic where social distancing is emphasised as one of the regulations, the lesson observations and interviews will be conducted remotely. You are requested to capture your lesson presentation on a video camera using your cell phone and send the recordings to the researcher or be observed using google classroom platform. The contents of the video will always be kept confidential, and no one will be invited in the google classroom besides the researcher. After analysing the lessons maximally, the video recordings will be deleted permanently. The interviews will be conducted telephonically, through Zoom or Microsoft Teams depending on your preference. You and the researcher will be the only people having access to any of the platforms. The interviews will be recorded, with your permission. The recordings will be deleted permanently after being maximally used.

I·am·looking·forward·to·your·participation·in·the·research·study.·I·request·you·to·sign·the·consent·form·provided·if·you·accept·my·request·to·participate.·Thank·you·in·advance·for·your·support·

Sincerely·yours,

Teresa·Ntsae·Mothudi

Please·complete·the·consent·form·below·

UNISA college of education

6.10 Appendix J: Consent Form for the Teacher

I, ... (Name and Surname), understand the context of the research study titled: **Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District.** I am aware that I should video record my lesson presentation and send to the researcher, furthermore, there will be audio recorded interviews telephonically, through Zoom or Microsoft Teams. I am aware that anonymity and confidentiality will be adhered to in this study. I am informed that I may withdraw my consent to participate at any time without penalty by advising the researcher. I agree on my free will to participate in the research study.

________________________ ____________________

Teacher's Signature Date

6.11 Appendix K: Letter to the Parent

Request to parents for permission to conduct research in your child's classroom

Date________________________

The Parent of grade 9 Mathematics Learner

_____________________________ school

Gauteng North District

Dear Parent/ Guardian

Re: Request for your child to participate in a research study

My name is Teresa Ntsae Mothudi, I am a Mathematics education student at the University of South Africa. I request you to allow your child to participate in a research study titled: **Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District.** Your child's role in the research study will be to participate in the lesson.
If you allow your child to participate in the research study, take note that the following ethical values will apply:

- your child's participation is voluntary.
- he/she may discontinue participation at any time if a need arises.
- he/she may also refrain from answering some interview questions when there is a need to do so.

Due to the Covid-19 pandemic where social distancing is emphasised as one of the regulations, the lesson observations and interviews will be conducted remotely. Teachers who give a consent to participate will be requested to capture their lesson presentation on a video camera using their cell phones and send the recordings to the researcher or they will be observed through google classroom platform. The contents of the video will always be kept confidential, and no one will be invited in the google classroom besides the researcher.

I am looking forward to your child's participation in the research study. I request you to sign the consent form provided if you give permission that your child may participate. Thank you in advance for your support

Yours sincerely

Teresa Ntsae Mothudi

Please complete the consent form below and return to school

6.12 Appendix L: Consent Form for the Parent

I…………………………………………………………………….,parent/guardian of …………………………………………………………… understand the context of the research study titled: **Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District.** I am aware that video and audio recordings of my child's participation will be conducted. I am aware that anonymity and confidentiality will be adhered to in this study. I am informed that I may withdraw my consent for my child to participate at any time without penalty by advising the researcher. I agree on my free will that my child will participate in the research study.

________________________ ____________________

Parent/ Guardian's Signature Date

6.13 Appendix M: Letter to the Grade 9 Learner

Dear grade 9 Learner

REQUEST FOR YOUR CONSENT TO PARTICIPATE IN A RESEARCH PROJECT

I am a mathematics subject advisor at Gauteng North District. I am studying for a Master's degree at the University of South Africa in the Department of Mathematics Education under the supervision of Ms SM Kodisang

I hereby ask you to take part in my research project. If you agree to take part, you will only sit in the class and continue with the lesson with your teacher. Due to the Covid-19 pandemic where social distancing is emphasised as one of the regulations, the lesson observations will be conducted remotely. Your teacher who gives a consent to participate will capture the lesson presentation on a video and send the recordings to the researcher or the teacher will use google classroom platform. The contents of the video will always be kept confidential, and no one will be invited in the google classroom besides the researcher. After analysing the lessons maximally, the video recordings will be deleted permanently.

My research topic is: **Grade 9 mathematics teachers' development of learner mathematical proficiency in linear equations, in Gauteng North District**

I hope that at the end of the research there may be answers to how mathematics teachers develop your mathematical proficiency in linear equations. If you take part in this research, I will only use the answers that you give to the teacher and the recorded videos. I promise not to disturb your teacher while teaching if they are teaching through google classroom.

If you agree to take part in this research, I promise that the videos will not be shown to anyone, and no one will know who took part. Your Mathematics teacher will also be asked to take part in the study. You are not forced to take part in the study. You can stop taking part at any time you want, and you will not be punished for stopping to take part.

Yours sincerely

Teresa Ntsae Mothudi

Please complete the consent form below and return to school

6.14 Appendix N: Consent Form for the Learner

13

I____ hereby agree to take part in this research. I am aware that the results of this research will be used to promote teaching and learning and may be used to help future learners. I am also aware that my teacher will video record the lesson during the presentation and the records will be kept confidentially.

In addition, I understand that I can decide to stop taking part in this study at any time and that I will not be punished.

Signed_________________________ Date ___________________

6.15 Appendix O: GDE Approval Letter

GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

8/4/4/1/2

GDE RESEARCH APPROVAL LETTER

Date:	24 August 2021
Validity of Research Approval:	08 February 2021– 30 September 2021 2021/203
Name of Researcher:	Mothudi TN
Address of Researcher:	1330 Pablo Street
	KarenPark
	Akasia
Telephone Number:	0762889591
Email address:	Teresa1979.phokwani@gmail.com/ Ntsae.phokwani@gauteng.gov.za
Research Topic:	Grade 9 mathematics teachers' development of learner mathematical proficiency in Linear equations: A case of selected schools in Gauteng North district.
Type of qualification	Masters in Mathematics Education
Number and type of schools:	4 Secondary Schools
District/s/HO	Gauteng North

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

25/08/2021

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

1

Making education a societal priority

Office of the Director: Education Research and Knowledge Management
7th Floor, 17 Simmonds Street, Johannesburg, 2001
Tel: (011) 355 0488
Email: Faith.Tshabalala@gauteng.gov.za
Website: www.education.gpg.gov.za

2. *The District/Head Office Senior Manager/s must be approached separately, and in writing, for permission to involve District/Head Office Officials in the project.*
3. ***Because of COVID 19 pandemic researchers can ONLY collect data online, telephonically or may make arrangements for Zoom with the school Principal. Requests for such arrangements should be submitted to the GDE Education Research and Knowledge Management directorate. The approval letter will then indicate the type of arrangements that have been made with the school.***
4. ***The Researchers are advised to make arrangements with the schools via Fax, email or telephonically with the Principal.***
5. *A copy of this letter must be forwarded to the school principal and the chairperson of the School Governing Body (SGB) that would indicate that the researcher/s have been granted permission from the Gauteng Department of Education to conduct the research study.*
6. *A letter / document that outline the purpose of the research and the anticipated outcomes of such research must be made available to the principals, SGBs and District/Head Office Senior Managers of the schools and districts/offices concerned, respectively.*
7. *The Researcher will make every effort obtain the goodwill and co-operation of all the GDE officials, principals, and chairpersons of the SGBs, teachers and learners involved. Persons who offer their co-operation will not receive additional remuneration from the Department while those that opt not to participate will not be penalised in any way.*
8. *Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal (if at a school) and/or Director (if at a district/head office) must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.*
9. *Research may only commence from the second week of February and must be concluded before the beginning of the last quarter of the academic year. If incomplete, an amended Research Approval letter may be requested to conduct research in the following year.*
10. *Items 6 and 7 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.*
11. *It is the researcher's responsibility to obtain written parental consent of all learners that are expected to participate in the study.*
12. *The researcher is responsible for supplying and utilising his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institutions and/or the offices visited for supplying such resources.*
13. *The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research report without the written consent of each of these individuals and/or organisations.*
14. *On completion of the study the researcher/s must supply the Director: Knowledge Management & Research with one Hard Cover bound and an electronic copy of the research.*
15. *The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned.*
16. *Should the researcher have been involved with research at a school and/or a district/head office level, the Director concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.*

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

....

Mr Gumani Mukatuni
Acting CES: Education Research and Knowledge Management

DATE: 25/08/2021

Office of the Director: Education Research and Knowledge Management
7th Floor, 17 Simmonds Street, Johannesburg, 2001
Tel: (011) 355 0488
Email: Faith.Tshabalala@gauteng.gov.za
Website: www.education.gpg.gov.za

6.16 Appendix P: Language Letter

Member South African Translators' Institute
www.language-services.online

P. O. Box 3172
Lyttelton South
0176

30 November 2022

TO WHOM IT MAY CONCERN

The dissertation "Grade 9 mathematics teachers' development of learner mathematical proficiency in Linear equations: A case of selected schools in Gauteng North District."

by Teresa Ntsae Mothudi has been proofread and edited for language by me.

I verify that it is ready for publication or public viewing regarding language and style and has been formatted per the prescribed style.

Please note that no view is expressed regarding the document's subject-specific technical content or changes after this letter's date.

Kind regards

Anna M de Wet

SATI MEMBER 1003422

BA (Afrikaans, English, Classical Languages) (Cum Laude), University of Pretoria.
BA Hons (Latin) (Cum Laude), University of Pretoria.
BA Hons (Psychology), University of Pretoria.

6.17 Appendix Q: Ethical Clearance

UNISA university of south africa

UNISA COLLEGE OF EDUCATION ETHICS REVIEW COMMITTEE

Date: 2021/06/09

Ref: **2021/06/09/36460125/39/AM**
Name: Ms TN Mothudi
Student No.:36460125

Dear Ms TN Mothudi

Decision: Ethics Approval from 2021/06/09 to 2024/06/09

Researcher(s): Name: Ms TN Mothudi
E-mail address: 36460125@mylife.unisa.ac.za
Telephone: 0762889591

Supervisor(s): Name: Mrs. SM Kodisang
E-mail address: kodissm@unisa.ac.za
Telephone: 012 429 3964

Name: Prof. FM Machaba
E-mail address: emachamf@unisa.ac.za
Telephone: 012 4298582

Title of research:

Grade 9 mathematics teachers' development of learner mathematical proficiency in Linear equations: A case of selected schools in Gauteng North district.

Qualification: MEd Mathematics Education

Thank you for the application for research ethics clearance by the UNISA College of Education Ethics Review Committee for the above mentioned research. Ethics approval is granted for the period 2021/06/09 to 2024/06/09.

*The **medium risk** application was reviewed by the Ethics Review Committee on 2021/06/09 in compliance with the UNISA Policy on Research Ethics and the Standard Operating Procedure on Research Ethics Risk Assessment.*

The proposed research may now commence with the provisions that:

1. The researcher will ensure that the research project adheres to the relevant guidelines set out in the Unisa Covid-19 position statement on research ethics attached.

University of South Africa
Preller Street, Muckleneuk Ridge, City of Tshwane
PO Box 392 UNISA 0003 South Africa
Telephone: +27 12 429 3111 Facsimile: +27 12 429 4150
www.unisa.ac.za

2. The researcher(s) will ensure that the research project adheres to the values and principles expressed in the UNISA Policy on Research Ethics.
3. Any adverse circumstance arising in the undertaking of the research project that is relevant to the ethicality of the study should be communicated in writing to the UNISA College of Education Ethics Review Committee.
4. The researcher(s) will conduct the study according to the methods and procedures set out in the approved application.
5. Any changes that can affect the study-related risks for the research participants, particularly in terms of assurances made with regards to the protection of participants' privacy and the confidentiality of the data, should be reported to the Committee in writing.
6. The researcher will ensure that the research project adheres to any applicable national legislation, professional codes of conduct, institutional guidelines and scientific standards relevant to the specific field of study. Adherence to the following South African legislation is important, if applicable: Protection of Personal Information Act, no 4 of 2013; Children's act no 38 of 2005 and the National Health Act, no 61 of 2003.
7. Only de-identified research data may be used for secondary research purposes in future on condition that the research objectives are similar to those of the original research. Secondary use of identifiable human research data requires additional ethics clearance.
8. No field work activities may continue after the expiry date **2024/06/09.** Submission of a completed research ethics progress report will constitute an application for renewal of Ethics Research Committee approval.

Note:

The reference number **2021/06/09/36460125/39/AM** *should be clearly indicated on all forms of communication with the intended research participants, as well as with the Committee.*

Kind regards,

Prof AT Motlhabane
CHAIRPERSON: CEDU RERC
motlhat@unisa.ac.za

Prof PM Sebate
EXECUTIVE DEAN
Sebatpm@unisa.ac.za

Approved - decision template – updated 16 Feb 2017

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6.18 Appendix R: Turnitin Summary

turnitin

Digital Receipt

This receipt acknowledges that Turnitin received your paper. Below you will find the receipt information regarding your submission.

The first page of your submissions is displayed below.

Submission author:	Teresa Ntsae MOTHUDI
Assignment title:	Complete dissertation/thesis for examination
Submission title:	Grade 9 mathematics teachers' development of learner mat...
File name:	Dessertation_for_Teresa_Mothudi_17_11_2022.docx
File size:	699.03K
Page count:	150
Word count:	43,675
Character count:	238,888
Submission date:	18-Nov-2022 10:14AM (UTC+0200)
Submission ID:	1957629491

Grade 9 mathematics teachers' development of learner mathematical proficiency in Linear equations: A case of selected schools in Gauteng North District.

by

MOTHUDI

TERESA NTSAE

STUDENT NUMBER

EMAIL-ADDRESS

A thesis submitted in fulfilment of the requirements for the degree of Master of Education in the Department of Mathematics Education, Faculty of Education at the University of South Africa.

SUPERVISOR

CO-SUPERVISOR

PROFESSOR MF MACHABA