Exploring teachers' mathematical meanings for teaching trigonometric ratios and functions to grade 10 learners in the Tshwane South District of Gauteng Province

by

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DECLARATION OF ORIGINALITY

I declare that the thesis entitled "Exploring teachers' mathematical meanings for teaching trigonometric concepts to Grade 10 learners in the Tshwane South District of Gauteng Province" is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete referencing. I further declare that I submitted the thesis to originality checking software and that it falls within the accepted parameters for originality. I further declare that I have not previously submitted this work, or part of it, for examination at Unisa for another qualification or at any other higher education institution.

....	14 November 2019
MR GT MPHUTHI	**DATE**

Dedication

I dedicate this study to my wife Thembisile, my mother Thobileng, my children Katleho, Nomvula, Zanokuhle and Njabulo, and my late brother Vusi.

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This MSc dissertation is a product of a long process of hard labour and extreme endurance. Without the support of some people, it would have been much more challenging to overcome the difficulties and stress in this process and to come up with this MSc dissertation. I would like to give my special thanks to these people.

Special thanks to my wife, Thembisile. I cannot thank you enough for what you have done to me. You are really a partner in love and in studying, and it is to you that I dedicated this work.

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ABSTRACT

This study aimed to explore the mathematical meanings possibly entertained by Grade 10 teachers on the particular trigonometric topics of ratios and functions. The quest to embark on this exploration was primarily triggered by the desire to enhance our understanding of and provide plausible explanations for learners' observed responses in relation to trigonometric problem-solving tasks. The study began by first addressing the key and contested conceptual polarisation of teacher knowledge and teacher knowing, which we consider carrying explicative power in allowing us to unpack the notion of mathematical meanings. Our research suggests that previous and accurate knowledge of these concepts is eminently important in enabling researchers to map out the exploration and study of teachers' so-called "mathematical meanings" concerning the teaching of trigonometry. Mason and Spence (1999, cited in Thompson, 2015) argue that "knowing is more useful for thinking about teaching and learning". In this study, this would constitute an act of knowing (by the teacher) what to teach, and how to teach. Hence the researcher views knowing as connoting the instructional activities of the knower, while knowledge connotes the acquired facts of the knower. Furthermore, the notion of mathematical meaning is conceived in our study as a reference to the conceptual images associated with mathematical constructs that eventually inform their instruction, the learning activities as chosen by classroom instructors (teachers), and how these intuitive images come to be communicated to and eventually received by learners.

To conduct this conceptual exploration systematically, the researcher followed a case study research approach that embraced elements of exploratory research design. This research design took inspiration from the qualitative research paradigm and was adopted to facilitate the researcher's intention to work with a small number of teachers to whom a trigonometric task was assigned, and who were later selected to participate in semi-structured interviews. Using convenience sampling procedures, the current study sampled 12 Grade 10 teachers who taught mathematics in the Tshwane South District in the Gauteng Province of South Africa. All teacher participants were given a trigonometry task to reflect on their conceptualisation and understanding of its mathematical content, which we considered to be an important, initial step in the effort to access these experts' implicit mathematical meanings. The assigned trigonometric task was drawn and adapted from Thompson's Mathematical Meanings for Teaching secondary mathematics (MMTsm) (Thompson, 2015), which proposes useful and authentic constructs to study respondents' mathematical meanings. The analysis of teachers' responses to the task led to a purposive sampling of two teachers to take part in

subsequent semi-structured interviews. The purpose of the interviews was to, (1) probe teachers further on their responses; (2) verify our understanding and interpretation of teachers' responses; and, (3) attempt to understand the argument that teachers' responses provided direct insight into their entertained mathematical meanings toward particular mathematics topics. The observations that emerged from this analysis seem to indicate that teachers will often entertain differing mathematical meanings toward a single mathematical topic, some of which meanings are neither coherent nor productive.

Key terms

Coherence

Conveyance of meaning

Mathematical meanings for teaching

Mathematical knowledge for teaching

Pedagogical content knowledge

Problem-solving

Secondary teachers

Trigonometric ratios

Trigonometric functions

Unit circle

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List of Abbreviations

ACE	Advanced Certificate in Education
CAPS	Curriculum Assessment Policy Statements
CH	Circle where the hypotenuse
CK	Content knowledge
DBE	Department of Basic Education
DE	Diploma in Education
DME	Department of Mathematics Education
ERIC	Education Resources Information Centre
FET	Further Education and Training
GET	Genetic Epistemology Theory
HoD	Heads of Department
IQ	Interview Question
MKT	Mathematical knowledge for teaching
MMTsm	Mathematical Meanings for Teaching Secondary Mathematics
NSC	National Senior Certificate
PCK	Pedagogical content knowledge
PEI	President's Education Initiative
REC	Research Ethical Committee
TSD	Tshwane South District
ZPD	Zone of proximal development

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CHAPTER ONE

A THEORETICAL OVERVIEW OF THE STUDY

1.1 INTRODUCTION

In 2015, the Department of Basic Education (DBE) released a National Diagnostic Report highlighting the importance of a teacher's content knowledge (CK) in facilitating teaching and learning activities, and further enhancing their students' understanding of the content of school subjects. The following educational highlights are acknowledged in the report:

> The deficiency in understanding *specific subject content* areas is a problem in many schools. The problem appears to be compounded by a shallow grasp of some of these content areas by teachers, or by teachers neglecting to cover some aspects of the curriculum (DBE, 2015, p. 5).

Shulman (1995) defines content knowledge or specific subject content as knowledge about the subject and its structure. In mathematics, CK can be viewed as knowledge of mathematics topics and how underlying mathematics concepts are connected and organised to present the subject matter at hand. The role of the teacher's CK in facilitating effective instruction[1] has been emphasised by many researchers (for example, see, Depaepe, Torbeyns, Vermeersch, Janssens, Janssen, Kelchtermans, Verschaffel & Van Dooren, 2015; Kleickmann, Ritchter, Kunter, Elsner, Besser, Krauss, Cheo & Baumert, 2015). Kleickmann *et al*. (2015) noted that teachers' content (subject) knowledge has the ability to affect the quality of their instruction and the consequent progress of learners significantly. Depaepe *et al*. (2015) argued that teachers' CK and learners' progress in school subjects are significantly positively associated, without of course downplaying the influence of the teacher's pedagogical content knowledge[2] (PCK). In light of these observations, it is crucial to seriously ponder the DBE's (2015) report

1. The word *instruction* was used in current study to refer to the classroom activities of teaching and learning in which a teacher is perceived as facilitating the learning process.
2. In this regard, pedagogical content knowledge (PCK) is conceived as a form of teacher's knowledge needed to transform content (subject) knowledge into forms that are pedagogically powerful and meaningful, and yet are adaptive to the variations in ability and background presented by the learners (see, An, Kulm & Wu, 2004; Shulman, 1987; Turnuklu & Yesildere, 2007).

on the influence of teachers' knowledge in determining the relative progress of their students in mathematics classrooms.

Ever since the *Equality of educational opportunity* study by Coleman, Campbell, Hobson, McPartland, Mood, Weinfeld and York (1966), researchers have agreed on a unique input of teacher knowledge to learner progress and scholastic performance (see, also, Hill, Rowan & Ball, 2005). The Centre for Development and Enterprise (CDE) (2014) conducted a study to investigate factors considered to have an influence on learner performance in mathematics. The study found that teachers' shallowness in subject knowledge (CK) and PCK of mathematics affected learner performance (CDE, 2014). In the process, researchers have identified a strong association between teacher knowledge and instruction (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep & Ball, 2008). Some of the studies that have been conducted have come to the conclusion that stronger teacher knowledge overwhelmingly precipitates positive educational outcomes for classroom instruction and learner progress (see, An, Kulm & Wu, 2004; CDE, 2014; Depaepe *et al*., 2015; Hauk, Toney, Jackson, Nair & Tsay, 2014; Hill *et al*., 2008; Kleickmann *et al*., 2015; Hill, Rowan & Ball, 2005; Turnuklu & Yesildere, 2007).

Despite the wealth of research on teacher knowledge and its documented influence on classroom instruction, learner performance in mathematics is continuously a subject of excessive concern globally. Most studies point to teachers' ignorance of CK as a source of educational problems for examples, see Depaepe *et al*., 2015; Hauk *et al*., 2014; Hill *et al*., 2008; Kleickmann *et al*., 2015). In South Africa, the poor or limited subject content knowledge of mathematics teachers seems to be a recurring challenge facing education. The problem has been described as one that is "assuming critical proportions" (Maree & Erasmus, 2007, p. 1). In 1999, the president of South Africa developed the President's Education Initiative (PEI) to look into the state of education delivery and concluded that the limited conceptual knowledge of teachers was one of the most critical challenges facing the South African education system (Taylor & Vinjevold, 1999).

In light of the abundance of well-documented research initiatives on teachers' CK and PCK and their correlation with educational rigorousness and success rates, we can ask the question: "*Should we continue to study the CK and PCK of instructors to determine if it has a predominant influence on instructional activities?*" The researcher has taken the view that most of the research presented so far has tended to value the measure rather than measuring the value of educational processes (Maphalala & Dhlamini, 2017). Thompson (2015) claimed that most of the existing studies on PCK and CK have primarily tended to assess teachers' "declarative knowledge" and its association to teaching instead of investigating the reasons which prompt teachers' decisions to teach (p. 1). While the current study has focused on teacher knowledge, it did not intend to assess or measure these forms of pedagogical knowledge.

Research to date acknowledges a deficiency in the literature on CK and PCK in failing to measure how teacher knowledge influences classroom instruction to enhance productive learning (Hill *et al.*, 2008; Thompson, 2015). This gap in the literature leads us to conclude that some other factor, rather than merely the measurement of teacher knowledge, needs to be studied in relation to teacher knowledge in the classroom. Perhaps an alternative angle of analysis could comprise a broader study of teachers' conceptualizations of the key aspects of mathematics knowledge which inform their regular classroom practices. In light of all this, the current study has attempted to probe the meanings that teachers potentially associate with their conceptual or content knowledge of key topics in mathematics. The researcher acknowledges that most teachers possess a minimal amount of CK and PCK needed to facilitate their instruction in a classroom setting, and no teachers were entirely ignorant of their professed domain.

As such, the current study has attempted to gain insight into what it means to say that teachers possess certain forms of knowledge. How do teachers internalize pockets of pedagogical knowledge needed to carry out their educational mandate to teach mathematics effectively? Broadly speaking, this study has investigated various kinds of meanings or conceptualizations embraced by mathematics teachers in relation to pedagogical knowledge; and how these meanings or conceptualizations have tended to influence instructors' preferred instructional practices in mathematics classrooms (see, Section 1.2.2). In particular, our study explored and examined the key mathematical meanings embraced by teachers when approaching the topic of *trigonometric ratios and functions* for Grade 10 learners; and how these meanings were eventually communicated to learners (see, Section 1.4).

1.2 MEASURING TEACHERS' MATHEMATICAL MEANINGS FOR TEACHING

In this section, the competing binary constructs of *teacher knowledge* and *teacher knowing* are introduced (see, Section 1.2.1). We approach these constructs in terms of how they can be positioned and contemplated to deepen our understanding of teachers' mathematical meanings of their reported knowledge of pedagogy and pedagogical techniques.

1.2.1 Explaining the notion of *teacher knowledge* further

Mason and Spence (1999, cited in Thompson, 2015) have attempted to distinguish between the concepts of *knowing* and *knowledge.* These authors have argued that "*knowing* is much more useful for thinking about teaching and learning than is *knowledge*" (p. 435). Mason and Spence (1999, cited in Thompson, 2015) noted that "knowing" connotes the activities of the knower, while "knowledge" connotes the facts themselves (p. 435). The latter could be associated with teacher knowledge, the measurement of which may not necessarily inform the inquirer much about the teachers' instructional (pedagogical) activities, especially in relation to the decisions and actions that characterize the teacher's classroom instruction (see, footnote 1). The decisions teachers make and actions they take on the basis of their discipline-specific knowledge largely influence the instructional methods they tend to prefer using in classroom settings (Thompson, 2015). In this way, a closer examination of the notion of *teacher knowing* may allow for a complete understanding of teacher actions in the classroom. Teacher actions can be thought of as the outcome of the instructional decisions teachers make in accordance with the domain-specific knowledge at their disposal.

1.2.2 Mathematical knowledge and examining the mathematical meaning

According to Thompson (2015), exploring the notion of a *teacher knowing* (as opposed to *teacher knowledge*, see Section 1.2.1) may bring the inquirer closer to understanding and explaining the sources of teachers' actions and decisions enacted in their instruction. This understanding could be viewed as a shift away from assessing teachers' PCK or CK, to allow for insights to emerge into what these forms of subject-specific knowledge mean for teachers. For instance, this shift allows the inquirer not necessarily to think about what teachers know about certain topics in trigonometry but to think about what teachers might mean when they say that they possess such knowledge. An acknowledgement of teachers' individualized and preferred conceptualizations (meanings) allows for a renewed appreciation of the key concepts associated with their subject-specific knowledge. The researcher in the current study adheres to the perspective that the meanings teachers entertain concerning their subject-specific knowledge could have a significant influence on what they, in turn, decide to convey to their learners, along with how these teachers position themselves instructionally when articulating these forms of knowledge to their learners (actions).

The following example illustrates the association between the notions of teacher's meanings of mathematical knowledge and the resulting instructional actions they take in classroom settings. In everyday life, people's decisions on certain activities are largely influenced by the meanings they attach to these life activities. For instance, a driver who was once involved in an accident may attach another meaning to driving a car, which could eventually influence their decisions and actions while driving. In the context of the current study, a task was given to mathematics teachers to examine the kinds of mathematical meanings these participants hold on a given concept in trigonometry; and, in addition, these meanings were eventually actualized by teachers as a function of observed classroom practices (Section 3.4.2.1).

According to Thompson (2015), teachers' mathematical meanings "constitute their images of the mathematics they teach and intend that students have" (p. 437). The current study thus probed teachers' understanding in terms of how teachers demonstrated their intent to transmit or impart their knowledge to learners (Section 3.4.2.2). This report provides a corpus of analyzed data that documents teachers' instructional decisions and the resulting actions they recounted to researchers. The results of this study provide a window unto the teachers' PCK that facilitated their teaching certain concepts in Grade 10 trigonometry (see, Section 3.4.2).

1.3 PROBLEM STATEMENT

Research on the association of teacher knowledge (conceptual knowledge) and the learning of trigonometry is sparse (Weber, 2008, 2005). The same can be said about the paucity of research on teachers' meanings regarding the conceptual knowledge they possess for teaching certain topics in trigonometry. There is, however, evidence to suggest that teachers' understanding of trigonometric functions and topics foundational to making sense of trigonometry lacks coherence (Fi, 2003; Moore, LaForest & Kim, 2012; Thompson, Carlson & Silverman, 2007; Weber, 2005). Moore, LaForest and Kim (2012) argue that teachers' poor grasp of trigonometric concepts is reflected in their learners "constructing disconnected understandings" (p. 1). Hence, trigonometry has been identified as a difficult topic for learners at school (Akkoc, 2008). Gür (2009, p. 68) confirms that "trigonometry is an area of mathematics that students believe to be particularly difficult and abstract compared with other subjects of mathematics."

According to Mji and Makgato (2006), examination results may provide a source for exploring difficulties pertaining to teaching and learning. Sasman (2011) analyzed learner responses in the 2009 to 2010 National Senior Certificate (NSC[3]) examinations and found that:

> Trigonometry was the most poorly answered section of Paper 2. There are still instances where some candidates showed no knowledge of the basics of trigonometry. It is a matter of concern that a learner can spend 3 years in the FET[4] band and ostensibly have learnt nothing in trigonometry (p. 10).

The DBE's National Diagnostic Reports for the years 2013, 2014, 2015 and 2016 show no improvement on certain aspects of trigonometry that were examined during the National Senior Certificate examinations of the years reflected in Table 1.1.

One could argue that learners' observed responses to the test items are largely a product of teacher knowledge and meanings conveyed to learners in the course of classroom instruction. Given this background, this research explored teachers' mathematical meanings of the content knowledge of trigonometry and how these meanings were conveyed to learners during trigonometric instruction. The study explored the actions and decisions that characterized teachers' instruction in Grade 10 trigonometry (Section 1.2.2).

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3. In South Africa, Grade 12 offers a national examination opportunity that South African secondary school learners have to pass in order to be admitted to a university. This examination is known as the National Senior Certificate (NSC) (Howie & Plomp, 2002).
4. Further Education and Training (FET) in the South African education system refers to the final senior secondary schooling phase that takes place just before university or tertiary education. It represents the final educational journey or lap in the basic education system in South Africa. FET covers the Grade 10 to Grade 12 levels. FET is preceded by the General Further and Education (GET) phase, which covers the learning grade levels 8 to 9.

Table 1. 1: Average performance (%) of learners in Grade 12 in trigonometry questions

Topic in Grade 12 trigonometry	**2012**	**2013**	**2014**	**2015**
	Average %	**Average %**	**Average %**	**Average %**
Trigonometric definitions	61.7	42.9	N/A	70
Simplify trigonometric expressions and identities	36.4	N/A	34	37.5
Solving trigonometric equations	N/A	24.8	37	65
Trigonometric functions and their graphs	24.0	34.8	N/A	42
Application of area, sine and cosine rules	31.5	N/A	57	41
Application of trigonometric concepts in solving problems including two- and three-dimensional problems	Not provided	12.5	N/A	39
	38.4	**28.75**	**42.67**	**49.08**

Source: *DBE (2016, 2015, 2014, 2013)*

1.4 THE AIM OF THE STUDY

The current study aimed to explore teachers' understandings of trigonometric ratios and functions by examining the mathematical meanings that teachers convey when they teach Grade 10 learners.

1.5 OBJECTIVES OF THE STUDY

The following objectives were set out in order to achieve the aim of the study:

- to initiate a productive dialogue of inquiry to reflect on existing research in relation to mathematics teachers' CK and PCK;
- to establish suitable instrument(s), if any, to explore teacher's meaning(s) concerning their knowledge of Grade 10 trigonometry;
- to examine mathematics teachers' actions needed to enhance productive instruction in Grade 10 trigonometry topics; and,
- to determine instructional decisions that teachers make to facilitate a Grade 10 trigonometry instruction.

1.6 RESEARCH QUESTIONS

The study was guided by the following research questions (RQs):

1.6.1 What mathematical meanings do teachers convey when teaching the topic of trigonometric ratios and functions to Grade 10 learners?

1.6.2 In what way(s) do teachers' knowledge-related decisions influence their instructional actions during a Grade 10 trigonometry lesson?

This study hypothesized that an inquiry into the decisions (RQ1) and actions (RQ2) of teachers that influence their instruction could provide useful insights into accessing and describing the pedagogic meanings they associate with their discipline-specific knowledge (Section 1.2.2).

1.7 SIGNIFICANCE OF THE STUDY

A search performed in ERIC (Education Resources Information Centre) research database of the words 'teacher', 'knowledge', 'understanding', 'trigonometric ratios and functions' in the South African context returned zero-article feedback. Many studies conducted in these areas have focused largely on the assessment of teachers' pedagogical content knowledge (PCK), mathematical knowledge for teaching (MKT) and the factors that influence the teaching and learning trigonometry (see, Sections 1.1 & 1.2). In light of these studies, we ask the question: What do we mean when we say that teachers *know* certain concepts in mathematics? The current study examines the meanings teachers associate with their CK pertaining to Grade 10 trigonometric topics elicited during interviews. The effort to uncover teachers' implicit, higher-order meanings associated with trigonometric concepts could allow them to improve their classroom instruction by revealing to them the latent concepts they associate with their CK.

1.8 DELIMITATION OF THE STUDY

According to Simon (2011, p. 2) "delimitations are those characteristics that limit the scope and define the boundaries of your study. The delimitations are in your control." Naturally, the act of instruction involves teachers and learners alike, with the relative success of the teacher's instruction being able to be assessed through examining the educational progress of their learners. Hence, one would have expected that this study would interact with both teachers and learners in an attempt to holistically examine teachers' latent associations concerning the knowledge they wish to convey to their learners. Given the complexity of the pedagogic questions under consideration and in light of time constraints, this study opted to focus exclusively on teachers rather than their learners. Future research at the doctoral level may wish to pay greater attention to the role of learners in the teaching process, to examine the extent to which different forms of teachers' meanings (if any) possibly contribute to productive learning. At the doctoral level, assessing learner engagement could constitute one of the principles aims of the study.

1.9 DEFINITION OF OPERATIONAL TERMS FOR THE PROPOSED STUDY

The following terms were defined for use in the current study:

1.9.1 Coherence

The notion of coherence is inherently linked to the discussion of the meanings teachers associate with their CK (Section 1.2.2). According to Thompson (2008), coherence results from "the development of meanings of each topic and the construction of contextual inter-relationships among them" (p. 47). In terms of the current study, the notion of coherence was conceptualized as referring to the understanding of how a teacher develops a consistently meaningful connection of differently positioned concepts and topics in mathematics to develop productive trigonometric instruction for Grade 10 students.

1.9.2 Mathematical meanings for teaching

This topic is given attention in Section 1.2.2 of this report. Teachers' mathematical meanings refer in this context to the images they entertain of the mathematical concepts that they wish to impart to their learners. The teachers' meanings "guide their instructional decisions and actions" (Thompson, 2015, p. 437) (see also, Section 1.2). This study examines the meanings teachers hold in relation to selected topics in Grade 10 trigonometry. In terms of Thompson's (2015) definitions, the current study explores teachers' mental images pertaining to selected trigonometric topics. In addition, teachers' instructional decisions and related actions were viewed as determining their perception of content (conceptual) knowledge (see, Sections 1.1, 1.2.1, 1.4 & 1.5).

1.10 THE DISSERTATION CHAPTER OUTLINE

The organization of the final dissertation chapters of the study is as follows:

Chapter One: Introduction

This chapter provides a theoretical overview of the study by firstly providing the contextual background of the study. The following issues are addressed: the problem in context, the statement of the problem, the study's aim and related objectives, the research questions, the rationale and significance of the study, and brief definitions of operational terms.

Chapter Two: Literature review and conceptual framework

This chapter discusses the current literature reflecting on the importance of trigonometry and explores the works of other researchers in relation to the instruction of high school trigonometry. Finally, the researcher used knowledge gained from the literature review to discuss a suitable conceptual framework for the study.

Chapter Three: Research methodology

This chapter reflects on the research design of the study, the research methodology and issues of instrumentation linked to data collection and analysis. In addition, pertinent ethical considerations are also addressed in this chapter.

Chapter Four: Data analysis

This chapter presents analyses of data and a discussion of the research findings.

Chapter Five: Summary, discussion and conclusion

This chapter presents a summary of the study in terms of how the study results were used to address the research questions, along with the aim and objectives of the study. The conclusion is presented in this chapter, and recommendations based on the findings of the study are presented.

CHAPTER TWO

LITERATURE REVIEW AND THEORETICAL PERSPECTIVE

2.1 INTRODUCTION

This study drew inspiration from Davis' (2011) conception that teacher knowledge is tacit, along with the meanings that are associated with this knowledge. Davis (2011) argues that teacher knowledge and associated meanings are difficult to share with others. Davis' (2011) claim that teacher knowledge and related meanings are tacit is basically the subject of this study. Scholars define tacit knowledge as skills, ideas and experiences that an individual possesses but which may not be communicated with ease (Chugh, 2015). This study explored the actual relevance of Davis' assertions in the real-life instructional context of Grade 10 trigonometry lessons. This study was conducted with teachers who taught mathematics in Grade 10 at the time at which this research was conducted (see, Section 3.5).

The constructivist claim is that instruction is enabled by social interaction and collaboration (Vygostky, 1978), with the latter factor carrying the implication that teacher effectiveness can be measured by the success with which they share meanings and communicate the tacit knowledge they possess with learners in socially-facilitated settings. Among other things, this chapter seeks to address the following issues: first, consistent with the aim of the study (Section 1.4), the researcher explored and examined the notion of *coherence* and *mathematical meanings* for teaching trigonometry in Grade 10 mathematics classrooms (Section 2.3, 2.4 & 2.8). Secondly, this chapter strives to present an overview of the literature on the challenges of teaching and learning trigonometry in relation to our notion of mathematical meanings. Thirdly, in this chapter, the researcher presents theoretical considerations that are linked to the framework that guided research activities in the current study.

2.2 TEACHER KNOWLEDGE

Teachers must generate instructional conditions that assist learners to experience productive classroom instruction and information sharing. Davis' (2011) conception that teacher knowledge is tacit highlights new teacher roles, allowing for a reconsideration of instructional approaches. The word *tacit* implies that teacher's knowledge is implicit; may not be seen or touched; and that this knowledge can be implied without being precisely instated (actions). Precision is difficult to achieve as it is determined by the response of the receiver. Therefore, the way teachers conduct themselves in the classroom is likely to be a function of their knowledge of the educational project at their disposal. The variables at play in the educational project are shown in Figure 2.1. Teacher's knowledge in Figure 2.1 is determined by the way in which the variables are integrated and collectively facilitated to generate productive learning. A productive teacher succeeds in effectively sharing their tacit knowledge with learners. This teacher can motivate learners to construct desirable mathematical meanings in their cognitive schema (see, Section 2.5).

Given that the mathematical performance of most learners has been declining persistently over the years (CDE, 2014), this study has attempted to explore questions such as, *does a poor performance in Grade 10 trigonometry highlight teachers' inability to share their tacit knowledge and mathematical meanings with learners*? For the researcher to provide a sound response to this question, the current study explored teachers' content knowledge, and how teachers generate and communicate the meaning, they associate with their knowledge to learners in order to promote mathematical learning and understanding. The following questions were important for this study, and are largely drawn from our guiding research questions (Section 1.6):

- *What kind of instructional actions do teachers adopt to communicate mathematical knowledge to learners*?
- *How do teachers facilitate their students' meaning-making processes*?

Teachers possess several types of knowledge. For instance, Thompson (2015) noted that it is possible to measure teachers' declarative content knowledge directly, but not possible to connect it to what the teacher decides to teach. If a teacher's understanding of trigonometry is flawed, then learners' understanding of trigonometry is likely to be unproductive or limited. How then should we unpack the teacher's tacit knowledge to access the mathematical meanings they convey with the intention of promoting learning (Davis, 2011)? Most of the literature reviewed in this study is informed by constructivism and social constructivist theory (see, Aljohani, 2017; Amineh & As, 2015; Mogashoa, 2014; Noh & Jin, 2017; Vygotsky, 1978), which regard teacher knowledge as an important ingredient in learning (Vygotsky, 1978). Tacit knowledge (Davis, 2011) and mathematical meanings (Thompson, 2015) make teachers appear as the sole owners of mathematical knowledge. Modern technology makes learning a social activity achievable through human interaction, the sharing of ideas and collaboration on various platforms provided by educational stakeholders. In and through this process, learning becomes a shared and negotiated enterprise, rather than a pre-determined event that is educationally pre-arranged and pre-packaged by teachers. This instructional view regarding learning is largely a custodian of constructivist thinking (Vygotsky, 1978).

Teachers should share their knowledge with learners, and, most importantly, guide learners to form mathematical meanings that are not only accurate but mostly coherent (Section 2.3). The Curriculum Assessment Policy Statements (CAPS) documents do not only emphasize content knowledge and pedagogical knowledge as prerequisites to the attainment of learning outcomes (Department of Basic Education [DBE], 2011) but also suggests that learning outcomes are achievable through motivating and making learners realize that they can accomplish the best learning outcomes even when working on their own (DBE, 2011). A teacher must be acquainted with the educational background of their learners to be effective instructors (Beswick, 2019). According to Beswick (2019), teachers should have a sense of what learners know and believe. Teachers who know their learners appropriate educational material in order to make their teaching meaningful to learners. Teachers who do not know their learners may limit their use of learner-oriented tasks to constrain learners' prospects of gaining new mathematical knowledge (Beswick, 2019).

It is the researcher's view that the limitations and flaws of tacit knowledge and mathematical meanings should be studied to furnish teachers with various teaching strategies that are learner-friendly. Basing curriculum practices excessively on tacit knowledge and teachers' mathematical meanings could compromise opportunities to address other areas of teacher professionalism, in which teachers need to be continuously supported.

2.3 COHERENCE IN THE CURRICULUM

In this section, the discussion and provision of the definition of *coherence* are advanced. Coherence is defined as the alignment of components in an educational system (Biggs, 2003, 1999). This description is examined to find out its implication for the instruction of trigonometry at the Grade 10 level. The discussions reflect largely on Davis' (2011) tacit knowledge and the principle of mathematical meaning advocated by Thompson (2015). Coherence is defined "as a state or situation in which all the parts or ideas fit together well so that they form a united whole" (Collins-COBUILD English Dictionary, 2001). The latter definition suggests that in a coherent system, ideas are logically connected, inter-dependent and inter-related.

The idea of interconnectedness relates to the notion of relational knowledge (Skemp, 1978; Van de Walle, Karp and Bay-Williams 2016). Skemp (1978) and Van de Walle *et al.* (2016) noted that the understanding of learning material is achievable when ideas are coherently connected. Piaget (1968) describes interconnectedness as the sorting out of schemas into a meaningful whole through a process of assimilation. Coherence means that all components in a system work in tandem with one another. Mathematical meanings and tacit knowledge are important in helping teachers to achieve the educational aspirations of the curriculum. The utilization of the term coherence in educational settings comes up mostly in discussions that aim to lend greater effectiveness and efficiency to the organization and implementation of a curriculum. Some scholars use the term coherence in various settings, with the usage of the term and its applicability explored in this report (for examples, see, Biggs, 2003, 1999; MacMahon & Thakore, 2006).

2.3.1 Exploring coherence as an alignment of components in an instructional system

Biggs (2003, 1999) indirectly uses the term coherence to highlight the importance of aligning instructional systems and selected curriculum objectives. Biggs (1999) emphasized the importance of aligning assessment and learning outcomes to guide learners and teachers to attaining improved, intended outcomes. Biggs (1999) replicates coherence as constructive alignment. Subsequently, MacMahon and Thakore (2006) describe the attainment of learning outcomes as "having coherence between assessment, teaching strategies and intended learning outcomes in an educational programme" (p. 10). It is then reasonable to conceive the notion of coherence as implying that all components in an instructional system, such as the curriculum and its intended outcomes, the teaching methods, the learning activities, the assessment tasks and resources to support learning and teaching activities, are all educationally aligned. However, it must be noted that the ability to align different parts of the curriculum is dependent on teachers' tacit knowledge and mathematical meanings.

Without taking into consideration this essential notion of coherence highlighted in the previous lines, an effective curriculum becomes problematic to interpret, comprehend and implement. Failing to consider the role of coherence will result in increased incompetency and ineffectiveness among teachers in their teaching, who may no longer have the instructional abilities to assist their learners. Some scholars have used the term alignment when referring to the notion of coherence in educational settings (see, Mhlolo, 2012; Mhlolo & Venkat, 2009; Wilson & Bertenthal, 2005). In this regard, coherence or alignment is presumed to be referring to the vertical and horizontal coordination of ideas in the curriculum (for examples, see, Confrey, Gianopulos, McGowan, Shah & Belcher, 2017; Mhlolo, 2012; Mhlolo & Venkat, 2009; Schimdt, Wang & MacKnight, 2005; Wilson & Bertenthal, 2005). For instance, Mhlolo and Venkat (2009) use the terms alignment and coherence interchangeably and go on to explain that, "curriculum alignment has to do with the degree of coherence or match between components of an instructional system" (Mhlolo & Venkat, 2009, p. 34).

Confrey *et al*. (2017) added that "vertical curricular coherence refers to progress across the grades, while horizontal coherence refers to coordination [within] a grade across teachers and schools, often involving governmental agencies as sources of authorized standards" (p. 718). This latter definition of coherence falls in line with Schimdt, Wang and McKnight's (2005) usage of the term alignment, with Schimdt *et al*. (2005) forwarding the claim that, in a school context, coherence refers to "the degree to which various policy instruments available to the system, for example, standards, textbooks, and assessments, accord with each other and with school practice" (p. 527). To illustrate Schimdt *et al*.'s (2005) definition of coherence, the coordination of educational concepts in Figure 2.1 shows a diagrammatic representation of the horizontal and vertical curricular coherence adapted from Webb (1977).

In Figure 2.1, the horizontal layer consists of three components; namely, educational policy, classroom instruction and student or learner achievements, and refers to the packaging of educational policies that are aligned to influence classroom instruction informed by teaching and learning activities. Teaching should be informed largely by educational policies or guidelines, while learning is mostly dependent on teaching. This horizontal alignment is essential to generate a productive teaching and learning system. Within a horizontal layer, a teacher should engage with educational policies meaningfully; that is, they incorporate, embrace and customize policy recommendations to actualize them in their mathematics classroom. The fact that the three components in Figure 2.1 are arranged horizontally may imply that three educational components must float at the same level, so the teacher should align and appropriate educational material (policy) to the level of the child (learner or student). In this way, teaching becomes effective in generating and facilitating meaningful learning.

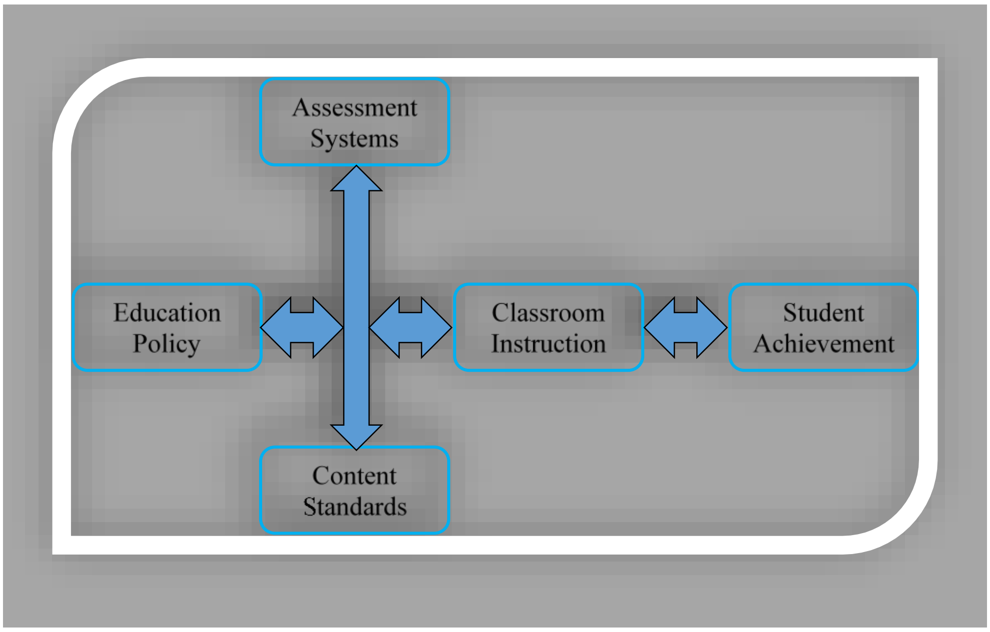


Figure 2. 1: Horizontal and vertical alignment within an education system
Source: *Webb (1997b)*

Concerning vertical alignment, assessment systems are used to evaluate accumulated knowledge, and to progress learners from one grade level to the next. The horizontal arrangement is a reminder that young learners must grow or advance upward, not only physically but educationally. The upward educational growth is achievable when intended forms of knowledge have been acquired. Assessment tasks that are intended for grade progression should be systematic, productive and must adhere to the attainment of well-pronounced sets of educational and grade-specific educational standards. Each grade level has a unique and appropriate pronouncement of expected assessment standards that are systematically articulated to enhance educational alignment and coherence. Therefore, the curriculum must be presented as a package that is simplified to comprehensible educational units. In Figure 2.1, the teacher is at the central stage, demonstrating and marshalling educational knowledge to learners; and demonstrating knowledge of the educational policy (curriculum) and knowledge of assessment strategies etc.

Therefore, an effective teacher will navigate freely and effectively, together with their learners, across the horizontal and vertical continuum model in Figure 2.1. The adequacy of expert knowledge in teaching is manifested through teacher's ability to keep the educational components in Figure 2.1 connected (aligned) and continuously at play (dynamic), and in tandem (coherent). This study conceptualized the teacher's instructional ability as a manifestation of their desirable *tacit knowledge* in mathematics classrooms and their ability to communicate this form of knowledge to learners effectively (see also, Section 2.2). The study explored this form of teachers' knowledge and the kinds of mathematical meanings that teachers, in turn, attach to these forms of knowledge. The study further explored if there were gaps or not between teachers' tacit knowledge and meanings they tended to manifest in relation to this knowledge. The study examined if there was a connection or a disconnect between teachers' tacit knowledge and the production of resulting meaning-making, instructional dispositions. It was the researcher's view that the outcome of the current study could be essential in the formulation of effective teacher development programs intended to capacitate classroom practitioners with the aim of elevating learner mathematical performance.

2.3.2 Coherence as strengthening connections between topics and sub-topics

In Section 2.3.1, the term *coherence* is unpacked. This term is unpacked in terms of this study as the alignment of components in a broader instructional system. These instructional components are demonstrated in Figure 2.1. Also, the researcher conceptualized the term coherence in relation to its function as a connector of topics and subtopics in the context of mathematical instruction (see, also, Larson, 2016; National Council for Teachers of Mathematics in the United States of America [NCTM], 2014), in which context the connection of mathematical concepts should be foregrounded. NCTM (2016) emphasizes that "connections are clear and receive emphasis from one year to the next, from one concept to another, and from one representation to another" (para. 1). Properly aligned connections may advance the intended coherence in the curriculum. Some authors agree that curricular coherence stimulates academic opportunities prompting learning (for examples, see, Boughey, 2018; Canrinus, Klette & Hammerness, 2017; Meij & Merx, 2018; Mhlolo, 2011). Boughey (2018) concluded that curriculum restructuring (e.g., the re-alignment of a curriculum to advance and strengthen its coherence) offers educational opportunities to enhance learners' mathematical success rates.

Curricular coherence allows for a curriculum package that is systematically aligned or arranged (Reeves & McAuliffe, 2012; Stols, 2013). Reeves and McAuliffe (2012) described the state of curricular coherence as the “extent to which curriculum topics and sub-topics are logically connected as they are introduced and presented to students within lessons, within each grade year and across different grades (p. 10)”. In terms of the aims of this study, teachers who pay special attention to the advancement of curricular coherence (educational or mathematical topics that are purposefully aligned and systematically connected) will enhance optimal mathematical understanding and the effective conveyance and transmission of mathematical meanings that are instructionally destined to learners (see, Section 1.4; see, also, Sections 1.9.1 & 1.9.2). This conception of coherence draws from the notion of curricular coherence, which emphasizes adherence to the underlying structure in a discipline (Schmidt *et al*., 2005).

While advancing the discussions on coherence and connections, the concept of *chunking* has been explored (Piaget, 1968). This subject is relevant to this discussion, as it talks about the breaking down of big ideas to meaningful and comprehensible units of knowledge. When big ideas are broken down, curriculum coherence should be preserved. Chunking is a conceptual construct that is fundamentally supported by Piaget (1968). While preparing their lessons, teachers with strong tacit knowledge are presumably able to break down the bigger whole into teachable units that are related (connected or coherent). Teachers require instructional support to break down the curriculum into teachable units that recognize learners’ experience and context. In this way, teaching becomes meaningful, as it acknowledges learner knowledge.

Teacher’s ability to utilize effective instructional skills that facilitate the presentation of educational material coherently will motivate learners to embrace the learning process meaningfully (Section 2.2). An effective teacher will recognize that learners bring into the classroom forms of knowledge that may be aligned to novel educational information. The new educational information could be ordered hierarchically in recognition of how learners acquire new mathematical concepts (see, Figure 2.1). This instructional method recognizes the importance of all stages of the teaching process. This style of question-posing, problem setting, and the utility of educational resources etc. should all consider various stages of learning, which cause learners to move from the known to the unknown and from simple to complex subject matter (Piaget, 1968). This style of teaching pays attention to the coherence and connectivity of educational concepts, which factors when they are addressed accordingly, help learners to embrace mathematical meaning properly.

Coherence, as the extent of the connection between topics and sub-topics, considers the learning of mathematical concepts from prior knowledge to more complex knowledge. By linking mathematical topics within and among mathematical domains (themes), mathematics takes shape as a unified discipline, rather than a mere collection of abstract topics. Coherence in the curriculum synchronizes mathematical content that is taught in schools; and, as learners progress from one grade to the next and bridge gaps in their knowledge, their mathematical meaning-making is also developed. This means that if learners recognize the importance of coherence in mathematical instruction, they, in turn, realize that mathematical knowledge is presented within a system that advocates connection and relatedness, which in turn suggests that desirable learning and the understanding of mathematics pave the way for a holistic educational experience.

2.3.3 Coherence as the fit of meanings and ideas developed within an educational system

Moore (2010) lamented the dearth of shared images to model the phenomenon of coherence. According to Moore (2010), a description of coherence must include more than a progressive list of topics and definitions. While identifying a list of topics is part of developing a coherent mathematical experience for learners, coherence as a process might be conceived largely as the development of meanings for each topic and the construction of contextual inter-relationships among them (Thompson, 2008, p. 47). The NCTM (2014) concurs that coherence in a curriculum should connect disparate mathematical concepts to consolidate learners' mathematical understanding and knowledge so that their application skills of mathematical concepts can be developed.

Thompson (2008, p. 32) asserted that the subject of coherence permeates most discussions in educational settings, with the coherence of a curriculum depending on the interplay of its constituent meanings. Thompson (2008) used the example of ratio, rate and proportion given by Schmidt, Houang and Cogan (2002) to highlight the development of meanings for each and the construction of contextual inter-relationships among them. When the focus is on coherence, the emphasis will not just be on doing and learning "mathematics", but rather on developing a scheme of understanding within which a variety of mathematical ideas are connected. The latter can serve as a conceptual structure for mathematics curricula and instruction (Silverman & Clay, 2009). To clarify an operational meaning of coherence, Thompson draws from Dewey (1910, cited in Thompson, 2013, p. 60):

> … coherence is a characteristic outcome of thinking - thinking leads to 'the organization of facts and conditions which, just as they stand, are isolated, fragmentary, and discrepant, the[ir] organization being effected through the introduction of connecting links, or middle terms'… (p. 79)

Paying attention to coherent mathematical ideas helps teachers to orient their mathematics instruction to the level of learners' conceptual understanding. This may be the case in the teaching of trigonometry, and particularly in Grade 10, which is a prominent subject of current research. In this way, learners' mathematical thinking is enhanced, and mathematical reasoning practices are developed so that the instruction of trigonometry can become a meaningful experience for students. For example, how should mathematics teachers look and respond to the tasks they assign their students? It's clear that teachers' feedback should help learners to realize the significance of coherent thinking in the presentation of mathematical ideas. Eggen and Kauchak (2014) have acknowledged that learners who receive feedback and assistance from their teachers or peers improve their performance in mathematics and are motivated to finish their tasks to a greater extent than those who do not receive the feedback. Therefore, teachers' feedback should present mathematical ideas simply, clearly and systematically (coherently) to help learners generate comprehensive knowledge of the subject matter.

Opportunities to provide scaffolding may be generated in educational settings where teachers share their mathematical tools with learners. Teachers may put together assignments that require learners to solve mathematics problems, so that that the misconceptions and errors of students with their origins in poor or disconnected instruction can be identified. Teachers should seize opportunities to re-align learners' mathematical dialogue and assist learners to develop a coherent and connected system of mathematical ideas. In a mathematics classroom, a learner may not be able to use a scientific calculator or apply a mathematics formula to compute trigonometric ratios. To assist the teacher, learners may use previously constructed mathematical pedagogies to help them to locate their preferred zone of proximal development

(ZPD) (see, Section 2.5). This process can be facilitated by teachers giving grade-appropriate examples and using appropriate illustrations and mnemonics (Vygotsky, 1978).

Vygotsky (1978) defined ZPD as the distance which separates what a given learner might achieve in their initial learning experience without any additional or special guidance and support from an instructor, and the level a learner could potentially reach in their education with a teacher's guidance and support (see, also, Section 2.5). The oscillation between these two theoretical stages in the learning process (i.e. actual development and potential development) may require young learners to seek out adult or peer guidance and assistance, which process is commonly known as scaffolding (Fani & Ghaemi, 2019; Sarker, 2019; Siyepu, 2013; Vygotsky, 1978). In scaffolding, a learner can improve their problem solving through shedding external or outside guidance and assistance (for examples, see, Brower, Woods, Jones, Park, Hu, Tandberg, Nix, Rahming & Martindale, 2018; Widjajanti, Nusantara, As'ari, Irawati, Haris, Akbar & Lusbiantoro, 2019).

The feedback provided by teachers to learners' mathematical problem-solving solutions constitutes yet another type of scaffolding. Learners who are constantly exposed to constructive feedback (appropriate ZPD and scaffolding) will provide correct mathematical misconceptions (to orient their mathematical concepts appropriately in the effort to achieve a state of mathematical coherence) and will be motivated to construct appropriate and desirable mathematical meanings (see, also, Siyepu, 2013). According to Siyepu (2013), such learners may exist in the first and second categories of Vygotsky's ZPD.

2.4 MEANINGS IN MATHEMATICS EDUCATION

The Merriam-Webster (n.d.) online dictionary was used in this study to provide the definitions of the term '*mean*' or '*meaning*', which form the primary variables that this study investigated (Sections 1.4 & 1.9.2). There are other variations to the definition of the term *meaning* offered by other dictionaries, but all have similar or almost the same understanding. Definitions 1(a) and 2 in Figure 2.2 explain 'mean' as the intention to express something one bears in mind, and mostly to signify what one is planning to convey. This explanation refers to the way in which information is embraced and incorporated into one's system of understanding and how this information is later conveyed to others. For example, if teachers have the ability to assign tasks and set targets, learners develop the motivation to be the source of information and share this information with peers. In this respect, peer-to-peer interaction may be critical, as learners might understand the language of their peers more thoroughly than the language of their teachers (Vygosky, 1978).

Adopting the colloquial and informal language of a learner's culture or community can likewise promote the assimilation and accommodation of mathematical concepts (Piaget, 1968). A teacher with correct mathematical meanings creates opportunities for learners to express their own understanding and apply this knowledge to their personal, individual life circumstances. If a gap in a learner's knowledge is identified, teachers are advised to modify their instruction or make use of teaching aids such as pictures or make demonstrations to simplify the concepts. Without a sound collection of mathematical meanings, a teacher may not be able to provide meaningful help to learners. The example in Figure 2.3 helps to clarify this assertion, supposing that the problem in Figure 2.3 was given to mathematics learners in Grade 10 at another school.

Definition of MEAN

meant \ˈment\; **meaning** \ˈmē-niŋ\

transitive verb

1 **a** : to have in the mind as a purpose : INTEND • she *means* to win —sometimes used interjectionally with *I*, chiefly in informal speech for emphasis • he throws, I *mean*, hard or to introduce a phrase restating the point of a preceding phrase

• we try to answer what we can, but I *mean* we're not God —Bobbie Ann Mason

b : to design for or destine to a specified purpose or future • I was *meant* to teach

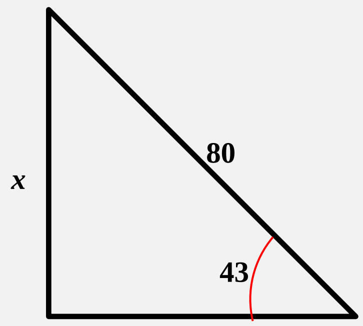
2 : to serve or intend to convey, show, or indicate : SIGNIFY • a red sky *means* rain

3 : to have importance to the degree of • health *means* everything

4 : to direct to a particular individual • His criticism was *meant* for all of us.

Figure 2. 2: The definition of the term 'meaning'
Source: *https://www.merriam-webster.com/dictionary/mean*

The teacher will make use of mathematical meanings to anticipate learners' problem-solving thoughts, as depicted in Figure 2.3. Subsequently, the teacher will determine what instructional action(s) is most appropriate to assist the learner's work in Figure 2.3. Through adopting a well-developed system of mathematical coherence, teachers are better able to identify gaps or mathematical disconnections that might have contributed to the learner in question's error when attempting to solve the problem in Figure 2.3. Universal systems of well-coordinated mathematical knowledge and ideas, coherently developed, serve as guiding tools for teachers that identify gaps and shortcomings in learner's problem-solving attempts. It is at this stage in the process that a teacher will be expected to regulate a meaning-making mechanism that guides them to make correct and meaningful interventions to assist.



What is *x*?

If the solution given by one of the learners was:

$$180 - 43 = 137$$
$$137 - 80 = 57$$
$$x = 57$$

Figure 2. 3: Example of a Grade 10 learner's work in trigonometry

While the learners' misunderstanding in Figure 2.3 could derive from the teacher's instruction, it is clear that the teacher can easily modify their instruction to suit the learning needs of their student. In this context, many scholars ascribe different denotations to the term *meaning* (see, Radford, 2008; Kaiser, 2008; Skovsmose, 2005; Suriakumaran, Duchhardt & Vollstedt, 2016; Thompson, 2013). Kaiser (2008) distinguishes between personal meaning and objective meaning to describe a continuum of viewpoints relating to the term 'mean'. According to Kaiser (2008), personal meaning is subjective, individualistic and private, implying that each person derives their specific meaning regarding a phenomenon, and attributes meaning on the basis of their personal understanding and experience. Given the fact that individuals often filter their teacher's instruction through their own personal interpretive framework and past experiences, teachers should be vigilant about demonstrating their knowledge and comprehension of their subject matter clearly and directly to their students.

Different kinds of meanings are personal and based on individual experience, relevance and context (Suriakumaran, Duchhardt & Vollstedt, 2016). Hence, a teacher must accommodate the possibility of shared and divergent interpretations of subject matter through striving to listen to learners' explanations. The teacher may use learners' interpretations to identify deficiencies in their pedagogy which inadvertently produce misconceptions and misunderstandings. In this study, the researcher interacted with teachers to explore their meanings of some trigonometric concepts (Section 3.6). For example, the researcher studied teachers' meanings of $\sin\theta$ and $\cos\theta$.

According to Kaiser (2008), the objective meaning is established via the interaction with the context, or teachers, parents or society without excluding learners' personal experiences. People acquire objective meaning through social interactions (Vygotsky, 1978). Objective meanings are socially negotiated and collectively adopted. Objective meaning highlights the importance of group work and collaboration. Kaiser (2008) agrees that learners' meaning "depends on the one hand on offers of meaning given by teachers, parents, and society as well as, on the other hand, on the students' personal experiences, abilities, dispositions, wishes and intentions" (p. 2). Skovesmose (2005, p. 85) argued that any theory of meaning ought to be considered in a broader context; that is, as a higher-level educational concept rather than a context- or domain-specific notion.

Furthermore, Skovesmose (2005) emphasized that "concepts are not delivered, they are constructed. The meaning of a concept can, therefore, be associated with what a person can do by means of the concept" (p. 85). For example, Skovesmose implies that, in order to understand the meaning of *sine of an angle* θ ($\sin\theta$), one must understand the meaning of an angle measure first (see, task 1.2 in Section 4.4.1.1.2). Radford, Schubring, and Seeger (2011) conceived the term *mean* in mathematics education as a semiotic concept in which teaching and learning "are considered as part[s] of [the] same process, connected by interrelated processes of signifying and meaning-making - that is to say, processes of communication and understanding" (p. 149). The authors further explain that:

> … meaning can come into presence through signs only - be they pointing gestures, utterances, texts, or succinct mathematical formulas. Meaning, that which we intend to express or communicate (for others and perhaps for ourselves too), has to be sign-ified. (p. 150)

Radford *et al*. (2011) pointed out the significance of signs for mathematics education because mathematics is a symbolic activity that teachers and learners facilitate through writing, talking, gesturing, and using other pointers. According to Thompson (2013), the notion of *meaning* is an intellectual attribute that is particular to individuals, whereas the practice of reflection and abstraction are widespread. Furthermore, meaning can be conceived as "what people intend to convey via an utterance, and what people imagine [is] being conveyed as they hear an utterance" (Thompson, 2013, p. 58). The researcher has observed that different explanations of the term *meaning* collectively render it an intention to convey an understanding residing in an individual's mind. Thompson (2013) conceded that "meaning resides in the minds of the person producing it and the person interpreting it" (p. 59). Thompson continues:

> Dewey also considered thinking to be the primary mechanism for the construction and refinement of meaning. That thinking both employs and expands notions, conceptions, is then simply saying that in inference and judgment we use meanings, and that this use also corrects and widens them. (Dewey, 1910, p. 125, cited in Thompson, 2013, p. 60)

The set of triangles used daily by learners holds special angles. If the classroom dialogue on mathematical meanings is limited, the teacher may not use mathematical tools in learners' possessions to explain special angles (Figure 2.4). Figure 2.4 displays a set of triangles, and Figure 2.5 illustrates how teachers might explain some triangle concepts using these set of triangles. The researcher agrees with Thompson (2015, 2014, 2013, 2008, 2007) that the mathematical knowledge that teachers value most resides in the realm of mathematical meanings; which is to say, the related conceptualized packages of mathematical images that they teach and intend their learners to embrace. The meanings that teachers possibly convey to learners form the basis of what eventually constitutes the meanings that learners formulate.

This formulation occurs in the process of learners striving to embrace and comprehend what the teacher communicates to them. In this context, meaningful understanding (comprehension) of newly learnt (communicated) mathematical concepts is contingent upon the coherence of essential mathematical ideas. It must be noted that ideas entail meanings; that is, when ideas are probed further, they tend to reveal meanings. Useful mathematical meanings influence the conceptual understanding of mathematical concepts (Thompson, 2008). It is essential for teachers and learners to build important and effective mathematical meanings in their endeavor to learn mathematics effectively (see, Section 2.6).

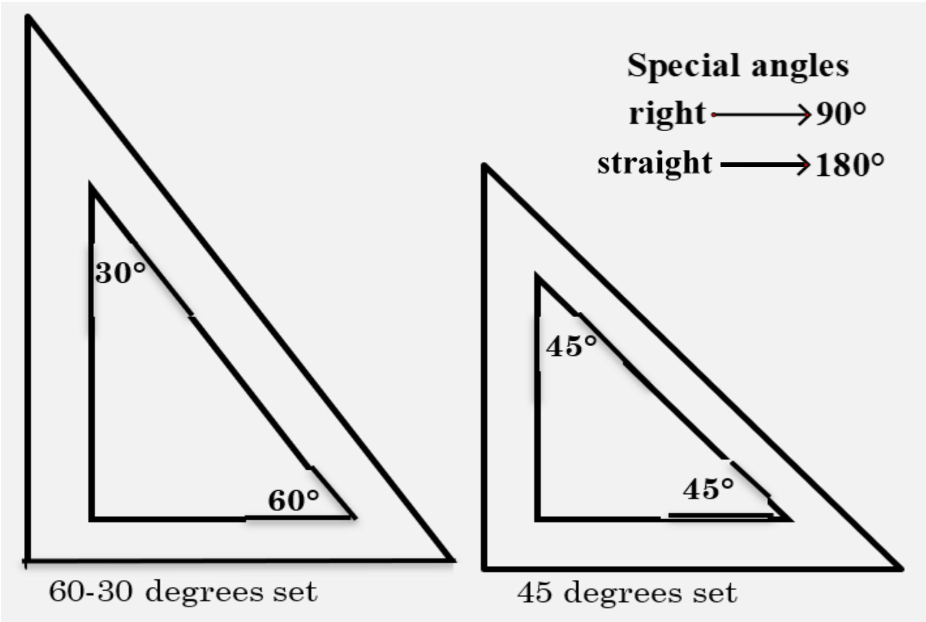


Figure 2. 4: The set of triangles that are usually in learners' possession

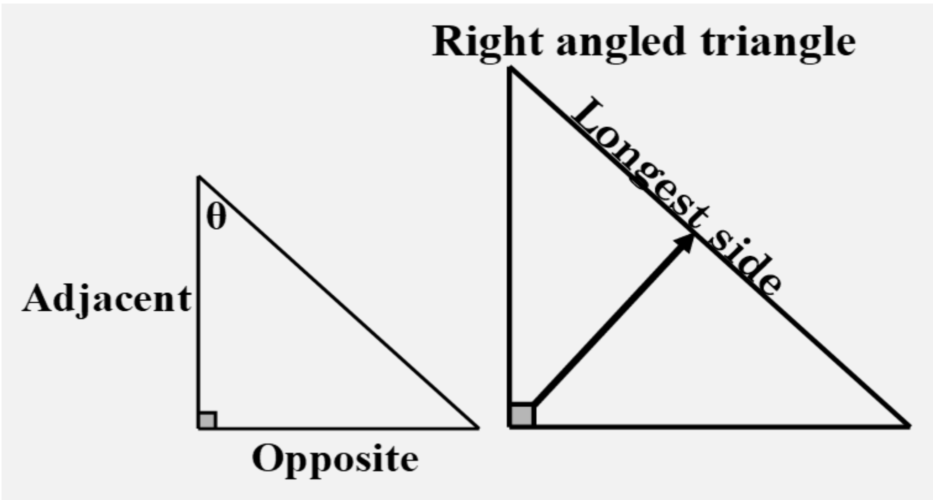


Figure 2. 5: Using the set of triangles to explain some concepts in a triangle

2.5 CONSTRUCTIVISM

A constructivist perspective on learning provides the foundation for this study, which allows for the understanding that humans are active participants in the creation of their knowledge; and, as such, one can regard all individuals, including teachers and students alike, as learners. The following passage provides a general description of constructivism drawn from Confrey (1990):

> Put into simple terms, constructivism can be described as essentially a theory about the limits of human knowledge, a belief that all knowledge is necessarily a product of our own cognitive acts. We can have no direct or unmediated knowledge of any external or objective reality. We construct our understanding through our experiences, and the character of our experience is influenced profoundly by our cognitive lenses". (p. 108)

Confrey (1990) seems to mean that constructivism advocates that new knowledge and meanings are created through the process of assimilation and accommodation. Assimilation occurs when an individual connects existing knowledge and personal experiences (Ernest, 2010). In the context of knowledge construction, assimilation denotes the utilization of an existing schema that gives meaning to one's experiences. Accommodation processes, on the other hand, occur when the assimilation process meets a challenge or is not successful. In other words, accommodation is the process of adjusting ideas that do not fit into existing schemas. Hence, prior knowledge is fundamental in the process of assimilation and accommodation. Put differently, constructivism is the theory that furnishes the claim that learners are active participants in the learning process (Ernest, 2010).

Constructivist theory negates the traditional view that conceives the mind of a child as a *tabula rasa*: a kind of empty vessel that teachers need to fill. Therefore, learners come to their classrooms carrying with them previously derived forms of knowledge, with the research indicating most of this knowledge deriving from the internet (see, Loan, 2012; Nickolas, 2013; Ngugi & Mberia, 2014). Thus, it's evident that there are many ways for learners to acquire knowledge beyond the formal context of classroom instructions, and teachers can take advantage of learners' pre-existing knowledge by placing them in groups of learning. In small-sized groups, learners collaborate and share information using, amongst other informative tools, technology. This view is in line with the social cognitive perspective in support of social interaction as a means of facilitating the production of knowledge (Nickolas, 2013).

Constructivists believe that learners do not acquire mathematical knowledge by only listening to teachers or reading textbooks. Learners construct mathematical knowledge by actively participating in the learning process. Therefore, teaching should support knowledge construction instead of merely communicating it (Hendry, 1996). In this way, teachers' meanings may precipitate mental disequilibrium in learner's mental reasoning and stimulate their quest to inquire more information. In group settings, learners share knowledge through accommodation (Vygotsky, 1978). To facilitate the imparting of mathematical meaning, astute teachers will adopt a discovery approach, opting to 'co-learn' with the learners, checking and examining their learners' mathematical reasoning in the process.

Social constructivism is predominantly based on Vygotsky's (1978) theoretical perspectives on learning. This theory prioritizes the social aspect of human interactions and recognizes that human intellectual development and social interactions are inter-connected. Knowledge is individually constructed but facilitates social connections between human beings in social interactions (Tobin, 1993). Vygotsky's (1978) theoretical perspective claims that interaction, scaffolding, mediation, feedback and apprenticeship are essential components of learning and knowledge acquisition. The teacher may not be the only one to possess mathematical meaning but should be familiar with various ways of learning to motivate and engage learners with different learning styles.

Vygotsky (1978) identified people's experience, language, objects, society and culture as essential components of knowledge construction in learning interfaces. Connections can be facilitated speedily when teachers identify the vocabulary that is central and uses language that is taken for granted to communicate important meanings. This implies that, as teachers become more confident, they strengthen and sharpen their tacit knowledge of their discipline (Davis, 2011). This end goal is achieved when learners come to observe what their mentors do and how these experiences are transmitted to them in words, pictures and meanings. For instance, matching games may allow learners to draw mental connections between angles and associated words, as in Figure 2.6, in which words are shown matching with corresponding geometric diagrams.

Examples that are used to teach trigonometry should be drawn from learners' experiences and should show relevance and sensitivity to learners' real world. This balance may be difficult to achieve if the teacher's mathematical meanings are largely detached from learners' immediate reality. Therefore, a teacher without a strong background in mathematical meanings may fail to advance mathematical connections with learners if they become too detached from their learners' reality. This concept lies at the heart of our study's exploration of teachers' mathematical meanings and how they influence the delivery of mathematical content knowledge (see, Section 1.6).

Types

- Acute
- Obtuse
- Reflex

Figure 2. 6: The names of special angles

There are myriad contextual variables influencing teachers' performance in the classroom, one of which is the environment in which the learning transpires (Vygotsky, 1978). The environment has been acknowledged as the primary contributor in children's intellectual development (Van de Walle, 2016), with intellectual development occurring through *internalization*, which is "the internal reconstruction of an external operation" (Thompson, 2014, p. 3).

2.5.1 The Zone of Proximal Development and its relevance to the current study

Vygotsky also introduced the notion of the Zone of Proximal Development (ZPD). According to Van de Walle (2016), "ZPD refers to a range of knowledge that may be out of reach for a person to learn on his or her own, but is accessible if the learner has support from peers or more knowledgeable others" (p. 51) (see, also, Section 2.3.3). ZPD is the difference between the level of a student's actual development and the more advanced level of potential development to which they tend, which reveals itself in the interactions between more and less accomplished participants. ZPD constitutes an essential component in children's intellectual development.

Figure 2.7 is a simple context-based mathematical diagram depicting an anti-clockwise turning fan. The idea that the diagram represents a scenario of a fan that is familiar to learners exemplifies the scaffolding effect; such that the diagram in Figure 2.7 introduces learners to special angles. Some teachers may construct an abstract mathematics scenario that obstructs learners' conceptualization of embedded mathematical knowledge and meanings, but teachers may use a familiar real-world scenario to make the seemingly abstract mathematical content in Figure 2.7 comprehensible nonetheless. The real-world scenario of a Chinese hand fan provides a useful scaffolding effect to help learners to relate (connect) mathematical concepts and formulate their own mathematical meanings with the mathematical content in Figure 2.7. Teachers must be tactful in their effort to convey mathematical meanings to learners who are naturally placed at different levels of ZPD.

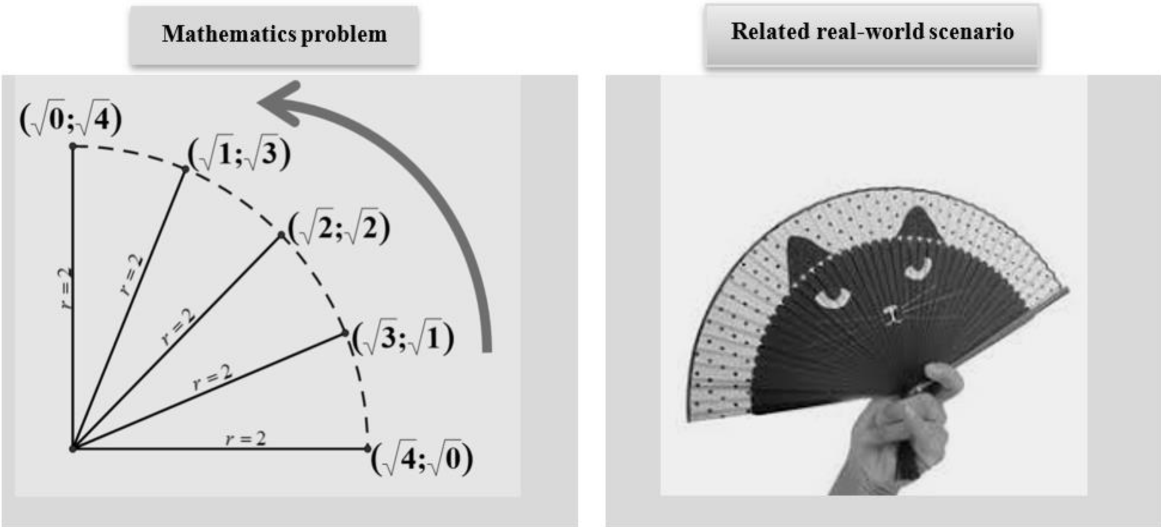


Figure 2. 7: Fan diagram introducing special angles and a related real-world scenario (scaffolding)

2.5.2 Three important stages of ZPD and their implication to the current study

According to Siyepu (2013, p. 5), in terms of ZPD, mathematics learners may be classified in three categories: namely, (1) those that can perform mathematics tasks independently; (2) those that can perform mathematics tasks with assistance; and, (3) those that may not succeed in performing mathematics task even after receiving assistance and guidance. The third group is beyond the ZDP and is completely and persistently unable to present a coherent mathematics dialogue (Siyepu, 2013).

In the current study, Siyepu's (2013) three categories of ZPD may be explained in terms of mathematical coherence or connectedness (Sections 2.3 & 1.9.1) and mathematical meaning making (Sections 2.3 & 1.9.2). Learners in the third category of ZPD struggle continuously to formulate a coherent mathematics dialogue, because they also fail to consolidate a basis from which to pursue their mathematical meaning-making. A learner in the first category of ZPD will need minimal assistance to optimize a sound formulation pertaining to the mathematical meaning and will then succeed in presenting a well-connected and coherent mathematics discourse. The learner in the second category of ZPD may need an extended session of scaffolding to optimize their level of mathematical meaning and subsequently be able to hold a coherent mathematical discourse.

2.5.3 Approaching mathematics as a socially negotiated enterprise

The constructivist approach posits that learning is an active constructive process, with the construction of knowledge being both individual and social. Individual and social views complement each other during teaching and learning. For example, learners may perform badly in a mathematics task not because they do not know related procedures, but because they do not have hands-on experience with calculators. Proper instructions and practical activities drawing from teachers' mathematical meaning become the necessary point of departure.

Explicit demonstrations accompanied by learners' familiarity with their mathematical tools (calculator) may speedily enhance proficiency and fluency, as in Figure 2.8. The teacher may have to illustrate and demonstrate their lessons and request students to solve specific problems to enhance their mastery of the subject matter through practice. The illustrations in Figure 2.8 demonstrate skills that are transferable to learners by an expert teacher who may also be eager to access learners' mathematical meanings related to the task at hand.

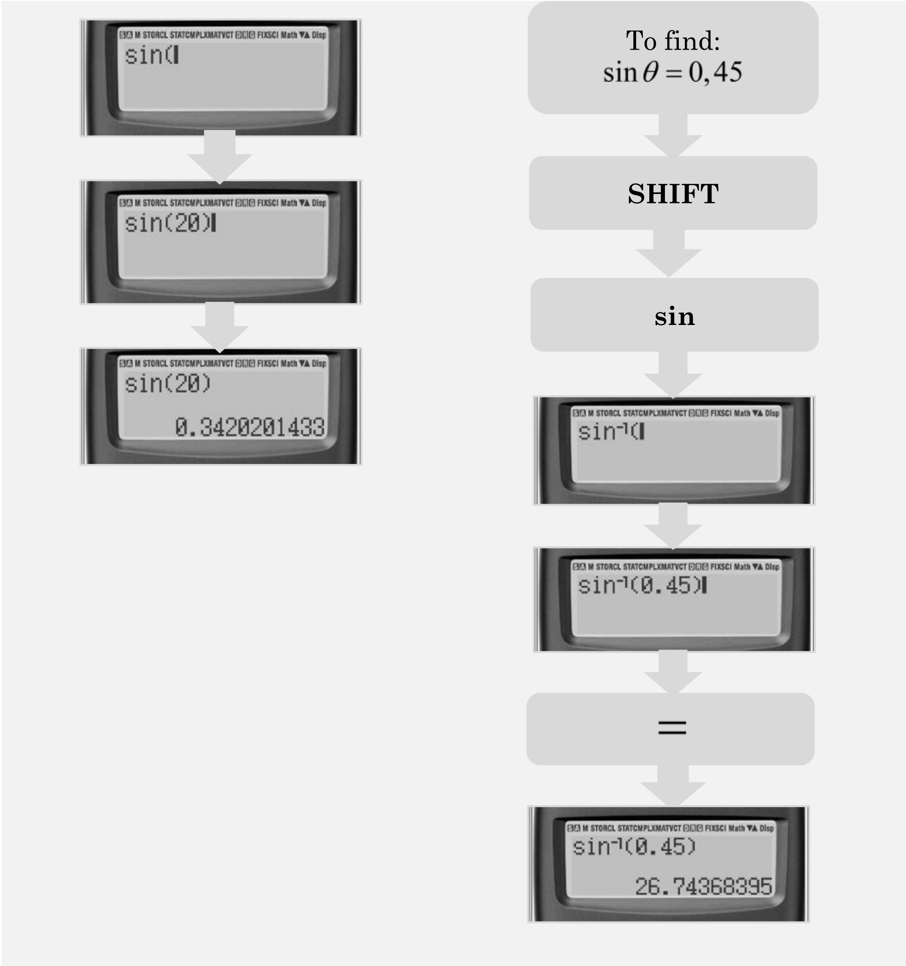


Figure 2. 8: The usage of a calculator that may provide mathematical exposure

Step 1: Use the calculator to find the value of different angles

Step2: Convert the given values into angles

Press second function 3

Figure 2. 9: A sample of a teacher's anticipated actions in relation to the task in Figure 2.8

The teacher is combining ideas of using a calculator and the identification of angles, and how these tools relate to the practice of angle measurement.

2.5.4 Tacit knowledge and constructivism

In Section 2.2, the subject of teacher knowledge was briefly discussed, in which an individual can acquire tacit knowledge through observing, imitating and practising what they see. In a school setting, a teacher may gain additional knowledge regarding trigonometry acquired through observing their coworkers or teachers who taught them. Piaget (1977) calls this process assimilation. According to Piaget (1977), assimilation accounts for the construction of meaning and knowledge understanding. Piaget's genetic epistemology tackles, to some extent, the learning of scientific concepts and does not address learning per se (see, Thompson, 2000; see, also, (see, Section 2.10). Through active participation, learning is rendered meaningful. In this case, learners *assimilate* and *accommodate*[5] new knowledge into a network of pre-existing cognitive structures. Thompson, Carlson and Silverman (2007) highlight that "to introduce coherence into one's meanings, necessarily requires a learner to reflect on the meanings she holds and to adjust them so that they are compatible in overlapping domains" (p. 416). The processes of assimilation and accommodation enable the learner to fit meanings into a mental schema. A mental schema is a structured pattern of thought that categorizes information and relationships among categories (Thompson *et al.*, 2007). For instance, a schema can be constructed through a process of naming the lines, angles and types of triangles. The task of applying names to the opposite, hypotenuse and adjacent sides may be based on work previously done. Figure 2.10 depicts three sides of a right-angled triangle XYZ and their related interior angles.

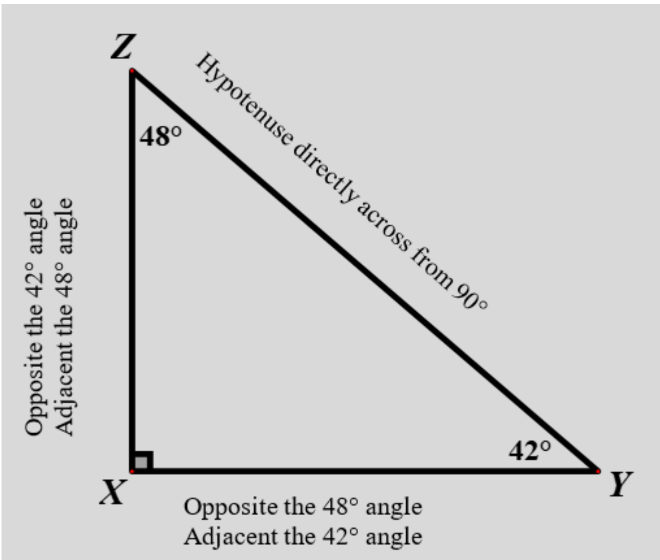


Figure 2. 10: The naming of the three sides of a right-angled triangle and the summary of interior angles

5. *Assimilate* and *accommodate* are used here in a Piagetian sense.

Mathematical terminology may be developed using Figure 2.10. For instance, the mathematical fact may be emphasized that for any right-angled triangle, the hypotenuse is the side that is directly across from the 90° angle. Subsequently, such a mathematical discourse in the mathematics classroom may initiate a meaning-making process. A teacher who allows learner participation can acquire new knowledge too, and teachers' tacit knowledge can be acquired through mentoring and coaching, and, in the process, may involve either leaners or peers, or both. The social constructivist theory emphasizes that people learn best when they share experiences (Vygotsky, 1978).

2.5.5 Mathematical knowledge and teaching

Teachers may lack mathematical content knowledge due to not enjoying exposure to good teachers (Ndlovu, Amin & Samuel, 2017; Odumosu, Olisama & Ogunsanya, 2018). Recently there has been a need to spur teachers' mathematical knowledge using technology (Nisson & Karlsson, 2019). Nisson and Karlsson (2019) argued that in recent times, technology has permeated various avenues of mathematics education, including classroom settings. Hence, there is a nascent need to provide avenues through which mathematics teachers can gain new knowledge of technology to ameliorate their mathematics instruction (Nisson & Karlsson, 2019; Umugiraneza, Bansilal & North, 2018).

Umugiraneza *et al*. (2018) found that most mathematics teachers do not use technology to enhance their teaching, and to contribute to learners' mathematical achievements. Today's children, rather, demonstrate the ability to acquire new educational knowledge when it is mediated through technology. Therefore, teachers' mathematics knowledge will be judged to be more learner-friendly in instances in which it incorporates elements of technology (see, Kim, Bae, Choi, Kim & Lim, 2019). Social contexts may facilitate teacher's tacit knowledge and mathematical meanings (Eggen & Kauchaki, 2014), and, when teaching and helping learners in classrooms to make better sense of trigonometric topics, teachers are well advised to make use of clues, such as those indicated in Figure 2.11.

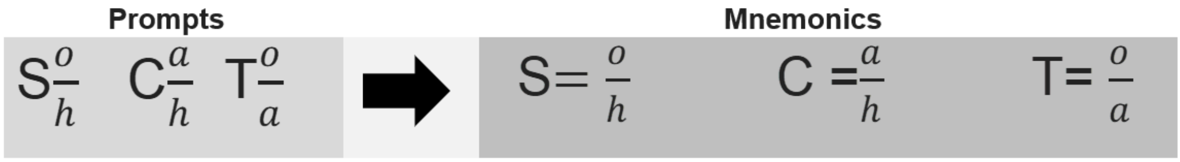


Figure 2. 11: Prompts and mnemonics which may be used to construct mathematical meanings

Teachers may use prompts and mnemonics to facilitate knowledge acquisition and meaning-making in trigonometry. The teacher may start by introducing prompts to learners before utilizing mnemonics, which are shown in an equation format in Figure 2.11. Procedural knowledge follows from the use of mnemonics, as shown in Figure 2.11, allowing educators to teach their learners the trigonometrical ratios effectively. In Figure 2.11, learners may use mnemonics to remember the definitions of the three basic trigonometric ratios, namely, $\sin\theta = \frac{\boldsymbol{o}pposite}{\boldsymbol{h}ypotenuse}$; $\cos\theta = \frac{\boldsymbol{a}djacent}{\boldsymbol{h}ypotenuse}$. Teaching under this paradigm can help learners to draw mathematical connections and derive important topic-specific meanings.

2.5.6 Mathematical meanings and constructivism

According to Thompson (2013), mathematical knowledge consists of three main components: (1) teacher's knowledge of their subject content; (2) teacher's knowledge of how to put together their classroom lessons (i.e., knowledge of teaching); and, (3) teacher's knowledge of how learning should take place to catalyze student learning (i.e., knowledge of learners). The last component is important, in so far as teachers with mathematical knowledge are advised to teach learners to enhance their learning experience (i.e., meaningful learning) (Thompson, 2013). Such a teacher is said to be an effective teacher, understood as an educator who supports, in this context, learners' mathematical reasoning (Brodie, 2013), guiding them properly to achieve new heights in mathematical meaning making. Piaget (1968) presented knowledge as a dynamic phenomenon, which is adaptive and an organized function of an organism. According to Piaget, knowledge is made up of schemas or mental slots that are interconnected (Piaget 1968), and the way in which mathematical meanings find residence in teacher's schema of knowledge will determine the strength with which the effectiveness of their pedagogy can be assessed.

The phrase '*mathematical meanings finding residence in schema*' in the preceding line refers to the strength of conceptual connectedness (relatedness) in one's cognitive schema, which is presumably the function of one's ability to make sense of incoming knowledge and eventually gain an understanding of a particular domain of knowledge (meaning making). In this respect, mathematical concepts that are loosely connected perturb efforts on the part of students to formulate mathematical sense and meaning. Strongly connected concepts convey a solid and rigid message; therefore, teaching ought to be organized in a way to vigorously penetrate related existing pockets of the schema to take up solid residence. A strong network of conceptual connectivity is a sign of deep understanding on the part of learners and a solid meaning-making event. How is the ability of a learner to draw a deeper understanding of a concept dependent on the efficacy of their instructor's lesson CK of trigonometry? A teacher with appropriate mathematical meanings is advised to consider the following advice to gauge the efficacy of their instruction for learners:

- A teacher will select the correct teaching and learning resources for the lesson. In this way a lesson is appropriated at the level of learners (knowledge of learners); and,
- A teacher will choose suitable real-world resources to enhance classroom instruction (see, Figure 2.7). The real-world scenario may provide the needed scaffolding to facilitate instruction. Reyes, Insorio, Ingreso, Hilario and Gutierrez (2019) referred to this teaching approach as the contextualization of mathematics. In this approach, learners are engaged to relate their real-world situations to mathematics (Reyes *et al*., 2019).

2.5.7 Teachers mathematical meanings and curriculum expectations

The curriculum in South Africa is based on addressing diversity to allow teachers to pay close attention to the possible misconceptions and errors of their learners. This process begins with learners identifying key terms whose meanings should be clarified in the context of everyday situations, and which are resolved in their understanding of trigonometry. For example, the right-angled triangle can be connected to a ladder put against the wall of a house to help someone climb on the roof of a house. The hypotenuse can be illustrated by attaching a rope to a point to demonstrate how builders mark the walls of a rondavel house or the horizon and rising sun. The teacher can decide on possible guiding questions and scaffolding to necessitate assimilation and accommodation. By engaging learner's reasoning, thinking and gaps in learners' knowledge will be identified in order to support learners appropriately and adequately. If a teacher does not possess this requisite knowledge, it means that their implementation of the curriculum is compromised. This study sought to determine if teachers' meaning can draw from examples that relate to learners' lives and, in the process, show how these meanings pertain to the curriculum itself. The separation between a teacher's interpretation of curricular content and requirements and the curriculum itself necessitates teachers to:

1. Have the ability to communicate their thinking in clear language that can be understood by learners;
2. Transform what they know into reality and practical situations that create meaning for the observer or listener;
3. Identify teaching points and list them clearly;
4. Identify resources to mediate the teaching of a concept;
5. Demonstrate CK clearly and effectively;
6. Create meaning and contextualize important concepts;
7. Allow learners to contribute their ideas and draw connections between curricular content and their personal lives and experiences;
8. Create contexts in which learners feel that new knowledge can be assimilated with past understanding;
9. Provide clear and simplified explanations for learners;
10. Cater to learners who are failing to understand concepts easily;
11. Identify gaps in learners' knowledge and find alternative methods to fill in these gaps;
12. Support learners so that they can come up with productive reasoning;
13. Demonstrate the ability to elicit and support learner teaching;
14. Diagnose the sources of confusion for learners; and,
15. Provide immediate feedback to prevent learners from becoming despondent or falling behind their peers.

2.6 KNOWLEDGE AND UNDERSTANDING

A deep understanding of mathematics is necessary to allow for responsible citizens in an increasingly globalized world. In this section, the researcher discusses the concepts of knowledge and understanding in the context of the current study.

2.6.1 Conceptual understanding

The construction of new ideas and knowledge requires that both teachers and learners be actively involved in the thinking process. Actively thinking in this context implies that teachers need to bridge knowledge gaps that exist, and which precipitate a disconnect between their own CK and classroom instruction. Learners' requirements need to drive teachers to craft instructions which draw connections between students' old and new knowledge, allowing them to reflect on their thinking and apply the most useful tools to understanding new concepts (Hiebert & Carpenter, 1992, p. 67). Questions which promote critical reflection among teachers and students alike are to be promoted to this end.

Teachers should be able to show how the *angle 1* and *angle 2* in Figure 2.12 are related, for instance, and indicate how this information pertains to the sub-field of trigonometry as a whole. The differences between these two angles indicated, and why this knowledge is important should indicate how teachers understand their own thought and knowledge processes. In Figure 2.13, the line nearest angle θ is adjacent, with its opposite angle the furthest line from angle θ, and the slanting ladder the hypotenuse. In this context, the teacher may ask:

- ✓ *What new ideas can I introduce?*
- ✓ *How is the idea of lines connected to angles?*
- ✓ *What resources can I use to support this idea?*
- ✓ *How does the knowledge I have address the gaps demonstrated by learners?*

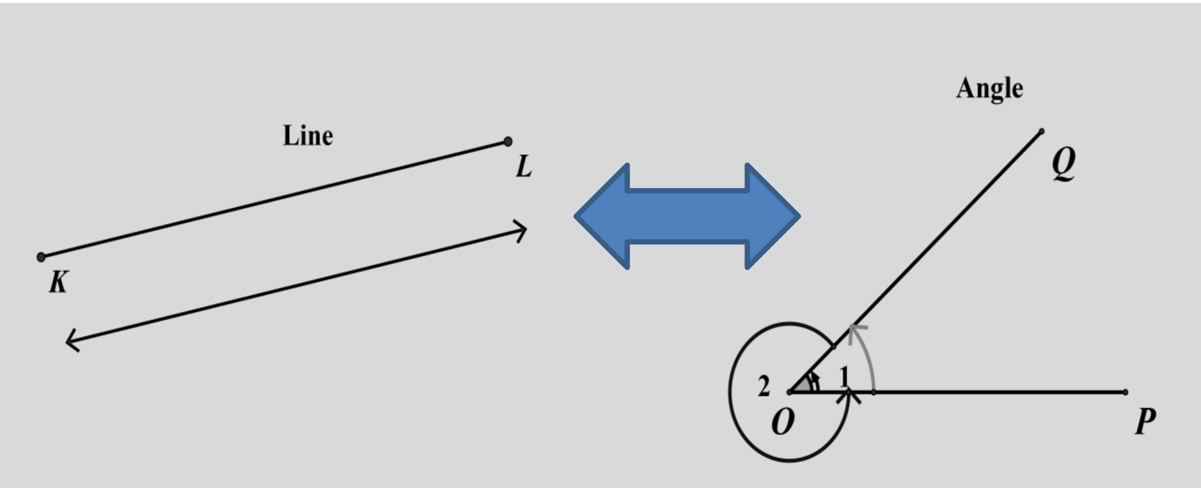


Figure 2. 12: Lines and angles

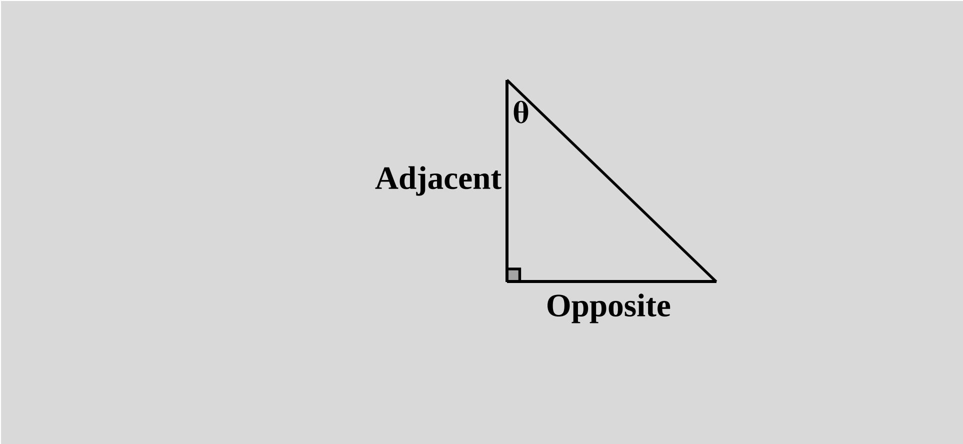


Figure 2. 13: A right-angled triangle

Conceptual understanding is attained through "logical connections constructed internally in the mind of an individual and existing as part of a network of ideas" (Hiebert & Carpenter, 1992, p. 67). In the process of learning, existing connections within the knowledge structures may be consolidated and new relationships constructed, and, consequently, meaningful understandings may be promoted. Hiebert and Carpenter (1992) posited that conceptual knowledge is not stored as an isolated piece of information, but a network of interconnected ideas. When a teacher creates a context to compare and contrast objects and form interrelationships between concepts, they show evidence of conceptual understanding.

2.6.2 Procedural knowledge and the teacher's mathematical meanings

Procedural understanding in mathematics is an understanding of the procedures that teachers use when carrying out computations in the effort to solve mathematical problems, being a set of rules and procedures (see, Rittle-Johnson, Schneider, & Star, 2015; Rittle-Johnson & Schneider, 2014; Van de Walle, 2016). Procedural knowledge is conceived as the use of known algorithms transferable to learners through various modes of presentation. Other researchers have cautioned that the possession of procedural understanding by teachers does not translate into sense-making and understanding (Orton, 1992; Vygotsky, 1967). In other words, teachers may follow a procedure without understanding why that procedure works or why it produces a particular set of results and transmit that knowledge as it is to the learners. Procedural understanding may not be enough even though it plays an important role in understanding significant aspects of mathematics and applications. It means procedural understanding and conceptual understanding are inherently connected, and therefore, must coexist in learners' minds.

2.6.3 Instrumental and relational knowledge and mathematical meanings

Skemp (1976) and Van de Walle (2016) seemingly agree that relational knowledge is a network of ideas connected coherently and that explanations concerning why these rules work, while vitally important in their own right, merely constitute knowledge based on rules that do not demonstrate an understanding of their own significance. Kilpatrick, Swafford and Findell (2001, p. 116) unveiled the basic definition of knowledge, with authors describing a five-strand proficiency explanation directly related to mathematics meanings. Meaning, mathematical concepts operations, and relations build up conceptual knowledge, such that carrying out procedures flexibly, accurately, efficiently and appropriately constitutes procedural knowledge. In this context, strategic knowledge encompasses the ability to formulate, represent and solve mathematics problems, while adaptive reasoning denotes the capacity of a learner for logical thought, reflection, explanation and justification. This last type of knowledge carries the implicit assumption that mathematics is inherently sensible and useful. The five strands of this so-called "proficiency explanation" are interwoven and interdependent, and comprise various mathematical meanings (NCTM, 2000; Skemp, 1976; Stylianides & Stylianides, 2007).

2.6.4 Declarative knowledge

Declarative knowledge refers to knowledge about things that anyone in a field can have access to, and which is usually expressed orally (Biggs & Tang, 2011). This knowledge is the equivalent of content knowledge, and is verifiable, replicable and logically consistent; is found in libraries, textbooks and on the internet; and refers to what teachers 'declare' in lectures. The limitation of declarative knowledge as a form of meaning is that mathematical reasoning that is independent does not support it. Most teachers who rely on this particular form of meaning become rigid in their instructional approach, which negatively impacts their ability to continue to develop mathematical reasoning and fluency in their work in the classroom. To enhance their mathematical meanings and perfect their practice, teachers need to participate in discussions that force them to apply what they know to solving tangible problems.

2.6.5 Functioning knowledge and mathematical meanings

Functioning knowledge is the "knowledge that informs action, where understanding underpins the performance. The teacher does not only receive pre-existing knowledge but is actively involved in putting knowledge to work" (Biggs & Tang, 2011, p. 82). Functioning knowledge depends on a deep understanding of theory and its application, and requires teachers to to be firmly grounded in theory and field-specific concepts in order to become fully competent instructors. Biggs and Tang (2011) assert that "functioning knowledge requires a solid foundation of declarative knowledge, but that is not to say that the declarative knowledge must be in place first" (p. 82). This assertion is consistent with Skemp's (1976) understanding of procedural and conceptual knowledge as being instrumental and relational.

2.6.6 The role of mathematical meaning in creating understanding

According to Thompson (2015, 2013), understanding something is congruent is integral to assimilate it into an appropriate schema (cognitive structure) (see, also, Piaget, 1968). It is the teacher's responsibility in this sense to facilitate assimilation, and the challenge for teachers is to continuously refine their understanding of CK in order to avoid associating incorrect meanings to new experiences and ideas. Teachers are very often unaware that they do not possess the correct mathematical meanings until they put them into practice. How often has a teacher given a class some similar problems to do (after demonstrating a particular number process on the board) only to find a number of children cannot solve the problems at hand? The children in this example thought that they possessed the right understanding of the material, but did not; and, as has been seen, most learners perform badly in trigonometry: a situation that forcers learners to reveal whether they have successfully acquired correct mathematical meanings and are capable of conveying them.

2.7 RESEARCH IN TRIGONOMETRY TEACHING

Trigonometry forms an integral component of high school mathematics. Trigonometry is a unit in mathematics that integrates algebraic techniques, geometrical realities and graphical interpretations (Orhun, 2004). The statement assumes the perspective that integration and coming to understand the process through which ideas are developed in trigonometry, can positively catalyze to learn other key concepts. The Cartesian plane and coordinates in Figure 2.14 can be transferred into knowledge on different graphs that can be organized in accordance with their functions, inequalities, position and movement, matrices and transformation. Good and adequate mathematical meanings in this way can facilitate the ability of learners to grapple with more complex concepts related to the Cartesian plane.

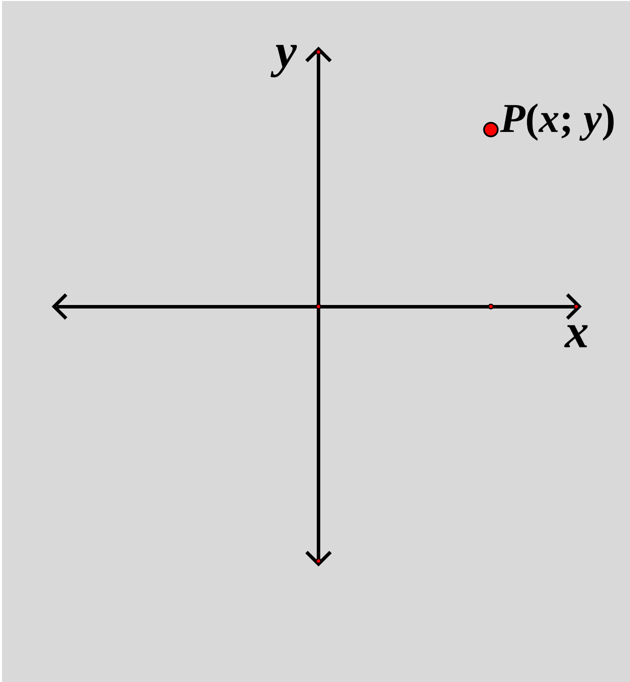


Figure 2. 14: A cartesian plane

Despite the importance of this section for allowing mathematics teachers to assess their own understanding of trigonometric functions and topics, it is a paradigm that is in many ways shallow and lacking in coherence (Fi, 2003; Moore *et al*., 2012; Thompson *et al*., 2007; Weber, 2005). Coherence constitutes an integral component of teachers' understanding and the meanings that they attach to their subject matter or content knowledge (Section 2.2.3). It can be concluded from this assertion that the shallowness in teachers' knowledge of trigonometry may have implications for teaching and learning full stop. Shallowness in a teacher's instruction negatively impacts the quality of classroom teaching and learning, and the DBE (2015, 2014, 2013) and various other reports have all identified trigonometry as one of the most failed topics in Grade 12.

Van Laren (2012) conducted a study with in-service teachers enrolled in an Advanced Certificate in Education (ACE[6]) programme offered at the University of KwaZulu-Natal. Van Laren explored mathematics teachers' challenges in relation to curricula in South Africa and how these changes tended to influence the teaching of trigonometry. Teachers were required

6. The Higher Education Institutions have offered an ACE programme to professionally qualified and practicing teachers who are in possession of a three-year teaching diploma to elevate teachers' content knowledge and improve mathematics teaching and learning.

to adopt an alternative approach such as the use of technology, collaboration, instructional coaching and interactive strategies to improve the instruction of trigonometry at the Grade 10 level, which also formed part of the ACE programme. Teachers identified learners' limited knowledge as stemming from the absence of essential content background knowledge on the part of teachers. Examples drawn from the theorem of Pythagoras was one of the key challenges they faced (Van Laren, 2012), with one of the participants claiming:

> Firstly, trigonometry ratios are a problem on their own. They confuse learners about their meaning. Learners firstly struggle with concepts of sin, cos, tan, etc. Even though explanations are there it might be a challenge still (Van Laren, 2012, p. 211).

This study was premised on this exact notion of mathematical meanings, which teachers are advised to subject to empirical study in order to verify how effective they implement in their classrooms the knowledge they claim to possess.

2.8 TRIGONOMETRY IN THE SOUTH AFRICAN MATHEMATICS CURRICULUM

This section aims to provide insights into the nature of trigonometry in the context of South Africa. The source of trigonometry is the Curriculum and Assessment Policy Statement (CAPS) document, with the CAPS document providing a framework for what teachers are expected to teach. The question is: *Do teachers' mathematical meanings fall in line with the expectations of the CAPS document*? Brodie (2011) identified trigonometry as a key problem area affecting learner performance, with the quality of a curriculum depending on the quality of the instruction implemented by teachers. In this sense, it is clear that the mathematical meanings of teachers often have a preponderate impact on learner performance, and it is for this reason that teachers' meanings are studied to help mitigate shortfalls in their teaching of trigonometry. Grade 10 trigonometry is part of the CAPS FET phase of the Mathematics curriculum. Table 2.1 provides a list of the main topics in the Further Education and Training (FET) Phase for a mathematics curriculum.

Table 2. 1: The main topics in the FET mathematics CAPS

1.	Functions
2.	Number Patterns, Sequences, Series
3.	Finance, growth and decay
4.	Algebra
5.	Differential Calculus
6.	Probability
7.	Euclidean Geometry and Measurement
8.	Analytical Geometry
9.	Trigonometry
10.	Statistics

Source: *DBE (2011, p. 9)*

The development of trigonometry knowledge in the South African curriculum is largely oriented towards procedural knowledge and is based less on conceptual knowledge (see, Pournara, 2001; see, also, Section 2.5.2). The teaching of trigonometry begins by addressing the ratio definition of sine, cosine and tangent functions based on a right-angled triangle (Figure 2.15). Pournara (2001) argued that such an approach is limiting and does not coherently support future learning of trigonometry for students. Kamber and Takaci (2018) agreed with Pournara (2001) that applying a right triangle trigonometry when solving problems related to determining the length of an unknown side of a right-angle triangle works perfectly. However, when it comes to estimating the value of a trigonometric function or drawing trigonometric graphs, the right triangle trigonometry may not be helpful. For example, Pournara (2001) found that learners had trouble when extending right triangle definitions to cover angles greater than 90°.

According to CAPS, it will be moving from *(a)* to *(b)* as illustrated in Figure 2.15 (the overview of Grade 10 trigonometry). One suspects that coherence in the CAPS document may be lacking, particularly in the trigonometry curriculum. The absence of clear articulation and demonstration of the connection between angle measure and the trigonometric ratios in the CAPS trigonometry's content clarification seems to confirm this suspicion. For example, in Grade 10, CAPS documents do not address the meaning of angle measure, and, as a result, trigonometric concepts may not be well comprehended beyond Grade 10 (DBE, 2016, 2015, 2014; Pournara, 2001; Sasman, 2011; Van Laren, 2012).

8. TRIGONOMETRY	
	(a) Definitions of the trigonometric ratios sin θ, cos θ and tan θ in a right-angled triangles. (b) Extend the definitions of sin θ, cos θ and tan θ to $0^o \leq \theta \leq 360^o$. (c) Derive and use values of the trigonometric ratios (without using a calculator for the special angles $\theta \in \{0^o;30^o;45^o;60^o;90^o\}$ (d) Define the reciprocals of trigonometric ratios.
	So Solve problems in two dimensions.

Figure 2. 15: An overview of Grade 10 trigonometry (CAPS)
Source: *DBE (2011)*

The shortcomings of right triangle trigonometry are that it promotes procedural knowledge, focusing on algorithmic methods and application procedures in trigonometry, depriving learners of the opportunity to understand the importance of the role of this angle in trigonometry. In calculus tasks such as approximating the derivative of $\sin x$, ratio understanding of sine does not make sense. A learner's knowledge of procedures in this instance may impede the development of their functional knowledge of the sine function. Kendal and Stacey (1998), found that right triangle trigonometry did help learners to learn trigonometry successfully. This study focused on Trigonometry as a sub-domain of mathematical knowledge because it is a critically important topic in the curriculum, and a problematic topic of teaching (DBE, 2016, 2015, 2014; Pournara, 2001; Sasman, 2011; Van Laren, 2012). The study sought to explore how teachers prepare learners for future trigonometry content knowledge.

2.9 TRIGONOMETRY UNDERSTANDING

Moore (2010) argues that both learners and teachers often do not construct meanings for trigonometric functions that include robust connections between the right-angled triangles and unit circle contexts. Therefore, it is vital that, before learners are required to learn the trigonometric ratios and functions of sine and cosine, that they are first equipped with foundational knowledge upon which new knowledge can be built. Firstly, the understanding and teaching of the topic proceeds from the definition of key terms in this context, such as that of ‘an angle’. According to Van de Walle (2016), an angle is an amount of turning, and the relative inclination to two rays and sharpness of an angular region. Assuming that an angle can be greater or smaller than 360 degrees, understanding a standard angle alone is not enough to conceptualize all angle situations, including angle measurement (Thompson, 2008). Proceeding beyond the examination of angles, the curriculum moves to the subject of Pythagoras’ theorem, trigonometric ratios and finally functions. In this way, students’ lack of foundational knowledge from the onset may produce later challenges in their ability to understand and master trigonometry.

Thompson (2008) explained that learners’ difficulty in learning trigonometry is owing to their failure to draw connection between related concepts, suggesting that teachers in high schools do not teach trigonometric concepts in the right order. Thompson sees lessons focused on angles, Pythagoras’ theorem, trigonometric ratios and finally functions as the right sequence in which learners are best advised to approach their study of trigonometry. Brodie (2011) argued that teachers tend to focus more on learning procedures instead of emphasizing the understanding of trigonometry in conceptual terms. However, it is imperative that teachers demonstrate that part of their mathematics meanings spread evenly from the conceptual to the procedural realms necessary for students to make sense of sine and cosine functions. Correct mathematical meaning is the basis from which learners can derive meaningful interpretations of CAPS and is a vital step in the process of learners’ developing thorough knowledge of their disciplines and its correct application. In turn, this understanding can facilitate a coherent understanding of concepts related to trigonometric ratios and functions.

2.9.1 How a protractor works

Angle measure in degrees is taught as a procedure and not as a measure. There is thus a need to develop angle measure in degrees, to provide a tangible context for this knowledge, so that it coheres with radian measure. According to Thompson (2008), one way to do this is to base the idea of degree on arc length. The property being measured then is the angle’s open-ness, and the method for measuring this open-ness is drawing a circle centered at the angle’s vertex and measuring the arc that the angle subtends in units of the arc; that is, $\frac{1}{360}$ the circle’s circumference. In this way, both degree and radian measures are measures of the same thing, the subtended arc (e.g., for an illustrated example, see Figure 2.16).

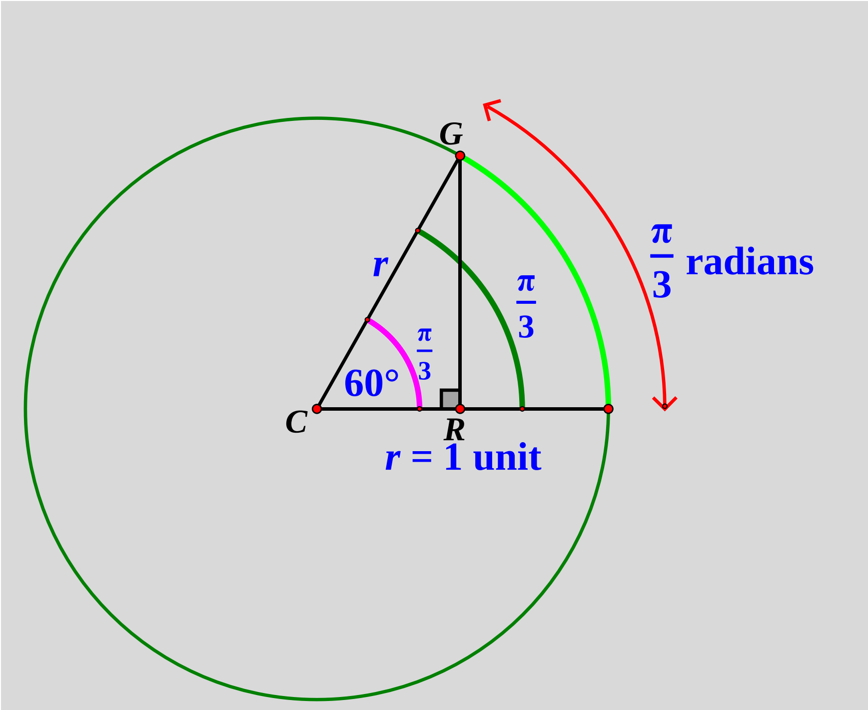


Figure 2. 16: Degree measure and radian measure representation

To understand the relationship between degree and radian measure, teachers must first understand the conventionality of both. The measured thing (the arc) is the same in both cases; its magnitude is simply cut into different numbers of segments according to the system being used. Curricular treatments of triangle trigonometry and of periodic functions that are coherent both within themselves and between the two developments derive their meanings of angle measurements as outlined, and would also draw from their mutual similarity, as similar triangles evince the same ratios. Thus, to know the ratio between the sides of one triangle and the sides of the other will reveal the ratios of corresponding sides of all similar triangles. Periodic functions would draw from the meaning of angle measure, from similarity, and should additionally highlight how one might think of varying an angle in order to measure it systematically. In this section, the researcher follows Thompson's (2008) conceptual analysis of a mathematical idea, with a greater focus placed on 'trigonometric understanding'. Thompson's (2008) conceptual analysis tool follows the given sequence:

- develop angle measure in degrees;
- determine whether the angle measure coheres with radian measure;
- clarify the idea of degrees as related to arc length;

- measure the angle's open-ness;
- center a circle at the angle's vertex;
- measure the arc formed by the angle; and,
- determine if the degree and radian measures are measures of the same thing (subtended arc).

The ideas that constitute Thompson's (2008) tool described in preceding sections should form part of the mathematical meanings that should be conveyed by the teacher for an understanding of the angle measure in trigonometry. If the ideas on the tool are not fully mastered teachers, who teach the angle measure are likely to create confusion at the entry point into the topic. Thompson (2008) stated three observations that have inspired the study that informs the decision to teach the concept of angle measure coherently from the beginning.

- ✔ If the idea of angle measure is used to develop the coherent meanings in trigonometry, it should be the basis for teaching trigonometry.
- ✔ Teachers' mathematical meanings are the reason our profession exists. Everything we do as mathematics educators, is directly or indirectly intended to improve meanings for the learning of mathematics.
- ✔ If the teacher's mathematical reasoning is distorted, the majority of learners are not expected to use these concepts in their thinking.

Thompson's (2008) observations raise a compelling need to explore teachers' mathematical meanings in order to establish where they may need help. Figure 2.17 illustrates trigonometric concepts revealed when teaching right triangle trigonometry and unit circle trigonometry as a coherent unit. In right triangle trigonometry, trigonometric concepts are defined as ratios of lengths of the sides in right-angled triangles. For example, the sine of an angle is equal to the ratio of the length of the 'opposite side' to the length of the hypotenuse side. Figure 2.17 shows the named triangle sides, with the trigonometric ratios indicated helping to establish a relationship between angles and side lengths. However, in this context, trigonometry is restricted to angles in degrees smaller than 90^o, such that, in unit circle trigonometry, trigonometric concepts are defined as coordinates, as is illustrated in Figure 2.17.

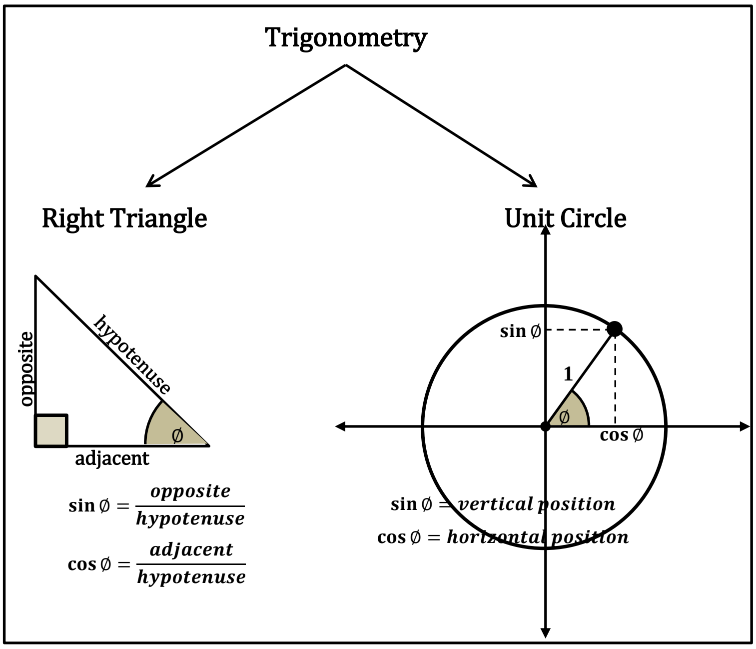


Figure 2. 17: Right-angled triangle and unit circle trigonometry

Unit circle trigonometry extends trigonometry to any angles and both negative and positive angles. Unit circle trigonometry provides a foundation for the definition of trigonometric functions. The meanings indicated in this context should provide for mathematical connections drawn between and among concepts; for example, the meaning of $cos\ cos\ \theta$ in right triangle trigonometry must be coherent with the meaning of $cos\ cos\ \theta$ in unit circle trigonometry. In conclusion, Thompson (2008) identified four ways in which conceptual analysis can be used:

- in building models of what learners know at specific moments in time, and what they are able to comprehend in specific situations;
- in describing ways of knowing that might be propitious for learners' mathematical learning;
- in describing ways of knowing that might be deleterious to learners' understanding of important ideas and in describing ways of knowing that might be problematic in specific situations; and,
- in analyzing the coherence, or fit, of various ways of understanding a body of ideas. Each is described regarding their meanings, and their meanings can then be inspected regarding their mutual compatibility and mutual support (Thompson, 2008, p. 59).

2.10 CONSIDERATIONS FOR FRAMEWORK

The study aimed to explore mathematical meanings for teaching that the researcher anticipated teachers imparted to learners when teaching trigonometric ratios and functions in Grade 10. This anticipation was a product of the claim that learners are constantly in pursuit of understanding what their teachers say and do in the classroom, and, in the process, learners construct their meanings. The researcher believes that the way in which a teacher understands a mathematics concept or idea is essential in determining the mathematical understanding that learners will eventually formulate (Byerly & Thompson, 2017). To explain and make sense of these educational constructs, the study adopted a theory of meanings inspired by genetic epistemology (Piaget & Duckworth, 1970; Thompson, 2015, 2013; Thompson & Draney, 2014). According to Piaget and Duckworth (1970), Genetic Epistemology Theory (GET) "deals with both the formation and the meaning of knowledge" (p. 12).

GET encompasses Piaget's theory of cognitive development, providing insights into how knowledge acquisition takes place through stages that determine one's thought processes and functionality (Figure 2.18). The researcher believes that, in accordance with the knowledge processes and stages presented in Figure 2.18, humans can systematically learn and grow intellectually, thus being capable of formulating their own meanings of domain-specific concepts and ideas. GET may be linked with constructivism, since both theories are commonly premised on the notion that individuals are continuously constructing and acquiring knowledge; which, in this study, refers to a stage of continued meaning making (Piaget & Duckworth, 1970). As people gain more information about the environment around them, knowledge creation and expansion take hold. Essentially, GET is the basis on which schemes can be assimilated by instructors, (Piaget & Duckworth, 1970; Thompson, 2015, 2013; Thompson & Draney, 2014) as is poignantly captured in Piaget and Duckworth (1970):

> To know is to assimilate reality into systems of transformations. To know is to transform reality in order to understand how a certain state is brought about. Knowledge then is a system of transformations that become progressively adequate (p. 15).

In terms of this explanation, meaning making is a product of one's understanding of knowledge assimilated into systems of transformation. Teachers who coherently understand an idea create greater anticipation for learners to learn the idea meaningfully. Meaningful learning is achieved when an incoming idea is in equilibrium with a learner's already assimilated ideas or schema, and this contributes to productive cognitive development (Figure 2.18). In this regard, the researcher views GET as providing a suitable theoretical passage to explain the notion of mathematical meanings in ways that adequately account for teachers' understanding of the ideas that they teach their learners.

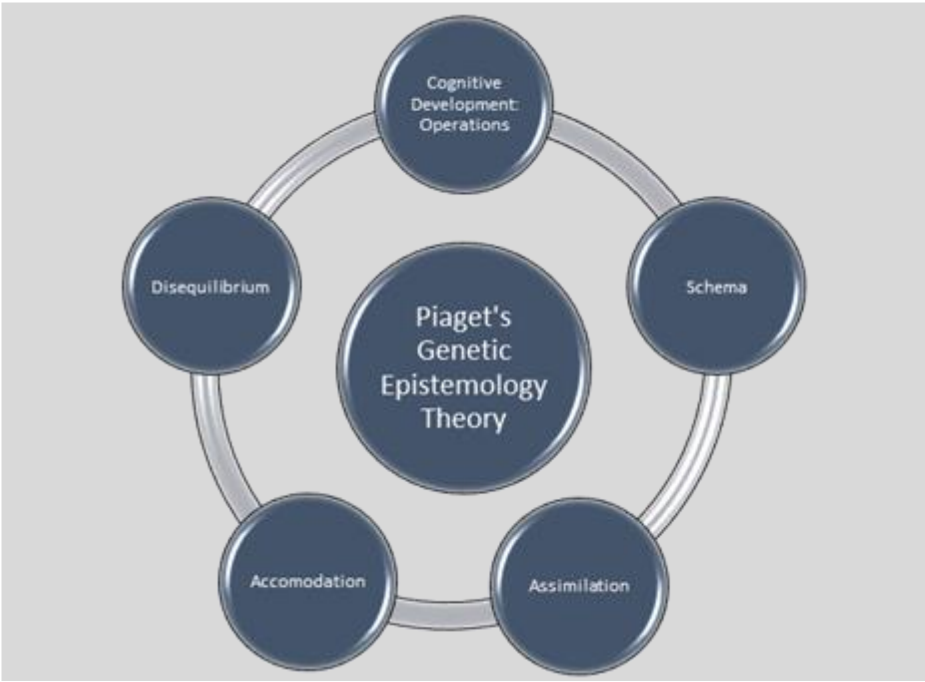


Figure 2. 18: Components of Piaget's Genetic Epistemology Theory (GET)

The researcher strongly supports the notion that an association between a teacher's mathematical meanings and, in turn, the meanings learners develop in the classroom exists. The researcher further acknowledges that the mathematical meanings formed by teachers and their learners may not necessarily be the same (Yoon, Byerley & Thompson, 2015). Learners turn to assimilate or make meaning of a teacher's ideas differently, such that meaningful learning prepares learners for productive assimilation and application of learnt ideas in various contexts. In line with these educational contributions, the researcher strongly believes that pursuing teachers' mathematical meanings and their perceived influence on mathematical learning has found a suitable abode within the theoretical confines of GET.

2.10.1 What constitutes teachers' inherent meanings?

The theory of meanings creates a link connecting meanings, understandings and ways of thinking between what teachers know and teach, and what their learners need to learn. Piaget's genetic epistemology relies on the notion of assimilation, in which the assimilation of a concept to a scheme connotes the understanding of that concept (Thompson, 2015, 2013; Thompson & Draney, 2014).

Table 2. 2: Definitions of understanding, meaning, and ways of thinking

CONSTRUCT	DEFINITION
Understanding (in the moment)	Cognitive state resulting from an assimilation
Meaning (in the moment)	The space of implications existing at the moment of understanding
Understanding (stable)	Cognitive state resulting from an assimilation to a scheme
Meaning (stable)	The space of implications that results from having assimilated to a scheme. The scheme is the meaning, what Harel previously called *way of understanding*
Ways of thinking	Habitual anticipation of specific meanings or ways of thinking in reasoning

Source: *Thompson et al. (2014)*

2.10.2 State of teachers' embodied notions of meaning

Studies show that teachers convey their meanings to learners deliberately or non-deliberately (Thompson, 2015, 2013; Thompson & Draney, 2014). It is reasonable to argue that learners construct meanings that resemble their teacher's meanings. Thompson and Draney (2014) emphasized, "… that teachers convey their personal meanings to students, whether teachers hold them tacitly or consciously, and it is meanings learners develop that are the foundation for their future uses of the mathematics they learn and are foundational for their future mathematical learning" (p. 2). Thompson (2015, p. 437) argued that "teachers' mathematical meanings guide their instructional decisions and actions"; which is to say, teachers with unproductive meanings can at times convey them to learners unthinkingly, thereby frustrating learners in the process (Izsak, 2012; Thompson, 2013).

2.10.3 How is it articulated at the classroom level?

This study aims to explore the mathematical meanings entertained by teachers, and which they convey when teaching trigonometric ratios and functions to Grade 10 learners. To gain access to teacher's mathematical meanings, the creation of theoretically informed assessment items was considered imperative. A theory of meaning inspired by genetic epistemology (Piaget, 1977; Thompson, 2015, 2013; Thompson and Draney, 2014) informed our current study. When someone's understanding in a given context is produced through assimilating to a scheme, the scheme in question becomes commensurate with its implied meaning. In other words, when someone thinks in a certain way when he or she is cognitively engaged in reasoning in a particular setting, then it means that the person routinely utilises specific meanings and styles of thinking. (Thompson and Draney, 2014).

2.10.4 Anticipated teachers' mathematical meanings for trigonometry in Grade 10

In this section, the researcher provides a synopsis of what could have been the anticipated mathematical meanings and ways of understanding or thinking that teachers most likely drew upon when responding to a given set of tasks (Appendix 1). According to Moore, LaForest and Kim (2016), "understanding is an in-the-moment cognitive state of equilibrium results from one's (successful) assimilation to a scheme. Meaning refers to the actions and schemes that an individual anticipates or enacts in the moment of understanding" (p. 223). Thompson (2013) explained that meaning is synonymous with an individual's understanding at a given moment, with their understanding being implied by discrete actions, schemes, and assimilation and accommodation processes. For example, a teacher understanding that the sine of an angle can be expressed as the ratio of the side opposite the angle divided by the hypotenuse side of a given right angle triangle would constitute one instance of "meaning" in this sense.

At this moment in time, the teacher can understand that, in any right-angle triangle, the sine of an angle is expressed by the ratio $\frac{opposite}{hypotenuse}$. This is one of the implications of the teacher's understanding at that moment (Yoon & Thompson, 2018). Thus, the teacher's "meaning in the moment of understanding is the space of implications of that understanding" (Yoon & Thompson, 2018, p. 2). Furthermore, a scheme is characterized as a structure comprising of mental actions emanating from assimilated previous experience. Consequently, assimilation generates schemes. Tallman and Frank (2018) suggested that it is possible to explain ways of thinking and ways of understanding in terms of assimilation and schemes. Tallman and Frank (2018) asserted that:

"one's way of understanding a particular mathematical concept is the scheme (i.e., organization of generalized mental actions) with which she assimilates experiences of the concept or idea. Ways of thinking, on the other hand, are the habitual forms of reasoning that govern the application of a variety of specific mathematical schemes (i.e., ways of understanding" (2018, p. 7).

What the assertion implies, in the context of this study, is that a scheme is the meaning of the understanding that the teacher construct in the moment, and ways of thinking alludes to a teacher having developed a pattern for utilizing specific meanings or reasoning about particular ideas. It is for the reasons discussed that the researcher presents the mathematical meanings for teaching trigonometric functions as representative of anticipated teacher's ways of thinking and ways of understanding the trigonometric concepts under consideration.

2.10.4.1 What is an angle?

Mathematically, an angle could be described as a rotation or turn. Often one can refer to the amount of rotation or turn using phrases such as *full turn*, *half turn*, and so on. Figure 2.19 shows diagrams of examples of angles and their descriptions in terms of the amount of turn they cover in their rotation or turning.

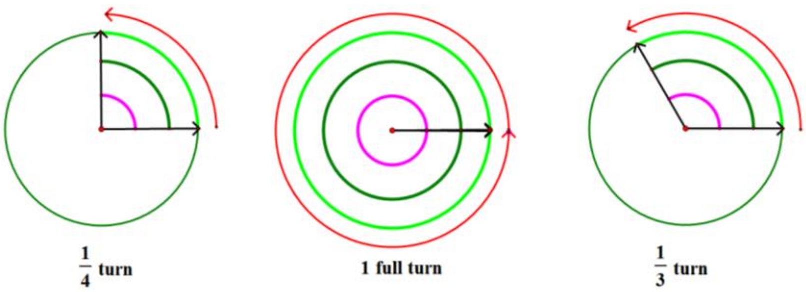


Figure 2. 19: Mathematical demonstration of angle formulation with varied turnings

Real life examples of angle formation could be observed in the swinging of doors and car wheels, producing differing quantities of turns or turning. For example, when the car is in motion, the wheels could make more than a full turn or one turn. However, a household door usually makes a turn or swinging that is less than a full turn. These two examples, the car wheels and a household door, may help us to perceive and project different formations and positioning of angles. The notion of angle and its formations in real-world scenarios thus demonstrates advanced knowledge and the applicability of mathematical knowledge promoting productive meaning-making of mathematical knowledge (meaningful learning). According to Wilson and Adams (1992, p. 10), when an object is turning, inherent in the dynamic situation of turning are the following four components:

- the point of turning;
- the initial side of the angle;
- the direction of the turn; and,
- the terminating side of the angle.

The diagram in Figure 2.20 illustrates Wilson and Adams' four points of dynamic turning, thus promoting desirable conceptualization and understanding of the concept of *angle formation*.

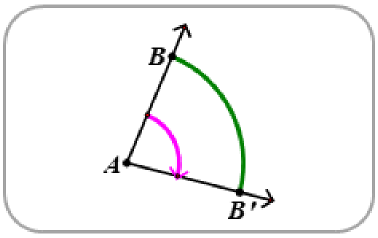


Figure 2. 20: Demonstrating the formation of an angle

With reference to Figure 2.20, by turning $\overrightarrow{AB}$ (ray AB) about the point A and landing at $\overrightarrow{AB'}$ the turning produces an angle known as $\angle BAB'$, with the notation "$\angle$" being a shorthand for an *angle*. A single-headed curved arrow in Figure 2.20 serves to demonstrate the direction and amount of turn of an angle. In this context, the meaning that the learners are supposed to capture when learning about the situation in Figure 2.20 is that *an angle is formed or is a product of turning an object or arm about a fixed point, which is A in Figure 2.20, and will be described by the amount of turning*[7] *from the initial position*. According to Wilson and Adams (1992),

7. In Figure 2.20 the turn is clockwise. In general, the turn can either be clockwise or anti-clockwise when an angle is formed.

perceiving an angle as a turn, as depicted in Figure 2.20, may assist in reversing a popular misconception among students that the length marks used to represent angles determine the size of the angle, and that one side in a situation of angle formation must be horizontal.

2.10.4.2 What does it mean to measure an angle?

It is the researcher's view that perceiving an angle as a turn, and further describing it as an amount of turn (Section 2.10.4.1), may promote subsequent understanding of the notion of *angle measure*. In Figure 2.19, we have observed that one full turn produces a complete circle that is centered at the point of turning. This implies that a circle is an angle with its vertex being the center, and with the turn amounting to one. Drawing a circle using a *mathematics compass*[8] demonstrates the notion that a circle is an angle formed by rotating the top of a compass, such that one complete turn produces a circle (see, Figure 2.21). In considering the knowledge that we acquire from our study of circles' perimeters may allow us to realize that one full turn in a circle will be equated to what is called the *circumference*[9] of a circle.

Having this knowledge and background may help us to realize that a circle is a useful tool needed to measure the amount of turn that we use to describe an angle. The turns that are considered to be less than one full turn, which is equivalent to the circumference of a circle, are multiplicatively described relative to the one full turn. Hence, it is educationally productive to use the idea of the amount of turn in a circle to illustrate the process of measuring an angle, which is explored in item 1.2 that was given as a task to teachers who participated in this study (Appendix 1). We see that, in Figure 2.22, the concept of a turn along the circumference is well depicted, revealing that a circle with a center at point *A* has two rays; namely, ray *AB* and ray AB' between which $\angle BAB'$ is formulated (constructed). In addition, Figure 2.22 introduces the concept of an *arc*[10], which stands at BB' in the diagram.

A central angle is an angle whose vertex is located at the center of a circle. We observe in Figure 2.22 that angle BAB' is the central angle, which has a vertex at A, and that the rays *AB*

---

8. A *mathematics compass* is used to draw perfect circles. The mathematics compass works with a sharpened pencil inserted into the cam lock. A sharp, pointed tip rests on the paper that the circle will be drawn on, and you rotate the top of the compass to create a circle of a certain diameter (Karimi, 2017).
9. The *circumference* of a circle is the distance around the circle and may be calculated by using the formula $C = 2\pi r$ or $C = \pi d$ where r is the radius, and d is the diameter of the circle, and π is a constant number and is also known as the ratio of a circle's circumference to its diameter.
10. An *arc* is a portion of the circumference of a circle and is named by its endpoints.

and AB' of the central angle BAB' intersect with the circle at two points; namely, B and B' respectively. These two points of contact determine the length of the arc BB', with the arc here being referred to as the *intercepted arc*[11] for the central angle BAB'.

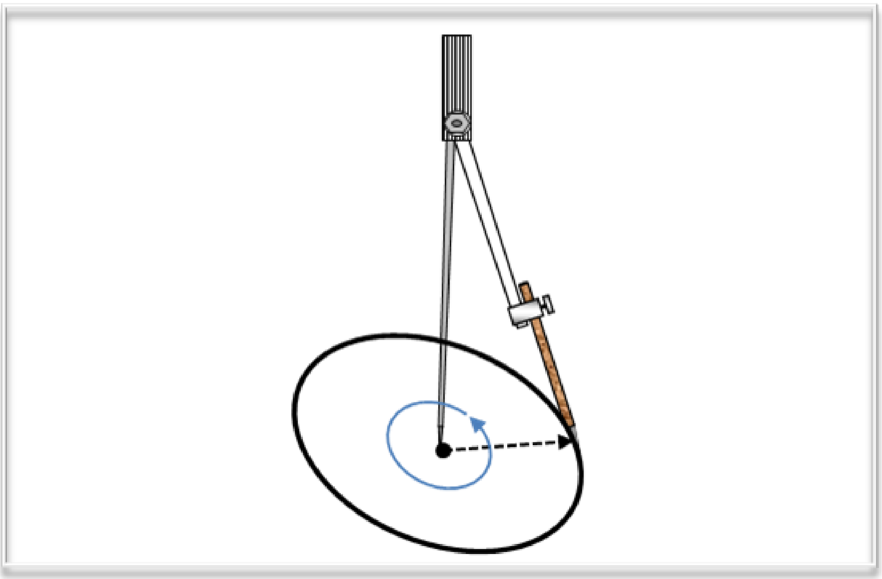


Figure 2. 21: A mathematics compass point turning anti-clockwise resulting in a circle

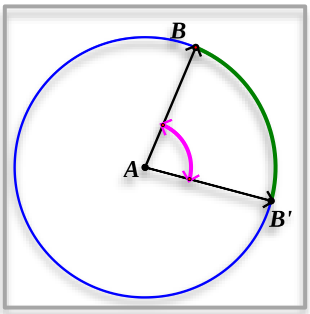


Figure 2. 22: Angle formulation in the context of a circle

The relationship between the amount of turn and the intercepted arc may be sought by asking

11. The intercepted arc is the part of the circle between the two sides of the central angle.

questions such as the following question:

- *How does the measure of the central angle (amount of turn) compare to the measure of its intercepted arc (arc length)?*

To answer the preceding question, we may have to divide the circle into *n* equal-size arcs in a way that each arc will be $\frac{1}{n}$ of the circle's circumference (hence, one full turn $= \frac{n}{n}$ of the circumference).

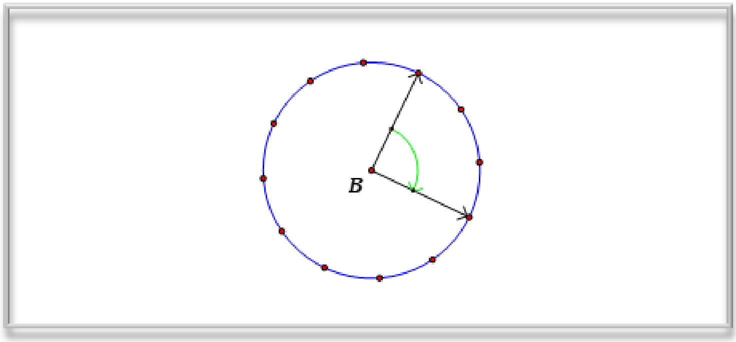


Figure 2. 23: Using the idea of a circumference and arc to explore the notion of angle measure

In Figure 2.23 the circle has been divided into 12 equal-size arcs. In this case, each arc will be $^1/_{12}$ of the circumference of a circle. such that the amount of turn (measure) of the central angle *B* will be, $\frac{3\ arcs}{12\ arcs} = \frac{3}{12}$ of the circumference of a circle. Accordingly, from the preceding explanations, we may learn and observe that the measure of a central angle is equal to the measure of its intercepted arc. In terms of the objectives of this study, the meaning that one may intend to convey to learners is that the angle measure is revealed in the process of determining the fractional amount of the circumference of the circle that is subtended by the central angle (Moore, 2013; Thompson, 2008). In other words, to measure an angle is tantamount to measuring the length of the arc that is subtended by the central angle.

2.10.4.3 What does it mean for an angle to have a measure of 1 degree?

Recalling our previous discussion of mathematics, we remember that, by convention, a degree is the most commonly used unit for measuring angles. In addition, it is conventional that one full turn is equal to the circumference of a circle, such that the relationship between the amount of turn and the degree measure of an angle can be expressed as one full turn equaling 360°, which in turn implies that the circumference of a circle is 360°. We give the degree measure of central angle *B* in Figure 2.23 as:

$$\frac{3}{12} \times 360° = 90°$$

It must be noted that the circle was divided into twelve equal-size arcs (Section 2.10.4.2), and that this mathematical statement means that the amount of turn can be expressed in degrees as well as in the fractional amount of the circle's circumference. The intended meaning to be conveyed to learners by teachers is that a degree (1°) is a measure of the arc length that equals $\frac{1}{360^{th}}$ of the circumference of the circle. Angle measure in degrees is viewed as the length of the arc subtended by the central angle, measured in arcs of the length $\frac{1}{360^{th}}$ of the circle's circumference (Thompson, 2008, 2007). See also how this knowledge is demonstrated in a protractor in Figure 2.24.

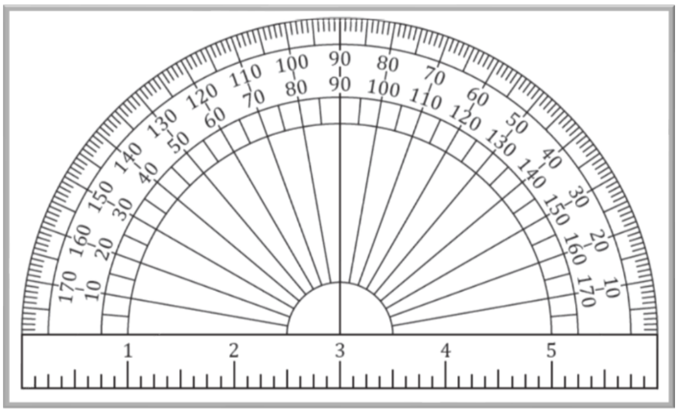


Figure 2. 24: A protractor

2.10.4.4 Using a protractor

In line with the preceding discussions, angle measure is conceived as an amount of turn (see, Sections 2.10.4.2 & 2.10.4.3; see, also, Figure 2.24). However, to measure a given angle means one is imposing an arc of a circle on the angle (Moore, 2013; Thompson, 2008, 2007). This idea is consistent with how a *protractor*[12] works or is used when measuring angles (see, Figure 2.23 & Section 2.8). The calibration on a protractor is made of arcs of the length $\frac{1}{360^{th}}$ of the circle's circumference, where each central angle subtends its intercepted arc. Thus, when using a protractor to find the measure of an angle, in principle one is measuring the amount of turn.

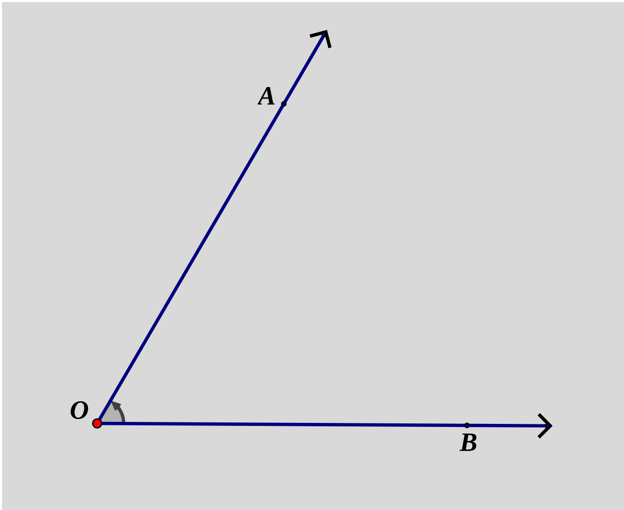


Figure 2. 25: Finding a measure of angle BOA

For example, when measuring angle *BOA* in Figure 2.25 using a protractor, you are placing a circle on top of an angle, with the angle measure of 60° read on a protractor being equivalent to the amount of turn of 60° which subtends the arc of the length $60° \times \frac{1}{360°} = \frac{1}{6^{th}}$ of the circle's circumference. Figure 2.26 illustrates the angle measure of 60° using a protractor, with Figure 2.26 showing how the protractor measures the angle *BOA*'s openness by imposing arc *BA* on angle *BOA*. Thus, the 60°-degrees angle measure is a process of partitioning the imposed arc using a specified unit length (Moore, 2013). It is important to highlight that angle measures of the intercepted arcs *DC* and *AB* in Figure 2.26 are equal, and both equal to 60° or $\frac{1}{6^{th}}$ of the circle's circumference, but that the magnitude of the arc lengths are not equal[13]. In other words, the magnitude of the arc length is multiplicatively dependent on the magnitude of the radius of

12. A protractor is an instrument that is used to measure angles.
13. The length of $arc\ DC = OD \times \angle COD$ (radians), and the length of $arc\ BA = OA\ \times\ \angle BOA$ (radians), but the fraction of the circumference subtended by the angle does not change.

a circle. Consequently, in Figure 2.26, the magnitude of the intercepted arc *DC* is multiplicatively related to *OD*, and the magnitude of the intercepted arc *BA* is multiplicatively related to *OA*.

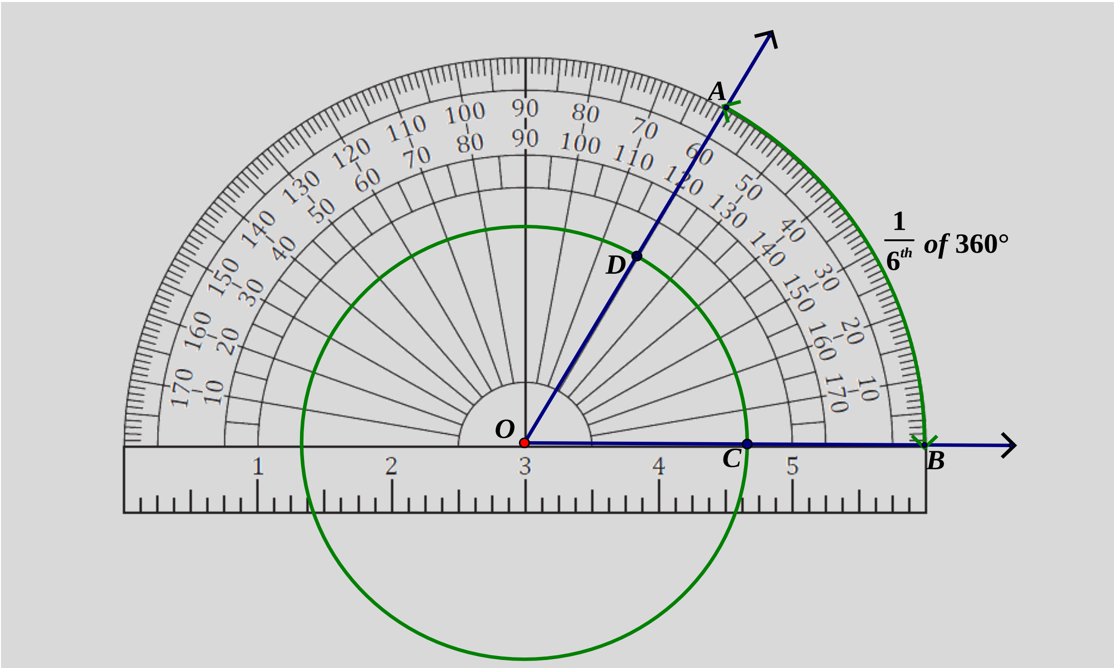


Figure 2. 26: Angle measure of 60° using a protractor

In line with the purpose of this study, the researcher's intention with the item 4 task given to teachers who participated in this study was to explore the teachers' meaning that they hold regarding the usage of a protractor when engaging in angle measure activities. An angle measure is an arc length's fraction of a circle's circumference (see, Sections 2.10.4.2 & 2.10.4.3; see, also, Appendix 1). The researcher's anticipated meanings relating to the usage of teachers' protractors carried the understanding that any given amount of rotation references an angle measure, and that the product which results from measuring an angle is the observed arc length. This understanding discourages the perception that an angle and the measure of its openness is a static object (the corner or vertex) or a fixed position on a circle (for example, the 60° mark on the protractor) (Moore, 2010). Instead, the openness of the angle varies in accordance with the corresponding arc length's fraction of the circle's circumference. This way of making sense of angle measure is important to facilitate reasoning about the unit circle and the periodic motion.

2.10.4.5 The meaning of the sine of an angle

The understanding of the notions of an angle and an angle measure motivated researchers to believe that ideas developed can leverage the comprehension of trigonometric functions, such as the *sine* of an angle (see, Sections 2.10.4.1 to 2.10.4.4). Bressound (2010) provided a historical account regarding the origins of the *sine* of an angle, and, by referring to Ptolemy's discourse on translating the length of a *chord*[14] into a *sine* function, Bressound (2010) explained that "if the chord subtends an arc length of 2θ and the radius of the circle is R, then half the chord length is $Rsin\theta$ " (p. 109). Therefore, the measure of the central angle is equivalent to the measure of the intercepted arc; in other words, to provide a tangible example to illustrate this point, the measure of angle WTX in Figure 2.27 is also 2θ.

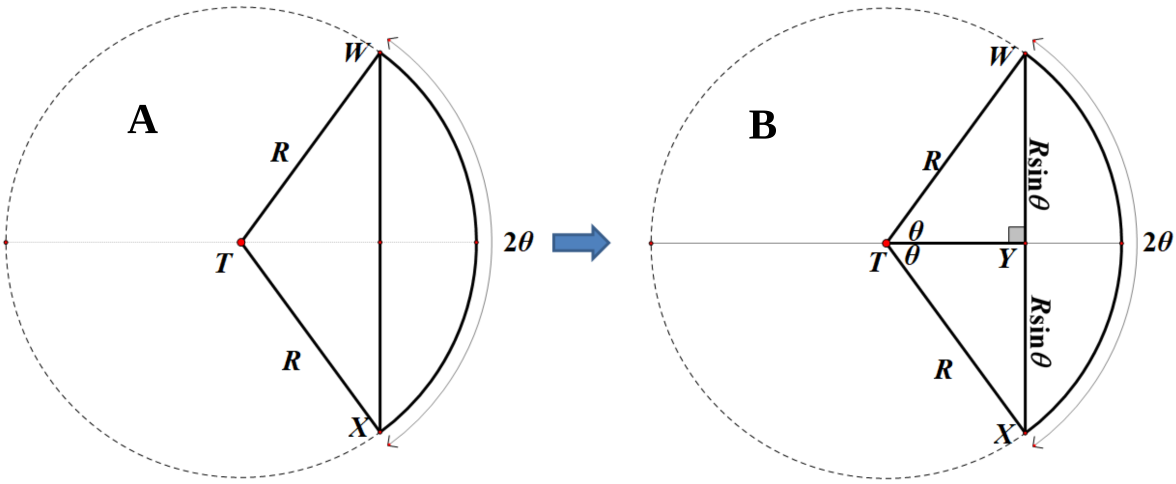


Figure 2. 27 When the chord intercepts an arc of 2θ, then its length is 2R sinθ, where R is the radius of the circle (Bressound, 2010, p. 109)

To illustrate Bressound's (2010) assertion, we may consider Figure 2.27, in which **A** shows a circle with centre T, chord WX subtending angle measure (arc length) 2 and a radius R. In Figure 2.27, **B** shows (for the same circle T) a chord WX, which is bisected by a line segment TY at point Y, suggesting that point Y is the midpoint of chord WX. Given the fact that a circle has a line of symmetry through its diameter, it should have a rotational symmetry at its centre too. It is therefore reasonable to make the following deduction:

14. A *chord* is a segment whose endpoints are any two points on the circumference of a circle.

> *The line segment joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.*

In Figure 2.27, **B** line segment, TY is perpendicular to chord WX. Then the *sine* of angle WTY (θ) is defined as the length of the line segment WY, and may be written as $Rsin\theta$. Thus, the direct mathematical implication of Bressound's explanation is that the *sine* of an angle refers to a length, and that its magnitude is multiplicatively reliant on the radius of the circle. A closer look at Figure 2.27 **B** reveals that the process of halving the chord and joining the midpoint of the chord with the centre of the circle results in the formation of two congruent right-angled triangles YTW and YTX. Hence, in triangle *YTW*, *WY* is the side opposite angle θ, *TY* is the side adjacent to angle θ, and *TW* is the hypotenuse side. Therefore, the *sine* of an angle of a right-angled triangle is made by the length of the side opposite the central angle.

Given Bresound's historical account, Thompson *et al*. (2007) explained the *sine* under consideration by picturing a right-angled triangle embedded in the circle, such that the hypotenuse of the triangle was commensurate with the radius of the circle, and the vertex of the angle in question would constitute the center of the circle (see, Figure 2.28). The *sine* of an angle refers to the percent of the length of the radius made by the length of the side opposite to the central angle in the embedded right-angle triangle.

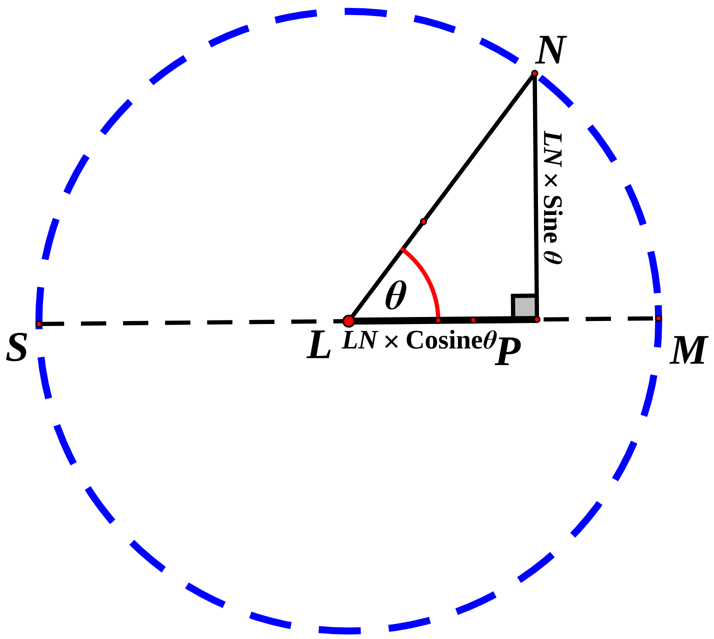


Figure 2. 28: The sine and cosine of angle θ

Similarly, the *cosine* of an angle refers to the percent of the length of the radius made by the length of the side adjacent to the central angle (Thompson *et al*., 2007, p. 417). Figure 2.28 shows a circle with centre *L*, and the embedded right triangle *LPN* with angle θ at *L*. The *sine* of θ is the percent of the length *LN* made by the length of *NP* and the *cosine* of θ is the percent of the length *LN* made by the length of *LP*. The meanings conveyed by Bressound (2010) and

Thompson *et al*. (2007) in relation to the *sine* of an angle are seemingly consistent with each other. For this study, the researcher conceived and embraced the meaning of the *sine* of an angle as the percent of the radius' length made by the side opposite the angle at the centre of a circle in the embedded right-angled triangle. The reason for choosing to adopt this position is that, at the time of conducting this study, the researcher believed that the adopted meaning would make it possible for the sine function to be explored as a function, with the output of the sine function being commensurate with the magnitude of the side opposite to the central angle, measured as a fraction of a radius. The meaning makes it possible to describe the *sine* function for angles greater than 90° or less than 0°.

Figure 2.29 shows a circle *A* with radii *AB* and *AB′*. Arm *AB′* is rotating anticlockwise from 0° to 360° and, in this case, diagrams **I - IX** represent the movement of rotation. As the arm *AB′* rotates, the central angle *BAB′* is formed, and the intercepted arc *BB′* is observed. Also, from the point *B′*, which is located on the circumference of the circle, a perpendicular line is shown to intersect with radius *AB* at the point *I*; thus, an embedded right-angled triangle *B′AI* can be observed. The magnitude of the length $B'I$, which is the sine of angle *BAB′*, is equal to zero for $B\hat{A}B' = 0°$ (diagram I), then increases from 0 to a certain fraction of the radius for $0° < B\hat{A}B' < 90°$ (diagram II), then equals the length of the radius, that is, equals 100% of the radius length for $B\hat{A}B' = 90°$ (diagram III). The magnitude of $B'I$ decreases for $90° < B\hat{A}B' < 180°$ (diagram IV) until it is equal to zero for $B\hat{A}B' = 180°$ (diagram V). The magnitude of the length $B'I$ decreases further for $180° < B\hat{A}B' < 270°$ (diagram VI), then equals the length of the radius for $B\hat{A}B' = 270°$ (diagram VII). In diagrams VIII and IX, the magnitude of the length of $B'I$ increases for $270° < B\hat{A}B' < 360°$ until it is equal to zero for $B\hat{A}B' = 360°$ (see, Figure 2.29).

Angles are divided into quadrants[15] to simplify classification according to size, so that, for any given angle, one can associate the quadrant in which the radius corresponds to its partner angle. Given that the circle is commonly divided into 360 angular degrees, the quadrants are named by 90º segments and are numbered anticlockwise.

15. A quadrant is one-quarter of a circle.

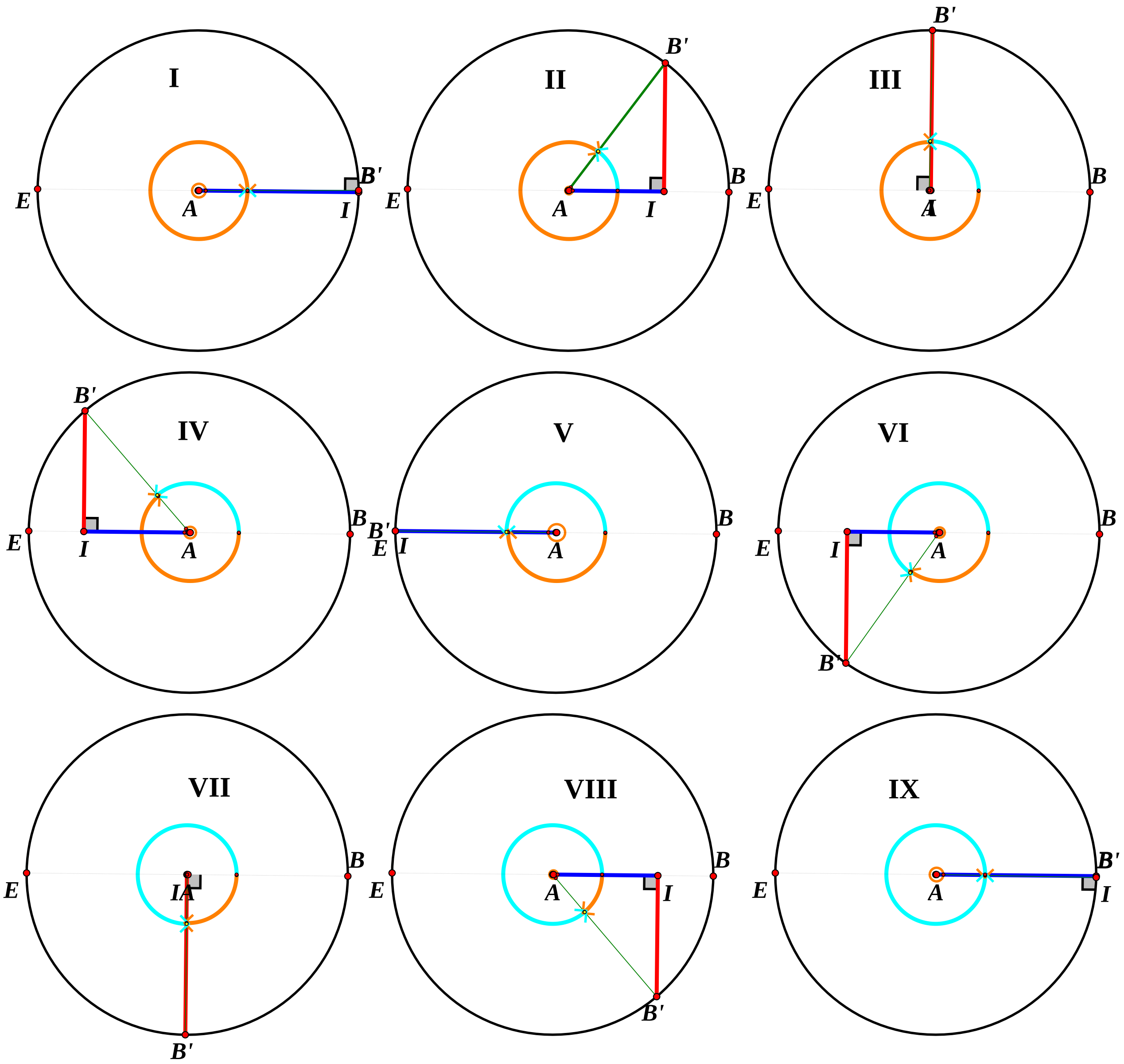


Figure 2. 29: Definition of the sine and cosine of an angle greater, less than or equal to 0°

The first quadrant has angles measuring 0º to 90º, the angle in diagram II of Figure 2.29. The second quadrant goes from 90º to 180º, the third goes from 180º to 270º, and the fourth goes from 270º to 360º. The related angles are in diagrams IV, VI and VIII of Figure 2.28, respectively. By convention, mathematicians represent the quadrant boundaries, with 0º or 360º going horizontally to the right (diagrams I and IX), 90º going vertically upward (diagram III), 180º going horizontally to the left (diagram V), and 270º going vertically downward (diagram VII), for a better illustration see Figure 2.30. The consecutively numbered curved arrows in

Figure 2.29 indicate angles in the first, second, third and fourth quadrants in that order. By employing the ideas of algebra pertaining to graphical work, the relationship between the definitions of trigonometric functions and the *rectangular coordinate system*[16] (see Figure 2.31) helps in developing the definitions for the trigonometric functions of the *general angle*[17].

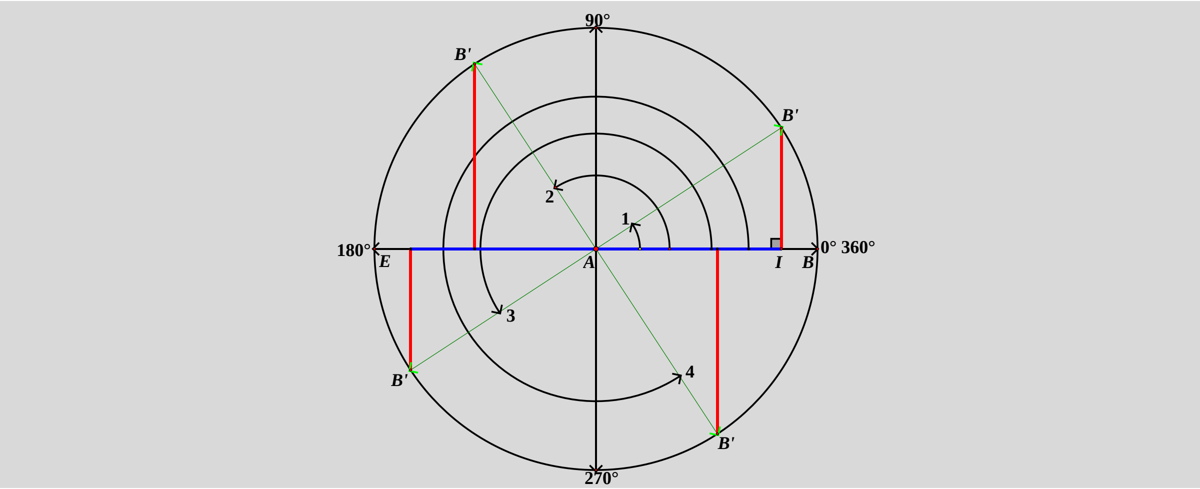


Figure 2. 30: Quadrants in a circle and an embedded right-angled triangle defining trigonometric functions

Figure 2.31 shows a rectangular coordinate system and point *Q*. A rectangular coordinate system is simply made by placing two number lines (one drawn vertically and the other drawn horizontally) at right angles with each other in such a way that they intersect at the zero marks. The horizontal number line is called the *x-axis*, and the vertical number line is called the *y-axis*, and the two axes intersect at the origin. Note that the axes in Figure 2.30 divide the plane into four quadrants, with point *Q* in the first quadrant, and the horizontal distance from the origin of point *Q* being 2 units and referred to as the *x* coordinate of the point. The vertical distance from the origin of point *Q* is 3 units and is called the *y* coordinate of the point, with the numbers in the ordered pair (2; 3) called the coordinates[18] of point (*Q*) and the coordinates of the point always written in the same order (with *x* coordinate first). The letter identifying the point is

16. Rectangular coordinates are also called Cartesian coordinates, named after the French mathematician René Descartes (1596-1650).
17. The general angle in the context of this study, is any angle not confined to the first quadrant. In other words, the general angle also refers to the angles of the size greater than 90° or less than 0°.
18. The ordered pair numbers are called rectangular coordinates, but by convention are simply called coordinates.

ordinarily written before the coordinates, as in Q (2; 3).

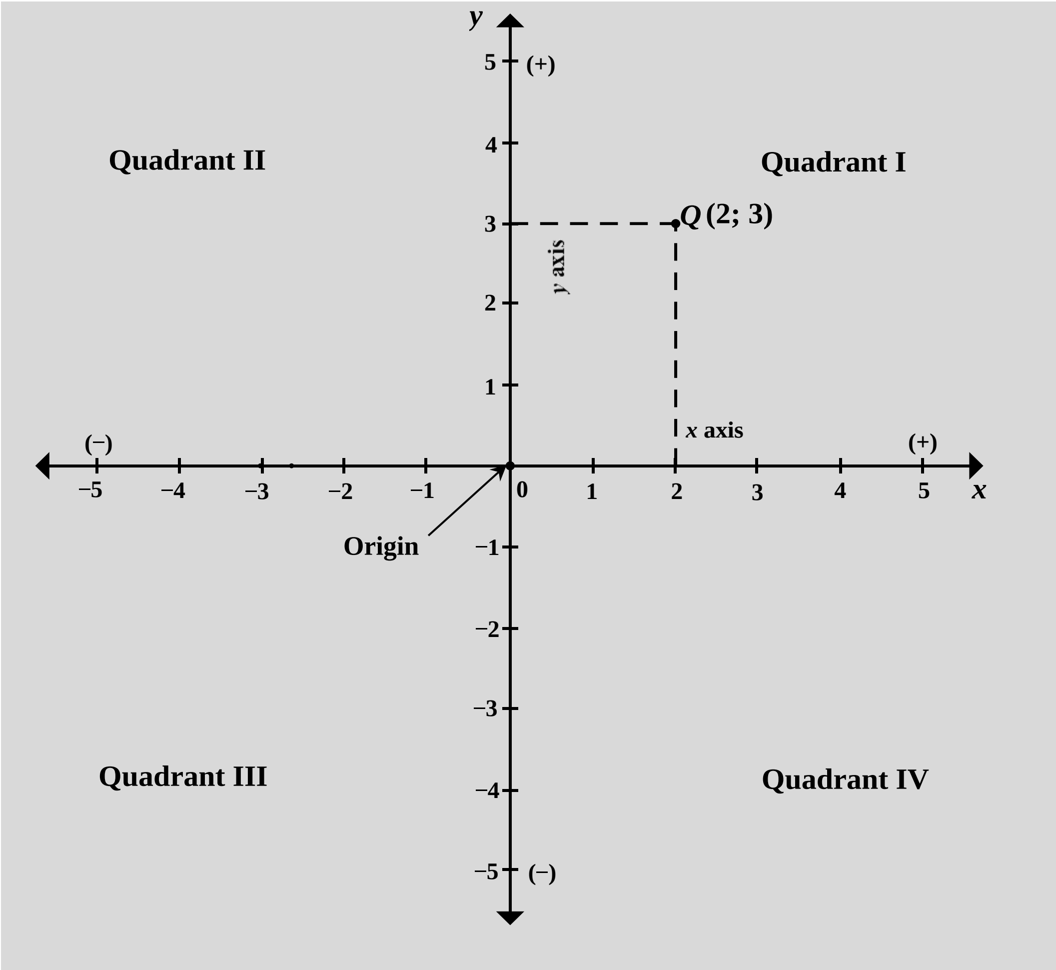


Figure 2. 31: The rectangular coordinate system

Thus, to explain the relationship between the trigonometric functions defined in a circle and the rectangular coordinate system, Figures 2.30 and 2.31 are merged as shown in Figure 2.32. Immediately it may be observed that the quadrant boundaries can be represented by the axes (see, Figure 2.32). Moving along the *x-axis* from the origin, positive to the right represents the 0° or 360° boundary and negative to the left represents the 180° boundary. Moving along the *y-axis* from the origin, positive up represents the 90° boundary, and negative down represent the 270° boundary, as shown in Figure 2.32. Once more, the horizontal distance from the origin of point B' is $rcos\ B\hat{A}\ B = AI$ units, and the vertical distance from the origin of point B' is $rsin\ B\hat{A}\ B = IB$ uni, where r is the radius of the circle, and is always positive and constant. In the context of this study, the radius r of the circle is the hypotenuse of the right-angled triangle embedded in a circle. The coordinates of the point B' can be written as $B'(\ rcos\ B\hat{A}\ B;$ $rsin\ B\hat{A}\ B)$. Progressively, the general angles are defined by a rotating radius, so that, for any given angle, one can associate the quadrant in which the angle's corresponding radius lies. That way, the sign of the trigonometric functions can be easily figured out for the various quadrants on the coordinate system.

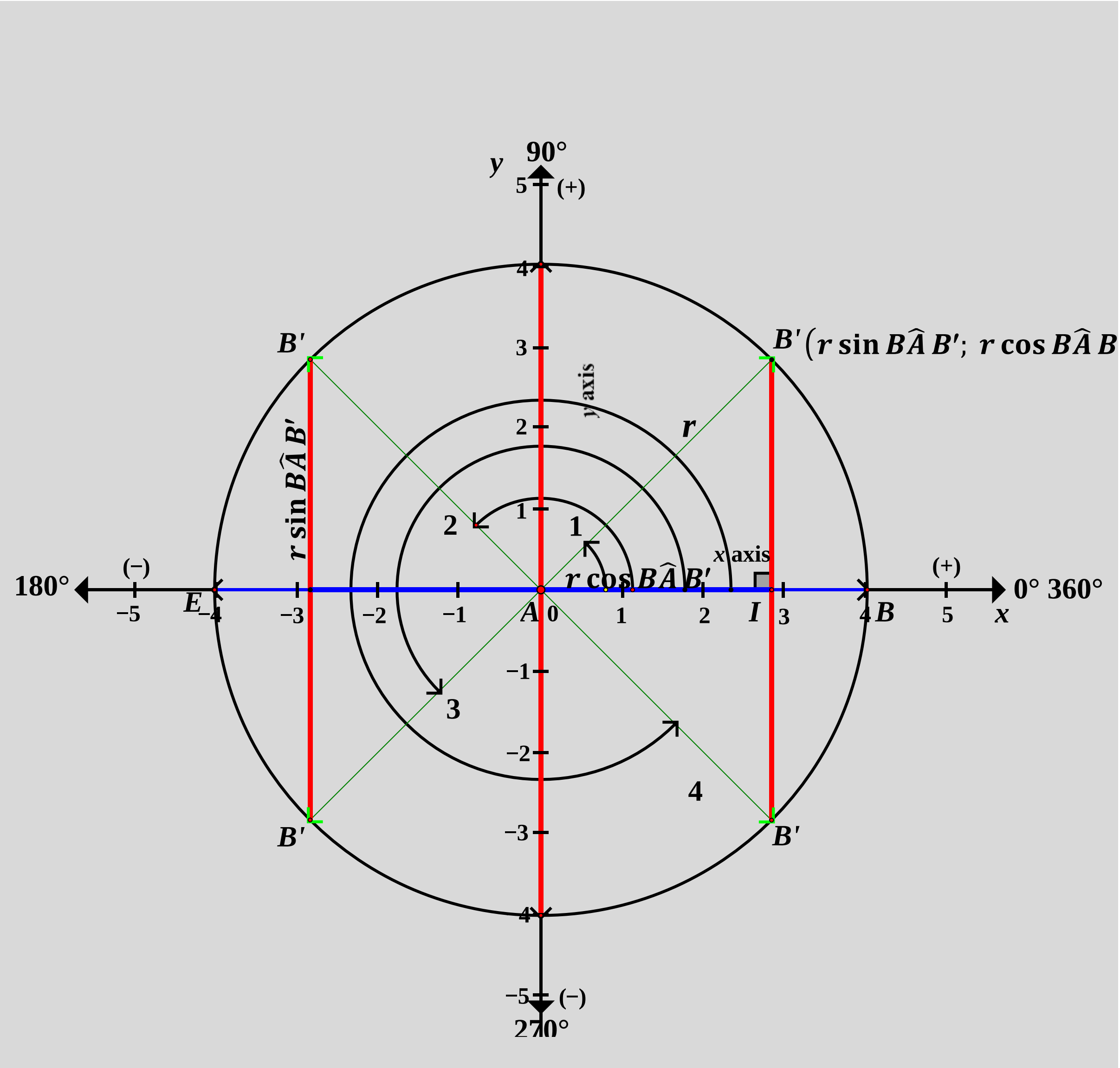


Figure 2. 32: The relationship between the trigonometric functions defined in a circle and the rectangular coordinate system

2.10.4.6 The meaning of $sin\ 30°$

One way to explain the meaning of $sin\ 30°$ is to construct an *equilateral triangle*[19] embedded in a circle of radius r units, as shown in Figure 2.33 **X**. In Figure 2.33, **Y** chord *IH* and the central angle *ICH* are bisected by diameter *KD*, resulting in the formation of two congruent 30°–60°–90° right angle triangles *CHR* and *CIR*. Thus, a right-angled triangle *CHR* embedded in the circle where the hypotenuse (*CH*) is the radius, and *RH* is the side opposite 30° (the angle at the centre) is observed in Figure 2.33 **Z**. Hence $RH = \frac{1}{2}r = r \sin 30°$, and by applying the Pythagoras Theorem[20], $CR = \frac{\sqrt{3}}{2}r = r \cos 30°$.

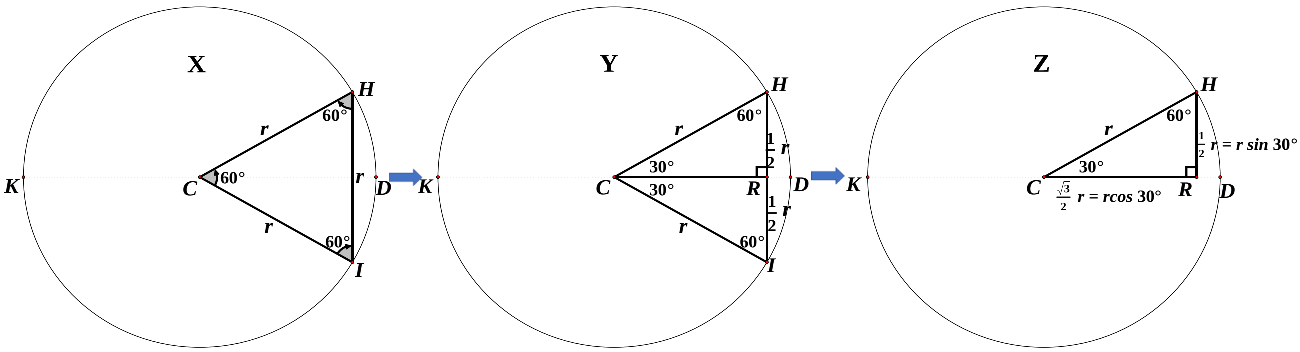


Figure 2. 33: Process that explains the meaning of sin 30° and cos 30°

In accordance with the discussion in Section 2.10.4.5, the probable meaning that the researcher anticipates teachers hold is that $sin\ 30°$ is a length that is 0.5 times as long as the radius of the circle, and constituted by the length of the side opposite the centre of the circle (see, Figure 2.34). In other words, $sin\ 30°$ is equal to 50% of the radius of the circle. Based on the meaning of the $sine\ 30°$, then an anticipated meaning of $sin\ 90°$ (Figure 2.35) is that it is the length that is 100% of the radius of the circle. In other words, it is the length $RH = r\ sin\ 90° = r$ (the hypotenuse CH of right angle triangle *CHR*). The expected meaning of $cos\ 100°$ is that it is the length that is 17% of the radius length made by the side adjacent 100° along the x axis in

19. An equilateral triangle is a triangle in which all three sides are equally long. It also has all equal angles, each 60°.
20. Pythagoras Theorem: In a right-angle triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.

the negative to the left direction from the origin (Figure 2.35). In other words, $CR = r\ cos\ 100° = -0.17r$.

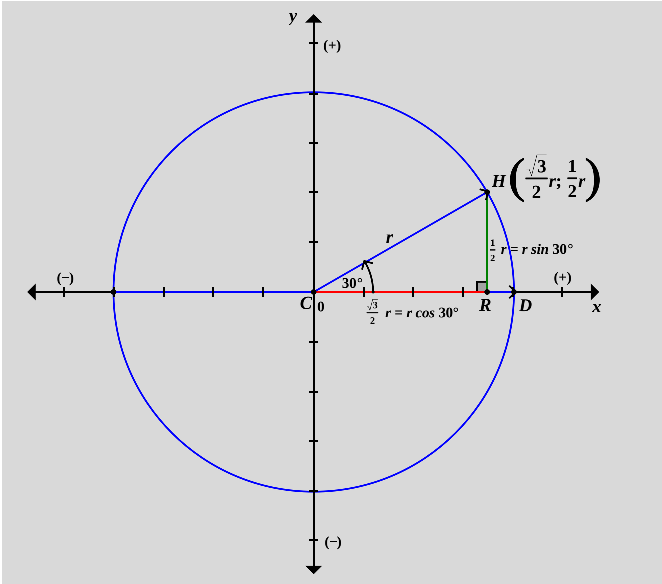


Figure 2. 34: Sin $\boldsymbol{30°}$

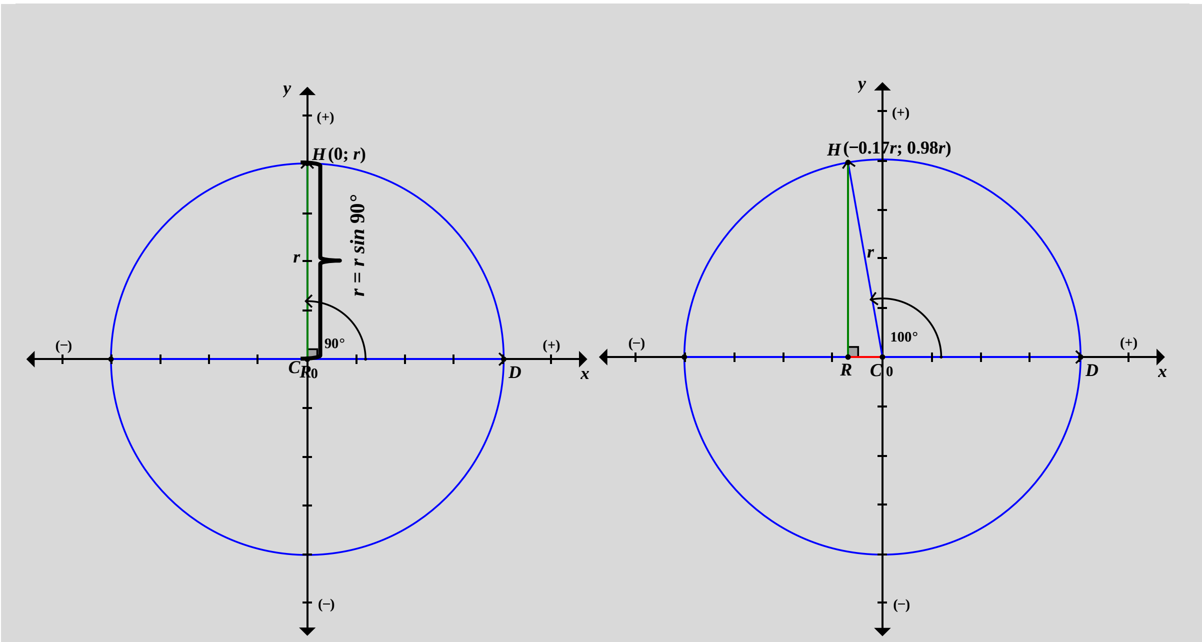


Figure 2. 35: Sin $\boldsymbol{90°}$ ***and cos*** $\boldsymbol{100°}$

2.10.4.7 Determining an output of the sine and cosine

Item 3 of this task explored teachers' understanding of what it means to determine the output of the sine and cosine functions (Appendix 1). Teachers were expected to use the idea of an arc length as a measure of an angle, and the sine and cosine functions as processes relating the values of the two measures (see, Section 2.10.4.1 to Section 2.10.4.6). Given the right-angled triangle *ABC*, to determine the output of the *sine* and *cosine* of angle *ABC*, teachers could use the length of the hypotenuse (*BA*) as the radius to draw circle *B* (Figure 2.36). Then, using the subtended arc length *DA*, the teachers could think of measuring each side of the triangle relative to the hypotenuse to apply the sine and cosine functions to the right-angled triangle embedded in a circle. In this case, for example, the output of the sine of angle *ABC* is the length made by the side opposite angle *ABC*, which is *CA* = 60% of the radius (hypotenuse *BA*) length.

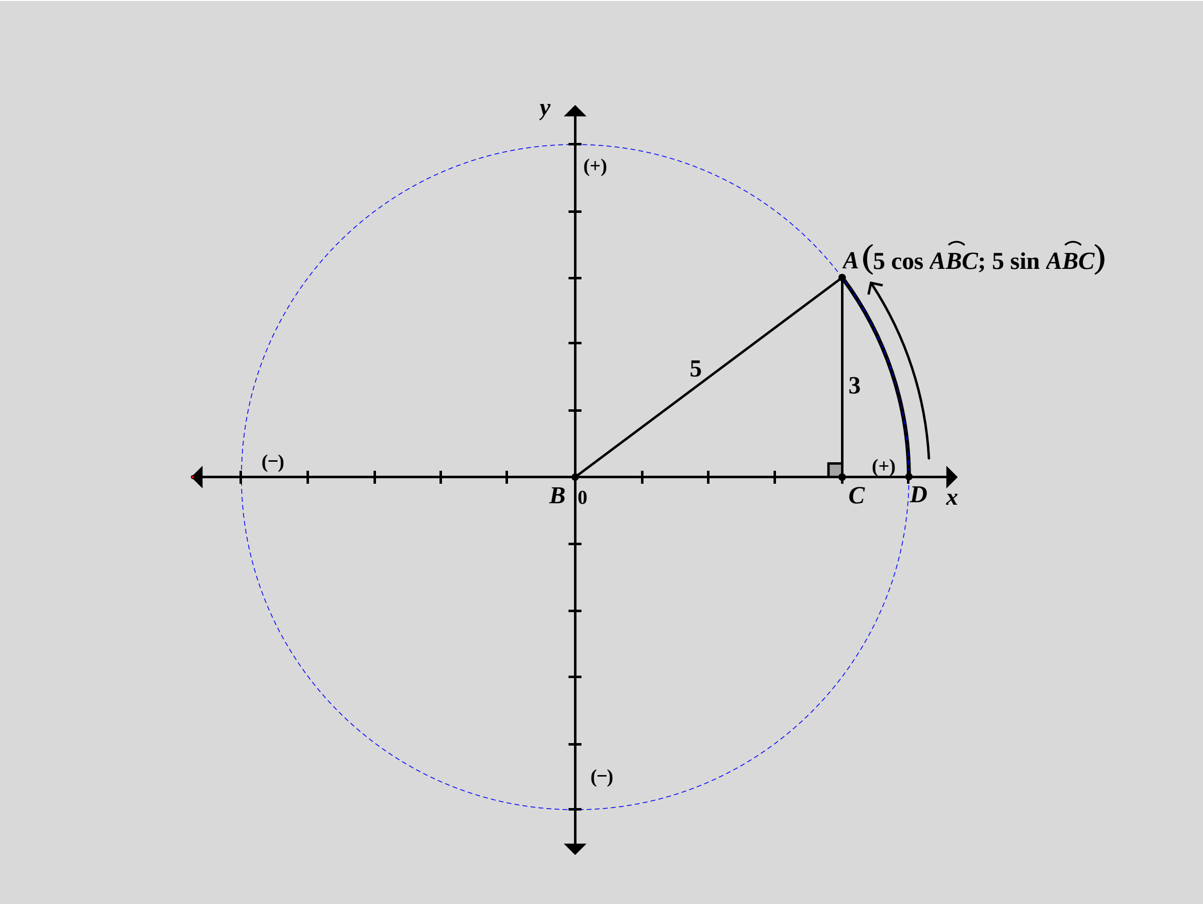


Figure 2. 36: Determining the output of sine and cosine of angle ABC

2.10.4.7 The meaning and the usage of the unit circle

Before addressing the meaning and use of the term "unit circle", the researcher explains that the radian[21] angle measure comprehension occupies a crucial role in the instruction of trigonometric functions and other related topics in mathematics. In Section 2.10.4.1, the meaning of an angle is discussed, and, in Section 2.10.4.2, the meaning of an angle measure is presented. In Section 2.10.4.3, the degree is introduced as the standard unit for the angular measure; however, the other unit that mathematicians use for angle measure is called the radian. According to Moore (2013), radian angle measures utilize the radius of a circle as a unit of measure when quantifying a subtended arc. For example, Figure 2.37 shows circle *C* with a central angle subtending an arc, whose length is equivalent to the radius of the circle. The angle measure of the central angle is then said to be one radian (see Figure 2.37).

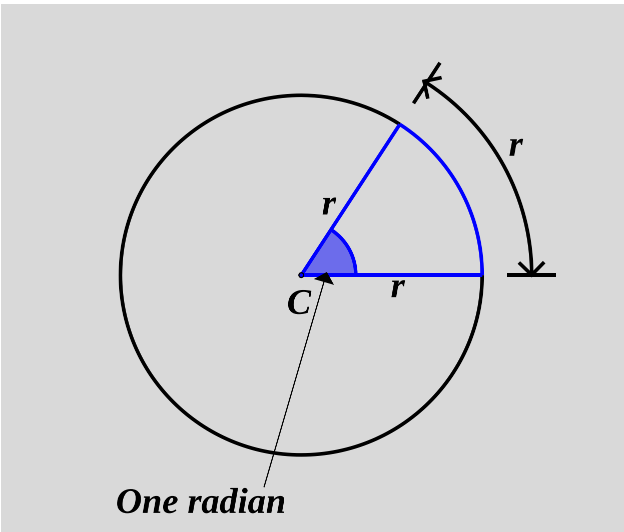


Figure 2. 37: The radian angle

Henceforth, an arc that is equal to twice the radius subtends a central angle that is two radians, and so on. Along these lines, it can be shown that an arc with a length of 2π times the radius subtends a central angle of 2π radians. Thus, a radian angle measure has a direct relationship

21. The radian unit is defined as the measure of an angle with its vertex at the centre of a circle and with an intercepted arc on the circle equal in length to the radius. In general, the radian is defined as the ratio of the length of arc on the circle that an angle intercepts on a circle to the length of its radius (Ewen & Nelson, 2011, p. 430). The radian angular measure is not taught in mathematics classes in the RSA public schools that follows or implements the CAPS policy.

with the length of the subtended arc and uses a part of the circle, the radius, as a unit[22]. It seems there are two angle measure meanings, degree and radian, that deal with the same concept, but are developed differently. For this study, the researcher agrees with Moore that in order to reconcile the two angle measure meanings, the angle measures must be viewed without considering units and must be accepted as "representative of the same quantitative relationship" (Moore, 2013, p. 227). Although some authors explain a radian angle as a dimensionless ratio and not a unit measure like a degree (for example, see, Calter & Calter, 2011). Mohr and Phillips (2015) maintained that a radian angle is a measurable attribute and that we can express the measured results, for example, in units of radians or revolutions. Hence, it is vital to put across a neutral angle measure meaning that encompasses both radian and degree angular measure meanings.

One way to achieve this reconciliation is to consider angle measure, radian and degree, as a process of establishing the number of parts that the central angle subtends out of the circle's circumference. In other words, we convey the length of the subtended arc as a fractional part of the circumference of the circle (see, Sections 2.10.4.2 & 2.10.4.3). Moreover, in the instance where a central angle measures 1°, then we say 1° out of a possible 360° is cut by the angle, so the length of the arc is $^{1}/_{360}{}^{\text{th}}$ of the circumference of any circle that has the centre at the vertex of the central angle. Equally, in 1 radian out of a possible 2π the angle cuts radians, so that the length of the arc is $^{1}/_{2\pi}$ of the circumference of any circle which has at its centre the vertex of the central angle. Accordingly, "radians and degrees measure the same quantity and are thus scaled versions of one another" (Moore, 2013, p. 227).

Radian and degree angle measures, to put it differently, are equivalent and proportional; moreover, the ratio of the length of an arc of a circle to the circumference of the circle equals the ratio of the degree or radian measure of the arc to 360°; hence, one can use the following proportion to show the equivalence of angle measures $\frac{\theta}{360°} = \frac{\varphi}{2\pi}$, where θ is measured in degrees and φ is measured in radians. Expressing angle measure as the length of a subtended arc and, subsequently, as the fraction of a circle's circumference is essential for facilitating

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22. Hence, this characteristic of a radian angle measure is the main reason why it is preferred in many applications, and useful in differentiating trigonometric functions. For example, the $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ when θ is measured in radians, whereas, the $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \frac{180}{\pi}$ when θ is measured in degrees.

reasoning about the unit circle. Moreover, the radian angle measure enables the comprehension of the unit circle because of its direct relationship characteristic with the radius of a circle. This direct relationship implies that the circumference of the circle can be measured in radii.

2.10.4.7.1 What is the meaning of the unit circle?

Some of the mathematics textbooks define the unit circle as the circle centered at the origin and having a radius equivalent to one ($r = 1$). However, these textbooks fail to explain what '1' represents or explain the relationship between '1' and the magnitude of the radius of a circle, in which the radius is not necessarily equal to '1' (Moore, 2013).

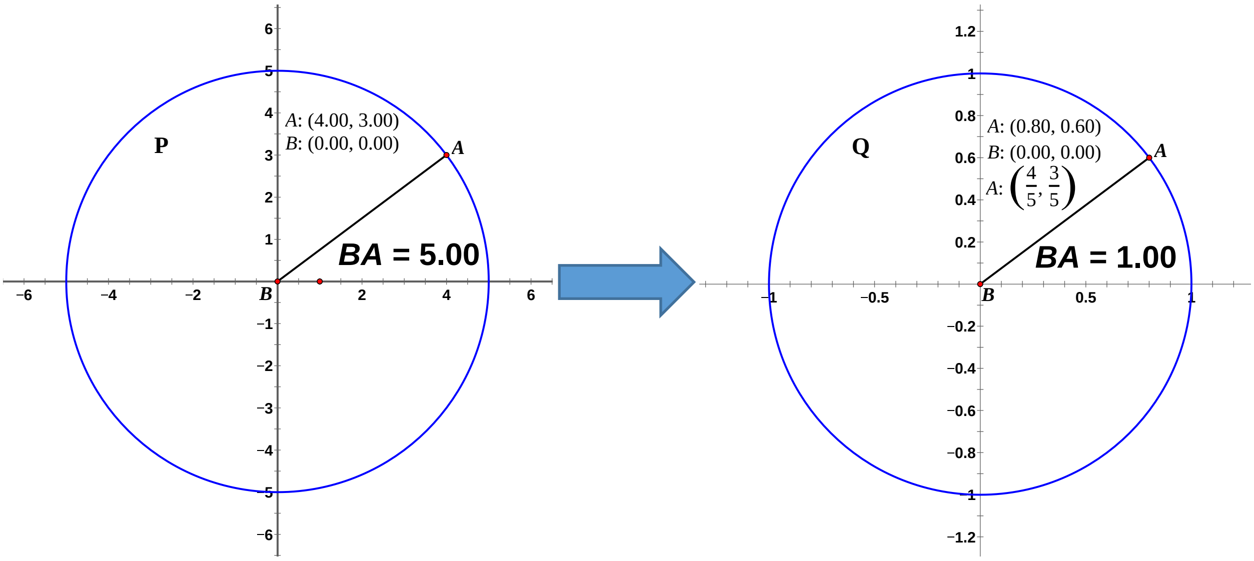

Figure 2. 38: Unitizing the circle using the radius as a unit of measure

The anticipated meaning of the unit circle that teachers should convey is that it is a circle resulting from the process of unitizing the magnitude of the radius. The process of unitizing the magnitude of the radius, in this case, means using the radius of the circle as a unit of measure (see, Moore, 2014, 2013, 2012, 2010; Moore, LaForest & Kim, 2016, 2012; Moore & LaForest, 2014; Thompson, 2008). For example, given a circle with a radius length of measure 5 *cm* (Figure 2.38 **P**), one can conceive 5 *cm* to mean 5 times as large as the magnitude of a centimeter. If one uses the radius as a unit magnitude, then measures in units of the radius will be $\frac{1}{5}$ times as large as corresponding measures in centimetres (Figure 2.38 **Q**).

To unitize a circle that has a measure of k radius length, one divides the measure by k[23], producing a radius length of one unit and all numerical values equivalent to those on the unit circle (see, Figure 2.38 **Q**). Thus, '1', in this instance, is a value that results after the measurement process of converting a given radius length to measure the radius as a unit magnitude. The researcher concurs with Moore *et al*. (2016) that the meaning of the unit circle founded on considering the radius of a circle as a unit magnitude necessitates useful, productive ideas that are advantageous in the learning of trigonometry. Moreover, Moore *et al*. elucidates that this meaning supports:

> (a) defining the unit circle in a way such that coordinates and angle measures on the unit circle are associated with a unit magnitude (i.e., the radius);
> (b) connecting the outputs of trigonometric functions to ratios and giving meaning to said ratios (i.e., measuring in radii); and,
> (c) constructing a meaning for the unit circle that encompasses a circle whose radius length is given in any unit other than radii (Moore *et al*., 2016, p. 225)

Moore *et al*.'s (2016) observations referred to earlier in this manuscript suggest that, by using the circle's radius length as a unit of measurement, where '1' is a value, trigonometric functions can be connected to any circle. This suggestion contrasts with the idea of relating trigonometric functions to only a circle of $r = 1$, where '1' is just a number. Furthermore, with the unit circle meaning that is under consideration being in place, it makes sense to define the sine and cosine functions as processes containing an input of angle measure. The meaning and the observation mentioned pave the way for the smooth development of the sine and cosine functions coherently in the context of the right-angled triangle embedded in the unit circle (Figure 2.39).

To elaborate, consider the right-angled triangle *ABD* in Figure 2.39, in which the sine and cosine functions have an input of central angle measure *CBA*, measured in degrees or radians, and output sides *AD* and *BD*, which have lengths in which their measures are given as percentages or fractions of the hypotenuse *BA*. The hypotenuse *BA*, in this context, is the radius

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23. Or one can multiply the radius by $\frac{1}{k}$.

of circle *B* (Figure 2.40). In the same vein, using the unit circle (Figure 2.40), the output of the sine function, is the *y* coordinate of the end-point *A* of arc *CA*, subtended by the central angle *CBA*, and the output of the cosine function is the *x* coordinate of the end-point *A* of arc *CA* subtended by the central angle *CBA*. Both the vertical (*y* coordinate) and the horizontal (*x* coordinate) distances are measured as a fraction of a unit (one) radius (Moore, 2010). Hence, the meaning of the unit circle that takes the radius as a unit measure, along with the ideas presented in the preceding paragraphs, allow for the coherent development of the *cosine* and *sine* functions (Figures 2.40).

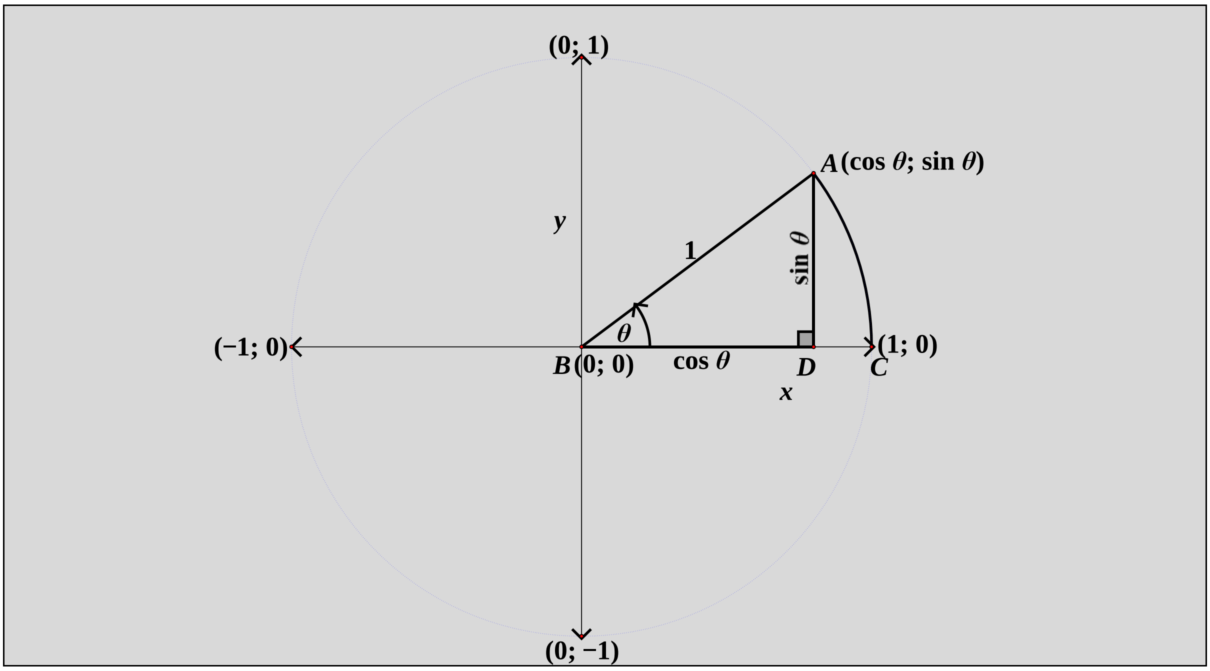

Figure 2. 39: Defining the trigonometric functions using the unit circle

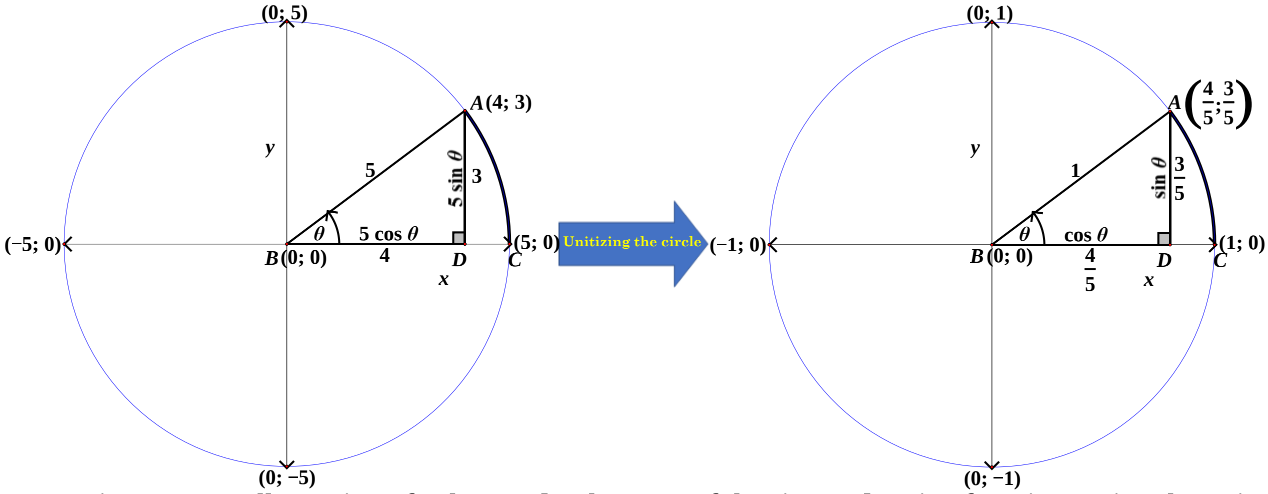

Figure 2. 40: Illustration of coherent development of the sine and cosine functions using the unit circle

Thus, for example, the outputs of the sine and cosine of angle measured at $\theta = 36.87°$ for the right-angle triangle *ABD* embedded in the unit circle as illustrated in Figure 2.40, are values $\left(\frac{3}{5}\ and\ \frac{4}{5}\right)$, and these values result from the process of multiplicative comparison. As such, one can define the $sin\ sin\ 36.87° \approx \frac{3}{5} = 0.60$ and $cos\ cos\ 36.87° \approx \frac{4}{5} = 0.80$. Moreover, the outputs of the sine and cosine of the inputted angle emerge naturally as ratios and are given as the portion of the unit radius respectively. Still, if the radius of the circle (that is, the hypotenuse of the right-angle triangle) increases, the outputs of the sine and cosine functions do not change (see Figure 2.41). The observation is consistent with Right Angle Triangle Similarity Theorem[24] from Euclidean Geometry. Thus, we deduce, if two triangles are similar, then their corresponding sides are in proportion, meaning the ratios of the corresponding sides are equivalent.

24. If an acute angle of one right angle triangle is congruent to an acute angle of another right-angled triangle, then the triangles are similar.

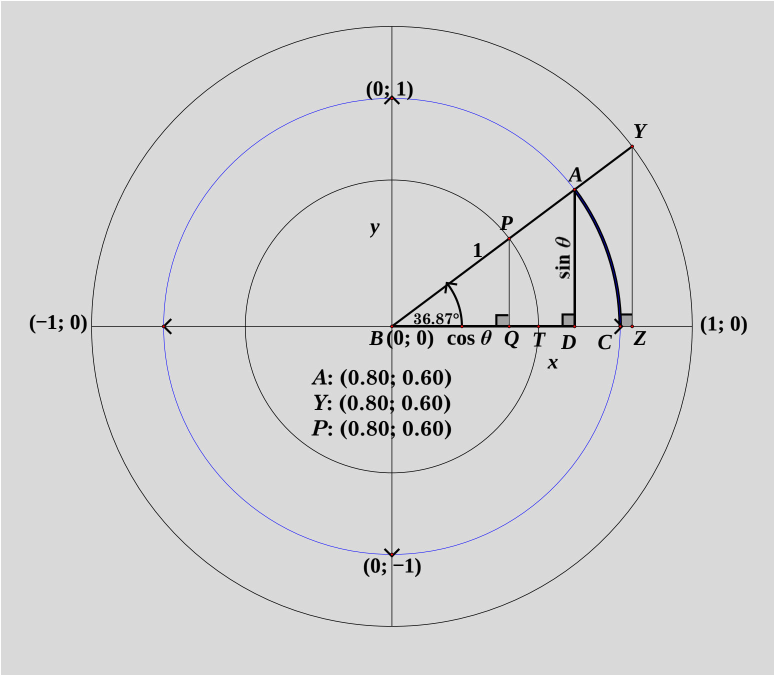

Figure 2. 41: The similarity property of trig functions

Additionally, if the angle measure varies and the radius remain as a unit ($r = 1$), then the output values of the sine and cosine functions change in tandem with the respective angle measures.

2.11 RESEARCHER' S CONCLUDING REFLECTIONS

This chapter is concluded by revisiting the notion of an effective teacher. This notion is presented in this section in terms of a general overview of the literature review, mainly in relation to "mathematical meanings", "tacit knowledge" and "coherence" (see, Sections 2.1; 2.2 & 2.3). According to Davis and Simmt (2006), the key metaphor for effective teaching and meaningful learning is maintaining coherence. An effective teacher is an instructor who is able to think like their learners, which, in the context of the current study, can be thought of as an effective teacher who does not adhere to rigid constructs of meaning-making, but is also able to anticipate how these conceptual meanings would be embraced and assimilated within varied systems of her learner's understandings. In this regard, teachers' mathematical meanings may not represent a rigid and non-conforming system of conceptual connectedness, but a preserved and trustfully sustained statement of mathematical constructs and logic.

The word "logic" is used in the preceding line in relation to the explanation in Section 2.2.2. A logical system addresses and largely manifests elements of coherence in which all connecting elements are systematically arranged and are intelligible. An effective teacher is able to contextualize and customize her pre-constructed mathematical meanings to formulations and constructs that are meaningfully accommodated to her learners' systems of understanding. Any form of mathematical instruction that satisfies this description has the potential to generate productive and meaningful learning experiences. Hence, this study was aimed at accessing teachers' systems of mathematical meaning-making in order to determine how these meanings are transmitted to learners in Grade 10 classrooms. Probably, the mode of meaning transference should throw insights into the level of effectiveness and productivity of mathematics instruction.

2.12 CONCLUSION

The review of related literature focuses on tacit knowledge and theories of meaning and their impact on teachers' understanding of what they must teach in mathematics. Several research studies that have been cited situate teachers' meaning within the context of trigonometry, the South African mathematics curriculum, and types of knowledge that form part of a teacher's PCK. The findings from the literature review show that learners are not performing well at grade twelve in concepts related to trigonometry. The study then goes on to explore the level of teachers' mathematical meanings at Grade 10 in the topic of trigonometry, in which context they are responsible for laying the foundation upon which learners can base their understanding of more complex concepts. Hence, establishing their level of embedded mathematical meanings will inform strategies that are aimed at improving coherence when teaching trigonometry professionally. The gaps to be filled by the present study have here been highlighted, and the next chapter will endeavor to discuss the research methodology we have chosen to adopt in the present study.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 INTRODUCTION

The current study was motivated by two central observations: (1) to identify and problematize the paucity of similar studies in South Africa focusing on mathematical meanings as they are conveyed by teachers to facilitate Grade 10 lessons in trigonometry; and, (2) the researcher's continued interest in the topic stemming from their observations of classroom practice[25] and exposure to literature addressing similar subjects. Mainly, the current study has explored mathematical meanings that are potentially held and conveyed by mathematics teachers when teaching topics of *trigonometric ratios and functions* in Grade 10. The setting for this investigation was six schools in the Tshwane[26] South District of Gauteng Province, in South Africa (see, Section 3.5).

This chapter will discuss the research methodology that the current study employed to facilitate this scientific enquiry, but we will first endeavor to define precisely what we mean when we refer to a *research methodology*. According to McMillan and Schumacher (2010), a research methodology is a systematic and purposeful adventure organized to generate data on a particular research problem, with research methods referring to the techniques used by the researcher to collect and analyze data. In this chapter, the researcher begins by restating the research questions, and subsequent discussions reflect on the philosophical position within which the study is located. The research design and approach that guided the study are also discussed in turn, with a section that focuses on the target population and sampling of the study also being presented. Subsequently, the researcher presents the data collection methods as well as data analysis techniques. In light of all this, the research that was conducted largely pertains to the qualitative issues of scientific rigor in a data collection process (i.e. namely credibility

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25. The phrase *classroom practice* was loosely used in this study as a reference to a mathematics teachers' observed pedagogical actions manifested in the classroom during a course of delivering and managing a Grade 10 lesson of trigonometry.
26. At the time of conducting the current study, the word *Tshwane* has been popularly adopted and used interchangeably with the old city term of *Pretoria*. The shift from Pretoria to Tshwane was politically motivated due to a notable change in government in South Africa in 1994. The word Tshwane was generally adopted, as it was perceived as more representative of the vernacular and colloquial term used by most people who stayed in the city of Pretoria and its surrounding townships during the regime change.

and trustworthiness), which issues were also presented within the confines of a qualitative research paradigm.

3.2 THE RESEARCH QUESTIONS

The current study was guided by the following research questions (RQs):

RQ1: What mathematical meanings do teachers convey when teaching a topic of trigonometric ratios and functions in Grade 10?

RQ2: In what way(s) do decisions related to teachers' knowledge influence their instructional actions during a Grade 10 trigonometry lesson?

3.3 THE RESEARCH PARADIGM

The researcher should subscribe to an understanding that, when undertaking a meaningful exploration of a qualitative research method, it is important to consider the underlying research conventions that a chosen research method should comply with (Willis & Jost, 2007). Thus, it is important to consider research approaches as being underpinned and influenced by different research paradigms (Cohen, Manion & Morrison, 2013). A research paradigm is defined by Richards and Schimidt (2013) as "a conceptual framework of beliefs, theoretical assumptions, accepted research methods, and standards that define legitimate work in a particular science or discipline" (p. 418). Okeke and van Wyk (2015) explained a research paradigm "as a sort of 'camp', to which a researcher belongs in terms of assumptions, propositions, thinking and approach to research" (p. 21).

Such a consideration provides a window of understanding into the researcher's beliefs about the nature of reality (ontology), and the nature of knowledge (epistemology) (Krauss, 2005). Positivist and interpretive research paradigms are the dominant paradigms that are largely discussed in the research community, with the current study being located within a qualitative interpretive paradigm. According to this interpretive paradigm, researchers understand and interpret their findings through the semantic lens that participants project onto the phenomena under consideration. In other words, researchers do not find knowledge, but they construct it, such that the ontological assumption that underpinned this current study's viewpoint was that reality is a social construct from which researchers themselves are by no means exempt.

According to this research approach, there is no single observable fact; preferably, there are multiple interpretations of a single event (see, Merriam, 2009; Okeke & van Wyk, 2015). Krauss (2005) summarized that “in general, qualitative research is based on a relativistic, constructive ontology that posits that there is no objective reality” (p. 760). In contrast to the interpretive paradigm, the positivist approach holds the view that researchers should strive to capture reality through multiple methods to approximate reality. Okeke and van Wyk (2015) noted that:

> positivists believe that reality is separate from the individual who observes it. They consider the subject (the researcher) and the object (the phenomenon being researched) to be separate, independent things (p. 23)

The epistemological stance that was adopted in this study assumes that researchers (in mathematics education) come to know reality thorough explorations of people’s understanding regarding the phenomena in question (see, Ernest 2016, 2012, 1997). Moreover, many qualitative researchers’ epistemological assumption is that “the best way to understand any phenomenon is to view it in its context” (Krauss, 2005, p. 759). Constructivism is closely associated with the interpretive paradigm, since it recognizes that truth is relative and largely dependent on an individual’s perspective (Goldin, 1990).

The perspective that social constructivism can be reduced to social interaction and communication, which is favored by the author of this manuscript, is perceived as such in order to provide an avenue through which the personal, subjective dimension of meaning making can be taken into account. The current study intended to access teachers’ forms of knowledge associated with mathematical meanings held by participants and, in the process, the study explored how this form of knowledge potentially held by the participants was eventually conveyed to learners when a selected topic of trigonometry in Grade 10 was taught. The researcher interacted with participants to examine the status of their mathematical meanings for teaching trigonometry, such that the exploration and understanding of participants’ ideas and perceptions fell within the confines of their social constructivist ideals (Cohen *et al.*, 2011; Ernest, 2012; Goldin, 1990).

3.4 THE RESEARCH DESIGN

McMillan and Schumacher (2014) describe a research design as "procedures for conducting the study, including when, from whom, and under what conditions the data will be collected" (p. 28). Dhlamini (2012) emphasizes that the research design is contingent upon the purpose of a research study, with most researchers regarding the research design as an architectural plan or a conceptual structure to map out and guide the research process. Most researchers predominantly use qualitative and quantitative research approaches, with the difference between the two approaches carrying significant implications for the nature of the design and the types of conclusions drawn from the study. Within the qualitative and quantitative research approaches, there are various research designs, and the qualitative research design emphasizes gathering data on naturally occurring phenomena, whereby most data gathered occur in the form of words. Qualitative research designs include the following research approaches: phenomenology, grounded theory, ethnography, case studies, and some critical studies. Quantitative research designs, however, emphasize objectivity in measuring and describing phenomena (McMillan & Schumacher, 2014), in which a quantitative research design is understood to maximize objectivity through using numbers, statistics, structure, and control. Quantitative research designs employ the following research approaches: survey, correlational designs, causal-comparative designs, experimental designs and critical studies.

The current study followed a case study approach that also embraces elements of an exploratory research design, which is mostly associated with a qualitative approach; that is, it is classified as a qualitative research method (Johnson & Christensen, 2012). Other researchers classify exploratory research as a mixed methods research approach (see, Creswell & Plano Clark, 2011; McMillan & Schumacher, 2014). In this context, Crewell and Plano Clark (2011) view an exploratory research design as comprising a two-phase sequential design "because the researcher starts by qualitatively exploring a topic before building to a second, quantitative phase" (p. 86). An exploratory research design is usually used when the researcher seeks "to learn about some little-known phenomenon by studying it in depth" (Fraenkel & Wallen, 2009, p. 431). In an exploratory research design, the researcher may identify key issues and conduct the inquiry to gain insights and understanding of the phenomenon under consideration.

Merriam (2009) described a case study as "research that involves detailed descriptions and analysis of systems that are bounded" (p. 40). Again, a case study is consistent with testing, observing and interviewing when collecting data. During a data collection process, the researcher may also be regarded as the primary data collection instrument; and, as such, a case study design allows the researcher to determine what kind of data to collect before the research commences, and to choose an analytical method that is suitable for obtaining insightful answers to their research questions. Okeke and van Wyk (2015, p. 172) delineated the rationale for using a case study design:

- to explore a general problem within a focused setting;
- to generate theoretical insights in developing new knowledge;
- to test existing theories in reference to the case at hand; and,
- to shed light on other similar cases to provide a level of generalization that takes into account relevant literature and contextualizes the case study in relation to the studies which precede it.

In the current study, a case study design was adopted to explore, interpret and describe the mathematical meanings teachers associate with the trigonometric concepts they teach in Grade 10. Mathematics teachers participated in the study and demonstrated the feature of bounded systems, thereby allowing researchers to involve themselves to a greater extent in the data collection process and function themselves as the study's primary data collection instruments.

3.5 POPULATION AND SAMPLING

The issue of sampling arises directly and naturally in the process of narrowing down and defining the population on which the research is based. Thus, the sample population accessed in the study will be described in the sections that follow, along with the sample and sampling procedures with which they were obliged to comply.

3.5.1 The population of the study

The population of a study denotes the larger group to which the researcher wishes to apply the results of their study. Johnson and Christensen (2012) defined the population as "the set of all elements, and the large group to which a researcher wants to generalize his or her sample results" (p. 218). The population in the current study comprises teachers who taught mathematics in Grade 10 in schools located in the Tshwane South District of the province of

Gauteng[27] in South Africa. At the time of conducting this research, the Tshwane South District consisted of 57 urban state-owned secondary schools that primarily offered mathematics as one of the subjects in Grade 10. At this time, the district was divided into five circuits, which catered for schools located in designated geographical areas.

It seemed that the number of schools in the circuits was not necessarily defined by a pattern, but a product of schools located in relatively close geographical proximities. The number of schools in these clusters always varied. Table 3.1 shows the number of schools in each circuit and the number of Grade 10 mathematics teachers in each district at the time of conducting the current study.

Table 3. 1: The current profile of schools whose teachers participated in the study

Circuit[28]	**Number of schools**	**Number of Grade 10 mathematics teachers**
TS-1	11	30
TS-2	9	17
TS-3	15	35
TS-4	9	20
TS-5	13	35
Total	**57**	**137**

The information in Table 3.1 was compiled from the sampling frame[29] that was supplied by the Department of Basic Education (DBE) in the Tshwane South District (TSD). The sampling frame consisted of all 57 schools in the district and the documented number of mathematics teachers in all schools at each grade level. Most of the schools in the sampling frame had an average of two to three Grade 10 mathematics teachers.

3.5.2 The sample of the study

According to Johnson and Christensen (2012), "a sample is a set of elements taken from a larger population according to certain rules" (p. 217). A sample is the sub-group on which a

27. Gauteng is one of the nine provinces in South Africa. The name Gauteng is a SeSotho word for gold, which is known to be an abundant natural resource in Gauteng.
28. The abbreviation TS in Table 3.1 stands for Tshwane South. TS-1, for instance, refers to Tshwane South Circuit 1, TS-2 refers to Tshwane South Circuit 2, and so on.
29. The phrase *sampling frame* refers to a list that consists of individuals who meets the description of the study population from who the eventual study sample will be drawn.

researcher obtains information (Fraenkel & Wallen, 2009). Ideally, researchers would prefer to study the entire population of interest; however, this is usually not feasible due to study populations often being too large and widely dispersed geographically (Fraenkel & Wallen, 2009, p. 90). For instance, the population of the study, which consisted of Grade 10 mathematics teachers, numbered 137 in total, and was widely spread out across five circuits (see, Table 3.1). It was not possible to reach out to all the teachers in the entire district given the time constraints associated with this type of study. Twelve teachers out of 137 Grade 10 teachers in the Tshwane South District were sampled for participation in the study (Section 3.5.3).

3.5.3 Sampling procedures

The definition of a sample in Section 3.5.2 highlights the fact that the group is selected "according to certain rules" (Johnson & Christensen, 2012, p. 217). The researcher employs non-random sampling techniques to select six schools to be investigated. The schools were selected from one circuit out of the five circuits in Table 3.1. Random sampling methods ensure representativeness of the population in most quantitative studies. Qualitative studies are primarily characterized by sampling techniques with a limited degree of representativeness (Figure 3.1), with several non-random sampling techniques being employed in qualitative research studies to facilitate planned research activities.

In this study, the researcher selected the schools using convenience sampling techniques (Gay, Mills & Airasian, 2011). A convenience sample is a group of people who are conveniently accessible and available for the study (Fraenkel & Wallen, 2009; Gay *et al*., 2011). To illustrate the value of using a convenience sampling technique, we may think of a university professor who might find it easier to work and study a group of students sitting in a relatively convenient and accessible location in the lecture room (see, Figure 3.2). According to Gay *et al*. (2011), convenience sampling techniques are used to find "whoever happens to be available at the time" (p. 140). In the case of a university professor, the students in the front row would be easily accessible and available for research (Figure 3.2).

It is usually difficult to secure the participation of schools in research studies due to schools' fear that the research activities might interfere with planned teaching and learning activities. The researcher then approached schools in which the relationship between the researcher and the schools already existed. Twelve teachers from the sampled schools participated in the study. Several studies have used the convenience sampling techniques to facilitate educational research studies (for examples, see, Dhlamini, 2012; Gweshe, 2015; Mji & Makgato, 2006). Convenience sampling techniques features have both advantages and disadvantages. For example, due to the reliance of the technique on available and willing participants, convenience sampling has disadvantages that might include sample bias and at times lack of representativeness (see, Gweshe, 2015; McMillan & Schumacher, 2010; see, also, Figure 3.1). According to Gweshe (2015), the strengths of a convenience sampling are that, (1) it is less costly; (2) it may secure higher participation; and, (3) it has a smaller attrition rate, which results in the withdrawal of participants from the research.

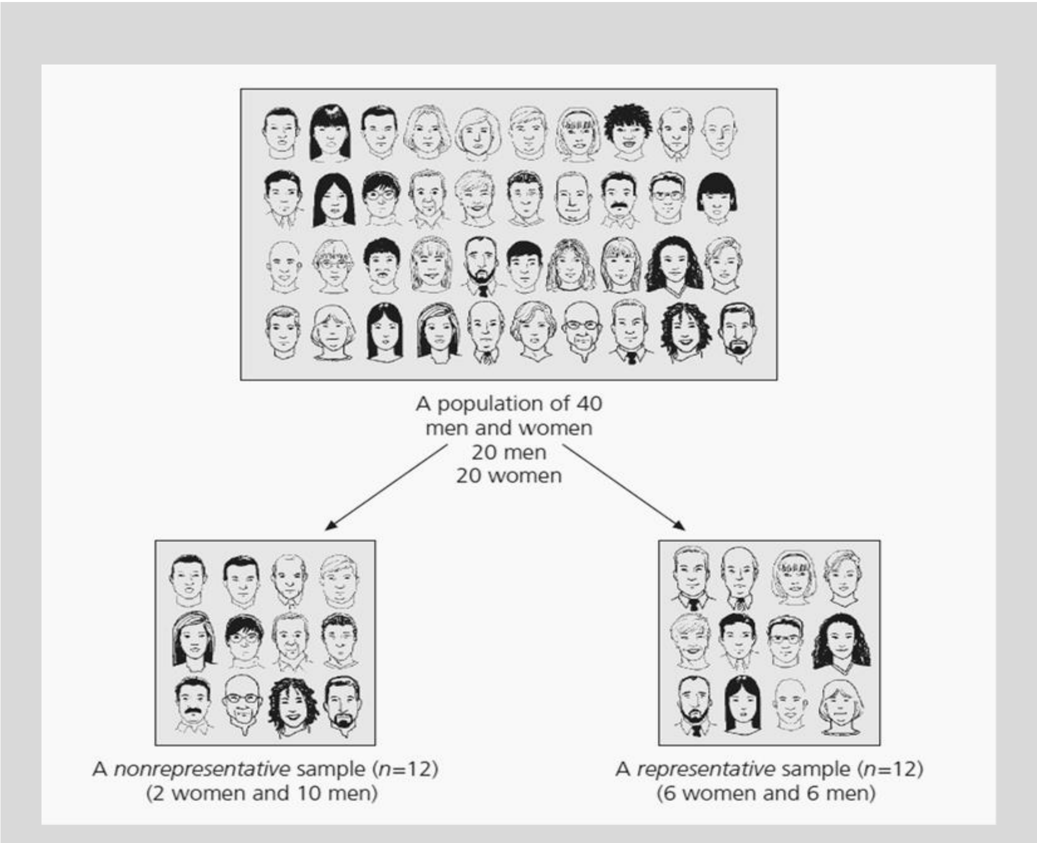


Figure 3. 1: Clarifying issues of representativeness and non-representativeness in research
Source: *Fraenkel and Wallen (2009, p. 92)*

After administering the *meanings-exploring task* on twelve teachers (Appendix 1), the researcher purposively sampled two teachers for participation in the follow-up semi-structured interviews (Appendix 2). The interviewees were sampled on the basis of their task responses that potentially generated probing possibilities to seek further clarity in an attempt to access and examine teachers' mathematical meanings. The choice of a sample size in this study was selected in keeping with the guideline that "the sample size depends on the purpose of the study and the nature of the population under scrutiny" (Cohen, Manion & Morrison, 2002, p. 93), and the fact that in qualitative studies "it is more likely that the sample size will be small" (p. 93). The study extensively examined teachers' meanings of their knowledge for teaching. Teachers were given a mathematical task or tasks on which they were probed in terms of how they understand (or the meaning(s) they attach to the task). The length and depth of the probing sessions varied from one teacher to another in light of the projected likelihood of varied responses from the respondents. As such, the nature of these research activities would not permit a relatively large sample to be investigated. The inherent implication of this was that the researcher could not generalize results concerning a study population derived from a qualitative study, but that the results could only be analyzed on a case by case basis.

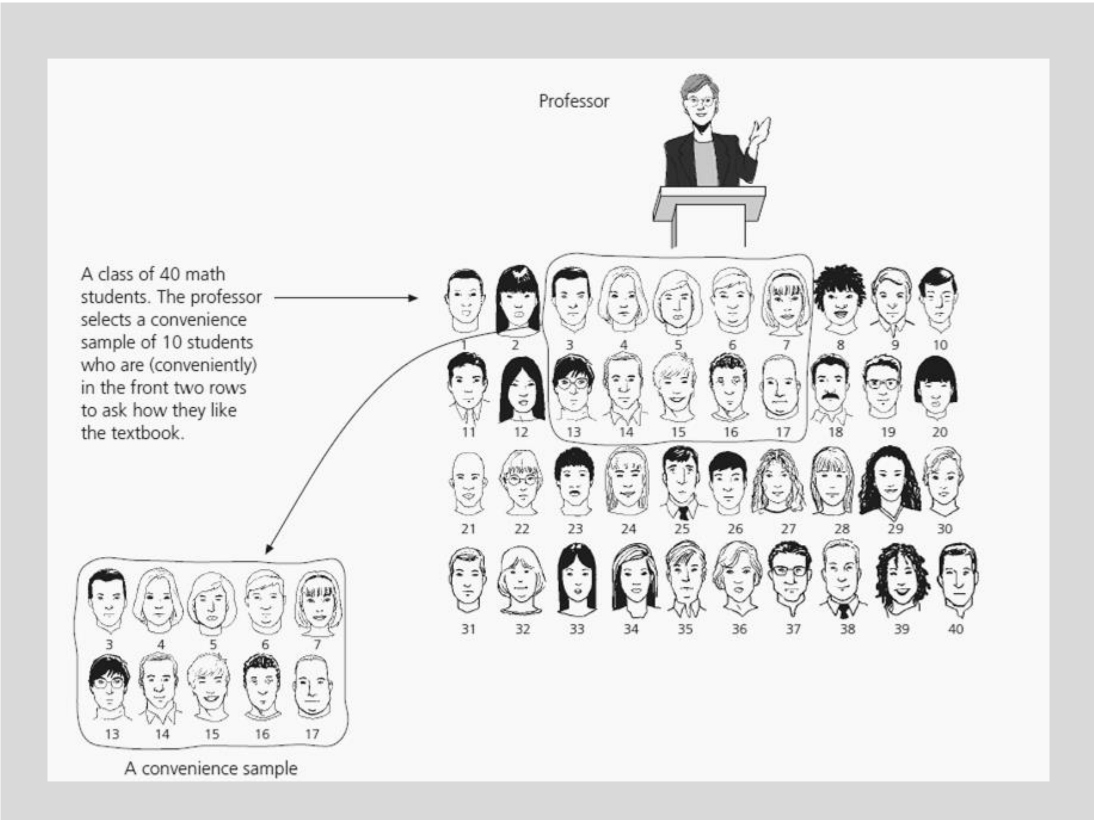


Figure 3. 2: A professor opting to employ a convenience sampling technique
Source: *Fraenkel and Wallen (2009, p. 98)*

3.6 INSTRUMENTATION

The term *instrumentation* refers to a detailed discussion of the methods of data collection, and the rationale for the choice and use of data collection instruments. According to Fraenkel and Wallen (2009), instrumentation is "the whole process of preparing to collect data" (p. 110). Such a process involves, therefore, not only the consideration of the instruments that are used to cull data and the process through which they were selected, but also the procedures that were employed to catalyze the creation of these instruments and how researchers would come to assess the rigor and trustworthiness of their data collection instruments.

3.6.1 Data collection instruments

The study has aimed to explore teachers' understanding of trigonometric ratios and functions by examining the mathematical meanings that teachers convey when they teach Grade 10 learners (Section 1.4). To operationalize our aims as researchers, we have:

- developed mathematical tasks pertaining to trigonometry to gauge teachers' responses to the task under consideration (Appendix 1); and,
- conducted semi-structured interviews with a purposive sample in mind (see, Appendix 2; see, also, Section, 3.5.3).

3.6.2 The purpose of the data collection instruments

In Section 1.2.2, it is explained that investigating teachers' mathematical meanings for the knowledge that they convey to their learners might involve an exercise in examining teachers' decisions in terms of how they think knowledge could be presented to their respective learners, and the kinds of instructional actions that teachers should articulate in facilitating student learning. These considerations influenced the purpose for which data collection instruments were constructed in the current study.

3.6.2.1 A trigonometry task

A trigonometry task that was given to teachers resembled Thompson's Mathematical Meanings for Teaching secondary mathematics (MMTsm) (Thompson, 2015). MMTsm is a 43-item diagnostic instrument developed for use in mathematics professional development (see, Thompson, 2015; see, also, Appendix 1). The task in the current study provided teachers with an opportunity to interpret a given scenario and respond to it in a manner that reflects on the meanings that they attached to certain mathematical concepts in the given task. All twelve teachers in the sample completed this task in writing. Which accorded the sample group of teachers with a place to write their responses (Appendix 1). In this way, the task would determine the extent to which teachers either distinguished or confounded meanings of certain trigonometric concepts in Grade 10.

The purpose of the trigonometry task was to expose teachers' conceptualizations of trigonometric concepts in relation to what they viewed as productive ways of handling these concepts instructionally. The items in the trigonometry task were designed to help the researcher gain insight into what constituted teachers' meanings in relation to the task at hand (Thompson, 2015). Items relating to the task probed teachers to provide meanings for the following concepts: angle measures, functions, and sine and cosine functions (see, Appendix 1).

3.6.2.2 Semi-structured interviews

According to Okeke and van Wyk (2015), an interview is "a face-to-face conversational engagement between two people where questions are asked by the interviewer in order to elicit responses that can be analyzed within qualitative research situations" (p. 297). Interviews are distinguished by how structured or unstructured they are (Gay *et al*., 2011; Merriam, 2009; Okeke & van Wyk, 2015). This study based its research on face-to-face semi-structured interviews, which generally consisted of a combination of structured and unstructured interviews. Given the exploratory nature of this study, semi-structured interviews were conducted to probe teachers further on their task responses (see, Appendix 2 & Section 3.6.2.1). The fundamental motivation in employing qualitative interviews in research was to strive to view the world through the participants' eyes; thus. the purpose of the semi-structured interviews was:

- to examine whether teachers understood the task item as being about what the researcher intended to do or not;
- to check if the items in the task would be able to draw the desirable responses; and,
- to gain insight into teachers' meanings.

Although the interview questions were predetermined, the researcher did not impose certain type of responses from the interviewees. The researcher strived to gain an understanding from the respondents' perspectives; and, in this regard, explanations of and justifications for their reasoning, as well as their level of understanding of trigonometric concepts, were elicited from the respondents through their responses.

3.6.3 The development of the data collection instruments

The task for teachers was drawn from the official DBE curriculum documents in the form of CAPS. The interview schedule was largely developed from participants' responses to the task. The items in the trigonometry task and the interviews were mainly constructed to address the aim, the objectives and the research questions of the study (see, Section 1.4; see, also, Section 1.5 & Section 1.6).

3.6.4 Dependability of the instruments

The notion of dependability is concerned with whether the researcher would obtain the same results if the researcher could use the same phenomenon twice or more (Kumar, 2011, p. 172). This condition might be viewed as synonymous with the quality of external validity in quantitative studies, in which the pre-condition of replicability is prioritized. External validity ensures that the results of the study are generalizable, with this feature hardly being viewed as possible in qualitative studies. Shenton (2004) argued that dependability in qualitative studies could be addressed when the research process is "reported in detail, thereby enabling a future researcher to repeat the work" (p. 71).

3.6.4.1 A trigonometry task

The researcher made special attempts to assess the dependability of the task under consideration and their accompanying probing questions. Firstly, all tasks were drawn from the state-approved textbooks, which in turn were validated. Secondly, the researcher adapted items largely from the Thompson's MMTsm, which is generally accepted as a valid tool for unearthing participants' meanings of mathematical concepts (Thompson, 2015). In addition, a panel of mathematics experts participated in the validation process of data collection instruments. The panel consisted of mathematics and mathematics education lectures, curriculum specialists and secondary school mathematics teachers. In this regard, each member of the panel provided commentary on the suitability of the task items and the interview schedule. The consultation with the experts was conducted in order to compute the task's possible intensities of dependability in exploring respondents' conceptual meanings.

Ideally, experts were asked first to ascertain whether the items were designed in such a way that made it possible for them to reveal their meanings when responding. Secondly, they were asked whether the items in question address the body of ideas that are encapsulated in the curriculum for which participating teachers are responsible. The comments made by experts on this subject highlighted the need for researchers to reformulate the task (i.e. review and recast certain task items), which suggestion was then forwarded to the research supervisor to arbitrate. The researcher responded to the supervisor's feedback, and this culminated in the production of a final revised draft of the task's items. The final draft of the trigonometry task consisted of 10 items addressing the following concepts: angle measure, functions, the sine and cosine functions, unit circle and right-angled triangle trigonometry (see, Appendix 1).

3.6.4.2 Semi-structured interviews

The literature review in Chapter 2 informed the process through which a semi-structured draft interview schedule was formulated, consisting of items addressing the issue of respondents' understanding of functions and trigonometric concepts. The researcher subjected the draft schedule to expert evaluation, and one of the lecturers in the Department of Mathematics Education (DME) at the University of South Africa participated in this process. The DME expert was asked to assess the issues in the interview schedule relating to its language, along with the clarity and comprehensibility of the items under review. The DME expert advised the researcher to rephrase certain items in the interview schedule, but nonetheless retain the originality of the interview schedule's items. At the end of this process, the revised draft of the interview schedule was generated consisting of newly reworded items, and the researcher then went on to conduct all their interviews with teachers using the interview schedule to preserve uniformity (Section 3.6.2.2).

3.6.5 Addressing issues of trustworthiness, triangulation, credibility, and transferability

Lincoln and Guba (1985) suggest that the trustworthiness of a research study is vital to evaluate its worth or soundness. Trustworthiness involves establishing credibility, transferability, dependability, and confirmability, with each of these terms being explained in the next sections in line with the way in which they were incipiently conceptualized and eventually operationalized in the current study to optimize the fidelity of the research process.

3.6.5.1 Dependability

Dependability, as we mentioned earlier, is concerned with whether researchers would obtain the same results if he/she could observe the same phenomenon more than once (Kumar, 2011, p. 172). This quality is difficult to achieve in qualitative studies, which often lack the apposite mechanisms to enhance their generalizability (external validity). Shenton (2004) argues that to address the dependability issue more directly, the research process in the study "should be reported in detail, thereby enabling a future researcher to repeat the work" (p. 71).

3.6.5.2 Credibility

Shenton (2004) emphasizes that, when addressing credibility in qualitative studies, one can ask the question: "How congruent are the findings with reality?" (p. 64). The triangulation of results can be an essential component in the effort to validate the trustworthiness (credibility) of a study's results. Also, techniques like member checking can be employed to authenticate the results. Which is to say, interview respondents are the best judges to determine if the research's findings truly reflect their views and feelings (Kumar, 2011).

3.6.5.3 Confirmability

This refers to the extent to which the research results can be "confirmed or corroborated by others" (Kumar, 2011, p. 172). According to Shenton (2004), the use of instruments for data collection should be viewed independently from human skill and perception. Among other options, in both the interviews and lesson observations, a question/item guide can be used to facilitate the data collection process, thus preventing human intervention to interfere too intensely with the research process. Kumar (2011) summarizes that confirmability "is only possible if both researchers follow the process in an identical manner for the results to be compared" (p. 172).

3.6.5.4 Transferability

Similar to the notion of dependability, transferability refers, according to Kumar (2011), to "the degree to which the results of qualitative research can be generalized or transferred to other contexts or settings" (p. 172). Transferability can be equated, in this sense, to the relative external validity of quantitative research, and the applicability of its research results to the broader community in which it was conducted. This effect is achievable in qualitative research studies when sufficient contextual information about the research site and its research activities are comprehensively publicized (Lincoln & Guba, 1985). One of the ways in which the trustworthiness of qualitative research can be established is through the use of multiple data collection methods (Okeke & van Wyk, 2015). To this end, the researcher utilized a task in trigonometry to elicit relevant data, along with semi-structured interviews.

The interviews provided an opportunity for researchers to verify teachers' understanding, as revealed by the trigonometric task (see, Section 3.6.4 for details on the development of the task). Enhancing the trustworthiness of the trigonometry task and the interview schedule, the researcher gave both the task and the interview schedule to a panel of reviewers to assess the credibility and transferability of items in each instrument. The panel consisted of high school mathematics teachers, a subject advisor from DBE and the university lecturer in mathematics education. During the administration of the final drafts of the interview schedule, the researcher asked teachers similar questions. To avoid bias and attempt to minimize subjectivity, the researcher did not allow his knowledge of trigonometry to influence the teachers' explanations.

Thus, the teachers were encouraged to explain their perspectives, and researchers refrained from providing leading questions to provide hints on whether participants were right or wrong. At the beginning of each interview session, teachers were encouraged to express their viewpoints during the interview without any reservations or fear. In this way, the researcher aimed to facilitate an informal atmosphere during the interviews. According to Cohen *et al*. (2011), triangulation is the use of two or more methods for data collection in the study of a particular aspect of human behavior. In the current study, triangulation was used to assess whether or not credibility and transferability were achieved through the use of semi-structured interviews and the trigonometric task, with the data under analysis being collected over an extended period of time.

3.7 DATA COLLECTION PROCESSES

3.7.1 Administering the trigonometry task on teachers

Administration of the trigonometry task was conducted in August 2016, at which time we incipiently invited all participating teachers to a meeting to discuss the study's procedure. Teachers were given the trigonometric task to complete at their convenience, and on each document, teachers were afforded a space to record their responses. After a week's time, researchers collected the completed tasks from individual teachers at their respective schools, and teacher responses were kept confidential for analysis.

3.7.2 Conducting the study interviews with teachers

Initially, the interviews were delayed because of the unforeseen circumstances and challenges that were beyond the control of the researcher. Ultimately the researcher conducted the interviews in January and February of 2018. After mutually agreeing on a time and place to meet, two teachers were interviewed separately at one of the schools which provided one cohort of participants (see, Section 3.5.2). In order to acquire valuable data from respondents, researchers refrained from interrupting respondents as they gave their explanations to the best of their ability. The interviews were recorded using audio equipment, and, during each session, researchers provided each teacher with a copy of the interview schedule and their responses from the trigonometric task. Each teacher was provided with their written responses to the trigonometric task during the interviews in the event that they were unable to remember the answers they had submitted.

3.8 DATA ANALYSIS

The researcher developed an open coding scheme to track teachers' responses and categorized ways of thinking, and then set about organizing the themes of these responses and the topics they address as constituting productive and unproductive meanings. In developing this tool to analyze the data of teachers' responses, the researcher drew inspiration from the constructs presented in Table 2.2 (Section 2.10.1). In addition, typological methods of analysis of the kind mentioned above were used to analyze the interview data (Section 3.4.2.2). Interview data was first transcribed using an audio recorder, producing three distinct transcripts in total. Afterwards, researchers read and reread the transcripts to develop intense familiarity with their content, after which point the data were categorized and organized into emerging themes and patterns of similarities. These themes and categories were then formulated to address the primary aim of the study (Section 1.4).

3.9 ETHICAL CONSIDERATION

McMillan and Schumacher (2010) note that educational research focusses primarily on human beings and, as a result, researchers are ethically responsible for protecting the rights and overall welfare of their study's participants. Before undertaking this study, the researcher applied to the Department of Basic Education (DBE) through the District Director to request permission to access and request schools to participate in the study (Appendix 6.3). The researcher also wrote consent letters to the principals of the schools and requested permission to conduct research (Appendix 6.4). After the DBE and principals granted the permissions, the researcher formulated request letters to select teachers, inviting them to participate in the proposed study (Appendix 6.6). All the teachers who agreed to participate in the study signed an informed consent form to confirm their participation (see, Appendix 6.7 & Appendix 6.8).

The consent letters to principals and teachers addressed issues pertaining to the nature of the research, provided a written guarantee concerning the confidentiality of the study and the anonymity of the participants (i.e. assured participants that their identities would be withheld), and informed participants that the results of the proposed study would be circulated to all participating schools. The results of the study were subsequently aggregated to protect the identities of participants and those of their schools. The letter clarified that participation in the proposed study was voluntary, and that participants would be free to withdraw at any stage of the research without incurring a penalty. Before the commencement of the study, the researcher applied for an ethical clearance certificate through the University of South Africa's (UNISA) Research Ethical Committee (REC).

3.10 CONCLUSION

This chapter has described the methodology and design of the research study, and began with restating restatement of the research problem being broached, its associated qualitative paradigm, and the research design that researchers believed is most appropriate to explore this topic. A qualitative case study, which is the research design that this study has employed, was also presented in detail in this chapter. The qualitative data collection methods have been discussed, along with the additional issues of the trustworthiness, dependability, credibility, and transferability of the study's data collection procedures. In closing, the study's ethical considerations have been highlighted. In Chapter 4, the researcher will present and analyze the results of the study at length.

CHAPTER FOUR

DATA ANALYSIS AND FINDINGS

4.1 INTRODUCTION

Chapter 3 provides a discussion of the research design, research methodology, and scientific procedures employed in this study for data collection. Section 3.6 laid out a comprehensive description of data collection tools and other related data sources for the study, and Section 3.8 presented a data analysis process. This chapter discusses the relevant data analysis and presents the findings of the study, which were gathered using qualitative methods (Section 4.2). The study aimed to explore teachers' understandings of trigonometric ratios and functions by examining the mathematical meanings that teachers convey when they teach Grade 10 learners (Section 1.4).

Twelve teachers actively employed as Grade 10 mathematics teachers participated in the study (Sections 3.5.2 & 3.5.3). The researcher used methods of qualitative analysis to make sense of teachers' responses to the task, which extended to an analysis of their projected mathematical meanings on selected mathematical topics. In addition, the analysis also helped to examine and make sense of teachers' instructional decisions that they manifest while conducting trigonometry lessons. The purpose of the data analysis was to use teachers' responses to answer research questions and, in the process, use this same conceptual framework to consider participants' responses.

4.2 DATA COLLECTION INSTRUMENTS AND RESEARCH QUESTIONS

The researcher administered a written task (trigonometric task) to the teachers who participated in the study, with subsequent semi-structured interviews being conducted to explore this data collection process further (Section 3.6.2.1). Table 4.1 shows the purpose of each data collection instrument and their observed relation to the research questions of the study (see, also, Section 3.6.2). Table 4.1 demonstrates that the data collection instruments were set out mainly to address discrete research questions.

Table 4. 1: Data collection instruments and related questions

Data collection instrument	Data source	Items related research questions
Written task	Teachers	1 and 2
Semi-structured interviews	Teachers	1 and 2

A written task set the stage to initiate the study-related activities, and this tool served to explore the meanings held by Grade 10 teachers on the subject of trigonometric functions and ratios (Section 3.6.2.1; Appendix 1). Subsequently, semi-structured interviews provided an opportunity for researchers to gain enhanced insights into teachers' knowledge of and their skills needed to facilitate classroom instruction related to trigonometric functions (Section 3.6.2.2; Appendix 2). Mainly, the researcher strove to gain insight into the analytical, meaning-making and interpretive actions of teachers when engaging with the written task of mathematical concepts relevant to Grade 10 trigonometry. Piaget's notion of assimilation to a scheme inspired the researcher to design a study that would be effective in eliciting insights into the mathematical CK of teachers (Section 2.4). It was the researcher's view that, in participating in the study, teachers would explicitly reveal conceptual meanings of mathematical knowledge that implicitly reside in their minds.

In keeping with Piaget's theoretical writings, the knowledge that respondents demonstrated was interpreted in relation to information assimilated to their cognitive schema and pertained directly to the mathematical concepts being treated in the study. In this regard, Piaget (1991) asserts that the meaning that is held in one's cognitive system is the function of what has been assimilated to a scheme. It is important to acknowledge that, as a distinct form of knowledge unto itself, a scheme represents the meaning that one already has; which is to say, a scheme constitutes an individual's understanding of assimilated information pertaining to a particular context or phenomenon (Piaget, 1991). Assimilation could be conceived as a process of taking in some information and attempting to attach a more consummate understanding to it.

4.3 ANALYSIS OF PARTICIPANTS' DEMOGRAPHIC VARIABLES

Twelve teachers who were teaching mathematics in Grade 10 participated in the study, which consisted of two females and ten males (Section 4.1; Figure 4.1). Participants were all, at the time the study was administered, employed as Grade 10 mathematics instructors, and participants were distinguished in terms of the characteristics of: (1) gender; (2) professional ranking; (3) mathematics qualifications; and, (4) teaching experience. The researcher would subsequently come to view these characteristics as essential factors determining the type of meanings that teachers held in relation to certain mathematical concepts. Figure 4.1 provides an outline of the distribution of these characteristics across the study's participants.

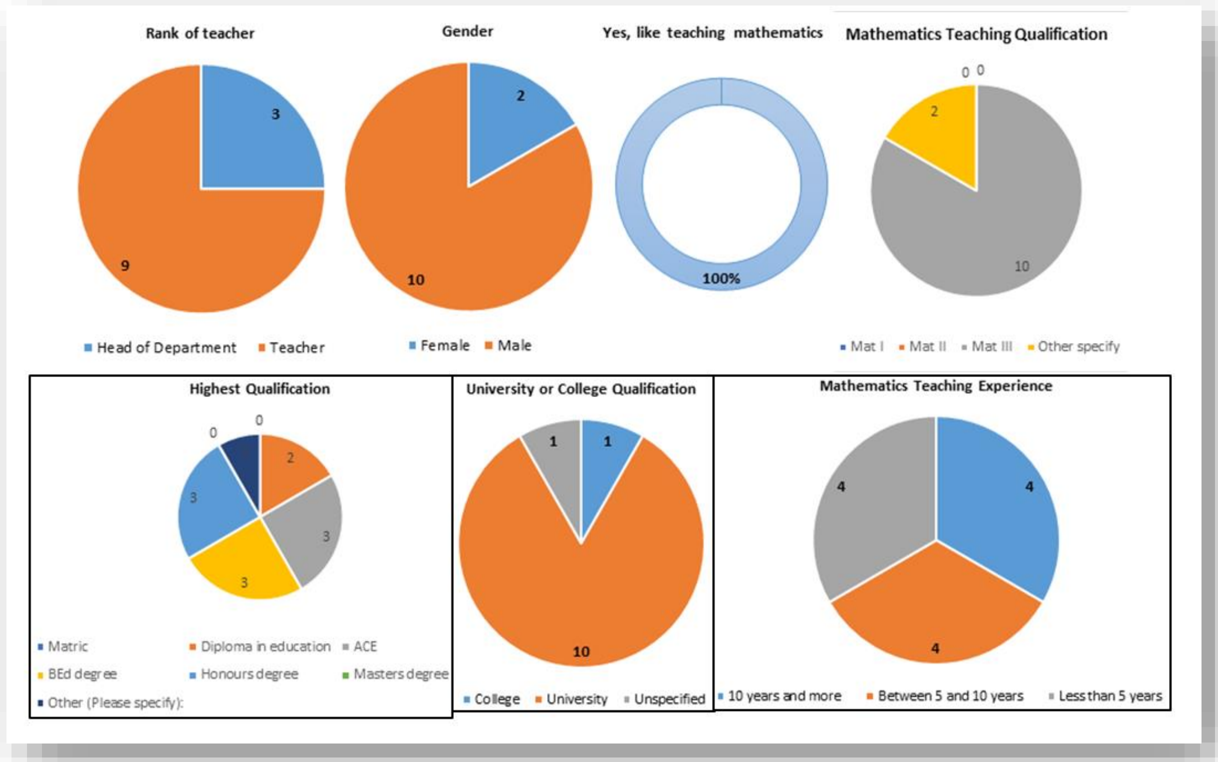


Figure 4.1: Distribution of participants' demographic variables and their analysis

Participants in Figure 4.1 were drawn from four schools in the Tshwane South District, Gauteng province (Section 3.5.3). This location provided a viable geographical context in which to facilitate planned meetings and interactions between researchers and study participants. In this process, it was easy for the researcher to access schools in this district because, at the time at which the study was conducted, the researcher was residing in the same locality as the study's participants. Participants in the study have been referred to as Teacher 1 (T1), T2, T3,…,T12 (see, Table 4.2). According to Figure 4.1, nine participants were regular teachers, and three were Heads of Department (HoD), though all 12 participants acknowledged that they enjoyed teaching Grade 10 mathematics, and possessed varied professional documents qualifying them to teach, such as matriculation certificates (Grade 12); Advanced certificates in Education (ACE), diplomas in Education, bachelor's degrees in education (BEd), honour's degrees, masters' degrees, or were in possession of what was classified as 'other' (Figure 4.1).

Also, in addressing the variable of participants' characteristics, the issue of varying professional experiences in teaching mathematics emerged and, in this regard, three categories were identified varying from five to 10 years of teaching experiences. A closer look into participants' characteristics in Figure 4.1 and Table 4.2 indicates that the study collected data from teachers who exhibited a significant variation in their professional qualities, with these variations being considered as essential in allowing them to access the mathematical meanings of individuals who differed significantly while teaching mathematics in Grade 10. Also, Table 4.2 provides additional information relating to participants' demographic backgrounds.

Table 4. 2: Participants' detailed demographics

Participants' gender	Participants' code names	Number of participants
Male	T1, T2, T3, T4, T6, T7, T9, T10, T11 & T12	10
Female	T5 & T8	2
Rank of a teacher		
Head of Department	T1, T3 & T12	3
Teacher	T2, T4, T5, T6, T7, T8, T9, T10 & T11	9
Highest qualification of a teacher		
Advanced Certificate in Education (ACE)	T8, T9 & T10	3
Diploma in Education (DE)	T6 & T7	2
BEd Degree	T1, T4 & T11	3
Honours Degree (Hons)	T2, T3 & T5	3
Other (BSc Degree)	T12	1
Mathematics teaching qualification		
Mathematics I	None	0
Mathematics II	None	0
Mathematics III	T1, T2, T3, T4, T6, T7, T8, T9, T11 & T12	10
Other (Unspecified)	T5 & T10	2
University or college qualification		
University	T1, T2, T3, T4, T6, T7, T8, T9, T11 & T12	10
College	T10	1
Did not specify	T5	1
Mathematics teaching experience		
10 years and more	T3, T8, T10 & T12	4
Between 5 and 10 years	T1, T4, T5 & T9	4
Less than 5 years	T2, T6, T7 & T11	4

Table 4. 3: Participants' professional qualifications and their teaching experiences

Teaching experience	***University education Mathematics III***	***Other qualification (not specified)***	***ACE***	***DE***	***BEd***	***Hons***	***Other***	***University***	***College***	***Unspecified***
10 years and more	T3, T8, T12	T10	T8, T10	0	0	T3	T12	T3, T8, T12	T10	0
Between 5 and 10 years	T1, T4, T9	T5	T9	0	T1, T4	T5	0	T1, T4, T9	0	T5
Less than 5 years	T2, T6, T7, T11	0	0	T6, T7	T11	T2	0	T2, T6, T7, T11	0	0
Total	**10**	**2**	**3**	**2**	**3**	**3**	**1**	**10**	**1**	**1**

Table 4.2 shows that, at the time of conducting this study, three participants were HoDs, and nine participants were ordinary teachers, although principals and deputy principals did not participate in the study. Table 4.2 and Table 4.3 reveal that, prior to undertaking the study, the participants had studied mathematics to levels 1 and 2 of higher education institutions; that is, mathematics 1 and mathematics 2. Table 4.3 indicates that, of the four teachers who possessed at least ten or more years of teaching experience, three had acquired university-level mathematics III, and the fourth teacher did not specify the mathematics level they obtained at college. In addition, of the four teachers who possessed teaching experience, falling between five and ten years, three had acquired a university-level mathematics III. In this category, the fourth teacher did not specify the mathematics teaching qualification they had obtained.

Teachers with less than five years of teaching experience had acquired a university-level mathematics III, and these observations suggest that, from a study sample of twelve teachers, two did not provide enough details on their tertiary mathematics education, while ten revealed that they possessed a third level university mathematics (MAT III). Furthermore, we observe from Tables 4.2 and 4.3 that participants' highest qualification included an Advanced Certificate in Education (ACE), a Diploma in Education (DE), a Bachelor of Education (BEd), Honours in educational degrees (Hons) and a Bachelor of Science (BSc). Our observation also reveals that, among the four teachers with more than ten years of teaching experience, two of them had ACE qualifications, one had an Honors degree while the last one possessed a BSc degree (Table 4.3).

We have noted that the ACE qualification that two of the study's participants possessed was obtained from the university and, in another case, from an educational college. According to

Tables 4.2 and 4.3, four teachers had teaching experience that fell between five years and 10 years, and, in this category, one had an ACE qualification obtained from the university, two had a BEd, and one had an Honours degree. Of the four teachers who possessed less than five years of teaching experience, two had DE qualifications obtained from university, one had a BEd, and one had an Honours degree.

Overall, the study's participants comprised three teachers with ACE qualifications, two teachers with DE qualifications, three teachers with Honours degrees and one teacher with a BSc degree. From these observations, it is reasonable to conclude that most of the teachers who participated in the study were qualified to teach mathematics in Grade 10, and, by expectation, would be able to teach the topic of trigonometric functions at this level. We have noted that the ACE qualification that two of the study participants possessed had been obtained from the university and in another case from an educational college. According to Tables 4.2 and 4.3, four teachers had teaching experience that fell between five years and 10 years, and in this category, one had an ACE qualification obtained from the university, two had a BEd, and one had an Honours degree. Of the four teachers with less than five years of teaching experience, two had a DE qualification obtained from the university, one had a BEd, and one had an Honours degree.

Overall, the study participants comprised three teachers with ACE qualification, two teachers with DE qualification, three teachers with Honours degree and one teacher with a BSc degree. From these observations, it is reasonable to conclude that most of the teachers who participated in the study were qualified to teach mathematics in Grade 10, and by expectation would be able to teach the topic of trigonometric functions at this level.

4.4 OBSERVED TEACHERS' MATHEMATICAL MEANINGS

In this section, the findings emanating from the trigonometry task will be presented, and this presentation will be followed by the analysis and presentation of the findings of the semi-structured interviews (see, Sections 3.6.1, 3.6.2.2 & 3.7.2). All twelve teachers participated in writing the trigonometric task (Section 3.7.1), and two teachers participated in the semi-structured interviews (Section 3.7.2). To interact meaningfully with teachers' responses in both the trigonometric task and the semi-structured interviews, Piaget's Genetic Epistemology Theory (GET) has been used (Section 2.9.1).

In addition, the ways of knowing that are largely espoused by Thompson (2015) and Mason and Spence (1999) have provided the framework on which this research design was formulated, to inform the articulation of the research process and to facilitate the analysis of data (see, Section 2.4). The analysis in this study has been facilitated in recognition of Moore's (2010) contribution that:

> Knowledge is gained through individual experiences, where the experiences are entirely unique to the individual. Also, this knowledge is not of anything; there is no one-to-one correspondence between what knowledge… [consists] of and the knowledge itself. Rather, knowledge is what comes together through the process of an individual altering his or her knowing (mental schema) in response to a cognitive perturbation or disequilibrium (p. 1)

The preceding quote prepares the analyzer to acknowledge that the nature of knowledge possessed by individuals varies from one individual to another, and so the sense-making of such knowledge stands in direct relation to each individual's frame of reference. With respect to this line of thinking, we should acknowledge that knowledge is received and processed differently across individuals who may be existing within the same space, or in different spaces. The analysis and subsequent interpretation of a study's results have primarily been influenced by the view that there is a multiplicity of perspectives. Drawing from Polzer, DeLaurentis and Fry (2007), the researcher has used the word '*perspective*' as referring to "an individual's version of operational context" (p. 1). In this context, the phrase *operational context* would be conceived as implying and emphasizing the condition of non-fixation or fluidity, which is non-static, and refusing to preserve the condition of rigidness.

In line with the study's results, the researcher anticipated embracing elements of variations in the ways in which study participants would interact and respond to reality. In particular, the researcher anticipated that the study participants could reveal and display a variety of meanings when firstly interacting with the trigonometric task (Sections 1.2 & 1.9.2), and later provide responses in semi-structured interviews. In line with the theoretical considerations of this study, teachers' responses, in both the tasks and semi-structured interviews at hand, would generally constitute their knowledge-to, referring to their active knowledge possibly influencing their classroom practices (Mason & Spence, 2000; Thompson, 2015, 2013).

4.4.1 Analysis of the trigonometry task

This section presents the analysis of the items of the trigonometry task in relation to teachers' responses (Appendix1). The word item here is used as a reference to questions in the trigonometric task that was administered to twelve teachers (Section 3.7.1). In this context, item 1 would be referring to question 1 in the trigonometry task, while a sub-question for this item would be presented as question 1.1, and so on. We observe from Table 4.4 that there were "18" items in the task, including primary tasks and related sub-tasks. Hence, a teacher who responds to all trigonometric tasks would have completed 18 tasks in all. Table 4.4 presents the analysis of how teachers responded to each question (item) of the task. In Table 4.4 and under the section "*Teacher's response to each item*", "1" refers to an item that was attempted, and "0" correlates with an item that was left blank. For instance, Teacher 1 (T1) did not do or did not respond to *item 10* of the task, while this teacher responded to all other items of the task. It may be seen that Table 4.4 provides information on: (1) the number of teachers who responded to each item ("Total 1"); and, (2) the number of items that each teacher has responded to ("Total 2").

We see from Table 4.4 that, of the study's twelve participants, five of them (T5, T6, T7, T9 & T12) responded to all items in the task. A closer look into five participants who responded to all task items reveals that: (1) TWO TEACHERS (T6 & T7) have less than five years teaching experience; (2) TWO TEACHERS (T5 & T9) have between five years and 10 years of teaching experience; and, (3) ONE TEACHER (T12) possesses more than 10 years of teaching experience (see, Tables 4.2 & 4.3). In terms of what the study aimed to investigate (i.e. namely, to explore teachers' understandings of trigonometric ratios and functions by examining the mathematical meanings that teachers convey when they teach Grade 10 learners, as was articulated in Section 1.4), there is not much that one can read in terms of teachers' serving (field related) experiences and the number of task items covered in answering.

These observations are also supported by the fact that, of the teachers who tended to respond to a relatively low number of items in Table 4.4, T2 and T3 (responding to 13 items) and T1 and T11 (responding to 14 items) fell randomly across the continuum, spanning from "10 *years and more*", "*between 5 years and 10 years*", and "*less than 5 years*", without generating any tangible pattern of analysis. However, one should note that a *13-items response* (T2 & T3) by a single participant constituted a 72% rate of task response (response rate). It may, therefore, be reasonable to judge that all study participants tended to demonstrate a great deal of willingness to use the task to share their mathematical meanings and knowledge as demonstrated by an averagely high number of items responded to in Table 4.4.

Table 4. 4: Number of items that each participant attempted

Task item	**Item content**	**Teacher's response to each item**												
		T1	**T2**	**T3**	**T4**	**T5**	**T6**	**T7**	**T8**	**T9**	**T10**	**T11**	**T12**	**Total 1**
1.1	Meaning of an angle and angle measure	1	1	1	1	1	1	1	1	1	0	1	1	**11**
1.2		1	1	1	1	1	1	1	1	1	1	1	1	**12**
1.3		1	0	0	1	1	1	1	1	1	1	1	1	**10**
2.1	Meaning of the sine and cosine of an angle	1	1	1	1	1	1	1	1	1	1	1	1	**12**
2.2		1	1	1	1	1	1	1	0	1	1	1	1	**11**
2.3		1	1	1	0	1	1	1	1	1	1	1	1	**11**
3		1	1	1	1	1	1	1	0	1	1	1	1	**11**
4		1	1	1	1	1	1	1	1	1	1	1	1	**12**
5		1	1	1	1	1	1	1	1	1	1	1	1	**12**
6		1	1	1	1	1	1	1	1	1	1	1	1	**12**
7.1	Meaning of unit circle	1	0	1	1	1	1	1	1	1	1	1	1	**11**
7.2		1	1	0	1	1	1	1	1	1	1	1	1	**11**
8.1	Solution of triangle	1	1	1	1	1	1	1	1	1	1	1	1	**12**
8.2		0	1	1	1	1	1	1	1	1	1	1	1	**11**
9		1	0	1	1	1	1	1	1	1	1	0	1	**10**
10.1	Application: 2-D problem	0	1	0	1	1	1	1	0	1	1	0	1	**8**
10.2		0	0	0	1	1	1	1	1	1	0	0	1	**7**
10.3		0	0	0	1	1	1	1	1	1	0	0	1	**7**
Total 2		**14**	**13**	**13**	**17**	**18**	**18**	**18**	**15**	**18**	**15**	**14**	**18**	

Using this line of analysis, the researcher further observes that teachers' qualifications in mathematics did not play a significant role in shaping and influencing participants' behaviour in responding to the trigonometric task items. In terms of the qualifications, five teachers who responded to all items in the task, T5, T6, T7, T9 and T12, were observably distributed as follows: (1) four teachers possessed mathematics III from the university; and, (2) one teacher did not specify the level of mathematics qualification. On the other hand, teachers who responded to fewer items, T1, T2, T3 and T11, all seemed to be in possession of mathematics 3, which is the highest level of teaching qualification a study participant can obtain (Tables 4.2 & 4.3). Arguably, the latter are presumably teachers with higher mathematical knowledge, and one could have expected them to have responded to all items with ease. In this instance, it is important to note that this study did not focus on measuring teachers' knowledge of mathematics, but, instead, the study tended to want to explore participants' meaning-making of their possessed mathematical knowledge, in relation to specific trigonometric concepts. This is in line with the study aim and objectives (Sections 1.4 & 1.5). Given this background, one could observe that the exposition of one's mathematical meanings may not be the function of one's level of discipline-specific knowledge or qualifications.

We close this section by advancing a useful commentary that the observations in Table 4.4 are specifically significant in demarcating and guiding the analysis process of this study and providing a thorough analysis of teachers' responses to individual task items in the sections that follow. Table 4.5 contains a summary of anticipated teachers' mathematical meanings conveyed when teaching Grade 10 learners (see, Section 2.10.4 for an elaborate discussion of meanings). It is important to indicate that the sections that follow provide an analysis of items 1, 2, 3, 4, 7 and 10 (10.1 and 10.2) only. The analysis of items 5, 6, 8, 9 and 10.3 was excluded in this report because the teachers' responses could not provide useful and relevant information to answer the research questions. We have just observed that teaching experience and levels of qualifications may not be a significant factor in shaping the exposition of teachers' mathematical meanings of Grade 10 trigonometric tasks that are treated in this study (Section 3.6.1).

4.4.1.1 Analysis of items 1 and 4: Definition of an angle and meaning of angle measure

Items 1 and 4 explored teachers' understanding of and the meanings they attached to the notions of *angles* and *angle measure* (Appendix 1). These task items were adapted from Moore (2010) (Section 3.6.2.1), and the inclusion of tasks 1 and 4 was motivated by their observed utility in supporting the development of important and critical trigonometric ideas. The knowledge of angles and the angle measuring process is understood to be a touchstone in mathematics for making subsequent important advances and headway in high school trigonometry. These sections of the task thus constitute the basis upon which important and advanced concepts in trigonometry can be understood and taught. Hence, gauging teachers' knowledge and understanding of these sections was critical in determining their likely ability to foster learners' meaning-making capacity within the context of the classroom.

Table 4. 5: Summary of anticipated teachers mathematical meanings conveyed when teaching grade 10 trigonometry

Concept	**Mathematical Meaning to be Conveyed**
Angle	an angle is formed or is a product of turning an object or arm about a fixed point, which is equivalent to the amount of turning from the initial position
Angle Measure	the angle measure is a process of determining the fractional amount of the circumference of the circle that is subtended by the central angle (Moore, 2013; Thompson, 2008). In other words, to measure an angle is tantamount to measuring the length of the arc that is subtended by the central angle.
Angle Measure of one degree	is a measure of the arc length that equals $\frac{1}{360^{th}}$ of the circle's circumference. An angle measure in degrees is viewed as the length of the arc subtended by the central angle, measured in arcs of the length $\frac{1}{360^{th}}$ of the circle's circumference
Protractor Usage	is the understanding that an amount of rotation references angle measure and the product resulting from the mathematical process of measuring an angle is the observed arc length.
Sine of an angle	is the percent of the radius' length made by the side opposite the angle at the centre of a circle in the embedded right-angle triangle.
Sin 30°	is a length that is 0.5 times as long as the radius of the circle made by the length of the side opposite the centre of the circle. In other words, $sin\ 30°$ is equal to 50% of the radius of the circle.
Sin 90°	is the length that is 100% of the radius of the circle made by the side opposite 90° (the side coinciding with the radius).
Cos 100°	denotes the length that is 17% of the radius length made by the side adjacent 100° along the x axis in the negative to the left direction from the origin
Output of Sin θ determination	uses the length of the hypotenuse as the radius to draw a circle, then measuring each side of the triangle relative to the hypotenuse in order to apply the sine and cosine functions to the right-angle triangle embedded in a circle, then the output of the sine (cosine) of angle ABC is the length made by the side opposite (adjacent) angle ABC
Unit Circle	is a circle resulting after the process of unitizing the magnitude of the radius. The process of unitizing the magnitude of the radius, in this case, means using the radius of the circle as a unit of measure
Unit Circle Usage	allows for the coherent development of the cosine and sine functions on the Cartesian coordinate system.

4.4.1.1.1 Observed teachers' meanings of an angle as explored in item 1.1 of the task

Item 1.1 of the trigonometric task that was administered to the teachers asked:

What is an angle?

In line with the intended formulation of the concept of an angle, the researcher observed that participants' responses in item 1.1 tended to fall in the categories of: (1) consistent; (2) inconsistent; (3) not making sense; (4) no response. This method of describing teachers' responses to the question of what constitutes an angle is in line with the discussion in Section 2.10.4.1 and Table 4.5. To facilitate the analysis of teachers' responses to item 1.1, the descriptors in Table 4.6 were formulated.

Table 4. 6: The framework of analysis for teachers' responses to item 1.1 of the trigonometric task

Item 1 response types	Explaining the type of response
Consistent	This category of response is perceived to be in line or replicates/ supports the angle definition given in Section 2.10.4.1. The definition in this regard must be seen to contain and promote aspects of mathematical knowledge and may be communicated to learners with ease instructionally.
Inconsistent	This type of response is seen to possess minimally some aspects of the angle definition that are presented in Section 2.10.4.1. However, some of the explanations captured in this response are not in full unison with the definition in Section 2.10.4.1, with some parts of this response being mathematically unsound and having no direct relation to advanced teaching and learning in mathematics.
Not making sense	This kind of response is completely out of line with the definition given in Section 2.10.4.1 and, in fact, would display no overlap with the definition in Section 2.10.4.1. The mathematical content that is carried in this response is incorrect and may not promote productive and meaningful learning of mathematics.
No response	No response provided

Using the framework of analysis indicated in Table 4.6, the analysis of teachers' responses to item 1.1 proceeded in line with Table 4.7.

4.4.1.1.2 Teachers' observed responses to item 1.2 of the task

Item 1.2 of the trigonometric task that was administered to the teachers asked:

What does it mean to measure an angle?

In line with the discussions in Section 2.10.4.2 and Table 4.5, the researcher anticipated that teachers' responses to item 1.2 of the task would embrace or take into consideration some mathematical concepts such as a circle, circumference, central angle, and arc. However, this expectation would also take into consideration the fact that the study respondents could not necessarily embrace these concepts entirely. In fact, a mere mention of one of these mathematical concepts would come close to the respondent's response to the desired response of the study. When analyzing teachers' responses to item 1.2, the researcher observed that teachers tended to provide responses that were almost like those given in item 1.1.

Table 4. 7: Analysis of teachers' responses to item 1.1 of the trigonometric task

Response classification	Types of responses generated by teachers	Teachers associated with response
Consistent responses	In terms of the descriptors in Table 4.5 the following responses are classified as being consistent: • Measure of the turn between the two arms/ lines • Amount of turn between two straight lines that have a common end point • A quantity used to account for the amount of rotation of a point around a fixed point. A quantity that accounts for the difference in orientation between two lines, rays or segments - which may or may not intersect	T2, T4, T12
Inconsistent responses	In terms of the descriptors in Table 4.5 the following responses are classified as being inconsistent: • Formed when two straight lines cross/ meet/ intersect at a point • Space between two lines • Figure measured or formed from two rays which start from a common point or that intersect • Space between two intersecting lines • It is made when two straight lines cross or meet each other at a point. • When two lines intersect at a certain point an angle between them is formed. • Space between two lines from a single point	T1, T3, T5, T7 T8, T9 T11
Responses not making sense	In terms of the descriptors in Table 4.5 the following responses are classified as not making sense: • Measure of the size of a line and the horizontal at the point of contact	T6
No response	No response given for item 1.1	T10

Teachers' responses to item 1.2 tended to generate the following categories: (1) *the anticipated responses*; (2) *responses perceived to be close to the anticipated one*; (3) *responses embracing anticipated mathematical concepts but not making sense*; (4) *responses replicating item 1.1 responses;* and, (5) *responses not related to the anticipated one*. The analysis of teachers' responses to item 1.2 is captured in Table 4.8.

Table 4.8 shows that, of the 12 teachers who provided responses to item 1.2, only one teacher provided an answer that was mathematically correct. Two teachers provided responses to item

1.2 that fell under category 2, implying that their responses were perceived to be close to the anticipated response. It is concerning that most of the teacher responses fell under category 5, in which cases the responses were classified as being in no sense related to the anticipated response and as such were misdirected. It is further concerning to conceive the types of mathematical meanings that teachers in category 5 generate and eventually convey to learners when they are called upon to deliver a Grade 10 trigonometry lesson on the topic of functions.

Table 4. 8: Analysis of teachers' responses to item 1.2 of the trigonometric task

Response category	Response categorization	Example of related response	Teachers
1	Response perceived to be the anticipated one	• Always involves the fraction of an arc length suspending the angle to the circumference of a circle of the same radius as the arc	T12
2	Response perceived to be close to the anticipated one	• Rotational measurement of how wide the segments are from each other at the point of intersection • Measuring the turn from one static arm to the turning arm	T1 T2
3	Response embracing anticipated mathematical concepts but not making sense	• How far a point has travelled around the circle. Describing the amount of openness of an angle's rays • Measuring how much a line has been rotated from its fixed position • Finding the magnitude/ size of an angle	T7 T9 T10
4	Responses similar to those of item 1.1	• Finding a distance between two straight lines that have a common end-point	T4
5	Responses not related to the anticipated one	• The distance between two lines • Measure the space between two lines • Determining the size of the measure of a degree without using calculations • The amount one line has been turned in relation to the other • Measuring the space between lines that are drawn from 1 point	T3 T5 T6 T8 T11

4.4.1.1.3 Teachers' observed responses to item 1.3 of the task

Item 1.3 of the trigonometric task that was given to teachers asked:

What does it mean for an angle to have a measure of 1 degree?

Item 1.3 tended to explore respondents' meanings and interpretations of the number of degrees (magnitude) allocated to an angle. In another sense, in this item, teachers would be expected to demonstrate how they would meaningfully figure out this kind of angle representation, and how they would allow this kind of mathematical knowledge to reside conceptually in their cognitive system. The responses in Table 4.9 constitute teachers' meanings of angle measurements of one degree (1°). In addition, the response classifications in Table 4.9 are used to analyze and interpret teachers' responses. Also, in line with the discussions in Section 2.10.4.3 and Table 4.5, the researcher anticipated that teachers' responses to item 1.3 of the task would comprise a description that related angle measure of one degree as a measure of the arc length that equals $\frac{1}{360^{th}}$ of the circle's circumference.

4.4.1.1.4 Analysis of teachers' responses to item 4 of the task

Item 4 of the task requires teachers to demonstrate and manifest their meanings (in the content of the mathematical task) in relation to the work done by student A (SA) and student B (SB). Item 4 was presented as follows:

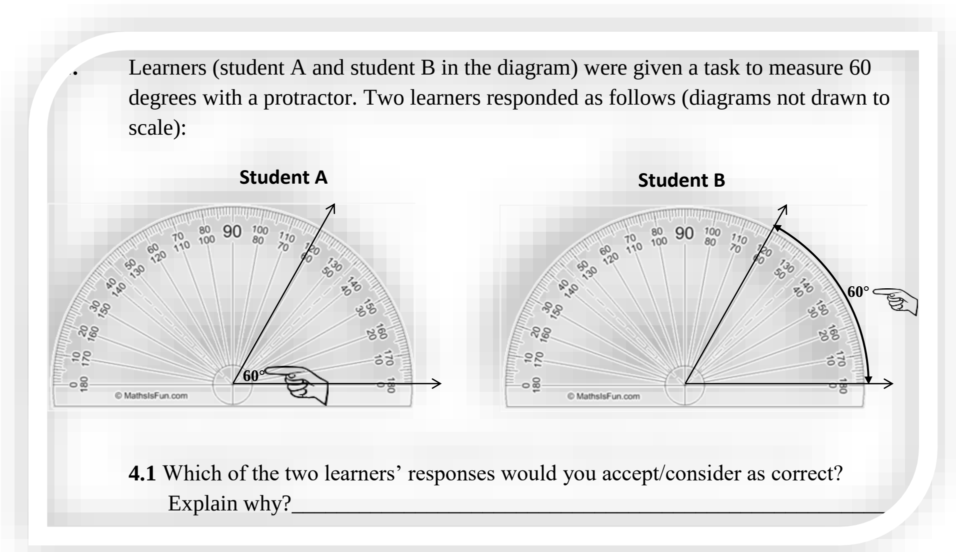


Figure 4. 2: Item 4 of trigonometric task that was given to student A and student B

Table 4. 9: Teachers' responses to item 1.3 of the trigonometric task

Teacher/ respondent	Type of response provided	What could a response imply in terms of the study aim and objectives (Sections 1.4 & 1.5), and also the categorizations in Tables 4.7 and 4.8?
T1	To move a terminal arm anticlockwise	This is a CATEGORY 3 response. Mathematical knowledge and concept are not accurately articulated, and not presented with clarity. Meanings made in this category are largely disoriented and lacking mathematical logicality.
T4	The distance between the two lines will be 1 degree that have the common end-point	This is a CATEGORY 3 response. Mathematical knowledge and concept are not accurately articulated, and not presented sequentially. Meanings made in this category are largely disoriented and lacking mathematical logicality.
T5	The unit of measurement of an angle is equal to $^{1}/_{360}$ of a circle's circumference	This is a CATEGORY 1 response. Teacher's response is perceived to be closer to the expected answer. In terms of Table 4.8 the teacher is capable of generating, and further conveying to learners, mathematical meanings that are educationally desirable and productive.
T6	A line moved in a measure of only a degree from the normal to the terminal point of the arm	This is a CATEGORY 4 response that is not related to the expected response. The response has no logic and is largely characterized with mathematical ambiguity. The meanings formed by this teacher are not mathematically coherent, and do not promote mathematical logicality.
T7	When its vertex is placed at the centre of a circle and has rays that cut off an arc that $^{1}/_{360}$ of the circumference	This is a CATEGORY 1 response. Teacher's response is perceived to be closer to the expected answer. In terms of Table 4.8 the teacher is capable of generating, and further convey to learners, mathematical meanings that are educationally desirable and productive.
T8	An angle made by $^{1}/_{360}{}^{th}$ part of a revolution, as there are 360 degrees in a circle	This is a CATEGORY 1 response. Teacher's response is perceived to be closer to the expected answer. In terms of Table 4.8 the teacher is capable of generating, and further convey to learners, mathematical meanings that are educationally desirable and productive.
T11	The space between two lines measures 1 degree if a protractor is used	This is a CATEGORY 2 response. The teacher is able to generate a set of mathematical concepts needed to formulate a desirable response of the task. However, these concepts are not properly presented to achieve coherence. This response is however closer to the expected answer. Meanings generated by a teacher in this category may not be fully accounted for mathematically.
T9	A line has been rotated by 1 degree from its chosen fixed position about its end-point	This is a CATEGORY 3 response. Mathematical knowledge and concept are not accurately articulated, and not presented with clarity. Meanings made in this category are largely disoriented and lacking mathematical logicality.
T10	Degree is a standard unit for angles	This is a CATEGORY 4 response that that is not related to the expected response. The response has no logic and is largely characterized with mathematical ambiguity. The meanings formed by this teacher are not mathematically coherent, and do not promote mathematical logicality.
T12	$^{1}/_{360}$ of a full rotation. When the arc subtending the angle is $^{1}/_{360}$ of the circumference of the associated circle of equal radius	This is a CATEGORY 1 response. Teacher's response is perceived to be closer to the expected answer. In terms of Table 4.8 the teacher is capable of generating, and further convey to learners, mathematical meanings that are educationally desirable and productive.

Table 4.10 presents the responses of SA and SB. The students' responses to item 4 of the task allowed the researcher to analyze teachers' responses in context (Figure 4.2; Table 4.10). To reflect meaningfully on teachers' responses to item 4, a line of reasoning was employed to analyze item 1.2 (Section 4.4.1.1.2). However, in item 4, teachers were expected to use their knowledge of angle measure to assess and reflect on a task that had been written by SA and SB embracing aspects of angle measure, suggesting that a slightly different lens could also be used to analyze item 4 responses. Looking at the description of students' work in item 4 would require teachers to demonstrate their understanding of angle measure, and how they would use a protractor to execute some mathematical activities embedded in the task.

Table 4. 10: The actual verbalization of SA and SB responses to item 4

Student A (SA)	Student B (SB)
Drew a horizontal line first, then put the protractor on top of the line such that the zero-degree line coincided with the horizontal line and one endpoint was at the centre of the semi-circle. Then the student read off 60^0 on the protractor and made a dot at 60^0. Then the student joined the dot at 60^0 and the end-point of the line at the centre of the semi-circle. The student pointed at the corner to indicate the angle measure of 60^0.	First, the student considered the fact that an angle is dynamic and not static. It is a measure of turn, and a protractor does not acknowledge this. To show the rotation, the student drew the arc in such a way that its length is intercepted between the reading of 0^0 and 60^0. The student pointed at the arc to indicate the angle measure of 60^0.

Item 4 would need teachers to identify, learners' problem-solving attempts and perspectives, what they would judge as acceptable and correct mathematical articulations for measuring the angle of 60° using a protractor. In this instance, and also in line with the discussion found in Section 2.10.4.4 and Table 4.5, the researcher anticipated that teachers would provide a task related response that projected some level of understanding that a protractor could be used to measure the amount of turn of 60°, which is conceived to subtend the arc of the length $60° \times \frac{1}{360°} = \frac{1}{6^{th}}$ of the circle's circumference. Teachers' responses to students' tasks are briefly summarized and analyzed in Table 4.11.

Table 4.10 presents the responses of SA and SB. The students' responses to item 4 of the task allowed the researcher to analyze teachers' responses in context (Figure 4.2; Table 4.10). To reflect meaningfully on teachers' responses to item 4, a line of reasoning was employed to analyze item 1.2 (Section 4.4.1.1.2). However, in item 4, teachers were expected to use their

knowledge of angle measure to assess and reflect on a task that had been written by SA and SB embracing aspects of angle measure, suggesting that a slightly different lens could also be used to analyze item 4 responses. Looking at the description of students' work in item 4 would require teachers to demonstrate their understanding of angle measure, and how they would use a protractor to execute some mathematical activities embedded in the task.

Table 4. 11: Analysis of teachers' responses to students' attempts and subsequent meanings projected (item 4)

Response category	Response categorization	Example of related response	Teachers
1	Response perceived to be the anticipated one	• Student B because angle is an indication of the measure of turns a line has made from its fixed position.	T9
2	Response perceived to be close to the anticipated one	• Student B because has indicated both the direction of where he started his measurement. • Student B because 60 degrees is the distance between the two lines. • Student B because shows that he measured the angle from the horizontal to the terminal exactly where the line intersects the tick at 60 degrees. • Student B because when measuring an angle, you start from zero degrees to 60 degrees. • Both solutions will be accepted since an angle is distance between two lines and both learners are indicating that 60 degrees correctly but writing 60 where they like.	T1 T4 T6, T7, T8 T10 T3
3	Response embracing anticipated mathematical concepts but not making sense	• Student A because the protractor is placed at zero on the other arm or line. • Student A because an angle is formed between two lines at the vertex. • Student A has a better grasp of the angle concept since she has not linked the angle to an arc length (without context) as has student B. • Student A, Angle is measured at a vertex. • Both responses because the learners determined the measure between to intersecting lines which they both got it to be 60 degrees. The angle is not the same as the arc that makes it special to be constant when it is in between two intersecting lines.	T2 T12 T11 T5
4	Responses not related to the anticipated one	• Both responses may imply misconceptions because "in the absence of further information about their thought processes these cannot be assumed"	T12
5	Responses not related to the anticipated one	• None of teachers' responses was identified to be falling within this level of categorization.	N/A

4.4.1.1.5 Analysis of item 2 of the trigonometric task

Item 2 of the task consisted of three questions; namely, 2.1, 2.2 and 2.3. The first question probed teachers' understanding of the sine of an angle. Questions 2.2 and 2.3 explored teachers' meanings in relation to sin 30° and other related trigonometric function (see, Figure 4.3).

4.4.1.1.5.1 Analysis of items 2.1 and item 2.2

To generate the intended responses to item 2.2 of Figure 4.3, teachers were supposed to be cognizant of a right triangle that is embedded in the circle, such that the hypotenuse is the radius, and 30° is the angle at the centre (see, Figure 4.4). From the diagram in Figure 4.4 it should be observed that Sin 30° is a length that is 0.5 times as long as the radius of the circle made by the length of the side opposite the centre of the circle. A teacher that would display this expected level of mathematical knowledge and understanding would be placed in category 1 of Table 4.11, and associated mathematical meanings possibly generated and associated with this response would be explained in terms of category 1 of Table 4.11.

Subsequently, in analyzing teachers' responses to items 2.1 and item 2.2, categories like those generated in Table 4.8 were used. Also, to anticipate and interpret the possible types of mathematical meanings generated by teachers' responses, categories like those generated in Table 4.9 are used (see this line of analysis in Table 4.12).

2. **2.1** Define the *sine* of an angle. ____________________________________

2.2 What is the meaning of sin30°? ____________________________________

2.3 Referring to the meaning of sin30° that you provided in 2.2, what is the meaning of *sin(90°)*?; and that of *cos(100°)*?

Figure 4. 3: Sub-questions of item 2 of the task

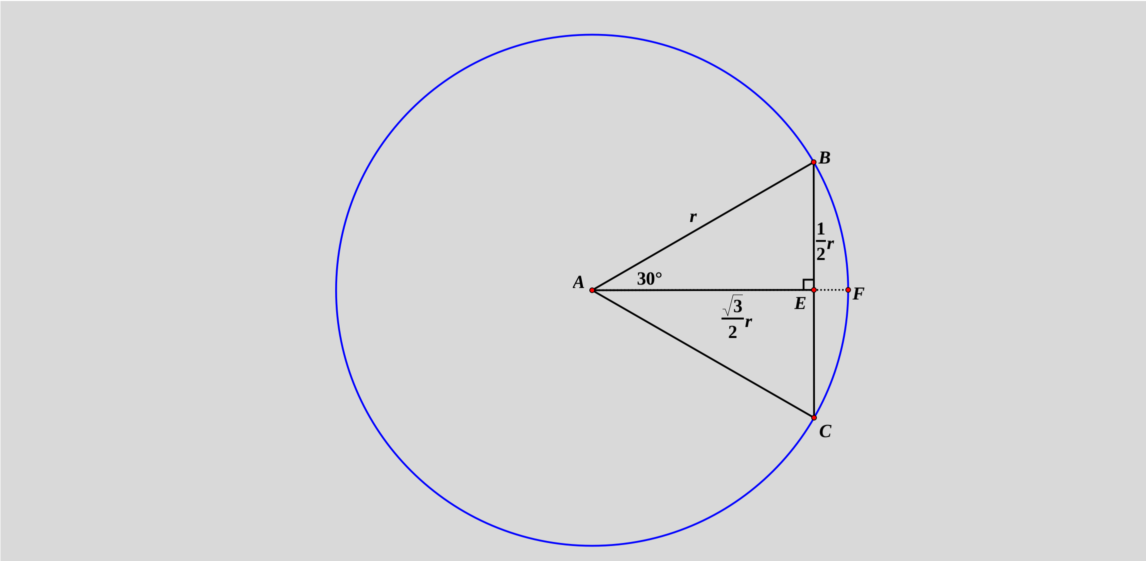

B
r
1
2r
A
30°
E
F
√3
2r
C

Table 4. 12: Item classification to analyze items 2.1 and 2.2 of the trigonometric task

Response category	**Response categorization**	**Example of related responses for item 2.1 and item 2.2 of trigonometric task**			
		Item 2.1		***Item 2.2***	
		Teacher	*Response generated*	*Teacher*	*Response generated*
1	Response perceived to be the anticipated one	None	N/A	None	N/A
2	Response perceived to be close to the anticipated one	T4	• Trigonometric function of an angle and it is the ration for the length of the opposite side over the longest side of the triangle	None	N/A
3	Response embracing anticipated mathematical concepts	T1 T2 T3 T5 T6 T7 T8 T11 T10 T9	• The relation of an angle and length of the sides • The side opposite to that angle divided by the side hypotenuse to the angle • The ratio of opposite side and hypotenuse side in a right-angled triangle • The ratio or division of the opposite side to the angle and the hypotenuse side of the angle • The ratio of the opposite side of an angle to the hypotenuse side in a right-angled triangle • Ratio of the length of the side that is opposite that angle to the length of the longest side of the triangle • The ratio opposite side of an angle to the hypotenuse • The ratio of opposite and hypotenuse sides with regard to that angle • Ratio of opposite side over hypotenuse • In a right-angled triangle, the sine of an angle is the ratio of the opposite to the angle to the side of the same triangle opposite to the angle of 90 degrees	T1 T2 T3 T4 T5 T6 T7 T10 T11 T9	• Means, at a measure of 30 degrees angle, what is the length or height of that angle. • Sine 30 degrees means that the angle between the opposite side and the hypotenuse side is a right-angled triangle with the opposite side being 1 unit and the hypotenuse side being 2 units. • Sin 30 degrees = $^1/_2$, meaning the ratio of sin 30 degrees is one to two. • Means that in a right-angled triangle ratio of the side opposite to 30 degrees to the longest side of the triangle is equal to ½. • It means the ratio between the opposite and the hypotenuse side is 1:2 or the opposite side=1 unit and the hypotenuse side=2 units. • The ratio of the opposite side to 30 degrees to hypotenuse side in a right-angled triangle which is ½. • Means the opposite side has a length of 1 and hypotenuse will be 2 hence sine 30^0 being ratio ½. • The ratio of the angle is 30 degrees. • The ratio of ½ (opposite and hypotenuse) with respect to that angle 30 degrees. • The ratio in a right-angled triangle of the side opposite to 30 degrees to the side opposite to 90 degrees.

		T12	• The construction of a ray from the origin of the Cartesian plain such that the angle measured anticlockwise from the positive x-axis to the ray is the angle in question. Select any coordinate pair on the ray that is not the origin. The sine of the angle will be the ratio $\frac{y}{\sqrt{(x^2+y^2)}}$	T12	• The ratio $\frac{y}{\sqrt{(x^2+y^2)}}$ when the ray is angled at 30 degrees
4	Response not related to the anticipated one	None	N/A	None	N/A
5	No attempt	None	N/A	T8	• No Response

To qualify and perceive teachers' responses in Table 4.12 in terms of the possible mathematical meanings, the researcher opted to generate Table 4.13, which tended to be almost like Table 4.9.

Table 4. 13: Possible mathematical meanings associated with teachers' responses to items 2.1 and 2.2

CATEGORY 1	In terms of exploring teachers' meanings, a response in this category could imply that the respondent can generate, and further convey to learners, mathematical meanings that are educationally desirable and productive. These meanings are coherent with desirable mathematical knowledge.
CATEGORY 2	In terms of exploring teachers' meanings, a response in this category could imply that the respondent's mathematical knowledge is not consistent and is still yet to strike coherence between the intended and non-productive knowledge. Meanings generated by a teacher in this category may not be fully accounted for mathematically.
CATEGORY 3	The teacher in this category has grabbed and internalized a variety of mathematical knowledge and concepts. However, knowledge generation and production in this category is less coherent and sequential; and may turn to lack the quality of mathematical sense-making. The teacher is incapable of utilizing and communicating mathematical knowledge productively. Meanings made in this category are largely incoherent, disoriented and lacking mathematical logicality.
CATEGORY 4	The teacher in this category has tendency to generate cognitive conflicts of mathematical knowledge and concepts. The correctness of mathematical knowledge generated by a teacher in this category is superficial and non-reliable. The meanings formed by this teacher are not mathematically coherent, and do not promote mathematical logicality.

Table 4. 14: Item classification to analyze item 2.3 of the trigonometric task

Response category	**Response categorization**	**Example of related teachers' responses for item 2.3 of trigonometric task**			
		Item 2.3 (Sine 90 degrees)		***Item 2.3 (Cos 100 degrees)***	
		Teacher	*Response generated*	*Teacher*	*Response generated*
1	Response perceived to be the anticipated one				
2	Response perceived to be close to the anticipated one	T3	• Sin 90 degrees = 1 since sin of an angle is given by y/y and at y-axis the y value can be read/ can be any number but zero. This is the reason why cos 90 degrees = 0 because on y-axis our x value is zero.	T3	• Cosine of an angle is given by x/r and with cos 100 degrees expect a negative solution since 100 is on third quad. X is negative according to definition of cos x/r.
3	Response embracing anticipated mathematical concepts but not making sense	T1 T2 T5 T6 T7 T8 T10 T11 T9	• Sine will be at the highest height and length. • Sine 90 degrees means measuring the angle between the two arms on a right-angled triangle whereby the angles between the two sides/arms is 90 degrees. • Sin (90) means the ratio of opposite side and hypotenuse side is 1:1. • 90 degrees is 1 because the side opposite to 90 degrees is the hypotenuse side. • $Sin90^0$ is 1/1 because the side opposite to 90^0 is the hypotenuse side. • Sin 90 degrees = ratio of a side opposite 90 degrees over the hypotenuse and cos 100°=? • Side of the angle at 90 degrees and its hypotenuse. • Ratio of opposite and hypotenuse for sine of 90 degrees which is 1 • Sine 90 degrees is the ratio of the side opposite to 90 degrees hence the value of ratio=1.	T9 T7 T6 T11	• Cos 100 degrees is the negative ratio of the side adjacent to the complement of 100 degrees, i.e. 80 degrees to the side opposite to 90 degrees. • Cos 100^0 is equal to zero. • cos (100^0) is equal to sin 10 degrees • Adjacent and hypotenuse for cos.

4	Response not related to the anticipated one	T12	• As before	None	N/A

4.4.1.1.5.2 Analysis of item 2.3

Table 4.14 has been generated to present the analysis of item 2.3 of the trigonometric task (Figure 4.3). The approach used to analyze item 2.3 is like the analysis procedures used in items 2.1 and 2.2; meaning, similar categories like those in Table 4.12 are generated to facilitate the analysis of item 2.3. To explain the possible mathematical meanings associated with teachers' responses in item 2.3, Table 4.14 was used.

4.4.1.1.6 Analysis of item 3 of the trigonometric task

Item 3 that was given to teachers also tended to address issues of mathematical meaning that most of the enquiry in the study addressed. Item 3 that was given to the participating teachers is presented in Figure 4.5 and the analysis of this item is presented in Table 4.15. It must be noted that to analyze and interpret teachers' responses to item 2.3, categories in Table 4.12 were used. The categories in Table 4.12 are explained in Table 4.13.

3. In triangle ABC what does it mean to determine the output of the *sine* and *cosine* of the measure of angle ABC *without* measuring the angle (diagram not drawn to scale).

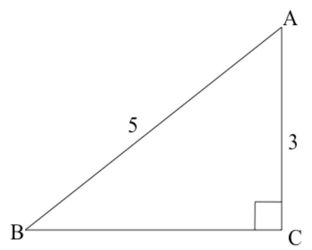


Figure 4. 5: Item 3 of the trigonometric task

Table 4. 15: Analysis of teachers' responses to item 3 of the trigonometric task

Teacher	Type of response	Response category	Possible associated mathematical meanings
T2	To determine the output of sine from the triangle means that the sine is determined from the length of the sides of a triangle. Also, for cosine means the same, i.e., sine = opposite side over hypotenuse side and cosine = adjacent side over hypotenuse side.	2	. In terms of exploring teachers' meanings, a response in this category could imply that the respondent's mathematical knowledge is not consistent and has still yet to strike coherence between the intended and non-productive knowledge. Meanings generated by a teacher in this category may not be fully accounted for mathematically.
T1	It means to get the ratio of the sides.	2	
T5	It means you determine the amount or the value that is equals to the amount of sin(angle) and cos(angle)	2	
T6	It means that we use the definitions of sine and cosine to determine the measure of angle ABC in terms of line ratios from trigonometric definition of the sine and cosine.	2	
T7	It means that we use the definitions of sine and cosine to determine the measure of angle ABC in terms of the ratios from trigonometric definitions of the sine and cosine of angles.	2	
T3	It won't be easy to get the output. Since we need to know the angle because the angle is the one that will tell us about the opposite and adjacent sides. Without knowing the reference angle, you can't determine sine and cosine in a right-angled triangle.	3	The teacher in this category has grabbed and internalized a variety of mathematical knowledge and concepts. However, knowledge generation and production in this category is less coherent and sequential; and may turn to lack the quality of mathematical sense-making.
T4	To find the angle of B, use sinB = 3/5 and for angle A, use cosA = 3/5. For C we can use sum of angle of a triangle or the sign at C means C = 90 degrees.	3	
T10	Sin B = 3/5 = 0,6. Cosine B = 4/5 = 0.8. Use calculator to find B. $x^2 = 5^2 - 3^2$; x= 25-9; x=4	3	
T11	Sin of angle ABC is the ratio of sides 3/5. Cos of ABC is the ratio is sides 4/5.	3	
T9	Output of sine means to find the ratio of side opposite to that specified; unknown angle to the side opposite to 90 degrees. Output of cosine means to find the ratio of the side to that specified angle to the side opposite 90 degrees.	3	
T12	It means determining the ratios AC: BA and BC: BA respectively; the numerical values can be determined (BC = 4 units by the Pythagorean Theorem.)	3	
	T8 did not respond		

4.4.1.1.7 Analysis of item 7 of the trigonometric task

In this section we analyze item 7 of the trigonometric task that was given to teachers. This item consisted of two sub-questions, namely items 7.1 and 7.2, and reads as follows:

7. **7.1** Explain the meaning of a unit circle? ____________________________________

____

7.2 How could you use the unit circle in trigonometry? ________________________

____

Figure 4. 6: Item 7 of the trigonometric task

To analyze teachers' responses to this category, the researcher employed similar strategies in which the classification or categorization of teachers' responses and attempts were examined to anticipate the possible mathematical meanings with which each response was associated (see, Table 4.16).

4.4.1.1.8 Analysis of item 10 of the trigonometric task

Item 10 of the task was a question adopted from a previous standardized state examination question paper for Grade 10. In this item, teachers were expected to respond to two sub-questions, namely item 10.1 and item 10.2, which had been built from the main question of this item. In this item, teachers were expected to demonstrate their understanding and apply their knowledge of trigonometry to solve the quiz embedded in the item (see, Figure 4.7).

Table 4. 16: Item classification to analyze item 7 of the trigonometric task

Response category	**Response categorization**	**Example of related teachers' responses for item 7 of trigonometric task**			
		Item 7.1 (unit circle)		***Item 7.2 (using a unit circle in trigonometry)***	
		Teacher	*Response generated*	*Teacher*	*Response generated*

1	Response perceived to be the anticipated one	T12	• Circle whose radius is 1 unit according to some length scale.	T12
				
				T6, T7
				
				T9
					• A few that come to mind:
					  ○ Deriving the Pythagorean Identities
					  ○ Representing the ratios sine, cosine and tangent (drew diagram)
					  ○ Defining radian measure "a truly unit less scale for measuring angles"
					• Introduce a lesson on trigonometric ratios using a radius of one unit and a point which will exactly give the radius on one unit so that I would use the unit circle to teach compound angles indirectly.
					• Knowing the exact values of sine and cosine of 30°, 45°, 60°, the unit circle can be used also to verify such results (drew diagram). $Sin\ (30°) = \frac{y}{r} \vert\ \frac{1}{2} = \frac{y}{1} \vert\ y = \frac{1}{2}$. $Cos\ (30°) = \frac{x}{r} \vert\ \frac{\sqrt{3}}{2} = \frac{x}{1} \vert\ x = \frac{\sqrt{3}}{2}$. This can also be used to calculate/confirm that $\sin^2\theta + \cos^2\theta = 1$, directly from the theorem of Pythagoras.

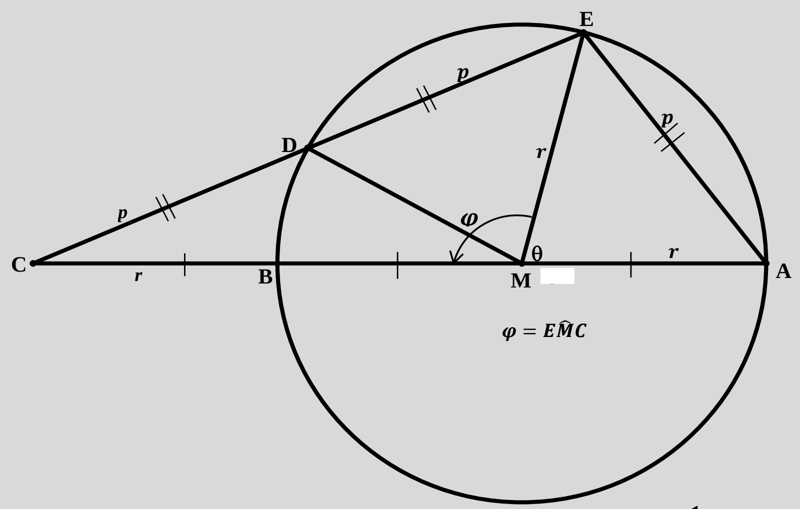
2	Response perceived to be close to the anticipated one	T5, T9 ,T10 T6, T7	• Circle with a radius that is equal to one unit. • Circle with a radius of one unit exactly. This circle is represented by the equation $x^2+y^2=1$.	T2 T5	• With Cartesian plane when introducing the trigonometric ratios/defining the trigonometric ratios in all four quadrants. • To determine the trig identity that states $\sin^2\theta + \cos^2\theta = 1$ by use of sine and cosine together with Pythagoras, because have constant radius which will act as hypotenuse side.
3	Response embracing anticipated mathematical concepts but not making	T1 T3 T4	• Same measure of the circumference of the circle from the centre when you are drawing it • Circle with the radius of 1 and the centre at the origin (0; 0). • Circle with a radius of 1 and the centre is at the origin in the Cartesian plain.	T1 T4 T8	• To label quadrant or even the special angles. • When looking for special angles, especially 0°, 90°, 180°, 270° and 360° degrees. • To calculate the values of e.g. sin (90°), cos (90°), tan (90°), sin (180°), sin (270°), cos (270°), cos (360°) etc. without using a calculator.

	sense	T8 T11	• Circle whose radius is one throughout from the centre to circumference at x=0°; 360° x= -180° on the x axis and y = (0°; 90°); (0°; 270°), 270°; 360°. • Circle with a radius of 1 and measurement of special angle.	T10 T9	• Make r=1 and show that cos0°=1, cos (90°) =0, cos (180°) = -1 etc. •
4	Response not related to the anticipated one Or No response	T2	• No response	T3	• No response

4.4.1.1.8.1 Analysis of item 10 of the trigonometric task

To analyze teachers' responses to item 10, a select group of responses have been sampled. In fact, it was observed that teachers employed a variety of problem solving strategies to respond to item 10. In item 10.1, all teachers provided suggestions reflecting on what they seemingly conceptualized as that which would inform learners' problem-solving decisions. In the researcher's view, teachers' responses to item 10 largely constituted teachers' interpretations and knowledge of learners' anticipated problem-solving actions. It must also be noted that, in the process of teachers providing responses to item 10.1, it was still be possible to observe a system of teachers' mathematical meanings that guided their anticipated learners' actions.

10. The following question was asked in a Grade 10 mathematics examination (see question at the foot of the diagram).



M is the centre of the circle. If MB = BC = r, show that $\cos\theta = \frac{1}{4}$

In relation to the diagram in Question 10 determine the following:

10.1 What important decisions should learners make in order to solve this task? ______

____

10.2 Develop a solution for this task. ______________________________________

____

Figure 4. 7: Item 10 of the trigonometric task administered to the teachers

4.4.1.1.8.2 Teachers' anticipated response to item 10.1

Given the fact that the problem in item 10.1 was first given to Grade 10 learners, the researcher expected teachers to suggest the application of mathematical meanings for teaching trigonometry addressed in Grade 10. These meanings would therefore take into consideration the understanding of $sin\, sin\, \theta$ as addressed in Section 2.10.4.3. Moreover, the researcher anticipated teachers' responses to include, amongst other things, the forms of mathematical knowledge listed in Figure 4.8.

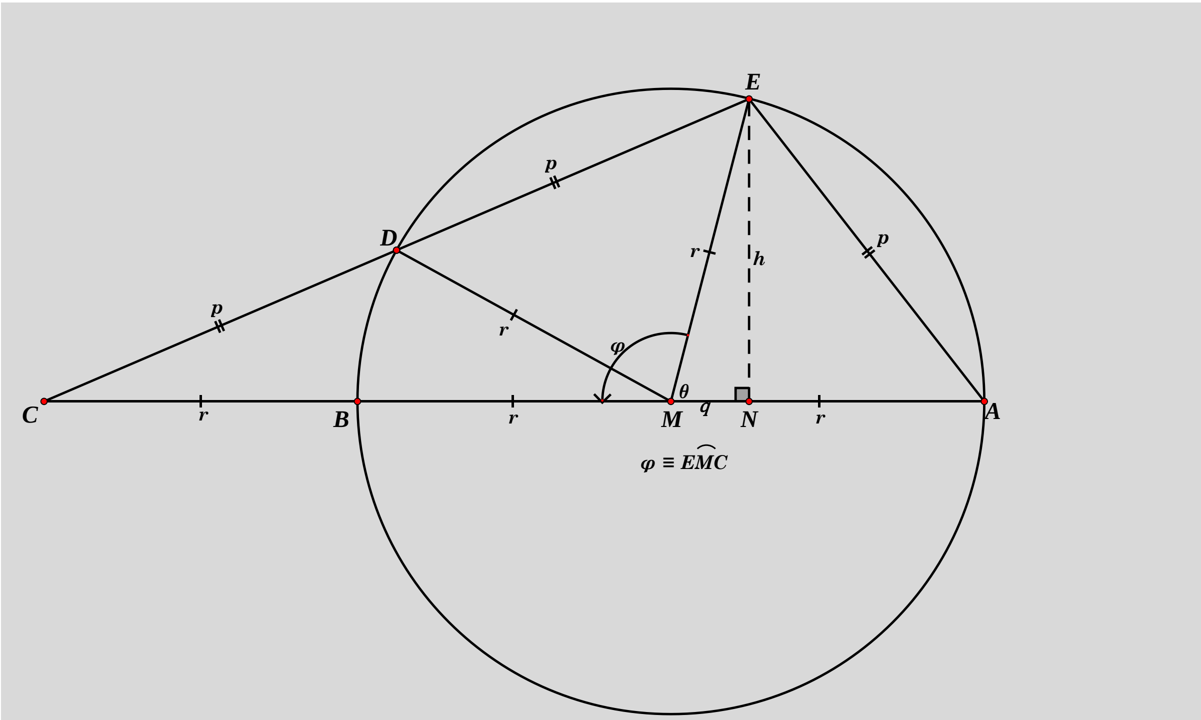


Figure 4. 8: Possible forms of mathematical knowledge associated with teachers responses of item 10.1

In Figure 4.8, some of the possible forms of mathematical knowledge possibly associated with teachers' responses of item 10.1 are:

- Identify three right angled triangles of which one has θ as the central angle and is embedded in the given circle,
- Identify the need to make use of the Pythagorean Theorem and set up four equations, and,

- Use algebra techniques to solve the equations simultaneously.

Table 4.17 provides information on how teachers' suggestions looked, and in addition, teachers' suggestions are viewed and analyzed in terms of mathematical meanings they possibly project or convey.

Table 4. 17: Analysis of teachers' responses to item 10.1 and related possible meanings embedded in responses

Teacher	Example of response	Response Category
T2	• The learner should consider that triangle DEM and triangle EMA are congruent. Meaning that EMA=DME= θ	3
T4	• They need to know how to apply cosine rule. But it's too long and difficult for Grade 10.	2
T5	• The learner must apply cycle theorem and show all sides and angles that are equal before attempting the question.	3
T6	• They must first decide whether the circle has the centre or not at M and decide on the relationship between the dimensions.	4
T7	• They must first decide whether the circle has the centre or not at M and then decide on the relationship between the dimensions.	4
T10	• BM=r, DM=R, AEC=180 degrees-θ	4
T9	○ They should recognize that if they can construct a line joining BD, then they can apply the mid-point theorem. ○ And that if they further construct a line joining A, and D, they will have congruent triangles, which will then lead them to having a right-angled triangle.	2
T12	• The following, non-exhaustive list: ▪ Identify right-angled triangle that has θ as one of its interior angle. ▪ Ensure that the sides of the triangle are either all in terms of r, or all in terms of p (to divide out common factors) ▪ Be prepared to "see" a triangle that is not shown (construct) ▪ Be prepared to establish angles not given in terms of θ ▪ Identify the need to make use of the midpoint theorem	2
T1, T3, T8 and T11	• No response	N/A

To qualify teachers' responses in Table 4.16 in terms of the possible mathematical meanings culminated in a diagram like Table 4.9 being generated. In item 10.2, teachers were expected to develop a correct solution for the task in 10.1. In this case, teachers were expected to reproduce and demonstrate their level of mathematical knowledge through answering the given task.

Table 4. 18: Possible mathematical meanings associated with teachers' responses to items 10.1

CATEGORY 1	In terms of exploring teachers' meanings, a response in this category could imply that the respondent is capable of generating, and further conveying to learners, mathematical meanings that are productive. These meanings are coherent with the desirable mathematical knowledge that is needed to successfully solve the problem.
CATEGORY 2	The teacher in this category has grabbed and internalized a variety of mathematical knowledge and concepts. For example, respondents have suggested the usage of the midpoint theorem or the cosine rule etc. However, knowledge generation and production in this category is less coherent with respect to the anticipated meanings. The ideas, nevertheless, would help in solving the problem successfully.
CATEGORY 3	In terms of exploring teachers' meanings, a response in this category could imply that the respondent's mathematical knowledge is not consistent and is still yet to strike coherence between the intended and non-productive knowledge. The teacher is, in this way, incapable of utilizing and communicating mathematical knowledge productively.
CATEGORY 4	The teacher in this category has the tendency to generate cognitive conflicts of mathematical knowledge and concepts and may in turn lack the quality of mathematical sense-making. The correctness of mathematical knowledge generated by a teacher in this category is superficial and confusing. The meanings formed by this teacher are not mathematically coherent, and do not promote mathematical logicality. Meanings generated by a teacher in this category may not be fully accounted for mathematically.

In ΔEMA, draw the altitude h, then

$$q = rcos\theta \ (A)$$
$$r^2 = h^2 + q^2 \ (B)$$
$$(2p)^2 = h^2 + (2r+q)^2$$
$$4p^2 = h^2 + 4r^2 + 4rq + q^2 \ (C)$$

and

$$p^2 = h^2 + (r-q)^2$$
$$p^2 = h^2 + r^2 - 2rq + q^2 \ (D)$$

Now substitute (D) in (C) and simplify to get

$$4rq = h^2 + q^2 \ (E)$$

Now substitute (B) in (E)

$$4rq = r^2$$
$$4q = r \ (F)$$

Now substitute (A) in (F)

$$4r\cos\theta = r$$
$$\cos\theta = \frac{1}{4}$$

Figure 4. 9: Anticipated response to item 10.2

These were some of the approaches that lead to the solution of the problem, but the researcher did not anticipate that these approaches would be relevant to teachers teaching trigonometry at the Grade 10 level. An example of these approaches has been given in Figure 4.10:

(1) Making use of the cosine rule procedure

To use the cosine rule we consider ΔEMA and ΔEMC. Now in ΔEMA, by using the cosine rule

$$p^2 = 2r^2 - 2.r^2 \cos\theta \cdots (1)$$

And in ΔEMC, where $\varphi = 180° - \theta$

$$(2p)^2 = (2r)^2 + r^2 - 2r.r\cos\varphi$$

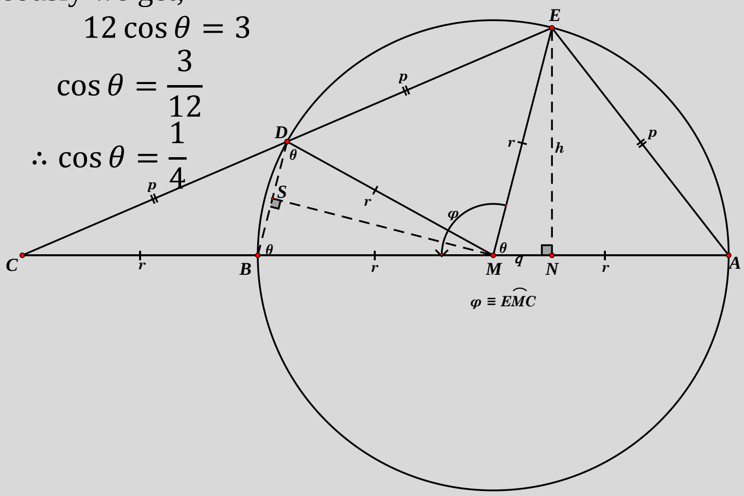
$$4p^2 = 5r^2 + 2r^2\cos\theta \cdots (2)$$

Solving (1) and (2) simultaneously we get,

$$12\cos\theta = 3$$

$$\cos\theta = \frac{3}{12}$$

$$\therefore \cos\theta = \frac{1}{4}$$



(2) Making use of the midpoint theorem

Solution:

Join BD (construction)

$BD = \frac{ME}{2} = \frac{r}{2}$ (points B and D are midpoints of MC and EC)

$D\hat{B}M = E\widehat{M}A$ (corresponding angles; $DB \parallel EM$; midpoint theorem)

Construct perpendicular bisector MS of BD with S on BD

$\Delta MDS \equiv \Delta MBS$ (SAS)

$BS = \frac{r}{4}$ $(BS = \frac{BD}{2})$

Thus $r\cos\theta = \frac{r}{4}$ (definition of the cosine of θ in ΔMBS)

$\therefore \cos\theta = \frac{1}{4}$

Figure 4. 10: Alternative problem-solving procedures to item 10.1

In addition, to examine teacher's responses or solutions to item 10.2, the researcher developed a rubric tool that guided the assessment of teachers' solutions. Table 4.19 shows the approach of teachers' responses to item 10.2, and whether this approach yielded successful or unsuccessful results (see, Table 4.19).

Table 4. 19: Teachers' approach to item 10.2 and the approach's success or not

Approach	Successful attempt	Unsuccessful attempt
Trigonometric function definition		
Cosine rule	T5, T6, T7, and T8	T4
Midpoint theorem	T9 and T12	
No attempt	T1, T2, T3, T10, and T11	

Successful attempts mean a teacher's solution is complete and correct, while unsuccessful attempts imply that the teacher's solution is incomplete or is incorrect. Table 4.19 shows that teachers' attempts to solve item 10.2, and largely accurately too, indicates that these were successful rather than unsuccessful attempts. It must be noted that, in Table 4.19, teachers' responses that were placed in the "No attempt" zone were neither classified as successful nor unsuccessful. In fact, the "*No attempt*" classification in Table 4.19 meant that teachers did not present the expected problem-solving attempts in this item.

4.5 ANALYSIS OF SEMI-STRUCTURED INTERVIEWS

Semi-structured interviews were conducted with two teachers participating in the study (see, Appendix 2; see, also, Section 3.6.2.2). The researcher used the codes Teacher 2 (T2) and T3 instead of using the respondents' actual names to respond to issues concerning confidentiality and anonymity (Section 3.9). The duration of each interview ranged from 50 minutes to one hour, with the researcher explaining that the purpose and main objectives of the semi-structured interviews to the prospective respondent were:

- To establish teachers' instructional *actions* that they viewed as essential to facilitate productive instruction in Grade 10 trigonometry classrooms; and,
- To gain more insights and more in-depth understanding of teachers' mathematical meanings of trigonometric concepts, especially those relating to the trigonometric task.

The next section discusses the participants' responses to some of the interview items.

4.5.1 Teachers' interview responses

The interviews payed attention to two issues that are mentioned in Section 4.5; namely, (1) to identify some of the instructional actions or classroom moves perceived to characterize a productive pedagogy in a Grade 10 trigonometry lesson; and, (2), to find out respondents' conceptualizations and meanings of some trigonometry concepts that this study addressed. Therefore, this section identified interview items and responses that are perceived to come close to addressing the two issues mentioned in the preceding lines.

4.5.1.1 Analysis of teachers' responses to interview item one

This interview question asked:

> *In your understanding, how should a teacher demonstrate his/ her understanding of trigonometric concepts in the classroom?*

In a response, the second respondent (T2) emphasized that the teacher's classroom actions or moves would be influenced by their understanding of certain concepts in trigonometry. Instead of highlighting the anticipated teacher's moves and actions, T2 opted to respond to the interview item by zooming into a specific topic in Grade 10 trigonometry and said:

T2: ……. *in that way it would be like teaching from a right-angled triangle*.

This type of response from T2 fell short of revealing some of the anticipated lesson moves or actions that the interview question intended to unravel.

T3 said, *"… when I put in the issue of how a bridge is designed*", T3 tended to identify and provide trigonometry-related examples in everyday contexts, such as the mathematics embedded in bridges. The response of T3 was seemingly a significant shift away from the anticipated response, which could have focused on manifesting the respondent's knowledge of teacher moves and actions characterizing teacher instruction when a topic of trigonometry was taught in a Grade 10 mathematics classroom.

Overall, the responses of T2 and T3 to the first interview item did not respond to the anticipated teacher actions that the first interview item was probing. These observations, in relation to teachers' responses to the first question of the interview, seemed to suggest that respondents could not project clear conceptualizations constituting desirable teaching actions in mathematics classrooms. Given that all teachers shifted from providing anticipated responses, the researcher attempted to probe teachers further. The researcher asked T3 further,

> So, you say, you need to demonstrate your understanding from….

In responding, T3 replied,

> **T3**: *Through the bridge, roof truss and those things…. So then now, the question should be… like your follow up question something there should be like the one of "what do you mean by sine of something"?*

It was interesting to observe that instead of responding to the issue of teacher actions, T3 opted to respond to the first question of the interview by reflecting on the issue of meanings, which was also at the heart of the study (see, Section 1.5). However, T3 was observed to revert to the matter of teacher actions later in their response. T3 said, "*…so before I can introduce trigonometry to Grade 10 I must show them or tell them about the application first, just to draw their attention*". This type of response implied that T3 held the view that teachers' meanings of mathematical concepts would be prioritized in the classroom as compared to learners' mathematical meanings. The use of phrases such as, "*I must show them…*" and "*I must tell them…*", suggested that T3 advocated a transmission mode of teaching, implying that the teacher's actions in the T3 mathematics lesson would be dominantly teacher-oriented or largely teacher-centered (see, Appendix 3).

In terms of the objectives of the current study, it is reasonable to assume that their responses implied that the teacher meanings of mathematical concepts were relatively prioritized over those of the learners. Teachers seemed to be trapped in the conventional classroom culture of always aspiring to impose their mathematical knowledge and meanings on their learners. This kind of teaching, and presumably learning, model, cannot be classified as 'developmental', because they perpetuate a single perspective o, learning mathematics, rather than promoting a multi-pronged approach in which learners' conceptualizations and views are taken into consideration.

In short, teachers' actions as implied by the responses of T3 to the first question of the interview were largely demonstrative. T3 uses the phrases, "*I can introduce*…", "*I must show them or tell them*…", "*I just start by saying*…", "*So for me to draw their interest, what I need to do is to show them*…". T3's last contribution to the first interview question goes by saying:

> **T3:** "…so by that particular chain, they must know if the chain can carry the weight of the engine as is and at what particular angle" (*the closing comments here referred to an example of engineering activities that T3 had given while responding to the first interview question*).

In the preceding comment, T3 emphasized that "*they must know*". However, a subsequent question would be,

> What kind of teacher actions that would lead leaners to *knowing*?

In terms of T3 responses, a teacher must impose their mathematical knowledge to their learners, as implied by respondent's (T3) phrases such as, "*I must show them or tell them*…", "*I just start by saying*…", and, unfortunately, these classroom actions presented a picture of a teacher-dominant instructional model.

4.5.1.2 Analysis of teachers' responses to interview item two

The second question in the semi-structured interviews addressed an important construct of the study; namely, the *meanings* (mathematical) that teachers presumably attached to a certain concept in mathematics. The question asked:

> *What is the meaning of an angle measure*?

T2 and T3 were keen to provide answers to the preceding interview question and, in responding to the question, teachers provided interestingly varying views on how they conceptualized the notion of '*angle measure*'. The following are actual teachers' responses:

> **T2:** *… if like I turn the arm I will have the right angle and if it's straight I will have the 180 which completes, which is 180 degrees…*

T3: *…. distance between two lines meaning now if I have a line and a line, then I'll have a distance between the two*

It seems the response of T2 relates more to a procedural model, which in turn tends to focus on procedures that measure angles in trigonometry. T2 holds mathematical meanings that are presumably tangible or demonstrable, such that, when T2 reflects on their mathematical meanings, they responded by relapsing into doing a demonstration, thus allowing learners to build or construct their own mathematical meanings based on their observations of what they would be seeing or observing in teacher's demonstrations. In addition, T2 was captured as saying, "… *I mean like where you demonstrate a half turn and full turn and so forth*…" (see, Appendix 3). A concept demonstration model is used by T2 to facilitate the unpacking of mathematical meanings.

Comparatively, T3 seems to follow a *concept definition model* when responding to the second question of the interview. This could be observable in phrases such as, "… *distance between two lines*…", and "…*vertices are two lines joined together*…". It must be noted that accuracies in presenting mathematical concepts are not the focus of this analysis, and the seeming reliance of T3 on the textbook definition of mathematical concepts is highlighted by the researcher's follow-up at certain stages of this conversation,

> *The angle in your definition or your meaning of an angle is that distance between two lines?*

The kinds of mathematical meanings that T3 projected in their response were far more conceptual than demonstrational. This aspect is observable in the T3 further response to the second question of the interview, "… *but now by the distance between that I was referring to this angle*…" In this case, one can see that T3 is a teacher who strives to explain mathematical terms theoretically, as opposed to making these terms demonstrable. Therefore, the meanings conveyed by T3 are largely conceptualized entities while those conveyed by T2 are tangible and demonstrable (see extended versions of teachers' responses in Appendix 3).

4.5.1.3 Analysis of teachers' responses to interview item three

The third question in the semi-structured interviews pertains to the concept of teacher actions in the classroom, while delivering a lesson on certain mathematical concepts. The question asked:

> *How should a concept of an angle measure be introduced to Grade 10 learners*?

In response to question three, the interview T2 tended to rely on a demonstration model. Some of T2 responses included phrases such as, "…*if I turn my arm, then I turn my arm like this*…", and, "*after that demonstration is when we take the protractor and measure the angles*". Overall, T2 emphasized the use of a protractor when introducing the trigonometry topic to Grade 10 learners. This line of reasoning is also manifested in the researcher's further probing of the response of T2,

> *… so you said using a protractor is how you introduce the concept of…*?

The contrasting teaching approaches were observed in teachers' responses to the third question of the interview. Unlike with T2, T3 emphasized the significance of learners' pre-knowledge as a point of departure in proceeding with the lesson. T3 emphasized, "…*In Grade 8 as well, that's when they start with the construction of the angles and stuff in Grade 9*". In this way, T3 seemed to recognize the fact that, at the entry of any mathematical lesson, teachers may not assume that learners are empty vessels, and that they are not in possession of knowledge fact in relation to the topic that would be taught in the classroom. In recognition of learner knowledge, T3 further commented,

> **T3:** *… So in Grade 10 we assume that now learners know how to..... Yeah, in Grade 9 they know how to use that particular protractor*.

These were interesting observations about two teachers. It seems that T2 would make demonstrable efforts to construct the intended mathematical concepts to their learners. This teacher, T2, will make use of real life materials to facilitate a mathematics instruction or lesson. In contrast, T3 will assume that learners come to the classroom with pre-possessed conceptualizations of mathematical concepts, which eventually inform the meanings they will convey to mathematical concepts. In terms of T3, it is important to recognize and build new knowledge on learners' pre-existing mathematical conceptualizations to correct and strengthen the accuracy of mathematical meanings that learners attach to mathematical concepts.

4.5.1.4 Analysis of teachers' responses to interview item four

The fourth question of the interviews explored some aspects of mathematical knowledge and was posed as follows:

> *What would you like your Grade 10 learners to understand in relation to Sin x and/ or Cos x*?

This interview item focused not only on issues of mathematical knowledge and understanding, but the question also addressed issues of sequencing and coherence (Section 1.9.1). Right at the onset, T2 provided the following elongated response:

> **T2:** *What I would like the learners to understand is what it will be like for Sine and Cos for them to be able to actually see the difference between the sine and the cos in terms of like how we say like when we draw the acute angle for instance on a Cartesian plane to say the first quadrant, second quadrant and the third and fourth. So, I would like the learners to understand how these trigonometric ratios differ in each of the four quadrants particularly on the right-angled triangle because like here a Grade 10 learner whereby any angles we are not dealing with, we are just dealing with the right-angled triangle.*

T3 also generated a seemingly lengthy response, and said:

> **T3:** … *So now with my Grade 10 definitely I will say to them before they can use these three trigonometric functions (referring to sine, cosine and tangent functions) they need to understand what I must have, a right-angled triangle, without that they cannot think of trig ratios*.

In their responses, it was not clear what T3 referred to with the phrase "… *to understand what I must have*,…". However, in the context of ongoing discussions between the researcher and T3, it is reasonable to assume that T3 probably referred to knowledge possession when using the phrase "…*what I must have*,…". This kind of knowledge possession seems to refer to knowledge of some aspects of a right-angled triangle. In our further analysis of T3 responses to the fourth question, it appears that the kind of mathematical knowledge that T3 deemed important would revolve around the sides of a right-angled triangle, and their associations. In the opening line of the response, T3 said, "*They should be having the same hypotenuse, I think*", without commenting further on the issue of 'sameness' in relation to a hypotenuse. This teacher went on to say:

> **T3:** … *Because we are looking we are looking at the definition of sin cos and tan. Opposite over hypotenuse, so if I don't have a right-angled triangle I'll never know which one is opposite, which one is hypotenuse*.

It is reasonable to see that both teachers (T2 & T3) would anticipate that, at the start of a trigonometric conversation in the classroom, learners would be expected to possess knowledge and understanding of three functions in trigonometry; namely, *sine*, *cosine* and *tangent*. According to T3, understanding of these three main trigonometric concepts would be demonstrated when learners were working with different types of triangles. In this regard T3 further replied:

> **T3:** *Because now that's where the confusion comes in when they go to Grade 11, you give them a scalene triangle to find one side, they'll go straight to one of the three trigonometric functions which are no longer applicable because we are no longer talking of a right-angled triangle.*

In the case of a preceding response, it seems that T3 implied that mathematical understanding would be demonstrated and manifested in the application of mathematical knowledge when a learner solves mathematical tasks.

4.5.1.5 Analysis of teachers' responses to interview item five

Item Five of the interview probed the respondents on the issue of conceptual understanding, namely, requesting them to provide a definition or description of a certain concept in mathematics. The item in this regard was posed as follows:

Describe sin x in your own words.

T2 quickly responded that,

> **T2:** "… *The sine of x, I would define it in terms of a, when looking at, in terms of the right-angled triangle where we have opposite over the hypotenuse, that is when we are looking at the right-angled triangle*"

While giving the preceding response, T2 continued to provide an explanation or description of "s*inx*", and the following are direct T2 utterances, "… *the sine of an angle on right-angled triangle is opposite over the adjacent*…", which appeared to be mathematically incorrect in light of the equation,

$$sinx = {}^{Opposite\ side}/_{Hypotenuse\ side}$$

The observed inconsistencies in the presentation of mathematical knowledge by T2 was interpreted by the researcher as reflecting the shaky foundation of their entertained mathematical meanings.

In comparison, T3 provided an almost similar response to T2's response when answering the interview's fifth question. T3 replied, "…*That's opposite over hypotenuse, …. …in any given right-angled triangle, the sinx is given by the ratio of opposite and hypotenuse*". The researcher requested T3 to unpack the concept of "ratio", which was featured in the later response provided by T3, which was seemingly long. T3 responded by saying:

> **T3:** … *we are saying opposite over…and those two things give us a ratio now we are dividing the 2. So that over that. So, if I say to you sinx is given by opposite over hypotenuse. Opposite over hypotenuse what is that? What is opposite over hypotenuse? It is the ratio in a way. It's 2 divided by 4 or 2 divided by 9.*

The phrase "*those two things*" in T3's response is used at the start of the reply, which is presumed to be referring to the two components or entities (parts) of a fraction; namely, the numerator and the denominator ($^{numerator}/_{denominator}$).

4.5.1.6 Analysis of teachers' responses to interview item six

The sixth question of the semi-structured interview tended to assess respondents' knowledge of the nature of *sin x,* explaining whether it is a function or not. The question was posed as follows:

> *Describe sin x in your own words. Is sin x a function or not? Please explain your answer?*

Apart from yielding a possible *Yes* or *No* response, the question instructed respondents to explain their initial responses. In an attempt to respond to the sixth question of the interview, T2 abruptly said, "*Yes* $sinx$ *is a function…*", and, in justification of this seemingly confident response, T2 continued by saying, "*…because if one equates* $sinx$ *with* y, *as* $y = sinx$, *ordered pairs can be formed and a graph can be sketched*". T2 continued to assert that equating *sin x* to *y* was a condition for classifying *sin x* as a function. T2 continued, "*… if we just say to me that Sine of x then like, there I don't see a function*". This line of thinking appeared to be a point of interest to the researcher, who then probed further by asking, "*… In other words, you are saying the Sine of x is not a function*?", seeking clarity that if the condition of equality is not attached to *sin x*, then the later would not be considered a function. T2 seemed to stand their ground by replying, "…, *that is why I said, only and only if you just equate that sine of x*".

To clarify their earlier response, T2 brought into the discussion the notion of *f(x)*. In this regard, T2 said: "*…maybe you say the f of x equals the sine of x then yeah, I'm having a function, then the sine of x will be a function*". The discussion between the researcher and T2 continued to the point at which T2 introduced the topic of a Cartesian Plain, in which context T2 attempted to strengthen their argument that *sin x* is a function. T2 argued:

> **T2:** *Now I'm looking at how do I define that sine x. then it will be like, on that Cartesian plane maybe I'm having some units up then the hypotenuse is low then I will have the sine of x equals to maybe a specific unit. Maybe suppose I have 1.5 units, the sine of x equals to 1.5, then that is not a function. The sine of x equals to just 1.5 then to me it looks like it stops there, it doesn't say anything. But once I say ok, here you have the, I'm equating that y equals to the sine of x whereby now one can plot on the Cartesian plane.*

T2 stood firm that *sin x* would not be a function if it is not depicted in an equation format. This is confirmed by the researcher. who finally probed T2, "*So, the sine of an angle is not a trigonometric function?*", and T2 immediately replied by saying, "*Yeah, yeah. But only according to my understanding*". In light of this discussion between the researcher and T2, it is reasonable to conclude that T2 did not hold a stabilized conception of *sin x* as a function. It becomes clear that the meanings that T2 attached to the notion of *sin x* would not have been built on a solid understanding and conceptualization of how should *sin x* be understood. It is therefore reasonable to argue that T2's meaning of the mathematical concept of *Sin x* was not rigid but somewhat fluid and lacking some aspects of coherence.

T3 was also brought in into the discussion and started by briefly responding: "*Yeah it is a function*". The researcher then requested T3 to motivate their response. T3 replied: "*As I have said that a relationship between…, that's now when I define the word function. The relationship between x and y*". The researcher probed, "*…So, what is y in this case*?". and immediately replied:

> **T3:** … we are talking of *sine y* would be the opposite (*here T3 was using the terminology from relating sides of a right-angled triangle*). But remember now that these things might change anyway, but in the first quadrant, your *y* would be what? Your opposite.

Comparatively, T3 seemed to rely heavily on the idea of the Cartesian Plane to strengthen their argument that *sin x* is a function.

Overall, both teachers, T2 and T3, strongly argued that *sin x* is a function, basing their rationale on what seemed to be individualized perspectives that they each had adopted. However, the researcher noted that, in both cases, teachers were not firmly grounded to their perspectives in pursuit of their arguments. When probed further, they each appeared to fall short. For instance, T2 did not display confidence when they ended the interview by saying, "*Yeah, yeah. But according to my understanding*": implying that they alone possessed what we might be referred to as 'localized knowledge', which may not hold when applied to diverse and divergent contexts and settings. T3 could also be classified in the same way.

4.5.1.7 Analysis of teachers' responses to interview item number seven

The seventh question of the interviews probed teachers in relation to teaching strategies that they could suggest when the concepts of *sin x* and *cos x* were to be taught to a Grade 10 mathematics classroom. The question was posed as follows:

> *What kind of teaching action(s) should a Grade 10 teacher adopt when teaching the mathematical concepts of sin x and cos x*?

In a simplified version, the researcher asked,

> *Explain your teaching approach that you could use to make your Grade 10 learners understand the trigonometric concepts of sin x and cos x*?"

At first, T2 gave the impression that they did not understand the research question when they responded by saying "*How will I teach the concept of…*". The researcher was quick to rephrase the research question to T2, but T2's response was "*There eish*". T2 was out of their depth and could not formulate the teaching approach needed for use to facilitate the apprehension of the trigonometric concepts of *sine x* and *cos x*. Given this situation, the researcher then decided to remind T2 about the written response to the research question (see, Figure 4.10).

> **Researcher (R):** *Initially you said you will define the trigonometry focusing on how the trigonometric ratio sine and cos are defined on triangles. If I understand you correctly, your approach as you stated it will initially involve using a right-angled triangle?*

T2 replied by saying, "*Yeah, I will be using right-angled triangle*".

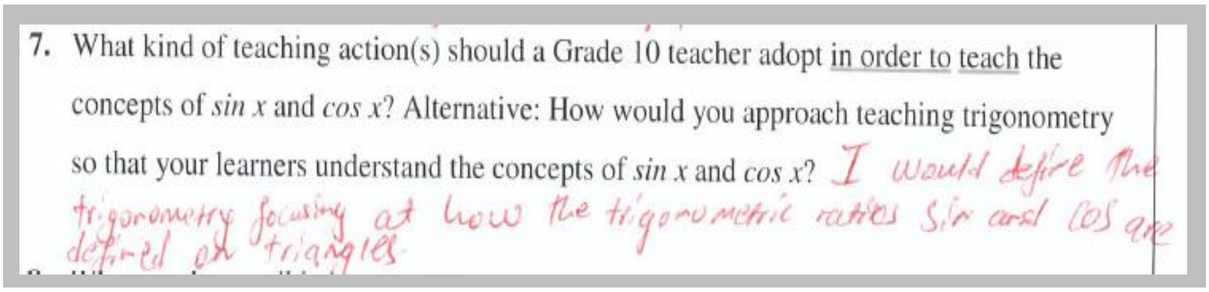

7. What kind of teaching action(s) should a Grade 10 teacher adopt in order to teach the concepts of *sin x* and *cos x*? Alternative: How would you approach teaching trigonometry so that your learners understand the concepts of *sin x* and *cos x*? I would define the trigonometry focusing at how the trigonometric ratios Sin and Cos are defined on triangles

Figure 4. 11: The written response of T2 to the research question 7

However, the response of T2 to question seven of the interviews was rather very telegraphic. The actual activities that would be actualized in the lesson are not spelt out, and, in fact, there is no discussion on the stages of the lesson and how each of these stages would be played out. Therefore, there is little to learn here in terms of the teachers' actions needed to facilitate Grade 10 instruction on *sin x* and c*os x*.

On the other hand, T3 responded by saying, "*I think they must stick to trigonometry*…". It was not clear what T3 meant by the phrase "*stick to trigonometry*". Later, after probing by the researcher, T3 echoed the sentiment of teaching trigonometry from right-angled triangle perspective. In other words, T3 suggested that Grade 10 learners will understand trigonometric concepts through mastering a right-angled triangle. T3 stressed the following:

> **T3:** So then now, if at the beginning if I didn't emphasize the issue of a right-angled triangle, there's no other way.

Seemingly the approach of T3 to teaching trigonometry to Grade 10 learners was the right-angled triangle approach[30]. T3, in trying to unpack the research question 7, pondered that, "… *So Grade 10s, why they are failing trigonometry* …". At this point, the researcher posed the following question to T3:

> **R:** Yeah. That's what one wants to establish. In fact, I'm trying to give a reason with my study, why they are failing. Because it starts there in Grade 10. So yeah,

30. The *right-angled triangle approach* to teaching trigonometry places a large emphasis on the knowledge of procedures, such as labeling triangles and computing ratios that may come at the expense of conceptual understanding of sine and cosine as functions (Kendal & Stacey, 1998).

we were discussing question 7. We were talking about question 7. What kind of teaching actions or alternatively, how will you approach teaching trigonometry so that your learners understand the concept of *sin(x)* and *cos(x)*?

To this question T3 provided the following lengthy response:

> **T3:** *Understanding of the learner, because you know what is happening here? Sometimes the educators, they become unfair to these kids because now I'll bring in the issue of the first quadrant. When I teach or when educators teach they'll focus mainly on the first quadrant and then they'll make sure that now they put in the angle down there, but come exams they take that particular right-angled triangle and put it in the second quadrant now they take the angle up there now the learners are totally confused. So, it becomes unfair to the kids*.

Unlike T2, T3 was able to indicate the activities that would be adopted in an attempt of helping learners understand the concept of *sine x* and *cos x*.

4.5.1.8 Analysis of teachers' responses to interview item number eight

The eighth question of the interview seemed to assess teachers' knowledge of learners' experienced difficulties and challenges when learning certain topics in Grade 10 trigonometry. The following is the format in which the question was posed to the respondents:

> *What are the possible learner difficulties in relation to solving trigonometric tasks in Grade 10*?

An alternative and simplified version of the eighth question of the interview would sound as follows:

> *In your understanding, what are some of the difficulties that learners in Grade 10 experience when they are solving trigonometry problems*?

It must be noted that, in some instances, an alternative version would be asked to assist the respondents to comprehend the primary question of the interview.

This is what T2 presented as a response to the eight question of the interview:

> **T2:** *Umm, the difficulties they have in solving trigonometric problems, one of the difficulties they have it's for them confusing this trig ratios, it's one of them. Like when looking at the right-angled triangle sometimes learners confuse each one of the opposite sides. That's one of the challenges they have. And the other thing, the other challenge they have, when the angle is drawn, maybe the other quadrants except the first quadrant there. Remember on the Cartesian plane there were we have the positive y-axis and the negative y axis, now learners tend to confuse the which one is positive y-axis and which one is negative y-axis.*

Mainly, T2 reflected on learner challenges pertaining to procedural knowledge; for example, identifying the sides, *opposite* and *adjacent*, in a right-angled triangle. Furthermore, according to T2, learners sometimes confuse "… *this trig ratios*…", implying that learners tend to define the *sine* of an angle as *adjacent* divided by the *hypotenuse* ($^{\text{adjacent}}/_{\text{hypotenuse}}$) and the *cosine* of an angle as *opposite* divided by the *hypotenuse* ($^{opposite}/_{hypotenuse}$).

In responding to the eighth research question, T3 said:

> **T3:** *Expose them to different types of questions. Like, yeah, another one comes in the exam. You used to give them a normal triangle even if you can move it in all the quadrants, it's fine but now here comes the exam one will just rotate that angle or the triangle maybe 45 degrees. Meaning now, it's no longer sitting this way but the 90 degrees is there at the corner, it's up there. Now some of the learners still get confused, so the types of learners… yes, they have brains but they don't think for themselves.*

In their response, T3 mentioned similar learner challenges that were highlighted by T2. However, T3 drew attention to learners who experienced difficulties when the orientation of the right-angled triangle was changed. In other words, T3 implied that learners would struggle to identify a right-angled triangle that is not positioned in a standard format.

4.5.1.9 Analysis of teachers' responses to interview item nine

The ninth question of the interview asked the teacher respondents to brainstorm teaching strategies which can ease or mitigate learner difficulties and challenges experienced when approaching specific topics in Grade 10 trigonometry. The researcher presented the question to the respondents in the following format:

> *What kind of action(s) would you employ in order to alleviate learners' difficulties and difficulties in Question 8*?

An alternative version of the ninth question of the interview read as follows:

> *What strategies do you use to help your learners with the difficulties mentioned in Q 8*?

In responding to the ninth question, T2 gave the following response

> **T2:** *To actually help learners with such difficulties, just by demonstrating it various examples and explaining to them like in terms of saying ok if this is the mistake that you have made, and this is how you were actually supposed to do. And then after that maybe give them another problem related*.

It seemed as though T2's strategy for dealing with leaner difficulties regarding the apprehension of basic trigonometric concepts involved demonstrating using different examples and providing explanations for incorrect attempts. Also, T2 recommended the usage of mnemonics such as *SOHCAHTOA*. T2 elaborated:

> **T2:** *Yeah, normally like as I indicated that you find that they are mixing those opposite and sometimes you say ok here is sine of x then you have indicated x as an angle on the right-angled triangle. Now the learner will just confuse to say ok, maybe the learner instead of saying the sine opposite over hypotenuse now the learner will tan, which is opposite over the adjacent. So that like acronym it actually like simplify the concept of sine is opposite side over the hypotenuse side and then the tan and the cos and so forth*.

According to T2, using mnemonics in a right-angled triangle context would help learners to address difficulties associated with learning trigonometric concepts.

Comparatively, T3 suggested that teachers should attend to learners' challenges and difficulties using a one-on-one approach; that is, through attending to learners on an individual basis. In addition, T3 recommended and emphasized the approach of labelling the right-angled triangles properly. In this regard, T3 elaborated:

> **T3:** *Oh yeah. Because now what makes these learners not to answer correctly. Number 1, as I've said, in class work you find educators they'll put in their reference angles down there and write 60 degrees, 1. Come exam time or you give them homework, now in the particular homework you're no longer there you've put the angle up there. So, the kids they fail now to separate which one is opposite, which one is adjacent. So, if they can master that, then it's done problem solved. If they master the labelling of triangles, right-angled triangles, problem solved.*

4.6 SUMMARY OF TEACHERS' INTERVIEW RESPONSES AND EMERGING CONSTRUCTS

Table 4.20 provides a summary of the key analysis of teachers' responses to the interview questions that are discussed in Sections 4.5.1.1 to 4.5.1.10. Subsequently, Table 4.21 provides a further summary of teachers' interview analysis and the researcher's overall impressions and interpretations of teachers' responses. Tables 4.20 and 4.21 serve as a tabulated summary of the concluding remarks of Chapter 4. In the next chapter, the concluding discussions are provided in line with the answers provided to the research questions of the study, in allowing them to respond to the study's objectives.

Table 4. 20: Summary of the analysis of teachers' interview responses

Interview questions	Key issues addressed	Aggregating teachers' responses	Section where teachers' responses are analyzed and discussed	Related research question (RQ)
1	Focusing on teacher actions during a Grade 10 trigonometry lesson	Initially unable to identify and unpack anticipated teacher actions Upon probing, teacher actions are presented as a set of teacher-cantered activities during the lesson Instances of knowledge imposition are observable Teacher meanings are prioritized and imposed on learners	4.5.1.1	RQ = 1.6.2
2	Unpacking teachers' mathematical meanings	Mathematical meanings are preferably demonstrable (concept demonstration model) A concept definition model is also projected Meanings are more conceptual than demonstrative	4.5.1.2	RQ = 1.6.1
3	introducing (teacher actions) a concept of angle measure to grade 10 learners	Demonstration model is used, and real-life examples are incorporated to facilitate understanding (eg., human arm, etc.) Recognition of learners' pre-existing mathematical conceptualizations and meanings	4.5.1.3	RQ = 1.6.2
4	Learner knowledge and understanding	Knowledge of basic mathematical concepts Understanding is demonstrated in the successful application of mathematical knowledge	4.5.1.4	
5	Conceptual definition	Shaky constructions of mathematical meanings by the teachers	4.5.1.5	RQ = 1.6.1
6	Knowledge of mathematical concepts	Sinx is conceptualized as a function when it is represented as an equation, such as y=sinx Using Cartesian Plane to argue that sinx is a function	4.5.1.6	RQ = 1.6.1

7	Teaching approach	Approach to teaching trigonometry to Grade 10 learners is the right-angled triangle approach	4.5.1.7	RQ = 1.6.2
8	Teachers’ knowledge of learners’ experienced difficulties and challenges	Learner challenges that pertain to procedural knowledge, for example, identifying the sides, opposite and adjacent, in a right-angled triangle. Confusing the trigonometric ratios Altering the orientation of the right-angled triangle	4.5.1.8	RQ = 1.6.1
9	Teaching strategies to address learners’ difficulties	Demonstrating using different examples and providing explanations for incorrect attempts Use mnemonics such as SOHCAHTOA Label the right-angled triangles properly Provide one-on-one teaching	4.5.1.9	RQ = 1.6.2

Table 4. 21: Summary of teachers' responses to interview questions (IQ) and related researcher's overall impressions and interpretations

<table>
<tr><th rowspan="2">Focus of Interview Question (IQ)</th><th colspan="2">Overall teachers' responses</th><th colspan="2">Researcher's impression and interpretation of teachers' responses</th><th rowspan="2">Related RQ</th></tr>
<tr><th>T2</th><th>T3</th><th>T2</th><th>T3</th></tr>
<tr><td rowspan="2">IQ1

Teacher moves & actions (Section 4.5.1.1)</td><td rowspan="2">Teacher moves influenced by teacher knowledge and understanding of mathematics</td><td rowspan="2">Teacher moves influenced by teacher's ability to connect real world examples to mathematics</td><td>T2 could not explicitly spell out anticipated teacher moves and actions</td><td>T3 shifted/ deviated from anticipated response</td><td rowspan="2">1.6.2</td></tr>
<tr><td colspan="2"><u>Overall impression</u>: Teachers responses did not list actual teacher actions/ moves. Teacher meanings prioritized over learners' meanings
<u>Interpretation</u>: Teachers did not have proper conceptualizations of expected teacher moves during a trigonometry lesson</td></tr>
<tr><td rowspan="2">IQ2

Teachers' meanings of mathematical concepts (Section 4.5.1.2)</td><td rowspan="2">Response depicted inclination towards a procedural model/ approach

Meanings are tangible or demonstratable

Learners construct own meanings</td><td rowspan="2">Followed a concept definition model (textbook definition-not own meaning)

Meanings are conceptual than demonstrational</td><td>T2 is demonstration orientated</td><td>T3 explains mathematical terms theoretically, and meaning construction is textbook reliant</td><td rowspan="2">1.6.1</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Teachers differed in their articulations of mathematical meanings
<u>Interpretation:</u> T2 demonstrates and allow learners to construct their own mathematical meanings; T2 is fixed to textbook meanings</td></tr>
<tr><td rowspan="2">IQ3

Presentation (actions) of mathematical concept (Section 4.5.1.3)</td><td rowspan="2">Reliance on demonstration model</td><td rowspan="2">Learner pre-knowledge used as departing point of lesson</td><td>T2 is demonstration orientated

Using a protector to demonstrate angle measure</td><td>Emphasis on recognition of learner knowledge</td><td rowspan="2">1.6.2</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Teachers differed in their classroom approaches to teach mathematical concept
<u>Interpretation:</u> T2 makes demonstratable efforts to build mathematical knowledge; T3 build from learners' knowledge</td></tr>
<tr><td rowspan="2">IQ4
Mathematical knowledge & understanding (sequencing/ coherence)</td><td rowspan="2">Emphasis on difference between sin and cos functions

Uses Cartesian plane & right-angled triangle to</td><td rowspan="2">Emphasis on knowledge of right-angled triangle to construct knowledge of trigonometric ratios

Used the phrase: "…what I</td><td>T2 prioritized learners' knowing of working with three trigonometric functions, and their applications in a variety of problems</td><td>Phrase used by T3 referred teacher's knowledge possession (knowledge of sides of right-angled triangle and their association)</td><td rowspan="2"></td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Both teachers shared similar knowledge and</td></tr>
</table>

<table>
<tr><td>(Section 4.5.1.4)</td><td>facilitate knowledge construction & understanding</td><td>must have…”</td><td colspan="2">understanding to teaching trigonometric functions
<u>Interpretation:</u> Knowledge of three basic trigonometric functions is prioritized</td><td></td></tr>
<tr><td rowspan="2">IQ5

Conceptual understanding of mathematical constructs (Section 4.5.1.5)</td><td rowspan="2">Understanding facilitated through knowledge of right-angled triangle</td><td rowspan="2">Uses the phrase “those two things” when referring to the parts of fraction (numerator & denominator) relating to ratios of trigonometric functions</td><td>T2 manifested inconsistencies in presenting mathematical knowledge and concepts</td><td>The phrase “those two things” seemingly referred to numerator and the denominator $(^{numerator}/_{denominator})$</td><td>1.6.1</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Both teachers presented similar responses
<u>Interpretation:</u> Teachers presented similar conceptual understanding of mathematical knowledge. T2 displayed a shaky construction of mathematical meanings</td><td>1.6.1</td></tr>
<tr><td rowspan="2">IQ6

Knowledge & description of a mathematical concept (Section 4.5.1.6)</td><td rowspan="2">T2 linked the notion of a function to the plotting of a graph

The quality of ‘equating’ (equation) used as a condition of being a function

The notion of f(x) to justify the condition of being a function</td><td rowspan="2">T3 agreed that sinx is a function

Emphasis on the relationship between x and y in y=sinx</td><td>T2 strongly explained the condition of functionality in terms of an equation</td><td>T3 relied heavily on the Cartesian plane to explain y=sinx</td><td rowspan="2">1.6.1</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Both teachers argued strongly that sin x is a function
<u>Interpretation:</u> Both teachers were not firmly grounded to their perspectives in pursuit of their arguments</td></tr>
<tr><td rowspan="2">IQ7
Teaching strategies for trigonometry (Section 4.5.1.7)</td><td rowspan="2">T2 initially did not spell out teaching strategies

No discussion of the stages of the lesson</td><td rowspan="2">T3 emphasized teacher understanding of learners as determining the teaching strategy</td><td>T2 observed not to understand the question and a reference to a written task is made

T2 relied on right-angled triangle approach</td><td>T3 emphasized activities to use in attempting to learners to understand mathematical concepts</td><td rowspan="2">1.6.2</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Both teachers were not explicit in their articulation of teaching strategies
<u>Interpretation:</u> There was no clarity in teachers’ articulations of teaching strategies needed to teach trigonometric functions</td></tr>
<tr><td>IQ8</td><td>T2 mentioned that learners confuse trigonometric ratios</td><td>T3 emphasized the importance of exposing</td><td>T2 had a clear understanding of learners’ challenges and difficulties</td><td>T3 emphasized the importance of orienting a right-angled triangle</td><td></td></tr>
</table>

<table>
<tr>
<td rowspan="2">Knowledge of learners’ difficulties (Section 4.5.1.8)</td>
<td rowspan="2">(confusion in mentioning sides of the sides of right-angled triangle)

T2 mentioned learners’ inability to represent trigonometric functions correctly in different quadrants of the cartesian plane</td>
<td rowspan="2">learners to different types of questions in the same topic

Right-angled triangle should be presented in different orientations to deepen learners’ understanding</td>
<td>in the specified topic of trigonometry</td>
<td></td>
<td rowspan="2">1.6.1</td>
</tr>
<tr>
<td colspan="2"><u>Overall impression:</u> Both teachers presented learning challenges that are almost similar
<u>Interpretation:</u> Different types of problems should be given to learners to expose them to various orientations of the problem</td>
</tr>
<tr>
<td rowspan="2">IQ9

Teaching strategies to address learning challenges (Section 4.5.1.9)</td>
<td rowspan="2">T2 used SOHCAHTOA as a tool to mitigate learning challenges and difficulties</td>
<td rowspan="2">T3 suggested that teachers should address learning challenges & difficulties using the one-on-one approach</td>
<td>T2 relied on demonstration instruction to address challenges & difficulties in learning mathematical concepts</td>
<td>T3 recommended individualized teaching approaches to assist learners to comprehend mathematics concepts</td>
<td>1.6.2</td>
</tr>
<tr>
<td colspan="2"><u>Overall impression:</u> Teachers suggested different teaching approaches in addressing and mitigating learning challenges & difficulties
<u>Interpretation:</u> It seems teachers replied on this interview item relying on their experiences and teaching strategies they have tested in their mathematics lessons</td>
<td>1.6.2</td>
</tr>
</table>

CHAPTER FIVE

SUMMARY, RECOMMENDATIONS AND CONCLUSION

5.1 INTRODUCTION

This chapter presents recommendations and a conclusion section to the study, opening with a restatement of the study's aim and objectives along with its research questions, and then endeavors to provide an overview of the findings. The chapter concludes with an extended discussion and recommendations for future research and practice.

5.2 THE AIM AND OBJECTIVE OF THE STUDY

This study was motivated by the observation that Grade 12 learners relentlessly performed poorly in trigonometry in the final examination (see, Section 1.3). The study aimed to explore teachers' understanding of trigonometric ratios and functions by examining the mathematical meanings that teachers convey when they teach Grade 10 learners (Section 1.4). To achieve the aim of the study, the researcher set out the following objectives (Section 1.5):

- to initiate a productive dialogue of inquiry to reflect on existing research in relation to the knowledge of mathematics teachers and related classroom instruction;
- to establish suitable instrument(s), if any, to explore teacher's meaning(s) of their subject matter knowledge in Grade 10 trigonometry;
- to examine mathematics teachers' actions needed to enhance productive instruction in Grade 10 trigonometry topics; and,
- to determine instructional decisions that teachers make to facilitate a Grade 10 trigonometry instruction.

The study explored the following research questions (Section 1.6):

1. What mathematical meanings do teachers convey when teaching a topic pertaining to trigonometric ratios and functions in Grade 10?

2. In what way(s) do teachers' knowledge-related decisions influence their instructional actions during a Grade 10 trigonometry instruction?

5.3 SUMMARY OF THE FINDINGS

In this section, the researcher presents a summary of their findings that falls in line with the study's research questions, in attempting to fulfill the objectives of the study. In this regard, the findings are constituting mainly by the mathematical meanings for teaching foundational concepts of trigonometry that participants possibly convey to their Grade 10 learners. The researcher's endeavors to describe teachers' thinking (mathematical meanings) is entrenched in Thompson and Harel's explanation of understanding, meaning and ways of thinking (Thompson, 2015; Thompson, Carlson, Byerly & Hatfield, 2014). The explanation of these concepts is informed by Piagetian notions of actions, schemes, assimilation and accommodation (see, Sections 2.10 & 4.4). In a similar vein, Thompson (2015) describes the notion of scheme and way of thinking as follows:

- A scheme denotes the meaning or understanding through which a person constructs in a given moment; for example, considering the dynamic meaning or understanding of angle as a rotation (or turn); and,
- A way of thinking refers to the process through which a person develops a pattern for utilizing specific meanings or ways of thinking in relation to particular ideas; for example, in considering the meaning of an angle measure that is founded upon the notion of angle as a turn to conceptualise openness as a changing magnitude that can be measured by arc length.

As such, in reviewing Thompson's (2015) rubric concerning schemes and modes of thought, the researcher concludes that the aim and objectives of the study have been achieved. Furthermore, it is evident that our two research questions have been answered, such that, in the next section, the researcher has put forward research findings emerging from the analysis of the trigonometric task at hand.

5.3.1.1 The trigonometry task

Table 5. 1: Summary of the research findings and the related RQ

Item content	Task item	Response category and categorization				Related RQ
		1. Response perceived to be the anticipated one	2. Response perceived to be close to the anticipated one	3. Response embracing anticipated mathematical concepts but not making sense	4. Responses not related to the anticipated one	
		Number of teachers in each category				
Meaning of angle and angle measure	1.1	3	7	1	1	1.6.1
	1.2	1	2	3	6	1.6.1
	1.3	4	1	3	4	1.6.1
	4	1	7	4	1	1.6.1 & 1.6.2
Meaning of the sine and cosine of an angle	2.1	0	1	11	0	1.6.1
	2.2	0	0	11	1	1.6.1
	2.3	0	1	9	2	1.6.2
	3	0	5	6	1	1.6.1
Meaning of the unit circle	7.1	1	5	5	1	1.6.1
	7.2	4	2	5	1	1.6.2
Application: 2-D problem	10.1	0	3	2	7	1.6.2
	10.2	0	4	2	5	1.6.2

In accordance with Section 4.4.1, Table 5.1 is generated to show the category and categorization of teachers' responses to the trigonometry task items. To elucidate teachers' responses in terms of the possible mathematical meanings generated in each category, the researcher used Table 5.2, which is like Table 4.18.

Table 5. 2: Possible mathematical meanings associated with teachers' responses to task items

CATEGORY	**1**	In terms of exploring teachers' meanings, a response in this category could imply that the respondent is capable of generating, and further convey to learners, mathematical meanings that are productive. These meanings are coherent with the desirable mathematical knowledge that is needed to solve the problem successfully.
	2	The teacher in this category has grabbed and internalized a variety of mathematical knowledge and concepts. For example, respondents have suggested the usage of the midpoint theorem or the cosine rule etc. However, knowledge generation and production in this category is less coherent with respect to the anticipated meanings. The ideas, nevertheless, would help in solving the problem successfully.
	3	In terms of exploring teachers' meanings, a response in this category could imply that the respondent's mathematical knowledge is not consistent and is still yet to strike coherence between the intended and non-productive knowledge. The teacher is incapable of utilizing and communicating mathematical knowledge productively
	4	The teacher in this category tends to generate cognitive conflicts of mathematical knowledge and concepts and may turn to lack the quality of mathematical sense-making. The correctness of mathematical knowledge generated by a teacher in this category is superficial and confusing. The meanings formed by this teacher are not mathematically coherent and do not promote mathematical logicality. Meanings generated by a teacher in this category may not be fully accounted for mathematically.

5.3.1.2 Conclusion and discussion of the trigonometry task

5.3.1.2.1 Definition of an angle and the meaning of angle measure

Teachers presented different meanings of angle measure and it would seem as though teachers' meanings relating to item 1 were confined to one aspect, thus allowing them to bypass the notion of *arc*. Our subsequent qualitative analysis further revealed that the meanings held by teachers show limited conceptual knowledge about the measure of 1 degree.

5.3.1.2.2 Definition and the meaning of the Sine of an angle

In this regard, it seems that the definition of the *sine* of an angle to which each teacher ascribed contained their own distinct meanings. It became progressively more evident from what teachers said that the mathematical meanings they presented were unique to each individual. It also became evident that teachers struggled to provide the meaning of sin 90° and cos 100° by referring to the meaning of sin 30°. Teachers held on to the right-angled triangle ratio meaning of the sine of an angle, even when it made no sense when the angle in question is greater than 90°.

5.3.1.2.3 Understanding and using the unit circle

Teachers provided declarative knowledge of unit circles (i.e. sought to describe a unit circle as a circle that has a radius of one), and that knowledge was found to be particularly predominantly in many secondary school mathematics textbooks and curricular statements. Teachers failed to interpret '1' as the result of a process of unitizing a circle. Furthermore, there were indications that teachers ascribed to mathematical meanings, but they lacked the mathematical language requisite to express what they understood regarding how the unit circle in trigonometry was to be used.

5.3.1.2.4 Application: 2-dimensional problem-solving

The research findings revealed that most teachers provided responses that were not related to the anticipated ones (see, Tables 5.1 & 5.2; see, also, Section 4.4.1.1.8.2). In addition, teachers used different problem-solving approaches to respond to their task items. In item 10.1, most teachers furnished problem-solving decisions that would not necessarily lead to successful solutions to the problem. On the other hand, in item 10.2, some of the teachers managed to solve the problem under consideration, but not by using the meanings of cos θ, as addressed in Section 2.10.4.3.

5.3.2.1 The semi-structured interviews

Table 5.3, which is derived from Table 4.21, provides a summary of the findings emanating from the analysis of teachers' interviews (see, Section 4.6). Also, Table 5.3 gives the researcher's overall impression and interpretation of teachers' responses.

Table 5. 3: Summary of teachers' responses to interview questions (IQ) and the researcher's overall impressions and interpretations

<table>
<tr><th rowspan="2">The focus of Interview Question (IQ)</th><th colspan="2">Researcher's impression and interpretation of teachers' responses</th><th rowspan="2">Related RQ</th></tr>
<tr><th>T2</th><th>T3</th></tr>
<tr><td rowspan="2">IQ1

Teacher moves & actions (Section 4.5.1.1)</td><td>T2 could not explicitly spell out anticipated teacher moves and actions</td><td>T3 shifted/ deviated from the anticipated response</td><td rowspan="2">1.6.2</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Teachers responses did not list actual teacher actions/ moves. Teacher meanings prioritized over learners' meanings
<u>Interpretation:</u> Teachers did not have proper conceptualizations of expected teacher moves during a trigonometry lesson</td></tr>
<tr><td rowspan="2">IQ2

Teachers' meanings of mathematical concepts (Section 4.5.1.2)</td><td>T2 is demonstration orientated</td><td>T3 explains mathematical terms theoretically, and meaning construction is textbook reliant</td><td rowspan="2">1.6.1</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Teachers differed in their articulations of mathematical meanings
<u>Interpretation:</u> T2 demonstrates and allow learners to construct their own mathematical meanings; T2 is fixed to textbook meanings</td></tr>
<tr><td rowspan="2">IQ3

Presentation (actions) of mathematical concept (Section 4.5.1.3)</td><td>T2 is demonstration orientated

Using a protector to demonstrate angle measure</td><td>Emphasis on recognition of learner knowledge</td><td rowspan="2">1.6.2</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Teachers differed in their classroom approaches to teach the mathematical concept
<u>Interpretation:</u> T2 makes demonstratable efforts to build mathematical knowledge; T3 build from learners' knowledge</td></tr>
<tr><td rowspan="2">IQ4
Mathematical knowledge & understanding (sequencing/ coherence) (Section 4.5.1.4)</td><td>T2 prioritized learners' knowledge of working with three trigonometric functions, and their applications in a variety of problems</td><td>A phrase used by T3 referred teacher's knowledge possession (knowledge of sides of right-angled triangle and their association)</td><td rowspan="2"></td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Both teachers shared similar knowledge and understanding of teaching trigonometric functions
<u>Interpretation:</u> Knowledge of three basic trigonometric functions is prioritized</td></tr>
<tr><td rowspan="2">IQ5

Conceptual understanding of mathematical constructs (Section 4.5.1.5)</td><td>T2 manifested inconsistencies in presenting mathematical knowledge and concepts</td><td>The phrase "those two things" seemingly referred to the numerator and the denominator ($^{numerator}/_{denominator}$)</td><td>1.6.1</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Both teachers presented similar responses
<u>Interpretation:</u> Teachers presented a similar conceptual understanding of mathematical knowledge. T2 displayed a shaky construction of mathematical meanings</td><td>1.6.1</td></tr>
<tr><td rowspan="2">IQ6

Knowledge & description of a mathematical concept (Section 4.5.1.6)</td><td>T2 strongly explained the condition of functionality in terms of an equation</td><td>T3 relied heavily on the Cartesian plane to explain y=sinx</td><td rowspan="2">1.6.1</td></tr>
<tr><td colspan="2"><u>Overall impression:</u> Both teachers argued strongly that sin x is a function
<u>Interpretation:</u> Both teachers were not firmly grounded to their perspectives in the pursuit of their arguments</td></tr>
</table>

IQ7 Teaching strategies for trigonometry (Section 4.5.1.7)	T2 observed not to understand the question and a reference to a written task is made T2 relied on a right-angled triangle approach	T3 emphasized activities to use in attempting to learners to understand mathematical concepts	1.6.2
	<u>Overall impression:</u> Both teachers were not explicit in their articulation of teaching strategies **<u>Interpretation:</u>** There was no clarity in teachers' articulations of teaching strategies needed to teach trigonometric functions		
IQ8 Knowledge of learners' difficulties (Section 4.5.1.8)	T2 had a clear understanding of learners' challenges and difficulties in the specified topic of trigonometry	T3 emphasized the importance of orienting a right-angled triangle	1.6.1
	<u>Overall impression:</u> Both teachers presented learning almost similar challenges **<u>Interpretation:</u>** Different types of problems should be given to learners to expose them to various orientations of the problem		
IQ9 Teaching strategies to address learning challenges (Section 4.5.1.9)	T2 relied on demonstration instruction to address challenges & difficulties in learning mathematical concepts	T3 recommended individualized teaching approaches to assist learners to comprehend mathematics concepts	1.6.2
	<u>Overall impression:</u> Teachers suggested different teaching approaches in addressing and mitigating learning challenges & difficulties **<u>Interpretation:</u>** It seems teachers replied on this interview item relying on their experiences and teaching strategies they have tested in their mathematics lessons		1.6.2

5.3.2.2 Conclusion and discussion for the semi-structured interviews

Generally, both teachers reiterated the meanings they exhibited when responding to written items. Teacher 2 and Teacher 3 seemed to view the sine and the cosine of an angle as algorithms grounded on ratios. For example, their understanding of and way of thinking about sine was confined to a right-angled triangle in which the 'sine of an angle equaled its opposite over hypotenuse'. Teacher 3 emphasized that, in order to apply the ratio meaning of the sine of an angle, learners had to be provided with a labelled right-angled triangle. The analysis also revealed that Teacher 2 and Teacher 3 do not have a function meaning for the sine of an angle. For example, according to Teacher 2 and Teacher 3, sin is not a function but $y = sin$ is a function. It was clear that both teachers did not think of sine or cosine as mathematical operations acting on angles; for example, operations like taking a square root of a number or logarithms.

5.5 LIMITATIONS

The sample of the study consisted of only twelve mathematics teachers from the Tshwane South District of Gauteng Province in the Republic of South Africa. Hence, the researcher cannot claim that the sample is representative of any specific population. Therefore, the researcher advises that the reader not generalize the results of the study inappropriately.

5.6 DISCUSSION AND CONCLUSION

Most responses to the trigonometry items considered by this study suggest that teachers adhered to meanings for angle and angle measure that are not of particular educational benefit to the formal instruction of trigonometric concepts and functions. This reality strongly suggests a need for greater attention to these ideas to be paid in in-service professional development programs for high school teachers. Few of the teachers surveyed were able to describe angle as a turn (rotation) and angle measure as quantifying the angle's openness. For many teachers in the study, angle and angle measure were likewise conceived of as the same thing, with teachers providing the same description for the two concepts. The researcher found that most teachers described angle and angle measure as the distance between two lines that are drawn from the same vertex. The evidence in the study's data shows that the static meaning of an angle is unproductive, since it does not support the understanding of angle measure.

The researcher agrees with several authors that the way teachers come to understand angle measure and how they teach it is ultimately not coherent and lacks conceptual meaning (see, Akkoc, 2008; Moore, La Forest & Kim, 2016; Tallman & Frank, 2018; Thompson, 2008; Thompson, Carlson & Silverman, 2007). Other researchers have documented that the source for making trigonometry to be a problematic subject emanates from the underdeveloped meanings of angle and angle measure (Moore, 2012, 2014; Weber, 2005). The data in this study reveals that many teachers teach meanings for the sine and cosine of angles that do not coherent with the meaning of angle measure, and several teachers presented the idea of angle measure that is isolated from the idea of trigonometry function. Moreover, teachers that provided a ratio understanding of a trigonometric function operating on an acute angle found it challenging to articulate the meaning of a trigonometric function operating on an obtuse angle. The exploratory research conducted as part of this study suggests that secondary teachers entertain meanings regarding angles and angle measures that are unproductive, and that their meanings for the sine and cosine are also weakly associated with a deficient understanding of right-angled triangle ratios.

It is essential to bring the reader's attention to the fact that this study is not principally concerned with the substantive misconceptions of teachers on their subject matter, as revealed in our collected data, but rather the ideas that these teachers entertain on the subject of mathematics, and the ideas they wish to impress on their students. Likewise, the reader ought not to perceive the findings of this study as constituting a criticism of teachers' capabilities, but rather as identifying the elements that together contribute to the ineffective instruction of angle and angle measures for improving learners' learning of trigonometry.

5.7 RECOMMENDATIONS FOR FUTURE PRACTICE

The findings of this study are crucial to improving the teaching and learning of trigonometry at the Grade 10 level. In this regard, it is imperative that teachers be subjected to a continuous, comprehensive professional development programme that:

- aims to help teachers develop and teach productive mathematical meanings; and,
- helps teachers select curriculum materials that cohere with their effort to re-conceptualise their mathematics regarding supporting learners' construction of coherent mathematical meanings.

5.8 SUGGESTIONS FOR FUTURE RESEARCH

The current study has implicit observable limitations, such as bringing to light findings from only a small number of participants in one district, and which are based on an unfamiliar, untested framework of understanding in South Africa. The researcher suggests that future research should address such limitations that are implicated in this study. The research reported in this study is considerably novel in South Africa and, in this regard, addresses the absence in the literature on the subject of the mathematical meanings teachers wished to convey when teaching foundational concepts in trigonometry. There is a considerable need for further research in this area to contribute to a growing body of literature that carries the potential to promote and enhance the productive teaching of trigonometry and other topics.

The study's participants were, as we have already mentioned, professional educators in South Africa with varying degrees of educational attainments, even though our text refers at various instances to learners (i.e. their perspective and experiences). It might be advisable for future researchers to extend the purview of their research to include learning contexts in which teachers taught foundational trigonometric ideas to their learners. I anticipate it will be fruitful to have further research conducted on the effects of teaching coherent mathematical ideas in trigonometry which is informed by the present framework the author has selected for this study.

REFERENCES

Akkoc, H. (2008). Pre-service mathematics teachers' concept images of radian. *International Journal of Mathematical Education in Science and Technology*, *39*(7), 857-878.

Aljohani, M. (2017). Principles of constructivism in foreign language teaching. *Journal of Literature and Art Studies, 7*(1), 97-107. doi: 10.17265/2159-5836/2017.01.013.

Amineh, R. J., & As, H. D. (2015). Review of constructivism and social constructivism. *Journal of Social Sciences, Literature and Languages, 1*(1), 9-16.

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education*, 7, 145-172.

Beswick, K. (2019). The role of knowledge and beliefs in helping learners to progress their mathematical understanding. *Journal of Mathematics Teacher Education, 22*(2), 125-128. https://doi.org/10.1007/s10857-019-09432-5.

Biggs J. (2003). *Teaching for Quality Learning at University*. (2nd Edition). Maidenhead: Society for Research into Higher Education, Open University Press.

Biggs, J. B. (1999). *Teaching for quality learning at university*. Buckingham: Open University Press.

Biggs, J. B., & Tang, C. (2011). *Teaching for quality learning at university: What the student does*. McGraw-hill education (UK).

Boughey, C. (2018). Using the curriculum to enhance teaching and learning. South African Journal of Science, 114(9/10), 1-3. https://doi. org/10.17159/sajs.2018/a0288.

Bressoud, D. M. (2010). Historical Reflections on Teaching Trigonometry. *MatheMaticS teacher, 104*(2), 106-112.

Brodie, K. (2011). Working with learners' mathematical thinking: Towards a language of description for changing pedagogy. *Teaching and Teacher Education*, *27*(1), 174-186.

Brodie, K. (2013). The power of professional learning communities. *Education as change*, *17*(1), 5-18.

Brower, R. L., Woods, C. S., Jones, T. B., Park, T. J., Hu, S., Tandberg, D. A., Nix, A. N., Rahming, S. G., & Martindale, S. K. (2018). Scaffolding mathematics remediation for academically at-risk students following developmental education reform in Florida. *Community College Journal of Research and Practice, 42*(2), 112-128, DOI: 10.1080/10668926.2017.1279089.

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior*, *48*, 168-193.

Calter, P. A., & Calter, M.A. (2011). Technical Mathematics with Calculus (6th Edition). Wiley

Canrinus, E. T., Klette, K., & Hammerness, K. (2017). Diversity in coherence: Strengths and opportunities of three programs. *Journal of Teacher Education*, *10*(3).

Centre for Development and Enterprise (CDE). (2014). What does research tell us about teachers, teaching and learner performance in mathematics? Johannesburg: Centre for Development and Enterprise.

Chugh, R. (2015). Do Australian universities encourage tacit knowledge transfer? In Proceedings of the 7th International Joint Conference on Knowledge Discovery, Knowledge Engineering and Knowledge Management, ISBN 978-989-758-158-8.

COBUILD., & University of Birmingham. (2001). *Collins COBUILD English dictionary for advanced learners*. Glasgow: HarperCollins. (available at https://www.collinsdictionary.com/dictionary/english/coherence, accessed on 25 March 2018

Cohen, L., Manion, L., & Morrison, K. (2013). *Research methods in education* (7th Ed.). Routledge.

Cohen, L., Manion, L., & Morrison, K. (2011). *Research Methods in Education* (7th Ed.). London: Routledge.

Cohen, L., Manion, L., & Morrison, K. (2002). *Research Methods in Education* (5th Ed.). London: Routledge.

Coleman, J. S., Campbell, E. Q., Hobson, C. J., McPartland, J., Mood, A. M., Weinfeld, F. D., & York, R. (1966). Equality of educational opportunity. *Washington, dc*, 1066-5684.

Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher & N. Noddings (Eds.), *Journal for Research in Mathematics Education Monographs*, Journal for Research in Mathematics Education Monographs (Vol. 4, pp. 107-122+195-210). Washington, DC: National Council of Teachers of Mathematics. Retrieved from http://www.jstor.org/stable/749916.

Confrey, J., Gianopulos, G., McGowan, W., Shah, M., & Belcher, M. (2017). Scaffolding learner-centered curricular coherence using learning maps and diagnostic assessments designed around mathematics learning trajectories. *ZDM*, *49*(5), 717-734.

Creswell, J.W., & Plano Clark, V.L. (2011). Designing and Conducting Mixed Methods Research (2nd Edition). SAGE

Davis, B. (2011). Mathematics teachers' subtle, complex disciplinary knowledge. *Science*, *332*(6037), 1506-1507.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational studies in mathematics*, *61*(3), 293-319.

Depaepe, F., Torbeyns, J., Vermeersch, N., Janssens, D., Janssen, R., Kelchtermans, G., Verschaffel, L., & Van Dooren, W. (2015). Teachers' content and pedagogical content knowledge on rational numbers: A comparison of prospective elementary and lower secondary school teachers. *Teaching and Teacher Education*, 47, 82-92.

Department of Basic Education. (2002). Revised national curriculum statement Grades R-12. Mathematics. Pretoria, DOE.

Department of Basic Education. (2011). Curriculum and assessment policy statement Grades R–12. Mathematics. Pretoria, DBE.

Department of Basic Education. (2013). National Senior Certificate Examination. National Diagnostic Report on learner performance 2012. Pretoria, DBE.

Department of Basic Education. (2014). National Senior Certificate Examination. National Diagnostic Report. Pretoria, DBE.

Department of Basic Education. (2015). Annual National Assessment 2014. Diagnostic Report. Pretoria, DBE.

Department of Basic Education. (2015). National Senior Certificate Examination 2014. Diagnostic Report. Pretoria, DBE.

Dewey, J. (1910). *How we think*. Boston, MA: D. C. Heath.

Dhlamini, J. J. (2012). *Investigating the effect of context-based problem solving instruction on the performance of learners.* Unpublished doctoral thesis. University of South Africa, Pretoria, South Africa. Available from http://hdl.handle.net/10500/9739.

Eggen, P., & Kauchaki, D. (2014). Introduction to teaching: Becoming a professional. 5th Edition, Pearson, Florida.

Ernest, P. (2010) Reflections on Theories of Learning. In: Sriraman B., English L. (eds) Theories of Mathematics Education. Advances in Mathematics Education. Springer, Berlin, Heidelberg

Ernest P *et al*. (2016). The philosophy of mathematics education. In: The Philosophy of Mathematics Education. ICME-13 Topical Surveys. Springer, Cham

Ernest, P. (1997). Chapter 3: The Epistemological Basis of Qualitative Research in Mathematics Education: A Postmodern Perspective. *Journal for Research in Mathematics Education. Monograph, 9*, 22-177. doi:10.2307/749945

Ernest, P. (2012). What is our First Philosophy in Mathematics Education? *For the Learning of Mathematics, 32*(3), 8-14.

Ewen, D., & Nelson, C.R. (2011). Elementary Technical Mathematics. Tenth Edition. Brooks/Cole, Cengage Learning.

Fani, T., & Ghaemi, F. (2019). Implications of Vygotsky's Zone of Proximal Development (ZPD) in teacher education: ZPTD and self-scaffolding. *Social and Behavioral Sciences, 1*(29), 1549-1554.

Fi, C. D. (2003). Preservice secondary school mathematics teachers' knowledge of trigonometry: Subject matter content knowledge, pedagogical content knowledge and envisioned pedagogy.

Fraenkel, J. R., & Wallen N. E. (2009). *How to design and evaluate research in education* (7th Ed.). McGraw-Hill: New York.

Gay, L. R., Mills, E., & Airasian, P. (2011). *Educational research: Competencies for analysis and application.* (10th Ed.). Pearson Education: Upper Saddle River, New Jersey.

Goldin, G. (1990). Chapter 3: Epistemology, Constructivism, and Discovery Learning in Mathematics. *Journal for Research in Mathematics Education. Monograph, 4*, 31-210. doi:10.2307/749911.

Gür, H. (2009). Trigonometry Learning. *New Horizons in Education*, *57*(1), 67-80.

Gweshe, L. G. (2015). *The effect of using a Computer Assisted Instruction on teaching Circle Geometry in Grade 11*. Unpublished masters' dissertation. University of South Africa, Pretoria, South Africa.

Hauk, S., Toney, A., Jackson, B., Nair, R., & Tsay, J. (2014). Developing a model of pedagogical content knowledge for secondary and post-secondary mathematics instruction. *Dialogic Pedagogy, 2*, 16-40.

Hendry, G. D. (1996). Constructivism and educational practice. *Australian Journal of Education*, *40*(1), 19-45.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding, In Grouws, D. A (ed). *Handbook of Research on Mathematics Teaching and Learning: A Project of the National Council of Teachers of Mathematics.* New York: Macmillan Publishing Company.

Hill, H. C., Blunk, M, L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American educational research journal*, *42*(2), 371-406.

Howie, S., & Plomp, T. (2002). Mathematical literacy of school leaving pupils in South Africa. *International Journal of Educational Development*, *22*(6), 603-615.

Izsák, A. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education, 43*, 391-427.

Johnson, B., & Christensen, L. (2012). *Educational Research, Quantitative, Qualitative, and Mixed*

Approaches. (4th Ed.). SAGE Publications, Inc.: NY.

Kaiser, G. (2008). Meaning in mathematics education reflections from various perspectives. In *Retrieved from Symposium on the Occasion of the 100th Anniversary of ICMI: http://www.unige.ch/math/EnsMath/Rome2008/WG5/Papers/KAIS. pdf*.

Kamber, D., & Takaci, D. (2018). On Problematic Aspects in Learning Trigonometry. *International Journal of Mathematical Education in Science and Technology*, *49*(2), 161-175.

Karimi, S. (2017). How to Use a Math Compass. Sciencing. Retrieved from https://sciencing.com/use-math-compass-4840414.html (Accessed 23/02/2019)

Kendal, M., & Stacey, K. (1998). Teaching trigonometry. *Australian Mathematics Teacher, The*, *54*(1), 34.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds). (2001). Adding It Up: Helping Children Learn Mathematics. Washington DC: National Academy Press.

Kim, D. J., Bae, S., Choi, S., Kim, H., & Lim, W. (2019). Creative character education in mathematics for prospective teachers. *Sustainability, 11*(1730), 1-16.

Kleickmann, T., Richter, D., Kunter, M., Elsner, J., Besser, M., Krauss, S., Cheo, M., & Baumert, J. (2015). Content knowledge and pedagogical content knowledge in Taiwanese and German mathematics teachers. *Teaching and Teacher Education*, 46, 115-126.

Krauss, S. E. (2005). Research paradigms and meaning making: A primer. *The qualitative report*, *10*(4), 758-770.

Kumar, R. (2011). Research Methodology: A Step-by-Step Guide for Beginners. 3rd Edition. Sage, New Delhi.

Larson, M. (2016). Curricular coherence in the age of open educational-resources. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Matt-Larson/Curricular-Coherence-in-the-Age-of-Open-Educational-Resources/. Accessed 31 March 2018.

Lincoln, Y.S., & Guba, E.G. (1985). Naturalistic inquiry. Beverly Hills, CA: Sage.

Loan, F. A. (2012). Impact of the Internet surfing on reading practices and choices**,** *Webology*, *9*(1).

Maphalala, M. C., & Dhlamini, J. J. (2017). Standardized testing: a case of annual national assessments in South African primary schools. *Gender and Behaviour*, *15*(4), 10334-10346.

Maree, J. G., & Erasmus, C. P. (2007). Mathematics skills of Tswana-speaking learners in the North West Province of South Africa. *Open University of Pretoria, South Africa.* pp.1-19.

Mason, J. & Spence, M. (1999). Beyond Mere Knowledge of Mathematics: The Importance of Knowing-To Act in the Moment. In Tirosh D. (Ed.) *Forms of Mathematical Knowledge: learning and teaching with understanding, Educational Studies in Mathematics*, *38*, 135-161.

doi: 10.1023/A:1003622804002.

McMahon, T., & Thakore, H. (2006) Achieving Constructive Alignment: Putting Outcomes First. The Quality of Higher Education v3: pp. 10-19.

McMillan, J. H., & Schumacher, S. (2014). *Research in education: Evidence-based inquiry*. Pearson Higher Ed.

McMillian, J. H., & Schumaker, S. (2010). *Research in education, evidence based inquiry* (8th Ed.). Boston: Pearson Education.

Mean. (n.d.). Retrieved April 6, 2018, from https://www.merriam-webster.com/dictionary/meanReferences

Meij, L. W., & Merx, S. (2018). Improving curriculum alignment and achieving learning goals by making the curriculum visible. *International Journal for Academic Development*, *23*(3), 219-231. DOI: 10.1080/1360144X.2018.1462187.

Merriam, S. B. (2009). *Qualitative Research A Guide to Design and Implementation* (Second Ed.). San Francisco, CA 94103-1741: Jossey-Bass.

Mhlolo, M. K. (2011). From coherence in theory to coherence in practice: A stock-take of the written, tested and taught National Curriculum Statement for Mathematics (NCSM) at further education and training (FET) level in South Africa. *Unpublished doctoral dissertation. University of the Witwatersrand, Johannesburg, South Africa. Available from http://wiredspace. wits. ac. za/bitstream/handle/10539/11274/Mhlolo% 20M% 20K% 20PhD% 20Final% 20Thesis. pdf*.

Mhlolo, M. K., & Venkat, H. (2009). Curriculum Coherence: An analysis of the National Curriculum Statement for Mathematics (NCSM) and the exemplar papers at Further Education and Training (FET) level in South Africa. *African Journal of Research in Mathematics, Science and Technology Education*, *13*(1), 33-49.

Mji, A., & Makgato, M. (2006). Factors associated with high school learners' poor performance: A spotlight on mathematics and physical science. *South African Journal of Education, 26*(2), 253-266.

Mogashoa, T. (2014). Constructivist theory in qualitative educational research. *American International Journal of Contemporary Research*, *4*(7), 51-59.

Mohr, P. J., & Phillips, W. D. (2014). Dimensionless units in the SI. *Metrologia*, *52*(1), 40.

Moore, K. C. (2010). The role of quantitative and covariational reasoning in developing precalculus students’ images of angle measure and central concepts of trigonometry. Paper presented at the Thirteenth Annual Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) Conference, Raleigh, NC: North Carolina State University.

Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context, 2*, 75-92.

Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*, *83*(2), 225-245.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, *45*(1), 102-138.

Moore, K. C., & LaForest, K. R. (2014). The circle approach to trigonometry. *Mathematics Teacher*, *107*(8), 616-623.

Moore, K. C., LaForest, K. R., & Kim, H. J. (2016). Putting the unit in pre-service secondary teachers' unit circle. *Educational Studies in Mathematics, 92*(2), 221-241.

Moore, K. C., LaForest, K., & Kim, H. J. (2012). The unit circle and unit conversions. *Proceedings of the Fifteenth Annual Conference on Research in Undergraduate Mathematics Education* (pp. 16-31). Portland, OR: Portland State University. Available at: http://sigmaa.maa.org/rume/crume2012/RUME_Home/Home.html.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Ndlovu, Z., Amin, N., & Samuel, M. A. (2017). Examining pre-service teachers' subject matter knowledge of school mathematics concepts. *Journal of Education*, *1*(70), 46-72.

Ngugi, M. N., & Mberia, H. K. (2014). The influence of the Internet surfing on the reading culture of secondary school teachers: A case study of newspaper readership in Kigumo Sub County, Kenya. *International Journal of Academic Research in Business and Social Sciences*, *4*(11), 96.

Nickolas, J. A. (2013). Integrating social media into the classroom curriculum About Campus;18(1)

Nisson, P., & Karlsson, G. (2019). Capturing student teachers' pedagogical content knowledge (PCK) using CoRes and digital technology. *International Journal of Science Education, 4*(41), 419-447.

Noh, M. A. D., & Jin, H. F. (2017). Social constructivism in social media: Facebook for teaching and learning purposes. *Malaysian Online Journal of Education, 1*(2), 59-67.

Odumosu, M. O., Olisama, O. V., & Ogunsanya, A. (2018). Teachers' content and pedagogical knowledge on students' achievement in algebra. *International Journal of Education and*

Research, *6*(3), 83-94.

Okeke, C., van Wyk, M. (2015). *Educational Research: An African approach*. Oxford University Press Southern Africa (Pty) Ltd.

Orhun, N. (2004). Student's mistakes and misconceptions on teaching of trigonometry. *Journal of Curriculum Studies, 32*(6), 797-820.

Orton, A. (1992): *Learning mathematics Issues, Theory and practice* (2nd Ed.). London: Cassell.

Piaget, J. (1968). *Six psychological studies*, D. Elkind, ed. New York: Vintage Books.

Piaget, J. (1977). *Psychology and epistemology: Towards a theory of knowledge*. New York, NY: Penguin.

Piaget, J., & Duckworth, E. (1970). Genetic epistemology. *American Behavioral Scientist*, *13*(3), 459-480.

Polzer, H., DeLaurentis, D., & Fry, D.N. (2007). Multiplicity of Perspectives, Context Scope, and Context Shifting Events. *2007 IEEE International Conference on System of Systems Engineering*, 1-6.

Pournara, C. (2001). *An investigation into learners' thinking in trigonometry at Grade 10 level*. Unpublished M.Ed. thesis, University of Witwatersrand.

Radford, L. (2008). Theories in mathematics education: A brief inquiry into their conceptual differences. *Report of Survey Team*, *7*.

Radford, L., Schubring, G., & Seeger, F. (2011). Signifying and meaning-making in mathematical thinking, teaching, and learning. *Educational Studies in Mathematics*, *77*(2), 149-156.

Reeves, C., & McAuliffe, S. (2012). Is curricular incoherence slowing down the pace of school mathematics in South Africa? A methodology for assessing coherence in the implemented curriculum and some implications for teacher education.

Reyes, J. D., Insorio, A. O., Ingreso, M. L. V., Hilario, F. F., & Gutierrez, C. R. (2019). Conception and Application of Contextualization in Mathematics Education. *International Journal of Educational Studies in Mathematics, 6*(1), 1-18.

Richards, J. C., & Schmidt, R. W. (2013). *Longman dictionary of language teaching and applied linguistics*. Routledge.

Rittle-Johnson, B., & Schneider, M. (2014). Developing conceptual and procedural knowledge of mathematics. *Oxford handbook of numerical cognition*, 1102-1118.

Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review*, *27*(4), 587-597.

Sarker, M. F. (2019). Zone of proximal development. *International Journal of Advancements in*

Research & Technology, *8*(1), 27-47. ISSN 2278-7763.

Sasman, M. (2011). Insights from NSC Mathematics Examinations. In *Venkat, H & Essien, A (eds), Proceedings of the 17th Annual Association for Mathematics Education of South Africa (AMESA) National Congress*, Gauteng. Johannesburg, volume 1, 2-13.

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525-559.

Schmidt, W., Houang, R., & Cogan, L. (2002). A coherent curriculum. *American Educator, Summer 2002*, 1-17.

Shenton, A.K. (2004). Strategies for Ensuring Trustworthiness in Qualitative Research Projects. Education for Information, 22, 63-75. https://doi.org/10.3233/EFI-2004-22201

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard educational review*, *57*(1), 1-23.

Shulman, L. S. (1995). The pedagogical colloquium: Three models. *AAHE Bulletin*, *41*(10), 6-9.

Silverman, J., & Clay, E. L. (2009). Supporting teachers' development of coherence through a focus on mathematical structure. In *31st Annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Atlanta, GA*.

Simon, M. (2011). Assumptions, limitations and delimitations. *Dissertation and scholarly research: Recipes for success. Seattle, WA: Dissertation Success, LLC. Available at www. dissertationrecipes. com*.

Siyepu, S. (2013). The zone of proximal development in the learning of mathematics. *South African Journal of Education*, *33*(2), 1-12. http://www.sajournalofeducation.co.za.

Skemp, R. (1978). Relational understanding and instrumental understanding', ArithmeticTeacher26, 9-15. *Skemp926Arithmetic Teacher1978*.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. Mathematics Teaching, 77, 20-26.

Skovsmose, O. (2005). Meaning in mathematics education. In *Meaning in mathematics education* (pp. 83-100). Springer, Boston, MA.

Stols, G. (2013). An investigation into the opportunity to learn that is available to Grade 12 mathematics learners. *South African Journal of Education*, *33*(1), 1-18. Retrieved March 31, 2018, from http://www.scielo.org.za/scielo.php?

Stylianides, A. J., & Stylianides, G. J. (2007). Learning mathematics with understanding: A critical consideration of the learning principle in the principles and standards for school mathematics. *The Montana Mathematics Enthusiast*, 4(1), 103-114.

Suriakumaran, N., Duchhardt, C., & Vollstedt, M. (2016). *Personal Meaning and Motivation when Learning Mathematics: A Theoretical Approach*. Verlag nicht ermittelbar.

Tallman, M. A., & Frank, K. M. (2018). Angle measure, quantitative reasoning, and instructional coherence: an examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. *Journal of Mathematics Teacher Education*, 1-27.

Taylor N & Vinjevold P (eds) 1999. Teaching and learning in South African schools. In: Taylor N & Vinjevold P (eds). *Getting learning right: report of the President's Education Initiative Research Project*. Braamfontein: Joint Education Trust.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In, *Proceedings of the annual meeting of the international group for the psychology of mathematics education* (Vol. 1, pp. 45-64). PME Morelia, Mexico.

Thompson, P. W. (2013). In the absence of meaning. In, K. Leatham (Ed.), *Vital directions for research in mathematics education.*

Thompson, P. W. (2014). Constructivism in mathematics education. In *Encyclopedia of Mathematics Education* (pp. 96-102). Springer Netherlands.

Thompson, P. W. (2015). Researching mathematical meanings for teaching. In English, L., & Kirshner, D. (Eds.), *Third Handbook of International Research in Mathematics Education* (pp. 435-461). London: Taylor and Francis.

Thompson, P. W., & Draney, K. A (2014). Methodology for investigating teachers'mathematical meanings for teaching. *Short oral communications*, 246. In Oesterle, S., Nicol, C., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting 6 - 169 of PME 38 and PME-NA 36, Vol. 6, p. 169. Vancouver, Canada: PME.

Thompson, P. W, Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing*. WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education*, *10*(4-6), 415-432.

Tobin, K. G. (1993). *The practice of constructivism in science education*. Psychology Press.

Turnuklu, E. B., & Yesildere, S. (2007). The pedagogical content knowledge in mathematics: Preservice primary mathematics teachers' perspectives in Turkey. *Issues in the*

Undergraduate Mathematics Preparation of School Teachers, 1, 1-13.

Umugiraneza, O., Bansilal, S., & North, D. (2018). Exploring teachers' use of technology in teaching and learning mathematics in KwaZulu-Natal schools. *Pythagoras, 39*(1), a342. https://doi.org/10.4102/pythagoras.v39il.342.

Van de Walle, J.A., Karp, K.S., & Bay-Williams, J.M. (Eds.). (2016). *Elementary and middle school mathematics: Teaching developmentally*. Boston, MA: Pearson Education.

Van Laren, L. (2012). Mathematics teachers at the intersection of policy interventions: What practicing teachers offer as challenges in teaching trigonometry. *Education as Change, 16*(2). 203-215.

Vygotsky, L. S. (1978). *Mind in society*. Boston: Harvard University Press.

Webb, N. L. (1997). Determining alignment of expectations and assessments in mathematics and science education. *NISE Brief 1(2)*. Available at http://facstaff.wcer.wisc.edu/normw/1997alignmentbrief.htm (accessed March 29, 2018).

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, *17*(3), 91-112.

Weber, K. (2008). Teaching trigonometric functions: Lessons learned from research. *Mathematics Teacher*, *102*(2), 144-150.

Widjajanti, K., Nusantara, T., As'ari, A. R., Irawati, S., Haris, Z. A., Akbar, D. N., & Lusbiantoro, R. (2019). Delaying scaffolding using GeoGebra: Improving the ability of vocational students to draw conclusions. *TEM Journal, 8*(1), 305-310. ISSN 2217-8309, DOI: 10.18421/TEM81-42.

Willis, J. W., & Jost, M. (2007). *Foundations of qualitative research: Interpretive and critical approaches*. Sage.

Wilson, M., & Berthenthal, M. (2005). Systems for state science assessment. Washington, DC: National Academy Press.

Wilson, P. S., & Adams, V. M. (1992). A Dynamic Way to Teach Angle and Angle Measure. *Arithmetic Teacher*, *39*(5), 6-13.

Yoon, H., & Thompson, P. (2018). What are conveyed meanings from a teacher to students? Conference: Proceedings of the 21th Annual Conference for Research in Undergraduate Mathematics Education., At San Diego, CA.

Yoon, H., Byerley, C., & Thompson, P. W. (2015). Teachers' meanings for average rate of change in USA and Korea. In T. Fukawa-Connelly (Ed.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*. Pittsburgh, PA: RUME.

APPENDICES SECTION

APPENDIX 1: The Trigonometry Task

Mathematical Meanings for Teaching Secondary School Trigonometry Diagnostic Tool (MMTsstDT)

School (Name) code : ________________

Teacher (Name) code: ________________

1. **1.1** What is an angle? ____

____

____

____

1.2 What does it mean to measure an angle? __________________________________

____

____

____

1.3 What explanation would you give to a learner who asks: What does it mean for an angle to have a measure of 1 degree?____________________________________

____

____

____

2. **2.1** Define the *sine* of an angle. ____

____

____

____

2.2 What is the meaning of sin30°? ____

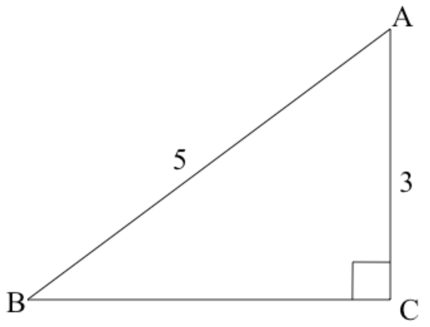
____

____

____

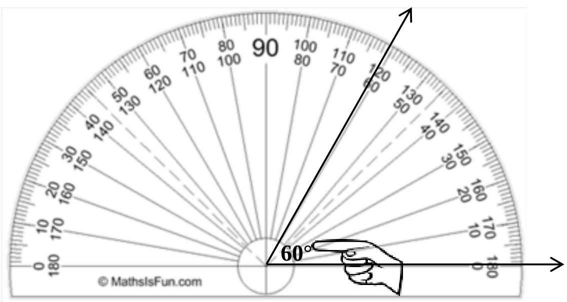
2.3 Referring to the meaning of sin30° that you provided in 2.2, what is the meaning of *sin(90°)*?; and that of *cos(100°)*?_____________________________________

3. In triangle ABC what does it mean to determine the output of the *sine* and *cosine* of the measure of angle ABC *without* measuring the angle (diagram not drawn to scale).

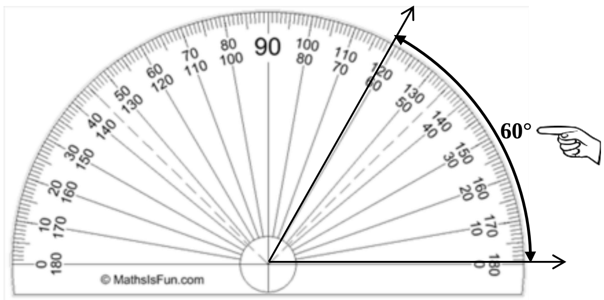


4. Learners (student A and student B in the diagram) were given a task to measure 60 degrees with a protractor. Two learners responded as follows (diagrams not drawn to scale):

Student A

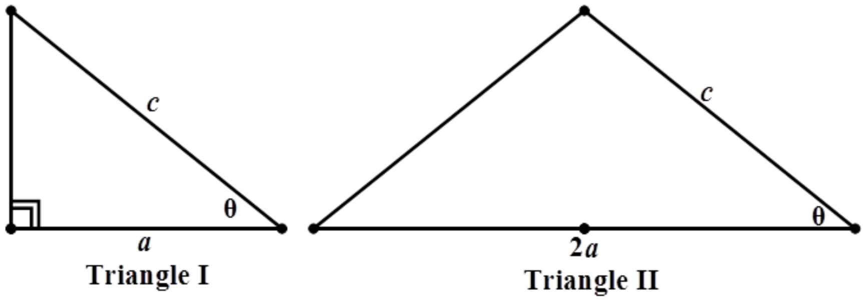


Student B



4.1 Which of the two learners' responses would you accept/consider as correct? Explain why?

5. You are given the scenario in Triangle I and Triangle II (diagrams not drawn to scale):



5.1 Is the meaning of $\sin\theta$ in Triangle I the same as the meaning of $\sin\theta$ in Triangle II? Explain: ______________________________

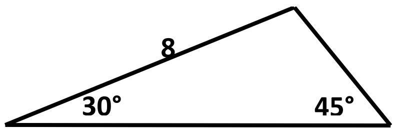
6. Draw two special triangles commonly used in Grade 10 trigonometry to calculate exact solutions. Label all angles with their measures. ______________________________

6.1 How would you introduce the notion of "exact solution" to your Grade 10 learners? ______________________________

7. **7.1** Explain the meaning of a unit circle? ______

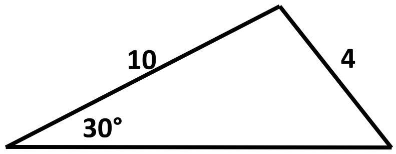
7.2 How could you use the unit circle in trigonometry? ______

8. **8.1** What kind of knowledge is needed to find all missing sides and angles in the given triangle (diagrams not to scale)?

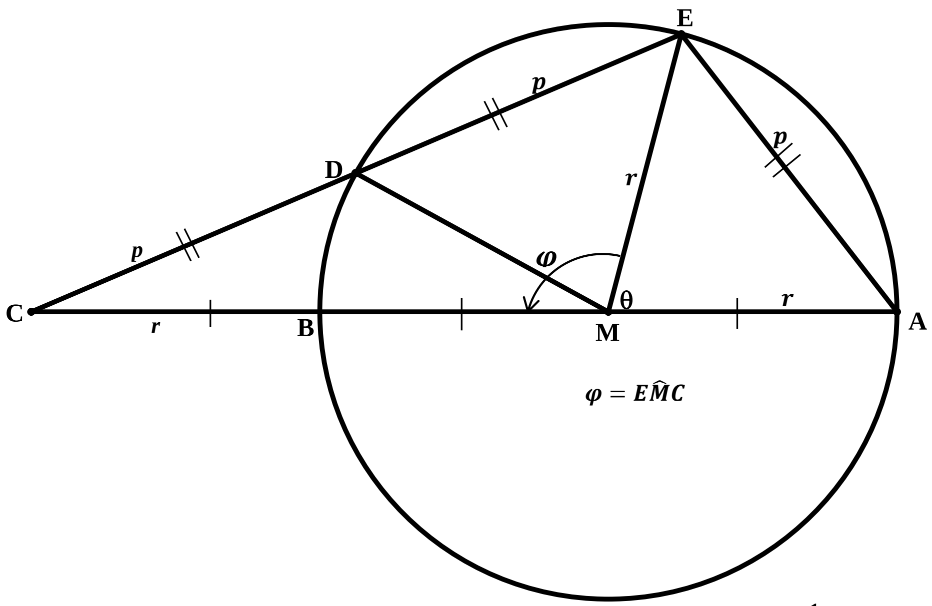


8.2 How will you decide to assist your Grade 10 learners to solve the task in 8.1? ___

9. Demonstrate the way(s) in which you will want your learners to find all missing sides and angles in the given triangle (diagrams not drawn to scale): Show all your work.



10. The following question was asked in a Grade 10 mathematics examination (see question at the foot of the diagram).



M is the centre of the circle. If MB = BC = r, show that $\cos\theta = \frac{1}{4}$

In relation to the diagram in Question 10 determine the following:

10.1 What important decisions should learners make in order to solve this task? _____

10.2 Develop a solution for this task.

10.3 Allocate marks for all your solution steps in 10.2.

THANK YOU!!

APPENDIX 2: A Semi-Structured Interview Schedule

Main objective: To determine teachers' instructional *actions* to facilitate productive instruction in trigonometry

Grade level : 10
Section : Trigonometry (mathematics)
School code : ____________________________
Teacher code : ____________________________

1. In your understanding how should a teacher demonstrate his/ her understanding of trigonometric concepts in the classroom?

2. What is the meaning of an angle measure?

3. How should a concept of an angle measure be introduced to Grade 10 learners? Alternative: How should the concept of angle measure be approached with Grade 10 learners?

4. What would you like your Grade 10 learners to understand in relation to *sin x* and or *cos x*?

5. Describe *sin x* in your own words.

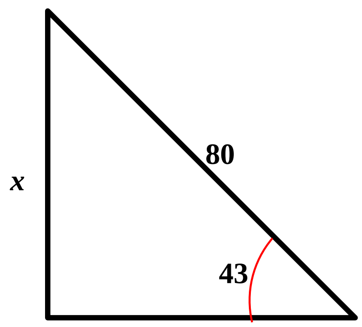
6. Is *sin x* a function or not? Please explain your answer?

7. What kind of teaching action(s) should a Grade 10 teacher adopt in order to teach the concepts of *sin x* and *cos x*? Alternative: How would you approach teaching trigonometry so that your learners understand the concepts of *sin x* and *cos x*?

8. What are the possible learner difficulties in relation to finding trigonometric solutions in Grade 10? Alternative: In your experience, what difficulties do Grade 10 learners have when they are solving trigonometry problems?

9. What kind of action(s) would you employ in order to alleviate learners' difficulties and difficulties in Question 7? Alternative: What strategies do you use to help your learners with the difficulties mentioned in Q 8?

10. Grade 10 learners at a certain school were asked to solve the given problem:



What is *x*?

The solution given by one of the learners was:

$180 - 43 = 137$

$137 - 80 = 57$

$\underline{x = 57}$

What kind of action(s) would you adopt in order to assist this learner? Alternative: How would you assist this learner to make better sense of the problem? or What misunderstandings does this learner have and how would you help this learner?

THANK YOU!!

APPENDIX 3: Teacher 2 And Teacher 3 Interview Transcripts

Teacher 2 Interview

Researcher: **The last time we met we did the semi structured interview and what I'm going to do is I'm just going read out your responses and then make a follow up. So there were some**

Teacher 2: Sort of probing?

Researcher: **Yes there were 10 questions on the semi structured interview you did not answer question 1. Which was, in your understanding how can a teacher demonstrate his or her understanding of trigonometric concepts in the classroom? If I may just ask why actually you did not…**

I think this….

Researcher: **I'm sorry Allen let me, just hang on I need (going through papers) yeah I just need to have you, you left it, you did not answer I actually want you to give a response.**

Teacher 2: The question by the way how you, a teacher should demonstrate their understanding of trigonometry.

Researcher: **Yeah how he or she should understand trig in the classroom.**

Teacher 2: Actually it was quite vague for me. It was a bit vague like how a teacher should demonstrate his understanding

Researcher: **Do you mind elaborating vague in terms of maybe you maybe asked to rephrase it…? What could be, or maybe if I may explain. Because in your understanding how should a teacher demonstrate his or her understanding of trigonometric concepts in the classroom?**

Teacher 2: What actually comes into my mind, I'll just say whatever comes into my mind.

Researcher: **Initially you said the question was vague.**

Teacher 2: Yeah it was vague, like I didn't know how to respond and describe the demonstration.

Researcher: **But do you think one can demonstrate the understanding?**

Teacher 2: Yeah, I think. I think yeah,

Researcher: **One can demonstrate the understanding?**

Teacher 2: Yeah

Researcher: **Let us not even talk about trig concepts, let's say one can ask you to demonstrate your understanding of factorization for instance. Will you be able to?**

Teacher 2: Yeah, I will be able to demonstrate but like. But the question actually like, when I think of it like, how can one demonstrate an understanding of trigonometric concepts? It actually troubles me, that question. like as a teacher I have demonstrate my understanding, let me just say from my perspective I can say I want to demonstrate how I understand trigonometry. Maybe in that way it would be like teaching from a right-angle triangle. That is the basic, because like trigonometry in my understanding actually stems from the right-angle triangle.

Researcher: **So that is how you demonstrate? Explaining trigonometry from a right-angel triangle?**

Teacher 2: Yeah, right from the foundation.

Researcher: **In your understanding trigonometry actually, it's foundation is from the right-angled triangle?**

Teacher 2: Yeah.

Researcher: **So here I will also ask question number 2 and here I asked what the meaning of angle measure.**

Teacher 2: Oh, the angle measure?

Researcher: **Yeah. And your response literally said; it was the measure of the turn or point around a point if you're moving or turning like point A B around point A. And on your questionnaire that I gave you initially. There's question 1.2 "What does it mean to measure an angle?" You said to measure an angle we measure the turn from one static arm to the turning of the arm between 2 lines. So, what you are actually implying here?**

Teacher 2: What I imply by measuring angles from turning an arm is like from the basic measuring of angle I took from like when you are using the none standard measuring of angle whereby we say if this is an arm. If like I turn the arm I will have the right angle and if it's straight I will have the 180 which completes, which is 180 degrees. But then in a sort of non-standard measure. Non-standard measure I mean like where you demonstrate a half turn and full turn and so forth. Thereafter is then that where you sort of. I will come to standard measures like using the protractor where I measure the angles, right angled triangle and so forth. Not triangle, right angle which is 90 degrees and 180 degrees and so forth.

Researcher: **I just want to say, do you think these two questions are the same. What is the meaning of angle measure? And what does it mean to measure an angle? Angle measure? What does it mean to measure an angle?**

Teacher 2: Yeah, they are the same.

Researcher: **They are the same?**

Teacher 2: Yeah.

Researcher: **Why do you say so**?

Teacher 2: An angle measure like we are measuring an angle. Like when we say an angle measure I think it's the measurement you get when you measure that angle, that's angle measure. And then measuring an angle that's the application, you are measuring whatever. As I indicated whatever turn you want to measure if like it's a full turn or half turn or you see.

Researcher: **If one would ask you to explain, what does it mean to measure an angle?**

Teacher 2: What does it mean to measure an angle? I will take it from my definition to say ok, an angle is a measure of a turn. Then from there if like we have to measure an angle, suppose maybe we are having, maybe I will just draw a line, two lines and then demonstrate to the person to say ok from the static point to between the two lines or the arms. From the one that is static to the one that is moving like we are measuring from. Maybe from the one that is static to the one that is moving. So we are checking whether this line which has moved or the arm which has moved it has moved to how many angles or much is the angle it has moved.

Researcher: So how should the concept of angles measure be approached or be introduced to grade 10 learners?

Teacher 2: To grade 10 learners, the concept can be introduced through. Before, I can say before when we measure the angle it starts with the non-standard measures. To say, ok, if I turn my arm, then if I turn my arm like this then we have a right angle and if I turn it like this then I have the straight angle. not the 180 because the 180 is when I will be using a protractor to measure. After that demonstration is when we take the protractor and measure the angles.

Researcher: Ok, so you said using a protractor is how you introduce the concept of…

Teacher 2: No, starting from demonstrating without using a protractor. Actually like in my, the pre-existing knowledge they have is like t they know all the types of angles.

Researcher: So in other words you're saying from geometry, Euclidean geometry?

Teacher 2: Yes. They already know those types of angles, the obtuse. They know all that when you measure an acute for instance it's like anything less than between 0 to 90 then it's an acute. Between 0 to less than 90 it's an acute, then I will take it from there to say ok, in the previous grades you actually learnt those different types of angles. Now from there if I have to teach them to measure, its then that we use a protractor.

Researcher: So how will you relate obtuse angle, for example? Or let me just simplify it, acute angle in terms of what you are saying with a trigonometric concept?

Teacher 2: Relating with the trigonometric concept?

Researcher: **Yes, now you are introducing your approach to your learners that you have obtuse and you have acute? How will that be helpful, let me put it like that, in understanding any trigonometric concept?**

Teacher 2: By the way trigonometric concepts. Can you rephrase the question?

Researcher: **How will it help. Here is a leaner, yeah it's an acute Saying to a learner here is an acute angle, how will that knowledge to say that an acute have in understanding any concept of trigonometry? Remember this is how you are introducing this concept or approach. Remember this is how you are introducing this concept or approach, so from a geometric perspective what I'm asking now is how will it help learners understand trigonometry?**

Teacher 2: Actually, like. I don't know if I got you correctly. If for instance, here is a learner and then from this learner I discover that ok, the learner at least has that key knowledge of the different types of angles. Now I want to introduce the concept of trigonometry to the learner. Then that now I will actually dwell on the triangle which will be the basic as I said. The foundation, the right-angled triangle hence like I want to introduce the trigonometry.

Researcher: **Ok, so if I hear you we are setting ourselves to acute**

Teacher 2: Ummm

Researcher: **That will fit in the right-angle?**

Teacher 2: Yeah exactly.

Researcher: **Ok, alright. Now if we go to question 4, what would you like your grade 10 learners to understand in relation to sine x and all**

Teacher 2: In relation to the sine of x?

Researcher: **Yeah, you shall respond to said angles either right angled triangle or any types of angles which are not right-angled triangles. Ok I just want a clarification there. You did say on a right-angled triangle, any types of sine which are not right-angled triangle.**

Teacher 2: By the way, here we are talking about grader 10 leaners. For a grade 10 learner like as the concept of sine and cos. By the way the question is how…?

Researcher: **What would you like your grade 10 learners to understand in relation to sine x?**

Teacher 2: Alright, in relation to sine x and then the cos ok. What I would like the learners to understand, well it will be like for sine and cos for them to be able to actually see the difference between the sine and the cos in terms of like how we say like when we draw the acute angle for instance on a cartesian plan to say the first quadrant, second quadrant and the third and fourth. So, I would like the learners to understand how these trig ratios differ in each of the four quadrants particularly on the right -angled triangle because like here a grade 10 learner whereby any angles we are not dealing with, we are just dealing with the right-angled triangle.

Researcher: **Will you say this question is also vague or does not make sense, I mean if I ask you what I'm trying to say in terms of sine x and cos x or in relation?**

Teacher 2: Ah, it's clear, it's clear, it's clear.

Researcher: **Ok let's move on. Describe sine x in your own words.**

Teacher 2: The sine of x in my own words. The sign of x I would define it in terms of a, when looking at, in terms of the right-angled triangle where we have opposite over the hypotenuse, that is when we are looking at the right-angled triangle.

Researcher: **So that would be it. Using your words for a given right-angled triangle you said sine x is actually opposite side of the triangle divided by the hypotonus side**

Teacher 2: Yeah

Researcher: **Ok. Now with that description if we are to go back to question 4, what would you like grade 10 learners to understand in relation to sine cos x? for instance you will want them to understand the opposite side of the triangle divided by the hypotonus will give you the sine of x or adjacent whatever that giving you the cos sine of x.**

Teacher 2: Yes, that it also gives you, like there on that one there I explained that I would like them to know that the sign of an angle on right-angled triangle is opposite over the adjacent. But then the other thing on that question I elaborated on it like more to say what I would like them to understand is like when, remember when I introduce trigonometry it will be on the cartesian plan whereby I will draw that right-angled triangle. Then the right-angled triangle, remember here I want to actually come to the issue of maybe like the sine of an angle when I drew the right-angled triangle on the first quadrant then it's a positive angle. and then when I go to the second quadrant then I also have the positive depending on like, because there we are using the opposite side over the hypotonus which now the opposite is still on the positive y axis and moving to the third quadrant, then now it changes that's negative y axis over the hypotonus which the hypotonus will always be a positive there.

Researcher: **Ok. So, you are now mixing, you are saying, because I didn't catch quite exactly the introduction of the cartesian, the co-ordinate in your description of your sine of x and also what you like your learners to understand in relation to sine of x. So, you mention now that you will be having the cartesian. So, what will be the purpose of the cartesian plan?**

Teacher 2: The purpose is like hence I'm saying like the purpose of the cartesian plan will be like as we say the sides opposite the hypotonus divided by the hypotonus. Then there the leaner will be able to see that if I do that on a cartesian plan the leaner will be able to see that ok if the opposite for instance in terms of the units, I'm having the opposite then the length of that opposite length is 3 units. Then how will I have the back sine in terms of the units. What will be the value of that side of an angle. or the other thing, it will also come to head when like I want the learner to find maybe the angle to say ok, now we want to find the angle between the hypotonus I mean the opposite and the hypotonus what will be the angle if like we are using the sine as a trig ratio.

Researcher: **So in other words if may go back to number 4 what would you like your grade 10 learners to understand in relation to sine x and or cos x. your intention you want them to understand as a trig ratio, trigonometry trig ratio where in 5 you say now when you describe it then you get that ration. That is your description, ok. Is sine x a function or not.**

Teacher 2: Sine x is a function

Researcher: **I know we spoke extensively about this and you did say sine x is a function.**

Teacher 2: Actually sine x is a function yeah, it is a function.

Researcher: **You said if and only if, that is putting my words. When you equate the sine of x to y or when you write it as y=sine x.**

Teacher 2: Yeah, yeah then we will be having the function. But if we just say to me that sine of x then like, there I don't see a function.

Researcher: **In other words you are saying the sine of x is not a function?**

Teacher 2: Yeah, that is why I said only and only if you just equate that sine of x. maybe you say the f of x equals the sine of x then yeah, I'm having a function, then the sine of x will be a function. But the sine of x per say like, when I look at it like here is the sign of x and then we said sine x is equals to the opposite over the hypotonus. Now I'm looking at how do I define that sine x. then it will be like, on that cartesian plan maybe I'm having some units up then the hypotonus is low then I will have the sine of x equals to maybe a specific unit. Maybe suppose I have 1.5 units, the sine of x equals to 1.5, then that is not a function. The sine of x equals to just 1.5 then to me it looks like it stops there, it doesn't say anything. But once I say ok, here you have the, I'm equating that y. y equals to the sine of x whereby now one can plot on the cartesian plan. One can plot maybe the co-ordinate there.

Researcher: **So, ok I know its just an example as you put it. So in other words you are saying you can not on a cartesian plan find sine of x is equal to 1.5?**

Teacher 2: On a cartesian plan?

Researcher: **Umm.**

Teacher 2: Yeah I can, I can find it.

Researcher: **And yet you are saying the sine of equalling to 1.5 is not a function.**

Teacher 2: Eish.

Researcher: **I'm trying to understand.**

Teacher 2: Yeah, neh.

Researcher: **So if I'm correct in your understanding sine of x is not a function unless you describe there is an equation or a formula x equals the sine of x as an equation.**

Teacher 2: Yes, yeah.

Researcher: **But the sine of x is not.**

Teacher 2: It's not.

Researcher: **So, even when I'm saying trig functions is in your understanding it's only y equals to, so, sine of an angle is not a trig function?**

Teacher 2: Yeah, yeah. But according to my understanding.

Researcher: **Ok, ok. Ok let's move on. What kind of teaching actions should a grade 10 teacher adopt in order to teach a concept of sine x? or perhaps if I may ask it differently. How will you approach teaching trigonometry so that your learners understand the concept of sine x?**

Teacher 2: How will I teach the concept of…

Researcher: **No, no. how will you approach teaching trigonometry so that your learners understand the concept of sine x and cos sine x?**

Teacher 2: There eish,

Researcher: **Initially what you said you will define the trigonometry focusing on how the trigonometric ratio sine and cos are defined on triangles. If I understand you your approach as you said it initially will be using right-angled triangle.**

Teacher 2: Yeah, I will be using right-angled triangle.

Researcher: So, let's move on. In your experience what difficulties do grade 10 learners have when they are solving trigonometry problems.

Teacher 2: Umm, the difficulties they have in solving trigonometric problems, one of the difficulties they have it's for them confusing this trig ratios, it's one of them. Like when looking at the right-angled triangle sometimes learners confuse each one of the opposite sides. That's one of the challenges they have. And the other thing, the other challenge they have, when the angle is drawn, maybe the other quadrants except the first quadrant there. Remember on the cartesian plan there were we have the positive y-axis and the negative y axis, now learners tend to confuse the which one is positive y-axis and which one is negative y-axis.

Researcher: So just moving on to the next question that relates to that. What kind of actions will you employ in order to avert learners' difficulties. An alternative question to that. What strategies do you use to help your learners the difficulties that you mentioned?

Teacher 2: To actually help learners with such difficulties, just by demonstrating it various examples and explaining to them like in terms of saying ok if this is the mistake that you have made, and this is how you were actually supposed to do. And then after that maybe give them another problem related.

Researcher: And also you suggested that you used an acronym I think that's what you said sohcahtoa

Teacher 2: Yeah, normally like as I indicated that you find that they are mixing those opposite and sometimes you say ok here is sine of x then you have indicated x as an angle on the right-angled triangle. Now the learner will just confuse to say ok, maybe the learner instead of saying the sine opposite over hypotonus now the learner will tan, which is opposite over the adjacent. So that like acronym it actually like simplify the concept of sine is opposite side over the hypotonus side and then the tan and the cos and so forth.

Researcher: Ah, I have question 10 here, I will not you know because it is part of that demonstration. I think we will stop it here even though on this other thing one would like to ask you more questions on the questionnaire.

Teacher 2: Yeah maybe if you like I need to clarify..

Researcher: **I will only ask you question…. that is question number last I can give it to you to look at and you did not actually attempt it. I just want to know, what are challenges or what do you think it was a challenge for you not to be able to answer this question.**

Teacher 2: Ok let me have a look at it. Question 9, is it 9?

Researcher: **Yes**

Teacher 2: Oh, the question is it this one?

Researcher: **No, no, no it's question 10.**

Teacher 2: Oh question 10. Let's see what's asked.

Researcher: **If I may ask you a specific question. Do you think the question is relevant for learners at grade 10 level? Now taking it to cognicents… the curriculum.**

Teacher 2: Like grade 10 I don't think it's relevant.

Researcher: **It's not appropriate for grade 10?**

Teacher 2: Not for grade 10. I think it's appropriate for grade 11. Because for grade 10 (reading to himself)

Researcher: **You don't have to solve it now. I wanted to understand after looking at it….**

Teacher 2: Develop a solution for a task, yeah I even found it challenging yeah.

Researcher: **Do you think that that question can be solved without using the sine and the cosine angle?**

Teacher 2: Without?

Researcher: **Ehh**

Teacher 2: Without using the sine and the cosine no. let me analyse it further.

Researcher: I'm just taking it from when you said it might not be appropriate for grade 10 because in grade 10 according to the curriculum there is no cosine or sine they only start to learn about those in grade 11.

Teacher 2: Yeah. I was just looking at like the use of geometry, if they use the concept of geometry to say ok, if now we are having these triangles maybe the congruency of the triangles will be…. I was just looking at it. Where is the centre? (talking to self)

Without that concept right-angled triangle for grade 10 this cannot be solved.

Researcher: Yeah I was just pulling a leg but putting that aside do you also see that the problem even with right-angled triangle trig it doesn't help here?

Teacher 2: Neh?

Researcher: No, I'm just commenting because if, let me say understand understanding the right-angled triangle do you think it will help solving this?

Teacher 2: Only if we have maybe we have these triangles that I see here maybe if they were right-angled triangles yeah. Maybe that could help. But these are just any triangles which need the concept of cosine and the sine. Ummm, I don't get it.

Researcher: No it's fine. Just on the side I once gave it to my post Matric and I saw that they solved it without using the sine and the cosine rule. It's only one person that manged and I said use the cosine and they said they can't find it. But now I believe some the can actually. But even with some other teachers they could not. We will just leave it.

What do you think is the problem if at grade 10 level or even grade 12 level or even post Matric or even teachers a problem like this can't be solved. What do you think could be the reason why they cant solve?

Teacher 2: The reason it can be like maybe failure to analyse the question because like your Matric geometry needs like to be analysed. You take every piece of information and put it together then you find the answer. But then if you can not analyse then you can't. but in this one even myself

I'm failing to actually figure out how I can actually come to respond to it.

Researcher: **You mean trigonometry?**

Teacher 2: Yeah

Researcher: **Well yeah, I can tell you the teachers that managed it they used mid-point error geometry. You can actually…**

Teacher 2: With a mid-point maybe…

Researcher: **My, my, my worry is because you have the cosine of a function why can't they using trig concept they can't solve it using trig or using a trig solution.**

Teacher 2: So you are saying….

Researcher: **No it's dynamic for them to think in terms of geometry using the mi-point. It's analytic enough as you were saying as also geometry will demand. But from understanding geometry why will be those impediments for them to be able to understand to solve this?**

Teacher 2: I mentioned the issue of failing to analyse then the other thing maybe even the knowledge gap, trigonometry….

Researcher: **What knowledge gap?**

Teacher 2: The knowledge gap can be like not having a full understanding of the concept which will then lead to failure to validate the answer.

Researcher: **Not having a full understanding of the trig concept?**

Teacher 2: Yeah.

Researcher: **What do you mean by that?**

Teacher 2: What do I mean by that hey

Researcher: **I'm also recording your tapping there (both laugh)**

I actually do agree with you when you say… I don't know what you mean by knowledge gap, but not understanding certain things. But I'm trying to find out what exactly is it that they don't. because they have learnt trigonometry and they can define the cosine of x and yet when it comes to problems like this they can't say what could be you know… but, in any case this is another level or something for you to ponder about yeah.

Basically, in terms of my research is one is attempting to access your understanding because you know what you are telling me or as we discuss its there and I want to look at that. At this level it's just to understand it not correct or do anything. But with a view of once I have understood that then one can be able to come back to you to say you know this is actually your understanding and that is why in an example you can't answer or when you teach learners cannot answer this. But in understanding his point of view once I have actually analysed what your responses I believe learners should because I believe any learner should be able to answer this even at grade 10 level. When they go to grade 11 when they learn about those other ways it will be effective and they can use that to answer this question but not that they are a requirement. But it's just that they are form of ways for them to be able to solve these problems.

Eh Proff thanks. The last time we did the semi structured interview and what I'm going to do, I'm just going to read your responses and then make a follow up.

Teacher 2: Ok

Teacher 3 Interview

Researcher: **When we met the last time you did answer a semi structured interview. I gave you that as a thing that you had to fill in first and I want to probe you on your responses that you gave, just to get some clarification as to what you mean by some other aspects. Ok let start. In question 1, the question was, in your understanding how should a teacher demonstrate his or her understanding of trigonometric concepts in the classroom?**

Teacher 3: Yeah, I need to get another tool, this one is not…

Researcher: **You response was; educator's understanding of trig concepts can be demonstrated by showing practical examples, how bridges are designed. So, I just want to know when you show learners how bridges are designed, how does that introduce the trigonometric concept to your grade 10 learners?**

Teacher 3 No. Remember that now when we talk of trigonometry, usually we are referring to some of the angles. So when I put in the issue of how a bridge is designed, was referring, in a way now, to a particular bridge. It has an angle, so you cannot just design a bridge for example with a straight line just going up without a particular angle. And that takes us far into the roof truss, that's where trigonometry is being used a lot, those angles and stuff because when they do the roof truss I think there is the 90 degrees, the 45 degrees and how steep the roof should be. That's where the trigonometry comes in.

Researcher: **So, what you are saying here in your understanding is that trigonometry should be introduced or… you understanding is in terms of explaining from a context?**

Teacher 3: Yeah.

Researcher: **From applying?**

Teacher 3: There is application, true.

Researcher: **So you say you need to demonstrate your understanding from…**

Teacher 3: Through the bridge, roof truss and those things. That is the application by the way. So then now, the question should be… like your follow up question something there should be like the one of "what do you mean by sine of something."

Researcher: **Yeah, we are coming to that.**

Teacher3: Yeah, so before I can introduce trigonometry to grade 10 I must show them or tell them about the application first just to draw their attention. Because if I just start by saying trigonometry, sine, cos, so forth and so on, automatically they'll lose interest. So for me to draw their interest, what I need to do is to show them the application and say "now look some of you will become civil engineers." And civil engineers are definitely going to need this section as I have mentioned in the two examples the bridge and the roof truss. And the now another one, the group of mechanical. I saw somewhere when they remove and engine mechanically, there's a particular chain. So that particular chain, they must know if the chain can carry the weight of the engine as is and at what particular angle. Those are the applications of trigonometry.

Researcher: **Ok. Thanks for that I think we can go to the second question. Remember as I explained initially that my focus of this research is just to explore your meanings of these trig concepts. Particularly the ones that you convey to grade 10s when you are teaching this concept. So I'm not looking at, or trying to judge or gage your understanding. But I'm extracting your understanding, whether how it is we will see when I have analysed when I'm coming back with feedback. At the end of the day what one wants to establish is to come up with some form of a program or module that will support teacher development or professional development in the teaching of this topic. So as we will probably in the analysis we will be able to tell whether your approach has this. Remember there is no wrong or right answer, we find some impediments because the background is that many learners actually fail trigonometry, and even when I engage with post matrics I found that they could not solve a circle problem, which we will come to. And also I looked at your response and one wants to understand why they can't actually engage in this and what is wrong. And also even the DBE it has, I haven't read the recent report, but looking at previous reports consecutively trigonometry was the most failed topic in matric. Now let's come to question 2, the meaning of angle measure. The question was; what is your meaning of angle measure? And your response is; it's to measure the distance between 2 lines and not line as seen but between vertices. Line not as seen and you talk about the distance between two lines.**

Teacher 3: Ok, distance between two lines meaning now if I have a line and a line, then I'll have a distance between the two. But now the reason why I have put the vertices, is because the vertices is two lines joined together but now by the distance between that I was referring to this angle.

Researcher: **The angle in your definition or your meaning of an angle is that distance between two lines?**

Teacher 3: Yes. Between 2 lines if you show it, but now if one reads like you were reading it when I said the distance between 2 lines, one can just draw 2 parallel lines. Then now it has a different meaning, if I have 2 parallel lines the distance between those two parallel lines, is it the angle? Then the answer would definitely be no. But once we put in the word vertex.

Researcher: **Between vertices?**

Teacher 2: Between vertices. Remember now what is a vertex?

Researcher: **Yeah, I will if I understand. Let me try and understand, what you mean here is you have vertex and you have 1 line going in another direction or you draw 2 lines from the same vertex or point and you say now the distance between 2 lines from the same point that is your angle? That is what you mean by angle measure? That distance?**

Teacher 3: Yeah.

Researcher: **Ok.**

Teacher 3: Yeah, now it sounds better because the distance between… no no, the line from the same point moving in different directions as I've said that it's 2 lines from the same point moving in different directions therefore now you have that particular distance between the two. But if you want to talk of, for example, the distance between the one that I have mentioned parallel lines. One would make it from there to there. If you ask is that an angle? The answer would be no.

Researcher: **So how will you represent that distance?**

Teacher 3: The distance? No you tell the pupils that the distance between 2 lines that's it. So now you need to know many things that's where Euclidian geometry comes in.

Researcher: **Yeah, I'm trying to imagine the distance between 2 lines from the same point going in different directions and making a space. So you say angle measure is the distance between that space?**

Teacher 3: Yeah.

Researcher: **Now I'm asking that distance, you know, between. How will you actually represent that**

distance? I'm sorry to confuse you.

Teacher 3: Maybe now it's because I've put in distance there, because now people will say distances are measured in metres and other units but when you put in angle, then you need to put in the degrees and stuff. But now once I measure the angle from the same… the angle of 2 lines from the same point therefore that's whereby I'm going to put in, because now I'm talking of the angle, the unit there should be degrees. But if I have those 2 lines like that, I cannot talk of any degrees I can only metres or millimetres in terms of the units.

Researcher: Ok let's come to question number 3. The question was; how should the concept of angle measure be introduce to grade 10? Alternatively how should the concept of angle measure be approached with grade 10 learners? And your response was; an angle measure is introduced in grade 9. So grade 10…

Teacher 3: They should be knowing how to…

Researcher: Know how to measure the angle? So if I understand you, you are saying the approach that you'll adopt is to measuring an angle? That is how it has to be introduced?

Teacher 3: Yeah. Because of that particular question. Personally I would say like if you're putting the grade 9. Because grade 9 what do they do? In grade 8 as well, that's when they start with the construction of the angles and stuff in grade 9. So in grade 10 we assume that now learners know how to.

Researcher: Ok. So hence that is why you're saying this is where the protractor can be brought in. So the way it's supposed to be approached, the issue of angle measure is using a protractor?

Teacher 3: Yeah, in grade 9 they know how to use that particular protractor.

Researcher: Ok. So far so good. So let's go to question 4. What would you like your grade 10 learners to understand in relation to sin(x) and or cos(x)?

Teacher 3: Sin(x) not?

Researcher: **Or cos(x).**

Teacher 3: Sin(x) or cos(x)?

Researcher: **Yeah. And. Sin(x) and or cos(x). Your initial response was that you want them to understand the relationship of sin(x), cos(x) is the same…**

Teacher 3: They should be having the same hypotenuse I think.

Researcher: **Yeah. Same hypotenuse in a right-angled triangle. The y share the same hypotenuse in a right angle triangle. The most important part about what learners should understand about sin(x) and cos (x) as trig functions can only be used on a right-angled triangle and they should know the position of the given angle and naming of all sides like opposite side, adjacent side. So would this if I understand you, is that you want your grade 10 to understand sin(x) and cos(x) only in terms of a right-angled triangle?**

Teacher 3: Not only.

Researcher: **Ok.**

Teacher 3: Because we have the other ones, the other trig functions like the tan. So then now when you start with them in grade 10 we do all 6. But from those 6, we should not confuse maths and technical maths.

Researcher: **It's mathematics.**

Teacher 3: Yeah. Because now what is happening in grade 10 is they do all 6. Yeah. In grade 11 they revise all 6 but when it comes to exams they ask of the 3, as well as in grade 12 they ask of the 3. The first 3, the sine, cos and tan. So now with my grade 10 definitely I will say to them before they can use these 3 trig functions they need to understand what I must have, a right-angled triangle, without that they cannot think of trig ratios. Because now that's where the confusion comes in when they go to grade 11, you give them a scalene triangle to find one side, they'll go straight to one of the 3 trig functions which are no longer applicable because we are no longer talking of a right-angled triangle. Now I'm not saying a scalene triangle can never be a right-angled triangle, I'm not saying that, it can be. So don't confuse me.

Researcher: **I understand a scalene…**

Teacher 3: Can be a right-angled triangle. Yeah it's just that now it's only a term that just came into my mind. Because the other one is the isosceles can also be a right-angled triangle.

Researcher: **So what you are trying to say is you might find that if learners are given a triangle that is not a right-angled triangle, yes we have a right-angled triangle and we have triangles which are acute angled triangle or obtuse, you know. So you're saying if given the 2, then the sine or the cosine is not…**

Teacher 3: No. acute can be right-angled triangle. Obtuse I'm just thinking.

Researcher: **No what I meant in terms of, yeah. I also don't think because remember scalene. The definition of scalene triangle is a triangle that has sides that are not equal, all the 3 sides are not equal. And we can have a right-angled triangle which falls under scalene.**

Teacher 3: Exactly.

Researcher: **But we have scalene, isosceles, equilateral. Ok. With whatever properties.**

Teacher 3: Yeah.

Researcher: **But also a scalene, you know, we can have a scalene which is an acute angle. Because here, your point is not right-angled triangle where you will… if I…if you. In my words you will have a problem in explaining the sine or the cosine or the tangent. So the term or the mane that you want is not… where if a triangle is not he right-angled triangle then you won't be able to explain the sine of theta.**

Teacher 3: Yeah. Because we are looking we are looking at the definition of sin cos and tan. Opposite over hypotenuse, so if I don't have a right-angled triangle I'll never know which one is opposite, which one is hypotenuse. So then I need that particular right-angled triangle.

Researcher: **So, in other words, your answer to this question is that you want your learners to**

understand sin(x) and, or cos(x) in a right-angled triangle?

Teacher 3: Yebo. In a right-angled tringle. And then now, again they must be aware that now, our reference angle can be moved. But we cannot move the what? The right angle. If you move the right angle, it means now you are transforming or you are…

Researcher: **What do you mean by move right angle?**

Teacher 3: No, no, no. For example, if maybe I have drawn a right-angled triangle on a chalkboard, then I turn it around.

Researcher: **Oh. You mean not moving the right-angled triangle, but you're changing the orientation of the right-angled triangle?**

Teacher 3: Yes.

Researcher: **Like flipping it or… horizontally, diagonally.**

Teacher 3: Yeah. Yeah because now if I don't… if I move it 1 learner will say this turn can be there. So if you are given and the right angle is up that means now there was some transformation that took place. But in a right-angle triangle, then 90 degrees will remain there, but these ones now that's where the reference angle comes in. Now, if I talk of sin(x) where is my reference angle? Because now the opposite and the adjacent are informed by the reference angle.

Researcher: **Ok. Ok, let's move on… yeah question 5, describe sin(x) in your own words.**

Teacher 3: Sin(x)? That's opposite over hypotenuse, that is our words. Our words, that is our words.

Researcher: **Yeah, you know, sin(x) is opposite over hypotenuse.**

Teacher 3: Yeah, opposite over… that is our own words.

Researcher: **But in your initial response you said sin(x) is given by the ratio of opposite and hypotenuse.**

Teacher 3: A what?

Researcher: **Ratio.**

Teacher 3: Oh, I said sin(x) is the ratio of what? Opposite and yeah. By that time I was normal, I think.

Researcher: **Why do you say ratio?**

Teacher 3: Yeah. Because now, remember we are saying we are saying opposite over…and those 2 things give us a ratio now we are dividing the 2. So that over that. So if I say to you sin(x) is given by opposite over hypotenuse. Opposite over hypotenuse what is that? What is opposite over hypotenuse? It is the ratio in a way. It's 2 divided by 4 or 2 divided by 9.

Researcher: **Oh. So you mean opposite divided by hypotenuse?**

Teacher 3: Yeah.

Researcher: **Ok. So that's what you mean by the ratio?**

Teacher 3: Yeah.

Researcher: **So by the way, this is what you tell your learners?**

Teacher 3: Yeah.

Researcher: **Your meanings reside in your mind?**

Teacher 3: Yeah.

Researcher: **Ok. Let's move on. Is sin(x) a function or not? Please explain, I know we discussed this extensively.**

Teacher 3: I didn't skip that question.

Researcher: **But your initial response sin(x) is a function one thing, the next it's not a function**

because when talking of a function, you talk of a relationship between x and y. So if y = sin(x) then we talk of a function?

Teacher 3: Yeah. We discussed that, I know, then we concluded and said it is a function.

Researcher: **Did we conclude that it is a function? Now, let me ask you. Honestly you, is sin(x) a function?**

Teacher 3: Yeah it is a function.

Researcher: **Why do you say it is a function?**

Teacher 3: As I have said that a relationship between… that's now when I define the word function. The relationship between x and a y.

Researcher: **So what is y in this case?**

Teacher 3: Remember, we are talking of what? In that case then because no we are talking of sine, your y would be the opposite. But remember now that these things might change anyway but in first quadrant, your y would be what? Your opposite. I know that now you'll ask me like where is the hypotenuse, because I will say hypotenuse is the r in the first quadrant. Now I know you are going to ask me about the x because I am saying the relationship between x and y. I know you'll get there. So maybe one needs to go back and change the definition of the function between x and y. maybe we can say a function is a ratio between a co-ordinate, if I may say it, the point, in a point that's made up of an x and a y. It can be formed by x and r. Yeah.

Researcher: **Now you're talking about first quadrant and your understanding, just to take you back, in your description of sin(x) which it was. And also even going a step back, in a right-angled triangle. So when you talk about first quadrant, how is that related to right-angled triangle?**

Teacher 3: In first quadrant, remember when we draw our right-angled triangle in first quadrant in a way, going back to the issue of the protractor. You place it, you measure the angle 60 there and then once you measure that, therefore now usually we are saying now to our learners. Because now some learners, after drawing, I'll use the hypotenuse line for now…sides, after drawing your 60 degrees, the hypotenuse side. So then now always when we do the construction, it should always be parallel to the y-axis not the x-axis that's where now the right angle comes in. Because I will be having my hypotenuse then I'll draw my line parallel my y-axis. Then if I conclude that, then I'll definitely be having a right-angled triangle. Because I have said 60 degrees then now that puts us it in the first quadrant.

Researcher: **When we started you also spoke of a right-angled triangle and there were no quadrants.**

Teacher 3: I never mentioned a quadrant.

Researcher: **You never mentioned a quadrant or x and y. you only spoke of hypotenuse, opposite over hypotenuse in relation to sine x.**

Teacher 3: Umm

Researcher: **Now you are talking about first quadrant and I'm trying to relate it to your…. Is it if I understand you, you are now only talking about the first quadrant. Now because we mentioned the word function, otherwise if the word function…**

Teacher 3: No I didn't mention function.

Researcher: **You mentioned quadrant**

Teacher 3: Not exactly, I knew you were going to mention it somewhere**.**

Researcher: **No I mean initially if…**

Teacher 3: Oh, ok

Researcher: **Let me ask it in this way this question. Suppose the first question was if sine x was a function and the second question was describe sine x in your own words. Were you going to…?**

Teacher 3: The quadrant?

Researcher: **Yeah.**

Teacher 3: Describe sine x in my own words…

Researcher: **Remember now you are describing it after you have declared if sine x. is sine x a function or not.**

Teacher 3: Yeah for now I would say the issue of first quadrant…

Researcher: **No, no let me just repeat the question. What I'm saying is that I just want you to tell me something because now I'm saying I'm twisting. Assume I came I ask you first if sine x is a function and you gave me you know…**

Teacher 3: Whatever definition

Researcher: **Yeah, and then I'm coming and saying subsequently to you now and ask you, you know, describe sine x. or if we go back to another question…**

Teacher 3: You know what? Sorry to cut you but now…

Researcher: **Because in question 4, what would you like your grade 10 learners to understand in relation to sine x and cos x. your response was that in terms of right-angled triangle and there was no mention of the quadrant. So I'm saying suppose that question came after.**

Teacher 3: No I was not going to mention again the first quadrant because now the issue of the quadrant by the way they come at a later stage.

Researcher: **Oh, ok.**

Teacher 3: Yeah because now when do we introduce the first quadrant and everything it's when now we are talking about the reduction formula and stuff. So the leaners must not go, first quadrant this is according to the definition that we learnt previously saying that now sine x is opposite over hypotenuse. Now that's when I will be dealing with the reduction. So the learners mustn't be like oh, why do we say sine x as the first quadrant is positive because now we will be using whatever variables that you will be having there saying that now, our y is positive our r is positive. And when you go to the second quadrant our y is still positive our r is still positive but remember that r will always remain positive. It is only x and y that will always change. Now why in the second quadrant we have cos, that's where we are using the previous definition. Because if I'm going to talk of when I introduce trigonometry in grade 10 then start putting in those quadrants and stuff I will be confusing them. I must make sure that now they understand the right-angled triangle, naming of it and which one is opposite and hypotenuse and if they master it then we take it further.

Researcher: **Ok. Let's move on'. Question 7 was, what kind of teaching actions would a grade 10 teacher adopt in order to teach concepts of sine x? Alternative to that, how would you approach teaching trigonometry so that your learners understand the concepts of sine x and cosine x?**

Teacher 3: I think they should stick to trigonometry, that's the key.

Researcher: **In your initial you said let learners know Cartesian plan, know how to label sides on a right-angled triangle.**

Teacher 3: Yes that's it. If they must then that's it and that's the basic then leaners should never fail to meet, remember we are talking of sine and cos, we are not talking trigonometry in general. Because once we talk of trig in general it means now that you have to put- because that's when now the first quadrant…

Researcher: **Do we have trigonometry general in grade 10?**

Teacher 3: They are dealing with right-angled triangle, they put in ahhh, grade. They are doing reduction formula in grade 10 if I'm not wrong.

Researcher: **I can tell you. I can tell.**

Teacher 3: Ah, ok.

Researcher: **Do you want?**

Teacher 3: Yes please. I also have it on my phone so that we can clarify that.

Researcher: **Ok trigonometry in grade 10. Definitions of trig ratios, sine theta cos theta and right-angled triangles. Extended definitions of sin theta and tan theta, greater than 0 less than 360, deriving and use of trig ratios without using a calculator… the special angles. So, yeah it's definitions of trig ratios sine theta, cos theta and tan theta in a right-angled triangle.**

Teacher 3: Yeah. I wanted to emphasize in a right-angled triangle.

Researcher: **Yeah, extended definitions of sine theta, cos theta and tan theta, yeah, to 360.**

Teacher 3: Yeah.

Researcher: **And derive and use values of trig without using a calculator.**

Teacher 3: Yeah, those are special angles.

Researcher: **For, special angles. And well, define reciprocals of trig ratios.**

Teacher 3: Yeah this is the other great part as I've mentioned.

Researcher: **And solve problems in 2 dimensions.**

Teacher 3: So then now, if at the beginning if I didn't emphasize the issue of a right-angled triangle, there's no other way. So grade 10s, why they are failing trigonometry…

Researcher: **Yeah. That's what one wants to establish. In fact I'm trying to give a reason with my study, why they are failing. Because it starts there in grade 10. So yeah, we were discussing question 7. We were talking about question 7. What kind of teaching actions or alternatively, how will you approach teaching trigonometry so that your learners understand the concept of sin(x) and cos(x)?**

Teacher 3 Understanding of the learner, because you know what is happening here? Sometimes the educators, they become unfair to these kids because now I'll bring in the issue of the first quadrant. When I teach or when educators teach they'll focus mainly on the first quadrant and then they'll make sure that now they put in the angle down there, but come exams they take that particular right-angled triangle and put it in the second quadrant now they take the angle up there now the learners are totally confused. So it becomes unfair to the kids.

Researcher: **Why is it? Why?**

Teacher 3: No educators are like that. All of them. Excluding me.

Researcher: **But why… why will learners be confused?**

Teacher 3: No, remember that now these learners, that is my observation, the learners usually they compare maths with their other subjects. If you give them 1+2, in an exam give them 2+1 then it's a problem to them. But now we need to expose these kids in all types in a classroom situation, all types of questions so now when they go to exams, they'll be ready for them exams. So most of them, they see things for the first time in an exam. Like as I've just made an example saying in a classroom you give them a question in the first quadrant, second come exam you give them a question in the second or third quadrant. Why didn't you show them that during class? Why did you hide that from them?

Researcher: **Ok. So in other words… in fact you are now answering question number 8. What are possible, or in your experience, difficulties that grade 10 learners have when they are solving trig problems? You said the problem lies with the educators? They don't…?**

Teacher 3: Expose them to different types of questions. Like, yeah, another one comes in the exam. You used to give them a normal triangle even if you can move it in all the quadrants, it's fine but now here comes the exam one will just rotate that angle or the triangle maybe 45 degrees. Meaning now, it's no longer sitting this way but the 90 degrees is there at the corner, it's up there. Now some of the learners still get confused, so the types of learners… yes, they have brains but they don't think for themselves.

Researcher: **That's another… we won't go into that. Let's go to question 9. What kind of actions will**

you employ in order to alleviate learners' difficulties? And difficulties same as question 8. Alternatively, what strategies do you use to help your learners with the difficulties that you just mentioned?

Teacher 3: Yeah. Usually what I think one on one basis it helps, number 1. What did I say? Or didn't I answer that?

Researcher: **You said nothing more but, labelling is important.**

Teacher 3: Oh yeah. Because now what makes these learners not to answer correctly. Number 1, as I've said, in class work you find educators they'll put in their reference angles down there and write 60 degrees, 1. Come exam time or you give them homework, now in the particular homework you're no longer there you've put the angle up there. So the kids they fail now to separate which one is opposite, which one is adjacent. So if they can master that, then it's done problem solved. If they master the labelling of triangles, right-angled triangles, problem solved.

Researcher: **So nothing else, just labelling?**

Teacher 3: If they master that.

Researcher: **And learners will be able to…**

Teacher 3: Exactly.

Researcher: **So by labelling, you mean labelling of a right-angled triangle?**

Teacher 3: A right-angled triangle, yes.

Researcher: **So that will solve the problem?**

Teacher 3: Exactly.

Researcher: **Ok.**

Teacher 3: Yeah. Because now these kids, if you'll just check their response you'll see that now, their opposite and their adjacent, they are mixing those things. So the issue of what? Labelling. Because I remember I once did it in one of the classes, I've given them a task then I labelled for them so I gave them 2 and then I wrote hypotenuse, I gave them 3 and wrote opposite, I gave them 4 and I said adjacent, they got it correct. But if you remove the labelling, then it's a mess.

Researcher: So what could be the problem with labelling? Why will it be…?

Teacher 3: Maybe it's the lack of understanding, the learners themselves.

Researcher: Oh.

Teacher 3: Meaning that they fail to understand when you say what is opposite. Maybe it's because of the definition. Maybe they couldn't the definition initially, that opposite side is that side. It's how we define opposite, it's how we define adjacent. But the hypotenuse, with my… hypotenuse they get it correct. All of them. The problem is the 2.

Researcher: The opposite and adjacent?

Teacher 3: Once you move that angle then it's a problem. You kill them.

Researcher: In a right-angled triangle?

Teacher 3: Yeah, in a right-angled triangle.

Researcher: Ok. Ok. Maybe let's move to question number last. Yeah. It was this problem, if you still remember. Remember the question was how a learner…

Teacher 3: Oh the learner… yeah, yeah. I remember. Yeah, I mean look. You know that thing, I mean it goes back to the previous… and I'm not sure what I wrote there but...

Researcher: Yeah. What you said was; by making sure they understand the units. Saying that angles are measured in degrees and length in centimetre, remind him he can't add…

Teacher 3: Unlike terms, definitions because no he's adding the metres with the degrees there. So those are totally unlike terms, of which this is the knowledge they learn in grade 8 and 9. Like you cannot add y and x, so what the learner is doing is adding x and y. I'm just putting y and x because those are the ones that are commonly used in simultaneous equations. So if the learner adds, remember that now 180 is a degree, I can say degree is a x, 45 is a degree I could call that an x. But now the difference between the 2 is what? It's also a degree, but going down I can say that is also an x. now going down the learner is also putting what? The degree and the metres, so he's adding unlike terms. So the learner never mastered the knowledge, in grade 8 and 9, of like and unlike terms. So if a learner understood that then, he was not going to touch the 2.

Researcher: **Ok.**

Teacher 3: He could have done something different but he was going to say; I'm not supposed to add unlike terms. That's it. So this learner never understood that.

Researcher: **Oh, ok. And… so in other words, in helping that learner, you will explain unlike terms? I see here you also mentioned the Pythagoras theorem.**

Teacher 3: Yeah.

Researcher: **And if given 1 angle and 1 side then…**

Teacher 3: Exactly, then now.

Researcher: **The functions, sin(x), cos(x), and tan(x).**

Teacher 3: Yeah, for sure. Meaning that now, what did I put? So what happened is number 1, I need to distinguish between like terms and like terms, 1. And then now I must put back the definition of sine, when to use sine, when to use cos, when to use tan. So that learner, again, it means he never understood the definition of sine and cos and how it can be used.

Researcher: **Ok. Ok. I think we will… Yeah I just want to take you back to this diagnostic tool that you did a few ages ago and there was a question there. Question 10 which you did not answer**

at all.

Teacher 3: Which one?

Researcher: **Related to these questions which is a 2 dimensional.**

Teacher 3: I didn't answer that one?

Researcher: **Yeah you did not answer this.**

Teacher 3: Why?

Researcher: **That's what I want you find out, why you did not answer or what could be challenges that you might have faced such that you could not…**

Teacher 3: Was this me?

Researcher: **Or maybe let me just phrase this question. Do you think that question is fair for grade 10 learners? Remember at grade 10 there are no sine or area rules that have been introduced.**

Teacher 3: When you ask the question of is it fair for grade 10. Then ill definitely say no. You know why? Because in grade 10 we are talking of a right-angled triangle. And it's not going to be easy for the child to… anyway it's not. Don they have a right-angle? I don't see it. Do they?

Researcher: **No I want you to tell me, in fact you did not answer. The 2 dimensional problem and my expectation is that they should be able to answer 2 dimensional problems.**

Teacher 3: Yeah but no, no, no this question was not fair for them.

Researcher: **Will not be fair for them?**

Teacher 3: Yeah. Let alone the question itself, but I'm looking at the diagram itself. Yes, it is 2 dimensional but for grade 10 it's not fair. They'll just look at it the way I did and then leave it.

Researcher: **Yeah, it's just that you also left it but you're not the only one.**

Teacher 3: Yeah. It's not fair. Because now you see, once you say show that cos(r)… cos theta is 1 over 4, for grade 10? No. you'll get it the way as neat as is.

Researcher: **As their teacher left it?**

Teacher 3: Yeah. Surely. Because now, definitely I need to go to grade 11 and borrow some knowledge to show that. Do you have the solution?

Researcher: **Well…I can. That is not the question.**

Teacher 3: Didn't you go to grade 11 or grade 12 and borrow the knowledge to solve this?

Researcher: **Uh...well now you're answering… I can solve it without going to grade 11 or 12.**

Teacher 3: Ok, let's see.

Researcher: **Well, I'll show you at some stage or just after this.**

Teacher 3: Oh, ok.

Researcher: **Yeah I can.**

Teacher 3: No let me… do I have the soft copy of this?

Researcher: **I don't know.**

Teacher 3: I want to take a picture of this then I'll just look at it again.

Researcher: **Just to taking you back again on this diagnostic tool that you once answered, particularly if we can look at question 5. You said these 2 triangles, the question was; is the meaning of sine theta in this triangle the same as the meaning of sine theta in triangle 1, the same as the meaning of sine theta in triangle 2? You said no.**

Teacher 3: The same meaning?

Researcher: Yeah.

Teacher 3: Yeah, I said no. before we can apply a trig ratio you need to have 1, a right-angled triangle. 2, a reference angle. Figure 1 does satisfy the condition, but figure 2 has a reference angle, no 90 degrees. So you didn't show me a 90 degrees there.

Researcher: Do you still say that…?

Teacher 3: There is no 90 degrees. I cannot just conclude, I should not assume, I should not use my naked eyes to say this is 90 degrees. It might look like, but it's not. So I still stay with this one.

Researcher: Oh, ok.

Teacher 3: Because that's where now our grade 10s get it wrong. Because now, my learners, I told them if they you don't see 90 degrees don't assume. Because in that particular one it might be 91. But if I put in that particular sign, now I'm telling them that this is 90 degrees. So, all my test, grade 10 if it didn't indicate 90 degrees, for moderation it goes back if I set it there will always be 90 degrees. There must be. Because we are talking of right-angle triangle, so once we talk of right-angled triangle, there should be 90 degrees. If you didn't indicate that 90 degrees, you're talking of another triangle now, yeah. So there, I'm still sticking to that.

Researcher: Can I ask you, I know I'm extending this a bit more, sorry about that? How will you define the altitude or the height of a triangle? Any triangle, including a right-angled triangle.

Teacher 3: Height?

Researcher: Yeah. Or the altitude.

Teacher 3: Or altitude? Including the right-angled triangle? Height of a right-angled triangle… I'm just trying to make some examples and stuff but ill get it. Ill get it.

Researcher: Yeah ok. Let us stop it here. This will be kept… in fact I did not mention your name in this

audio. It will just be me and I'll make sure that this does not identify you by any mean. No mention of the school. But thank you I will go analyse and I will call a meeting when I'm ready with the report.

Teacher 3: And this solution? Let me take a picture then I'll look at it again using grade 10 knowledge.

Researcher: **Other teachers actually have attempted it. Others did solve it correctly.**

APPENDIX 4: Teachers' Demographics

Teacher number | | | | | | |

TEACHERS' DEMOGRAPHICS

Section A: GENERAL INFORMATION

1 Rank of Teacher

Deputy Principal	Head of Department	Teacher

2 Gender

Male	Female

3 What is your highest qualification?

Matric	
Diploma in education	
Advance certificate in education	
BEd degree	
Honours degree	
Masters degree	
Other (Please specify):	

4 What are your Mathematics teaching qualifications?

Mat I	Mat II	Mat III	Other specify

____

5 Specify whether it is a university or college qualification

College	University

6 Mathematics Teaching Experience

10 Years and More	Between 5 and 10 Years	Less than 5 Years

7 Do you like to teach Mathematics?

Yes	No

____

APPENDIX 5: Sample of Teacher Responses to The Trigonometry Task

Teacher 1

1\. **1.1** What is an angle? ~~it is~~ An angle is formed When two Segments intersect and at the point of Intersection an angle is form, Which will then be the measure of how wide twose Segments are from each other.

1.2 What does it mean to measure an angle? It means a rotational measurement of how wide the segments are from each other at the point of Intersections

1.3 What explanation would you give to a learner who asks: What does it mean for an angle to have a measure of 1 degree? Is to move a terminal arm anticlockwise.

Teacher 2

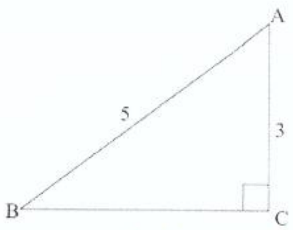
2.1 Define the *sine* of an angle. The Sine of an angle ~~is th.~~ Can be defined as the Side opposite to that angle divided by the Side ~~adjacent~~ Hypotenus to the angle.

2.2 What is the meaning of sin30°? Sine 30° means that the angle between the opposite Side and the hypoterus Side in a right angled triangle with the opposite side being 1 unit and the hypotenus side being 2 units

2.3 Referring to the meaning of sin30° that you provided in 2.2, what is the meaning of *sin(90°)*?; and that of *cos(100°)*? Sine 90° Means measuring the angle between the two arms of side on a right angled triangle whereby the angles between the two Sides/arms is 90°

Teacher 5

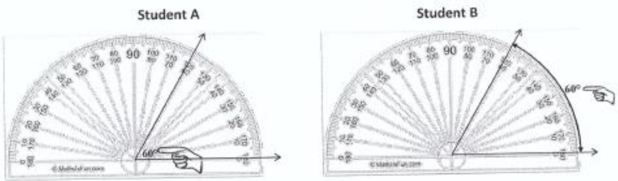
3\. In triangle ABC what does it mean to determine the output of the *sine* and *cosine* of the measure of angle ABC *without* measuring the angle (diagram not drawn to scale).



It means you determine the amount ~~of~~ or the value that is equals to the amount of Sin(angle) and cos(angle),

Teacher 7

4\. Learners (student A and student B in the diagram) were given a task to measure 60 degrees with a protractor. Two learners responded as follows (diagrams not drawn to scale):



4.1 Which of the two learners' responses would you accept/consider as correct? Explain why? Student B because he is accurate and Student A is ~~accur~~ precise, Since student B shows that he measured the angle from the horizontal to the terminal exactly where the line intersect the tick at 60°

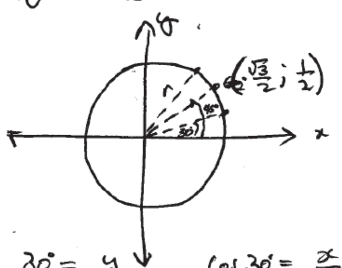
Teacher 8

7. **7.1** Explain the meaning of a unit circle? Circle whose radius is one throughout from the centre to circumference at $x = 0; 360°$, and $x = -180$ on the x-axis and y: $(0, 90°)$, ~~(0, 270)~~, $270°$; $360°$??

Teacher 9

7.2. Knowing ~~take~~ exact values of sine & cosine of $30°$; $45°$; $60°$ the unit circle can be used also to verify such results

e.g.



$\sin 30° = \frac{y}{r}$ $\qquad$ $\cos 30° = \frac{x}{r}$

$\frac{1}{2} = \frac{y}{1}$ $\qquad$ $\frac{\sqrt{3}}{2} = \frac{x}{1}$

$y = 1.$ $\qquad$ $\therefore x = \frac{\sqrt{3}}{2}$

This can also be used to calculate / confirm that $\sin^2\theta + \cos^2\theta = 1$; ~~diff~~ directly from the theorem of Pythagoras: $x^2 + y^2 = r^2$

Teacher 6

10.1 What important decisions should learners make in order to solve this task? They must first decide whether the circle has the centre or not a N1, and decide on the relationship between the dimensions.

10.2 Develop a solution for this task. $AM = MB = BC = r$ (Given)

$\therefore CM = r + r = 2r$

$ME = DM = r$ ...... (radii)

$CE = 2p$ so: $p^2 = r^2 + r^2 - 2r \cdot r \cos\theta$

$\therefore p^2 = 2r^2(1 - \cos\theta)$

$(2p)^2 = (2r)^2 + r^2 - 2(2 \cdot r \cdot r) \cos\beta$ but $\cos\beta = \cos(180° - \theta)$

$\cos\beta = -\cos\theta$

$4p^2 = (5 + 4\cos\theta)r^2$

$4p^2 = 8r^2(1 - \cos\theta)$

$8r^2(1 - \cos\theta) = r^2(5 + 4\cos\theta)$

$3 = 12\cos\theta \therefore \cos\theta = \frac{1}{4}$

10.3 Allocate marks for all your solution steps in 10.2. (7)

APPENDIX 6: Consent Letters

Letter of permission application to the school principal

P.O. Box 42693
Boordfontein
0201
Date: __________

Dear Principal,

Re: Request for permission to conduct research in your school

My name is Gabriel Mphuthi. I am registered with the Institute of Science and Technology (ISTE) at the University of South Africa (UNISA) for a Master of Science programme with a specialization in Mathematics Education. As a requirement to complete my degree, I am expected to do research with schools to improve aspects of teaching and learning. I have been granted permission by the Gauteng Department of Education to conduct my research in six schools in the Tshwane South District, and I have selected your school to participate in this research. The title of my research is: *Exploring teachers' mathematical meanings for teaching trigonometric ratios and functions to Grade 10 learners*. My research supervisor is Dr Joseph J Dhlamini who is a lecturer in the Department of Mathematics Education at the University of South Africa (UNISA).

The research aims to explore teachers' understandings of trigonometric ratios and functions by examining the mathematical meanings that teachers convey when they teach Grade 10 learners. To operationalize this aim, the researcher will: (1) administer a trigonometrical task on which teachers' comments will be assessed; and, (2) conduct semi-structured interviews with a purposive sample of teachers. I, therefore, request the participation of Grade 10 mathematics teachers from your school. The research will be interactive. I will give a task to teachers in which they will reflect on their understanding of the task, and the interviews will serve as a follow-up to probe teachers further on the meanings the hold and also on how they teach specific trigonometry topics to learners.

The benefit of teachers to participate in this research will be to assist them to improve their teaching of specific topics in mathematics. The participation of teachers in this research is, however voluntary in that they may decide to withdraw their participation at any stage of the study without a penalty. Teachers may decline to answer any specific questions. If you choose to permit me to conduct this research in your school, I will first convene a meeting with Grade 10 teachers to explain the aim of the study and the nature of their participation in the research. The real names of the school and those of participating teachers will not be used for reporting purposes. The pseudonyms will be used instead. Also, the researcher will verify all teachers' responses with them before the final report is submitted to the University for Examination purposes. The final report of the findings of the research will be availed to all participating schools. There are no known risks to you for assisting in this study.

Should you require further information, please do not hesitate to contact me by email at g.mphuthi@absamail.co.za or 082 334 2716. I have enclosed a form which you may use to respond to this request.

Yours sincerely

Gabriel Mphuthi

Informed consent form for the principal's response

Dear Mr G Mphuthi

I, ____, the principal of, ____________________________________ high/ secondary school, acknowledge the receipt of the letter of request for my school to participate in your research. I have read and understood the content of the letter that explains your research, which is entitled, *Exploring teachers' mathematical meanings for teaching trigonometric ratios and functions to Grade 10 learners*. I have also understood the aim of your research.

I, therefore, ***give consent***/ ***do not give consent*** that my school will take part in your research.

Principal signature : ______________________________

Date : ______________________________

Researcher signature : ______________________________

Date : ______________________________

Letter of participation request to the teachers

P.O. Box 42693
Boordfontein
0201
Date: ____________

TO: Grade 10 Mathematics Teacher

Dear Sir/ Madam

My name is Gabriel Mphuthi. I am registered with the Institute of Science and Technology (ISTE) at the University of South Africa (UNISA) for a Master of Science programme with a specialization in Mathematics Education. As a requirement to complete my degree, I am expected to do research with schools to improve aspects of teaching and learning. I have developed an interest in the research that examines the influence of teachers' knowledge on the performance of learners in mathematics. The title of my research is: *Exploring teachers' mathematical meanings for teaching trigonometric ratios and functions to Grade 10 learners*. My research supervisor is Dr Joseph J Dhlamini who is a lecturer in the Department of Mathematics Education at the University of South Africa (UNISA).

I have been granted permission by the Gauteng Department of Education to conduct my research in six schools in the Tshwane South District, and I have selected your school to participate in this research. Your participation as a Grade 10 mathematics teacher will contribute to the improvement of teaching in line with teachers' knowledge of school mathematics. If you decide to participate in this research, you will be asked to complete:

1. A mathematics task to reflect on your understanding and meaning(s) of certain concepts in Grade 10 trigonometry. I will take some notes as you respond to the task. I will only ask clarity seeking questions and will not ask questions about procedures or definitions.
2. You may also be selected for the interview in which you may be asked to explain some of your task responses, and also reflect on your teaching of these concepts. The interview will be audio-recorded and transcribed for analysis. I may also take note of your responses and actions during the interview.

Participation in this research is voluntary. Your decision whether or not to participate will not adversely affect you in any way. The use of pseudonyms will protect you from ever being identified and thus ensure confidentiality. The data and interpretations will be available in the form of a dissertation, and anyone will have access to that document, but your identity will not be apparent or disclosed. There are no incentives given to those who will participate in the research. There are no foreseen risks to those who decide to participate in this research.

In case you decide to participate in this research, the researcher will call a meeting in which the objectives of the research will further be explained. Every person in the meeting will be given a chance to ask questions. You are free to contact me on the following number in case of further questions: 082 334 2716.

Please sign and complete the consent slip below, and return it to me if you agree.

Your assistance is greatly appreciated.

Yours sincerely

Gabriel Mphuthi

An informed response form for Grade 10 Mathematics Teachers

Dear Mr G Mphuthi

I, __________________________, the Grade 10 mathematics teacher from, __________________________ high/ secondary school, acknowledge the receipt of the letter of request for my participation in your research. I have read and understood the content of the letter that explains your research, which is entitled, *Exploring teachers' mathematical meanings for teaching trigonometric ratios and functions to Grade 10 learners*. I have also understood the aim of your research.

☐ Yes, I wish to participate in your research. Also, note my responses to my involvement in the following research activities:

- I will participate in the trigonometric task:

YES	NO

- I will agree to participate in the interview session if I am selected:

YES	NO

- I may be audio recorded if I am selected for the interviews:

YES	NO

☐ No, I do not wish to participate in your research.

Signature: ________________________________ Date: ______________________________

(Please Sign)

Contact Information:

(1) Telephone number: ________________________________

(2) Email: ____

Letter of consent for the teachers

Research Title: *Exploring teachers' mathematical meanings for teaching trigonometric ratios and functions to Grade 10 learners*

Researcher: Mr Gabriel Mphuthi

Purpose

This is a research study. The aim of the study is to explore teachers' understandings of trigonometric ratios and functions by examining the mathematical meanings that teachers convey when they teach Grade 10 learners. We are inviting you to participate in this research study because you are a secondary mathematics teacher.

Procedures

If you agree to participate, your involvement will last for no more than three hours at three separate meetings. However, if you are selected to participate in the interview portion of the study, then you will be interviewed for an additional hour. The following procedures are involved in this study. If you decide to participate in the study, here are some of the things you may be asked to:

1) Complete a subject-area diagnostic task that will last about one hour to one and a half-hour. I will take some notes as you take the test. I will only answer clarification questions and will not answer questions about procedures or definitions.
2) Based on the initial analysis of the subject-matter diagnostic assessment, participants will be selected for a one-on-one interview. If you are selected for the interview, then you will be interviewed for about an hour. The interview will be audiotaped, transcribed and interpreted. I will also take note of your responses and actions during the interview.

Risks

The possible risks associated with participating in this research are as follows: There are no foreseeable risks associated with participation in this research.

Benefits

There may be no personal benefit for participating in this study. However, it is hoped that, in the future, society and the mathematics education field could benefit from this study by gaining a better understanding of teachers' knowledge of school mathematics.

Costs and compensation

You will not have any costs for participating in this research project. You will not be compensated for participating in this research project.

Confidentiality

Records of participation in this research project will be kept confidential to the extent permitted by law. However, federal government regulatory agencies and the University Institutional Review Committee (a committee that reviews and approves research studies) may inspect and copy records pertaining to this research. It is possible that these records could contain information that personally identifies you. In the event of any report or publication from this study, your identity will not be disclosed. Results will be reported in a summarized manner in such a way that you cannot be identified.

If you are selected for the interview

Audio or visual recording

By initialing in the space provided, you verify that you have been told that audio recordings will be generated during the course of this study. The audiotapes will serve as a reference and verification tools for the researcher's notes and your responses to interview questions and activities. The audiotapes will be retained until after the final acceptance and publication of the dissertation; at which time they will be destroyed.

_________________ Participant's initials

Voluntary participation

Taking part in this research study is voluntary. You may choose not to take part at all. If you agree to participate in this study, you may stop participating at any time. If you decide not to take part, or if you stop participating at any time, your decision will not result in any penalty or loss of benefits to which you may otherwise be entitled.

Questions

Questions are encouraged. If you have any questions about this research project, please contact: Mr Gabriel Mphuthi, 082 334 2716; or email: g.mphuthi@absamail.co.za

Your signature indicates that this research study has been explained to you that your questions have been answered and that you agree to take part in this study. You will receive a copy of this form.
Subject's Name (printed): ____

______________________________ ____________________
(Signature of Subject) (Date)

Researcher statement

I have discussed the above points with the participant or, where appropriate, with the participant's legally authorized representative. It is my opinion that the participant understands the risks, benefits, and procedures involved with participation in this research study.

____ ______________________
(Signature of Person who Obtained Consent) (Date)

APPENDIX 7: Gauteng Department of Education Research Approval Letter

For administrative use:
Reference no. D2017 / 359 A

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

GDE AMENDED RESEARCH APPROVAL LETTER

Date:	9 November 2016
Validity of Research Approval:	6 February 2017 to 30 September 2017
Previous GDE Research Approval letter reference number	D2016 / 316 dated 16 October 2015
Name of Researcher:	Mphuthi G.T.
Address of Researcher:	P.O. Box 42693; Boordfontein; 0201
Telephone / Fax Number/s:	012 549 3771; 082 334 2716; 086 640 6973
Email address:	Emphutg@unisa.ac.za; g.mputhi@absamail.co.za
Research Topic:	Exploring teachers' mathematical meanings for teaching Trigometric Ratios and Functions to Grade 10 learners in the Tshwane South District of Gauteng Province
Number and type of schools:	SIX Secondary Schools
District/s/HO	Tshwane South

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved. A separate copy of this letter must be presented to the Principal, SGB and the relevant District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted. However participation is VOLUNTARY.

The following conditions apply to GDE research. The researcher has agreed to and may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

CONDITIONS FOR CONDUCTING RESEARCH IN GDE

2016/11/10

1

Making education a societal priority

Office of the Director: Education Research and Knowledge Management ER&KM)

1. The District/Head Office Senior Manager/s concerned, the Principal/s and the chairperson/s of the School Governing Body (SGB.) must be presented with a copy of this letter.
2. The Researcher will make every effort to obtain the goodwill and co-operation of the GDE District officials, principals, SGBs, teachers, parents and learners involved. Participation is voluntary and additional remuneration will not be paid;
3. Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal and/or Director must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.
4. Research may only commence from the second week of February and must be concluded by the end of the THIRD quarter of the academic year. If incomplete, an amended Research Approval letter may be requested to conduct research in the following year.
5. Items 3 and 4 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.
6. It is the researcher's responsibility to obtain written consent from the SGB/s; principal/s, educator/s, parents and learners, as applicable, before commencing with research.
7. The researcher is responsible for supplying and utilizing his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institution/s, staff and/or the office/s visited for supplying such resources.
8. The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research title, report or summary.
9. On completion of the study the researcher must supply the Director: Education Research and Knowledge Management, with electronic copies of the Research Report, Thesis, Dissertation as well as a Research Summary (on the GDE Summary template). Failure to submit your Research Report, Thesis, Dissertation and Research Summary on completion of your studies / project – a month after graduation or project completion - may result in permission being withheld from you and your Supervisor in future.
10. The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned;
11. Should the researcher have been involved with research at a school and/or a district/head office level, the Director/s and school/s concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

....

Dr David Makhado

Director: Education Research and Knowledge Management

DATE: 2016/11/10 ....

2

Making education a societal priority

Office of the Director: Education Research and Knowledge Management ER&KM)

APPENDIX 8: Tshwane South District Research Support Letter

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

Enquiries: S.G Ngwenya
Tel: (012) 401 6331/ 012 401 6385
Fax (012) 401 6386
Email: Sello.Ngwenya@gauteng.gov.za
Ref: Private Sector Partnerships (14/1/1/1)

RESEARCH

TO: **MR MPHUTHI G (RESEARCHER)**

FROM: **MRS H E KEKANA**
DIRECTOR: TSHWANE SOUTH DISTRICT

DATE: **28 JULY 2016**

SUBJECT: **RESEARCH APPROVAL – MR MPHUTHI G**

Mr G Mphuthi
P.O Box 42693
Boordfontein
0212
Telephone – 012 529 3771
Mobile: 082 884 2716;
E-mail: Emphutg@unisa.ac.za; gmphuthi@absamail.co.za

RESEARCH APPROVAL – MR MPHUTHI G

We have received a letter approving your research from Head Office and we grant you permission to access the research sites subject to the approval of the School Governing Bodies of the targeted schools. The title of your research is "Exploring teachers' mathematical meanings for teaching trigonometric ratios to Grade 10 learners". You are expected to adhere strictly to the conditions given by Head Office Research and Knowledge Management Office.

Kindly communicate with the principal and/or SGB of the schools you have selected for your research regarding your schedule.

Please submit your report including findings and recommendations to the district office and to Head Office Research and Knowledge management Office at least two weeks after concluding your research. You may be requested to participate in the Department of Education's mini-research conference to discuss your findings and recommendations with departmental officials and other researchers.

Page 1 of 2

Making education a societal priority

Office of the District Director: Tshwane South
(Mamelodi/Eersterust/Pretoria East/Pretoria South/Atteridgeville/Laudium)
President Towers building, 265 Pretorius Street, Pretoria, 0002
Private Bag X198, Pretoria, 0001 Tel: (012) 401 6317; Fax: (012) 401 6318
Website: www.education.gpg.gov.za

±

CONDITIONS

1. This programme is a support programme and may not supersede the Departmental programmes, projects and priorities. In case competition for accommodation arises, departmental programmes will take precedence.
2. The safety and wellness of all learners and school staff must be prioritised at all cost.
3. Teaching and learning time may not be interfered with.
4. The Department or schools may not be held liable for any payments incurred in this programme.
5. The programme will be self-sustaining and no costs will be charged to the Education Department at any level, learners and/ or their parents or the schools.
6. The participants may not be exposed to any harm or any life-endangering situations and they may withdraw at any time from participation without giving any reason/s.

Our support will be rescinded in case you contravene the terms and conditions indicated herein or any departmental regulations and laws or if at any point risk to the reputation of the district, the GDE or the education Department becomes imminent.

Yours sincerely

Mrs H.E Kekana
Director: Tshwane South District
Date: 01 /09 /2016

Office of the District Director: Tshwane South
(Mamelodi/Eersterust/Pretoria East/Pretoria South/Atteridgeville/Laudium)
President Towers building, 265 Pretorius Street, Pretoria, 0002
Private Bag X198, Pretoria, 0001 Tel: (012) 401 6317; Fax: (012) 401 6318
Website: www.education.gpg.gov.za

APPENDIX 9: Ethics Clearance Certificate

UNISA college of science, engineering and technology

Dear Gabriel T Mphuthi (7744528)

Date: 2016-01-26

Application number: 2015_CGS/ISTE_013

REQUEST FOR ETHICAL CLEARANCE: (Exploring teachers` mathematical meanings for teaching trigonometric ratios and functions at Grade 10)

The College of Science, Engineering and Technology's (CSET) Research and Ethics Committee has considered the relevant parts of the studies relating to the abovementioned research project and research methodology and is pleased to inform you that ethical clearance is granted for your research study as set out in your proposal and application for ethical clearance.

Therefore, involved parties may also consider ethics approval as granted. However, the permission granted must not be misconstrued as constituting an instruction from the CSET Executive or the CSET CRIC that sampled interviewees (if applicable) are compelled to take part in the research project. All interviewees retain their individual right to decide whether to participate or not.

We trust that the research will be undertaken in a manner that is respectful of the rights and integrity of those who volunteer to participate, as stipulated in the UNISA Research Ethics policy. The policy can be found at the following URL:
http://cm.unisa.ac.za/contents/departments/res_policies/docs/ResearchEthicsPolicy_apprvCounc_21Sept07.pdf

Please note that the ethical clearance is granted for the duration of this project and if you subsequently do a follow-up study that requires the use of a different research instrument, you will have to submit an addendum to this application, explaining the purpose of the follow-up study and attach the new instrument along with a comprehensive information document and consent form.

Yours sincerely

Prof Ernest Mnkandla
Chair: College of Science, Engineering and Technology Ethics Sub-Committee

Prof IOG Moche
Executive Dean: College of Science, Engineering and Technology

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2016 -01- 26
OFFICE OF THE EXECUTIVE DEAN
College of Science, Engineering and Technology

University of South Africa
College of Science, Engineering and Technology
The Science Campus
C/o Christiaan de Wet Road and Pioneer Avenue, Florida Park, Roodepoort
Private Bag X6, Florida, 1710
www.unisa.ac.za/cset

UNISA college of science, engineering and technology

APPENDIX 10: Language Editing Certificate

Editing Press

Editorial Certificate

This certificate is to affirm that Editing Press Inc., comprising faculty and postgraduates from the Universities of Oxford and Cambridge, has edited, to the best of its members' abilities, the work entitled

EXPLORING TEACHERS' MATHEMATICAL MEANINGS FOR TEACHING TRIGONOMETRIC RATIOS AND FUNCTIONS TO GRADE 10 LEARNERS IN THE TSHWANE SOUTH DISTRICT OF GAUTENG PROVINCE

by

Gabriel Tshepo Mphuthi

This certificate is issued without prejudice to the author on

3 November 2019

Charles Anderson
Director of Academic Editing
Editing Press Inc.
www.editing.press