REMOTENESS IN THE CATEGORY OF LOCALES

by

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Declaration

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I declare that the above thesis is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

I further declare that I submitted the thesis to the appropriate originality checking software and that it falls within the accepted requirements for originality.

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Abstract

This thesis is concerned with the study of remote sublocales which are a pointfree version of remote subsets which we define using van Mill's definition of a remote collection. Unlike in van Mill's case though, the remote sublocales and remote subsets in this thesis do not necessarily need to be closed and no separation axioms are imposed on locales and spaces. We characterize both of these concepts and further show that in a T_1-space, the collection of isolated points is the largest remote subset of the space. Using the motivation that remote points were initially introduced with respect to a dense subspace of the Stone-Čech compactification, we introduce and study properties of some versions of remote sublocales called remote (resp. *remote) from a dense sublocale.

We also examine localic maps that preserve and reflect remote sublocales and their versions. We prove that the localic maps whose image functions send remote sublocales to remote sublocales are precisely those with weakly open left adjoints. We also use the result about the reflection of remote sublocales to prove that the Booleanization of a locale is the largest remote sublocale of the locale, a result with no pointset topological counterpart. For the preservation and reflection of sublocales that are remote from dense sublocales, we use the Stone extension, realcompact reflector and the Lindelöf reflector as particular cases.

Veksler defined a maximal nowhere dense subset of a Tychonoff space as a closed nowhere dense subset which is not a nowhere dense subset of any closed nowhere dense subset of the space; it is called homogeneous maximal nowhere dense in case all of its regular-closed subsets are maximal nowhere dense in the space. We introduce pointfree versions of (homogeneous) maximal nowhere dense subsets and examine a relationship between the introduced sublocales and remote sublocales where we show, among other results, that every closed nowhere dense sublocale which is *remote from its supplement is maximal nowhere dense. Regarding preservation and reflection of (homogeneous) maximal nowhere dense sublocales, we show that every open localic map that sends dense elements to dense elements preserves and reflects maximal nowhere dense sublocales, and if such a localic map is further injective, then it sends homogeneous maximal nowhere dense sublocales back and forth.

In the category of bilocales, we provide a comprehensive study of (i, j)-nowhere dense sublocales and subsequently introduce (i, j)-remote sublocales and prove that the (i, j)-remote sublocales of a bilocale whose i-part coincides with the total part of the bilocale are precisely the sublocales that are (i, j)-remote from dense subbilocales. For a bilocale (L, L_1, L_2), we introduce and study the sublocale $\text{Rem}_B L$ which is the collection of all elements of L inducing the closed (i, j)-remote sublocales of L.

Keywords: sublocale, localic map, compactification, bilocale, subbilocale, remote point, remote sublocale, nowhere dense sublocale, maximal nowhere dense sublocale, subbilocale, dense subbilocale, (i, j)-nowhere dense sublocale, (i, j)-remote sublocale

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Publications

Articles extracted from this thesis:

1. M.S. Nxumalo, *On sublocales that miss every nowhere dense sublocale*, Quaest. Math., (2023)(Under Review)
2. M.S. Nxumalo, *On maximal nowhere dense sublocales*, Appl. Gen. Topology, (2023)(Under Review)

Basic Notations

$\beta\mathbb{R}$: The Stone-Čech compactification of the set of real numbers $\mathbb{R}$.

$\mathfrak{O}X$: Locale of opens of a space X.

a^*: The pseudocomplement of a.

$\mathfrak{B}L$: Booleanization of a locale L.

$\widetilde{S}$: Sublocale induced by a subset S.

$\mathrm{Nd}(L)$: The largest nowhere dense sublocale of a locale L.

$\mathcal{S}(L)$: Collection of all sublocales of a locale L.

$\mathcal{S}_{\mathrm{rem}}(L)$: Collection of all remote sublocales of a locale L.

$\mathfrak{ND}(L)$: Collection of all nowhere dense sublocales of a locale L.

$\mathcal{S}_{\mathrm{rem}}(L \ltimes S)$: Collection of sublocales of L which are remote from $S \in \mathcal{S}(L)$.

${}^*\mathcal{S}_{\mathrm{rem}}(L \ltimes S)$: Collection of sublocales of L which are *remote from $S \in \mathcal{S}(L)$.

$\mathrm{Rem}(L)$: All elements of a locale L inducing closed remote sublocales.

$\mathrm{Rem}(L \ltimes S)$: Collection of all elements of a locale L inducing closed sublocales which are remote from $S \in \mathcal{S}(L)$.

${}^*\mathrm{Rem}(L \ltimes S)$: The set of elements of a locale L inducing closed sublocales which are *remote from $S \in \mathcal{S}(L)$.

$\mathrm{Rs}(L \ltimes S)$: The largest sublocale of a locale L remote from $S \in \mathcal{S}(L)$.

${}^*\mathrm{Rs}(L \ltimes S)$: The largest sublocale of a locale L *remote from $S \in \mathcal{S}(L)$.

(L, L_1, L_2): Bilocale.

$a^\bullet$: The bilocale pseudocomplement of a.

$\mathrm{Rem}_B L$: Collection of all elements of L inducing the closed (i, j)-remote sublocales of (L, L_1, L_2).

$\mathrm{Obj}(\mathcal{A})$: Collection of all objects of a category $\mathcal{A}$.

Top: Category of topological spaces whose morphisms are continuous maps.

Loc: Category of locales whose morphisms are localic maps.

CRegLoc: Category of completely regular locales whose morphisms are localic maps.

$\mathbf{CFLoc}_R$: Category of locales which are also coframes whose morphisms are Rem-maps.

$\mathbf{BooLoc}_R$: Full subcategory of $\mathbf{CFLoc}_R$ whose objects are Boolean locales.

BiLoc: Category of bilocales whose morphisms are bilocalic maps.

RemBiLoc$_R$: Category of bilocales (L, L_1, L_2) inducing $\text{Rem}_B L$.

TBiLoc: Full subcategory of **BiLoc** whose objects are bilocales (L, L_1, L_2) satisfying the condition that each L_2-dense member of L_1 is complemented in L.

BiCFLoc: Full subcategory of **BiLoc** where objects are bilocales whose total parts are coframes.

RemBiLoc: Full subcategory of **BiLoc** whose objects are bilocales (L, L_1, L_2) giving rise to the sublocale $\text{Rem}_B L$.

BiCFLoc$_R$: Subcategory of **BiCFLoc** whose morphisms are **Rem**$_B$-maps.

RemBiLoc$_R$: Subcategory of **RemBiLoc** whose morphisms are **Rem**$_B$-maps.

RemBiLoc$_{RB}$: Full subcategory of **RemBiLoc**$_R$ whose objects are bilocales (L, L_1, L_2) with 1 the only L_2-dense element of L_1.

Contents

Chapter 1

Introduction and Preliminaries

1.1 A brief history on remote points and remote collections

The notion of a remote point was first introduced by Fine and Gillman [27] in 1962, as a point $p \in \beta\mathbb{R}$ that is not in the closure of any discrete subset of $\mathbb{R}$. Subsequent to that, other authors considered remote points of arbitrary Tychonoff spaces (see, for instance, Mandelker [45] and [46]). Van Douwen [14], in 1981, undertook a systematic study of remote points and gave several characterizations. He referred to remote points as points of $\beta X \smallsetminus X$ missing the closure of every nowhere dense subset of a Tychonoff space X, while those missing the closure of every discrete subset of X as far points. It was shown by Woods [60] in 1971 that these notions are equivalent in the context of metric space with no isolated points. Van Mill [48], in 1982, introduced the notion of a *remote collection* as a collection of closed subsets of a Tychonoff space in which some member of the collection misses every nowhere dense subset of the space.

In the pointfree setting, Dube [17], in 2009, approached the characterization of remote points provided by van Douwen in [14] with a definition heavily dependent on points of completely regular locales. Subsequent to that, Dube and Mugochi [21], in 2015, generalized the notion of remote points to arbitrary extensions of locales, and their work depended on points. The study of remoteness in pointfree topology which does not rely on points is introduced here for the first time. We aim to introduce remote sublocales using van Mill's concept of a remote collection. This will be achieved through defining a remote subset from a singleton remote collection, and transfer this notion to locales to get remote sublocales. Remote sublocales are extensions of remote points that were initially introduced by Dube in [17]. We do not claim that the study of these objects is a completely new work. This thesis contributes to the theory of sublocales. We wish to adapt most of the work done in [17] and [21] to sublocales, with no reference to points whatsoever. This is not an outrageous idea. In support of this, it suffices to mention that the concept of remainder preservation (which refers to points) was first considered for locales in [22]. A much improved view which deals only with sublocales and not points was presented in [26].

The work done in [17] only focuses on investigating certain frame homomorphisms such that the Stone extension transfers remote points back and forth. In this work, we do not only consider the Stone extension, but also the realcompact reflector and the Lindelöf reflector.

To our knowledge, little to none has been done regarding remoteness in the categories of bitopological spaces and bilocales. In this thesis, we include results about bilocalic remoteness.

1.2 Synopsis of the thesis

This thesis is organized as follows. The last six sections of this chapter provide some background results that are needed throughout the thesis.

The second chapter introduces the concept of a remote sublocale. The first section of this chapter begins with a definition of a remote subset constructed from van Mill's definition of a remote collection. The remote subsets introduced are not necessarily closed and no separation axiom is assumed on the space. This is then followed by a definition of a remote sublocale which turns out to be conservative in locales, in the sense that a subset is remote if and only if the sublocale it induces is remote in the locale of opens. We show that in T_1-spaces, the collection of isolated points is the largest remote subset. Since the concept of remoteness was initially introduced with respect to a dense subspace of the Stone-Čech compactification, we also introduce and study some properties of sublocales that are remote (resp. *remote) from dense sublocales. The chapter ends with two sections that respectively consider algebraic aspects of remoteness and remoteness in binary coproducts.

Chapter 3 discusses preservations and reflections of the concepts of remoteness introduced in Chapter 2. The first section focuses on examining localic maps that send remote sublocales back and forth. It turns out that the localic maps whose image functions send remote sublocales to remote sublocales are precisely those with weakly open left adjoints. We use a result about the reflection of remote sublocales to prove that the Booleanization of a locale is the largest remote sublocale of the locale, a result which does not have a counterpart in spaces. The last section focuses on localic maps preserving remote sublocales from a dense sublocale. Particular cases in this section are the Stone extension, realcompact reflector and the Lindelöf reflector. We prove that under certain conditions, the preservation of remote sublocales is equivalent to the preservation of sublocales that are remote from a dense sublocale.

Since nowhere density plays a crucial role in the study of remote sublocales, in Chapter 4, we introduce some variants of nowhere dense sublocales, particularly maximal nowhere dense and homogeneous maximal nowhere dense sublocales. These were initially introduced in spaces, so we show that their localic definitions are conservative in locales and prove some properties of these sublocales. We further examine a relationship between (homogeneous) maximal nowhere dense sublocales and remote sublocales via the introduction of (almost) inaccessible sublocales. We show that every closed nowhere dense sublocale which is *remote from its supplement is maximal nowhere dense. The chapter ends with a study of localic maps that preserves and reflects (homogeneous) maximal nowhere dense sublocales. It is apparent that every open localic map that sends dense elements to dense elements preserves and reflects maximal nowhere dense sublocales. If such a localic map is further injective, then it sends homogeneous maximal nowhere dense sublocales back and forth.

In the last chapter, Chapter 5, we transfer the notion of remoteness to bilocales. The first section provides a comprehensive study of (i, j)-nowhere dense sublocales which are localic counterparts of (τ_i, τ_j)-nowhere dense subsets. Unlike in locales, certain conditions must be imposed on a bilocale for (i, j)-nowhere dense sublocales to be characterized by the smallest dense sublocale of the total part of the bilocale. The second section introduces (i, j)-remote sublocales. We show that for a bilocale (L, L_1, L_2), if $L_i = L$, then the (i, j)-remote sublocales of (L, L_1, L_2) are precisely the sublocales that are (i, j)-remote from dense subbilocales of (L, L_1, L_2). Unlike in the case of locales, the preservation of (i, j)-remoteness fails to be characterized using the preservation of (i, j)-nowhere dense sublocales. The last section of this chapter studies the sublocale $\mathrm{Rem}_B L$ of a bilocale (L, L_1, L_2) which is the collection of all elements of L inducing the closed (i, j)-remote sublocales of (L, L_1, L_2). We prove that every bilocale (L, L_1, L_2) in which either all L_2-dense members of L_1 are complemented in L or the total part L of (L, L_1, L_2) is also a coframe, induces the sublocale $\mathrm{Rem}_B L$. We consider the category $\mathbf{RemBiLoc}_R$ whose objects are bilocales (L, L_1, L_2) inducing the sublocale $\mathrm{Rem}_B L$, and morphisms are Rem_B-maps, and show that there is a natural transformation from the functor $\mathrm{Rem}_B : \mathbf{RemBiLoc}_R \to \mathbf{Loc}$ to the forgetful functor $G : \mathbf{RemBiLoc}_R \to \mathbf{Loc}$.

1.3 Locales and localic maps

We direct the reader to [50] and [26] for more details on meanings presented in this section.

A *locale* (or frame) is a complete lattice L satisfying the following distributive law:

$$x \wedge \bigvee A = \bigvee \{x \wedge a : a \in A\}$$

for all $A \subseteq L$ and $x \in L$. The *top* element and the *bottom* element of a locale L will be denoted by 1_L and 0_L, respectively, with subscripts dropped when L is clear from the context. A subset F of a locale L closed under finite meets and arbitrary joins in L is called a *subframe* of L. By a *point* of L we mean an element $p < 1$ such that $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$, for all $a, b \in L$.

The *pseudocomplement* of an element $a \in L$ is denoted by

$$a^* := \bigvee \{x \in L : a \wedge x = 0\}.$$

An element $a \in L$ is said to be

1. *dense* if $a^* = 0$.

2. *compact* if $a \leq \bigvee A$ for each $A \subseteq L$ implies that $a \leq \bigvee B$ for some finite $B \subseteq A$.

3. *complemented* if $a \vee a^* = 1$.

4. *rather below* an element $b \in L$, denoted $a \prec b$, if $a^* \vee b = 1$.

5. *completely below* $b \in L$, denoted by $a \prec\!\!\prec b$, if there is a sequence (x_q) of elements of L indexed by $\mathbb{Q} \cap [0,1]$ such that $a_0 = a$, $a_1 = b$ and $x_q \prec x_r$ whenever $q < r$.

A locale L is

1. *compact* if 1 is compact.

2. *Boolean* in case every element is complemented.

3. *regular* provided that for every $a \in L$,

$$a = \bigvee \{x \in L : x \prec a\}.$$

4. *completely regular* if for all $a \in L$,

$$a = \bigvee \{x \in L : x \prec\!\!\prec a\}.$$

A *frame homomorphism* is a map between locales which preserves binary meets (including the top element) and arbitrary joins (including the bottom element). It is called *dense* in case it maps only the bottom element to the bottom element, a *quotient map* provided it is surjective, and an *extension* if it is a dense quotient map. Associated with a frame homomorphism $h : M \to L$ is its infima preserving right adjoint $h_* : L \to M$ given by

$$h_*(x) = \bigvee \{a \in M : h(a) \leq x\}.$$

It is usually called a *localic map*. Left adjoints of localic maps, say g, are represented by g^*. In this thesis, h is assumed to be a frame homomorphism whose right adjoint will always be represented by f. Similarly, f will be a localic map whose left adjoint will always be h. By a *closed* frame homomorphism, we mean a frame homomorphism $h : M \to L$ such that

$$f(x \vee h(y)) = f(x) \vee y$$

for all $x \in L$ and $y \in M$. A localic map $f : L \to M$ is *dense* if $f(0) = 0$.

A *sublocale* of a locale L is a subset $S \subseteq L$ such that:

1. S is closed under all meets, and

2. $x \to s \in S$, for all $x \in L$ and $s \in S$,

where $\to$ is a Heyting operation on L satisfying:

$$a \leq b \to c \quad \text{if and only if} \quad a \wedge b \leq c$$

for every $a, b, c \in L$. The smallest sublocale of L is the sublocale $\mathsf{O} = \{1\}$. A sublocale S of L is *void* if $S = \mathsf{O}$ and *non-void* if $S \neq \mathsf{O}$. Sublocales are locales in their own rights. The collection $\mathcal{S}(L)$ of all sublocales of L is a lattice where $0_{\mathcal{S}(L)} = \mathsf{O}$, $1_{\mathcal{S}(L)} = L$, meet operation is given by intersection and join operation is given by

$$\bigvee_{i \in I} S_i = \Big\{ \bigwedge M : M \subseteq \bigcup_{i \in I} S_i \Big\}.$$

The collection $\mathcal{S}(L)$ is a *coframe* in the sense that, for every $S \in \mathcal{S}(L)$ and every collection $\{A_i\} \subseteq \mathcal{S}(L)$,

$$S \vee \bigcap A_i = \bigcap (S \vee A_i)\,.$$

We shall use the prefix S- for localic properties defined on a sublocale S of L. A sublocale $S \subseteq L$ *misses* a sublocale T of L if $S \cap T = \mathsf{O}$. It is *complemented* if it has a complement in $\mathcal{S}(L)$, and *linear* if

$$S \cap \bigvee \{C_i : i \in I\} = \bigvee \{S \cap C_i : i \in I\}$$

for each family $\{C_i : i \in I\} \subseteq \mathcal{S}(L)$.

Every sublocale S of L has a *supplement*, denoted by $S^{\#}$ or $L \smallsetminus S$, where

$$L \smallsetminus S = \bigvee \{A \in \mathcal{S}(L) : A \cap S = \mathsf{O}\}.$$

The *closed* and the *open* sublocales induced by $a \in L$ are the sublocales

$$\mathfrak{c}(a) = \{x \in L : a \leq x\} \quad \text{and} \quad \mathfrak{o}(a) = \{x \in L : a \to x = x\} = \{a \to x : x \in L\},$$

respectively, and are complements of each other. We shall write A sublocale is *clopen* if it is both closed and open. Just like in spaces and subspaces, open sublocales in a sublocale S are the $\mathfrak{o}_S(a) = S \cap \mathfrak{o}(a)$, and similarly we have the closed sublocale of S, $\mathfrak{c}_S(a) = S \cap \mathfrak{c}(a)$. The *closure* of $S \in \mathcal{S}(L)$ is denoted by

$$\mathrm{cl}(S) = \overline{S} = \mathfrak{c}\big(\bigwedge S\big) = \bigcap \{\mathfrak{c}(x) : S \subseteq \mathfrak{c}(x)\}$$

and its *interior* is denoted by

$$\operatorname{int}(S) = \mathfrak{o}\big(\bigwedge(L \smallsetminus S)\big) = \bigvee\{\mathfrak{o}(x) : \mathfrak{o}(x) \subseteq S\}.$$

By a *regular-closed* sublocale we mean a sublocale which is the closure of some open sublocale.

Here are some properties of open and closed sublocales.

- $\mathfrak{o}(0) = \mathfrak{c}(1) = \mathsf{O}$ and $\mathfrak{c}(0) = \mathfrak{o}(1) = L$.
- $\mathfrak{c}(x) \subseteq \mathfrak{o}(a)$ if and only if $x \vee a = 1$.
- $\mathfrak{o}(x) \subseteq \mathfrak{c}(a)$ if and only if $x \wedge a = 0$.
- $\mathfrak{o}(x) \cap \mathfrak{o}(a) = \mathfrak{o}(x \wedge a)$ and $\bigvee_{i\in I} \mathfrak{o}(x_i) = \mathfrak{o}\big(\bigvee_{i\in I} x_i\big)$.
- $\mathfrak{c}(x) \vee \mathfrak{c}(a) = \mathfrak{c}(x \wedge a)$ and $\bigcap_{i\in I} \mathfrak{c}(x_i) = \mathfrak{c}\big(\bigvee_{i\in I} x_i\big)$.
- $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and $(\mathfrak{c}(a))^\circ = \mathfrak{o}(a^*)$.

A sublocale $S \subseteq L$ is said to be *dense* if $\overline{S} = L$. This is equivalent to requiring S to contain 0. The subset

$$\mathfrak{B}(L) = \{x \to 0 : x \in L\} = \{x \in L : x = x^{**}\}$$

of L is the least dense sublocale of L, usually called the *Booleanization* of L. A sublocale is *nowhere dense* if it misses the smallest dense sublocale.

For a localic map $f : L \to M$, the *localic image function* $f[-] : \mathcal{S}(L) \to \mathcal{S}(M)$, given by

$$f[S] = \{f(x) : x \in S\}$$

is the left adjoint of the *localic preimage function* $f_{-1}[-] : \mathcal{S}(M) \to \mathcal{S}(L)$ given by

$$f_{-1}[T] = \bigvee\{A \in \mathcal{S}(L) : A \subseteq f^{-1}(T)\}.$$

For a localic map $f : L \to M$, $x \in M$ and $A \in \mathcal{S}(L)$,

$$f_{-1}[\mathfrak{c}_M(x)] = \mathfrak{c}_L(h(x)); \quad f_{-1}[\mathfrak{o}_M(x)] = \mathfrak{o}_L(h(x)) \quad \text{and} \quad f[\overline{A}] \subseteq \overline{f[A]}.$$

A localic map $f : L \to M$ is dense if and only if $f[L]$ is dense in M.

An alternative representation of a sublocale of L is given by the notion of a *nucleus* which is defined as a mapping $\nu : L \to L$ such that

1. $a \leq \nu(a)$,

2. $a \leq b \Longrightarrow \nu(a) \leq \nu(b)$,

3. $\nu\nu(a) = \nu(a)$, and

4. $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$

for every $a, b \in L$. The set $\mathrm{Fix}(\nu) = \{a \in L : \nu(a) = a\}$ is a locale with meets in L. For a sublocale $S \subseteq L$ there is the quotient map $\nu_S : L \to S$ defined by

$$\nu_S(a) = \bigwedge\{s \in S : a \leq s\}.$$

Open sublocales and closed sublocales of a sublocale S of L are given in terms of nucleus as

$$\mathfrak{o}_S(\nu_S(a)) = S \cap \mathfrak{o}(a) \quad \text{and} \quad \mathfrak{c}_S(\nu_S(a)) = S \cap \mathfrak{c}(a),$$

respectively, for $a \in L$. For any $S \in \mathcal{S}(L)$ and $x \in L$, $S \subseteq \mathfrak{o}(x)$ if and only if $\nu_S(x) = 1$. The joins in a sublocale S of a locale L are given by

$$\bigvee^{S} x_i = \nu_S\left(\bigvee x_i\right).$$

A noteworthy result about dense sublocales is that pseudocomplementation in a dense sublocale is precisely that in the locale. This is so because, if A is a dense sublocale of L and $x \in A$, then writing x^{*A} and $\to_A$ for, respectively, the pseudocomplement of x in A and the Heyting operation in A, we have the equalities

$$x^{*A} = x \to_A 0_A = x \to 0_L = x^*.$$

1.4 Category theory

Our main reference for this section is [34].

By a *category* we mean a quadruple $\mathcal{A} = (\mathrm{Obj}(\mathcal{A}), \mathrm{hom}, id, \circ)$ where

1. $\mathrm{Obj}(\mathcal{A})$ is a class whose members are called *objects* of $\mathcal{A}$ or simply $\mathcal{A}$-*objects*,

2. for each pair (A, B) of $\mathcal{A}$-objects, $\hom_{\mathcal{A}}(A, B)$ is a set whose members are called *$\mathcal{A}$-morphisms from A to B*, indicated in terms of arrows by writing $f : A \to B$ with A the *domain* of f (notation: $\mathrm{dom}(f)$) and B the *codomain* of f (notation: $\mathrm{codom}(f)$),

3. for each $A \in \mathrm{Obj}(\mathcal{A})$, $id_A : A \to A$ is an $\mathcal{A}$-morphism called the *$\mathcal{A}$-identity morphism*, and

4. there is a composition law associating with each pair (f, g) of $\mathcal{A}$-morphisms satisfying $\mathrm{dom}(f) = \mathrm{codom}(f)$ an $\mathcal{A}$-morphism $g \circ f : \mathrm{dom}(f) \to \mathrm{codom}(g)$, satisfying:

   (a) $h \circ (g \circ f) = (h \circ g) \circ f$ (associativity) whenever the compositions are defined.

   (b) $\mathcal{A}$-identities act as identities with respect to composition, i.e., for $\mathcal{A}$-morphisms $f : A \to B$, we have $id_B \circ f = f$ and $f \circ id_A = f$.

   (c) the sets $\hom_{\mathcal{A}}(A, B)$ are pairwise disjoint.

A morphism f of a category $\mathcal{A}$ is said to be an *isomorphism* provided that there is another morphism g of $\mathcal{A}$ such that $f \circ g$ and $g \circ f$ are $\mathcal{A}$-identity morphisms.

We shall use **Loc** to represent the category whose objects are locales and morphisms are localic maps. We denote by **Frm** the category whose objects are frames and morphisms are frame homomorphisms.

By the *opposite category* of a category $\mathcal{A}$ we mean a category having the same class of objects and morphisms as $\mathcal{A}$, but with reversed directions of morphisms. It is often denoted by $\mathcal{A}^{op}$. **Loc** is the opposite category of **Frm**.

A *functor* is a function $F : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ that assigns to each $\mathcal{A}$-object A a $\mathcal{B}$-object FA and to each $\mathcal{A}$-morphism $f : A \to A'$ a $\mathcal{B}$-morphism $F(f) : FA \to FA'$, in such a way that F preserves composition and identity morphisms. A construction with the properties similar like those of a functor, only modified by reversing the directions of morphisms is called a *contravariant functor*. By an *endofunctor* we refer to a functor that maps objects and morphisms from one category back to the same category. We shall use $id_{\mathcal{A}}$ to indicate the *identity functor* from category $\mathcal{A}$ to itself. A functor $F : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ is said to be *faithful* if for all $A, A' \in \mathrm{Obj}(\mathcal{A})$, the restriction $F : \hom_{\mathcal{A}}(A, A') \to \hom_{\mathcal{B}}(FA, FA')$ is injective.

For functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$, we define a *natural transformation* $\omega : F \to G$ as a function that assigns to each $\mathcal{A}$-object A a $\mathcal{B}$-morphism $\omega_A : FA \longrightarrow GA$ in such a way that the following condition holds: For every $g \in \hom_{\mathcal{A}}(A, A')$, the following square commutes:

$$\begin{array}{ccc} FA & \xrightarrow{\omega_A} & GA \\ {\scriptstyle F(g)}\downarrow & & \downarrow{\scriptstyle G(g)} \\ FA' & \xrightarrow[\omega_{A'}]{} & GA' \end{array} \tag{1.4.1}$$

If all the ω_A's are isomorphisms, then ω is said to be a *natural isomorphism*. We say that two functors are *naturally isomorphic* if there exists a natural isomorphism between them.

By a *monad* on a category $\mathcal{A}$ we mean a triple (T, η, μ) where $T : \mathcal{A} \to \mathcal{A}$ is a functor, and $\eta : id_{\mathcal{A}} \to T$ and $\mu : T \circ T \to T$ are natural transformations such that the diagrams

$$\begin{array}{ccc} T \circ T \circ T & \xrightarrow{T\mu} & T \circ T \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T \circ T & \xrightarrow[\mu]{} & T \end{array} \tag{1.4.2}$$

and

$$\begin{array}{ccccc} T & \xrightarrow{T\mu} & T \circ T & \xleftarrow{\eta T} & T \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle id} & \\ & & T & & \end{array} \tag{1.4.3}$$

commute. The opposite of a monad is a *comonad*.

A category $\mathcal{B}$ is called a *subcategory* of a category $\mathcal{A}$ in case

1. $\mathrm{Obj}(\mathcal{B}) \subseteq \mathrm{Obj}(\mathcal{A})$,

2. $\hom_{\mathcal{B}}(B, B') \subseteq \hom_{\mathcal{A}}(B, B')$ for every $B, B' \in \mathrm{Obj}(\mathcal{B})$,

3. for each $\mathcal{B}$-object B, the $\mathcal{A}$-identity morphism on B is the $\mathcal{B}$-identity morphism on B, and

4. the composition law in $\mathcal{B}$ is the restriction of the composition law in $\mathcal{A}$ to $\mathcal{B}$-morphisms.

If a category $\mathcal{B}$ is a subcategory of $\mathcal{A}$ with the condition that $\hom_{\mathcal{B}}(B, B') = \hom_{\mathcal{A}}(B, B')$ for every $B, B' \in \mathrm{Obj}(\mathcal{B})$, then $\mathcal{B}$ is said to be a *full subcategory* of $\mathcal{A}$.

A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is called a *reflective subcategory* of $\mathcal{A}$ provided that for every $A \in \mathrm{Obj}(\mathcal{A})$, there is a $B \in \mathrm{Obj}(\mathcal{B})$ and a $\mathcal{B}$-morphism $r_B : A \to B$ such that for every $\mathcal{A}$-morphism $f : A \to B'$, where $B' \in \mathrm{Obj}(\mathcal{B})$, there is a unique $\mathcal{B}$-morphism $f' : B \to B'$ such that $f' \circ r_B = f$. The morphism r_B is called the $\mathcal{B}$-reflection for A. The opposite of the notion of a reflective subcategory of a category $\mathcal{A}$ is *coreflective subcategory* of category $\mathcal{A}$ and the opposite of the concept of the $\mathcal{B}$-reflection for A is called the *$\mathcal{B}$-coreflection* for A.

1.5 Compactifications

We refer to [50], [9] and [23] for the general theory of compactifications of locales.

A *compactification* of a locale L is a pair (M, h) where:

1. M is a compact and regular locale, and
2. h is a dense and onto frame homomorphism from M to L.

An *ideal* of a locale L is a subset I of L such that:

1. $0 \in I$,
2. I is closed under binary joins, and
3. $a \leq b$ implies $a \in I$, for all $a \in L$ and each $b \in I$.

A *filter* F is defined dually and is *proper* if $0 \notin F$, otherwise it is *improper*. An ideal I of a locale L is *regular* (resp. *completely regular*) if for every $a \in I$, there is $b \in I$ such that $a \prec b$ (resp. $a \prec\!\!\prec b$). The collection $\mathcal{CRJ}L$ of all completely regular ideals of a locale L, ordered by inclusion, is a compact and completely regular locale where $0_{\mathcal{CRJL}} = \{0\}$, $1_{\mathcal{CRJL}} = L$, meet is given by intersection and join is given by

$$\bigvee_{i \in I} I_i = \Big\{ \bigvee F : F \text{ is finite and } F \subseteq \bigcup_{i \in I} I_i \Big\}.$$

The *Stone-Čech compactification* of a locale L is the pair $(\beta L, \beta_L)$, where $\beta L = \mathcal{CRJ}L$ and β_L is the join map $\beta L \to L$ which is an extension. The right adjoint of β_L is given by the localic embedding

$$r_L : x \mapsto \{y \in L : y \prec\!\prec x\}.$$

For any localic map $f : L \to M$ between completely regular locales L and M, there is a localic map $\beta(f) : \beta L \to \beta M$, called the *Stone extension*, defined by

$$\beta(f) : I \mapsto \bigvee\{J \in \beta M : h(J) \subseteq I\}$$

with its left adjoint $\beta(h) : \beta M \to \beta L$ given by

$$\beta(h) : J \mapsto \{x \in L : x \leq h(y) \text{ for some } y \in J\}.$$

[7] *Cozero elements* of a locale L are precisely those elements $a \in L$ such that

$$a = \bigvee\{x_n : x_n \prec\!\prec a\}$$

for some sequence (x_n) in L, or equivalently,

$$a = \bigvee\{a_n : a_n \prec\!\prec a_{n+1}\}$$

for some sequence (a_n) in L.

The *cozero part* of L, denoted by $\operatorname{Coz} L$, is the collection of all cozero elements of L.

Let L be a completely regular locale. An ideal of $\operatorname{Coz} L$ is a *$\sigma-$ideal* if it is closed under countable joins. The locale of σ-ideals of $\operatorname{Coz} L$, denoted by λL, is the *regular Lindelöf reflection* of L and the extension $\lambda_L : \lambda L \to L$ defined by $I \mapsto \bigvee I$, is the *regular Lindelöf coreflection map* whose right adjoint is called the *regular Lindelöf reflection map*, [44]. This is a special case of a more general result concerning κ-locales (see [43]). We define the extension $k_L : \beta L \to \lambda L$ by $I \mapsto \langle I\rangle_\sigma$, where $\langle\cdot\rangle$ signifies σ-ideal generation in $\operatorname{Coz} L$.

For any $x \in L$, let

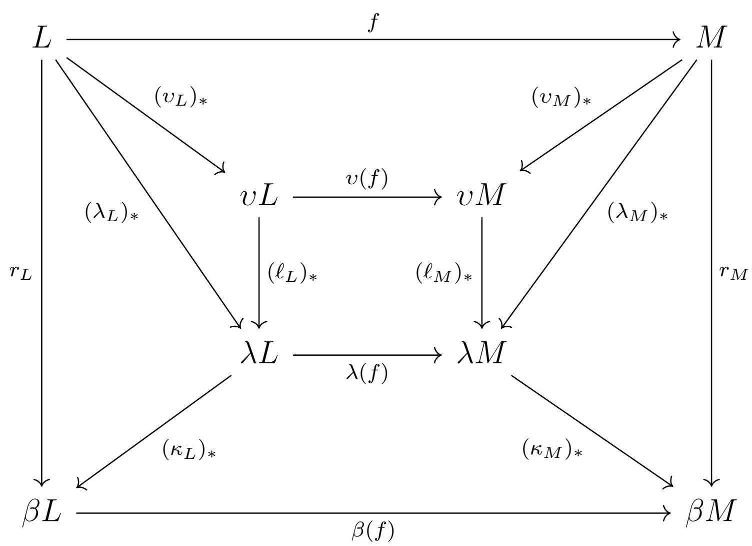
$$[x] = \{a \in \operatorname{Coz}\ L : a \leq x\}.$$

The map

$$\ell : \lambda L \to \lambda L, I \mapsto \left[\bigvee I\right] \wedge \bigwedge\{P \in Pt(\lambda L) : I \leq P\}$$

is a nucleus. The locale υL defined to be $\mathrm{Fix}(\ell)$ is the *realcompact reflection* of L with the *realcompact reflection map* given by the join map $\upsilon_L : \upsilon L \to L$ which is an extension. The extension $\lambda L \to \upsilon L$ effected by ℓ is denoted by ℓ_L. See [8] for the construction of the realcompact reflection.

When βL is regarded as the locale of regular ideals of $\mathrm{Coz}\, L$, we get the following commuting diagram in the category **CRegLoc** of completely regular locales whose morphisms are localic maps between them. It puts into perspective the collection of the maps described above.



(1.5.1)

For any completely regular locale L and $x \in L$, we have

$$\ell_L([x])(\lambda_L)_*(x) = (\upsilon_L)_*(x) = (\ell_L)_*([x]) = [x].$$

By a γ-*lift* we mean the localic morphism $\gamma(f) : \gamma L \to \gamma M$, where $\gamma \in \{\beta, \lambda, \upsilon\}$.

For $\gamma \in \{\beta, \lambda, \upsilon\}$, the assignment

$$\gamma : \mathbf{CRegLoc} \to \mathbf{CRegLoc}$$

$$f \mapsto \gamma(f)$$

is a functor.

1.6 Remote points

For detailed meanings of the notions introduced in this section, we refer to [14] and [18].

In point-set topology, a point $p \in \beta X \setminus X$, where βX is the Stone-Čech compactification of a completely regular space X, is *remote* if $p \notin \overline{N}^{\beta X}$, for every nowhere dense set $N \subseteq X$.

In point-free setting, for a completely regular locale L, a point $I \in \beta L$ is a *remote point* if for each nowhere dense quotient map $h : M \to L$, $I \vee r_M(h_*(0)) = 1_{\beta M}$, where h is *nowhere dense* if for every non-zero $x \in M$ there exists a non-zero $y \leq x$ in M such that $h(y) = 0$.

1.7 Binary coproducts of locales

See [50, 10, 13] as references for binary coproducts of locales.

For any two locales L and M, we construct the binary coproduct $L \oplus M$ as follows. Consider the set

$$\mathcal{D}(L \times M) = \{U \subseteq L \times M : {\downarrow} U = U \neq \emptyset\},$$

ordered by inclusion. A member U of $\mathcal{D}(L \times M)$ is said to be *saturated* in case

1. $\{a\} \times B \subseteq U$ implies $(a, \bigvee B) \in U$ for every $a \in L$, $B \subseteq M$, and

2. $A \times \{b\} \subseteq U$ implies $(\bigvee A, b) \in U$ for all $A \subseteq L$, $b \in M$.

For each $a \in L$, $b \in M$, the set

$$a \oplus b = \{(0, b), (a, 0)\} \cup {\downarrow}(a, b)$$

is the least saturated member of $\mathcal{D}(L \times M)$ containing ${\downarrow}(a, b)$. The collection

$$L \oplus M = \{U \in \mathcal{D}(L \times M) : U \text{ is saturated}\}$$

is a locale and the maps

$$q_L : L \to L \oplus M, a \mapsto a \oplus 1$$

and

$$q_M : M \to L \oplus M, b \mapsto 1 \oplus b$$

are frame homomorphisms which are usually called *coproduct injections*.

The top element and bottom element of $L \oplus M$ are denoted by $1_{L \oplus M}$ and $0_{L \oplus M}$, respectively. For any $a, c \in L$ and $b, d \in M$, if $0_{L \oplus M} \neq a \oplus b \leq c \oplus d$, then $a \leq c$ and $b \leq d$. For every $(a, b) \in L \times M$, $a \oplus b = 0_{L \oplus M}$ if and only if $a = 0$ or $b = 0$. Consequently, $(a \oplus b)^* = (a^* \oplus 1) \vee (1 \oplus b^*)$.

1.8 Bilocales

We recall some terminology from [5, 51].

A *bilocale* is a triple (L, L_1, L_2) where L_1, L_2 are subframes of a locale L and for all $a \in L$,

$$a = \bigvee\{a_1 \wedge a_2 : a_1 \in L_1, a_2 \in L_2 \text{ and } a_1 \wedge a_2 \leq a\}.$$

We call L the *total part* of (L, L_1, L_2), and L_1, L_2 the first and second parts, respectively. We use the notations L_i, L_j to denote the first or second parts of (L, L_1, L_2), always assuming that $i, j = 1, 2$, $i \neq j$.

For $c \in L_i$ we denote

$$c^\bullet = \bigvee\{x \in L_j : x \wedge c = 0\}.$$

A *subbilocale* of a bilocale (L, L_1, L_2) is a triple (S, S_1, S_2) where S is a sublocale of L and

$$S_i = \nu_S[L_i] \quad \text{for} \quad i = 1, 2.$$

A *biframe homomorphism* (or *biframe map*) $h : (M, M_1, M_2) \to (L, L_1, L_2)$ is a frame homomorphism $h : M \to L$ for which

$$h(M_i) \subseteq L_i \quad (i = 1, 2).$$

Chapter 2

On Remote Sublocales

This chapter aims to introduce and provide a study of a concept of remoteness in locales. The information in this chapter forms part of the research paper: M.S. Nxumalo, *On sublocales that miss every nowhere dense sublocale*, Quaest. Math., (2023)(Under Review).

2.1 Introducing remote sublocales

The objective in this section is to extend to pointfree topology the notion of a remote subset introduced by van Mill in [48]. Unlike in [48], where remoteness is defined only for Tychonoff spaces and remote subspaces are required to be closed, we broaden the scope by working in arbitrary spaces and do not require remote subspaces to be closed.

Recall that a subspace of a topological space is called *nowhere dense* if its closure has an empty interior. In the introduction, we recalled van Mill's definition of a *remote collection* which is a collection $\mathcal{F}$ of closed subsets of a Tychonoff space X where every nowhere dense subset of X misses some member of $\mathcal{F}$. In the event that $\mathcal{F}$ is a singleton, say $\mathcal{F} = \{A\}$, this then reduces to saying the closed set A is remote. Since van Mill does not explicitly mention the remoteness of arbitrary subsets of arbitrary spaces, we formulate the following definition.

Definition 2.1.1. A subset A of a space X is *remote* if $A \cap N = \emptyset$ for every nowhere dense $N \subseteq X$.

We shall extend this concept to locales by merely replacing subspaces with sublocales, and then show that the extension is conservative, in the sense that a subset A of a space X is remote if and only if the induced sublocale of $\mathfrak{O}X$ is remote.

Definition 2.1.2. We say a sublocale of a locale L is *remote* if it misses every nowhere dense sublocale of L.

Before we proceed, we give examples of remote sublocales and show that unlike in the case of nowhere denseness (recall that, in [53], Plewe shows that the closure of a nowhere dense sublocale is nowhere dense), the closure of a remote sublocale need not be remote.

We need the following lemma which will also be used elsewhere. In [53], Plewe observed that a complemented sublocale is nowhere dense if and only if its closure has a void interior. Since a sublocale of a nowhere dense sublocale is nowhere dense, it follows then that a sublocale is nowhere dense if and only if its closure is nowhere dense.

Lemma 2.1.3. *A sublocale N of a locale L is nowhere dense iff $\bigwedge N$ is dense in L. In particular, $\mathfrak{c}(a)$ is nowhere dense iff a is dense.*

Proof.

$$\begin{aligned} N \text{ is nowhere dense} &\iff \mathfrak{c}\left(\bigwedge N\right) \text{ is nowhere dense} \\ &\iff \mathfrak{c}\left(\bigwedge N\right)^{\circ} = \mathsf{O} \\ &\iff \mathfrak{o}\left(\left(\bigwedge N\right)^{*}\right) = \mathfrak{o}(0) \\ &\iff \left(\bigwedge N\right)^{*} = 0, \end{aligned}$$

which proves the lemma. □

A consequence of this lemma which we are going to use further below, is the following corollary.

Corollary 2.1.4. *A sublocale N of L is nowhere dense iff $L \smallsetminus \overline{N}$ is dense.*

Proof. By Lemma 2.1.3, N is nowhere dense if and only if $\bigwedge N$ is dense, which is true if and only if $\mathfrak{o}(\bigwedge N)$ is dense. So, from the equalities

$$L \smallsetminus \overline{N} = L \smallsetminus \mathfrak{c}\left(\bigwedge N\right) = \mathfrak{o}\left(\bigwedge N\right),$$

it follows that N is nowhere dense if and only if $L \smallsetminus \overline{N}$ is dense. □

Observation 2.1.5. From Corollary 2.1.4, we also observe that an open sublocale is dense if and only if its supplement is nowhere dense. Indeed, $\mathfrak{o}(x)$ is dense if and only if x is dense if and only if $\mathfrak{c}(x)$ is nowhere dense if and only if $L \smallsetminus \mathfrak{o}(x)$ is nowhere dense.

Another consequence of Lemma 2.1.3 is the following result.

Corollary 2.1.6. *Every closed remote sublocale is clopen.*

Proof. Consider an arbitrary closed sublocale $\mathfrak{c}(x)$ which is remote. Since $\mathfrak{c}(x \vee x^*)$ is nowhere dense, $\mathfrak{c}(x) \cap \mathfrak{c}(x \vee x^*) = \mathsf{O}$, which implies $\mathfrak{c}(x \vee x \vee x^*) = \mathfrak{c}(1)$, whence $x \vee x^* = 1$ showing that x is complemented, and hence $\mathfrak{c}(x)$ is clopen. □

Using Lemma 2.1.3, we prove the following proposition, which will enable us to present the examples alluded to above.

Proposition 2.1.7. *A locale is remote as a sublocale of itself iff it is Boolean.*

Proof. ($\Longrightarrow$): Let L be a locale which is a remote sublocale of itself. If $a \in L$ is dense and $a < 1$, then the nowhere dense sublocale $\mathfrak{c}(a)$ is non-void, which is not possible since L misses every nowhere dense sublocale of itself. Thus $a = 1$. Hence for any $x \in L$, $x \vee x^* = 1$ because $x \vee x^*$ is dense. This shows that L is Boolean.

($\Longleftarrow$): Suppose that L is Boolean. First, observe that a locale M is Boolean if and only if the void sublocale is its only nowhere dense sublocale. Indeed, if M is Boolean, then $\mathfrak{c}_M(a)$ is nowhere dense if and only if $a = 1$. On the other hand, for any $a \in M$, $a \vee a^*$ is dense, hence $\mathfrak{c}(a \vee a^*)$ is nowhere dense, making $\mathfrak{c}_M(a \vee a^*) = \mathsf{O}$, hence $a \vee a^* = 1$, whence M is Boolean. Therefore, the only nowhere dense sublocale of L is O. Thus L misses every nowhere dense sublocale of itself, making it a remote sublocale of itself. □

This naturally raises a question as to when a space is remote as a subspace of itself. We answer that in Proposition 2.1.9 below. Recall, for instance, from [41], that a space is *almost discrete* if each closed subset is open (equivalently, each open subset is closed). We start by showing that almost discrete spaces are precisely those whose locales of opens are Boolean.

Proposition 2.1.8. *A space X is almost discrete if and only if $\mathfrak{O}X$ is Boolean.*

Proof. ($\Longrightarrow$): Assume X is almost discrete and let $U \in \mathfrak{O}X$. Then $U = \overline{U}$, so $U \vee U^* = U \cup (X \smallsetminus \overline{U}) = X = 1_{\mathfrak{O}X}$, where U^* is the pseudocomplement in $\mathfrak{O}X$ of U given by

$$U^* = X \smallsetminus \overline{U} = (X \smallsetminus U)^\circ,$$

showing that $\mathfrak{O}X$ is Boolean.

($\Longleftarrow$): Suppose that $\mathfrak{O}X$ is Boolean and let U be an open subset of X. Then $U \in \mathfrak{O}X$ and so, in light of $\mathfrak{O}X$ being Boolean, $X = U \vee U^* = U \cup (X \smallsetminus \overline{U})$. Since $\overline{U} \cap (X \smallsetminus \overline{U}) = \emptyset$, it follows that $\overline{U} \subseteq U$ so that $U = \overline{U}$. Thus U is closed. $\square$

Proposition 2.1.9. *A space X is remote as a subset of itself iff it is almost discrete.*

Proof. ($\Longrightarrow$): Let $A \subseteq X$ be closed. Then $X \smallsetminus A$ is open so that $\overline{X \smallsetminus A} \cap A$ is nowhere dense. Since X is remote, we have that $\emptyset = X \cap \overline{X \smallsetminus A} \cap A = A \cap \overline{X \smallsetminus A}$. Therefore

$$\begin{aligned} A &\subseteq X \smallsetminus \overline{X \smallsetminus A} \\ &= \operatorname{int}(X \smallsetminus (X \smallsetminus A)) = \operatorname{int}(A). \end{aligned}$$

Thus A is open. Hence X is almost discrete.

($\Longleftarrow$): Recall from [40] that the only nowhere dense subset of an almost discrete space is the empty set. So X misses every nowhere dense subset of itself, making it a remote subset of itself. $\square$

We note some examples.

Example 2.1.10. (1) The sublocales O and $\mathfrak{B}L$ are remote sublocales. This shows, among other things, that a locale always has a dense remote sublocale.

(2) Every sublocale contained in a remote sublocale is remote.

Remark 2.1.11. The closure of a remote sublocale is not necessarily remote. For instance, if L is not Boolean, then $\mathfrak{B}L$ is a remote sublocale whose closure is not remote because $\overline{\mathfrak{B}L} = L$ and, as observed above, a locale is remote if and only if it is Boolean.

We are now going to show that in the class of T_D-spaces localic remoteness is "conservative" in the sense that a subset of a T_D-space is remote if and only if the sublocale it induces is remote. Firstly, we note that the following proposition holds by virtue of Plewe's results recalled above.

Proposition 2.1.12. *A sublocale is remote iff it misses every closed nowhere dense sublocale.*

Secondly, we need to know that a subspace of a T_D-space is nowhere dense if and only if the sublocale it induces is nowhere dense. We have not seen a proof of this in the literature, so we furnish one. We require two lemmas, the first of which is proved in [24, Lemma 3.5] but for Tychonoff spaces; however a closer look at the proof given in that paper shows that the result holds for all T_D-spaces. Before we state the first lemma, we recall that if X is a topological space and $A \subseteq X$, then the induced sublocale $\widetilde{A}$ is the set

$$\widetilde{A} = \{\operatorname{int}((X \smallsetminus A) \cup G) : G \in \mathfrak{O}X\}.$$

In particular, if A is closed, then

$$\widetilde{A} = \{(X \smallsetminus A) \cup G : G \in \mathfrak{O}X\}.$$

The lemma states the following.

Lemma 2.1.13. *[24] For any subset S of a T_D-space X, $\overline{\widetilde{S}} = \widetilde{\overline{S}}$.*

The second lemma was mentioned in [1] without a proof. We furnish its proof.

Lemma 2.1.14. *[1] If K is a closed subset of a space X, then $\widetilde{K} = \mathfrak{c}_{\mathfrak{O}X}(X \smallsetminus K)$.*

Proof. By what we observed above,

$$\widetilde{K} = \{(X \smallsetminus K) \cup G : G \in \mathfrak{O}X\} \subseteq \mathfrak{c}_{\mathfrak{O}X}(X \smallsetminus K).$$

On the other hand, if $V \in \mathfrak{c}_{\mathfrak{O}X}(X \smallsetminus K)$, then $X \smallsetminus K \subseteq V$ so that $V = (X \smallsetminus K) \cup V$. This makes $V \in \widetilde{K}$. Thus $\mathfrak{c}_{\mathfrak{O}X}(X \smallsetminus K) \subseteq \widetilde{K}$. □

Lemma 2.1.15. *A subset S of a T_D-space X is nowhere dense iff $\widetilde{S}$ is a nowhere dense sublocale of $\mathfrak{O}X$.*

Proof. We first prove the result for closed subsets. So let K be a closed subset of X. Since $\widetilde{K} = \mathfrak{c}_{\mathfrak{O}X}(X \smallsetminus K)$, $\operatorname{int} \widetilde{K} = \mathfrak{o}_{\mathfrak{O}X}\big((X \smallsetminus K)^*\big)$, and so

$$
\begin{aligned}
\widetilde{K} \text{ is nowhere dense} \quad &\iff \quad \mathfrak{o}_{\mathfrak{O}X}\big((X \smallsetminus K)^*\big) = \mathsf{O} \\
&\iff \quad (X \smallsetminus K)^* = 0_{\mathfrak{O}X} \\
&\iff \quad X \smallsetminus \overline{X \smallsetminus K} = \emptyset \\
&\iff \quad K^\circ = \emptyset \\
&\iff \quad K \text{ is nowhere dense.}
\end{aligned}
$$

Therefore, for any $S \subseteq X$,

$$
\begin{aligned}
S \text{ is nowhere dense} \quad &\iff \quad \overline{S} \text{ is nowhere dense} \\
&\iff \quad \widetilde{\overline{S}} \text{ is nowhere dense} \qquad \text{since } \overline{S} \text{ is closed} \\
&\iff \quad \overline{\widetilde{S}} \text{ is nowhere dense} \qquad \text{by Lemma 2.1.13} \\
&\iff \quad \widetilde{S} \text{ is nowhere dense.}
\end{aligned}
$$

This proves the proposition. ☐

As in [50], for any topological space X and $x \in X$, we set

$$\widetilde{x} = X \smallsetminus \overline{\{x\}},$$

and recall that $\{\widetilde{x}, 1_{\mathfrak{O}X}\}$ is a sublocale of $\mathfrak{O}X$. In general, if p is a point in a locale L, then $\{p, 1\}$ is a sublocale of L, and these are what are called the *one-point sublocales* of L. In [49], the authors show that if X is a topological space and $Y \subseteq X$, then

$$\widetilde{Y} = \bigvee \big\{ \{\widetilde{y}, 1_{\mathfrak{O}X}\} : y \in Y \big\},$$

where the join is calculated in $\mathcal{S}(\mathfrak{O}X)$. This says that the locale $\widetilde{Y}$ is covered by the one-point sublocales of $\mathfrak{O}X$ associated with the elements of Y.

In the upcoming proof we shall use the fact that complemented sublocales are linear, [35, 36]. We shall also use [50, Proposition VI. 1.3.1], which states that if X is a T_D-space, then for any $x \in X$ and $A \subseteq X$, $\widetilde{x} \in \widetilde{A}$ if and only if $x \in A$.

Theorem 2.1.16. *A subset of a T_D-space is remote iff the sublocale it induces is remote.*

Proof. ($\Longrightarrow$): Let S be a subset of a T_D-space X. We suppose, first, that S is remote, and prove that $\widetilde{S}$ is remote. In accordance with the definition, we need to show that $\widetilde{S}$ misses every nowhere dense sublocale of $\mathfrak{O}X$. By Proposition 2.1.12, it suffices to show that $\widetilde{S}$ misses every closed nowhere dense sublocale of $\mathfrak{O}X$. So, let B be a closed nowhere dense sublocale of $\mathfrak{O}X$. Then there is an open set $U \subseteq X$ such that, in light of Lemma 2.1.14,

$$B = \mathfrak{c}_{\mathfrak{O}X}(U) = \mathfrak{c}_{\mathfrak{O}X}(X \smallsetminus \big(X \smallsetminus U)\big) = \widetilde{X \smallsetminus U}$$

since $X \smallsetminus U$ is closed in X. By Lemma 2.1.15, $X \smallsetminus U$ is nowhere dense in X, and so $S \cap (X \smallsetminus U) = \emptyset$. Note that, by [49, Proposition VI. 1.3.1], if $s \in S$, then $\widetilde{s} \notin \widetilde{X \smallsetminus U}$, and so $\widetilde{X \smallsetminus U} \cap \{\widetilde{s}, 1_{\mathfrak{O}X}\} = \mathsf{O}$. Now,

$$\begin{aligned}
\widetilde{S} \cap B = \widetilde{X \smallsetminus U} \cap \widetilde{S} &= \widetilde{X \smallsetminus U} \cap \bigvee\big\{\{\widetilde{s}, 1_{\mathfrak{O}X}\} : s \in S\big\} \\
&= \bigvee\big\{\widetilde{X \smallsetminus U} \cap \{\widetilde{s}, 1_{\mathfrak{O}X}\} : s \in S\big\} \quad \text{since } \widetilde{X \smallsetminus U} \text{ is complemented} \\
&= \mathsf{O},
\end{aligned}$$

showing that $\widetilde{S}$ is remote.

($\Longleftarrow$): Suppose that $\widetilde{S}$ is remote. Let K be a nowhere dense subset of X. Then $\overline{K}$ is nowhere dense in X which implies that $\widetilde{\overline{K}}$ is a nowhere dense sublocale of $\mathfrak{O}X$, and so $\widetilde{S} \cap \widetilde{\overline{K}} = \mathsf{O}$. If there was an element $w \in S \cap \overline{K}$, then, for the one-point sublocale $\{\widetilde{w}, 1_{\mathfrak{O}X}\}$ we would have $\widetilde{\overline{K}} \cap \{\widetilde{w}, 1_{\mathfrak{O}X}\} \neq \mathsf{O}$, and so, by the calculation observed in the preceding paragraph of the proof, we would have $\widetilde{\overline{K}} \cap \widetilde{S} \neq \mathsf{O}$. It follows therefore that $S \cap \overline{K} = \emptyset$ which implies that $S \cap K = \emptyset$. Thus S is remote. □

Observation 2.1.17. Inside the proof of the forward direction of Theorem 2.1.16, we deduce that for a T_D-space X and subsets A and B of X where A is either closed or open, $A \cap B = \emptyset$ if and only if $\widetilde{A} \cap \widetilde{B} = \mathsf{O}$.

We introduce the following notations, for any locale L. We set

$$\mathcal{S}_{\mathrm{rem}}(L) = \{S \in \mathcal{S}(L) : S \text{ is a remote sublocale}\},$$

and for elements we set

$$\mathrm{Rem}(L) = \{a \in L : \mathfrak{c}(a) \in \mathcal{S}_{\mathrm{rem}}(L)\}.$$

We aim to characterize remote sublocales in Theorem 2.1.18 below. We first give a characterization with no restriction on sublocales and, subsequently, a characterization restricted to complemented sublocales. Towards that end, we remind the reader of the following notation from [25]:

$$\mathrm{Nd}(L) = \bigvee\{S \in \mathcal{S}(L) : S \text{ is nowhere dense}\} = \bigvee\{\mathfrak{c}(x) : x \text{ is a dense element of } L\}.$$

In [25], the authors further observe that

$$\mathrm{Nd}(L) = L \smallsetminus \mathfrak{B}L.$$

We shall at times use this description of the sublocale $\mathrm{Nd}(L)$.

Theorem 2.1.18. *Let L be a locale and $A \in \mathcal{S}(L)$. The following statements are equivalent.*

1. *$A \in \mathcal{S}_{rem}(L)$.*
2. *For all nowhere dense $N \in \mathcal{S}(L)$, $A \cap \overline{N} = \mathsf{O}$.*
3. *A is contained in every dense open sublocale of L.*
4. *For all dense $a \in L$, $\nu_A(a) = 1$.*
5. *For every open sublocale $U \in \mathcal{S}(L)$, $A \subseteq \overline{U}$ implies $A \subseteq U$.*
6. *For each $N \in \mathcal{S}(L)$, $\mathfrak{B}L \cap \overline{N \cap A} = \mathsf{O}$ implies $A \cap N = \mathsf{O}$.*

Proof. (1) $\Longrightarrow$ (2): Follows since every closure of a nowhere dense sublocale is nowhere dense.

(2) $\Longrightarrow$ (3): Follows since $\mathfrak{o}(x)$ is dense if and only if x is dense if and only if $\mathfrak{c}(x)$ is nowhere dense.

(3) $\Longrightarrow$ (4): True because $\nu_A(x) = 1$ if and only if $A \subseteq \mathfrak{o}(x)$.

(4) $\Longrightarrow$ (5): Let $U = \mathfrak{o}_L(x)$ be such that $A \subseteq \overline{\mathfrak{o}_L(x)}$. Then $A \cap \mathfrak{o}_L(x^*) = \mathsf{O}$ which implies $\mathfrak{o}_A(\nu_A(x^*)) = A \cap \mathfrak{o}_L(x^*) = \mathsf{O}$. We get that $\nu_A(x^*) = 0_A$. Because $x \vee x^*$ is dense in L, it follows that

$$\nu_A(x \vee x^*) = \nu_A(x) \vee \nu_A(x^*) = 1.$$

Therefore $\nu_A(x) = 1$. Thus $A \subseteq \mathfrak{o}(x)$.

(5) $\Longrightarrow$ (6): Let $N \in \mathcal{S}(L)$ be such that $\mathfrak{B}L \cap \overline{N \cap A} = \mathsf{O}$. Then $\mathfrak{B}L \subseteq L \smallsetminus \overline{N \cap A}$ so that $L \smallsetminus \overline{N \cap A}$ is dense hence $A \subseteq \overline{L \smallsetminus \overline{N \cap A}}$. The hypothesis in (5) implies that $A \subseteq L \smallsetminus \overline{N \cap A}$. Since $L \smallsetminus \overline{N \cap A}$ is complemented,

$$\mathsf{O} = A \cap L \smallsetminus (L \smallsetminus \overline{N \cap A}) = A \cap \overline{N \cap A} \supseteq (N \cap A) \cap \overline{N \cap A} = N \cap A.$$

(6) $\Longrightarrow$ (1): Let C be a nowhere dense sublocale of L. Then $A \cap C$ is a nowhere dense sublocale of L. Since this implies $\overline{A \cap C}$ is a nowhere dense sublocale of L, we have $\mathfrak{B}L \cap \overline{A \cap C} = \mathsf{O}$. It follows that $A \cap C = \mathsf{O}$. $\square$

In view of the equivalence of (1) and (3) in the foregoing theorem, we have that:

> *A sublocale of L is remote if and only if it is contained in the intersection of all dense open sublocales of L.*

In the event of complemented sublocales, we also have the following characterization.

Proposition 2.1.19. *A complemented sublocale of a locale L is remote iff it misses* $\mathrm{Nd}(L)$.

Proof. Recall that the closed nowhere sublocales of L are precisely the sublocales $\mathfrak{c}(x)$, for x dense in L. Now, if A is a complemented sublocale of L, then

$$\begin{aligned} A \cap \mathrm{Nd}(L) = \mathsf{O} &\iff A \cap \bigvee \{\mathfrak{c}(x) : x \text{ is dense in } L\} = \mathsf{O} \\ &\iff \bigvee \{A \cap \mathfrak{c}(x) : x \text{ is dense in } L\} = \mathsf{O} \\ &\iff A \cap \mathfrak{c}(x) = \mathsf{O} \text{ for all dense } x \in L \\ &\iff A \text{ is remote} \end{aligned}$$

which proves the result. $\square$

Next, we show that every locale has the largest remote sublocale. In the first section of Chapter 3, we will show that this largest remote sublocale is precisely the Booleanization of a locale.

Proposition 2.1.20. *The join of remote sublocales is remote.*

Proof. Let $\{A_i : i \in I\}$ be a family of remote sublocales and N a closed nowhere dense sublocale of L. Then since complemented sublocales are linear,

$$N \cap \bigvee_i A_i = \bigvee_i (N \cap A_i) = \mathsf{O},$$

which proves that $\bigvee_i A_i$ is remote. □

This result enables us to show that $\mathcal{S}_{\text{rem}}(L)$, partially ordered by inclusion, is a coframe. In this regard, recall that a subset S of a coframe H is called a *subcolocale* of H if it closed under joins and if for all $s \in S$ and $x \in H$, $s \smallsetminus x \in S$, where

$$s \smallsetminus x = \bigwedge \{a \in H : s \leq a \vee x\}.$$

Since for any sublocales S and T, the sublocale $S \smallsetminus T$ is contained in S, and since a sublocale smaller than a remote sublocale is remote, we deduce from Proposition 2.1.20 the following result.

Proposition 2.1.21. *For any locale L, $\mathcal{S}_{rem}(L)$ is a coframe.*

We have just mentioned that in Chapter 3 we will show that for any locale L, $\mathfrak{B}L$ is the largest remote sublocale of L. In the case of fit locales, we can actually deduce this result from the characterization that a sublocale is remote if and only if it is contained in the intersection of all dense open sublocales. To do this, we recall that a locale is *fit* if and only if every sublocale is a meet of open sublocales.

Corollary 2.1.22. *If L is a fit locale, then $\mathfrak{B}L$ is the largest remote sublocale of L.*

Proof. Since L is fit, $\mathfrak{B}L$ is an intersection of open, hence dense open, sublocales. But every dense sublocale of L contains $\mathfrak{B}L$, hence

$$\mathfrak{B}L = \bigcap \{U \in \mathcal{S}(L) : U \text{ is open and dense}\},$$

that is $\mathfrak{B}L$ is the intersection of *all* open dense sublocales of L. So, by the equivalence of conditions (1) and (3) in Theorem 2.1.18, a sublocale is remote if and only if it is contained in $\mathfrak{B}L$. Since $\mathfrak{B}L$ is remote, it follows that $\mathfrak{B}L$ is the largest remote sublocale of L. □

Observation 2.1.23. The above corollary tells us that, $\mathfrak{B}L$ is the only dense and remote sublocale of a fit locale L. This is because, if A is dense and remote, then its density gives $\mathfrak{B}L \subseteq A$ and its remoteness implies $A \subseteq \mathfrak{B}L$.

The equivalences (1), (2), (3) and (5) of Theorem 2.1.18 can be obtained for T_D-spaces as a corollary to Theorem 2.1.18. It turns out, however, that these equivalences hold even for spaces that are not T_D, as we show in the next theorem.

Theorem 2.1.24. *Let X be a space and $S \subseteq X$. The following statements are equivalent.*

1. *S is remote.*
2. *$S \cap \overline{N} = \emptyset$ for each nowhere dense $N \subseteq X$.*
3. *S is contained in every dense open subset of X.*
4. *If $S \subseteq \overline{U}$, then $S \subseteq U$ for each open subset $U \subseteq X$.*

Proof. (1) $\Longrightarrow$ (2): Let $N \subseteq X$ be nowhere dense. Then $\overline{N}$ is nowhere dense. It follows that $S \cap \overline{N} = \emptyset$ as required.

(2) $\Longrightarrow$ (3): Let U be a dense and open subset of X. Since the complement of a dense open set is nowhere dense, we have that $\overline{X \smallsetminus U} = X \smallsetminus U$ is nowhere dense. By condition (2), $S \cap (X \smallsetminus U) = \emptyset$ so that $S \subseteq U$.

(3) $\Longrightarrow$ (4): Let $U \subseteq X$ be open such that $S \subseteq \overline{U}$. Since $\overline{(X \smallsetminus \overline{V}) \cup V} = X$ for every open $V \subseteq X$, by condition (3), $S \subseteq (X \smallsetminus \overline{U}) \cup U$. Therefore $S = S \cap \overline{U} \subseteq U$.

(4) $\Longrightarrow$ (1): Let $N \subseteq X$ be nowhere dense. Then $X \smallsetminus \overline{N}$ is open and dense. We get that $S \subseteq \overline{X \smallsetminus \overline{N}}$. By (4), $S \subseteq X \smallsetminus \overline{N} \subseteq X \smallsetminus N$. Therefore $S \cap N = \emptyset$ making S remote. □

As in locales, from the equivalence of (1) and (3) in the foregoing theorem, we have that:

> *A subset of a space X is remote if and only if it is contained in the intersection of all dense open subsets of X.*

Just like in locales, the union of remote subsets is remote. This is because, for any collection $\{A_i : i \in I\}$ of remote subsets of X and any nowhere dense $N \subseteq X$,

$$N \cap \bigcup_{i \in I} A_i = \bigcup_i \{N \cap A_i : i \in I\} = \emptyset.$$

This tells us that every space has the largest remote subset. In T_1-spaces, we can actually identify this subset. Recall that a point $x \in X$ is called an *isolated point* in case $\{x\}$ is an open subset of X. We denote by $\text{Iso}(X)$ the set of all isolated points of X.

Proposition 2.1.25. *In a T_1-space X, $\text{Iso}(X)$ is the largest remote subset of X.*

Proof. We start by showing that x is isolated if and only if $\{x\}$ is remote. Indeed, suppose that x is isolated and choose a closed nowhere dense $N \subseteq X$. Then $X \smallsetminus N$ is dense in X. But the nonempty set $\{x\}$ is open, so $\{x\} \cap (X \smallsetminus N) \neq \emptyset$, which implies that $\{x\} \cap N = \emptyset$. Therefore $\{x\}$ is remote. Conversely, suppose that $\{x\}$ is remote. Then $\{x\}$ misses every nowhere dense subset of X. This means that $\{x\}$ is not nowhere dense in X. Observe that,

$$\begin{aligned} \{x\} \text{ is not nowhere dense } &\iff X \smallsetminus \{x\} \text{ is not dense} \\ &\iff \{x\} \text{ is open} \\ &\iff x \text{ is isolated} \end{aligned}$$

which proves the converse.

Since every subset contained in a remote subset is remote, we get that, if $R \subseteq X$ is remote, then $\{x\}$ is remote for each $x \in R$. Therefore each $x \in R$ belongs to $\text{Iso}(X)$, making $R \subseteq \text{Iso}(X)$. Because $\{x\}$ is remote for each $x \in \text{Iso}(X)$, we have that $\bigcup_{x \in \text{Iso}(X)} \{x\} = \text{Iso}(X)$ is remote. Thus $\text{Iso}(X)$ is the largest remote subset of X. $\square$

Corollary 2.1.26. *If a T_1-space has no isolated points, then the empty set is the only remote subset of the space.*

We observe from Proposition 2.1.25 that

> *In a T_1-space, a subset is remote if and only if it consists entirely of isolated points.*

From the above paragraph, we get that remote subsets of a Tychonoff space are precisely the remote subsets of its Stone-Čech compactification. This follows since the isolated points of the Stone-Čech compactification of a Tychonoff space are precisely the isolated points of the underlying space.

Remark 2.1.27. In a T_1-space X, the pair $(\mathrm{Iso}(X), \mathbb{S}_{\mathrm{rem}}(X))$, where $\mathbb{S}_{\mathrm{rem}}(X)$ is the collection of all remote subsets of X, is a discrete topological space. This follows since every subset of $\mathrm{Iso}(X)$ is a remote subset of X, hence belonging to $\mathbb{S}_{\mathrm{rem}}(X)$.

Having characterized closed remote sublocales, it makes sense to characterize elements that induce closed remote sublocales. That is, we seek to have other descriptions of the elements that belong to $\mathrm{Rem}(L)$. The descriptions follow easily from the fact that $\mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathfrak{c}(a \vee b)$ and $\mathsf{O} = \mathfrak{c}(1)$.

Proposition 2.1.28. *The following are equivalent for an element $a \in L$.*

(a) $a \in \mathrm{Rem}(L)$.

(b) $a \vee d = 1$ *for every dense* $d \in L$.

(c) $d^* \leq a$ *implies* $d \vee a = 1$ *for every* $d \in L$.

The result in Proposition 2.1.28(b) enables us to show that $\mathrm{Rem}(L)$ is a filter in L. Indeed, (i) $1 \in \mathrm{Rem}(L)$ (ii) if $a \leq b$ and $a \in \mathrm{Rem}(L)$, then $b \in \mathrm{Rem}(L)$, and (iii) if $a, b \in \mathrm{Rem}(L)$ and d is dense, then

$$(a \wedge b) \vee d = (a \vee d) \wedge (b \wedge d) = 1,$$

showing that $a \wedge b \in \mathrm{Rem}(L)$.

Observation 2.1.29. For any locale L, the set $\mathrm{Rem}(L) \cup \{0\}$ is a subframe of L, as one checks routinely.

In the following corollary, we give yet another characterization of when a locale L is Boolean in terms of $\mathrm{Rem}(L)$.

Corollary 2.1.30. *A locale L is Boolean iff* $L = \mathrm{Rem}(L)$.

Proof. ($\Longrightarrow$) : Suppose that L is Boolean. Then the only dense element of L is the top element, so the bottom element of L joins all dense elements of L at the top, hence $0 \in \mathrm{Rem}(L)$. Since $\mathrm{Rem}(L)$ is a filter, $\mathrm{Rem}(L)$ is then an improper filter, i.e., $\mathrm{Rem}(L) = L$.

($\Longleftarrow$) : Assume that $L = \mathrm{Rem}(L)$ and let $x \in L$. Then $x \in \mathrm{Rem}(L)$. But $x \vee x^*$ is dense, so by Proposition 2.1.28, $x \vee x \vee x^* = 1$, i.e., $x \vee x^* = 1$, showing that L is Boolean. □

Another corollary of Proposition 2.1.28 is the following. In the proof we shall use the following lemma.

Lemma 2.1.31. *A closed sublocale of a locale which is also a coframe is itself a coframe.*

Proof. Let L be a locale which is a coframe and $\mathfrak{c}(x) \in \mathcal{S}(L)$. Then, for each $a, b_i \in \mathfrak{c}(x)$,

$$\begin{aligned}
a \vee_{\mathfrak{c}(x)} \left(\bigwedge b_i\right) &= \nu_{\mathfrak{c}(x)}\left(a \vee \left(\bigwedge b_i\right)\right) \text{ since } \bigvee_S d_i = \nu_S\left(\bigvee d_i\right) \text{ for all } d_i \in L, S \in \mathcal{S}(L)\\
&= \nu_{\mathfrak{c}(x)}\left(\bigwedge (a \vee b_i)\right) \text{ since } L \text{ is a coframe}\\
&= \nu_{\mathfrak{c}(0_{\mathfrak{c}(x)})}\left(\bigwedge (a \vee b_i)\right) \text{ since } \mathfrak{c}(x) \text{ is closed}\\
&= 0_{\mathfrak{c}(x)} \vee \left(\bigwedge (a \vee b_i)\right) \text{ since } \nu_{\mathfrak{c}(d)}(c) = d \vee c \text{ for all } c, d \in L\\
&= \bigwedge \left(0_{\mathfrak{c}(x)} \vee (a \vee b_i)\right) \text{ since } L \text{ is a coframe}\\
&= \bigwedge \left(\nu_{\mathfrak{c}(x)}(a \vee b_i)\right)\\
&= \bigwedge \left(a \vee_{\mathfrak{c}(x)} b_i\right).
\end{aligned}$$

Thus $\mathfrak{c}(x)$ is a coframe. □

Corollary 2.1.32. *If L is a locale which is also a coframe, then* $\mathrm{Rem}(L)$ *is a sublocale of L. Furthermore,* $\mathrm{Rem}(L)$ *is a closed remote sublocale and hence a coframe.*

Proof. Let $\{a_i : i \in I\}$ be a subset of $\mathrm{Rem}(L)$. For any dense $d \in L$,

$$d \vee \bigwedge a_i = \bigwedge (d \vee a_i) = 1$$

hence $\bigwedge a_i \in \mathrm{Rem}(L)$. Next, if $x \in L$ and $a \in \mathrm{Rem}(L)$, then, in light of the fact that $a \leq x \to a$, the element $x \to a$ joins every dense element of L at the top, showing that $x \to a \in \mathrm{Rem}(L)$. Therefore $\mathrm{Rem}(L)$ is a sublocale of L.

$\mathrm{Rem}(L)$ is a remote sublocale of L: Consider any closed sublocale $\mathfrak{c}(x)$ which is nowhere dense. Then x is dense. Thus, if $a \in \mathfrak{c}(x) \cap \mathrm{Rem}(L)$, then $x \leq a$ and $x \vee a = 1$ which implies $a = 1$, showing that $\mathfrak{c}(x) \cap \mathrm{Rem}(L) = \mathsf{O}$. Therefore $\mathrm{Rem}(L)$ is a remote sublocale.

$\mathrm{Rem}(L)$ is closed: Let $x \in \overline{\mathrm{Rem}\, L}$ and choose a dense $y \in L$. Then $0_{\mathrm{Rem}\, L} = \bigwedge \mathrm{Rem}\, L \leq x$. Since $0_{\mathrm{Rem}\, L} \in \mathrm{Rem}\, L$, $y \vee 0_{\mathrm{Rem}\, L} = 1$ so that $y \vee x = 1$. Thus $x \in \mathrm{Rem}\, L$ making $\mathrm{Rem}\, L$ a closed sublocale of L. □

We have, so far, introduced what we can refer to as remoteness on a single locale. In what follows, we consider sublocales of a locale which are remote from the locale's dense sublocales. This is supported by the reason that remoteness already present in the literature is for points of the Stone-Čech compactification βL with respect to underlying locale L.

The following variants of remoteness shall be frequently referred to as remoteness from a dense sublocale, to differentiate them from the remoteness introduced in Definition 2.1.2.

Definition 2.1.33. Let $S \subseteq L$ be a dense sublocale of L. Then

1. $T \in \mathcal{S}(L)$ is *remote from* S if $T \cap \overline{N}^L = \mathsf{O}$ for every S-nowhere dense $N \in \mathcal{S}(S)$.
2. A sublocale $T \subseteq L \smallsetminus S$ is **remote from* S if $T \cap \overline{N}^L = \mathsf{O}$ for every S-nowhere dense $N \in \mathcal{S}(S)$.

We shall drop the superscript L from the closure of N in the above definition to mean that the closure is taken in the whole locale L.

The motivation for Definition 2.1.33(2) is that remote points were initially defined in spaces to be points belonging to $\beta X \smallsetminus X$ and not contained in the closure of any nowhere dense subset of X.

In the case of a single locale L, T being remote in L is equivalent to T missing the closure of every nowhere dense sublocale of L, which suggests that the author in [48] still had in mind the definition of a remote point which involves missing the closure of a nowhere dense subset when he formulated the definition of a remote collection. This is not the case for T being remote from a dense sublocale. So, the choice of missing $\overline{N}$ instead of just N is to align with the idea of missing the closure of a nowhere dense subset from remote points.

For a sublocale S of a locale L, set

$$\mathcal{S}_{\text{rem}}(L \ltimes S) = \{A \in \mathcal{S}(L) : A \text{ is remote from } S\},$$

$$^{*}\mathcal{S}_{\text{rem}}(L \ltimes S) = \{A \in \mathcal{S}(L) : A \text{ is *remote from } S\},$$

and for elements, set

$$\text{Rem}(L \ltimes S) = \{a \in L : \mathfrak{c}_L(a) \in \mathcal{S}_{\text{rem}}(L \ltimes S)\},$$

$$^{*}\text{Rem}(L \ltimes S) = \{a \in L : \mathfrak{c}_L(a) \in {}^{*}\mathcal{S}_{\text{rem}}(L \ltimes S)\}.$$

Since the set $\mathcal{S}_{\text{rem}}(L \ltimes S)$ does not restrict where its members come from, we have the following result.

Proposition 2.1.34. *For every dense sublocale S of a locale L, ${}^{*}\mathcal{S}_{rem}(L \ltimes S) \subseteq \mathcal{S}_{rem}(L \ltimes S)$.*

Observation 2.1.35. For a non-void Boolean locale L, we have that ${}^{*}\mathcal{S}_{\text{rem}}(L \ltimes L) \subset \mathcal{S}_{\text{rem}}(L \ltimes L)$. This is because $L \in \mathcal{S}_{\text{rem}}(L \ltimes L)$ but L is not contained in $L \smallsetminus L = \mathsf{O}$.

In an attempt to obtain an equality in Proposition 2.1.34, we start by recalling from [53] that a sublocale is *rare* if its supplement is the whole locale. Restricting our sublocales to dense and rare sublocales yields the following proposition which is easy to prove. Sublocales which are simultaneously dense and rare do exist. For instance, recall that Plewe in [53] defines a locale to be *dense in itself* if every Boolean sublocale has a dense supplement. He then shows that a locale is dense in itself if and only if its Booleanization is rare. The locale $\mathfrak{O}(\mathbb{R})$, where $\mathbb{R}$ is the set of real numbers, is an example of a dense in itself locale. This is motivated by the fact that the space $\mathbb{R}$ is dense in itself (because it has no isolated points) and since, according to [53], every sober space is dense in itself if and only if its locale of opens is dense in itself, the real line being sober and dense in itself makes $\mathfrak{O}(\mathbb{R})$ dense in itself.

Proposition 2.1.36. *If S is a dense and rare sublocale of a locale L, then ${}^{*}\mathcal{S}_{rem}(L \ltimes S) = \mathcal{S}_{rem}(L \ltimes S)$.*

We collect into one proposition some observations about the variants of remoteness introduced in Definition 2.1.33. Observe that if K is a dense sublocale of L, then $\mathfrak{B}L = \mathfrak{B}K$ since, as presented in the preliminaries, $x^{*K} = x^{*}$ for every $x \in K$. As a result of this, a sublocale of K is nowhere dense in K if and only if it is nowhere dense in L.

Proposition 2.1.37. *Let S be a dense sublocale of a locale L. Each of the following statements holds.*

1. $\mathcal{S}_{rem}(L \ltimes \mathfrak{B}L) = \mathcal{S}(L)$.

2. ${}^{*}\mathcal{S}_{rem}(L \ltimes \mathfrak{B}L) = \{T \in \mathcal{S}(L) : T \subseteq L \smallsetminus \mathfrak{B}L\}$.

3. *For each* $A, B \in \mathcal{S}(L)$, *if* $A \subseteq B$ *and* $B \in \mathcal{S}_{rem}(L \ltimes S)$, *then* $A \in \mathcal{S}_{rem}(L \ltimes S)$.

4. $\mathcal{S}_{rem}(L) \subseteq \mathcal{S}_{rem}(L \ltimes S)$ *for every dense* $S \in \mathcal{S}(L)$.

5. *For any sublocale* T *of* L *with* $S \subseteq T$, $\mathcal{S}_{rem}(L \ltimes T) \subseteq \mathcal{S}_{rem}(L \ltimes S)$ *and* ${}^{*}\mathcal{S}_{rem}(L \ltimes T) \subseteq {}^{*}\mathcal{S}_{rem}(L \ltimes S)$.

6. *If* $A \subseteq L$ *is remote from* S, *then* $A \cap S \in \mathcal{S}_{rem}(L)$.

7. $S \in \mathcal{S}_{rem}(L \ltimes S)$ *iff* $S = \mathfrak{B}L$.

Proof. (1) If $A \in \mathcal{S}(L)$ and $N \in \mathcal{S}(\mathfrak{B}L)$ is nowhere dense in $\mathfrak{B}L$, then $N = \mathsf{O}$ which implies that $A \cap \overline{N} = \mathsf{O}$. Thus $A \in \mathcal{S}_{\text{rem}}(L \ltimes \mathfrak{B}L)$. Hence $\mathcal{S}(L) \subseteq \mathcal{S}_{\text{rem}}(L \ltimes \mathfrak{B}L)$, making $\mathcal{S}_{\text{rem}}(L \ltimes \mathfrak{B}L) = \mathcal{S}(L)$ since the other containment always holds.

(2) Can be deduced from (1).

(3) This is trivial.

(4) Follows since a remote sublocale of L misses the closure of every nowhere dense sublocale of L, and hence the closure of every nowhere dense sublocale of S since S is a dense sublocale of L.

(5) Let $T \in \mathcal{S}(L)$ and $S \subseteq T$, so that T is also dense in L. That $\mathcal{S}_{\text{rem}}(L \ltimes T) \subseteq \mathcal{S}_{\text{rem}}(L \ltimes S)$ follows because every S-nowhere dense sublocale is T-nowhere dense, which follows since S is dense in T. The containment ${}^{*}\mathcal{S}_{\text{rem}}(L \ltimes T) \subseteq {}^{*}\mathcal{S}_{\text{rem}}(L \ltimes S)$ uses the facts that $L \smallsetminus T \subseteq L \smallsetminus S$ and $\mathcal{S}_{\text{rem}}(L \ltimes T) \subseteq \mathcal{S}_{\text{rem}}(L \ltimes S)$.

(6) Assume that $A \in \mathcal{S}_{\text{rem}}(L \ltimes S)$ and let N be nowhere dense in L. Since $N \cap S \subseteq N$, $N \cap S$ is nowhere dense in L which in turn makes it S-nowhere dense. By hypothesis, $A \cap \overline{S \cap N} = \mathsf{O}$. This makes $(A \cap S) \cap N = \mathsf{O}$. Thus $A \cap S \in \mathcal{S}_{\text{rem}}(L)$.

(7) ($\Longrightarrow$) : Let N be S-nowhere dense. Then since $S \in \mathcal{S}_{\text{rem}}(L \ltimes S)$, $S \cap \overline{N} = \mathsf{O}$ which implies that $\mathsf{O} = S \cap N = N$. This makes S Boolean. So, $S = \mathfrak{B}L$ because the only dense Boolean sublocale of L is $\mathfrak{B}L$.

($\Longleftarrow$) : Follows since $\mathfrak{B}L \in \mathcal{S}_{\text{rem}}(L)$, hence by (4), $S = \mathfrak{B}L \in \mathcal{S}_{\text{rem}}(L \ltimes S)$. $\square$

Observation 2.1.38. (1) From Proposition 2.1.37(2), observe that when L is not dense in itself, we get another case where ${}^*\mathcal{S}_{\text{rem}}(L \ltimes S) \neq \mathcal{S}_{\text{rem}}(L \ltimes S)$. This is because we have that $L \neq L \smallsetminus \mathfrak{B}L$ so that by Proposition 2.1.37(1), $L \in \mathcal{S}_{\text{rem}}(L \ltimes \mathfrak{B}L)$ but $L \notin {}^*\mathcal{S}_{\text{rem}}(L \ltimes \mathfrak{B}L)$.

(2) Using Proposition 2.1.37(1) and the fact that $L \in \mathcal{S}_{\text{rem}}(L)$ if and only if L is Boolean (from Corollary 2.1.30), it is easy to see that for a non-Boolean locale L, $L \in \mathcal{S}_{\text{rem}}(L \ltimes \mathfrak{B}L)$ but $L \notin \mathcal{S}_{\text{rem}}(L)$. This is a particular case where we do not have equality in Proposition 2.1.37(4). However, $\mathcal{S}_{\text{rem}}(L \ltimes L) = \mathcal{S}_{\text{rem}}(L)$.

We note some examples.

Example 2.1.39. (1) Recall from preliminaries the notion of a remote point of βL. A point $I \in \beta L$ is remote if and only if $\mathfrak{c}(I)$ is remote from L. This follows since, for any dense $x \in L$, $I \vee r_L(x) = \top$ if and only if $\mathfrak{c}(I) \cap \mathfrak{c}(r_L(x)) = \mathsf{O}$.

(2) In **Top** (the category of topological spaces whose morphisms are continuous functions between them), we say that $A \subseteq X \smallsetminus Y$, where Y is a dense subspace of X, is **remote from Y* in case $A \cap \overline{N}^X = \emptyset$ for any nowhere dense subset N of Y. For a Tychonoff space X, a point $p \in \beta X \smallsetminus X$ is remote if and only if $\{p\}$ is *remote from X. To see this, observe that a point $p \in \beta X \smallsetminus X$ if and only if $\{p\} \subseteq \beta X \smallsetminus X$, and also, for a nowhere dense subset N of X, p is a remote point if and only if $p \notin \overline{N}^{\beta X}$ if and only if $\{p\} \cap \overline{N}^{\beta X} = \mathsf{O}$ if and only if $\{p\}$ is *remote from X.

The following proposition shows that $\mathfrak{B}L$ is the only dense sublocale from which L is remote.

Proposition 2.1.40. *Let S be a dense sublocale of L. The following statements are equivalent.*

1. *L is remote from S.*

2. S is a Boolean algebra.

3. $S = \mathfrak{B}L$.

Proof. (1) $\Longrightarrow$ (2): We start by showing that O is the only S-nowhere dense sublocale of S. Let $N \in \mathcal{S}(S)$ be S-nowhere dense. Since L is remote from S, we have that $\mathsf{O} = L \cap \overline{N}$, which implies that $N = \mathsf{O}$. This makes O the only S-nowhere dense sublocale of S, hence S is a Boolean algebra.

(2) $\Longrightarrow$ (3): The only dense Boolean sublocale of any locale is its Booleanization.

(3) $\Longrightarrow$ (1): Follows from Proposition 2.1.37(1). □

Observation 2.1.41. A locale is dense in itself if and only if it is *remote from its Booleanization. To verify this, observe that L is dense in itself if and only if $\mathfrak{B}L$ is rare if and only if $L \subseteq L \smallsetminus \mathfrak{B}L$ if and only if $L \in \{T \in \mathcal{S}(L) : T \subseteq L \smallsetminus \mathfrak{B}L\} = {}^*\mathcal{S}_{\text{rem}}(L \ltimes \mathfrak{B}L)$, where the last equality holds by Proposition 2.1.37(2).

We characterize members of $\mathcal{S}_{\text{rem}}(L \ltimes S)$ and ${}^*\mathcal{S}_{\text{rem}}(L \ltimes S)$. The proof of the following proposition relies on the fact that

$$\overline{\mathfrak{c}_S(a)} = \mathfrak{c}\Big(\bigwedge(\mathfrak{c}_S(a))\Big) = \mathfrak{c}(a)$$

for each $a \in S \in \mathcal{S}(L)$. We shall only prove the equivalence of statements (2) and (3) regarding the set $\mathcal{S}_{\text{rem}}(L \ltimes S)$.

Proposition 2.1.42. *Let $S \in \mathcal{S}(L)$ be dense in a locale L and $A \in \mathcal{S}(L)$ (resp. $A \in \mathcal{S}(L \smallsetminus S)$). The following statements are equivalent.*

1. *$A \in \mathcal{S}_{rem}(L \ltimes S)$ (resp. $A \in {}^*\mathcal{S}_{rem}(L \ltimes S)$).*

2. *$A \cap \overline{N} = \mathsf{O}$ for each S-closed and S-nowhere dense N.*

3. *For all S-dense $x \in S$, $A \cap \mathfrak{c}(x) = \mathsf{O}$.*

4. *For all S-dense $x \in S$, $A \subseteq \mathfrak{o}(x)$.*

5. *For every S-dense $x \in S$, $\nu_A(x) = 1$.*

Proof. (2) $\Longrightarrow$ (3): Let $x \in S$ be S-dense. Then $\mathfrak{c}_S(x)$ is S-closed and S-nowhere dense. By (2), $\mathsf{O} = A \cap \overline{\mathfrak{c}_S(x)} = A \cap \mathfrak{c}(x)$.

(3) $\Longrightarrow$ (2): If $N \in \mathcal{S}(S)$ is S-closed and S-nowhere dense, then $\bigwedge N$ is S-dense. It follows from (3) that $\mathsf{O} = A \cap \mathfrak{c}\left(\bigwedge N\right) = A \cap \overline{N}$ which proves the result. □

Comment 2.1.43. The equivalence of (4) in Proposition 2.1.42 tells us that, A is remote (resp. *remote) from S if and only if it is contained in every open dense sublocale of L induced by an element of S. This is reminiscent of the characterization of the remote sublocales of L as precisely those that are contained in every dense sublocale of L.

Proposition 2.1.42 leads us to another example of a sublocale of L which is remote from a dense sublocale S.

Example 2.1.44. If S is a dense sublocale of L, then the sublocale $L \smallsetminus \overline{\mathrm{Nd}(S)}$ is remote from S. To see this, choose an S-dense $x \in S$, then $\mathfrak{c}_S(x)$ is S-nowhere dense and contained in $\mathrm{Nd}(S)$, so that $\mathfrak{c}(x) = \overline{\mathfrak{c}_S(x)} \subseteq \overline{\mathrm{Nd}(S)}$. Therefore $\mathfrak{c}(x) \cap (L \smallsetminus \overline{\mathrm{Nd}(S)}) = \mathsf{O}$. By Proposition 2.1.42(3), $L \smallsetminus \overline{\mathrm{Nd}(S)}$ is remote from S.

Concerning the above example, there is a case where $L \smallsetminus \overline{\mathrm{Nd}(S)}$ is different from O. Firstly, recall from [15] that if $\mathfrak{c}_S(a)$ is nowhere dense in a dense sublocale S of L, then $\mathfrak{c}(a)$ is nowhere dense in L. Now, consider a dense sublocale $S \in \mathcal{S}(L)$ where $\mathrm{Nd}(S)$ is S-nowhere dense and $L \neq \mathsf{O}$ (for instance, a locale whose Booleanization is complemented, see [25, Corollary 4.16]). Since $\mathrm{Nd}(S)$ is the largest S-nowhere dense sublocale and its closure in S is S-nowhere dense, $\mathrm{Nd}(S) = \overline{\mathrm{Nd}(S)}^{S}$ making it S-closed nowhere dense. Because S is dense in L, $\mathrm{Nd}(S)$ is nowhere dense in L so that $\overline{\mathrm{Nd}(S)}$ is nowhere dense in L. Therefore $L \neq \overline{\mathrm{Nd}(S)}$ which means that $L \smallsetminus \overline{\mathrm{Nd}(S)} \neq \mathsf{O}$.

Just like in Proposition 2.1.20, from Proposition 2.1.42 one can deduce that every locale has the largest sublocale which is remote (resp. *remote) from a given dense sublocale. We formalise this deduction in the following proposition.

Proposition 2.1.45. *Let $S \in \mathcal{S}(L)$ be dense in a locale L. The join of sublocales of L remote (resp. *remote) from S is remote (resp. *remote) from S.*

For $S \in \mathcal{S}(L)$ dense in L, set

$$\mathrm{Rs}(L \ltimes S) = \bigvee \{A \in \mathcal{S}(L) : A \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)\}$$

and

$$^{*}\mathrm{Rs}(L \ltimes S) = \bigvee \{A \in \mathcal{S}(L) : A \in {}^{*}\mathcal{S}_{\mathrm{rem}}(L \ltimes S)\}.$$

Observation 2.1.46. (1) Since, according to Example 2.1.39(3), $\mathfrak{B}L$ is remote from S, we have that $\mathfrak{B}L \subseteq \mathrm{Rs}(L \ltimes S)$, which makes $\mathrm{Rs}(L \ltimes S)$ a dense sublocale of L.

(2) In any locale L, $^{*}\mathrm{Rs}(L \ltimes \mathfrak{B}L) = L \smallsetminus \mathfrak{B}L$. This follows since, by Proposition 2.1.37(2), $L \smallsetminus \mathfrak{B}L \in {}^{*}\mathcal{S}_{\mathrm{rem}}(L \ltimes \mathfrak{B}L)$ making $L \smallsetminus \mathfrak{B}L \subseteq {}^{*}\mathrm{Rs}(L \ltimes \mathfrak{B}L)$. Also all *remote sublocales (including $^{*}\mathrm{Rs}(L \ltimes \mathfrak{B}L)$) belong to the set $\{T \in \mathcal{S}(L) : T \subseteq L \smallsetminus \mathfrak{B}L\}$.

We noticed in Example 2.1.44 that $L \smallsetminus \overline{\mathrm{Nd}(S)}$ is remote from a dense sublocale S of L. In the following result, we give a necessary and sufficient condition for it to be precisely $\mathrm{Rs}(L \ltimes S)$. Let us start by recalling that an extension map sends dense elements to dense elements. This was stated in [10] and later proved in [12] in terms of weakly open maps, which we shall discuss in the next chapter.

Proposition 2.1.47. *Let S be a dense sublocale of a locale L. The following statements are equivalent.*

1. $\mathrm{Rs}(L \ltimes S) = L \smallsetminus \overline{\mathrm{Nd}(S)}$.
2. $\mathrm{Nd}(S)$ *is S-nowhere dense.*

Proof. (1) $\Longrightarrow$ (2): Assume that $\mathrm{Rs}(L \ltimes S) = L \smallsetminus \overline{\mathrm{Nd}(S)}$. Since $\mathfrak{B}L \subseteq \mathrm{Rs}(L \ltimes S)$, we have that $\mathfrak{B}L \subseteq L \smallsetminus \overline{\mathrm{Nd}(S)}$. Therefore $\mathfrak{B}L \cap \overline{\mathrm{Nd}(S)} = \mathsf{O}$ which implies that $\mathfrak{B}L \cap \mathrm{Nd}(S) = \mathsf{O}$. Therefore $\mathrm{Nd}(S)$ is nowhere dense making $\bigwedge \mathrm{Nd}(S)$ dense in L. But S is dense, so ν_S is an extenstion so that $\nu_S(\bigwedge \mathrm{Nd}(S)) = \bigwedge \mathrm{Nd}(S)$ is S-dense. By Lemma 2.1.3, $\mathrm{Nd}(S)$ is S-nowhere dense.

(2) $\Longrightarrow$ (1): Since $L \smallsetminus \overline{\mathrm{Nd}(S)} \subseteq \mathrm{Rs}(L \ltimes S)$, we need to only show that $\mathrm{Rs}(L \ltimes S) \subseteq L \smallsetminus \overline{\mathrm{Nd}(S)}$. $\mathrm{Nd}(S)$ being S-nowhere dense implies that $\mathrm{Rs}(L \ltimes S) \cap \overline{\mathrm{Nd}(S)} = \mathsf{O}$. This gives $\mathrm{Rs}(L \ltimes S) \subseteq L \smallsetminus \overline{\mathrm{Nd}(S)}$. □

We observed above that $\mathcal{S}_{\text{rem}}(L) = \mathcal{S}_{\text{rem}}(L \ltimes L)$. In Chapter 5, we will give a case where the equality $\mathcal{S}_{\text{rem}}(L) = \mathcal{S}_{\text{rem}}(L \ltimes S)$ holds even for sublocales $S \neq L$. For now, we only establish a relationship between $\mathcal{S}_{\text{rem}}(S)$ and $\mathcal{S}_{\text{rem}}(L \ltimes S)$ which, incidentally, is one of the main results within this section.

Proposition 2.1.48. *Let S be a dense sublocale of a locale L. Then*

$$\mathcal{S}(S) \cap \mathcal{S}_{rem}(L \ltimes S) = \mathcal{S}_{rem}(S).$$

Proof. $\mathcal{S}(S) \cap \mathcal{S}_{\text{rem}}(L \ltimes S) \subseteq \mathcal{S}_{\text{rem}}(S)$: Let $A \in \mathcal{S}(S) \cap \mathcal{S}_{\text{rem}}(L \ltimes S)$ and choose an S-nowhere dense $N \in \mathcal{S}(S)$. Then $A \cap \overline{N} = \mathsf{O}$ which implies that $A \cap N = \mathsf{O}$. Thus $A \in \mathcal{S}_{\text{rem}}(S)$.

$\mathcal{S}_{\text{rem}}(S) \subseteq \mathcal{S}(S) \cap \mathcal{S}_{\text{rem}}(L \ltimes S)$: Let $A \in \mathcal{S}_{\text{rem}}(S)$ and choose an S-nowhere dense N. Then $\overline{N}^S$ is S-nowhere dense so that $\mathsf{O} = A \cap \overline{N}^S = A \cap \overline{N} \cap S = A \cap \overline{N}$. Thus $A \in \mathcal{S}(S) \cap \mathcal{S}_{\text{rem}}(L \ltimes S)$. □

Below, we consider a relationship between $\text{Rem}(L \ltimes S)$ and $\text{Rem}(S)$. We start by characterizing members of $\text{Rem}(L \ltimes S)$ just like we did in Proposition 2.1.28. We also include the characterization of members of $^*\text{Rem}(L \ltimes S)$.

Proposition 2.1.49. *Let $S \in \mathcal{S}(L)$ be dense.*

1. *$a \in \text{Rem}(L \ltimes S)$ iff $a \vee x = 1$ for all dense $x \in S$.*

2. *For $\mathfrak{c}(a) \subseteq L \smallsetminus S$, $a \in {}^*\text{Rem}(L \ltimes S)$ iff $a \vee x = 1$ for all dense $x \in S$.*

We also note the following proposition about members of the set $\text{Rem}(L \ltimes S)$.

Proposition 2.1.50. *Let $S \in \mathcal{S}(L)$ be dense. If $a \in S \cap \text{Rem}(L \ltimes S)$, then $x^* \leq a$ implies $a \vee x = 1$ for all $x \in S$.*

Proof. Let $a \in S \cap \text{Rem}(L \ltimes S)$ and choose $x \in S$ such that $x^* \leq a$. Since $x \vee x^*$ is dense in L and ν_S is an extension, $\nu_S(x \vee x^*) = \nu_S(x) \vee \nu_S(x^*)$ is dense in S. It follows from Proposition 2.1.49(1) that $a \vee \nu_S(x) \vee \nu_S(x^*) = 1$. But $x^* \leq a$, so $\nu_S(x^*) \leq \nu_S(a)$ implying that

$$1 = a \vee \nu_S(x) \vee \nu_S(a) = \nu_S(x) \vee \nu_S(a) = x \vee a,$$

where the latter equality holds since $x, a \in S$. □

Observation 2.1.51. The preceding result also holds for $a \in {}^*\mathrm{Rem}(L \ltimes S) \cap S$. When S is complemented, we get that $S \cap (L \smallsetminus S) = \mathsf{O}$ which implies $a = 1$ making the statement of Proposition 2.1.50 trivial for the case of ${}^*\mathrm{Rs}(L \ltimes S)$.

For the following proof, we need a condition on a sublocale S such that the localic embedding $i : S \hookrightarrow L$ is *uplifting* (according to [20], a localic map $f : L \to M$ is uplifting if $x \vee y = 1$ implies $f(x) \vee f(y) = 1$). This is rather a weaker condition of a *flat sublocale* which was defined by Johnstone in [39] as a sublocale of a locale which is a sublattice of the locale. So, we shall say that S is *weakly flat* in case the localic embedding $i : S \hookrightarrow L$ is uplifting.

Proposition 2.1.52. *Let $S \in \mathcal{S}(L)$ be dense and weakly flat. Then* $S \cap \mathrm{Rem}(L \ltimes S) = \mathrm{Rem}(S)$.

Proof. Observe that

$$\begin{aligned} x \in S \cap \mathrm{Rem}(L \ltimes S) \quad &\iff \quad x \in S \text{ and } x \vee y = 1 \text{ for all } S\text{-dense } y \in S \\ &\iff \quad x \in S \text{ and } x \vee_S y = 1 \text{ for all } S\text{-dense } y \in S, \\ &\qquad\quad \text{since } S \text{ is weakly flat} \\ &\iff \quad x \in \mathrm{Rem}(S), \end{aligned}$$

which proves the result. □

2.2 Algebraic approach to remoteness

We now wish to consider an algebraic notion of remoteness motivated by remoteness of sublocales.

Recall from Theorem 2.1.18 that a sublocale A of L is remote precisely when the corresponding frame surjection $\nu_A : L \to A$ sends every dense element to the top. This motivates the following definition which applies to frame homomorphisms which are not necessarily quotient maps.

Definition 2.2.1. We say a frame homomorphism $h : M \to L$ is *heavy* if $h(a) = 1$ for every dense $a \in M$.

For the following characterization of heavy quotient maps, we start by recalling from [18] that a quotient map $g : M \to L$ is nowhere dense if and only if $g_*(0_L)$ is dense in M.

Proposition 2.2.2. *Let $h : M \to L$ be a quotient map. The following statements are equivalent.*

1. *h is heavy.*
2. *$h_*[L]$ is a remote sublocale of M.*
3. $\mathrm{Fix}(h_* h) \subseteq \bigwedge\{\mathrm{Fix}(\nu_{\mathfrak{o}(x)}) : x$ *is dense in* $M\}$.
4. *$h_*[L] \cap \alpha_*[N] = \mathsf{O}$ for every nowhere dense quotient map $\alpha : M \to N$.*

Proof. (1) $\Longrightarrow$ (2): Let $a \in M$ be dense in M. We show that $\nu_{h_*[L]}(a) = 1$. Let $x \in h_*[L]$ be such that $a \leq x$. Then $h(a) \leq h(x)$. Since h is heavy, $h(a) = 1$ so that $h(x) = 1$. Therefore $x = 1$ making $\bigwedge\{z \in h_*[L] : a \leq z\} = \nu_{h_*[L]}(a) = 1$. It follows from Theorem 2.1.18(4) that $h_*[L]$ is remote.

(2) $\Longrightarrow$ (3): Let $y \in \mathrm{Fix}(h_* h)$ and z be a dense element of M. Then $y = h_*(h(y))$, making $y \in h_*[L]$. Since $h_*[L]$ is a remote sublocale of M and $\mathfrak{o}(z)$ is a dense sublocale of M, by Theorem 2.1.18, $h_*[L] \subseteq \mathfrak{o}(z)$. We get that $y \in \mathfrak{o}(z)$, i.e., $y = z \to y = \nu_{\mathfrak{o}(z)}(y)$. Thus $y \in \bigwedge\{\mathrm{Fix}(\nu_{\mathfrak{o}(x)}) : x$ is dense in $M\}$.

(3) $\Longrightarrow$ (4): Let $\alpha : M \to N$ be a nowhere dense quotient map. Then $\alpha_*(0_N)$ is dense in M. Choose $x \in h_*[L] \cap \alpha_*[N]$. Then $x = h_*(b)$ and $x = \alpha_*(c)$ for some $b \in L$ and $c \in N$. Therefore $x = h_*(b) = \alpha_*(c) \geq \alpha_*(0_N)$ making x dense in M. Since $h_*(h(h_*(h(a)))) = h_*(h(a))$ for every $a \in M$, $h_*(h(x)) \in \mathrm{Fix}(h_* h)$. It follows from condition (3) that $h_*(h(x)) \in \mathrm{Fix}(\nu_{\mathfrak{o}(x)})$, i.e., $\nu_{\mathfrak{o}(x)}(h_*(h(x))) = h_*(h(x))$. Therefore $x \to h_*(h(x)) = h_*(h(x))$ which implies that

$$h_*(h(x) \to h(x)) = h_*(h(x)).$$

But $a \to a = 1$ for every a, so $h_*(1) = h_*(h(x) \to h(x)) = h_*(h(x))$ which implies $h(x) = 1$. Therefore $x = 1$ so that $h_*[L] \cap \alpha_*[N] = \mathsf{O}$ as required.

(4) $\Longrightarrow$ (1): Let $x \in M$ be dense. Then $\mathfrak{c}(x)$ is M-nowhere dense which implies that the left adjoint $i^* : M \to \mathfrak{c}(x)$ of the localic embedding map $i : \mathfrak{c}(x) \hookrightarrow M$ is a nowhere dense

quotient map. By (4), $\mathsf{O} = h_*[L] \cap (i^*)_*[\mathfrak{c}(x)] = h_*[L] \cap \mathfrak{c}(x)$. Observe that $h_*(h(x)) \in h_*[L]$ and $h_*(h(x)) \in \mathfrak{c}(x)$ where the latter statement follows since $a \leq h_*(h(a))$ for every $a \in M$. Therefore $h_*(h(x)) = 1$ making $h(x) = 1$. Thus h is heavy. □

Remark 2.2.3. We note that, since a heavy frame homomorphsim $h : M \to L$ sends dense elements to the top element, then $x \vee x^*$ being dense implies that $1 = h(x \vee x^*) = h(x) \vee h(x^*)$ for all $x \in M$. Since $h(x) \wedge h(x^*) = h(x \wedge x^*) = 0$, $h(x)$ is a complemented element. Therefore h sends every element to a complemented element. It follows from [39] that there is a unique frame homomorphism $k : \mathcal{S}(M)^{\mathrm{op}} \to L$ such that $k \circ \mathfrak{c} = h$, where $\mathfrak{c} : M \to \mathcal{S}(M)^{\mathrm{op}}$ is given by $a \mapsto \mathfrak{c}(a)$.

We close this section with an introduction and a discussion of some variant of heavy homomorphisms in line with the definition of sublocales that are remote from dense sublocales which was introduced in Definition 2.1.33. We showed in Proposition 2.1.42 that for any dense sublocale S of a locale L, a sublocale A of L is remote from S if and only if the corresponding quotient map $\nu_A : L \to A$ sends all S-dense elements to the top. This motivates the following definition.

Definition 2.2.4. Let $h : M \to L$ be a frame homomorphism. A frame homomorphism $g : M \to T$ is said to be *h-weakly heavy* in case $g(h_*(a)) = 1$ for every L-dense a.

We shall drop the prefix h- when there is no danger of confusion.

We consider a relationship between heavy frame homomorphisms and weakly heavy frame homomorphisms. The first result explores heaviness on a single frame homomorphism.

Proposition 2.2.5. *Let $h : M \to L$ be a frame homomorphism.*

1. *If h_* sends dense elements to dense elements, then h is heavy implies h is weakly heavy.*

2. *If h is weakly open, then h is weakly heavy implies h is heavy.*

Proof. (1) Let $x \in L$ be L-dense. Since h_* sends dense elements to dense elements, $h_*(x)$ is dense in M. It follows that $h(h_*(x)) = 1$ making h a weakly heavy frame homomorphism.

(2) Let $x \in M$ be dense. The weakly openness of h implies that $h(x)$ is L-dense. Therefore $h(h_*(h(x))) = 1$ so that $h(x) = 1$. Thus h is heavy. □

Observation 2.2.6. We also get that if a frame homomorphism $h : M \to L$ is heavy and h_* sends dense elements to dense elements, then L is Boolean. To verify this, let $x \in L$ be dense. We show that $x = 1$. Since h_* sends dense elements to dense elements, $h_*(x)$ is dense in M. Heaviness of h implies $h(h_*(x)) = 1$ making $x = 1$. Thus L is Boolean.

Proposition 2.2.7. *Let $h : M \to L$ and $g : M \to T$ be frame homomorphisms.*

1. *If h_* sends dense elements to dense elements, then h is heavy implies g is weakly heavy.*
2. *If h is injective and weakly open, then g is weakly heavy implies g is heavy.*

Proof. (1) Let $x \in L$ be L-dense. Since, by Observation 2.2.6, L is Boolean, $x = 1$. Therefore $g(h_*(x)) = g(1) = 1$ making g weakly heavy.

(2) Let $x \in M$ be dense. Then $h(x)$ is dense in L by weakly openness of h. Since g is weakly heavy, $g(h_*(h(x))) = 1$. Because h is injective, $1 = g(x) = g(h_*(h(x)))$. Thus g is heavy. □

Next, we characterize weakly heavy quotient maps. The proof follows a similar sketch of the proof of Proposition 2.2.2 taking into account that if $h : M \to L$ is a quotient map, then $h_* : L \to h_*[L]$ is a frame isomorphism, and hence sends dense elements back and forth. We shall only prove the equivalence of (1) and (2).

Proposition 2.2.8. *Let $h : M \to L$ be an extension and consider a quotient map $g : M \to T$. The following statements are equivalent.*

1. *g is h-weakly heavy.*
2. *$g_*[T]$ is remote from $h_*[L]$.*
3. $\mathrm{Fix}(g_*g) \subseteq \bigwedge\{\mathrm{Fix}(\nu_{\mathfrak{o}(x)}) : x$ *is dense in* $h_*[L]\}$.
4. *For every nowhere dense quotient $\alpha : L \to N$, $\mathfrak{c}[(\alpha h)_*(0_N)] \cap g_*[T] = \mathsf{O}$.*

Proof. (1) $\Longrightarrow$ (2): Let $a \in h_*[L]$ be dense in $h_*[L]$. We show that $\nu_{g_*[T]}(a) = 1$. Choose $x \in g_*[T]$ such that $a \leq x$. Then $a = h_*(y)$ for some $y \in L$. Such y is clearly dense in L.

We get that $g(a) = g(h_*(y)) \leq g(x)$. Since g is weakly heavy, $1 = g(h_*(y)) = g(a)$ so that $g(x) = 1$. Therefore $x = 1$ which implies $\nu_{g_*[T]}(a) = 1$. Thus $g_*[T]$ is remote from $h_*[L]$ by Proposition 2.2.2(4).

(2) $\implies$ (1): Let $x \in L$ be L-dense. Since h is a quotient map, $h_* : L \to h_*[L]$ is a frame isomorphism making $h_*(x)$ dense in $h_*[L]$. By condition (2), $\nu_{g_*[T]}(h_*(x)) = 1$. Since $g_*(g(h_*(x))) \in g_*[T]$ and $h_*(x) \leq g_*(g(h_*(x)))$, $g_*(g(h_*(x))) \in \{t \in g_*[T] : h_*(x) \leq t\}$ so that

$$1 = \bigwedge \{t \in g_*[T] : h_*(x) \leq t\} = \nu_{g_*[T]}(h_*(x)) \leq g_*(g(h_*(x))).$$

Therefore $g(h_*(x)) = g(g_*(g(h_*(x)))) = 1$ making g weakly heavy. □

2.3 Remoteness in binary coproducts

We close this chapter with a short discussion of remoteness in binary coproducts. We have expended a great deal of effort in trying to prove the result we present below for arbitrary coproducts, but our efforts were not successful. The main difficulty seems to emanate from the behaviour of coproduct injections in arbitrary coproducts; even coproducts of countably many locales. We thus leave the general case as an (at the moment) unresolved matter, which we plan to pursue in another undertaking.

Our focus will be on members of the set $\mathrm{Rem}(L \oplus M)$. We start by recalling from preliminaries that a frame homomorphism $h : M \to L$ is closed if $f(a \vee h(b)) = f(a) \vee b$ for every $a \in L$, $b \in M$. We also recall from [21, Proposition 4.3.] that if an element $a \in L$ is dense in L, then $a \oplus 1$ is dense in $L \oplus M$. To see this, using the fact that $(x \oplus y)^* = (x^* \oplus 1) \vee (1 \oplus y^*)$ for all $x \in L$, $y \in M$ (from [10]), we get that

$$(a \oplus 1)^* = (a^* \oplus 1) \vee (1 \oplus 1^*) = (0 \oplus 1) \vee (1 \oplus 0) = 0_{L \oplus M}.$$

Proposition 2.3.1. *Let L and M be locales such that the coproduct injection $L \xrightarrow{q_L} L \oplus M$ is closed. If for any $x \in L$ and $y \in M$, $x \oplus y \in \mathrm{Rem}(L \oplus M)$, then $x \in \mathrm{Rem}(L)$. Consequently, if $L \xrightarrow{q_L} L \oplus M \xleftarrow{q_M} M$ are closed coproduct injections, then for any $x \in L$ and $y \in M$, $x \oplus y \in \mathrm{Rem}(L \oplus M)$ implies $x \in \mathrm{Rem}(L)$ and $y \in \mathrm{Rem}(M)$.*

Proof. Let $x \in L$, $y \in M$ and assume that $x \oplus y \in \mathrm{Rem}(L \oplus M)$. For any dense $a \in L$, we get that $q_L(a) = a \oplus 1$ is dense in $L \oplus M$. Since $x \oplus y \in \mathrm{Rem}(L \oplus M)$, we get that $(x \oplus y) \vee (a \oplus 1) = 1_{L \oplus M}$, which implies that $(x \oplus y) \vee q_L(a) = 1_{L \oplus M}$. Therefore in light of q_L being a closed homomorphism, $(q_L)_*(x \oplus y) \vee a = 1$. If $a = 1$, then we are done. If $a \neq 1$, then

$$(q_L)_*(x \oplus y) = \bigvee\{b \in L : q_L(b) \leq x \oplus y\} = \bigvee\{b \in L : b \oplus 1 \leq x \oplus y\} \neq 0.$$

Therefore there exists $b \neq 0$ in L such that $0_{L \oplus M} \neq b \oplus 1 \leq x \oplus y$, which implies that $b \leq x$ and $y = 1$. Therefore

$$1 = (q_L)_*(x \oplus y) \vee a = (q_L)_*(x \oplus 1) \vee a = (q_L)_*(q_L(x)) \vee a = x \vee a,$$

where the latter equality holds since q_L is injective. Thus $x \in \mathrm{Rem}(L)$. Similarly, $y \in \mathrm{Rem}(M)$ whenever q_M is closed. □

Remark 2.3.2. We note that there are cases where the coproduct injections are closed. For instance, according to [13], the injection q_L is closed whenever either M is compact or L is a coframe.

Chapter 3

Preservation and Reflection of Remoteness

The work presented in this chapter is part of the research paper: M.S. Nxumalo, *On sublocales that miss every nowhere dense sublocale*, Quaest. Math., (2023)(Under Review).

3.1 Pushing forward and pulling back of remote sublocales

In this section, we focus on describing localic maps that send the sublocales introduced in Definition 2.1.2 back and forth. We shall say that a localic map preserves sublocales with property P if it takes a sublocale with property P to a sublocale with property P.

We begin this section by considering a localic map f such that $f[-]$ preserves remote sublocales.

Recall from [11] that a frame homomorphism $h : M \to L$ is *weakly open* (or *skeletal*) if $h(x^{**}) \leq h(x)^{**}$ for every $x \in M$. We shall frequently make use of the following equivalent condition of a weakly open map which was proved in the cited paper:

h sends dense elements to dense elements.

We have the following characterization of weakly open maps in terms of nowhere dense sublocales.

Lemma 3.1.1. *A frame homomorphism $h : M \to L$ is weakly open iff $f_{-1}[-]$ sends nowhere dense sublocales to nowhere dense sublocales.*

Proof. ($\Longrightarrow$): Assume that $h : M \to L$ is weakly open and let $N \in \mathcal{S}(M)$ be nowhere dense. By Lemma 2.1.3, $\bigwedge N$ is dense. The hypothesis implies that $h\left(\bigwedge N\right)$ is dense in L. Therefore $\mathfrak{c}\left(h\left(\bigwedge N\right)\right) = f_{-1}[\overline{N}]$ is nowhere dense. But $f_{-1}[N] \subseteq f_{-1}[\overline{N}]$, so $f_{-1}[N]$ is nowhere dense.

($\Longleftarrow$): Let $x \in M$ be dense. Then $\mathfrak{c}(x)$ is nowhere dense. By hypothesis, $f_{-1}[\mathfrak{c}(x)]$ is nowhere dense. But $f_{-1}[\mathfrak{c}(x)] = \mathfrak{c}(h(x))$, so $h(x)$ is dense in L, as required. □

We note that the forward implication of the previous lemma provides a different way of proving the result by Stephen in her doctoral thesis [57, Lemma 4.2.20(2)], where she showed that, if h is dense and onto, then $f_{-1}[-]$ preserves nowhere dense sublocales. This follows since, according to [10], every dense and onto frame homomorphism is weakly open.

We now characterize localic maps whose localic image functions preserve remote sublocales. As in [50], if S is a sublocale of L, we shall write $j_S : S \to L$ for the mapping $x \mapsto x$.

Theorem 3.1.2. *Let $f : L \to M$ be a localic map. The following statements are equivalent:*

1. *$f[-]$ preserves remote sublocales.*
2. *$f[\mathfrak{B}L]$ is a remote sublocale of M.*
3. *$f_{-1}[-]$ preserves nowhere dense sublocales.*
4. *$f_{-1}[-]$ preserves closed nowhere dense sublocales.*
5. *h is weakly open.*
6. *There is a unique localic map $\mathfrak{B}f : \mathfrak{B}L \to \mathfrak{B}M$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{B}L & \xrightarrow{\mathfrak{B}f} & \mathfrak{B}M \\ {\scriptstyle j_{\mathfrak{B}L}}\downarrow & & \downarrow{\scriptstyle j_{\mathfrak{B}M}} \\ L & \xrightarrow[f]{} & M \end{array} \tag{3.1.1}$$

commutes.

Proof. (1) $\Longrightarrow$ (2): Since $\mathfrak{B}L \in \mathcal{S}(L)$ is remote, it follows from (1) that $f[\mathfrak{B}L] \in \mathcal{S}(M)$ is remote.

(2) $\Longrightarrow$ (3): If N is nowhere dense in M, then $f[\mathfrak{B}L] \cap N = \mathsf{O}$ since $f[\mathfrak{B}L]$ is remote, by (2). Therefore

$$\mathsf{O} = f_{-1}[\mathsf{O}] = f_{-1}[f[\mathfrak{B}L] \cap N] = f_{-1}[f[\mathfrak{B}L]] \cap f_{-1}[N] \supseteq \mathfrak{B}L \cap f_{-1}[N],$$

making $f_{-1}[N]$ nowhere dense in L.

(3) $\Longrightarrow$ (4): Trivial.

(4) $\Longrightarrow$ (5): Similar to Lemma 3.1.1.

(5) $\Longleftrightarrow$ (6): Follows from [10, Proposition 1.1.].

(6) $\Longrightarrow$ (1): Let $A \in \mathcal{S}_{\text{rem}}(L)$. If $\mathfrak{o}(a)$ is dense in M, then

$$\begin{aligned}
\mathfrak{B}M \subseteq \mathfrak{o}(a) \quad &\Longrightarrow \quad \mathfrak{B}f[\mathfrak{B}L] \subseteq \mathfrak{o}(a) \quad \text{since} \quad \mathfrak{B}f[\mathfrak{B}L] \subseteq \mathfrak{B}M \\
&\Longrightarrow \quad j_{\mathfrak{B}M}[\mathfrak{B}f[\mathfrak{B}L]] \subseteq \mathfrak{o}(a) \\
&\Longrightarrow \quad f[j_{\mathfrak{B}L}[\mathfrak{B}L]] \subseteq \mathfrak{o}(a) \quad \text{by commutativity of diagram 3.1.1} \\
&\Longrightarrow \quad f[\mathfrak{B}L] \subseteq \mathfrak{o}(a) \\
&\Longrightarrow \quad \mathfrak{B}L \subseteq f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(h(a)).
\end{aligned}$$

We get that the open sublocale $f_{-1}[\mathfrak{o}(a)]$ is dense in L, so $A \subseteq f_{-1}[\mathfrak{o}(a)]$ since A is remote. Therefore $f[A] \subseteq \mathfrak{o}(a)$, showing that $f[A]$ is contained in every open dense sublocale, and therefore is remote. □

Remark 3.1.3. In [38], Johnstone gives other characterizations (in terms of sublocales) of localic maps whose left adjoints are weakly open homomorphisms. None of the characterizations in the foregoing theorem appears in Johnstone's cited paper except that condition (2) in Theorem 3.1.2 is a re-wording of Johnstone's condition that states that f restricts to a localic map $\mathfrak{B}L \to \mathfrak{B}M$.

We include a case where a localic map sends members of $\text{Rem}(L)$ to members of $\text{Rem}(M)$. Recall from [47] that a frame homomorphism $h : M \to L$ is said to be *weakly closed* in case $a \vee h(b) = 1$ implies $f(a) \vee b = 1$ for every $a \in L$ and $b \in M$. This is clearly a weakening of the condition that defines closed homomorphisms, but agrees with it if the domain of the homomorphism is regular, as shown in [18].

Proposition 3.1.4. *Let $f : L \to M$ be a localic map. If h is weakly open and weakly closed, then $f[\mathrm{Rem}(L)] \subseteq \mathrm{Rem}(M)$.*

Proof. Let $x \in \mathrm{Rem}(L)$ and choose a dense $y \in M$. By hypothesis, $h(y)$ is dense in L, making $x \vee h(y) = 1$. But h is weakly closed, so $f(x) \vee y = 1$. Thus $f(x) \in \mathrm{Rem}(M)$. □

In the following result we show that if the left adjoint of an injective localic map is heavy, then the localic map preserves remote sublocales.

Proposition 3.1.5. *Let $f : L \to M$ be an injective localic map whose left adjoint is heavy. Then $f[-]$ preserves remote sublocales. Moreover, if L is Boolean, then h is heavy iff $f[-]$ preserves remote sublocales.*

Proof. Assume that $h : M \to L$ is heavy. It follows from Proposition 2.2.2 that $f[L]$ is a remote sublocale of M. Therefore the localic image of every sublocale of L (including remote sublocales of L) under f is a remote sublocale of M. Thus $f[-]$ preserves remote sublocales.

For the special case, if L is Boolean, then $L = \mathfrak{B}L$. So, $f[-]$ preserves remote sublocales implies $f[L] = f[\mathfrak{B}L]$ is a remote sublocale of M, making h heavy. □

In the special case of the preceding proposition we saw the impact of a Boolean domain of a localic map. In the following proposition, we consider a Boolean codomain of a localic map.

Proposition 3.1.6. *If $f : L \to M$ is a localic map and M is Boolean, then $f[-]$ preserves remote sublocales.*

Proof. Follows since in a Boolean locale, every sublocale is remote. □

We move to giving conditions on localic maps such that their localic preimage functions preserve remote sublocales. En route to that, we record the following result.

Lemma 3.1.7. *Let $f : L \to M$ be a localic map. Then f sends dense elements to dense elements iff $f[-]$ preserves nowhere dense sublocales.*

Proof. ($\Longrightarrow$): Suppose that f sends dense elements to dense elements and let N be a nowhere dense sublocale of L. Then $\bigwedge N$ is dense making $f[\bigwedge N] = \bigwedge f[N]$ dense in M, by hypothesis. So, by Lemma 2.1.3, $\mathfrak{c}\left(\bigwedge f[N]\right)$ is a nowhere dense sublocale of M. Because $\mathfrak{c}\left(\bigwedge f[N]\right) = \overline{f[N]} \supseteq f[N]$, we get that $f[N]$ is a nowhere dense sublocale of M.

($\Longleftarrow$): Recall that $\overline{f[\mathfrak{c}(a)]} = \mathfrak{c}(f(a))$ for every $a \in L$, because

$$\begin{aligned} \overline{f[\mathfrak{c}(a)]} &= \mathfrak{c}\Big(\bigwedge f[\mathfrak{c}(a)]\Big) \\ &= \mathfrak{c}\left(f\left(\bigwedge \mathfrak{c}(a)\right)\right) \text{ because } f \text{ preserves meets} \\ &= \mathfrak{c}(f(a)). \end{aligned}$$

Now, if $x \in L$ is dense, then $\mathfrak{c}(x)$ is a nowhere dense sublocale of L which, by hypothesis, implies that $f[\mathfrak{c}(x)]$ is a nowhere dense sublocale of M. But the closure of a nowhere dense sublocale is nowhere dense, so $\overline{f[\mathfrak{c}(x)]} = \mathfrak{c}(f(x))$ is nowhere dense, making $f(x)$ dense. □

The proof of Proposition 3.1.8(1) below follows since the preimage function $f_{-1}[-]$ of a localic map f preserves meets and O. We only prove Proposition 3.1.8(2) since the proof of (1) is along the lines of the proof of the implication (2) $\Longrightarrow$ (3) in Theorem 3.1.2.

Proposition 3.1.8. *Let $f : L \to M$ be a localic map that sends dense elements to dense elements. Then:*

1. *The preimage function $f_{-1}[-] : \mathcal{S}(M) \to \mathcal{S}(L)$ preserves remote sublocales.*

2. *$h(\mathrm{Rem}(M)) \subseteq \mathrm{Rem}(L)$.*

Proof. (2) Let $y \in \mathrm{Rem}(M)$ and choose a dense $x \in L$. By hypothesis, $f(x)$ is dense in M. Therefore $y \vee f(x) = 1$, making $1 = h(y \vee f(x)) = h(y) \vee h(f(x)) \leq h(y) \vee x$. Thus $h(y) \in \mathrm{Rem}(L)$. □

We give some examples.

Example 3.1.9. (1) Let N be a nowhere dense sublocale of L. The map $j_N : N \to L$ sends dense elements to dense elements. Indeed, if $x \in N$ is N-dense, then $\mathfrak{c}_N(x)$ is N-nowhere dense. Because every sublocale contained in a nowhere dense sublocale is nowhere dense, we have that $\mathfrak{c}_N(x)$ is nowhere dense in L. It follows that $\overline{\mathfrak{c}_N(x)}$ is nowhere dense in L, i.e., $\mathfrak{c}(x)$ is nowhere dense. Thus $x = j_N(x)$ is dense in L.

(2) Using Proposition 3.1.8(1), it follows that the localic preimage function $(j_N)_{-1}[-]$ induced by the localic embedding in (1) preserves remote sublocales. However, we can show this without relying on the fact that j_N sends dense elements to dense elements. Indeed, for each $A \in \mathcal{S}(N)$ and each $B \in \mathcal{S}_{\text{rem}}(L)$, we have that $B \cap A = \mathsf{O}$. Therefore

$$\mathsf{O} = (j_N)_{-1}[B \cap A] = (j_N)_{-1}[B] \cap (j_N)_{-1}[A] = (j_N)_{-1}[B] \cap A.$$

Thus $(j_N)_{-1}[B] \in \mathcal{S}_{\text{rem}}(N)$.

(3) In a given localic map $f : L \to M$ with Boolean locale M, L is Boolean if and only if $f_{-1}[M]$ is a remote sublocale of L. This follows since $L = f_{-1}[M]$.

(4) In **Top**, the author of [32] defined a continuous mapping from a space X onto a space Y as *closed irreducible* if the image under f of every proper closed subset of X is a proper closed subset of Y. The author further proved that if a continuous map $f : X \to Y$ is closed irreducible, then the image of every closed nowhere dense subset of X under f is closed and nowhere dense in Y. We claim that the preimage of a remote subset of Y under a closed irreducible continuous map $f : X \to Y$ is a remote subset of X. Indeed, let A be a remote subset of Y and F be nowhere dense in X. Then $\overline{F}$ is nowhere dense. It follows that $f[\overline{F}]$ is nowhere dense (and closed) in Y. Therefore $A \cap f[\overline{F}] = \emptyset$ so that $f^{-1}[A] \cap \overline{F} = \emptyset$. This implies that $f^{-1}[A] \cap F = \emptyset$. Thus $f^{-1}[A]$ is a remote subset of X.

We characterize remote-preserving localic maps below in terms of $\mathfrak{B}L$. We start by giving Proposition 3.1.10 below which shows that $\mathfrak{B}L$ is the largest remote sublocale of a locale L. Recall that in Chapter 2 we showed that the join of remote sublocales of any locale is a remote sublocale, hence every locale has the largest remote sublocale. Hence this proposition identifies $\mathfrak{B}L$ as the largest sublocale of L.

Proposition 3.1.10. *Let L be a locale. Then $\mathfrak{B}L$ is the largest remote sublocale of L.*

Proof. Let $A \in \mathcal{S}(L)$ be the largest remote sublocale of L which exists according to Proposition 2.1.20. Since $\mathfrak{B}L$ is remote, we have that $\mathfrak{B}L \subseteq A$, making A dense. We show that A is a

Boolean algebra. Since the map $j_A : A \to L$ sends dense elements to dense elements, it follows from Proposition 3.1.8 that $(j_A)_{-1}[A]$ is a remote sublocale of A. Since $A = (j_A)_{-1}[A]$, A is a remote sublocale of itself. It follows from Proposition 2.1.7 that A is Boolean. But $\mathfrak{B}L$ is the only dense and Boolean sublocale of L, so $A = \mathfrak{B}L$. □

Therefore we have the following proposition which characterizes remote sublocales in terms of the smallest dense sublocale.

Proposition 3.1.11. *A sublocale is remote if and only if it is contained in the smallest dense sublocale.*

Although the result in the foregoing proposition informs us where to locate all the remote sublocales of a given locale, it does not give a fuller description of them. We shall now describe these sublocales. Towards that end, we recall (from our main reference [50]) that for every $a \in L$, the set

$$\mathfrak{b}(a) = \{x \to a : x \in L\}$$

is a sublocale of L, and furthermore:

- $\mathfrak{b}(a)$ is the smallest sublocale of L containing a. "Smallest" here means that $\mathfrak{b}(a)$ is contained in every sublocale of L which contains a.
- A sublocale of L is Boolean if and only if it is of the form $\mathfrak{b}(x)$ for some $x \in L$.

Theorem 3.1.12. *For any locale L,*

$$\mathcal{S}_{rem}(L) = \{\mathfrak{b}(x^*) : x \in L\}.$$

Proof. Let A be a remote sublocale of L. Then by Proposition 3.1.11, $A \subseteq \mathfrak{B}L$. Since every sublocale of a Boolean locale is Boolean, A is a Boolean sublocale of L and so there is an $a \in L$ such that $A = \mathfrak{b}(a)$. Since $a \in \mathfrak{b}(a)$, it follows that $a \in \mathfrak{B}L$, and so $a = a^{**}$. This proves the containment $\subseteq$ in the claimed equality.

To reverse the containment, let $x \in L$ and consider the sublocale $\mathfrak{b}(x^*)$ of L. Since x^* belongs to $\mathfrak{B}L$ and $\mathfrak{b}(x^*)$ is the smallest sublocale containing x^*, we have $\mathfrak{b}(x^*) \subseteq \mathfrak{B}L$, and so

$\mathfrak{b}(x^*)$ is a remote sublocale of L by Proposition 3.1.11. This proves the containment $\supseteq$, and hence establishes the claimed equality. □

Observation 3.1.13. (1) Among other things, Proposition 3.1.10 tells us that a sublocale is nowhere dense if and only if it misses every remote sublocale, because

$$\begin{aligned} S \text{ is nowhere dense in } L \quad &\iff \quad S \cap \mathfrak{B}L = \mathsf{O} \\ &\iff \quad S \cap R = \mathsf{O} \text{ for every remote sublocale } R. \end{aligned}$$

(2) For a dense sublocale S of L we have $\mathrm{Rs}(L \ltimes S) \cap S = \mathfrak{B}L$. To see this, we have that $\mathfrak{B}L \subseteq \mathrm{Rs}(L \ltimes S) \cap S$ since both $\mathrm{Rs}(L \ltimes S)$ and S are dense in L, where density of $\mathrm{Rs}(L \ltimes S)$ was recorded in Observation 2.1.46. For the other containment, recall from Proposition 2.1.37 (6) that for a locale M and a dense sublocale T of M, $A \in \mathcal{S}_{\mathrm{rem}}(M \ltimes T)$ implies $A \cap T \in \mathcal{S}_{\mathrm{rem}}(M)$. Since $\mathrm{Rs}(L \ltimes S) \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)$, it follows that $\mathrm{Rs}(L \ltimes S) \cap S \in \mathcal{S}_{\mathrm{rem}}(L)$. Now, $\mathfrak{B}L$ being the largest remote sublocale of L gives $\mathrm{Rs}(L \ltimes S) \cap S \subseteq \mathfrak{B}L$. Thus $\mathrm{Rs}(L \ltimes S) \cap S = \mathfrak{B}L$.

Remark 3.1.14. In Chapter 2 we commented that for fit locales, the smallest dense sublocale of a locale is the only sublocale which is simultaneously dense and remote. At that stage, we did not have enough machinery to relax the fitness condition. We are now able to remove this condition and state that:

> *For any locale L, $\mathfrak{B}L$ is the only sublocale which is simultaneously dense and remote.*

Remark 3.1.15. In any T_1-space X, $\mathrm{Iso}(X)$ is a remote subset of X, hence $\widetilde{\mathrm{Iso}(X)}$ is a remote sublocale of $\mathfrak{O}X$, and therefore $\widetilde{\mathrm{Iso}(X)} \subseteq \mathfrak{B}(\mathfrak{O}X)$. Thus, if $\mathrm{Iso}(X)$ is dense in X, then $\widetilde{\mathrm{Iso}(X)}$ is dense in $\mathfrak{O}X$, hence $\mathfrak{B}(\mathfrak{O}X) \subseteq \widetilde{\mathrm{Iso}(X)}$, implying that $\widetilde{\mathrm{Iso}(X)} = \mathfrak{B}(\mathfrak{O}X)$. This is one of those instances where the smallest dense sublocale of a spatial locale is induced by a subspace.

We use Proposition 3.1.10 to characterize remote preserving localic maps in terms of the least dense sublocale.

Theorem 3.1.16. *Let $f : L \to M$ be a localic map. The following statements are equivalent.*

1. *$f_{-1}[-]$ preserves remote sublocales.*

2. $f_{-1}[\mathfrak{B}M]$ is a remote sublocale of L.

Proof. We only show that (2) $\Longrightarrow$ (1): Let $B \in \mathcal{S}_{\text{rem}}(M)$. Since $\mathfrak{B}M$ is the largest member of $\mathcal{S}_{\text{rem}}(M)$, we have that $B \subseteq \mathfrak{B}M$. Therefore $f_{-1}[B] \subseteq f_{-1}[\mathfrak{B}M]$. Since, by hypothesis, $f_{-1}[\mathfrak{B}M] \in \mathcal{S}_{\text{rem}}(L)$, using the fact that a sublocale of a remote sublocale is remote, we get that $f_{-1}[B] \in \mathcal{S}_{\text{rem}}(L)$. □

We proved in Corollary 2.1.32 that Rem(L) is a remote sublocale whenever the locale L is a coframe. Since we now know that $\mathfrak{B}L$ is the largest remote sublocale of a locale L, Rem$(L) \subseteq \mathfrak{B}L$ making Rem(L) Boolean because every sublocale of a Boolean locale is Boolean. Since a locale M is Boolean if and only if $M = \text{Rem}(M)$, Rem(Rem$(L)) = \text{Rem}(L)$. Unlike in the situation of Pel$^{\infty}(L)$ given by Dube in [17], the sequence of Rem$^{\delta}(L)$'s, where δ is an ordinal, stabilizes at Rem(L).

Definition 3.1.17. Define a Rem-*map* to be a localic map $f : L \to M$ such that $f[\text{Rem}(L)] \subseteq \text{Rem}(M)$.

Denote by $\mathbf{CFLoc}_R$ the category of locales which are also coframes whose morphisms are Rem-maps, and by $\mathbf{BooLoc}_R$ the full subcategory of $\mathbf{CFLoc}_R$ whose objects are Boolean locales.

We show that $\mathbf{BooLoc}_R$ is a coreflective subcategory of $\mathbf{CFLoc}_R$. That is, for each $L \in \text{Obj}(\mathbf{CFLoc}_R)$, we shall find a Boolean locale P and a Rem-map $c : P \to L$ such that for every Rem-map $f : N \to L$, from a Boolean locale N, there exists a unique Rem-map $f' : N \to P$ such that $c \circ f' = f$.

Proposition 3.1.18. *$\mathbf{BooLoc}_R$ is a coreflective subcategory of $\mathbf{CFLoc}_R$.*

Proof. Let $L \in \text{Obj}(\mathbf{CFLoc}_R)$. Since Rem$(L)$ is a remote sublocale, it is therefore contained in the largest remote sublocale $\mathfrak{B}L$ of L making it a Boolean locale so that it belongs to $\mathbf{BooLoc}_R$.

The map $c = j_{\text{Rem}(L)} : \text{Rem}(L) \to L$ is a Rem-map. Indeed,

$$c[\text{Rem}(\text{Rem}(L))] = \text{Rem}(\text{Rem}(L)) \subseteq \text{Rem}(L)$$

making c a Rem-map.

Let $f : N \to L$ be a Rem-map where N is a Boolean locale. Then $N = \mathrm{Rem}(N)$. So, there is a localic map, say $f' : N \to \mathrm{Rem}(L)$, which maps as f. This localic map is clearly a Rem-map satisfying $c \circ f' = f$.

f' is unique: Let $k : N \to \mathrm{Rem}(L)$ be a Rem-map such that $c \circ k = f$. Then, for each $x \in N$, $k(x) = c(k(x)) = f(x) = c(f'(x)) = f'(x)$. Thus $k = f'$ so that f' is unique. □

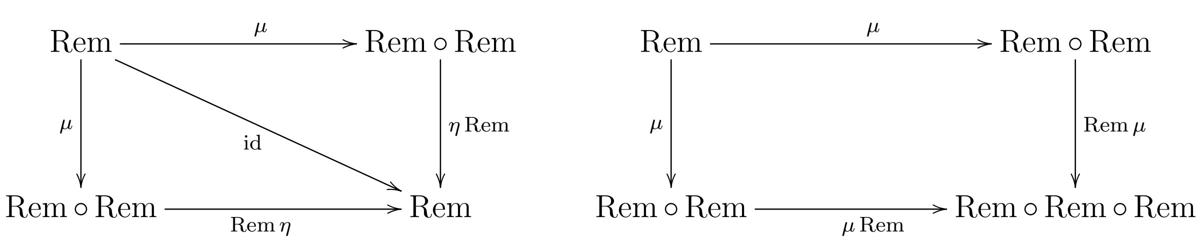
Recall that if $f : L \to M$ is a localic map with $S \in \mathcal{S}(L)$ and $T \in \mathcal{S}(M)$ such that $f[S] \subseteq T$, then the restriction map $f_{|S} : S \to T$ is a localic map. It is clear that for a Rem-map $f : L \to M$, where $L, M \in \mathrm{Obj}(\mathbf{CFLoc}_R)$, the function $f_{|\mathrm{Rem}(L)} : \mathrm{Rem}(L) \to \mathrm{Rem}(M)$ is a localic map because $f[\mathrm{Rem}(L)] \subseteq \mathrm{Rem}(M)$, $\mathrm{Rem}(L) \in \mathcal{S}(L)$ and $\mathrm{Rem}(M) \in \mathcal{S}(M)$. Using this argument, one can easily show that there is an endofunctor $\mathrm{Rem} : \mathbf{CFLoc}_R \to \mathbf{CFLoc}_R$ which maps each Rem-map $f : L \to M$ to the restriction map $f_{|\mathrm{Rem}(L)} : \mathrm{Rem}(L) \to \mathrm{Rem}(M)$. We formalize this in the following result.

Proposition 3.1.19. *The assignment*

$$\mathrm{Rem} : \boldsymbol{CFLoc}_R \to \boldsymbol{CFLoc}_R, \ (f : L \to M) \mapsto (f_{|\mathrm{Rem}(L)} : \mathrm{Rem}(L) \to \mathrm{Rem}(M))$$

is an endofunctor.

The fact that $\mathrm{Rem}(L) = \mathrm{Rem}(\mathrm{Rem}(L)) = \mathrm{Rem}(\mathrm{Rem}(\mathrm{Rem}(L)))$ for every $L \in \mathrm{Obj}(\mathbf{CFLoc}_R)$ leads to the realization that the endofunctor $\mathrm{Rem} : \mathbf{CFLoc}_R \to \mathbf{CFLoc}_R$ forms a *comonad*. That is, there are natural transformations $\eta : \mathrm{Rem} \to \mathrm{id}_{\mathbf{CFLoc}_R}$ and $\mu : \mathrm{Rem} \to \mathrm{Rem} \circ \mathrm{Rem}$ such that the following diagrams commute:



(3.1.2)

Proposition 3.1.20. *The triple* $(\mathrm{Rem}, \eta, \mu)$ *where*

1. Rem *is the endofunctor* $\mathrm{Rem} : \boldsymbol{CFLoc}_R \to \boldsymbol{CFLoc}_R$,

2. $\eta : \mathrm{Rem} \to \mathrm{id}_{\boldsymbol{CFLoc}_R}$ *is a function that assigns to each* $L \in \mathrm{Obj}(\boldsymbol{CFLoc}_R)$ *the map* $\eta_L = j_{\mathrm{Rem}(L)} : \mathrm{Rem}(L) \to L$, *and*

3. $\mu : \mathrm{Rem} \to \mathrm{Rem} \circ \mathrm{Rem}$ *is a map assigning to each* $L \in \mathrm{Obj}(\boldsymbol{CFLoc}_R)$ *the map* $\mu_L = \mathrm{id}_{\mathrm{Rem}(L)} : \mathrm{Rem}(L) \to \mathrm{Rem}(\mathrm{Rem}(L))$

is a comonad.

Proof. $\eta : \mathrm{Rem} \to \mathrm{id}_{\mathbf{CFLoc}_R}$ is a natural transformation: Let $L \in \mathrm{Obj}(\mathbf{CFLoc}_R)$ and define $\eta_L = j_{\mathrm{Rem}}(L)$. It is clear that η_L is a localic map and $\eta_L[\mathrm{Rem}(\mathrm{Rem}(L))] = \eta_L[\mathrm{Rem}(L)] \subseteq \mathrm{Rem}(L)$, making η_L a Rem-map. For each Rem-map $f : L \to M$, the diagram

$$\begin{array}{ccc} \mathrm{Rem}(L) & \xrightarrow{\eta_L} & \mathrm{id}_{\mathbf{CFLoc}_R}(L) = L \\ \Big\downarrow{\scriptstyle \mathrm{Rem}(f)} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathbf{CFLoc}_R}(f) = f} \\ \mathrm{Rem}(M) & \xrightarrow[\eta_M]{} & \mathrm{id}_{\mathbf{CFLoc}_R}(M) = M \end{array} \tag{3.1.3}$$

commutes. Indeed, for each $x \in \mathrm{Rem}(L)$,

$$f(\eta_L(x)) = f(x) = \mathrm{Rem}(f)(x) = \eta_M(\mathrm{Rem}(f)(x)).$$

Thus η is a natural transformation.

$\mu : \mathrm{Rem} \to \mathrm{Rem} \circ \mathrm{Rem}$ is a natural transformation: Choose $L \in \mathrm{Obj}(\mathbf{CFLoc}_R)$ and let μ_L be the localic map $\mathrm{id}_{\mathrm{Rem}(L)} : \mathrm{Rem}(L) \to \mathrm{Rem}(\mathrm{Rem}(L))$ which exists because $\mathrm{Rem}(\mathrm{Rem}(L)) = \mathrm{Rem}(L)$. Then $\mu_L[\mathrm{Rem}(\mathrm{Rem}(L))] = \mu_L[\mathrm{Rem}(L)] = \mathrm{Rem}(L) = \mathrm{Rem}(\mathrm{Rem}(\mathrm{Rem}(L)))$ so that μ_L is a Rem-map. For each $f \in \hom_{\mathbf{CFLoc}_R}(L, M)$ where $M \in \mathrm{Obj}(\mathbf{CFLoc}_R)$, we have that

$$\begin{aligned} \mathrm{Rem}(\mathrm{Rem}(f)) &= \mathrm{Rem}(f_{|\mathrm{Rem}(L)}) \\ &= (f_{|\mathrm{Rem}(L)})_{|\mathrm{Rem}(\mathrm{Rem}(L))} \\ &= (f_{|\mathrm{Rem}(L)})_{|\mathrm{Rem}(L)} \\ &= f_{|\mathrm{Rem}(L)} \\ &= \mathrm{Rem}(f). \end{aligned}$$

Therefore the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Rem}(L) & \xrightarrow{\mu_L} & \mathrm{Rem}(\mathrm{Rem}(L)) = \mathrm{Rem}(L) \\ \Big\downarrow{\scriptstyle \mathrm{Rem}(f)} & & \Big\downarrow{\scriptstyle \mathrm{Rem}(\mathrm{Rem}(f))=\mathrm{Rem}(f)} \\ \mathrm{Rem}(M) & \xrightarrow[\mu_M]{} & \mathrm{Rem}(\mathrm{Rem}(M)) = M \end{array} \tag{3.1.4}$$

Thus μ is a natural transformation.

$\eta\,\mathrm{Rem}\ \circ\ \mu = \mathrm{id}$: Let $L \in \mathrm{Obj}(\mathbf{CFLoc}_R)$. Then

$$(\eta\,\mathrm{Rem})_L \circ \mu_L = \eta_{\mathrm{Rem}(L)} \circ \mathrm{id}_{\mathrm{Rem}(L)} = j_{\mathrm{Rem}(\mathrm{Rem}(L))} = \mathrm{id}_{\mathrm{Rem}(L)}\,.$$

$\mathrm{Rem}\,\eta\ \circ\ \mu = \mathrm{id}$: For each $M \in \mathrm{Obj}(\mathbf{CFLoc}_R)$,

$$(\mathrm{Rem}\,\eta)_M \circ \mu_M = \mathrm{Rem}(\eta_M) \circ \mathrm{id}_{\mathrm{Rem}(M)} = (\eta_M)_{|\mathrm{Rem}(\mathrm{Rem}(M))} = (j_{\mathrm{Rem}(M)})_{|\mathrm{Rem}(\mathrm{Rem}(M))} = \mathrm{id}_{\mathrm{Rem}(M)}\,.$$

$\mathrm{Rem}\,\mu\ \circ\ \mu = \mu\,\mathrm{Rem}\ \circ\ \mu$: Let $N \in \mathrm{Obj}(\mathbf{CFLoc}_R)$. Then

$$\begin{aligned} (\mathrm{Rem}\,\mu)_N \circ \mu_N &= \mathrm{Rem}(\mu_N) \circ \mathrm{id}_{\mathrm{Rem}(N)} \\ &= (\mu_N)_{|\mathrm{Rem}(\mathrm{Rem}(N))} \circ \mathrm{id}_{\mathrm{Rem}(N)} \\ &= \mathrm{id}_{\mathrm{Rem}(\mathrm{Rem}(N))} \circ \mathrm{id}_{\mathrm{Rem}(N)} \\ &= \mu_{\mathrm{Rem}(N)} \circ \mathrm{id}_{\mathrm{Rem}(N)} \\ &= (\mu\,\mathrm{Rem})_N \circ \mu_N. \end{aligned}$$

Thus $(\mathrm{Rem}, \eta, \mu)$ is a comonad. □

We close this section with a description of localic maps $f : L \to M$ such that:

1. $A \in \mathcal{S}_{\mathrm{rem}}(L)$ if and only if $f[A] \in \mathcal{S}_{\mathrm{rem}}(M)$;
2. $x \in \mathrm{Rem}(L)$ if and only if $f(x) \in \mathrm{Rem}(M)$;
3. $f_{-1}[A] \in \mathcal{S}_{\mathrm{rem}}(M)$ if and only if $A \in \mathcal{S}_{\mathrm{rem}}(M)$; and
4. $h(y) \in \mathrm{Rem}(L)$ if and only if $y \in \mathrm{Rem}(M)$.

We have the following results.

Proposition 3.1.21. *Let $f : L \to M$ be localic map such that f sends dense elements to dense elements and h is weakly open. Then:*

1. *$A \in \mathcal{S}_{rem}(L)$ iff $f[A] \in \mathcal{S}_{rem}(M)$ for each $A \in \mathcal{S}(L)$.*

2. *If f is closed, then $x \in \mathrm{Rem}(L)$ iff $f(x) \in \mathrm{Rem}(M)$ for all $x \in L$.*

Proof. The forward implications of (1) and (2) follow from Theorem 3.1.2 and Proposition 3.1.4, respectively. We only prove the reverse direction of (1). The proof for (2) follows a similar sketch. Let $A \in \mathcal{S}(L)$ be such that $f[A] \in \mathcal{S}_{\mathrm{rem}}(M)$ and let $\mathfrak{o}(x)$ be dense in L. Then x is dense in L. Since f sends dense elements to dense elements, $f(x)$ is dense in M so that $\mathfrak{o}(f(x))$ is also dense in M. The remoteness of $f[A]$ gives $f[A] \subseteq \mathfrak{o}(f(x))$ which implies that

$$A \subseteq f_{-1}[f[A]] \subseteq f_{-1}[\mathfrak{o}(f(x))] = \mathfrak{o}(h(f(x))) \subseteq \mathfrak{o}(x).$$

Thus A is contained in every open dense sublocale of L, making it a remote sublocale of L. □

Proposition 3.1.22. *Let $f : L \to M$ be localic map such that $f[-]$ is surjective, f sends dense elements to dense elements and h is weakly open. Then:*

1. *$f_{-1}[B] \in \mathcal{S}_{rem}(L)$ iff $B \in \mathcal{S}_{rem}(M)$ for all $B \in \mathcal{S}(M)$.*

2. *$h(y) \in \mathrm{Rem}(L)$ iff $y \in \mathrm{Rem}(M)$ for each $y \in M$.*

Proof. The reverse directions of both (1) and (2) were proved in Proposition 3.1.8. For the forward directions, we only give a proof for (1). Let $B \in \mathcal{S}(M)$ be such that $f_{-1}[B] \in \mathcal{S}_{\mathrm{rem}}(L)$ and choose a dense sublocale $\mathfrak{o}(x)$ of M. Since h is weakly open and x is dense in M, $h(x)$ is dense in L so that $\mathfrak{o}(h(x))$ is a dense sublocale of L. It follows that $f_{-1}[B] \subseteq \mathfrak{o}(h(x))$ because $f_{-1}[B]$ is a remote sublocale of L. Therefore

$$B = f[f_{-1}[B]] \subseteq f[\mathfrak{o}(h(x))] = f[f_{-1}[\mathfrak{o}(x)]] \subseteq \mathfrak{o}(x).$$

Thus $B \in \mathcal{S}_{\mathrm{rem}}(M)$. □

Here is an example of a localic map with the features hypothesized above which is not an isomorphism.

Example 3.1.23. Consider a Boolean locale L with a subframe $M \neq L$ and M not isomorphic to L. For instance, the two-element chain as a subframe of the four-element Boolean algebra. Let $f : L \to M$ be the localic map whose left adjoint is the identical embedding $M \hookrightarrow L$. Then f is surjective and not an isomorphism. We claim that $f[-] : \mathcal{S}(L) \to \mathcal{S}(M)$ is surjective. To see this, consider any sublocale T of M. Since M is Boolean, T is a closed sublocale of M and hence $T = \mathfrak{c}(x)$ for some $x \in M$. Since f is onto, there is $a \in L$ such that $f(a) = x$. In light of all sublocales of M being closed,

$$f[\mathfrak{c}(a)] = \overline{f[\mathfrak{c}(a)]} = \mathfrak{c}(f(a)) = \mathfrak{c}(x) = T,$$

showing that $f[-]$ is surjective. Since the only dense element in any Boolean locale is the top element, both f and its left adjoint send dense elements to dense elements, hence the left adjoint of f is weakly open. This is also an example of a localic map that is not an isomorphism yet it is closed, sends dense elements to dense elements and has a weakly open left adjoint.

Comment 3.1.24. In [20], the authors show that for any localic map $f : L \to M$, if $f[-] : \mathcal{S}(L) \to \mathcal{S}(M)$ is surjective, then f is also surjective. They however do not comment about surjectivity of localic image function if the underlying localic map is surjective. The calculation in the foregoing example shows that if a surjective localic map has a Boolean codomain, then the induced localic image function is also surjective.

3.2 Pushing forward and pulling back of sublocales which are remote (resp. *remote) from dense sublocales

This section focuses on discussing preservation and reflection of sublocales that are remote (resp. *remote) from dense sublocales. For the rest of this section, we regard βL as the locale of regular ideals of $\operatorname{Coz} L$ so that the diagram 1.5.1 commutes.

Consider a commuting diagram

$$\begin{array}{ccc} S & \xrightarrow{\ g\ } & T \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\omega} \\ L & \xrightarrow[\ f\ ]{} & M \end{array} \tag{3.2.1}$$

where S, T, L and M are locales, f and g are localic maps and the downward morphisms are dense injective localic maps. Our discussion will make use of the information provided in diagram 3.2.1. We commented in the preliminaries that a localic map $k : P \to Q$ is dense if and only if $k[P]$ is a dense sublocale of Q. So, $\alpha[S]$ and $\omega[T]$ are dense sublocales of L and M, respectively. Since for an extension $v : W \to Y$, $v_* : Y \to v_*[Y]$ is an isomorphism, we will sometimes write S and T for the sublocales $\alpha[S]$ and $\omega[T]$, respectively.

A particular case of the situation depicted in diagram 3.2.1 is that of γ-lifts given in diagram 1.5.1.

For *remoteness, we note that, $A \cap \alpha[S] = \mathsf{O}$ implies $A \subseteq L \smallsetminus \alpha[S]$ but $A \subseteq L \smallsetminus \alpha[S]$ does not always imply that A misses $\alpha[S]$ unless $\alpha[S]$ is complemented. We will sometimes treat these cases differently.

We start by recording a description of localic maps that preserve remoteness from a dense sublocale. For the following result, we recall from [26] that a localic map $f : L \to M$ *takes A-remainder to B-remainder* if $f[L \smallsetminus A] \subseteq M \smallsetminus B$ where $A \in \mathcal{S}(L)$, $B \in \mathcal{S}(M)$. We shall write $f : L \to M$ takes S-remainder to T-remainder to mean that f takes $\alpha[S]$-remainder to $\omega[T]$-remainder.

Proposition 3.2.1. *Assume that g^* in diagram 3.2.1 is weakly open and $f^* \circ \omega = \alpha \circ g^*$. Then*

1. $f[\mathcal{S}_{rem}(L \ltimes S)] \subseteq \mathcal{S}_{rem}(M \ltimes T)$.
2. *If f^* is weakly closed, then* $f[\mathrm{Rem}(L \ltimes S)] \subseteq \mathrm{Rem}(M \ltimes T)$.
3. *Suppose that f takes S-remainder to T-remainder, then*
   (a) $f[{}^*\mathcal{S}_{rem}(L \ltimes S)] \subseteq {}^*\mathcal{S}_{rem}(M \ltimes T)$.
   (b) *If f^* is weakly closed, then* $f[{}^*\mathrm{Rem}(L \ltimes S)] \subseteq {}^*\mathrm{Rem}(M \ltimes T)$.

Proof. (1) Let $A \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)$ and choose an $\omega[T]$-dense $y \in \omega[T]$. Then $y = \omega(t)$ for some $t \in T$. Because g^* is weakly open, $g^*(t)$ is S-dense so that $\alpha(g^*(t))$ is $\alpha[S]$-dense since $\alpha : S \to \alpha[S]$ is an isomorphism. Therefore $\mathsf{O} = A \cap \mathfrak{c}_L(\alpha(g^*(t)) = A \cap \mathfrak{c}_L(f^*(\omega(t))$ where the latter equality follows since $f^* \circ \omega = \alpha \circ g^*$. Therefore $A \subseteq \mathfrak{o}(f^*(\omega(t))) = f_{-1}[\mathfrak{o}(\omega(t))] = f_{-1}[\mathfrak{o}(y)]$.

We get that $f[A] \subseteq f[f_{-1}[\mathfrak{o}(y)]] \subseteq \mathfrak{o}(y)$. This tells us that $f[A]$ is contained in every open sublocale induced by an $\omega[T]$-dense element, so by Proposition 2.1.42(3), $f[A] \in \mathcal{S}_{\text{rem}}(M \ltimes T)$.

(2) Let $x \in \text{Rem}(L \ltimes S)$ and choose an $\omega[T]$-dense $t \in \omega[T]$. Therefore $t = \omega(y)$ for some $y \in T$. Since g^* is weakly open, $g^*(y)$ is S-dense, making $\alpha(g^*(y))$ $\alpha[S]$-dense. It follows that $\alpha(g^*(y)) \vee x = 1$. Because $\alpha \circ g^* = f^* \circ \omega$, $f^*(\omega(y)) \vee x = 1$. The weakly closedness of f^* implies that $1_M = \omega(y) \vee f(x) = t \vee f(x)$. Thus $f(x) \in \text{Rem}(M \ltimes T)$.

(3) With the assumption that f takes S-remainder to T-remainder, it is clear that $A \subseteq L \smallsetminus \alpha[S]$ implies $f[A] \subseteq f[L \smallsetminus \alpha[S]] \subseteq M \smallsetminus \omega[T]$ for all $A \in \mathcal{S}(L)$. This together with (1) and (2) show that (3)(a) and (3)(b) hold. □

Observation 3.2.2. (1) In terms of γ-lifts, the condition $f^* \circ \omega = \alpha \circ g^*$ on f resembles that of a *γ-map* which was defined in [23] as a frame homomorphism $t : M \to L$ that satisfies $\gamma(t) \circ (\gamma_M)_* = (\gamma_L)_* \circ t$.

(2) For Proposition 3.2.1(3), in the case where $\alpha[S]$ is complemented, we replace f takes S-remainder to T-remainder with the condition that f is injective and $f[\alpha[S]] = \omega[T]$. From this we get that $A \subseteq L \smallsetminus \alpha[S]$ implies $A \cap \alpha[S] = \mathsf{O}$. Therefore $\mathsf{O} = f[\mathsf{O}] = f[A \cap \alpha[S]] = f[A] \cap f[\alpha[S]]$ so that $f[A] \subseteq M \smallsetminus f[\alpha[S]] = M \smallsetminus \omega[T]$.

We digress to explore localic maps with the properties given in Proposition 3.2.1(1) and Proposition 3.2.1(3)(a). We shall say that g in diagram 3.2.1 is *f-remote preserving* if $f[\mathcal{S}_{\text{rem}}(L \ltimes S)] \subseteq \mathcal{S}_{\text{rem}}(M \ltimes T)$ and *f-*remote preserving* if $f[{}^*\mathcal{S}_{\text{rem}}(L \ltimes S)] \subseteq {}^*\mathcal{S}_{\text{rem}}(M \ltimes T)$. In the case of γ-lifts, we shall say that f is *γ-remote preserving* if $\gamma(f)\,[\mathcal{S}_{\text{rem}}(\gamma L \ltimes L)] \subseteq \mathcal{S}_{\text{rem}}(\gamma M \ltimes M)$ and *γ-*remote preserving* provided that $\gamma(f)\,[{}^*\mathcal{S}_{\text{rem}}(\gamma L \ltimes L)] \subseteq {}^*\mathcal{S}_{\text{rem}}(\gamma M \ltimes M)$.

Since $\mathfrak{B}L \in \mathcal{S}_{\text{rem}}(L \ltimes S)$ for every dense sublocale S of L, in the next result, we characterize f-remote preserving maps in terms of the Booleanization of a locale. We also include, in the same result, a characterization in terms of the largest sublocale remote from a given dense sublocale. We recall that if $w : P \to Q$ is a dense injective localic map, then for all $x \in P$, x is P-dense if and only if $w(x)$ is L-dense.

Theorem 3.2.3. *Suppose that $f^* \circ \omega = \alpha \circ g^*$. The following statements are equivalent.*

1. *g is f-remote preserving.*

2. $f[\mathfrak{B}L] \in \mathcal{S}_{rem}(M \ltimes T)$.

3. $f[\mathrm{Rs}(L \ltimes S)] \in \mathcal{S}_{rem}(M \ltimes T)$.

4. $f[\mathrm{Rs}(L \ltimes S)] \subseteq \mathrm{Rs}(M \ltimes T)$.

Proof. (1) $\Longrightarrow$ (2): Since $\mathfrak{B}L$ is remote from every dense sublocale of L and $\alpha[S]$ is a dense sublocale of L, $\mathfrak{B}L$ is remote from $\alpha[S]$. By (1), $f[\mathfrak{B}L] \in \mathcal{S}_{\mathrm{rem}}(M \ltimes T)$.

(2) $\Longrightarrow$ (3): Let $a \in \omega[T]$ be $\omega[T]$-dense. Then $a = \omega(x)$ for some $x \in T$. By hypothesis, $f[\mathfrak{B}L] \subseteq \mathfrak{o}(\omega(x))$. Therefore $\mathfrak{B}L \subseteq f_{-1}[\mathfrak{o}(\omega(x))] = \mathfrak{o}[f^*(\omega(x))]$. Since $f^* \circ \omega = \alpha \circ g^*$, $\mathfrak{B}L \subseteq \mathfrak{o}[\alpha(g^*(x))]$, making $\alpha(g^*(x))$ L-dense so that $g^*(x)$ is S-dense and hence $\alpha(g^*(x))$ is $\alpha[S]$-dense. But $\mathrm{Rs}(L \ltimes S)$ is remote from $\alpha[S]$, so $\mathrm{Rs}(L \ltimes S) \subseteq \mathfrak{o}[\alpha(g^*(x))]$. Therefore

$$f[\mathrm{Rs}(L \ltimes S)] \subseteq f[\mathfrak{o}(\alpha(g^*(x)))] = f[\mathfrak{o}(f^*(\omega(x)))] = f[f_{-1}[\mathfrak{o}(\omega(x))]] \subseteq \mathfrak{o}(\omega(x)) = \mathfrak{o}(a).$$

Thus $f[\mathrm{Rs}(L \ltimes S)] \in \mathcal{S}_{\mathrm{rem}}(M \ltimes T)$.

(3) $\Longrightarrow$ (4): Follows since $\mathrm{Rs}(M \ltimes T)$ is the largest sublocale remote from $\omega[T]$.

(4) $\Longrightarrow$ (1): Let $A \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)$. Then $A \subseteq \mathrm{Rs}(L \ltimes S)$ so that $f[A] \subseteq f[\mathrm{Rs}(L \ltimes S)]$. But $f[\mathrm{Rs}(L \ltimes S)] \subseteq \mathrm{Rs}(M \ltimes T)$, so $f[A] \subseteq f[\mathrm{Rs}(M \ltimes T)]$. Since sublocales contained in members of $\mathcal{S}_{\mathrm{rem}}(M \ltimes T)$ are remote from $\omega[T]$, $f[A]$ is remote from $\omega[T]$. Thus g is f-remote preserving. ∎

We give the following characterization of f-*remote preserving maps.

Proposition 3.2.4. *Assume that the downward embeddings α and ω in diagram 3.2.1 are dense and $f^* \circ \omega = \alpha \circ g^*$. The following statements are equivalent.*

1. *g is f-*remote preserving.*

2. *$f[{}^*\mathrm{Rs}(L \ltimes S)]$ is *remote from $\omega[T]$.*

3. *$f[{}^*\mathrm{Rs}(L \ltimes S)] \subseteq {}^*\mathrm{Rs}(M \ltimes T)$.*

Proof. (1) $\Longrightarrow$ (2): Follows since ${}^*\mathrm{Rs}(L \ltimes S) \in {}^*\mathcal{S}_{\mathrm{rem}}(L \ltimes S)$.

(2) $\Longrightarrow$ (3): Trivial.

(3) $\Longrightarrow$ (1): Proof is similar to that of Theorem 3.2.3 (4) $\Longrightarrow$ (1). □

In Proposition 3.2.6 below, we explore a relationship between f-remote preservation and preservation of remote sublocales. We give the following lemma which will be useful in proving the result.

Lemma 3.2.5. *The following statements hold.*

1. $A \in \mathcal{S}_{rem}(S)$ *iff* $\alpha[A] \in \mathcal{S}_{rem}(L \ltimes S)$.
2. $A \in \mathcal{S}_{rem}(L \ltimes S)$ *implies* $(\alpha)_{-1}[A] \in \mathcal{S}_{rem}(S)$.

Proof. (1) Recall from Proposition 2.1.48 that $\mathcal{S}_{\mathrm{rem}}(K) = \mathcal{S}_{\mathrm{rem}}(L \ltimes K) \cap \mathcal{S}(K)$ for each dense $K \in \mathcal{S}(L)$. Therefore $\alpha[A] \in \mathcal{S}_{\mathrm{rem}}(\alpha[S])$ if and only if $\alpha[A] \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S) \cap \mathcal{S}(\alpha[S])$. Since $\alpha : S \to \alpha[S]$ is an isomorphism, it is easy to see that $A \in \mathcal{S}_{\mathrm{rem}}(S)$ if and only if $\alpha[A] \in \mathcal{S}_{\mathrm{rem}}(\alpha[S])$ if and only if $\alpha[A] \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S) \cap \mathcal{S}(\alpha[S])$.

(2) Let $x \in S$ be S-dense. Then $\alpha(x)$ is $\alpha[S]$-dense. It follows that $A \cap \mathfrak{c}_L(\alpha(x)) = \mathsf{O}$. Therefore

$$\mathsf{O} = (\alpha)_{-1}[A] \cap (\alpha)_{-1}[\mathfrak{c}_L(\alpha(x))] = (\alpha)_{-1}[A] \cap \mathfrak{c}_S((\alpha)^*(\alpha(x))) = (\alpha)_{-1}[A] \cap \mathfrak{c}_S(x)$$

proving the result. □

Proposition 3.2.6. *Assume that $f^* \circ \omega = \alpha \circ g^*$. Then g is f-remote preserving iff $g[-]$ preserves remote sublocales.*

Proof. ($\Longrightarrow$) : Since $\alpha[S]$ is dense in L, $\mathfrak{B}\alpha[S] = \mathfrak{B}L$ making $\mathfrak{B}\alpha[S] = \alpha[\mathfrak{B}S]$ remote from $\alpha[S]$. Because g is f-remote preserving, we have that $f[\alpha[\mathfrak{B}S]] \in \mathcal{S}_{\mathrm{rem}}(M \ltimes T)$ which implies that $\omega[g[\mathfrak{B}S]] \in \mathcal{S}_{\mathrm{rem}}(M \ltimes T)$ since $\omega^* \circ f = g \circ \alpha^*$. It follows from Lemma 3.2.5(2) that $(\omega)_{-1}[\omega[g[\mathfrak{B}S]]] \in \mathcal{S}_{\mathrm{rem}}(T)$. But $g[\mathfrak{B}S] \subseteq (\omega)_{-1}[\omega[g[\mathfrak{B}S]]]$, so $g[\mathfrak{B}S] \in \mathcal{S}_{\mathrm{rem}}(T)$. By Theorem 3.1.2(2), g preserves remote sublocales.

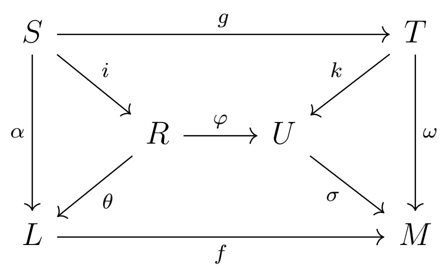
($\Longleftarrow$) : We show that $f[\mathfrak{B}L]$ is remore from $\omega[T]$. Since $\mathfrak{B}L \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)$, it follows from Lemma 3.2.5(2) that $(\alpha)_{-1}[\mathfrak{B}L] \in \mathcal{S}_{\mathrm{rem}}(S)$. By hypothesis, $g[(\alpha)_{-1}[\mathfrak{B}L]] \in \mathcal{S}_{\mathrm{rem}}(T)$. By Lemma 3.2.5(1), $\omega[g[(\alpha)_{-1}[\mathfrak{B}L]]] \in \mathcal{S}_{\mathrm{rem}}(M \ltimes T)$ which implies that $f[\alpha[(\alpha)_{-1}[\mathfrak{B}L]]] \in$

$\mathcal{S}_{\text{rem}}(M \ltimes T)$ since $f \circ \alpha = \omega \circ g$. But $\mathfrak{B}L = \mathfrak{B}\alpha[S] \subseteq \alpha[S]$ and using the fact that $\alpha : S \to \alpha[S]$ is an isomorphism,

$$f[\alpha[(\alpha)_{-1}[\mathfrak{B}L]]] = f[\alpha[(\alpha)_{-1}[\mathfrak{B}\alpha[S]]]] = f[\mathfrak{B}\alpha[S]] = f[\mathfrak{B}L] \in \mathcal{S}_{\text{rem}}(M \ltimes T)$$

as required. □

Consider a commuting diagram



(3.2.2)

where S, T, R, U, L and M are locales, the downward arrows are dense injective localic maps and the horizontal arrows are localic maps. We find a relationship between f-remote preservation and φ-remote preservation. En route to that, we give the following lemma.

Lemma 3.2.7. *From diagram 3.2.2, $\theta[\mathcal{S}_{rem}(R \ltimes S)] \subseteq \mathcal{S}_{rem}(L \ltimes S)$.*

Proof. We have that α is dense since it is the composite of two dense localic maps i and θ. Let $A \in \mathcal{S}_{\text{rem}}(R \ltimes S)$ and choose an $\alpha[S]$-dense $y \in \alpha[S]$. Then $y = \alpha(x)$ for some $x \in S$. Since $\alpha : S \to \alpha[S]$ is an isomorphism, x is S-dense so that $i(x)$ is $i[S]$-dense. Therefore $A \cap \mathfrak{c}_R(i(x)) = \mathsf{O}$. Observe that $\theta[A] \cap \mathfrak{c}_L(\alpha(x)) = \mathsf{O}$. To see this, let $a \in \theta[A] \cap \mathfrak{c}_L(\alpha(x))$. Then $a = \theta(b)$ for some $b \in A$ and $\alpha(x) \leq a$. We have that

$$i(x) = \theta^*(\theta(i(x))) = \theta^*(\alpha(x)) \leq \theta^*(\theta(b)) = b$$

since θ is injective and $\alpha = \theta \circ i$. Therefore $b \in A \cap \mathfrak{c}_R(i(x))$ which implies $b = 1$ so that $a = \theta(b) = 1$. Thus $\theta[A] \cap \mathfrak{c}_L(\alpha(x)) = \mathsf{O}$. Hence $\theta[A] \in \mathcal{S}_{\text{rem}}(L \ltimes S)$. □

Since $\theta[R] \subseteq L$, we have that

$$\{B \in \mathcal{S}(\theta[R]) : B \cap \alpha[S] = \mathsf{O}\} \subseteq \{C \in \mathcal{S}(L) : C \cap \alpha[S] = \mathsf{O}\}$$

so that

$$\theta[R] \smallsetminus \alpha[S] = \bigvee\{B \in \mathcal{S}(\theta[R]) : B \cap \alpha[S] = \mathsf{O}\} \subseteq \bigvee\{C \in \mathcal{S}(L) : C \cap \alpha[S] = \mathsf{O}\} = L \smallsetminus \alpha[S].$$

As a result of this and Lemma 3.2.7, we have the following result.

Lemma 3.2.8. *From diagram 3.2.2,* $\theta[{}^{*}\mathcal{S}_{rem}(R \ltimes S)] \subseteq {}^{*}\mathcal{S}_{rem}(L \ltimes S)$.

Observation 3.2.9. In light of the preceding two lemmas and the relationship between the β, υ and λ extensions depicted in diagram 1.5.1, we have

$$\mathcal{S}_{\text{rem}}(\upsilon L \ltimes L) \subseteq \mathcal{S}_{\text{rem}}(\lambda L \ltimes L) \subseteq \mathcal{S}_{\text{rem}}(\beta L \ltimes L)$$

and

$${}^{*}\mathcal{S}_{\text{rem}}(\upsilon L \ltimes L) \subseteq {}^{*}\mathcal{S}_{\text{rem}}(\lambda L \ltimes L) \subseteq {}^{*}\mathcal{S}_{\text{rem}}(\beta L \ltimes L).$$

Proposition 3.2.10. *If g in diagram 3.2.2 is f-remote preserving, then it is φ-remote preserving.*

Proof. Let $A \in \mathcal{S}_{\text{rem}}(R \ltimes S)$. It follows from Lemma 3.2.7 that $\theta[A] \in \mathcal{S}_{\text{rem}}(L \ltimes S)$. Since g is f-remote preserving, $f[\theta[A]] \in \mathcal{S}_{\text{rem}}(M \ltimes T)$ making $\sigma[\varphi[A]] \in \mathcal{S}_{\text{rem}}(M \ltimes T)$. By Lemma 3.2.5(2), $(\sigma)_{-1}[\sigma[\varphi[A]]] \in \mathcal{S}_{\text{rem}}(U) \subseteq \mathcal{S}_{\text{rem}}(U \ltimes T)$. Since $\varphi[A] \subseteq (\sigma)_{-1}[\sigma[\varphi[A]]]$ and a sublocale of any member of $\mathcal{S}_{\text{rem}}(U \ltimes T)$ belongs to $\mathcal{S}_{\text{rem}}(U \ltimes T)$, we have that $\varphi[A] \in \mathcal{S}_{\text{rem}}(U \ltimes T)$. Thus g is φ-remote preserving. □

Observation 3.2.11. Recall from [26] that given a localic map $f : L \to M$ and any $K \in \mathcal{S}(L)$, $f[L \smallsetminus K] \subseteq M \smallsetminus f[K]$ whenever $K = f_{-1}[J]$ for some $J \in \mathcal{S}(M)$. Since for the *remoteness case of Proposition 3.2.10 we need $\varphi[R \smallsetminus i[S]] \subseteq U \smallsetminus k[T]$, we assume that $i[S] = \varphi_{-1}[k[T]]$ and $\varphi[-]$ is surjective in diagram 3.2.2. Then

$$\varphi[R \smallsetminus i[S]] \subseteq U \smallsetminus \varphi[i[S]] = U \smallsetminus \varphi[\varphi_{-1}[k[T]]] = U \smallsetminus k[T]$$

so that $A \in {}^{*}\mathcal{S}_{\text{rem}}(R \ltimes S)$ implies $\varphi[A] \in {}^{*}\mathcal{S}_{\text{rem}}(U \ltimes T)$. This approach also helps in verifying *remoteness cases of Corollary 3.2.13 and Proposition 3.2.15(2) and (3) below.

Observation 3.2.12. The converse of Proposition 3.2.10 holds if $\alpha[-]$ is surjective (hence an isomorphism). Indeed, assume that g is φ-remote preserving and let $A \in \mathcal{S}_{\text{rem}}(L \ltimes S)$. By Lemma 3.2.5(2), $\alpha_{-1}[A] \in \mathcal{S}_{\text{rem}}(S)$. Therefore $i[\alpha_{-1}[A]] \in \mathcal{S}_{\text{rem}}(R \ltimes S)$ by Lemma 3.2.5(1). Since g is φ-remote preserving, $\varphi[i[\alpha_{-1}[A]]] \in \mathcal{S}_{\text{rem}}(U \ltimes T)$. By Lemma 3.2.7, $\sigma[\varphi[i[\alpha_{-1}[A]]]] \in \mathcal{S}_{\text{rem}}(M \ltimes T)$ so that $f[\theta[i[\alpha_{-1}[A]]]] \in \mathcal{S}_{\text{rem}}(M \ltimes T)$ because $f \circ \theta = \sigma \circ \varphi$. Since $\theta \circ i = \alpha$ and

$\alpha[-]$ is surjective, $f[\theta[i[\alpha_{-1}[A]]]] = f[\alpha[\alpha_{-1}[A]]] = f[A] \in \mathcal{S}_{\text{rem}}(M \ltimes T)$. Thus g is f-remote preserving.

Corollary 3.2.13. *We have that*

$$\beta\text{-remote preserving} \implies \lambda\text{-remote preserving} \implies \upsilon\text{-remote preserving.}$$

Observation 3.2.14. To get the reverse directions of Corollary 3.2.13, we observe that the morphisms υ_L, β_L and λ_L are isomorphisms whenever L is compact. The case of λ_L follows since, according to [37], λ_L is injective (hence an isomorphism) whenever L is Lindelöf. Because every compact locale is Lindelöf, we have that λ_L is an isomorphism whenever L is compact.

We end our digression with the following result.

Proposition 3.2.15. *Consider a commuting diagram*

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ & \searrow^{t} & \downarrow{\scriptstyle \varphi} \\ & & N \end{array} \tag{3.2.3}$$

where f, g and t are localic maps and L, M and N are locales.

1. *If φ and f are γ-remote preserving, then t is γ-remote preserving.*

2. *If t is γ-remote preserving, φ sends elements to dense elements, then f is γ-remote preserving.*

3. *If t is γ-remote preserving and $A \subseteq \gamma(f)[\mathfrak{B}_{\gamma L}]$ for all $A \in \mathcal{S}_{rem}(\gamma M \ltimes M)$, then φ is γ-remote preserving.*

Proof. (1) For each $A \in \mathcal{S}(\gamma L)$, we have

$$\begin{aligned} A \in \mathcal{S}_{\text{rem}}(\gamma L \ltimes L) &\implies \gamma(f)[A] \in \mathcal{S}_{\text{rem}}(\gamma M \ltimes M) \quad \text{since } f \text{ is } \gamma\text{-remote preserving} \\ &\implies \gamma(\varphi)[\gamma(f)[A]] \in \mathcal{S}_{\text{rem}}(\gamma N \ltimes N) \text{ since } \varphi \text{ is } \gamma\text{-remote preserving} \\ &\implies \gamma(\varphi \circ f)[A] \in \mathcal{S}_{\text{rem}}(\gamma N \ltimes N) \quad \text{since } \gamma \text{ is a functor} \\ &\implies \gamma(t)[A] \in \mathcal{S}_{\text{rem}}(\gamma N \ltimes N). \end{aligned}$$

(2) Let $A \in \mathcal{S}_{\text{rem}}(\gamma L \ltimes L)$ and choose dense $x \in M$. Since t is γ-remote preserving, we have that $\gamma(t)[A] \in \mathcal{S}_{\text{rem}}(\gamma N \ltimes N)$. Since φ sends dense elements to dense elements, we have that $\varphi(x)$ is dense in N. It follows that $\gamma(t)[A] \subseteq \mathfrak{o}((\gamma_N)_*(\varphi(x)))$. But $(\gamma_N)_* \circ \varphi = \gamma(\varphi) \circ (\gamma_M)_*$, so $\gamma(t)[A] \subseteq \mathfrak{o}(\gamma(\varphi)((\gamma_M)_*(x)))$. Therefore $\gamma(\varphi)_{-1}[\gamma(t)[A]] \subseteq \mathfrak{o}((\gamma_M)_*(x))$ making $\gamma(\varphi)_{-1}[\gamma(t)[A]]$ remote from M. Since $\gamma(\varphi)[\gamma(f)[A]] = \gamma(t)[A]$ implies $\gamma(f)[A] \subseteq \gamma(\varphi)_{-1}[\gamma(t)[A]]$, we have that $\gamma(f)[A]$ is remote from M.

(3) Let $A \in \mathcal{S}_{\text{rem}}(\gamma M \ltimes M)$. Then $A \subseteq \gamma(f)[\mathfrak{B}_{\gamma L}]$ which implies that

$$\gamma(\varphi)[A] \subseteq \gamma(\varphi)[\gamma(f)[\mathfrak{B}_{\gamma L}]] = \gamma(t)[\mathfrak{B}_{\gamma L}].$$

But $\gamma(t)[\mathfrak{B}_{\gamma L}]$ is remote from N, so $\gamma(\varphi)[A]$ is remote from N. It follows that φ is γ-remote preserving. □

Observation 3.2.16. We commented in Observation 3.2.11 that the approach given in that observation can be used to prove the *remoteness case of Proposition 3.2.15(2) and (3). The *remoteness case of Proposition 3.2.15(1) follows the similar sketch of the proof of Proposition 3.2.15(1) where γ-remote preserving is replaced by γ-*remote preserving.

We return to descriptions of localic maps that reflect and preserve the variants of remoteness introduced in Definition 2.1.33.

Proposition 3.2.17. *Assume that the morphism g in diagram 3.2.1 sends dense elements to dense elements. Then*

1. $f[A] \in \mathcal{S}_{rem}(M \ltimes T)$ *implies* $A \in \mathcal{S}_{rem}(L \ltimes S)$ *for every* $A \in \mathcal{S}(L)$.
2. $f(x) \in \text{Rem}(M \ltimes T)$ *implies* $x \in \text{Rem}(L \ltimes S)$ *for all* $x \in L$.
3. *Suppose that* $\omega[T]$ *is a complemented sublocale of* M *and* $f_{-1}[\omega[T]] = \alpha[S]$. *Then*
    (a) $f[A] \in {}^*\mathcal{S}_{rem}(M \ltimes T)$ *implies* $A \in {}^*\mathcal{S}_{rem}(L \ltimes S)$ *for every* $A \in \mathcal{S}(L)$.
    (b) $f(x) \in {}^*\text{Rem}(M \ltimes T)$ *implies* $x \in {}^*\text{Rem}(L \ltimes S)$ *for all* $x \in L$.

Proof. (1) Assume that $f[A] \in \mathcal{S}_{\text{rem}}(M \ltimes T)$ and let $a \in \alpha[S]$ be $\alpha[S]$-dense. Then $a = \alpha(x)$ for some $x \in S$ where such x is S-dense. Since g sends dense elements to dense elements,

$g(x)$ is T-dense so that $\omega(g(x))$ is $\omega[T]$-dense. It follows that $f[A] \subseteq \mathfrak{o}(\omega(g(x)))$ which implies $f[A] \subseteq \mathfrak{o}(f(\alpha(x)))$ because $k \circ g = f \circ \alpha$. Therefore

$$A \subseteq f_{-1}[f[A]] \subseteq f_{-1}[\mathfrak{o}(f(\alpha(x)))] = \mathfrak{o}(h(f(\alpha(x)))) \subseteq \mathfrak{o}(\alpha(x)) = \mathfrak{o}(a).$$

Thus $A \in \mathcal{S}_{\text{rem}}(L \ltimes S)$.

(2) Follows similar sketch of the proof of (1).

(3) We only show that $f[A] \subseteq M \smallsetminus \omega[T]$ implies $A \subseteq L \smallsetminus \alpha[S]$. Observe that for complemented $\omega[T]$ in M with $f_{-1}[\omega[T]] = \alpha[S]$,

$$\begin{aligned} f[A] \subseteq M \smallsetminus \omega[T] &\iff f[A] \cap \omega[T] = \mathsf{O} \\ &\implies A \cap f_{-1}[\omega[T]] = \mathsf{O} \\ &\iff A \cap \alpha[S] = \mathsf{O} \\ &\implies A \subseteq L \smallsetminus \alpha[S]. \end{aligned}$$

for all $A \in \mathcal{S}(L)$. □

Proposition 3.2.18. *Suppose that the localic map g in diagram 3.2.1 sends dense elements to dense elements. Then*

1. $f_{-1}[\mathcal{S}_{rem}(M \ltimes T)] \subseteq \mathcal{S}_{rem}(L \ltimes S)$.
2. $h[\mathrm{Rem}(M \ltimes T)] \subseteq \mathrm{Rem}(L \ltimes S)$.
3. *If $f_{-1}[\omega[T]] = \alpha[S]$ and $\omega[T]$ is complemented in M, then:*
   (a) $f_{-1}[{}^*\mathcal{S}_{rem}(M \ltimes T)] \subseteq {}^*\mathcal{S}_{rem}(L \ltimes S)$.
   (b) $h[{}^*\mathrm{Rem}(M \ltimes T)] \subseteq {}^*\mathrm{Rem}(L \ltimes S)$.

Proof. (1) Let $A \in \mathcal{S}_{\text{rem}}(M \ltimes T)$ and choose an S-dense $a \in S$. Then $a = \alpha(x)$ for some $x \in S$ which is S-dense. $g(x)$ is T-dense because g sends dense elements to dense elements. It follows that $A \subseteq \mathfrak{o}(\omega(g(x)))$ since $\omega(g(x))$ is $\omega[T]$-dense and $A \in \mathcal{S}_{\text{rem}}(M \ltimes T)$. Therefore

$$f_{-1}[A] \subseteq f_{-1}[\mathfrak{o}(\omega(g(x)))] = \mathfrak{o}(h(\omega(g(x)))) = \mathfrak{o}(h(f(\alpha(x)))) \subseteq \mathfrak{o}(\alpha(x)) = \mathfrak{o}(x)$$

making $f_{-1}[A] \in \mathcal{S}_{\text{rem}}(L \ltimes S)$.

(2) Proof follows similar sketch of the proof of (1).

(3) Assume that $f_{-1}[\omega[T]] = \alpha[S]$ and $\omega[T]$ is complemented in M. We only show that $A \subseteq M \smallsetminus \omega[T]$ implies $f_{-1}[A] \subseteq L \smallsetminus \alpha[S]$ which is needed for both 3(a) and 3(b). We have that $A \subseteq M \smallsetminus \omega[T]$ gives $A \cap \omega[T] = \mathsf{O}$. Therefore $\mathsf{O} = f_{-1}[A] \cap f_{-1}[\omega[T]] = f_{-1}[A] \cap \alpha[S]$, which implies that $f_{-1}[A] \subseteq L \smallsetminus \alpha[S]$. □

Proposition 3.2.19. *Suppose g^* in diagram 3.2.1 is a weakly open map.*

1. *The following hold for $\alpha \circ g^* = f^* \circ \omega$ and surjective $f[-]$.*
   - *(a) $f_{-1}[A] \in \mathcal{S}_{rem}(L \ltimes S)$ implies $A \in \mathcal{S}_{rem}(M \ltimes T)$ for all $A \in \mathcal{S}(M)$.*
   - *(b) If f takes S-remainder to T-remainder, then $f_{-1}[A] \in {}^*\mathcal{S}_{rem}(L \ltimes S)$ implies $A \in {}^*\mathcal{S}_{rem}(M \ltimes T)$ for every $A \in \mathcal{S}(M)$.*
2. *The following statements hold for either weakly closed f^* and surjective g or $\alpha \circ g^* = f^* \circ \omega$ and surjective f.*
   - *(a) For each $x \in M$, $f^*(x) \in \mathrm{Rem}(L \ltimes S)$ implies $x \in \mathrm{Rem}(M \ltimes T)$.*
   - *(b) If $f[L \smallsetminus \alpha[S]] \subseteq M \smallsetminus \omega[T]$, then $f^*(x) \in {}^*\mathrm{Rem}(L \ltimes S)$ implies $x \in {}^*\mathrm{Rem}(M \ltimes T)$ for all $x \in M$.*

Proof. (1) Assume that $\alpha \circ g^* = f^* \circ \omega$ and surjective $f[-]$.

(a) Let $A \in \mathcal{S}(M)$ be such that $f_{-1}[A] \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)$ and choose an $\omega[T]$-dense $b \in \omega[T]$. Then $b = \omega(x)$ for some $x \in T$ with x a T-dense element. The weakly openness of g^* implies that $g^*(x)$ is S-dense so that $\alpha(g^*(x))$ is $\alpha[S]$-dense. Therefore

$$\mathsf{O} = f_{-1}[A] \cap \mathfrak{c}(\alpha(g^*(x))) = f_{-1}[A] \cap \mathfrak{c}(f^*(\omega(x))) = f_{-1}[A] \cap f_{-1}[\mathfrak{c}(\omega(x))] = f_{-1}[A \cap \mathfrak{c}(\omega(x))].$$

Since $f[-]$ is surjective, $\mathsf{O} = f[f_{-1}[A \cap \mathfrak{c}(\omega(x))]] = A \cap \mathfrak{c}(\omega(x)) = A \cap \mathfrak{c}(b)$. Thus $A \in \mathcal{S}_{\mathrm{rem}}(M \ltimes T)$.

(b) We only show that $f_{-1}[A] \subseteq L \smallsetminus \alpha[S]$ implies $A \subseteq L \smallsetminus \alpha[S]$. Observe that

$$f_{-1}[A] \subseteq L \smallsetminus \alpha[S] \implies f[f_{-1}[A]] \subseteq f[L \smallsetminus \alpha[S]] \implies A \subseteq M \smallsetminus \omega[T].$$

(2) (a) Suppose that f^* is weakly closed, g is surjective and let $T \in T$ be T-dense. Then $t = \omega(b)$ for some $b \in T$ which is T-dense. Then $g^*(b)$ is S-dense since g^* is weakly open. Because $f^*(x) \in \mathrm{Rem}(L \ltimes S)$, we get that

$$f^*(x) \vee \alpha(g^*(b)) = 1. \tag{3.2.4}$$

The weakly closedness of f^* gives $x \vee f(\alpha(g^*(b))) = 1$. Therefore

$$1 = x \vee \omega(g(g^*(b))) = x \vee \omega(b) = x \vee t$$

where the second equality holds since g is surjective. Thus $x \in \mathrm{Rem}(M \ltimes T)$.

Assume that $\alpha \circ g^* = f^* \circ \omega$ and f is surjective. Then from equation 3.2.4, we get that $f^*(x) \vee f^*(t) = 1$ which implies that

$$f(f^*(x \vee t)) = f(f^*(x) \vee f^*(t)) = f(1) = 1$$

so that by surjectivity of f, $x \vee t = 1$ making $x \in \mathrm{Rem}(M \ltimes T)$.

(b) Can be deduced from 1(b) and 2(a). $\square$

Chapter 4

Maximal Nowhere Density

In [59], Veksler introduced a notion of a maximal nowhere dense subset. Since nowhere density is of paramount importance when discussing remoteness, we dedicate this chapter to introducing and studying maximal nowhere dense sublocales from Veksler's classical definition, and also establishing a relationship between maximal nowhere density and remoteness.

This chapter has four sections. The first section discusses maximal nowhere dense sublocales, the second section discusses homogeneous maximal nowhere density which is a variant of maximal nowhere density that was also introduced by Veksler in the cited article. In Section 4.3, we examine a relationship between the introduced notions of maximal nowhere density and remoteness. The last section shows how localic maps transfer (homogeneous) maximal nowhere dense sublocales back and forth.

This chapter forms part of the results in the paper: M.S. Nxumalo, *On maximal nowhere dense sublocales*, Appl. Gen. Topology, (2023)(Under Review).

4.1 Maximal nowhere dense sublocales

This section introduces a localic notion of maximal nowhere density from that of Veksler. We will further show that the localic definition of maximal nowhere density is conservative in locales and finally discuss some properties of maximal nowhere dense sublocales.

Let us recall from [59] that a closed nowhere dense subset F of a Tychonoff space X is *maximal nowhere dense* if there is no closed nowhere dense $K \subseteq X$ such that F is nowhere dense in K.

We broaden our study to arbitrary nowhere dense subsets of any topological space. We give the following definition.

Definition 4.1.1. A nowhere dense subset N of a topological space X is *maximal nowhere dense* in case there is no nowhere dense subset K of X such that N is nowhere dense in K.

We define maximal nowhere dense sublocales by replacing subsets with sublocales from Definition 4.1.1.

Definition 4.1.2. Let L be a locale. A nowhere dense sublocale N of L is *maximal nowhere dense (m.n.d)* if there is no nowhere dense sublocale S of L such that N is nowhere dense in S.

Denote by $\mathfrak{M}(L)$ the collection of all maximal nowhere dense sublocales of a locale L.

In what follows, we consider some examples. We remind the reader that the only locale that is nowhere dense in itself is the trivial one. That is, L is nowhere dense as a sublocale of itself if and only if $L = \{1\}$.

Example 4.1.3. (1) In a non-Boolean strongly submaximal locale L (according to [19], a locale is *strongly submaximal* if each of its dense sublocales is open), $\mathrm{Nd}(L)$ is maximal nowhere dense. To see this, we start by showing that in a strongly submaximal locale L, $\mathrm{Nd}(L)$ is nowhere dense. Indeed, observe that in a strongly submaximal locale L, the dense sublocale $\mathfrak{B}L$ is open (in particular, complemented), so

$$\begin{aligned}\mathfrak{B}L \cap \mathrm{Nd}(L) &= \mathfrak{B}L \cap \bigvee\{S : S \text{ is nowhere dense}\}\\ &= \bigvee\{\mathfrak{B}L \cap S : S \text{ is nowhere dense}\}\\ &= \bigvee\{\mathsf{O}\} = \mathsf{O}\end{aligned}$$

making $\mathrm{Nd}(L)$ nowhere dense.

Now, let $A \in \mathcal{S}(L)$ be nowhere dense such that $\mathrm{Nd}(L)$ is nowhere dense in A. By the nature of $\mathrm{Nd}(L)$, $A = \mathrm{Nd}(L)$, making $\mathrm{Nd}(L)$ nowhere dense as a sublocale of itself. Hence $\mathrm{Nd}(L) = \mathsf{O}$. This means that O is the only nowhere dense sublocale of L, which contradicts that L is non-Boolean.

(2) O is not maximal nowhere dense. This follows since O is nowhere dense as a sublocale of itself. As a result, we get that a Boolean locale does not have a maximal nowhere dense sublocale. This also tells us that N in Definition 4.1.2 cannot be open. Otherwise, by Proposition 2.1.4, $L \smallsetminus \overline{N}$ is dense making $L \smallsetminus N$ dense because $L \smallsetminus \overline{N} \subseteq L \smallsetminus N$. Since N is non-void (since it is maximal nowhere dense making it different from O) and open, we must have that $(L \smallsetminus N) \cap N \neq \mathsf{O}$ which is not possible because N is complemented.

(3) A non-closed maximal nowhere dense sublocale: Consider the set $X = \{a, b, c, d\}$ endowed with $\mathfrak{O}X = \{\emptyset, X, \{a\}, \{a, c\}\}$. We have that $\mathfrak{B}\tau = \{\emptyset, X\}$ and $\widetilde{\{c\}} = \{\{a\}, X\}$. Therefore $\widetilde{\{c\}}$ is a non-closed nowhere dense sublocale of $\mathfrak{O}X$. The only nowhere dense sublocales of $\mathfrak{O}X$ containing $\widetilde{\{c\}}$ are $\{\{a\}, \{a, c\}, X\}$ and $\widetilde{\{c\}}$. It is easy to check that the sublocale $\widetilde{\{c\}}$ is not nowhere dense in any of the nowhere dense sublocales of $\mathfrak{O}X$, making it a non-closed maximal nowhere dense sublocale.

In Proposition 4.1.5 below, we give a characterization of maximal nowhere dense sublocales some part of which will be used in calculations that involve maximal nowhere dense sublocales.

Recall from the preliminaries that for a sublocale A of L, x^{*A} denotes the pseudocomplement of an $x \in A$, calculated in A.

For use below, we prove the following lemma.

Lemma 4.1.4. *A sublocale N of a locale L is nowhere dense in $K \in \mathcal{S}(L)$ iff N is nowhere dense in $\overline{K}$.*

Proof. Recall that $\mathfrak{B}S = \mathfrak{B}L$ for every dense $S \in \mathcal{S}(L)$. Because every sublocale is dense in its closure, we have $N \cap \mathfrak{B}K = N \cap \mathfrak{B}\overline{K}$, which implies N is nowhere dense in K if and only if N is nowhere dense in $\overline{K}$. □

We shall use $\mathfrak{N}\mathfrak{D}(L)$ to denote the collection of all nowhere dense sublocales of a locale L.

Proposition 4.1.5. *Let N be a nowhere dense sublocale of a locale L. The following statements are equivalent.*

1. *N is maximal nowhere dense.*

2. There is no closed nowhere dense sublocale $\mathfrak{c}(y)$ of L in which N is a nowhere dense sublocale of $\mathfrak{c}(y)$.

3. $\overline{N}$ is maximal nowhere dense.

4. $\overline{N}$ is not a nowhere dense sublocale of any closed nowhere dense sublocale of L.

5. There is no dense $y \in L$ such that $y \leq \bigwedge N$ and $(\bigwedge N)^{*\mathfrak{c}(y)} = y$.

Proof. (1) $\implies$ (2): Follows since there is no nowhere dense (particularly, closed nowhere dense) sublocale A of L in which $N \in \mathfrak{ND}(A)$.

(2) $\implies$ (3): Suppose that there is a nowhere dense sublocale K such that $\overline{N}$ is nowhere dense in K. Since every sublocale of a nowhere dense sublocale is nowhere dense, N is nowhere dense in K. By Lemma 4.1.4, N is nowhere dense in the closed nowhere dense sublocale $\overline{K}$, which contradicts the hypothesis in condition (2). Thus $\overline{N}$ is maximal nowhere dense.

(3) $\implies$ (4): Trivial.

(4) $\implies$ (5): Let $y \in L$ be dense such that $y \leq \bigwedge N$ and $(\bigwedge N)^{*\mathfrak{c}(y)} = y$. This means that $\bigwedge N$ is $\mathfrak{c}(y)$-dense which implies that $\mathfrak{c}_{\mathfrak{c}(y)}(\bigwedge N)$ is $\mathfrak{c}(y)$-nowhere dense. Because $\mathfrak{c}_{\mathfrak{c}(y)}(\bigwedge N) = \overline{N} \cap \mathfrak{c}(y) = \overline{N}$, we have that $\overline{N}$ is nowhere dense in the closed sublocale $\mathfrak{c}(y)$, which contradicts the hypothesis.

(5) $\implies$ (1): Assume that N is nowhere dense in a nowhere dense sublocale K of L. By Lemma 4.1.4, N is nowhere dense in the closed nowhere dense sublocale $\overline{K}$. Therefore $\bigwedge K$ is a dense element of L such that $\bigwedge K \leq \bigwedge N$ and $(\bigwedge N)^{*\overline{K}} = \bigwedge K$, which is a contradiction. □

In terms of closed nowhere dense sublocales, we have the following result which holds since for every $a, b \in L$,

$$\begin{aligned} a^{*\mathfrak{c}(b)} &= \bigvee_{\mathfrak{c}(b)} \{x \in \mathfrak{c}(b) : a \wedge x = 0_{\mathfrak{c}(b)} = b\} \\ &= \nu_{\mathfrak{c}(b)} \left(\bigvee \{x \in \mathfrak{c}(b) : a \wedge x = b\} \right) \\ &= b \vee \left(\bigvee \{x \in L : a \wedge x = b\} \right) \\ &= b \vee (a \to b) \\ &= a \to b. \end{aligned}$$

Corollary 4.1.6. *Let L be a locale and $\mathfrak{c}(x) \in \mathcal{S}(L)$. Then $\mathfrak{c}(x)$ is maximal nowhere dense if and only if there is no dense $y \in L$ such that $y \leq x$ and $x \to y = y$.*

Proposition 4.1.5 suggests that, when doing calculations about maximal nowhere dense sublocales, there is no loss of generality with restricting to closed nowhere dense sublocales.

We give the following result regarding binary intersections of induced sublocales which will be used below. Recall from [1] that $\mathfrak{o}(U) = \widetilde{U}$ for all $U \in \mathfrak{O}X$.

Lemma 4.1.7. *Let X be a space. For any $A, B, C \subseteq X$ with B either open or closed, $A \cap B \subseteq C$ implies $\widetilde{A} \cap \widetilde{B} \subseteq \widetilde{C}$.*

Proof. Assume that $A \cap B \subseteq C$. Then $A \subseteq C \cup (X \smallsetminus B)$. Because $\widetilde{S \cup T} = \widetilde{S} \vee \widetilde{T}$ for all $S, T \subseteq X$, we have that $\widetilde{A} \subseteq \widetilde{C} \vee \widetilde{X \smallsetminus B}$. If B is open, then $\widetilde{X \smallsetminus B} = \mathfrak{c}(B)$ so that

$$\begin{aligned} \widetilde{A} \subseteq \widetilde{C} \vee \mathfrak{c}(B) &\implies \widetilde{A} \cap \mathfrak{o}(B) \subseteq \widetilde{C} \quad \text{since } \mathfrak{c}(B) \text{ is complemented} \\ &\implies \widetilde{A} \cap \widetilde{B} \subseteq \widetilde{C} \quad \text{since } \mathfrak{o}(B) = \widetilde{B}. \end{aligned}$$

If B is closed, then $\widetilde{X \smallsetminus B} = \mathfrak{o}(X \smallsetminus B)$. Therefore

$$\begin{aligned} \widetilde{A} \subseteq \widetilde{C} \vee \mathfrak{o}(X \smallsetminus B) &\implies \widetilde{A} \subseteq \widetilde{C} \vee (\mathfrak{O}X \smallsetminus \mathfrak{c}(X \smallsetminus B)) \\ &\implies \widetilde{A} \cap \mathfrak{c}(X \smallsetminus B) \subseteq \widetilde{C} \\ &\implies \widetilde{A} \cap \widetilde{B} \subseteq \widetilde{C} \quad \text{since } \mathfrak{c}(X \smallsetminus B) = \widetilde{B}. \end{aligned}$$

In both cases, $A \cap B \subseteq C$ implies $\widetilde{A} \cap \widetilde{B} \subseteq \widetilde{C}$. □

Using the preceding result, we show below that a subset is nowhere dense in a subspace of a T_D-space precisely when the sublocale it induces is nowhere dense in the sublocale induced by the subspace. We shall make use of [3, Proposition 4.1.] which states that a sublocale is nowhere dense if and only if its closure has a void interior. Incidentally, this result generalizes a similar one of Plewe's which is stated only for complemented sublocales. We remind the reader that open sublocales of a sublocale S of a locale L are the $\mathfrak{o}_S(a) = S \cap \mathfrak{o}(a)$ for $a \in S$. So, in $\mathfrak{O}X$, open sublocales of $S \in \mathcal{S}(\mathfrak{O}X)$ are the $\mathfrak{o}_S(U) = S \cap \mathfrak{o}(U) = S \cap \widetilde{U}$ for $U \in S$.

Lemma 4.1.8. *Let X be a T_D-space and $F \subseteq X$. Then $A \subseteq F$ is F-nowhere dense iff $\widetilde{A}$ is $\widetilde{F}$-nowhere dense.*

Proof. ($\Longrightarrow$) : Let $U \in \widetilde{F}$ be such that $\mathfrak{o}_{\widetilde{F}}(U) \subseteq \overline{\widetilde{A}}^{\widetilde{F}}$. Then $U \in \mathfrak{O}X$ and $\widetilde{U} \cap \widetilde{F} \subseteq \overline{\widetilde{A}}^{\widetilde{F}} = \overline{\widetilde{A}} \cap \widetilde{F}$ so that $\widetilde{U \cap F} \subseteq \widetilde{U} \cap \widetilde{F} \subseteq \widetilde{\overline{A}}$. Therefore $U \cap F \subseteq \overline{A}$ making $U \cap F \subseteq \overline{A}^{F}$. Since A is F-nowhere dense, $U \cap F = \emptyset$. By Observation 2.1.17, $\widetilde{U} \cap \widetilde{F} = \mathsf{O}$ making $\mathrm{int}_{\widetilde{F}}\left(\overline{\widetilde{A}}^{\widetilde{F}}\right) = \mathsf{O}$. Thus $\widetilde{A}$ is $\widetilde{F}$-nowhere dense.

($\Longleftarrow$) : Let $U \in \mathfrak{O}X$ be such that $U \cap F \subseteq \overline{A}^{F} = \overline{A} \cap F$. Then $U \cap F \subseteq \overline{A}$. Since U is open, it follows from Lemma 4.1.7 that $\widetilde{U} \cap \widetilde{F} \subseteq \widetilde{\overline{A}}$ which gives $\widetilde{U} \cap \widetilde{F} \subseteq \widetilde{\overline{A}} \cap \widetilde{F}$. Because $\widetilde{A}$ is $\widetilde{F}$-nowhere dense, $\widetilde{U} \cap \widetilde{F} = \mathsf{O}$ implying that $U \cap F = \emptyset$. Therefore A is F-nowhere dense. □

We are now in a position to show that the notion of maximal nowhere density introduced in Definition 4.1.2 is conversative in locales.

Proposition 4.1.9. *Let X be a T_D-space. A subset F of X is maximal nowhere dense in X iff $\widetilde{F} \in \mathfrak{M}(\mathfrak{O}X)$.*

Proof. ($\Longrightarrow$) We prove this by contradiction. Let $F \subseteq X$ be maximal nowhere dense. It follows from Proposition 4.1.5 that $\overline{F}$ is maximal nowhere dense. We will show that $\overline{\widetilde{F}} \in \mathfrak{M}(\mathfrak{O}X)$ which, by Proposition 4.1.5, will imply that $\widetilde{F} \in \mathfrak{M}(\mathfrak{O}X)$. Suppose that there is a closed $\mathcal{S} \in \mathfrak{N}\mathfrak{D}(\mathfrak{O}X)$ such that $\overline{\widetilde{F}} \in \mathfrak{N}\mathfrak{D}(\mathcal{S})$, i.e., $\widetilde{\overline{F}} \in \mathfrak{N}\mathfrak{D}(\mathcal{S})$. Because $\mathcal{S}$ is a closed sublocale of $\mathfrak{O}X$, choose a closed set $K \subseteq X$ such that $\mathcal{S} = \mathfrak{c}(X \smallsetminus K) = \widetilde{K}$. It follows from Lemma 4.1.8 that $\overline{F}$ is K-nowhere dense making F to be K-nowhere dense which is a contradiction. Thus $\overline{\widetilde{F}} \in \mathfrak{M}(\mathfrak{O}X)$ which implies that $\widetilde{F} \in \mathfrak{M}(\mathfrak{O}X)$.

($\Longleftarrow$) Suppose that $\widetilde{F}$ is maximal nowhere dense in $\mathfrak{O}X$ but F is not maximal nowhere dense in X. Then F is nowhere dense in some nowhere dense subset K of X. It follows from Lemma 4.1.8 that $\widetilde{F}$ is nowhere dense in the nowhere dense sublocale $\widetilde{K}$ of $\mathfrak{O}X$, contradicting that $\widetilde{F}$ is maximal nowhere dense in $\mathfrak{O}X$. Thus F is maximal nowhere dense in X. $\square$

We close this section with a consideration of some results about maximal nowhere dense sublocales.

For the proof of Proposition 4.1.11(2), we start by recalling from [26] that for any sublocale S of L,

$$\mathrm{int}(S) = \mathfrak{o}\left(\bigwedge(L \smallsetminus S)\right) = L \smallsetminus \mathfrak{c}\left(\bigwedge(L \smallsetminus S)\right) = L \smallsetminus \overline{L \smallsetminus S}.$$

The above can also be applied to all complemented sublocales of a locale, as shown below.

Lemma 4.1.10. *Let L be a locale. Then a sublocale F of a complemented sublocale A of L is A-nowhere dense if and only if $A \subseteq \overline{A \cap (L \smallsetminus \overline{F})}$.*

Proof. We have that

$$\begin{aligned}
F \text{ is nowhere dense in } A \quad &\Longleftrightarrow \quad \mathrm{int}_A\left(\overline{F}^A\right) = \mathsf{O} \\
&\Longleftrightarrow \quad \mathrm{int}_A\left(\overline{F} \cap A\right) = \mathsf{O} \\
&\Longleftrightarrow \quad A \smallsetminus \overline{(A \smallsetminus (\overline{F} \cap A))}^A = \mathsf{O} \\
&\Longleftrightarrow \quad A \smallsetminus \left(\overline{A \smallsetminus (\overline{F} \cap A)} \cap A\right) = \mathsf{O} \\
&\Longleftrightarrow \quad A \cap \left(L \smallsetminus \left(\overline{A \smallsetminus (\overline{F} \cap A)} \cap A\right)\right) = \mathsf{O} \\
&\Longleftrightarrow \quad A \cap \left(\left(L \smallsetminus \overline{(A \smallsetminus (\overline{F} \cap A))}\right) \vee (L \smallsetminus A)\right) = \mathsf{O} \\
&\Longleftrightarrow \quad \left(A \cap \left(L \smallsetminus \overline{(A \smallsetminus (\overline{F} \cap A))}\right)\right) \vee (A \cap (L \smallsetminus A)) = \mathsf{O} \\
&\Longleftrightarrow \quad A \cap \left(L \smallsetminus \overline{(A \cap (L \smallsetminus (\overline{F} \cap A)))}\right) = \mathsf{O} \\
&\Longleftrightarrow \quad A \cap \left(L \smallsetminus \overline{(A \cap ((L \smallsetminus \overline{F}) \vee (L \smallsetminus A)))}\right) = \mathsf{O} \\
&\Longleftrightarrow \quad A \cap \left(L \smallsetminus \overline{(A \cap (L \smallsetminus \overline{F})) \vee (A \cap (L \smallsetminus A))}\right) = \mathsf{O} \\
&\Longleftrightarrow \quad A \cap \left(L \smallsetminus \overline{(A \cap (L \smallsetminus \overline{F}))}\right) = \mathsf{O} \\
&\Longleftrightarrow \quad A \cap \left(L \smallsetminus \overline{(A \cap (L \smallsetminus \overline{F}))}\right) = \mathsf{O} \\
&\Longleftrightarrow \quad A \subseteq \overline{A \cap (L \smallsetminus \overline{F})}
\end{aligned}$$

which proves the result. □

Proposition 4.1.11. *Let L be a locale and F a non-void nowhere dense sublocale of L. Then*

1. *If $A \in \mathfrak{ND}(L)$, $F \in \mathfrak{M}(L)$ and $F \subseteq A$, then $A \in \mathfrak{M}(L)$.*

2. *If $F \cap (L \smallsetminus \overline{N}) \neq \mathsf{O}$ for all $N \in \mathfrak{ND}(L \smallsetminus F)$, then $F \in \mathfrak{M}(L)$.*

Proof. (1) If $A \in \mathfrak{ND}(L)$, $F \in \mathfrak{M}(L)$, $F \subseteq A$ and $N \in \mathfrak{ND}(L)$ such that $A \in \mathfrak{ND}(N)$, then $F \in \mathfrak{ND}(N)$, which is a contradiction. Thus $A \in \mathfrak{M}(L)$.

(2) Assume that there is a nowhere dense sublocale $\mathfrak{c}(x)$ such that $F \in \mathfrak{ND}(\mathfrak{c}(x))$. We get that $\mathfrak{c}(x) \cap \left(L \smallsetminus \overline{\mathfrak{c}(x) \cap (L \smallsetminus F)}\right) = \mathsf{O}$. Observe that $\mathfrak{c}(x) \cap (L \smallsetminus F) \in \mathfrak{ND}(L \smallsetminus F)$. This is so because for each $a \in L$, $\mathfrak{o}(a) \cap (L \smallsetminus F) \subseteq \overline{(L \smallsetminus F) \cap \mathfrak{c}(x)}$ implies $\mathfrak{o}(a) \cap (L \smallsetminus \overline{F}) \subseteq \mathfrak{c}(x)$. But $\mathfrak{c}(x) \in \mathfrak{ND}(L)$ and $\mathfrak{c}(a) \cap (L \smallsetminus \overline{F})$ is open in L so $\mathfrak{c}(a) \cap (L \smallsetminus \overline{F}) = \mathsf{O}$. Therefore $\mathfrak{o}(a) \subseteq \overline{F}$. Since $\overline{F}$ is nowhere dense in L, $\mathfrak{o}(a) = \mathsf{O}$ making $\mathfrak{o}(a) \cap (L \smallsetminus F) = \mathsf{O}$ as required. Now, by hypothesis, $F \cap (L \smallsetminus \overline{\mathfrak{c}(x) \cap (L \smallsetminus F)}) \neq \mathsf{O}$. That is, $\mathfrak{c}(x) \cap (L \smallsetminus \overline{(L \smallsetminus F) \cap \mathfrak{c}(x)}) \neq \mathsf{O}$ since $F \subseteq \mathfrak{c}(x)$. This is a contradiction. Thus $F \in \mathfrak{M}(L)$. □

Observation 4.1.12. Using the fact that a finite join of nowhere dense sublocales is nowhere dense, Proposition 4.1.11(1) tells us that any finite join of maximal nowhere dense sublocales is maximal nowhere dense.

In what follows, we show that a nowhere dense sublocale which is maximal, in the usual sense of not being contained in any other nowhere dense sublocale other than itself, exists precisely when a locale has the largest nowhere dense sublocale.

Let us call a nowhere dense sublocale N of a locale L *strongly maximal nowhere dense* if, for any nowhere dense sublocale A, $N \subseteq A$ implies $A = N$.

Proposition 4.1.13. *Let L be a locale. The following statements are equivalent.*

1. *L has a strongly maximal nowhere dense sublocale.*

2. $\mathrm{Nd}(L)$ *is nowhere dense.*

If L is non-Boolean, this is further equivalent to:

3. Nd(L) *is maximal nowhere dense.*

Proof. (1) $\implies$ (2): Let $A \in \mathcal{S}(L)$ be strongly maximal nowhere dense. We show that Nd$(L) \subseteq A$ which will make Nd(L) nowhere dense. Choose a nowhere dense $N \in \mathcal{S}(L)$. Then $\overline{N}$ and $\overline{A}$ are nowhere dense sublocales making $\overline{N} \vee \overline{A}$ nowhere dense. But $A \subseteq \overline{N} \vee \overline{A}$ and A is strongly maximal nowhere dense, so $A = \overline{N} \vee \overline{A}$. Therefore $N \subseteq \overline{N} \vee \overline{A} = A$. Since N was arbitrary, Nd$(L) = \bigvee\{S \in \mathcal{S}(L) : S \in \mathfrak{ND}(L)\} \subseteq A$. Thus Nd$(L) \subseteq A$ implying that Nd(L) is nowhere dense.

(2) $\implies$ (1): If Nd(L) is nowhere dense, then there is no other nowhere dense sublocale containing Nd(L) other than itself. Thus Nd(L) is a strongly maximal nowhere dense sublocale of L.

Assume that L is non-Boolean. The equivalence (2) $\iff$ (3) follows since Nd(L) contains every nowhere dense sublocale of L and Nd$(L) \neq \mathsf{O}$. □

4.2 Homogeneous maximal nowhere dense sublocales

Related to a maximal nowhere dense subset is a homogeneous maximal nowhere dense subset which was defined for spaces by Veksler in [59] as a closed nowhere dense subset F of a Tychonoff space X in which each non-empty F-regular-closed subset is maximal nowhere dense in X. In this thesis, we do not only focus on Tychonoff spaces, but all topological spaces.

We extend Veksler's definition of a homogeneous maximal nowhere dense subset of any topological space to locales as follows. We start by reminding the reader that we use prefix S-for a localic property defined on a sublocale S.

Definition 4.2.1. A closed nowhere dense sublocale N of a locale L is *homogeneous maximal nowhere dense (h.m.n.d)* if each non-void N-regular-closed sublocale is maximal nowhere dense in L.

Without the closedness requirement in Definition 4.2.1, we give the following definition.

Definition 4.2.2. A nowhere dense sublocale F of a locale L is *strongly homogeneous maximal nowhere dense* if each non-void F-regular-closed sublocale is maximal nowhere dense in L.

Denote by $\mathfrak{HM}(L)$ the collection of homogeneous maximal nowhere dense sublocales of a locale L.

Observation 4.2.3. We note that regular-closed sublocales of a closed nowhere dense sublocale $\mathfrak{c}(x)$ of L are of the form $\mathfrak{c}(a \to x)$ for some $a \in L$. Indeed, A is $\mathfrak{c}(x)$-regular-closed if and only if $A = \overline{\mathfrak{o}(a) \cap \mathfrak{c}(x)} \cap \mathfrak{c}(x)$ for some $a \in L$. Therefore

$$\begin{aligned}
\overline{\mathfrak{o}(a) \cap \mathfrak{c}(x)} \cap \mathfrak{c}(x) &= \mathfrak{c}\left(\bigwedge (\mathfrak{o}(a) \cap \mathfrak{c}(x))\right) \cap \mathfrak{c}(x) \\
&= \mathfrak{c}\left(\bigwedge \left(\mathfrak{c}_{\mathfrak{o}(a)}(\nu_{\mathfrak{o}(a)}(x))\right)\right) \cap \mathfrak{c}(x) \\
&= \mathfrak{c}\left(\nu_{\mathfrak{o}(a)}(x)\right) \cap \mathfrak{c}(x) \\
&= \mathfrak{c}\left(a \to x\right) \cap \mathfrak{c}(x) \\
&= \mathfrak{c}\left(a \to x\right) \quad \text{since } x \leq a \to x.
\end{aligned}$$

In light of Observation 4.2.3 and the characterizations of maximal nowhere dense sublocales given in Proposition 4.1.5, we get the following characterizations of homogeneous maximal nowhere dense sublocales. The proof is straightforward and shall be omitted.

Proposition 4.2.4. *Let L be a locale and $x \in L$. The following statements are equivalent.*

1. $\mathfrak{c}(x) \in \mathfrak{HM}(L)$.
2. *For each $a \in L$, $\mathfrak{c}(a \to x) \in \mathfrak{M}(L)$.*
3. *For each $a \in L$, there is no dense $y \in L$ such that $y \leq a \to x$ and $(a \to x)^{*\mathfrak{c}(y)} = y$.*

In the next result, we show that the definition of a homogeneous maximal nowhere dense sublocale given in Definition 4.2.1 is conservative in locales. Prior to that, we give the following lemma which is related to Lemma 4.1.8. The notation $\widetilde{A}$ of a sublocale induced by a subset A of a space X fails to display for long equations. To address this issue, we adopt [49]'s notation S_A of a sublocale induced by a subset A of a space X which we shall use interchangeably with the usual notation $\widetilde{A}$.

Lemma 4.2.5. *Let X be a T_D-space, $U \in \mathfrak{O}X$, $K \subseteq X$ and F a closed subset of X such that $U \cap F \subseteq K$. Then $U \cap F$ is K-nowhere dense iff $\widetilde{U} \cap \widetilde{F}$ is $\widetilde{K}$-nowhere dense.*

Proof. $(\Longrightarrow)$: Since U is open in X, $\widetilde{U} \cap \widetilde{F} \subseteq \widetilde{K}$ by Lemma 4.1.7. Let $V \in \mathfrak{O}X$ be such that $\widetilde{V} \cap \widetilde{K} \subseteq \overline{\widetilde{U} \cap \widetilde{F}} \cap \widetilde{K}$. Then $\widetilde{V \cap K} \subseteq \overline{\widetilde{U} \cap \widetilde{F}}$. Since

$$
\begin{aligned}
\overline{\widetilde{U} \cap \widetilde{F}} &= \overline{\mathfrak{o}(U) \cap \mathfrak{c}(X \smallsetminus F)} \\
&= \mathfrak{c}\left(\bigwedge (\mathfrak{o}(U) \cap \mathfrak{c}(X \smallsetminus F))\right) \\
&= \mathfrak{c}(U \to (X \smallsetminus F)) \\
&= \mathfrak{c}(\mathrm{int}(X \smallsetminus (U \cap F))) \\
&= S_{(X \smallsetminus \mathrm{int}(X \smallsetminus (U \cap F)))} \\
&= S_{\left(\overline{X \smallsetminus (X \smallsetminus (U \cap F))}\right)} \\
&= \widetilde{\overline{U \cap F}},
\end{aligned}
$$

we have that $V \cap K \subseteq \overline{U \cap F} \cap K$. Since $U \cap F$ is K-nowhere dense and $V \cap K$ is K-open, $V \cap K = \emptyset$ which gives $\widetilde{V} \cap \widetilde{K} = \mathsf{O}$. Thus $\widetilde{U} \cap \widetilde{F}$ is $\widetilde{K}$-nowhere dense.

$(\Longleftarrow)$: Let $V \in \mathfrak{O}X$ be such that $V \cap K \subseteq \overline{U \cap F} \cap K$. Since V is open and $\widetilde{A \cap B} \subseteq \widetilde{A} \cap \widetilde{B}$ for all $A, B \subseteq X$, it follows from Proposition 4.1.7 that

$$\widetilde{V} \cap \widetilde{K} \subseteq \widetilde{\overline{U \cap F}} = \overline{\widetilde{U \cap F}} \subseteq \overline{\widetilde{U} \cap \widetilde{F}}$$

making $\widetilde{V} \cap \widetilde{K} \subseteq \overline{\widetilde{U} \cap \widetilde{F}} \cap \widetilde{K}$. Therefore $\mathsf{O} = \widetilde{V} \cap \widetilde{K} \supseteq \widetilde{V \cap K}$ making $V \cap K = \emptyset$. Thus $U \cap F$ is K-nowhere dense. □

Observation 4.2.6. Observe that for closed subsets A and B of a T_D-space X,

$$\widetilde{A} \cap \widetilde{B} = \mathfrak{c}(X \smallsetminus A) \cap \mathfrak{c}(X \smallsetminus B) = \mathfrak{c}((X \smallsetminus A) \cup (X \smallsetminus B)) = \mathfrak{c}(X \smallsetminus (A \cap B)) = \widetilde{A \cap B}.$$

So, for closed subsets F and C of X, it follows from Lemma 4.1.8 that $F \cap C$ is K-nowhere dense if and only if $\widetilde{F \cap C} = \widetilde{F} \cap \widetilde{C}$ is $\widetilde{K}$-nowhere dense.

Proposition 4.2.7. *Let X be T_D-space. A closed set $F \subseteq X$ is homogeneous maximal nowhere dense iff $\widetilde{F} \subseteq \mathfrak{O}X$ is homogeneous maximal nowhere dense.*

Proof. ($\Longrightarrow$) : Suppose that $F \subseteq X$ is homogeneous maximal nowhere dense in X. Let S be a non-void $\widetilde{F}$-regular-closed sublocale and assume that there is a closed sublocale K of $\mathfrak{O}X$ such that $S \in \mathfrak{ND}(K)$. Since both S and K are closed in $\mathfrak{O}X$, where closedness of S follows since $\widetilde{F}$ is closed in $\mathfrak{O}X$, we have $S = \widetilde{A}$ and $K = \widetilde{E}$ for some closed sets $A, E \subseteq X$. Both A and E are non-empty. Because $\widetilde{A}$ is non-void regular-closed in $\widetilde{F}$, there is a non-empty $U \in \mathfrak{O}X$ such that $\widetilde{A} = \overline{\widetilde{F} \cap \widetilde{U}} \cap \widetilde{F} = \overline{\widetilde{F} \cap \widetilde{U}}$. The sublocale $\widetilde{F} \cap \widetilde{U}$ is clearly non-void. Since $\widetilde{F} \cap \widetilde{U} \subseteq \widetilde{A}$ and $\widetilde{A}$ is nowhere dense in $\widetilde{E}$, we must have that $\widetilde{F} \cap \widetilde{U}$ is nowhere dense in $\widetilde{E}$. By Proposition 4.2.5, $F \cap U$ is E-nowhere dense. It follows from Observation 2.1.17 that $F \cap U \neq \emptyset$. Since E is closed, $\overline{F \cap U}^E = \overline{F \cap U} \subseteq E$ making $\overline{F \cap U} \in \mathfrak{ND}(E)$. Since subsets contained in a nowhere dense subset are nowhere dense, the non-empty F-regular-closed subset $\overline{F \cap U} \cap F$ is E-nowhere dense which is a contradiction. Thus $\widetilde{F} \in \mathfrak{HM}(\mathfrak{O}X)$.

($\Longleftarrow$) : Suppose that $\widetilde{F}$ is homogeneous maximal nowhere dense in $\mathfrak{O}X$. Let $U \in \mathfrak{O}X$ be such that $\overline{U \cap F} \cap F$ is non-empty and K-nowhere dense for some closed nowhere dense $K \subseteq X$. Observe that $U \cap F \subseteq K$ and since U is open, it follows from Proposition 4.1.7 that $\widetilde{U} \cap \widetilde{F} \subseteq \widetilde{K}$. Because K is closed, we have that $\widetilde{K} = \overline{\widetilde{K}}$ so that $\overline{\widetilde{U} \cap \widetilde{F}} \cap \widetilde{F} \subseteq \widetilde{K}$. It is easy to see that the sublocale $\overline{\widetilde{U} \cap \widetilde{F}} \cap \widetilde{F}$ is $\widetilde{F}$-regular-closed and non-void. Since $\overline{U \cap F} \cap F$ is nowhere dense in K, we have that $U \cap F$ is also K-nowhere dense. It follows from Lemma 4.2.5 that $\widetilde{U} \cap \widetilde{F}$ is $\widetilde{K}$-nowhere dense. This makes $\overline{\widetilde{U} \cap \widetilde{F}} \cap \widetilde{F}$ a $\widetilde{K}$-nowhere dense sublocale which is a contradiction. Thus F is homogeneous maximal nowhere dense. □

The following result tells us that homogeneous maximal nowhere density is regular-closed hereditary.

Proposition 4.2.8. *Let L be a locale and F be a closed nowhere dense sublocale of L. If $F \in \mathfrak{HM}(L)$ and A is a non-void F-regular-closed sublocale, then $A \in \mathfrak{HM}(L)$.*

Proof. Let N be a non-void regular-closed sublocale of A and suppose that there is $B \in \mathfrak{ND}(L)$ such that $N \in \mathfrak{ND}(B)$. Because A is F-regular-closed and N is A-regular-closed, $A = \overline{\mathfrak{o}(x) \cap F} \cap F$ and $N = \overline{\mathfrak{o}(y) \cap A} \cap A$ for some $x, y \in L$. Since both F and A are closed, $A = \overline{\mathfrak{o}(x) \cap F}$ and $N = \overline{\mathfrak{o}(y) \cap A}$ so that $N = \overline{\mathfrak{o}(y) \cap \overline{\mathfrak{o}(x) \cap F}}$. Therefore

$$\overline{\mathfrak{o}(y) \cap \mathfrak{o}(x) \cap F} = \overline{\mathfrak{o}(y \wedge x) \cap F} \subseteq N.$$

The sublocale $\overline{\mathfrak{o}(y) \cap \mathfrak{o}(x) \cap F} \neq \mathsf{O}$, otherwise $\mathfrak{o}(y) \cap \mathfrak{o}(x) \cap F = \mathsf{O}$ making $\mathfrak{o}(y) \cap \overline{\mathfrak{o}(x) \cap F} = \mathsf{O}$ which is not possible. Since N is nowhere dense in B, we get that $\overline{\mathfrak{o}(y \wedge x) \cap F}$ is nowhere dense in B making $\overline{\mathfrak{o}(y \wedge x) \cap F} \cap F \in \mathfrak{ND}(B)$. This is not possible because $\overline{\mathfrak{o}(y \wedge x) \cap F} \cap F$ is non-void and regular-closed in F which must be maximal nowhere dense in L. Thus $A \in \mathfrak{HM}(L)$. ☐

We close this section by considering a relationship between maximal nowhere dense sublocales and (strongly) homogeneous maximal nowhere dense sublocales.

Proposition 4.2.9. *Every homogeneous maximal nowhere dense (resp. strongly homogeneous maximal nowhere dense) sublocale is maximal nowhere dense.*

Proof. Follows since every locale is a regular-closed sublocale of itself. ☐

4.3 Remoteness and maximal nowhere density

The aim of this section is to explore a relationship between remote sublocales and (homogeneous) maximal nowhere dense sublocales.

We begin by introducing inaccessible and almost inaccessible sublocales from Veksler's notions of inaccessible points and almost inaccessible points. To do this, we shall start by transferring inaccessible points from spaces to locales.

Recall from [59] that a point $x \in E \subseteq X$ is E-inaccessible (resp. almost E-inaccessible) if $x \notin \overline{N}$ (resp. $x \notin \mathrm{int}_E(\overline{N} \cap E)$) for all $(X \smallsetminus E)$-closed nowhere dense N.

Our journey to introducing localic notions of inaccessible and almost inaccessible points will start with inaccessible points and end with almost inaccessible points.

In a Tychonoff space X, we have that $x \notin \overline{N}$ if and only if $\overline{\{x\}} \cap \overline{N} = \emptyset$ if and only if $(X \smallsetminus \overline{\{x\}}) \cup (X \smallsetminus \overline{N}) = X$ for every $x \in X$, $N \subseteq X$. So the definition of an E-inaccessible point $x \in E \subseteq X$ is equivalent to:

$(X \smallsetminus \overline{\{x\}}) \cup (X \smallsetminus \overline{N}) = X$ *for all* $(X \smallsetminus E)$*-closed nowhere dense* N.

Recall that $X \smallsetminus \overline{A} = 0_{\widetilde{A}}$ for any subset A of a space X. This and the preceding paragraph motivate the following localic definition of an inaccessible point.

Definition 4.3.1. A point p of a sublocale S of a completely regular locale L is *S-inaccessible* if for each $(L \smallsetminus S)$-closed nowhere dense sublocale N, $0_N \vee p = 1$, where the join is calculated in L.

Recall from [2, Lemma 3.2.1] that for a subset A of a T_D-space X,

$$\bigvee\{\{X \smallsetminus \overline{\{x\}}, X\} : x \notin A\} = \widetilde{X \smallsetminus A}$$

is the supplement of $\widetilde{A}$, i.e., $\widetilde{X \smallsetminus A} = \widetilde{X} \smallsetminus \widetilde{A}$.

According to [21, 54], a regular locale is T_1 in the sense that every point is a maximal element. Hence a point p of a regular locale L has a property that $a \vee p = 1$ if and only if $a \nleq p$ for every $a \in L$.

In what follows, we show that a point x of a Tychonoff space X is inaccessible if and only if $\widetilde{x}$ is inaccessible, where $\widetilde{x}$ is the point of $\mathfrak{O}X$ induced by $x \in X$. We shall need the following lemma.

Lemma 4.3.2. *Let X be a topological space, F a closed subset of X and $A \subseteq X$. Then* $\bigwedge(\widetilde{F} \cap \widetilde{A}) = \operatorname{int}((X \smallsetminus A) \cup (X \smallsetminus F))$

Proof. We have that $X \smallsetminus F \subseteq (X \smallsetminus A) \cup (X \smallsetminus F)$, making

$$X \smallsetminus F = \operatorname{int}(X \smallsetminus F) \subseteq \operatorname{int}((X \smallsetminus A) \cup (X \smallsetminus F))$$

so that $\operatorname{int}((X \smallsetminus A) \cup (X \smallsetminus F)) \in \widetilde{F}$. Also, since $X \smallsetminus F$ is open, $\operatorname{int}((X \smallsetminus A) \cup (X \smallsetminus F)) \in \widetilde{A}$. Therefore $\operatorname{int}((X \smallsetminus A) \cup (X \smallsetminus F)) \in \widetilde{F} \cap \widetilde{A}$, making $\bigwedge(\widetilde{F} \cap \widetilde{A}) \leq \operatorname{int}((X \smallsetminus A) \cup (X \smallsetminus F))$.

On the other hand, let $V \in \widetilde{F} \cap \widetilde{A}$. Then $X \smallsetminus F \subseteq V$ and $V = \operatorname{int}((X \smallsetminus A) \cup G)$ for some $G \in \mathfrak{O}X$. We get that

$$\begin{aligned}
\operatorname{int}((X \smallsetminus A) \cup (X \smallsetminus F)) &\subseteq \operatorname{int}((X \smallsetminus A) \cup V) \\
&= \operatorname{int}\left((X \smallsetminus A) \cup \operatorname{int}((X \smallsetminus A) \cup G)\right) \\
&\subseteq \operatorname{int}((X \smallsetminus A) \cup G) \\
&= V.
\end{aligned}$$

Therefore $\operatorname{int}((X\smallsetminus A)\cup(X\smallsetminus F)) \leq \bigwedge(\widetilde{F}\cap\widetilde{A})$ and hence $\bigwedge(\widetilde{F}\cap\widetilde{A}) = \operatorname{int}((X\smallsetminus A)\cup(X\smallsetminus F))$. $\square$

Proposition 4.3.3. *Let X be a Tychonoff space and $x \in E \subseteq X$. Then x is E-inaccessible iff $\widetilde{x}$ is $\widetilde{E}$-inaccessible.*

Proof. $(\Longrightarrow)$: Let K be an $(\widetilde{X} \smallsetminus \widetilde{E})$-closed nowhere dense sublocale. Choose a closed subset F of X such that $K = \widetilde{F} \cap (\widetilde{X} \smallsetminus \widetilde{E})$. Since $\widetilde{X} \smallsetminus \widetilde{E} = \widetilde{X \smallsetminus E}$, $K = \widetilde{F} \cap \widetilde{X \smallsetminus E} \supseteq S_{(F\cap(X\smallsetminus E))}$. It follows from Lemma 4.1.8 that $F \cap (X \smallsetminus E)$ is an $(X \smallsetminus E)$-closed nowhere dense subset. Since x is E-inaccessible,

$$\left(X \smallsetminus \overline{\{x\}}\right) \cup \left(X \smallsetminus \overline{F \cap (X \smallsetminus E)}\right) = X.$$

Because $\widetilde{x} = X \smallsetminus \overline{\{x\}}$ is a point and every completely regular locale is regular and hence T_1, it follows that $X \smallsetminus \overline{F \cap (X \smallsetminus E)} \nleq \widetilde{x}$. Since $X \smallsetminus \overline{F \cap (X \smallsetminus E)} = \operatorname{int}(E \cup (X \smallsetminus F))$ and $\operatorname{int}(E \cup (X \smallsetminus F)) = \bigwedge(\widetilde{F} \cap \widetilde{X \smallsetminus E})$ by Lemma 4.3.2, $\bigwedge(\widetilde{F} \cap \widetilde{X \smallsetminus E}) \nleq \widetilde{x}$ so that

$$\bigwedge(\widetilde{F} \cap \widetilde{X \smallsetminus E}) \vee \widetilde{x} = 0_K \vee \widetilde{x} = 1_{\mathfrak{O}X}$$

because $\mathfrak{O}X$ is T_1. Thus $\widetilde{x}$ is a $\widetilde{E}$-inaccessible point.

$(\Longleftarrow)$: Let C be an $(X \smallsetminus E)$-closed nowhere dense subset. Set $C = F \cap (X \smallsetminus E)$ for some closed $F \subseteq X$. If follows from Proposition 4.2.5 that $\widetilde{F} \cap \widetilde{X \smallsetminus E}$ is $(\widetilde{X \smallsetminus E})$-closed nowhere dense. But $\widetilde{X} \smallsetminus \widetilde{E} = \widetilde{X \smallsetminus E}$, so the $(\widetilde{X} \smallsetminus \widetilde{E})$-closed sublocale $\widetilde{F} \cap (\widetilde{X} \smallsetminus \widetilde{E})$ is $(\widetilde{X} \smallsetminus \widetilde{E})$-nowhere dense. By hypothesis, $\widetilde{x} \vee 0_K = 1_{\mathfrak{O}X}$. By Lemma 4.3.2, $0_K = X \smallsetminus \overline{F \cap (X \smallsetminus E)}$. Therefore $\left(X \smallsetminus \overline{\{x\}}\right) \cup \left(X \smallsetminus \overline{F \cap (X \smallsetminus E)}\right) = X$ which implies that

$$\emptyset = \overline{\{x\}} \cap \overline{F \cap (X \smallsetminus E)} = \{x\} \cap \overline{F \cap (X \smallsetminus E)}.$$

Thus $x \notin \overline{F \cap (X \smallsetminus E)} = \overline{C}$, making x E-inaccessible. $\square$

To transfer almost inaccessibility to locales, we recall that for $x \in E \subseteq X$,

$$\begin{aligned} x \notin \operatorname{int}_E(\overline{N} \cap E) \quad &\Longleftrightarrow \quad x \notin E \cap \left(X \smallsetminus \overline{E \cap (X \smallsetminus \overline{N})}\right) \\ &\Longleftrightarrow \quad x \in \overline{E \cap (X \smallsetminus \overline{N})}. \end{aligned}$$

The above equivalence motivates the following localic definition of an almost inaccessible point.

Definition 4.3.4. A point p of a sublocale S of a completely regular locale L is *almost inaccessible* if for each $(L \smallsetminus S)$-closed nowhere dense sublocale N, $p \in \mathrm{cl}_S\left(S \cap \left(L \smallsetminus \overline{N}\right)\right)$.

In what follows we prove that a point x of X is almost inaccessible precisely when $\widetilde{x}$ is almost inaccessible.

Proposition 4.3.5. *Let X be a Tychonoff space. A point x of a subset E of X is almost E-inaccessible iff $\widetilde{x}$ is almost $\widetilde{E}$-inaccessible.*

Proof. $(\Longrightarrow)$: It is clear that $\widetilde{x}$ is a point belonging to the closed sublocale $\widetilde{\overline{E}} = \overline{\widetilde{E}}$. Choose an $(\widetilde{X} \smallsetminus \widetilde{E})$-closed nowhere dense K. Then $K = \widetilde{F} \cap (\widetilde{X} \smallsetminus \widetilde{E})$ for some closed $F \subseteq X$. Since $\widetilde{X} \smallsetminus \widetilde{E} = \widetilde{X \smallsetminus E}$, $K = \widetilde{F} \cap \widetilde{X \smallsetminus E}$. It follows from Proposition 4.2.5 that $F \cap (X \smallsetminus E)$ is $(X \smallsetminus E)$-closed nowhere dense. Therefore $x \notin \mathrm{int}_E(\overline{F \cap (X \smallsetminus E)} \cap E)$ which means $x \in \overline{E \cap (X \smallsetminus \overline{F \cap (X \smallsetminus E)})}$. We show that $\widetilde{x} \in \overline{\widetilde{E} \cap (\widetilde{X} \smallsetminus \overline{K})}$. Let $U \in \mathfrak{O}X$ be such that $\widetilde{E} \cap (\widetilde{X} \smallsetminus \overline{K}) \subseteq \mathfrak{c}(U)$. Then $\widetilde{E} \cap \left(\widetilde{X} \smallsetminus \overline{\widetilde{F} \cap (\widetilde{X} \smallsetminus \widetilde{E})}\right) \subseteq \mathfrak{c}(U)$, i.e., $\widetilde{E} \cap \left(\widetilde{X} \smallsetminus \overline{\widetilde{F} \cap \widetilde{X \smallsetminus E}}\right) \subseteq \mathfrak{c}(U)$. We get that

$$\begin{aligned}
\widetilde{E} &\subseteq \mathfrak{c}(U) \vee \overline{\widetilde{F} \cap \widetilde{X \smallsetminus E}} \\
&= \mathfrak{c}(U) \vee \mathfrak{c}\left(\bigwedge\left(\widetilde{F} \cap (\widetilde{X \smallsetminus E})\right)\right) \\
&= \mathfrak{c}(U) \vee \mathfrak{c}(X \smallsetminus \overline{F \cap (X \smallsetminus E)}) \quad \text{since } \bigwedge\left(\widetilde{F} \cap (\widetilde{X \smallsetminus E})\right) = X \smallsetminus \overline{F \cap (X \smallsetminus E)} \\
&= \mathfrak{c}\left(U \cap (X \smallsetminus \overline{F \cap (X \smallsetminus E)})\right) \\
&= S_{\left((X \smallsetminus U) \cup \overline{F \cap (X \smallsetminus E)}\right)} \quad \text{by Lemma 2.1.14.}
\end{aligned}$$

Therefore $E \subseteq (X \smallsetminus U) \cup \overline{F \cap (X \smallsetminus E)}$ so that $E \cap (X \smallsetminus \overline{F \cap (X \smallsetminus E)}) \subseteq X \smallsetminus U$. Because U is open, $X \smallsetminus U$ is closed, making $x \in \overline{E \cap (X \smallsetminus \overline{F \cap (X \smallsetminus E)})} \subseteq X \smallsetminus U$ since U was arbitrary. Therefore $\widetilde{x} \in \widetilde{X \smallsetminus U} = \mathfrak{c}(U)$ implying that $\widetilde{x} \in \overline{\widetilde{E} \cap (\widetilde{X} \smallsetminus \overline{K})}$. Thus

$$\widetilde{x} \in \overline{\widetilde{E} \cap (\widetilde{X} \smallsetminus \overline{K})} \cap \widetilde{E} = cl_{\widetilde{E}}\left(\widetilde{E} \cap (\widetilde{X} \smallsetminus \overline{K})\right)$$

which means that $\widetilde{x}$ is almost $\widetilde{E}$-inaccessible.

$(\Longleftarrow)$: Let N be an $(X \smallsetminus E)$-closed nowhere dense subset and set $N = F \cap (X \smallsetminus E)$ for some closed $F \subseteq X$. It follows from Proposition 4.2.5 that $\widetilde{F} \cap \widetilde{X \smallsetminus E} = \widetilde{F} \cap (\widetilde{X} \smallsetminus \widetilde{E})$ is $(\widetilde{X \smallsetminus E} = \widetilde{X} \smallsetminus \widetilde{E})$-closed nowhere dense. Therefore

$$\widetilde{x} \in \mathrm{cl}_{\widetilde{E}}\left(\widetilde{E} \cap \left(\widetilde{X} \smallsetminus \overline{\widetilde{F} \cap (\widetilde{X} \smallsetminus \widetilde{E})}\right)\right).$$

We show that $x \in \overline{E \cap (X \smallsetminus \overline{N})}$. Let K be a closed set such that $E \cap (X \smallsetminus \overline{N}) \subseteq K$. Then $E \subseteq \overline{N} \cup K$ so that

$$\widetilde{E} \subseteq \widetilde{\overline{N}} \vee \widetilde{K} = \overline{\widetilde{N}} \vee \widetilde{K} = \overline{S_{(F \cap (X \smallsetminus E))}} \vee \widetilde{K} \subseteq \overline{\widetilde{F} \cap \left(\widetilde{X} \smallsetminus \widetilde{E}\right)} \vee \widetilde{K}.$$

Therefore

$$\widetilde{E} \cap \left(\widetilde{X} \smallsetminus \overline{\widetilde{F} \cap \left(\widetilde{X} \smallsetminus \widetilde{E}\right)}\right) \subseteq \widetilde{K}.$$

Because $\widetilde{K}$ is a closed sublocale,

$$\mathrm{cl}_{\widetilde{E}} \left(\widetilde{E} \cap \left(\widetilde{X} \smallsetminus \overline{\widetilde{F} \cap (\widetilde{X} \smallsetminus \widetilde{E})}\right)\right) = \widetilde{E} \cap \overline{\left(\widetilde{E} \cap \left(\widetilde{X} \smallsetminus \overline{\widetilde{F} \cap (\widetilde{X} \smallsetminus \widetilde{E})}\right)\right)} \subseteq \widetilde{K}$$

so that $\widetilde{x} \in \widetilde{K}$. Therefore $x \in K$. Thus $x \in \overline{E \cap (X \smallsetminus \overline{N})}$ which implies $x \in \mathrm{cl}_E \left(E \cap (X \smallsetminus \overline{N})\right)$. As a result, $x \notin \mathrm{int}_E \left(E \cap \overline{N}\right)$. Hence x is almost E-inaccessible. □

In terms of sublocales, we define inaccessibility and almost inaccessibility on arbitrary locales. We give the following definition.

Definition 4.3.6. Let S be a sublocale of L. A sublocale $T \in \mathcal{S}(S)$ is *S-inaccessible* (resp. *almost S-inaccessible*) if for all $(L \smallsetminus S)$-nowhere dense sublocale N, $T \cap \overline{N} = \mathsf{O}$ (resp. $T \subseteq \mathrm{cl}_S \left(S \cap \left(L \smallsetminus \overline{N}\right)\right)$).

We introduce the following notations for any locale L and $S \in \mathcal{S}(L)$:

$$\mathcal{S}_{\mathrm{Inac}}(S) = \{A \in \mathcal{S}(L) : A \text{ is } S\text{-inaccessible}\},$$

and

$$\mathcal{S}_{\mathrm{Ainac}}(S) = \{A \in \mathcal{S}(L) : A \text{ is almost } S\text{-inaccessible}\}.$$

We shall drop the prefix S- if the sublocale is clear from the context.

In what follows, we characterize inaccessible sublocales. The proof is similar to that of Proposition 2.1.42 and shall be omitted.

Proposition 4.3.7. *The following are equivalent for a sublocale of S of L.*

1. *$T \in \mathcal{S}(S)$ is S-inaccessible.*

2. $T \cap \mathfrak{c}(x) = \mathsf{O}$ *for each* $(L \smallsetminus S)$*-dense* $x \in L \smallsetminus S$.

3. $T \subseteq \mathfrak{o}(x)$ *for every* $(L \smallsetminus S)$*-dense* $x \in L \smallsetminus S$.

4. $\nu_T(x) = 1$ *for each* $(L \smallsetminus S)$*-dense* $x \in L \smallsetminus S$.

For sublocales F and A of a locale L, we have that

$$F \subseteq \mathrm{cl}_F(F \cap A) \iff F = \mathrm{cl}_F(F \cap A) \iff F \cap A \text{ is } F\text{-dense.}$$

As a result of this, we have the following observation regarding almost inaccessible sublocales.

Observation 4.3.8. Let F be a sublocale of a locale L. The following statements are equivalent.

1. $F \in \mathcal{S}_{\mathrm{Ainac}}(F)$.

2. $F = \mathrm{cl}_F(F \cap (L \smallsetminus \overline{N}))$ for every $(L \smallsetminus F)$-nowhere dense N.

3. $F \cap (L \smallsetminus \overline{N})$ is F-dense for every $(L \smallsetminus F)$-nowhere dense N.

We give the following lemma which we shall use below.

Lemma 4.3.9. *Let L be a locale, $N \in \mathcal{S}(L)$, S a complemented sublocale of L and $T \in \mathcal{S}(S)$. Then $T \subseteq \mathrm{cl}_S(S \cap \left(L \smallsetminus \overline{N}\right))$ iff $T \cap \mathrm{int}_S\left(S \cap \overline{N}\right) = \mathsf{O}$.*

Proof. We have that

$$\begin{aligned} T \subseteq \mathrm{cl}_S(S \cap \left(L \smallsetminus \overline{N}\right)) &\iff T \subseteq \overline{S \cap \left(L \smallsetminus \overline{N}\right)} \\ &\iff T \cap \left(L \smallsetminus \overline{S \cap (L \smallsetminus \overline{N})}\right) = \mathsf{O} \\ &\iff T \cap S \cap \left(L \smallsetminus \overline{S \cap (L \smallsetminus \overline{N})}\right) = \mathsf{O} \\ &\iff T \cap \mathrm{int}_S\left(S \cap \overline{N}\right) = \mathsf{O} \end{aligned}$$

which proves the result. □

The preceding lemma leads us to the following characterization of almost inaccessible sublocales of complemented sublocales. We only prove the equivalences (2) $\iff$ (3).

Proposition 4.3.10. *The following are equivalent for a complemented sublocale S of a locale L and $T \in \mathcal{S}(S)$.*

1. *T is almost S-inaccessible.*

2. *$T \cap \operatorname{int}_S(S \cap \overline{N}) = \mathsf{O}$ for each $(L \smallsetminus S)$-nowhere dense sublocale N.*

If in particular, S is closed, this is further equivalent to:

3. *$a \to (\bigwedge S) \leq \bigwedge T$ for every $(L \smallsetminus S)$-dense a.*

Proof. (2) $\Longleftrightarrow$ (3): Let a be $(L \smallsetminus S)$-dense. Then $\mathfrak{c}_{(L\smallsetminus S)}(a)$ is $(L \smallsetminus S)$-nowhere dense. By (2),

$$\begin{aligned}
T \cap \operatorname{int}_S(S \cap \overline{\mathfrak{c}_{(L\smallsetminus S)}(a)}) = \mathsf{O} \quad &\Longleftrightarrow \quad T \subseteq \operatorname{cl}_S\left(S \cap \left(L \smallsetminus \overline{\mathfrak{c}_{(L\smallsetminus S)}(a)}\right)\right) \text{ since } S \text{ is complemented}\\
&\Longleftrightarrow \quad T \subseteq \overline{S \cap \left(L \smallsetminus \overline{\mathfrak{c}_{(L\smallsetminus S)}(a)}\right)}\\
&\Longleftrightarrow \quad T \subseteq \overline{S \cap (L \smallsetminus \mathfrak{c}(a))} \quad \text{since } \overline{\mathfrak{c}_{(L\smallsetminus S)}(a)} = \mathfrak{c}(a)\\
&\Longleftrightarrow \quad T \subseteq \mathfrak{c}\left(\bigwedge (S \cap \mathfrak{o}(a))\right)\\
&\Longleftrightarrow \quad T \subseteq \mathfrak{c}\left(a \to \left(\bigwedge S\right)\right) \quad \text{since } S \text{ is closed}\\
&\Longleftrightarrow \quad \overline{T} \subseteq \mathfrak{c}\left(a \to \left(\bigwedge S\right)\right)\\
&\Longleftrightarrow \quad a \to \left(\bigwedge S\right) \leq \bigwedge T.
\end{aligned}$$

Starting the above argument with an $(L \smallsetminus S)$-nowhere dense N and using the fact that N is $(L \smallsetminus S)$-nowhere dense if and only if $\bigwedge N$ is $(L \smallsetminus S)$, gives the desired equivalence. □

Next, we collect into one proposition some results about inaccessible sublocales and almost inaccessible sublocales.

Proposition 4.3.11. *Let L be a locale and $S \in \mathcal{S}(L)$.*

1. *Every S-inaccessible sublocale is almost S-inaccessible.*

2. *If $A \in \mathcal{S}(L)$ is S-inaccessible (resp. almost S-inaccessible) and $\mathcal{S}(L) \ni B \subseteq A$, then B is S-inaccessible (resp. almost S-inaccessible).*

3. *If S is open in L, then S is S-inaccessible.*

4. *L is L-inaccessible and hence by (2) every sublocale of L is L-inaccessible.*

5. *If S is complemented, then every sublocale T of $L \smallsetminus S$ which is open in L is $(L \smallsetminus S)$-inaccessible.*

6. *A join of S-inaccessible (resp. almost S-inaccessible) sublocales is S-inaccessible (resp. almost S-inaccessible).*

7. *If S is open, then $S \cap \mathrm{Rs}(L \ltimes S)$ is S-inaccessible. Moreover, if S is dense and open, then $\mathfrak{B}L$ is S-inaccessible.*

Proof. (1) Let $T \in \mathcal{S}(S)$ be such that T is S-inaccessible. Then $T \cap \overline{N} = \mathsf{O}$ for all $N \in \mathfrak{ND}(L \smallsetminus S)$, which implies that $T \subseteq L \smallsetminus \overline{N}$. Therefore $T \subseteq S \cap (L \smallsetminus \overline{N}) \subseteq \mathrm{cl}_S(S \cap (L \smallsetminus \overline{N}))$ which proves the result.

(2) Straightforward.

(3) Assume that S is open in L and choose $N \in \mathfrak{ND}(L \smallsetminus S)$. It is clear that $\overline{N} \subseteq L \smallsetminus S$ since $L \smallsetminus S$ is closed. Because S is complemented, we have that $S \cap (L \smallsetminus S) = \mathsf{O}$ so that $S \cap \overline{N} = \mathsf{O}$. Thus $S \in \mathcal{S}_{\mathrm{Inac}}(S)$.

(4) Because L is open as a sublocale of itself, it follows from (3) that L is L-inaccessible. Therefore, by (2), every sublocale of L is L-inaccessible.

(5) Let T be a sublocale of $L \smallsetminus S$ which is open in L. We must show that $T \cap \overline{N} = \mathsf{O}$ for every $(L \smallsetminus (L \smallsetminus S))$-nowhere dense N, i.e., for every S-nowhere dense N. Since $T \subseteq L \smallsetminus S$, $T \cap S = \mathsf{O}$ because S is complemented. So, for any S-nowhere dense N, $T \cap N = \mathsf{O}$. But T is open in L so $T \cap \overline{N} = \mathsf{O}$.

(6) We only verify the case of S-inaccessible. Let $U_i \in \mathcal{S}_{\mathrm{Inac}}(S)$ (for $i \in I$) and choose $N \in \mathfrak{ND}(L \smallsetminus S)$. Since $\overline{N}$ is complemented, $\overline{N} \cap \bigvee U_i = \bigvee \left(\overline{N} \cap U_i\right) = \bigvee\{\mathsf{O}\} = \mathsf{O}$. Thus $\bigvee U_i \in \mathcal{S}_{\mathrm{Inac}}(S)$.

(7) Since S is open and hence complemented, we have that

$$S \cap \mathrm{Rs}(L \ltimes S) = S \cap \bigvee\{A \in \mathcal{S}(L) : A \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)\} = \bigvee\{A \cap S : A \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)\}.$$

Because a join of S-inaccessible sublocales is S-inaccessible, it suffices to show that each sublocale of the form $S \cap A$ for some $A \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)$ is S-inaccessible. Since $S \in \mathcal{S}_{\mathrm{Inac}}(S)$ and every sublocale contained in an inaccessible sublocale is inaccessible, we get that sublocales of the from $S \cap A$ for some $A \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)$ are S-inaccessible. Thus $S \cap \mathrm{Rs}(L \ltimes S)$ is S-inaccessible.

Since, according to Observation 3.1.13(2), $\mathfrak{B}L = T \cap \mathrm{Rs}(L \ltimes T)$ for any dense $T \in \mathcal{S}(L)$, we have that $\mathfrak{B}L = S \cap \mathrm{Rs}(L \ltimes S)$ is S-inaccessible for dense and open S. □

Observation 4.3.12. Proposition 4.3.11(4) differentiates between remoteness and inaccessibility. Recall that a necessary and sufficient condition for a locale L to be remote as a sublocale of itself is that it must be Boolean. Yet, every locale L (not necessarily Boolean) is L-inaccessible as a sublocale of itself.

Remark 4.3.13. We note from the preceding observation that since every sublocale S of a locale L is a locale in its own right, it is therefore S-inaccessible as a sublocale of itself. However, in this thesis, the notion $S \in \mathcal{S}_{\mathrm{Inac}}(S)$ for $S \in \mathcal{S}(L)$, read as S is inaccessible as a sublocale of L with respect to itself, shall mean $S \cap \overline{N} = \mathsf{O}$ for every $(L \smallsetminus S)$-nowhere dense sublocale N. This also applies to $S \in \mathcal{S}_{\mathrm{Ainac}}(S)$.

We note some examples.

Example 4.3.14. (1) In a completely regular locale L, a point p of L is $\mathfrak{c}(p)$-inaccessible (resp. almost $\mathfrak{c}(p)$-inaccessible) if and only if $\mathfrak{c}(p)$ is $\mathfrak{c}(p)$-inaccessible (resp. almost $\mathfrak{c}(p)$-inaccessible).

(2) [58] In **Top**, we have that a point $p \in \beta X \smallsetminus X$, where X is Tychonoff, is remote if and only if p is $(\beta X \smallsetminus X)$-inaccessible.

Observe that for each dense and complemented $S \in \mathcal{S}(L)$, $T \in \mathcal{S}(L \smallsetminus S)$ is $(L \smallsetminus S)$-inaccessible if and only if $T \cap \overline{N} = \mathsf{O}$ for every $(L \smallsetminus (L \smallsetminus S))$-nowhere dense sublocale N, i.e. for every S-nowhere dense sublocale N. This shows that sublocales of $L \smallsetminus S$ which are *remote from a dense and complemented sublocale S of L are precisely the $(L \smallsetminus S)$-inaccessible sublocales. We formalise this in the following proposition.

Proposition 4.3.15. *A sublocale $T \in \mathcal{S}(L \smallsetminus S)$ where S is dense and complemented in a locale L, is $(L \smallsetminus S)$-inaccessible iff it is *remote from S.*

In the following result, we codify the variants of remoteness and inaccessibility for dense and complemented sublocales.

Proposition 4.3.16. *Let L be a locale, S a dense and complemented sublocale of L and $T \in \mathcal{S}(L \smallsetminus S)$. We have the following situation where arrows indicate implications:*

$$\boxed{(1)\ T \in \mathcal{S}_{rem}(L).} \longrightarrow \boxed{(2)\ T \in \mathcal{S}_{rem}(L \ltimes S).} \longleftrightarrow \boxed{(3)\ T \in {}^{*}\mathcal{S}_{rem}(L \ltimes S).}$$

$$\boxed{(3)\ T \in {}^{*}\mathcal{S}_{rem}(L \ltimes S).} \updownarrow \boxed{(4)\ T \in \mathcal{S}_{\text{Inac}}(L \smallsetminus S).} \downarrow \boxed{(5)\ T \in \mathcal{S}_{\text{Ainac}}(L \smallsetminus S).}$$

Proof. (1) $\Longrightarrow$ (2): Follows from Proposition 2.1.37(4).

(2) $\Longleftrightarrow$ (3): This is a combination of Proposition 2.1.34 and the fact that $T \subseteq L \smallsetminus S$.

(3) $\Longleftrightarrow$ (4): Follows from Proposition 4.3.15.

(4) $\Longrightarrow$ (5): For each S-nowhere dense N, we have that $T \cap \overline{N} = \mathsf{O}$ by (4). Therefore $T \subseteq (L \smallsetminus S) \cap (L \smallsetminus \overline{N})$ which implies that $T \subseteq \mathrm{cl}_{L \smallsetminus S}\big((L \smallsetminus S) \cap (L \smallsetminus \overline{N})\big)$, as required. □

We give the following theorem in which some of its statements prepare us for a relationship between maximal nowhere density and remoteness in Proposition 4.3.23.

Theorem 4.3.17. *Let L be a locale and F be a non-void and closed nowhere dense sublocale of L. Then each of the following statements holds.*

1. *If $F \in \mathcal{S}_{\text{Ainac}}(F)$, then $F \in \mathfrak{M}(L)$.*
2. *If L is compact, then $F \in \mathfrak{M}(L)$ implies that there is $x \in F$ such that $x \notin \mathrm{int}_F(\overline{N} \cap F)$ for every $(L \smallsetminus F)$-nowhere dense N.*
3. *$F \in \mathfrak{H}\mathfrak{M}(L)$ implies that every sublocale of F is almost F-inaccessible.*

Proof. (1) Let $F = \mathfrak{c}(b)$ for some $b \in L$ and choose $\mathfrak{c}(c) \in \mathfrak{N}\mathfrak{D}(L)$ such that $F \subseteq \mathfrak{c}(c)$. We show that F is not nowhere dense in $\mathfrak{c}(c)$. Observe that $\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b) \in \mathfrak{N}\mathfrak{D}(\mathfrak{o}(b))$. Indeed, if

$$\mathfrak{o}(x) \cap \mathfrak{o}(b) \subseteq \overline{\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b)}^{\mathfrak{o}(b)} = \overline{\mathfrak{c}(c) \cap \mathfrak{o}(b)} \cap \mathfrak{o}(b) = \mathfrak{c}(c) \cap \mathfrak{o}(b),$$

then $\mathfrak{o}(x) \cap \mathfrak{o}(b) = \mathfrak{o}(x \wedge b) \subseteq \mathfrak{c}(c)$. But $\mathfrak{c}(c) \in \mathfrak{ND}(L)$, so $\mathfrak{o}(x \wedge b) = \mathsf{O}$. Thus $\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b) \in \mathfrak{ND}(\mathfrak{o}(b))$.

Since $\mathfrak{c}(b) \in \mathcal{S}_{\text{Ainac}}(\mathfrak{c}(b))$, we have that

$$\mathfrak{c}(b) = \text{cl}_{\mathfrak{c}(b)} \left(\mathfrak{c}(b) \cap \left(L \smallsetminus \overline{(\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b))} \right) \right) = \text{cl}_{\mathfrak{c}(b)} \left(\mathfrak{c}(b) \smallsetminus \overline{(\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b))} \right).$$

Because $\mathfrak{c}(b)$ is non-void, $\mathfrak{c}(b) \smallsetminus \overline{(\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b))} \neq \mathsf{O}$. Since $\mathfrak{c}(b) \subseteq \mathfrak{c}(c)$, $\mathfrak{c}(c) \smallsetminus \overline{(\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b))} \neq \mathsf{O}$.

We must have that $\mathfrak{c}(b) \notin \mathfrak{ND}(\mathfrak{c}(c))$, otherwise

$$\begin{aligned} \mathsf{O} &= \text{int}_{\mathfrak{c}(c)} \left(\overline{\mathfrak{c}(b)}^{\mathfrak{c}(c)} \right) \\ &= \mathfrak{c}(c) \cap \left(L \smallsetminus \overline{(\mathfrak{c}(c) \cap L \smallsetminus \overline{\mathfrak{c}(b)})} \right) \\ &= \mathfrak{c}(c) \cap \left(L \smallsetminus \overline{(\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b))} \right) \\ &= \mathfrak{c}(c) \smallsetminus \overline{(\mathfrak{c}(c) \smallsetminus \mathfrak{c}(b))} \end{aligned}$$

which is not possible. So there is no closed $K \in \mathfrak{ND}(L)$ such that $F \in \mathfrak{ND}(K)$. Thus $F \in \mathfrak{M}(L)$.

(2) Assume that L is compact, $F = \mathfrak{c}(b) \in \mathfrak{M}(L)$ and suppose that for each $x \in F$, there is an $\mathfrak{o}(b)$-nowhere dense N_x such that $x \in \text{int}_{\mathfrak{c}(b)}(\mathfrak{c}(b) \cap \overline{N_x})$. Set $\text{int}_{\mathfrak{c}(b)}(\mathfrak{c}(b) \cap \overline{N_x}) = \mathfrak{o}(a_x) \cap \mathfrak{c}(b)$. Then

$$\mathfrak{c}(b) \subseteq \bigvee_{x \in \mathfrak{c}(b)} (\mathfrak{o}(a_x) \cap \mathfrak{c}(b)) \subseteq \mathfrak{o}\left(\bigvee_{x \in \mathfrak{c}(b)} a_x \right).$$

Therefore $b \vee \left(\bigvee_{x \in \mathfrak{c}(b)} a_x \right) = 1$. By compactness of L, there is a finite set $B \subseteq \mathfrak{c}(b)$ such that

$b \vee \left(\bigvee_{x\in B} a_x\right) = 1$. We get that

$$\begin{aligned}
\mathfrak{c}(b) &\subseteq \mathfrak{c}(b) \cap \mathfrak{o}\left(\bigvee_{x\in B} a_x\right) \\
&= \mathfrak{c}(b) \cap \bigvee_{x\in B} \mathfrak{o}(a_x) \\
&= \bigvee_{x\in B} (\mathfrak{c}(b) \cap \mathfrak{o}(a_x)) \\
&= \bigvee_{x\in B} \left(\operatorname{int}_{\mathfrak{c}(b)}(\mathfrak{c}(b) \cap \overline{N_x})\right) \\
&\subseteq \bigvee_{x\in B} \left(\mathfrak{c}(b) \cap \overline{N_x}\right) \\
&= \mathfrak{c}(b) \cap \bigvee_{x\in B} \overline{N_x} \\
&\subseteq \bigvee_{x\in B} \overline{N_x} \\
&= \overline{\bigvee_{x\in B} N_x}.
\end{aligned}$$

Observe that $N_x \in \mathfrak{ND}(L)$. This is so because N_x is nowhere dense in a dense sublocale $\mathfrak{o}(b)$ of L making it nowhere dense in L. Therefore $\overline{N_x} \in \mathfrak{ND}(L)$. Since finite joins of closed nowhere dense sublocales are nowhere dense, $\bigvee_{x\in B} \overline{N_x} = \overline{\bigvee_{x\in B} N_x}$ is nowhere dense in L. We show that $\mathfrak{c}(b)$ is nowhere dense in $\overline{\bigvee_{x\in B} N_x}$ which will contradict that $\mathfrak{c}(b) \in \mathfrak{M}(L)$.

Set $A = \overline{\bigvee_{x\in B} N_x}$. Observe that

$$\begin{aligned}
\operatorname{int}_A\left(\overline{\mathfrak{c}(b)}^A\right) &= A \cap \left(L \smallsetminus \overline{(A \cap L \smallsetminus \overline{\mathfrak{c}(b)})}\right) \\
&= A \cap \left(L \smallsetminus \overline{(A \cap \mathfrak{o}(b)}\right) \\
&= A \cap \left(L \smallsetminus \overline{\left(\overline{\bigvee_{x\in B} N_x} \cap \mathfrak{o}(b)\right)}\right) \\
&\subseteq A \cap \left(L \smallsetminus \overline{\left(\left(\bigvee_{x\in B} N_x\right) \cap \mathfrak{o}(b)\right)}\right) \\
&= A \cap \left(L \smallsetminus \overline{\bigvee_{x\in B} N_x}\right) \quad \text{since} \quad \bigvee_{x\in B} N_x \subseteq \mathfrak{o}(b) \\
&= \mathsf{O}.
\end{aligned}$$

Thus $\mathfrak{c}(b)$ is nowhere dense in A which is a contradiction.

(3) Suppose that there is $B \in \mathcal{S}(F)$ such that $B \notin \mathcal{S}_{\text{Ainac}}(F)$. Then, by Proposition 4.3.10, $B \cap \text{int}_F(\overline{N} \cap F) \neq \mathsf{O}$ for some $N \in \mathfrak{N}\mathfrak{D}(L \smallsetminus F)$. We get that $\text{int}_F(\overline{N} \cap F) \neq \mathsf{O}$. Set $A = \text{int}_F(\overline{N} \cap F)$. Then $\overline{A}^F = \overline{A}$ is a non-void F-regular-closed sublocale. Since $F \in \mathfrak{H}\mathfrak{M}(L)$, $\overline{A}$ is maximal nowhere dense in L. We show that $\overline{A}$ is nowhere dense in $\overline{N}$ which will contradict that it is a maximal nowhere dense sublocale. It is clear that $\overline{A} \subseteq \overline{N}$. Furthermore, observe that

$$\begin{aligned}
\text{int}_{\overline{N}}\left(\overline{A}^{\overline{N}}\right) &= \text{int}_{\overline{N}}(\overline{A}) \\
&= \overline{N} \cap \left(L \smallsetminus \overline{\overline{N} \cap L \smallsetminus \overline{A}}\right) \\
&\subseteq \overline{N} \cap \left(L \smallsetminus \overline{\overline{N} \cap L \smallsetminus F}\right) \quad \text{since} \quad \overline{A} \subseteq F \\
&\subseteq \overline{N} \cap \left(L \smallsetminus \overline{N \cap L \smallsetminus F}\right) \\
&= \overline{N} \cap \left(L \smallsetminus \overline{N}\right) \quad \text{since} \quad N \subseteq L \smallsetminus F \\
&= \mathsf{O}.
\end{aligned}$$

Thus $\overline{A}$ is nowhere dense in $\overline{N}$ which is not possible. Hence every sublocale of F belongs to $\mathcal{S}_{\text{Inac}}(F)$. □

If we consider F-clopen sublocales, we get the following result.

Proposition 4.3.18. *Let F be a non-void nowhere dense sublocale of L. If F is homogeneous maximal nowhere dense, then each F-clopen sublocale is almost inaccessible as a sublocale of L with respect to itself.*

Proof. Let A be an F-clopen sublocale and assume that $A \cap \text{int}_A(\overline{N} \cap A) \neq \mathsf{O}$ for some $N \in \mathfrak{N}\mathfrak{D}(L \smallsetminus A)$. Then $\text{int}_A(\overline{N} \cap A)$ is F-open so that $\overline{\text{int}_A(\overline{N} \cap A)} \cap F = \overline{\text{int}_A(\overline{N})}$ is a non-void regular-closed sublocale of F. By hypothesis, $\overline{\text{int}_A(\overline{N} \cap A)}$ is maximal nowhere dense. Following the argument used in last part of the proof of Theorem 4.3.17(3) and using the fact that A is F-closed, we get that $\overline{\text{int}_A(\overline{N} \cap A)}$ is nowhere dense in $\overline{N}$ which is not possible. □

We note the following example.

Example 4.3.19. (1) If X is a Hausdorff space with no isolated point, then every one-point sublocale of $\mathfrak{O}X$ which is almost inaccessible as a sublocale of $\mathfrak{O}X$ with respect to itself is maximal nowhere dense. To see this, it suffices to show that such sublocales are non-void closed nowhere dense in $\mathfrak{O}X$. Since every Hausdorff space is sober, each point p of $\mathfrak{O}X$ is of the form $p = X \smallsetminus \overline{\{x\}}$ for some $x \in X$. Applying Hausdorffness again gives $p = X \smallsetminus \{x\}$ which is open and dense in X, making $\{x\}$ closed nowhere dense in X. It follows from Lemma 2.1.15 that $\widetilde{\{x\}}$ is closed and nowhere dense in $\mathfrak{O}X$. But $\widetilde{\{x\}} = \{X \smallsetminus \{x\}, 1_{\mathfrak{O}X}\} = \{p, 1_{\mathfrak{O}X}\}$, so the one-point sublocale $\{p, 1_{\mathfrak{O}X}\}$ is non-void closed nowhere dense. Now, if such a one-point sublocale $\{p, 1_{\mathfrak{O}X}\}$ is an almost inaccessible sublocale of L with respect to itself, it follows from Proposition 4.3.17(1) that $\{p, 1_{\mathfrak{O}X}\}$ is maximal nowhere dense.

(2) The sublocales described in (1) are homogeneous maximal nowhere dense. This is so because for any point $p \in \mathfrak{O}X$, $\{p, 1_{\mathfrak{O}X}\}$ is the only non-void sublocale contained in $\{p, 1_{\mathfrak{O}X}\}$. Therefore all non-void $\{p, 1_{\mathfrak{O}X}\}$-regular-closed sublocales are maximal nowhere dense in $\mathfrak{O}X$.

We include the following result where we make use of Theorem 4.3.17(3) to show that a homogeneous maximal nowhere dense sublocale is regular-closed in every complemented nowhere dense sublocale containing it.

Proposition 4.3.20. *Let L be a locale and F a non-void closed nowhere dense sublocale of L. If $F \in \mathfrak{HM}(L)$ and $F \subseteq A$, where A is a complemented nowhere dense sublocale of L, then F is an A-regular-closed sublocale.*

Proof. Assume that $F \neq \overline{\mathrm{int}_A(F)}$. Because it is always true that $\overline{\mathrm{int}_A(F)} \subseteq F$, this assumption says that $F \nsubseteq \overline{\mathrm{int}_A(F)}$. Therefore the F-open sublocale $F \smallsetminus \overline{\mathrm{int}_A(F)} = F \cap \left(L \smallsetminus \overline{\mathrm{int}_A(F)}\right)$ is non-void. Also, $F \smallsetminus \overline{\mathrm{int}_A(F)} \subseteq \overline{A \smallsetminus F}$. Indeed,

$$\begin{aligned}
F \smallsetminus \overline{\mathrm{int}_A(F)} &\subseteq F \cap (L \smallsetminus \mathrm{int}_A(F)) \\
&= F \cap \Big(L \smallsetminus \Big(A \cap (L \smallsetminus \overline{A \cap (L \smallsetminus F)})\Big)\Big) \\
&= F \cap \Big((L \smallsetminus A) \vee (L \smallsetminus (L \smallsetminus \overline{A \cap (L \smallsetminus F)}))\Big) \\
&= F \cap \Big((L \smallsetminus A) \vee \overline{A \cap (L \smallsetminus F)}\Big) \\
&= (F \cap (L \smallsetminus A)) \vee (F \cap \overline{A \cap (L \smallsetminus F)}) \\
&= \mathsf{O} \vee (F \cap \overline{A \cap (L \smallsetminus F)}) \text{ since } A \text{ is complemented and } F \subseteq A \\
&= F \cap \overline{A \cap (L \smallsetminus F)} \\
&\subseteq \overline{A \cap (L \smallsetminus F)} = \overline{A \smallsetminus F}.
\end{aligned}$$

This makes $\left(F \smallsetminus \overline{\mathrm{int}_A(F)}\right) \cap \mathrm{int}_F(F \cap \overline{A \smallsetminus F}) \neq \mathsf{O}$. Observe that $A \smallsetminus F \in \mathfrak{ND}(L \smallsetminus F)$. To see this, let U be an open sublocale of $L \smallsetminus F$ contained in $\overline{A \smallsetminus F}^{(L \smallsetminus F)} = \overline{A \smallsetminus F} \cap (L \smallsetminus F)$. Then $U \subseteq \overline{A}$. But $A \in \mathfrak{ND}(L)$ and an open sublocale of $L \smallsetminus F$ is open in L, we have that $U = \mathsf{O}$. Thus $A \smallsetminus F \in \mathfrak{ND}(L \smallsetminus F)$. We have found a sublocale $F \smallsetminus \overline{\mathrm{int}_A(F)}$ of F and $A \smallsetminus F \in \mathfrak{ND}(L \smallsetminus F)$ such that $\left(F \smallsetminus \overline{\mathrm{int}_A(F)}\right) \cap \mathrm{int}_F\left(F \cap \overline{A \smallsetminus F}\right) \neq \mathsf{O}$, i.e, a sublocale of F which is not almost F-inaccessible. By Theorem 4.3.17(3), F is not homogeneous maximal nowhere dense, which is a contradiction. □

In what follows, we characterize locales in which every non-void nowhere dense sublocale is maximal nowhere dense. Recall from [19] that the *boundary* of a sublocale S of a locale L is given by $\mathrm{bd}(S) = \overline{S} \smallsetminus \mathrm{int}(S)$ and a sublocale S of L is *preopen* if $S \subseteq \mathrm{int}\,(\overline{S})$. For an open sublocale $\mathfrak{o}(x) \in \mathcal{S}(L)$, $\mathrm{bd}(\mathfrak{o}(x)) = \mathfrak{c}(x \vee x^*)$ which is nowhere dense. Call a sublocale S of a locale L *semi-open* in case $S \subseteq \overline{\mathrm{int}(S)}$ and *α-open* provided that $S \subseteq \mathrm{int}(\overline{\mathrm{int}(S)})$.

Theorem 4.3.21. *Let L be a locale. The following statements are equivalent.*

1. *Every non-void nowhere dense sublocale of L is maximal nowhere dense.*
2. *Every non-void closed nowhere dense sublocale is maximal nowhere dense.*
3. *Every non-void open sublocale induced by a non-complemented element of L has a maximal nowhere dense boundary.*

4. *Every non-void closed nowhere dense sublocale of L is homogeneous maximal nowhere dense.*

5. *Every non-void closed nowhere dense sublocale is almost inaccessible as a sublocale of L with respect to itself.*

6. *Every non-void closed nowhere dense sublocale is inaccessible as a sublocale of L with respect to itself.*

7. *Every non-void preopen sublocale of a closed nowhere dense sublocale is maximal nowhere dense.*

8. *Every non-void semi-open sublocale of a closed nowhere dense sublocale is maximal nowhere dense.*

9. *Every non-void α-open sublocale of a closed nowhere dense sublocale is maximal nowhere dense.*

Proof. (1) $\Longleftrightarrow$ (2): Follows from Proposition 4.1.5.

(2) $\Longrightarrow$ (3): Let $\mathfrak{o}(x) \in \mathcal{S}(L)$ be non-void with x non-complemented. Then $x \vee x^* \neq 1$ making $\mathrm{bd}(\mathfrak{o}(x)) = \mathfrak{c}(x \vee x^*)$ a non-void closed nowhere dense sublocale. It follows from (2) that $\mathrm{bd}(\mathfrak{o}(x))$ is maximal nowhere dense.

(3) $\Longrightarrow$ (4): Let $\mathfrak{c}(x)$ be a non-void nowhere dense sublocale of L and choose $y \in L$ such that $\overline{\mathfrak{o}(y) \cap \mathfrak{c}(x)} \cap \mathfrak{c}(x) \neq \mathsf{O}$. Then

$$\overline{\mathfrak{o}(y) \cap \mathfrak{c}(x)} \cap \mathfrak{c}(x) = \overline{\mathfrak{o}(y) \cap \mathfrak{c}(x)} = \mathfrak{c}(y \to x) \neq \mathsf{O}$$

implying that $y \to x \neq 1$. But $\mathfrak{c}(y \to x) \subseteq \mathfrak{c}(x) \in \mathfrak{ND}(L)$, so $\mathfrak{c}(y \to x) \in \mathfrak{ND}(L)$ making $\mathfrak{o}(y \to x)$ non-void, open and

$$(y \to x) \vee (y \to x)^* = (y \to x) \vee 0 = y \to x \neq 1.$$

It follows from (3) that

$$\mathrm{bd}(\mathfrak{o}(y \to x)) = \mathfrak{c}((y \to x) \vee (y \to x)^*) = \mathfrak{c}(y \to x) = \overline{\mathfrak{o}(y) \cap \mathfrak{c}(x)}$$

is maximal nowhere dense. Thus $\mathfrak{c}(x)$ is homogeneous maximal nowhere dense.

(4) $\Longrightarrow$ (5): Follows from Theorem 4.3.17(3).

(5) $\Longrightarrow$ (6): Let $\mathfrak{c}(x)$ be a non-void nowhere dense sublocale and assume that $\overline{N}\cap\mathfrak{c}(x)\neq \mathsf{O}$ for some $N\in\mathfrak{ND}((L\smallsetminus\mathfrak{c}(x))=\mathfrak{o}(x))$. Since $\mathfrak{o}(x)$ is dense in L, N is nowhere dense in L so that the non-void sublocale $\overline{N}\cap\mathfrak{c}(x)$ is closed nowhere dense in L. It follows from (5) that $\overline{N}\cap\mathfrak{c}(x)$ is almost $\big(\overline{N}\cap\mathfrak{c}(x)\big)$-inaccessible. Observe that $N\in\mathfrak{ND}(L\smallsetminus(\overline{N}\cap\mathfrak{c}(x)))$. To see this, let $a\in L$ be such that $\mathfrak{o}(a)\cap\big(L\smallsetminus(\overline{N}\cap\mathfrak{c}(x))\big)\subseteq\overline{N}\cap\big(L\smallsetminus(\overline{N}\cap\mathfrak{c}(x))\big)$. Then $\mathfrak{o}(a)\cap\big(L\smallsetminus(\overline{N}\cap\mathfrak{c}(x))\big)\subseteq\overline{N}$. Because $N\in\mathfrak{ND}(L)$ and $\mathfrak{o}(a)\cap\big(L\smallsetminus(\overline{N}\cap\mathfrak{c}(x))\big)$ is open, $\mathfrak{o}(a)\cap\big(L\smallsetminus(\overline{N}\cap\mathfrak{c}(x))\big)=\mathsf{O}$. Therefore $\mathfrak{o}(a)\subseteq\overline{N}\cap\mathfrak{c}(x)$ making $\mathfrak{o}(a)=\mathsf{O}$ since $\overline{N}\cap\mathfrak{c}(x)=\overline{\overline{N}\cap\mathfrak{c}(x)}$ is nowhere dense in L. Thus $N\in\mathfrak{ND}(L\smallsetminus(\overline{N}\cap\mathfrak{c}(x)))$.

Therefore $\overline{N}\cap\mathfrak{c}(x)\subseteq\overline{\big(\overline{N}\cap\mathfrak{c}(x)\big)\cap\big(L\smallsetminus\overline{N}\big)}=\overline{\mathsf{O}}=\mathsf{O}$ which is not possible. Thus $\mathfrak{c}(x)\cap\overline{N}=\mathsf{O}$ making $\mathfrak{c}(x)\in\mathcal{S}_{\text{Inac}}(\mathfrak{c}(x))$.

(6) $\Longrightarrow$ (7): Let F be closed nowhere dense and A be a non-void F-preopen sublocale. Then $\overline{A}$ is closed nowhere dense in L and non-void. It follows from (6) that $\overline{A}\in\mathcal{S}_{\text{Inac}}(\overline{A})$. Because $\mathcal{S}_{\text{Inac}}(\overline{A})\subseteq\mathcal{S}_{\text{Ainac}}(\overline{A})$ by Proposition 4.3.11(1), it follows from Theorem 4.3.17(1) that $\overline{A}$ is maximal nowhere dense. Because $\overline{A}\subseteq F$, it follows from Proposition 4.1.11(1) that F is maximal nowhere dense.

(7) $\Longrightarrow$ (8): Let F be a closed nowhere sublocale of L and A a non-void F-semi-open sublocale. Then $\overline{\operatorname{int}_F(A)}^F=\overline{\operatorname{int}_F(A)}\neq\mathsf{O}$ since $\mathsf{O}\neq A\subseteq\overline{\operatorname{int}_F(A)}^F$ and F is closed. Since every open sublocale is preopen, $\operatorname{int}_F(A)$ is preopen, non-void and nowhere dense because it is contained in the nowhere dense sublocale $\overline{\operatorname{int}_F(A)}^F$ which is nowhere dense by virtue of being a sublocale of the nowhere dense sublocale F. It follows from (6) that $\operatorname{int}_F(A)$ is maximal nowhere dense. By Proposition 4.1.5, $\overline{\operatorname{int}_F(A)}$ is maximal nowhere dense. Observe that $\overline{A}=\overline{\operatorname{int}_F(A)}$. Indeed, $\operatorname{int}_F(A)\subseteq A\subseteq\overline{A}$ implying that $\overline{\operatorname{int}_F(A)}\subseteq\overline{A}$. On the other hand, $A\subseteq\overline{\operatorname{int}_F(A)}$ implies $\overline{A}\subseteq\overline{\operatorname{int}_F(A)}$. Thus $\overline{A}=\overline{\operatorname{int}_F(A)}$. Therefore $\overline{A}$ is maximal nowhere dense. Applying Proposition 4.1.5 yields A is maximal nowhere dense.

(8) $\Longrightarrow$ (9): Let F be a closed nowhere sublocale of L and $A\in\mathcal{S}(F)$ be such that $\mathsf{O}\neq A\subseteq\operatorname{int}_F\Big(\overline{\operatorname{int}_F(A)}^F\Big)$. Then A is nowhere dense in L and $\mathsf{O}\neq A\subseteq\operatorname{int}_F\Big(\overline{\operatorname{int}_F(A)}^F\Big)\subseteq\overline{\operatorname{int}_F(A)}$ so that A is non-void F-semi-open. By (7), A is maximal nowhere dense.

(9) $\Longrightarrow$ (1): Let F be a non-void nowhere dense sublocale of L. Then $\overline{F}$ is non-void nowhere dense. Since every locale is α-open as a sublocale of itself, $\overline{F}$ is $\overline{F}$-α-open. It follows from (8) that $\overline{F}$ is maximal nowhere dense. By Proposition 4.1.5, F is maximal nowhere dense. $\square$

In the following example we show that there is a locale having the properties described in Theorem 4.3.21.

Example 4.3.22. Consider the three-element chain $\mathbf{3} = \{1, 0, a\}$. Clearly, $\mathbf{3}$ is non-Boolean and the only non-void closed nowhere dense sublocale of $\mathbf{3}$ is $\mathfrak{c}(a)$ which is maximal nowhere dense because it is not nowhere dense as a sublocale of itself.

Using Proposition 4.3.16 and the fact that if $S \in \mathcal{S}(L)$ is open and dense, then $S^{\#} = L \smallsetminus S$ is nowhere dense, we get the following result about remoteness and maximal nowhere density.

Proposition 4.3.23. *Let $S \neq L$ be an open dense sublocale of L.*

1. *If $S^{\#} \in {}^{*}\mathcal{S}_{rem}(L \ltimes S)$, then $S^{\#} \in \mathfrak{M}(L)$.*
2. *If $S^{\#} \in \mathfrak{H}\mathfrak{M}(L)$, then every $S^{\#}$-remote sublocale is *remote from S.*

Proof. (1) If $S^{\#} \in {}^{*}\mathcal{S}_{\mathrm{rem}}(L \ltimes S)$, then, by Proposition 4.3.16, $S^{\#} \in \mathcal{S}_{\mathrm{Ainac}}(S^{\#})$. It follows from Theorem 4.3.17(1) that $S^{\#} \in \mathfrak{M}(L)$.

(2) Let $A \in \mathcal{S}_{\mathrm{rem}}(S^{\#})$ and choose an S-nowhere dense N. Since, by Theorem 4.3.17(3), sublocales of homogeneous maximal nowhere dense sublocales are almost inaccessible as sublocales of L with respect to themselves, $S^{\#}$ is almost $S^{\#}$-inaccessible, i.e., $\mathrm{int}_{S^{\#}}(S^{\#} \cap \overline{N}) = \mathsf{O}$. Since $S^{\#} \cap \overline{N} = \overline{S^{\#} \cap \overline{N}}^{S^{\#}}$, we get that $S^{\#} \cap \overline{N}$ is $S^{\#}$-nowhere dense. Because $A \in \mathcal{S}_{\mathrm{rem}}(S^{\#})$,

$$\mathsf{O} = A \cap S^{\#} \cap \overline{N} = A \cap \overline{N}.$$

Thus $A \in \mathcal{S}_{\mathrm{rem}}(L \ltimes S)$ making $A \in {}^{*}\mathcal{S}_{\mathrm{rem}}(L \ltimes S)$ since $A \subseteq S^{\#}$. $\square$

4.4 Preservation and reflection of maximal nowhere density

We end this chapter with a discussion of localic maps that send maximal nowhere density back and forth. We shall also include results about inaccessibility.

Proposition 4.4.1. *Let $f : L \to M$ be a localic map such that both f and h send dense elements to dense elements. Then f reflects maximal nowhere dense sublocales.*

Proof. Let $N \in \mathcal{S}(M)$ be maximal nowhere dense in M. Then $\overline{N} \in \mathfrak{M}(M)$. Since h is weakly open, it follows from Theorem 3.1.2 that f reflects closed nowhere dense sublocales so that $f_{-1}[\overline{N}]$ (which is equal to $\mathfrak{c}(h(\bigwedge N))$) is nowhere dense in L. It is left to show that it is maximal nowhere dense. Suppose not, that is, there exists a nowhere dense sublocale $\mathfrak{c}(b)$ of L such that $f_{-1}[\overline{N}]$ is nowhere dense in $\mathfrak{c}(b)$. Since f sends dense elements to dense elements, $f(b)$ is dense in M making $\mathfrak{c}(f(b))$ nowhere dense in M. The sublocale $\overline{N}$ is nowhere dense in the nowhere dense sublocale $\mathfrak{c}(f(b))$. Indeed, if $\mathfrak{o}(x) \cap \mathfrak{c}(f(b)) \subseteq \overline{N}^{\mathfrak{c}(f(b))} = \overline{N} \cap \mathfrak{c}(f(b))$ for some $x \in M$, then $\mathfrak{o}(x) \cap \mathfrak{c}(f(b)) \subseteq \overline{N}$ so that

$$\mathfrak{o}(h(x)) \cap \mathfrak{c}(b) \subseteq \mathfrak{o}(h(x)) \cap \mathfrak{c}(h(f(b))) = f_{-1}[\mathfrak{o}(x) \cap \mathfrak{c}(f(b))] \subseteq f_{-1}[\overline{N}].$$

Therefore $\mathfrak{o}(h(x)) \cap \mathfrak{c}(b) \subseteq f_{-1}[\overline{N}] \cap \mathfrak{c}(b)$. Since $f_{-1}[\overline{N}] \in \mathfrak{N}\mathfrak{D}(\mathfrak{c}(b))$ and $\mathfrak{o}(h(x)) \cap \mathfrak{c}(b)$ is open in $\mathfrak{c}(b)$, $\mathfrak{o}(h(x)) \cap \mathfrak{c}(b) = \mathsf{O}$ which implies that $\mathfrak{c}(b) \subseteq \mathfrak{c}(h(x)) = f_{-1}[\mathfrak{c}(x)]$. Therefore $f[\mathfrak{c}(b)] \subseteq f[f_{-1}[\mathfrak{c}(x)]] \subseteq \mathfrak{c}(x)$ implying that $\mathfrak{c}(f(b)) = \overline{f[\mathfrak{c}(b)]} \subseteq \overline{\mathfrak{c}(x)} = \mathfrak{c}(x)$. This makes $\mathfrak{c}(f(b)) \cap \mathfrak{o}(x) = \mathsf{O}$. Therefore $\overline{N}$ is nowhere dense in $\mathfrak{c}(f(b))$ which contradicts that $\overline{N}$ is maximal nowhere dense in M. Therefore $f_{-1}[N]$ is maximal nowhere dense in L. □

In the next result, we discuss localic maps that preserve maximal nowhere dense sublocales. We recall from [52] that if a localic map $f : L \to M$ is open, then $f_{-1}[\overline{A}] = \overline{f_{-1}[A]}$ for each $A \in \mathcal{S}(M)$. We also note that an open localic map has a weakly open left adjoint. Indeed, assume that $f : L \to M$ is an open localic map, let $x \in M$ be dense and choose $y \in L$ such that $h(x) \wedge y = 0$. Then $\mathfrak{o}(y) \subseteq \mathfrak{c}(h(x))$ so that $f[\mathfrak{o}(y)] \subseteq f[\mathfrak{c}(h(x))] = f[f_{-1}[\mathfrak{c}(x)]] \subseteq \mathfrak{c}(x)$. But $\mathfrak{c}(x)$ is nowhere dense and $f[\mathfrak{o}(y)]$ is open by openness of f, we get that $f[\mathfrak{o}(y)] = \mathsf{O}$. Therefore $\mathfrak{o}(y) \subseteq f_{-1}[f[\mathfrak{o}(y)]] = f_{-1}[\mathsf{O}] = \mathsf{O}$. Thus $y = 0$ making $h(x)$ dense in L.

Observation 4.4.2. Not every open localic map sends dense elements to dense elements. Consider the localic map $f : L \to \mathbf{2}$ where L is non-Boolean. Since $\mathbf{2}$ is Boolean, every sublocale of $\mathbf{2}$ is open making the localic image of each open sublocale of L to be open in $\mathbf{2}$. Hence f is open. However, f does not send all dense elements to dense elements since the only element (dense) of L that is mapped to $1_{\mathbf{2}}$ (the only dense element of $\mathbf{2}$) is 1. But 1 is not the only dense element of L otherwise L is Boolean.

Proposition 4.4.3. *Let $f : L \to M$ be an open localic map that sends dense elements to dense elements. Then f preserves maximal nowhere dense sublocales.*

Proof. Let $N \in \mathfrak{M}(L)$. We show that $\overline{f[N]} \in \mathfrak{M}(M)$. It follows from Lemma 3.1.7 that $f[N]$ is nowhere dense in M so that $\overline{f[N]} = \mathfrak{c}\,(f(\bigwedge N))$ is nowhere dense in M. Suppose that $\mathfrak{c}\,(f(\bigwedge N)) \in \mathfrak{ND}(\mathfrak{c}(y))$ for some $\mathfrak{c}(y) \in \mathfrak{ND}(M)$. By Lemma 4.1.10, $\mathfrak{c}(y) \subseteq \overline{\mathfrak{c}(y) \cap \mathfrak{o}\,(f(\bigwedge N))}$. Therefore $f_{-1}[\mathfrak{c}(y)] = \mathfrak{c}(h(y)) \subseteq f_{-1}\left[\overline{\mathfrak{c}(y) \cap \mathfrak{o}\,(f(\bigwedge N))}\right]$. By openness of f,

$$\begin{aligned}
\mathfrak{c}(h(y)) &\subseteq \overline{f_{-1}\left[\mathfrak{c}(y) \cap \mathfrak{o}\left(f(\textstyle\bigwedge N)\right)\right]} \\
&= \overline{\mathfrak{c}(h(y)) \cap \mathfrak{o}\left(h\left(f\left(\textstyle\bigwedge N\right)\right)\right)} \\
&\subseteq \overline{\mathfrak{c}(h(y)) \cap \mathfrak{o}\left(\textstyle\bigwedge N\right)} \\
&= \overline{\mathfrak{c}(h(y)) \cap \left(L \smallsetminus \overline{N}\right)}.
\end{aligned}$$

Therefore $\mathfrak{c}(h(y)) \cap \left(L \smallsetminus \overline{\mathfrak{c}(h(y)) \cap \left(L \smallsetminus \overline{N}\right)}\right) = \mathsf{O}$. This makes $N \in \mathfrak{ND}(\mathfrak{c}(h(y)))$, where $\mathfrak{c}(h(y)) \in \mathfrak{ND}(L)$, contradicting that $N \in \mathfrak{M}(L)$. Therefore $\overline{f[N]}$ is maximal nowhere dense so that by Proposition 4.1.5, $f[N]$ is maximal nowhere dense sublocales. □

Since in Proposition 4.4.1 we only needed a condition that both f and h send dense elements to dense elements and because the left adjoint of an open localic map is weakly open, we have the following result.

Corollary 4.4.4. *Every open localic map that sends dense elements to dense elements preserves and reflects maximal nowhere dense sublocales.*

In the next result, we discuss preservation and reflection of strongly homogeneous maximal nowhere dense sublocales by localic maps.

Proposition 4.4.5. *Let $f : L \to M$ be an open localic map that sends dense elements to dense elements.*

1. *Then f preserves strongly homogeneous maximal nowhere dense sublocales.*

2. *If f is injective, then it reflects (strongly) homogeneous maximal nowhere dense sublocales.*

Proof. (1) Let F be a strongly homogeneous maximal nowhere dense sublocale of L and choose a non-void sublocale $\overline{\mathfrak{o}(y) \cap f[F]} \cap f[F]$ where $y \in M$. Such a sublocale is $f[F]$-regular-closed. The F-regular-closed sublocale $\overline{\mathfrak{o}(h(y)) \cap F} \cap F$ is non-void otherwise, $\mathfrak{o}(h(y)) \cap F = \mathsf{O}$ so that $f[F] \subseteq f[\mathfrak{c}(h(y))] = f[f_{-1}[\mathfrak{c}(y)]] \subseteq \mathfrak{c}(y)$. Therefore $f[F] \cap \mathfrak{o}(y) = \mathsf{O}$ which is not possible. Since F is strongly h.m.n.d, $\overline{\mathfrak{o}(h(y)) \cap F} \cap F$ is m.n.d. Because open localic maps that send dense elements to dense elements preserve maximal nowhere dense sublocales (by Proposition 4.4.3), $f\left[\overline{\mathfrak{o}(h(y)) \cap F} \cap F\right]$ is m.n.d in M. Since

$$\begin{aligned} f\left[\overline{\mathfrak{o}(h(y)) \cap F} \cap F\right] &\subseteq f\left[\overline{\mathfrak{o}(h(y)) \cap F}\right] \cap f[F] \\ &\subseteq \overline{f[\mathfrak{o}(h(y)) \cap F]} \cap f[F] \\ &\subseteq \overline{f[\mathfrak{o}(h(y))] \cap f[F]} \cap f[F] \\ &\subseteq \overline{\mathfrak{o}(y) \cap f[F]} \cap f[F] \end{aligned}$$

and because $\overline{\mathfrak{o}(y) \cap f[F]} \cap f[F]$ is nowhere dense in M, it follows from Proposition 4.1.11(2) that $\overline{\mathfrak{o}(y) \cap f[F]} \cap f[F]$ is m.n.d. Thus $f[F]$ is strongly homogeneous maximal nowhere dense in M.

(2) We only prove reflection of strongly homogeneous maximal nowhere dense sublocales. That of homogeneous maximal nowhere dense sublocales follows the same sketch. Let K be a strongly homogeneous maximal nowhere dense sublocale of M and consider a non-void sublocale $\overline{\mathfrak{o}(x) \cap f_{-1}[K]} \cap f_{-1}[K]$ where $x \in L$. We must show that this $f_{-1}[K]$-regular-closed sublocale is m.n.d. We have that $\overline{\mathfrak{o}(f(x)) \cap K} \cap K$ is a non-void K-regular-closed sublocale. To see that it is non-void, observe that having $\overline{\mathfrak{o}(f(x)) \cap K} \cap K = \mathsf{O}$ implies that $\mathfrak{o}(f(x)) \cap K = \mathsf{O}$ so that

$$\mathsf{O} = f_{-1}[\mathfrak{o}(f(x))] \cap f_{-1}[K] = \mathfrak{o}(h(f(x))) \cap f_{-1}[K] = \mathfrak{o}(x) \cap f_{-1}[K]$$

where the latter equality follows from injectivity of f. This cannot be true, so $\overline{\mathfrak{o}(f(x)) \cap K} \cap K$ is non-void. Since K is strongly homogeneous maximal nowhere dense, $\overline{\mathfrak{o}(f(x)) \cap K} \cap K$ is m.n.d in M. Because open localic maps are weakly open and f sends dense elements to dense elements, it follows from Proposition 4.4.1 that $f_{-1}[\overline{\mathfrak{o}(f(x)) \cap K} \cap K]$ is m.n.d. Observe that

$$f_{-1}[\overline{\mathfrak{o}(f(x)) \cap K} \cap K] = f_{-1}[\overline{\mathfrak{o}(f(x)) \cap K}] \cap f_{-1}[K] = \overline{f_{-1}[\mathfrak{o}(f(x))] \cap f_{-1}[K]} \cap f_{-1}[K]$$

where the latter equality follows from openness of f. By injectivity of f,

$$f_{-1}[\overline{\mathfrak{o}(f(x)) \cap K} \cap K] = \overline{\mathfrak{o}(x) \cap f_{-1}[K]} \cap f_{-1}[K]$$

making $\overline{\mathfrak{o}(x) \cap f_{-1}[K]} \cap f_{-1}[K]$ m.n.d. in L. Thus $f_{-1}[K]$ is strongly homogeneous maximal nowhere dense in L. □

Observation 4.4.6. For the preservation of homogeneous maximal nowhere dense sublocales, the localic map f in Proposition 4.4.5 must also preserve closed sublocales. That is, it must also be closed which is a rather too stringent condition.

Open localic maps also allow us to study, under certain conditions, preservation and reflection of inaccessible and almost inaccessible sublocales as presented below.

Proposition 4.4.7. *Let $f : L \to M$ be an open and injective localic map. Then for all open $S \in \mathcal{S}(L)$,*

1. $f[\mathcal{S}_{\text{Inac}}(S)] \subseteq \mathcal{S}_{\text{Inac}}(f[S])$, *and*
2. $f[\mathcal{S}_{\text{Ainac}}(S)] \subseteq \mathcal{S}_{\text{Ainac}}(f[S])$.

Proof. (1) Let $S = \mathfrak{o}(x)$ for some $x \in L$ and choose $A \in \mathcal{S}_{\text{Inac}}(S)$. Observe that $f[A] \subseteq f[S]$. To show that $f[A]$ is $f[S]$-inaccessible, select an $(M \smallsetminus f[S])$-nowhere dense sublocale N. Since f is open, we have that $f[S]$ is open so that $f[S] = \mathfrak{o}(y)$ for some $y \in M$. Now, $N \subseteq \mathfrak{c}(y)$ which implies that $f_{-1}[N] \subseteq f_{-1}[\mathfrak{c}(y)] = \mathfrak{c}(h(y))$. Observe that $\mathfrak{c}(h(y)) = \mathfrak{c}(x)$. To see this, let

$a \in \mathfrak{c}(h(y))$. Then

$$\begin{aligned} h(y) \leq a &\implies y \leq f(h(y)) \leq f(a) \\ &\implies \mathfrak{o}(y) \subseteq \mathfrak{o}(f(a)) \\ &\implies f[\mathfrak{o}(x)] \subseteq \mathfrak{o}(f(a)) \quad \text{since } f[\mathfrak{o}(x)] = f[S] = \mathfrak{o}(y) \\ &\implies \mathfrak{o}(x) \subseteq f_{-1}[f[\mathfrak{o}(x)]] \subseteq f_{-1}[\mathfrak{o}(f(a))] = \mathfrak{o}(h(f(a))) \subseteq \mathfrak{o}(a) \\ &\implies \mathfrak{c}(a) \subseteq \mathfrak{c}(x) \end{aligned}$$

so that $a \in \mathfrak{c}(x)$. On the other hand, for $a \in \mathfrak{c}(x)$, we have

$$\begin{aligned} x \leq a &\implies \mathfrak{o}(x) \subseteq \mathfrak{o}(a) \\ &\implies f[\mathfrak{o}(x)] \subseteq f[\mathfrak{o}(a)] \\ &\implies \mathfrak{o}(y) \subseteq f[\mathfrak{o}(a)] \quad \text{since } f[\mathfrak{o}(x)] = f[S] = \mathfrak{o}(y) \\ &\implies \mathfrak{o}(h(y)) = f_{-1}[\mathfrak{o}(y)] \subseteq f_{-1}[f[\mathfrak{o}(a)]] = \mathfrak{o}(a) \quad \text{since } f \text{ is injective} \\ &\implies \mathfrak{c}(a) \subseteq \mathfrak{c}(h(y)) \end{aligned}$$

making $a \in \mathfrak{c}(h(y))$.

Therefore $f_{-1}[N] \subseteq \mathfrak{c}(x)$. We show that $f_{-1}[N] \in \mathfrak{ND}(L \smallsetminus S)$. Since $N \in \mathfrak{ND}(M \smallsetminus f[S])$, i.e., $M \smallsetminus f[S] = \mathfrak{c}(y) \subseteq \overline{\mathfrak{c}(y) \cap M \smallsetminus \overline{N}}$, we have $\mathfrak{c}(y) = \overline{\mathfrak{c}(y) \cap \mathfrak{o}\left(\bigwedge N\right)}$. Therefore

$$\begin{aligned} \mathfrak{c}(h(y)) = f_{-1}[\mathfrak{c}(y)] &\subseteq f_{-1}\left[\overline{\mathfrak{c}(y) \cap \mathfrak{o}\left(\bigwedge N\right)}\right] \\ &= \overline{f_{-1}\left[\mathfrak{c}(y) \cap \mathfrak{o}\left(\bigwedge N\right)\right]} \quad \text{since } f \text{ is open} \\ &= \overline{\mathfrak{c}(h(y)) \cap \mathfrak{o}\left(h\left(\bigwedge N\right)\right)} \\ &= \overline{\mathfrak{c}(h(y)) \cap \left(L \smallsetminus \mathfrak{c}\left(h\left(\bigwedge N\right)\right)\right)} \\ &= \overline{\mathfrak{c}(h(y)) \cap \left(L \smallsetminus f_{-1}[\overline{N}]\right)} \\ &= \overline{\mathfrak{c}(h(y)) \cap \left(L \smallsetminus \overline{f_{-1}[N]}\right)} \quad \text{since } f \text{ is open.} \end{aligned}$$

Therefore $\mathfrak{c}(h(y)) \cap \left[L \smallsetminus \overline{\mathfrak{c}(h(y)) \cap \left(L \smallsetminus \overline{f_{-1}[N]}\right)}\right] = \mathsf{O}$ making $f_{-1}[N] \in \mathfrak{ND}(\mathfrak{c}(h(y)))$. Because $\mathfrak{c}(h(y)) = \mathfrak{c}(x) = L \smallsetminus S$, we have that

$$f_{-1}[N] \in \mathfrak{ND}(L \smallsetminus S). \tag{4.4.1}$$

Since $A \in \mathcal{S}_{\mathrm{Inac}}(S)$, $A \cap \overline{f_{-1}[N]} = \mathsf{O}$. Because f is open, $A \cap f_{-1}[\overline{N}] = \mathsf{O}$. Since $f_{-1}[\overline{N}] = \mathfrak{c}(h(\bigwedge N))$, $A \subseteq \mathfrak{o}(h(\bigwedge N)) = f_{-1}[\mathfrak{o}(\bigwedge N)]$ so that $f[A] \subseteq f[f_{-1}[\mathfrak{o}(\bigwedge N)]] \subseteq \mathfrak{o}(\bigwedge N)$. Therefore $\mathsf{O} = f[A] \cap \mathfrak{c}(\bigwedge N) = f[A] \cap \overline{N}$. Thus $f[A] \in \mathcal{S}_{\mathrm{Inac}}(f[S])$.

(2) Choose $A \in \mathcal{S}_{\mathrm{Ainac}}(S)$ and an $(M \smallsetminus f[S])$-nowhere dense sublocale N. (4.4.1) still holds since it does not require A to be S-inaccessible. Therefore $A \subseteq S \cap \overline{S \cap L \smallsetminus \overline{f_{-1}[N]}}$ so that $A \subseteq S \cap \overline{S \cap f_{-1}[\mathfrak{o}(\bigwedge N)]}$ because f is open. We get that

$$\begin{aligned}
f[A] &\subseteq f\left[S \cap \overline{S \cap f_{-1}\left[[\mathfrak{o}\left(\bigwedge N\right)\right]}\right] \\
&\subseteq f[S] \cap f\left[\overline{S \cap f_{-1}\left[\mathfrak{o}\left(\bigwedge N\right)\right]}\right] \\
&\subseteq f[S] \cap \overline{f\left[S \cap f_{-1}\left[\mathfrak{o}\left(\bigwedge N\right)\right]\right]} \\
&\subseteq f[S] \cap \overline{f[S] \cap f\left[f_{-1}\left[\mathfrak{o}\left(\bigwedge N\right)\right]\right]} \\
&\subseteq f[S] \cap \overline{f[S] \cap \mathfrak{o}\left(\bigwedge N\right)} \\
&= f[S] \cap \overline{f[S] \cap \left(M \smallsetminus \mathfrak{c}\left(\bigwedge N\right)\right)} \\
&= f[S] \cap \overline{f[S] \cap (M \smallsetminus \overline{N})}.
\end{aligned}$$

Thus $f[A] \in \mathcal{S}_{\mathrm{Ainac}}(f[S])$. □

Proposition 4.4.8. *Let $f : L \to M$ be localic map such that both f and h send dense elements to dense elements and let $T \in \mathcal{S}(M)$ be closed nowhere dense. Then*

1. *$f_{-1}[\mathcal{S}_{\mathrm{Inac}}(T)] \subseteq \mathcal{S}_{\mathrm{Inac}}(f_{-1}[T])$, and*
2. *If f is open, then $f_{-1}[\mathcal{S}_{\mathrm{Ainac}}(T)] \subseteq \mathcal{S}_{\mathrm{Inac}}(f_{-1}[T])$.*

Proof. (1) Let $T \in \mathcal{S}(M)$ be closed and nowhere dense and choose $A \in \mathcal{S}_{\mathrm{Inac}}(T)$. Then $f_{-1}[A] \subseteq f_{-1}[T]$. Let $N \in \mathfrak{ND}(L \smallsetminus f_{-1}[T])$. Since, by [26], $P \smallsetminus g_{-1}[C] \subseteq g_{-1}[R \smallsetminus C]$ for every localic map $g : P \to R$ with $C \in \mathcal{S}(R)$, we get that $N \subseteq L \smallsetminus f_{-1}[T] \subseteq f_{-1}[M \smallsetminus T]$. Therefore $f[N] \subseteq f[f_{-1}[M \smallsetminus T]] \subseteq M \smallsetminus T$. We show that $f[N] \in \mathfrak{ND}(M \smallsetminus T)$. Because h is weakly open, it follows from Theorem 3.1.2 that $f_{-1}[-] : \mathcal{S}(M) \to \mathcal{S}(L)$ preserves closed nowhere dense sublocales so that $f_{-1}[T]$ is closed nowhere dense in L. By Corollary 2.1.4, $L \smallsetminus f_{-1}[T]$ is open and dense. Now, N being nowhere dense in $L \smallsetminus f_{-1}[T]$ implies N is nowhere dense in L. Since f sends dense elements to dense elements, it follows from Lemma 3.1.7 that $f[-]$ preserves nowhere dense sublocales so that $f[N]$ is nowhere dense in M. Observe that $f[N]$ is nowhere dense in $M \smallsetminus T$. To see this, let $y \in M$ be such that $\mathfrak{o}(y) \cap (M \smallsetminus T) \subseteq \overline{f[N]} \cap (M \smallsetminus T)$. Then $\mathfrak{o}(y) \cap (M \smallsetminus T) \subseteq \overline{f[N]}$. Because $M \smallsetminus T$ is open in M and $f[N] \in \mathfrak{N}\mathfrak{D}(M)$, we have that $\mathfrak{o}(y) \cap (M \smallsetminus T) = \mathsf{O}$ making $\mathfrak{o}(y) \subseteq T$. But T is nowhere dense in M, so $\mathfrak{o}(y) = \mathsf{O}$ implying that

$$f[N] \in \mathfrak{N}\mathfrak{D}(M \smallsetminus T). \tag{4.4.2}$$

S-inaccessibility of A implies $A \cap \overline{f[N]} = \mathsf{O}$. Therefore

$$\mathsf{O} = f_{-1}\left[A \cap \overline{f[N]}\right] = f_{-1}[A] \cap f_{-1}\left[\overline{f[N]}\right] = f_{-1}[A] \cap \mathfrak{c}\left(h\left(f\left(\bigwedge N\right)\right)\right) \supseteq f_{-1}[A] \cap \overline{N}.$$

Thus $f_{-1}[A] \in \mathcal{S}_{\mathrm{Inac}}(f_{-1}[T])$.

(2) Assume that f is open. Set $T = \mathfrak{c}(b)$ for some $b \in M$ and choose $A \in \mathcal{S}_{\mathrm{Ainac}}(T)$. (4.4.2) still holds, so

$$A \subseteq \mathfrak{c}(b) \cap \overline{\mathfrak{c}(b) \cap \left(M \smallsetminus \overline{f[N]}\right)} = \mathfrak{c}(b) \cap \overline{\mathfrak{c}(b) \cap \mathfrak{o}\left(\bigwedge f[N]\right)}.$$

Therefore

$$\begin{aligned}
f_{-1}[A] &\subseteq \mathfrak{c}(h(b)) \cap f_{-1}\left[\overline{\mathfrak{c}(b) \cap \mathfrak{o}\left(\bigwedge f[N]\right)}\right] \\
&= \mathfrak{c}(h(b)) \cap \overline{f_{-1}\left[\mathfrak{c}(b) \cap \mathfrak{o}\left(\bigwedge f[N]\right)\right]} \quad \text{since } f \text{ is open} \\
&= \mathfrak{c}(h(b)) \cap \overline{\mathfrak{c}(h(b)) \cap \mathfrak{o}\left(h\left(f\left(\bigwedge N\right)\right)\right)} \\
&\subseteq \mathfrak{c}(h(b)) \cap \overline{\mathfrak{c}(h(b)) \cap \mathfrak{o}\left(\bigwedge N\right)} \\
&= f_{-1}[T] \cap \overline{f_{-1}[T] \cap \left(L \smallsetminus \overline{N}\right)} \\
&= \mathrm{cl}_{f_{-1}[T]}\left(f_{-1}[T] \cap \left(L \smallsetminus \overline{N}\right)\right).
\end{aligned}$$

Thus $f_{-1}[A] \in \mathcal{S}_{\mathrm{Ainac}}(f_{-1}[T])$. □

Chapter 5

Remoteness in Bilocales

In mathematics (particularly, frame theory), it is prevalent to want to know how an introduced notion fits in other settings. In this chapter, we present an extension of remoteness in the category of bilocales.

5.1 $(i, j)-$nowhere dense sublocales

We devote this section to introducing (i, j)-nowhere dense sublocales from (i, j)-nowhere dense subspaces and studying some of their properties. (i, j)-nowhere dense sublocales will be used in the theory of (i, j)-remote sublocales that will follow after this section.

Recall from [30] that for a bitopological space (bispace in short) (X, τ_1, τ_2), where int_{τ_i} and cl_{τ_i} (for $i = 1, 2$) denote the τ_i-interior and τ_i-closure, respectively, a subset A of X is (τ_i, τ_j)-*nowhere dense* in X if $\mathrm{int}_{\tau_j}(\mathrm{cl}_{\tau_i}(A)) = \emptyset$ $(i \neq j)$. We will extend this notion to locales and explore some of its bilocalic properties.

We recall the following notions of bilocales from Chapter 1. A *bilocale* is a triple (L, L_1, L_2) where L_1, L_2 are subframes of a locale L and for all $a \in L$,

$$a = \bigvee \{a_1 \wedge a_2 : a_1 \in L_1, a_2 \in L_2 \text{ and } a_1 \wedge a_2 \leq a\}.$$

We call L the *total part* of (L, L_1, L_2), and L_1, L_2 the first and second parts, respectively. We use the notations L_i, L_j to denote the first or second parts of (L, L_1, L_2), always assuming that $i, j = 1, 2$, $i \neq j$.

For $c \in L_i$ we denote

$$c^{\bullet} = \bigvee\{x \in L_j : x \wedge c = 0\}.$$

For a bilocalic notion of (τ_i, τ_j)-nowhere density, we introduce bilocalic counterparts of the notions of closure and interior. Let (L, L_1, L_2) be a bilocale. In [51], the authors introduced the following notation for a sublocale $S \subseteq L$:

$$\mathrm{cl}_i(S) = \bigcap\{\mathfrak{c}(a) : a \in L_i, S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}\left(\bigvee\{a \in L_i : S \subseteq \mathfrak{c}(a)\}\right) \quad (i = 1, 2).$$

We define $\mathrm{int}_i(S)$ as follows:

$$\mathrm{int}_i(S) = \bigvee\{\mathfrak{o}(a) : a \in L_i, \mathfrak{o}(a) \subseteq S\} \quad (i = 1, 2).$$

We shall refer to these concepts as *bilocale closure* and *bilocale interior*, respectively. Throughout this thesis, we assume that $i \neq j \in \{1, 2\}$, unless otherwise stated.

We define an (i, j)-nowhere dense sublocale as follows.

Definition 5.1.1. Let (L, L_1, L_2) be a bilocale. A sublocale S of L is (i, j)-*nowhere dense* if $\mathrm{int}_j(\mathrm{cl}_i(S)) = \mathsf{O}$ $(i \neq j \in \{1, 2\})$.

Our discussion of (i, j)-nowhere density involves bilocale interiors and bilocale closures. Before we consider the properties of bilocale closures and bilocale interiors which will be useful below, we collect some properties of the bilocale pseudocomplement in the following proposition. We remark that some of these might be part of bilocale folklore.

Proposition 5.1.2. *Let (L, L_1, L_2) be a bilocale and $i \neq j \in \{1, 2\}$. Then*

1. $0^{\bullet} = 1_{L_i}$.
2. *For every $a \in L_i$, $a \wedge a^{\bullet} = 0$.*
3. *$a \wedge b = 0$ iff $a \leq b^{\bullet}$ for all $a \in L_j, b \in L_i$.*
4. *$a \leq b$ implies $b^{\bullet} \leq a^{\bullet}$ for all $a, b \in L_i$.*
5. *For each $a \in L_i$, $a \leq a^{\bullet\bullet}$.*

6. *For each* $a \in L_i$, $a^\bullet = a^{\bullet\bullet\bullet}$.

7. $(a \vee b)^\bullet = a^\bullet \wedge b^\bullet$ *for every* $a, b \in L_i$.

Proof. (1) By definition of a bilocale pseudocomplement,

$$0^\bullet = \bigvee\{x \in L_i : x \wedge 0 = 0\} = 1_{L_i}.$$

(2) If $a \in L_i$, then

$$a \wedge a^\bullet = a \wedge \bigvee\{x \in L_j : x \wedge a = 0\} = \bigvee\{a \wedge x : x \in L_j, x \wedge a = 0\} = 0$$

where the second equality follows since L is a locale.

(3) Let $a \in L_j$ and $b \in L_i$.

($\Longrightarrow$): If $a \wedge b = 0$, then $a \in \{x \in L_j : x \wedge b = 0\}$. So, $a \leq b^\bullet = \bigvee\{x \in L_j : x \wedge b = 0\}$.

($\Longleftarrow$): Assume that $a \leq b^\bullet$. Since $b^\bullet$ is the largest element in L_j missing b, a must also miss b.

(4) Let $a, b \in L_i$ be such that $a \leq b$ and let $c = b^\bullet$. Then $c \leq b^\bullet$ and $c \in L_j$. It follows from (3) that $c \wedge b = 0$ so that $c \wedge a = 0$. Therefore $c \leq a^\bullet$. Thus $b^\bullet \leq a^\bullet$.

(5) For each $a \in L_i$, we have that $a^{\bullet\bullet} = \bigvee\{x \in L_i : x \wedge a^\bullet = 0\}$. But $a \wedge a^\bullet = 0$ and $a \in L_i$ so $a \leq a^{\bullet\bullet}$.

(6) Let $a \in L_i$. From (5) we have that $a \leq a^{\bullet\bullet}$ so that by (4), $a^{\bullet\bullet\bullet} \leq a^\bullet$. For the other inequality, we have from (2) that $a^\bullet \wedge (a^\bullet)^\bullet = 0$. Now that $a^\bullet \in L_j$ and $(a^\bullet)^\bullet \in L_i$, application of (3) gives $a^\bullet \leq (a^\bullet)^{\bullet\bullet} = a^{\bullet\bullet\bullet}$. Thus $a^\bullet = a^{\bullet\bullet\bullet}$.

(7) Let $a, b \in L_i$ and $y = (a \vee b)^\bullet$. Since L_i is a subframe, $a \vee b \in L_i$ so that $y \in L_j$. It follows from (3) that $y \wedge (a \vee b) = 0$, i.e., $(y \wedge a) \vee (y \wedge b) = 0$. Therefore $y \wedge a = 0$ and $y \wedge b = 0$ making $y \leq a^\bullet$ and $y \leq b^\bullet$, by (3). Thus $y \leq a^\bullet \wedge b^\bullet$.

On the other hand, we have that

$$\begin{aligned}(a \vee b) \wedge (a^\bullet \wedge b^\bullet) &= ((a \vee b) \wedge a^\bullet) \wedge ((a \vee b) \wedge b^\bullet) \\ &= ((a \wedge a^\bullet) \vee (b \wedge a^\bullet)) \wedge ((a \wedge b^\bullet) \vee (b \wedge b^\bullet)) \\ &= (b \wedge a^\bullet) \wedge (a \wedge b^\bullet) \\ &= 0.\end{aligned}$$

Therefore $a^\bullet \wedge b^\bullet \leq (a \vee b)^\bullet$. Thus $(a \vee b)^\bullet = a^\bullet \wedge b^\bullet$. □

We discuss some properties of bilocale closure and bilocale interior in the next result.

Proposition 5.1.3. *Let (L, L_1, L_2) be a bilocale and $S, T \in \mathcal{S}(L)$. The following statements hold for $i \neq j \in \{1, 2\}$.*

1. *[51] $S \subseteq \overline{S} \subseteq \mathrm{cl}_i(S)$.*
2. *If $T \subseteq S$, then $\mathrm{cl}_i(T) \subseteq \mathrm{cl}_i(S)$.*
3. *$\mathrm{cl}_i(\mathrm{cl}_i(S)) = \mathrm{cl}_i(S)$.*
4. *$\mathfrak{c}(a) = \mathrm{cl}_i(\mathfrak{c}(a))$ for every $a \in L_i$.*
5. *$\mathrm{int}_i(S) = \mathfrak{o}\left(\bigvee\{a \in L_i : \mathfrak{o}(a) \subseteq S\}\right)$.*
6. *$\mathrm{int}_i(S) \subseteq \mathrm{int}(S) \subseteq S$.*
7. *If $T \subseteq S$, then $\mathrm{int}_i(T) \subseteq \mathrm{int}_i(S)$.*
8. *$\mathrm{int}_i(\mathrm{int}_i(S)) = \mathrm{int}_i(S)$.*
9. *$\mathfrak{o}(a) = \mathrm{int}_i(\mathfrak{o}(a))$ for every $a \in L_i$.*
10. *For each $a \in L_i$, $\mathfrak{c}(a^\bullet) = \mathrm{cl}_j(\mathfrak{o}(a))$.*
11. *For each $a \in L_i$, $\mathfrak{o}(a^\bullet) = \mathrm{int}_j(\mathfrak{c}(a))$.*
12. *For each $a \in L_i$, $\mathrm{cl}_j(\mathfrak{o}(a)) = L \smallsetminus \mathrm{int}_j(\mathfrak{c}(a))$.*
13. *For each $a \in L_i$, $\mathrm{int}_j(\mathfrak{c}(a)) = L \smallsetminus \mathrm{cl}_j(\mathfrak{o}(a))$.*

14. For each $a \in L$*,* $L \smallsetminus \mathrm{int}_i(\mathfrak{o}(a)) = \mathrm{cl}_i(\mathfrak{c}(a))$*.*

15. For each $a \in L$*,* $L \smallsetminus \mathrm{cl}_i(\mathfrak{c}(a)) = \mathrm{int}_i(\mathfrak{o}(a))$*.*

Proof. (1) Observe that $\{a \in L_i : S \subseteq \mathfrak{c}(a)\} \subseteq \{b \in L : S \subseteq \mathfrak{c}(b)\}$. Therefore

$$\bigvee\{a \in L_i : S \subseteq \mathfrak{c}(a)\} \leq \bigvee\{b \in L : S \subseteq \mathfrak{c}(b)\}$$

so that

$$S \subseteq \overline{S} = \mathfrak{c}\left(\bigvee\{b \in L : S \subseteq \mathfrak{c}(b)\}\right) \subseteq \mathfrak{c}\left(\bigvee\{a \in L_i : S \subseteq \mathfrak{c}(a)\}\right) = \mathrm{cl}_i(S).$$

(2) Let $T \subseteq S$. Then

$$\{a \in L_i : S \subseteq \mathfrak{c}(a)\} \subseteq \{b \in L_i : T \subseteq \mathfrak{c}(b)\} \Longrightarrow \bigvee\{b \in L_i : S \subseteq \mathfrak{c}(b)\} \leq \bigvee\{a \in L_i : T \subseteq \mathfrak{c}(a)\}.$$

Therefore

$$\mathrm{cl}_i(T) = \mathfrak{c}\left(\bigvee\{a \in L_i : T \subseteq \mathfrak{c}(a)\}\right) \subseteq \mathrm{cl}_i(S) = \mathfrak{c}\left(\bigvee\{a \in L_i : S \subseteq \mathfrak{c}(a)\}\right).$$

(3) It suffices to show that $\mathrm{cl}_i(\mathrm{cl}_i(S)) \subseteq \mathrm{cl}_i(S)$:

$$\begin{aligned}
\mathrm{cl}_i(\mathrm{cl}_i(S)) &= \mathrm{cl}_i\left(\mathfrak{c}\left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right)\right) \\
&= \mathfrak{c}\left(\bigvee\{y \in L_i : \mathfrak{c}\left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right) \subseteq \mathfrak{c}(y)\}\right) \\
&= \mathfrak{c}\left(\bigvee\{y \in L_i : y \leq \bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\}\right).
\end{aligned}$$

Observe that

$$\{x \in L_i : S \subseteq \mathfrak{c}(x)\} \subseteq \{y \in L_i : y \leq \bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\}$$

which implies

$$\mathrm{cl}_i(\mathrm{cl}_i(S)) = \mathfrak{c}\left(\bigvee\{y \in L_i : y \leq \bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\}\right) \subseteq \mathfrak{c}\left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right) = \mathrm{cl}_i(S).$$

(4) We have

$$\mathrm{cl}_i(\mathrm{cl}(a)) = \mathfrak{c}\left(\bigvee\{x \in L_i : \mathfrak{c}(a) \subseteq \mathfrak{c}(x)\}\right) = \mathfrak{c}\left(\bigvee\{x \in L_i : x \leq a\}\right) = \mathfrak{c}(a)$$

for all $a \in L_i$.

(5) We have

$$\mathrm{int}_i(S) = \bigvee\{\mathfrak{o}(a) : a \in L_i, \mathfrak{o}(a) \subseteq S\} = \mathfrak{o}\left(\bigvee\{a \in L_i : \mathfrak{o}(a) \subseteq S\}\right).$$

(6) Observe that $\{\mathfrak{o}(a) : a \in L_i, \mathfrak{o}(a) \subseteq S\} \subseteq \{\mathfrak{o}(b) : b \in L, \mathfrak{o}(b) \subseteq S\}$. Therefore,

$$\mathrm{int}_i(S) = \bigvee\{\mathfrak{o}(a) : a \in L_i, \mathfrak{o}(a) \subseteq S\} \subseteq \bigvee\{\mathfrak{o}(b) : \mathfrak{o}(b) \subseteq S\} = \mathrm{int}_i(S) \subseteq S.$$

(7) Similar to the proof of (2).

(8) Similar to the proof of (3).

(9) Similar to the proof of (4).

(10) We have that

$$\begin{aligned}
\mathrm{cl}_j(\mathfrak{o}(a)) &= \mathfrak{c}\left(\bigvee\{x \in L_j : \mathfrak{o}(a) \subseteq \mathfrak{c}(x)\}\right) \\
&= \mathfrak{c}\left(\bigvee\{x \in L_j : x \wedge a = 0\}\right) \\
&= \mathfrak{c}(a^\bullet)
\end{aligned}$$

for each $a \in L_i$.

(11) Similar to the proof of (10).

(12) Combination of (10) and (11).

(13) Similar to the argument of (12).

(14) Observe that

$$\begin{aligned}
L \smallsetminus \mathrm{int}_i(\mathfrak{o}(a)) &= L \smallsetminus \mathfrak{o}\left(\bigvee\{x \in L_i : \mathfrak{o}(x) \subseteq \mathfrak{o}(a)\}\right) \\
&= L \smallsetminus \mathfrak{o}\left(\bigvee\{x \in L_i : \mathfrak{c}(a) \subseteq \mathfrak{c}(x)\}\right) \\
&= \mathfrak{c}\left(\bigvee\{x \in L_i : \mathfrak{c}(a) \subseteq \mathfrak{c}(x)\}\right) \\
&= \mathrm{cl}_i(\mathfrak{c}(a))
\end{aligned}$$

for all $a \in L$.

(15) Similar to the proof of (14). □

We consider i-dense sublocales. Recall from [55] that a subset A of a bitopological space (X, τ_1, τ_2) is *i-dense* if $\mathrm{cl}_{\tau_i}(A) = X$. This recalled notion motivates the following definition of an i-dense sublocale.

Definition 5.1.4. A sublocale A of a bilocale (L, L_1, L_2) is *i-dense* if $\mathrm{cl}_i(A) = L$.

We work towards showing that a subset A of a bitopological space (X, τ_1, τ_2) is i-dense if and only if $\widetilde{A}$ is i-dense.

Recall from [42] that given a topological property P, a bitopological space (X, τ_1, τ_2) is sup-P if $(X, \tau_1 \vee \tau_2)$ has property P. We say that (X, τ_1, τ_2) is sup-T_D if $(X, \tau_1 \vee \tau_2)$ is T_D.

In bilocalic terms, we denote the sublocale induced by a subset A of X as follows:

$$\widetilde{A} = \{\mathrm{int}_{\tau_1 \vee \tau_2}((X \smallsetminus A) \cup G) : G \in \tau_1 \vee \tau_2\}.$$

We shall denote by τ the topology $\tau_1 \vee \tau_2$.

Lemma 5.1.5 below provides a useful property of bilocale closure.

Lemma 5.1.5. *Let A be a subset of a* sup-T_D*-bispace* (X, τ_1, τ_2). *Then* $\widetilde{\mathrm{cl}_{\tau_i}(A)} = \mathrm{cl}_i(\widetilde{A})$ *for* $i = 1, 2$.

Proof. Since $A \subseteq \mathrm{cl}_{\tau_i}(A)$, $\widetilde{A} \subseteq \widetilde{\mathrm{cl}_{\tau_i}(A)}$. Because $\mathrm{cl}_{\tau_i}(A)$ is τ-closed, $\widetilde{A} \subseteq \mathfrak{c}(X \smallsetminus \mathrm{cl}_{\tau_i}(A)) = \widetilde{\mathrm{cl}_{\tau_i}(A)}$ by Lemma 2.1.14. By Proposition 5.1.3(2), $\mathrm{cl}_i(\widetilde{A}) \subseteq \mathrm{cl}_i(\mathfrak{c}(X \smallsetminus \mathrm{cl}_{\tau_i}(A)))$. Since $X \smallsetminus \mathrm{cl}_{\tau_i}(A) \in \tau_i$, it follows from Proposition 5.1.3(4) that $\mathrm{cl}_i(\mathfrak{c}(X \smallsetminus \mathrm{cl}_{\tau_i}(A))) = \mathfrak{c}(X \smallsetminus \mathrm{cl}_{\tau_i}(A))$ which gives

$$\mathrm{cl}_i(\widetilde{A}) \subseteq \mathfrak{c}(X \smallsetminus \mathrm{cl}_{\tau_i}(A)) = \widetilde{\mathrm{cl}_{\tau_i}(A)}.$$

On the other hand, since $\mathrm{cl}_i(\widetilde{A}) = \mathfrak{c}\left(\bigvee\{U \in \tau_i : \widetilde{A} \subseteq \mathfrak{c}(U)\}\right)$, set $\bigvee\{U \in \tau_i : \widetilde{A} \subseteq \mathfrak{c}(U)\} = V$ for some $V \in \tau_i$. Then $\widetilde{A} \subseteq \mathfrak{c}(V) = \widetilde{X \smallsetminus V}$ implying that $A \subseteq X \smallsetminus V$. Therefore $\mathrm{cl}_{\tau_i}(A) \subseteq \mathrm{cl}_{\tau_i}(X \smallsetminus V) = X \smallsetminus V$ since $X \smallsetminus V$ is τ_i-closed. We get $V \subseteq X \smallsetminus \mathrm{cl}_{\tau_i}(A)$ so that $\mathfrak{c}(V) \subseteq \mathfrak{c}(X \smallsetminus \mathrm{cl}_{\tau_i}(A))$. Therefore $\mathrm{cl}_i(\widetilde{A}) \subseteq \widetilde{\mathrm{cl}_{\tau_i}(A)}$.

Thus $\mathrm{cl}_i(\widetilde{A}) = \widetilde{\mathrm{cl}_{\tau_i}(A)}$. □

As a result of Lemma 5.1.5, we have the following proposition which shows that in the class of sup-T_D-bispaces, the definition of i-density given in Definition 5.1.4 is conservative in bilocales in a sense that a subset A of a sup-T_D-bispace (X, τ_1, τ_2) is i-dense precisely when $\widetilde{A}$ is i-dense in the bilocale (τ, τ_1, τ_2).

Proposition 5.1.6. *Let A be a subset of a* sup-T_D*-bispace* (X, τ_1, τ_2). *Then A is i-dense iff $\widetilde{A}$ is i-dense.*

Proof. A subset A of X is i-dense if and only if $\text{cl}_{\tau_i}(A) = X$ if and only if $\widetilde{\text{cl}_{\tau_i}(A)} = \widetilde{X}$ if and only if $\text{cl}_i(\widetilde{A}) = \widetilde{X}$ if and only if $\widetilde{A}$ is i-dense. □

We give an elementary notion of i-density.

Definition 5.1.7. Define an element $x \in L_j$ of a bilocale (L, L_1, L_2) to be L_i*-dense* (or just i*-dense*) if $x^\bullet = 0$.

The following result gives a characterization of i-dense elements.

Proposition 5.1.8. *Let (L, L_1, L_2) be a bilocale and $x \in L_j$. Then the following statements are equivalent.*

1. *x is i-dense.*
2. *$\mathfrak{o}(x)$ is i-dense.*
3. *For all $a \in L_i$, $a \wedge x = 0$ implies $a = 0$.*

Proof. (1) $\iff$ (2): Observe that for any $x \in L_j$,

$$\begin{aligned} x^\bullet = 0 \quad &\iff \quad \mathfrak{c}(x^\bullet) = L \\ &\iff \quad \text{cl}_i(\mathfrak{o}(x)) = L \quad \text{since} \quad \mathfrak{c}(x^\bullet) = \text{cl}_i(\mathfrak{o}(x)) \text{ from Proposition 5.1.3(10)} \\ &\iff \quad \mathfrak{o}(x) \quad \text{is} \quad j\text{-dense.} \end{aligned}$$

(2) $\implies$ (3): If $a \in L_i$ such that $x \wedge a = 0$, then $\mathfrak{o}(x) \subseteq \mathfrak{c}(a)$ which implies that

$$L = \mathfrak{c}(0) = \text{cl}_i(\mathfrak{o}(x)) \subseteq \text{cl}_i(\mathfrak{c}(a)) = \mathfrak{c}(a).$$

Thus $a = 0$.

(3) $\implies$ (1): Recall that $x^\bullet \wedge x = 0$ by Proposition 5.1.2(2). The hypothesis gives $x^\bullet = 0$. Thus x is i-dense. □

Observation 5.1.9. Every $y \in L_i$ such that $x \leq y$ for some i-dense $x \in L_j$ is i-dense. Indeed, if $a \in L_i$ is such that $a \wedge y = 0$, then $a \wedge x = 0$ so that $a = 0$ because x is i-dense.

In the next result, we show that in the class of sup-T$_D$-bispaces, the definition of (i,j)-nowhere density given in Definition 5.1.1 is conservative in bilocales. Recall that for each $U \in \tau_i$,

$$\begin{aligned} U^{\bullet} &= \bigvee\{G \in \tau_j : G \cap U = \emptyset\} \\ &= \bigvee\{G \in \tau_j : G \subseteq X \smallsetminus U\} \\ &= \operatorname{int}_{\tau_j}(X \smallsetminus U) \\ &= X \smallsetminus \operatorname{cl}_{\tau_j}(U). \end{aligned}$$

Proposition 5.1.10. *Let (X, τ_1, τ_2) be a* sup-T_D*-bispace. A subset $A \subseteq X$ is (τ_i, τ_j)-nowhere dense iff $\widetilde{A}$ is (i,j)-nowhere dense.*

Proof. Observe that

$$\begin{aligned} \operatorname{int}_{\tau_j}(\operatorname{cl}_{\tau_i}(A)) = \emptyset &\iff X \smallsetminus \operatorname{int}_{\tau_j}(\operatorname{cl}_{\tau_i}(A)) = X \\ &\iff \operatorname{cl}_{\tau_j}(X \smallsetminus \operatorname{cl}_{\tau_i}(A)) = X \\ &\iff X \smallsetminus \operatorname{cl}_{\tau_j}(X \smallsetminus \operatorname{cl}_{\tau_i}(A)) = \emptyset \\ &\iff (X \smallsetminus \operatorname{cl}_{\tau_i}(A))^{\bullet} = \emptyset \quad \text{since} \quad U^{\bullet} = X \smallsetminus \operatorname{cl}_{\tau_j}(U) \quad \text{for all} \quad U \in \tau_i \\ &\iff \mathfrak{o}((X \smallsetminus \operatorname{cl}_{\tau_i}(A))^{\bullet}) = \mathsf{O} \\ &\iff \operatorname{int}_j(\mathfrak{c}(X \smallsetminus \operatorname{cl}_{\tau_i}(A))) = \mathsf{O} \quad \text{from Proposition 5.1.3(9)} \\ &\iff \operatorname{int}_j\left(\widetilde{\operatorname{cl}_{\tau_i}(A)}\right) = \mathsf{O} \quad \text{since} \quad \operatorname{cl}_{\tau_i}(A) \quad \text{is} \quad \tau-\text{closed} \\ &\iff \operatorname{int}_j\left(\operatorname{cl}_i(\widetilde{A})\right) = \mathsf{O} \quad \text{since} \quad \widetilde{\operatorname{cl}_{\tau_i}(A)} = \operatorname{cl}_i(\widetilde{A}) \end{aligned}$$

which proves the result. ∎

The following result gives a characterization of (i,j)-nowhere dense sublocales.

Theorem 5.1.11. *Let (L, L_1, L_2) be a bilocale and $S \in \mathcal{S}(L)$. The following statements are equivalent.*

1. S *is* (i,j)*-nowhere dense.*

2. $L \smallsetminus \mathrm{cl}_i(S)$ *is* j*-dense.*

3. $(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\})^{\bullet} = 0.$

4. $\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}$ *is* j*-dense.*

5. $\mathrm{cl}_i(S)$ *is* (i,j)*-nowhere dense.*

Proof. (1) $\Longleftrightarrow$ (2): We have that

$$\begin{aligned}
\mathrm{int}_j(\mathrm{cl}_i(S)) = \mathsf{O} &\iff \mathfrak{o}\left(\bigvee\{a \in L_j : \mathfrak{o}(a) \subseteq \mathrm{cl}_i(S)\}\right) = \mathsf{O}\\
&\iff \mathfrak{c}\left(\bigvee\{a \in L_j : \mathfrak{o}(a) \subseteq \mathrm{cl}_i(S)\}\right) = L\\
&\iff \mathfrak{c}\left(\bigvee\{a \in L_j : (L \smallsetminus \mathrm{cl}_i(S)) \subseteq \mathfrak{c}(a)\}\right) = L,\\
&\qquad \text{since } \mathrm{cl}_i(S) \text{ is complemented.}\\
&\iff \mathrm{cl}_j(L \smallsetminus \mathrm{cl}_i(S)) = L \quad \text{by Proposition 5.1.3(4).}
\end{aligned}$$

(2) $\Longleftrightarrow$ (3): Observe that

$$\begin{aligned}
\mathrm{cl}_j(L \smallsetminus \mathrm{cl}_i(S)) = L &\iff \mathrm{cl}_j\left(L \smallsetminus \mathfrak{c}\left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right)\right) = L\\
&\iff \mathrm{cl}_j\left(\mathfrak{o}\left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right)\right) = L\\
&\iff \mathfrak{c}\left(\left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right)^{\bullet}\right) = L \quad \text{by Proposition 5.1.3(10)}\\
&\iff \left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right)^{\bullet} = 0.
\end{aligned}$$

(3) $\Longleftrightarrow$ (4): Follows from definition of j-density.

(4) $\iff$ (5): We have that

$$\begin{aligned}
\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\} \text{ is } j\text{-dense} &\iff \bigvee\{x \in L_i : \mathrm{cl}_i(S) \subseteq \mathrm{cl}_i(\mathfrak{c}(x)) = \mathfrak{c}(x)\} \text{ is } j\text{-dense} \\
&\iff \bigvee\{x \in L_i : \mathfrak{o}(x) \subseteq L \smallsetminus \mathrm{cl}_i(S)\} \text{ is } j\text{-dense} \\
&\iff \mathrm{cl}_j\left(\mathfrak{o}\left(\bigvee\{x \in L_i : \mathfrak{o}(x) \subseteq L \smallsetminus \mathrm{cl}_i(S)\}\right)\right) = L \\
&\iff \mathrm{cl}_j\left(\mathfrak{o}\left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right)\right) = L \\
&\iff \mathrm{cl}_j\left(L \smallsetminus \mathfrak{c}\left(\bigvee\{x \in L_i : S \subseteq \mathfrak{c}(x)\}\right)\right) = L \\
&\iff \mathrm{cl}_j\left(L \smallsetminus \mathrm{cl}_i(S)\right) = L \\
&\iff L \smallsetminus \mathrm{cl}_j\left(L \smallsetminus \mathrm{cl}_i(S)\right) = \mathsf{O} \\
&\iff \mathfrak{o}\left(\bigvee\{a \in L_j : L \smallsetminus \mathrm{cl}_i(S) \subseteq \mathfrak{c}(a)\}\right) = \mathsf{O} \\
&\iff \mathfrak{o}\left(\bigvee\{a \in L_j : \mathfrak{o}(a) \subseteq \mathrm{cl}_i(S)\}\right) = \mathsf{O} \\
&\iff \mathrm{int}_j\big(\mathrm{cl}_i(S)\big) = \mathsf{O} \\
&\iff \mathrm{int}_j\big(\mathrm{cl}_i(\mathrm{cl}_i(S))\big) = \mathsf{O} \quad \text{since } \mathrm{cl}_i(\mathrm{cl}_i(S)) = \mathrm{cl}_i(S) \\
&\iff \mathrm{cl}_i(S) \text{ is } (i,j)\text{-nowhere dense.}
\end{aligned}$$

(5) $\iff$ (1): Follows since $\mathrm{cl}_i(\mathrm{cl}_i(S)) = \mathrm{cl}_i(S)$. □

In terms of closed sublocales, we get the following characterization of (i, j)-nowhere dense sublocales.

Corollary 5.1.12. *An element $a \in L_i$ is j-dense iff $\mathfrak{c}(a)$ is (i, j)-nowhere dense.*

We give more properties of (i, j)-nowhere dense sublocales.

Proposition 5.1.13. *Let (L, L_1, L_2) be a bilocale and $S \in \mathcal{S}(L)$. The following statements hold.*

1. *S is (i, j)-nowhere dense whenever $\mathrm{cl}_i(S) \cap \mathfrak{B}L = \mathsf{O}$.*

2. *Let $T \in \mathcal{S}(L)$. If S is (i, j)-nowhere dense and $T \subseteq S$, then T is (i, j)-nowhere dense.*

Proof. (1) Observe that

$$\begin{aligned}
\mathrm{cl}_i(S) \cap \mathfrak{B}L = \mathsf{O} &\iff \mathfrak{B}L \subseteq L \smallsetminus \mathrm{cl}_i(S) \text{ since } \mathrm{cl}_i(S) \text{ is complemented}\\
&\iff \overline{L \smallsetminus \mathrm{cl}_i(S)} = L\\
&\implies \mathrm{cl}_j(L \smallsetminus \mathrm{cl}_i(S)) = L \quad \text{by Proposition 5.1.3(1)}\\
&\iff \mathrm{int}_j(\mathrm{cl}_i(S)) = \mathsf{O}
\end{aligned}$$

where the latter equivalence can be deduced from the proof of Proposition 5.1.11(4) $\iff$ (5).

(2) If $T \subseteq S$, then $\mathrm{cl}_i(T) \subseteq \mathrm{cl}_i(S)$ by Proposition 5.1.3(2). This implies that $\mathrm{int}_j(\mathrm{cl}_i(T)) \subseteq \mathrm{int}_j(\mathrm{cl}_i(S))$. But $\mathrm{int}_j(\mathrm{cl}_i(S)) = \mathsf{O}$, so $\mathrm{int}_j(\mathrm{cl}_i(T)) = \mathsf{O}$. □

To characterize (i,j)-nowhere dense sublocales in terms of $\mathfrak{B}L$, we recall from [29] that a bilocale (L, L_1, L_2) is *balanced* if $x \in L_1$ implies $x^* \in L_2$ and $x \in L_2$ implies $x^* \in L_1$.

In a balanced bilocale (L, L_1, L_2), $a^* = a^\bullet$ for all $a \in L_i$. Indeed, it is clear that $a^\bullet \leq a^*$. Furthermore, if $y = a^*$, then $y \in L_j$ and $y \wedge a = 0$. Therefore $y \in \{x \in L_j : a \wedge x = 0\}$. Thus

$$y = a^* \leq \bigvee \{x \in L_j : a \wedge x = 0\} = a^\bullet.$$

Proposition 5.1.14. *Let (L, L_1, L_2) be a balanced bilocale and $N \in \mathcal{S}(L)$. Then $N \in \mathcal{S}(L)$ is (i,j)-nowhere dense iff $\mathfrak{B}L \cap \mathrm{cl}_i(N) = \mathsf{O}$.*

Proof. For each $N \in \mathcal{S}(L)$, we have that

$$\begin{aligned}
N \text{ is } (i,j)\text{-nowhere dense} &\iff \left(\bigvee\{x \in L_i : N \subseteq \mathfrak{c}(x)\}\right)^\bullet = 0 \text{ by Theorem 5.1.11}\\
&\iff \left(\bigvee\{x \in L_i : N \subseteq \mathfrak{c}(x)\}\right)^* = 0 \text{ since } (L, L_1, L_2) \text{ is balanced}\\
&\iff \mathfrak{o}\left(\bigvee\{x \in L_i : N \subseteq \mathfrak{c}(x)\}\right) \text{ is dense}\\
&\iff \mathfrak{B}L \subseteq \mathfrak{o}\left(\bigvee\{x \in L_i : N \subseteq \mathfrak{c}(x)\}\right)\\
&\iff \mathfrak{B}L \cap \mathfrak{c}\left(\bigvee\{x \in L_i : N \subseteq \mathfrak{c}(x)\}\right) = \mathsf{O}\\
&\iff \mathfrak{B}L \cap \mathrm{cl}_i(N) = \mathsf{O}
\end{aligned}$$

which proves the result. □

We recalled in preliminaries the frame-theoretic definition of a nowhere dense quotient map which was defined by Dube in [16]. This notion was motivated by Plewe's definition of a nowhere dense sublocale. In bilocales, we could only get the following result in an attempt to get a notion of an (i,j)-nowhere dense biframe homomorphism in such a way that a sublocale S of a bilocale $(L.L_1,L_2)$ is (i,j)-nowhere dense precisely when the biframe map $\nu_S:(L,L_1,L_2)\to(S,S_1,S_2)$ is (i,j)-nowhere dense.

Proposition 5.1.15. *Let (L,L_1,L_2) be a bilocale and $S\in\mathcal{S}(L)$. If S is (i,j)-nowhere dense, then for each non-zero $x\in L_j$, there exists a non-zero $y\in L$ with $y\le x$ such that $\nu_S(y)=0_S$.*

Proof. Let x be a non-zero element of L_j. Since S is (i,j)-nowhere dense, the L_i-element

$$\bigvee\{a\in L_i:S\subseteq\mathfrak{c}(a)\}=\bigvee\left\{a\in L_i:a\le\bigwedge S\right\}$$

is j-dense by Theorem 5.1.11. It follows from Proposition 5.1.8 that $\bigvee\{a\in L_i:a\le\bigwedge S\}\wedge x\neq 0$. Set $y=\bigvee\{a\in L_i:a\le\bigwedge S\}\wedge x$. Then $y\le x$, $y\neq 0$ and

$$\nu_S(y)=\nu_S\left(\bigvee\left\{a\in L_i:a\le\bigwedge S\right\}\wedge x\right)=\nu_S\left(\bigvee\left\{a\in L_i:a\le\bigwedge S\right\}\right)\wedge\nu_S(x)\le 0_S$$

which proves the result. □

We show below that the converse of Proposition 5.1.15 is not always true.

Example 5.1.16. Let (X,τ_1,τ_2) be a bitopological space where $X=\{a,b,c\}$, $\tau_1=\{\emptyset,X,\{a\}\}$ and $\tau_2=\{\emptyset,X,\{a,b\}\}$. Then the triple $(\tau=\{\emptyset,X,\{a\},\{a,b\}\},\tau_1,\tau_2)$ is a bilocale and the sublocale $S=\{X,\{a\}\}$ is not $(1,2)$-nowhere dense but for each non-empty $A\in\tau_2$, there is a non-empty $B\in\tau$ such that $B\subseteq A$ and $\nu_S[B]=0_S=\{a\}$.

We introduce certain types (i,j)-nowhere dense sublocales which give the converse of Proposition 5.1.15. In light of Proposition 5.1.8(3), we give the following definition.

Definition 5.1.17. Let (L,L_1,L_2) be a bilocale.

1. An element $y\in L$ is said to be *almost i-dense* in case $y\wedge a=0$ implies $a=0$ for all $a\in L_i$.

2. A sublocale of L is *almost i-dense* if it meets every non-void open sublocale induced by an element of L_i.

3. A sublocale N of L is *almost (i,j)-nowhere dense* if 0_N is almost j-dense.

We give the following results, some of which will be used below.

Proposition 5.1.18. *Let (L, L_1, L_2) be a bilocale.*

1. *For both elements and sublocales, i-density implies almost i-density.*

2. *An element $y \in L_j$ is i-dense if and only if it is almost i-dense.*

3. *Every $y \in L$ such that $x \leq y$ for some i-dense (resp. almost i-dense) $x \in L_j$ is almost i-dense.*

4. *Every element that is dense in the total part of a bilocale is almost i-dense.*

5. *(i,j)-nowhere density implies almost (i,j)-nowhere density.*

6. *A sublocale $N \subseteq L$ is almost (i,j)-nowhere dense iff $\overline{N}$ is almost (i,j)-nowhere dense.*

7. *For each $N \in \mathcal{S}(L)$, if $\mathrm{cl}_i(N)$ is almost (i,j)-nowhere dense, then N is almost (i,j)-nowhere dense.*

Proof. (1) The proof for elements is clear.

For sublocales, let $S \in \mathcal{S}(L)$ be i-dense. We show that for every $x \in L_i$, $S \cap \mathfrak{o}(x) = \mathsf{O}$ implies $\mathfrak{o}(x) = \mathsf{O}$. Let $x \in L_i$. Then

$$\begin{aligned} S \cap \mathfrak{o}(x) = \mathsf{O} \quad &\Longrightarrow \quad S \subseteq \mathfrak{c}(x) \\ &\Longrightarrow \quad L = \mathrm{cl}_i(S) \subseteq \mathrm{cl}_i(\mathfrak{c}(x)) = \mathfrak{c}(x) \\ &\Longrightarrow \quad \mathfrak{o}(x) = \mathsf{O}. \end{aligned}$$

(2) Trivial.

(3) Let $y \in L$ be such that $x \leq y$ for some i-dense $x \in L_j$. If $z \in L_i$ is such that $y \wedge z = 0$, then $x \wedge z = 0$. Since x is i-dense, $z = 0$, making y almost i-dense.

The other part follows since by (2), almost i-dense = i-dense for all $x \in L_j$.

(4) Let $x \in L$ be dense in L and choose $a \in L_i$ such that $x \wedge a = 0$. Then $a \leq x^* = 0$, making $a = 0$. Thus x is almost i-dense.

(5) Let $S \in \mathcal{S}(L)$ be (i,j)-nowhere dense. It follows from Theorem 5.1.11 that $\bigvee\{a \in L_i : S \subseteq \mathfrak{c}(a)\}$ is j-dense. Since $\bigvee\{a \in L_i : S \subseteq \mathfrak{c}(a)\} = \bigvee\{a \in L_i : a \leq 0_S\} \leq 0_S$, it follows from (3) that 0_S is almost j-dense. Thus S is almost (i,j)-nowhere dense.

(6) Follows since $0_N = 0_{\overline{N}}$ for every $N \in \mathcal{S}(L)$.

(7) Because $0_{\mathrm{cl}_i(N)} = \bigvee\{a \in L_i : a \leq 0_N\} \leq 0_N$, it follows from (3) that 0_N is almost j-dense whenever $0_{\mathrm{cl}_i(N)}$ is. Thus if $\mathrm{cl}_i(N)$ is almost (i,j)-nowhere dense, then N is almost (i,j)-nowhere dense. ☐

We get the following result where the left to right implication is proved similarly to Proposition 5.1.15. We shall therefore only prove the converse.

Proposition 5.1.19. *Let (L, L_1, L_2) be a bilocale and $S \in \mathcal{S}(L)$. Then S is almost (i,j)-nowhere dense iff for each non-zero $x \in L_j$, there exists a non-zero $y \in L$ with $y \leq x$ such that $\nu_S(y) = 0_S$.*

Proof. ($\Longleftarrow$): We show that 0_S is almost j-dense, i.e., $0_S \wedge x \neq 0$ for every non-zero $x \in L_j$. Let $x \in L_j$ be such that $x \neq 0$. By hypothesis, there is a non-zero $y \in L$ with $y \leq x$ such that $\nu_S(y) = 0_S$. Because $y \leq \nu_S(y)$ and $y \wedge x \neq 0$, we have that $0_S \wedge x \neq 0$. Thus 0_S is almost j-dense making S almost (i,j)-nowhere dense. ☐

Observation 5.1.20. The combination of Proposition 5.1.19 and Proposition 5.1.15 tells us that (i,j)-nowhere density implies almost (i,j)-nowhere density.

5.2 (i,j)-remote sublocales

The aim of this section is to introduce and study (i,j)-remote sublocales. These are bilocale counterparts of remote sublocales introduced in Chapter 2.

We give a definition of an (i,j)-remote sublocale.

Definition 5.2.1. Let (L, L_1, L_2) be a bilocale. A sublocale $S \subseteq L$ is (i, j)-*remote* if $N \cap S = \mathsf{O}$ for every (i, j)-nowhere dense sublocale N of L.

In bispaces, we shall say that a subset A of a bispace (X, τ_1, τ_2) is (τ_i, τ_j)-*remote* in case $N \cap A = \emptyset$ for all (τ_i, τ_j)-nowhere dense subset N of X.

We consider some examples.

Example 5.2.2. (1) The void sublocale is (i, j)-remote.

(2) In a balanced bilocale (L, L_1, L_2), $\mathfrak{B}L$ is (i, j)-remote. This follows from Proposition 5.1.14.

(3) In a *symmetric bilocale*, which was defined in [4] as a bilocale (L, L_1, L_2) in which $L = L_1 = L_2$, the sublocale $\mathfrak{B}L$ is (i, j)-remote.

(4) If A is an (i, j)-remote sublocale of L, then every sublocale of L contained in A is (i, j)-remote.

Next, we define sublocales that are (i, j)-remote from a dense subbilocale. Recall from Chapter 1 that a *subbilocale* of a bilocale (L, L_1, L_2) is a triple (S, S_1, S_2) where S is a sublocale of L and

$$S_i = \nu_S[L_i] \quad \text{for} \quad i = 1, 2.$$

We shall say that (S, S_1, S_2) is a *dense subbilocale* of L in case S is dense in L. To avoid confusion, we shall use subscripts i_S and j_S for subframes of a sublocale S.

Definition 5.2.3. Let (S, S_1, S_2) be a dense subbilocale of a bilocale (L, L_1, L_2). A sublocale $T \subseteq L$ is (i, j)-*remote from* (S, S_1, S_2) if $\mathrm{cl}_i(N) \cap T = \mathsf{O}$ for every (i_S, j_S)-nowhere dense $N \in \mathcal{S}(S)$, for $i \neq j \in \{1, 2\}$.

We combine into one theorem results about both the notions of (i, j)-remoteness given in Definition 5.2.1 and Definition 5.2.3. In the same theorem, we give a condition such that all the results are equivalent.

We start by observing that, for a subbilocale (S, S_1, S_2) of a bilocale (L, L_1, L_2), $\mathrm{cl}_i(\mathfrak{c}_S(x)) =$

$\text{cl}_i(\mathfrak{c}(x))$ for every $x \in S_i$. Indeed,

$$\begin{aligned}
\text{cl}_i(\mathfrak{c}_S(x)) &= \mathfrak{c}\left(\bigvee\{y \in L_i : \mathfrak{c}_S(x) \subseteq \mathfrak{c}(y)\}\right) \\
&= \mathfrak{c}\left(\bigvee\{y \in L_i : \overline{\mathfrak{c}_S(x)} \subseteq \mathfrak{c}(y)\}\right) \\
&= \mathfrak{c}\left(\bigvee\{y \in L_i : \mathfrak{c}(x) \subseteq \mathfrak{c}(y)\}\right) \quad \text{since} \quad \overline{\mathfrak{c}_S(x)} = \mathfrak{c}(x) \\
&= \text{cl}_i(\mathfrak{c}(x)).
\end{aligned}$$

We remind the reader that for any dense sublocale S of a locale L, $0_S = 0_L$ and hence $\nu_S(x) = 0_S$ if and only if $x = 0_L$.

The following lemma will be useful below.

Lemma 5.2.4. *Let (S, S_1, S_2) be a dense subbilocale of a bilocale (L, L_1, L_2). An element y of L_i is j-dense iff $\nu_S(y)$ is j-dense$_S$.*

Proof. ($\Longrightarrow$): Let $y \in L_i$ be a j-dense element. Since $\nu_S[L_i] = S_i$, $\nu_S(y) \in S_i$. Now, choose $a \in S_j$ such that $a \wedge \nu_S(y) = 0$. Then $a = \nu_S(x)$ for some $x \in L_j$. Therefore

$$0 = \nu_S(x) \wedge \nu_S(y) = \nu_S(x \wedge y) \geq x \wedge y.$$

Since y is j-dense and $x \in L_j$, $x = 0$ so that $\nu_S(x) = 0$. Thus $\nu_S(y)$ is j-dense$_S$.

($\Longleftarrow$): Let $y \in L_i$ be such that $\nu_S(y)$ is j-dense$_S$ and choose $a \in L_j$ such that $a \wedge y = 0$. Then $0 = \nu_S(a \wedge y) = \nu_S(a) \wedge \nu_S(y)$. But $\nu_S(y)$ is j-dense$_S$, so $\nu_S(a) = 0$ by Proposition 5.1.8(3). Since $a \leq \nu_S(a)$, $a = 0$ making y j-dense. □

Theorem 5.2.5. *Let (S, S_1, S_2) be a dense subbilocale of a bilocale (L, L_1, L_2) and let $A \in \mathcal{S}(L)$. Consider the following statements.*

1. *A is (i, j)-remote.*
2. *$A \cap \text{cl}_i(C) = \mathsf{O}$ for every (i, j)-nowhere dense $C \in \mathcal{S}(L)$.*
3. *$A \cap \overline{N} = \mathsf{O}$ for every (i, j)-nowhere dense N.*
4. *$A \cap \mathfrak{c}(x) = \mathsf{O}$ for each j-dense $x \in L_i$.*
5. *$A \subseteq \mathfrak{o}(a)$ for every j-dense $a \in L_i$.*

6. $\nu_A(d) = 1$ *for every* j*-dense* $d \in L_i$*.*

7. A *is* (i,j)*-remote from* S*.*

8. $A \cap \mathrm{cl}_i(\mathfrak{c}(c)) = \mathsf{O}$ *for each* j*-dense*$_S$ $c \in S_i$*.*

9. $A \subseteq \mathrm{int}_i(\mathfrak{o}(p))$ *for every* j*-dense*$_S$ $p \in S_i$*.*

Then $1 \iff 2 \iff 3 \iff 4 \iff 5 \iff 6 \implies 7 \iff 8 \iff 9$*. Moreover, all these statements are equivalent whenever* $S_i = L_i$ *(with* i *not neccessarily equal to* j*).*

Proof. (1) $\iff$ (2): Follows since a sublocale C of L is (i,j)-nowhere dense if and only if $\mathrm{cl}_i(C)$ is (i,j)-nowhere dense.

(2) $\implies$ (3): Let $N \in \mathcal{S}(L)$ be (i,j)-nowhere dense. Since $\overline{N} \subseteq \mathrm{cl}_i(N)$, it follows from Proposition 5.1.13(2) that $\overline{N}$ is (i,j)-nowhere dense. Therefore $A \cap \overline{N} = \mathsf{O}$.

(3) $\implies$ (4): Let $x \in L_i$ be j-dense. It follows that $\mathfrak{c}(x)$ is (i,j)-nowhere dense. By (3),

$$\mathsf{O} = A \cap \overline{\mathfrak{c}(x)} = A \cap \mathfrak{c}(x).$$

(4) $\iff$ (5): Follows since $A \cap \mathfrak{c}(y) = \mathsf{O}$ if and only if $A \subseteq \mathfrak{o}(y)$ for all $A \in \mathcal{S}(L)$ and every $y \in L$.

(5) $\implies$ (6): Let $d \in L_i$ be j-dense. By (5), $A \subseteq \mathfrak{o}(d)$. Since $\nu_B(a) = 1$ if and only if $B \subseteq \mathfrak{o}(a)$ for every $a \in L, B \in \mathcal{S}(L)$, $\nu_A(d) = 1$.

(6) $\implies$ (1): Let $N \in \mathcal{S}(L)$ be (i,j)-nowhere dense. It follows from Theorem 5.1.11 that $\bigvee\{a \in L_i : N \subseteq \mathfrak{c}(a)\}$ is a j-dense element of L_i. By hypothesis, $\nu_A(\bigvee\{a \in L_i : N \subseteq \mathfrak{c}(a)\}) = 1$ which implies that $A \subseteq \mathfrak{o}\left(\bigvee\{a \in L_i : N \subseteq \mathfrak{c}(a)\}\right)$. Therefore

$$\mathsf{O} = A \cap \mathfrak{c}\left(\bigvee\{a \in L_i : N \subseteq \mathfrak{c}(a)\}\right) = A \cap \mathrm{cl}_i(N).$$

(6) $\implies$ (7): Choose an (i_S, j_S)-nowhere dense $N \in \mathcal{S}(S)$. Then $\mathrm{cl}_{i_S}(N)$ is (i_S, j_S)-nowhere dense. But $\mathrm{cl}_{i_S}(N) = \mathfrak{c}_S\big(\bigvee_S \{b \in S_i : N \subseteq \mathfrak{c}_S(b)\}\big)$, so set $c = \bigvee_S \{b \in S_i : N \subseteq \mathfrak{c}_S(b)\}$. Then $c \in S_i$ and j-dense$_S$. Since $S_i = \nu_S[L_i]$, there is $y \in L_i$ such that $\nu_S(y) = c$. It follows from

Lemma 5.2.4 that y is j-dense. By (6),

$$\begin{aligned}
\nu_A(y) = 1 \quad &\iff \quad A \subseteq \mathfrak{o}(y) \text{ since } \nu_B(a) = 1 \text{ if and only if } B \subseteq \mathfrak{o}(a) \text{ for every } a \in L, B \in \mathcal{S}(L) \\
&\implies \quad A \cap \mathfrak{c}(y) = \mathsf{O} \\
&\iff \quad A \cap \mathrm{cl}_i(\mathfrak{c}(y)) = \mathsf{O} \quad \text{by Proposition 5.1.3(4)} \\
&\implies \quad A \cap \mathrm{cl}_i(\mathfrak{c}(\nu_S(y))) = \mathsf{O} \quad \text{since } \mathfrak{c}(\nu_S(y)) \subseteq \mathfrak{c}(y) \\
&\implies \quad A \cap \mathrm{cl}_i(\mathfrak{c}_S(c)) = \mathsf{O} \\
&\iff \quad A \cap \mathrm{cl}_i(\mathrm{cl}_{i_S}(\mathfrak{c}_S(c))) = \mathsf{O} \quad \text{by Proposition 5.1.3(4)} \\
&\iff \quad A \cap \mathrm{cl}_i(\mathrm{cl}_{i_S}(\mathrm{cl}_{i_S}(N))) = \mathsf{O} \\
&\implies \quad A \cap \mathrm{cl}_i(N) = \mathsf{O}.
\end{aligned}$$

(7) $\implies$ (8): Let $c \in S_i$ be j-dense$_S$. Then $\mathfrak{c}_S(c)$ is (i_S, j_S)-nowhere dense. By (7),

$$\mathsf{O} = \mathrm{cl}_i(\mathfrak{c}_S(c)) \cap A = \mathrm{cl}_i(\mathfrak{c}(c)) \cap A.$$

(8) $\iff$ (9): We have that $A \cap \mathrm{cl}_i(\mathfrak{c}(x)) = \mathsf{O}$ if and only if $A \subseteq L \smallsetminus \mathrm{cl}_i(\mathfrak{c}(x)) = \mathrm{int}_i(\mathfrak{o}(x))$ for all $A \in \mathcal{S}(L)$ and every $x \in L$.

(9) $\implies$ (7): Let $N \in \mathcal{S}(S)$ be (i_S, j_S)-nowhere dense. We get that $\bigvee_S\{a \in S_i : N \subseteq \mathfrak{c}_S(a)\}$ is a j-dense$_S$ element of S_i. By (9), $A \subseteq \mathrm{int}_i\left(\mathfrak{o}\left(\bigvee_S\{a \in S_i : N \subseteq \mathfrak{c}_S(a)\}\right)\right)$ which implies that

$$\mathsf{O} = A \cap \left(L \smallsetminus \mathrm{int}_i \left(\mathfrak{o} \left(\bigvee_S \{a \in S_i : N \subseteq \mathfrak{c}_S(a)\} \right)\right)\right) = A \cap \mathrm{cl}_i \left(\mathfrak{c} \left(\bigvee_S \{a \in S_i : N \subseteq \mathfrak{c}_S(a)\} \right)\right)$$

where the last equality follows from Proposition 5.1.3(14). But

$$\mathrm{cl}_i \left(\mathfrak{c}_S \left(\bigvee_S \{a \in S_i : N \subseteq \mathfrak{c}_S(a)\} \right)\right) \subseteq \mathrm{cl}_i \left(\mathfrak{c} \left(\bigvee_S \{a \in S_i : N \subseteq \mathfrak{c}_S(a)\} \right)\right),$$

so

$$\mathsf{O} = A \cap \mathrm{cl}_i \left(\mathfrak{c}_S \left(\bigvee_S \{a \in S_i : N \subseteq \mathfrak{c}_S(a)\} \right)\right) = A \cap \mathrm{cl}_i\left(\mathrm{cl}_{i_S}(N)\right) \supseteq A \cap \mathrm{cl}_i(N).$$

For the special case, assume that $S_i = L_i$. We prove (7) $\implies$ (4). For each j-dense $a \in L_i$, we have that $a \in S_i$ and a is j-dense$_S$, so that $\mathfrak{c}(a)$ is (i_S, j_S)-nowhere dense. By (7), $\mathrm{cl}_i(\mathfrak{c}_S(a)) \cap A = \mathsf{O}$ which implies that $\mathsf{O} = \mathrm{cl}_i(\mathfrak{c}(a)) \cap A = \mathfrak{c}(a) \cap A$. □

We give a neccessary and sufficient condition for the total part of a bilocale to be (i, j)-remote as a sublocale of itself.

Proposition 5.2.6. *Let (L, L_1, L_2) be a bilocale. The following statements are equivalent.*

1. *L is (i, j)-remote as a sublocale of itself.*

2. *1 is the only j-dense element of L_i.*

Proof. (1) $\Longrightarrow$ (2): Let $x \in L_i$ be j-dense. We show that $x = 1$. Since L is (i, j)-remote as a sublocale of itself, $L \cap \mathfrak{c}(x) = \mathsf{O}$ by Theorem 5.2.5(3). Therefore $\mathfrak{c}(x) = \mathsf{O}$ making $x = 1$.

(2) $\Longrightarrow$ (1): Let $x \in L_i$ be j-dense. By (2), $x = 1$ so that $\mathfrak{c}(x) = \mathsf{O}$. Therefore $L \cap \mathfrak{c}(x) = \mathsf{O}$. Thus L is (i, j)-remote as a sublocale of itself. □

Next, we discuss (i, j)-remoteness of closed sublocales.

Proposition 5.2.7. *Let (L, L_1, L_2) be a bilocale and $a \in L$. Then $\mathfrak{c}(a)$ is (i, j)-remote iff $a \vee x = 1$ for every j-dense $x \in L_i$.*

Proof. For each j-dense $x \in L_i$, Theorem 5.2.5 implies that

$$\begin{aligned} \mathfrak{c}(a) \cap \mathrm{cl}_i(\mathfrak{c}(x)) = \mathsf{O} \quad &\Longleftrightarrow \quad \mathfrak{c}(a) \cap \mathfrak{c}(x) = \mathsf{O} \\ &\Longleftrightarrow \quad \mathfrak{c}(a \vee x) = \mathsf{O} \\ &\Longleftrightarrow \quad a \vee x = 1 \end{aligned}$$

which proves the result. □

Following Proposition 5.2.7, we get the following result for the case of sublocales that are (i, j)-remote from a dense subbilocale.

Proposition 5.2.8. *Let (S, S_1, S_2) be a dense subbilocale of a bilocale (L, L_1, L_2) and $a \in L$. Then $\mathfrak{c}(a)$ is (i, j)-remote from S iff $a \vee x = 1$ for every j-dense$_S$ $x \in S_i$.*

In what follows, we move the introduced notions of (i, j)-remoteness to bispaces.

Proposition 5.2.9. *Let (X, τ_1, τ_2) be a* sup-T_D *bitopological space and $A \subseteq X$. Then $\widetilde{A}$ is (i,j)-remote iff A is (τ_i, τ_j)-remote, where $i \neq j \in \{1,2\}$.*

Proof. $(\Longrightarrow)$: Let $N \subseteq X$ be (τ_i, τ_j)-nowhere dense. By Proposition 5.1.10, $\widetilde{N}$ is (i,j)-nowhere dense in τ. By hypothesis, $\widetilde{A} \cap \mathrm{cl}_i(\widetilde{N}) = \mathsf{O}$. But $\mathrm{cl}_i(\widetilde{N}) = \widetilde{\mathrm{cl}_{\tau_i}(N)}$, so $\widetilde{A} \cap \widetilde{\mathrm{cl}_{\tau_i}(N)} = \mathsf{O}$. Therefore $S_{(A \cap \mathrm{cl}_{\tau_i}(N))} = \mathsf{O}$ making $A \cap \mathrm{cl}_{\tau_i}(N) = \emptyset$.

$(\Longleftarrow)$: Let F be an (i,j)-nowhere dense sublocale of τ. By Theorem 5.1.11, $\mathrm{cl}_i(F)$ is (i,j)-nowhere dense in τ. But $\mathrm{cl}_i(F)$ is τ-closed, so there is a τ-closed $B \subseteq X$ such that $\mathrm{cl}_i(F) = \widetilde{B}$. It follows from Proposition 5.1.10 that B is (τ_i, τ_j)-nowhere dense. Therefore $B \cap A = \emptyset$. By Observation 2.1.17,

$$\mathsf{O} = \widetilde{B} \cap \widetilde{A} = \mathrm{cl}_i(F) \cap \widetilde{A}$$

which implies that $F \cap \widetilde{A} = \mathsf{O}$ as required. □

In bispaces, recall that the authors of [31] define a *subbispace* of a bispace (X, τ_1, τ_2) as a triple $(S, S_{\tau_1}, S_{\tau_2})$ where S_{τ_i} is a topology induced on $S \subseteq X$ by τ_i, for $i \in \{1,2\}$. For a bispace (X, τ_1, τ_2), we shall say that $A \subseteq X$ is (τ_i, τ_j)*-remote from a dense subbispace* $(S, S_{\tau_1}, S_{\tau_2})$ of (X, τ_1, τ_2) in case A misses the τ_i-closure of every (S_{τ_i}, S_{τ_j})-nowhere dense $N \subseteq S$. To transfer the notion of a sublocale which is (i,j)-remote from a dense subbilocale, we start by showing that (i,j)-nowhere density of subbilocales is conservative in bilocales. For a bispace (X, τ_1, τ_2) and $S \subseteq X$, set $\nu_{\widetilde{S}}[\tau_1] = \widetilde{S}_1$ and $\nu_{\widetilde{S}}[\tau_2] = \widetilde{S}_2$. It is clear that the triple $(\widetilde{S}, \widetilde{S}_1, \widetilde{S}_2)$ is a subbilocale of (τ, τ_1, τ_2).

Lemma 5.2.10. *Let (S, S_1, S_2) be a subbilocale of a bilocale (L, L_1, L_2). Then $\mathrm{cl}_{i_S}(N) = \mathrm{cl}_i(N) \cap S$ for each $N \in \mathcal{S}(S)$, where $i \in \{1,2\}$.*

Proof. Let $x \in \mathrm{cl}_{i_S}(N)$ and choose $\mathfrak{c}(a) \in \mathcal{S}(L)$ where $a \in L_i$ and $N \subseteq \mathfrak{c}(a)$. Then $N \subseteq \mathfrak{c}(a) \cap S = \mathfrak{c}_S(\nu_S(a))$ and $\nu_S(a) \in S_i$ since $\nu_S[L_i] = S_i$. Therefore $x \in \mathfrak{c}_S(\nu_S(a)) \subseteq \mathfrak{c}(a)$. Thus $x \in \mathrm{cl}_i[N] \cap S$.

On the other hand, let $x \in \mathrm{cl}_i[N] \cap S$ and $a \in S_i$ be such that $N \subseteq \mathfrak{c}_S(a)$. Since $S_i = \nu_S[L_i]$, $a = \nu_S(y)$ for some $y \in L_i$. Because $\mathfrak{c}_S(a) = \mathfrak{c}(a) \cap S$,

$$N \subseteq \mathfrak{c}(a) \cap S = \mathfrak{c}(\nu_S(y)) \cap S \subseteq \mathfrak{c}(\nu_S(y)) \subseteq \mathfrak{c}(y).$$

Therefore $x \in \mathfrak{c}(y)$ making $x \in \mathfrak{c}(y) \cap S = \mathfrak{c}_S(\nu_S(y)) = \mathfrak{c}_S(a)$. Thus $x \in \mathrm{cl}_{i_S}(N)$. □

Proposition 5.2.11. *Let (X, τ_1, τ_2) be a sup-T_D-bispace and $S \subseteq X$. Then $N \subseteq S$ is (S_{τ_i}, S_{τ_j})-nowhere dense iff $\widetilde{N} \subseteq \widetilde{S}$ is $(i_{\widetilde{S}}, j_{\widetilde{S}})$-nowhere dense, where $i \neq j \in \{1, 2\}$.*

Proof. ($\Longrightarrow$): We show that

$$\bigvee_{\widetilde{S}} \left\{ \mathfrak{o}_{\widetilde{S}}(U) : U \in \widetilde{S}_j, \mathfrak{o}_{\widetilde{S}}(U) \subseteq \mathrm{cl}_{i_{\widetilde{S}}}[\widetilde{N}] \right\} = \mathsf{O}.$$

Let $U \in \widetilde{S}_j$ be such that $\mathfrak{o}_{\widetilde{S}}(U) \subseteq \mathrm{cl}_{i_{\widetilde{S}}}\left(\widetilde{N}\right)$. Because $\widetilde{S}_j = \nu_{\widetilde{S}}[\tau_j]$, $U = \nu_{\widetilde{S}}(V)$ for some $V \in \tau_j$. We get that $\mathfrak{o}_{\widetilde{S}}(U) = \mathfrak{o}_{\widetilde{S}}(\nu_{\widetilde{S}}(V)) = \mathfrak{o}(V) \cap \widetilde{S} = \widetilde{V} \cap \widetilde{S}$. Since $\mathrm{cl}_{i_{\widetilde{S}}}\left(\widetilde{N}\right) = \mathrm{cl}_i(\widetilde{N}) \cap \widetilde{S} = \widetilde{\mathrm{cl}_{\tau_i}(N)} \cap \widetilde{S}$ by Lemma 5.2.10, $\widetilde{V} \cap \widetilde{S} \subseteq \widetilde{\mathrm{cl}_{\tau_i}(N)} \cap \widetilde{S}$. Therefore $\widetilde{V \cap S} \subseteq \widetilde{\mathrm{cl}_{\tau_i}(N)}$ making $V \cap S \subseteq \mathrm{cl}_{\tau_i}(N) \cap S$. But $V \in \tau_j$, so $V \cap S \in S_{\tau_j}$. Therefore $V \cap S = \emptyset$. Because $V \in \tau$, it follows from Observation 2.1.17 that $\mathsf{O} = \widetilde{V} \cap \widetilde{S} = \mathfrak{o}_{\widetilde{S}}(U)$ making $\mathrm{int}_{j_{\widetilde{S}}}\left(\mathrm{cl}_{i_{\widetilde{S}}}\left(\widetilde{N}\right)\right) = \mathsf{O}$. Thus $\widetilde{N}$ is $(i_{\widetilde{S}}, j_{\widetilde{S}})$-nowhere dense.

($\Longleftarrow$): Let $U \in S_{\tau_j}$ be such that $U \subseteq \mathrm{cl}_{\tau_i}(N)$. Then $U = V \cap S$ for some $V \in \tau_j$. Since V is τ-open, it follows from Lemma 4.1.7 that $\widetilde{V} \cap \widetilde{S} \subseteq \widetilde{\mathrm{cl}_{\tau_i}(N)} = \mathrm{cl}_i(\widetilde{N})$. Therefore

$$\mathfrak{o}_{\widetilde{S}}(\nu_{\widetilde{S}}(V)) = \mathfrak{o}(V) \cap \widetilde{S} = \widetilde{V} \cap \widetilde{S} \subseteq \mathrm{cl}_i(\widetilde{N}) \cap \widetilde{S} = \mathrm{cl}_{i_{\widetilde{S}}}(\widetilde{N}).$$

Such $\nu_{\widetilde{S}}(V)$ belongs to $\nu_{\widetilde{S}}[\tau_j] = \widetilde{S}_j$. Since $\widetilde{N}$ is $(i_{\widetilde{S}}, j_{\widetilde{S}})$-nowhere dense,

$$\mathsf{O} = \mathfrak{o}_{\widetilde{S}}(\nu_{\widetilde{S}}(V)) = \widetilde{V} \cap \widetilde{S} \supseteq \widetilde{V \cap S}.$$

Therefore $\emptyset = V \cap S = U$. Thus N is (S_{τ_i}, S_{τ_j})-nowhere dense. □

Now, we show that the definition of a sublocale (i, j)-remote from a dense subbilocale is conservative in bilocales. The proof is similar to that of Proposition 5.2.9, taking into account the result proved in Proposition 5.2.11. It shall be omitted.

Proposition 5.2.12. *Let (X, τ_1, τ_2) be a sup-T_D-bispace and $S \subseteq X$. Then $A \subseteq X$ is (τ_i, τ_j)-remote from $(S, S_{\tau_1}, S_{\tau_2})$ iff $\widetilde{A} \in \mathcal{S}(\tau)$ is (i, j)-remote from $(\widetilde{S}, \widetilde{S}_1, \widetilde{S}_2)$.*

We close this section with a short discussion on how (i, j)-remote sublocales are sent back and forth by bilocalic maps as defined below. We start by recalling that a *biframe map* $h : (M, M_1, M_2) \to (L, L_1, L_2)$ is a frame homomorphism $h : M \to L$ for which

$$h(M_i) \subseteq L_i \quad (i = 1, 2).$$

Definition 5.2.13. We call $f : (L, L_1, L_2) \to (M, M_1, M_2)$ a *bilocalic map* if (i) $f : L \to M$ is a localic map, (ii) $f[L_i] \subseteq M_i$ for $i = 1, 2$, and (iii) $f^* : (M, M_1, M_2) \to (L, L_1, L_2)$ is a biframe map, where $f^* : M \to L$ is the left adjoint of f.

For a bilocalic map $f : (L, L_1, L_2) \to (M, M_1, M_2)$, the localic map $f : L \to M$ is called the *total part* of $f : (L, L_1, L_2) \to (M, M_1, M_2)$.

For a bilocalic map $f : (L, L_1, L_2) \to (M, M_1, M_2)$, $f[-] : \mathcal{S}(L) \to \mathcal{S}(M)$ and $f_{-1}[-] : \mathcal{S}(M) \to \mathcal{S}(L)$ are respectively the usual localic image and localic preimage functions induced by the total part of f.

We have not seen the above introduced concept of a bilocalic map in the literature.

Example 5.2.14. For a locale L and $S \in \mathcal{S}(L)$, the localic embedding map $(S, S, S) \hookrightarrow (L, L, L)$ is a bilocalic map.

Proposition 5.2.15. *Let $f : (L, L_1, L_2) \to (M, M_1, M_2)$ be a bilocalic map. Consider the following statements:*

1. *f^* sends j-dense elements to j-dense elements.*

2. *$f_{-1}[-] : \mathcal{S}(M) \to \mathcal{S}(L)$ preserves (i, j)-nowhere dense sublocales.*

3. *$f[-] : \mathcal{S}(L) \to \mathcal{S}(M)$ preserves (i, j)-remote sublocales.*

Then for $i \neq j \in \{1, 2\}$, (1) $\iff$ (2) $\implies$ (3). Moreover, if (L, L_1, L_2) is balanced, then (1) $\iff$ (2) $\iff$ (3).

Proof. (1) $\implies$ (2): Let $A \in \mathcal{S}(M)$ be (i, j)-nowhere dense. Then $\bigvee\{x \in M_i : A \subseteq \mathfrak{c}(x)\} \in M_i$ is j-dense. It follows that $f^*\big(\bigvee\{x \in M_i : A \subseteq \mathfrak{c}(x)\}\big)$ is j-dense and $f^*\big(\bigvee\{x \in M_i : A \subseteq \mathfrak{c}(x)\}\big) \in L_i$ because f^* is a biframe homomorphism. Therefore

$$\mathfrak{c}\left(f^*\left(\bigvee\{x \in M_i : A \subseteq \mathfrak{c}(x)\}\right)\right) = f_{-1}\left[\mathfrak{c}\left(\bigvee\{x \in M_i : A \subseteq \mathfrak{c}(x)\}\right)\right] = f_{-1}[\mathrm{cl}_i(A)]$$

is (i, j)-nowhere dense by Proposition 5.1.12. Because $f_{-1}[A] \subseteq f_{-1}[\mathrm{cl}_i(A)]$, it follows from Proposition 5.1.13(2) that $f_{-1}[A]$ is (i, j)-nowhere dense.

(2) $\implies$ (1): Let $a \in M_i$ be j-dense. By Corollary 5.1.12, $\mathfrak{c}(a)$ is (i,j)-nowhere dense. By hypothesis, $f_{-1}[\mathfrak{c}(a)]$ is (i,j)-nowhere dense. But $f_{-1}[\mathfrak{c}(a)] = \mathfrak{o}(f^*(a))$, so $\mathfrak{c}(f^*(a))$ is (i,j)-nowhere dense, making $f^*(a) \in L_i$ j-dense by Proposition 5.1.12.

(2) $\implies$ (3): Let $A \in \mathcal{S}(L)$ be (i,j)-remote and choose an (i,j)-nowhere dense sublocale N of M. Set $\mathrm{cl}_i(N) = \mathfrak{c}(a)$ for some $a \in M_i$. By (2),

$$\mathrm{cl}_i(f_{-1}[\mathfrak{c}(a)]) \cap A = \mathrm{cl}_i(\mathfrak{c}(f^*(a))) \cap A = \mathsf{O}.$$

But $f^*(a) \in L_i$, so $\mathfrak{c}(f^*(a)) \cap A = \mathsf{O}$. Clearly $\mathfrak{c}(a) \cap f[A] = \mathsf{O}$. Thus $\mathrm{cl}_i(N) \cap f[A] = \mathsf{O}$.

For the special case, we prove (3) $\implies$ (1). Assume that (L, L_1, L_2) is balanced, $f[-]$ preserves (i,j)-remote sublocales and let $a \in M_i$. It follows from Example 5.2.2(2) that $\mathfrak{B}L$ is (i,j)-remote. Since (i,j)-remote sublocales are contained in every open sublocale induced by L_j-elements and $f[-]$ preserves (i,j)-remote sublocales, $f[\mathfrak{B}L] \subseteq \mathfrak{o}(a)$. Therefore $\mathfrak{B}L \subseteq f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$, making the L_i-element $f^*(a)$ a dense element of L. By Proposition 5.1.18(4), $f^*(a)$ is almost j-dense so that it is j-dense by Proposition 5.1.18(2). □

Proposition 5.2.16. *Let $f : (L, L_1, L_2) \to (M, M_1, M_2)$ be a bilocalic map that sends j-dense elements to j-dense elements. Then $f_{-1}[-]$ preserves (i,j)-remote sublocales.*

Proof. Let $A \in \mathcal{S}(M)$ be (i,j)-remote and choose a j-dense $x \in L_i$. Since $f[L_i] \subseteq M_i$ and f sends j-dense elements to j-dense elements, $f(x) \in M_i$ is j-dense. Therefore $A \cap \mathfrak{c}(f(x)) = \mathsf{O}$ which implies that $f_{-1}[A] \cap \mathfrak{c}(x) = \mathsf{O}$, as required. □

To give an example of a bilocalic map f such that $f_{-1}[-]$ preserves (i,j)-remote sublocales, we consider the following notations.

Definition 5.2.17. Let $f : (L, L_1, L_2) \to (M, M_1, M_2)$ be a bilocalic map. We call f:

1. *Dense* if its total part is dense.
2. *Injective* if the restrictions $f|_{L_i} : L_i \to M_i$ are injective maps, for $i \in \{1, 2\}$.

It is clear that an injective bilocalic map has an injective total part.

Example 5.2.18. For a dense and injective bilocalic map $f : (L, L_1, L_2) \to (M, M_1, M_2)$, $f_{-1}[-]$ preserves (i, j)-remote sublocales. To verify this, it suffices to show that f sends j-dense elements to j-dense elements. Let $y \in L_i$ be j-dense and choose $x \in M_i$ such that $x \wedge f(y) = 0$. Then

$$f^*(x) \wedge y = f^*(x) \wedge f^*(f(y)) = f^*(x \wedge f(y)) = f(0) = 0.$$

Since y is j-dense, $f^*(x) = 0$. Therefore $x = f(f^*(x)) = 0$. Thus $f(y)$ is j-dense. It follows from Proposition 5.2.16 that $f_{-1}[-]$ preserves (i, j)-remote sublocales.

5.3 A sublocale $\mathrm{Rem}_B L$

In this section, we construct a sublocale from a collection of elements inducing (i, j)-remote sublocales. Some properties of this sublocale will be studied.

For a bilocale (L, L_1, L_2), set

$$\mathrm{Rem}_B L = \{x \in L : \mathfrak{c}(x) \text{ is } (1,2)\text{-remote}\} = \{x \in L : x \vee a = 1 \text{ for every 2-dense } a \in L_1\}.$$

In the following result, we put some restrictions on (L, L_1, L_2) so that $\mathrm{Rem}_B L$ is a sublocale of L.

Proposition 5.3.1. *Let (L, L_1, L_2) be a bilocale such that:*

1. *L is a coframe, or*
2. *Every L_2-dense member of L_1 is complemented in L.*

Then $\mathrm{Rem}_B L$ is a sublocale of L.

Proof. We only verify (2). Let each $a_\alpha \in \mathrm{Rem}_B L$ and $y \in L_1$ be 2-dense. Since y is complemented in L,

$$y \vee \bigwedge a_\alpha = \bigwedge (y \vee a_\alpha) = \bigwedge \{1\} = 1$$

so that $\bigwedge a_\alpha \in \mathrm{Rem}_B L$.

Furthermore, let $x \in L$, $a \in \text{Rem}_B L$ and $y \in L_1$ be L_2-dense. Then $y \vee a = 1$. Since $a \leq x \to a$, we have that $y \vee (x \to a) = 1$. Thus $x \to a \in \text{Rem}_B L$.
Hence $\text{Rem}_B L$ is a sublocale of L. □

Comment 5.3.2. The idea of having a subframe L_i of L whose members are complemented in L is not outrageous. Zarghani et. al. in [62] and [61] define a *topoframe* as a pair (L, τ) where L is a frame and τ a subframe of L all of whose elements are complemented in L.

Denote by:

1. **BiLoc** the category of bilocales whose morphisms are bilocalic maps.
2. **TBiLoc** the full subcategory of **BiLoc** whose objects are bilocales (L, L_1, L_2) satisfying the condition that each L_2-dense member of L_1 is complemented in L.
3. **BiCFLoc** the full subcategory of **BiLoc** where objects are bilocales whose total parts are coframes.
4. **RemBiLoc** the full subcategory of **BiLoc** whose objects are bilocales (L, L_1, L_2) giving rise to the sublocale $\text{Rem}_B L$.

Observation 5.3.3. Proposition 5.3.1 tells us that **TBiLoc** and **BiCFLoc** are full subcategories of **RemBiLoc**.

We consider some examples.

Example 5.3.4. (1) Consider the bitopological space (X, τ_1, τ_2), where $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b\}\}$. It is clear that all τ_2-dense members of τ_1 are complemented in $\tau = \{\emptyset, X, \{a\}, \{b\}, \{b, c\}, \{a, b\}\}$ and $\text{Rem}_B \tau = \mathfrak{c}(\{a\})$ is a sublocale of τ.

(2) Consider the bitopological space (X, τ_1, τ_2), where $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{b, c\}, \{b\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. It is clear that all τ_2-dense members of τ_1 are complemented in

$$\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}$$

and $\text{Rem}_B \tau = \{X\}$.

(3) The total parts of the bilocales in (1) and (2) are coframes. In fact, in both (1) and (2) we considered the category **TBiLoc** $\cap$ **BiCFLoc**. For a different perspective, consider the bilocale $(\mathfrak{O}\mathbb{R}, L_1 = \{\emptyset, \mathbb{R}\}, L_2 = \mathfrak{O}\mathbb{R})$. We have that $(\mathfrak{O}\mathbb{R}, L_1, L_2) \notin \mathrm{Obj}(\textbf{BiCFLoc})$ (see, for instance, [33]). Since $1 = \mathbb{R}$ is the only L_2-dense element of L_1 and 1 is complemented in L, $(\mathfrak{O}\mathbb{R}, L_1, L_2) \in \mathrm{Obj}(\textbf{TBiLoc})$.

(4) Consider the bitopological space (X, τ_1, τ_2), where $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{c\}, \{a, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b, c\}\}$. Clearly,

$$\tau = \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

is a coframe so that $(\tau, \tau_1, \tau_2) \in \mathrm{Obj}(\textbf{BiCFLoc})$, but $\{c\}$ is a τ_2-dense member of τ_1 which is not complemented in τ, making $(\tau, \tau_1, \tau_2) \notin \mathrm{Obj}(\textbf{TBiLoc})$.

(5) A bilocale is *Boolean*, [56], if $x \prec_i x$ for each $x \in L_i$, $i = 1, 2$, i.e., there is $c \in L_j$ $(i \neq j)$ such that $x \wedge c = 0$ and $x \vee c = 1$. This tells us that each $x \in L_i$, $i = 1, 2$, is complemented in L. As a result of this, every Boolean bilocale is an object of **TBiLoc** and hence of **RemBiLoc**. A symmetric Boolean bilocale is an object of **TBiLoc** $\cap$ **BiCFLoc**. This is so because every Boolean locale is a coframe.

(6) Recall from [28] that the collection $\mathfrak{C}L$ of all congruences on the locale L form a locale. The triple $(\mathfrak{C}L, \nabla_L, \Delta_L)$, where $\nabla_L = \{\nabla_a : a \in L\}$ and Δ_L is the subframe of $\mathfrak{C}L$ generated by $\{\Delta_a : a \in L\}$, is a bilocale called the *congruence bilocale* of L. The congruence bilocale of a locale is an object of **TBiLoc**. This follows since every member of ∇_L is complemented in $\mathfrak{C}L$.

(7) For a symmetric bilocale (L, L_1, L_2), $\mathrm{Rem}_B L = L$ if and only if L is Boolean. The verification is similar to that of Corollary 2.1.30.

In what follows, we consider conditions such that the bilocale of ideals of a locale L induce $\mathrm{Rem}_B L$. Recall from [4] that the triple $(\mathfrak{J}L, (\mathfrak{J}L)_1, (\mathfrak{J}L)_2)$, where $\mathfrak{J}L$ is the locale of all ideals of L and $(\mathfrak{J}L)_i$ $(i = 1, 2)$ is the subframe of $\mathfrak{J}L$ consisting of all ideals $J \subseteq L$ generated by $J \cap L_i$, is a bilocale called the *ideal bilocale*. A locale is called *Noetherian* if all of its elements are compact. All ideals in the following result are in the total part of the bilocale.

Proposition 5.3.5. *A bilocale (L, L_1, L_2) is an object of **TBiLoc** only if* $(\mathfrak{J}L, (\mathfrak{J}L)_1, (\mathfrak{J}L)_2) \in$

Obj(***TBiLoc***). *Moreover, if L is Noetherian, then (L, L_1, L_2) is an object of **TBiLoc** iff* $(\mathfrak{J}L, (\mathfrak{J}L)_1, (\mathfrak{J}L)_2) \in$ Obj(***TBiLoc***)

Proof. Let $x \in L_1$ be L_2-dense. Since $a \leq x \in L_1 \cap {\downarrow}x$ for each $a \in {\downarrow}x$ and ${\downarrow}x \in \mathfrak{J}L$, ${\downarrow}x \in (\mathfrak{J}L)_1$. The ideal ${\downarrow}x$ is $(\mathfrak{J}L)_2$-dense. Indeed, let $J \in (\mathfrak{J}L)_2$ be such that $J \wedge {\downarrow}x = 0_{\mathfrak{J}L}$. If $a \in J$, then $a \leq b$ for some $b \in J \cap L_2$. But $(\bigwedge J) \wedge x = 0$ and $b \leq \bigvee J$, so $b \wedge x = 0$ making $b = 0$ since x is L_2-dense. Therefore $a = 0$. Thus $J = 0_{\mathfrak{J}L}$ so that ${\downarrow}x$ is $(\mathfrak{J}L)_2$-dense. By hypothesis, ${\downarrow}x$ is complemented in $\mathfrak{J}L$, i.e., there is $I \in \mathfrak{J}L$ such that $I \wedge {\downarrow}x = 0_{\mathfrak{J}L}$ and $I \vee {\downarrow}x = 1_{\mathfrak{J}L} = {\downarrow}1$. Since $I \subseteq {\downarrow}(\bigvee I)$, $(\bigvee I) \vee x = 1$. Since we also have that $(\bigvee I) \wedge x = 0$, x is complemented in L. Thus $(L, L_1, L_2) \in$ Obj(**TBiLoc**).

For the special case, let $J \in (\mathfrak{J}L)_1$ be $(\mathfrak{J}L)_2$-dense. Recall from [6] that a locale is Noetherian if and only if each ideal is principal. So, ${\downarrow}\bigvee J = J$, making $\bigvee J \in J$. Therefore $\bigvee J \leq x$ for some $x \in L_1 \cap J$, making $\bigvee J = x \in L_1$. Observe that $\bigvee J$ is L_2-dense. To see this, choose $y \in L_2$ such that $y \wedge (\bigvee J) = 0$. Then $({\downarrow}\bigvee J) \cap {\downarrow}y = 0_{\mathfrak{J}L}$. But $J \subseteq {\downarrow}(\bigvee J)$, so $J \cap {\downarrow}y = 0_{\mathfrak{J}L}$, so that ${\downarrow}y = 0_{\mathfrak{J}L}$. Therefore $y = 0$ and hence $\bigvee J$ is L_2-dense. Since $(L, L_1, L_2) \in$ Obj(**TBiLoc**), $\bigvee J$ is complemented in L, i.e., there is $a \in L$ such that $a \vee (\bigvee J) = 1$ and $a \wedge (\bigvee J) = 0$. Therefore $1_{\mathfrak{J}L} = ({\downarrow}\bigvee J) \vee {\downarrow}a = J \vee {\downarrow}a$ and $0_{\mathfrak{J}L} = ({\downarrow}\bigvee J) \wedge {\downarrow}a = J \wedge {\downarrow}a$. Therefore J is complemented in $\mathfrak{J}L$. Thus $(\mathfrak{J}L, (\mathfrak{J}L)_1, (\mathfrak{J}L)_2) \in$ Obj(**TBiLoc**). □

Proposition 5.3.6. *Let* $(L, L_1, L_2) \in$ Obj(***RemBiLoc***). *Then* $\mathrm{Rem}_B L$ *is a closed sublocale of* L.

Proof. We show that for every $A \in \mathcal{S}(L)$, $A \subseteq \mathrm{Rem}_B L$ implies $\overline{A} \subseteq \mathrm{Rem}_B L$. Assume that $A \subseteq \mathrm{Rem}_B L$ and let $x \in \overline{A}$ and $y \in L_1$ be 2-dense. Since $\bigwedge A \in \mathrm{Rem}_B L$, $(\bigwedge A) \vee y = 1$. But $\bigwedge A \leq x$, so $x \vee y = 1$. Thus $x \in \mathrm{Rem}_B L$. Consequently, $\overline{\mathrm{Rem}_B L} \subseteq \mathrm{Rem}_B L$, making $\mathrm{Rem}_B L$ a closed sublocale. □

Observation 5.3.7. Since $\mathfrak{B}L$ is seldomly complemented, Proposition 5.3.6 tells us that $\mathrm{Rem}_B L$ is not always the same as $\mathfrak{B}L$. This is also confirmed by Example 5.3.4(1) where $\mathfrak{B}\tau = \{\emptyset, X, \{a\}, \{b, c\}\} \neq \mathrm{Rem}_B \tau$. Noting the fact that $\mathfrak{B}\tau \cap \mathrm{Rem}_B \tau = \{\{a\}, X\}$, we also get that $\mathrm{Rem}_B L$ is not always nowhere dense. Since $0_{\mathrm{Rem}_B \tau} = \{a\}$ does not meet the non-zero $\{b\} \in \tau_2$, $\mathrm{Rem}_B \tau$ is also not almost $(1,2)$-nowhere dense (hence, not $(1,2)$-nowhere dense).

Lastly, $\text{Rem}_B\tau$ is not remote because it does not miss the τ-nowhere dense sublocale $\mathfrak{c}(\{a,b\})$.

We showed earlier that every closed sublocale of a locale which is also a coframe is a coframe. As a result of this, we have that if $(L, L_1, L_2) \in \text{Obj}(\textbf{BiCFLoc})$, then $\text{Rem}_B L$ is closed and hence a coframe. We get the following immediate result.

Proposition 5.3.8. *If* $(L, L_1, L_2) \in \text{Obj}(\textbf{\textit{BiCFLoc}})$*, then* $(\text{Rem}_B L, \nu_{(\text{Rem}_B L)}[L_1], \nu_{(\text{Rem}_B L)}[L_2])$ *is an object of* ***RemBiLoc****.*

In the following result, we give neccessary and sufficient conditions for $\text{Rem}_B L$ to be the whole locale.

Proposition 5.3.9. *Let* $(L, L_1, L_2) \in \text{Obj}(\textbf{\textit{RemBiLoc}})$*. The following statements are equivalent.*

1. $\text{Rem}_B L = L$.

2. L *is* $(1,2)$*-remote as a sublocale of itself.*

3. 1 *is the only* 2*-dense element of* L_1.

4. $0 \in \text{Rem}_B L$.

Proof. (1) $\Longrightarrow$ (2): Let $x \in L_1$ be 2-dense. Since $0 \in \text{Rem}_B L$, we have that $0 \vee x = 1$, implying that $x = 1$. Thus 1 is the only 2-dense element of L_1.

(2) $\Longleftrightarrow$ (3): Follows from Proposition 5.2.6.

(3) $\Longrightarrow$ (4): Since 1 is the only 2-dense element of L_1, we have that $0 \vee x = 1$ for every 2-dense $x \in L_1$, making $0 \in \text{Rem}_B L$.

(4) $\Longrightarrow$ (1): Follows since $\text{Rem}_B L$ is closed. □

Observation 5.3.10. One should not confuse Example 5.3.4(5) with Proposition 5.3.9. The conditions in Proposition 5.3.9 do not always imply that the total part L is Boolean. For instance, consider the set $X = \{a,b,c\}$ endowed with topologies $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{b\}\}$. We have that $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and the only τ_2-dense member of τ_1 is

X which is complemented in τ making $(\tau, \tau_1, \tau_2) \in \mathrm{Obj}(\mathbf{TBiLoc})$. It follows from Proposition 5.3.9 that $\mathrm{Rem}_B\tau = \tau$, but τ is not Boolean since, for instance, the element $\{a\}$ is not complemented.

We consider a condition on $(L, L_1, L_2) \in \mathrm{Obj}(\mathbf{RBiLoc})$ such that $\mathrm{Rem}_B L$ is a remote sublocale of L.

Proposition 5.3.11. *Let (L, L_1, L_2) be a bilocale. If*

1. $(L, L_1, L_2) \in \mathrm{Obj}(\boldsymbol{BiCFLoc})$, *OR*

2. $(L, L_1, L_2) \in \mathrm{Obj}(\boldsymbol{TBiLoc})$ *with* $L_1 = L$,

then $\mathrm{Rem}_B L$ *is a remote sublocale of* L.

Proof. The case of $(L, L_1, L_2) \in \mathrm{Obj}(\mathbf{BiCFLoc})$ follows from Corollary 2.1.32.

For $(L, L_1, L_2) \in \mathrm{Obj}(\mathbf{TBiLoc})$, let $y \in L$ be dense. By Proposition 5.1.18(4), y is almost 2-dense. But $y \in L_1$, so y is 2-dense by Proposition 5.1.18(2). It follows from Theorem 5.2.5 that $\mathfrak{c}(x) \subseteq \mathfrak{o}(y)$ for every $x \in \mathrm{Rem}_B L$. That is $x \in \mathfrak{o}(y)$ for all $x \in \mathrm{Rem}_B L$. Therefore $\mathrm{Rem}_B L \subseteq \mathfrak{o}(y)$. Thus $\mathrm{Rem}_B L$ is a remote sublocale of L. □

Observation 5.3.12. (1) The converse of Proposition 5.3.11 is not always true. Example 5.3.4(1)-(4) are counter examples.

(2) Since $\mathfrak{B}L$ is the largest remote sublocale of a locale L, Proposition 5.3.11 tells us that $\mathrm{Rem}_B L \subseteq \mathfrak{B}L$ whenever $L_1 = L$.

Since, according to [50], the frame homomorphism $\nabla_L : L \to \mathfrak{C}L$ is an isomorphism if and only if L is Boolean, we have the following result which holds since $\mathrm{Rem}_B L$ is remote and hence Boolean for $(L, L_1, L_2) \in \mathrm{Obj}(\mathbf{BiCFLoc})$.

Corollary 5.3.13. *For every $(L, L_1, L_2) \in \mathrm{Obj}(\boldsymbol{BiCFLoc})$, $\nabla_{\mathrm{Rem}_B L} : \mathrm{Rem}_B L \to \mathfrak{C}\,\mathrm{Rem}_B L$ is an isomorphism.*

In light of Example 5.3.4(7), we have the following result.

Proposition 5.3.14. *Let* $(L, L_1, L_2) \in \text{Obj}(\textbf{\textit{RemBiLoc}})$. *Then*

$$(\text{Rem}_B L, \text{Rem}_B L, \text{Rem}_B L) = (L, L, L)$$

iff L *is Boolean.*

Proposition 5.3.15. *Let* $(L, L_1, L_2) \in \text{Obj}(\textbf{\textit{RemBiLoc}})$. *Then* $\nu_S[\text{Rem}_B L] \subseteq \text{Rem}_B S$ *for every dense subbilocale* $(S, S_1, S_2) \in \text{Obj}(\textbf{\textit{RemBiLoc}})$ *of* (L, L_1, L_2).

Proof. Let $x \in \text{Rem}_B L$ and choose a 2_S-dense $y \in S_i$. Then $y = \nu_S(a)$ for some $a \in L_1$. It follows from Lemma 5.2.4 that a is 2-dense. Therefore $x \vee a = 1$ so that

$$1 = \nu_S(x \vee a) = \nu_S(x) \vee_S \nu_S(a) = \nu_S(x) \vee_S y.$$

Thus $\nu_S(x) \in \text{Rem}_B S$. □

Observation 5.3.16. If $(L, L_1, L_2) \in \text{Obj}(\mathbf{BiCFLoc})$, then all open subbilocales and closed subbilocales of (L, L_1, L_2) are objects of **BiCFLoc**. The case of closed subbilocales was verified in Chapter 2 where we showed that every closed sublocale of a locale that is a coframe is itself a coframe. We verify the case of open subbilocales. Choose $x \in L$. We show that $\mathfrak{o}(x)$ is a coframe. Let $a_\alpha \in \mathfrak{o}(x)$ for each α and $y \in \mathfrak{o}(x)$. Then

$$\begin{aligned}
y \vee_{\mathfrak{o}(x)} \bigwedge a_\alpha &= \nu_{\mathfrak{o}(x)}\left(y \vee \bigwedge a_\alpha\right) \\
&= \nu_{\mathfrak{o}(x)}\left(\bigwedge (y \vee a_\alpha)\right) \text{ since } L \text{ is a coframe} \\
&= x \to \bigwedge (y \vee a_\alpha) \\
&= \bigwedge (x \to (y \vee a_\alpha)) \text{ since } b \to \bigwedge c_\alpha = \bigwedge (b \to c_\alpha), \text{ for all } b, c_\alpha \in L \\
&= \bigwedge (\nu_{\mathfrak{o}(x)}(y \vee a_\alpha)) \\
&= \bigwedge (y \vee_{\mathfrak{o}(x)} a_\alpha).
\end{aligned}$$

Recall from [51] that, for any subframe L' and sublocale S of a locale L, there is a frame homomorphism

$$L' \xrightarrow{\subseteq} L \xrightarrow{\nu_S} S.$$

For a closed sublocale S, say $S = \mathfrak{c}(x)$ for some $x \in L$, the above frame homomorphism $\nu_S \circ \subseteq : L' \to S$ takes each $a \in L'$ to $a \vee x$. We give the following result.

Proposition 5.3.17. *Let* $(L, L_1, L_2) \in \mathrm{Obj}(\textbf{\textit{RemBiLoc}})$*. Then the frame homomorphism* $\alpha : L_1 \to \mathrm{Rem}_B L$*, defined by* $x \mapsto x \vee 0_{\mathrm{Rem}_B L}$*, sends* L*-dense elements of* L_1 *to the top.*

Proof. Let $x \in L_1$ be L-dense. It follows from Proposition 5.1.18(4) that x is almost 2-dense so that it is 2-dense by Proposition 5.1.18(2). Since $0_{\mathrm{Rem}_B L} \in \mathrm{Rem}_B L$, $1 = x \vee 0_{\mathrm{Rem}_B L} = \alpha(x)$. □

In Proposition 5.3.18 below, we give conditions on a bilocalic map $f : (L, L_1, L_2) \to (M, M_1, M_2)$ such that the restriction map between $\mathrm{Rem}_B L$ and $\mathrm{Rem}_B M$ is a localic map. By a *weakly closed biframe homomorphism* we mean a biframe map whose total part is weakly closed.

Proposition 5.3.18. *Let* $(L, L_1, L_2), (M, M_1, M_2) \in \mathrm{Obj}(\textbf{\textit{RemBiLoc}})$ *and* $f : (L, L_1, L_2) \to (M, M_1, M_2)$ *be a bilocalic map such that* $h : (M, M_1, M_2) \to (L, L_1, L_2)$ *is weakly closed and sends* 2*-dense elements to* 2*-dense elements. Then*

$$f_{|\mathrm{Rem}_B L} : \mathrm{Rem}_B L \to \mathrm{Rem}_B M$$

is a localic map.

Proof. It suffices to show that $f[\mathrm{Rem}_B L] \subseteq \mathrm{Rem}_B M$. Choose $x \in \mathrm{Rem}_B L$ and let $y \in M_1$ be 2-dense. Since h sends 2-dense elements to 2-dense elements, $h(y) \in L_1$ is 2-dense. Therefore $h(y) \vee x = 1$. But h is weakly closed so $y \vee f(x) = 1$. Thus $f(x) \in \mathrm{Rem}_B M$. □

Definition 5.3.19. Call a bilocalic map $f : (L, L_1, L_2) \to (M, M_1, M_2)$ a ***Rem***$_B$*-map* if $f[\mathrm{Rem}_B L] \subseteq \mathrm{Rem}_B M$.

Example 5.3.20. The bilocalic maps described in the statement of Proposition 5.3.18 are **Rem**$_B$-maps.

Denote by **BiCFLoc**$_R$ and **RemBiLoc**$_R$ the subcategories of **BiCFLoc** and **RemBiLoc**, respectively, whose morphisms are **Rem**$_B$-maps.

There is a functor between **RemBiLoc**$_R$ and **Loc**, as one checks routinely.

Proposition 5.3.21. *The assignment*

$$\mathrm{Rem}_B : \textbf{\textit{RemBiLoc}}_R \to \textbf{\textit{Loc}},$$

$$(L, L_1, L_2) \mapsto \mathrm{Rem}_B L,$$

$$\mathrm{Rem}_B(f : (L, L_1, L_2) \to (M, M_1, M_2)) = f_{|\mathrm{Rem}_B L}$$

*with the usual composition in **Loc**, is a functor.*

Using the fact that each $\mathrm{Rem}_B L \in \mathrm{Obj}(\mathbf{BooLoc})$ for every $(L, L_1, L_2) \in \mathrm{Obj}(\mathbf{BiCFLoc})$, we have the following result.

Proposition 5.3.22. *The assignment* $\mathrm{Rem}'_B : \boldsymbol{BiCFLoc}_R \to \boldsymbol{BooLoc}$ *mapping as* Rem_B *is a functor.*

Recall from [5] that there is a faithful functor $U : \mathbf{BiFrm} \to \mathbf{Frm}$ which takes the total part. It is clear that there is also a functor $F : \mathbf{BiLoc} \to \mathbf{Loc}$ behaving the same as the functor U. Furthermore, one can easily see that there is a functor $G : \mathbf{RemBiLoc}_R \to \mathbf{Loc}$ which maps as F.

In what follows, we show that there is a natural transformation from Rem_B to G.

Proposition 5.3.23. *There is a natural transformation* $\eta : \mathrm{Rem}_B \to G$.

Proof. Let $(L, L_1, L_2) \in \mathrm{Obj}(\mathbf{RemBiLoc}_R)$ and $\eta_{(L,L_1,L_2)}$ be the map $j_{\mathrm{Rem}_B L} : \mathrm{Rem}_B L \to L$. The map $\eta_{(L,L_1,L_2)}$ is clearly a localic map. Now, choose $f : (L, L_1, L_2) \to (M, M_1, M_2) \in \mathrm{Morp}(\mathbf{RemBiLoc}_R)$. Then the diagram

$$\begin{array}{ccc} \mathrm{Rem}_B L & \xrightarrow{\eta_{(L,L_1,L_2)}} & L \\ \downarrow{\scriptstyle \mathrm{Rem}_B(f)} & & \downarrow{\scriptstyle G(f)} \\ \mathrm{Rem}_B M & \xrightarrow[\eta_{(M,M_1,M_2)}]{} & M \end{array} \tag{5.3.1}$$

commutes. Indeed, for each $x \in \mathrm{Rem}_B L$,

$$G(f)(\eta_{(L,L_1,L_2)}(x)) = G(f)(x) = f(x) = \eta_{(M,M_1,M_2)}(f(x)) = \eta_{(M,M_1,M_2)}(\mathrm{Rem}_B(f)(x))$$

which proves the result. □

Denote by $\mathbf{RemBiLoc}_{RB}$ the full subcategory of $\mathbf{RemBiLoc}_R$ whose objects are bilocales (L, L_1, L_2) with 1 the only L_2-dense element of L_1.

Proposition 5.3.24. *The assignment* $\mathrm{Rem}_{RB} : \boldsymbol{RemBiLoc}_{RB} \to \boldsymbol{Loc}$, *where* $\mathrm{Rem}_{RB}\, L = \mathrm{Rem}_B L$ *and* $\mathrm{Rem}_{RB}(f) = \mathrm{Rem}_B(f)$, *is a faithful functor.*

Proof. We only prove faithfulness: Let $f, g : (L, L_1, L_2) \to (M, M_1, M_2) \in \mathrm{Morp}(\mathbf{RemBiLoc}_{RB})$ be such that $\mathrm{Rem}_{RB}(f) = \mathrm{Rem}_{RB}(g)$. Since, by Proposition 5.3.9, $\mathrm{Rem}_B L = L$ and $\mathrm{Rem}_B\, M = M$, we have that

$$\text{total part of } f = f_{|\mathrm{Rem}_B L} = \mathrm{Rem}_{RB}(f) = \mathrm{Rem}_{RB}(g) = g_{|\mathrm{Rem}_B L} = \text{total part of } g.$$

So that $f = g$, making Rem_{RB} faithful. □

Consider the functor $\hat{G} : \mathbf{RemBiLoc}_{RB} \to \mathbf{Loc}$ which maps as $G : \mathbf{RemBiLoc}_R \to \mathbf{Loc}$. We have the following result.

Proposition 5.3.25. *The functors* $\hat{G}$ *and* Rem_{RB} *are naturally isomorphic.*

Proof. Consider the natural transformation $\omega : \mathrm{Rem}_{RB} \to \hat{G}$ which maps as the natural transformation $\eta : \mathrm{Rem}_B \to G$ given in Proposition 5.3.23. Since $L = \mathrm{Rem}_R\, L = \mathrm{Rem}_{RB}\, L$ for every $(L, L_1, L_2) \in \mathrm{Obj}(\mathbf{RemBiLoc}_{RB})$, each component $\omega_{(L,L_1,L_2)} = j_{\mathrm{Rem}_B L} : \mathrm{Rem}_B L \to L$ is an isomorphism. Thus ω is a natural isomorphism. □

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