CONTRIBUTIONS TO THE THEORY OF NEARNESS IN POINTFREE TOPOLOGY

by

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In times of great difficulty, it is our wisdom to keep our spirits calm, quiet, and sedate, for then we are in the best frame both to do our own work, and to consider the work of God...

- John Wesley (1703 - 1791)

Abstract

We investigate quotient-fine nearness frames, showing that they are reflective in the category of strong nearness frames, and that, in those with spatial completion, any near subset is contained in a near grill. We construct two categories, each of which is shown to be equivalent to that of quotient-fine nearness frames. We also consider some subcategories of the category of nearness frames, which are co-hereditary and closed under coproducts. We give due attention to relations between these subcategories. We introduce totally strong nearness frames, whose category we show to be closed under completions. We investigate N-homomorphisms and remote points in the context of totally bounded uniform frames, showing the role played by these uniform N-homomorphisms in the transfer of remote points, and their relationship with C^*-quotient maps. A further study on grills enables us to establish, among other things, that grills are precisely unions of prime filters. We conclude the thesis by showing that the lattice of all nearnesses on a regular frame is a pseudo-frame, by which we mean a poset pretty much like a frame except for the possible absence of the bottom element.

Contents

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Glossary

Categories:

Name	Objects	Morphisms	Page
AuNFrm	Almost uniform nearness frames	Uniform frame homomorphisms	63
Compl	Pairs (CL, L)	Droppable uniform frame homomorphisms	38
ConNFrm	Constrained nearness frames	Uniform frame homomorphisms	55
CntrNFrm	Controlled nearness frames	Uniform frame homomorphisms	59
CozNFrm	Cozero nearness frames	Uniform frame homomorphisms	73
CStrNFrm	Completely strong nearness frames	Uniform frame homomorphisms	67
Ext	Dense onto frame homomorphisms	Pairs of frame homomorphisms	34
FiNFrm	Fine nearness frames	Uniform frame homomorphisms	16
Frm	Frames	Frame homomorphisms	4
HZdNFrm	H-zero-dimensional nearness frames	Uniform frame homomorphisms	50
IntNFrm	Interpolative nearness frames	Uniform frame homomorphisms	63
Loc	Locales	Locale Maps	5
LfNFrm	Locally fine nearness frames	Uniform frame homomorphisms	22
NFrm	Nearness frames	Uniform frame homomorphisms	9
$\mathbf{NFrm}_\ell$	Nearness frames	Liftable uniform frame homomorphisms	21
QfNFrm	Quotient-fine nearness frames	Uniform frame homomorphisms	18
SmNFrm	Smooth nearness frames	Uniform frame homomorphisms	78

Name	Objects	Morphisms	Page
StrNFrm	Strong nearness frames	Uniform frame homomorphisms	11
TbNFrm	Totally bounded nearness frames	Uniform frame homomorphisms	75
Top	Topological spaces	Continuous maps	4
TStrNFrm	Totally strong nearness frames	Uniform frame homomorphisms	82
UCCNFrm	Uniformly Čech-complete nearness frames	Uniform frame homomorphisms	61
UniFrm	Uniform frames	Uniform frame homomorphisms	10
UnNFrm	Uniformly normal nearness frames	Uniform frame homomorphisms	67
UpnNFrm	Uniformly prenormal nearness frames	Uniform frame homomorphisms	67
UsCCNFrm	Uniformly Strongly Čech-complete nearness frames	Uniform frame homomorphisms	62
ZdNFrm	Uniformly zero-dimensional nearness frames	Uniform frame homomorphisms	46

Stone Separation Lemma [30]: *Let L be a distributive lattice, and suppose $I \subseteq L$ is an ideal and $F \subseteq L$ a filter such that $I \cap F = \emptyset$. Then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap F = \emptyset$.*

Axiom of Countable Dependent Choice (CDC) [40]: *If R is a binary relation on a set A such that, for each $x \in A$, $(x, y) \in R$ for some $y \in A$, and if p is an element of A, there exists a sequence $x_0, x_1, \ldots$ in A such that*

$$x_0 = p \ \text{ and } \ (x_n, x_{n+1}) \in R$$

for all $n = 0, 1, \ldots$.

Chapter 1

Introduction and preliminaries

1.1 A history of nearness in classical and pointfree topology

The concept of *nearness* in spaces was first introduced by H. Herrlich [31] in 1974 as an axiomatization of the concept of nearness between arbitrary collections of sets. Such a development can be envisaged from the fact that one can obtain *topological spaces* via axiomatizing the concept of nearness between a point x and a set A - namely, the requirement that "x belongs to the closure of A". By further axiomatizing the concept of nearness between two sets one obtains *proximity spaces*. Also, what are dubbed *contiguity spaces* arise from an axiomatization of nearness between finite collections of sets. Hence, in that sense, nearness spaces evolved naturally. For a study guide to nearness spaces we recommend [48].

Uniform spaces (which are a special type of nearness spaces) were first introduced by means of *entourages* by A. Weil [55] in 1937. The approach by means of *covers* was introduced by J.W. Tukey [52] in 1940, and well marketed especially in 1964 by J.R. Isbell, who in [37] states that if entourages and uniform covers are each used where "most convenient" in the study of uniform spaces "the result is that Tukey's system of uniform covers is used nine-tenths of the time". Quasi-uniform spaces (which, colloquially, are uniform spaces that lack symmetry) were first defined in terms of entourages by L.

Nachbin [46] in 1948. A cover-like approach for quasi-uniform spaces was given by T. Gantner and R. Steinlage [29] in 1972.

The study of topological properties from a lattice-theoretic viewpoint was initiated by H. Wallman [53] in 1938. The term *frame* was introduced by C.H. Dowker in 1966 and brought to the fore in the article [21] co-authored with D. Papert. The dual notion *locale* was introduced by J.R. Isbell in 1972 in the ground-breaking paper *Atomless Parts Of Spaces* [38]. In the words of B. Banaschewski [6], Isbell was able to put "the precise relationship between frames and spaces into categorical perspective". Locales have sometimes been regarded as generalized topological spaces, and the terms *pointless thinking* and *pointfree topology* have been used in relation to the categories **Loc** (of locales) and **Frm** (of frames) respectively. Indeed there are those (like B. Banaschewski [6]) who maintain that **Frm** is the context in which the actual constructions of topological concepts are done, whereas others (like P.T. Johnstone [41]) maintain that frame theory is lattice theory applied to topology and locale theory is topology itself. For an expository reference to frames/locales we recommend [39].

The concept of *nearness frame* was first announced to a group of mathematicians by B. Banaschewski (based on joint work with A. Pultr) in 1990 in a series of lectures delivered at the University of Cape Town. These lectures culminated with the 1996 article [16].

1.2 Synopsis of the thesis

Why study nearness frames (or nearness spaces), since it has been shown (see [16, Lemma 1]) that every *regular frame* admits a *nearness*? Well, regularity is indeed a much older and well understood concept. However, nearness was not introduced for the purposes of studying regularity. An analogy can be drawn from uniform spaces having been introduced as a generalization of metric spaces and as a topological study of completeness, only to find that topological spaces arising from uniform spaces are the completely regular ones. One could then hardly say what is the use as complete regularity is a well understood property. The study of nearness frames has more to do with tackling properties of *covers* of certain frames, which turn out to be the regular ones, than it is about the underlying frames.

As the title suggests, this thesis is indeed a contribution to the theory of nearness frames, through a study of over nine specific types of nearness frames, and numerous properties pertaining to elements of the underlying frames and morphisms between the nearness frames, giving due attention to relations between some of the categories concerned. Chapter 1 is essentially an introduction to frames and the structure of a nearness on a frame. Here we present the relevant definitions pertaining to frames, nearness frames and uniform frames, and outline the requisite background for the ensuing chapters. Some of the definitions are highlighted at certain instances in the body of the thesis for purposes of quick reference, and ensuring smooth-flowing arguments. As for standard references to the categorical notions used in the thesis, we recommend [42], [34] or the more recent [1] with an updated version made available online by the authors at http://katmat.math.uni-bremen.de/acc.

In Chapter 2 we introduce quotient-fine nearness frames as those which are quotients of fine ones. We characterize them as precisely those nearness frames whose completions are fine. We show that the subcategory they form is reflective in the category of strong nearness frames. We also consider briefly quotient-fine nearness frames with spatial completions; and show that in each such nearness frame, any near subset is contained in a near grill. We end the chapter by constructing two categories (**Ext** and **Compl**) each of which we show to be equivalent to the category of quotient-fine nearness frames.

In Chapter 3 we consider some subcategories of nearness frames which are co-hereditary and closed under coproducts (or countable coproducts in one instance) and characterize quotient-fine nearness frames among these. We introduce totally strong nearness frames, whose category we show is closed under completions.

In Chapter 4 we investigate, in the context of nearness frames, the notions of N-homomorphisms and remote points introduced in [27]. Our typical nearness frames in the chapter are the totally bounded uniform frames. We show how the uniform N-homomorphisms are related to C^*-quotient maps, and the role they play in the transfer of remote points.

Chapter 5 consists of a miscellany of unrelated results, commencing with a study on grills, initiated in Chapter 2 and establish, among other things, that grills are precisely unions of prime filters. We conclude the chapter by showing that the lattice of all nearnesses on a (regular) frame is a pseudo-frame, by which we mean a poset pretty much like a frame except for the possible absence of the bottom element.

1.3 Frames

In this section we review definitions pertaining to frames which we will need in the sequel. A *frame* L is a complete lattice satisfying the infinite distributive law: for any $a \in L$ and any $S \subseteq L$,

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

Thus, one envisages a frame L as having the lattice structure $(L, \wedge, \bigvee, 0, 1)$, where 0 is the *bottom* element, and 1 is the *top* element. A *frame homomorphism* (or *frame map*) between frames L and M is a map $h : L \longrightarrow M$ which preserves finite meets and arbitrary joins. In that case $h(0) = 0$ and $h(1) = 1$. We write **Frm** for the category of frames and frame homomorphisms. By a *subframe* P of a frame L, we mean $P \subseteq L$ where P is itself a frame under the same operations ($\wedge$ and $\bigvee$) as in L, with $0, 1 \in P$.

In our discussions that follow L will always be a frame unless otherwise stated.

A typical example of a frame is a topology $\mathcal{O}X$ on a set X. If $f : X \longrightarrow Y$ is a continuous map between topological spaces, then $f^{-1} : \mathcal{O}Y \longrightarrow \mathcal{O}X$ is a frame homomorphism. Clearly, this establishes a *contravariant functorial* relationship between the category **Top** of topological spaces and continuous maps and the category **Frm** as illustrated below:

$$\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}$$

$$\begin{array}{ccc} X & & \mathcal{O}X \\ \downarrow{\scriptstyle f} & & \uparrow{\scriptstyle f^{-1} = \mathcal{O}f} \\ Y & & \mathcal{O}Y \end{array}$$

Corresponding to any frame homomorphism $h : L \longrightarrow M$ is a map $h_* : M \longrightarrow L$, known as the *right adjoint* of h, which is not necessarily a frame homomorphism, but preserves arbitrary meets, and is defined by

$$h_*(y) = \bigvee \{x \in L \mid h(x) \leq y\}.$$

The following property holds for every $x \in L$ and every $y \in M$:

$$h(x) \leq y \iff x \leq h_*(y).$$

A frame homomorphism $h : L \longrightarrow M$ is *dense* if for every $a \in L$, $h(a) = 0$ implies $a = 0$. This holds if and only if $h_*(0) = 0$. A frame homomorphism $h : L \longrightarrow M$ is onto if and only if $hh_* = \mathrm{id}_M$.

The dual of **Frm** is the category **Loc** of *locales* and locale maps, an act of turning the arrows around and with far-reaching consequences. For an enlightening discussion on this note see [41].

We say that a subset $S \subseteq L$ *generates* L if for every element $x \in L$,

$$x = \bigvee \{s \in S \mid s \leq x\}.$$

We will occasionally be making use of pseudocomplements in our discussion, and here we give some definitions.

(1) Let $a \in L$. The element $\bigvee\{x \in L \mid a \wedge x = 0\} \in L$ is called the *pseudocomplement* of a and is denoted by a^*. We note that $a \wedge a^* = 0$. However $a \vee a^* = 1$ does not hold in general.

(2) In the case where $a \vee a^* = 1$, we say a is *complemented.*

(3) $a \in L$ is called *dense* if $a^* = 0$.

(4) For every $a \in L$, $a \leq a^{**}$ always holds. If $a = a^{**}$, then a is called a *regular* element.

A frame is *zero-dimensional* if every element is the join of complemented elements below it. If all the elements of a frame are regular, then the frame is called *Boolean.*

We call $D \subseteq L$ a *downset* if $x \in D$ and $y \leq x$ implies $y \in D$, and we call $U \subseteq L$ an *upset* if $u \in U$ and $u \leq v$ implies $v \in U$. For any $a \in L$, we write

$${\downarrow} a = \{x \in L \mid x \leq a\}, \text{ which is a downset,}$$

and

$$\uparrow a = \{y \in L \mid a \le y\}, \text{ which is an upset.}$$

We note that $\downarrow a$ is a frame whose bottom is $0 \in L$ and top is a. Similarly, $\uparrow a$ has $1 \in L$ as its top and a as its bottom.

We call $J \subseteq L$ an *ideal* if it satisfies:

i1. $0 \in J$.

i2. $b \in J$ and $a \le b$ implies $a \in J$. (i.e. J is a downset).

i3. $a, b \in J$ implies $a \vee b \in J$.

A subset $F \subseteq L$ is called a *filter* if it satisfies the properties:

f1. $0 \notin F$ and $1 \in F$.

f2. $a \in F$ and $a \le b$ implies $b \in F$. (i.e. F is an upset).

f3. $a, b \in F$ implies $a \wedge b \in F$.

$F \subseteq L$ is called a *prime filter* if it is a filter and satisfies:

$$a \vee b \in F \text{ implies } a \in F \text{ or } b \in F.$$

A filter $U \subseteq L$ is called an *ultrafilter* if for any filter $F \subseteq L$, whenever $U \subseteq F$, then $U = F$.

In the following lemma we collect some results concerning filters, where item (iii) is a characterization of ultrafilters shown in [26].

Lemma 1.3.1 *In a given frame L we have the following:*

(i) Every ultrafilter is a prime filter.

(ii) Every ultrafilter contains all dense elements in L.

(iii) A filter $F \subseteq L$ is an ultrafilter iff for each $a \in L$, either $a \in F$ or $a^* \in F$.

An *extension* of a frame L is a dense onto homomorphism $h : M \longrightarrow L$. By abuse of language, we say an extension $h : M \longrightarrow L$ of L has a property Ω of frames if the frame M has the property Ω.

A result often used in frame theory is that every frame homomorphism $h : L \longrightarrow M$ has a *dense-onto factorization*

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[rr, bend right, "h"'] & \uparrow h_*(0) \arrow[r, "\bar{h}"] & M \end{tikzcd} ,$$

where φ is the onto homomorphism $x \mapsto x \vee h_*(0)$ and $\bar{h}$ the dense homomorphism mapping as h.

The notion of regularity in frames plays a major role in the theory of nearness frames. First, recall the *well inside* or *rather below* relation $\prec$ on a frame L defined by: $y \prec x$ iff there is $z \in L$ (called a *separating element*) such that $y \wedge z = 0$ and $x \vee z = 1$. We say a frame L is *regular* if every $x \in L$ is expressible as

$$x = \bigvee \{y \in L \mid y \prec x\} .$$

Next, we have the notion of complete regularity, which is defined by means of scales in a frame. By a *scale* in a frame L we mean a countable (rational-number) indexed subset

$$\{c_q \mid q \in \mathbb{Q} \cap [0,1]\} = (c_q)$$

of L such that whenever $p < q$, then $c_p \prec c_q$. We define the *completely below* relation $\prec\!\prec$ on L by: $a \prec\!\prec b$ if there is a scale (c_q) such that $a \leq c_0$ and $c_1 \leq b$. We say L is *completely regular* if every $x \in L$ is expressible as

$$x = \bigvee \{y \in L \mid y \prec\!\prec x\} .$$

Remark 1.3.2 Following the practice in [5] and [49], in place of (c_q), we resort to the labeling (c_{nk}), where $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$, especially in cases where we need to invoke the axiom of Countable Dependent Choice (CDC). So $a \prec\!\prec b$ in a frame L in case there exists a family (c_{nk}) of elements of L such that

$$c_{00} = a,\ c_{01} = b,\ c_{nk} = c_{n+1\,2k},\ c_{nk} \prec c_{nk+1}$$

for all $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$. One says that (c_{nk}) is an *interpolating* sequence (relative to the relation $\prec$) between a and b.

A frame L is *normal* if for any elements a, $b \in L$, if $a \vee b = 1$, then there are elements c, $d \in L$ such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$.

By a *cover* A of a frame L we mean a subset of L such that $\bigvee A = 1$. We write $\mathrm{Cov}(L)$ for the set of all covers of the frame L. The frame L is *compact* if for any $A \in \mathrm{Cov}(L)$, there is a finite $F \subseteq A$ in $\mathrm{Cov}(L)$.

By a *compactification* of L we mean a dense onto frame homomorphism $h : M \longrightarrow L$ with M being a compact regular frame and L completely regular. It is customary to denote a compactification by $\gamma L \longrightarrow L$, taking γ to be the compactification homomorphism. The Stone-Čech compactification of L is normally denoted by $\beta L \longrightarrow L$ (or simply βL).

A cover C in a frame L is said to be *locally finite* if there exists $D \in \mathrm{Cov}(L)$ such that for every $y \in D$, the set

$$\{x \in C \mid x \wedge y \neq 0\}$$

is finite. One says that the cover D *finitizes* the cover C [51]. L is said to be *paracompact* if every cover $A \in \mathrm{Cov}(L)$ has a locally finite refinement. It should be noted here that compactness implies paracompactness.

1.4 Nearness frames

Regarding nearness frames, Banaschewski [6] writes:

> In classical topology, the concepts of uniformity and nearness are entities assigned to a specified set. In the context of frames the specified object (namely a frame) is already a "topology", and consequently nearnesses and uniformities become additional structures on a frame.

In this section we lay out the necessary terminology for these *structured frames*.

For covers A and B in a frame L we say A *refines* B and write $A \leq B$ if for every $a \in A$, there exists $b \in B$ such that $a \leq b$. We write $\mathrm{FCov}(L)$ for the collection of all covers of L refined by some finite cover.

The *star* of $x \in L$ with respect to a cover A of L is the element

$$Ax = \bigvee \{a \in A \mid a \wedge x \neq 0\}.$$

Further, we write $AB = \{Ax \mid x \in B\}$ and $A \wedge B = \{a \wedge b \mid a \in A,\ b \in B\}$ each of which is a cover of L if A and B are both covers. We say A *star-refines* B, written $A \leq^* B$, if $AA \leq B$.

Given a collection $\mu \subseteq \mathrm{Cov}(L)$, we say $x \in L$ is *μ-strongly below* $y \in L$, written $x \lhd_\mu y$ (or simply $x \lhd y$) if there is a cover $A \in \mu$ such that $Ax \leq y$.

We may now state the definition of nearness frames.

Definition 1.4.1 A nonempty collection $\mu \subseteq \mathrm{Cov}(L)$ is called a *nearness* on L if the following hold:

n1. Whenever $A \in \mu$ refines $B \in \mathrm{Cov}(L)$, then $B \in \mu$.

n2. Whenever A, $B \in \mu$, then $A \wedge B \in \mu$.

n3. Every $x \in L$ can be expressed as

$$x = \bigvee \{y \in L \mid y \lhd_\mu x\}.$$

This property is referred to as the *admissibility property.*

In the case where μ is a nearness on L, we refer to $\lhd_\mu$ as the *uniformly below* relation on L, oftentimes dropping the index and simply writing $\lhd$ when the nearness on L is understood. The pair (L, μ) is called a *nearness frame*, and members of μ are called *uniform covers.*

A map $h : (L, \mu) \longrightarrow (M, \eta)$ between nearness frames is called a *uniform frame homomorphism* if it is a frame homomorphism and for every $A \in \mu,\ h[A] \in \eta$. We write **NFrm** for the resulting category of nearness frames and uniform frame homomorphisms. Throughout the thesis, whenever we define a subcategory of **NFrm** it is understood that the morphisms of the subcategory are the uniform frame homomorphisms.

A nearness μ on a frame L is said to be *generated* by $\nu \subseteq \mu$ if for every $A \in \mu$ there exists $B \in \nu$ such that $B \leq A$.

Let (L,μ) be a nearness frame. We say the nearness μ on L is *induced* by an extension $h : M \longrightarrow L$ if

$$\mu = h[\mathrm{Cov}(M)] = \{h[C] \mid C \in \mathrm{Cov}(M)\}.$$

Now, if (X,ξ) is a nearness space with $x \in X$ and A, $B \subseteq X$, we recall that $x \lhd_\xi A$ iff $\{X-\{x\}, A\} \in \xi$ iff $x \in \mathrm{int}_\xi(A)$, and that $B \lhd_\xi A$ iff $\{X-B, A\} \in \xi$ (see, for example, [32] and [48]). We say that (X,ξ) is *framed* if whenever $x \lhd_\xi A$ in (X,ξ), there exists $B \subseteq X$ such that $x \lhd_\xi B \lhd_\xi A$. We further take note of the observation in [36] that a nearness space (X,ξ) is framed iff the family μ of open uniform covers in (X,ξ) is a nearness on the associated frame $\mathcal{O}X$ of open subsets of X.

Definition 1.4.2 A nearness is called a *uniformity* if every uniform cover has a uniform star refinement.

The pair $(L,\mathcal{U})$ is a *uniform frame* if $\mathcal{U}$ is a uniformity on L. By definition, the notion of nearness is weaker than that of uniformity. We write **UniFrm** for the category of uniform frames and uniform frame homomorphisms.

We shall frequently use the following properties of the relation $\lhd$.

(1) If $x \lhd y$, $a \leq x$ and $y \leq b$, then $a \lhd b$.

(2) If $x \lhd y$ and $a \lhd b$, then $x \wedge a \lhd y \wedge b$ and $x \vee a \lhd y \vee b$.

(3) If μ is a uniformity, then $x \lhd y$ implies $x \lhd z \lhd y$ for some $z \in L$.

Concerning the transfer of nearnesses and uniformities via frame homomorphisms, the following result appears in [6].

Lemma 1.4.3 *If (L,μ) is a nearness (or uniform) frame and $h : L \longrightarrow M$ an onto frame homomorphism, then the set*

$$\eta = \{h[A] \mid A \in \mu\} = h[\mu]$$

is a nearness (or uniformity) on the frame M.

The following results appear in [4]:

Lemma 1.4.4 *(i) Every frame has a nearness iff it is regular.*

(ii) If L is a compact regular frame, then $\mathrm{Cov}(L)$ *is the unique nearness on L, and, in fact,* $\mathrm{Cov}(L)$ *is a uniformity on L.*

As a consequence of the above lemma, all frames in this thesis are taken to be regular. In contrast to Lemma 1.4.4(i), regarding which frames admit uniformities, we have the following lemma with two results which appear in [49] and [51], respectively.

Lemma 1.4.5 *(i) A frame has a uniformity iff it is completely regular.*

(ii) If L is a regular frame, then $\mathrm{Cov}(L)$ *is a uniformity iff the frame L is paracompact.*

A nearness frame (L, μ) is said to be *fine* if $\mu = \mathrm{Cov}(L)$. Fine nearness frames have their classical counterparts in topological nearness spaces.

Let (L, μ) be a nearness frame and $C \in \mu$. We observe that the set

$$\check{C} = \{z \in L \mid z \lhd y \text{ for some } y \in C\}$$

is a cover of L (not necessarily a uniform cover). Whenever $\check{C} \in \mu$ for any $C \in \mu$, we say that (L, μ) is a *strong* nearness frame. Strong nearness frames, which have their classical counterparts in regular nearness spaces, play an important role in this thesis.

We write **StrNFrm** for the category of strong nearness frames and uniform frame homomorphisms. It should be immediately clear that any uniform frame is a strong nearness frame; for whenever $B \leq^* C$ in a uniformity $\mathcal{U}$, we have $B \leq \check{C}$ since $Bx \leq y$ implies $x \lhd y$. Therefore $\check{C} \in \mathcal{U}$, so that $\mathcal{U}$ is strong.

The following definitions standardize the terminology we will be using concerning uniform frame homomorphisms throughout this thesis. Let (L, μ) and (M, η) be nearness frames and $h : L \longrightarrow M$ a uniform frame map. Then:

(i) h is a *surjection* if it is onto and $\eta = \{h[A] \mid A \in \mu\}$.

(ii) We shall often refer to surjections as *quotient maps*. In this case we refer to the nearness frame (M, η) as a *quotient* of (L, μ).

(iii) We say that h is a *strict surjection* if it is a dense surjection and the image of η under the right adjoint $h_* : M \longrightarrow L$, $h_*[\eta] = \{h_*[C] \mid C \in \eta\}$, generates the nearness μ.

In the above definitions we have adopted the terminology in [6]. We note that there is a slight variation in the manner in which the terms *surjection* and *quotient map* are defined in [4].

The following results appear in [6]:

Lemma 1.4.6 *(i) If (L, μ) is a strong nearness frame, then any dense surjection $h : (L, \mu) \longrightarrow (M, \eta)$ is strict.*

(ii) A quotient of a strong nearness frame is strong.

(iii) If $h : (L, \mu) \longrightarrow (M, \eta)$ is a dense surjection, then (L, μ) is strong iff (M, η) is strong.

(iv) There are nearness frames (L, μ) where a dense surjection $h : L \longrightarrow M$ is not necessarily strict.

Let $h : (L, \mu) \longrightarrow (M, \eta)$ be a uniform frame homomorphism. We will also need the following results which appear in [6].

Lemma 1.4.7 *(i) If $a \lhd b$ in L, then $h(a) \lhd h(b)$ in M.*

(ii) If h is a dense surjection, then $a \lhd b$ in L implies $h_*h(a) \leq b$.

(iii) If h is a strict surjection, then $x \lhd y$ in M iff $h_*(x) \lhd h_*(y)$ in L.

(iv) If h is a strict surjection, then for any $a \in L$ and any $x \in M$, we have $a \lhd h_*(x)$ in L iff $h(a) \lhd x$ in M.

A nearness frame (L, μ) is said to be *complete* if every strict surjection $h : (M, \eta) \longrightarrow (L, \mu)$ is an isomorphism. A *completion* of (L, μ) is a strict surjection $h : (K, \nu) \longrightarrow (L, \mu)$, where (K, ν) is a complete nearness frame. A uniform frame homomorphism $h : K \longrightarrow L$ is called a *weak completion* if it is a dense surjection with (K, ν) being complete.

The following results appear in [6].

Lemma 1.4.8 *(i) Every fine nearness frame is complete.*

(ii) Every compact nearness frame is complete.

(iii) Every nearness frame has a unique completion.

(iv) The completion of a strong nearness frame is strong.

Given a nearness frame (L, μ), we denote its completion by $(CL, C\mu)$, often referring to the strict surjection $\gamma_L : CL \longrightarrow L$ as the *completion map.* According to the construction given in [6], CL is the frame generated by the downsets $\downarrow a$ for $a \in L$. The completion map $\gamma_{_L} : CL \longrightarrow L$, is defined by

$$\gamma_{_L}(D) = \bigvee D,$$

for each $D \in CL$, and is *universal* in the sense that for any strict surjection $h : M \longrightarrow L$, there exists a strict surjection $g : CL \longrightarrow M$ such that $hg = \gamma_{_L}$. The right adjoint

$$(\gamma_{_L})_* : L \longrightarrow CL$$

has the property that for every $a \in L$,

$$(\gamma_{_L})_*(a) = \downarrow a = \bigvee\{(\gamma_{_L})_*(x) \mid x \lhd_\mu a\}$$

Using the abbreviation $r_{_L}$ for the right adjoint, the nearness $C\mu$ is the one generated by the collection

$$\{r_{_L}[A] \mid A \in \mu\}.$$

Let (L, μ) be a nearness frame. Write $a \lhd\!\lhd b$, to be read "a is uniformly completely below b", if there is an interpolating sequence (c_{nk}) in L between a and b, where

$$c_{00} = a,\ c_{01} = b,\ \ c_{nk} = c_{n+1\,2k},\ \text{and}\ c_{nk} \lhd c_{n\,k+1}$$

for all $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$.

We say a nearness frame (L, μ) is *interpolative* or has the *interpolation property* if, for every $a, b \in L,\ a \lhd b$ implies $a \lhd c \lhd b$ for some $c \in L$. An *almost uniform* nearness frame is one which is strong and interpolative. Clearly, uniform frames are almost uniform nearness frames.

A nearness frame (L, μ) is *totally bounded* if every $A \in \mu$ is refined by some finite $B \in \mu$. In the following lemma we gather some results from [25] about totally bounded nearness frames.

Lemma 1.4.9 *(i) A nearness frame is totally bounded iff every uniform cover has a finite uniform subcover.*

(ii) A totally bounded nearness frame is strong iff it is uniform.

(iii) Every nearness frame (L, μ) has a totally bounded coreflection given by (L, μ_T), where

$$\mu_T = \{A \in \mu \mid B \leq A \text{ for some finite } B \in \mu\}.$$

Let (L, μ) be a nearness frame and (L, μ_T) its totally bounded coreflection. For any a, $b \in L$, we write $a \lhd_T b$ if there exists $C \in \mu_T$ such that $Ca \leq b$. The following lemma contains results which appear in [25], and will come in handy in some instances in our discussion.

Lemma 1.4.10 *Let (L, μ) be a nearness frame, and a, $b \in L$. Then*

(i) $a \lhd_T b$ iff $a \lhd b$.

(ii) If (L, μ_T) is strong, then $a \lhd b$ iff $a \lhd\!\lhd b$.

Concerning binary coproducts of nearness frames, let (L, μ) and (M, η) be nearness frames. Then their coproduct is the nearness frame

$$(L \oplus M, \mu \oplus \eta),$$

where $L \oplus M$ is the coproduct in **Frm**, generated by members of the form $a \oplus b$ as specified in [24]. Let $A \in \mu$ and $B \in \eta$, and form the set

$$A \oplus B = \{a \oplus b \mid a \in A,\, b \in B\},$$

which is a cover of $L \oplus M$. Then

$$\mu \oplus \eta = \{C \in \mathrm{Cov}(L \oplus M) \mid A \oplus B \leq C, \text{ for some } A \in \mu \text{ and } B \in \eta\}.$$

For further explanations see [24], and for coproducts of arbitrary families of nearness frames see [50]. We write $(\oplus_i L_i, \oplus_i \mu_i)$ for the coproduct of a family $\{(L_i, \mu_i)\}_{i \in I}$ of nearness frames. For each $i \in I$,

$$\iota_i : L_i \longrightarrow \oplus_i L_i$$

is the coproduct injection. The frame $\oplus_i L_i$ is generated by elements of the form

$$\oplus_i a_i = \bigwedge_i \iota_i(a_i)\,,$$

where the $a_i \in L_i$ are such that only finitely many of them are not equal to 1. Each $A \in \oplus_i \mu_i$ is refined by a cover of the form $\oplus_i A_i$, where the $A_i \in \mu_i$ are such that only finitely many of them are nontrivial (i.e. $\neq \{1\}$). The results in the following lemma appear in [50].

Lemma 1.4.11 *The elements $\oplus_i a_i$ have the following properties:*

(i) $\oplus_i a_i = 0$ *iff* $a_i = 0$ *for some* $i \in I$.

(ii) $0 \neq \oplus_i a_i \leq \oplus_i b_i$ *iff for all* $i \in I$, $a_i \leq b_i$.

(iii) $0 \neq \oplus_i a_i \lhd \oplus_i b_i$ *in* $(\oplus_i L_i, \oplus_i \mu_i)$ *iff for all* $i \in I$, $0 \neq a_i \lhd b_i$ *in* (L_i, μ_i).

Let (L, μ) be a nearness frame. Then a subset $A \subseteq L$ is said to be *near* (introduced in [22]) if for any $C \in \mu$, there is $x \in C$ such that for every $a \in A$, $x \wedge a \neq 0$. The following characterization of near subsets appears in [22].

Lemma 1.4.12 *Let (L, μ) be a nearness frame. Then $A \subseteq L$ is a near subset if and only if the set*

$$A^* = \{a^* \mid a \in A\} \notin \mu.$$

Given a nearness frame (L, μ), a nonempty subset $A \subseteq L$ is called a *cluster* if it is a near subset, and whenever $A \subseteq B \subseteq L$ with B near, then $A = B$. Thus, a cluster is a *maximal near subset* of L.

Chapter 2

Quotient-fine nearness frames

In this chapter we introduce quotient-fine nearness frames as quotients of fine nearness frames. We characterize them as nearness frames whose completions are fine. We show that the subcategory of quotient-fine nearness frames resides reflectively in that of strong nearness frames. We consider briefly quotient-fine nearness frames with spatial completions, showing that in each such nearness frame, any near subset is contained in a near grill. We end the chapter by constructing two categories, namely **Ext** and **Compl**, each of which we show to be equivalent to that of quotient-fine nearness frames.

2.1 The reflectiveness of QfNFrm in StrNFrm

In this section we define quotient-fine nearness frames, showing that their subcategory is reflective in **StrNFrm**. We characterize quotient-fine nearness frames which are separable, and those which are uniformly paracompact. We also consider those which have spatial completions. We end the section by introducing f-fine nearness frames, and providing a way of constructing certain types of quotient-fine nearness frames which are f-fine.

First, we recall that (L, μ) is a fine nearness frame if $\mu = \mathrm{Cov}(L)$. We write **FiNFrm** for the category of fine nearness frames. We immediately observe that **FiNFrm** is reflective in **NFrm**, with

$$(L, \mu) \xrightarrow{\mathrm{id}_L} (L, \mathrm{Cov}(L))$$

being the reflection arrow for any nearness frame (L, μ). To see this, note that for any uniform frame homomorphism $h : (L,\mu) \longrightarrow (M, \mathrm{Cov}(M))$, the following diagram commutes:

$$\begin{tikzcd}
(L,\mu) \arrow[rr, "\mathrm{id}_L"] \arrow[dr, "h"'] & & (L, \mathrm{Cov}(L)) \arrow[dl, "\bar{h}=h"] \\
& (M, \mathrm{Cov}(M)) &
\end{tikzcd}$$

Clearly $\bar{h} = h$ is uniform and is the unique map making the triangle commutative.

The following result is indicated as trivial in [4], but nevertheless, we present a proof for it.

Proposition 2.1.1 *The category* **FiNFrm** *resides in* **StrNFrm**.

Proof: Let $(L, \mathrm{Cov}(L))$ be a fine nearness frame and let $A \in \mathrm{Cov}(L)$. We need to show that the set

$$\check{A} = \{x \in L \mid x \lhd a, \text{ for some } a \in A\}$$

belongs to $\mathrm{Cov}(L)$. Now $a \prec b$ iff $a \lhd b$ since the nearness on L is $\mathrm{Cov}(L)$. Given $a \in A$, let

$$C_a = \{x \in L \mid x \prec a\} = \{x \in L \mid x \lhd a\}.$$

Note that $a = \bigvee C_a$, and $C_a \subseteq \check{A}$. So we have

$$1 = \bigvee A = \bigvee_{a \in A} C_a \leq \bigvee \check{A}.$$

This means $\check{A} \in \mathrm{Cov}(L)$ as required. ∎

The argument used in establishing the observation that fine nearness frames are reflective in the category of nearness frames, can be applied in establishing the following result by simply replacing (L,μ) in the stated diagram with a strong nearness frame.

Corollary 2.1.2 **FiNFrm** *resides reflectively in* **StrNFrm**.

Definition 2.1.3 A nearness frame (L,μ) is called *quotient-fine* (the term *subfine* is used in [25]) if there is a fine nearness frame $(M, \mathrm{Cov}(M))$ and an onto frame homomorphism $h : M \longrightarrow L$ such that $\mu = \{h[C] \mid C \in \mathrm{Cov}(M)\}$. Thus, a nearness frame is quotient-fine if it is the quotient of a fine nearness frame.

In this chapter we shall frequently use the abbreviation *q-fine* in place of quotient-fine.

Remark 2.1.4 Clearly every fine nearness frame is q-fine with the *strongness* property on fine nearness frames being transferred to the q-fine ones via the surjections. However, not every q-fine nearness frame is fine. To see this, take any noncompact completely regular frame and endow it with the nearness it inherits from its Stone-Čech compactification.

We denote the category of q-fine nearness frames by **QfNFrm**. The following result characterizes when a q-fine nearness frame is fine.

Proposition 2.1.5 *Let* (L, μ) *be a q-fine nearness frame via the uniform frame homomorphism* $h : M \longrightarrow L$. *Then* (L, μ) *is fine iff whenever* $A \in \mathrm{Cov}(L)$, *we have* $h_*[A] \in \mathrm{Cov}(M)$.

Proof: ($\Rightarrow$) Let $A \in \mathrm{Cov}(L)$. Then $A = h[C]$, for some $C \in \mathrm{Cov}(M)$. For any $c \in C$, we have $c \leq h_*h(c) = h_*[h(c)]$, by definition of the right adjoint h_*. This implies that $C \leq h_*[h[C]] = h_*[A]$. Consequently $h_*[A] \in \mathrm{Cov}(M)$.

($\Leftarrow$) Given the condition, let $A \in \mathrm{Cov}(L)$. Then $h_*[A] \in \mathrm{Cov}(M)$. Since h is onto, we have $h[h_*[A]] = A$. Hence $A \in \mu$, so that (L, μ) is fine. ■

It has now become clear that **QfNFrm** $\subseteq$ **StrNFrm**. We will show that **QfNFrm** resides reflectively in **StrNFrm**. To facilitate our demonstration, we state and prove the following result which shows that in Definition 2.1.3, the map h may be taken to be dense; in which case the fine nearness frame $(M, \mathrm{Cov}(M))$ will be a completion of (L, μ).

Lemma 2.1.6 *The following are equivalent for a nearness frame* (L, μ):

(1) (L, μ) *is quotient-fine.*

(2) There is a dense onto homomorphism $h : M \longrightarrow L$ *such that* $\mu = h[\mathrm{Cov}(M)]$.

(3) The completion of (L, μ) *is fine.*

Proof: (1) $\Rightarrow$ (2): By (1) there is an onto homomorphism $g : K \longrightarrow L$ such that $\mu = g[\mathrm{Cov}(K)]$. Consider the dense-onto factorization

$$K \xrightarrow{\varphi} \uparrow g_*(0) \xrightarrow{\bar{g}} L$$

where φ is the map $x \mapsto x \vee g_*(0)$ and $\bar{g}$ maps as g. Since g is onto, $\bar{g}$ is also onto, so that it is dense onto. Any cover of $\uparrow g_*(0)$ is a cover of K; so $\bar{g}[\mathrm{Cov}(\uparrow g_*(0))] \subseteq g[\mathrm{Cov}(K)]$. On the other hand, let $C \in \mathrm{Cov}(K)$. Then the set $D = \{g_*(0) \vee c \mid c \in C\}$ is a cover of $\uparrow g_*(0)$ such that $g[C] = \bar{g}[D]$. This shows that $g[\mathrm{Cov}(K)] \subseteq \bar{g}[\mathrm{Cov}(\uparrow g_*(0))]$, and hence equality. Thus, $\mu = \bar{g}[\mathrm{Cov}(\uparrow g_*(0))]$.

(2) $\Rightarrow$ (3): Since M with its fine nearness is complete, it suffices to show that h is a strict surjection. Note that h is a surjection, and therefore $h_*[U]$ is a uniform cover of M for each uniform cover U of L. Now let C be a uniform cover of M. Then $h[\check{C}]$ is a uniform cover of L, and so, in light of denseness of h, $h_*h[\check{C}] \leq C$. Thus, h is a strict surjection, and hence M is the completion of L.

(3) $\Rightarrow$ (1): This is so because the map $\gamma_L : CL \longrightarrow L$ is a surjection. ∎

Remark 2.1.7 (1) From Lemma 2.1.6, we deduce that a q-fine nearness frame is fine iff it is complete.

(2) The following example is worth noting. In [6, Section 5.1] it is shown that the completion of a nearness frame is compact iff the nearness frame is totally bounded and uniform. Thus, in view of Lemma 1.4.9(ii), *every totally bounded strong nearness frame is q-fine.*

Proposition 2.1.8 **QfNFrm** *resides reflectively in* **StrNFrm**.

Proof: Let (L, μ) be a strong nearness frame. We need to construct a q-fine reflection $(QL, Q\mu)$ for (L, μ). Let $\gamma_L : CL \longrightarrow L$ be the completion of (L, μ). Equip the frame CL with the fine nearness $\mathrm{Cov}(CL)$.

Put

$$QL = L \ \text{ and } \ Q\mu = \{\gamma_L[D] \mid D \in \mathrm{Cov}(CL)\}.$$

First, we show that $(QL, Q\mu)$ is a q-fine nearness frame. But this should be immediate, since $(CL, \mathrm{Cov}(CL))$ is fine, and $\gamma_L : CL \longrightarrow QL$ is dense, onto with $Q\mu = \gamma_L[\mathrm{Cov}(CL)]$.

Next, we show that $(QL, Q\mu)$ is indeed a reflection of (L, μ), with

$$\mathrm{id}_L : (L, \mu) \longrightarrow (QL, Q\mu)$$

as the reflection arrow. For any uniform frame homomorphism $h : (L, \mu) \longrightarrow (K, \eta)$, where (K, η) is a q-fine nearness frame, we need a unique uniform frame homomorphism

$$\bar{h} : (QL, Q\mu) \longrightarrow (K, \eta)$$

such that the diagram

$$\begin{tikzcd}
(L,\mu) \arrow[rr, "\mathrm{id}_L"] \arrow[dr, "h"'] & & (QL, Q\mu) \arrow[dl, "\bar{h}"] \\
& (K,\eta) &
\end{tikzcd}$$

commutes. First, to see that id_L is a uniform map, let $A \in \mu$. Now $(\gamma_L)_*[A] \in C\mu$, since γ_L is a strict map, and note that $C\mu \subseteq \mathrm{Cov}(CL)$. Therefore

$$\mathrm{id}_L[A] = A = \gamma_L(\gamma_L)_*[A] \in \gamma_L[\mathrm{Cov}(CL)] = Q\mu.$$

Second, put $\bar{h} = h$. This makes $\bar{h}$ a frame homomorphism. To see that $\bar{h}$ is uniform, consider the completion $\gamma_K : (CK, C\eta) \longrightarrow (K, \eta)$ of (K, η). Since (L, μ) and (K, η) are strong nearness frames, we have the following commutative diagram (see [6]):

$$\begin{tikzcd}
(CL, C\mu) \arrow[r, "Ch"] \arrow[d, "\gamma_L"'] & (CK, C\eta) \arrow[d, "\gamma_K"] \\
(L,\mu) \arrow[r, "h"'] & (K,\eta)
\end{tikzcd}$$

i.e. $h\gamma_L = \gamma_K(Ch)$. Therefore, for any uniform cover $\gamma_L[D]$ of QL, where $D \in \mathrm{Cov}(CL)$, we have

$$\bar{h}[\gamma_L[D]] = \bar{h}\gamma_L[D] = h\gamma_L[D] = \gamma_K(Ch)[D] \in \eta.$$

Lastly, it is trivial that $\bar{h}(\mathrm{id}_L) = h$ and that $\bar{h}$ is unique. ∎

We thus have a functor

$$\mathbf{StrNFrm} \xrightarrow{Q} \mathbf{QfNFrm}$$

$$\begin{array}{ccc} (L,\mu) & & (QL,Q\mu) \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle Qh} \\ (M,\eta) & & (QM,Q\eta) \end{array}$$

where $Qh(x) = h(x)$. The only instance when the strongness of (L,μ) was used in the proof to Proposition 2.1.8 was when we lifted h to completions. Thus the category **StrNFrm** can be replaced by the larger category $\mathbf{NFrm}_\ell$ of nearness frames and *liftable* uniform homomorphisms.

The following result is clear from the construction above.

Corollary 2.1.9 *If* (L,μ) *is a complete nearness frame, then its q-fine reflection is* $(L, \mathrm{Cov}(L))$.

Let C denote the completion functor on **StrNFrm**. In the next result we show that the functors C and Q commute on **StrNFrm**.

Corollary 2.1.10 *For any strong nearness frame* (L,μ), $QC(L,\mu) = CQ(L,\mu)$.

Proof: By construction, and in the light of the preceding corollary, we have

$$QC(L,\mu) = Q(CL, C\mu) = (CL, \mathrm{Cov}(CL)).$$

On the other hand, note that the map

$$(CL, \mathrm{Cov}(CL)) \longrightarrow (L, \gamma_L[\mathrm{Cov}(CL)]),$$

mapping as γ_L, is a strict surjection; a consequence of which is that $(CL, \mathrm{Cov}(CL))$ is the completion of $(L, \gamma_L[\mathrm{Cov}(CL)]) = Q(L,\mu)$. Therefore

$$CQ(L,\mu) = C(QL, Q\mu) = (CL, \mathrm{Cov}(CL)) = Q(CL, C\mu) = QC(L,\mu)$$

as claimed. ■

Quotient-fine nearness frames, as we have defined them, can be thought of as frame analogues of what Bentley and Herrlich [18] call subtopological nearness spaces, if one restricts to what are called regular nearness spaces. Indeed a regular nearness space is subtopological if and only if its completion is topological - a result in line with one of the characterizations in Lemma 2.1.6.

Although there are similarities between the two, there are also differences. One such difference regards subtopological coreflections of uniform spaces and q-fine reflections of uniform frames. Whereas the subtopological coreflection of a uniform space need not be uniform [18, Example 10], the q-fine reflection of a uniform frame is always uniform as we show below. Recall from [6] and [14] that *a completely regular frame L is paracompact iff it admits a complete uniformity iff* $\mathrm{Cov}(L)$ *is a uniformity.* In consequence we have

Corollary 2.1.11 *The q-fine reflection of a uniform frame is also a uniform frame.*

Proof: Let (L, μ) be a uniform frame, and $(CL, C\mu)$ its completion, which is also a uniform frame [6]. Since the frame CL is paracompact, $\mathrm{Cov}(CL)$ is a uniformity. Now the q-fine reflection of (L, μ) is

$$Q(L, \mu) = (QL, Q\mu) = (L, \gamma_{_L}[\mathrm{Cov}(CL)]).$$

The nearness $\gamma_{_L}[\mathrm{Cov}(CL)]$ is a uniformity since the completion arrow $\gamma_{_L}$ is onto [6]. Hence $(QL, Q\mu)$ is a uniform frame. ∎

We recall that a nearness frame (L, μ) is *locally fine* if for any cover $A \in \mu$ and a family of covers $\{B_a \mid a \in A\} \subseteq \mu$, we have that

$$\{a \wedge b \mid a \in A \text{ and } b \in B_a\} \in \mu.$$

We adopt the notation $A \wedge (B_a)_{_A}$ for the latter cover. We write **LfNFrm** for the category of locally fine nearness frames.

Proposition 2.1.12 *(i) Every fine nearness frame is locally fine.*

(ii) Every q-fine nearness frame is locally fine.

Proof: (i) Given a fine nearness frame $(L, \mathrm{Cov}(L))$, let A be a cover and $\{B_a \mid a \in A\}$ be a family of covers. Then

$$\bigvee [A \wedge (B_a)_A] = \bigvee A \wedge \bigvee_{a \in A} \bigvee B_a = 1 \wedge 1 = 1.$$

So therefore $A \wedge (B_a)_A \in \mathrm{Cov}(L)$.

(ii) Let (L, ν) be a q-fine nearness frame. Then exists a dense onto uniform frame homomorphism $h : (M, \mathrm{Cov}(M)) \longrightarrow (L, \nu)$ with $\nu = h[\mathrm{Cov}(M)]$. Let $A \in \nu$ and $\{B_a \mid a \in A\}$ a family of uniform covers of L. Now we have $U,\ D_a \in \mathrm{Cov}(M)$ such that $A = h[U]$ and $B_a = h[D_a]$ for each $a \in A$.

Then

$$A \wedge (B_a)_A = h[U] \wedge (h[D_a])_A = h[U \wedge (D_a)_A] \in \nu.$$

Hence (L, ν) is locally fine. ∎

We have now established the categorical inclusions:

$$\mathbf{FiNFrm} \subseteq \mathbf{QfNFrm} \subseteq \mathbf{LfNFrm}.$$

Zenk [57] shows that a nearness frame is locally fine iff it is q-fine, bearing in mind that his nearness frames are interpolative. Dube [23] has shown that locally fine nearness frames are reflective in the category of nearness frames.

Recall that a frame L is said to be *Lindelöf* if every cover of L has a countable subcover. A nearness frame (L, μ) is *separable* [45] if every uniform cover is refined by a countable uniform cover. Keeping in mind that the completion of a q-fine nearness frame is fine we obtain the following result.

Proposition 2.1.13 *A q-fine nearness frame (L, μ) is separable iff the underlying frame CL of its (fine) completion is Lindelöf.*

Proof: ($\Leftarrow$) Suppose CL is Lindelöf. Let $A \in \mu$, and let $\gamma_L : CL \longrightarrow L$ be the completion arrow. Now there exists $C \in \mathrm{Cov}(CL)$ such that $\gamma_L[C] = A$, since γ_L is a surjection. Also, since CL is Lindelöf, there exists a countable $D \in \mathrm{Cov}(CL)$ such that $D \subseteq C$. Then $\gamma_L[D]$ is a countable uniform cover of L which refines $\gamma_L[C] = A$. Hence (L, μ) is separable.

($\Rightarrow$) Let (L, μ) be separable and $C \in \mathrm{Cov}(CL)$. Since γ_L is strict, there exists $A \in \mu$ such that $(\gamma_L)_*[A] \leq C$. By the separable property, there exists a countable $B \in \mu$ such that $B \leq A$. So $(\gamma_L)_*[B] \in \mathrm{Cov}(CL)$ is countable and $(\gamma_L)_*[B] \leq C$. For each $b \in B$, take $c_b \in C$ such that $(\gamma_L)_*(b) \leq c_b$. Then $\bar{C} = \{c_b \mid b \in B\} \subseteq C$ is a countable subcover. ■

Using the above result and in the light of [45, Proposition 4.1.1] we deduce the following corollary.

Corollary 2.1.14 *A q-fine nearness frame is separable iff its completion is separable.*

In a frame L,

(i) a subset $A \subseteq L$ is said to *finitize* a subset $B \subseteq L$ [23] if for each $a \in A$ the set $B_a = \{b \in B \mid a \wedge b \neq 0\}$ is finite.

(ii) a subset $S \subseteq L$ is called *locally finite* [14] if there is a cover $C \in \mathrm{Cov}(L)$ that finitizes it. Such a cover C is referred to as a *witness* for S in [14].

By restricting to uniform covers in the above definitions, we have the following definitions as they appear in [23]:

(a) Let (L, μ) be a nearness frame. Then a subset $S \subseteq L$ is said to be *uniformly locally finite* if there is a uniform cover $C \in \mu$ finitizing it.

(b) A nearness frame is *uniformly paracompact* if every uniform cover is refined by a uniformly locally finite uniform cover.

Remark 2.1.15 In [23] it is observed that a fine nearness frame $(L, \mathrm{Cov}(L))$ is uniformly paracompact iff the underlying (regular) frame L is paracompact.

Proposition 2.1.16 *A q-fine nearness frame (L, μ) is uniformly paracompact iff the underlying frame CL of its completion is paracompact.*

Proof: ($\Leftarrow$) Let $\gamma_{_L} : CL \longrightarrow L$ be the completion arrow, with CL being a paracompact frame. We need to show that (L, μ) is uniformly paracompact. Let $A \in \mu$. Then $\gamma_{_L}[C] = A$ for some $C \in \mathrm{Cov}(CL)$. So there is a locally finite $D \in \mathrm{Cov}(CL)$ such that $D \leq C$. Note that this implies $\gamma_{_L}[D] \leq \gamma_{_L}[C] = A$. We show that $\gamma_{_L}[D]$ is a uniformly locally finite refinement of A.

Let $E \in \mathrm{Cov}(CL)$ be a cover which finitizes D. So for each $x \in E$, the set

$$D_x = \{d \in D \mid x \wedge d \neq 0\}$$

is finite. Then $\gamma_{_L}[E] \in \mu$ finitizes $\gamma_{_L}[D]$ since for each $\gamma_{_L}(x) \in \gamma_{_L}[E]$, we have

$$\{\gamma_{_L}(d) \mid d \in D,\ \gamma_{_L}(x) \wedge \gamma_{_L}(d) = \gamma_{_L}(x \wedge d) \neq 0\} \subseteq \{\gamma_{_L}(d) \mid d \in D,\ x \wedge d \neq 0\} = \gamma_{_L}[D_x]$$

which is finite.

($\Rightarrow$) Suppose (L, μ) is uniformly paracompact. Let $C \in \mathrm{Cov}(CL)$. Then since $\gamma_{_L}$ is strict, there is $A \in \mu$ such that $\gamma_*[A] \leq C$. Let $B \leq A$ be a uniformly locally finite refinement. Then $\gamma_*[B] \leq C$ and we show that $\gamma_*[B]$ is locally finite. In a similar argument as above, if $F \in \mu$ finitizes B, one easily realizes that $\gamma_*[F]$ finitizes $\gamma_*[B]$, and the proof ends. ■

Combining Remark 2.1.15 and Proposition 2.1.16, we establish the following result.

Corollary 2.1.17 *A q-fine nearness frame is uniformly paracompact iff its completion is uniformly paracompact.*

In [45] Naidoo shows that the functor which takes a nearness frame to its separable coreflection preserves surjections. We have not succeeded in determining whether the q-fine reflection functor

$$\mathbf{StrNFrm} \xrightarrow{Q} \mathbf{QfNFrm}$$

preserves surjections. However, given a surjection $h : (L, \mu) \longrightarrow (M, \eta)$ between strong nearness frames, we have the following condition on the lifted homomorphism

$$Ch : CL \longrightarrow CM$$

which is equivalent to

$$Qh : QL \longrightarrow QM$$

being a surjection.

We note that an onto uniform frame homomorphism $h : (L, \mu) \longrightarrow (M, \nu)$ is a surjection iff for any $C \in \nu$, there is $A \in \mu$ such that $h[A] \leq C$.

Proposition 2.1.18 *Let $h : (L, \mu) \longrightarrow (M, \eta)$ be a surjection between strong nearness frames. Then $Qh : QL \longrightarrow QM$ is a surjection iff every cover of CM is refined by the image under Ch of some cover of CL.*

Proof: $(\Rightarrow)$ Suppose Qh is a surjection. Let $C \in \mathrm{Cov}(CM)$. Since $\gamma_{_M}[\check{C}] \in \eta$ and $Q\eta = \gamma_{_M}[\mathrm{Cov}(CM)]$, $\gamma_{_M}[\check{C}] \in Q\eta$. However, since Qh is a surjection and $Q\mu = \gamma_{_L}[\mathrm{Cov}(CL)]$, there is $D \in \mathrm{Cov}(CL)$ such that $(Qh)[\gamma_{_L}[D]] \leq \gamma_{_M}[\check{C}]$. Bear in mind that the following squares commute:

$$\begin{tikzcd} CL \arrow[r, "Ch"] \arrow[d, "\gamma_L"'] & CM \arrow[d, "\gamma_M"] \\ L \arrow[r, "h"'] \arrow[d, "\mathrm{id}_L"'] & M \arrow[d, "\mathrm{id}_M"] \\ QL \arrow[r, "Qh"'] & QM \end{tikzcd}$$

So

$$h\gamma_{_L}[D] \leq \gamma_{_M}[\check{C}] \qquad (\dagger)$$

Since $\gamma_{_M}(Ch) = h\gamma_{_L}$, we have from $(\dagger)$ that

$$\gamma_{_M}[(Ch)[D]] \leq \gamma_{_M}[\check{C}],$$

which, in turn, implies that

$$(Ch)[D] \leq (\gamma_{_M})_* \gamma_{_M}[\check{C}] \leq C.$$

$(\Leftarrow)$ Conversely, suppose the given condition holds. First, Qh is onto, since it maps as h which is onto. Second, let A be a uniform cover of QM. Then $A = \gamma_{_M}[U]$ for some $U \in \mathrm{Cov}(CM)$. By the hypothesis, we have $(Ch)_*[U] \in \mathrm{Cov}(CL)$ so that $\gamma_{_L}[(Ch)_*[U]]$ is a uniform cover of QL. But

$$(Qh)[\gamma_{_L}[(Ch)_*[U]]] = h\gamma_{_L}[(Ch)_*[U]] = \gamma_{_M}(Ch)[(Ch)_*[U]] \leq \gamma_{_M}[U] = A.$$

Hence Qh is a surjection. ∎

By a *point* or *prime element* $p \in L$ we mean an element of L with the following properties: (i) $p \neq 1$ and (ii) $x \wedge y \leq p$ in L implies $x \leq p$ or $y \leq p$. We say the frame L is *spatial* or *has enough points* if every member of L is a meet of points above it. [Thus for every $x \in L$ we have $x = \bigwedge\{p \in L \mid p \text{ is a point and } x \leq p\}$].

Now recall that an element $m \in L$ is *maximal* if $m \neq 1$ and $m \leq t \neq 1$ implies $m = t$. We gather some known facts about points and spatiality in the following lemma (see, for example, [39]).

Lemma 2.1.19 *(i) Every compact regular frame is spatial.*

(ii) If L is regular, then $p \in L$ is a point iff p is a maximal element.

(iii) If L is spatial, then for any $x \in L$, if $x \neq 1$, then there is a point $p \in L$ such that $x \leq p$.

We observed in Remark 2.1.7(1) the (rather obvious) result that a quotient-fine nearness frame is fine if and only if it is complete. In the case of nearness frames with spatial completion we will show that a condition (called Cauchy completeness) which is generally weaker than completeness is equivalent to fineness. A filter F of a nearness frame (L, μ) is called a *regular Cauchy filter* if for any $A \in \mu$, $A \cap F \neq \emptyset$ and for each $x \in F$, there is $y \in F$ such that $y \triangleleft x$. A nearness frame (L, μ) is *Cauchy complete* if every regular Cauchy filter meets every cover of L. Completeness implies Cauchy completeness, but the converse does not hold.

It shown in [36] that a strong nearness frame is Cauchy complete if and only if every Cauchy filter converges.

A neat characterization of Cauchy completeness in terms of homomorphisms, and which we note here as a lemma, is given in [6].

Lemma 2.1.20 *A nearness frame (L, μ) is Cauchy complete iff any homomorphism $CL \longrightarrow \mathbf{2}$ factors through $\gamma_L : CL \longrightarrow \mathbf{2}$, where $\mathbf{2}$ denotes the two-element frame.*

We now give a characterization of q-fine nearness frames with spatial completions.

Proposition 2.1.21 *A q-fine nearness frame with a spatial completion is fine iff it is Cauchy complete.*

Proof: ($\Rightarrow$) If (L, μ) is fine, then it is complete and therefore Cauchy complete.

($\Leftarrow$) Conversely, let (L, μ) be a Cauchy complete q-fine nearness frame with a spatial completion. Suppose on the contrary that L is not fine. Then L is not complete. Thus $\gamma_L : CL \longrightarrow L$ is not codense, that is, there is $a \neq 1$ in CL such that $\gamma_L(a) = 1$. By spatiality, let p be a point of CL such that $a \leq p$. Next, let $\xi : CL \longrightarrow \mathbf{2}$ be the homomorphism determined by p; namely

$$\xi(x) = 0 \text{ iff } x \leq p.$$

By Cauchy completeness, there is a homomorphism $g : L \longrightarrow \mathbf{2}$ such that $g\gamma_L = \xi$. This leads to a contradiction, since $\xi(a) = 0$ and $g\gamma_L(a) = g(1) = 1$. Therefore L is fine. ∎

Recall that a nearness frame (L, μ) is said to be *finitely fine* if every cover of L that is refined by a finite cover is uniform. Note that this means $\mu = \mathrm{FCov}(L)$. We introduce a weaker condition in the following definition.

Definition 2.1.22 A nearness frame (L, μ) is *f-fine* if $\mu \supseteq \mathrm{FCov}(L)$.

Clearly, every finitely fine nearness frame is f-fine but not vice versa, since any non-compact fine nearness frame is f-fine but fails to be finitely fine. We aim to provide a way of constructing certain types of q-fine nearness frames which are f-fine. Let $\mathfrak{X}$ be a set of prime filters of a regular frame L. Define a collection $\mathfrak{N}_{\mathfrak{X}}$ of covers of L by

$$\mathfrak{N}_{\mathfrak{X}} = \{C \in \mathrm{Cov}(L) \mid C \cap F \neq \emptyset \text{ for each } F \in \mathfrak{X}\}.$$

Proposition 2.1.23 $\mathfrak{N}_{\mathfrak{X}}$ *is a nearness on the regular frame L.*

Proof: First, let C, $D \in \mathfrak{N}_{\mathfrak{X}}$. Then for $F \in \mathfrak{X}$, take $c \in C \cap F$ and $d \in D \cap F$. Then $c \wedge d \in (C \wedge D) \cap F$. (Note that $c \wedge d \in F$ since F is a filter). Therefore $C \wedge D \in \mathfrak{N}_{\mathfrak{X}}$.

Second, let $C \in \mathfrak{N}_{\mathfrak{X}}$, and suppose D is a cover of L such that $C \leq D$. Then for $F \in \mathfrak{X}$, let $c \in C \cap F$. Then there is $d \in D$ such that $c \leq d$, and since F is an upset, $d \in F$. Therefore $D \cap F \neq \emptyset$, so that $D \in \mathfrak{N}_{\mathfrak{X}}$. So far we have shown that $\mathfrak{N}_{\mathfrak{X}}$ is a filter under the refinement order in $\mathrm{Cov}(L)$.

Third, we show the admissibility property indirectly. Let $A \in \mathrm{FCov}(L)$. Then A is refined by a finite subcover $B = \{b_1, b_2, \ldots, b_n\}$. For any prime filter $F \in \mathfrak{X}$, we have $\bigvee B = 1 \in F$; and so F contains at least one of the $b_k's$ in B since F is prime. This means $F \cap B \neq \emptyset$, so that $F \cap A \neq \emptyset$. Therefore $A \in \mathfrak{N}_{\mathfrak{X}}$. Consequently $\mathrm{FCov}(L) \subseteq \mathfrak{N}_{\mathfrak{X}}$. Hence $\mathfrak{N}_{\mathfrak{X}}$ inherits the admissibility property from $\mathrm{FCov}(L)$. ∎

Now a filter $F \subseteq L$ is *completely prime* if for any $S \subseteq L$, $\bigvee S \in F$ implies $S \cap F \neq \emptyset$.

Proposition 2.1.24 *Let L be a regular frame and $\mathfrak{N}_{\mathfrak{X}}$ as defined above. Then:*

(i) If $\mathfrak{X}$ and $\mathfrak{Y}$ are sets of prime filters, then $\mathfrak{X} \subseteq \mathfrak{Y}$ implies $\mathfrak{N}_{\mathfrak{X}} \supseteq \mathfrak{N}_{\mathfrak{Y}}$.

(ii) If $\mathfrak{Y}$ contains only completely prime filters, then $\mathfrak{N}_{\mathfrak{X}} = \mathfrak{N}_{\mathfrak{X} \cup \mathfrak{Y}}$.

Proof: (i) Given $\mathfrak{X} \subseteq \mathfrak{Y}$, let $C \in \mathfrak{N}_{\mathfrak{Y}}$. Then $C \cap F \neq \emptyset$ for each $F \in \mathfrak{Y}$, which includes all the F's in $\mathfrak{X}$. Therefore $C \in \mathfrak{N}_{\mathfrak{X}}$.

(ii) Suppose the filters in $\mathfrak{Y}$ are only completely prime ones. Now since $\mathfrak{X} \subseteq \mathfrak{X} \cup \mathfrak{Y}$, we have $\mathfrak{N}_{\mathfrak{X}} \supseteq \mathfrak{N}_{\mathfrak{X} \cup \mathfrak{Y}}$ from (i). As for the other inclusion, any cover C of L has a nonempty intersection with each completely prime filter F, since $\bigvee C = 1 \in F$. In particular $\mathfrak{N}_{\mathfrak{X}} \subseteq \mathfrak{N}_{\mathfrak{X} \cup \mathfrak{Y}}$, completing the proof. ∎

Now let us recall how the strict extension of a (regular) frame L determined by a set of filters is constructed (see [10]). Given a set $\mathfrak{X}$ of filters of L and a subset A of L, let

$$\mathfrak{X}_A = \{F \in \mathfrak{X} \mid A \cap F \neq \emptyset\}.$$

View the powerset $\mathfrak{P}(\mathfrak{X})$ as a Boolean frame, and let $\tau_{\mathfrak{X}} L$ be the subframe of $L \times \mathfrak{P}(\mathfrak{X})$ given by

$$\tau_{\mathfrak{X}} L = \left\{ \left(\bigvee A, \mathfrak{X}_A \right) \mid A \subseteq L \right\},$$

and $\tau : \tau_{\mathfrak{X}}L \longrightarrow L$ be the dense onto homomorphism defined by

$$\tau\left(\bigvee A, \mathfrak{X}_A\right) = \bigvee A.$$

Then $\tau : \tau_{\mathfrak{X}}L \longrightarrow L$ is the *strict extension of L determined by $\mathfrak{X}$*. Now a filter $F \subseteq L$ is *regular* if for each $a \in F$, there exists $b \in F$ such that $b \prec a$.

The following lemma appears in [11, Proposition 9].

Lemma 2.1.25 *If $\mathfrak{X}$ consists of prime filters, then $\tau_{\mathfrak{X}}L$ is regular iff each filter in $\mathfrak{X}$ is regular.*

Proposition 2.1.26 *Let $\mathfrak{X}$ be a set of regular prime filters of L, and endow L with the nearness $\mathfrak{N}_{\mathfrak{X}}$. Then L is q-fine and f-fine.*

Proof: We have already seen that $\mathrm{FCov}(L) \subseteq \mathfrak{N}_{\mathfrak{X}}$; and so L is f-fine. Next, we show that L is q-fine. Let $\tau : \tau_{\mathfrak{X}} \longrightarrow L$ be the strict extension determined by $\mathfrak{X}$. We will show that L's nearness $\mathfrak{N}_{\mathfrak{X}} = \tau[\mathrm{Cov}(\tau_{\mathfrak{X}}L)]$, and the proof ends since $\tau_{\mathfrak{X}}L$ is a regular frame and τ is an onto uniform frame homomorphism.

Let $\mathcal{C}$ be a cover of $\tau_{\mathfrak{X}}L$. So there is a family $\{A_\lambda \mid \lambda \in \Lambda\}$ of subsets of L such that

$$\mathcal{C} = \left\{\left(\bigvee A_\lambda, \mathfrak{X}_{A_\lambda}\right) \mid \lambda \in \Lambda\right\};$$

so that

$$\tau[\mathcal{C}] = \left\{\bigvee A_\lambda \mid \lambda \in \Lambda\right\}.$$

Since $\mathcal{C}$ is a cover of $\tau_{\mathfrak{X}}L$, we have that

$$\bigcup_{\lambda \in \Lambda} \mathfrak{X}_{A_\lambda} = \mathfrak{X}.$$

Let $F \in \mathfrak{X}$. Then there exists $\kappa \in \Lambda$ such that $F \in \mathfrak{X}_{A_\kappa}$. Thus, $A_\kappa \cap F \neq \emptyset$, and hence $\bigvee A_\kappa \in F$ since F is an upset. This shows that $\tau[\mathcal{C}]$ meets every filter in $\mathfrak{X}$ so that $\tau[\mathcal{C}] \in \mathfrak{N}_{\mathfrak{X}}$, and hence $\tau[\mathrm{Cov}(\tau_{\mathfrak{X}}L)] \subseteq \mathfrak{N}_{\mathfrak{X}}$. Next, let $C \in \mathfrak{N}_{\mathfrak{X}}$. For any $c \in C$, abbreviate $\mathfrak{X}_{\{c\}}$ as $\mathfrak{X}_c$. Define a subset $\mathcal{C}$ of $\tau_{\mathfrak{X}}L$ by

$$\mathcal{C} = \{(c, \mathfrak{X}_c) \mid c \in C\}.$$

We claim that $\mathcal{C} \in \mathrm{Cov}(\tau_{\mathfrak{X}} L)$. Since $\bigvee C = 1$, we need only show that $\bigcup\{\mathfrak{X}_c \mid c \in C\} \supseteq \mathfrak{X}$. Let $F \in \mathfrak{X}$. Since $C \in \mathfrak{N}_{\mathfrak{X}},\ C \cap F \neq \emptyset$. This shows that $F \in \mathfrak{X}_a$ for some $a \in C$, establishing the claim. So $\mathcal{C} \in \mathrm{Cov}(\tau_{\mathfrak{X}} L)$. But

$$\tau[\mathcal{C}] = \{\tau(c, \mathfrak{X}_c) \mid c \in C\} = C;$$

so $C \in \tau[\mathrm{Cov}(\tau_{\mathfrak{X}} L)]$, showing that $\mathfrak{N}_{\mathfrak{X}} \subseteq \tau[\mathrm{Cov}(\tau_{\mathfrak{X}} L)]$, and hence equality holds. Therefore L is q-fine. ∎

It is worth remarking that if (L, μ) is a nearness frame with a spatial completion, then the underlying frame of the completion of L is isomorphic to the frame $\tau_{\mathfrak{X}} L$, where $\mathfrak{X}$ is the set of non-convergent regular Cauchy filters of L. The reason is a combination of Lemmas 3 and 4 in [10]. We close this section by showing that all finitely fine nearness frames can be constructed in exactly the same way.

Proposition 2.1.27 *A nearness frame (L, μ) is finitely fine iff its nearness $\mu = \mathfrak{N}_{\mathfrak{X}}$, where $\mathfrak{X}$ is the set of all prime filters of L.*

Proof: It clearly suffices to show that if $\mathfrak{X}$ is the set of all prime filters of L, then $\mathfrak{N}_{\mathfrak{X}} = \mathrm{FCov}(L)$. The inclusion $\supseteq$ holds trivially. For the reverse inclusion, suppose, on the contrary that $C \in \mathfrak{N}_{\mathfrak{X}} \setminus \mathrm{FCov}(L)$. Then the set $J = \{\bigvee S \mid S \text{ is a finite subset of } C\}$ is a proper ideal of L containing C. By the dual version of Stone's Separation Lemma (see the Glossary or [30, Theorem 15]), there is a prime filter F disjoint from J. So $C \cap F = \emptyset$ contradicting the fact that $C \in \mathfrak{N}_{\mathfrak{X}}$. Hence the desired result holds. ∎

2.2 The role played by grills

In regular nearness spaces there is a characterization of subtopological ones that says *a nearness space (X, μ) is subtopological if and only if every near collection of subsets is contained in a near grill* (see [18]). A close scrutiny of the validating arguments shows that what makes the characterization valid is that, talking frame-theoretically, the frame $\mathfrak{B}(X)$ of subsets of X is Boolean, and furthermore, the near collections are allowed to contain any type of subset and not just the open ones. We have not been able to obtain a satisfactory analogue of the cited topological theorem. Nevertheless, we have some noteworthy results. In this short section our aim is to show that if a nearness frame is q-fine and its completion has enough points, then every near subset is contained in a near grill. First, we recall some definitions.

In a frame L, a nonempty $G \subseteq L$ is called a *grill* if it satisfies:

g1. $0 \notin G$.

g2. $a \in G$ and $a \leq b$ implies $b \in G$. (i.e. G is an upset).

g3. $a \vee b \in G$ implies $a \in G$ or $b \in G$.

An element $a \in L$ is called an *atom* if $a \neq 0$ and $y \leq a$ implies $y = 0$ or $y = a$. L is said to be *atomic* if for every $x \in L$,

$$x = \bigvee\{a \in L \mid a \text{ is an atom and } a \leq x\}.$$

We note that in a Boolean frame an element is a point iff its complement is an atom [39]. We also note that if the completion of a nearness frame (or a q-fine nearness frame for that matter) has enough points, it does not follow that the nearness frame has enough points. This is illustrated by the following example.

Example 2.2.1 Let L be a Boolean frame with no atoms. Then L has no points. Consider the Stone-Čech compactification $\sigma : \beta L \longrightarrow L$ of L, and endow L with the nearness $\sigma[\mathrm{Cov}(\beta L)]$. Then L is a q-fine nearness frame and its completion has enough points.

Proposition 2.2.2 *Let (L, μ) be a q-fine nearness frame with a spatial completion. Then any near subset of L is contained in a near grill.*

Proof: Let $h : M \longrightarrow L$ be a completion with M spatial. Since h is a surjection, $\mu = h[\mathrm{Cov}(M)]$. Let $A \subseteq L$ be near. Our aim is to construct a near grill $G \supseteq A$. Making use of Lemma 1.4.12, we begin by claiming that

$$\bigvee\{h_*(a^*) \mid a \in A\} = \bigvee h_*[A^*] \neq 1.$$

If not, then $h_*[A^*] \in \mathrm{Cov}(M)$, and hence $hh_*[A^*] = A^* \in \mu$, which contradicts the cited lemma.

Next, since M has enough points, there is a point $p \in M$ such that

$$\bigvee \{h_*(a^*) \mid a \in A\} \leq p.$$

We put

$$G = \{x \in L \mid h_*(x^*) \leq p\}.$$

Clearly, $A \subseteq G$. We show that G is near. Suppose G is not near. Then there is a cover U of M such that $h[U] = G^*$. This then implies that $U \leq h_*[G^*]$, and hence

$$1 = \bigvee U \leq \bigvee \{h_*(x^*) \mid x \in G\} \leq p$$

which of course is false.

Finally, we show that G is a grill. First, since $h_*(0^*) = h_*(1) = 1$, we have $0 \notin G$. Second, let $a \in G$ with $a \leq b$. Then $h_*(b^*) \leq h_*(a^*) \leq p$, so that $b \in G$, and therefore G is an upset. Third, suppose $u \vee v \in G$. Then

$$h_*(u^*) \wedge h_*(v^*) = h_*(u^* \wedge v^*) = h_*((u \vee v)^*) \leq p,$$

which implies $h_*(u^*) \leq p$ or $h_*(v^*) \leq p$ since p is a point. Thus $u \in G$ or $v \in G$, and the proof ends. ∎

For any frame L, the collection of $\mathcal{J}L$ of all ideals of L is a frame [39]. The join map $h : \mathcal{J}L \longrightarrow L$, which takes each ideal to its join in L, is a frame homomorphism. The following result appears in [6].

Lemma 2.2.3 *For any finitely fine Boolean nearness frame (L, μ), the join map $h : \mathcal{J}L \longrightarrow L$ is a completion.*

In view of Proposition 2.2.2, we obtain the following.

Corollary 2.2.4 *Any near subset of a finitely fine Boolean nearness frame is contained in a maximal near grill.*

Proof: Let (L, μ) be a Boolean finitely fine nearness frame. Then, being compact, the completion of L is fine; and so L is a q-fine nearness frame with a spatial completion. Therefore, by Proposition 2.2.2, every near subset is contained in a near grill. It remains to show maximality.

Keeping the above notation, let A, p and G be as in the proof of Proposition 2.2.2, and $h : \mathcal{J}L \longrightarrow L$ the join map. We show that G is maximal. Let H be a near grill with $G \subseteq H$, and take $a \in H$. Suppose on the contrary $a \notin G$. Then $h_*(a^*) \not\leq p$, and since h is dense,

$$h_*(a^*) \wedge h_*(a^{**}) = h_*(0) = 0 \leq p.$$

So $h_*(a^{**}) \leq p$ since p is a point.

This implies $a^* \in G \subseteq H$, so that both a and a^* are in H. But now $\{a,\ a^*\}$ is a uniform cover each of whose members does not meet at least one member of H, which contradicts the fact that H is near. Hence the desired result holds. ∎

2.3 The categories Ext and Compl

In this section we define the categories **Ext** and **Compl** and show their equivalence to **QfNFrm**. For convenience of notation, in this section, we may simply refer to a typical nearness frame by L instead of the usual pair (L, μ).

The category **Ext** is defined as follows:

(a) The objects are dense onto frame homomorphisms $h : M \longrightarrow L$, which we shall, at times, write as (h, M, L).

(b) A morphism $(\alpha, \beta) : (h_1, M_1, L_1) \longrightarrow (h_2, M_2, L_2)$ between two objects is a pair of frame homomorphisms $\alpha : M_1 \longrightarrow M_2$ and $\beta : L_1 \longrightarrow L_2$ such that the diagram

$$\begin{tikzcd} M_1 \arrow[r, "\alpha"] \arrow[d, "h_1"'] & M_2 \arrow[d, "h_2"] \\ L_1 \arrow[r, "\beta"'] & L_2 \end{tikzcd}$$

commutes.

(c) If $(\alpha, \beta) : (h_1, M_1, L_1) \longrightarrow (h_2, M_2, L_2)$ and $(\gamma, \delta) : (h_2, M_2, L_2) \longrightarrow (h_3, M_3, L_3)$ are morphisms, the composite $(\gamma, \delta) \circ (\alpha, \beta)$ is the morphism

$$(\gamma, \delta) \circ (\alpha, \beta) = (\gamma \circ \alpha, \delta \circ \beta) : (h_1, M_1, L_1) \longrightarrow (h_3, M_3, L_3).$$

In order to define the category **Compl**, we take a cue from spaces (see [18]). First we need some background concerning drawing up a frame version of the restriction of a function $f : X \longrightarrow Y$ to a function $f_{|A} : A \longrightarrow B$, where $f[A] \subseteq B$. This is provided by [47, Proposition 7.1.2] in the following way.

For any frame homomorphism $g : N \longrightarrow K$, we have

$$\mathrm{Fix}(g_* g) = \{x \in N \mid g_* g(x) = x\} = g_*[K];$$

so that $g_*[K]$ is a frame. Thus if g is onto, then the map $g_*[K] \longrightarrow K$, mapping as g, is an isomorphism whose inverse is the map $K \longrightarrow g_*[K]$ given by $a \mapsto g_*(a)$. Let $h : L \longrightarrow M$ be a frame homomorphism, and suppose $\alpha : L \longrightarrow A$ and $\beta : M \longrightarrow B$ are onto homomorphisms. Suppose, further that $h_*[\beta_*[B]] \subseteq \alpha_*[A]$. By the result cited from [47], there is a (necessarily unique) frame homomorphism $h' : \alpha_*[A] \longrightarrow \beta_*[B]$ whose right adjoint is $h_{*|\beta_*[B]}$, the restriction of h_* to $\beta_*[B]$. Define $\check{h} : A \longrightarrow B$ to be the composite

$$A \xrightarrow{\alpha_*} \alpha_*[A] \xrightarrow{h'} \beta_*[B] \xrightarrow{\beta} B\ ,$$

where α_* and β in the composition are actually appropriate restrictions of the same-named homomorphisms. Note that $\check{h}$ makes the diagram

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\alpha"'] & M \arrow[d, "\beta"] \\ A \arrow[r, "\check{h}"'] & B \end{tikzcd}$$

commute. This is seen by taking right adjoints, keeping in mind that, in the composition that makes up $\check{h}$, $\alpha_* \alpha = \mathrm{id}_A$:

$$(\check{h}\alpha)_* = (\beta h' \alpha_* \alpha)_* = (\beta h')_* = h'_* \beta_* = h_* \beta_* = (\beta h)_*\, ,$$

implying that $\check{h}\alpha = \beta h$.

In analogy with "liftable" homomorphisms between nearness frames, that is, those which can be lifted to completions, we introduce "droppable" homomorphisms.

Let L and M be nearness frames. A homomorphism $h : CL \longrightarrow CM$ is said to be *droppable* if

$$h_*[(\gamma_M)_*[M]] \subseteq (\gamma_L)_*[L],$$

where $\gamma_L : CL \longrightarrow L$ and $\gamma_M : CM \longrightarrow M$ are the completion arrows. The homomorphism $\check{h} : L \longrightarrow M$ defined above will then be called the *drop* of h.

Remark 2.3.1 If a homomorphism $g : CM \longrightarrow CL$ is droppable with drop h, then h is liftable, and, by the denseness of γ_L, the lift of h is g. Thus, dropping and lifting takes us back where we started. The following lemma shows that the other way round also holds.

Lemma 2.3.2 *Suppose $h : M \longrightarrow L$ is liftable to completions with lift g. Then g is droppable and its drop is h.*

Proof: Consider the commutative diagram

$$(\dagger) \qquad \begin{tikzcd} CM \arrow[r, "g"] \arrow[d, "\gamma_M"'] & CL \arrow[d, "\gamma_L"] \\ M \arrow[r, "h"'] & L \end{tikzcd}$$

In this case $\gamma_L g = h\gamma_M$, and hence

$$(\gamma_L g)_* = (h\gamma_M)_* = (\gamma_M)_* h_*.$$

Since $h_*[L] \subseteq M$, it follows that

$$(\gamma_L g)_*[L] = (\gamma_M)_* h_*[L] \subseteq (\gamma_M)_*[M]\,,$$

showing that g is droppable.

Next, the drop of g is the composition

$$M \xrightarrow{(\gamma_M)_*} (\gamma_M)_*[M] \xrightarrow{g'} (\gamma_L)_*[L] \xrightarrow{\gamma_L} L\,,$$

whose right adjoint we will show to coincide with that of h. Keep in mind that if $x \in (\gamma_L)_*[L]$, then $g'_*(x) = g_*(x)$. For any $a \in L$,

$$(\gamma_L g'(\gamma_M)_*)_* (a) = \gamma_M \left(g'_* \left((\gamma_L)_*(a)\right)\right) = \gamma_M \left(g_* \left((\gamma_L)_*(a)\right)\right).$$

On the other hand, from commutativity of the diagram (†), we have $\gamma_L g = h\gamma_M$, so that for each $a \in L$,

$$g_*(\gamma_L)_*(a) = (\gamma_M)_* h_*(a)\,,$$

and hence

$$\gamma_M \left(g_*((\gamma_L)_*(a))\right) = h_*(a)\,,$$

since γ_M is onto. Thus h and the drop of g have identical right adjoints, and are therefore the same homomorphism. ∎

Corollary 2.3.3 *A homomorphism $g : CM \longrightarrow CL$ is droppable iff there is a homomorphism $h : M \longrightarrow L$ that makes the diagram*

$$\begin{tikzcd} CM \arrow[r, "g"] \arrow[d, "\gamma_L"'] & CL \arrow[d, "\gamma_L"] \\ M \arrow[r, "h"'] & L \end{tikzcd}$$

commute. Furthermore, if the homomorphism h exists, then it is the drop of g.

Proof: We observed above that the drop of g makes the diagram commute, so the forward implication holds. Conversely, if h makes the diagram commute, then it is liftable with lift g. So, by the lemma, g is droppable and h is the drop of g. ∎

Corollary 2.3.4 *If $g : CM \longrightarrow CL$ is a droppable uniform homomorphism, then its drop is also uniform.*

Proof: Denote the drop of g by $\breve{g}$, and let μ and η be the respective nearnesses on L and M, so that $C\mu$ and $C\eta$ are the respective nearnesses on the completions. Given $A \in \eta$, we have $(\gamma_M)_*[A] \in C\eta$, which implies $g(\gamma_M)_*[A] \in C\mu$, and hence $\gamma_L g(\gamma_M)_*[A] \in \mu$. But

$\gamma_L g = \breve{g}\gamma_M$ implies $\gamma_L g(\gamma_M)_* = \breve{g}$, since γ_M is onto. Thus, $\breve{g}$ sends uniform covers to uniform covers. ■

We are now able to define the category **Compl**:

(a) Objects are pairs (CL, L) consisting of a quotient-fine nearness frame and its completion.

(b) A morphism $f : (CM, M) \longrightarrow (CL, L)$ between objects is a droppable uniform homomorphism $f : CM \longrightarrow CL$.

(c) If $f : (CM, M) \longrightarrow (CL, L)$ and $g : (CL, L) \longrightarrow (CK, K)$ are morphisms in **Compl**, then the composition $g \circ f : CM \longrightarrow CK$ is a droppable uniform homomorphism. We then define the composite $g \circ f : (CM, M) \longrightarrow (CK, K)$ in **Compl** to be the droppable $g \circ f$.

Corollary 2.3.3 makes it immediate that composition in **Compl** is well defined. Next, we introduce four functors, with accompanying diagrams showing the mappings for clarity:

(E) $\mathbf{QfNFrm} \xrightarrow{\mathsf{E}} \mathbf{Ext}$ sends a q-fine nearness frame L to $CL \xrightarrow{\gamma_L} L$, and a uniform homomorphism $h : L \longrightarrow M$ between q-fine nearness frames to the pair of arrows (Ch, h), where Ch is the lift of h to completions. Thus,

$$\mathbf{QfNFrm} \xrightarrow{\quad\mathsf{E}\quad} \mathbf{Ext}$$

$$\begin{tikzcd} L \arrow[d, "h"'] & & CL \arrow[r, "\gamma_L"] \arrow[d, "Ch"] & L \arrow[d, "h"] \\ M \arrow[r, phantom, "\longmapsto" near start] & & CM \arrow[r, "\gamma_M"'] & M \end{tikzcd}$$

(F) $\mathbf{Ext} \xrightarrow{\mathsf{F}} \mathbf{QfNFrm}$ sends an object (h, M, L) of **Ext** to the q-fine nearness frame $(L, h[\mathrm{Cov}(M)])$, and a morphism $(\alpha, \beta) : (h, M, L) \longrightarrow (g, N, K)$ to the uniform homomorphism $\beta : (L, h[\mathrm{Cov}(M)]) \longrightarrow (K, g[\mathrm{Cov}(N)])$. Thus,

$$\mathbf{Ext} \xrightarrow{\quad\mathsf{F}\quad} \mathbf{QfNFrm}$$

$$\begin{array}{ccc} (h, M, L) & & (L, h[\mathrm{Cov}(M)]) \\ \Big\downarrow{\scriptstyle (\alpha,\beta)} & \longmapsto & \Big\downarrow{\scriptstyle \beta} \\ (g, N, K) & & (K, g[\mathrm{Cov}(N)]) \end{array}$$

(G) $\mathbf{Compl} \xrightarrow{\mathsf{G}} \mathbf{Ext}$ sends an object (CL, L) of $\mathbf{Compl}$ to $CL \xrightarrow{\gamma_L} L$, and a morphism $g : (CM, M) \longrightarrow (CL, L)$ to the pair $(g, \breve{g})$, where $\breve{g}$ is the drop of g. Thus,

$$\mathbf{Compl} \xrightarrow{\quad\mathsf{G}\quad} \mathbf{Ext}$$

$$\begin{array}{ccc} (CM, M) & & CM \xrightarrow{\gamma_M} M \\ \Big\downarrow{\scriptstyle g} & \longmapsto & \Big\downarrow{\scriptstyle g} \qquad \Big\downarrow{\scriptstyle \breve{g}} \\ (CL, L) & & CL \xrightarrow[\gamma_L]{} L \end{array}$$

(H) $\mathbf{Ext} \xrightarrow{\mathsf{H}} \mathbf{Compl}$ sends an object (h, M, L) of $\mathbf{Ext}$ to (CL, L), where L is endowed with the nearness $\mathfrak{N}L = h[\mathrm{Cov}(M)]$. Its action on morphisms needs some elaboration. Recall that a completion of a nearness frame is unique only up to isomorphism. What this means is that if $h : M \longrightarrow L$ is a completion of L, then there is a unique isomorphism $\varphi_M^L : CL \longrightarrow M$ such that the triangle

$$\begin{tikzcd} CL \arrow[rr, "\gamma_L"] \arrow[dr, "\varphi_M^L"'] & & L \\ & M \arrow[ur, "h"'] & \end{tikzcd}$$

commutes. Thus, given a morphism $(\alpha, \beta) : (h, M, L) \longrightarrow (g, N, K)$ of $\mathbf{Ext}$, we have a commutative square

$$\begin{tikzcd} CL \arrow[r, "u"] \arrow[d, "\gamma_L"'] & CK \arrow[d, "\gamma_K"] \\ L \arrow[r, "\beta"'] & K \end{tikzcd}$$

where u is the droppable uniform homomorphism with its drop $\breve{u} = \beta$ and $u = (\varphi_N^K)^{-1}\alpha\varphi_M^L$. The action of H on (α, β) is then defined by $\mathsf{H}(\alpha, \beta) = (\varphi_N^K)^{-1}\alpha\varphi_M^L$.

The diagrams below present a clearer picture:

$$\mathbf{Ext} \xrightarrow{\quad\quad\quad\quad \mathsf{H} \quad\quad\quad\quad} \mathbf{Compl}$$

$$\begin{tikzcd}
M \arrow[r, "h"] \arrow[d, "\alpha"'] & L \arrow[d, "\beta"] \\
N \arrow[r, "g"'] & K
\end{tikzcd}
\longmapsto
\begin{tikzcd}
 & & CL \arrow[dll, bend right, "\varphi^{L}_{M}"'] \arrow[dl, "\gamma_L"'] \arrow[ddd, "\mathsf{H}(\alpha,\beta)"] \\
M \arrow[r, "h"] \arrow[d, "\alpha"] & L \arrow[d, "\beta"] & \\
N \arrow[r, "g"'] & K & \\
 & & CK \arrow[ull, bend left, "\varphi^{K}_{N}"] \arrow[ul, "\gamma_K"]
\end{tikzcd}$$

Note that $\mathsf{H}(\alpha, \beta)$ is indeed uniform, since α is uniform because M and N are viewed as fine nearness frames.

We now establish the categorical equivalences as promised at the beginning of this section.

Proposition 2.3.5 *The pair* (E, F) *is an equivalence of categories. Thus,* **Ext** *and* **QfNFrm** *are equivalent.*

Proof: We shall construct a pair of natural isomorphisms

$$\xi : 1_{\mathbf{QfNFrm}} \longrightarrow \mathsf{F} \circ \mathsf{E} \text{ and } \zeta : 1_{\mathbf{Ext}} \longrightarrow \mathsf{E} \circ \mathsf{F}\,,$$

and the proof ends. Now for any object L and any morphism $h : L \longrightarrow M$ of **QfNFrm**, we have

$$(\mathsf{F} \circ \mathsf{E})(L) = \mathsf{F}(CL, L) = L \text{ and } (\mathsf{F} \circ \mathsf{E})(h) = \mathsf{F}(Ch, h) = h.$$

For each object L of **QfNFrm**, let ξ_L be the identity homomorphism id_L. We claim that $(\xi_L)_{L\in\mathbf{QfNFrm}}$ defines a natural transformation

$$\xi : 1_{\mathbf{QfNFrm}} \longrightarrow \mathsf{F} \circ \mathsf{E}.$$

Let $h : L \longrightarrow M$ be a morphism of **QfNFrm**. In view of the above calculations, the square

$$\begin{tikzcd}
L \arrow[r, "\mathrm{id}_L"] \arrow[d, "h"'] & L \arrow[d, "h"] \\
M \arrow[r, "\mathrm{id}_M"'] & M
\end{tikzcd}$$

clearly, commutes. Furthermore, ξ_L is an isomorphism for each object L of **QfNFrm**.

Next, for any object (h, M, L) and any morphism $(\alpha, \beta) : (h, M, L) \longrightarrow (g, N, K)$ of **Ext**, we have

$$(\mathsf{E} \circ \mathsf{F})(h, M, L) = \mathsf{E}(L, h[\mathrm{Cov}(M)]) = (\gamma_{_L}, CL, L)$$

and

$$(\mathsf{E} \circ \mathsf{F})(\alpha, \beta) = \mathsf{E}(\beta) = (C\beta, \beta).$$

Recall the notation in the definition of the functor H above. For any object (h, M, L) of **Ext**, let $\zeta_{(h,M,L)}$ - which we abbreviate as ζ_h - be the morphism

$$((\varphi^{^L}_{_M})^{-1}, \mathrm{id}_L) : (h, M, L) \longrightarrow (\gamma_{_L}, CL, L)$$

of **Ext**, which, in fact, is an **Ext**-isomorphism. We claim that $(\zeta_h)_{h \in \mathbf{Ext}}$ defines a natural transformation

$$\zeta : 1_{\mathbf{Ext}} \longrightarrow \mathsf{E} \circ \mathsf{F}.$$

To prove the claim, let $(\alpha, \beta) : (h, M, L) \longrightarrow (g, N, K)$ be a morphism of **Ext**. We must show that the diagram

$$\begin{tikzcd}
(h, M, L) \arrow[r, "\zeta_h"] \arrow[d, "(\alpha,\beta)"'] & (\gamma_{_L}, CL, L) \arrow[d, "(C\beta,\beta)"] \\
(g, N, K) \arrow[r, "\zeta_g"'] & (\gamma_{_K}, CK, K)
\end{tikzcd}$$

commutes. This means we must show that

$$\left((\varphi^{^K}_{_N})^{-1} \circ \alpha,\ \mathrm{id}_K \circ \beta\right) = \left(C\beta \circ (\varphi^{^L}_{_M})^{-1}, \beta \circ \mathrm{id}_L\right),$$

that is

$$(\varphi^{^K}_{_N})^{-1} \circ \alpha = C\beta \circ (\varphi^{^L}_{_M})^{-1} \ \text{ and } \ \mathrm{id}_K \circ \beta = \beta \circ \mathrm{id}_L.$$

The latter is immediate. The former follows from the commutativity of the diagram

$$\begin{tikzcd}
 & & CL \arrow[dll, bend right, "\varphi^L_M"'] \arrow[dl, "\gamma_L"] \arrow[ddd, "C\beta"] \\
M \arrow[r, "h"] \arrow[d, "\alpha"'] & L \arrow[d, "\beta"] & \\
N \arrow[r, "g"'] & K & \\
 & & CK \arrow[ull, bend left, "\varphi^K_N"] \arrow[ul, "\gamma_K"']
\end{tikzcd}$$

since $\varphi^L_{_M}$ and $\varphi^K_{_N}$ are isomorphisms. ■

Remark 2.3.6 If we had wanted only to know that the categories **QfNFrm** and **Ext** are equivalent without exhibiting the attendant natural transformations, we could have done so by showing that the functor F is full, faithful and isomorphism-dense, where "isomorphism-dense" means each object of the codomain is isomorphic to the image of some object of the domain. The proof would have gone as follows:

(a) F *is full*: Let $h : M \longrightarrow L$ and $g : N \longrightarrow K$ be objects of **Ext**. We abbreviate these objects as h and g. We must show that the map

$$\mathsf{F}_{h,g} : \mathrm{Hom}_{\mathbf{Ext}}(h, g) \longrightarrow \mathrm{Hom}_{\mathbf{QfNFrm}}(L, K)$$

is onto. So let $f : L \longrightarrow K$ be a uniform homomorphism. View M and N as fine nearness frames and recall that the nearness on L, $\mathfrak{N}L = h[\mathrm{Cov}(M)]$ and similarly for K. Since L and K are strong, and $h : M \longrightarrow K$ and $g : N \longrightarrow K$ are (isomorphic to) completions of L and K respectively, the homomorphism f is liftable to a uniform homomorphism $\bar{f} : M \longrightarrow N$. As a consequence, $(\bar{f}, f)$ is an element of $\mathrm{Hom}_{\mathbf{Ext}}(h, g)$ mapped to f by $\mathsf{F}_{h,g}$.

(b) F *is faithful*: We show that $\mathsf{F}_{h,g}$ is one-to-one. Let (α, β) and (ι, κ) be arrows in $\mathrm{Hom}_{\mathbf{Ext}}(h, g)$ such that $\mathsf{F}_{h,g}(\alpha, \beta) = \mathsf{F}_{h,g}(\iota, \kappa)$. Then we have the commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{h} & L \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ N & \xrightarrow[g]{} & K \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \xrightarrow{h} & L \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\kappa} \\ N & \xrightarrow[g]{} & K \end{array}$$

with $\beta = \kappa$. Then $g\alpha = \beta h = \kappa h = g\iota$, whence $\alpha = \iota$ since g is monic as it is dense. Therefore $(\alpha, \beta) = (\iota, \kappa)$.

(c) F is *isomorphism-dense*. Let L be a q-fine nearness frame. Then $\gamma_{_L} : CL \longrightarrow L$ is an element of **Ext** so that $\mathsf{F}(\gamma_{_L} : CL \longrightarrow L) = L$ and the proof ends.

The functor F is not an isomorphism. If it were, then its object-function would be bijective by [34, Proposition 14.3]. But clearly its object-function is not bijective. For instance, given any regular frame L, let $j : L \oplus \mathbf{2} \longrightarrow L$ be an isomorphism $a \oplus 1 \mapsto a$. Then $\mathsf{F}(L \xrightarrow{\ \mathrm{id}_L\ } L) = \mathsf{F}(L \oplus \mathbf{2} \xrightarrow{\ j\ } L)$.

Proposition 2.3.7 *The pair* (G, H) *is an equivalence of categories. Thus,* **Compl** *and* **Ext** *are equivalent.*

Proof: We shall construct a pair of natural isomorphisms

$$\vartheta : 1_{\mathbf{Ext}} \longrightarrow \mathsf{G} \circ \mathsf{H} \text{ and } \varrho : 1_{\mathbf{Compl}} \longrightarrow \mathsf{H} \circ \mathsf{G}.$$

Now for any object (h, M, L) of **Ext**, we have

$$(\mathsf{G} \circ \mathsf{H})(h, M, L) = \mathsf{G}(CL, L) = (\gamma_{_L}, CL, L),$$

and for any morphism $(\alpha, \beta) : (h, M, L) \longrightarrow (g, N, K)$ of **Ext**, we have

$$(\mathsf{G} \circ \mathsf{H})(\alpha, \beta) = \mathsf{G}\left((\varphi^K_{_N})^{-1}\alpha\varphi^L_{_M} : CL \longrightarrow CK\right) = \left((\varphi^K_{_N})^{-1}\alpha\varphi^L_{_M},\, \beta\right).$$

As in the previous proof (see Proposition 2.3.5), for any object (h, M, L) of **Ext**, we let $\vartheta_{(h,M,L)}$ - which again we abbreviate as ϑ_h - be the **Ext**-isomorphism given by

$$\left((\varphi^L_{_M})^{-1},\, \mathrm{id}_L\right) : (h, M, L) \longrightarrow (\gamma_{_L}, CL, L).$$

Let $(\alpha, \beta) : (h, M, L) \longrightarrow (g, N, K)$ be a morphism of **Ext**. The diagram

$$\begin{tikzcd}
(h, M, L) \arrow[r, "\vartheta_h"] \arrow[d, "(\alpha,\beta)"'] & (\gamma_{_L}, CL, L) \arrow[d, "((\varphi^K_{_N})^{-1}\alpha\varphi^L_{_M}, \beta)"] \\
(g, N, K) \arrow[r, "\vartheta_g"'] & (\gamma_{_K}, CK, K)
\end{tikzcd}$$

commutes because

$$\left((\varphi^K_{_N})^{-1}\alpha\varphi^L_{_M},\, \beta\right) \circ \left((\varphi^L_{_M})^{-1}, \mathrm{id}_L\right) = \left((\varphi^K_{_N})^{-1}\alpha, \mathrm{id}_K\beta\right) = \left((\varphi^K_{_N})^{-1}, \mathrm{id}_K\right) \circ (\alpha, \beta).$$

Consequently, ϑ is a natural isomorphism.

Next, for any object (CL, L) of **Compl**, we have

$$(\mathsf{H} \circ \mathsf{G})(CL, L) = \mathsf{H}(\gamma_{_L}, CL, L) = (CL, L),$$

and for any morphism $f : (CM, M) \longrightarrow (CL, L)$ of **Compl**, we have

$$(\mathsf{H} \circ \mathsf{G})(f) = \mathsf{H}(f, \check{f}) = (\varphi^{M}_{CM})^{-1} f \varphi^{L}_{CL} = \mathrm{id}_{CM}^{-1} \cdot f \cdot \mathrm{id}_{CL} = f.$$

Given an object (CL, L) of **Compl**, let $\varrho_{(CL,L)}$ be the morphism

$$\varrho_{(CL,L)} : (CL, L) \longrightarrow (CL, L)$$

defined by the uniform isomorphism $\mathrm{id}_{CL} : CL \longrightarrow CL$. Then $\varrho_{(CL,L)}$ is an isomorphism in **Compl**. We show that this defines a natural transformation as desired. Let $f : (CM, M) \longrightarrow (CL, L)$ be a morphism in **Compl**. Then the diagram

$$\begin{tikzcd}
(CM, M) \arrow[r, "\varrho_{(CM,M)}"] \arrow[d, "f"'] & (\mathsf{H} \circ \mathsf{G})(CM, M) \arrow[d, "(\mathsf{H}\circ\mathsf{G})(f)"] \\
(CL, L) \arrow[r, "\varrho_{(CL,L)}"'] & (\mathsf{H} \circ \mathsf{G})(CL, L)
\end{tikzcd}$$

commutes, since it reduces to the diagram

$$\begin{tikzcd}
(CM, M) \arrow[r, "\mathrm{id}_{CM}"] \arrow[d, "f"'] & (CM, M) \arrow[d, "f"] \\
(CL, L) \arrow[r, "\mathrm{id}_{CL}"'] & (CL, L)
\end{tikzcd}$$

which is clearly commutative. This concludes the proof. ∎

Because the relation “is equivalent to” is an equivalence relation on the conglomerate of all categories [34, Proposition 14.9], we have that:

Corollary 2.3.8 *Any two of the categories* **Compl**, **Ext** *and* **QfNFrm** *are equivalent.*

Chapter 3

Some subcategories closed under coproducts

In this chapter we consider certain subcategories of **NFrm** which we show to be closed under the formation of quotients, completions and coproducts. To be more precise, in one instance we show the subcategory to be closed under countable coproducts. We also characterize quotient-fine nearness frames that reside inside these subcategories. The characterizations are of the form we describe below.

For some of the subcategories **A** considered here, we show that the following are equivalent for a nearness frame (L, μ):

(1) (L, μ) is quotient-fine and is in **A**.

(2) The completion of (L, μ) is fine and is in **A**.

(3) The nearness on L is induced by an extension with a certain property.

We shall say that **A** is *co-hereditary* in **NFrm** if it is closed under quotients, and *coproductive* in **NFrm** if it is closed under formation of coproducts.

We conclude the chapter with the introduction of a subcategory of **StrNFrm** which contains both the uniformly prenormal and the almost uniform nearness frames; and is closed under the formation of completions.

3.1 Zero-dimensionality in structured frames

In [44] McKee defines a nearness space (X, ξ) to be zero-dimensional if every uniform cover $\mathcal{U}$ is refined by a uniform cover $\mathcal{V}$ such that $\{V, X \setminus V\}$ is a uniform cover for each $V \in \mathcal{V}$. A further observation given in [44] is that zero-dimensional nearness spaces are regular. Consequently, employing the terminology of S.S. Hong and Y.K. Kim [36], zero-dimensional nearness spaces are framed. Now given a framed nearness space (X, ξ), for any open subset V of X, $\{V, X \setminus V\}$ is a uniform cover of X if and only if $\{V, V^*\}$ is a uniform cover of the nearness frame $\mathcal{O}X$. This, in turn, is true if and only if V is uniformly below itself in the nearness frame $\mathcal{O}X$. We therefore formulate the following definition.

Definition 3.1.1 A nearness frame (L, μ) is *uniformly zero-dimensional* if for any cover $A \in \mu$, there is a cover $B \in \mu$ refining A and with the property that for each $b \in B$, $b \lhd b$. We write **ZdNFrm** for the category of uniformly zero-dimensional nearness frames.

Now if $V \in \mathcal{O}X$, with X being a topological space, then $V^* = \mathrm{int}(X - V)$ is the pseudocomplement of V in $\mathcal{O}X$. So a framed nearness space (X, ξ) is zero-dimensional iff the associated nearness frame $(\mathcal{O}X, \mu)$ is uniformly zero-dimensional. We use the term "uniformly zero-dimensional" in order to distinguish between this new concept and the usual zero-dimensionality of frames defined by stipulating that every element be a join of complemented elements.

Proposition 3.1.2 *If a nearness frame (L, μ) is uniformly zero-dimensional, then its underlying frame L is zero-dimensional.*

Proof: Let $a \in L \setminus \{0\}$. Now $a = \bigvee\{x \in L \mid x \lhd a\}$ by the admissibility property of μ. We need to show that a can be expressed as a join of complemented elements below it in L. Suppose $x \lhd a$. This implies that $\{x^*, a\} \in \mu$. Since (L, μ) is uniformly zero-dimensional, there is $B \in \mu$ refining $\{x^*, a\}$ and having the property that $\{b, b^*\} \in \mu$ for every $b \in B$.

Now we have that $x \leq Bx = \bigvee\{y \in B \mid y \wedge x \neq 0\}$. Also $y \wedge x \neq 0$ implies $y \nleq x^*$, so that $y \leq a$, since B refines $\{x^*, a\}$. Consequently

$$a = \bigvee\{x \in L \mid x \lhd a\} \leq \bigvee\{y \in B \mid y \wedge x \neq 0\} \leq a.$$

Since each element of B is complemented, it follows that a is expressible as a join of complemented elements below it, so that L is zero-dimensional. ∎

The reverse implication in the above proposition holds if we require that the nearness frame be fine as shown in the result below.

Proposition 3.1.3 *A fine nearness frame (L, μ) is uniformly zero-dimensional iff the underlying frame L is zero-dimensional.*

Proof: The implication $\Rightarrow$ follows from the above proposition. As for the converse, suppose L is a zero-dimensional frame, and let $A \in \mu$. Then for each $a \in A$, put

$$B_a = \{x \in L \mid x \leq a \text{ and } x \text{ is complemented}\}.$$

Then, since (L, μ) is fine, for every $x \in B_a$, $\{x, x^*\}$ is a uniform cover, and, since L is zero-dimensional, $\bigvee B_a = a$. Consequently the set $B = \bigcup_{a \in A} B_a$ is a uniform cover refining A, with each of its elements uniformly below itself. Thus, (L, μ) is uniformly zero-dimensional. ∎

Definition 3.1.4 Let (L, μ) be a nearness frame. We say $B \in \mu$ *strongly refines* $A \in \mu$ and write $B \lhd A$ if for each $b \in B$, there exists $a \in A$ such that $b \lhd a$. Note that $B \lhd A$ implies $B \leq A$, since $b \lhd a$ implies $b \leq a$.

Before showing the containment **ZdNFrm** $\subseteq$ **StrNFrm**, we will need the following characterization of strong nearness frames.

Lemma 3.1.5 (L, μ) *is a strong nearness frame if and only if for each $A \in \mu$, there exists $B \in \mu$ such that $B \lhd A$.*

Proof: ($\Rightarrow$) Suppose (L, μ) is strong. Then, by definition, given $A \in \mu$, we have that the cover

$$\check{A} = \{b \in L \mid \exists\, a \in A,\ b \lhd a\}$$

belongs to μ. Then clearly, $\check{A} \lhd A$.

($\Leftarrow$) Conversely, suppose the given condition holds. Let $C \in \mu$. Then by the given condition, let B be a uniform cover strongly refining C. If $b \in B$, then there exists $c \in C$ such that $b \lhd c$, which implies $b \leq c$. Consequently, B refines $\check{C}$. Hence $\check{C} \in \mu$, so that (L, μ) is strong. ∎

Proposition 3.1.6 *Every uniformly zero-dimensional nearness frame is strong.*

Proof: Suppose (L, μ) is uniformly zero-dimensional. Let $A \in \mu$. By the hypothesis, there exists $B \in \mu$ such that $B \leq A$, with $b \lhd b$ for every $b \in B$. To complete the proof, we show that B strongly refines A. But this is clear, since for each $b \in B$, there exists $a \in A$ such that $b \lhd b \leq a$, so that $b \lhd a$. Consequently, $B \lhd A$. ∎

Proposition 3.1.7 **ZdNFrm** *is co-hereditary and coproductive in* **NFrm**.

Proof: First, we show that **ZdNFrm** is co-hereditary. Let (L, μ) and (M, η) be nearness frames, where (L, μ) is uniformly zero-dimensional, and let $h : L \longrightarrow M$ be an onto homomorphism such that $\eta = h[\mu]$. To see that (M, η) is uniformly zero-dimensional, let $A \in \eta$ and $B \in \mu$ be such that $A = h[B]$. Since (L, μ) is uniformly zero-dimensional, there exists $C \in \mu$ refining B, with the property that for every $c \in C$, $c \lhd c$. In this case $h[C]$ is a uniform cover of M refining $h[B]$, and $h(c) \lhd h(c)$ for every $c \in C$. Hence (M, η) is uniformly zero-dimensional.

Second, we show that **ZdNFrm** is closed under formation of coproducts. Let $\{(L_i, \mu_i)\}_{i \in I}$ be a family of uniformly zero-dimensional nearness frames. Let $(\oplus_i L_i, \oplus_i \mu_i)$ be the coproduct of the said family, and $A \in \oplus_i \mu_i$. Pick $B_i \in \mu_i$, where finitely many of the B_i's are nontrivial, such that $\oplus_i B_i \leq A$. Let $B_{i_1}, \ldots, B_{i_m}$ be the nontrivial uniform covers.

For each $k \in \{1, \ldots, m\}$, let C_{i_k} be a uniform cover of L_{i_k} refining B_{i_k} such that each element of C_{i_k} is uniformly below itself. For each $i \notin \{i_1, \ldots, i_m\}$, let $C_i = \{1\}$. Then $\oplus_i C_i \in \oplus_i \mu_i$. Observe that $\oplus_i C_i \leq \oplus_i B_i$.

Next, let $\oplus_i c_i \in \oplus_i C_i$. Since $c_{i_k} \lhd c_{i_k}$ for all $k = 1, \ldots, m$ and $1 \lhd 1$, we conclude, by Lemma 1.4.11(iii), that $\oplus_i c_i \lhd \oplus_i c_i$. Hence $(\oplus_i L_i, \oplus_i \mu_i)$ is uniformly zero-dimensional. ∎

Lemma 3.1.8 *If a strong nearness frame has a dense quotient which is uniformly zero-dimensional, then the nearness frame itself is uniformly zero-dimensional.*

Proof: Let $h : (L,\mu) \longrightarrow (M,\eta)$ be a dense surjection with (L,μ) strong and (M,η) uniformly zero-dimensional. Then, by Lemma 1.4.6(i), h is a strict surjection. Let $A \in \mu$. Pick a uniform cover C of M such that $h_*[C] \leq A$. Since (M,η) is uniformly zero-dimensional, there exists a uniform cover D of M which refines C and each of whose elements is uniformly below itself. Then $h_*[D]$ is a uniform cover of L refining A. By Lemma 1.4.7(iii), every element of $h_*[D]$ is uniformly below itself. Therefore (L,μ) is uniformly zero-dimensional. ∎

As a consequence of **ZdNFrm** being co-hereditary and Lemma 3.1.8 we have the following result.

Corollary 3.1.9 *A nearness frame is uniformly zero-dimensional iff its completion is uniformly zero-dimensional.*

The next result, which is in the format mentioned in the introduction to this chapter, is a frame version of the main result of [44], and is easily deducible from the above results.

Proposition 3.1.10 *The following are equivalent for a nearness frame (L,μ):*

(1) (L,μ) is quotient-fine and uniformly zero-dimensional.

(2) The completion of (L,μ) is both fine and uniformly zero-dimensional.

(3) The nearness on L is induced by a zero-dimensional extension.

Proof: (1)⇒(2): Let (L,μ) be quotient-fine and uniformly zero-dimensional. Then, by the above corollary, its completion is uniformly zero-dimensional. The completion is fine, by Lemma 2.1.6.

(2)⇒(3): Suppose the completion $(CL, C\mu)$ satisfies condition (2). Then $C\mu = \mathrm{Cov}(CL)$. By Proposition 3.1.3, the frame CL is zero-dimensional. Now the completion map $\gamma_{_L} : CL \longrightarrow L$ is dense onto, and since (L,μ) is quotient-fine,

$$\mu = \gamma_{_L}[C\mu] = \gamma_{_L}[\mathrm{Cov}(CL)].$$

Hence (3) holds.

(3)$\Rightarrow$(1): Let $h : M \longrightarrow L$ be a dense onto frame homomorphism such that the frame M is zero-dimensional, and $\mu = h[\mathrm{Cov}(M)]$. Then (L, μ) is quotient-fine since it is a dense quotient of a fine nearness frame. By Proposition 3.1.3, $(M, \mathrm{Cov}(M))$ is uniformly zero-dimensional. Hence, by Proposition 3.1.7, (L, μ) is uniformly zero-dimensional. ∎

We now want to contrast uniform zero-dimensionality with the frame-theoretic version of what Herrlich [33] calls zero-dimensionality of nearness spaces defined by requiring that every uniform cover be refined by a uniform partition. We therefore formulate the following definition. Recall that a *partition* of a frame L is a cover P by complemented elements such that $a \wedge b = 0$ for all distinct a, $b \in P$. Note that in fact the word "complemented" can be omitted in the definition of a partition because if P is a cover of L and $a \wedge b = 0$ for all distinct a, $b \in P$, then each element of P is complemented. Indeed, given any $a \in P$,

$$a \wedge \bigvee (P \setminus \{a\}) = \bigvee \{a \wedge x \mid x \in P,\ x \neq a\} = 0 \text{ and } a \vee \bigvee (P \setminus \{a\}) = 1.$$

Definition 3.1.11 A nearness frame is *H-zero-dimensional* if every uniform cover is refined by a uniform partition. We write **HZdNFrm** for the category of H-zero-dimensional nearness frames.

Clearly, any partition star-refines itself. Thus, any H-zero-dimensional nearness frame is actually a uniform frame. This definition is not new. It is precisely the definition of what are called *transitive* uniform frames in [7]. Because we want to emphasize the inherent zero-dimensionality in these nearness frames, we shall use the adjective in the definition rather than the one in [7].

Proposition 3.1.12 *If a nearness frame (L, μ) is H-zero-dimensional, then it is uniformly zero-dimensional.*

Proof: Suppose (L, μ) is H-zero-dimensional. Let $A \in \mu$ and P a uniform partition refining A. Let $x \in P$. We infer that P refines $\{x,\ x^*\}$, since whenever $a \neq x$ in P we have $a \wedge x = 0$, so that $a \leq x^*$. Hence $\{x, x^*\} \in \mu$ so that (L, μ) is uniformly zero-dimensional. ∎

The converse of the above result is false; that is, a uniformly zero-dimensional nearness frame need not be H-zero-dimensional. The following example substantiates this contention.

Example 3.1.13 As mentioned in [7, page 40], the frame $L = \mathcal{O}X$, where X is the Gleason cover of the Tychonoff plank, is regular and extremally disconnected but not normal. Thus, L is zero-dimensional since, for any $a \in L$, $x \prec a$ implies $x^{**} \leq a$, so that

$$a = \bigvee \{x \in L \mid x \prec a\} = \bigvee \{x^{**} \mid x \prec a\},$$

showing that a is a join of complemented elements. Therefore, in view of Proposition 3.1.3, $(L, \mathrm{Cov}(L))$ is a uniformly zero-dimensional nearness frame. On the other hand though, $(L, \mathrm{Cov}(L))$ is not H-zero-dimensional, for if it were, then it would be a uniform frame, and hence L would be paracompact, and therefore normal.

The above example shows that the two notions of uniform zero-dimensionality do not coincide; the latter being strictly stronger in general. There are instances where they agree. To show that we shall require the following lemma from [7].

Lemma 3.1.14 *In any frame L, any locally finite (and therefore any finite) cover by complemented elements is refined by a partion.*

As observed in the proof of [7, Proposition 2.3], if A is a finite cover by complemented elements and P is a partition refining A, then the set $\{\tilde{x} \mid x \in P\}$, where, for each $x \in P$,

$$\tilde{x} = \bigvee \{y \in P \mid A \cap {\downarrow} x = A \cap {\downarrow} y\}$$

is a finite partition refining A.

Proposition 3.1.15 *A finitely fine nearness frame is uniformly zero-dimensional iff it is H-zero-dimensional.*

Proof: In light of Proposition 3.1.12, we need only prove the left-to-right implication. So, let L be a uniformly zero-dimensional nearness frame. Let A be a finite uniform cover. By hypothesis, there is a uniform cover B which refines A and is such that $b \lhd b$ for each

$b \in B$. Since L is finitely fine, B has a finite subset B' which is a uniform cover. Each element of B' is complemented, so by Lemma 3.1.14 and the discussion following it, there is a finite partition P which refines B', and hence A. Since L is finitely fine and P is finite, P is a uniform partition refining A. Since finite uniform covers generate the nearness of L, it follows that every uniform cover of L is refined by a uniform partition. Therefore L is H-zero-dimensional. ∎

In order for us to characterize quotient-fine H-zero-dimensional nearness frames in the manner described at the beginning of the chapter, we shall need the following preliminary results.

Lemma 3.1.16 *A quotient of an H-zero-dimensional nearness frame is H-zero-dimensional.*

Proof: Let (L, μ) be a nearness frame and $h : L \longrightarrow M$ a quotient map. Suppose A is a uniform cover of M. Then, since h is a surjection, pick $B \in \mu$ such that $A = h[B]$. Since (L, μ) is H-zero-dimensional, there is a uniform partition P of L such that P refines A. Therefore $h[P]$ is a uniform partition of M refining A. Consequently, $(M, h[\mu])$ is H-zero-dimensional. ∎

Lemma 3.1.17 *Let $h : (L, \mu) \longrightarrow (M, \eta)$ be a strict surjection. Then (L, μ) is H-zero-dimensional iff (M, η) is H-zero-dimensional. Hence, a nearness frame is H-zero-dimensional iff its completion has the same feature.*

Proof: The one implication follows from Lemma 3.1.16 Conversely, let A be a uniform cover of L. Then $h_*[B]$ refines A for some uniform cover B of M. By hypothesis, there is a uniform partition P of M which refines B. Since h is a strict surjection, $h_*[P]$ is a cover of L. Let x and y be distinct elements of $h_*[P]$. Choose $a, b \in P$ such that $x = h_*(a),\ y = h_*(b)$. Then a and b are distinct, and therefore, in view of h being dense,

$$x \wedge y = h_*(a) \wedge h_*(b) = h_*(a \wedge b) = h_*(0) = 0.$$

Thus, $h_*[P]$ is a uniform partition of L refining A. ∎

To obtain a characterization of quotient-fine H-zero-dimensional nearness frames, we first recall from [7, Proposition 2.6] that

> *a completely regular frame is paracompact and strongly zero-dimensional iff every cover is refined by a partition.*

Proposition 3.1.18 *The following are equivalent for a nearness frame* (L, μ)*:*

(1) (L, μ) *is quotient-fine and H-zero-dimensional.*

(2) The completion of (L, μ) *is fine and H-zero-dimensional.*

(3) The nearness on L is induced by a paracompact and strongly zero-dimensional extension.

Proof: (1)⇒(2): Suppose (L, μ) is quotient-fine and H-zero-dimensional. Then its completion is H-zero-dimensional, by Lemma 3.1.17, and fine, by Lemma 2.1.6.

(2)⇒(3): Suppose condition (2) holds. Let $(CL, C\mu)$ be the completion. Then $C\mu = \mathrm{Cov}(CL)$, and, since the completion is H-zero-dimensional, every cover of the frame CL is refined by a partition. So by the result in [7] stated above, CL is paracompact and strongly zero-dimensional. Since the completion map $\gamma_{_L} : CL \longrightarrow L$ is dense onto, and $\mu = \gamma_{_L}[C\mu]$, we deduce that (3) holds.

(3)⇒(1): Let $h : M \longrightarrow L$ be a dense onto frame homomorphism, where M is a paracompact, strongly zero-dimensional frame and $\mu = h[\mathrm{Cov}(M)]$. Then (L, μ) is quotient-fine, and by the cited result in [7], $(M, \mathrm{Cov}(M))$ is H-zero-dimensional. Consequently, by Lemma 3.1.16, (L, μ) is H-zero-dimensional. ∎

Proposition 3.1.19 **HZdNFrm** *is co-hereditary and coproductive in* **NFrm**.

Proof: The co-hereditary property is Lemma 3.1.16. To show coproductivity, let $\{(L_i, \mu_i)\}_{i \in I}$ be a family of H-zero-dimensional nearness frames. Let $\oplus_i A_i$ be a basic uniform cover of $\oplus_i L_i$ with $A_{i_1}, \ldots, A_{i_m}$ being the only nontrivial uniform covers. For each $k \in \{1, \ldots, m\}$, let P_{i_k} be a uniform partition of L_{i_k} which refines A_{i_k}. For $i \notin \{i_1, \ldots, i_m\}$ let $P_i = \{1\}$. Then $\oplus_i P_i \in \oplus_i \mu_i$ and $\oplus_i P_i \leq \oplus_i A_i$.

It remains to show that $\oplus_i P_i$ is a partition of $\oplus_i L_i$. Let $\oplus_i p_i$ and $\oplus_i q_i$ be distinct elements of $\oplus_i P_i$. Since $p_j = 1 = q_j$ for $j \notin \{1, \ldots, m\}$ there exists $l \in \{1, \ldots, m\}$ such that $p_{i_l} \neq q_{i_l}$. So p_{i_l} and q_{i_l} are distinct elements of the partition P_{i_l}, and this implies $p_{i_l} \wedge q_{i_l} = 0$. Therefore $\oplus_i p_i \wedge \oplus_i q_i = \oplus_i (p_i \wedge q_i) = 0$. Thus, $\oplus_i P_i$ is a partition, and the proof ends. ∎

3.2 Čech-complete nearness frames

In this section, following the discussion of Bentley and Hunsaker [20] on Čech-complete nearness spaces, we introduce Čech-complete and strongly Čech-complete frames, leading us to explore the properties of constrained and controlled nearness frames. We end the section by introducing uniformly Čech-complete and uniformly strongly Čech-complete nearness frames, showing their relationship with the constrained and controlled ones.

Let L be a frame, $F \subseteq L$ a filter, and $\mathcal{N} \subseteq \mathrm{Cov}(L)$. Then we say:

(a) F *clusters* if for every cover $A \in \mathrm{Cov}(L)$, there exists $a \in A$ such that for every $x \in F$, $a \wedge x \neq 0$.

(b) F is $\mathcal{N}$-Cauchy if for every $C \in \mathcal{N}$, $F \cap C \neq \emptyset$. (In other words, F meets every cover in $\mathcal{N}$).

(c) F *converges* if F meets every cover of L. (Thus, every convergent filter is $\mathcal{N}$-Cauchy).

(d) $\mathcal{N}$ is *complete* if every $\mathcal{N}$-Cauchy filter converges.

(e) $\mathcal{N}$ is *weakly complete* if every $\mathcal{N}$-Cauchy filter clusters.

Definition 3.2.1 (i) A frame L is *Čech-complete* if there is a countable collection $\mathcal{N} \subseteq \mathrm{Cov}(L)$ which is weakly complete. L is *strongly Čech-complete* if it has a countable complete collection of covers.

(ii) A nearness frame (L, μ) is said to be *constrained* if there is a countable collection $\mathcal{N} \subseteq \mu$ such that every $\mathcal{N}$-Cauchy filter is a near subset. In this case we say $\mathcal{N}$

constrains L. We write **ConNFrm** for the resulting subcategory of constrained nearness frames.

In order to establish our first result in this section, we shall need to observe the following:

Lemma 3.2.2 *In a fine nearness frame* $(L, \mathrm{Cov}(L))$*, a filter* $F \subseteq L$ *is near iff it clusters.*

Proof: ($\Rightarrow$) Let $F \subseteq L$ be a near filter, and let $A \in \mathrm{Cov}(L)$. Then, by definition, there exists $x \in A$ such that for every $y \in F$, $x \wedge y \neq 0$. This implies that F clusters.

($\Leftarrow$) Conversely, if $F \subseteq L$ clusters, then, clearly, F is a near subset, since the nearness here is the whole of $\mathrm{Cov}(L)$. ∎

Proposition 3.2.3 *A frame* L *is Čech-complete iff* $(L, \mathrm{Cov}(L))$ *is a constrained nearness frame.*

Proof: ($\Rightarrow$) Suppose L is Čech-complete. We show that $(L, \mathrm{Cov}(L))$ is constrained. By hypothesis, let $\mathcal{N} \subseteq \mathrm{Cov}(L)$ be a weakly complete countable collection, and let $F \subseteq L$ be an $\mathcal{N}$-Cauchy filter. Then F clusters, since $\mathcal{N}$ is weakly complete. So, by the above lemma, F is near, and hence, $(L, \mathrm{Cov}(L))$ is constrained.

($\Leftarrow$) Conversely, let $\mathcal{M} \subseteq \mathrm{Cov}(L)$ be a countable collection constraining L. Then every $\mathcal{M}$-Cauchy filter $F \subseteq L$ is near, and, by the above lemma, F clusters. This implies $\mathcal{M}$ is weakly complete, so that L is Čech-complete. ∎

Proposition 3.2.4 *Suppose* $h : (M, \eta) \longrightarrow (L, \mu)$ *is a dense surjection between nearness frames. Then* (M, η) *is constrained iff* (L, μ) *is constrained.*

Proof: ($\Leftarrow$) Suppose $\mathcal{N}$ constrains L. Then we claim that

$$\mathcal{M} = \{h_*[A] \mid A \in \mathcal{N}\}$$

constrains M. Let G be any $\mathcal{M}$-Cauchy filter, and put $F = h[G]$. Then, since h is dense, F is a filter in L which is $\mathcal{N}$-Cauchy. So F is near. Let $U \in \eta$. Then $h[U] \in \mu$, and so there is $u \in U$ such that $h(u) \wedge x \neq 0$ for every $x \in F$. This implies $u \wedge y \neq 0$ for every $y \in G$. Hence G is near.

($\Rightarrow$) Conversely, suppose $\mathcal{K}$ constrains M. For each $U \in \mathcal{K}$, let $A_U \in \mu$ be such that $h_*[A_U]$ refines U. Put

$$\mathcal{H} = \{A_U \mid U \in \mathcal{K}\}.$$

We claim that $\mathcal{H}$ constrains L. To see this, let F be an $\mathcal{H}$-Cauchy filter in L. Put

$$G = \{z \in M \mid z \geq h_*(x) \text{ for some } x \in F\}.$$

Then G is a $\mathcal{K}$-Cauchy filter in M, and is therefore near. If $A \in \mu$, then there is $a \in A$ such that $h_*(a) \wedge z \neq 0$ for every $z \in G$, since $h_*(A) \in \eta$. Then for any $x \in F$, $a \wedge x \neq 0$ since h is dense. Hence F is near. ∎

Since the completion map $\gamma_{_L} : CL \longrightarrow L$ is a dense surjection, we deduce from the above proposition the following result.

Corollary 3.2.5 *A nearness frame is constrained iff its completion is constrained.*

We have the following characterization for constrained quotient-fine nearness frames.

Proposition 3.2.6 *The following are equivalent for a nearness frame (L, μ):*

(1) (L, μ) is quotient-fine and constrained.

(2) The completion of (L, μ) is both fine and constrained.

(3) The nearness on L is induced by a Čech-complete extension.

Proof: (1)$\Rightarrow$(2): Let (L, μ) be quotient-fine and constrained. Then, by the above corollary, the completion $(CL, C\mu)$ is constrained. The completion is fine, by Lemma 2.1.6.

(2)$\Rightarrow$(3): Suppose the completion $(CL, C\mu)$ is fine and constrained. Then $C\mu = \mathrm{Cov}(CL)$, and, by Proposition 3.2.3, the frame CL is Čech-complete. Since the completion map $\gamma_{_L} : CL \longrightarrow L$ is dense onto and $\mu = \gamma_{_L}[C\mu]$, we deduce (3) holds.

(3)⇒(1): Let $h : M \longrightarrow L$ be a dense onto frame homomorphism, where M is a Čech-complete frame and $\mu = h[\mathrm{Cov}(M)]$. Then (L, μ) is quotient-fine, and, by Proposition 3.2.3, $(M, \mathrm{Cov}(M))$ is constrained. Consequently, by Proposition 3.2.4, (L, μ) is constrained. ∎

In the following example we shall need to observe the following:

Lemma 3.2.7 *(i) Every strongly Čech-complete frame is Čech-complete.*

(ii) If L is compact regular, then it is Čech-complete.

Proof: (i) Clear from the definitions.

(ii) Given L is compact regular, let $A \in \mathrm{Cov}(L)$ and take $\mathcal{N} = \{A\}$. Since L is compact regular, every filter $F \subseteq L$ clusters (see [35, Corollary 1.5]). In particular if F is an $\mathcal{N}$-Cauchy filter, then it clusters, so that $\mathcal{N}$ is weakly complete, and consequently, L is Čech-complete. ∎

Example 3.2.8 We wish to note that the underlying frame of a constrained nearness frame may fail to be Čech-complete. To see this, consider the set $\mathbb{Q}$ of rationals with its usual topology $L = \mathcal{O}\mathbb{Q}$. So L is not a Čech-complete frame. Let $h : \beta L \longrightarrow L$ be the Stone-Čech compactification of L, and equip L with the nearness

$$\mu = \{h[A] \mid A \in \mathrm{Cov}(\beta L)\}.$$

Now, since βL is compact and regular, it is also Čech-complete, and therefore constrained (being a fine nearness frame). In addition, $h : \beta L \longrightarrow L$ is the completion of (L, μ). So (L, μ) is constrained, even though L fails to be Čech-complete.

In the next set of results we aim to show that constrainedness is preserved under countable coproducts. We make use of ultrafilters to achieve that. First, we note the following result.

Lemma 3.2.9 *If $h : L \longrightarrow M$ is a dense frame homomorphism, and $U \subseteq M$ an ultrafilter, then $h^{-1}[U] \subseteq L$ is an ultrafilter of L.*

Proof: Let $a \in L$. Then, since h is dense, it preserves pseudocomplements (that is $h(a^*) = h(a)^*$). This implies, by Lemma 1.3.1(iii), either $h(a) \in U$ or $h(a^*) \in U$. Consequently $a \in h^{-1}[U]$ or $a^* \in h^{-1}[U]$, and therefore $h^{-1}[U]$ is an ultrafilter. ∎

Let (L, μ) be a nearness frame. We say a filter $F \subseteq L$ is *Cauchy* if $F \cap C \neq \emptyset$ for each $C \in \mu$. We observe that Cauchy filters are near subsets. To see this, if $F \subseteq L$ is Cauchy and $U \in \mu$, then there exists $x \in F \cap U$. Since F is a filter, $x \wedge y \neq 0$ for each $y \in F$, so that F is near.

To show that the coproduct of countably many constrained nearness frames is constrained, we shall need to observe the following:

Lemma 3.2.10 *An ultrafilter U in a nearness frame (L, μ) is a near subset iff it is Cauchy.*

Proof: If U is Cauchy, then clearly, it is near (see [22]). Conversely, suppose U is near, and let $C \in \mu$. Take $c \in C$ such that $c \wedge x \neq 0$ for each $x \in U$. Since U is an ultrafilter, this implies $c \in U$. ∎

Our argument for the following result is modeled along that of [20, Proposition 9].

Proposition 3.2.11 ConNFrm *is countably coproductive in* **NFrm**.

Proof: Let $\{(L_n, \mu_n)\}_{n \in \mathbb{N}}$ be a countable family of constrained nearness frames. We show that the coproduct $(\oplus_n L_n, \oplus_n \mu_n)$ is also constrained. For each $n \in \mathbb{N}$, let $\mathcal{N}_n \subseteq \mu_n$ be a countable collection constraining L_n. Define a countable collection $\mathcal{N}$ of uniform covers of $\oplus_n L_n$ by

$$\mathcal{N} = \{\oplus_n C_n \mid \text{for some } k \in \mathbb{N},\, n \leq k \text{ implies } C_n \in \mathcal{N}_n, \text{ and } C_n = \{1\} \text{ otherwise}\}\,.$$

We show that $\mathcal{N}$ constrains $\oplus_n L_n$. Let $\mathsf{F} \subseteq \oplus_n L_n$ be an $\mathcal{N}$-Cauchy filter in $\oplus_n L_n$. We shall show that F is near, and the proof ends. Let G be an ultrafilter containing F. Then G is also $\mathcal{N}$-Cauchy. Let

$$(\iota_n : L_n \longrightarrow \oplus_n L_n)_n$$

be the coproduct injections. Since the ι_n are dense, we have that, for each $n \in \mathbb{N}$,

$$\iota_n^{-1}[\mathsf{G}] = \{a \in L_n \mid \iota_n(a) \in \mathsf{G}\}$$

is an ultrafilter in L_n. We show that, for each $n \in \mathbb{N}$, $\iota_n^{-1}[\mathsf{G}]$ is an $\mathcal{N}_n$-Cauchy filter. Fix $n \in \mathbb{N}$. For each $m \leq n$ pick any cover $C_m \in \mathcal{N}_m$, and for $m > n$, let $C_m = \{1\}$. Then $\oplus_m C_m \in \mathcal{N}$. Since G is $\mathcal{N}$-Cauchy, it meets $\oplus_m C_m$. Pick an element

$$\oplus_m c_m \in \mathsf{G} \cap (\oplus_m C_m).$$

Since $\iota_n(c_n) = \oplus_m b_m$, where $b_n = c_n$ and $b_m = 1$ for $m \neq n$, we have that

$$\oplus_m c_m \leq \iota_n(c_n).$$

So $\iota_n(c_n) \in \mathsf{G}$, since G is an upset. This implies $c_n \in \iota_n^{-1}[\mathsf{G}]$, and therefore $\iota_n^{-1}[\mathsf{G}]$ meets C_n. Since C_n was arbitrarily chosen, we conclude that $\iota_n^{-1}[\mathsf{G}]$ is near.

Next, we show that G is near. Let A be a uniform cover of $\oplus_n L_n$. Find $\oplus_n A_n \leq \mathsf{A}$, where $A_{n_1} \in \mu_{n_1}, \ldots, A_{n_k} \in \mu_{n_k}$ is a finite subsequence of the A_n's and $A_n = \{1\}$ for $n \notin \{n_1, \ldots, n_k\}$. Since the $\iota_n^{-1}[\mathsf{G}]$ are near ultrafilters, they are Cauchy and hence, there exists $x_n \in A_n$, for every n, such that $x_n \in \iota_n^{-1}[\mathsf{G}]$. This implies $\iota_n(x_n) \in \mathsf{G}$. Since G is a filter, it follows that

$$\bigwedge_n \iota_n(x_n) = \oplus_n x_n \in \mathsf{G},$$

where the meet is in fact a finite meet (since only finitely many of the x_n's are nontrivial). Since $\oplus_n A_n \leq \mathsf{A}$, there exists $\mathbf{a} \in \mathsf{A}$ such that $\oplus_n x_n \leq \mathbf{a}$. Since G is a filter and $\oplus_n x_n \in \mathsf{G}$, it follows that $\mathbf{a} \in \mathsf{G}$. Thus, $\mathbf{a} \in \mathsf{A} \cap \mathsf{G}$. Therefore G is near, and we conclude that F is also near. ∎

Definition 3.2.12 A nearness frame (L, μ) is said to be *controlled* if there is a countable collection $\mathcal{N} \subseteq \mu$ such that every $\mathcal{N}$-Cauchy filter is Cauchy. In this case we say $\mathcal{N}$ *controls* L. We write **CntrNFrm** for the resulting category of controlled nearness frames.

Remark 3.2.13 Since Cauchy filters are near, as observed earlier, controlled nearness frames are constrained. Thus, **CntrNFrm** $\subseteq$ **ConNFrm**.

To note an easy example of controlled nearness frames, we recall that a nearness frame (L, μ) is said to be of *countable type* if the nearness μ is generated by a countable collection of covers of L. So every nearness frame of countable type is controlled by a countable base for the nearness.

Proposition 3.2.14 *A frame L is strongly Čech-complete iff $(L, \mathrm{Cov}(L))$ is a controlled, fine nearness frame.*

Proof: ($\Rightarrow$) Given L is strongly Čech-complete, let $\mathcal{N} \subseteq \mathrm{Cov}(L)$ be a countable complete collection, and $F \subseteq L$ an $\mathcal{N}$-Cauchy filter. Then F converges. This implies that $F \cap C \neq \emptyset$ for every $C \in \mathrm{Cov}(L)$, so that F is a Cauchy filter. Hence $\mathcal{N}$ controls L.

($\Leftarrow$) Conversely, suppose $(L, \mathrm{Cov}(L))$ is controlled. Let $\mathcal{M} \subseteq \mathrm{Cov}(L)$ be a countable collection controlling L, and let $F \subseteq L$ be an $\mathcal{M}$-Cauchy filter. Then for every $A \in \mathrm{Cov}(L)$, $F \cap A \neq \emptyset$, since $\mathcal{M}$ controls L. But this means F converges. Hence $\mathcal{M}$ is complete, so that L is strongly Čech-complete. ∎

Proposition 3.2.15 *If $h : (M, \eta) \longrightarrow (L, \mu)$ is a surjection of nearness frames, then (M, η) is controlled iff (L, μ) is controlled.*

Proof: ($\Rightarrow$) Suppose $\mathcal{A} \subseteq \eta$ controls M. We show that

$$\mathcal{B} = \{h[A] \mid A \in \mathcal{A}\}$$

controls L. Let F be a $\mathcal{B}$-Cauchy filter in L. Then the set

$$G = \{z \in M \mid z \geq h_*(x) \text{ for some } x \in F\}$$

is an $\mathcal{A}$-Cauchy filter, and hence is Cauchy. Let $U \in \mu$ and pick a $V \in \eta$ such that $h[V]$ refines U. Choose $v \in V \cap G$ and $x \in F$ such that $h_*(x) \leq v$. Also choose $u \in U$ such that $h(v) \leq u$. Then $x = hh_*(x) \leq h(v) \leq u$, and it follows that $u \in F \cap U$. So F is Cauchy.

($\Leftarrow$) Conversely, suppose $\mathcal{N} \subseteq \mu$ controls L. Put

$$\mathcal{M} = \{h_*[C] \mid C \in \mathcal{N}\},$$

and let G be an $\mathcal{M}$-Cauchy filter of M. Then $F = h[G]$ is an $\mathcal{N}$-Cauchy filter of L. By an argument similar to the above, we see that G is a Cauchy filter. ∎

In the above proof, we did not require that h be strict. Only the onto property of h was required. It then follows that the category **CntrNFrm** is co-hereditary. We also deduce from the above proposition that **CntrNFrm** is closed under completions as stated in the following corollary.

Corollary 3.2.16 *A nearness frame is controlled iff its completion is controlled.*

Proposition 3.2.17 *The following are equivalent for a nearness frame (L, μ):*

(1) (L, μ) is quotient-fine and controlled.

(2) The completion of (L, μ) is both fine and controlled.

(3) The nearness on L is induced by a strongly Čech-complete extension.

Proof: (1)⇒(2): Suppose (L, μ) is quotient-fine and controlled. By the corollary above, the completion $(CL, C\mu)$ is controlled, and, by Lemma 2.1.6, the completion is fine.

(2)⇒(3): Suppose $(CL, C\mu)$ is fine and controlled. Then, by Proposition 3.2.14, CL is a strongly Čech-complete frame. Since $C\mu = \mathrm{Cov}(CL)$ and the completion map γ_L is dense onto with $\mu = \gamma_L[C\mu]$, we deduce that (3) holds.

(3)⇒(1). Suppose $h : M \longrightarrow L$ is dense onto with $\mu = h[\mathrm{Cov}(M)]$ and M being a strongly Čech-complete frame. Then, by Proposition 3.2.14, $(M, \mathrm{Cov}(M))$ is controlled. (L, μ) is quotient-fine since h is a quotient map, and, by Proposition 3.2.15, (L, μ) is controlled. ∎

Definition 3.2.18 A nearness frame (L, μ) is called *uniformly Čech-complete* if there exists a countable collection $\mathcal{N} \subseteq \mu$ which is weakly complete. We write **UCCNFrm** for the resulting category.

Remark 3.2.19 Consequently, from the definitions, if (L, μ) is a uniformly Čech-complete nearness frame, then the underlying frame L is Čech-complete. Notice that if (L, μ) is a fine nearness frame, then the converse holds.

Proposition 3.2.20 *Every uniformly Čech-complete nearness frame is constrained. Thus,* **UCCNFrm** $\subseteq$ **ConNFrm**.

Proof: Given (L, μ) is uniformly Čech-complete, let $\mathcal{N} \subseteq \mu$ be a countable weakly complete collection, and let $F \subseteq L$ be an $\mathcal{N}$-Cauchy filter. Take $C \in \mu$. Then, since F clusters, there exists $x \in C$ such that for all for all $y \in F$, $x \wedge y \neq 0$. This implies F is near. Hence $\mathcal{N}$ constrains L. ∎

Definition 3.2.21 A nearness frame (L, μ) is called *uniformly strongly Čech-complete* if there exists a countable collection $\mathcal{N} \subseteq \mu$ which is complete. We write **UsCCNFrm** for the resulting category.

We observe that every uniformly strongly Čech-complete nearness frame is necessarily uniformly Čech-complete, and its underlying frame is strongly Čech-complete. Since every filter $F \subseteq L$ which converges is clearly a Cauchy filter, the following result is immediate.

Proposition 3.2.22 *Every uniformly strongly Čech-complete nearness frame is controlled.*

We, therefore, have established the following containments among the subcategories discussed in this section:

$$\mathbf{UsCCNFrm} \subseteq \mathbf{UCCNFrm} \subseteq \mathbf{ConNFrm}$$

and

$$\mathbf{UsCCNFrm} \subseteq \mathbf{CntrNFrm} \subseteq \mathbf{ConNFrm}.$$

3.3 Almost uniform nearness frames

Our attempts at establishing whether or not the category of strong nearness frames is coreflective in the category of nearness frames were not successful. We have however established that almost uniform frames are coreflective in the category of interpolative nearness frames - a result presented shortly. But first we establish that **StrNFrm** is closed under coproducts.

Proposition 3.3.1 *The category* **StrNFrm** *is coproductive in* **NFrm**.

Proof: Let $\{(L_i, \mu_i)\}_{i\in I}$ be a family of strong nearness frames. We show that the coproduct $(\oplus_i L_i, \oplus_i \mu_i)$ is also a strong nearness frame. Let $A \in \oplus_i \mu_i$. Then A is refined by a uniform cover of the form $\oplus_i A_i$, where only finitely many of the A_i are nontrivial, say $A_{i_1}, \ldots, A_{i_m}$. We construct a strong refinement of $\oplus_i A_i$ as follows: for each $i \in \{i_1, \ldots, i_m\}$, let B_i be a uniform cover of L_i strongly refining A_i. For all the other i's, take $B_i = \{1\}$. Then, as a consequence of Lemma 1.4.11(iii), $\oplus_i B_i$ is a uniform cover of $\oplus_i L_i$ strongly refining $\oplus_i A_i$, and hence A, so that $(\oplus_i L_i, \oplus_i \mu_i)$ is strong. ■

Write **IntNFrm** for the category of interpolative nearness frames and **AuNFrm** for the category of almost uniform nearness frames.

Proposition 3.3.2 *The category* **AuNFrm** *is coproductive in* **NFrm**.

Proof: Let $(L_i, \mu_i)_{i\in I}$ be a family of almost uniform frames. We show that the coproduct $(\oplus_i L_i, \oplus_i \mu_i)$ is almost uniform. Now each of the L_i's is strong, so the coproduct $\oplus_i L_i$ is strong, by the above proposition. We need only show that this coproduct is interpolative.

Suppose $\oplus_i a_i \lhd \oplus_i b_i$ in $\oplus_i L_i$. If $\oplus_i a_i = 0$, then we have $0 \lhd 0 \lhd \oplus_i b_i$. Suppose $0 \neq \oplus_i a_i$. Then, by Lemma 1.4.11(iii), $0 \neq a_i \lhd b_i$ for all $i \in I$. For every i, find $c_i \in L_i$ such that $a_i \lhd c_i \lhd b_i$, since the L_i are interpolative. Then, by the cited lemma, $\oplus_i a_i \lhd \oplus_i c_i \lhd \oplus_i b_i$. Hence $\oplus_i L_i$ is interpolative. ■

We observe that the above result and the proof of it, actually establishes that **AuNFrm** is coproductive in **IntNFrm**.

Before we state the next result, it is necessary to note that if $a \lhd b$ in any nearness frame (L, μ), then $b^* \lhd a^*$. This is so, since $a \lhd b$ implies $\{a^*, b\} \in \mu$, so that $\{a^*, b\}b^* = a^*$.

Lemma 3.3.3 *Let (L, μ) be an interpolative nearness frame, and put*

$$\tilde{\mu} = \{A \in \mu \mid B \lhd_\mu A \textit{ for some } B \in \mu\}.$$

Then $\tilde{\mu}$ is an almost uniform nearness on L.

Proof: First, given $A, B \in \tilde{\mu}$ we have $C, D \in \mu$ such that $C \lhd_\mu A$ and $D \lhd_\mu B$. This implies $C \wedge D \lhd_\mu A \wedge B$, so that $A \wedge B \in \tilde{\mu}$. If $A \leq C \in \mathrm{Cov}(L)$ with $A \in \tilde{\mu}$, then $C \in \mu$, since μ is a nearness and $A \in \mu$. So $C \in \tilde{\mu}$, since $B \lhd_\mu A \leq C$ implies $B \lhd_\mu C$. Hence $\tilde{\mu}$ is a filter relative to refinement ordering.

Second, we show that $\tilde{\mu}$ is interpolative. To do that, we first establish that, for any $a, b \in L$,

$$(\ddagger) \qquad\qquad a \lhd_\mu b \text{ implies } a \lhd_{\tilde{\mu}} b.$$

Since μ is interpolative, we can pick $c, d \in L$ such that $a \lhd_\mu c \lhd_\mu d \lhd_\mu b$. Then $\{c^*, d\}$ and $\{a^*, b\}$ are both in μ, and $\{c^*, d\} \lhd_\mu \{a^*, b\}$ because $c^* \lhd_\mu a^*$ and $d \lhd_\mu b$. It follows therefore that $\{a^*, b\} \in \tilde{\mu}$; a consequence of which is that $a \lhd_{\tilde{\mu}} b$.

Now suppose $x \lhd_{\tilde{\mu}} y$. Then $x \lhd_\mu y$ since $\tilde{\mu} \subseteq \mu$. Take $z \in L$ such that $x \lhd_\mu z \lhd_\mu y$. Therefore, by $(\ddagger)$, $x \lhd_{\tilde{\mu}} z \lhd_{\tilde{\mu}} y$. Hence $\tilde{\mu}$ is interpolative.

Third, we show that $\tilde{\mu}$ has the strong property. Let $A \in \tilde{\mu}$. Then there exists $B \in \mu$ such that $B \lhd_\mu A$, since (L, μ) is strong. For each $b \in B$, take $a_b \in A$ and $c_b \in L$ such that $b \lhd_\mu c_b \lhd_\mu a_b$. Form the set $C = \{c_b \mid b \in B\}$, and note that $B \lhd_\mu C$, so that $C \in \tilde{\mu}$. Also $C \lhd_\mu A$. It follows then from $(\ddagger)$ that $C \lhd_{\tilde{\mu}} A$. Consequently, $\tilde{\mu}$ has the strong property.

Lastly, admissibility follows from $(\ddagger)$ since, as μ is a nearness, for any $a \in L$,

$$a = \bigvee \{x \in L \mid x \lhd_\mu a\} \leq \bigvee \{x \in L \mid x \lhd_{\tilde{\mu}} a\} \leq a. \ \blacksquare$$

Proposition 3.3.4 **AuNFrm** *is a coreflective subcategory of* **IntNFrm**. *In particular, if (L, μ) is an interpolative nearness frame, then $(L, \tilde{\mu})$ is its almost uniform coreflection with the identity map id_L being the coreflection arrow.*

Proof: Let $h : (M, \eta) \longrightarrow (L, \mu)$ be a uniform frame homomorphism with (M, η) almost uniform. We need to produce a unique uniform homomorphism $\bar{h} : (M, \eta) \longrightarrow (L, \tilde{\mu})$ such that the triangle

$$\begin{tikzcd} (L, \tilde{\mu}) \arrow[rr, "\mathrm{id}_L"] & & (L, \mu) \\ & (M, \eta) \arrow[ul, "\bar{h}"] \arrow[ur, "h"'] & \end{tikzcd}$$

commutes. We define $\bar{h}$ by $\bar{h}(x) = h(x)$. Then $\bar{h}$ is a frame homomorphism. To see that it is uniform, let $D \in \eta$. Since h is a uniform homomorphism, $h[D] \in \mu$. Since η is strong, $\check{D} \in \eta$ and $h[\check{D}] \lhd_\mu h[D]$. Consequently, $h[D] \in \tilde{\mu}$. But $\bar{h}[D] = h[D]$; so $\bar{h}$ is uniform. Clearly, $\bar{h}$ makes the triangle above commute, and since id_L in monic, the uniqueness of $\bar{h}$ follows. ∎

Remark 3.3.5 In the language of Zenk [57], the above proposition proves that strong nearness frames (as he defines them) form a coreflective subcategory of the category of admissible nearness frames.

Now let

$$\mathbf{IntNFrm} \xrightarrow{\mathsf{A}} \mathbf{AuNFrm}$$

be the functor resulting from the almost uniform coreflection established above, and

$$\mathbf{NFrm} \xrightarrow{\mathsf{T}} \mathbf{NFrm}$$

the coreflection functor which sends a nearness frame (L, μ) to its totally bounded coreflection (L, μ_T). We aim to show that if (L, μ) is a nearness frame for which $\mathsf{T}(L, \mu)$ is strong, then $\mathsf{AT}(L, \mu) = \mathsf{TA}(L, \mu)$. In order that this makes sense, (L, μ) and $\mathsf{T}(L, \mu)$ must be interpolative so that $\mathsf{A}(L, \mu)$ and $\mathsf{A}(\mathsf{T}(L, \mu))$ are defined. But, by Lemma 1.4.10, if $\mathsf{T}(L, \mu)$ is strong, then both (L, μ) and $\mathsf{T}(L, \mu)$ are interpolative. In fact, by Lemma 1.4.9(ii), $\mathsf{T}(L, \mu)$ is uniform.

Proposition 3.3.6 *Let (L, μ) be a nearness frame such that $\mathsf{T}(L, \mu)$ is strong. Then $\mathsf{AT}(L, \mu) = \mathsf{TA}(L, \mu)$.*

Proof: Since $\mathsf{T}(L,\mu)$ is almost uniform,

$$\mathsf{AT}(L,\mu) = \mathsf{T}(L,\mu) = (L,\mu_T).$$

On the other hand

$$\mathsf{T}(\mathsf{A}(L,\mu)) = (L,(\tilde{\mu})_T),$$

and so we need to show that $\mu_T = (\tilde{\mu})_T$. Clearly $(\tilde{\mu})_T \subseteq \mu_T$. Now let $A \in \mu_T$, and $B \in \mu$ finite such that $B \leq A$. Then $B \in \mu_T$. But μ_T is strong, so there exists $C \in \mu_T$ such that

$$C \lhd_\mu B \leq A.$$

This shows that A and B are in $\tilde{\mu}$, and hence A is refined by some finite cover in $\tilde{\mu}$, and so $A \in (\tilde{\mu})_T$. Thus $\mu_T \subseteq (\tilde{\mu})_T$, and hence equality. ■

We give an example to show that A and T do not commute on **IntNFrm**. Notice that if (L,μ) is interpolative, then $\mathsf{T}(L,\mu)$ is interpolative, since $\lhd_T$ coincides with $\lhd_\mu$, by Lemma 1.4.10(i).

Example 3.3.7 Let L be a non-normal, completely regular frame where $\prec$ coincides with $\prec\!\prec$. View $(L,\mathrm{Cov}(L))$ as a fine nearness frame. Then $(L,\mathrm{Cov}(L))$ is almost uniform and $\mathsf{A}(L,\mathrm{Cov}(L)) = (L,\mathrm{Cov}(L))$. Since L is not normal, it has a finite cover which does not have a finite star-refinement (see [51]). By Lemma 1.4.9(ii), it follows that L has a finite uniform cover which does not have a finite uniform strong refinement. Thus, $\mathsf{T}(L,\mathrm{Cov}(L)) = \mathsf{T}(\mathsf{A}(L,\mathrm{Cov}(L)))$ is not strong. But $\mathsf{A}(\mathsf{T}(L,\mathrm{Cov}(L)))$ is strong, so $\mathsf{AT} \neq \mathsf{TA}$.

3.4 Normality

In this section we consider a hierarchy of four subcategories

$$\mathbf{UnNFrm} \subseteq \mathbf{UpnNFrm} \subseteq \mathbf{CStrNFrm} \subseteq \mathbf{StrNFrm}$$

of **NFrm**, each with the property that the underlying frame of a nearness frame in it is completely regular. The first was briefly studied by Dube [22] in his PhD thesis, the next two were defined in [25] under appellations different from those we shall give them.

We propose to change the names of the first three subcategories above to be in line with our general nomenclature. Given a nearness frame (L,μ) and a cover A of L, we set

$$A^{\mathrm{cs}} = \{x \in L \mid x \lhd\!\lhd a \text{ for some } a \in A\}.$$

Definition 3.4.1 We say a nearness frame is:

(1) *uniformly normal* if it is strong and the totally bounded coreflection of its completion is also strong.

(2) *uniformly prenormal* if it is strong and its totally bounded coreflection is also strong.

(3) *completely strong* if for every uniform cover A, A^{cs} is also a uniform cover.

The resulting subcategories are, respectively, denoted by **UnNFrm**, **UpnNFrm**, and **CStrNFrm**.

We will need the following result in some of our arguments in the sequel. The proof for item (ii) is modeled along that of [49, Lemma 1.5].

Lemma 3.4.2 *In a nearness frame (L,μ),*

(i) $a \lhd\!\lhd b$ implies $a \lhd b$.

(ii) $a \lhd b$ implies $a \lhd\!\lhd b$, given that (L,μ) is interpolative.

Let (L,μ) be a nearness frame and A, $B \in \mu$. We say B *completely refines* A and write $B \lhd\!\lhd A$ if for any $b \in B$, there exists $a \in A$ such that $b \lhd\!\lhd a$. The following characterization of completely strong nearness frames appears in [25].

Lemma 3.4.3 *A nearness frame is completely strong if and only if every uniform cover is completely refined by a uniform cover.*

We observe from Lemma 3.4.2(i) that, given uniform covers A and B in a nearness frame (L,μ), if $B \lhd\!\lhd A$, then $B \lhd A$. Consequently completely strong nearness frames are strong. Thus, **CStrNFrm** $\subseteq$ **StrNFrm**. The containments

$$\mathbf{UnNFrm} \subseteq \mathbf{UpnNFrm} \subseteq \mathbf{CStrNFrm}$$

are established in [22].

Completely strong nearness frames are called "uniformly completely regular" in [25], but we introduced them here in the said nomenclature due to their relationship with strong nearness frames by drawing analogies from completely regular frames and their relationship with the regular ones.

Remark 3.4.4 (1) What we have called uniformly prenormal nearness frames are called "uniformly normal" in [25]. The drawback of this is that, unlike in the case of Proposition 3.1.10, we cannot characterize a quotient-fine "uniformly normal" nearness frame (as defined in [25]) similarly by inserting "normal extension" where we have "zero-dimensional extension"; however with "uniformly normal" as defined here, we can.

(2) The term "prenormal" is borrowed from [17] where it is used (without the modifier "uniformly") to define nearness spaces which are regular and whose totally bounded reflections (or "contigual reflections" as Bentley terms them) are also regular. Thus, a prenormal nearness space X (which is therefore framed) is prenormal if and only if $\mathcal{O}X$ is a uniformly prenormal nearness frame.

We recite the following results from [22] and [25] for future use.

Lemma 3.4.5 *The following statements hold for nearness frames.*

(1) A complete nearness frame is uniformly normal iff it is uniformly prenormal.

(2) Every uniform frame is uniformly normal.

(3) If (L, μ) is a nearness frame, then (L, μ_T) is strong iff for every finite $A \in \mu$, there exists a finite $B \in \mu$ such that $B \lhd_\mu A$.

Since every completely regular frame admits a uniformity, a uniform frame whose underlying frame is not normal is an example of a uniformly normal nearness frame whose underlying frame is not normal. In the case of fine nearness frames, we have that the nearness frame is uniformly normal (or uniformly prenormal) if and only if its underlying frame is normal. In the proof of this we use the following characterization of normal regular frames proved in [51].

Lemma 3.4.6 *The following are equivalent for a regular frame L:*

(1) L is normal.

(2) Every finite cover of L has a finite star-refinement.

(3) Every finite cover of L has a star-refinement.

Proposition 3.4.7 *A fine nearness frame is uniformly prenormal iff its underlying frame is normal.*

Proof: ($\Rightarrow$) Consider a fine nearness frame $(L, \mathrm{Cov}(L))$, and assume it is uniformly prenormal. Then $(L, (\mathrm{Cov}(L))_T)$ is strong, and hence uniform, by Lemma 1.4.9(ii). Let A be a finite cover of L. Then A is a finite uniform cover of $(L, \mathrm{Cov}(L))$, and therefore has a star-refinement. Therefore L is normal, by Lemma 3.4.6.

($\Leftarrow$) Let L be a normal frame. We must show that $(L, \mathrm{Cov}(L))$ is uniformly normal. Since it is strong, we must show that $(L, (\mathrm{Cov}(L))_T)$ is strong. Let A be a finite uniform cover of $(L, \mathrm{Cov}(L))$. By Lemma 3.4.6, there is a finite cover B of L which star-refines A. Thus, B is a finite uniform cover of $(L, \mathrm{Cov}(L))$ which strongly refines A. Therefore $(L, (\mathrm{Cov}(L))_T)$ is strong. ■

Corollary 3.4.8 *A fine nearness frame is uniformly normal iff its underlying frame is normal.*

Proof: This follows from the foregoing proposition, the fact that a fine nearness frame is complete, and Lemma 3.4.5(1). ■

We now establish preliminary results that will culminate with our goal of characterizing uniformly normal quotient-fine nearness frames in the manner described at the beginning of the chapter. We start with the following observation.

Lemma 3.4.9 *Suppose $g : (N, \mu) \longrightarrow (K, \nu)$ is a surjection. Then $g : (N, \mu_T) \longrightarrow (K, \nu_T)$ is also a surjection.*

Proof: Clearly, $g : (N, \mu_T) \longrightarrow (K, \nu_T)$ is uniform because it takes finite μ-covers to finite ν-covers, and finite uniform covers generate the nearnesses in both the domain and codomain. Let A be a finite uniform cover of K. Since g is a surjection, there exists $B \in \mu$ such that $g[B] \leq A$. For each $a \in A$, let b_a be the element of N given by

$$b_a = \bigvee \{x \in B \mid g(x) \leq a\}.$$

Define the set $B' \subseteq N$ by

$$B' = \{b_a \mid a \in A\}.$$

Then B' is a uniform cover of N since it is refined by B. Thus, B' is a uniform cover of (N, μ_T). Furthermore, $g[B'] \leq A$. Therefore every finite uniform cover of K is refined by the image of some finite uniform cover of N. This proves that g is a surjection. ∎

Next, we show that uniform normality is inherited by completions.

Lemma 3.4.10 *The completion of a uniformly normal nearness frame is uniformly normal.*

Proof: If (L, μ) is uniformly normal, then it is strong and hence so is $(CL, C\mu)$. By definition of uniform normality, $(CL, (C\mu)_T)$ is strong. Thus, $(CL, C\mu)$ is uniformly prenormal, and hence uniformly normal by Lemma 3.4.5(1). ∎

In [6] it is shown, via closed quotients, that the completion functor which takes a strong nearness frame to its completion preserves surjections. We shall need this result, but we give a more direct proof of it here.

Lemma 3.4.11 *Any surjection* $(L, \mu) \xrightarrow{h} (M, \nu)$ *between strong nearness frames lifts to a surjection* $(CL, C\mu) \xrightarrow{Ch} (CM, C\nu)$ *between the completions.*

Proof: Commutativity of the diagram below

$$\begin{tikzcd} CL \arrow[r, "Ch"] \arrow[d, "\gamma_L"'] & CM \arrow[d, "\gamma_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

already holds since the nearness frames are strong [6]. So we have $h\gamma_{_L} = \gamma_{_M}(Ch)$. Let $A \in C\nu$. Then, since $(CM, C\nu)$ is strong, there exists $A' \in C\nu$ such that $A' \lhd A$. Since h is a surjection, there exists $B \in \mu$ such that $h[B] \leq \gamma_{_M}[A']$. Also, since $\gamma_{_L}$ is a surjection, there exists $D \in C\mu$ such that $\gamma_{_L}[D] \leq B$. Then

$$\gamma_{_M}(Ch)[D] = h\gamma_{_L}[D] \leq h[B] \leq \gamma_{_M}[A'].$$

Since $A' \lhd A$ and $\gamma_{_M}$ is a dense surjection, together with Lemma 1.4.7(ii), we conclude that

$$(Ch)[D] \leq (\gamma_{_M})_*\gamma_{_M}[A'] \leq A.$$

Consequently, Ch is a surjection. ∎

Lemma 3.4.12 **UnNFrm** *is co-hereditary.*

Proof: Let $h : (L, \mu) \longrightarrow (M, \eta)$ be a surjection with (L, μ) uniformly normal. Since (L, μ) is strong, we have that (M, η) is also strong by Lemma 1.4.6(ii). So, by Lemma 3.4.11, the lift of h, namely $Ch : CL \longrightarrow CM$, is a surjection. By Lemma 3.4.9, $(CM, (C\eta)_T)$ is a quotient of the strong nearness frame $(CL, (C\mu)_T)$, and is therefore itself strong. Therefore (M, η) is uniformly normal. ∎

We finally arrive at the main goal, namely:

Proposition 3.4.13 *The following are equivalent for a nearness frame (L, μ):*

(1) (L, μ) is quotient-fine and uniformly normal.

(2) The completion of (L, μ) is fine and uniformly normal.

(3) The nearness on L is induced by a normal extension.

Proof: (1)⇒(2): Let (L, μ) be quotient-fine and uniformly normal. Then, by Lemma 3.4.10, its completion is uniformly normal. This completion is fine, by Lemma 2.1.6.

(2)⇒(3): Suppose $(CL, C\mu)$ is fine and uniformly normal. Let $\gamma_L : CL \longrightarrow L$ be the completion map. Then

$$\mu = \gamma_L[C\mu] = \gamma_L[\mathrm{Cov}(CL)].$$

Since CL is normal, by Corollary 3.4.8, and γ_L is a dense onto map, we deduce (3).

(3)⇒(1): Let $h : (M, \mathrm{Cov}(M)) \longrightarrow (L, \mu)$ be a dense onto homomorphism such that $\mu = h[\mathrm{Cov}(M)]$, with M being a normal frame. Then (L, μ) is quotient-fine since $(M, \mathrm{Cov}(M))$ is fine, and $(M, \mathrm{Cov}(M))$ is uniformly normal, by Corollary 3.4.8. Hence (L, μ) is uniformly normal, by Lemma 3.4.12. ∎

3.5 Cozero nearness frames

Bentley, Herrlich and Ori [19] have defined a zero space in terms of zero sets of uniformly continuous functions. This can be translated to a definition in terms of co-zero sets. In this section we define a co-zero nearness frame and study its properties.

Let L be a frame. Recall (from [8] or [9]) that an element $a \in L$ is called a *cozero element* if it is a join of countably many members of L that are completely below it; i.e.

$$a = \bigvee \{a_n \in L \mid a_n \prec\!\prec a,\ n = 1, 2, \ldots\}.$$

The collection $\mathrm{Coz}(L)$ of all cozero elements of L is a sub-σ-frame of L (i.e, a sublattice closed under countable joins and satisfying the frame distributivity law for countable joins), and it is completely regular in the sense that every $c \in \mathrm{Coz}(L)$ is a countable join of members of $\mathrm{Coz}(L)$ completely below it (see [8]).

Likewise we adopt the above terminology for nearness frames in the following definition.

Definition 3.5.1 Let (L, μ) be a nearness frame. Then $a \in L$ is a *uniformly cozero element* if

$$a = \bigvee \{a_n \in L \mid a_n \lhd\!\lhd a,\ n = 1, 2, \ldots\}.$$

We write $\mathrm{Coz}_\mu(L)$ for the collection of all uniformly cozero elements of L.

Since $x \lhd\!\lhd y$ implies $x \prec\!\prec y$, we note that $\mathrm{Coz}_\mu(L) \subseteq \mathrm{Coz}(L)$. Note that $0, 1 \in \mathrm{Coz}_\mu(L)$, since $0 \lhd\!\lhd 0$ and $1 \lhd\!\lhd 1$. The following result is modeled along that given in [8, Corollary 1].

Lemma 3.5.2 *In a nearness frame* (L, μ), *if* $a \lhd\!\lhd b$, *then there is* $c \in \mathrm{Coz}_\mu(L)$ *such that* $a \lhd\!\lhd c \lhd\!\lhd b$.

Proof: Given $a \lhd\!\lhd b$, let (c_q) be a scale between a and b. We take

$$c = \bigvee \left\{ c_{q_n} \mid \frac{q}{2^n} < \frac{1}{2} \right\}.$$

Then c is the desired uniformly cozero element. ∎

Remark 3.5.3 We observe that since $a \lhd\!\lhd b$ iff $a \lhd b$ in an almost uniform nearness frame (L, μ), we have from the above lemma that $\lhd$ interpolates via uniformly cozero elements in an almost uniform nearness frame.

Let (L, μ) be a nearness frame, and $a \in \mathrm{Coz}_\mu(L)$. Then

$$a = \bigvee \{a_n \in L \mid a_n \lhd\!\lhd a,\ n = 1, 2, \ldots\}.$$

For each $a_n \lhd\!\lhd a$, let $c_n \in \mathrm{Coz}_\mu(L)$ be such that $a_n \lhd\!\lhd c_n \lhd\!\lhd a$. Then

$$a = \bigvee \{c_n \mid c_n \lhd\!\lhd a,\ n = 1, 2, \ldots\}.$$

Thus, every uniformly cozero element is a countable join of uniformly cozero elements uniformly completely below it. Suppose $\{u_k \mid k = 1, 2, \ldots\}$ is a countable family of uniformly cozero elements. Then it is evident, from Definition 3.5.1, that the join $a = \bigvee u_k$ is a countable join of elements uniformly completely below it. Thus $\mathrm{Coz}_\mu(L)$ is closed under countable joins. Also, $\mathrm{Coz}_\mu(L)$ is closed under binary meets, since this is a consequence of the fact that if a, $b \in \mathrm{Coz}_\mu(L)$, then $x_n \lhd\!\lhd a$ and $y_m \lhd\!\lhd b$ imply that $x_n \wedge y_m \lhd\!\lhd a \wedge b$. We have therefore shown that:

Proposition 3.5.4 $\mathrm{Coz}_\mu(L)$ *is a sub-*σ*-frame of* L.

Definition 3.5.5 Let (L, μ) be a nearness frame. Then any $B \in \mu$ such that $B \subseteq \mathrm{Coz}_\mu(L)$ is called a *uniformly cozero cover*. We say (L, μ) is a *cozero nearness frame* if every uniform cover is refined by a uniformly cozero cover, and write **CozNFrm** for the resulting category of cozero nearness frames.

Given a nearness frame (L, μ) and a cover A of L, we set

$$A^{\mathrm{coz}} = \{c \in \mathrm{Coz}_\mu(L) \mid c \leq a \text{ for some } a \in A\}.$$

First, we note that $A^{\mathrm{cs}} \leq A^{\mathrm{coz}}$; for if $x \in A^{\mathrm{cs}}$, then $x \lhd\!\lhd a$ for some $a \in A$. So, by Lemma 3.5.2, there exists $c \in \mathrm{Coz}_\mu(L)$ such that $x \lhd\!\lhd c \lhd\!\lhd a$. Thus, $x \leq c \leq a$ and $c \in A^{\mathrm{coz}}$. Second, we observe that (L, μ) *is a cozero nearness frame if and only if for every* $A \in \mu$, $A^{\mathrm{coz}} \in \mu$.

Proposition 3.5.6 *If* (L, μ) *is a cozero nearness frame, then the underlying frame* L *is completely regular.*

Proof: Let $a \in L$. By admissibility, we have

$$a = \bigvee \{x \in L \mid x \lhd a\}.$$

For $x \lhd a$, take $A \in \mu$ such that $Ax \leq a$. Since (L, μ) is a cozero nearness frame, $A^{\mathrm{coz}} \in \mu$ with $A^{\mathrm{coz}} \leq A$, so

$$A^{\mathrm{coz}}x \leq Ax \leq a.$$

Now $A^{\mathrm{coz}}x$ is a join of uniformly cozero elements, and therefore a join of cozero elements. Thus, a is a join of cozero elements, and therefore L is completely regular by [8, Proposition 1]. ∎

In the next result we establish that **CStrNFrm** $\subseteq$ **CozNFrm**.

Proposition 3.5.7 *A completely strong nearness frame is a cozero nearness frame. A fine cozero nearness frame is completely strong.*

Proof: Suppose (L, μ) is completely strong, and let $A \in \mu$. Then there exists $B \in \mu$ such that for each $b \in B$, there is $a_b \in A$ with $b \lhd\!\lhd a_b$. By Lemma 3.5.2, for each $b \in B$, let $c_b \in \mathrm{Coz}_\mu(L)$ be such that $b \lhd\!\lhd c_b \lhd\!\lhd a_b$. Form the set $C = \{c_b \mid b \in B\}$. Then $C \in \mu$ since B refines C, and $C \subseteq \mathrm{Coz}_\mu(L)$. Furthermore, C refines A; therefore (L, μ) is a cozero nearness frame.

Now let (L, μ) be a fine cozero nearness frame. Then $\lhd$ coincides with $\prec$, and hence $\lhd\!\lhd$ coincides with $\prec\!\prec$; a consequence of which is that $\mathrm{Coz}_\mu(L) = \mathrm{Coz}(L)$. Now let A be a uniform cover of L. Since L is completely regular, by Proposition 3.5.6, the set

$$B = \{c \in \mathrm{Coz}(L) \mid c \prec\!\prec a \text{ for some } a \in A\}$$

is a cover of L, and therefore a uniform cover of L which completely refines A. So (L, μ) is completely strong. ∎

As a corollary, since **UnNFrm** $\subseteq$ **UpnNFrm** $\subseteq$ **CStrNFrm** $\subseteq$ **CozNFrm**, we recover the following results from [25].

Corollary 3.5.8 *If a nearness frame is uniformly normal or uniformly prenormal or completely strong, then its underlying frame is completely regular.*

Proposition 3.5.9 *A totally bounded nearness frame is completely strong iff it is a strong, cozero nearness frame.*

Proof: ($\Rightarrow$) Every completely strong nearness frame is strong, as observed in the previous section, and is a cozero nearness frame, by Proposition 3.5.7.

($\Leftarrow$) Every totally bounded, strong nearness frame is uniform, by Lemma 1.4.9(ii), and every uniform frame is uniformly normal, by Lemma 3.4.5(2); and therefore completely strong, by the containments stated just before Corollary 3.5.8. ∎

We now have the following hierarchy

$$\mathbf{UniFrm} \subseteq \mathbf{UnNFrm} \subseteq \mathbf{UpnNFrm} \subseteq \mathbf{CStrNFrm} \subseteq \mathbf{CozNFrm} \cap \mathbf{StrNFrm}.$$

Since every totally bounded strong nearness frame is uniform, if we let **TbNFrm** be the subcategory of **NFrm** consisting of all totally bounded nearness frames, we have that

$$\begin{aligned} \mathbf{TbNFrm} \cap \mathbf{UniFrm} &= \mathbf{TbNFrm} \cap \mathbf{UnNFrm} \\ &= \mathbf{TbNFrm} \cap \mathbf{UpnNFrm} \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{TbNFrm} \cap \mathbf{CStrNFrm} \\
&= \mathbf{TbNFrm} \cap \mathbf{StrNFrm} \\
&= \mathbf{TbNFrm} \cap \mathbf{CozNFrm} \cap \mathbf{StrNFrm}.
\end{aligned}
$$

Proposition 3.5.10 *Let L be a regular frame such that $(L, (\mathrm{Cov}(L))_T)$ is a cozero nearness frame. Then L is a normal frame.*

Proof: Let $a \vee b = 1$ in L. Since $\{a, b\} \in (\mathrm{Cov}(L))_T$ and $(L, (\mathrm{Cov}(L))_T)$ is a cozero nearness frame, there exists a uniform cover $B \subseteq \mathrm{Coz}_T(L)$ (i.e. B consists of uniformly cozero elements relative to $(L, (\mathrm{Cov}(L))_T)$) such that $B \leq \{a, b\}$. Since $(L, (\mathrm{Cov}(L))_T)$ is totally bounded, there exists a finite cover $B' \subseteq B$ such that $B' \leq \{a, b\}$. Since

$$B' \subseteq \mathrm{Coz}_T(L) \subseteq \mathrm{Coz}(L),$$

and letting

$$z = \bigvee \{x \in B' \mid x \leq a\} \text{ and } w = \bigvee \{x \in B' \mid x \leq b\},$$

we have that z, $w \in \mathrm{Coz}(L)$, $z \leq a$, $w \leq b$ and $z \vee w = 1$. This proves that L is normal, by [3, Corollary 8.3.2]. ∎

Since a uniform frame homomorphism preserves $\lhd$, it clearly preserves $\lhd\!\lhd$. Consequently, if $h : (L, \mu) \longrightarrow (M, \eta)$ is a uniform frame homomorphism and a is a uniformly cozero element of L, then $h(a)$ is a uniformly cozero element of M. This leads to the following result.

Proposition 3.5.11 *If $h : (L, \mu) \longrightarrow (M, \eta)$ is a quotient map and (L, μ) is a cozero nearness frame, then so is (M, η).*

Proof: Let A be a uniform cover of M. Find a uniform cover B of L such that $h[B]$ refines A. Let C be a uniform cover of L consisting of uniformly cozero elements such that C refines A. Then $h[C]$ is a uniform cover of M consisting of uniformly cozero elements (by the observation above) which refines A. ∎

Since the completion map is a quotient map, we deduce that if the completion of a nearness frame is cozero, then so is the nearness frame.

Lemma 3.5.12 *Let $(L_1, \mu_1), \ldots, (L_k, \mu_k)$ be a finite collection of nearness frames. Suppose a_i is a uniformly cozero element of L_i for $i = 1, \ldots, k$. Then $a_1 \oplus \cdots \oplus a_k$ is a uniformly cozero element of $L_1 \oplus \cdots \oplus L_k$.*

Proof: For each i, let $\iota_i : L_i \longrightarrow L_1 \oplus \cdots \oplus L_k$ be the coproduct inclusion. Then, in light of ι_i being a uniform homomorphism, $\iota_i(a_i)$ is a uniformly cozero element, as observed above. But

$$a_1 \oplus \cdots \oplus a_k = \iota_1(a_1) \wedge \cdots \wedge \iota_k(a_k)\,;$$

therefore $a_1 \oplus \cdots \oplus a_k$ is a uniformly cozero element as it is a meet of finitely many uniformly cozero elements. ∎

Since in a coproduct $(\oplus_i L_i, \oplus_i \mu_i)$ of an arbitrary family of nearness frames, we have that a typical member $\oplus_i a_i$ only has finitely many of the a_i's nontrivial (i.e. $a_i \neq 1$ for finitely many i's), we deduce the following result:

Corollary 3.5.13 *If $(\oplus_i L_i, \oplus_i \mu_i)$ is a coproduct of nearness frames and $a_{i_k} \in L_{i_k}$ are uniformly cozero elements for $k = 1, \ldots, n$, then $\oplus_i a_i$, where $a_i = 1$ for $i \notin \{i_1, \ldots, i_n\}$, is a uniformly cozero element.*

Consequently, we establish that

Proposition 3.5.14 **CozNFrm** *is coproductive in* **NFrm**.

Proof: Let $(L_i, \mu_i)_{i \in I}$ be a family of cozero nearness frames. To see that the coproduct $(\oplus_i L_i, \oplus_i \mu_i)$ is a also a cozero nearness frame, let $A \in \mu$. We need a uniformly cozero cover $B \in \oplus_i \mu_i$ refining A. Let $\oplus_i A_i \in \oplus_i \mu_i$ be such that $\oplus_i A_i \leq A$ with finitely many of the A_i's nontrivial, say $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$. For each $i \in \{i_1, i_2, \ldots, i_n\}$, let B_i be a uniformly cozero cover in L_i refining A_i. For $i \notin \{i_1, i_2, \ldots, i_n\}$ take $B_i = \{1\}$. Then put $B = \oplus_i B_i$. Then B refines A, and by the preceding corollary, $B \subseteq \mathrm{Coz}_\mu(L)$. Hence the desired result holds. ∎

3.6 Smooth nearness frames

Smooth nearness frames were introduced in [12] as an ad-hoc means to studying completion in nearness frames. In this section we investigate some properties of these nearness frames, culminating in showing that the smooth property is not changed under completions.

Call a nearness frame (L, μ) *smooth* if for each uniform cover C, the set

$$C^s = \{x \in L \mid x^{**} \leq y \text{ for some } y \in C\}$$

is also a uniform cover. Write **SmNFrm** for the full subcategory of **NFrm** defined by smooth nearness frames.

Remark 3.6.1 Note here that since $x \lhd y$ implies $x^{**} \leq y$, we have $\check{C} \subseteq C^s$, so that, as observed in [12],

Every strong nearness frame is smooth. However, a smooth nearness frame need not be strong.

Thus, **StrNFrm** $\subsetneq$ **SmNFrm**.

It is clear that quotient-fine (and, hence, fine) nearness frames are smooth, and that most of the subcategories considered in this chapter lie in **SmNFrm**.

Let (L, μ) be a nearness frame. If A is a uniform cover, then the set

$$A^{**} = \{x^{**} \in L \mid x \in A\}$$

is also a uniform cover (since A refines A^{**}).

The following characterization of smooth nearness frames follows naturally.

Lemma 3.6.2 *A nearness frame (L, μ) is smooth iff for each $A \in \mu$, there exists $B \in \mu$ such that B^{**} refines A.*

Proof: $(\Rightarrow)$ Suppose (L, μ) is smooth. Let $A \in \mu$. Then

$$A^s = \{x \in L \mid x^{**} \leq a, \text{ some } a \in A\} \in \mu.$$

Since (L, μ) is smooth, by the hypothesis, $(A^s)^{**}$ is a uniform cover (as observed above) of the desired kind refining A.

($\Leftarrow$) Conversely, assume that the condition holds. For $A \in \mu$, let B^{**} refine A for some $B \in \mu$. Then $B \subseteq A^s$, so that $B \leq A^s$. Consequently, $A^s \in \mu$, so that (L, μ) is smooth. ■

Example 3.6.3 It is worth noting here that the cover A^{**}, as introduced above, is not necessarily the same as the set

$$A^r = \{x \in A \mid x = x^{**}\}$$

of all regular elements in A. As an example, let (L, μ) be a nearness frame where L is a compact non-Boolean frame. (Trivially, every nearness on a Boolean frame is smooth). Let A be a uniform cover of L consisting only of regular elements and let $x \in L$ be a non-regular element such that $x^{**} \notin A$. Put $B = A \cup \{x\}$. Then $B \in \mu$ and $B^{**} = A \cup \{x^{**}\}$, but

$$B^r = \{y \in B \mid y = y^{**}\} = A.$$

In order to show that **SmNFrm** is coproductive, we shall need the following lemma.

Lemma 3.6.4 *Let $\oplus_i L_i$ be the coproduct of a family $\{L_i\}_{i \in I}$ of frames. Then for each element $\oplus_i a_i \in L$,*

$$(\oplus_i a_i)^{**} = \oplus_i (a_i^{**}).$$

Proof: Let $\iota_i : L_i \longrightarrow \oplus_i L_i$ be the ith coproduct injection. We first show that

$$(\dagger) \qquad (\oplus_i a_i)^* = \bigvee_i \iota_i(a_i^*).$$

By definition, for each index k, and for any $x \in L_k$, $\iota_k(x) = \oplus_i b_i$, where $b_k = x$ and $b_i = 1$ for $i \neq k$. Now if $\oplus_i c_i$ is any element of $\oplus_i L_i$ such that

$$(\oplus_i a_i) \wedge (\oplus_i c_i) = 0,$$

then

$$\oplus_i (a_i \wedge c_i) = 0,$$

so that $a_k \wedge c_k = 0$ for some index k. This implies $c_k \leq a_k^*$. Consequently

$$\oplus_i c_i \leq \iota_k(a_k^*) \leq \bigvee_i \iota_i(a_i^*).$$

Since the elements $\oplus_i x_i$ generate the frame $\oplus_i L_i$, it follows that if any element of $\oplus_i L_i$ does not meet $\oplus_i a_i$, then it is below $\bigvee_i \iota_i(a_i^*)$. Therefore

$$(\oplus_i a_i)^* \leq \bigvee_i \iota_i(a_i^*).$$

But, by applying the infinite distributive law,

$$(\oplus_i a_i) \wedge \left(\bigvee_i \iota_i(a_i^*) \right) = 0.$$

Therefore $\bigvee_i \iota_i(a_i^*)$ is the largest element of $\oplus_i L_i$ disjoint from $\oplus_i a_i$. Thus, (†) holds.

Second, we apply (†) and the fact that for each index k,

$$(\iota_k(x))^* = \iota_k(x^*),$$

to obtain

$$(\oplus_i a_i)^{**} = \left(\bigvee_i \iota_i(a_i^*) \right)^* = \bigwedge_i (\iota_i(a_i^*))^* = \bigwedge_i \iota_i(a_i^{**}) = \oplus_i(a_i^{**}),$$

establishing the desired result. ■

Proposition 3.6.5 SmNFrm *is coproductive in* **NFrm**.

Proof: Let $\{(L_i, \mu_i)\}_{i \in I}$ be a family of smooth nearness frames. We show that their coproduct $(\oplus_i L_i, \oplus_i \mu_i)$ is also smooth. To see that, let $A \in \mu$, and let $\oplus_i A_i \in \mu$ be a refinement of A. Let $A_{i_1}, \ldots, A_{i_m}$ be the nontrivial covers among the covers A_i's.

We construct a uniform cover of the form B^{**} refining A as follows: For each $i \in \{i_1, \ldots, i_m\}$, let $B_i^{**} \in \mu_i$ refine A_i. We let $B_i^{**} = \{1\}$ for the other i's. Then, making use of the above lemma,

$$B^{**} = (\oplus_i B_i)^{**} = \oplus_i B_i^{**}$$

refines $\oplus_i A_i$ which refines A. Hence the desired result follows. ■

In [12] it is shown that any dense surjection $h : (L, \mu) \longrightarrow (M, \eta)$ with (L, μ) smooth is in fact a strict surjection, and consequently any weak completion $h : (L, \mu) \longrightarrow (M, \eta)$, where (L, μ) is smooth, becomes a completion. Here we show the following result.

Proposition 3.6.6 *If $h : (L, \mu) \longrightarrow (M, \eta)$ is a dense surjection, then (L, μ) is smooth iff (M, η) is smooth.*

Proof: ($\Rightarrow$) Suppose (L, μ) is smooth. Let $C \in \eta$. To show that (M, η) is smooth, we need $D \in \eta$ such that $D^{**} \leq C$. Since $h_*[C] \in \mu$, there exists $B \in \mu$ such that $B^{**} \leq h_*[C]$, since (L, μ) is smooth. Since h preserves pseudocomplements, being a dense onto map, we have $h[B^{**}] = h[B]^{**}$. Thus, $h[B]$ is a uniform cover of M such that $h[B]^{**}$ refines C, and therefore (M, η) is smooth.

($\Leftarrow$) Suppose (M, η) is smooth. Let A be a uniform cover of L. Since h is a strict surjection, there is a uniform cover B of M such that $h_*[B] \leq A$. Since (M, η) is smooth, by the hypothesis, B^{**} is a uniform cover of M. Therefore $h_*[B^{**}]$ is a uniform cover of L since h is a strict surjection. Since h is a dense onto homomorphism, h_* commutes with pseudocomplementation; so that

$$h_*(b^{**}) = h_*(b)^{**}$$

for each $b \in B$, and hence $h_*[B^{**}] = h_*[B]^{**}$. But now $h_*[B]^{**}$ refines A^{**}; therefore A^{**} is also a uniform cover, and hence (L, μ) is smooth. ∎

The following corollary is evident from the above result, since the completion map is a strict surjection.

Corollary 3.6.7 *A nearness frame is smooth iff its completion is smooth.*

3.7 Totally strong nearness frames

By imposing a stronger refinement ordering on uniform covers, in particular, one which uses scales in the manner in which the completely below relation is defined, we introduce, in this section, a type of nearness frames called the totally strong ones and establish that their category, namely **TStrNFrm**, is closed under completions, and that the inclusions **UpnNFrm** $\subseteq$ **TStrNFrm** $\subseteq$ **StrNFrm** and **AuNFrm** $\subseteq$ **TStrNFrm** hold.

Definition 3.7.1 Let (L, μ) be a nearness frame, and A, $B \in \mu$. Write $A \lhd\!\lhd_s B$ if there is an interpolating sequence of uniform covers (C_{nk}) between A and B, where

$$C_{00} = A,\ C_{01} = B,\ \ C_{nk} = C_{n+1\,2k},\ \text{and}\ C_{nk} \lhd C_{n\,k+1}$$

for all $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$. In this case we say A *scale refines* B. We call a nearness frame *totally strong* if every uniform cover is scale refined by a uniform cover.

Clearly, if $A \lhd\!\lhd_s B$, then $A \lhd B$. Consequently, every totally strong nearness frame is strong. We write **TStrNFrm** for the category of totally strong nearness frames.

In order to show that every almost uniform nearness frame is totally strong, we need the following result which shows that interpolation in the underlying frame L is transferred to its nearness μ.

Lemma 3.7.2 *Suppose (L, μ) is an interpolative nearness frame, and suppose A, $B \in \mu$ with $A \lhd B$. Then there exists $C \in \mu$ such that $A \lhd C \lhd B$.*

Proof: Let A, $B \in \mu$ be such that $A \lhd B$. Then for each $a \in A$, there exists $b_a \in B$ such that $a \lhd b_a$. Since (L, μ) is interpolative, there exists $c_a \in L$ such that $a \lhd c_a \lhd b_a$. Form the set

$$C = \{c_a \in L \mid a \in A\}.$$

Then C is a uniform cover, since A refines it. Furthermore $A \lhd C \lhd B$ by the way C is constructed. ∎

Proposition 3.7.3 *If (L, μ) is almost uniform, then it is totally strong.*

Proof: Let $B \in \mu$. Since (L, μ) is strong, there exists $A \in \mu$ such that $A \lhd B$. By Lemma 3.7.2, $\lhd$ interpolates in μ, since (L, μ) is interpolative. Therefore $A \lhd\!\lhd_s B$, since, by CDC, a scale of uniform covers witnessing this can be constructed in the same manner as done in [49, Lemma 1.5]. ∎

As an observation from the above two results, it should be evident that if (L,μ) is a strong nearness frame with the property that whenever $A \lhd B$ in μ, there exists $C \in \mu$ such that $A \lhd C \lhd B$, then (L,μ) is totally strong. In our next set of results we aim to show that the category **TStrNFrm** is closed under completions. Our proof will be facilitated by noting the following: if $h : (L,\mu) \to (M,\eta)$ is a uniform frame homomorphism and A scale refines B in L, then $h[A]$ scale refines $h[B]$ in M, for if (C_{nk}) is a scale of uniform covers of L witnessing $A \lhd\!\lhd_s B$, then clearly $(h[C_{nk}])$ is a scale of uniform covers of M witnessing $h[A] \lhd\!\lhd_s h[B]$. On the other hand, if h is a strict surjection and U scale refines V in M, then $h_*[U]$ scale refines $h_*[V]$ in L, for if (W_{nk}) is a scale of uniform covers of M witnessing $U \lhd\!\lhd_s V$, then, by the strictness of h, $(h_*[W_{nk}])$ is a scale of uniform covers of L witnessing $h_*[U] \lhd\!\lhd_s h_*[V]$.

Lemma 3.7.4 *Let $h : (L,\mu) \longrightarrow (M,\eta)$ be a strict surjection. Then (L,μ) is totally strong iff (M,η) is totally strong.*

Proof: ($\Rightarrow$) Suppose (L,μ) is totally strong and let U be a uniform cover of M. Then, by strictness, $h_*[U]$ is a uniform cover of L. Since (L,μ) is totally strong, there is a uniform cover A of L that scale refines $h_*[U]$. By what we have observed above, $h[A]$ is a uniform cover of M scale refining $h[h_*[U]] = U$. Therefore (M,η) is totally strong.

($\Leftarrow$) Conversely, suppose (M,η) is totally strong and let A be a uniform cover of L. By strictness there is a uniform cover U of M such that $h_*[U] \leq A$. Since (M,η) is totally strong, there is a uniform cover V of M which scale refines U. Then $h_*[V]$ is a uniform cover of L scale refining $h_*[U]$, and hence scale refining A. Therefore (L,μ) is totally strong. ∎

Since completion maps are strict surjections, we deduce the following result.

Corollary 3.7.5 *A nearness frame is totally strong if and only if its completion is totally strong.*

Next, we establish that **UpnNFrm** $\subseteq$ **TStrNFrm**.

Proposition 3.7.6 *Every uniformly prenormal nearness frame is totally strong.*

Proof: Let (L, μ) be uniformly prenormal, and $A \in \mu$. Since (L, μ) is strong, let $B \in \mu$ be such that $B \lhd A$. Then $B \lhd_T A$, by Lemma 1.4.10(i). The totally bounded coreflection (L, μ_T) is strong, and so is uniform, by Lemma 1.4.9(ii), and therefore $\lhd_T$ interpolates. So we can use the interpolation to obtain an interpolating sequence (C_{nk}) of uniform covers between B and A (see for example the argument used in proving Proposition 3.7.3). Hence (L, μ) is totally strong. ■

Chapter 4

N-homomorphisms and remote points

In this chapter we investigate - in the context of nearness frames - the notions of N-homomorphism and remote points introduced for frames in [27] . Throughout this chapter we shall be working with totally bounded uniform frames, unless otherwise specified. Our notation for a typical such frame is $(L,\mathcal{U})$, where $\mathcal{U}$ is the uniformity on the frame L. However, in this chapter, we shall relax that requirement, denoting a totally bounded uniform frame by its underlying frame. If we have not named a uniformity in question when talking about a totally bounded uniform frame L, we shall, at times, write $\mathcal{U}L$ for the uniformity.

4.1 N-homomorphisms

The term "N-homomorphism" is adapted in [27] for frames from the classical "N-map" introduced in [56]. In this section we adopt the same terminology for nearness frames, calling the said maps "uniform N-homomorphisms" and discuss their relationship with the C^*-quotient maps introduced in [3].

Let (L, μ) be a nearness frame and $(CL, C\mu)$ its completion. We shall denote the top and bottom of CL by $\top = L$ and $\bot = \{0\}$, respectively. An ideal $J \subseteq L$ is called *regular* if for each $x \in J$, there exists $y \in J$ such that $x \lhd y$. The class $\mathfrak{R}L$ of all regular ideals in

L is a subframe of the frame $\mathfrak{I}L$ of all ideals (see [6], [13]). The building blocks of $\mathfrak{R}L$, as given in [13], are worth noting here:

(a) $\perp = \{0\}$ and $\top = L$ are both regular ideals.

(b) For any regular ideals I and J, we have that $I \wedge J = I \cap J$ and

$$I \vee J = \{a \vee b \mid a \in I,\ b \in J\}$$

are regular ideals.

(c) Any directed union of regular ideals is a regular ideal. Thus, directed joins in $\mathfrak{R}L$ are unions.

The following results appear in [4]:

Lemma 4.1.1 *(i) A nearness frame is totally bounded and uniform iff its completion is compact.*

(ii) If L is a totally bounded uniform frame, then the join map

$$\bigvee : \mathfrak{R}L \longrightarrow L$$

is a completion map, which is also a compactification of the frame L.

The following result is stated in [13].

Lemma 4.1.2 *Suppose L is a totally bounded uniform frame. Then*

(i) For each $a \in L$, the set

$$\nabla a = \{x \in L \mid x \lhd a\}$$

is a regular ideal.

(ii) The map $r : L \longrightarrow \mathfrak{R}L$ given by $r(a) = \nabla a$ is the right adjoint of the join map $\bigvee : \mathfrak{R}L \longrightarrow L$.

We note the following diagram, where the arrow $L \xrightarrow{h} M$ lifts to the completions $\mathfrak{R}L \xrightarrow{\bar{h}} \mathfrak{R}M$ for totally bounded, uniform frames L and M. γ_L and γ_M are the respective join maps.

$$\begin{tikzcd}
\mathfrak{R}L \arrow[r, "\bar{h}"] \arrow[d, shift left, "\gamma_L"] & \mathfrak{R}M \arrow[d, shift right, "\gamma_M"'] \\
L \arrow[r, "h"'] \arrow[u, bend left, "r_L"] & M \arrow[u, bend right, "r_M"']
\end{tikzcd}$$

The following set of results, which implicitly appear in [13] but the proofs are not explicitly given, establish how $\bar{h}$ maps; in particular that, for each $I \in \mathfrak{R}L$,

$$\bar{h}(I) = \{x \in M \mid x \leq h(s) \text{ for some } s \in I\}.$$

Lemma 4.1.3 *For any $I \in \mathfrak{R}L$, the set*

$$\bar{I} = \{x \in M \mid x \leq h(s) \textit{ for some } s \in I\}$$

is a regular ideal in $\mathfrak{R}M$.

Proof: First, note that since $0 \in I$ and $0 = h(0)$, we have $0 \in \bar{I}$. Second, let $x \in \bar{I}$ and $y \leq x$. Then clearly $y \in \bar{I}$, since $x \leq h(s)$ for some $s \in I$ implies $y \leq h(s)$. So $\bar{I}$ is a downset. Third, let x, $y \in \bar{I}$, and let s, $t \in I$ be such that $x \leq h(s)$ and $y \leq h(t)$. Then $x \vee y \leq h(s) \vee h(t) = h(s \vee t)$ and $s \vee t \in I$. This implies that $x \vee y \in \bar{I}$. We have now shown that $\bar{I}$ is an ideal.

Lastly, to see that $\bar{I}$ is regular, let $x \in \bar{I}$ and $s \in I$ such that $x \leq h(s)$. Since I is regular, there exists $t \in I$ such that $s \lhd t$, and by interpolation in L, we have $u \in L$ such that $s \lhd u \lhd t$. So we have $x \leq h(s) \lhd h(u) \lhd h(t)$, which implies that $x \lhd h(u)$ and $h(u) \in \bar{I}$. Hence $\bar{I}$ is regular. ∎

In order to show that the map $\bar{h}$ is a frame homomorphism we will need the following result from frame theory.

Lemma 4.1.4 *If A and B are frames and $g : A \longrightarrow B$ is a map with the properties:*

(a) $g(0) = 0$ *and* $g(1) = 1$.

(b) $g(a_1 \vee a_2) = g(a_1) \vee g(a_2)$.

(c) $g(a_1 \wedge a_2) = g(a_1) \wedge g(a_2)$.

(d) $g\left(\bigvee_{\lambda\in\Lambda} a_\lambda\right) = \bigvee_{\lambda\in\Lambda} g(a_\lambda)$ *for any updirected collection* $(a_\lambda)_{\lambda\in\Lambda} \subseteq A$,

then g *is a frame homomorphism.*

Proposition 4.1.5 *The map* $\bar{h} : \mathfrak{R}L \longrightarrow \mathfrak{R}M$ *defined by:*

$$\bar{h}(I) = \{x \in M \mid x \leq h(s) \textit{ for some } s \in I\}$$

for each $I \in \mathfrak{R}L$, *is a frame homomorphism.*

Proof: We show that $\bar{h}$, as now given, satisfies the properties in the above lemma. We keep in mind that the map $h : L \longrightarrow M$ is a (uniform) frame homomorphism.

(a) Now

$$\begin{aligned} \bar{h}(\perp) = \bar{h}(\{0\}) &= \{x \in M \mid x \leq h(s) \text{ for some } s \in \{0\}\} \\ &= \{x \in M \mid x \leq h(0) = 0\} \\ &= \{0\} = \perp, \end{aligned}$$

and

$$\begin{aligned} \bar{h}(\top) = \bar{h}(L) &= \{x \in M \mid x \leq h(s) \text{ for some } s \in L\} \\ &= \{x \in M \mid x \leq h(1) = 1\} \\ &= M = \top. \end{aligned}$$

(b) We show that $\bar{h}(I \vee J) = \bar{h}(I) \vee \bar{h}(J)$. Let $x \vee y \in \bar{h}(I) \vee \bar{h}(J)$. Then $x \vee y \leq h(s) \vee h(t) = h(s \vee t)$ for some $s \in I$, $t \in J$. Since $s \vee t \in I \vee J$, we deduce that $x \vee y \in \bar{h}(I \vee J)$, showing the one inclusion. As for the other inclusion, let $z \in \bar{h}(I \vee J)$. Then $z \leq h(p \vee q) = h(p) \vee h(q)$ for some $p \vee q \in I \vee J$. Since I and J are regular ideals, there exist $a \in I$ and $b \in J$ such that $p \lhd a$ and $q \lhd b$. This implies $h(p) \lhd h(a)$ and $h(q) \lhd h(b)$, so that $h(p) \in \bar{h}(I)$ and $h(q) \in \bar{h}(J)$. Therefore $z \leq h(p) \vee h(q) \in \bar{h}(I) \vee \bar{h}(J)$, and hence $z \in \bar{h}(I) \vee \bar{h}(J)$.

(c) We show that $\bar{h}(I \wedge J) = \bar{h}(I) \wedge \bar{h}(J)$. Now $\wedge$ in $\mathfrak{R}L$ and $\mathfrak{R}M$ is intersection. So the result follows immediately since

$$z \in \bar{h}(I \wedge J) \;\Leftrightarrow\; z \leq h(s) \text{ for some } s \in I \cap J \;\Leftrightarrow\; z \leq h(s)$$

for $s \in I$ and $s \in J \;\Leftrightarrow\; z \in \bar{h}(I) \wedge \bar{h}(J)$.

(d) We show that

$$\bar{h}(\bigvee_{\lambda\in\Lambda} I_\lambda) = \bigvee_{\lambda\in\Lambda} \bar{h}(I_\lambda)$$

for an updirected family $(I_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{R}L$. Now in $\mathfrak{R}L$ the join of an updirected family of regular ideals is simply their union, and, by (b), the family $(\bar{h}(I_\lambda))_{\lambda\in\Lambda}$ is also updirected. Let $z \in \bar{h}(\bigvee_\lambda I_\lambda)$. Then $z \leq h(s)$ for some $s \in \bigcup_\lambda I_\lambda$, which implies $s \in I_\kappa$ for some $\kappa \in \Lambda$. In that case $z \in \bar{h}(I_\kappa)$, and consequently $z \in \bigcup_\lambda \bar{h}(I_\lambda)$. On the other hand, if $w \in \bigcup_\lambda \bar{h}(I_\lambda)$, then $w \in \bar{h}(I_\iota)$ for some $\iota \in \Lambda$. This implies $w \leq h(t)$ for some $t \in I_\iota$; and since $t \in \bigcup_\lambda I_\lambda$, we conclude that $w \in \bar{h}(\bigcup_\lambda I_\lambda)$. Thus, $\bar{h}$ is indeed a frame homomorphism.
■

Proposition 4.1.6 *The map $\bar{h} : \mathfrak{R}L \longrightarrow \mathfrak{R}M$ now established is uniform.*

Proof: The uniformity on $\mathfrak{R}M$, $\mathfrak{U}(\mathfrak{R}M)$, is generated by the collection

$$\{(\gamma_{_M})_*[A] \mid A \in \mathfrak{U}M\} = \{r_{_M}[A] \mid A \in \mathfrak{U}M\}.$$

Let $\mathcal{C} \in \mathfrak{U}(\mathfrak{R}L)$. We show that $\bar{h}[\mathcal{C}]$ is a uniform cover by showing that it is refined by $r_{_M}[A]$ for some $A \in \mathfrak{U}M$. Let $C \in \mathfrak{U}L$ be such that $r_{_L}[C] \leq \mathcal{C}$. Since $\bar{h}$ is a frame homomorphism, it is order-preserving, so that

$$\bar{h}[r_{_L}[C]] \leq \bar{h}[\mathcal{C}].$$

Let

$$\check{C} = \{x \in L \mid x \lhd c \text{ for some } c \in C\}.$$

Then $\check{C} \in \mathfrak{U}L$ and $\check{C} \lhd C$ since L is strong. This implies $h[\check{C}] \in \mathfrak{U}M$ since h is a uniform homomorphism. We end the proof by showing that

$$r_{_M}[h[\check{C}]] \leq \bar{h}[\mathcal{C}].$$

Let $y \in r_{_M}h(x)$, where $x \in \check{C}$. Then, by definition of $r_{_M}$, $y \lhd h(x)$, with $x \lhd c$ for some $c \in C$. Since $h(x) \lhd h(c)$, we have $y \lhd h(c)$. This implies $y \in \bar{h}(r_{_L}(c))$, so that $r_{_M}h(x) \subseteq \bar{h}r_{_L}(c)$. Hence

$$r_{_M}[h[\check{C}]] \leq \bar{h}[r_{_L}[C]] \leq \bar{h}[\mathcal{C}]\,,$$

and this ends the proof. ■

Proposition 4.1.7 *The diagram*

$$\begin{tikzcd} \mathfrak{R}L \arrow[r, "\bar{h}"] \arrow[d, "\gamma_L"'] & \mathfrak{R}M \arrow[d, "\gamma_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

commutes, where $\bar{h}$ is as above.

Proof: Given $I \in \mathfrak{R}L$, we establish that $h\gamma_{_L}(I) = \gamma_{_M}\bar{h}(I)$. Now

$$h\gamma_{_L}(I) = h(\bigvee I) = \bigvee h[I]$$

and

$$\gamma_{_M}\bar{h}(I) = \bigvee\{x \in M \mid x \leq h(s) \text{ for some } s \in I\} \leq \bigvee h[I] = h(\bigvee I).$$

On the other hand, for any $u \in I$, we have $h(u) \in \bar{h}(I)$ since $h(u) \leq h(u)$. Thus, $\bigvee h[I] \leq \bigvee \bar{h}(I)$, establishing the desired result. ■

We follow a similar approach as in [27] in drafting the following definition.

Definition 4.1.8 A uniform frame homomorphism $h : L \longrightarrow M$ is a *uniform N-homomorphism* if for any $a \in L$ and any $u \in M$, whenever $u \lhd h(a)$ in M there exists $x \in L$ such that $x \lhd a$ in L and $u \leq h(x)$.

We have the following characterization for uniform N-homomorphisms.

Proposition 4.1.9 *A uniform frame homomorphism $h : L \longrightarrow M$ is a uniform N-homomorphism iff for all $a \in L$, $\bar{h}r_{_L}(a) = r_{_M}h(a)$.*

Proof: ($\Rightarrow$) Suppose h is a uniform N-homomorphism. Let $a \in L$. We show that $\bar{h}r_{_L}(a) = r_{_M}h(a)$. The inclusion $\bar{h}r_{_L}(a) \subseteq r_{_M}h(a)$ always holds; for if $x \in \bar{h}r_{_L}(a)$, then $x \leq h(s)$ for some $s \lhd a$ in L. This implies $x \lhd h(a)$, so that $x \in r_{_M}h(a)$. On the other hand, suppose $y \in r_{_M}h(a)$ so that $y \lhd h(a)$. Then, by the hypothesis, there is $b \lhd a$ in L such that $y \leq h(b)$. This implies $y \in \bar{h}r_{_L}(a)$, giving the other inclusion.

($\Leftarrow$) Conversely, suppose the equality $\bar{h}r_{_L} = r_{_M}h$ holds. Let $a \in L$ and $u \in M$ be such that $u \lhd h(a)$. Then $u \in r_{_M}h(a) = \bar{h}r_{_L}(a)$. So $u \leq h(s)$, for some $s \in r_{_L}(a)$; that is $u \leq h(s)$, for some $s \lhd a$. Hence h is a uniform N-homomorphism. ∎

Proposition 4.1.10 *Let $h : L \longrightarrow M$ be a uniform N-homomorphism. Then h is a surjection iff $\bar{h}$ is a surjection.*

Proof: The implication ($\Rightarrow$) always holds by [6, Corollary 6.1], where h need not be a uniform N-homomorphism. (Refer to Lemma 3.4.11, where we provide an alternative proof to the cited result).

($\Leftarrow$) Suppose $\bar{h}$ is a surjection. First, we show that h is onto. Now $\gamma_{_M}\bar{h} = h\gamma_{_L}$ always holds for completions, and implies $\bar{h}_*r_{_M} = r_{_L}h_*$, on taking right adjoints. Also $\bar{h}r_{_L} = r_{_M}h$, since h is a uniform N-homomorphism. Let $b \in M$. We show that $hh_*(b) = b$. Since $r_{_M}(b) \in \mathfrak{R}M$ and $\bar{h}$ is onto, there exists $I \in \mathfrak{R}L$ such that $r_{_M}(b) = \bar{h}(I)$. This implies

$$\bar{h}_*\bar{h}(I) = \bar{h}_*r_{_M}(b) = r_{_L}h_*(b).$$

Again, this implies

$$\bar{h}(I) = \bar{h}\bar{h}_*\bar{h}(I) = \bar{h}r_{_L}h_*(b) = r_{_M}hh_*(b).$$

Therefore $r_{_M}(b) = r_{_M}hh_*(b)$; so that $\gamma_{_M}r_{_M}(b) = \gamma_{_M}r_{_M}hh_*(b)$. Consequently, since $\gamma_{_M}$ is onto (so that $\gamma_{_M}r_{_M} = \mathrm{id}_M$), $b = hh_*(b)$. Thus h is onto.

Second, we show that uniform covers of M are refined by images under h of uniform covers of L. Let A be a uniform cover of M. Since $\bar{h}$ is a surjection, there exists a uniform cover C of L such that $\bar{h}r_{_L}[C] \leq r_{_M}[A]$. This implies $r_{_M}h[C] \leq r_{_M}[A]$, since $\bar{h}r_{_L} = r_{_M}h$ (h being a uniform N-homomorphism). So $\gamma_{_M}r_{_M}h[C] \leq \gamma_{_M}r_{_M}[A]$. Therefore $h[C] \leq A$, since $\gamma_{_M}r_{_M} = \mathrm{id}_M$. Thus, h is a surjection. ∎

In [3], the concept of a C^*-quotient map is introduced, which captures the spatial notion of a C^*-embedded subspace. One of the characterizations of C^*-quotient maps established in [3, Theorem 7.1.1] is that a quotient map $h : L \longrightarrow M$ is a C^*-quotient map iff whenever $a \prec\!\!\prec b$ in M, there exist $u \prec\!\!\prec v$ in L such that $a \leq h(u)$ and $h(v) \leq b$. Adapting this characterization to uniform frames, we formulate the following definition

Definition 4.1.11 A surjection $h : L \longrightarrow M$ is called a *uniform C^*-quotient map* if whenever $u \lhd v$ in M then there are elements $a \lhd b$ in L such that $u \leq h(a)$ and $h(b) \leq v$.

Lemma 4.1.12 *An onto uniform N-homomorphism $h : L \longrightarrow M$ is a uniform C^*-quotient map.*

Proof: Let $u \lhd v$ in M. Since h is onto, take $a, b \in L$ such that $u = h(a)$ and $v = h(b)$. So $u \lhd h(b)$ in M, and since h is a uniform N-homomorphism, there is $x \in L$ such that $x \lhd b$ in L and $u \leq h(x)$. This implies $u \leq h(x) \lhd h(b) = v$. Consequently h is a uniform C^*-quotient map. ∎

Proposition 4.1.13 *If $h : L \longrightarrow M$ is a dense surjection such that $\varphi : \uparrow (h\gamma_L)_*(0) \longrightarrow M$ is a completion of M, then h is a uniform C^*-quotient map.*

Proof: Suppose the conditions of the hypothesis hold. Since $\gamma_L : \mathfrak{R}L \longrightarrow L$ is dense, the composite map $h\gamma_L$ is dense. So $\uparrow (h\gamma_L)_*(0) = \mathfrak{R}L$, so that the hypothesis actually says the composite map $h\gamma_L : \mathfrak{R}L \longrightarrow M$ is a completion of M. To show that h is a uniform C^*-quotient map, let $u \lhd v$ in M. We must find $a \lhd b$ in L such that $u \leq h(a) \lhd h(b) \leq v$. Since $h\gamma_L : \mathfrak{R}L \longrightarrow M$ is a dense surjection, we have

$$(h\gamma_L)_*(u) \lhd (h\gamma_L)_*(v) \text{ in } \mathfrak{R}L.$$

That is,

$$(\gamma_L)_* h_*(u) \lhd (\gamma_L)_* h_*(v) \text{ in } \mathfrak{R}L,$$

and therefore

$$\gamma_L \left((\gamma_L)_* h_*(u) \right) \lhd \gamma_L \left((\gamma_L)_* h_*(v) \right) \text{ in } L$$

since γ_L is a uniform homomorphism. Thus, $h_*(u) \lhd h_*(v)$ in L since $\gamma_L (\gamma_L)_* = \mathrm{id}_{\mathfrak{R}L}$. Therefore taking $a = h_*(u)$ and $b = h_*(v)$ establishes the result since $hh_* = \mathrm{id}_L$. ∎

4.2 Remote points

In this section we discuss remote points. First we take a brief survey of some results we will need, keeping the same notation as used in the preceding section.

Lemma 4.2.1 *Given a totally bounded uniform frame L, we have the following properties for any a, $b \in L$ and I, $J \in \mathfrak{R}L$:*

(a) $\bigvee r(a) = a$.

(b) $J = \bigcup\{r(a) \mid a \in J\}$.

(c) $a \lhd b$ *implies* $r(a) \prec r(b)$.

(d) $r(a^*) = r(a)^*$.

(e) $J^* = r((\bigvee J)^*)$.

(f) $I \prec J$ *implies* $\bigvee I \in J$.

(g) $r(a) \prec J$ *implies* $a \in J$.

Proof: (a), (b) and (c) appear in [6] and [13].

(d) This is true since r is the right adjoint of a dense onto frame homomorphism $\bigvee$ (see [2]).

(e) Here we show that $r((\bigvee J)^*)$ is the pseudocomplement of J. First, let

$$x \in J \wedge r((\bigvee J)^*) = J \cap r((\bigvee J^*).$$

Then $x \in J$ and $x \in r((\bigvee J)^*)$. So $x \lhd (\bigvee J)^*$. Since $x \leq \bigvee J$, we have $x \leq (\bigvee J) \wedge (\bigvee J)^* = 0$. Therefore $x = 0$. This means $J \wedge r((\bigvee J)^*) = \perp = \{0\}$.

Second, suppose $K \in \mathfrak{R}L$ is such that $J \wedge K = J \cap K = \{0\}$. Let $x \in K$. Since K is regular, we can pick $y \in K$ such that $x \lhd y$. Then for any $t \in J$, $t \wedge y = 0$, and so $y \wedge \bigvee J = \bigvee_{t \in J}(y \wedge t) = 0$. This means $y \leq (\bigvee J)^*$, so that $x \lhd (\bigvee J)^*$. Therefore $x \in r((\bigvee J)^*)$, so that $K \subseteq r((\bigvee J)^*)$. Thus, the desired result $r((\bigvee J)^*) = J^*$ follows.

(f) Suppose $I \prec J$, and let $K \in \mathfrak{R}L$ be such that $I \cap K = \{0\}$ and $K \vee J = L$. Let $x \in K$ and $y \in J$ be such that $x \vee y = 1$. Now we have

$$0 = \bigvee\{0\} = \bigvee(I \wedge K) = \bigvee I \wedge \bigvee K.$$

This implies that $\bigvee I \wedge x = 0$; so that $\bigvee I \leq y \in J$. Consequently $\bigvee I \in J$.

(g) This is a particular case of (f) since, by (a), $a = \bigvee r(a) \in J$. ∎

Proposition 4.2.2 *If $J \in \mathfrak{R}L$ contains a dense element, then $J = \top$.*

Proof: Suppose $a \in J$ is dense (i.e. $a^* = 0$). Let $b \in J$ be such that $a \lhd b$. Then $\{a^*, b\}$ is a uniform cover of L, and this forces $b = 1$, since $a^* = 0$. Therefore $1 \in J$, and since J is a downset we have $J = L = \top$. ∎

To define remote points we will need the notion of nowhere dense quotients, which we define shortly.

Definition 4.2.3 A quotient map $h : L \longrightarrow M$ is said to be *nowhere dense* if $x \neq 0$ in L implies there is $y \leq x$ in L such that $y \neq 0$ and $h(y) = 0$.

The following handy characterizations of nowhere dense quotients appear in [27].

Lemma 4.2.4 *(i) A closed quotient $\varphi : L \longrightarrow \uparrow a$ is nowhere dense if and only if a is dense.*

(ii) A quotient $h : L \longrightarrow M$ is nowhere dense iff $h_*(0) \in L$ is a dense element.

Definition 4.2.5 A point $P \in \mathfrak{R}L$ is said to be *remote* if for each nowhere dense quotient $h : L \longrightarrow M$, $P \vee r(h_*(0)) = \top$.

Our next result brings a connection between remote points and ultrafilters.

Proposition 4.2.6 *Let $P \in \mathfrak{R}L$ be a point. Then if the set*

$$F = \{a \in L \mid r(a) \vee P = \top\}$$

is an ultrafilter in L, then P is a remote point.

Proof: Let $h : L \longrightarrow M$ be a nowhere dense quotient. Then $h_*(0)$ is a dense element, by Lemma 4.2.4(ii), and therefore belongs to the ultrafilter F. Hence by the property of membership to F, P is a remote point. ∎

The following result characterizes remote points in $\mathfrak{R}L$.

Proposition 4.2.7 *$P \in \mathfrak{R}L$ is a remote point iff for each dense $a \in L$, $P \vee r(a) = \top$.*

Proof: ($\Rightarrow$) Let P be a remote point and $a \in L$ a dense element. Then by Lemma 4.2.4(i), the closed quotient $L \xrightarrow{\varphi} \uparrow a$ is nowhere dense. So we have $\varphi_*(0_{\uparrow a}) = \varphi_*(a) = a$. Therefore, since P is remote, we have $\top = P \vee r(\varphi_*(a)) = P \vee r(a)$.

($\Leftarrow$) Conversely, suppose the condition holds. Let $h : L \longrightarrow M$ be a nowhere dense quotient. Then $h_*(0)$ is dense, and so by the hypothesis, $P \vee r(h_*(0)) = \top$, implying that P is remote. ∎

The following result characterizes remote points for a totally bounded uniform frame L where the right adjoint

$$r : L \longrightarrow \mathfrak{R}L$$

preserves disjoint binary joins; namely $r(a \vee b) = r(a) \vee r(b)$ whenever $a \wedge b = 0$ in L. The proof of the following lemma coincides with that of [27, Proposition 3.3].

Lemma 4.2.8 *Let L be a totally bounded uniform frame, where the right adjoint preserves disjoint binary joins as indicated above, and let I be a point of $\mathfrak{R}L$. Then the following are equivalent:*

(1) I is a remote point.

(2) $I \vee r(a) = \top$, whenever $a \in L$ is dense.

(3) $r(a) \leq I$ *implies* $a \in I$*, for any* $a \in L$.

(4) $r(a^*) \leq I$ *implies* $r(a) \vee I = \top$*, for any* $a \in L$.

(5) $J^* \leq I$ *implies* $r(\bigvee J) \vee I = \top$ *for any* $J \in \mathfrak{R}L$.

(6) The set $F = \{a \in L \mid r(a) \vee I = \top\}$ *is an ultrafilter in* L.

Our next three results focus on the transfer of remote points. In particular, given a quotient map $h : L \longrightarrow M$, where L and M are such that the respective right adjoints r_L and r_M preserve disjoint binary joins, and a point I of $\mathfrak{R}M$, we find conditions on h such that if $\bar{h}_*(I)$ is a remote point of $\mathfrak{R}L$, then I is a remote point of $\mathfrak{R}M$, and vice versa. One of the conditions involves what are called assertive homomorphisms in [27].

Definition 4.2.9 A uniform frame homomorphism $h : L \longrightarrow M$ is called *uniformly assertive* if whenever $a \lhd b$ in M, then $h_*(a) \lhd h_*(b)$ in L.

Clearly, by Lemma 1.4.7(iii), any dense surjection is uniformly assertive, since it is strict, by Lemma 1.4.6(i) (bearing in mind that our nearness frames in this chapter have the strong property).

Proposition 4.2.10 *Let* $h : L \longrightarrow M$ *be a surjection and* I *a point of* $\mathfrak{R}M$ *such that* $\bar{h}_*(I)$ *is a remote point of* $\mathfrak{R}L$*. Then any one of the following conditions implies that* I *is a remote point:*

(a) h *is a uniform N-homomorphism.*

(b) h *is uniformly assertive.*

(c) h *is dense.*

Proof: (a) Suppose h is a uniform N-homomorphism. We use condition (3) in Lemma 4.2.8 to show that I is a remote point. Let $a \in M$ be such that $r_M(a) \leq I$. Then $\bar{h}_* r_M(a) \leq \bar{h}_*(I)$. Now $\gamma_M \bar{h} = h\gamma_L$, since h lifts to $\bar{h}$; so that $\bar{h}_* r_M = r_L h_*$. This implies that $r_L(h_*(a)) \leq \bar{h}_*(I)$. Since $\bar{h}_*(I)$ is a remote point, we have that $h_*(a) \in \bar{h}_*(I)$, by condition (3) in Lemma 4.2.8. This implies $h_*(a) \lhd s$, for some $s \in \bar{h}_*(I)$, so that we can write $r_L(h_*(a)) \lhd \bar{h}_*(I)$. Consequently,

$$(\dagger) \qquad\qquad \bar{h}r_L h_*(a) \lhd \bar{h}\bar{h}_*(I) \leq I.$$

Since h is a uniform N-homomorphism, we have that $\bar{h}r_L = r_M h$, so that, from (†), we have $r_M hh_*(a) \lhd I$, which implies $r_M(a) \lhd I$, since h is onto. Thus, $a \in I$, so that, by Lemma 4.2.8, I is a remote point.

(b) Suppose h is uniformly assertive. Once again, we begin with an element $a \in M$ for which $r_M(a) \leq I$. We aim to show that $a \in I$. We note (†) still holds, here, since it does not require h to be a uniform N-homomorphism. We show that $r_M(a) \leq \bar{h}r_L h_*(a)$. Let $x \in r_M(a)$. Then $x \lhd a$. Then, since h is uniformly assertive, $h_*(x) \lhd h_*(a)$, so that $h_*(x) \in r_L(h_*(a))$. But $x = hh_*(x)$, since h is onto. In particular, $x \leq h(h_*(x))$, so that $x \in \bar{h}(r_L h_*(a))$. Consequently, from (†), we have $r_M(a) \lhd I$, so that $a \in I$. Thus, I is remote, by Lemma 4.2.8.

(c) If h is dense, then it is uniformly assertive, by the remark stated just before the statement of this proposition. Consequently, condition (c) becomes condition (b), and the desired result follows. ■

Proposition 4.2.11 *If $h : L \longrightarrow M$ is a uniformly assertive uniform N-homomorphism, and I a remote point of $\mathfrak{R}M$, then $\bar{h}_*(I)$ is a remote point of $\mathfrak{R}L$.*

Proof: Let $a \in L$ be such that $r_L(a) \leq \bar{h}_*(I)$. We aim to show that $a \in \bar{h}_*(I)$. Now we have

$$\bar{h}r_L(a) \leq \bar{h}\bar{h}_*(I) \leq I,$$

so that $\bar{h}r_L(a) \leq I$. Since h is a uniform N-homomorphism, $r_M(h(a)) \leq I$. This implies $h(a) \in I$, since I is a remote point. Choose $s \in I$ such that $h(a) \lhd s$. Then, $a \leq h_*h(a) \lhd h_*(s)$, since h is uniformly assertive. So $a \in r_L(h_*(s))$.

Let $y \in \bar{h}(r_L h_*(s))$. Then $y \leq h(t)$ for some $t \lhd h_*(s)$. Thus, $y \leq h(t) \leq s$ so that $y \in I$. This shows that $\bar{h}(r_L h_*(s)) \leq I$, and hence $r_L h_*(s) \subseteq \bar{h}_*(I)$. Consequently $a \in \bar{h}_*(I)$, and the result follows. ■

In general, if $h : L \longrightarrow M$ is an onto homomorphism and p a point of L, then it does not follow that $h(p)$ is a point of M. If however $h(p) \neq 1$, then it is a point. For, if $h(p) \leq z \neq 1$, then $p \leq h_*(z) \neq 1$, so that, by maximality of p, $p = h_*(z)$, and hence $h(p) = hh_*(z) = z$.

Proposition 4.2.12 *If $h : L \longrightarrow M$ is an onto uniform N-homomorphism, and I a remote point of $\mathfrak{R}L$ such that $\bar{h}(I) \neq \top$, then $\bar{h}(I)$ is a remote point of $\mathfrak{R}M$.*

Proof: Let $x \in M$ be such that $r_M(x) \leq \bar{h}(I)$. We need to show that $x \in \bar{h}(I)$. First, we claim that $r_L h_*(x) \leq I$. If not, then, since I is a point, $r_L h_*(x) \vee I = \top$, so that $\bar{h} r_L h_*(x) \vee \bar{h}(I) = \top$. Since h is a uniform N-homomorphism, we have $r_M h h_*(x) \vee \bar{h}(I) = \top$ which implies $r_M(x) \vee \bar{h}(I) = \top$, which is false. Second, since I is a remote point and $r_L h_*(x) \leq I$, we have that $h_*(x) \in I$. Pick $y \in I$ such that $h_*(x) \lhd y$. Since $x = hh_*(x)$ and $h_*(x) \in r_L(y) \subseteq I$, it follows that

$$x \in \bar{h}(r_L(y)) \subseteq \bar{h}(I),$$

as required. ∎

Chapter 5

Odds and Ends

This chapter consists of two sections which are not related, but nevertheless offer a miscellany of results as a contribution towards the theory of nearness frames.

In Chapter 2 we mentioned grills briefly. In this chapter we revisit them and establish, among other things, that grills are precisely unions of prime filters. We also show that, similar to the spatial case, there are instances where preservation of near subsets by the right adjoint characterizes uniform frame homomorphisms.

We conclude the chapter by showing that the lattice of all nearnesses on a frame is a pseudo-frame, by which we mean a partially ordered set defined exactly like a frame except that it need not have a bottom.

5.1 Grills and clusters

Grills and clusters play an important role in the theory of nearness spaces (see for example [32] and [48]). We have already made reference to grills in our discussion focussed on quotient-fine nearness frames in Chapter 2. This section is devoted to investigating these concepts in the wider context of the theory of nearness frames. In particular, our discussion centers on establishing the interconnections between the notions *grill, cluster* and *near* subset. Grills are known to be dual to filters. Recall from Chapters 1 and 2 that, in a nearness frame (L, μ), a near subset $A \subseteq L$ has the property that every uniform cover of L has an element which meets every element of A, a cluster is a maximal near subset, and a grill $G \subseteq L$ is an upset such that $0 \notin G$ and if $a \vee b \in G$, then $a \in G$ or $b \in G$.

In the theory of nearness spaces, uniformly continuous maps can be characterized by near subsets namely: a function $f : X \longrightarrow Y$ between nearness spaces is uniformly continuous iff for every near subcollection $\mathcal{A} \subseteq \mathcal{P}X$, the collection $\{f[A] \mid A \in \mathcal{A}\}$ is near in Y. Here we show that the property of being a near subset characterizes uniform frame homomorphisms only in certain instances as specified by the following result.

Proposition 5.1.1 *If (M, ν) is a strong nearness frame, (L, μ) an arbitrary nearness frame, and $h : M \longrightarrow L$ a dense onto frame homomorphism, then h is uniform iff h_* preserves near subsets.*

Proof: ($\Rightarrow$) Suppose the hypothesis holds, with h being a uniform frame homomorphism. Let A be a near subset of L. We show that $h_*[A]$ is a near subset of M. If $C \in \nu$, then $h[C] \in \mu$. So there exists $c \in C$ such that $h(c) \wedge a \neq 0$ for each $a \in A$, since A is near. This implies $c \wedge h_*(a) \neq 0$ since h is onto. [To see this, suppose $c \wedge h_*(a) = 0$. Then $0 = h(c \wedge h_*(a)) = h(c) \wedge hh_*(a) = h(c) \wedge a$ giving a contradiction]. Therefore $h_*[A]$ is near.

($\Leftarrow$) Conversely, suppose h_* preserves near subsets. We show that h is uniform. Let $C \in \nu$, and suppose on the contrary $h[C] \notin \mu$. Since (M, ν) is strong, we have that

$$\check{C} = \{x \in M \mid \exists c \in C,\ x \lhd c\} \in \nu.$$

Now for any $x \in \check{C}$, $x^{**} \leq c$ for some $c \in C$ (see Remark 3.6.1). Thus, in line with our supposition, $h[\check{C}^{**}] \notin \mu$. Since h is dense onto, $h[\check{C}^{**}] = h[\check{C}^*]^* \notin \mu$ and hence $h[\check{C}^*]$ is near by Lemma 1.4.12. So, by the hypothesis $h_*h[\check{C}^*]$ is near. Since $\check{C} \in \nu$, we can find $x \in \check{C}$ which meets with every element of $h_*h[\check{C}^*]$. But $h_*(h(x)^*)$ is an element of $h_*h[\check{C}^*]$ and

$$h(x \wedge h_*(h(x)^*)) = h(x \wedge h_*h(x^*)) = h(x) \wedge h(x^*) = 0,$$

which implies $x \wedge h_*(h(x)^*) = 0$ by denseness. So we have a contradiction. Hence the desired result holds. ∎

Proposition 5.1.2 *Every cluster is a grill.*

Proof: Let (L, μ) be a nearness frame and $C \subseteq L$ a cluster. Now, since C is near, we have $0 \notin C$. Suppose $a \in C$ or $b \in C$. Then $C \cup \{a \vee b\}$ is near, and consequently $a \vee b \in C$, since C is a maximal near subset.

On the other hand, suppose $a \vee b \in C$ with $a \notin C$ and $b \notin C$. Then since C is a cluster, $C \cup \{a\}$ and $C \cup \{b\}$ are not near and so $D = C^* \cup \{a^*\} \in \mu$ and $E = C^* \cup \{b^*\} \in \mu$ (from Lemma 1.4.12). But $D \wedge E \leq C^* \cup \{a^* \wedge b^*)$ since $a^* \wedge b^* = (a \vee b)^* \in C^*$. So $C^* \in \mu$, implying that C is not near. This is a contradiction. Hence the result holds. ∎

Next, we show that every near subset is contained in some grill. We will need the following characterization of grills.

Lemma 5.1.3 *The following are equivalent for a given nonempty subset G of a frame L:*

(1) G is a grill.

(2) $L \setminus G$ is an ideal.

(3) $G = \bigcup\{F \subseteq G \mid F$ is a prime filter$\}$.

Proof: (1)⇒(2): Suppose G is a grill. Then $L \setminus G$ is nonempty since $0 \in L \setminus G$. Let $a \leq b \in L \setminus G$. Then $b \notin G$ and, since G is an upset, $a \notin G$. So $a \in L \setminus G$. Next, suppose a, $b \in L \setminus G$. Since G is a grill, we cannot have $a \vee b \in G$ (otherwise $a \in G$ or $b \in G$). Therefore $a \vee b \in L \setminus G$, and consequently, $L \setminus G$ is an ideal.

(2)⇒(3): Suppose that $L \setminus G$ is an ideal. If

$$x \in \bigcup\{F \subseteq G \mid F \text{ is a prime filter}\},$$

then trivially, $x \in G$, giving the inclusion $\supseteq$. As for the other inclusion, suppose $y \in G$. By the hypothesis, $L \setminus G$ is a downset. So G is an upset, implying that $\uparrow y \subseteq G$. Therefore $\uparrow y$ is a filter disjoint from the ideal $L \setminus G$. By invoking the dual version of Stone's Separation Lemma (a statement of which is in the Glossary, or see [30, Theorem 15]), there is a prime filter F containing $\uparrow y$ and disjoint from $L \setminus G$. Hence $y \in F \subseteq G$, so that the inclusion $\subseteq$ holds. Hence (3) holds.

(3)$\Rightarrow$(1): Suppose the equality in (3) holds. Then G is an upset which does not contain 0, since each F in the union has these properties. Suppose $a \vee b \in G$. Then $a \vee b \in F$ for some prime filter $F \subseteq G$. By the definition of prime filters, $a \in F$ or $b \in F$ so that $a \in G$ or $b \in G$. ∎

Proposition 5.1.4 *In any nearness frame* (L, μ)*, every near subset is contained in a grill.*

Proof: Let A be a near subset of L, and pick any $C \in \mu$. Choose $c \in C$ such that $c \wedge a \neq 0$ for all $a \in A$. This implies $a \not\leq c^*$ for each $a \in A$. [Note that if $a \leq c^*$, then $c \wedge a \leq c \wedge c^* = 0$, so that $c \wedge a = 0$ which gives a contradiction]. So we have $A \subseteq L \setminus \downarrow c^*$. Now $\downarrow c^*$ is an ideal, so that $L \setminus \downarrow c^*$ is a grill by the above lemma. ∎

Let (L, μ) be a nearness frame and $A \subseteq L$. We use the notation

$$\sec A = \{x \in L \mid x \wedge a \neq 0,\ \forall a \in A \setminus \{0\}\}.$$

We say A is *semi-Cauchy* in case $\sec A$ is near.

A nearness frame (L, μ) is said to be *separated* if whenever a subset $A \subseteq L$ is both near and semi-Cauchy, then the set $\{s \in L \mid A \cup \{s\} \text{ is near}\}$ is near. It is shown in [22] that strong nearness frames are separated. Our next result identifies separated nearness frames among the arbitrary ones.

Proposition 5.1.5 *If* (L, μ) *is a nearness frame in which every near subset is contained in a unique cluster, then* (L, μ) *is separated.*

Proof: Given the hypothesis, let $A \subseteq L$ be near and semi-Cauchy, and let C be the unique cluster with $A \subseteq C$. Put $S = \{s \in L \mid A \cup \{s\} \text{ is near}\}$. For each $s \in S$, let C_s be the unique cluster such that $A \cup \{s\} \subseteq C_s$. Then $A \subseteq C_s$ for each $s \in S$. So $S \subseteq C$. Since C is near, we have that S is near, and therefore (L, μ) is separated. ∎

Remark 5.1.6 We note that if (L, μ) is a nearness frame, $A \subseteq L$ is near and

$$C = \{c \in L \mid A \cup \{c\} \text{ is near}\}$$

is near, then C is the unique cluster containing A. To see this let $B \supseteq A$ be near. Then for each $b \in B$, we have $A \cup \{b\} \subseteq B$ and so $A \cup \{b\}$ is near. In that case $b \in C$, so that $B \subseteq C$.

Recall that a Boolean nearness frame is one where the underlying frame L is Boolean.

Proposition 5.1.7 *If (L, μ) is a Boolean separated nearness frame, then every near grill in L is contained in a unique cluster.*

Proof: Let $G \subseteq L$ be a near grill. We first show that G is semi-Cauchy, and then use the above remark to draw our conclusion. So we begin by showing that $\sec G$ is near.

Suppose on the contrary that $\sec G$ is not near. Then the set $\{a^* \mid a \in \sec G\} \in \mu$, and so there exists $b \in \sec G$ such that $b^* \wedge x \neq 0$ for each $x \in G$, since G is near. But for each $x \in G$, we have $x = (x \wedge b) \vee (x \wedge b^*)$, since L is Boolean. Since G is a grill, we should have $x \wedge b \in G$ or $x \wedge b^* \in G$. But we cannot have $x \wedge b \in G$ since $b^* \wedge (x \wedge b) = 0$. So $x \wedge b^* \in G$; which contradicts the fact that $b \in \sec G$, as $b \wedge (x \wedge b^*) = 0$. Therefore $\sec G$ is near, so that G is semi-Cauchy.

Since (L, μ) is separated, the set $C = \{c \in L \mid G \cup \{c\} \text{ is near}\}$ is near. So, by Remark 5.1.6, C is the unique cluster containing G. ■

Our next result shows that clusters are preserved by dense surjections. We shall need the following result appearing in [22].

Lemma 5.1.8 *A surjection $h : (M, \nu) \to (L, \mu)$ is dense iff for every near subset A of M, $h[A]$ is a near subset of L.*

Proposition 5.1.9 *Let $h : (M, \nu) \longrightarrow (L, \mu)$ be a dense surjection. If $C \subseteq M$ is a cluster, then $h[C]$ is a cluster in L.*

Proof: Suppose $C \subseteq M$ is a cluster. Since C is near, we have, by Lemma 5.1.8, that $h[C]$ is near. Now suppose $h[C] \subseteq D$ for some $D \subseteq L$ which is near. We show that $D \subseteq h[C]$, which will show that $h[C]$ is a maximal near subset. Let $d \in D$, and choose $b \in M$ such that $h(b) = d$. Let $A \in \nu$. Then $h[A] \in \mu$. Since D is near, there exists $a \in A$ such that $h(a) \wedge x \neq 0$ for each $x \in D$. In particular, $0 \neq h(a) \wedge h(b) = h(a \wedge b)$, which implies $a \wedge b \neq 0$. Consider any element $c \in C$. Since $h[C] \subseteq D$, $h(a) \wedge h(c) \neq 0$, which implies $a \wedge c \neq 0$. Thus, a meets every element of $C \cup \{b\}$, which implies that $C \cup \{b\}$ is near, and therefore $b \in C$ as C is a maximal near subset. Thus, $d = h(b) \in h[C]$. Hence $h[C] = D$, as required. ∎

If denseness is dropped in the above proposition, then $h[C]$ can fail to be a cluster mainly because $h[C]$ need not be near when C is near, as shown by the following example.

Example 5.1.10 Let $\mathbf{4} = \{0, a, a^*, 1\}$ be the Boolean algebra of four elements and $\mathbf{2}$ the two-element chain. Regard these frames as fine nearness frames. Let $h : \mathbf{4} \longrightarrow \mathbf{2}$ be the frame homomorphism given by

$$0 \mapsto 0, \quad a \mapsto 0, \quad a^* \mapsto 1, \quad 1 \mapsto 1.$$

Then h is a nondense surjection. The set $C = \{a, 1\}$ is a cluster in $\mathbf{4}$ for which $h[C]$ is not near, and therefore not a cluster.

5.2 The lattice of nearnesses on a frame

Let us recall that a partially ordered set H is said to be a *preframe* if every directed subset of H has a join, every finite subset of H has a meet, and binary meets distribute over all directed joins. Here we define a partially ordered set P to be a *pseudo-frame* in case every nonempty subset of P has a join, every finite subset of P has a meet (and hence P has the top element) and binary meets are distributive over joins. Thus, a pseudo-frame is exactly like a frame except for the possible absence of the bottom element. Clearly, every pseudo-frame is a preframe.

The following example, kindly communicated to us by one of the examiners of the thesis, shows that a pseudo-frame need not be a frame.

Example 5.2.1 Consider the set of integers with their usual order, with a new top added. This set satisfies the condition that all non-empty subsets have joins, but it does not have a bottom element.

We shall show that the lattice $\mathcal{N}L$ of all nearnesses on a frame L is a pseudo-frame. We note that in [57], Zenk defines a "nearness" on a frame L to be merely a filter of covers without imposing the admissibility condition. When (the stronger version of) admissibility is imposed, he talks of an admissible nearness. In [57, Lemma 8] he shows that, under subset inclusion, the poset of "nearnesses" is a frame. Our proof bears absolutely no resemblance to that of Zenk's.

Lemma 5.2.2 *Let L be a regular frame, and $\mathcal{N}$ a nonempty collection of covers of L. Form the collection*

$$\hat{\mathcal{N}} = \{A \subseteq L \mid A \textit{ is the meet of finitely many covers in } \mathcal{N}\}.$$

Then $\hat{\mathcal{N}}$ is closed under $\wedge$ (finite meets).

Proof: Let A, $B \in \hat{\mathcal{N}}$. Write $A = A_1 \wedge \cdots \wedge A_n$ and $B = B_1 \wedge \cdots \wedge B_m$, where all the A_k's and B_k's are members of $\mathcal{N}$. Then

$$A \wedge B = (A_1 \wedge \cdots \wedge A_n) \wedge (B_1 \wedge \cdots \wedge B_m)$$

is clearly a meet of finitely many members of $\mathcal{N}$. Therefore $A \wedge B \in \hat{\mathcal{N}}$. ■

Lemma 5.2.3 *Let L, $\mathcal{N}$ and $\hat{\mathcal{N}}$ be as in the above lemma. Then the set*

$$\bar{\hat{\mathcal{N}}} = \{C \in \mathrm{Cov}(L) \mid A \leq C \textit{ for some } A \in \hat{\mathcal{N}}\}$$

is a pre-nearness on L.

Proof: First, let $A \in \bar{\hat{\mathcal{N}}}$ and $C \in \mathrm{Cov}(L)$ such that $A \leq C$. Then we have $B \in \hat{\mathcal{N}}$ such that $B \leq A$. In that case $B \leq C$; so that $C \in \bar{\hat{\mathcal{N}}}$. Second, let A, $B \in \bar{\hat{\mathcal{N}}}$ and C, $D \in \hat{\mathcal{N}}$ such that $C \leq A$ and $D \leq B$. This implies $C \wedge D \in \hat{\mathcal{N}}$ by Lemma 5.2.2, and $C \wedge D \leq A \wedge B$. Therefore $A \wedge B \in \bar{\hat{\mathcal{N}}}$, and so the result holds. ■

It is clear from the definitions given above that $\mathcal{N} \subseteq \hat{\mathcal{N}} \subseteq \bar{\hat{\mathcal{N}}}$.

Lemma 5.2.4 *Let $(\mu_\alpha)_{\alpha \in \Lambda}$ be a nonempty family of nearnesses on a frame L. Then*

$$\mu = \bigvee_\alpha \mu_\alpha = \left(\bigcup_\alpha \mu_\alpha \right)^{\bar{\wedge}}$$

is the join of the family (μ_α) in the poset $\mathcal{N}L$ of all nearnesses on L ordered by inclusion.

Proof: μ is a pre-nearness on L, by Lemma 5.2.3, and since $\mu_\alpha \subseteq \bigcup_\alpha \mu_\alpha \subseteq \mu$ for every α, and since each μ_α is admissible, we conclude that μ is admissible; so that μ is a nearness. Next, suppose $\mu_\alpha \subseteq \nu$ for every α, where ν is a nearness on L. Then $\bigcup_\alpha \mu_\alpha \subseteq \nu$, and consequently $\mu \subseteq \nu$. Thus, μ is the join of the family (μ_α). ∎

The preceding lemma shows that every nonempty subset of $\mathcal{N}L$ has a join; the consequence of which is that every nonempty subset of $\mathcal{N}L$ has a meet. Note though that the lemma does not explicitly indicate how binary meets are computed in $\mathcal{N}L$. They are set intersections as we show next.

Lemma 5.2.5 *If μ and η are nearnesses on a frame L, then the intersection $\mu \cap \eta$ is a nearness on L.*

Proof: The filter properties of $\mu \cap \eta$ are quite clear: A, $B \in \mu \cap \eta$ implies $A \wedge B \in \mu$ and $A \wedge B \in \eta$. Therefore $A \wedge B \in \mu \cap \eta$. Also, if $A \in \mu \cap \eta$ and $A \leq C \in \mathrm{Cov}(L)$, then $C \in \mu$ and $C \in \eta$; so that $C \in \mu \cap \eta$.

To show admissibility, denote by $\lhd$ the uniformly below relation relative to $\mu \cap \eta$. Let $a \in L$. Then

$$a = \bigvee \{x \in L \mid x \lhd_\mu a\} \text{ and } a = \bigvee \{y \in L \mid y \lhd_\eta a\}.$$

Therefore, by the frame distributive law,

$$a = \bigvee \{x \wedge y \mid x \lhd_\mu a \text{ and } y \lhd_\eta a\}.$$

Now if $x \lhd_\mu a$ and $y \lhd_\eta a$, then

$$\{x^*, a\} \in \mu \text{ and } \{y^*, a\} \in \eta.$$

Since $x^* \leq (x \wedge y)^*$ and $y^* \leq (x \wedge y)^*$, we have that $\{(x \wedge y)^*, a\} \in \mu \cap \eta$ as it is refined by a cover in μ and also by a cover in η. Thus, $x \wedge y$ is an element of L such that $x \wedge y \lhd a$. This shows that

$$a = \bigvee\{x \wedge y \mid x \lhd_\mu a \text{ and } y \lhd_\eta a\} \leq \bigvee\{z \in L \mid z \lhd a\} \leq a,$$

and hence $\mu \cap \eta$ is admissible. ∎

We now state our main result in this section.

Proposition 5.2.6 *The lattice $\mathcal{N}L$ of all nearnesses on a frame is a pseudo-frame.*

Proof: Let $\mu \in \mathcal{N}L$ and $\{\nu_i \mid i \in I\} \subseteq \mathcal{N}L$. The required distributivity, namely $\bigvee(\mu \wedge \nu_i) \leq \mu \wedge \bigvee \nu_i$, always holds. So we must show that $\mu \wedge \bigvee \nu_i \leq \bigvee(\mu \wedge \nu_i)$. Let $A \in \mu \wedge \bigvee \nu_i$. Since $\wedge$ is intersection in $\mathcal{N}L$, $A \in \mu$ and $A \in \bigvee \nu_i$. By Lemma 5.2.3, there are finitely many indices $i_1, \ldots, i_m$ and covers $A_{i_1} \in \nu_{i_1}, \ldots A_{i_m} \in \nu_{i_m}$ such that

$$(\dagger) \qquad\qquad A_{i_1} \wedge \cdots \wedge A_{i_m} \leq A.$$

For each $k \in \{1, \ldots, m\}$, $A \cup A_{i_k} \in \mu \cap \nu_{i_k}$ since the cover $A \cup A_{i_k}$ is refined both by A (which is in μ) and by A_{i_k} (which is in ν_{i_k}). Consequently,

$$(A \cup A_{i_1}) \wedge \cdots \wedge (A \cup A_{i_m}) \in (\mu \cap \nu_{i_1}) \vee \cdots \vee (\mu \cap \nu_{i_m}) \leq \bigvee(\mu \wedge \nu_i).$$

We claim that $(A \cup A_{i_1}) \wedge \cdots \wedge (A \cup A_{i_m})$ refines A. To show this, let $x \in (A \cup A_{i_1}) \wedge \cdots \wedge (A \cup A_{i_m})$. Then

$$x = x_1 \wedge \cdots \wedge x_m$$

for some elements $x_1 \in A \cup A_{i_1}, \ldots, x_m \in A \cup A_{i_m}$. If $x_\ell \in A$ for some $\ell \in \{1, \ldots, m\}$, then $x \leq x_\ell \in A$. If, on the other hand, $x_k \in A_{i_k}$ for each $k \in A_{i_k}$, then

$$x \in A_{i_1} \wedge \cdots \wedge A_{i_m},$$

and hence, from $(\dagger)$, $x \leq a$ for some $a \in A$. This shows that every element of $(A \cup A_{i_1}) \wedge \cdots \wedge (A \cup A_{i_m})$ is below some element of A. Therefore $A \in \bigvee(\mu \wedge \nu_i)$; which shows that $\mu \wedge \bigvee \nu_i \leq \bigvee(\mu \wedge \nu_i)$, and hence equality. ∎

Clearly, the top element of $\mathcal{N}L$ is $\mathrm{Cov}(L)$. It is not clear how the bottom, in the instances where it exists, can be explicitly described. Note though that if L is compact, then $\mathcal{N}L$ is a one-element frame, and hence $0_{\mathcal{N}L} = \mathrm{Cov}(L)$. The converse actually also holds. That is, if $\mathcal{N}L$ is a one-element frame, then L is compact. For, $\mathrm{Cov}(L) = (\mathrm{Cov}(L))_T$ implies every cover has a finite subcover.

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