Operational matrices for solving variable order differential equations

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MAHLULI NAISBITT NCUBE

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SUPERVISOR: PROF H JAFARI

CO-SUPERVISOR: DR ZI ALI

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DECLARATION

Name: MAHLULI NAISBITT NCUBE

Student number: 46565876

Degree: Doctor of Philosophy in Applied Mathematics

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Abstract

In this thesis, we extensively explore the role of matrices as substitutes for derivative and integral operators. By expressing an approximate solution of a partial differential in an implicit form involving polynomials, we demonstrate how to deduce novel composite operational matrices. We also show how to utilise the laws of matrix multiplication to come up with a single matrix that performs the role of differentiation and integration. In conjunction with the Garlekin technique, we apply these composite matrices to numerically solve partial differential equations. Through practical examples, we prove that these composite operational matrices are convenient in approximating the solution of partial differential equations using a computer algebra system like Mathematica.

Keywords: Variable order differential equations, Operational matrices, Caputo fractional derivative, Approximate solution, Garlekin technique, Polynomials, Composite derivative matrix, Composite integral matrix, Matrix multiplication, Associative Law, Commutative Law.

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The following published papers are extracts from the thesis.

(1) Tajadodi H, Jafari H and **Ncube M.N** (2022). Genocchi polynomials as a tool for solving a class of fractional optimal control problems, *Int. J. Optim. Contr* 12(2).

(2) Zhang A, Ganji R.M, Jafari H, **Ncube M.N** and Agamalieva L (2022). Numerical Solution of Distributed-Order Integro-Differential Equations, *Fractals* 30(5).

(3) Meddahi M, Jafari H and **Ncube M.N** (2021). New general integral transform via Atangana-Baleanu derivatives, *Adv. Diff. Eqns*, 1:1-14.

(4) Can N.H, Jafari H and **Ncube M.N** (2020). Fractional calculus in data fitting, *Alexandria. Engr. Journ.*, 59(5):3269-3274.

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Chapter 1

Introductory Remarks

1.1 Introduction

The Matrix, a versatile concept in mathematics is a rectangular array of symbols, numbers or expressions arranged in the form of rows and columns. The size of the matrix is described in the form of the number of rows and columns that it has, starting with the rows and then columns. For example, a 3×4 matrix implies it has 3 rows and 4 columns. Positioning of the elements in the matrix is very important as they determine the uniqueness of the matrix. The conventional way of describing the position of an element in the matrix is a_{ij}, this tells us that an element a occupies the i^{th} row and j^{th} column. Changing positions of the elements in a matrix will change the whole matrix, and therefore changing the intended results it sets to accomplish.

Matrices have varied applications in mathematics. In linear algebra, matrices represent coefficients of variables in a system of equations. In this situation, with the set up matrix, we can apply the Gauss-Jordan elimination [38, 39] or the Gauss elimination [38] to find the solutions of the equation. This is a classical example where the concept of the matrix can be of use in solving a system of equations.

In numerical analysis, matrices play various roles. A specific example is in the finite element method where the concept of the matrix is applied in the creation of the stiffness matrix [2]. This stiffness matrix is in the form of a system of equations that needs to be solved so as to approximate the solutions of partial differential equations of elliptic nature.

In geometrical transformations, matrices play the role of mappings. We have matrices for rotation both clockwise and counter clockwise, for reflection, enlargement and stretches. The coordinates of the object have to be in the form of a matrix so that we are able to multiply it with the applicable transformation matrix to get the coordinates of the image.

An analogy of geometric transformations in calculus is differentiation and integration. These two operators take the original function and transform it into a different state. Therefore, we can also understand differentiation and integration as mappings. In this way, we can comprehend that it is feasible to represent these two operations with matrices. However, with differentiation and integration, we are mapping both functions and real numbers, unlike in geometrical transformations where we deal with real numbers.

In this thesis, our main aim is to explore the role of matrices as substitutes for differentiation and integration. We are going to refer to these matrices as operational matrices, since they operate on a function to change its form. Furthermore, we intend to investigate the application of operational matrices in the solution of differential equations (DEs).

DEs are mathematical models that represent the rate of change of one or more variables with respect to the other variable(s). Therefore, DEs play a pivotal role in modelling systems that exhibit change over time. Such systems include, weather patterns, chemical reactions, derivative markets in finance and many other systems. The classical differential equations are the ones with an order of the derivative being an integer. However, it is also possible, although not common in introductory texts of calculus to have the integer order derivative extended to accommodate fractions resulting in a fractional differential equation (FDE). Therefore a FDE is a generalisation that includes an integer order differential equation.

The definition of the fractional derivative is an area under active research. There are many

definitions of the fractional derivative being proposed by researchers, with each definition possessing some form of deficiency. When it comes to applications, the common fractional derivatives that researchers make use of are, the Caputo [1, 37], the Caputo-Fabrizio [28] and the Atangana-Baleanu [29].
To some extent, the results from different fractional derivatives are not the same, although the deviation is not that much. A common example is the comparison of the results from the Caputo and Caputo-Fabrizio. It is noticed that the past events have a lesser influence on the current events when applying the former derivative than the later. Therefore, there is a general consensus that the practical situation under consideration dictates the choice of the fractional derivative.

It has been witnessed through several different experimental observations that the results from the fractional derivative outperform those from the classical derivative. The theoretical explanation for this important observation is that the fractional part of the FDE captures some important practical information that the classical derivative is incapable of. This has promoted the widespread use of the fractional derivative as researchers are able to build a detailed mathematical model that yield more realistic results.

Of late, there have been suggestions to further extend the fractional derivative by replacing the fraction in the derivative with a function. This function can be of one or several variables. This type of derivative becomes more complex than both the integer and fractional order. Due to the complexity that comes with this development in terms of computations, there has been a noticeable reluctance in pursuing research along these lines.

Although the fractional derivative might result in a robust mathematical model that yields more desirable results compared to its integer counterpart, one major drawback is that it is more cumbersome to solve. This is due to an extra parameter that is added, and allowed to vary within a specified range. Thus careful consideration should be taken in choosing a solution method that will lessen the computational difficulties, but at the same time achieving accurate results. It has to be pointed out that there are no special solution methods reserved for fractional differential equations. Any solution method that can be used for integer order differential equations can be successfully implemented in a fractional derivative setting.

Most mathematical models that capture real life situations are complex, and in the context of differential equations, they are best represented in the form of non linear DEs and Partial differential equations (PDEs). Notable examples include, the Van der Pol equation that has applications in electrical circuits, the heat equation that finds applications in probability, and the Navier-Stokes equations that model viscous flow.

1.2 Definitions and notations

Analytical Solution- This is the solution obtained when a differential equation is solved leading to the dependent variable being expressed as a function of independent variable(s) in an algebraic equation. The solution is expressed in closed form.
Numerical Solution- This is an approximate solution to a differential equation. In most cases, the solution is not in closed form.
Steady state solution- This is a solution that is independent of time.
Garlekin methods- These are a class of methods that convert a continuous operator problem, for example a differential equation to a discrete problem.
Operational matrix- This is a matrix that represents a derivative or integral operator.

We now explain some notation that we will be using in this research.

Table 1.1: Table showing important notations and meaning.

Notation	Meaning
$n \in \mathbb{N}$	n is the element of natural numbers.
$n \in \mathbb{N}_0$ or $n \in \mathbb{N} \cup 0$	n is the element of natural numbers including zero.
$x \in \mathbb{R}$	x is the element of Real numbers.
$\mathbb{R}^n$	$x = (x_1, x_2, ..., x_n) : x_m \in \mathbb{R}; m = 1, 2, ..., n$
$x \in (a, b)$	$x \in \mathbb{R} : a < x < b$
$x \in [a, b]$	$x \in \mathbb{R} : a \leq x \leq b$
$x \in [a, b)$	$x \in \mathbb{R} : a \leq x < b$
$x \in (a, b]$	$x \in \mathbb{R} : a < x \leq b$

Given matrices A, B and C having the dimensions $M \times N$, then the following matrix operations hold,

(i) $A + B = B + A$, matrix addition is commutative.

(ii) $AB \neq BA$, in general matrix multiplication is not commutative, however, we do have some exceptions.

(iii) $A(B + C) = AB + AC$, distributive.

(iv) $A^n = A \times A \times A \times \cdots \times A$, n times.

(v) A^{-1}, denotes the inverse of matrix A.

(vi) $A^0 = \mathtt{I}$, where $\mathtt{I}$ denotes the identity matrix.

(vii) A^T denotes the transpose of matrix A, the rows change to columns and the columns change to rows.

(viii) $(AB)^T = B^T A^T$.

(ix) $\mathtt{I}^T = \mathtt{I}$, the transpose of an identity matrix is an identity matrix.

If y is a function of t, then,

$y' = \frac{dy}{dt}$,
$y'' = \frac{d^2y}{dt^2}$.

If y is a function of two variables, x and t, then,
$y_t = \frac{\partial y}{\partial t} = \partial_t y, \quad y_x = \frac{\partial y}{\partial x} = \partial_x y,$
$y_{tt} = \frac{\partial^2 y}{\partial t^2} = \partial_{tt} y, \quad y_{xx} = \frac{\partial^2 y}{\partial x^2} = \partial_{xx} y.$

1.3 Outline of the Thesis

The research is divided into six chapters as follows. The current chapter introduces the thesis to the reader, we also provide some terminology and shorthand notations that we use throughout the thesis. In chapter 2, we do a review of the necessary literature and indicate how this research fit into the existing body of knowledge. Chapter 3 focuses on reviewing the operational matrices of derivative and integral. In addition to this, we suggest another way of deducing an integral matrix using integral transforms. In chapter 4, we present completely new results. We explain how partial differential equations lend themselves to matrices that are functions of other matrices. In chapter 5, we deal with applications of the proposed concepts. In particular, we use the diffusion equation as our case study. The summary of our research findings are found in chapter 6, this includes, our results, the challenges encountered during research and directions for future research.

Chapter 2

Literature Review

Surprisingly, not much ground is being covered in the use of operational matrices in partial differential equations (PDEs). More focus in the literature is on the ordinary differential equations (ODEs). Considering the importance of PDEs in mathematics and real world applications, it is essential that the use of operational matrices in PDEs be brought on par with ODEs. This thesis seeks to make a contribution that addresses the issue of this gap.

Operational matrix is a broad term that basically encompasses two operations, namely, the differential and integral operator matrices. The terms differentiation matrix and the derivative matrix are also used in place of the differential operator matrix. Also, the integral operator matrix is known as the integration matrix.

On their own, operational matrices do not possess the capability of solving differential equations. They are used in conjunction with other methods, in particular numerical techniques. Thus, at least for now any method that involves the application of operational matrices will be a numerical method.

There are many numerical methods that are successfully employed to approximate the solution of DEs and PDEs. Some of the most common numerical methods include, the finite difference [1,2], the finite element [2], Galerkin methods [16], collocation [7,11,17], Adomian decomposition [3], iterative method [5], Homotopy perturbation [4]. There has also been successful attempts in applying hybrid techniques [32]. In such situations, the integral operator is replaced with an integral transform, and another numerical technique is applied to complete the solution procedure. In the Laplace Adomian hybrid method, the Laplace transform is coupled with the Adomian decomposition [21]. A combination of the Laplace transform and the Homotopy technique yields another common hybrid method [6].

However, even though we are spoilt for choice of the numerical techniques at our disposal, the nature of the problem to solve and the given conditions limits us to particular solution methods.

In the application of the operational matrices to solve differential equations, the two most common used techniques are the collocation and the Galerkin methods. The most sensible explanation for this might be, the nature of the operational matrices facilitate the use of these two methods. The role of these two techniques in the solution process is to construct a system of equations. The solutions of the system of these equations play a pivotal role in the approximate solution of the DEs.

Polynomials are the backbone for the construction of operational matrices. The elements of an operational matrix are dependent on the polynomial under consideration. Thus different polynomials will yield operational matrices with different elements. Among the common polynomials that have been used in the construction of operational matrices are, Chebyshev [36], Bernstein [12, 18, 35], Genocchi [19] and Bernoulli [16]. A special note on the connection between Euler and Bernoulli polynomials is provided in [34].

Rada, Kazemb, Shabanc and Paranda deduce integral, derivative and product operational matrices from the Bernoulli polynomials [16]. In conjunction with the Garlekin method, they apply these operational matrices to approximate the solutions of variable coefficient non linear and linear ODEs. The authors provide a clear explanation of what constitutes an operational matrix of product. They clearly illustrate that there are certain terms in an equation that determine if the solution procedure warrants a product operational matrix. The Bernoulli polynomials are written in the form of a product of two matrices, this feature is exhibited by almost all the polynomials and its very helpful in simplifying calculations.

Isah and Chang Phang construct the derivative operational matrix based on Genocchi polynomials [11]. They use this matrix with the collocation technique to numerically solve delay differential equations. The authors do not only make a comparison of their results with analytic solutions, but they also do so against the solutions from other numerical techniques. They also use their solutions to emphasize that an increase in the dimensional size of the operational matrix brings along with it an improvement in the results.

Isah, Chang Phang and Piau Phang use the same technique as in [11] to numerically solve ODEs of fractional order [17]. Since the computations involving FDEs are cumbersome, the authors choose to use particular numerical values to represent fractional derivatives rather a variable. This decision has a tremendous effect on reducing computational difficulties. The authors use evidence from the results to argue that the first few number of polynomials are adequate to yield accurate solutions.

Liu, Li and Wu make use of Chebyshev polynomials of the second kind to construct operational matrices. The main intention of building these matrices is to apply them in ODEs of variable order [8]. They use the collocation method to create a system consisting of algebraic equations in their numerical solution.

Saadatmandi and Dehghan demonstrate how to use the Legendre polynomials in the numerical solution of constant coefficient fractional ordinary differential equations. They apply the collocation and tau methods as their numerical techniques [10]. The authors provide a splendid explanation on how to transition from the operational matrix with integer order to that with fractional order.

Ganji and Jafari utilise the Jacobi polynomials to numerically solve variable order differential equations [7]. To be more specific, they use the operational matrices derived from the shifted Chebyshev and Legendre polynomials as these are known to be the special cases of the Jacobi polynomials. It is shown that different functions such as, exponential, trigonometric, quadratic and linear can be taken as the order of the derivative. The authors compare their solutions with analytic ones, and they illustrate that the first few polynomials are enough to attain good approximations.

Phang Chang et al apply an operational matrix deduced from Legendre polynomials to numerically solve constant coefficient partial differential equations [22]. Yang, Ma and Wang use the same operational matrix, but they approximate the solution of PDEs with variable coefficients [24].

Yin, Song, Lu and Leng employ the Legendre wavelets in conjunction with Laplace transform to numerically ODEs that are both non linear and linear [20]. An operator that consists of the Laplace transform together with its associated inverse is discussed, in fact, this operator resembles an integral operator.

Rani and Mishra couple the Adomian decomposition method together with Laplace transform to solve non linear ordinary differential equations [21]. They make use of the integral operational matrix constructed from the Bernoulli polynomials in their solution process.

Most of the literature pertaining the application of operational matrices is related to ODEs. When it comes to constructing operational matrices for partial derivatives, researchers have up to this date been very cautious not to deviate much from the formulation used in the ordinary differential equations. With this traditional formulation, the spatial and time variable are explicitly constructed. The consequence of this approach implies the operational matrices representing the space and time variables are written separately, see [27, 30, 40]. Thus, the application of operational matrices on ODEs is carried over to PDEs. There is some comfort drawn from such a decision in the sense that one is manoeuvring in a simpler and familiar territory.
In this research, we break away from this culture in ways that we will clarify later and investigate the implications of our decision on operational matrices. In brief, we will attempt to amalgamate the spatial and time variable with the aim of creating an operational matrix in which these two variables coexist.

In the context of this research, the term variable order implies that the order of the differential equation can be integer, fractional and at certain times a function.

Chapter 3

Operational Matrices

Abstract

In this chapter, we provide a detailed explanation on how to go about deducing operational matrices from polynomials. The theory concerning the extraction of these matrices from polynomials is provided. Specific examples are given to show the solution procedure in the application of operational matrices. Since this chapter introduces the reader to operational matrices and lays important foundation for the next chapter, we will limit the scope of our work to a single variable.

3.1 Introduction

The contents of this chapter pertains to introducing the concept of the operational matrix, specifically, we dwell on the differentiation and integral operational matrices. We describe in great depth how to deduce the operational matrices from polynomials and subsequently apply them in the numerical solution of differential equations. There are many types of polynomials from which the operational matrices can be deduced, but we will concentrate on the Shifted Legendre polynomials.

When it comes to an integral operational matrix, we show that, besides deducing this matrix through direct integration, it is possible to derive it from integral transforms. In particular, we will show how to achieve this through the use of two integral transforms, the generalised integral transform that was recently introduced by Jafari and the Laplace transform. Furthermore, we will discuss about the difference between two types of integral operational matrices.

We will demonstrate that, given a fractional differential equation, one has the option of either using the derivative or the integral operational matrix. In this regard, we will investigate how utilising different operational matrices impact the solutions of a fractional differential equation.

We use the Garlekin technique coupled with operational matrices to approximate the solution of ODEs. To test the reliability of our results, we compare them against the analytical solutions, perform absolute error calculations and convergence analysis.

3.2 Fractional calculus and the General Jafari transform

We discuss the basic definitions of fractional calculus and a generalised transform introduced by Jafari. We will also include the Laplace transform in our discussion so that we can demonstrate how the General Jafari transform (GJT) is related to the Laplace transform.

Caputo's concept of defining the fractional derivative is arguably the most used in applications, we give its definition below.

Definition 3.2.1. *According to Caputo, the fractional derivative with order μ has the following definition [1],*

$$\mathcal{D}_t^{\mu} y(t) = \begin{cases} y^{(p)}(t) & if \quad \mu = p; \\ \frac{1}{\Gamma(p-\mu)} \int\limits_0^t \frac{y^{(p)}(s)ds}{(t-s)^{\mu-p+1}} & if \quad \mu \in (p-1, p]. \end{cases} \tag{3.1}$$

$\Gamma[.]$ represents Euler's Gamma function. Gamma function's purpose here is to accommodate the factorial calculations of non integer numbers. The useful connection between the factorial and Gamma function is given below [1],

$$\Gamma[n+1] = n!. \tag{3.2}$$

We are able to evaluate the factorial of non integer terms by utilising the Gamma function on the left hand side of (3.2).
Another important concept that makes use of the Gamma function in fractional calculus is the Mittag-Leffler function. This is regarded as a general function, imposing particular conditions on it yields common functions like the *cosine*, *sine* and exponential functions. We give the definition of this function below.
Definition 3.2.2. *We define Mittag-Leffler function having two parameters as [1],*

$$E_{\mu,\nu}(t) = \sum_{k=0}^{\infty} \frac{t^{\mu k}}{\Gamma(\mu k + \nu)}, \quad t, \mu, \nu \in (0, \infty). \tag{3.3}$$

The anti derivative of (3.1) is referred to as the Riemann-Lioville fractional integral, we provide its formal definition below.

Definition 3.2.3. *The fractional integral having order* μ *and operating upon* $y(t)$ *known as the Riemann-Liouville can be represented in the form [1],*

$$\tilde{\mathcal{I}}^{\mu} y(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \int\limits_0^t \frac{y(s)ds}{(t-s)^{1-\mu}}, & if \quad \mu > 0, \quad t > 0 \quad ; \\ y(t), & if \quad \mu = 0. \end{cases} \tag{3.4}$$

We need to state the properties of the integral given in (3.4). These properties describe how to execute the functions of this integral [1].

(i) $\tilde{\mathcal{I}}^{\mu} \tilde{\mathcal{I}}^{\alpha} y(t) = \tilde{\mathcal{I}}_t^{\mu+\alpha} y(t), \quad \mu, \alpha \in (0, \infty).$

(ii) $\tilde{\mathcal{I}}^{\mu} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)} t^{\mu+\nu}, \quad \nu \in (-1, \infty), \quad \mu, t \in (0, \infty).$

(iii) $\tilde{\mathcal{I}}^{\mu} \mathcal{D}^{\mu} y(t) = y(t) - \sum_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0), \quad \mu \in (p-1, p],$
$\mathcal{D}^{\mu}$ is the Caputo fractional derivative that we previously defined.

Property (*iii*) above serves to demonstrate that the integral in (3.4) is applicable to the Caputo fractional derivative [1].

Integral transforms can be viewed as an alternative to integration. The Laplace transform is most probably the widely used of all the integral transforms. Its formal definition is given below.

Definition 3.2.4. *The Laplace transform of the function $y(t)$ is defined as [2],*

$$\mathcal{L}[y(t)] = Y(s) = \int_0^\infty e^{-st} y(t)dt, \quad s > 0. \tag{3.5}$$

The Laplace transform will be defined only if the integral on the right hand side of (3.5) exists.

There is a generalised integral transform that was discovered by Jafari. This General Jafari transform (GJT) is a transform that incorporates many different integral transforms. Imposing certain conditions on the GJT yields other integral transforms. We give the definition of this transform below.

Definition 3.2.5. *We define the GJT of the function $y(t)$, $\mathcal{T}[s]$, as [14, 15],*

$$\mathrm{T}[y(t)] = \mathcal{T}[s] = r(s) \int_0^\infty e^{-w(s)t} y(t)dt, \quad t \geq 0, \quad w(s) > 0, \quad r(s) \neq 0. \tag{3.6}$$

The above integral transform is based on the assumption that the integral defined above exists. If $r(s) = 1$ and $w(s) = s$, then the GJT reduces to the Laplace transform [14]. We refer the reader to [14, 15] for information on how the GJT is associated with other integral transforms.

Integral transforms are also applicable to special functions and derivatives with fractional order [1]. In the next theorems, we state without proof, the Laplace transforms of the special function in (3.3) and the Caputo fractional derivative.

The Laplace transform of (3.3) is proved in [1], we give the result below.

Definition 3.2.6. *The special function in* (3.3) *has it's Laplace transform given as [1],*

$$\mathcal{L}[t^{\nu-1} E_{\mu,\nu}(\mp c^2 t^\mu)] = \frac{s^{\mu-\nu}}{s^\mu \pm c^2}, \quad c \in \mathbb{R} \quad \mu, \nu \in (0, \infty), \quad s^\mu > |c|. \tag{3.7}$$

Definition 3.2.7. *We can effect the Laplace transform in* (3.5) *on the Caputo's definition of the fractional derivative in* (3.1) *such that [1],*

$$\mathcal{L}[\mathcal{D}_t^{\mu} y(t)] = s^{\mu} Y(s) - \sum_{m=0}^{p-1} s^{-1+\mu-m} y^{(m)}(0), \quad p-1 < \mu \le p, \quad p \in \mathbb{N}. \tag{3.8}$$

Theorem 3.2.1. *The GJT of a p order integer derivative is stated as,*

$$\mathrm{T}[y^{(p)}(t)] = w^{p}(s)\mathcal{T}(s) - r(s)\sum_{m=0}^{p-1} w^{p-1-m}(s) y^{(m)}(0). \tag{3.9}$$

The proof of (3.9) is found in [14].

It follows that we can deduce the GJT of (3.1) by replacing n with μ in (3.9) . The result is written in the corollary below.

Corollary 3.2.1. *The GJT of the fractional derivative definition given in* (3.1) *can be stated as,*

$$\mathrm{T}[\mathcal{D}_t^{\mu} y(t)] = w^{\mu}(s)\mathcal{T}(s) - r(s)\sum_{m=0}^{p-1} w^{\mu-1-m}(s) y^{(m)}(0), \quad p-1 < \mu \le p, \quad p \in \mathbb{N}. \tag{3.10}$$

Table 3.1 gives results of applying the GJT and Laplace transforms to some common functions and of the first order derivative [1, 14].

Table 3.1: Table showing GJT and Laplace transforms of basic functions.

	y(t)	$\mathcal{T}(s)$	$Y(s)$
functions and their respective transforms	1	$\frac{r(s)}{w(s)}$	$\frac{1}{s}$
	$\sin(ct)$	$\frac{cr(s)}{w(s)^2+c^2}$	$\frac{c}{s^2+c^2}$
	t	$\frac{r(s)}{w^2(s)}$	$\frac{1}{s^2}$
	e^t	$\frac{r(s)}{w(s)-1}$	$\frac{1}{s-1}$
	t^{μ}	$\frac{\Gamma[\mu+1]r(s)}{w^{\mu+1}(s)}$	$\frac{\Gamma[\mu+1]}{s^{\mu+1}}$
First derivative	$y'(t)$	$w(s)\mathcal{T}(s) - r(s)y(0)$	$sY(s) - y(0)$

We illustrate how to apply the GJT to find the solution of an ODE.

Example 3.1. *Suppose we are presented with the following ODE,*

$$y'(t) - y(t) = 0, \tag{3.11}$$
$$y(0) = 1,$$

whose analytic solution is $y(t) = e^t$.

We begin by effecting the GJT on each term in (3.11),

$$\mathrm{T}[y'(t)] - \mathrm{T}[y(t)] = 0,$$

applying (3.9) in the above equation gives,

$$w(s)\mathcal{T}(s) - r(s)y(0) - \mathcal{T}(s) = 0.$$

Substituting for the initial condition and then solving for $\mathcal{T}(s)$ yields,

$$\mathcal{T}(s) = \frac{r(s)}{w(s) - 1}$$

To get the solution $y(t)$, we take the inverse GJT as,

$$\mathrm{T}^{-1}\mathcal{T}(s) = \mathrm{T}^{-1}\left(\frac{r(s)}{w(s) - 1}\right),$$

Referring to the Table 3.1, we can tell that,

$$y(t) = e^t. \tag{3.12}$$

3.3 Shifted Legendre polynomials and operational matrices

We want to demonstrate how operational matrices of the derivative and integral can be deduced from Shifted Legendre polynomials.

Definition 3.3.1. *The n^{th} degree Shifted Legendre polynomials (SLPs) on* $[0, 1]$ *are defined as [13],*

$$\mathcal{P}_n(t) = \sum_{i=0}^{n} \omega_{n,i} t^i, \tag{3.13}$$

where $\omega_{n,i}$ are numbers generated as,

$$\omega_{n,i} = \frac{(-1)^{n+1}(n+i)!}{(n+i)!(i!)^2}.$$

We can easily generate the first few polynomials from (3.13) as follows,

$$\mathcal{P}_0(t) = 1, \tag{3.14}$$

$$\mathcal{P}_1(t) = -1 + 2t, \tag{3.15}$$

$$\mathcal{P}_2(t) = 1 - 6t + 6t^2, \tag{3.16}$$

$$\mathcal{P}_3(t) = -1 + 12t - 30t^2 + 20t^3. \tag{3.17}$$

The next two definitions will be essential in the derivation of operational matrices. In the next definition, we arrange the shifted Legendre polynomials in the form of a column matrix.

Definition 3.3.2. *We define a column matrix of SLPs, $\mathbb{P}(t)$, in the form,*

$$\mathbb{P}(t) = [\mathcal{P}_0(t), \mathcal{P}_1(t), \mathcal{P}_2(t), ..., \mathcal{P}_n(t)]^T, \quad n \in \mathbb{N}, \tag{3.18}$$

the superscript T denotes the transpose.

Definition 3.3.3. *We define a column vector of polynomials as [16],*

$$\mathcal{M}(t) = [1, t, t^2, ..., t^n]^T, \quad n \in \mathbb{N}. \tag{3.19}$$

There is a very useful relation between (3.18) and (3.19), we elaborate this in the next section.

3.3.1 Deducing the derivative operational matrices

We describe in detail how to deduce the operational matrices from the SLPs.

The following Lemma states the relation between $\mathbb{P}(t)$, and the column matrix $\mathcal{M}(t)$.

Lemma 3.3.1. *The SLPs given in* (3.18) *can be represented as a product of matrices, such that,*

$$\mathbb{P}(t) = \mathbb{A}\mathcal{M}(t), \tag{3.20}$$

$\mathbb{A}$ *is an* $(n+1)\times(n+1)$ *matrix whose entries are the coefficients of the SLPs.*

We note that since most polynomials can be expressed in the form (3.20), then (3.20) is applicable to other polynomials. The exception being trigonometric, hyperbolic and exponential polynomials.
Having introduced the necessary definitions pertaining to the operational matrices, we can now state without proof the following result that is crucial throughout the thesis.

Theorem 3.3.1. *An* $(n+1)\times(n+1)$ *operational matrix of the derivative,* $\tilde{\mathcal{D}}$*, can be written in place of the usual derivative operator such that [13],*

$$\frac{d\mathbb{P}(t)}{dt} = \tilde{\mathcal{D}}\mathbb{P}(t), \tag{3.21}$$

for a detailed discussion on how to compute the matrix $\tilde{\mathcal{D}}$, we refer the reader to [13].

There is another different way of presenting the operational matrix besides the form in (3.21). We explain how to get the elements of this matrix in the theorem below.

Theorem 3.3.2. *The derivative operational matrix,* $\mathcal{D}$*, with order* $(n+1)\times(n+1)$ *deduced from* $\mathcal{M}(t)$ *can be written as,*

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n & 0 \end{pmatrix}. \tag{3.22}$$

Proof. Differentiating the variable t on both sides of (3.20),

$$\begin{aligned}
\frac{d\mathbb{P}(t)}{dt} &= \frac{d\Big(\mathbb{A}\mathcal{M}(t)\Big)}{dt} \\
&= \mathbb{A}\frac{d\Big(\mathcal{M}(t)\Big)}{dt} \\
&= \mathbb{A}\begin{pmatrix} 0 \\ 1 \\ 2t \\ \vdots \\ nt^{n-1} \end{pmatrix} \\
&= \mathbb{A}\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n & 0 \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix}
\end{aligned}$$

Thus, we can replace the derivative operator with the matrix, $\mathcal{D} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n & 0 \end{pmatrix}$. □

We proved for the derivative of the first order. The next corollary explains how we cater for other derivatives higher than one.

Corollary 3.3.1. *The j^{th} order derivative for $j \in \mathbb{N}_0$ is given as,*

$$\frac{d^j\mathbb{P}(t)}{dt^j} = \mathbb{A}\mathcal{D}^j\mathcal{M}(t). \tag{3.23}$$

The consequences of this corollary imply that, to obtain the j^{th} derivative, the matrix $\mathcal{D}$ has to multiply itself j times.

We explain two important consequences from (3.21) and (3.22) . Firstly, the operational matrix $\tilde{\mathcal{D}}$ in (3.21) is a mapping that acts specifically on the SLPs. Thus, $\tilde{\mathcal{D}}$ cannot be necessarily used on other polynomials. Secondly, we can apply $\mathcal{D}$ to any polynomial that can be written in the form (3.20).

We have discussed the application of the derivative matrix pertaining to the integer order. We can adopt the same principle to the fractional derivatives. The following theorem details how to deduce the fractional derivative matrix.

Theorem 3.3.3. *The differentiation operational matrix, $\mathcal{D}_t^{\mu}$, with order $(n+1) \times (n+1)$ deduced from $\mathcal{M}(t)$ can be written as,*

$$\mathcal{D}_t^{\mu} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} & 0 & \dots & 0 & 0 \\ 0 & \frac{2t^{1-\mu}}{\Gamma[3-\mu]} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\Gamma[n]t^{1-\mu}}{\Gamma[n+1-\mu]} & 0 \end{pmatrix}. \tag{3.24}$$

Proof. Applying the operator $\mathcal{D}_t^\mu$ as defined in (3.1) on both sides of (3.20) with $p = 1$,

$$
\begin{aligned}
\mathcal{D}_t^\mu \mathbb{P}(t) &= \mathcal{D}_t^\mu \mathbb{A}\mathcal{M}(t) \\
&= \mathbb{A}\mathcal{D}_t^\mu \mathcal{M}(t) \\
&= \mathbb{A}\begin{pmatrix} 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} \\ \frac{t^{2-\mu}}{\Gamma[3-\mu]} \\ \vdots \\ \frac{\Gamma[n]t^{n-\mu}}{\Gamma[n+1-\mu]} \end{pmatrix} \\
&= \mathbb{A}\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} & 0 & \dots & 0 & 0 \\ 0 & \frac{2t^{1-\mu}}{\Gamma[3-\mu]} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\Gamma[n]t^{1-\mu}}{\Gamma[n+1-\mu]} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix}.
\end{aligned} \tag{3.25}
$$

Thus, we can replace the fractional derivative operator by the matrix,

$$
\mathcal{D}_t^\mu = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} & 0 & \dots & 0 & 0 \\ 0 & \frac{2t^{1-\mu}}{\Gamma[3-\mu]} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\Gamma[n]t^{1-\mu}}{\Gamma[n+1-\mu]} & 0 \end{pmatrix}.
$$

□

Corollary 3.3.2. *If $\mu = 1$, the matrix $\mathcal{D}_t^\mu$ simplifies to its integer equivalence $\mathcal{D}$.*

Corollary 3.3.3. *We can infer from the results of the previous theorem that the $(\mu j)^{th}$ order derivative for $j \in \mathbb{N}$ can be expressed as,*

$$
\mathcal{D}_t^{\mu j}\mathbb{P}(t) = \mathbb{A}\mathcal{D}_t^{\mu j}\mathcal{M}(t). \tag{3.26}
$$

We will now apply the matrix (3.24) to numerically solve a fractional ODE.

Example 3.2. *Suppose we have a fractional differential equation,*

$$
\begin{aligned}
&\mathcal{D}_t^\mu y(t) - y(t) = 0, \quad \mu \in (0, 1], \\
&y(0) = 1.
\end{aligned} \tag{3.27}
$$

We will first solve (3.27) analytically before we go into detail concerning its approximation.

Introducing the Laplace transform on each term of (3.27),

$$\mathcal{L}[\mathcal{D}_t^{\mu} y(t)] - \mathcal{L}[y(t)] = 0 \tag{3.28}$$

Applying (3.8) in (3.28),

$$s^{\mu} Y(s) - s^{\mu-1} y(0) - Y(s) = 0. \tag{3.29}$$

Substituting for $y(0)$ and making $Y(s)$ the subject of the formula,

$$Y(s) = \frac{s^{\mu-1}}{s^{\mu} - 1}. \tag{3.30}$$

To recover the original function t, we impose inverse Laplace transform on (3.30),

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left(\frac{s^{\mu-1}}{s^{\mu} - 1}\right). \tag{3.31}$$

With reference to (3.3) and (3.7), (3.31) becomes,

$$y(t) = E_{\mu,1} = \sum_{k=0}^{\infty} \frac{t^{\mu k}}{\Gamma[\mu k + 1]}, \tag{3.32}$$

which is the analytical solution of (3.27). For practical purposes, we have to terminate terms generated from (3.32) at some point. Fortunately, we get good solutions for the first few terms.

If $\mu = 1$, then (3.32) becomes,

$$y(t) = e^{t}. \tag{3.33}$$

Now, we attempt to approximate the solution of (3.27), which we assume takes the form,

$$\begin{aligned} y(t) &= \gamma^{T} \mathbb{P}(t) \\ &= \gamma^{T} \mathbb{A} \mathcal{M}(t), \end{aligned} \tag{3.34}$$

with $\gamma = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \dots & \gamma_n \end{pmatrix}^{T}$ and $\mathcal{M}(t)$ is given in (3.19). If we take $n = 2$, then, (3.34) becomes,

$$y(t) = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -6 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \tag{3.35}$$

also, $\mathcal{D}_t^\mu$ becomes,

$$\mathcal{D}_t^\mu = \begin{pmatrix} 0 & 0 & 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} & 0 & 0 \\ 0 & \frac{2t^{1-\mu}}{\Gamma[3-\mu]} & 0 \end{pmatrix}. \tag{3.36}$$

Substituting (3.34) in (3.27) implies,

$$\begin{aligned} \mathcal{D}_t^\mu \gamma^T \mathbb{A}\mathcal{M}(t) - \gamma^T \mathbb{A}\mathcal{M}(t) &= 0 \\ \gamma^T \mathbb{A}\mathcal{D}_t^\mu \mathcal{M}(t) - \gamma^T \mathbb{A}\mathcal{M}(t) &= 0. \end{aligned} \tag{3.37}$$

Using (3.35) on (3.37), we get,

$$\begin{aligned} \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -6 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} \\ \frac{2t^{1-\mu}}{\Gamma[3-\mu]} \end{pmatrix} - \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -6 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} &= 0 \\ -\gamma_0 + \gamma_1 - t(2\gamma_1 - 6\gamma_2) + \frac{t^{1-\mu}(2\gamma_1 - 6\gamma_2)}{\Gamma[2-\mu]} - \gamma_2 - 6\gamma_2 t^2 + \frac{12t^{2-\mu}\gamma_2}{\Gamma[3-\mu]} &= 0. \end{aligned} \tag{3.38}$$

We define the residual from (3.38) as,

$$R_2(t) = -\gamma_0 + \gamma_1 - t(2\gamma_1 - 6\gamma_2) + \frac{t^{1-\mu}(2\gamma_1 - 6\gamma_2)}{\Gamma[2-\mu]} - \gamma_2 - 6\gamma_2 t^2 + \frac{12t^{2-\mu}\gamma_2}{\Gamma[3-\mu]}, \tag{3.39}$$

the subscript 2 in R denotes that $n = 2$. We create two equations from (3.39) using the Galerkin [16],

$$\int_0^1 R_2(t)\mathcal{M}_0(t)dt = 0, \tag{3.40}$$

and,

$$\int_0^1 R_2(t)\mathcal{M}_1(t)dt = 0. \tag{3.41}$$

Utilising the initial condition $y(0) = 1$ in (3.35) implies that,

$$\gamma_0 - \gamma_1 + \gamma_2 = 1. \tag{3.42}$$

We then solve (3.40)–(3.42) for γ_0, γ_1 and γ_2 and then substitute these values in (3.35) to get the approximate solution of (3.27). We get the following approximate solutions for different values of μ.

If $\mu = 1$,

$$y(t) = \frac{12}{7} + \frac{6}{7}(-1 + 2t) + \frac{1}{7}(1 - 6t + 6t^2). \tag{3.43}$$

If $\mu = 0.8$,

$$\begin{aligned} y(t) &= 1.9985553721927587 + 1.1394258704072866(-1 + 2t) \\ &+ 0.14087049821452802(1 - 6t + 6t^2). \end{aligned} \tag{3.44}$$

If $\mu = 0.5$,

$$\begin{aligned} y(t) &= 2.8643539855722806 + 1.9630958204704716(-1 + 2t) \\ &+ 0.0987418348981914(1 - 6t + 6t^2). \end{aligned} \tag{3.45}$$

Figures 3.1–3.3 depicts the plots of the approximate and analytical solutions together with the associated absolute errors. The absolute error is computed as $y_{er} = | y_{anal} - y_{approx} |$, with y_{anal} and y_{approx} being the analytic and approximate solutions of (3.27) respectively.

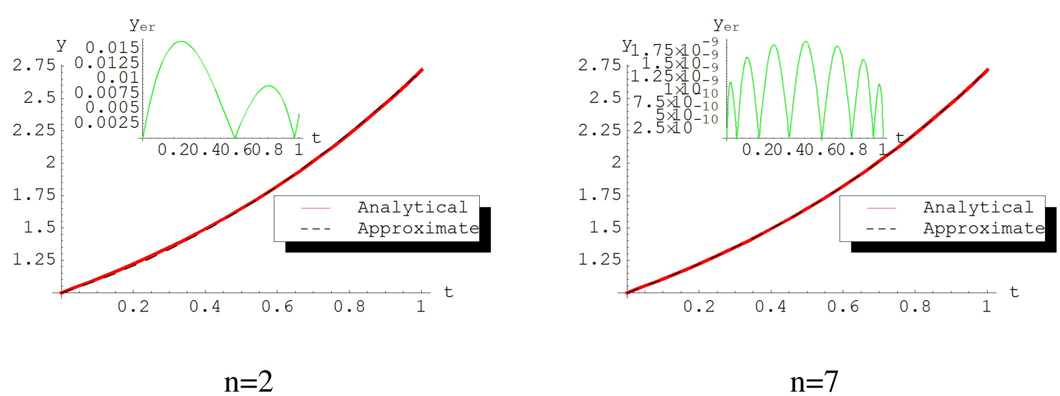
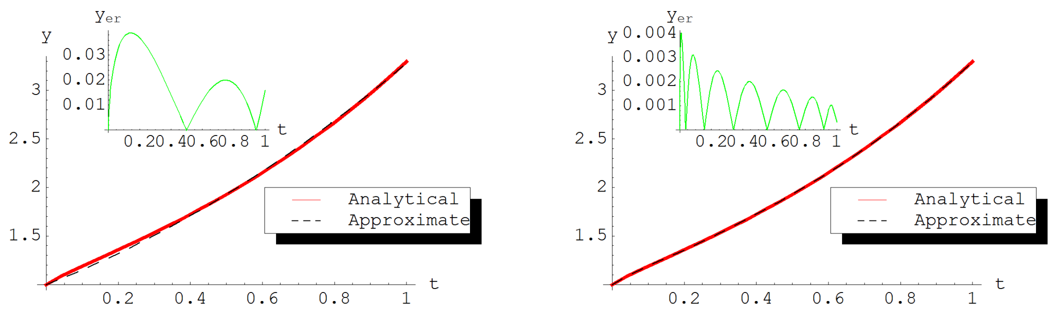
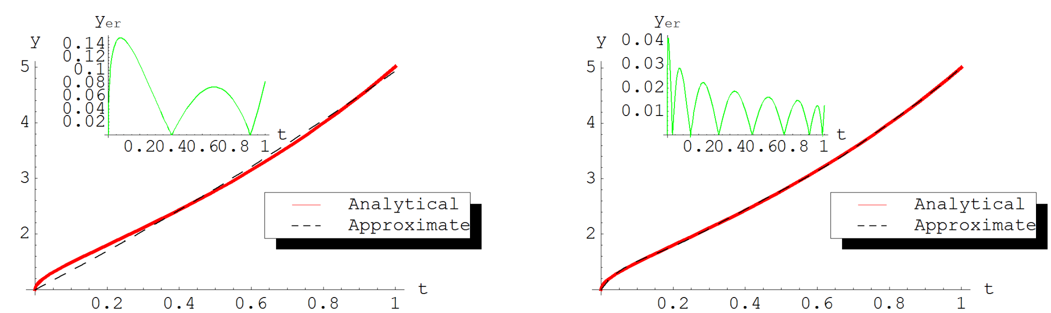


Figure 3.1: Approximate solutions of (3.27) using the derivative operational matrix (3.24) compared with the analytic solution for $\mu = 1$.



n=2 n=7

Figure 3.2: Approximate solutions of (3.27) using the derivative operational matrix (3.24) compared with the analytic solution for $\mu = 0.8$.



n=2 n=7

Figure 3.3: Approximate solutions of (3.27) using the derivative operational matrix (3.24) compared with the analytic solution for $\mu = 0.5$.

We take note from Figure 3.1–3.3 that as the number of polynomials used increase, the accuracy of the approximate solution improves. This result is true for all values of μ that we used. Also, we notice that the approximate solution tend to be better as $\mu \to 1$.

We have discussed about the derivative operational matrices and their involvement in the approximate solution of differential equations. We now turn our attention to integral operational matrices.

3.3.2 Deducing the integral operational matrices

The integral operational matrix acts as a substitute for an integral operator. It is of necessity that we first discuss about function approximation before attempting to discuss the integral operational matrices. The next theorems discusses the details of function approximation.

Theorem 3.3.4. *A function $f(t)$ can be written as a product of some $1 \times (n+1)$ matrix and a column of polynomials as [16],*

$$f(t) \approx v\mathcal{M}(t), \tag{3.46}$$

$v = [v_0, v_1, v_2, ..., v_n]$ *are the unknown coefficients that are to be calculated.*

The theorem below details how to compute the entries of v.

Theorem 3.3.5. *The entries of v stated in* (3.46) *are calculated as [16],*

$$v = \langle f(t), \mathcal{M}^T(t)\rangle\langle \mathcal{M}(t), \mathcal{M}^T(t)\rangle^{-1}, \tag{3.47}$$

provided the matrix $\langle \mathcal{M}(t), \mathcal{M}^T(t)\rangle$ is invertible,
where,

$$\langle f(t), \mathcal{M}^T(t)\rangle = \int_0^1 f(t)\mathcal{M}^T(t)dt \quad \text{and} \quad \langle \mathcal{M}(t), \mathcal{M}^T(t)\rangle = \int_0^1 \mathcal{M}(t)\mathcal{M}^T(t)dt.$$

We explain how to deduce integral operational matrices in the following theorems.

Theorem 3.3.6. *The matrix of integral, $\tilde{I}$, with order $(n+1)\times(n+1)$ deduced from $\mathcal{M}(t)$ can be written as,*

$$\tilde{I} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \\ v_0 & v_1 & v_2 & \dots & v_n \end{pmatrix} \tag{3.48}$$

$v = \begin{pmatrix} v_0 & v_1 & v_2 & \dots & v_n \end{pmatrix}$ *are the coefficients that we discussed in the previous theorem.*

Proof. Taking the integral of $\mathcal{M}(t)$,

$$
\begin{aligned}
\int_0^t \mathcal{M}(\tau)d\tau &= \begin{pmatrix} t \\ \frac{t^2}{2} \\ \frac{t^3}{3} \\ \vdots \\ \frac{t^{n+1}}{n+1} \end{pmatrix} \\
&\approx \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \\ v_0 & v_1 & v_2 & \dots & v_n \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} \\
&= \tilde{\mathcal{I}}\mathcal{M}(t).
\end{aligned}
$$

□

Next, we explain the derivation of the fractional integral matrix $\tilde{\mathcal{I}}^{\mu}$.

Theorem 3.3.7. *The matrix of integral, $\tilde{\mathcal{I}}^{\mu}$, with order $(n+1)\times(n+1)$ deduced from $\mathcal{M}(t)$ can be written as,*

$$
\tilde{\mathcal{I}}^{\mu} = \begin{pmatrix} 0 & \frac{t^{\mu-1}}{\Gamma[\mu+1]} & 0 & \dots & 0 \\ 0 & 0 & \frac{t^{\mu-1}}{\Gamma[\mu+2]} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma[n]t^{\mu-1}}{\Gamma[\mu+n]} \\ v_0 & v_1 & v_2 & \dots & v_n \end{pmatrix}. \tag{3.49}
$$

Proof. Applying (3.4) on $\mathcal{M}(t)$,

$$
\begin{aligned}
\frac{1}{\Gamma(\mu)}\int_0^t \frac{\mathcal{M}(s)ds}{(t-s)^{1-\mu}} &= \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]} \\ \vdots \\ \frac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]} \end{pmatrix} \\
&\approx \begin{pmatrix} 0 & \frac{t^{\mu-1}}{\Gamma[\mu+1]} & 0 & \dots & 0 \\ 0 & 0 & \frac{t^{\mu-1}}{\Gamma[\mu+2]} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma[n]t^{\mu-1}}{\Gamma[\mu+n]} \\ v_0 & v_1 & v_2 & \dots & v_n \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} \\
&= \tilde{\mathcal{I}}^{\mu}\mathcal{M}(t).
\end{aligned}
$$

Thus the proof is complete. □

Corollary 3.3.4. *If $\mu = 1$, the matrix $\tilde{\mathcal{I}}^{\mu}$ simplifies to its integer equivalence $\tilde{\mathcal{I}}$.*

We note that in matrices $\tilde{\mathcal{I}}^{\mu}$ and $\tilde{\mathcal{I}}$, we have to compute the numerical values of the entries of the last row using (3.47) as,

$$v = \left\langle \frac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]}, \mathcal{M}^T(t) \right\rangle \left\langle \mathcal{M}(t), \mathcal{M}^T(t) \right\rangle, \tag{3.50}$$

and

$$v = \left\langle \frac{t^{n+1}}{n+1}, \mathcal{M}(t) \right\rangle \left\langle \mathcal{M}(t), \mathcal{M}^T(t) \right\rangle. \tag{3.51}$$

The functions $f(t) = \frac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]}$ and $f(t) = \frac{t^{n+1}}{n+1}$ are the results of fractional and integer integration of the last term of $\mathcal{M}(t)$ respectively.

It is possible to determine another matrix of integration that does not involve the approximation of the last row in matrices $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{I}}^{\mu}$. More details are provided in the theorem below.

Theorem 3.3.8. *The matrix $\mathcal{I}$ that replaces the integral operator can be written as,*

$$\mathcal{I} = \begin{pmatrix} t & 0 & 0 & \dots & 0 \\ 0 & \frac{t}{2} & 0 & \dots & 0 \\ 0 & 0 & \frac{t}{3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{t}{n+1} \end{pmatrix}. \tag{3.52}$$

Proof.

$$\begin{aligned} \int_0^t \mathcal{M}(\tau)d\tau &= \begin{pmatrix} t & 0 & 0 & \dots & 0 \\ 0 & \frac{t}{2} & 0 & \dots & 0 \\ 0 & 0 & \frac{t}{3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{t}{n+1} \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} \\ &= \begin{pmatrix} t \\ \frac{t^2}{2} \\ \frac{t^3}{3} \\ \vdots \\ \frac{t^{n+1}}{n+1} \end{pmatrix}, \end{aligned}$$

this is the result that we expect from direct integration of $\mathcal{M}(t)$. □

We can tell from the previous theorem what the entries of the fractional integration matrix will be.

Corollary 3.3.5. *The fractional integral matrix $\mathcal{I}^{\mu}$ can be written as,*

$$\mathcal{I}^{\mu} = \begin{pmatrix} \frac{t^{\mu}}{\Gamma[1+\mu]} & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma[2]t^{\mu}}{\Gamma[2+\mu]} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma[3]t^{\mu}}{\Gamma[3+\mu]} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma[n+1]t^{\mu}}{\Gamma[n+1+\mu]} \end{pmatrix}. \tag{3.53}$$

We have shown how the integral operational matrices can be deduced from the polynomials by direct integration. We can deduce the same integral operational matrices through the use of integral transforms. In our case, we will demonstrate how to achieve this using the GJT and the Laplace transform.

Theorem 3.3.9. *The* $(n\times 1)\times(n\times 1)$ *operational matrix* $\mathcal{I}^{\mu}$ *in* (3.53) *can be deduced from the GJT.*

Proof. Taking the GJT of $\mathbb{P}(t)$,

$$\begin{aligned}
\mathrm{T}[\mathbb{P}(t)] &= \mathbb{A}\mathrm{T}[\mathcal{M}(t)] \\
&= \mathbb{A}\mathrm{T}\begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} \\
&= \mathbb{A}\begin{pmatrix} \frac{r(s)}{w(s)} \\ \frac{r(s)}{w^2(s)} \\ \frac{\Gamma[3]r(s)}{w^3(s)} \\ \vdots \\ \frac{\Gamma[n+1]r(s)}{w^{n+1}(s)} \end{pmatrix}.
\end{aligned}$$

Multiplying both sides by $\frac{1}{w^n(s)}$,

$$\frac{1}{w^n(s)}\mathrm{T}[\mathbb{P}(t)] = \mathbb{A}\frac{1}{w^n(s)}\begin{pmatrix} \frac{r(s)}{w(s)} \\ \frac{r(s)}{w^2(s)} \\ \frac{\Gamma[3]r(s)}{w^3(s)} \\ \vdots \\ \frac{\Gamma[n+1]r(s)}{w^{n+1}(s)} \end{pmatrix}$$

$$= \mathbb{A}\begin{pmatrix} \frac{r(s)}{w^{n+1}(s)} \\ \frac{r(s)}{w^{n+2}(s)} \\ \frac{\Gamma[3]r(s)}{w^{n+3}(s)} \\ \vdots \\ \frac{\Gamma[n+1]r(s)}{w^{2n+1}(s)} \end{pmatrix}.$$

Taking the inverse of the GJT,

$$
\begin{aligned}
\mathrm{T}^{-1}\left[\frac{1}{w^n(s)}\mathrm{T}[\mathbb{P}(t)]\right] &= \mathrm{T}^{-1}\mathbb{A}\begin{pmatrix} \frac{r(s)}{w^{n+1}(s)} \\ \frac{r(s)}{w^{n+2}(s)} \\ \frac{\Gamma[3]r(s)}{w^{n+3}(s)} \\ \vdots \\ \frac{\Gamma[n+1]r(s)}{w^{2n+1}(s)} \end{pmatrix} \\
&= \mathbb{A}\begin{pmatrix} \mathrm{T}^{-1}\left(\frac{r(s)}{w^{n+1}(s)}\right) \\ \mathrm{T}^{-1}\left(\frac{r(s)}{w^{n+2}(s)}\right) \\ \mathrm{T}^{-1}\left(\frac{\Gamma[3]r(s)}{w^{n+3}(s)}\right) \\ \vdots \\ \mathrm{T}^{-1}\left(\frac{\Gamma[n+1]r(s)}{w^{2n+1}(s)}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{t^n}{\Gamma[n+1]} \\ \frac{t^{n+1}}{\Gamma[n+2]} \\ \frac{2t^{n+2}}{\Gamma[n+3]} \\ \vdots \\ \frac{\Gamma[n+1]t^n}{\Gamma[2n+1]} \end{pmatrix} \\
&= \begin{pmatrix} \frac{t^n}{\Gamma[1+n]} & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma[2]t^n}{\Gamma[2+n]} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma[3]t^n}{\Gamma[3+n]} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma[n+1]t^n}{\Gamma[n+1+n]} \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix}
\end{aligned}
$$

If we consider the fractional integral, then n has to change to μ, thus we get the matrix $\mathcal{I}^\mu$ in (3.53). □

We again show that the same result can be achieved using the Laplace transform in the theorem below.

Theorem 3.3.10. *An operational matrix $\mathcal{I}^\mu$ in* (3.53) *can be deduced from the Laplace transform.*

Proof. Taking the Laplace transform of $\mathbb{P}(t)$,

$$\begin{aligned}
\mathcal{L}[\mathbb{P}(t)] &= \mathbb{A}\mathcal{L}[\mathcal{M}(t)] \\
&= \mathbb{A}\mathcal{L}\begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} \\
&= \mathbb{A}\begin{pmatrix} \frac{1}{s} \\ \frac{1}{s^2} \\ \frac{\Gamma[3]}{s^3} \\ \vdots \\ \frac{\Gamma[n+1]}{s^{n+1}} \end{pmatrix}.
\end{aligned}$$

Multiplying both sides by $\frac{1}{s^n}$,

$$\begin{aligned}
\frac{1}{s^n}\mathrm{T}[\mathbb{P}(t)] &= \mathbb{A}\frac{1}{s^n}\begin{pmatrix} \frac{1}{s} \\ \frac{1}{s^2} \\ \frac{\Gamma[3]}{s^3} \\ \vdots \\ \frac{\Gamma[n+1]}{s^{n+1}} \end{pmatrix} \\
&= \mathbb{A}\begin{pmatrix} \frac{1}{s^{n+1}} \\ \frac{1}{s^{n+2}} \\ \frac{\Gamma[3]}{s^{n+3}} \\ \vdots \\ \frac{\Gamma[n+1]}{s^{2n+1}} \end{pmatrix}.
\end{aligned}$$

We recover the original function through the application of inverse Laplace transform,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^n}\mathcal{L}[\mathbb{P}(t)]\right] &= \mathcal{L}^{-1}\mathbb{A}\begin{pmatrix} \frac{1}{s^{n+1}} \\ \frac{1}{s^{n+2}} \\ \frac{\Gamma[3]}{s^{n+3}} \\ \vdots \\ \frac{\Gamma[n+1]}{s^{2n+1}} \end{pmatrix} \\
&= \mathbb{A}\begin{pmatrix} \mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) \\ \mathcal{L}^{-1}\left(\frac{1}{s^{n+2}}\right) \\ \mathcal{L}^{-1}\left(\frac{\Gamma[3]}{s^{n+3}}\right) \\ \vdots \\ \mathcal{L}^{-1}\left(\frac{\Gamma[n+1]}{s^{2n+1}}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{t^n}{\Gamma[n+1]} \\ \frac{t^{n+1}}{\Gamma[n+2]} \\ \frac{2t^{n+2}}{\Gamma[n+3]} \\ \vdots \\ \frac{\Gamma[n+1]t^n}{\Gamma[2n+1]} \end{pmatrix} \\
&= \begin{pmatrix} \frac{t^n}{\Gamma[1+n]} & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma[2]t^n}{\Gamma[2+n]} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma[3]t^n}{\Gamma[3+n]} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma[n+1]t^n}{\Gamma[n+1+n]} \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix}
\end{aligned}$$

If we consider the fractional integral, then n becomes μ, thus yielding the matrix $\mathcal{I}^{\mu}$ in (3.53). □

There is some connection that is worth noting between the two transforms in deducing the integral transforms.

The operator $T^{-1}\Big[\frac{1}{w^n(s)}T[\mathbb{P}(t)]\Big]$ in the GJT is given as,

$$T^{-1}\Big[\frac{1}{w^n(s)}T[\mathbb{P}(t)]\Big] = T^{-1}\mathbb{A}\begin{pmatrix} \frac{r(s)}{w^{n+1}(s)} \\ \frac{r(s)}{w^{n+2}(s)} \\ \frac{\Gamma[3]r(s)}{w^{n+3}(s)} \\ \vdots \\ \frac{\Gamma[n+1]r(s)}{w^{2n+1}(s)} \end{pmatrix}. \tag{3.54}$$

The operator $\mathcal{L}^{-1}\Big[\frac{1}{s^n}\mathcal{L}[\mathbb{P}(t)]\Big]$ in the Laplace transform is given as,

$$\mathcal{L}^{-1}\Big[\frac{1}{s^n}\mathcal{L}[\mathbb{P}(t)]\Big] = \mathcal{L}^{-1}\mathbb{A}\begin{pmatrix} \frac{1}{s^{n+1}} \\ \frac{1}{s^{n+2}} \\ \frac{\Gamma[3]}{s^{n+3}} \\ \vdots \\ \frac{\Gamma[n+1]}{s^{2n+1}} \end{pmatrix}. \tag{3.55}$$

If $w(s) = s$ and $r(s) = 1$ in (3.54), then we get the result in (3.55), confirming that the Laplace transform is a particular case of the GJT.

3.3.3 Approximating the solutions of fractional ODEs.

We demonstrate how the operational matrix of integration can be used in the approximation of fractional differential equations, we consider (3.27) as our case study.

We assume the approximate solution of (3.27) takes the form (3.34), if $n = 2$, then,

$$\langle \mathcal{M}(t), \mathcal{M}^T(t)\rangle = \int_0^1 \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & t & t^2 \end{pmatrix} dt = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix},$$

therefore,

$$\langle \mathcal{M}(t), \mathcal{M}^T(t)\rangle^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}.$$

We need to approximate $\frac{2t^{2+\mu}}{\Gamma[3+\mu]}$, thus,

$$\left\langle \frac{2t^{2+\mu}}{\Gamma[3+\mu]}, \mathcal{M}^T(t) \right\rangle = \frac{2}{\Gamma[3+\mu]} \begin{pmatrix} \frac{1}{\Gamma[3+\mu]} & \frac{1}{\Gamma[(4+\mu]} & \frac{1}{\Gamma[5+\mu]} \end{pmatrix}.$$

Hence,

$$\begin{aligned} v &= \begin{pmatrix} v_0 & v_1 & v_2 \end{pmatrix} \\ &= \left\langle \frac{2t^{2+\mu}}{\Gamma(3+\mu)}, \mathcal{M}^T(t) \right\rangle \langle \mathcal{M}(t), \mathcal{M}^T(t) \rangle^{-1} \\ &= \begin{pmatrix} \frac{2\left(\frac{9}{3+\mu} - \frac{36}{4+\mu} + \frac{30}{5+\mu}\right)}{\Gamma(3+\mu)} & \frac{2\left(\frac{-36}{3+\mu} + \frac{192}{4+\mu} - \frac{180}{5+\mu}\right)}{\Gamma(3+\mu)} & \frac{2\left(\frac{30}{3+\mu} - \frac{180}{4+\mu} + \frac{180}{5+\mu}\right)}{\Gamma(3+\mu)} \end{pmatrix}. \end{aligned}$$

The integral operational matrix that acts upon $\mathcal{M}(t)$ is thus given as,

$$\tilde{\mathcal{I}}^{\mu} = \begin{pmatrix} 0 & \frac{t^{\mu-1}}{\Gamma[1+\mu]} & 0 \\ 0 & 0 & \frac{t^{\mu-1}}{\Gamma[2+\mu]} \\ \frac{2\left(\frac{9}{3+\mu} - \frac{36}{4+\mu} + \frac{30}{5+\mu}\right)}{\Gamma[3+\mu]} & \frac{2\left(\frac{-36}{3+\mu} + \frac{192}{4+\mu} - \frac{180}{5+\mu}\right)}{\Gamma[3+\mu]} & \frac{2\left(\frac{30}{3+\mu} - \frac{180}{4+\mu} + \frac{180}{5+\mu}\right)}{\Gamma[3+\mu]} \end{pmatrix}. \tag{3.56}$$

Taking the fractional integral of each term in (3.27),

$$\tilde{\mathcal{I}}^{\mu}\mathbb{D}_t^{\mu} y(t) - \tilde{\mathcal{I}}^{\mu} y(t) = 0,$$

$$y(t) - y(0) - \tilde{\mathcal{I}}^{\mu} y(t) = 0. \tag{3.57}$$

Substituting for the initial condition in (3.57) yields,

$$y(t) - 1 - \tilde{\mathcal{I}}^{\mu} y(t) = 0. \tag{3.58}$$

Substituting (3.34) in (3.58),

$$\begin{aligned} \gamma^T \mathbb{A}\mathcal{M}(t) - 1 - \tilde{\mathcal{I}}^{\mu}\left(\gamma^T \mathbb{A}\mathcal{M}(t)\right) &= 0 \\ \gamma^T \mathbb{A}\mathcal{M}(t) - 1 - \gamma^T \mathbb{A}\tilde{\mathcal{I}}^{\mu}\mathcal{M}(t) &= 0 \end{aligned} \tag{3.59}$$

We define the residual from (3.59) as , $R_2(t) = \gamma^T \mathbb{A}\mathcal{M}(t) - 1 - \gamma^T \mathbb{A}\tilde{\mathcal{I}}\mathcal{M}(t)$, thus, we create two equations from,

$$\int_0^1 R_2(t)\mathcal{M}_i(t)dt, \quad i = 0, 1,$$

and the third equation is obtained from the initial condition as,

$$y(0) = \gamma^T \mathbb{A}\mathcal{M}(0) = 1.$$

The three equations that we have just set up are,

$$\begin{aligned} -& \quad 1 + \left(1 - \frac{1}{\Gamma[2+\mu]}\right)\gamma_0 + \left(\frac{1}{\Gamma[2+\mu]} - \frac{2}{\Gamma[3+\mu]}\right)\gamma_1 - \frac{\gamma_2}{\Gamma[2+\mu]} + \frac{6\gamma_2}{\Gamma[3+\mu]} \\ -& \quad \frac{12\gamma_2}{\Gamma[6+\mu]}\left(20 + 9\mu + \mu^2\right) = 0, \end{aligned} \tag{3.60}$$

$$\begin{aligned} \frac{2}{(1+\mu)\Gamma[2+\mu]}\left(-\gamma_0 + \gamma_1 - \gamma_2\right) \quad &+ \quad \frac{2}{(3+\mu)\Gamma[2+\mu]}\left(-(7+\mu)\gamma_1 + (15+\mu)\gamma_2\right) \\ &+ \quad \frac{2}{\Gamma[3+\mu]}\left(\gamma_1 - 3\gamma_2\right) \\ &+ \quad \frac{\gamma_0}{2+\mu} + \frac{\gamma_1}{3} - \frac{12(2+\mu)\gamma_2}{\Gamma[5+\mu]} = 0, \end{aligned} \tag{3.61}$$

and

$$\gamma_0 - \gamma_1 + \gamma_2 - 1 = 0. \tag{3.62}$$

With the assistance of Mathematica, we solve (3.60)-(3.62) to get the unknowns γ_0, γ_1 and γ_2, we do this for specific values of μ. After computing the numerical values of γ_0, γ_1 and γ_2, we substitute them in (3.34) to get the approximate solution of (3.27).

If $\mu = 1$, we get,

$$y(t) = \frac{67}{39} + \frac{11}{13}(-1 + 2t) + \frac{5}{39}(1 - 6t + 6t^2). \tag{3.63}$$

If $\mu = 0.8$, then,

$$\begin{aligned} y(t) \quad &= \quad 2.0045218327422374 + 1.1157231198255249(-1 + 2t) \\ &+ \quad 0.1112012870832878(1 - 6t + 6t^2). \end{aligned} \tag{3.64}$$

If $\mu = 0.5$,

$$\begin{aligned} y(t) \quad &= \quad 2.874998700129662 + 1.9077563386708574(-1 + 2t) \\ &+ \quad 0.032757638541195414(1 - 6t + 6t^2). \end{aligned} \tag{3.65}$$

In Figures 3.4–3.6, we plot the approximate solutions of (3.27) versus the analytic solution together with the associated absolute errors, we do this for different combination pairs of μ and n.

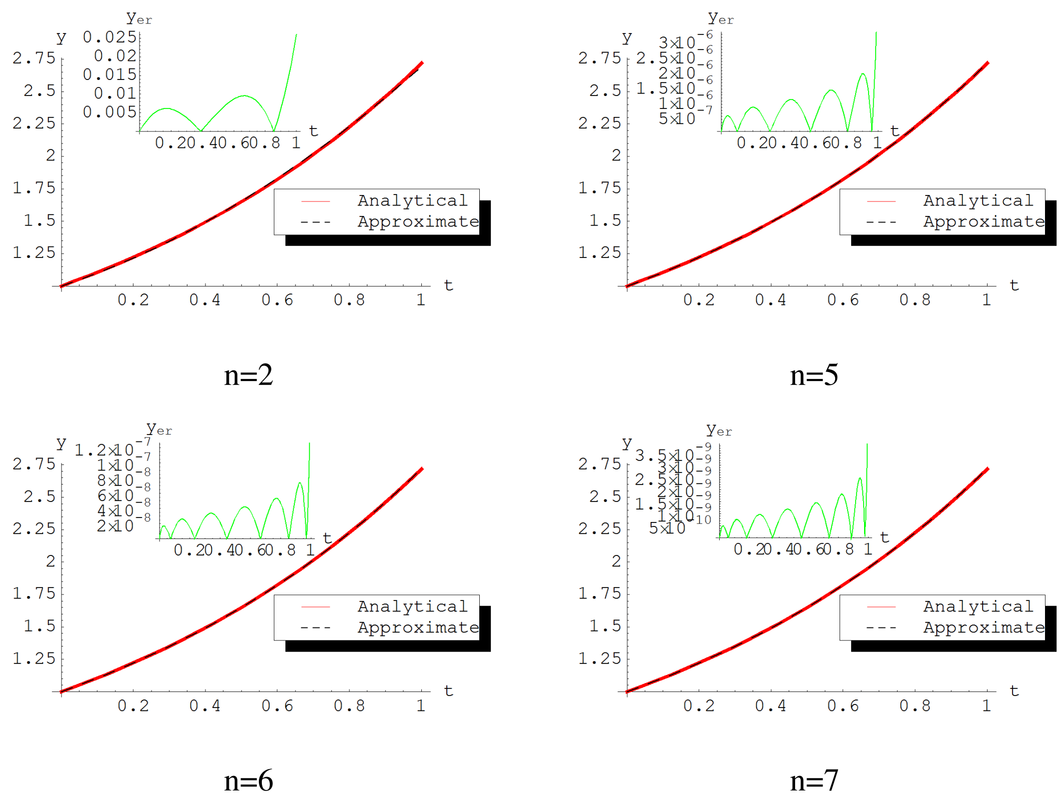


Figure 3.4: Approximate solutions of (3.27) using the integral operational matrix (3.49) compared with the analytic solution for $\mu = 1$.

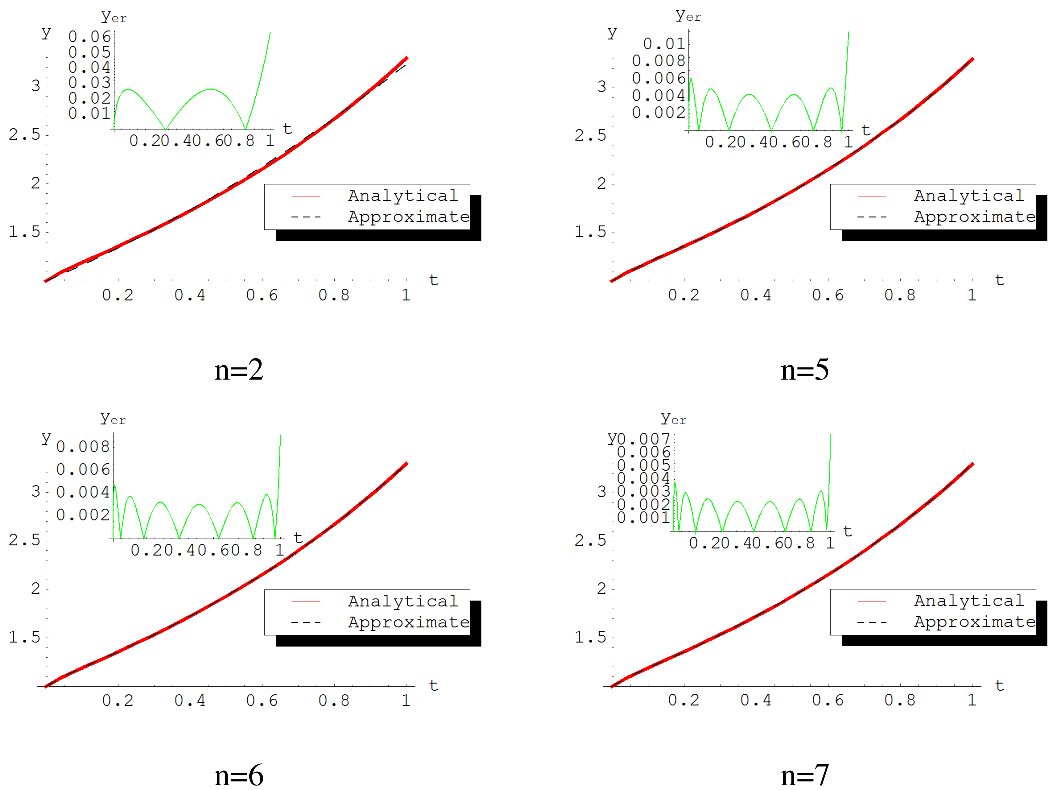


Figure 3.5: Approximate solutions of (3.27) using the integral operational matrix (3.49) compared with the analytic solution for $\mu = 0.8$.

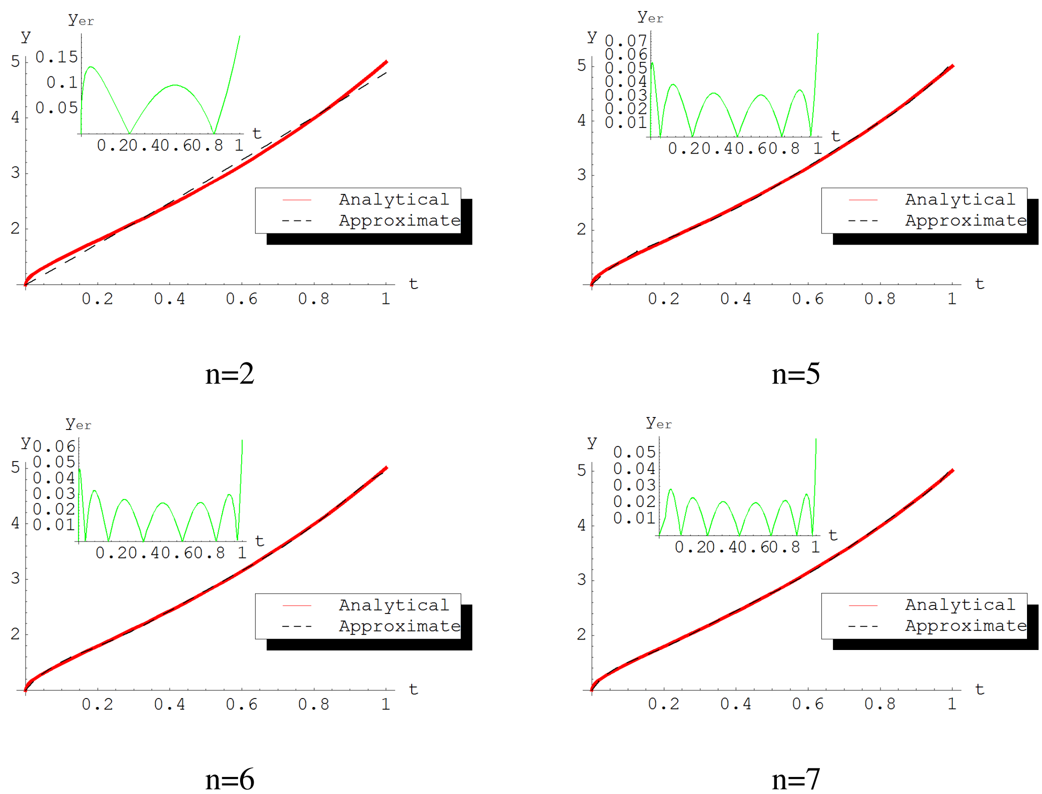


Figure 3.6: Approximate solutions of (3.27) using the integral operational matrix (3.49) compared with the analytic solution for $\mu = 0.5$.

We notice two important observations from Figures 3.4–3.6. Firstly, as we increase the value of n, the number of polynomials used, then the accuracy of the approximate solution improves, this result is consistent regardless the value of μ. Secondly, as the value of μ approaches 1, the approximate solution improves.

We have managed to use the matrix $\tilde{\mathcal{I}}$ to approximate the solution of (3.27). We repeat the whole process using the integral matrix $\mathcal{I}$ and we show the results below.

If $\mu = 1$ with $n = 2$, we get,

$$y(t) = \frac{67}{39} + \frac{11}{13}(-1 + 2t) + \frac{5}{39}(1 - 6t + 6t^2), \tag{3.66}$$

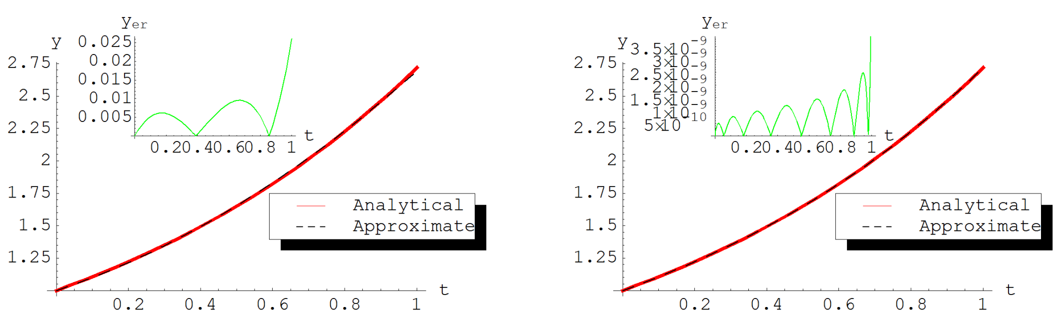
$\mu = 0.8$, then,

$$\begin{aligned} y(t) &= 2.004521832742238 + 1.1157231198255249(-1+2t) \\ &+ 0.11120128708328729(1-6t+6t^2), \end{aligned} \tag{3.67}$$

and $\mu = 0.5$, then,

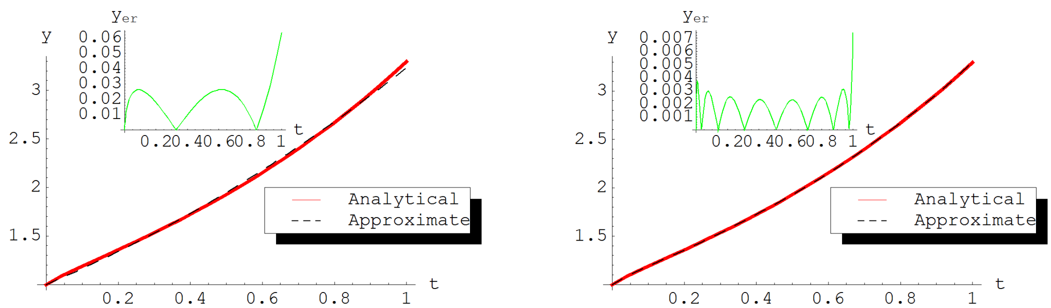
$$\begin{aligned} y(t) &= 2.8749987001296615 + 1.9077563386708567(-1+2t) \\ &+ 0.0327576385411953(1-6t+6t^2). \end{aligned} \tag{3.68}$$

We notice from (3.66)–(3.68) that the results from the matrix $\mathcal{I}$ are exactly the same as those from the matrix $\tilde{\mathcal{I}}$ for $\mu = 1$. We also observe that there is an insignificant difference in results from $\mathcal{I}$ and $\tilde{\mathcal{I}}$ for values of μ not equal to 1, in particular, for $\mu = 0.5$ and 0.8. In Figures 3.7–3.9, we plot our results for different values of n and μ together with the absolute errors.



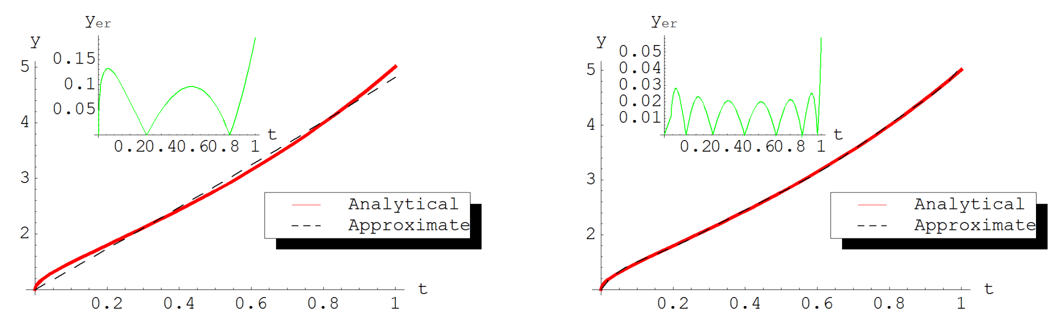
n=2 n=7

Figure 3.7: Approximate solutions of (3.27) using the integral operational matrix (3.53) compared with the analytic solution for $\mu = 1$.



n=2 n=7

Figure 3.8: Approximate solutions of (3.27) using the integral operational matrix (3.53) compared with the analytic solution for $\mu = 0.8$.



n=2 n=7

Figure 3.9: Approximate solutions of (3.27) using the integral operational matrix (3.53) compared with the analytic solution for $\mu = 0.5$.

We can conclude that, in general, the results from both the matrices $\tilde{\mathcal{I}}$ and $\mathcal{I}$ are the same. The only noticeable difference is when $n = 2$ and $\mu = 1$, with the matrix $\mathcal{I}$ giving better results than $\tilde{\mathcal{I}}$.

We now want to investigate convergence of the results from the operational matrices $\tilde{\mathcal{I}}$, $\mathcal{I}$ and $\mathcal{D}_t^{\mu}$. We will first discuss a theorem that guarantees convergence of solutions generated from (3.34).

The following definition will form a foundation for the next theorem.

Definition 3.3.4. *We define the function $\upsilon_n(t)$, the difference between two consecutive terms generated from* (3.34) *as,*

$$\upsilon_n(t) = y_{n+1}(t) - y_n(t), \quad n = 1, 2, \ldots \tag{3.69}$$

In [26], the authors discuss a theorem that guarantees convergence of a series solution. We can adopt this theorem in our case on the basis that as we change the value of n in (3.34), we are generating a sequence.

The following theorem guarantees the convergence of the solutions generated from (3.34) [26].

Theorem 3.3.11. *The sequence, $\upsilon_n, \upsilon_{n+1}, \upsilon_{n+2}, \ldots$, will converge whenever,* $\| \upsilon_n \| > \| \upsilon_{n+1} \| > \| \upsilon_{n+2} \| > \ldots$

In our case, the norms in the above theorem are computed over the interval [0, 1] as,

$$\| \upsilon_n \| = \sqrt{\int_0^1 | \upsilon_n(t) |^2 \, dt}, \tag{3.70}$$

of course one can use any integration limits that suits a particular situation.

The consequence of the above theorem is that as we increase the value of n, then we expect the magnitude of the differences between two consecutive terms to become smaller and smaller if the results converge.

The other important information that we can deduce from this theorem concerns the rate of convergence. This concept is particularly useful when comparing the convergence of different numerical methods and it can play a decisive role in selecting the best method. The smaller the values of $\| \upsilon_n \|$, then the faster the convergence of a method. Generally, it might be best to choose a numerical method with a faster rate of convergence as this implies that we get closer to the desired solution faster.

We compute the values of $\| \upsilon_n \|$ for different combination sets of n and μ using results computed from matrices $\tilde{\mathcal{I}}$, $\mathcal{I}$ and $\mathcal{D}_t^{\mu}$. The convergence results are displayed in Tables 3.2–3.4.

Table 3.2: Convergence results for different values of n for $\mu = 1$.

$\parallel \upsilon \parallel$	Differential matrix ($\mathcal{D}^{\mu}$)	Integral matrix ($\tilde{\mathcal{I}}$)	Integral matrix ($\mathcal{I}$)
$\parallel \upsilon_2 \parallel$	0.09115778913	0.02185080461	0.02185080461
$\parallel \upsilon_3 \parallel$	0.022377038351	0.01931698320	0.01931698320
$\parallel \upsilon_4 \parallel$	0.00488583557	0.00432128028	0.00432128028
$\parallel \upsilon_5 \parallel$	0.00098452564	0.00088266313	0.00088266313
$\parallel \upsilon_6 \parallel$	0.00018447595	0.00016691349	0.00016691349

Table 3.3: Convergence results for different values of n for $\mu = 0.8$.

$\parallel \upsilon \parallel$	Differential matrix ($\mathcal{D}^{\mu}$)	Integral matrix ($\tilde{\mathcal{I}}$)	Integral matrix ($\mathcal{I}$)
$\parallel \upsilon_2 \parallel$	0.14181076161	0.14498498599	0.14498498599
$\parallel \upsilon_3 \parallel$	0.07418689351	0.08648991335	0.08648991335
$\parallel \upsilon_4 \parallel$	0.06162478449	0.07196032536	0.07196032536
$\parallel \upsilon_5 \parallel$	0.05057729135	0.05997744431	0.05997744430
$\parallel \upsilon_6 \parallel$	0.04294762124	0.05144070680	0.05144070552

Table 3.4: Convergence results for different values of n for $\mu = \frac{1}{2}$.

$\| \upsilon \|$	Differential matrix ($\mathcal{D}^{\mu}$)	Integral matrix ($\tilde{\mathcal{I}}$)	Integral matrix ($\mathcal{I}$)
$\| \upsilon_2 \|$	0.26983933473	0.29023758456	0.29023758456
$\| \upsilon_3 \|$	0.19587578646	0.21846510279	0.21846510279
$\| \upsilon_4 \|$	0.16705724682	0.18783810524	0.18783810524
$\| \upsilon_5 \|$	0.14545274053	0.16471559804	0.16471559807
$\| \upsilon_6 \|$	0.12952705220	0.14740940791	0.14740940802

There is a lot of interesting information that we can deduce from Tables 3.2–3.4. We realise that our approximate solutions converge to the analytic solution as number of polynomials used increase, this result is consistent for all the three matrices.

There is faster rate of convergence for $\mu = 1$ compared with other values of μ, this also is true for all the three matrices.

The results from the integral matrices converge faster than the derivative matrix when $\mu = 1$. We observe the exact opposite when $\mu = 0.8$ and $\mu = 0.5$, the derivative matrix converges faster than the integral matrices.

We also observe that, both integral matrices, $\tilde{\mathcal{I}}$ and $\mathcal{I}$ yield the same convergence results regardless of the value of μ. In instances where the integral matrices give different convergence results, these results tend to be very close and we have very few occasions of these scenarios.

We have up to now compared our approximate solutions against analytic solutions. We now want to compare our approximate solution (3.66) with the results from the Adams-Bashforth-Moulton method (ABM), a numerical technique described in [24]. Note that in the table, we name our suggested approach the Garlekin, this is because it is the numerical method that we used in conjunction with the operational matrices to approximate the solution of (3.27).

Table 3.5: Comparison of the approximate solutions of (3.27) using the Galerkin and the ABM for $\mu = 1$.

t	Garlekin	ABM	Analytic
0.1	1.1	1.105170833	1.105170918
0.2	1.2153846153846153	1.22041776	1.221402758
0.3	1.3461538461538463	1.347699705	1.349858808
0.4	1.4923076923076921	1.489408691	1.491824698

The information in Table 3.5 indicate that the results from our suggested technique and from the ABM are in close agreement. It is imperative to take into account that the approach of these two techniques is very different. The ABM uses the step size to calculate solutions at particular points of the independent variable. The smaller the step size, the more cumbersome the ABM gets, but then the more accurate the results become. With the Garlekin, we observed from the previous examples that, it is increasing the number of polynomials that improves the solutions.

There is no conventional rule which links the number of polynomials in the Garlekin technique and the step size in the ABM. This makes it a bit tricky to compare the two methods which might be the reason for the differences in the solutions in Table 3.5.

However, we think that the step size and the number of polynomials used to produce the results in Table 3.5 are a close match.

3.4 Applying operational matrices to the fractional Van der Pol equation

Non linear ODEs are of immense importance in the field of Applied Mathematics, an example of such an equation is the Van der Pol. This differential equation has yielded impressive results in various fields such as in electrical circuits, science, technology and biology [25].

One of the challenges when using the Van der Pol equation in modelling is the attainment of its solutions due to the non linear terms it possesses. Thus many researchers resort to numerical methods to approximate its solution.

We will apply techniques in the previous section to approximate the solution of the Van der Pol equation, but we make a minor change in an effort to lessen the computational difficulties. We will do this by dropping the matrix $\mathbb{A}$ in (3.34), this is equivalent to taking this matrix as an identity. It implies that we will approximate our solution using $\mathcal{M}(t)$ polynomials. Thus our approximate solution of the Van der Pol equation will take the form,

$$y(t) = \gamma^T \mathcal{M}(t). \tag{3.71}$$

The forced Van der Pol differential equation of fractional order can be written as [25],

$$\begin{aligned} &\mathcal{D}_t^{\mu} y(t) - \sigma(1 - y^2(t))y'(t) + y(t) + \rho y^3(t) = 0, \qquad \mu \in (1, 2] \\ &y(0) = y_0, \quad y'(0) = y'_0, \end{aligned} \tag{3.72}$$

σ and ρ represent the parameters whose numerical values vary depending on the situation under consideration. We can explicitly write (3.72) in the form,

$$\begin{aligned} &\mathcal{D}_t^{\mu} y(t) = \sigma y'(t) - \sigma y^2(t)y'(t) - y(t) - \rho y^3(t), \\ &y(0) = y_0, \quad y'(0) = y'_0. \end{aligned} \tag{3.73}$$

Applying the fractional integral on each term in (3.73),

$$\mathcal{I}^{\mu}[\mathcal{D}_t^{\mu} y(t)] = \mathcal{I}^{\mu}[\sigma y'(t)] - \mathcal{I}^{\mu}[\sigma y^2(t)y'(t)] + \mathcal{I}^{\mu}[y(t)] + \mathcal{I}^{\mu}[\rho y^3(t)],$$

$$y(t) - y(0) - y'(0) = \mathcal{I}^{\mu}[\sigma y'(t)] - \mathcal{I}^{\mu}[\sigma y^2(t)y'(t)] + \mathcal{I}^{\mu}[y(t)] + \mathcal{I}^{\mu}[\rho y^3(t)]. \tag{3.74}$$

Substituting for the initial conditions and solving for $y(t)$ in (3.74) yields,

$$y(t) = y_0 + y'_0 + \mathcal{I}^{\mu}[\sigma y'(t)] - \mathcal{I}^{\mu}[\sigma y^2(t)y'(t)] + \mathcal{I}^{\mu}[y(t)] + \mathcal{I}^{\mu}[\rho y^3(t)], \tag{3.75}$$

Considering the particular case of (3.73) as given in [25], $\sigma = 0.1, \rho = 0.01, y(0) = 2$ and $y'(0) = 0$. Thus, (3.75) becomes,

$$\begin{aligned} y(t) &= 2 + \mathcal{I}^{\mu}[0.1y'(t)] - \mathcal{I}^{\mu}[0.1y^2(t)y'(t)] + \mathcal{I}^{\mu}[y(t)] + \mathcal{I}^{\mu}[0.01y^3(t)] \\ &= 2 + 0.1\mathcal{I}^{\mu}[y'(t)] - 0.1\mathcal{I}^{\mu}[y^2(t)y'(t)] + \mathcal{I}^{\mu}[y(t)] + 0.01\mathcal{I}^{\mu}[y^3(t)]. \end{aligned} \tag{3.76}$$

Next, we assume that the approximate solution of (3.76) takes the form (3.71) and we replace the derivative operator by the matrix $\mathcal{D}$, thus from (3.76), we get,

$$\begin{aligned}\gamma^T\mathcal{M}(t) &= 2+0.1\mathcal{I}^\mu\mathcal{D}\Big(\gamma^T\mathcal{M}(t)\Big)-0.1\mathcal{I}^\mu\Big[\Big(\gamma^T\mathcal{M}(t)\Big)^2\mathcal{D}\Big(\gamma^T\mathcal{M}(t)\Big)\Big]\\ &+ \mathcal{I}^\mu[\gamma^T\mathcal{M}(t)]+0.01\mathcal{I}^\mu\Big[\Big(\gamma^T\mathcal{M}(t)\Big)^3\Big].\end{aligned} \tag{3.77}$$

If we take $n = 2$, (3.77) becomes,

$$\begin{aligned}\begin{pmatrix}\gamma_0 & \gamma_1 & \gamma_2\end{pmatrix}\begin{pmatrix}1\\ t\\ t^2\end{pmatrix} &= 2+0.1\begin{pmatrix}\gamma_0 & \gamma_1 & \gamma_2\end{pmatrix}\mathcal{I}^\mu\mathcal{D}\begin{pmatrix}1\\ t\\ t^2\end{pmatrix}\\ &- \frac{1}{10}\mathcal{K}_1\mathcal{I}^\mu\mathcal{D}\begin{pmatrix}1\\ t\\ t^2\\ t^3\\ t^4\\ t^5\end{pmatrix}+\begin{pmatrix}\gamma_0 & \gamma_1 & \gamma_2\end{pmatrix}\mathcal{I}^\mu\begin{pmatrix}1\\ t\\ t^2\end{pmatrix}\\ &+ \frac{1}{100}\mathcal{K}_2\mathcal{I}^\mu\begin{pmatrix}1\\ t\\ t^2\\ t^3\\ t^4\\ t^5\\ t^6\end{pmatrix},\end{aligned} \tag{3.78}$$

where,

$$\mathcal{K}_1=\begin{pmatrix}\gamma_0^2\gamma_1 & 2\gamma_0\gamma_1^2+2\gamma_0^2\gamma_2 & \gamma_1^3+6\gamma_0\gamma_1\gamma_2 & 4\gamma_1^2\gamma_2+4\gamma_0\gamma_2^2 & 5\gamma_1\gamma_2^3 & 2\gamma_2^3\end{pmatrix},$$

and

$$\mathcal{K}_2=\begin{pmatrix}\gamma_0^3 & 3\gamma_0^2\gamma_1 & 3\gamma_0\gamma_1^2+3\gamma_0^2\gamma_2 & \gamma_1^3+6\gamma_0\gamma_1\gamma_2 & 3\gamma_1^2\gamma_2+3\gamma_0\gamma_2^2 & 3\gamma_1\gamma_2^2 & \gamma_2^3\end{pmatrix}.$$

We will now write down the derivative and integral matrices that we will make use of in (3.78).

The derivative matrices that we are going to utilise in (3.78) are given as,

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{pmatrix}.$$

Substituting these matrices in (3.78) gives,

$$\begin{aligned} \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} &= 2 + 0.1 \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \mathcal{I}^{\mu} \begin{pmatrix} 0 \\ 1 \\ 2t \end{pmatrix} \\ &- \frac{1}{10} \mathcal{K}_1 \mathcal{I}^{\mu} \begin{pmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \\ 4t^3 \\ 5t^4 \end{pmatrix} + \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \mathcal{I}^{\mu} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \\ &+ \frac{1}{100} \mathcal{K}_2 \mathcal{I}^{\mu} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \end{pmatrix}. \end{aligned} \tag{3.79}$$

Then the integral matrices $\mathcal{I}^{\mu}$ will be as follows,

$$\begin{aligned}\mathcal{I}^{\mu}\begin{pmatrix}0\\1\\2t\end{pmatrix} &= \begin{pmatrix}0 & 0 & 0\\0 & \frac{t^{\mu}}{\Gamma[\mu+1]} & 0\\0 & 0 & \frac{t^{\mu}}{\Gamma[\mu+2]}\end{pmatrix}\begin{pmatrix}0\\1\\2t\end{pmatrix}\\ &= \begin{pmatrix}0\\ \frac{t^{\mu}}{\Gamma[\mu+1]}\\ \frac{t^{\mu+1}}{\Gamma[\mu+2]}\end{pmatrix}.\end{aligned} \tag{3.80}$$

$$\begin{aligned}\mathcal{I}^{\mu}\begin{pmatrix}1\\t\\t^2\\t^3\\t^4\\t^5\end{pmatrix} &= \begin{pmatrix}\frac{t^{\mu}}{\Gamma[\mu+1]} & 0 & 0 & 0 & 0 & 0\\0 & \frac{t^{\mu}}{\Gamma[\mu+2]} & 0 & 0 & 0 & 0\\0 & 0 & \frac{2t^{\mu}}{\Gamma[\mu+3]} & 0 & 0 & 0\\0 & 0 & 0 & \frac{6t^{\mu}}{\Gamma[\mu+4]} & 0 & 0\\0 & 0 & 0 & 0 & \frac{24t^{\mu}}{\Gamma[\mu+5]} & 0\\0 & 0 & 0 & 0 & 0 & \frac{120t^{\mu}}{\Gamma[\mu+6]}\end{pmatrix}\begin{pmatrix}1\\t\\t^2\\t^3\\t^4\\t^5\end{pmatrix}\\ &= \begin{pmatrix}\frac{t^{\mu}}{\Gamma[\mu+1]}\\ \frac{t^{\mu+1}}{\Gamma[\mu+2]}\\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]}\\ \frac{6t^{\mu+3}}{\Gamma[\mu+4]}\\ \frac{24t^{\mu+4}}{\Gamma[\mu+5]}\\ \frac{120t^{\mu+5}}{\Gamma[\mu+6]}\end{pmatrix}.\end{aligned} \tag{3.81}$$

$$\begin{aligned}\mathcal{I}^{\mu}\begin{pmatrix}1\\t\\t^2\end{pmatrix} &= \begin{pmatrix}\frac{t^{\mu}}{\Gamma[\mu+1]} & 0 & 0\\0 & \frac{t^{\mu}}{\Gamma[\mu+2]} & 0\\0 & 0 & \frac{2t^{\mu}}{\Gamma[\mu+3]}\end{pmatrix}\begin{pmatrix}1\\t\\t^2\end{pmatrix}\\ &= \begin{pmatrix}\frac{t^{\mu}}{\Gamma[\mu+1]}\\ \frac{t^{\mu+1}}{\Gamma[\mu+2]}\\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]}\end{pmatrix}.\end{aligned} \tag{3.82}$$

$$
\begin{aligned}
\mathcal{I}^{\mu}\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \end{pmatrix} &= \begin{pmatrix}
\frac{t^{\mu}}{\Gamma[\mu+1]} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{t^{\mu}}{\Gamma[\mu+2]} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2t^{\mu}}{\Gamma[\mu+3]} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{6t^{\mu}}{\Gamma[\mu+4]} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{24t^{\mu}}{\Gamma[\mu+5]} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{120t^{\mu}}{\Gamma[\mu+6]} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{720t^{\mu}}{\Gamma[\mu+7]}
\end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \end{pmatrix} \\
&= \begin{pmatrix}
\frac{t^{\mu}}{\Gamma[\mu+1]} \\
\frac{t^{\mu+1}}{\Gamma[\mu+2]} \\
\frac{2t^{\mu+2}}{\Gamma[\mu+3]} \\
\frac{6t^{\mu+3}}{\Gamma[\mu+3]} \\
\frac{24t^{\mu+4}}{\Gamma[\mu+5]} \\
\frac{120t^{\mu+5}}{\Gamma[\mu+6]} \\
\frac{720t^{\mu+6}}{\Gamma[\mu+7]}
\end{pmatrix}. \qquad (3.83)
\end{aligned}
$$

Substituting (3.80)-(3.83) into (3.79),

$$
\begin{aligned}
\begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} &= 2 + 0.1 \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \end{pmatrix} \\
&- \frac{1}{10} \mathcal{K}_1 \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]} \\ \frac{6t^{\mu+3}}{\Gamma[\mu+4]} \\ \frac{24t^{\mu+4}}{\Gamma[\mu+5]} \\ \frac{120t^{\mu+5}}{\Gamma[\mu+6]} \end{pmatrix} + \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]} \end{pmatrix} \\
&+ \frac{1}{100} \mathcal{K}_2 \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]} \\ \frac{6t^{\mu+3}}{\Gamma[\mu+3]} \\ \frac{24t^{\mu+4}}{\Gamma[\mu+5]} \\ \frac{120t^{\mu+5}}{\Gamma[\mu+6]} \\ \frac{720t^{\mu+6}}{\Gamma[\mu+7]} \end{pmatrix}. \qquad (3.84)
\end{aligned}
$$

We derive the residual $R_2(t)$ from (3.84) as,

$$\begin{aligned} R_2(t) &= \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} - 2 - 0.1 \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \end{pmatrix} \\ &+ \frac{1}{10}\mathcal{K}_1 \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]} \\ \frac{6t^{\mu+3}}{\Gamma[\mu+4]} \\ \frac{24t^{\mu+4}}{\Gamma[\mu+5]} \\ \frac{120t^{\mu+5}}{\Gamma[\mu+6]} \end{pmatrix} - \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]} \end{pmatrix} \\ &- \frac{1}{100}\mathcal{K}_2 \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} \\ \frac{t^{\mu+1}}{\Gamma[\mu+2]} \\ \frac{2t^{\mu+2}}{\Gamma[\mu+3]} \\ \frac{6t^{\mu+3}}{\Gamma[\mu+3]} \\ \frac{24t^{\mu+4}}{\Gamma[\mu+5]} \\ \frac{120t^{\mu+5}}{\Gamma[\mu+6]} \\ \frac{720t^{\mu+6}}{\Gamma[\mu+7]} \end{pmatrix}. \end{aligned} \tag{3.85}$$

We create a single equation from (3.85),

$$\int_0^1 R_2(t)\mathcal{M}_0(t)dt = 0, \tag{3.86}$$

and two equations from the initial conditions,

$$\begin{aligned} \gamma^T \mathcal{M}(0) &= 2, \\ \gamma_0 &= 2, \end{aligned} \tag{3.87}$$

$$\begin{aligned} \gamma^T \mathcal{M}'(0) &= 0, \\ \gamma_1 &= 0. \end{aligned} \tag{3.88}$$

Solving (3.86)-(3.88) for the unknowns with $\mu = 2$ yields the results shown in the Appendix A1 . We note that there are several combinations of solutions emanating from

(3.86)-(3.88), but there is only one set of combination composed of real numbers only, the rest have complex numbers as one of the solution. We choose the combination set with real numbers only.
Thus, the approximate solution of (3.72) for $\mu = 2$ is,

$$y(t) = 2 - 0.9343075950635839t^2. \tag{3.89}$$

If we take $n = 3$ and $\mu = 2$, the combination of solutions is given in Appendix A2. We choose solutions that consists of only real numbers, thus we get the approximate solution as,

$$y(t) = 2 - 1.0521192366798373t^2 + 0.1617057020400526t^3. \tag{3.90}$$

The authors in [25] make use of the restarted Adomian decomposition method to approximate the solution of (3.72), they compared their results with the Adomian decomposition method (ADCMP).
We use the results from the ADCMP in [25] to compare with the results from our proposed scheme, the graphical comparison is shown in Figure 3.10 for different values of n.

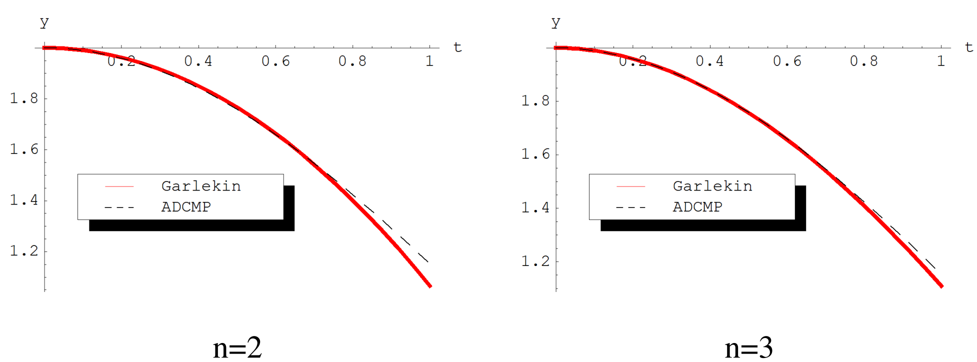


Figure 3.10: Comparison of the Garlekin technique and the ADCMP.

In figure 3.10, the values of n are specifically for the Garlekin technique, the method we used with the operational matrices. Concerning the ADCMP, a series approximation with four terms was used, we therefore believe that this is a fair comparison of the two techniques, particularly when $n = 3$ for the Garlekin. We observe that as n increases, the two methods tend to agree. Unfortunately, there is no analytical solution of (3.72) that we can use to compare with our proposed scheme.

3.5 Conclusion

We have managed to show how to deduce the operational matrices that represent both the derivative and integral. Different ways are explained on how to go about deducing the integration matrix. We demonstrated that it is possible depending on the set up of an ordinary differential equation to use only derivative operational matrix or an integral operational matrix. We had to show that results from these two operational are very close, suggesting that result wise, it does not matter which operational matrix is used. Comparing our results against the analytic solutions revealed that only the first few polynomials are enough to give good approximations.

Due to the nature of the Van der pol equation, we had to apply both the derivative and operational matrices. We compared our results with the Adomian decomposition method, and the results were in close agreement.

The results from this chapter, in particular the theoretical aspects will form a basis of the work in the next chapter. We will show how the matrices attained in this chapter act as building blocks of more larger matrices. These larger matrices will act as the substitutes for both the partial integral and derivative operators.

Chapter 4

Composite operational matrices

Abstract

This chapter constitutes our main contribution to the existing knowledge of operational matrices. We show how it is possible to build large operational matrices that accommodate more than one variable, in particular we focus on two variables. We will refer to these matrices as composite matrices, since the entries of these matrices are also matrices. The matrices from the previous chapter are the components of the matrices that we are going to construct. In addition to the discussion of these composite operational matrices, we describe the solution procedure that we follow in the application of these matrices to solve PDEs.

4.1 Introduction

As we mentioned in the introduction of this research, most of the work on operational matrices has been on ordinary differential equations, most probably this is due to the fact that it is much easier to deal with computations in ODEs than PDEs. Of course there has been noticeable achievements on the application of operational matrices on PDEs, but there is no doubt that more needs to be done along these lines. In this chapter, we bring a new dimension that reveals the effectiveness of operational matrices on PDEs.

By writing our approximate solution in an implicit format, we illustrate how partial derivatives and integrals lend themselves to composite operational matrices. The entries of these composite matrices are themselves matrices.
We will show how the commutativity and associativity law of matrix multiplication applies in the context of operational matrices. Of more interest to us in this work will be the associativity law, using this law, we will be able to construct a single matrix that performs the same duty as both the differential and integral matrices.

After the construction of the operational matrices, we apply them in the approximate solution of the initial boundary value problems (IBVPs). We write our approximate solution as a product consisting of polynomials and unknown coefficients. We will have to be able to find numerical values of the coefficients for us to be able to obtain a function that approximate the solution of the IBVP. To achieve this goal, we create a system of equations involving these unknown coefficients, and then solve for the unknowns. These system of equations are deduced from the initial and boundary conditions and then supplemented with those from the Garlekin technique.

Every numerical method should undergo tests for accuracy of its results so that we can rely on it. We will discuss a theorem that guarantees convergence of our approximate solution.

4.2 Derivative and integral composite operational matrices

The following definitions concern the polynomials and column matrices that we will make use of in the next two chapters. Some of these definitions are similar to the ones we used in the previous chapter, we will state them again for the sake of convenience.

Definition 4.2.1. *We define polynomials, $\mathcal{M}(t)$ and $\mathcal{M}(x)$ as,*

$$\mathcal{M}(t) = \begin{pmatrix} 1 & t & t^2 & \dots & t^n \end{pmatrix}^T, \tag{4.1}$$

$$\mathcal{M}(x) = \begin{pmatrix} 1 & x & x^2 & \dots & x^n \end{pmatrix}^T, \quad n \in \mathbb{N}. \tag{4.2}$$

We can conglomerate the matrices, $\mathcal{M}(t)$ and $\mathcal{M}(x)$ into a single matrix. We describe how to achieve this goal in the next theorem.

Theorem 4.2.1. *We define a matrix $\mathcal{M}(x,t)$, a composition of $\mathcal{M}(t)$ and $\mathcal{M}(x)$ in an implicit format as,*

$$\mathcal{M}(x,t) = \begin{pmatrix} \mathcal{M}(t) & x\mathcal{M}(t) & x^2\mathcal{M}(t) & \dots & x^n\mathcal{M}(t) \end{pmatrix}^T. \tag{4.3}$$

Proof.

$$\mathcal{M}^T(t)\mathcal{M}^T(x) = \begin{pmatrix} 1 & t & t^2 & \dots & t^n \end{pmatrix}\begin{pmatrix} 1 & x & x^2 & \dots & x^n \end{pmatrix}. \tag{4.4}$$

Expanding the right hand side,

$$\begin{pmatrix} 1 & t & \dots & t^n & x(1 & t & \dots & t^n) & \dots & x^n(1 & t & \dots & t^n) \end{pmatrix}.$$

Thus, we have,

$$\mathcal{M}^T(t)\mathcal{M}^T(x) = \begin{pmatrix} \mathcal{M}(t) & x\mathcal{M}(t) & x^2\mathcal{M}(t) & \dots & x^n\mathcal{M}(t) \end{pmatrix}, \tag{4.5}$$

Taking the transpose on both sides,

$$\begin{aligned}
\left(\mathcal{M}^T(t)\mathcal{M}^T(x)\right)^T &= \begin{pmatrix} \mathcal{M}(t) & x\mathcal{M}(t) & x^2\mathcal{M}(t) & \dots & x^n\mathcal{M}(t) \end{pmatrix}^T \\
\mathcal{M}(x)\mathcal{M}(t) &= \begin{pmatrix} \mathcal{M}(t) & x\mathcal{M}(t) & x^2\mathcal{M}(t) & \dots & x^n\mathcal{M}(t) \end{pmatrix}^T \\
\mathcal{M}(t)\mathcal{M}(x) &= \begin{pmatrix} \mathcal{M}(t) & x\mathcal{M}(t) & x^2\mathcal{M}(t) & \dots & x^n\mathcal{M}(t) \end{pmatrix}^T.
\end{aligned}$$

Thus, we can write,

$$\mathcal{M}(x,t) = \mathcal{M}(t)\mathcal{M}(x) = \begin{pmatrix} \mathcal{M}(t) & x\mathcal{M}(t) & x^2\mathcal{M}(t) & \dots & x^n\mathcal{M}(t) \end{pmatrix}^T.$$

□

It is possible use the previous theorem to write (4.3) in another form with different elements. We state this in the following corollary.

Corollary 4.2.1. *We can also write $\mathcal{M}(x,t)$ as,*

$$\mathcal{M}(x,t) = \begin{pmatrix} \mathcal{M}(x) & t\mathcal{M}(x) & t^2\mathcal{M}(x) & \dots & t^n\mathcal{M}(x) \end{pmatrix}^T. \tag{4.6}$$

We now want to discuss how to replace the derivative and integral operators acting on (4.3) and (4.6) with matrices.

In the next theorem, we show how to replace a derivative operator acting on (4.3) with a matrix.

Theorem 4.2.2. *The composite derivative operational matrix $\mathbf{D}_x$ that acts upon* (4.3) *can be written as,*

$$\mathbf{D}_x = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathtt{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathtt{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & n\mathtt{I} & \mathbf{0} \end{pmatrix}, \tag{4.7}$$

where $\mathtt{I} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ is an identity matrix and $\mathbf{0}$ represents a zero matrix whose dimensions are the same as those of $\mathtt{I}$.

Proof.

$$\begin{aligned}
\partial_x \mathcal{M}(x,t) &= \begin{pmatrix} 0 & \mathcal{M}(t) & 2x\mathcal{M}(t) & \dots & nx^{n-1}\mathcal{M}(t) \end{pmatrix}^T \\
&= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathtt{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathtt{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & n\mathtt{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathcal{M}(t) \\ x\mathcal{M}(t) \\ x^2\mathcal{M}(t) \\ \vdots \\ x^n\mathcal{M}(t) \end{pmatrix} \\
&= \mathbf{D}_x \mathcal{M}(x,t).
\end{aligned}$$

□

Adopting the same concept, we can deduce the second order composite derivative matrix.

Corollary 4.2.2.

$$\mathbf{D}_{xx} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2\mathtt{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 6\mathtt{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 12\mathtt{I} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & n(n-1)\mathtt{I} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{4.8}$$

The same concept of the operational matrices in the integer order can be extended to encompass operational matrices that represent the fractional derivatives. The next theorem serves to illustrate this concept. We will particularly dwell on the time variable, but the same result is applicable on the spatial variable.

Theorem 4.2.3. *The composite operational matrix* $\mathbf{D}_t^{\mu}$ *that acts upon* (4.3) *can be written* *as,*

$$\mathbf{D}_t^{\mu} = \begin{pmatrix} \mathcal{D}_t^{\mu} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{D}_t^{\mu} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{D}_t^{\mu} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{D}_t^{\mu} \end{pmatrix}, \tag{4.9}$$

where $\mathcal{D}_t^{\mu}$ *is the operational matrix given in* (3.24) *and* $\mathbf{0}$ *is the zero matrix whose dimensions are similar to those of* $\mathcal{D}_t^{\mu}$ *.*

Proof.

$$\mathcal{D}_t^{\mu}\mathcal{M}(x,t) = \Big(\mathcal{D}_t^{\mu}\mathcal{M}(t) \quad x\mathcal{D}_t^{\mu}\mathcal{M}(t) \quad x^2\mathcal{D}_t^{\mu}\mathcal{M}(t) \quad \dots \quad x^n\mathcal{D}_t^{\mu}\mathcal{M}(t)\Big)^T, \tag{4.10}$$

Applying Theorem 3.3.3 on (4.10),

$$\begin{aligned} \mathcal{D}_t^{\mu}\mathcal{M}(x,t) &= \left(\begin{pmatrix} 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} \\ \frac{t^{2-\mu}}{\Gamma[3-\mu]} \\ \vdots \\ \frac{\Gamma[n]t^{n-\mu}}{\Gamma[n+1-\mu]} \end{pmatrix} \; x\begin{pmatrix} 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} \\ \frac{t^{2-\mu}}{\Gamma[3-\mu]} \\ \vdots \\ \frac{\Gamma[n]t^{n-\mu}}{\Gamma[n+1-\mu]} \end{pmatrix} \; x^2\begin{pmatrix} 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} \\ \frac{t^{2-\mu}}{\Gamma[3-\mu]} \\ \vdots \\ \frac{\Gamma[n]t^{n-\mu}}{\Gamma[n+1-\mu]} \end{pmatrix} \dots x^n\begin{pmatrix} 0 \\ \frac{t^{1-\mu}}{\Gamma[2-\mu]} \\ \frac{t^{2-\mu}}{\Gamma[3-\mu]} \\ \vdots \\ \frac{\Gamma[n]t^{n-\mu}}{\Gamma[n+1-\mu]} \end{pmatrix}\right)^T \\ &= \begin{pmatrix} \mathcal{D}_t^{\mu} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{D}_t^{\mu} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{D}_t^{\mu} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{D}_t^{\mu} \end{pmatrix} \begin{pmatrix} \mathcal{M}(t) \\ x\mathcal{M}(t) \\ x^2\mathcal{M}(t) \\ \vdots \\ x^n\mathcal{M}(t) \end{pmatrix} \\ &= \mathbf{D}_t^{\mu}\mathcal{M}(x,t). \end{aligned} \tag{4.11}$$

□

The composite matrices $\mathbf{D}_x$, $\mathbf{D}_{xx}$ and $\mathbf{D}_t^{\mu}$ are not unique, the entries in these matrices are dependent on the arrangement of entries in the matrix $\mathcal{M}(x,t)$.

Next, we demonstrate how to write $\mathbf{D}_x$, $\mathbf{D}_{xx}$ and $\mathbf{D}_t^{\mu}$ in different formats.

Theorem 4.2.4. *The composite matrix $\mathbf{D}_x$ that acts upon $\mathcal{M}(x,t)$ can be written as,*

$$\mathbf{D}_x = \begin{pmatrix} \mathcal{D}_x & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{D}_x & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{D}_x & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{D}_x \end{pmatrix}, \tag{4.12}$$

where $\mathcal{D}_x$ is the matrix defined in (3.22) and $\mathbf{0}$ is the zero matrix with the same dimensions as $\mathcal{D}_x$.

Proof. Rewriting $\mathcal{M}(x,t)$ as,
$\mathcal{M}(x,t) = \begin{pmatrix} \mathcal{M}(x) & t\mathcal{M}(x) & t^2\mathcal{M}(x) & \dots & t^n\mathcal{M}(x) \end{pmatrix}^T.$

Performing the first derivative of $\mathcal{M}(x,t)$ on x,

$$\begin{aligned}
\partial_x \mathcal{M}(x,t) &= \begin{pmatrix} \mathcal{M}'(x) & t\mathcal{M}'(x) & t^2\mathcal{M}'(x) & \dots & t^n\mathcal{M}'(x) \end{pmatrix}^T \\
&= \left(\begin{pmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ nx^{n-1} \end{pmatrix} \; t\begin{pmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ nx^{n-1} \end{pmatrix} \; t^2\begin{pmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ nx^{n-1} \end{pmatrix} \; \dots \; t^n\begin{pmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ nx^{n-1} \end{pmatrix} \right)^T \\
&= \begin{pmatrix} \mathcal{D}_x & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{D}_x & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{D}_x & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{D}_x \end{pmatrix} \begin{pmatrix} \mathcal{M}(x) \\ t\mathcal{M}(x) \\ t^2\mathcal{M}(x) \\ \dots \\ t^n\mathcal{M}(x) \end{pmatrix} \\
&= \mathbf{D}_x \mathcal{M}(x,t).
\end{aligned} \tag{4.13}$$

□

By the same principle, we can deduce the matrix $\mathbf{D}_{xx}$.

Corollary 4.2.3.

$$\mathbf{D}_{xx} = \begin{pmatrix} \mathcal{D}_{xx} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{D}_{xx} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{D}_{xx} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{D}_{xx} \end{pmatrix} \tag{4.14}$$

The next theorem explains how we can write the matrix $\mathbf{D}_t^{\mu}$ with components different from the ones in (4.9).

Theorem 4.2.5. *The composite derivative matrix* $\mathbf{D}_t^{\mu}$ *that acts upon* $\mathcal{M}(x,t)$ *can also be written as,*

$$\mathbf{D}_t^{\mu} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \frac{\Gamma[2]t^{-\mu}}{\Gamma[2-\mu]}\mathtt{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{\Gamma[3]t^{-\mu}}{\Gamma[3-\mu]}\mathtt{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \frac{\Gamma[n+1]t^{-\mu}}{\Gamma[n+1-\mu]}\mathtt{I} \end{pmatrix}, \tag{4.15}$$

where $\mathtt{I} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$ *is the identity matrix. The subscript t in* $\mathbf{D}_t^{\mu}$ *emphasizes that we are taking the derivative with respect to t.*

Proof. Taking $\mathcal{M}(x,t) = \begin{pmatrix} \mathcal{M}(x) & t\mathcal{M}(x) & t^2\mathcal{M}(x) & \dots & t^n\mathcal{M}(x) \end{pmatrix}^T$, then,

$$\begin{aligned}
\mathcal{D}_t^{\mu}\mathcal{M}(x,t) &= \begin{pmatrix} \mathcal{D}_t^{\mu}\mathcal{M}(x) & \mathcal{D}_t^{\mu}t\mathcal{M}(x) & \mathcal{D}_t^{\mu}t^2\mathcal{M}(x) & \dots & \mathcal{D}_t^{\mu}t^n\mathcal{M}(x) \end{pmatrix}^T \\
&= \begin{pmatrix} \mathbf{0} & \frac{t^{1-\mu}}{\Gamma[2-\mu]}\mathcal{M}(x) & \frac{2t^{2-\mu}}{\Gamma[3-\mu]}\mathcal{M}(x) & \dots & \frac{\Gamma[n+1]t^{n-\mu}}{\Gamma[n+1-\mu]}\mathcal{M}(x) \end{pmatrix}^T \\
&= \begin{pmatrix} \mathbf{0} & \frac{\Gamma[2]t^{-\mu}}{\Gamma[2-\mu]}t\mathcal{M}(x) & \frac{\Gamma[3]t^{-\mu}}{\Gamma[3-\mu]}t^2\mathcal{M}(x) & \dots & \frac{\Gamma[n+1]t^{-\mu}}{\Gamma[n+1-\mu]}t^n\mathcal{M}(x) \end{pmatrix}^T \\
&= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \frac{\Gamma[2]t^{-\mu}}{\Gamma[2-\mu]}\mathtt{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{\Gamma[3]t^{-\mu}}{\Gamma[3-\mu]}\mathtt{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \frac{\Gamma[n+1]t^{-\mu}}{\Gamma[n+1-\mu]}\mathtt{I} \end{pmatrix} \begin{pmatrix} \mathcal{M}(x) \\ t\mathcal{M}(x) \\ t^2\mathcal{M}(x) \\ \vdots \\ t^n\mathcal{M}(x) \end{pmatrix} \\
&= \mathbf{D}_t^{\mu}\mathcal{M}(x,t). \qquad (4.16)
\end{aligned}$$

□

The composite matrices are also applicable to integral operators, the next two theorems describe how to go about deducing these matrices.

Theorem 4.2.6. *The composite integral operational matrix $\mathbf{I}_t^{\mu}$ that acts upon $\mathcal{M}(x,t)$ can be written as,*

$$\mathbf{I}_t^{\mu} = \begin{pmatrix} \mathcal{I}^{\mu} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{I}^{\mu} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{I}^{\mu} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{I}^{\mu} \end{pmatrix}, \qquad (4.17)$$

$\mathcal{I}^{\mu}$ is the fractional integral operational matrix (3.53) *that we discussed in chapter 3 and* $\mathbf{0}$ *is the zero matrix whose dimensions are the same as those of $\mathcal{I}^{\mu}$. The subscript t in $\mathbf{I}_t^{\mu}$ emphasises that we are integrating with respect to the variable t.*

Proof. We first evaluate the fractional integral of $\mathcal{M}(t)$,

$$\frac{1}{\Gamma[\mu]}\int_0^t \frac{\mathcal{M}(s)}{(t-s)^{1-\mu}}ds = \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} & \frac{t^{\mu+1}}{\Gamma[\mu+2]} & \frac{2t^{\mu+2}}{\Gamma[\mu+3]} & \dots & \frac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]} \end{pmatrix}^T. \tag{4.18}$$

We let $\lambda(t) = \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]} & \frac{t^{\mu+1}}{\Gamma[\mu+2]} & \frac{2t^{\mu+2}}{\Gamma[\mu+3]} & \dots & \frac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]} \end{pmatrix}^T$ for our convenience.

Computing the fractional integral of $\mathcal{M}(x,t) = \begin{pmatrix} \mathcal{M}(t) & x\mathcal{M}(t) & x^2\mathcal{M}(t)\dots x^n\mathcal{M}(t) \end{pmatrix}^T$,

$$\begin{aligned}
\frac{1}{\Gamma[\mu]}\int_0^t \frac{\mathcal{M}(x,s)}{(t-s)^{1-\mu}}ds &= \begin{pmatrix} \lambda(t) & x\lambda(t) & x^2\lambda(t) & \dots & x^n\lambda(t) \end{pmatrix}^T \\
&= \begin{pmatrix} \mathcal{I}^{\mu} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{I}^{\mu} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{I}^{\mu} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{I}^{\mu} \end{pmatrix} \begin{pmatrix} \mathcal{M}(t) \\ x\mathcal{M}(t) \\ x^2\mathcal{M}(t) \\ \dots \\ x^n\mathcal{M}(t) \end{pmatrix} \\
&= \mathbf{I}_t^{\mu}\mathcal{M}(x,t).
\end{aligned} \tag{4.19}$$

□

As we have seen with the composite derivative matrix, the integral composite matrix $\mathbf{I}_t^{\mu}$ is also not unique. In the next theorem, we show how this is possible.

Theorem 4.2.7. *The composite integral operational matrix $\mathbf{I}_t^{\mu}$ that acts upon $\mathcal{M}(x,t)$ can be presented in the form,*

$$\mathbf{I}_t^{\mu} = \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]}\mathtt{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \frac{t^{\mu}}{\Gamma[\mu+2]}\mathtt{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2t^{\mu}}{\Gamma[\mu+3]}\mathtt{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \frac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]}\mathtt{I} \end{pmatrix}, \tag{4.20}$$

where $\mathtt{I} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$ *is the identity matrix.*

Proof. We evaluate the fractional integral of $\mathcal{M}(x,t)$,

$$\begin{aligned}
\frac{1}{\Gamma[\mu]}\int_0^t \frac{\mathcal{M}(x,s)}{(t-s)^{1-\mu}}ds &= \begin{pmatrix} \frac{t^{\mu}\mathcal{M}(x)}{\Gamma[\mu+1]} & \frac{t^{\mu+1}\mathcal{M}(x)}{\Gamma[\mu+2]} & \frac{2t^{\mu+2}\mathcal{M}(x)}{\Gamma[\mu+3]} & \dots & \frac{\Gamma[n+1]t^{\mu+n}\mathcal{M}(x)}{\Gamma[\mu+n+1]} \end{pmatrix}^T \\
&= \begin{pmatrix} \frac{t^{\mu}\mathcal{M}(x)}{\Gamma[\mu+1]} & \frac{t^{\mu}t\mathcal{M}(x)}{\Gamma[\mu+2]} & \frac{2t^{\mu}t^{2}\mathcal{M}(x)}{\Gamma[\mu+3]} & \dots & \frac{\Gamma[n+1]t^{\mu}t^{n}\mathcal{M}(x)}{\Gamma[\mu+n+1]} \end{pmatrix}^T \\
&= \begin{pmatrix} \frac{t^{\mu}}{\Gamma[\mu+1]}\mathtt{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \frac{t^{\mu}}{\Gamma[\mu+2]}\mathtt{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2t^{\mu}}{\Gamma[\mu+3]}\mathtt{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \frac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]}\mathtt{I} \end{pmatrix} \begin{pmatrix} \mathcal{M}(x) \\ t\mathcal{M}(x) \\ t^2\mathcal{M}(x) \\ \dots \\ t^n\mathcal{M}(x) \end{pmatrix} \\
&= \mathbf{I}_t^{\mu}\mathcal{M}(x,t). \qquad (4.21)
\end{aligned}$$

□

In the next theorem, we show how to apply the associativity law of matrix multiplication to the composite matrices $\mathbf{D}_x$, $\mathbf{D}_{xx}$ and $\mathbf{I}_t^{\mu}$.

Theorem 4.2.8. *According the associativity law of matrix multiplication,*

$$\begin{aligned}
\mathbf{I}_t^{\mu}\Big(\mathbf{D}_x\mathcal{M}(x,t)\Big) &= \Big(\mathbf{I}_t^{\mu}\mathbf{D}_x\Big)\mathcal{M}(x,t) \\
&= \mathbf{C}_{tx}\mathcal{M}(x,t). \qquad (4.22)
\end{aligned}$$

Proof. We have,

$$\begin{aligned}
\mathcal{M}(x,t) &= \Big(\mathcal{M}(t) \quad x\mathcal{M}(t) \quad x^2\mathcal{M}(t) \quad \dots \quad x^n\mathcal{M}(t)\Big)^T, \\
\mathbf{D}_x\mathcal{M}(x,t) &= \Big(\mathbf{0} \quad \mathcal{M}(t) \quad 2x\mathcal{M}(t) \quad \dots \quad nx^{n-1}\mathcal{M}(t)\Big)^T, \\
\mathbf{I}_t^{\mu}\Big(\mathbf{D}_x\mathcal{M}(x,t)\Big) &= \Big(\mathbf{0} \quad \lambda(t) \quad 2x\lambda(t) \quad \dots \quad nx^{n-1}\lambda(t)\Big)^T.
\end{aligned}$$

Considering the right hand side,

$$\begin{aligned}
\mathbf{I}_t^{\mu}\mathbf{D}_x &= \begin{pmatrix} \mathcal{I}^{\mu} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{I}^{\mu} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{I}^{\mu} & \dots & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{I}^{\mu} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathtt{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathtt{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & n\mathtt{I} & \mathbf{0} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathcal{I}^{\mu}\mathtt{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathcal{I}^{\mu}\mathtt{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & n\mathcal{I}^{\mu}\mathtt{I} & \mathbf{0} \end{pmatrix} \\
&= \mathbf{C}_{tx}. \qquad (4.23)
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbf{C}_{tx}\mathcal{M}(x,t) &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathcal{I}^{\mu}\mathtt{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathcal{I}^{\mu}\mathtt{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & n\mathcal{I}^{\mu}\mathtt{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathcal{M}(t) \\ x\mathcal{M}(t) \\ x^2\mathcal{M}(t) \\ \dots \\ x^n\mathcal{M}(t) \end{pmatrix} \\
&= \Big(0 \quad \lambda(t) \quad 2x\lambda(t) \quad \dots \quad nx^{n-1}\lambda(t)\Big)^T, \qquad (4.24)
\end{aligned}$$

thus the proof is complete. □

In the next theorem, we only change the elements of $\mathcal{M}(x,t)$, and show that we achieve the same result as in the previous theorem.

Theorem 4.2.9. *If we take* $\mathcal{M}(x,t) = \Big(\mathcal{M}(x) \quad t\mathcal{M}(x) \quad t^2\mathcal{M}(x,t) \quad \dots \quad t^n\mathcal{M}(x)\Big)$*, we can achieve the result proven in the previous theorem.*

Proof. We have,

$$\begin{aligned}
\mathbf{D}_x\mathcal{M}(x,t) &= \Big(\mathcal{M}'(x) \quad t\mathcal{M}'(x) \quad t^2\mathcal{M}'(x) \quad \dots \quad t^n\mathcal{M}'(x)\Big)^T, \\
\mathbf{I}_t^{\mu}\Big(\mathbf{D}_x\mathcal{M}(x,t)\Big) &= \Big(\tfrac{t^{\mu}}{\Gamma[\mu+1]}\mathcal{M}'(x) \quad \tfrac{t^{\mu+1}}{\Gamma[\mu+2]}\mathcal{M}'(x) \quad \tfrac{2t^{\mu+2}}{\Gamma[\mu+3]}\mathcal{M}'(x) \quad \dots \quad \tfrac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]}\mathcal{M}'(x)\Big)^T.
\end{aligned}$$

Considering the right hand side,

$$\begin{aligned}
\mathbf{I}_t^{\mu}\mathbf{D}_x &= \begin{pmatrix}
\frac{t^{\mu}}{\Gamma[\mu+1]}\mathtt{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \frac{t^{\mu}}{\Gamma[\mu+2]}\mathtt{I} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \frac{2t^{\mu}}{\Gamma[\mu+3]}\mathtt{I} & \dots & \mathbf{0} \\
\vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \frac{\Gamma[n+1]t^{\mu}}{\Gamma[\mu+n+1]}\mathtt{I}
\end{pmatrix}
\begin{pmatrix}
\mathcal{D}_x & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathcal{D}_x & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathcal{D}_x & \dots & \mathbf{0} \\
\vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathcal{D}_x
\end{pmatrix} \\
&= \begin{pmatrix}
\frac{t^{\mu}}{\Gamma[\mu+1]}\mathtt{I}\mathcal{D}_x & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \frac{t^{\mu}}{\Gamma[\mu+2]}\mathtt{I}\mathcal{D}_x & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \frac{2t^{\mu}}{\Gamma[\mu+3]}\mathtt{I}\mathcal{D}_x & \dots & \mathbf{0} \\
\vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \frac{\Gamma[n+1]t^{\mu}}{\Gamma[\mu+n+1]}\mathtt{I}\mathcal{D}_x
\end{pmatrix} \\
&= \mathbf{C}_{tx}. \qquad (4.25)
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbf{C}_{tx}\mathcal{M}(x,t) &= \begin{pmatrix}
\frac{t^{\mu}}{\Gamma[\mu+1]}\mathtt{I}\mathcal{D}_x & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \frac{t^{\mu}}{\Gamma[\mu+2]}\mathtt{I}\mathcal{D}_x & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \frac{2t^{\mu}}{\Gamma[\mu+3]}\mathtt{I}\mathcal{D}_x & \dots & \mathbf{0} \\
\vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \frac{\Gamma[n+1]t^{\mu}}{\Gamma[\mu+n+1]}\mathtt{I}\mathcal{D}_x
\end{pmatrix}
\begin{pmatrix}
\mathcal{M}(x) \\ t\mathcal{M}(x) \\ t^2\mathcal{M}(x) \\ \dots \\ t^n\mathcal{M}(x)
\end{pmatrix} \\
&= \Big(\tfrac{t^{\mu}}{\Gamma[\mu+1]}\mathcal{M}'(x) \quad \tfrac{t^{\mu+1}}{\Gamma[\mu+2]}\mathcal{M}'(x) \quad \tfrac{2t^{\mu+2}}{\Gamma[\mu+3]}\mathcal{M}'(x) \quad \dots \quad \tfrac{\Gamma[n+1]t^{\mu+n}}{\Gamma[\mu+n+1]}\mathcal{M}'(x)\Big)^T \qquad (4.26)
\end{aligned}$$

thus the proof is complete. □

Corollary 4.2.4. *By the same token,*

$$\begin{aligned} \mathbf{I}_t^{\mu}\Big(\mathbf{D}_{xx}\mathcal{M}(x,t)\Big) &= \Big(\mathbf{I}_t^{\mu}\mathbf{D}_{xx}\Big)\mathcal{M}(x,t) \\ &= \mathbf{C}_{txx}\mathcal{M}(x,t). \end{aligned} \tag{4.27}$$

The previous two theorems were concerned about the associativity law of matrix multiplication. The important conclusion we deduce from this result is that we can combine two matrices, the composite integral and composite derivative into a single operational matrix. We can then use this single matrix to achieve the same goal as applying the integral and derivative matrices separately.

Another important observation that we deduce from the previous two theorems is commutativity in matrix multiplication. In general, multiplication of matrices is non commutative, although there are some exceptions. In our case, from the previous theorem, matrix multiplication is commutative. This is due to the presence of identity and zero matrices. The next corollary summarises this important observation.

Corollary 4.2.5.

$$\mathbf{I}_t^{\mu}\mathbf{D}_x = \mathbf{D}_x\mathbf{I}_t^{\mu},$$

and

$$\mathbf{I}_t^{\mu}\mathbf{D}_{xx} = \mathbf{D}_{xx}\mathbf{I}_t^{\mu}.$$

In the context of derivative and integral operators, which is our case here, there is a simpler explanation that we can offer on why matrix multiplication is commutative. It does not matter whether one starts with integration or differentiation of a function, the end result will be the same. We have to emphasize that this observation is based on two variables, with one variable to be differentiated and the other to be integrated. Therefore, it will be a whole different matter when one considers a single variable.

4.3 Methodology

4.3.1 Description of the numerical method

We now use the concepts developed in the preceding section to construct a numerical method. We will demonstrate how to apply this numerical technique to approximate the solution of the IBVPs. Thereafter, we discuss theoretical aspects of convergence.

Our main intent is to approximate solution of the IBVP,

$$\mathcal{D}_t^{\mu} y = y_{xx}, \quad \mu \in (p-1, p], \quad p \in \mathbb{N}, \quad (x,t) \in [0,1] \times [0,1], \tag{4.28}$$

$$y^{(q)}(x,0) = f(x), \quad q = 0, 1, ..., p-1, \tag{4.29}$$

$$y(0,t) = \beta(t), \tag{4.30}$$

$$y(1,t) = \rho(t), \tag{4.31}$$

with $y = y(x,t)$.

We write approximate solution of (4.28)–(4.31) as a product,

$$y(x,t) \approx \gamma^T \mathcal{M}(x,t), \tag{4.32}$$

where γ^T is given as,

$$\gamma^T = \begin{pmatrix} \gamma_{00} & \gamma_{10} & \dots & \gamma_{n0} & \gamma_{01} & \gamma_{11} & \dots & \gamma_{n1} & \dots & \gamma_{0n} & \gamma_{1n} & \dots & \gamma_{nn} \end{pmatrix}$$

and $\mathcal{M}(x,t)$ is given in (4.6).

The main task in hand is to calculate the numerical values of γ in (4.32), we intend to achieve this goal in two steps. Firstly, we utilise the initial and boundary conditions and thereafter, we make use of the Garlekin technique.

We write both the initial and boundary conditions in Taylor series form expanded about the point $x = 0$ and $t = 0$,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \epsilon_n, \tag{4.33}$$

$$\beta(t) = \beta(0) + t\beta'(0) + \frac{t^2}{2!}\beta''(0) + \cdots + \frac{t^n}{n!}\beta^{(n)}(0) + \varepsilon_n, \tag{4.34}$$

and

$$\rho(t) = \rho(0) + t\rho'(0) + \frac{t^2}{2!}\rho''(0) + \cdots + \frac{t^n}{n!}\rho^{(n)}(0) + \varrho_n, \tag{4.35}$$

where ϵ_n, ϱ_n and ε_n denote the truncation errors.

Imposing the initial condition on the approximate solution (4.32),

$$\begin{aligned} \gamma^T \mathcal{M}(x,0) &= \gamma^T \mathcal{M}(x) \\ &= \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \cdots + \gamma_{n0}x^n \\ &= f(x). \end{aligned} \tag{4.36}$$

Equating the coefficients of (4.33) without the error term with those of (4.36),

$$f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) = \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \cdots + \gamma_{n0}x^n, \tag{4.37}$$

gives us, $\gamma_{00} = f(0)$, $\gamma_{10} = f'(0)$, $\gamma_{20} = \frac{f''(0)}{2!}$ and $\gamma_{n0} = \frac{f^{(n)}(0)}{n!}$.

Similarly, imposing the boundary condition on (4.32),

$$\begin{aligned} \gamma^T \mathcal{M}(0,t) &= \gamma^T \mathcal{M}(t) \\ &= \gamma_{00} + \gamma_{01}t + \gamma_{02}t^2 + \cdots + \gamma_{0n}t^n \\ &= \beta(t). \end{aligned} \tag{4.38}$$

Equating the coefficients of (4.34) without the error term with those of (4.38),

$$\beta(0) + t\beta'(0) + \frac{t^2}{2!}\beta''(0) + \cdots + \frac{t^n}{n!}\beta^{(n)}(0) = \gamma_{00} + \gamma_{01}t + \gamma_{02}t^2 + \cdots + \gamma_{0n}t^n, \tag{4.39}$$

gives us, $\gamma_{00} = \beta(0)$, $\gamma_{01} = \beta'(0)$, $\gamma_{02} = \frac{\beta''(0)}{2!}$ and $\gamma_{0n} = \frac{\beta^{(n)}(0)}{n!}$.

Also,

$$
\begin{aligned}
\gamma^T \mathcal{M}(1,t) &= \gamma^T \mathcal{M}(t) \\
&+ \gamma_{00} + \gamma_{10} + \gamma_{20} + \cdots + \gamma_{n0} \\
&+ \gamma_{01}t + \gamma_{11}t + \gamma_{21}t + \cdots + \gamma_{n1}t \\
&+ \gamma_{02}t^2 + \gamma_{12}t^2 + \gamma_{22}t^2 + \cdots + \gamma_{n2}t^2 \\
&+ \qquad\qquad \vdots \\
&+ \gamma_{0n}t^n + \gamma_{1n}t^n + \gamma_{2n}t^n + \cdots + \gamma_{nn}t^n \\
&= \rho(t).
\end{aligned} \tag{4.40}
$$

Equating the coefficients of (4.35) without the error term with those of (4.40), we get the equations,

$$
\begin{aligned}
\rho(0) &= \gamma_{00} + \gamma_{10} + \gamma_{20} + \cdots + \gamma_{n0} \\
\rho'(0) &= \gamma_{01} + \gamma_{11} + \gamma_{21} + \cdots + \gamma_{n1} \\
\frac{1}{2}\rho''(0) &= \gamma_{02} + \gamma_{12} + \gamma_{22} + \cdots + \gamma_{n2} \\
\vdots &= \qquad\qquad \vdots \\
\frac{1}{n!}\rho^{(n)}(0) &= \gamma_{0n} + \gamma_{1n} + \gamma_{2n} + \cdots + \gamma_{nn}
\end{aligned} \tag{4.41}
$$

We have completed the first stage of computing the numerical values of γ, we point out an essential result from this. We realise that from the initial condition $\gamma_{00} = f(0)$, whilst from the boundary condition, (4.30), $\gamma_{00} = \beta_0$. What this communicates is that for this technique to work, the numerical values of $f(0)$ and β_0 should be equal.

The first stage of computing the numerical values of γ does not give all the values that we need. To get the rest of the numerical values of γ, we move to the next stage where we use the Garlekin method.
Replacing $\mathcal{D}_t^{\mu}$ with the composite matrix $\mathbf{D}_t^{\mu}$ and introducing the composite matrix $\mathbf{D}_{xx}$ on the right hand side of (4.28) gives,

$$
\mathbf{D}_t^{\mu} y(x,t) = \mathbf{D}_{xx} y(x,t). \tag{4.42}
$$

Substituting for $y(x,t)$ in (4.42) using (4.32),

$$\begin{aligned}\mathbf{D}_t^{\mu} y(x,t) &= \mathbf{D}_{xx} y(x,t)\\ \mathbf{D}_t^{\mu}\Big(\gamma^T \mathcal{M}(x,t)\Big) &= \mathbf{D}_{xx}\Big(\gamma^T \mathcal{M}(x,t)\Big)\\ \gamma^T \mathbf{D}_t^{\mu} \mathcal{M}(x,t) &= \gamma^T \mathbf{D}_{xx} \mathcal{M}(x,t). \end{aligned} \tag{4.43}$$

We then deduce the residual $\mathcal{R}(x,t)$ from (4.43),

$$\mathcal{R}(x,t) = \gamma^T \mathbf{D}_t^{\mu} \mathcal{M}(x,t) - \gamma^T \mathbf{D}_{xx} \mathcal{M}(x,t). \tag{4.44}$$

Applying the Garlekin technique using (4.44) and polynomials from $\mathcal{M}(x,t)$, we create equations,

$$\begin{aligned}\int_0^1\int_0^1 \mathcal{R}(x,t)t\mathcal{M}(x)dxdt &= 0\\ \int_0^1\int_0^1 \mathcal{R}(x,t)t^2\mathcal{M}(x)dxdt &= 0\\ \vdots &= \vdots\\ \int_0^1\int_0^1 \mathcal{R}(x,t)t^n\mathcal{M}(x)dxdt &= 0. \end{aligned} \tag{4.45}$$

There are a couple of important points that we need to emphasize about the equations created in (4.45). Firstly, we get n equations from each equation, for example, $\int_0^1\int_0^1 \mathcal{R}(x,t)t\mathcal{M}(x)dxdt = 0$, will generate equations,

$\int_0^1\int_0^1 \mathcal{R}(x,t)xtdxdt = 0$, $\quad \int_0^1\int_0^1 \mathcal{R}(x,t)x^2tdxdt = 0$, $\quad \dots \quad$, $\int_0^1\int_0^1 \mathcal{R}(x,t)x^ntdxdt = 0$.

Secondly, the number of equations generated from (4.45) are more than enough to supplement the equations generated from (4.41). In this research, we are going to use the first equations starting from the top in (4.45).

If we use the composite differential and integral matrices, we first have to write (4.28) as an integral equation. We do this by taking the fractional integral (3.4) on both sides of

(4.28)

$$y(x,t) - \sum_{q=0}^{p-1} y^{(q)}(x,0) = \mathbf{I}_t^{\mu} y_{xx}. \tag{4.46}$$

We then substitute (4.32) in (4.46) and introduce the differential composite matrix $\mathbf{D}_{xx}$,

$$\gamma^T \mathcal{M}(x,t) - \sum_{q=0}^{p-1} \gamma^T \mathcal{M}^{(q)}(x,0) = \mathbf{I}_t^{\mu} \mathbf{D}_{xx}\Big(\gamma^T \mathcal{M}(x,t)\Big)$$

$$\gamma^T \mathcal{M}(x,t) - \sum_{q=0}^{p-1} \gamma^T \mathcal{M}^{(q)}(x,0) = \gamma^T \Big(\mathbf{I}_t^{\mu} \mathbf{D}_{xx}\Big) \mathcal{M}(x,t). \tag{4.47}$$

Applying (4.27) and substituting for the initial condition in (4.47),

$$\gamma^T \mathcal{M}(x,t) - f(x) = \gamma^T \mathbf{C}_{txx} \mathcal{M}(x,t). \tag{4.48}$$

Note that in (4.48), we can choose to use $f(x)$ in either its original form or in its series form. Since $\mathcal{M}(x,t)$ is written as polynomials, then we will prefer to use $f(x)$ in the series form.
We deduce the residual function from (4.48) as,

$$R(x,t) = \gamma^T \mathcal{M}(x,t) - f(x) - \gamma^T \mathbf{C}_{txx} \mathcal{M}(x,t). \tag{4.49}$$

We can then use the residual $R(x,t)$ from (4.49) to substitute in place of $\mathcal{R}(x,t)$ in (4.45) to create equations to supplement those generated from (4.41).

Note that the residuals from (4.44) and (4.49) are not the same, therefore, we expect different results from using them.

4.3.2 Convergence and rates of convergence

In using (4.32) to approximate (4.28)-(4.31), there is no rule that specifies the value of n, this value of n stipulates how many polynomials are to be used for approximation. In as far as the aforementioned numerical approach is concerned, any number of polynomials that yield a good approximation is acceptable. Since there will be a different approximate solution for each value of n, then we need to investigate the behaviour of the solution as we change the value of n. One way of doing this is to find out if the approximate solution does converge as we increase the number of polynomials used. We will briefly discuss a concept that guarantee convergence of the approximate solution.

We will use the convergence concepts developed in the previous chapter, the only difference is that we are now dealing with two independent variables instead of one. The following definition forms the basis for the theorem that guarantees convergence.

Definition 4.3.1. *We define the function, $\upsilon_n(x,t)$, the difference between two consecutive functions from* (4.32) *as,*

$$\upsilon_n(x,t) = y_{n+1}(x,t) - y_n(x,t), \quad n = 1, 2, \dots \tag{4.50}$$

The next theorem guarantees convergence of the different solution generated from (4.32), we state this theorem without proof. We write $\upsilon_n(x,t) = \upsilon_n$ for simplicity.

Theorem 4.3.1. *The sequence, $\upsilon_n, \upsilon_{n+1}, \upsilon_{n+2}, \dots$, will converge whenever [26],*
$\| \upsilon_n \| \geq \| \upsilon_{n+1} \| \geq \| \upsilon_{n+2} \| \geq \dots$

The norms in the above theorem are computed over the interval [0, 1] as,

$$\| \upsilon_n \| = \sqrt{\int_0^1 \int_0^1 | \upsilon_n(x,t) |^2 \, dxdt}. \tag{4.51}$$

As we discussed the rate of convergence in the previous chapter for a one independent variable scenario, the implications are the same for a two independent variable situation. The smaller the values of $\| \upsilon_n \|$, then the faster the rate of convergence of a numerical method.

4.4 Conclusion

Besides the success we had in constructing the composite matrices, there is so much that we learnt about these matrices. The idea of the commutativity law in matrix multiplication is satisfied in a way that can be easily explained in the context of integration and multiplication. The product of composite matrices using the commutativity law shows us that it makes no difference which one comes first, integrating or differentiating a function, the result is the same. Also, the product of composite integral and derivative operational matrices to get one composite matrix means that one is now dealing with a single matrix that performs two functions. This is certainly an advantage that minimises the number of computations. The methodology used in the application of the composite operational matrices together with the Garlekin method in solving the PDEs is discussed. Thereafter, the theory related to analysing the convergence of the results from this solution procedure is explained. The methodology described in this chapter is put to test in the next chapter. Using this methodology, we attempt to approximate the solutions of the heat and wave equations.

Chapter 5

Applications

Abstract

The purpose of this chapter is to test the theoretical concepts that we developed in the previous chapter. We choose the heat and wave equations as our case studies. For the heat equation, we compare our practical results against the analytic solutions. Then, for the wave equation, we compare our results, firstly, against the analytic solutions and secondly, against the results from Legendre operational matrix where the Collocation technique was used as a numerical method. We also test for convergence of the results to ascertain the reliability of our methodology.

5.1 Introduction

Our main intention in this chapter is to put the theoretical concepts we developed in the previous chapter into practice. We apply our suggested numerical technique to approximate the solution of a diffusion equation.

In particular, we focus on the heat and wave equations. The heat equation is a partial differential equation (PDE) that models the diffusion of heat in a rod or space. This is one of the mostly studied equations in both pure and applied mathematics. Along with its variants, the heat equation is immensely important in various fields. Some of the com-

mon examples which utilise the variant of the heat equation are the Black-Scholes PDE in mathematical finance and the Schrödinger equation in quantum mechanics.
The wave equation is a PDE that finds practical applications mostly in classical physics. In particular, this equation is of vital importance in situations that exhibit wave propagation. Thus, this equation is important in electromagnetism, sound waves, acoustics, fluid dynamics and various other fields.
As a way of ascertaining accuracy and consistency of the results from this novel scheme, we compare our results against analytical solutions. We do this through the computations of absolute errors and performance of convergence analysis.

5.2 Approximate solution of the heat equation

We intend to approximate the solution of the following [27],

$$\begin{aligned} \mathcal{D}_t^{\mu} y &= \frac{1}{2}x^2 y_{xx}, \quad \mu \in (0,1], \quad (x,t) \in [0,1]\times[0,1], && (5.1)\\ y(x,0) &= x^2, && (5.2)\\ y(0,t) &= 0, && (5.3)\\ y(1,t) &= e^t. && (5.4) \end{aligned}$$

If the value of $\mu = 1$, then the analytical solution of (5.1)-(5.4) is $y(x,t) = x^2 e^t$.

We assume the approximate solution of (5.1)-(5.4) takes the form (4.32). If we take $n = 3$, then, from (4.32), we get,

$$\begin{aligned} y(x,t) &= \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{30}x^3 + \gamma_{11}xt + \gamma_{12}xt^2 + \gamma_{13}xt^3 + \gamma_{21}x^2t \\ &+ \gamma_{22}x^2t^2 + \gamma_{23}x^2t^3 + \gamma_{31}x^3t + \gamma_{32}x^3t^2 + \gamma_{33}x^3t^3 + \gamma_{01}t + \gamma_{02}t^2 + \gamma_{03}t^3. \quad (5.5) \end{aligned}$$

We can deduce from (5.2) and (5.5) that,

$$y(x,0) = \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{30}x^3 = x^2,$$

giving us $\gamma_{00} = 0, \quad \gamma_{10} = 0, \quad \gamma_{20} = 1$ and $\gamma_{30} = 0$.
Again, from (5.3) and (5.5), we get,

$$y(0,t) = \gamma_{00} + \gamma_{01}t + \gamma_{02}t^2 + \gamma_{03}t^3 = 0,$$

yielding $\gamma_{00} = 0, \quad \gamma_{01} = 0, \quad \gamma_{02} = 0$ and $\gamma_{03} = 0$. Also, from (5.4) and (5.5), we get,

$$\begin{aligned} y(1,t) &= \gamma_{00} + \gamma_{10} + \gamma_{20} + \gamma_{30} + \gamma_{11}t + \gamma_{12}t^2 + \gamma_{13}t^3 + \gamma_{21}t + \gamma_{22}t^2 + \gamma_{23}t^3 \\ &+ \gamma_{31}t + \gamma_{32}t^2 + \gamma_{33}t^3 + \gamma_{01}t + \gamma_{02}t^2 + \gamma_{03}t^3 \qquad (5.6) \\ &= e^t \\ &\approx 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3. \end{aligned}$$

Since we are already aware of the values of $\gamma_{00}, \gamma_{10}, \gamma_{20}, \gamma_{30}, \gamma_{01}, \gamma_{02}$ and γ_{03}. We substitute these values in (5.6) to get,

$$\begin{aligned} y(1,t) &= 1 + \gamma_{11}t + \gamma_{12}t^2 + \gamma_{13}t^3 + \gamma_{21}t + \gamma_{22}t^2 + \gamma_{23}t^3 \\ &+ \gamma_{31}t + \gamma_{32}t^2 + \gamma_{33}t^3 \\ &= e^t \\ &\approx 1 + t + \frac{1}{2}t + \frac{1}{6}t^3. \end{aligned}$$

Equating the coefficients of t in the above equation implies that,

$$\gamma_{11} + \gamma_{21} + \gamma_{31} = 1 \qquad (5.7)$$

$$\gamma_{12} + \gamma_{22} + \gamma_{32} = \frac{1}{2} \qquad (5.8)$$

$$\gamma_{13} + \gamma_{23} + \gamma_{33} = \frac{1}{6}. \qquad (5.9)$$

We have so far made use of the initial and boundary conditions to deduce some numerical values of γ, substituting these in (5.5) gives,

$$\begin{aligned} y(x,t) &= x^2 + \gamma_{11}xt + \gamma_{12}xt^2 + \gamma_{13}xt^3 + \gamma_{21}x^2t + \gamma_{22}x^2t^2 + \gamma_{23}x^2t^3 \\ &+ \gamma_{31}x^3t + \gamma_{32}x^3t^2 + \gamma_{33}x^3t^3, \qquad (5.10) \end{aligned}$$

as the approximate solution of (5.1)-(5.4).

In matrix form, (5.10) can be written as,

$$y(x,t) = \begin{pmatrix} 1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \begin{pmatrix} x^2 \\ xt \\ xt^2 \\ xt^3 \\ x^2t \\ x^2t^2 \\ x^2t^3 \\ x^3t \\ x^3t^2 \\ x^3t^3 \end{pmatrix}, \tag{5.11}$$

with, $\gamma^T = \begin{pmatrix} 1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}$ and $\mathcal{M}(x,t) = \begin{pmatrix} x^2 \\ xt \\ xt^2 \\ xt^3 \\ x^2t \\ x^2t^2 \\ x^2t^3 \\ x^3t \\ x^3t^2 \\ x^3t^3 \end{pmatrix}$.

Replacing the derivative operators with composite derivative matrices in (5.1), we get,

$$\begin{aligned} \mathbf{D}_t^{\mu} y &= \frac{1}{2}x^2 \mathbf{D}_{xx} y, \\ \gamma^T \mathbf{D}_t^{\mu} \mathcal{M}(x,t) &= \frac{1}{2}x^2 \gamma^T \mathbf{D}_{xx} \mathcal{M}(x,t) \end{aligned} \tag{5.12}$$

We then create the residual $\mathcal{R}(x,t)$, such that,

$$\mathcal{R}(x,t) = \gamma^T \mathbf{D}_t^{\mu} \mathcal{M}(x,t) - \frac{1}{2}x^2 \gamma^T \mathbf{D}_{xx} \mathcal{M}(x,t). \tag{5.13}$$

We then apply the Garlekin method to create a system of equations,

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) xt dx dt = 0 \tag{5.14}$$

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) xt^2 dx dt = 0 \tag{5.15}$$

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) xt^3 dx dt = 0 \tag{5.16}$$

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) x^2 t dx dt = 0 \tag{5.17}$$

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) x^2 t^2 dx dt = 0 \tag{5.18}$$

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) x^2 t^3 dx dt = 0 \tag{5.19}$$

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) x^3 t dx dt = 0 \tag{5.20}$$

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) x^3 t^2 dx dt = 0 \tag{5.21}$$

$$\int_0^1 \int_0^1 \mathcal{R}(x,t) x^3 t^3 dx dt = 0 \tag{5.22}$$

Combining equations (5.7)–(5.9) and (5.14)-(5.22), we have enough number of equations to enable us to compute the remaining numerical values of γ in (5.11). We solve these equations with the help of Mathematica, and for $\mu = 1$, we get,
$\gamma_{11} = \frac{3056}{226001}, \gamma_{12} = -\frac{11502}{226001}, \gamma_{13} = \frac{6511}{226001}, \gamma_{21} = \frac{221410}{226001}, \gamma_{22} = \frac{512245}{904004}, \gamma_{23} = \frac{587545}{2712012}, \gamma_{31} = \frac{1535}{226001}, \gamma_{32} = -\frac{14235}{904004}$ and $\gamma_{33} = -\frac{71225}{904004}$.
Thus, the approximate solution becomes,

$$\begin{aligned} y(x,t) &= x^2 + \frac{3056}{226001}tx - \frac{11502}{226001}t^2 x + \frac{6511}{226001}t^3 x + \frac{221410}{226001}tx^2 + \frac{512245}{904004}t^2 x^2 \\ &+ \frac{587545}{2712012}t^3 x^2 + \frac{1535}{226001}x^3 t - \frac{14235}{904004}x^3 t^2 - \frac{71225}{904004}x^3 t^3. \end{aligned} \tag{5.23}$$

If we use the composite integral and differential matrices, we will first substitute for the derivative operator on the right hand side of (5.1) with a composite derivative matrix and thereafter, we introduce the composite integral matrix on both sides, such that,

$$\begin{aligned} \mathbf{I}_t^{\mu} y &= \mathbf{I}_t^{\mu}\Big(\frac{1}{2}x^2\mathbf{D}_{xx}y\Big) \\ y(x,t)-y(x,0) &= \frac{1}{2}x^2\mathbf{I}_t^{\mu}\mathbf{D}_{xx}y(x,t) \\ y(x,t)-f(x) &= \frac{1}{2}x^2\mathbf{C}_{txx}y(x,t) \end{aligned} \tag{5.24}$$

Substituting (5.11) and for the initial condition in (5.24),

$$y(x,t)-x^2=\frac{1}{2}x^2\mathbf{C}_{txx}y(x,t), \tag{5.25}$$

giving the residual,

$$R(x,t)=y(x,t)-x^2-\frac{1}{2}x^2\mathbf{C}_{txx}y(x,t). \tag{5.26}$$

Note that, the residual created from the composite differential matrices, (5.13), is different from the one created from the composite integral and differential matrices, (5.26). Thus, in this respect, we expect the approximate solutions from $\mathcal{R}(x,t)$ and $R(x,t)$ to be different.

To get the approximate solution from the composite integral and differential matrices, we have to solve equations (5.7)–(5.9) and (5.14)–(5.22). However, we have to take note that we now use the residual $R(x,t)$ instead of $\mathcal{R}(x,t)$ in (5.14)–(5.22). After solving for γ and substituting in (5.10), we attain the following approximate solution,

$$\begin{aligned} y(x,t) &= x^2+\frac{5570}{1004147}tx-\frac{62145}{2008294}t^2x+\frac{31905}{2008294}t^3x+\frac{3984733}{4016588}tx^2+\frac{4304393}{8033176}t^2x^2 \\ &+ \frac{712516}{3012441}t^3x^2+\frac{9575}{4016588}x^3t-\frac{39225}{8033176}x^3t^2-\frac{86100}{1004147}x^3t^3. \end{aligned} \tag{5.27}$$

Figure5.1 depicts the diagrammatic representation of the analytic solution and approximate solutions from the composite derivative matrices. The diagrams are constructed for different combination sets of n and μ.

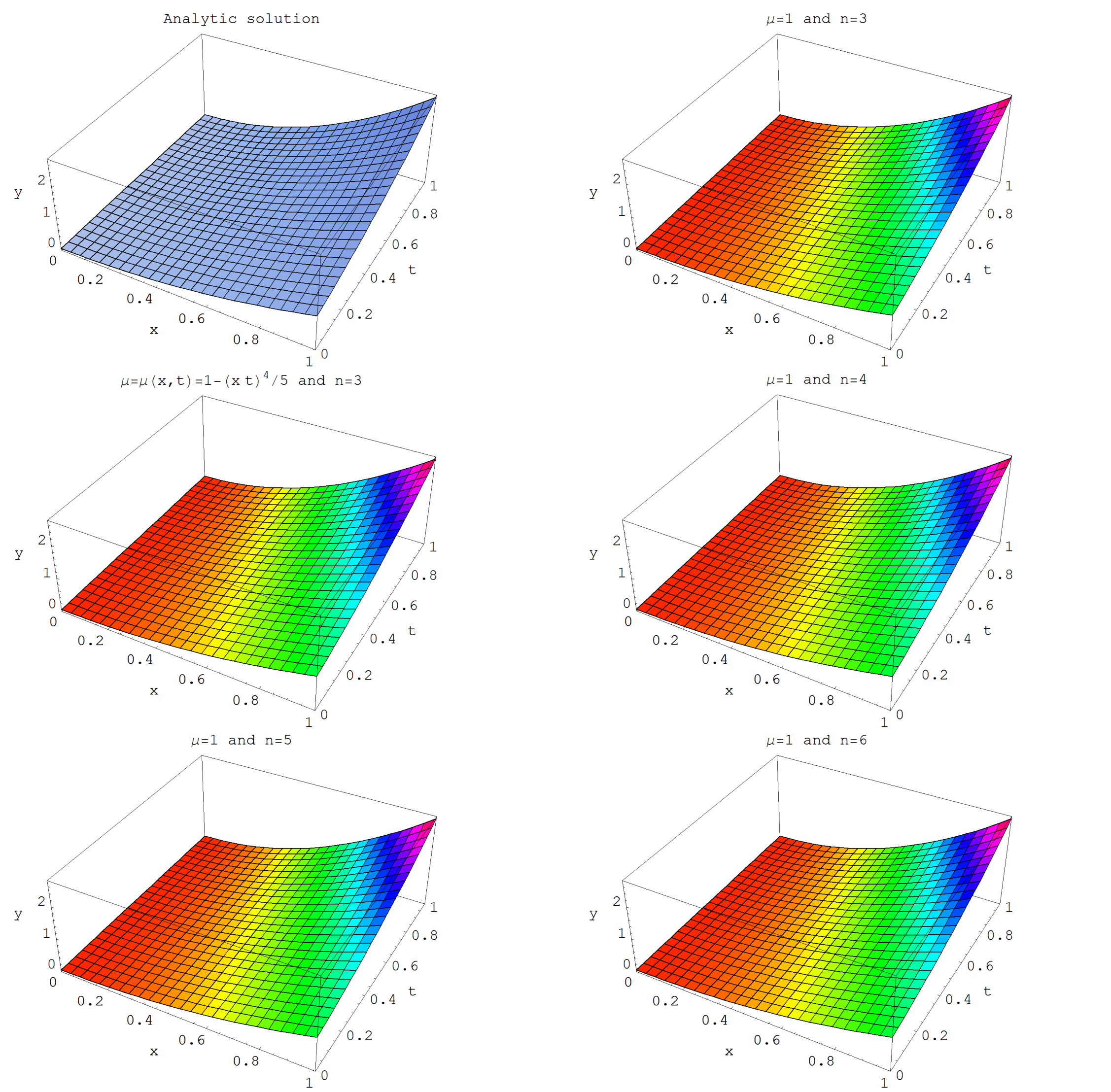


Figure 5.1: Approximate solutions of (5.1)-(5.4) from the composite derivative matrices for different combination sets of μ and n are compared against the analytic solution.

We note from Figure 5.1 that solutions from our technique are very close to the analytic solution. We also note that, we have one diagram with $\mu = \mu(x,t) = 1 - \frac{x^4t^4}{5}$, thus μ is a function. This means that it is possible to take the order of a derivative to be a function. However, we have to stress that this concept of taking the order of a derivative as a function is in its infancy, therefore there is nothing much that we can say about it. The main challenge one encounters in the function derivative order is the enormous sizes of the values of γ^T. That is the reason why we were only able to do the calculations for $n = 3$, which means few numbers of polynomials used, thus lessening the computational difficulty.

To get a much clear view of the relation between the analytic and approximate solution, we compute the absolute errors y_{er},
$y_{er} =| y_{anal}(x,t) - y_{appr}(x,t) |$, where y_{anal} and y_{appr} are the analytical and approximate solutions respectively.
Figure 5.2 shows the absolute errors for various combinations of n and μ.

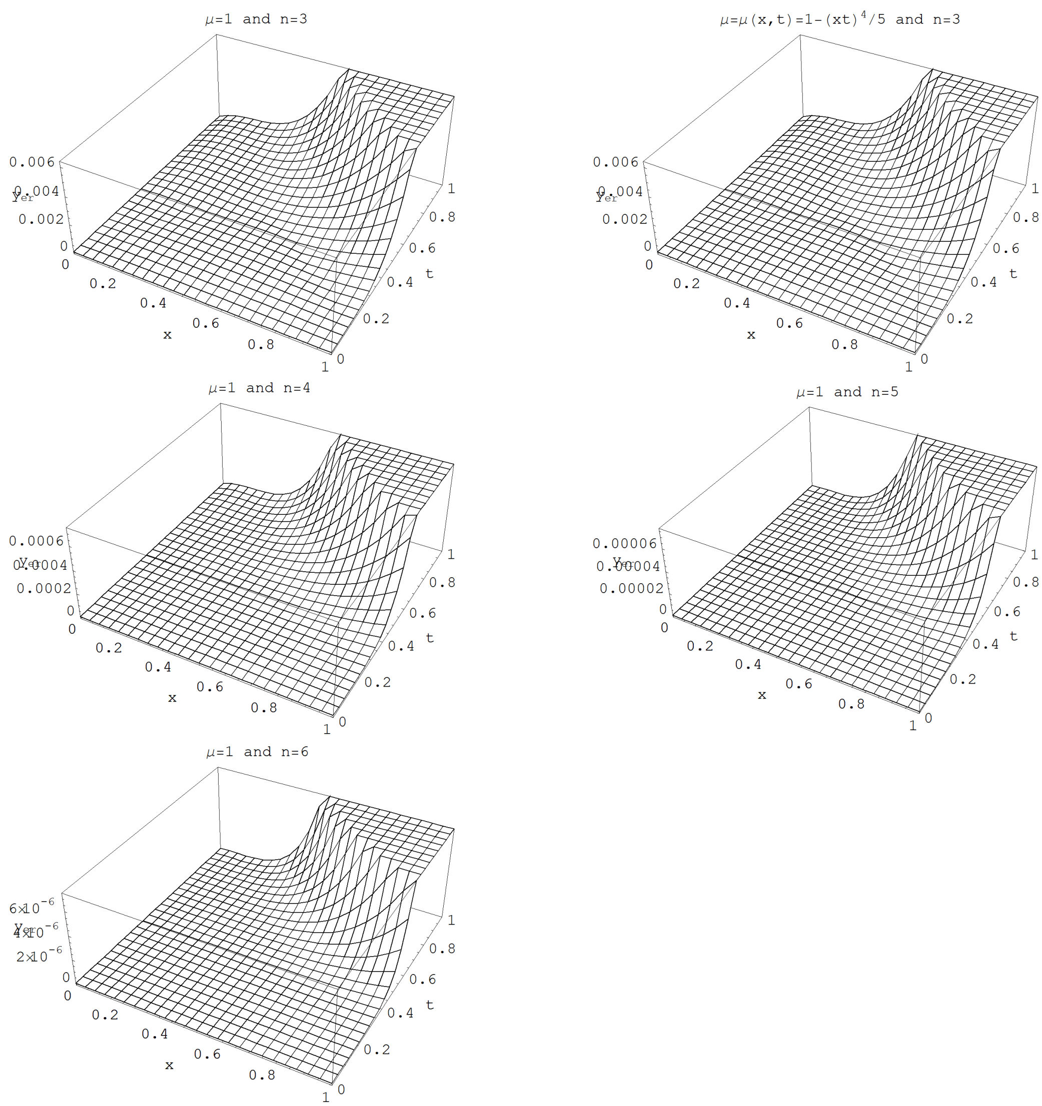


Figure 5.2: Error analysis of the results from Figure 5.1

It is apparent from Figure 5.2 that our approximate solution improves with increasing number of polynomials used.

In Figure 5.3, we depict approximate solutions resulting from the composite integral and differential matrices. Then in Figure 5.4, we depict the corresponding absolute errors. We do this for various combination sets of n and μ.

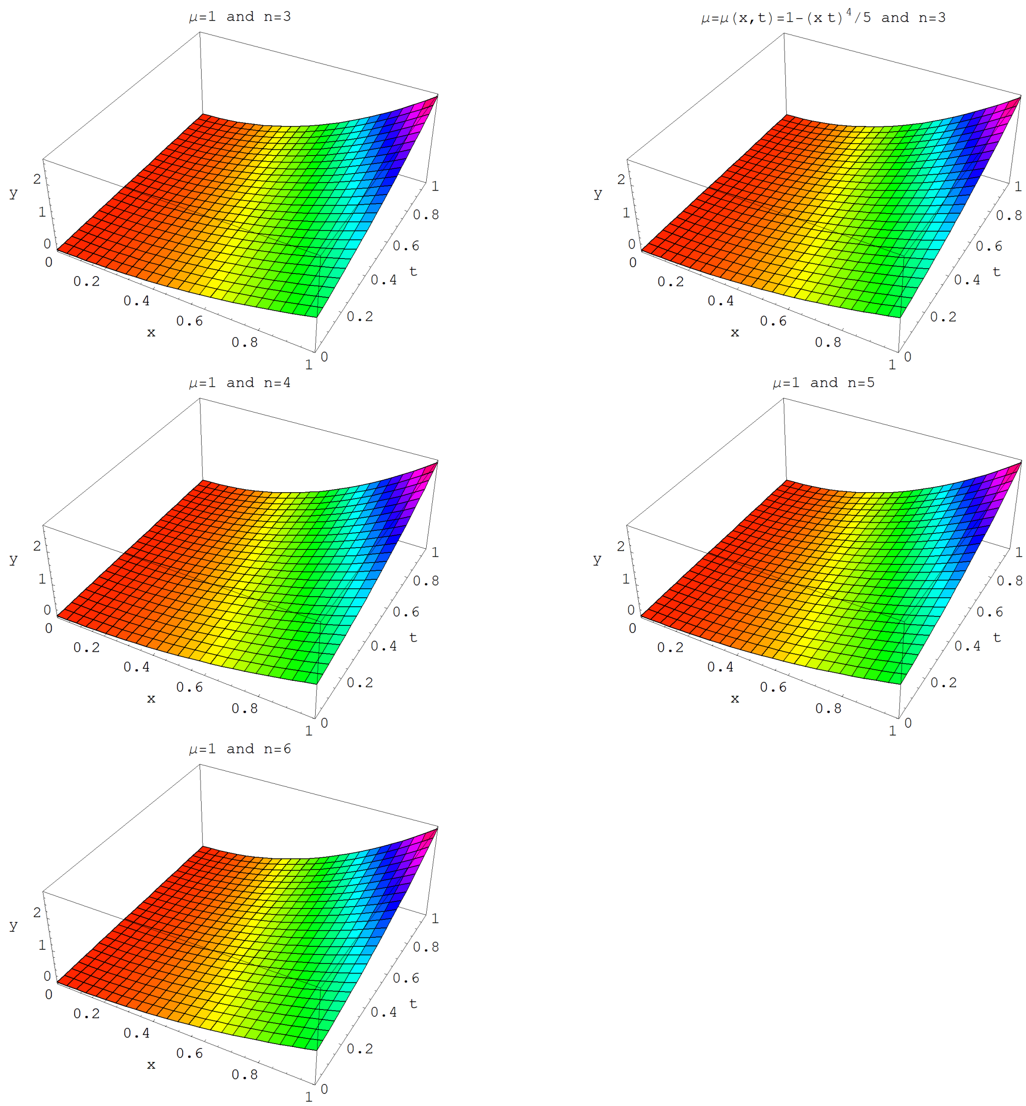


Figure 5.3: Approximate solutions of (5.1)-(5.4) from the composite integral and derivative matrices for various combination sets of μ and n.

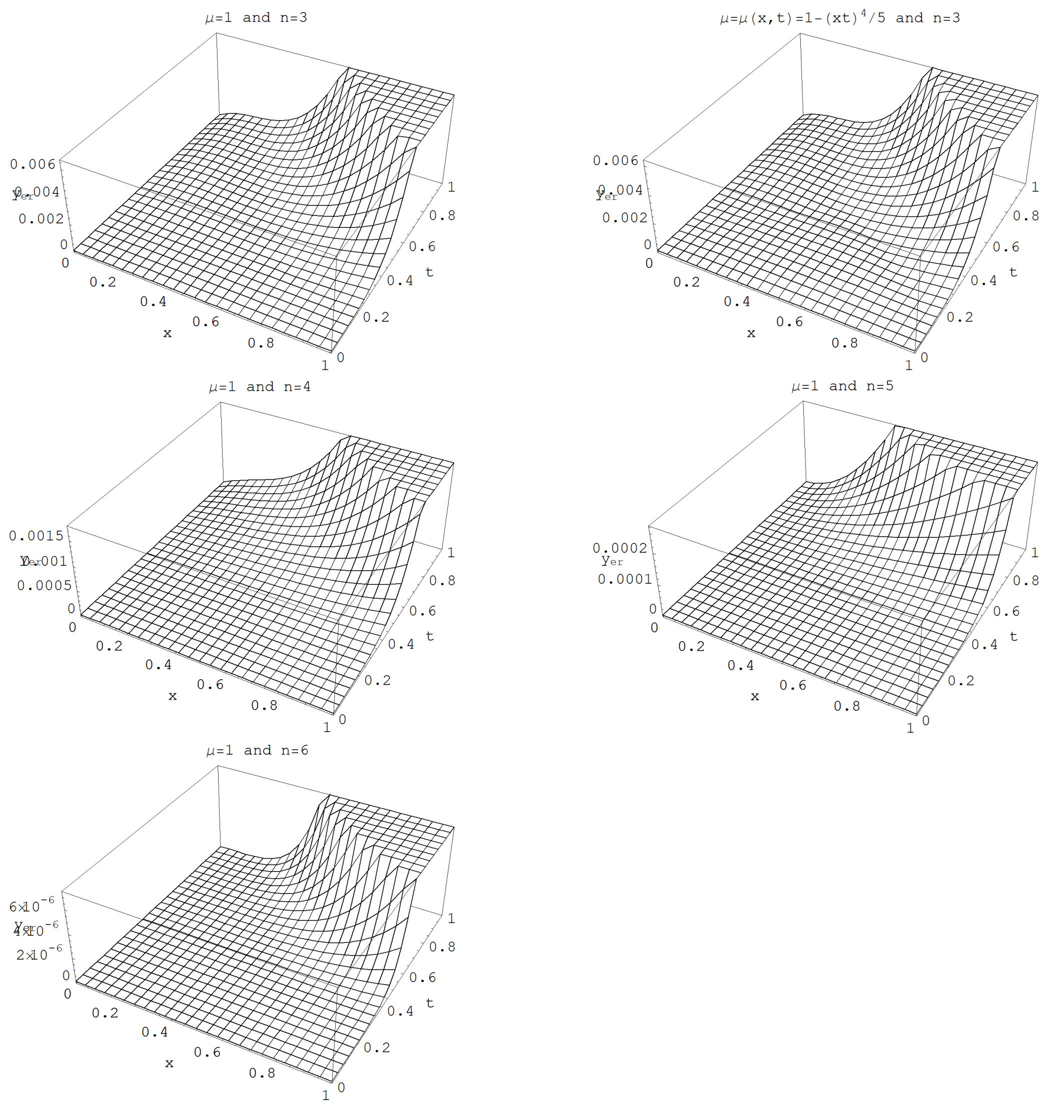


Figure 5.4: Error analysis of the results from Figure 5.3.

One striking observation from Figures 5.1–5.4 is that it does not make any difference whether we use only composite derivative matrices or both the integral and derivative composite operational to approximate the solution of (5.1)-(5.4). The results look the same from the diagrams.
To corroborate this important observation, we show results from the convergence analysis in Table 5.1.

Table 5.1: Convergence results of (5.1)–(5.4) from composite operational matrices.

	$\mathbf{D}_t^{\mu}$ and $\mathbf{D}_{xx}$	$\mathbf{I}_t^{\mu}\mathbf{D}_{xx}$
$\parallel \upsilon_3 \parallel$	0.004785699922751591	0.004521594889883231
$\parallel \upsilon_4 \parallel$	0.0008391604751856951	0.0010672764814549862
$\parallel \upsilon_5 \parallel$	0.00012478378609025523	0.00018117011293624844.

In Table 5.1, the middle column are the results from the composite derivative matrices, then the last column are the results from the composite integral and derivative matrices.
We can tell from Table 5.1 that as we increase the value of n, then $\parallel \upsilon \parallel$ gets smaller, implying the difference between consecutive terms of $y(x, t)$ become smaller and smaller, thus the approximate solution converges.
The results from Table 5.1 between the composite differential and composite integral and differential matrices are very close, suggesting that it does not matter much which option we choose in approximating solution of (5.1)-(5.4).

5.3 Approximate solution of the wave equation

We intend to approximate the solution of the wave equation [23],

$$\mathcal{D}_t^{\mu} y = y_{xx}, \quad \mu \in (1,2], \quad (x,t) \in [0,1] \times [0,1], \tag{5.28}$$

$$y(x,0) = \sin(x), \tag{5.29}$$

$$y_t(x,0) = 0, \tag{5.30}$$

$$y(0,t) = 0, \tag{5.31}$$

$$y(1,t) = \sin(1)\cos(t). \tag{5.32}$$

If the value of $\mu = 2$, then the analytical solution of (5.28)-(5.32) is $y(x,t) = \sin(x)\cos(t)$. We assume the approximate solution of (5.28)-(5.32) takes the form of (4.32), thus for $n = 3$, we get (5.5).
We can tell from (5.29) and (5.5) that,

$$y(x,0) = \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{30}x^3 \tag{5.33}$$

$$= \sin(x)$$

$$\approx x - \frac{1}{6}x^3. \tag{5.34}$$

Equating the coefficients of x in (5.33) and (5.34), gives, $\gamma_{00} = 0, \gamma_{10} = 1, \gamma_{20} = 0$ and $\gamma_{30} = -\frac{1}{6}$.
Imposing the initial condition (5.30) on (5.5) implies,

$$y_t(x,0) = \gamma_{11}x + \gamma_{21}x^2 + \gamma_{31}x^3 + \gamma_{01} = 0, \tag{5.35}$$

thus, we are able to deduce from (5.35) that, $\gamma_{11} = 0, \gamma_{21} = 0, \gamma_{31} = 0$ and $\gamma_{01} = 0$.
Imposing the condition (5.31) on (5.5) means,

$$y(0,t) = \gamma_{00} + \gamma_{01}t + \gamma_{02}t^2 + \gamma_{03}t^3 = 0, \tag{5.36}$$

thus, we can deduce that, $\gamma_{00} = 0, \gamma_{01} = 0, \gamma_{02} = 0$ and $\gamma_{03} = 0$.
We also use (5.32) and (5.5) to get,

$$y(1,t) = \gamma_{00} + \gamma_{10} + \gamma_{20} + \gamma_{30} + \gamma_{11}t + \gamma_{12}t^2 + \gamma_{13}t^3 + \gamma_{21}t + \gamma_{22}t^2 + \gamma_{23}t^3$$

$$+ \gamma_{31}t + \gamma_{32}t^2 + \gamma_{33}t^3 + \gamma_{01}t + \gamma_{02}t^2 + \gamma_{03}t^3. \tag{5.37}$$

$$= \sin(1)\cos(t). \tag{5.38}$$

Since, we already know some numerical values of γ^T, we substitute these in (5.37). We also expand $\cos(t)$ in (5.38) about the point $t = 0$. Thus, from (5.37) and (5.38), we get,

$$\begin{aligned} y(1,t) &= \frac{5}{6} + \gamma_{12}t^2 + \gamma_{13}t^3 + \gamma_{22}t^2 + \gamma_{23}t^3 \\ &+ \gamma_{32}t^2 + \gamma_{33}t^3. \qquad (5.39) \\ &= \sin(1) - \sin(1)\frac{1}{2}t^2. \qquad (5.40) \end{aligned}$$

Equating the coefficients of t in (5.39) and (5.40) yields the equations,

$$\frac{5}{6} = \sin(1), \qquad (5.41)$$

$$\gamma_{12} + \gamma_{22} + \gamma_{32} = -\frac{\sin(1)}{2}, \qquad (5.42)$$

$$\gamma_{13} + \gamma_{23} + \gamma_{33} = 0. \qquad (5.43)$$

There are a couple of important points that we need to mention in (5.41)-(5.43). Since, (5.5) is an approximate solution, in (5.41), the left hand side is approximately equal to the right hand side. Also, in (5.43), we will assume the solution to be $\gamma_{12} = \gamma_{23} = \gamma_{33} = 0$.

Thus, up to this far, we have attained some values of γ^T, we substitute these in (5.5) such that,

$$y(x,t) = x - \frac{1}{6}x^3 + \gamma_{12}xt^2 + \gamma_{22}x^2t^2 + \gamma_{32}x^3t^2. \qquad (5.44)$$

In matrix form, (5.44) can be written as,

$$y(x,t) = \begin{pmatrix} 1 & -\frac{1}{6} & \gamma_{12} & \gamma_{22} & \gamma_{32} \end{pmatrix} \begin{pmatrix} x \\ x^3 \\ xt^2 \\ x^2t^2 \\ x^3t^2 \end{pmatrix}, \qquad (5.45)$$

with $\gamma^T = \begin{pmatrix} 1 & -\frac{1}{6} & \gamma_{12} & \gamma_{22} & \gamma_{32} \end{pmatrix}$ and $\mathcal{M}(x,t) = \begin{pmatrix} x \\ x^3 \\ xt^2 \\ x^2t^2 \\ x^3t^2 \end{pmatrix}$.

We replace the derivative operators with the composite derivative matrices in (5.28) such that,

$$\begin{aligned} \mathbf{D}_t^{\mu} y &= \mathbf{D}_{xx} y, \\ \gamma^T \mathbf{D}_t^{\mu} \mathcal{M}(x,t) &= \gamma^T \mathbf{D}_{xx} \mathcal{M}(x,t). \end{aligned} \tag{5.46}$$

The residual from (5.46) is given as,

$$\mathcal{R}(x,t) = \gamma^T \mathbf{D}_t^{\mu} \mathcal{M}(x,t) - \gamma^T \mathbf{D}_{xx} \mathcal{M}(x,t). \tag{5.47}$$

We then create a system of equations from the residual (5.47) as,

$$\int_0^1 \mathcal{R}(x,t) x t^2 dx dt = 0, \tag{5.48}$$

$$\int_0^1 \mathcal{R}(x,t) x^2 t^2 dx dt = 0, \tag{5.49}$$

$$\int_0^1 \mathcal{R}(x,t) x^3 t^2 dx dt = 0. \tag{5.50}$$

Solving a system of equations (5.42), (5.43) and (5.48), we are able to get the numerical values of the remaining unknowns in (5.44). Through the assistance of Mathematica with the value of $\mu = 2$, we get the values, $\gamma_{12} = -\frac{135}{304}$, $\gamma_{22} = -\frac{5}{228}$ and $\gamma_{32} = \frac{15}{304}$. Therefore, we get the approximate solution,

$$y(x,t) = x - \frac{x^3}{3} - \frac{135}{304} x t^2 - \frac{5}{228} x^2 t^2 + \frac{15}{304} x^3 t^2. \tag{5.51}$$

If we use the composite differential and integral matrices, then the residual becomes,

$$\begin{aligned} R(x,t) &= y(x,t) - \sin(x) - \mathbf{I}_t^{\mu} \mathbf{D}_{xx} y(x,t), \\ &= \gamma^T \mathcal{M}(x,t) - \left(x - \frac{x^3}{6} \right) - \gamma^T \mathbf{C}_{txx} \mathcal{M}(x,t). \end{aligned} \tag{5.52}$$

We formulate equations as in (5.48)-(5.50), but we use the residual $R(x,t)$ given in (5.52). Solving a system of equations (5.42), (5.43) and (5.48) with residual $R(x,t)$ and $\mu = 2$ gives us $\gamma_{12} = -\frac{1375}{3216}$, $\gamma_{22} = -\frac{245}{1072}$ and $\gamma_{32} = \frac{385}{1608}$. Thus, our approximate solution is,

$$y(x,t) = x - \frac{x^3}{3} - \frac{1375}{3216} x t^2 - \frac{1375}{3216} x^2 t^2 + \frac{385}{1608} x^3 t^2. \tag{5.53}$$

We depict plots of the analytical solution and approximate solutions of (5.28)-(5.32) obtained from the composite derivative matrices in Figure 5.5 for various combination sets of n and μ.

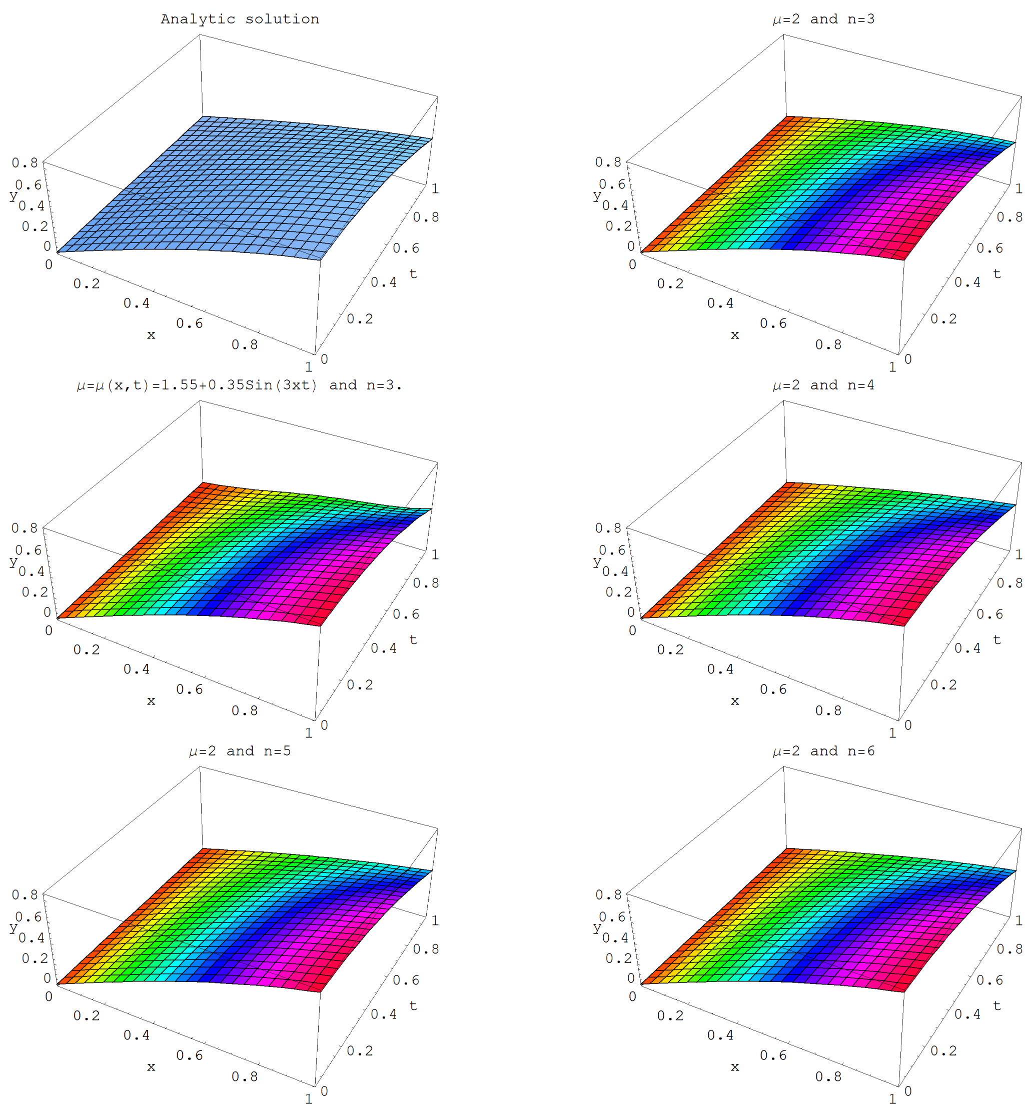


Figure 5.5: Approximate solutions of (5.28)-(5.32) from the composite derivative matrices for different combination sets of μ and n are compared against the analytic solution.

We observe from Figure 5.5 that the results from the analytical and approximate solutions are very close. A diagrammatic presentation of the absolute errors is shown in Figure 5.6 for various values of n with $\mu = 2$.

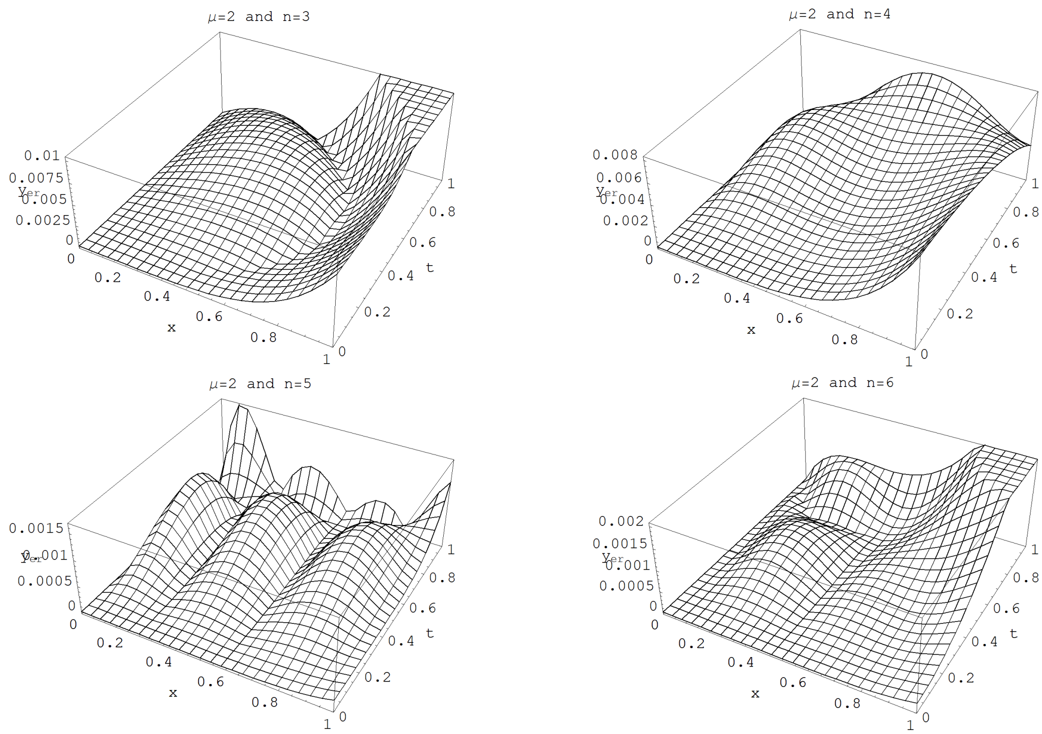


Figure 5.6: Error analysis of the results from Figure 5.5 for different values of n with $\mu = 2$.

The results in Figure 5.6 indicate that as we increase the number of polynomials for approximations, the accuracy improves.

In Figure 5.7, we plot the approximate solutions of (5.28)-(5.32) obtained from the composite differential and integral matrices. In Figure 5.8, we plot the associated absolute errors.

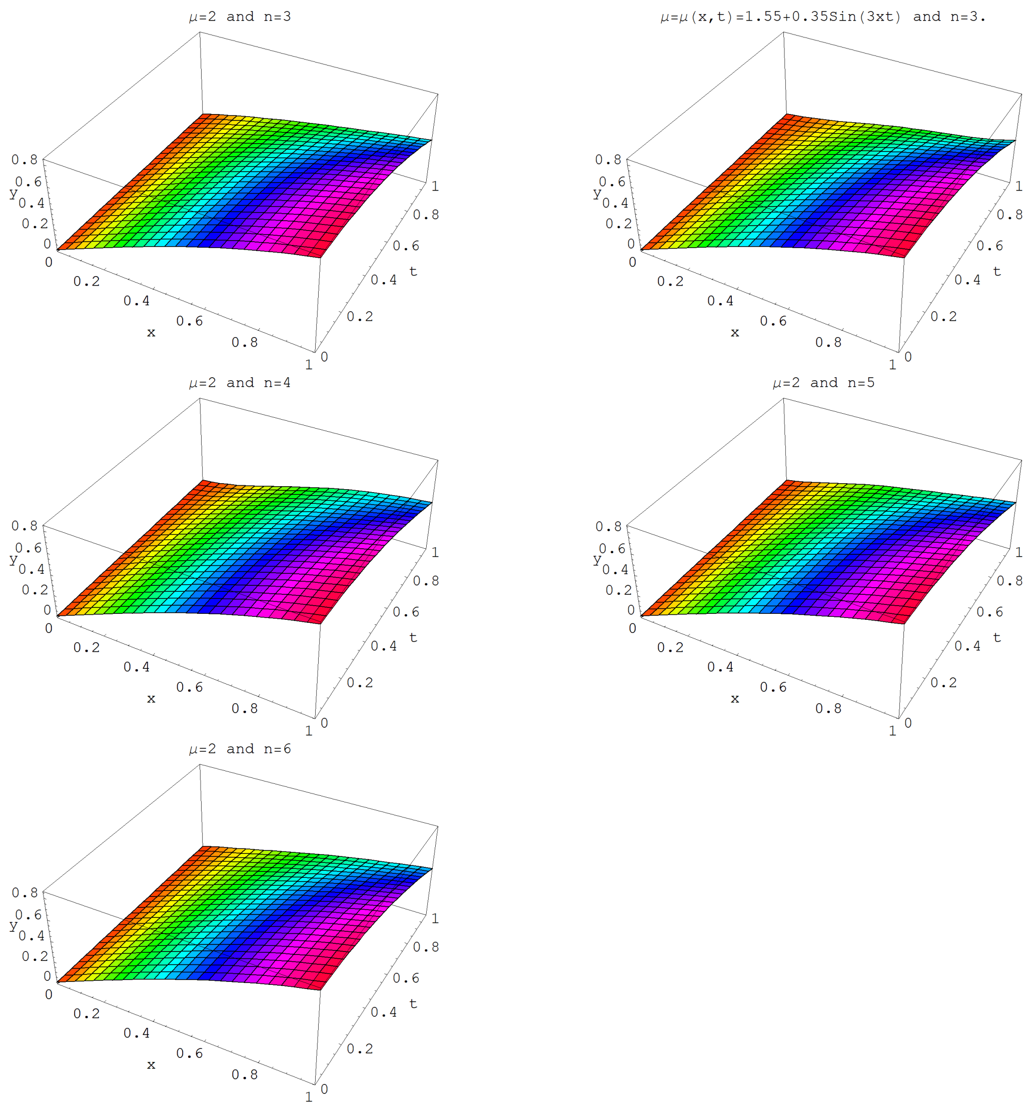


Figure 5.7: Approximate solutions of (5.28)-(5.32) from the composite integral and derivative matrices for different combination sets of μ and n.

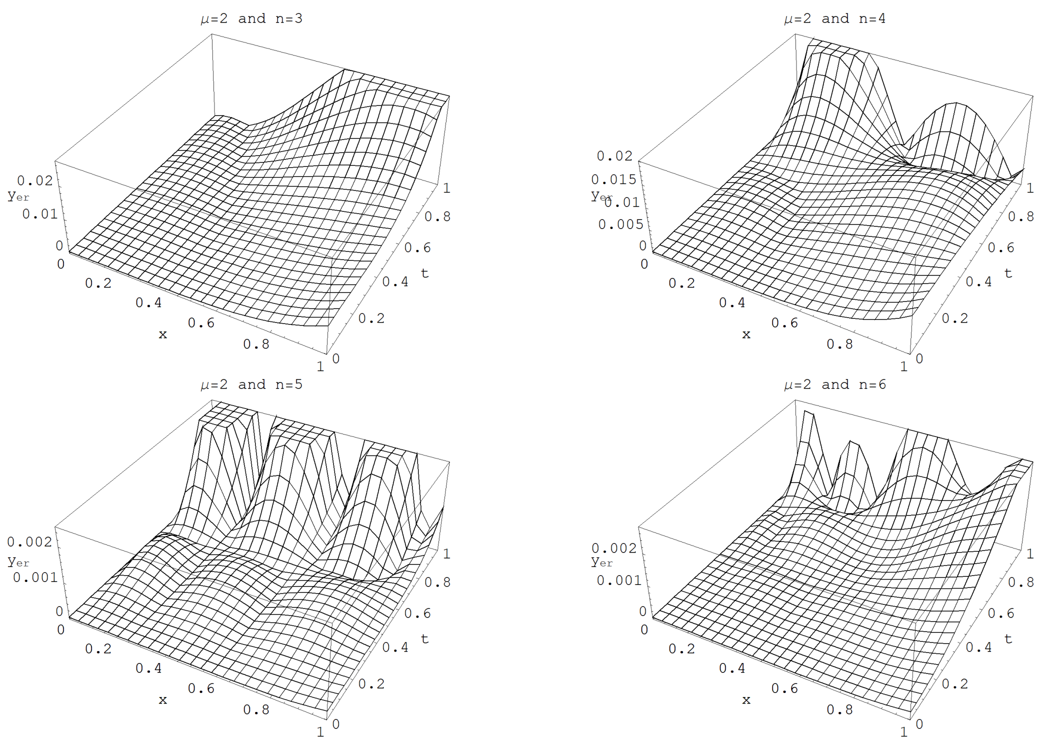


Figure 5.8: Error analysis of the results from Figure 5.7 for different values of n with $\mu = 2$.

We note from Figure 5.7–5.8 that the approximate solutions are very close to the analytic solution, also, as we increase the value of n, the accuracy of the numerical approach improves.
Comparing results from Figure 5.6–5.8, we realise that it does not make any difference if one uses composite derivative or composite derivative and integral matrices, the results are very close. To emphasize this important observation, we perform convergence analysis of the results in Table 5.2.

Table 5.2: Convergence results of (5.28)–(5.32) from composite operational matrices.

	$\mathbf{D}_t^{\mu}$ and $\mathbf{D}_{xx}$	$\mathbf{I}_t^{\mu}\mathbf{D}_{xx}$
$\lVert \upsilon_3 \rVert$	0.005006360460823758	0.010041424385944648
$\lVert \upsilon_4 \rVert$	0.003769572156940589	0.006657994660073802
$\lVert \upsilon_5 \rVert$	0.0007407665777154791	0.0017596044581037545

We notice from Table 5.2 that the convergence results from the composite differential matrices and composite integral and differential are close. However, for this example, we seem to be converging faster to the analytic solution through the use of the composite derivative matrices. The sensible explanation for this observation might be in the construction of the residuals. When using the derivative operational matrices, the residual function, 5.47, does not involve the initial condition. But, when using the composite integral and derivative matrices, we have to utilise the approximate initial condition in constructing the residual (5.52).

In [22], the authors use the Legendre operational matrix together with the collocation technique to approximate the solution of fractional differential equations. We use the technique described in [22] to approximate the solution of (5.28)-(5.32). We then make a comparison of the results from this technique against our own results. The comparisons are given in the form of diagrams in Figure 5.9.

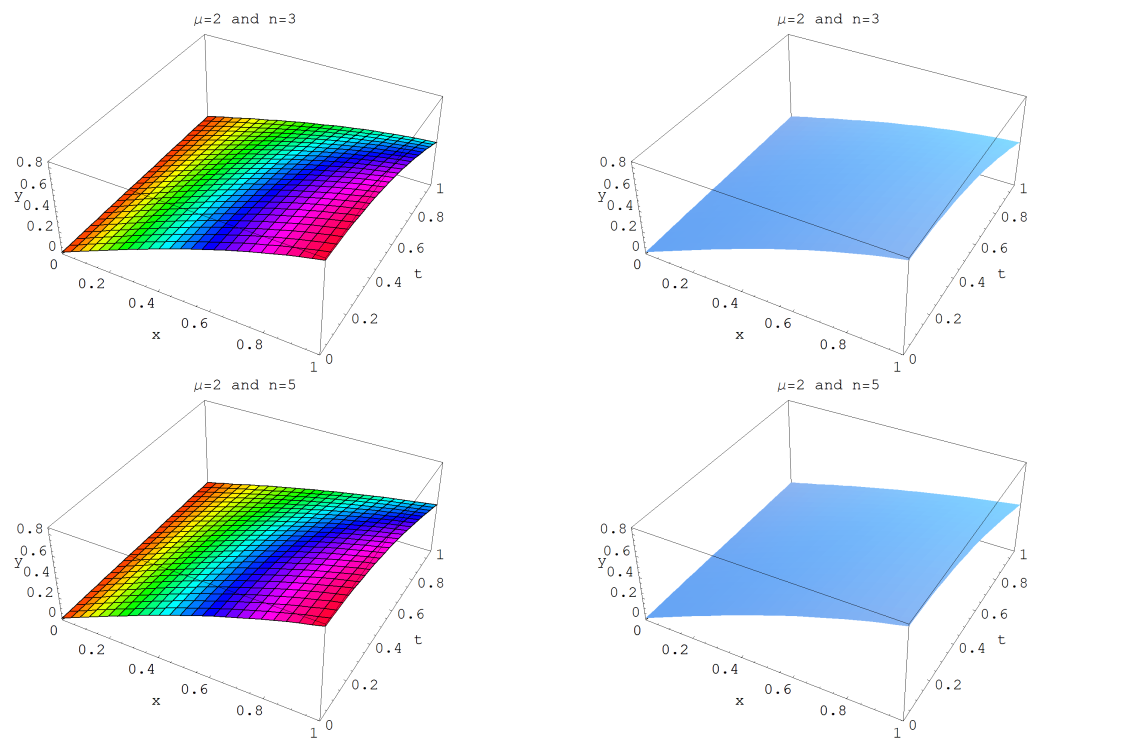


Figure 5.9: Comparison of the results, multli-coloured diagrams are results from our technique and blue colour are the results from the Legedre operational matrix using the collocation.

We observe from Figure 5.9 that results from our technique and those from the method described in [22] are in close agreement.

5.4 Conclusion

The results from our suggested numerical approach converge to the analytic solutions as we increase the number of polynomials. A more crucial observation of our results point out to the fact that we only need just the first few polynomials to get good results. Thus this approach has fast rate of convergence. We observe that the approximation of the initial condition does compromise the accuracy of the results. This is clearly evident from the convergence results of the approximate solution of the wave equation where the composite integral operational was applied. However, in general, we do observe that it does not make much difference whether one chooses to use differential operational matrices or a combination of the integral and derivative operational matrices.

Chapter 6

Summary

In the first chapter, we introduced the thesis by stating our research objectives and giving sufficient background. In addition, we gave the necessary mathematical definitions and some notations that were useful throughout the thesis.

In the second chapter, we solely focused our attention to the review of the literature relevant to our research. We discuss the developments of the use of operational matrices and explain how the objectives of our research contribute to the existing body of knowledge.

In chapter three, we introduce the idea of an operational matrix, in particular, we explain how operational matrices can be deduced from polynomials. We discuss how matrices can function as differential and integral operators. To put the theory of operational matrices into practice, we apply them in the approximate solution of ordinary differential equations. Of prominent importance in this chapter is the application of operational matrices in the approximate solution of the Vander Pol differential equation.

The fourth chapter is reserved to the contribution that we make to the existing knowledge of the operational matrices. We deviate from the tradition of writing an approximate solution of a partial differential equation in a specific way, and by so doing we unravel new notion of constructing operational matrices. This discovery leads us to realise that the partial derivatives and integrals can be represented by operational matrices whose entries are themselves matrices. We point out clearly how this new knowledge perfectly fits in the already existing notion of operational matrices.

In chapter five, we apply the knowledge developed in chapter four to solve practical problems. Specifically, we approximate the solutions of the heat and wave equations. We demonstrate that a small increment in the dimensions of the operational matrices drastically improves the accuracy of our approximate solution. Also, the results reveal that operational matrices with low dimensions are sufficient to give good approximations.

When we have a function derivative order, the calculations become challenging. In fact, we were not able to write results from Mathematica when solving variable order differential equations due to their considerable length. This difficulty prevented us from doing calculations with operational matrices of higher dimensions. Also the variable order concept is a relatively new idea that is still under development, because of this, there are few results in literature that we were able to use for comparison with our work. Mostly, we had to compare the results from our work with those from integer order derivatives.

There are numerous directions of future research that we had to pick up in the course of doing this research, but we will mention only two of them that we think are worth pursuing. Firstly, perhaps, to reduce the large volume of the results we experienced from variable order differential equations, other numerical techniques besides the Garlekin can be explored. Secondly, we managed to develop the new concept of the composite operational matrices and used it in conjunction with the Garlekin technique to approximate the solutions of PDEs. In our solution process, we had to approximate the initial and boundary conditions using the Taylor expansion. We chose to do this for the sake of consistency since we were using polynomials for our approximations. However, it will be worth finding out what is the effect on the approximate solution when one chooses to use the given initial conditions as functions without approximating them.

```
{{γ2 → -5.184459105694016` - 24.007554165344665` ⅈ, γ0 → 2.`, γ1 → 0.`},
 {γ2 → -5.184459105694016` + 24.007554165344665` ⅈ, γ0 → 2 → 2.`, γ1 → 0.`},
 {γ2 → -0.9343075950635839`, γ0 → 2.`, γ1 → 0.`}}
```

```
{{γ2 → -80.05833087020494` - 109.8153787495283` ⅈ,
  γ3 → 119.7654599121798` + 100.81576905305629` ⅈ, γ0 → 2.`, γ1 → 0.`},
 {γ2 → -80.05833087020494` + 109.8153787495283` ⅈ,
  γ3 → 119.7654599121798` - 100.81576905305629` ⅈ, γ0 → 2.`, γ1 → 0.`},
 {γ2 → -36.00128586033101` - 164.12747533815937` ⅈ,
  γ3 → 39.626957908282925` + 220.3144440239099` ⅈ, γ0 → 2.`, γ1 → 0.`},
 {γ2 → -36.00128586033101` + 164.12747533815937` ⅈ,
  γ3 → 39.626957908282925` - 220.3144440239099` ⅈ, γ0 → 2.`, γ1 → 0.`},
 {γ2 → -1.0521192366798373`, c3 → 0.1617057020400526`, c0 → 2.`, c1 → 0.`},
 {γ2 → 5.477418057270716` - 37.49060249939382` ⅈ,
  γ3 → -10.402140680299341` + 68.81312144858306` ⅈ, γ0 → 2.`, γ1 → 0.`},
 {γ2 → 5.477418057270716` + 37.49060249939382` ⅈ,
  γ3 → -10.402140680299341` - 68.81312144858306` ⅈ, γ0 → 2.`, γ1 → 0.`},
 {γ2 → 28.965398746885466` - 130.81784748004927` ⅈ,
  γ3 → -71.45186919442399` + 132.72530574090675` ⅈ, γ0 → 2.`, γ1 → 0.`},
 {γ2 → 28.965398746885466` + 130.81784748004927` ⅈ,
  γ3 → -71.45186919442399` - 132.72530574090675` ⅈ, γ0 → 2.`, γ1 → 0.`}}
```

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