IDEALS OF FUNCTION RINGS ASSOCIATED WITH SUBLOCALES

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Declaration

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I declare that *Ideals of Function Rings Associated with Sublocales* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references. This thesis has not been submitted for examination or degree at any other university.

Signed:  Dorca Nyamusi Stephen Date:15.03.2021

Abstract

The ring of real-valued continuous functions on a completely regular frame L is denoted by $\mathcal{R}L$. As usual, βL denotes the Stone-Čech compactification of L. In the thesis we study ideals of $\mathcal{R}L$ induced by sublocales of βL. We revisit the notion of purity in this ring and use it to characterize basically disconnected frames. The socle of the ring $\mathcal{R}L$ is characterized as an ideal induced by the sublocale of βL which is the join of all nowhere dense sublocales of βL.

A localic map $f\colon L \to M$ induces a ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ by composition, where $h\colon M \to L$ is the left adjoint of f. We explore how the sublocale-induced ideals travel along the ring homomorphism $\mathcal{R}h$, to and fro, via expansion and contraction, respectively.

The socle of a ring is the sum of its minimal ideals. In the literature, the socle of $\mathcal{R}L$ has been characterized in terms of atoms. Since atoms do not always exist in frames, it is better to express the socle in terms of entities that exist in every frame. In the thesis we characterize the socle as one of the types of ideals induced by sublocales.

A classical operator invented by Gillman, Henriksen and Jerison in 1954 is used to create a homomorphism of quantales. The frames in which every cozero element is complemented (they are called P-frames) are characterized in terms of some properties of this quantale homomorphism. Also characterized within the category of quantales are localic analogues of the continuous maps of R.G. Woods that characterize normality in the category of Tychonoff spaces.

Keywords Frame, locale, sublocale, ideal, quantale, ring of real-valued continuous functions.

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Dedication

To my son, Steve Nyarige, I LOVE YOU.

Chapter 1

Introduction and preliminaries

Then main aim in this chapter is to recall the concepts that will be central to the study in the rest of the thesis. We however do not recall all the concepts here; others will be recalled as and when needed. It is also here that we fix notation. Our references for frames are [29] and [35].

1.1 A brief history of ideals induced by sublocales

For a Tychonoff space X, the ideals $\boldsymbol{O}^p$ and $\boldsymbol{M}^p$ associated with a point $p \in \beta X$ are studied in detail in the Gillman-Jerison text [24]. They appear to first have been considered by Gillman, Henriksen and Jerison in [23].

In their study of functions with compact support in [28], Johnson and Mandelker generalized these types of ideals so that they are indexed by all subsets (instead of just points, or singleton) of βX. In [18], Dube extended the idea of Johnson and Mandelker to locales. With each sublocale A of βL, he defined the ideals $\boldsymbol{M}^A$ and $\boldsymbol{O}^A$ by

$$\boldsymbol{M}^A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathrm{cl}_{\beta L}(r_L(\mathrm{coz}\,\alpha))\}$$

and

$$\boldsymbol{O}^A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathrm{int}_{\beta L}\,\mathrm{cl}_{\beta L}(r_L(\mathrm{coz}\,\alpha))\},$$

where r_L denotes the right adjoint of the join map $\bigvee\colon \beta L \to L$. He used these ideals mainly for purposes of studying what he called P-sublocales, but he did not explore the properties of these ideals that we do in the thesis.

1.2 Synopsis of the thesis

In Chapter 1 we recall most of the background material from frames and locales that we shall need for the rest of the thesis. There are no proofs in this chapter because what appears in it is already available in the literature. It is also in this chapter that we fix notation.

All our frames in the thesis are assumed to be completely regular, except in a few instances where we explicitly state that complete regularity is not assumed. Similarly, all spaces are Tychonoff, which is to say they are completely regular and Hausdorff. We start Chapter 2 by recalling how the $\boldsymbol{O}$- and $\boldsymbol{M}$-ideals of the ring $\mathcal{R}L$ of real-valued continuous functions on a frame L are defined. As done in the paper where these ideals were introduced, we use the same notation $\boldsymbol{O}^A$ and $\boldsymbol{M}^A$ as in spaces. There is no danger of confusion because the index (which is the superscript, in this case) makes its clear where the ideal resides.

Although the ring $C(X)$ of real-valued continuous functions on a Tychonoff space X is isomorphic to $\mathcal{R}(\Omega(X))$ via the isomorphism $\varphi_X\colon C(X) \to \mathcal{R}(\Omega(X))$ that sends an $f \in C(X)$ to the element of $\mathcal{R}(\Omega(X))$ that maps as f^{-1}, it is not immediate what, for a subset $A \subseteq \beta X$, the image of the ideal of $C(X)$ associated with A looks like. In Chapter 2 we show how the $\boldsymbol{O}$- and $\boldsymbol{M}$-ideals of $C(X)$ are related to those of $\mathcal{R}(\Omega(X))$ via this ring isomorphism. We also consider some basic properties of these ideals.

Still within Chapter 2, we revisit purity of ideals of $\mathcal{R}L$. It is apposite to mention that pure ideals of $\mathcal{R}L$ were shown in [16] to be precisely the $\boldsymbol{O}$-ideals associated with closed sublocales of βL. Since different sublocales of βL can induce the same ideal, it does not mean that a non-closed sublocale cannot induce a pure ideal. We characterize when an arbitrary sublocale induces a pure ideal. A closed sublocale of L need not be closed in βL, and hence need not induce a pure ideal. We characterize the fames L for which every closed sublocale of L induces a pure ideal. The chapter closes with a characterization of basically disconnected frames via purity.

In Chapter 3 we study how an ideal associated with a sublocale travels forwards and backwards across a ring homomorphism induced by a localic map. More precisely, let $f\colon L \to M$ be a localic map and let $h\colon M \to L$ be its left adjoint. Then, exactly as in spaces, f has a Stone extension $\beta f\colon \beta L \to \beta M$. So, given a sublocale A of βL, we have its direct image $\beta f[A]$ which is a sublocale of βM. Thus, we have the ideals $\boldsymbol{O}^{\beta f[A]}$ and $\boldsymbol{M}^{\beta f[A]}$ of $\mathcal{R}M$. Oppositely, given a sublocale B of βM, we have its pullback $(\beta f)_{-1}[B]$ which is a sublocale of βL, and the ideals of $\mathcal{R}L$ associated with this sublocale.

For the ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$, we have the ideal $\mathcal{R}h^{-1}[\boldsymbol{O}^A]$ which we then compare with the ideal $\boldsymbol{O}^{\beta f[A]}$, and similarly for the $\boldsymbol{M}$-ideals. On the other hand, we compare the ideal $\boldsymbol{O}^{(\beta f)_{-1}[B]}$ to the ideal of $\mathcal{R}L$ generated by $\mathcal{R}h[\boldsymbol{O}^B]$, and similarly for the $\boldsymbol{M}$-ideals. There are (somewhat) expected containments and some rather surprising inequalities. Regarding equalities, the localic versions of functions that were introduced by R.G. Woods [41] in his study of normality in Tychonoff spaces play a rather unexpected role. In this regard, we have actually started the chapter by developing some results concerning such localic maps. In the last section of the chapter, the localic results are interpreted in $C(X)$.

The main gist of Chapter 4 is about a new look at the socle of $\mathcal{R}L$. Unlike in previous papers such as Dube [15] where the socle of $\mathcal{R}L$ was first was studied, here we show that it equals the sublocale-induced ideal $\boldsymbol{O}^{\mathrm{Nd}(\beta L)}$, where $\mathrm{Nd}(\beta L)$ denotes the join of all nowhere dense sublocales of βL. Thus characterized, it is then easy to give criteria, in terms of sublocales, of when the socle is zero and also for when it is an essential ideal. The latter is best achieved by computing its annihilator. It is for this reason that annihilation of ideals is treated first within the chapter.

In Chapter 5, we show how an operator that was introduced by Gillman, Henriksen and Jerison in 1954 [23] can be used in our context to create a quantale homomorphism. To recall, for any ideal I of $C(X)$, the authors of [23] set

$$\Delta(I) = \bigcap\{\mathrm{cl}_{\beta X}\, Z(f) \mid f \in I\},$$

and use this merely as a notation of convenience. We show how to make a similarly defined Δ, with domain the lattice of ideals of $\mathcal{R}L$ and codomain the frame $\mathcal{S}(\beta L)^{\mathrm{op}}$ of sublocales of βL a homomorphism of quantales. We study the ramifications of this, and, along the way, characterize P-frames using an adjunction that arises from the quantale homomorphism. We also characterize some of the localic versions Woods' maps mentioned earlier within the category of quantales.

1.3 Frames and their homomorphisms

A *frame* is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for every $a \in L$ and $S \subseteq L$. We denote the bottom element and the top element of L by 0_L and 1_L, respectively. We drop the subscript if it is not necessary to specify the frame. If X is a topological space, the frame of its open sets is denoted by $\Omega(X)$.

A mapping $h\colon M \to L$ between frames is called a *frame homomorphism* if it preserves joins and binary meets. In particular, frame homomorphisms preserve the top and the bottom elements. The category of frames and their homomorphisms is denoted **Frm**.

Every frame homomorphism $h\colon M \to L$ has a *right adjoint*, which is a mapping $h_*\colon L \to M$ given by

$$h_*(a) = \bigvee \{u \in M \mid h(u) \leq a\}.$$

The right adjoint is exactly the categorical right adjoint if h is viewed as a functor between posets. It is thus uniquely determined by

$$h(x) \leq y \qquad \Longleftrightarrow \qquad x \leq h_*(y).$$

Some of the properties of h_* are:

- h is surjective iff $h \circ h_* = \mathrm{id}_L$ iff h_* is injective.
- h is injective iff $h_* \circ h = \mathrm{id}_M$ iff h_* is surjective.

A frame homomorphism $h\colon M \to L$ is called *dense* if, for any $a \in M$, the equality $h(a) = 0$ implies $a = 0$. This is so precisely when $h_*(0) = 0$.

The *pseudocomplement* of an element $a \in L$ is the element

$$a^* = \bigvee \{x \in L \mid a \wedge x = 0\}.$$

The unary operation $(-)^*$ satisfies several properties, including, for every $a, b \in L$ and any family $(a_i \mid i \in I)$ of elements of L:

$$a \leq b \implies b^* \leq a, \qquad a \leq a^{**}, \qquad \Big(\bigvee_i a_i\Big)^* = \bigwedge_i a_i^*.$$

An element a is said to be:

- *regular* if $a = a^{**}$;
- *complemented* if $a \vee a^* = 1$;
- *dense* if $a^* = 0$.

If every element of L is complemented, then L is called a *Boolean frame.* Boolean frames are precisely the complete Boolean algebras. It is easy to check that a frame is Boolean if and only if each of its elements is regular.

An element $a \in L$ is *rather below* an element $b \in L$, written $a \prec b$, if there exists an $s \in L$ (called a *separating element*) such that

$$a \wedge s = 0 \quad \text{ and } \quad s \vee b = 1.$$

One checks routinely that $a \prec b$ if and only if $a^* \vee b = 1$. If there is a sequence $(x_r \mid r \in \mathbb{Q} \cap [0,1])$ of elements of L, indexed by the rational numbers in the unit interval $[0,1]$, such that $a = x_0$, $b = x_1$ and $x_r \prec x_s$ whenever $r < s$, then a is said to be *completely below* b, written $a \prec\!\!\prec b$. A frame is *regular* (resp. *completely regular*) if each of its elements is the join of those that are rather below (resp. completely below) it. Writing it out for the latter case, L is completely regular if

$$\forall a \in L, \qquad a = \bigvee \{x \in L \mid x \prec\!\!\prec a\}.$$

All frames in this thesis are assumed to be completely regular unless if it is specifically stated otherwise. Likewise, all topological spaces are Tychonoff unless stated otherwise. We write **CRFrm** for the full subcategory of **Frm** consisting of completely regular frames.

A *prime element* of L is an element $p < 1$ such that $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. We denote by $\mathrm{Pt}(L)$ the set of prime elements of L. By the distributive law, $p \in \mathrm{Pt}(L)$ if and only if $p < 1$ and $x \wedge y = p$ implies $x = p$ or $y = p$. In regular frames, prime elements are precisely the elements that are maximal strictly below the top. If X is T_1-space, then

$$\mathrm{Pt}(\Omega(X)) = \big\{X \smallsetminus \{x\} \mid x \in X\big\}.$$

1.4 The ring $\mathcal{R}L$ and the cozero map

Good references for this subsection are [5] and [6]. See also [35, Chapter XIV]. The *frame of reals*, denoted $\mathfrak{L}(\mathbb{R})$, is the (completely regular) frame generated by the ordered pairs (p, q) of rational numbers subject to the relations

(R1) $(p, q) \wedge (s, t) = (p \vee s, q \wedge t)$

(R2) $(p, q) \vee (s, t) = (p, t)$ whenever $p \leq s < q \leq t$

(R3) $(p, q) = \bigvee\{(s, t) \mid p < s < t < q\}$

(R4) $1_{\mathfrak{L}(\mathbb{R})} = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

The ring $\mathcal{R}L$ has as its elements frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, with operations induced by those of $\mathbb{Q}$ as an ℓ-ring. We denote the zero of this ring and its identity by $\mathbf{0}$ and $\mathbf{1}$, respectively. For any Tychonoff space X, $C(X) \cong \mathcal{R}(\Omega(X))$.

The *cozero map* of L is the mapping

$$\operatorname{coz}\colon \mathcal{R}L \to L \quad \text{defined by} \quad \operatorname{coz}\alpha = \alpha(-, 0) \vee \alpha(0, -),$$

where

$$(-, 0) = \bigvee\{(p, 0) \mid p < 0\} \quad \text{and} \quad (0, -) = \bigvee\{(0, q) \mid q > 0\}.$$

The assignment $L \mapsto \mathcal{R}L$ is functorial. For any frame homomorphism $h\colon M \to L$, the ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ is given by $(\mathcal{R}h)(\alpha) = h \circ \alpha$, and satisfies

$$\operatorname{coz}\big((\mathcal{R}h)(\alpha)\big) = h(\operatorname{coz}\alpha)$$

for every $\alpha \in \mathcal{R}M$.

We catalogue in a proposition some of the properties of the cozero map that we shall freely use, sometimes without comment. The proofs can be found in [4].

Proposition 1.4.1. *The following hold for any $\alpha, \beta \in \mathcal{R}L$.*

(1) $\operatorname{coz}\alpha = \operatorname{coz}(\alpha^2)$.

(2) $\operatorname{coz}\alpha = 0$ *iff* $\alpha = \mathbf{0}$.

(3) $\operatorname{coz}\beta = 1$ *iff* β *is invertible in* $\mathcal{R}L$.

(4) $\operatorname{coz}(\alpha\beta) = \operatorname{coz}\alpha \wedge \operatorname{coz}\beta$.

(5) $\operatorname{coz}(\alpha + \beta) \leq \operatorname{coz}\alpha \vee \operatorname{coz}\beta$.

(6) $\operatorname{coz}(\alpha^2 + \beta^2) = \operatorname{coz}\alpha \vee \operatorname{coz}\beta$.

An element $c \in L$ is called a *cozero element* if $c = \operatorname{coz}\gamma$ for some $\gamma \in \mathcal{R}L$. The lattice of all cozero elements of L is called the *cozero part* of L and is denoted by $\operatorname{Coz} L$. It is closed under finite meets and countable joins. Furthermore, if L is completely regular, then, for any $a \in L$,

$$a = \bigvee\{c \in \operatorname{Coz} L \mid c \leq a\} = \bigvee\{c \in \operatorname{Coz} L \mid c \prec\!\!\prec a\}.$$

Here are some characterizations of cozero elements sourced from [6, Proposition 1].

Proposition 1.4.2. *The following are equivalent for any* $a \in L$.

(1) $a \in \operatorname{Coz} L$.

(2) *There is a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* L *such that* $x_n \prec\!\!\prec a$ *for every* n *and* $a = \bigvee_n x_n$.

(3) *There is a sequence* $(a_n)_{n\in\mathbb{N}}$ *in* L *such that* $a_n \prec\!\!\prec a_{n+1}$ *for every* n *and* $a = \bigvee_n a_n$.

The following properties of the completely below relation will be put to good use in many instances. They are also sourced from [6].

Proposition 1.4.3. *The following hold in any completely regular frame* L.

(1) *If* $a \prec\!\!\prec b$, *then there exists* $c \in \operatorname{Coz} L$ *such that* $a \prec\!\!\prec c \prec\!\!\prec b$.

(2) *If* $a \prec\!\!\prec b$, *then there exists* $s \in \operatorname{Coz} L$ *such that* $a \wedge s = 0$ *and* $s \vee b = 1$.

1.5 The Stone-Čech Compactification

A frame L is *compact* if for every $S \subseteq L$ with $\bigvee S = 1$, there is a finite $T \subseteq S$ with $\bigvee T = 1$. Compact completely regular frames form a coreflective subcategory **KCRFrm** of **CRFrm**.

The coreflection of L, denoted βL, is called the *Stone-Čech compactification* of L. One way of constructing it, that we shall adopt throughout, is as follows.

An ideal I of $\operatorname{Coz} L$ is called *completely regular* if for every $u \in I$ there exists $v \in I$ such that $u \prec\!\!\prec v$. Then βL is the lattice of all completely regular ideals of $\operatorname{Coz} L$. The mapping

$$j_L \colon \beta L \to L \quad \text{defined by} \quad j_L(I) = \bigvee I$$

is dense onto, and is the coreflection map to L from compact completely regular frames. Its right adjoint, denoted r_L, is given by

$$r_L(a) = \{c \in \operatorname{Coz} L \mid c \prec\!\!\prec a\}.$$

Every element of βL is expressible as a join (and, in fact, a union) of ideals of the form $r_L(a)$. Namely,

$$I = \bigvee_{u \in I} r_L(u) = \bigcup_{u \in I} r_L(u);$$

the join coinciding with the union because it is a join of a directed collection. Some other properties of the mapping $r_L \colon L \to \beta L$ that we shall frequently use are:

(a) If $a \prec\!\!\prec b$ in L, then $r_L(a) \prec\!\!\prec r_L(b)$ in βL.

(b) If $a \wedge b = 0$, then $r_L(a \vee b) = r_L(a) \vee r_L(b)$.

(c) If $c, d \in \operatorname{Coz} L$, then $r_L(c \vee d) = r_L(c) \vee r_L(d)$.

(d) For any $I \in \beta L$, $I^* = r_L((\bigvee I)^*)$.

Every frame homomorphism $h \colon M \to L$ has the *Stone extension*, which is the unique frame homomorphism $\beta h \colon \beta M \to \beta L$ making the diagram

$$\begin{array}{ccc} \beta M & \xrightarrow{\beta h} & \beta L \\ {\scriptstyle j_M}\downarrow & & \downarrow{\scriptstyle j_L} \\ M & \xrightarrow{h} & L \end{array}$$

commute. Its action on elements I of βM is given by

$$(\beta h)(I) = \{c \in \operatorname{Coz} L \mid c \leq h(u) \text{ for some } u \in I\}.$$

1.6 Sublocales and localic maps

A frame is also called a *locale*, especially when frame homomorphisms are not considered as part of the discussion. Every frame is a Heyting algebra, with the Heyting implication explicitly given by

$$a \to b = \bigvee \{x \in L \mid a \wedge x \leq b\}.$$

A *sublocale* of a frame L is a subset $A \subseteq L$ such that

- for every $A \subseteq S$, $\bigwedge A \in S$, and
- for every $a \in L$ and $s \in S$, $a \to s \in S$.

A sublocale is a frame in its own right, with meets (and hence the partial order) calculated in L. The lattice of all sublocales of L is denoted by $\mathcal{S}(L)$. The meet in this lattice is intersection, and the join of any collection $\{S_i \mid i \in I\} \subseteq \mathcal{S}(L)$ is given by

$$\bigvee_i S_i = \Big\{ \bigwedge M \mid M \subseteq \bigcup_i S_i \Big\}.$$

Partially ordered by inclusion, $\mathcal{S}(L)$ is a *coframe*, which is to say for any $S \in \mathcal{S}(L)$ and any family $(S_i \mid i \in I)$ of sublocales, the distributive law below holds:

$$S \vee \bigwedge_i S_i = \bigwedge_i (S \vee S_i).$$

The smallest sublocale of L is $\{1\}$, and is usually denoted by O. It is called the *void sublocale*. If T and S are sublocales, we say T *misses* S, or T and S are *disjoint*, if $S \cap T = \mathsf{O}$.

A sublocale of L is *complemented* if it has a complement in $\mathcal{S}(L)$. Complemented sublocales are *linear*, meaning that if C is a complemented sublocale, then

$$C \cap \bigvee \{S_i \mid i \in I\} = \bigvee \{C \cap S_i \mid i \in I\},$$

for any family $(S_i \mid i \in I)$ of sublocales. In fact, complemented sublocales are precisely the linear ones. Unlike the lattice of subspaces of a topological space, $\mathcal{S}(L)$ is not always a Boolean algebra. Thus, in general, not every sublocale has a complement. However, every sublocale S has a *supplement* (which is dual to pseudocomplement in frames), denoted $L \smallsetminus S$ or $S^{\#}$, and given by

$$L \smallsetminus S = \bigcap \{T \in \mathcal{S}(L) \mid T \vee S = L\} = \bigvee \{R \in \mathcal{S}(L) \mid R \cap S = \mathsf{O}\}.$$

The *open* and the *closed* sublocales corresponding to each $a \in L$ are, respectively, the sublocales

$$\mathfrak{o}_L(a) = \{a \to x \mid x \in L\} = \{x \mid x = a \to x\} \quad \text{and} \quad \mathfrak{c}_L(a) = {\uparrow} a = \{x \in L \mid x \geq a\}.$$

We shall at times drop the subscript if there is only one frame under discussion. If $a \in \operatorname{Coz} L$, we say $\mathfrak{o}(a)$ is a *cozero-sublocale*, and $\mathfrak{c}(a)$ is a *zero-sublocale.*

Some of the properties of open and closed sublocales that we shall freely use are:

- $\mathfrak{o}(0) = \mathfrak{c}(1) = \mathsf{O}$ and $\mathfrak{o}(1) = \mathfrak{c}(0) = L$.
- $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ iff $a \vee b = 1$.
- $\mathfrak{o}(a) \subseteq \mathfrak{c}(b)$ iff $a \wedge b = 0$.
- $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$ and $\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$.
- $\bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}\Big(\bigvee_i a_i\Big)$ and $\bigcap_i \mathfrak{c}(a_i) = \mathfrak{c}\Big(\bigvee_i a_i\Big)$.

The *closure* of a sublocale S of L, denoted $\overline{S}$ or $\mathrm{cl}_L(S)$, and its *interior*, denoted S° or $\mathrm{int}_L(S)$, are the sublocales

$$\overline{S} = \bigcap\{\mathfrak{c}(a) \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}\Big(\bigwedge S\Big) \quad \text{and} \quad S^\circ = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\} = \mathfrak{o}\Big(\bigwedge(L \smallsetminus S)\Big).$$

In particular, $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$. A sublocale S of L is *dense* if $\overline{S} = L$. This is the case if and only if the bottom element of S is the bottom element of L. Every frame has the smallest dense sublocale, denoted $\mathfrak{B}L$, and called the *Booleanization* of L. As a set,

$$\mathfrak{B}L = \{a \in L \mid a = a^{**}\} = \{b^* \mid b \in L\},$$

and joins in $\mathfrak{B}L$ are given by

$$\bigvee^{\mathfrak{B}L} S = \Big(\bigvee S\Big)^{**}$$

for any $S \subseteq \mathfrak{B}L$. The mapping

$$\mathsf{b}_L \colon L \to \mathfrak{B}L \quad \text{given by} \quad \mathsf{b}_L(x) = x^{**}$$

is a dense onto frame homomorphism, whose right adjoint is the identical embedding $\mathfrak{B}L \rightarrowtail L$.

A mapping $f \colon L \to M$ is called a *localic map* if for every $a \in L$, $b \in M$, and $S \subseteq L$, (L1) $f(\bigwedge S) = \bigwedge f[S]$ (and, in particular, $f(1) = 1$),

(L2) $f(f^*(b) \to a) = b \to f(a)$, and

(L3) $f(a) = 1$ implies $a = 1$.

We shall write $f^*\colon M \to L$ for the left adjoint of f. Of course, f^* is a frame homomorphism, and if h is a frame homomorphism, h_* is a localic map. A localic map $f\colon L \to M$ gives rise to two mappings

$$f[-]\colon \mathcal{S}(L) \to \mathcal{S}(M) \quad \text{and} \quad f_{-1}[-]\colon \mathcal{S}(M) \to \mathcal{S}(L)$$

given by

$$f[S] = \{f(x) \mid x \in S\} \quad \text{and} \quad f_{-1}[T] = \bigvee\{A \in \mathcal{S}(L) \mid A \subseteq f^{-1}[T]\}.$$

We have that $f[-]$ preserves all joins, and $f_{-1}[-]$ preserves all meets (recall that they are intersections) and all binary joins, which then makes the mapping

$$f_{-1}[-]\colon \mathcal{S}(M)^{\mathrm{op}} \to \mathcal{S}(L)^{\mathrm{op}}$$

a frame homomorphism whose right adjoint is the mapping $f[-]$.

For any $b \in M$,

$$f_{-1}[\mathfrak{o}_M(b)] = \mathfrak{o}_L(f^*(b)) \quad \text{and} \quad f_{-1}[\mathfrak{c}_M(b)] = \mathfrak{c}_L(f^*(b)).$$

Remark 1.6.1. Finally, let us mention that we shall consistently write a frame homomorphism as $h\colon M \to L$, with M as domain and L as codomain. The reason is that in the Stone-Čech compactification of L, the frame homomorphism $j_L\colon \beta L \to L$ (a mapping which plays a most crucial role here) maps into L, and the corresponding localic map $r_L\colon L \to \beta L$ maps out of L. Since most results we refer to in the literature have homomorphisms $L \to M$, we trust that the reader will note the swopping of the domain and codomain.

Chapter 2

Ideals induced by sublocales

Our aim in this chapter is to show how the ideals of $C(X)$ associated with subspaces of βX are related to those of $\mathcal{R}L$ associated with sublocales of βL.

This will be useful in subsequent chapters. We also present some basic properties of the latter types of ideals, and undertake a detailed study of purity in the ring $\mathcal{R}L$.

2.1 Relating the $\boldsymbol{O}$- and $\boldsymbol{M}$-ideals of $C(X)$ to those of $\mathcal{R}(\Omega(X))$

In this section we recall the definitions of $\boldsymbol{O}$- and $\boldsymbol{M}$-ideals of $C(X)$, and relate them to those of $\mathcal{R}(\Omega(X))$. The ideal $\boldsymbol{O}^A$ is defined by

$$\boldsymbol{O}^A = \{f \in C(X) \mid A \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f)\}.$$

These ideals are special cases of the ideals $\boldsymbol{O}^p$, for $p \in \beta X$, that are studied extensively in [24]. They were introduced in [28], where the authors also define the ideal

$$\boldsymbol{M}^A = \{f \in C(X) \mid A \subseteq \operatorname{cl}_{\beta X} Z(f)\}.$$

It is easy to see that $\boldsymbol{O}^A = \bigcap_{p\in A} \boldsymbol{O}^p$ and $\boldsymbol{M}^A = \bigcap_{p\in A} \boldsymbol{M}^p$.

Taking a cue from this, in [18] the author defines for each sublocale A of βL the ideals $\boldsymbol{O}^A$ and

$\boldsymbol{M}^A$ of $\mathcal{R}L$ by setting

$$\boldsymbol{O}^A = \{\alpha \in \mathcal{R}L \mid A \subseteq \operatorname{int}_{\beta L} \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha))\} = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)\}$$

and

$$\boldsymbol{M}^A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha))\}.$$

It is clear that for any sublocale A of βL, $\boldsymbol{O}^A \subseteq \boldsymbol{M}^A = \boldsymbol{M}^{\overline{A}}$, and for any sublocales A and B of βL with $A \subseteq B$, $\boldsymbol{O}^B \subseteq \boldsymbol{O}^A$, and similarly for the $\boldsymbol{M}$-ideals. Furthermore, for any family $(A_k \mid k \in K)$ of sublocales of βL,

$$\boldsymbol{O}^{\bigvee_{k\in K} A_k} = \bigcap_{k\in K} \boldsymbol{O}^{A_k} \qquad \text{and} \qquad \boldsymbol{M}^{\bigvee_{k\in K} A_k} = \bigcap_{k\in K} \boldsymbol{M}^{A_k}.$$

Note that if A is a closed sublocale of βL, say $A = \mathfrak{c}_{\beta L}(I)$ for some $I \in \beta L$, then

$$\boldsymbol{O}^A = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \in I\} = \operatorname{coz}^{-1}[I] \qquad \text{and} \qquad \boldsymbol{M}^A = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\}.$$

We comment that usage of the same symbols $\boldsymbol{M}$ and $\boldsymbol{O}$ in both $C(X)$ and $\mathcal{R}L$ will not lead to confusion because the superscript will always make the context clear.

Recall that a *point* of a frame L is an element p such that $p < 1$ and whenever $x \wedge y \leq p$ then $x \leq p$ or $y \leq p$. We denote by $\operatorname{Pt}(L)$ the set of all points of L. A *one-point* sublocale of L is a sublocale of the form $\{p, 1\}$ for some $p \in \operatorname{Pt}(L)$. Let X be a Tychonoff space. We use the notation of [35] that if $A \subseteq X$ and $x \in X$, then $\widetilde{A}$ denotes the sublocale of $\Omega(X)$ induced by A, and $\tilde{x}$ denotes the point $X \smallsetminus \{x\}$ of $\Omega(X)$. Since Tychonoff spaces are sober and satisfy the T_D-axiom, every spatial sublocale of $\Omega(X)$ is of the form $\widetilde{A}$ for some $A \subseteq X$. Furthermore, for any $A \subseteq X$, $\widetilde{A}$ is the join of its one-point sublocales; that is,

$$\widetilde{A} = \bigvee \{\{\tilde{x}, 1\} \mid x \in A\}.$$

We recall from [5] that if X is a Tychonoff space, then the map

$$\varphi_X : C(X) \to \mathcal{R}(\Omega(X)) \qquad \text{given by} \qquad \varphi_X(f) = \Omega(f)$$

is a ring isomorphism. For use below, we relate the $\boldsymbol{O}$- and $\boldsymbol{M}$-ideals of $C(X)$ to those of $\mathcal{R}(\Omega(X))$ via this isomorphism. Usage of the same symbols in $C(X)$ and $\mathcal{R}(\Omega(X))$ will not lead to confusion because the superscript makes the context clear. We view X as a subspace of βX

and consider the identical embedding $i_X\colon X \to \beta X$. The right adjoint of the induced frame homomorphism $\Omega(i_X)\colon \Omega(\beta X) \to \Omega X$ maps thus:

$$\Omega(i_X)_*(U) = \beta X \smallsetminus \mathrm{cl}_{\beta X}(X \smallsetminus U).$$

Since $\Omega(\beta X)$ is a compact regular frame, we can view the Stone-Čech compactification of the frame $\Omega(X)$ as being given by the dense-onto frame homomorphism $\Omega(i_X)\colon \Omega(\beta X) \to \Omega(X)$. So, in the r_L-notation for the right adjoint of $\beta L \to L$, we have $r_{\Omega(X)} = \Omega(i_X)_*$.

For any $p \in \beta X$, the one-point sublocale $\{\tilde{p}, 1\}$ of $\Omega(\beta X)$ is the closed sublocale $\mathfrak{c}_{\Omega(\beta X)}(\tilde{p})$, hence, for any $\alpha \in \mathcal{R}(\Omega(X))$,

$$\alpha \in \boldsymbol{M}^{\{\tilde{p},1\}} \quad \text{iff} \quad \Omega(i_X)_*(\mathrm{coz}\,\alpha) \subseteq \tilde{p}.$$

Lemma 2.1.1. *For any $p \in \beta X$, $\varphi_X[\boldsymbol{M}^p] = \boldsymbol{M}^{\{\tilde{p},1\}}$.*

Proof. Let $f \in C(X)$, and note that $\mathrm{coz}(\Omega(f)) = f^{-1}(\mathbb{R} \smallsetminus \{0\}) = X \smallsetminus Z(f)$. Thus, in view of the definition of the ideal $\boldsymbol{M}^p$, we have

$$\begin{aligned} f \in \boldsymbol{M}^p \quad &\text{iff} \quad p \in \mathrm{cl}_{\beta X} Z(f) \\ &\text{iff} \quad \beta X \smallsetminus \mathrm{cl}_{\beta X} Z(f) \subseteq \beta X \smallsetminus \{p\} \\ &\text{iff} \quad \Omega(i_X)_*(\mathrm{coz}(\varphi_X(f))) \subseteq \tilde{p} \\ &\text{iff} \quad \varphi_X(f) \in \boldsymbol{M}^{\{\tilde{p},1\}}, \end{aligned}$$

which proves the result. □

In [14, Lemma 5.3(2)] it is shown that, exactly as in $C(X)$, for any frame L, any $\alpha \in \mathcal{R}L$, and any $\mathfrak{p} \in \mathrm{Pt}(\beta L)$, $\alpha \in \boldsymbol{O}^{\{\mathfrak{p},1\}}$ if and only if $\alpha\gamma = \mathbf{0}$ for some $\gamma \notin \boldsymbol{M}^{\{\mathfrak{p},1\}}$. We therefore have the following corollary, because $\varphi_X\colon C(X) \to \mathcal{R}(\Omega(X))$ is an isomorphism.

Corollary 2.1.2. *For any $p \in \beta X$, $\varphi_X[\boldsymbol{O}^p] = \boldsymbol{O}^{\{\tilde{p},1\}}$.*

Coming to ideals associated with subspaces, we have the following.

Corollary 2.1.3. *For any $A \subseteq \beta X$, $\varphi_X[\boldsymbol{M}^A] = \boldsymbol{M}^{\widetilde{A}}$ and $\varphi_X[\boldsymbol{O}^A] = \boldsymbol{O}^{\widetilde{A}}$.*

Proof. Since $\boldsymbol{M}^A = \bigcap_{p\in A} \boldsymbol{M}^p$, and since the set-function φ_X is a bijection, we have

$$\varphi_X[\boldsymbol{M}^A] = \varphi_X\Big[\bigcap_{p\in A} \boldsymbol{M}^p\Big] = \bigcap_{p\in A} \varphi_X[\boldsymbol{M}^p] = \bigcap_{p\in A} \boldsymbol{M}^{\{\tilde{p},1\}} = \boldsymbol{M}^{\bigvee\{\{\tilde{p},1\}\mid p\in A\}} = \boldsymbol{M}^{\tilde{A}}.$$

The other equality is shown similarly. □

2.2 Basic properties

The $\boldsymbol{O}$-ideals and $\boldsymbol{M}$-ideals come with a host of containments that always hold. For instance, for any sublocale A of βL and any sublocale S of L, we always have the containments

$$\boldsymbol{O}^A \subseteq \boldsymbol{M}^A, \quad \boldsymbol{O}^{\bar{A}} \subseteq \boldsymbol{O}^A \subseteq \boldsymbol{O}^{A^\circ}, \quad \boldsymbol{M}^{A^\circ} \subseteq \boldsymbol{M}^A$$

and, similarly,

$$\boldsymbol{O}_S \subseteq \boldsymbol{M}_S, \quad \boldsymbol{O}_{\mathrm{cl}_L(S)} \subseteq \boldsymbol{O}_S \subseteq \boldsymbol{O}_{\mathrm{int}_L(S)}, \quad \boldsymbol{M}_{\mathrm{int}_L(S)} \subseteq \boldsymbol{M}_S,$$

which is achieved via the following definition. Please note the slight change in notation – sublocales appearing as subscripts and not superscripts.

Definition 2.2.1. For any sublocale S of L, we define the ideals $\boldsymbol{O}_S$ and $\boldsymbol{M}_S$ of $\mathcal{R}L$ to be the ideals $\boldsymbol{O}_S = \boldsymbol{O}^{r_L[S]}$ and $\boldsymbol{M}_S = \boldsymbol{M}^{r_L[S]}$.

In this section we explore a little more the consequences of requiring some containments that always hold to be actually equalities. As has been demonstrated elsewhere (in spaces and in locales), this is not a gratuitous exploration. For instance:

(a) In [33], Mandelker defines a subset A of βX to be *round* in case $\boldsymbol{O}^A = \boldsymbol{M}^A$, and then develops an interesting theory around round subsets.

(b) In [18], Dube defines a closed sublocale A of L to be a *P-sublocale* if $\boldsymbol{M}_A = \boldsymbol{O}_A$. He then goes on to show that for basically disconnected frames these P-sublocales have some rather unexpected properties.

We shall see in the next section that the containment $\boldsymbol{O}^{\bar{A}} \subseteq \boldsymbol{O}^A$ is an equality precisely when the ideal $\boldsymbol{O}^A$ is pure. This however does not tell us about the localic properties of sublocales A for which $\boldsymbol{O}^A = \boldsymbol{O}^{\bar{A}}$.

Observation 2.2.2. If U is an open sublocale of βL, then $\boldsymbol{O}^U = \boldsymbol{O}^{\overline{U}^\circ} = \boldsymbol{M}^{\overline{U}^\circ} = \boldsymbol{M}^U$.

Proof. Pick $I \in \beta L$ such that $U = \mathfrak{o}(I)$. Now, for any $\alpha \in \mathcal{R}L$, we have

$$\begin{aligned} \alpha \in \boldsymbol{M}^{\mathfrak{o}(I)} \quad &\text{iff} \quad \mathfrak{o}(I) \subseteq \mathfrak{c}(r_L(\operatorname{coz}\alpha)) \\ &\text{iff} \quad I \wedge r_L(\operatorname{coz}\alpha) = 0_{\beta L} \\ &\text{iff} \quad I^{**} \wedge r_L(\operatorname{coz}\alpha) = 0_{\beta L} \\ &\text{iff} \quad \mathfrak{o}(I^{**}) \subseteq \mathfrak{c}(r_L(\operatorname{coz}\alpha)) \\ &\text{iff} \quad \alpha \in \boldsymbol{M}^{\mathfrak{o}(I^{**})}, \end{aligned}$$

which then says $\boldsymbol{M}^U = \boldsymbol{M}^{\overline{U}^\circ}$. The rest follows because, for any open sublocale V of βL, $\boldsymbol{O}^V = \boldsymbol{M}^V$, as one checks easily. □

In part of the proof of the first proposition we shall use the fact that if I and J are elements of βL with $I \prec J$, then $\bigvee I \in J$ (see [14, p. 156]).

As we shall shortly see, this fails for open sublocales. Also, the $\boldsymbol{O}$-version of the equivalence in the preceding paragraph is false; only one implication holds. In the proof that follows, we shall use the notion of *P-element.* To recall, an element $a \in L$ is called a *P-element* if the associated closed sublocale is a P-sublocale.

Proposition 2.2.3. *Let L be a completely regular frame.*

(a) *If A is a closed sublocale of βL with $\boldsymbol{O}^A = \boldsymbol{O}^{A^\circ}$, then A is regular-closed. The converse does not hold.*

(b) *If A is a closed sublocale of L with $\boldsymbol{O}_A = \boldsymbol{O}_{\operatorname{int}_L(A)}$, then A is regular-closed. The converse does not hold.*

(c) *If U is an open sublocale of βL, then $\boldsymbol{O}^U = \boldsymbol{O}^{\overline{U}}$ iff $\overline{U}$ is a round sublocale of βL.*

(d) *If U is an open sublocale of L, then $\boldsymbol{O}_U = \boldsymbol{O}_{\operatorname{cl}_L(U)}$ iff $\operatorname{cl}_L(U)$ is a P-sublocale of L.*

Proof. (a) Pick $I \in \beta L$ with $A = \mathfrak{c}_{\beta L}(I)$. Then $A^\circ = \mathfrak{o}_{\beta L}(I^*)$. Let $c \in I^{**}$. Pick $\gamma \in \mathcal{R}L$ such that $c = \operatorname{coz}\gamma$. We show that $\gamma \in \boldsymbol{O}^{A^\circ}$. Since $I^* \wedge I^{**} = 0_{\beta L}$ and since $r_L(c) \leq I^{**}$, we have $I^* \wedge r_L(c) = 0_{\beta L}$, which implies $I^* \leq r_L(c^*)$, and hence $\mathfrak{o}_{\beta L}(I^*) \subseteq \mathfrak{o}_{\beta L}(r_L(c^*))$. This containment implies $\gamma \in \boldsymbol{O}^{A^\circ}$. Therefore, by hypothesis, $\gamma \in \boldsymbol{O}^A$, so that $\mathfrak{c}_{\beta L}(I) \subseteq \mathfrak{o}_{\beta L}(r_L(c^*))$, whence

$$r_L(c)^* \vee I = r_L(c^*) \vee I = 1_{\beta L},$$

that is, $r_L(c) \prec I$, and hence $c \in I$ because $\bigvee r_L(c) = c$. Therefore $I^{**} \subseteq I$, and hence $I = I^{**}$. Therefore A is a regular-closed sublocale of βL.

(b) Pick $a \in L$ such that $A = \mathfrak{c}(a)$. Let $\gamma \in \mathcal{R}L$ be such that $\operatorname{coz}\gamma \le a^{**}$. We show that $\gamma \in \boldsymbol{O}_{\operatorname{int}_L(A)}$. Recall that $\operatorname{int}_L(\mathfrak{c}(a)) = \mathfrak{o}(a^*)$. Since $\operatorname{coz}\gamma \le a^{**}$, we have $a^* \le (\operatorname{coz}\gamma)^*$, so that $\mathfrak{o}(a^*) \subseteq \mathfrak{o}((\operatorname{coz}\gamma)^*)$, implying $\gamma \in \boldsymbol{O}_{\operatorname{int}_L(A)}$. So, by hypothesis, $\gamma \in \boldsymbol{O}_{\mathfrak{c}(a)}$, which says $\mathfrak{c}(a) \subseteq \mathfrak{o}((\operatorname{coz}\gamma)^*)$, so that $a \vee (\operatorname{coz}\gamma)^* = 1$, and hence $\operatorname{coz}\gamma \le a$. Thus, by complete regularity, $a^{**} \le a$, and hence $a = a^{**}$. This shows that A is regular-closed.

(c) Assume, first, that $\boldsymbol{O}^U = \boldsymbol{O}^{\overline{U}}$. Then

$$\boldsymbol{O}^{\overline{U}} = \boldsymbol{O}^U = \boldsymbol{M}^U = \boldsymbol{M}^{\overline{U}},$$

which shows that $\overline{U}$ is a round sublocale of βL. Conversely, if $\overline{U}$ is round, then $\boldsymbol{O}^{\overline{U}} = \boldsymbol{M}^{\overline{U}}$, and so,

$$\boldsymbol{O}^{\overline{U}} \subseteq \boldsymbol{O}^U = \boldsymbol{M}^U = \boldsymbol{M}^{\overline{U}} = \boldsymbol{O}^{\overline{U}},$$

proving the result.

(d) Assume, first, that $\boldsymbol{O}_U = \boldsymbol{O}_{\operatorname{cl}_L(U)}$. Pick $u \in L$ such that $U = \mathfrak{o}_L(u)$, so that $\operatorname{cl}_L(U) = \mathfrak{c}_L(u^*)$. To show that $\operatorname{cl}_L(U)$ is a P-sublocale of L, it suffices to show that u^* is a P-element of L. Consider any $c \in \operatorname{Coz} L$ with $c \le u^*$, and pick $\gamma \in \mathcal{R}L$ such that $c = \operatorname{coz}\gamma$. Then $u \le u^{**} \le (\operatorname{coz}\gamma)^*$, and so $\mathfrak{o}_L(u) \subseteq \mathfrak{o}_L((\operatorname{coz}\gamma)^*)$, which says $\gamma \in \boldsymbol{O}_{\mathfrak{o}_L(u)}$, and so, by hypothesis, $\gamma \in \boldsymbol{O}_{\mathfrak{c}_L(u^*)}$. Thus, $\mathfrak{c}_L(u^*) \subseteq \mathfrak{o}_L(c^*)$, which implies $c^* \vee u^* = 1$, that is, $c \prec u^*$. Therefore u^* is a P-element, and hence $\operatorname{cl}_L(U)$ is a P-sublocale of L.

Conversely, assume that $\operatorname{cl}_L(U)$ is a P-sublocale of L, and consider any $\gamma \in \boldsymbol{O}_{\mathfrak{o}_L(u)}$. Then $\boldsymbol{O}_{\mathfrak{o}_L(u)} \subseteq \boldsymbol{O}_{\mathfrak{o}_L((\operatorname{coz}\gamma)^*)}$, and so $u \le (\operatorname{coz}\gamma)^*$, from which we get $\operatorname{coz}\gamma \le u^*$. Since u^* is a P-element as $\mathfrak{c}_L(u^*)$ is a P-sublocale, we have $\operatorname{coz}\gamma \prec u^*$, and so $(\operatorname{coz}\gamma)^* \vee u^* = 1$, which implies $\mathfrak{c}_L(u^*) \subseteq \mathfrak{o}_L((\operatorname{coz}\gamma)^*)$, whence $\gamma \in \boldsymbol{O}_{\mathfrak{c}_L(u^*)}$. Therefore $\boldsymbol{O}_U \subseteq \boldsymbol{O}_{\operatorname{cl}_L(U)}$, and hence $\boldsymbol{O}_U = \boldsymbol{O}_{\operatorname{cl}_L(U)}$ since the opposite inclusion always holds. □

Here is an example showing that the converse to part (a) of this proposition does not hold.

Example 2.2.4. Let $L = \Omega(\mathbb{R})$, and consider the element $a = (0,1)$ of L. Since $a = a^{**}$, $r_L(a) = r_L(a^{**}) = r_L(a)^{**}$, and so the closed sublocale $A = \mathfrak{c}_{\beta L}(r_L(a))$ of βL is a regular-closed. Since every open set in $\mathbb{R}$ is a cozero-set, $a \in \operatorname{Coz} L$, and so there exists some $\alpha \in \mathcal{R}L$ such that

$a = \operatorname{coz}\alpha$. Now, $\alpha \in \boldsymbol{O}^{A^\circ}$ since $A^\circ = \mathfrak{o}_{\beta L}(r_L(a^*)) = \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)$. On the other hand, $\alpha \notin \boldsymbol{O}^A$, otherwise we would have $\mathfrak{c}_{\beta L}(r_L(a)) \subseteq \mathfrak{o}_{\beta L}(r_L(a^*))$, which would imply $r_L(a) \vee r_L(a^*) = 1_{\beta L}$, whence we would have $a \vee a^* = 1$, which is false because $a \vee a^* = (0,1) \cup \big((-\infty,0)\cup(0,\infty)\big) \neq \mathbb{R}$.

As an application, we have the following characterization of Boolean frames.

Corollary 2.2.5. *The following are equivalent for a frame L.*

(1) *L is Boolean.*

(2) *$\boldsymbol{O}_A = \boldsymbol{O}_{\operatorname{int}_L(A)}$ for every sublocale A of L.*

(3) *$\boldsymbol{O}_A = \boldsymbol{O}_{\operatorname{int}_L(A)}$ for every closed sublocale A of L.*

Proof. If L is Boolean, then every sublocale of L is open, and hence equals its interior. Therefore (1) implies (2). It is trivial that (2) implies (3). If (3) holds, then the proposition says every closed sublocale of L is regular-closed, which says $a = a^{**}$ for every $a \in L$, and this is known to be equivalent to L being Boolean. □

In [9, Theorem 2.6], the author proves that if A is a closed subset of βX, then the ideal $\boldsymbol{M}^A$ is finitely generated if and only if A is open. We have the following similar result, but for the ideal $\boldsymbol{O}^A$. Recall that r_L preserves disjoint binary joins.

Theorem 2.2.6. *The following are equivalent for a closed sublocale A of βL.*

(1) *$\boldsymbol{O}^A$ is finitely generated.*

(2) *$\boldsymbol{O}^A$ is a principal ideal generated by an idempotent.*

(3) *A is open.*

Proof. (1) ⇒ (3): Take $I \in \beta L$ such that $A = \mathfrak{c}_{\beta L}(I)$. Suppose that there are finitely many elements $\alpha_1, \dots, \alpha_n$ in $\mathcal{R}L$ such that $\boldsymbol{O}^A = \langle \alpha_1, \dots, \alpha_n \rangle$. By [13, Lemma 4.4],

$$\bigvee\{\operatorname{coz}\alpha \mid \alpha \in \boldsymbol{O}^A\} = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in I\} = \operatorname{coz}\alpha_1 \vee \cdots \vee \operatorname{coz}\alpha_n.$$

Set $\alpha_0 = \alpha_1^2 + \cdots + \alpha_n^2$. Since $\alpha_0 \in \boldsymbol{O}^A$, we have $\operatorname{coz}\alpha_0 \in I$, and so, $\operatorname{coz}\alpha_0 = \bigvee I \in I$. Since I is a regular ideal of $\operatorname{Coz} L$, there exists $d \in I$ such that $\operatorname{coz}\alpha_0 \prec\!\!\prec d$. Since $\operatorname{coz}\alpha_0 = \bigvee I$, we must have $d = \operatorname{coz}\alpha_0$. This certainly implies $I = r_L(d)$, whence

$$I \vee I^* = r_L(d) \vee r_L(d^*) = r_L(d \vee d^*) = 1_{\beta L}$$

because $d \vee d^* = 1$. This proves that A is clopen.

(3) ⇒ (2): If A is clopen, there exists a complemented element I of βL such that $A = \mathfrak{c}_{\beta L}(I)$. Then $I \in \operatorname{Coz}\beta L$, and so the element $c = \bigvee I$ belongs to $\operatorname{Coz} L$. Furthermore, c is a complemented element of L because frame homomorphisms send complemented elements to complemented elements. By [16, Proposition 2.2], there is an idempotent γ in $\mathcal{R}L$ such that $c = \operatorname{coz}\gamma$. We show that $\boldsymbol{O}^A = \langle\gamma\rangle$. Since $I \prec I$, $\bigvee I \in I$, that is $c \in I$, and therefore $\gamma \in \boldsymbol{O}^A$. Now let $\alpha \in \boldsymbol{O}^A$. Then $\operatorname{coz}\alpha \in I$, and so $\operatorname{coz}\alpha \leq \operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\gamma$, which means that α is a multiple of γ. In all then, $\boldsymbol{O}^A = \langle\gamma\rangle$.

(2) ⇒ (1): This is trivial. □

Remark 2.2.7. The requirement that A be a closed sublocale cannot be relaxed. We shall see in Lemma 4.2.2 of Chapter 4 that the ideal $\boldsymbol{O}^A$ is the zero ideal precisely when A is a dense sublocale. So, for instance, the ideal $\boldsymbol{O}^{\mathfrak{B}(\Omega(\mathbb{R}))}$ of $\mathcal{R}(\Omega(\mathbb{R}))$ is finitely generated, but $\mathfrak{B}(\Omega(\mathbb{R}))$ is not open.

Since $\boldsymbol{M}^A = \boldsymbol{M}^{\bar{A}}$ for any sublocale A of βL, and since, for any $I \in \beta L$,

$$\bigvee\{\operatorname{coz}\alpha \mid \alpha \in \boldsymbol{M}^{\mathfrak{c}_{\beta L}(I)}\} = \bigvee_{\alpha \in I} \operatorname{coz}\alpha$$

also by [16, Proposition 2.2], a proof as the foregoing one, with minor modification, enables us to state the following.

Theorem 2.2.8. *The following are equivalent for a sublocale A of βL.*

(1) $\boldsymbol{M}^A$ *is finitely generated.*

(2) $\boldsymbol{M}^A$ *is a principal ideal generated by an idempotent.*

(3) $\bar{A}$ *is open.*

2.3 Purity revisited

An ideal I of a ring A is *pure* (Johnstone [29] says "neat") if for every $u \in I$ there is a $v \in I$ such that $u = uv$. We denote the set of all pure ideals of A by $\mathrm{PId}(A)$. In [29, Proposition V. 2.8], Johnstone proves that $\mathrm{PId}(A)$ is a frame. The *m-operator* on the set of ideals of A is defined by

$$mI = \{u \in I \mid u = uv \text{ for some } v \in I\}.$$

If Q is an ideal of $\mathcal{R}L$, then

$$mQ = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\gamma \text{ for some } \gamma \in Q\},$$

and Q is pure if and only if $mQ = Q$ [17, Corollary 3.3].

As the heading suggests, our aim in this section is to present further results concerning purity in function rings. Among other things, we present a transparent description of pure ideals of the subring $\mathcal{R}^*L$ of $\mathcal{R}L$ consisting of bounded elements. A thorough search in the literature has revealed that pure ideals of this subring have hitherto not been described.

For the record, in [16] the pure ideals of $\mathcal{R}L$ are fully described as

$$\mathrm{PId}(\mathcal{R}L) = \{\boldsymbol{O}^A \mid A \text{ is a closed sublocale of } \beta L\}.$$

Since for any $I \in \beta L$ and any $\alpha \in \mathcal{R}L$, $\alpha \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)}$ if and only if $\operatorname{coz}\alpha \in I$, it is clear that the mapping $A \mapsto \boldsymbol{O}^A$ is injective on closed sublocales of βL. It is however not necessarily injective on all sublocales of βL (the reader may peak ahead to Lemma 4.2.2 to see that $\boldsymbol{O}^A$ is the zero ideal for any dense sublocale A of βL). So it is possible for $\boldsymbol{O}^A$ to be pure even if A is not a closed sublocale of βL.

The upcoming lemma characterizes when the ideals $\boldsymbol{O}^A$ and $\boldsymbol{M}^A$ are pure. We will put it to good use on a number of occasions. The characterization that $\boldsymbol{O}^A$ is pure if and only if $\boldsymbol{O}^A = \boldsymbol{O}^{\bar{A}}$, which we include as part of this lemma, is also observed in [19].

Lemma 2.3.1. *If A is a sublocale of βL, then $m\boldsymbol{O}^A = m\boldsymbol{M}^A = \boldsymbol{O}^{\bar{A}}$. Hence, $\boldsymbol{O}^A$ is pure iff $\boldsymbol{O}^A = \boldsymbol{O}^{\bar{A}}$, and $\boldsymbol{M}^A$ is pure iff $\boldsymbol{M}^A = \boldsymbol{O}^{\bar{A}}$.*

Proof. Since $\boldsymbol{O}^{\bar{A}} \subseteq \boldsymbol{O}^A \subseteq \boldsymbol{M}^A$, applying the m-operator and keeping in mind that $\boldsymbol{O}^{\bar{A}}$ is pure since $\bar{A}$ is a closed sublocale, we obtain

$$\boldsymbol{O}^{\bar{A}} = m\boldsymbol{O}^{\bar{A}} \subseteq m\boldsymbol{O}^A \subseteq m\boldsymbol{M}^A.$$

Now let $\alpha \in m\boldsymbol{M}^A$, and select $\gamma \in \boldsymbol{M}^A$ with $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\gamma$. Then $r_L(\operatorname{coz}\alpha) \prec r_L(\operatorname{coz}\gamma)$, so that $r_L(\operatorname{coz}\alpha)^* \vee r_L(\operatorname{coz}\gamma) = 1_{\beta L}$, and hence (in light of the fact that $\gamma \in \boldsymbol{M}^A = \boldsymbol{M}^{\bar{A}}$)

$$\bar{A} \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\gamma)) \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*),$$

which implies $\alpha \in \boldsymbol{O}^{\bar{A}}$, showing that $m\boldsymbol{M}^A \subseteq \boldsymbol{O}^{\bar{A}}$, and hence we have the three claimed equalities. The latter assertions follow from this. □

Localic characterizations of sublocales A of βL for which $\boldsymbol{O}^A$ is pure follow. As in spaces, we say a sublocale B is a *neighborhood* of a sublocale A if the interior of B contains A.

Theorem 2.3.2. *The following are equivalent for a sublocale A of βL.*

(1) *$\boldsymbol{O}^A$ is pure.*

(2) *Every zero-sublocale of βL which is a neighborhood of A is also a neighborhood of $\bar{A}$.*

(3) *Whenever A misses the closure of some cozero-sublocale of βL, then $\bar{A}$ also misses the closure of that cozero-sublocale.*

Proof. (2) ⇔ (3): This equivalence follows from the fact that, for any *complemented* sublocale S of any frame M, $\operatorname{int}(M \smallsetminus S) = M \smallsetminus \operatorname{cl} S$, as can be deduced from [21, Eq. (4.3)], and zero-sublocales are exactly the complements of cozero-sublocales.

(1) ⇒ (2): Suppose that $\boldsymbol{O}^A$ is pure, so that $\boldsymbol{O}^A = \boldsymbol{O}^{\bar{A}}$ by Lemma 2.3.1. Consider any $J \in \operatorname{Coz}(\beta L)$ with $A \subseteq \operatorname{int}_{\beta L} \mathfrak{c}_{\beta L}(J)$. Then $A \subseteq \mathfrak{o}_{\beta L}(J^*)$. Since $J \in \operatorname{Coz}(\beta L)$, $\bigvee J \in \operatorname{Coz} L$. Pick $\alpha \in \mathcal{R}L$ such that $\bigvee J = \operatorname{coz}\alpha$. Then $J^* = r_L(\operatorname{coz}\alpha)^*$. Thus, $A \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)$, which implies $\alpha \in \boldsymbol{O}^A$, and hence $\alpha \in \boldsymbol{O}^{\bar{A}}$, by hypothesis, whence $\bar{A} \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*) = \operatorname{int}_{\beta L} \mathfrak{c}_{\beta L}(J)$.

(2) ⇒ (1): Suppose that A has the hypothesized feature. To prove that $\boldsymbol{O}^A$ is pure, we need only show that $\boldsymbol{O}^A \subseteq \boldsymbol{O}^{\bar{A}}$. Let $\alpha \in \boldsymbol{O}^A$. Then $A \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)$. Put $c = \operatorname{coz}\alpha$, and find a sequence $(c_n)_{n\in\mathbb{N}}$ of cozero elements of L such that $c_n \prec\!\!\prec c_{n+1}$ for every n, and $c = \bigvee_n c_n$. Since $r_L(c_n) \prec\!\!\prec r_L(c_{n+1})$ for each n, the element $J = \bigcup_n r_L(c_n)$ is a cozero element of βL with

$$J^* = r_L\Big(\Big(\bigvee J\Big)^*\Big) = r_L\Big(\Big(\bigvee_n c_n\Big)^*\Big) = r_L(c^*).$$

Thus, $A \subseteq \mathfrak{o}_{\beta L}(J^*) = \operatorname{int}_{\beta L} \mathfrak{c}_{\beta L}(J)$. Since $\mathfrak{c}_{\beta L}(J)$ is a zero-sublocale of βL, the hypothesis implies that $\bar{A} \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)$. Therefore $\alpha \in \boldsymbol{O}^{\bar{A}}$, which establishes the desired containment. Therefore $\boldsymbol{O}^A$ is pure, by Lemma 2.3.1. □

Recall from [35, Proposition VI.2.2.1] that each spatial sublocale of a sober space is induced by a subspace. Recall also that complemented sublocales of a spatial locale are spatial [35, Proposition VI.3.3]. Thus, zero-sublocales and cozero-sublocales of βX are precisely the sublocales induced by the zero-sets and cozero-sets of βX, respectively. Since purity is preserved (and therefore also reflected) by ring isomorphisms, we therefore have the following corollary.

Corollary 2.3.3. *For a Tychonoff space X, the following are equivalent for a subset A of βX.*

(1) $\boldsymbol{O}^A$ *is pure.*

(2) *Every zero-set of βX which is a neighborhood of A is also a neighborhood of $\bar{A}$.*

(3) *Whenever A misses the closure of some cozero-set of βX, then $\bar{A}$ also misses the closure of that cozero-set.*

Let us pause for a moment for some bookkeeping. The ideals $\boldsymbol{O}^A$ and $\boldsymbol{M}^A$ are indexed by sublocales of βL. So, when we do not view L as a sublocale of βL, then, strictly speaking, for a sublocale S of L we cannot speak of $\boldsymbol{O}^S$ or $\boldsymbol{M}^S$. It is however desirable to have similar ideals indexed by sublocales of L, in such a way that when we do view L as a sublocale of βL, so that a sublocale of L is then a sublocale of βL, then the two concepts agree.

A pleasant observation from [18] is that, for S a sublocale of L, the ideals $\boldsymbol{O}_S$ and $\boldsymbol{M}_S$ can be described solely in terms of L without invoking βL, as follows:

$$\boldsymbol{O}_S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{o}_L((\operatorname{coz}\alpha)^*)\} \quad \text{and} \quad \boldsymbol{M}_S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{c}_L(\operatorname{coz}\alpha)\}.$$

Now, viewing L as a sublocale of βL, if S is a closed sublocale of L, it does not follow that S is a closed sublocale of βL. We can thus not simply deduce that the ideal $\boldsymbol{O}_S$ is pure. We shall see that if L is normal, then $\boldsymbol{O}_S$ is pure for every closed sublocale S of L. This will be via a characterization of the frames L for which $\boldsymbol{O}_S$ is pure for each closed sublocale S of L. Towards that end, let us say a frame L is *coz-interpolative* if whenever a cozero element of L is rather below some element of L, then it is completely below that element. This strange-sounding name is justified by the fact that the definition says if c is a cozero element and $c \prec a$, then the relation $\prec$ is interpolative between c and a. Normal frames are coz-interpolative. Here is an example of a non-normal coz-interpolative frame.

Example 2.3.4. Let L be a non-normal basically disconnected frame (see, for instance, the space described in [24, Problem 6Q]). Then L is coz-interpolative. To verify this, consider any $c \in \operatorname{Coz} L$ and $a \in L$ with $c \prec a$. Then $c^{**} \prec a$. Since c^{**} is complemented as L is basically disconnected, we have $c \leq c^{**} \prec\!\!\prec a$, as desired.

Before we proceed to the result for which we have introduced coz-interpolative frames, let us observe that for *F-frames* (recall from [4] that these are the frames L such that if $a \wedge b = 0$ in $\operatorname{Coz} L$, then there exist $u, v \in \operatorname{Coz} L$ such that $u \vee v = 1$ and $a \wedge u = b \wedge v = 0$) the concept of being coz-interpolative is expressible in terms of sublocales.

Theorem 2.3.5. *Consider the following conditions on a frame L.*

(1) *L is coz-interpolative.*

(2) *Whenever a zero-sublocale Z of L is a neighborhood of a closed sublocale A of L, there is a cozero-sublocale C of L such that $A \subseteq C \subseteq Z$.*

Condition (1) *implies condition* (2)*, and the two conditions are equivalent if L is an F-frame.*

Proof. Suppose that L is coz-interpolative. Let Z be a zero-sublocale of L which is a neighborhood of a closed sublocale A of L. Pick $c \in \operatorname{Coz} L$ and $a \in L$ with $Z = \mathfrak{c}_L(c)$ and $A = \mathfrak{c}_L(a)$. Then $\mathfrak{c}_L(a) \subseteq \operatorname{int}_L(\mathfrak{c}_L(c))$, which says $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(c^*)$, so that $c^* \vee a = 1$, and hence $c \prec a$. Since L is coz-interpolative, we therefore have $c \prec\!\!\prec a$. By [6, Corollary 3], there is a *cozero* separating element, that is, an $s \in \operatorname{Coz} L$ such that $c \wedge s = 0$ and $s \vee a = 1$. Thus, $\mathfrak{o}_L(c) \cap \mathfrak{o}_L(s) = \mathsf{O}$ and $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(s)$. The former implies $\mathfrak{o}_L(s) \subseteq \mathfrak{c}_L(c)$. Therefore the sublocale $C = \mathfrak{o}_L(s)$ is a cozero-sublocale of L with $A \subseteq C \subseteq Z$. Thus, condition (1) implies condition (2).

Now assume that L is an F-frame satisfying condition (2). Consider $c \in \operatorname{Coz} L$ and $a \in L$ with $c \prec a$. Then $c^* \vee a = 1$, which implies $\mathfrak{c}_L(a) \subseteq \operatorname{int}_L(\mathfrak{c}_L(c))$. Thus, $\mathfrak{c}_L(c)$ is a zero-sublocale which is a neighborhood of the closed sublocale $\mathfrak{c}_L(a)$. By condition (2), there is a cozero element d of L such that $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(d) \subseteq \mathfrak{c}_L(c)$. Consequently, $a \vee d = 1$ and $d \wedge c = 0$. Since c and d are cozero elements and L is an F-frame, there exist u and v in $\operatorname{Coz} L$ with $u \vee v = 1$ and $c \wedge u = 0 = d \wedge v$. Therefore c is rather below v in the lattice $\operatorname{Coz} L$, which, again by [6, Corollary 3], implies $c \prec\!\!\prec v$. We however have $v \leq a$ because $a \vee d = 1$ and $v \wedge d = 0$; so in all then $c \prec\!\!\prec a$, which proves that L is coz-interpolative. □

Remark 2.3.6. Recall from [7] that a frame is called an *Oz-frame* if the pseudocomplement of every cozero element is a cozero element. See also [25] for some interesting characterizations of these frames. In terms of sublocales, L is an Oz-frame precisely when the interior of every zero-sublocale is a cozero-sublocale. Therefore every Oz-frame satisfies condition (2) of Theorem 2.3.5.

Now here is a characterization of the frames L for which every ideal of the form $\boldsymbol{O}_S$ is pure for every closed sublocale S of L.

Theorem 2.3.7. *The ideal $\boldsymbol{O}_B$ of $\mathcal{R}L$ is pure for every closed sublocale B of L iff L is coz-interpolative.*

Proof. Let $a \in L$ and $\alpha \in \mathcal{R}L$. From the definition, we know that $\alpha \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))}$ if and only if $\operatorname{coz}\alpha \prec\!\!\prec a$. On the other hand,

$$\begin{aligned} \alpha \in \boldsymbol{O}_{\mathfrak{c}_L(a)} \quad &\text{iff} \quad \mathfrak{c}_L(a) \subseteq \mathfrak{o}_L((\operatorname{coz}\alpha)^*) \\ &\text{iff} \quad a \vee (\operatorname{coz}\alpha)^* = 1 \\ &\text{iff} \quad \operatorname{coz}\alpha \prec a. \end{aligned}$$

Therefore $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))} \subseteq \boldsymbol{O}_{\mathfrak{c}_L(a)}$. So, if L is coz-interpolative, then $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))} = \boldsymbol{O}_{\mathfrak{c}_L(a)}$, which then proves that $\boldsymbol{O}_B$ is pure for every closed sublocale B of L.

Conversely, suppose that $\boldsymbol{O}_B$ is pure for every closed sublocale B of L. Let $a \in L$ and $c \in \operatorname{Coz} L$ be such that $c \prec a$. Pick $\gamma \in \mathcal{R}L$ with $\operatorname{coz}\gamma = c$. Now, $\operatorname{coz}\gamma \prec a$ implies $\gamma \in \boldsymbol{O}_{\mathfrak{c}_L(a)}$ by the calculation above. Since $\boldsymbol{O}_{\mathfrak{c}_L(a)} = \boldsymbol{O}^{r_L[\mathfrak{c}_L(a)]}$, and since our hypothesis says this ideal is pure, we have $\boldsymbol{O}^{r_L[\mathfrak{c}_L(a)]} = \boldsymbol{O}^{\overline{r_L[\mathfrak{c}_L(a)]}}$ by Lemma 2.3.1. Since $\bigwedge \mathfrak{c}_L(a) = a$, and since r_L preserves meets, we have $\bigwedge r_L[\mathfrak{c}_L(a)] = r_L(a)$, which then implies

$$\overline{r_L[\mathfrak{c}_L(a)]} = \mathfrak{c}_{\beta L}(r_L(a)).$$

Thus, $\gamma \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))}$, which implies $\operatorname{coz}\gamma \prec\!\!\prec a$; showing that L is coz-interpolative. □

Let us interpret this result in spaces. Recall that if X is a Tychonoff space, then for any $U, V \in \Omega(X)$, $U \prec V$ if and only if $\overline{U} \subseteq V$, and $U \prec\!\!\prec V$ if and only if there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \notin V$. We therefore have the following corollary.

Corollary 2.3.8. *The ideal* $\boldsymbol{O}_A$ *of* $C(X)$ *is pure for every closed subset* A *of* X *iff whenever the closure of a cozero-set of* X *is contained in some open subset of* X*, then the cozero-set and the complement of that open set are completely separated.*

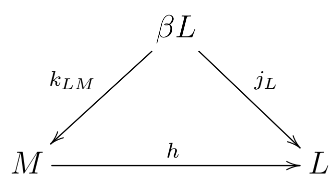
Another immediate corollary to Theorem 2.3.7 is the following. Recall that a topological space is normal if and only if the frame of its open subsets is normal.

Corollary 2.3.9. *If* L *is normal frame, then* $\boldsymbol{O}_B$ *is pure for every closed sublocale* B *of* L*. If* X *is a normal space, then* $\boldsymbol{O}_B$ *is pure for every closed subset* B *of* X*.*

Recall that the subcategory of completely regular Lindelöf frames resides coreflectively in **CRFrm** [31]. A σ*-ideal* of $\operatorname{Coz} L$ is a lattice ideal closed under countable joins. We denote by λL the frame of σ-ideals of $\operatorname{Coz} L$. The mapping $\lambda_L \colon \lambda L \to L$ that sends an ideal to its join is the coreflection map to L from Lindelöf completely regular frames. It is a dense C-quotient map(see [4] for the notion of C- and C^*-quotients), and therefore the induced ring homomorphism $\mathcal{R}(\lambda_L) \colon \mathcal{R}(\lambda L) \to \mathcal{R}L$ is an isomorphism. Since regular Lindelöf frames are normal, the direct images under this isomorphism of the pure ideals $\boldsymbol{O}_B$ of $\mathcal{R}(\lambda L)$, for B a closed sublocale of λL, are pure ideals of $\mathcal{R}L$. We wish to describe them in terms of the associated closed sublocales of βL. We do so by first proving a more general result. Recall that a frame homomorphism $h \colon M \to L$ is said to be *coz-surjective* if for every $c \in \operatorname{Coz} L$ there is a $d \in \operatorname{Coz} M$ such that $h(d) = c$.

Recall the Stone extension of a frame homomorphism from Section 1.5 of Chapter 1.

If $h \colon M \to L$ is a dense C^*-quotient map, then βh is an isomorphism [10, Corollary 2.2], and so we have the commuting triangle



where the morphism k_{LM} is defined by $k_{LM} = j_M \circ (\beta h)^{-1}$.

For use in the upcoming proof, recall from the definition that if $I \in \beta L$ and $\alpha \in \mathcal{R}L$, then

$$\alpha \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} \iff \operatorname{coz} \alpha \in I.$$

Theorem 2.3.10. *If $h\colon M \to L$ is a dense C-quotient map out of a normal frame, then, for any $a \in M$,*

$$(\mathcal{R}h)[\boldsymbol{O}_{\mathfrak{c}_M(a)}] = \boldsymbol{O}^{\mathfrak{c}_{\beta L}((\beta h)(r_M(a)))} = \boldsymbol{O}^{\mathfrak{c}_{\beta L}((k_{LM})_*(a))},$$

where k_{LM} is as defined above.

Proof. The second claimed equality follows from the fact that

$$(k_{LM})_* = \big(j_M \circ (\beta h)^{-1}\big)_* = \beta h \circ r_M$$

because the right adjoint of an isomorphism is its inverse.

Now, to the first equality. For brevity, let us write $I_a = (\beta h)(r_M(a))$. We start by showing that $(\mathcal{R}h)[\boldsymbol{O}_{\mathfrak{c}_M(a)}] \subseteq \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I_a)}$. Let $\alpha \in \boldsymbol{O}_{\mathfrak{c}_M(a)}$. Then $\operatorname{coz}\alpha \prec a$, which then implies $\operatorname{coz}\alpha \prec\!\!\prec a$ since M is normal. Thus, $\operatorname{coz}\alpha \in r_M(a)$. Since $\operatorname{coz}\big((\mathcal{R}h)(\alpha)\big) = h(\operatorname{coz}\alpha)$, it follows that $(\mathcal{R}h)(\alpha) \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I_a)}$. This proves that $(\mathcal{R}h)[\boldsymbol{O}_{\mathfrak{c}_M(a)}] \subseteq \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I_a)}$.

For the reverse inclusion, let us first show that for any $u, v \in \operatorname{Coz} M$,

$$h(u) \prec\!\!\prec h(v) \quad\Longrightarrow\quad u \prec\!\!\prec v. \tag{†}$$

Since $h(u)$ and $h(v)$ are cozero elements, we know from [6, Corollary 3] that there is a separating *cozero* element s in L such that $h(u) \wedge s = 0$ and $s \vee h(v) = 1$. Since h is a C-quotient map, it is coz-surjective, and so there is a $c \in \operatorname{Coz} M$ with $h(c) = s$. Thus, $h(u \wedge c) = 0$ and $h(c \vee v) = 1$, which, by density and coz-codensity of h, implies $u \wedge c = 0$ and $c \vee v = 1$, so that $u \prec\!\!\prec v$ by [6, Corollary 3] again.

Now, consider any $\gamma \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I_a)}$. Then $\operatorname{coz}\gamma \in I_a$, and so there is a $c \in \operatorname{Coz} M$ such that $c \prec\!\!\prec a$ and $\operatorname{coz}\gamma \prec\!\!\prec h(c)$. Since h is a C-quotient map, we can find $\tilde{\gamma} \in \mathcal{R}M$ such that $h \circ \tilde{\gamma} = \gamma$. Then $h(\operatorname{coz}\tilde{\gamma}) \prec\!\!\prec h(c)$, and so by (†), $\operatorname{coz}\tilde{\gamma} \prec\!\!\prec c$, whence $\operatorname{coz}\tilde{\gamma} \prec a$, which is to say $\tilde{\gamma} \in \boldsymbol{O}_{\mathfrak{c}_M(a)}$. Since $\gamma = (\mathcal{R}h)(\tilde{\gamma})$, this shows that $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I_a)} \subseteq (\mathcal{R}h)[\boldsymbol{O}_{\mathfrak{c}_M(a)}]$, and we thus have the claimed equality. □

Applied to $\lambda_L\colon \lambda L \to L$, this theorem takes the form described in the following corollary. It is not difficult to show that, putting $M = \lambda L$, $(k_{LM})_*(J) = \bigcup_{u \in J} r_L(u)$, for every $J \in \lambda L$.

Corollary 2.3.11. *For any $J \in \lambda L$, $\mathcal{R}(\lambda_L)[\boldsymbol{O}_{\mathfrak{c}_{\lambda L}(J)}] = \boldsymbol{O}^{\mathfrak{c}_{\beta L}\left(\bigcup_{u \in J} r_L(u)\right)}$.*

A more localic (as opposed to frame-theoretic) statement of Theorem 2.3.10 is worth recording. Recall that a localic map $f\colon L \to M$ is called *dense* if its left adjoint $f^*\colon M \to L$ is dense. Let us say f is a *localic C-embedding* if f^* is a C-quotient map. We mentioned above the Stone extension of a frame homomorphism. Similarly, there is a Stone extension of a localic map (see, for instance, [21]), also denoted βf.

Corollary 2.3.12. *If $f\colon L \to M$ is a dense localic C-embedding with M normal, then*

$$\mathcal{R}(f^*)[\boldsymbol{O}_A] = \boldsymbol{O}^{\overline{(\beta f)_{-1}[r_M[A]]}} = \boldsymbol{O}^{(\beta f)_{-1}\left[\overline{r_M[A]}\right]},$$

for any closed sublocale A of M.

Proof. Observe that if $g\colon H \to K$ is a localic isomorphism, then $\overline{g_{-1}[T]} = g_{-1}[\overline{T}]$, for any sublocale T of K. So the second equality in the statement of the corollary holds because βf is an isomorphism. Now, pick $a \in M$ such that $A = \mathfrak{c}_M(a)$. As observed in the proof of Theorem 2.3.7, $\overline{r_M[\mathfrak{c}_M(a)]} = \mathfrak{c}_{\beta M}(r_M(a))$, so

$$(\beta f)_{-1}[\overline{r_M[\mathfrak{c}_M(a)]}] = (\beta f)_{-1}[\mathfrak{c}_{\beta M}(r_M(a))] = \mathfrak{c}_{\beta L}((\beta f)^*(r_L(a))),$$

and hence the first equality in the statement of the corollary follows from Theorem 2.3.10 because $(\beta f)^* = \beta(f^*)$. ☐

Let us go back to Theorem 2.3.7. It tells us that if L is coz-interpolative then each ideal of the form $\boldsymbol{O}_B$, for B a closed sublocale of L, is pure. This, however, does not mean that all pure ideals of $\mathcal{R}L$ for such a frame L are of this kind. For that to be the case, the frame needs to be even more restricted, as the next theorem shows.

Theorem 2.3.13. $\mathrm{PId}(\mathcal{R}L) = \{\boldsymbol{O}_B \mid B$ *is a closed sublocale of* $L\}$ *iff L is compact.*

Proof. Assume first that $\mathrm{PId}(\mathcal{R}L) = \{\boldsymbol{O}_B \mid B$ is a closed sublocale of $L\}$. We prove that L is compact by showing that the localic map $r_L\colon L \to \beta L$ is an isomorphism. Since it is always injective, we need only show that it is surjective. Consider any $I \in \beta L$. Since $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)}$ is pure, the hypothesis furnishes an $a \in L$ such that $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \boldsymbol{O}_{\mathfrak{c}_L(a)}$. Since $\boldsymbol{O}_{\mathfrak{c}_L(a)} = \boldsymbol{O}^{r_L[\mathfrak{c}_L(a)]}$, and since this ideal is pure, Lemma 2.3.1 ensures that $\boldsymbol{O}^{r_L[\mathfrak{c}_L(a)]} = \boldsymbol{O}^{\overline{r_L[\mathfrak{c}_L(a)]}}$. But now, as observed in the proof of Theorem 2.3.7, $\overline{r_L[\mathfrak{c}_L(a)]} = \mathfrak{c}_{\beta L}(r_L(a))$; so we have $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))}$, whence

$\mathfrak{c}_{\beta L}(I) = \mathfrak{c}_{\beta L}(r_L(a))$ because these sublocales are closed. Therefore $I = r_L(a)$, which shows that r_L is surjective, as desired.

Conversely, assume that L is compact. Then (since our frames are completely regular) L is normal, and so, by Corollary 2.3.9, $\boldsymbol{O}_B$ is a pure ideal of $\mathcal{R}L$ for each closed sublocale B of L. On the other hand, if H is a pure ideal of $\mathcal{R}L$ then there is an $I \in \beta L$ such that $H = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)}$. Since L is compact, $r_L \colon L \to \beta L$ is surjective, and so there is an $a \in L$ such that $I = r_L(a)$. Note, as well, that the surjectivity of r_L implies $r_L[\mathfrak{c}_L(a)] = \mathfrak{c}_{\beta L}(r_L(a))$. In consequence,

$$H = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))} = \boldsymbol{O}^{r_L[\mathfrak{c}_L(a)]} = \boldsymbol{O}_{\mathfrak{c}_L(a)},$$

which then proves that $\mathrm{PId}(\mathcal{R}L) = \{\boldsymbol{O}_B \mid B \text{ is a closed sublocale of } L\}$. $\square$

Corollary 2.3.14. *The pure ideals of $C(X)$ are precisely the ideals $\boldsymbol{O}_B$ for B a closed subset of X iff X is compact.*

We mentioned at the beginning of the section that one of our aims is to give a description of pure ideals of $\mathcal{R}^*L$. We now embark on that, but first we recall some background.

As shown in [4], $j_L \colon \beta L \to L$ is C^*-quotient map. Since j_L is dense, the ring homomorphism $\mathcal{R}(j_L) \colon \mathcal{R}(\beta L) \to \mathcal{R}L$ it induces is injective. Since it maps into $\mathcal{R}^*L$ (as βL is compact), when its codomain is restricted to $\mathcal{R}^*L$, we have the ring isomorphism

$$\phi_L \colon \mathcal{R}(\beta L) \to \mathcal{R}^*L \quad \text{given by} \quad \phi_L(f) = j_L \circ f.$$

Now, in view of this ring isomorphism,

$$\mathrm{PId}(\mathcal{R}^*L) = \{\phi_L[J] \mid J \in \mathrm{PId}(\mathcal{R}(\beta L))\}$$

because, clearly, an onto ring homomorphism sends pure ideals to pure ideals. We however seek a more transparent description of pure ideals of $\mathcal{R}^*L$.

So far, given a frame L, we have dealt only with ideals of $\mathcal{R}L$. To prove the next result, we shall simultaneously deal with ideals of $\mathcal{R}L$ and of $\mathcal{R}(\beta L)$. To avoid possible confusion, we shall use a different symbol for the $\boldsymbol{O}$-ideals in $\mathcal{R}(\beta L)$. Also, we shall use notation that distinguishes between the cozero maps, and write $\mathrm{Coz} \colon \mathcal{R}(\beta L) \to \beta L$ for the cozero map on $\mathcal{R}(\beta L)$. Although this notation is identical to the one used for the cozero part of a frame, there is no danger of ambiguity. For any sublocale A of βL, we set

$$\boldsymbol{\mathcal{O}}_A = \{f \in \mathcal{R}(\beta L) \mid A \subseteq \mathfrak{o}_{\beta L}((\mathrm{Coz}\, f)^*)\}.$$

For any $f \in \mathcal{R}(\beta L)$, we have $\bigvee \operatorname{Coz} f = \operatorname{coz}(j_L \circ f)$. In part of the proof below we shall use the fact that if I and J are elements of βL with $I \prec J$, then $\bigvee I \in J$ (see [14, p. 156]).

Theorem 2.3.15. *An ideal of $\mathcal{R}^*L$ is pure iff it is of the form $\{\alpha \in \mathcal{R}^*L \mid \operatorname{coz}\alpha \in I\}$ for some $I \in \beta L$.*

Proof. As already mentioned above, an onto ring homomorphism sends pure ideals to pure ideals. So, if $\psi\colon A \to B$ is a ring isomorphism, then the pure ideals of B are precisely the images of the pure ideals of A under ψ. Now, for the isomorphism $\phi_L\colon \mathcal{R}(\beta L) \to \mathcal{R}^*L$ mentioned above, and taking into account the result in Theorem 2.3.13, we have

$$\operatorname{PId}(\mathcal{R}^*L) = \{\phi_L[\boldsymbol{\mathcal{O}}_{\mathfrak{c}_{\beta L}(I)}] \mid I \in \beta L\}.$$

We claim that, for any $I \in \beta L$,

$$\phi_L[\boldsymbol{\mathcal{O}}_{\mathfrak{c}_{\beta L}(I)}] = \{\alpha \in \mathcal{R}^*L \mid \operatorname{coz}\alpha \in I\}. \tag{‡}$$

To verify this, let $f \in \boldsymbol{\mathcal{O}}_{\mathfrak{c}_{\beta L}(I)}$. Then $\mathfrak{c}_{\beta L}(I) \subseteq \mathfrak{o}_{\beta L}((\operatorname{Coz} f)^*)$, which implies $(\operatorname{Coz} f)^* \vee I = 1_{\beta L}$, and hence $\operatorname{Coz} f \prec I$, whence $\bigvee \operatorname{Coz} f \in I$, that is, $\operatorname{coz}(\phi_L(f)) \in I$. Consequently, $\phi_L(f)$ is an element of $\mathcal{R}^*L$ whose cozero element belongs to I. This proves the inclusion $\subseteq$ in (‡). For the reverse inclusion, consider any $\alpha \in \mathcal{R}^*L$ with $\operatorname{coz}\alpha \in I$. Let $\tilde{\alpha}$ be the function in $\mathcal{R}(\beta L)$ such that $j_L \circ \tilde{\alpha} = \alpha$. Then $\tilde{\alpha}$ has the property that

$$\operatorname{Coz}\tilde{\alpha} \subseteq r_L\Big(\bigvee \operatorname{Coz}\tilde{\alpha}\Big) = r_L(\operatorname{coz}\alpha) \prec I$$

because $\mathrm{id} \leq h_* \circ h$ for every frame homomorphism h, and whenever $c \in I \in \beta L$, then $c \prec\!\!\prec d$ for some $d \in I$, so that $r_L(c) \prec r_L(d) \subseteq I$. Thus, $(\operatorname{Coz}\tilde{\alpha})^* \vee I = 1_{\beta L}$, which implies $\mathfrak{c}_{\beta L}(I) \subseteq \mathfrak{o}_{\beta L}(\operatorname{Coz}\tilde{\alpha})^*$, so that $\tilde{\alpha} \in \boldsymbol{\mathcal{O}}_{\mathfrak{c}_{\beta L}(I)}$. Since $\phi_L(\tilde{\alpha}) = \alpha$, we have shown the inclusion $\supseteq$ in (‡). Since βL is compact, its pure ideals are precisely the ideals $\boldsymbol{\mathcal{O}}_A$, for A a closed sublocale of βL, as shown in Theorem 2.3.13. The result therefore follows because $\phi_L\colon \mathcal{R}(\beta L) \to \mathcal{R}^*L$ is an isomorphism. ∎

Let us restate this result differently using the language of contraction of ideals. Recall that if $\phi\colon A \to B$ is a ring homomorphism and I is an ideal of B, then the ideal $\phi^{-1}[I]$ of A is called the *contraction* of I.

Corollary 2.3.16. *The pure ideals of $\mathcal{R}^*L$ (resp. $C^*(X)$) are precisely the contractions to $\mathcal{R}^*L$ (resp. $C^*(X)$) of the pure ideals of $\mathcal{R}L$ (resp. $C(X)$).*

Phrased this way, the reader may wonder if we could not have derived this by first arguing that, in function rings, contractions of pure ideals are pure ideals. We could not because (even in function rings) purity generally does not survive contraction, as the example below shows.

Example 2.3.17. Let $L = \Omega(\mathbb{R})$, and put $a = (0, 1)$. Note that $a = a^{**}$. Since every element of L is a cozero element, there is an $\alpha \in \mathcal{R}L$ such that $\operatorname{coz}\alpha = a$. Denote by $\mathsf{b}\colon L \to \mathfrak{B}L$ the Booleanization map $\mathsf{b}(x) = x^{**}$. Since b is dense, the ring homomorphism $\mathcal{R}\mathsf{b}\colon \mathcal{R}L \to \mathcal{R}(\mathfrak{B}L)$ is injective, and so its image is a function ring which is a subring of $\mathcal{R}(\mathfrak{B}L)$. Let H be the principal ideal of $\mathcal{R}(\mathfrak{B}L)$ generated by $\mathsf{b}\circ\alpha$. Since $\mathfrak{B}L$ is Boolean, and hence a P-frame, H is a pure ideal in $\mathcal{R}(\mathfrak{B}L)$ by [11, Corollary 3.10]. We claim that the ideal $(\mathcal{R}\mathsf{b})^{-1}[H]$ of $\mathcal{R}L$ is not pure. If it were, then since $\alpha \in (\mathcal{R}\mathsf{b})^{-1}[H]$, there would be an element $\gamma \in (\mathcal{R}\mathsf{b})^{-1}[H]$ such that $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\gamma$. The relation $\gamma \in (\mathcal{R}\mathsf{b})^{-1}[H]$ implies $\mathsf{b}\circ\gamma \in H$, and hence $\mathsf{b}\circ\gamma$ is a multiple of $\mathsf{b}\circ\alpha$. Now, in light of the cozero element of a product being below the cozero element of each factor, we would have

$$a \prec\!\!\prec (\operatorname{coz}\gamma)^{**} = \mathsf{b}(\operatorname{coz}\gamma) = \operatorname{coz}(\mathsf{b}\circ\gamma) \leq \operatorname{coz}(\mathsf{b}\circ\alpha) = (\operatorname{coz}\alpha)^{**} = a,$$

which would imply $a \prec\!\!\prec a$, which is of course false.

2.4 Characterizing basic disconnectedness

In Theorem 2.3.7 we characterized the frames L for which $\boldsymbol{O}_B$ is pure for every closed sublocale B of L. It is thus natural to seek a "companion" characterization with open sublocales in the place of closed ones. That will be the content of our next result. We shall approach it slightly differently from the previous case.

In [1], there are characterizations of basically disconnected spaces X in terms properties of pure ideals of $C(X)$. One such is that X is basically disconnected if and only if $\boldsymbol{O}^A$ is a pure ideal of $C(X)$ for every subspace of A of βX. Now, a topological space, when viewed as a frame, can have more sublocales than subspaces. So it is reasonable to wonder if replacing "subspace" with "sublocale" in the result of [1] just recited does not invalidate one of the implications.

We shall see that it does not. This we shall actually do by characterizing basically disconnected frames in several ways, including that L is basically disconnected if and only if $\boldsymbol{O}^A$ is pure for every sublocale of βL. One other characterization requires knowledge about the frame $\mathcal{S}_{\mathfrak{c}}(L)$, associated with any given frame L, defined in [36] by

$$\mathcal{S}_{\mathfrak{c}}(L) = \{S \in \mathcal{S}(L) \mid S \text{ is a join of closed sublocales of } L\}.$$

Since our frames are completely regular (and hence fit), each member of $\mathcal{S}_{\mathfrak{c}}(L)$ is actually a join of complemented sublocales. Following Isbell [27], we shall thus say members of $\mathcal{S}_{\mathfrak{c}}(L)$ are the *smooth* sublocales. We must point out that in [27] this descriptor is used not only for the fit case.

There are several characterizations of basically disconnected frames in [4] and [16]. The following ones are new, and they both extend and supplement the spatial ones in [1]. Recall from [3, Corollary to Lemma 1.9] that r_L preserves disjoint binary joins; that is, if $a \wedge b = 0$, then $r_L(a \vee b) = r_L(a) \vee r_L(b)$.

Theorem 2.4.1. *The following conditions are equivalent for L.*

(1) *L is basically disconnected.*

(2) *$\boldsymbol{O}^A$ is pure for every sublocale A of βL.*

(3) *The intersection of any collection of pure ideals of $\mathcal{R}L$ is pure.*

(4) *$\boldsymbol{O}^A$ is pure for every smooth sublocale A of βL.*

(5) *$\boldsymbol{O}^A$ is pure for every open sublocale A of βL.*

(6) *$\boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(a))}$ is pure for every $a \in L$.*

(7) *$\boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(c^*))}$ is pure for every $c \in \operatorname{Coz} L$.*

Proof. (1) $\Rightarrow$ (2): Suppose that L is basically disconnected, and let A be a sublocale of βL. We show that $\boldsymbol{O}^A \subseteq \boldsymbol{O}^{\bar{A}}$. Let $\gamma \in \boldsymbol{O}^A$. For brevity, write $c = \operatorname{coz}\gamma$. Then $A \subseteq \mathfrak{o}_{\beta L}(r_L(c^*))$. Since L is basically disconnected, $c^* \vee c^{**} = 1$, and since $c^* \wedge c^{**} = 0$, we have

$$r_L(c^*) \vee r_L(c^{**}) = r_L(c^* \vee c^{**}) = 1_{\beta L}.$$

Consequently, $\mathfrak{c}_{\beta L}(r_L(c^{**})) \subseteq \mathfrak{o}_{\beta L}(r_L(c^*))$, and hence

$$\bar{A} \subseteq \overline{\mathfrak{o}_{\beta L}(r_L(c^*))} = \mathfrak{c}_{\beta L}(r_L(c^{**})) \subseteq \mathfrak{o}_{\beta L}(r_L(c^*)) = \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\gamma)^*),$$

which implies $\gamma \in \boldsymbol{O}^{\bar{A}}$. Thus, $\boldsymbol{O}^A \subseteq \boldsymbol{O}^{\bar{A}}$, and therefore $\boldsymbol{O}^A = \boldsymbol{O}^{\bar{A}}$, showing that $\boldsymbol{O}^A$ is pure.

(2) $\Rightarrow$ (3): Assume that (2) holds, and let $\{Q_\lambda\}$ be a family of pure ideals of $\mathcal{R}L$. For each index λ, there is a closed sublocale K_λ of βL such that $Q_\lambda = \boldsymbol{O}^{K_\lambda}$. This then implies

$$\bigcap_\lambda Q_\lambda = \bigcap_\lambda \boldsymbol{O}^{K_\lambda} = \boldsymbol{O}^{\bigvee_\lambda K_\lambda},$$

showing that $\bigcap_\lambda Q_\lambda$ is pure in light of the hypothesis in (2).

(3) $\Rightarrow$ (4): Assume that (3) holds. If A is smooth, then there is a collection $\{K_\lambda\}$ of closed sublocales of βL such that $A = \bigvee_\lambda K_\lambda$. Thus, $\boldsymbol{O}^A = \bigcap_\lambda \boldsymbol{O}^{K_\lambda}$, which is an intersection of pure ideals, and hence $\boldsymbol{O}^A$ is pure.

(4) $\Rightarrow$ (5): This holds because open sublocales in subfit frames (and hence in completely regular frames) are smooth.

(5) $\Rightarrow$ (6) $\Rightarrow$ (7): These implications are trivial.

(7) $\Rightarrow$ (1): Assume that (7) holds, and let $c \in \operatorname{Coz} L$. Then the ideal $\boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(c^*))}$ is pure, by hypothesis, and so

$$\boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(c^*))} = \boldsymbol{O}^{\overline{\mathfrak{o}_{\beta L}(r_L(c^*))}} = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(c^{**}))}.$$

Take $\gamma \in \mathcal{R}L$ with $c = \operatorname{coz}\gamma$. Then $\gamma \in \boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(c^*))}$, which then implies $\gamma \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(c^{**}))}$. The latter says $\mathfrak{c}_{\beta L}(r_L(c^{**})) \subseteq \mathfrak{o}_{\beta L}(r_L(c^*))$, which implies $r_L(c^{**}) \vee r_L(c^*) = 1_{\beta L}$, whence, on taking joins, we obtain $c^{**} \vee c^* = 1$. Therefore L is basically disconnected. □

Corollary 2.4.2. *The ideal $\boldsymbol{O}_U$ is pure for every open sublocale U of L iff L is basically disconnected.*

Proof. If L is basically disconnected, then, by the theorem above, $\boldsymbol{O}_U$ is pure for every open sublocale U because $\boldsymbol{O}_U = \boldsymbol{O}^{r_L[U]}$, and $r_L[U]$ is a sublocale of βL.

Conversely, observe that for any $a \in L$ and $\gamma \in \mathcal{R}L$,

$$\begin{aligned}
\gamma \in \boldsymbol{O}_{\mathfrak{o}_L(a)} \quad & \text{iff} \quad \mathfrak{o}_L(a) \subseteq \mathfrak{o}_L((\operatorname{coz}\gamma)^*) \\
& \text{iff} \quad a \leq (\operatorname{coz}\gamma)^* \\
& \text{iff} \quad r_L(a) \leq r_L(\operatorname{coz}\gamma)^* \\
& \text{iff} \quad \mathfrak{o}_{\beta L}(r_L(a)) \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\gamma)^*) \\
& \text{iff} \quad \gamma \in \boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(a))},
\end{aligned}$$

so that $\boldsymbol{O}_{\mathfrak{o}_L(a)} = \boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(a))}$. It therefore follows from the implication (6) $\Rightarrow$ (1) in Theorem 2.4.1 that if $\boldsymbol{O}_U$ is pure for every open sublocale U of L then L is basically disconnected. □

We have deliberately understated the result in this corollary because we wanted to present it as the "open analogue" of Theorem 2.3.7. A more comprehensive result characterizes basically disconnected frames in terms of ideals associated with sublocales of L (and not of βL as above) as follows. The proof is a mere adaptation of the corresponding results in Theorem 2.4.1, and we therefore omit it.

Corollary 2.4.3. *The following are equivalent for* L*.*

(1) L *is basically disconnected.*

(2) $\boldsymbol{O}_S$ *is pure for every sublocale of* L*.*

(3) $\boldsymbol{O}_S$ *is pure for every* $S \in \mathcal{S}_{\mathfrak{c}}(L)$*.*

(4) $\boldsymbol{O}_U$ *is pure for every open sublocale of* U *of* L*.*

Chapter 3

Pulling and pushing the sublocale-induced ideals

A localic map $f\colon L \to M$ gives rise to a ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$, where h is the left adjoint of f. Our aim in this chapter is to study the contractions and extensions of the $\boldsymbol{O}$- and $\boldsymbol{M}$-ideals along the induced ring homomorphism.

3.1 Indispensable localic maps

We are not using "indispensable" as a mathematical adjective describing localic maps with some feature, but rather it has its everyday use, meaning that the localic maps we are presenting here are indispensable for the work that lies ahead. So, this section is preparatory for the main objective of pushing forward and pulling back ideals along ring homomorphisms induced by localic maps. We introduce localic maps that will play a pivotal role in Sections 3.2 and 3.3.

To start, recall that in Chapter 1 we presented the Stone-Čech compactification in the category **CRFrm**. The outlook in this chapter is more localic, so let us recall the *Stone extension* of a localic map $f\colon L \to M$. It is the unique localic map $\beta f\colon \beta L \to \beta M$ that makes the diagram

$$\begin{array}{ccc} \beta L & \xrightarrow{\beta f} & \beta M \\ {\scriptstyle r_L}\uparrow & & \uparrow{\scriptstyle r_M} \\ L & \xrightarrow{f} & M \end{array} \tag{3.1.1}$$

commute. Of course, dually, for every frame homomorphism $h\colon M \to L$ there is a unique frame homomorphism $\beta h\colon \beta M \to \beta L$ making a square in **CRFrm** (which appears Section 1.5 of Chapter 1) similar to that in Diagram (3.1.1) commute. We recalled in Chapter 1 how the frame homomorphism βh maps. The localic map βf maps as follows:

$$(\beta f)(J) = \bigvee\{I \in \beta M \mid f^*[I] \subseteq J\}.$$

It should be clear that if f is a localic map, then $(\beta f)^* = \beta(f^*)$.

Direct calculation shows that for any frame homomorphism $h\colon M \to L$ and any $a \in M$,

$$(\beta h)(r_M(a)) \subseteq r_L(h(a)). \tag{3.1.2}$$

In the next two sections we are going to encounter a number of cases where certain properties are characterized by the containment in (3.1.2) being actually an equality, either for all elements of M or all elements of some suitable subset of M.

Localic maps $f\colon L \to M$ with the property that $(\beta f)^*(r_M(a)) = r_L(f^*(a))$ for every $a \in M$ have ancestry in classical topology. To recall, Woods [41] calls a surjective continuous map $k\colon X \to Y$ an N-map if $\mathrm{cl}_{\beta X}\, k^{-1}[F] = (k^\beta)^{-1}[\mathrm{cl}_{\beta Y}\, F]$ for every closed subset F of Y, where $k^\beta\colon \beta X \to \beta Y$ is the Stone extension of k. If the equality $\mathrm{cl}_{\beta X}\, k^{-1}[Z] = (k^\beta)^{-1}[\mathrm{cl}_{\beta Y}\, Z]$ holds for each zero-set Z of Y, then Woods says the function k is a WN-map. We want to extend this to localic maps, and relax the surjectivity constraint that Woods imposed.

Let $f\colon L \to M$ be a (not necessarily surjective) localic map. Consider Diagram (3.1.1) above, and split it into the following wedges:

$$\begin{array}{ccc} \beta L & & \\ {\scriptstyle r_L}\uparrow & & \\ L & \xrightarrow{f} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \beta L & \xrightarrow{\beta f} & \beta M \\ & & \uparrow{\scriptstyle r_M} \\ & & M \end{array}$$

Let F be a closed sublocale of M. Using the wedge on the left, pull F back along f to obtain the closed sublocale $f_{-1}[F]$ of L, and then push this closed sublocale upwards along r_L to obtain the (not necessarily closed) sublocale $r_L[f_{-1}[F]]$ of βL, and then (to keep things closed) take the closure in βL to obtain $\mathrm{cl}_{\beta L}(r_L[f_{-1}[F]])$. Now do similarly along the wedge on the right (keeping things closed) to end up with $(\beta f)_{-1}[\mathrm{cl}_{\beta M}(r_M[F])]$. We shall be concerned with several cases where these two processes culminate in the same sublocale.

For a given localic map $f\colon L \to M$, let us agree to call the equality

$$\mathrm{cl}_{\beta L}(r_L[f_{-1}[B]]) = (\beta f)_{-1}[\mathrm{cl}_{\beta M}(r_M[B])] \tag{WE}$$

the *Woods equality*. We shall be interested in cases where (WE) holds for each sublocale in the following classes of closed sublocales:

- $\mathsf{K} = \{$all closed sublocales$\}$;
- $\mathsf{Z} = \{$all zero-sublocales$\}$;
- $\overline{\mathsf{C}} = \{$closures of cozero-sublocales$\}$.

We are now going to define certain types of localic maps in terms of the Woods equality, and give them names that accord with the ones Woods used in spaces.

Definition 3.1.1. We say a localic map $f\colon L \to M$ is:

(a) an *N-map* if its Woods equality holds for every sublocale in K;

(b) a *WN-map* if its Woods equality holds for every sublocale in Z; and

(c) a $\overline{\mathrm{C}}$*-map* if its Woods equality holds for every sublocale in $\overline{\mathsf{C}}$.

In calculations, it shall be useful to have characterizations of these maps in terms of elements. For better visual clarity, we use the overline in the upcoming proof to denote closure.

Lemma 3.1.2. *A localic map $f\colon L \to M$ is:*

(a) *an N-map iff $(\beta f^*)(r_M(a)) = r_L(f^*(a))$ for every $a \in M$.*

(b) *a WN-map iff $(\beta f^*)(r_M(a)) = r_L(f^*(a))$ for every $a \in \mathrm{Coz}\, M$.*

(c) *a* $\overline{\mathrm{C}}$*-map iff* $(\beta f^*)(r_M(c^*)) = r_L(f^*(c^*))$ *for every* $c \in \operatorname{Coz} M$*.*

Proof. We prove only (a) as the other proofs are similar. Observe, first, that for any frame H and any $x \in H$,

$$\overline{r_H[\mathfrak{c}_H(x)]} = \mathfrak{c}_{\beta H}(r_H(x))$$

because

$$\bigwedge r_H[\mathfrak{c}_H(x)] = r_H\Big(\bigwedge \mathfrak{c}_H(x)\Big) = r_H(x).$$

Now let $K = \mathfrak{c}_M(a)$, for some $a \in M$. Then

$$\overline{r_L[f_{-1}[\mathfrak{c}_M(a)]]} = \overline{r_L[\mathfrak{c}_M(f^*(a))]} = \mathfrak{c}_{\beta L}(r_L(f^*(a))),$$

and

$$(\beta f)_{-1}\left[\overline{r_M[\mathfrak{c}_M(a)]}\right] = (\beta f)_{-1}[\mathfrak{c}_{\beta M}(r_M(a))] = \mathfrak{c}_{\beta L}\big((\beta f^*)(r_M(a)\big).$$

Therefore the Woods equality holds for K if and only if $(\beta f)^*(r_M(a)) = r_L(f^*(a))$. □

This is as good a time as any to mention that frame homomorphisms with the properties characterizing N-maps and WN-maps in Lemma 3.1.2 were considered in [12], but without the motivation provided here. We retain the names they were given in that paper, and say a frame homomorphism is an *N-homomorphism* if its right adjoint is an N-map, a *WN-homomorphism* if its right adjoint is a WN-map, and a $\overline{\mathrm{C}}$*-homomorphism* if its right adjoint is a $\overline{\mathrm{C}}$-map.

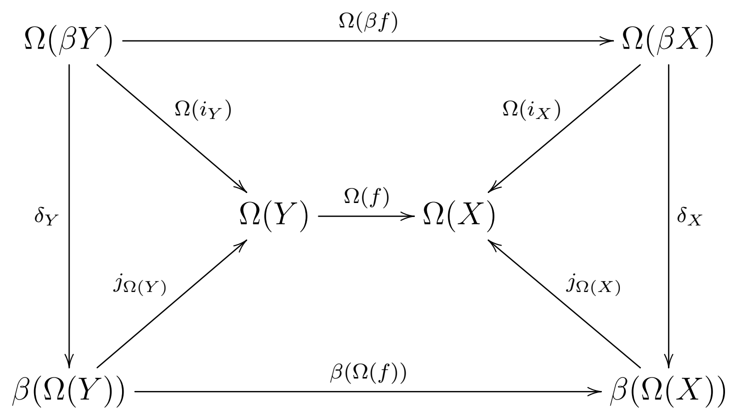
For use in Section 3.4 where we will consider $C(X)$, let us extend Woods' terminology and say a continuous function $f\colon X \to Y$ is an *N-map* (resp, a *WN*-map) if it satisfies the conditions of Woods, but without being necessarily surjective. We shall need to know that a continuous function $f\colon X \to Y$ is an N-map (resp. WN-map, resp. $\overline{\mathrm{C}}$-map) if and only if the localic map it induces is of the same type. This is not obvious, so we present a proof, but only for $\overline{\mathrm{C}}$-maps as the other assertions can be proved similarly.

Given a Tychonoff space X, as in Chapter 2 we view X as a subspace of βX, and consider the identical embedding $i_X\colon X \to \beta X$. The right adjoint of the induced frame homomorphism $\Omega(i_X)\colon \Omega(\beta X) \to \Omega X$ maps thus:

$$\Omega(i_X)_*(U) = \beta X \smallsetminus \mathrm{cl}_{\beta X}(X \smallsetminus U).$$

Again, it will be convenient to view the Stone-Čech compactification of the frame $\Omega(X)$ to be given by the dense-onto frame homomorphism $\Omega(i_X)\colon \Omega(\beta X) \to \Omega(X)$. So, as mentioned before, in the r_L-notation for the right adjoint of $\beta L \to L$, we have $r_{\Omega(X)} = \Omega(i_X)_*$.

A morphism $f\colon X \to Y$ in **Tych** gives rise to the digram



(3.1.3)

in **CRFrm** constructed as follows. For the upper trapezium, first use $f\colon X \to Y$ to form the **Tych**-version of Diagram (3.1.1), and then apply to it the contravariant functor $\Omega\colon$ **Tych** $\to$ **CRFrm**. For the lower trapezium, first apply Ω to $f\colon X \to Y$, and then form the **CRFrm**-version of Diagram (3.1.1). The triangles exist because, for any $L \in$ **CRFrm**, $j_L\colon \beta L \to L$ is the coreflection map to L from compact completely regular frames. In fact, δ_X and δ_Y are isomorphisms, as is well known. Since the trapeziums and the triangles commute, and since $j_{\Omega(X)}$ is a monomorphism because dense homomorphisms are monic in **CRFrm**, it follows that the outer square commutes.

In the upcoming proof we shall twice use the set-theoretic fact that if $g\colon A \to B$ is a function and $S \subseteq B$, then $g^{-1}[B \smallsetminus S] = A \smallsetminus g^{-1}[S]$.

Proposition 3.1.3. *A continuous map is a* $\overline{\mathrm{C}}$*-map (resp. an N-map, resp. a WN-map) iff the localic map it induces is a* $\overline{\mathrm{C}}$*-map (resp. an N-map, resp. a WN-map).*

Proof. Let $f\colon X \to Y$ be a continuous map between Tychonoff spaces. By Lemma 3.1.2(c), it suffices to show that f is a $\overline{\mathrm{C}}$-map if and only if $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ is a $\overline{\mathrm{C}}$-homomorphism. Let C be a cozero-set of Y, and set $U = Y \smallsetminus \mathrm{cl}_Y\, C$. Let us express the complements of the sets

$\mathrm{cl}_{\beta X}[f^{-1}[\mathrm{cl}_Y C]]$ and $(\beta f)^{-1}[\mathrm{cl}_{\beta Y}(\mathrm{cl}_Y C)]$ in terms of induced frame homomorphisms.

$$\begin{aligned}\beta X \smallsetminus \mathrm{cl}_{\beta X}[f^{-1}[\mathrm{cl}_Y C]] &= \beta X \smallsetminus \mathrm{cl}_{\beta X}\, f^{-1}[Y \smallsetminus U] \\ &= \beta X \smallsetminus \mathrm{cl}_{\beta X}(X \smallsetminus f^{-1}[U]) \\ &= \beta X \smallsetminus \mathrm{cl}_{\beta X}\big(X \smallsetminus (\Omega(f))(U)\big) \\ &= \Omega(i_X)_*((\Omega(f))(U)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\beta X \smallsetminus (\beta f)^{-1}[\mathrm{cl}_{\beta Y}(\mathrm{cl}_Y C)] &= (\beta f)^{-1}[\beta Y \smallsetminus \mathrm{cl}_{\beta Y}(Y \smallsetminus U)] \\ &= \Omega(\beta f)\big(\Omega(i_Y)_*(U)\big).\end{aligned}$$

Therefore

$$\mathrm{cl}_{\beta X}(f^{-1}[\mathrm{cl}_Y C]) = (\beta f)^{-1}[\mathrm{cl}_{\beta Y}(\mathrm{cl}_Y C)] \quad \text{iff} \quad \Omega(i_X)_*\big((\Omega(f))(U)\big) = \Omega(\beta f)\big(\Omega(i_Y)_*(U)\big).$$

From the commutativity of Diagram (3.1.3) and the fact that δ_X and δ_Y are isomorphisms, we have

$$\Omega(i_X)_* = \delta_X^{-1} \circ r_{\Omega(X)}, \qquad \Omega(i_Y)_* = \delta_Y^{-1} \circ r_{\Omega(Y)}, \qquad \Omega(\beta f) = \delta_X^{-1} \circ \beta(\Omega(f)) \circ \delta_Y,$$

and so, in light of the preceding calculation, the equality

$$\Omega(i_X)_*\big((\Omega(f))(U)\big) = \Omega(\beta f)\big(\Omega(i_Y)_*(U)\big)$$

is equivalent to the equality

$$\Big(\delta_X^{-1} \circ r_{\Omega(X)}\Big)\big((\Omega(f))(U)\big) = \Big(\delta_X^{-1} \circ \beta(\Omega(f)) \circ \delta_Y\Big)\big(\Omega(i_Y)_*(U)\big),$$

which, in turn, is equivalent to

$$r_{\Omega(X)}\big((\Omega(f))(U)\big) = \beta(\Omega(f))(r_{\Omega(Y)}(U))$$

because $\delta_Y \circ \Omega(i_Y)_* = r_{\Omega(Y)}$. Since

$$\{Y \smallsetminus \mathrm{cl}_Y C \mid C \text{ is a cozero-set of } Y\} = \{c^* \mid c \in \mathrm{Coz}(\Omega(Y))\},$$

it then follows that f is a $\overline{\mathrm{C}}$-map if and only if $\Omega(f)$ is a $\overline{\mathrm{C}}$-map. □

The next type of localic maps that will play a role below are best defined in terms of their left adjoints. Recall from [8] that a frame homomorphism $h\colon M \to L$ is said to be *nearly open* if $h(a^*) = h(a)^*$ for every $a \in L$. We weaken this.

Definition 3.1.4. A frame homomorphism $h\colon M \to L$ is *nearly coz-open* if $h(c^*) = h(c)^*$ for every $c \in \operatorname{Coz} M$. If the left adjoint of a localic map is nearly coz-open, we shall also say the localic map itself is nearly coz-open.

This is a proper weakening of near openness, as the following example shows.

Example 3.1.5. For any frame L, denote by $\vartheta_L\colon L \to \mathcal{S}(L)^{\mathrm{op}}$ the frame homomorphism given by $a \mapsto \mathfrak{c}_L(a)$. Since, for any $a \in L$, $\mathfrak{c}_L(a)^* = \mathfrak{o}_L(a)$ in $\mathcal{S}(L)^{\mathrm{op}}$, it follows that $\vartheta_L(a^*) = \vartheta_L(a)^*$ if and only if a is complemented. Therefore ϑ_L is nearly open if and only if L is Boolean. Recall that a *P-frame* is one in which every cozero element is complemented. Therefore ϑ_L is nearly coz-open if and only if L is a P-frame. Thus, for any P-frame L which is not Boolean, ϑ_L is a nearly coz-open homomorphism which is not nearly open.

The following lemma will be used in the next section. Recall that if $I \in \beta L$, then the pseudocomplement of I is given by $I^* = r_L(a^*)$ where $a = \bigvee I$.

Lemma 3.1.6. *If $f\colon L \to M$ is a localic map, then βf is nearly coz-open iff f is a nearly coz-open $\overline{\mathrm{C}}$-map.*

Proof. We conduct the proof in **CRFrm**. So let $h\colon M \to L$ be a frame homomorphism, and consider any $I \in \beta M$. Since $j_L \circ \beta h = h \circ j_M$, $\bigvee(\beta h)(I) = h\big(\bigvee I\big)$, and so, setting $a = \bigvee I$, we have

$$(\beta h)(I)^* = r_L(h(a)^*) \qquad \text{and} \qquad (\beta h)(I^*) = (\beta h)(r_M(a^*)). \tag{3.1.4}$$

Now assume that h is a nearly coz-open $\overline{\mathrm{C}}$-homomorphism, and let $I \in \operatorname{Coz}(\beta M)$. Put $a = \bigvee I$. Then $a \in \operatorname{Coz} M$, and so

$$(\beta h)(r_M(a^*)) = r_L(h(a^*)) = r_L(h(a)^*);$$

the first equality arising from h being a $\overline{\mathrm{C}}$-homomorphism, and the second because h is nearly coz-open. In light of the equalities in (3.1.4), we therefore have that βh is nearly coz-open.

Conversely, assume that βh is nearly coz-open. We show first that h is nearly coz-open. So, let $a \in \operatorname{Coz} M$. By [6, Corollary 5], there is an $I \in \operatorname{Coz}(\beta M)$ such that $\bigvee I = a$. Since βh is nearly coz-open, $(\beta h)(I^*) = (\beta h)(I)^*$, which implies $r_L(h(a)^*) = (\beta h)(r_M(a^*))$, as observed in (3.1.4). Taking joins, and invoking the equality $j_L \circ \beta h = h \circ j_M$, yields

$$h(a)^* = \bigvee r_L(h(a)^*) = \bigvee (\beta h)(r_M(a^*)) = h\Big(\bigvee r_M(a^*)\Big) = h(a^*),$$

which shows that h is nearly coz-open. Next, to show that h is a $\overline{\mathrm{C}}$-homomorphism, given $c \in \operatorname{Coz} M$, we must show that $(\beta h)(r_M(c^*)) = r_L(h(c^*))$. But this follows as in the near coz-openness case, with further utilization of the fact that $h(c^*) = h(c)^*$. $\square$

Although we shall not need the following result, we present it because it gives a characterization of nearly open maps that is not stated in [8]. In spaces, nearly open maps were defined by Pták [39] by a condition equivalent to saying $f\colon X \to Y$ is nearly open if and only if for every open set $U \subseteq X$, $f[U] \subseteq \operatorname{int} \overline{f[U]}$.

Proposition 3.1.7. *Suppose that $f\colon L \to M$ is a localic map, and write h for its left adjoint. Then the following statements are equivalent.*

(1) *f is nearly open.*

(2) *For every open sublocale U of L, $f[U] \subseteq \operatorname{int} \overline{f[U]}$.*

(3) *For every $a \in L$, $a \le h(f(a^*)^*)$.*

Proof. (2) $\Leftrightarrow$ (3): Let us observe that, for any $a \in L$,

$$\overline{f[\mathfrak{o}_L(a)]} = \mathfrak{c}_M\Big(\bigwedge f[\mathfrak{o}_L(a)]\Big) = \mathfrak{c}_M\Big(f\Big(\bigwedge \mathfrak{o}_L(a)\Big)\Big) = \mathfrak{c}_M(f(a^*)),$$

and therefore $\operatorname{int} \overline{f[\mathfrak{o}_L(a)]} = \mathfrak{o}_M(f(a^*)^*)$. Since for any $S \in \mathcal{S}(L)$ and $T \in \mathcal{S}(M)$, $f[S] \subseteq T$ if and only if $S \subseteq f_{-1}[T]$, we therefore have

$$\begin{aligned} f[\mathfrak{o}_L(a)] \subseteq \operatorname{int} \overline{f[\mathfrak{o}_L(a)]} \quad &\text{iff} \quad \mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(f(a^*)^*)] \\ &\text{iff} \quad \mathfrak{o}_L(a) \subseteq \mathfrak{o}_L\big(h(f(a^*)^*)\big) \\ &\text{iff} \quad a \le h(f(a^*)^*). \end{aligned}$$

This proves the equivalence of statements (2) and (3).

(3) ⇒ (1): Assume that (3) holds. In accordance with the definition, we must show that h is nearly open. Let $b \in M$. Putting $a = h(b)^*$, the foregoing equivalence says

$$h(b)^* \leq h(f(h(b)^{**})^*).$$

Since $b \leq f(h(b)) \leq f(h(b))^{**}$, we have $f(h(b)^{**})^* \leq b^*$, and so

$$h(b)^* \leq h(f(h(b)^{**})^*) \leq h(b^*),$$

whence we deduce that $h(b)^* = h(b^*)$. Therefore f is nearly open.

(1) ⇒ (3): Assume f is nearly open. We show that $a \leq h(f(a^*)^*)$, for every $a \in L$. Since h is nearly open, $h(f(a^*)^*) = h(f(a^*))^*$, and so, in view of the fact that $h(f(a^*)) \leq a^*$, we have

$$a \leq a^{**} \leq h(f(a^*))^* = h(f(a^*)^*),$$

which then shows that (1) implies (3). □

In the proof of the implication (3) ⇒ (1), we chose a to be $h(b)^*$. Now, if $b \in \operatorname{Coz} M$, it does not follow that $h(b)^* \in \operatorname{Coz} L$. Therefore when we refer to nearly coz-open maps, the corresponding implication does not follow from this one. It still holds though, as we now show.

Proposition 3.1.8. *The following are equivalent for any localic map $f\colon L \to M$ between completely regular frames.*

(1) *f is nearly coz-open.*

(2) *For every cozero-sublocale C of L, $f[C] \subseteq \operatorname{int} \overline{f[C]}$.*

(3) *For every $c \in \operatorname{Coz} L$, $c \leq h(f(c^*)^*)$; where h denotes the left adjoint of f.*

Proof. The equivalence of statements (2) and (3) and that statement (1) implies statement (3) are proved as in the previous proposition.

(3) ⇒ (1): Let $u \in \operatorname{Coz} M$. We must show that $h(u)^* \leq h(u^*)$; and it is here that we use complete regularity. Consider any $c \in \operatorname{Coz} L$ with $c \leq h(u)^*$. Then $h(u)^{**} \leq c^*$, and so

$$u \leq f(h(u)) \leq f(h(u)^{**}) \leq f(c^*),$$

which implies $f(c^*)^* \leq u^*$, whence (by invoking the inequality $c \leq h(f(c^*)^*)$ which holds by the current hypothesis) we obtain

$$c \leq h(f(c^*)^*) \leq h(u^*)$$

which implies $h(u)^* \leq h(u^*)$, by complete regularity, and thence $h(u)^* = h(u^*)$, as required $\square$

3.2 Pulling back

Given a localic map $f\colon L \to M$, we have the ring homomorphism $\mathcal{R}f^*\colon \mathcal{R}M \to \mathcal{R}L$. So if A is a sublocale of L, we have the ideal $\boldsymbol{M}_A$ of $\mathcal{R}L$ which we can then pull back to the ideal $(\mathcal{R}f^*)^{-1}[\boldsymbol{M}_A]$ of $\mathcal{R}M$. We also have the ideal $\boldsymbol{M}_{f[A]}$ of $\mathcal{R}M$. It turns out that these two ideals coincide. To see this, note that if $\alpha \in \mathcal{R}M$ then

$$\begin{aligned}
\alpha \in \boldsymbol{M}_{f[A]} \quad &\text{iff} \quad f[A] \subseteq \mathfrak{c}_M(\operatorname{coz}\alpha) \\
&\text{iff} \quad A \subseteq f_{-1}[\mathfrak{c}_M(\operatorname{coz}\alpha)] \\
&\text{iff} \quad A \subseteq \mathfrak{c}_L(f^*(\operatorname{coz}\alpha)) \\
&\text{iff} \quad A \subseteq \mathfrak{c}_L(\operatorname{coz}(\mathcal{R}f^*)(\alpha)) \\
&\text{iff} \quad (\mathcal{R}f^*)(\alpha) \in \boldsymbol{M}_A \\
&\text{iff} \quad \alpha \in (\mathcal{R}f^*)^{-1}[\boldsymbol{M}_A],
\end{aligned}$$

which then shows that $\boldsymbol{M}_{f[A]} = (\mathcal{R}f^*)^{-1}[\boldsymbol{M}_A]$.

Now let $S \subseteq \beta L$ be a sublocale. We then have the ideals $\boldsymbol{M}^{(\beta f)[S]}$ and $(\mathcal{R}f^*)^{-1}[\boldsymbol{M}^S]$ of $\mathcal{R}M$. Calculating as above, we have that for any $\alpha \in \mathcal{R}M$,

$$\alpha \in \boldsymbol{M}^{(\beta f)[S]} \quad \text{iff} \quad S \subseteq \mathfrak{c}_{\beta L}\big((\beta f^*)(r_M(\operatorname{coz}\alpha))\big) \tag{3.2.1}$$

while, on the other hand,

$$\alpha \in (\mathcal{R}f^*)^{-1}[\boldsymbol{M}^S] \quad \text{iff} \quad S \subseteq \mathfrak{c}_{\beta L}\big(r_L(f^*(\operatorname{coz}\alpha))\big). \tag{3.2.2}$$

Now, since $(\beta f^*)(r_M(\operatorname{coz}\alpha)) \leq r_L(f^*(\operatorname{coz}\alpha))$, as elements of βL, we have

$$\mathfrak{c}_{\beta L}\big(r_L(f^*(\operatorname{coz}\alpha))\big) \subseteq \mathfrak{c}_{\beta L}\big((\beta f^*)(r_M(\operatorname{coz}\alpha))\big).$$

Consequently, we deduce from (3.1.2) in Section 3.1 and (3.2.1) that

$$(\mathcal{R}f^*)^{-1}[\boldsymbol{M}^S] \subseteq \boldsymbol{M}^{(\beta f)[S]}.$$

It appears from (3.2.1) and (3.2.2) that this containment is an equality precisely when the containment in (3.1.2) is an equality for every $c \in \operatorname{Coz} M$, that is, precisely when f is a WN-map. We show in the following theorem (which also includes the results we have just observed) that this is indeed the case.

Theorem 3.2.1. *Let $f\colon L \to M$ be a localic map, with left adjoint h.*

(a) $(\mathcal{R}h)^{-1}[\boldsymbol{M}_A] = \boldsymbol{M}_{f[A]}$ *for every sublocale A of L.*

(b) $(\mathcal{R}h)^{-1}[\boldsymbol{M}^S] \subseteq \boldsymbol{M}^{(\beta f)[S]}$ *for every sublocale S of βL.*

(c) *The following are equivalent.*

  (i) $\boldsymbol{M}^{(\beta f)[S]} = (\mathcal{R}h)^{-1}[\boldsymbol{M}^S]$ *for every sublocale S of βL.*

  (ii) $\boldsymbol{M}^{(\beta f)[S]} = (\mathcal{R}h)^{-1}[\boldsymbol{M}^S]$ *for every closed sublocale U of βL.*

  (iii) *f is WN-map.*

Proof. Only (c) needs to be proved. It is trivial that condition (i) implies condition (ii). It follows from the equivalences in (3.2.1) and (3.2.2) above that condition (iii) implies condition (i). Now suppose that condition (ii) holds. Let $a \in \operatorname{Coz} M$, and choose $\alpha \in \mathcal{R}M$ such that $\operatorname{coz}\alpha = a$. Define the closed sublocale K of βL to be

$$K = \mathfrak{c}_{\beta L}\big((\beta h)(r_M(\operatorname{coz}\alpha))\big).$$

Then, by the equivalence in (3.2.1) above, $\alpha \in \boldsymbol{M}^{(\beta f)[K]}$, and hence $\alpha \in (\mathcal{R}h)^{-1}[\boldsymbol{M}^K]$, by hypothesis. Thus, by the equivalence in (3.2.2) above,

$$\mathfrak{c}_{\beta L}\big((\beta h)(r_M(\operatorname{coz}\alpha))\big) \;=\; K \;\subseteq\; \mathfrak{c}_{\beta L}\big(r_L(h(\operatorname{coz}\alpha))\big)$$

which implies $r_L(h(\operatorname{coz}\alpha)) \leq (\beta h)(r_M(\operatorname{coz}\alpha))$, and hence equality because the opposite inequality always holds. Therefore $(\beta h)(r_M(c)) = r_L(h(c))$ for every $c \in \operatorname{Coz} M$, which shows that condition (ii) implies condition (iii). □

Next, we look at $\boldsymbol{O}$-ideals. As in the previous case, we first prove a "containment result", and then characterise when it is an equality.

Theorem 3.2.2. *Let $f\colon L \to M$ be a localic map, with left adjoint h.*

(a) $\boldsymbol{O}^{(\beta f)[S]} \subseteq (\mathcal{R}h)^{-1}[\boldsymbol{O}^S]$ *for every sublocale S of βL.*

(b) *The following are equivalent.*

  (i) $\boldsymbol{O}^{(\beta f)[S]} = (\mathcal{R}h)^{-1}[\boldsymbol{O}^S]$ *for every sublocale S of βL.*

  (ii) $\boldsymbol{O}^{(\beta f)[U]} = (\mathcal{R}h)^{-1}[\boldsymbol{O}^U]$ *for every open sublocale U of βL.*

  (iii) *βf is nearly coz-open.*

Proof. (a) Let $\alpha \in \mathcal{R}M$, and, for brevity, put $a = \operatorname{coz}\alpha$. Now, for any $S \in \mathcal{S}(\beta L)$,

$$\begin{aligned} \alpha \in \boldsymbol{O}^{(\beta f)[S]} \quad & \text{iff} \quad (\beta f)[S] \subseteq \mathfrak{o}_{\beta M}(r_M(a^*)) \\ & \text{iff} \quad S \subseteq (\beta f)_{-1}[\mathfrak{o}_{\beta M}(r_M(a^*))] \\ & \text{iff} \quad S \subseteq \mathfrak{o}_{\beta L}\big((\beta h)(r_M(a^*))\big) \qquad \text{since } (\beta f)^* = \beta(f^*) = \beta h. \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha \in (\mathcal{R}h)^{-1}[\boldsymbol{O}^S] \quad & \text{iff} \quad (\mathcal{R}h)(\alpha) \in \boldsymbol{O}^S \\ & \text{iff} \quad S \subseteq \mathfrak{o}_{\beta L}\big(r_L(\operatorname{coz}((\mathcal{R}h)(\alpha))^*\big) \\ & \text{iff} \quad S \subseteq \mathfrak{o}_{\beta L}(r_L(h(a)^*)). \end{aligned}$$

Now, since

$$(\beta h)(r_M(a^*)) \,\le\, r_L(h(a^*)) \,\le\, r_L(h(a)^*),$$

we have $\mathfrak{o}_{\beta L}\big((\beta h)(r_M(a^*))\big) \subseteq \mathfrak{o}_{\beta L}\big(r_L(h(a)^*)\big)$, which then shows that $\boldsymbol{O}^{(\beta f)[S]} \subseteq (\mathcal{R}h)^{-1}[\boldsymbol{O}^S]$.

(b) It is trivial that (i) implies (ii).

(ii) $\Rightarrow$ (iii): Suppose that $\boldsymbol{O}^{(\beta f)[U]} = (\mathcal{R}h)^{-1}[\boldsymbol{O}^U]$ for every open sublocale U of βL. By Lemma 3.1.6, it suffices to show that f is a nearly coz-open $\overline{\mathrm{C}}$-map. Working with its left adjoint, we prove first that h is nearly coz-open. Let $c \in \operatorname{Coz} M$, and pick $\gamma \in \mathcal{R}M$ such that $\operatorname{coz}\gamma = c$. Let U be the open sublocale $U = \mathfrak{o}_{\beta L}\big(r_L(h(c)^*)\big)$ of βL. Then $U = \mathfrak{o}_{\beta L}\big(r_L(\operatorname{coz}(\mathcal{R}h)(\gamma))^*\big)$, which then says $(\mathcal{R}h)(\gamma) \in \boldsymbol{O}^U$, and hence $\gamma \in (\mathcal{R}h)^{-1}[\boldsymbol{O}^U]$. By hypothesis, we then have $\gamma \in \boldsymbol{O}^{(\beta f)[U]}$, and so $(\beta f)[U] \subseteq \mathfrak{o}_{\beta M}\big(r_M(c^*)\big)$, whence

$$\mathfrak{o}_{\beta L}\big(r_L(h(c)^*)\big) \;=\; U \;\subseteq\; (\beta f)_{-1}[\mathfrak{o}_{\beta M}\big(r_M(c^*)\big)] \;=\; \mathfrak{o}_{\beta L}\big((\beta h)(r_M(c^*))\big).$$

From this, we deduce that

$$r_L(h(c)^*) \subseteq (\beta h)(r_M(c^*)) \subseteq r_L(h(c^*)), \tag{†}$$

which implies $h(c)^* \leq h(c^*)$, and hence $h(c)^* = h(c^*)$. Therefore h is nearly coz-open. Thus, the containments in (†) are equalities, and so $(\beta h)(r_L(u^*)) = r_L(h(u^*))$ for every $u \in \operatorname{Coz} M$, which says f is a $\overline{\mathrm{C}}$-map by Lemma 3.1.2(c). It therefore follows from Lemma 3.1.6 that βf is nearly coz-open.

(iii) ⇒ (i): Suppose that βf is nearly coz-open. Then, by Lemma 3.1.6, f is nearly coz-open and is a $\overline{\mathrm{C}}$-map, which implies that, for any $a \in \operatorname{Coz} M$,

$$(\beta h)(r_M(a^*)) = r_L(h(a^*)) = r_L(h(a)^*).$$

Therefore the equivalences in the proof of part (a) show that $\boldsymbol{O}^{(\beta f)[S]} = (\mathcal{R}h)^{-1}[\boldsymbol{O}^S]$ for every sublocale S of βL. □

From part (a) of this theorem we obtain the following corollary.

Corollary 3.2.3. *For any localic map $f\colon L \to M$, $\boldsymbol{O}_{f[A]} \subseteq (\mathcal{R}f^*)^{-1}[\boldsymbol{O}_A]$ for every $A \in \mathcal{S}(L)$.*

Proof. Since Diagram (3.1.1) commutes, $(\beta f)[r_L[A]] = r_M[f[A]]$. So, by part (a) of Theorem 3.2.2,

$$\boldsymbol{O}_{f[A]} = \boldsymbol{O}^{r_M[f[A]]} = \boldsymbol{O}^{(\beta f)[r_L[A]]} \subseteq (\mathcal{R}f^*)^{-1}[\boldsymbol{O}^{r_L[A]}] = (\mathcal{R}f^*)^{-1}[\boldsymbol{O}_A],$$

which proves the result. □

The localic maps for which this containment is always an equality are precisely the nearly coz-open ones, as the following result shows.

Theorem 3.2.4. *If $f\colon L \to M$ is localic map, with left adjoint h, then $\boldsymbol{O}_{f[A]} = (\mathcal{R}h)^{-1}[\boldsymbol{O}_A]$ for every sublocale A of L iff f is nearly coz-open.*

Proof. Calculations similar to the ones above show that if $a = \operatorname{coz} \alpha$, then

$$\alpha \in \boldsymbol{O}_{f[A]} \quad \text{iff} \quad A \subseteq \mathfrak{o}_L(h(a^*)),$$

while, on the other hand,

$$\alpha \in (\mathcal{R}h)^{-1}[\boldsymbol{O}_A] \quad \text{iff} \quad A \subseteq \mathfrak{o}_L(h(a)^*).$$

Now, if f is nearly coz-open, then $h(c^*) = h(c)^*$ for every $c \in \operatorname{Coz} M$, so the observations above show that $\boldsymbol{O}_{f[A]} = (\mathcal{R}h)^{-1}[\boldsymbol{O}_A]$. For the converse, assume that $\boldsymbol{O}_{f[A]} = (\mathcal{R}f^*)^{-1}[\boldsymbol{O}_A]$ for every $A \in \mathcal{S}(L)$, and let $c \in \operatorname{Coz} M$. Since $h(c^*) \leq h(c)^*$, we need only show the opposite inequality. Set $A = \mathfrak{o}_L(h(c)^*)$. Pick $\gamma \in \mathcal{R}M$ with $c = \operatorname{coz} \gamma$. Since $\operatorname{coz}(\mathcal{R}h(\gamma)) = h(\operatorname{coz} \gamma)$, we have $(\mathcal{R}h)(\gamma) \in \boldsymbol{O}_A$, so that $\gamma \in (\mathcal{R}h)^{-1}[\boldsymbol{O}_A]$, and hence $\gamma \in \boldsymbol{O}_{f[A]}$, by hypothesis. The latter implies $f[A] \subseteq \mathfrak{o}_M(c^*)$, whence

$$\mathfrak{o}_L(h(c)^*) = A \subseteq f_{-1}[\mathfrak{o}_M(c^*)] = \mathfrak{o}_L(h(c^*)),$$

implying $h(c)^* \leq h(c^*)$, thence equality. Therefore f is nearly coz-open. □

3.3 Pushing forward

In the previous section, given a localic map $f \colon L \to M$, we started with a sublocale S of βL, formed the ideal $\boldsymbol{O}^S$ of $\mathcal{R}L$, pulled it back along the ring homomorphism $\mathcal{R}f^* \colon \mathcal{R}M \to \mathcal{R}L$, and then compared the resulting ideal with the one obtained by first pushing the sublocale S along βf and then computing the ideal $\boldsymbol{O}^{(\beta f)[S]}$.

In this section we perform the "dual" process. Namely, we start with a sublocale T of βM, pull it back along βf to form the ideal $\boldsymbol{O}^{(\beta f)_{-1}[T]}$ of $\mathcal{R}L$. On the other hand, we push the ideal $\boldsymbol{O}^T$ of $\mathcal{R}M$ forward along the ring homomorphism $\mathcal{R}f^* \colon \mathcal{R}M \to \mathcal{R}L$ to obtain the ideal generated by the image $(\mathcal{R}f^*)[\boldsymbol{O}^T]$, and then compare the two ideals.

In what follows, we write $\langle H \rangle$ for the ideal generated by a set H. We recall from [14, Lemma 4.4] that if γ and δ are elements of $\mathcal{R}L$ such that $\operatorname{coz} \gamma \prec\!\!\prec \operatorname{coz} \delta$, then γ is a multiple of δ.

Theorem 3.3.1. *Let $f \colon L \to M$ be a localic map.*

(a) $\langle (\mathcal{R}f^*)[\boldsymbol{O}^T] \rangle \subseteq \boldsymbol{O}^{(\beta f)_{-1}[T]}$*, for every sublocale T of βM.*

(b) $\langle(\mathcal{R}f^*)[\boldsymbol{O}^K]\rangle = \boldsymbol{O}^{(\beta f)_{-1}[K]}$, *for every closed sublocale K of βM.*

(c) $\langle(\mathcal{R}f^*)[\boldsymbol{O}_B]\rangle \subseteq \boldsymbol{O}_{f_{-1}[B]}$, *for every sublocale B of M.*

Proof. (a) Let $\gamma \in \boldsymbol{O}^T$, and put $c = \operatorname{coz}\gamma$. Write h for f^*. Then $T \subseteq \mathfrak{o}_{\beta M}(r_M(\operatorname{coz}\gamma)^*)$, which implies

$$\begin{aligned}(\beta f)_{-1}[T] &\subseteq (\beta f)_{-1}[\mathfrak{o}_{\beta M}\big(r_M(c^*)\big)]\\ &= \mathfrak{o}_{\beta L}\big((\beta h)(r_M(c^*)\big)\\ &\subseteq \mathfrak{o}_{\beta L}\big(r_L(h(c^*))\big) \qquad \text{since } \beta h\circ r_M \leq r_L\circ h\\ &\subseteq \mathfrak{o}_{\beta L}\big(r_L(h(c)^*)\big)\\ &= \mathfrak{o}_{\beta L}\big(r_L(\operatorname{coz}((\mathcal{R}h)(\gamma))^*)\big).\end{aligned}$$

Therefore $(\mathcal{R}h)(\gamma) \in \boldsymbol{O}^{(\beta f)_{-1}[B]}$, from which we deduce that $(\mathcal{R}f^*)[\boldsymbol{O}^T] \subseteq \boldsymbol{O}^{(\beta f)_{-1}[T]}$. The result therefore follows because $\boldsymbol{O}^{(\beta f)_{-1}[T]}$ is an ideal.

(b) Taking into cognisance the result in (a), we need only show that if K is a closed sublocale of βM, then $\boldsymbol{O}^{(\beta f)_{-1}[K]} \subseteq \langle(\mathcal{R}f^*)[\boldsymbol{O}^K]\rangle$. Pick $I \in \beta M$ with $K = \mathfrak{c}_{\beta M}(I)$, so that

$$(\beta f)_{-1}[K] = \mathfrak{c}_{\beta L}((\beta h)(I)).$$

Let $\alpha \in \boldsymbol{O}^{(\beta f)_{-1}[K]}$. Then $r_L(\operatorname{coz}\alpha) \prec\!\!\prec (\beta h)(I)$, which implies $r_L(\operatorname{coz}\alpha)^* \vee (\beta h)(I) = 1_{\beta L}$. Since $I = \bigvee_{u\in I} r_M(u)$, we therefore have

$$r_L(\operatorname{coz}\alpha)^* \vee \bigvee_{u\in I}(\beta h)(r_M(u)) = 1_{\beta L},$$

and so, by compactness of βL, there are finitely many elements $u_1, \dots, u_n$ in I such that

$$r_L(\operatorname{coz}\alpha)^* \vee \Big((\beta h)(r_M(u_1)) \vee \cdots \vee (\beta h)(r_M(u_n))\Big) = 1_{\beta L}.$$

Putting $c = u_1 \vee \cdots \vee u_n$, we have that $c \in I$ and

$$r_L(\operatorname{coz}\alpha)^* \vee (\beta h)(r_M(c)) = 1_{\beta L}.$$

Therefore $r_L(\operatorname{coz}\alpha) \prec\!\!\prec (\beta h)(r_M(c))$. On applying the join map $j_L\colon \beta L \to L$, we obtain

$$\operatorname{coz}\alpha = j_L(r_L(\operatorname{coz}\alpha)) \prec\!\!\prec \big(j_L\circ(\beta h)\big)(r_L(h(c))) = (h\circ j_M)(r_M(c)) = h\Big(\bigvee r_M(c)\Big) = h(c).$$

Pick $\gamma \in \mathcal{R}M$ with $\operatorname{coz}\gamma = c$. Then we have

$$\operatorname{coz}\alpha \prec\!\!\prec h(\operatorname{coz}\gamma) = \operatorname{coz}((\mathcal{R}h)(\gamma)),$$

which implies α is a multiple of $(\mathcal{R}h)(\gamma)$. But now $\gamma \in \boldsymbol{O}^{\mathfrak{c}_{\beta M}(I)}$ because $\operatorname{coz}\gamma \in I$; so $(\mathcal{R}h)(\gamma) \in (\mathcal{R}h)[\boldsymbol{O}^K]$, which then implies $\alpha \in \langle(\mathcal{R}h)[\boldsymbol{O}^K]\rangle$. Therefore $\boldsymbol{O}^{(\beta f)_{-1}[K]} \subseteq \langle(\mathcal{R}f^*)[\boldsymbol{O}^K]\rangle$, hence we have equality by part (a).

(c) We use the result in (a). Since $(\beta f) \circ r_L = r_M \circ f$, and since $f[f_{-1}[B]] \subseteq B$, for any sublocale B of M, we have

$$(\beta f)[r_L[f_{-1}[B]]] = r_M[f[f_{-1}[B]]] \subseteq r_M[B],$$

which implies $r_L[f_{-1}[B]] \subseteq (\beta f)_{-1}[r_M[B]]$, and hence $\boldsymbol{O}^{(\beta f)_{-1}[r_M[B]]} \subseteq \boldsymbol{O}^{r_L[f_{-1}[B]]}$. By (a), we therefore have

$$(\mathcal{R}f^*)[\boldsymbol{O}_B] = (\mathcal{R}f^*)[\boldsymbol{O}^{r_M[B]}] \subseteq \boldsymbol{O}^{(\beta f)_{-1}[r_M[B]]} \subseteq \boldsymbol{O}^{r_L[f_{-1}[B]]} = \boldsymbol{O}_{f_{-1}[B]},$$

from which the result follows. □

Part (b) of this theorem says the containment in part (a) is always an equality when restricted to closed sublocales of βM. One may thus wonder if the containment in part (c) is always an equality when restricted to closed sublocales of M. We shall see that it is not. In fact, when M satisfies a certain property (we shall introduce it shortly) strictly weaker than normality, we shall characterise the localic maps $f\colon L \to M$ for which the containment in part (c) is an equality on closed sublocales of M.

As in [20], we say a frame M is *coz-interpolative* in case for any $c \in \operatorname{Coz} M$ and any $m \in M$, the relation $c \prec m$ implies $c \prec\!\!\prec m$. Every normal frame is coz-interpolative, but the frame of open subsets of the space described in [24, Problem 6Q] is non-normal and coz-interpolative, as observed in [20].

In the proof of the next theorem we shall use the following result which appears as [12, Lemma 4.2]. We restate it using terminology introduced above.

Lemma 3.3.2. *A frame homomorphism $h\colon M \to L$ is a* K*-homomorphism iff for every $y \in L$ and $a \in M$, $y \prec\!\!\prec h(a)$ implies $y \leq h(s)$ for some $s \prec\!\!\prec a$ in M.*

Observe that the requirement that $y \leq h(s)$ in this characterization can be replaced with $y \prec\!\!\prec h(s)$ because if $y \prec\!\!\prec h(a)$, we can interpolate to obtain $z \in L$ such that $y \prec\!\!\prec z \prec\!\!\prec h(a)$, and then apply the lemma to the relation $z \prec\!\!\prec h(a)$.

Theorem 3.3.3. *If $f\colon L \to M$ is a localic map with M coz-interpolative, then $\langle(\mathcal{R}f^*)[\boldsymbol{O}_B]\rangle = \boldsymbol{O}_{f_{-1}[B]}$ for every closed sublocale B of M iff f is a* K*-map.*

Proof. ($\Rightarrow$): Suppose that $\langle(\mathcal{R}f^*)[\boldsymbol{O}_B]\rangle = \boldsymbol{O}_{f_{-1}[B]}$ for every closed sublocale of B of M. We use Lemma 3.3.2 to prove that f^* is an K-homomorphism. Write h for f^*, and consider any $a \in M$ and $y \in L$ such that $y \prec\!\!\prec h(a)$. The hypothesis says $\boldsymbol{O}_{f_{-1}[\mathfrak{c}_M(a)]} = \langle(\mathcal{R}h)[\boldsymbol{O}_{\mathfrak{c}_M(a)}]\rangle$; that is, $\boldsymbol{O}_{\mathfrak{c}_L(h(a))} = \langle(\mathcal{R}h)[\boldsymbol{O}_{\mathfrak{c}_M(a)}]\rangle$. Pick $c \in \operatorname{Coz} L$ such that $y \prec\!\!\prec c \prec\!\!\prec h(a)$, and then choose $\gamma \in \mathcal{R}L$ with $\operatorname{coz}\gamma = c$. Then $\gamma \in \boldsymbol{O}_{\mathfrak{c}_L(h(a))}$, and so there exist functions $\alpha_1, \ldots, \alpha_n$ in $\mathcal{R}M$ and functions $\delta_1, \ldots, \delta_n$ in $\boldsymbol{O}_{\mathfrak{c}_M(a)}$ such that

$$\gamma = \alpha_1 \cdot (\mathcal{R}h)(\delta_1) + \cdots + \alpha_n \cdot (\mathcal{R}h)(\delta_n).$$

Applying the cozero map $\operatorname{coz}\colon \mathcal{R}L \to L$ to this yields

$$\begin{aligned} c \;=\; \operatorname{coz}\gamma \;&\leq\; \operatorname{coz}((\mathcal{R}h)(\delta_1)) \vee \cdots \vee \operatorname{coz}((\mathcal{R}h)(\delta_n)) \\ &=\; h(\operatorname{coz}(\delta_1)) \vee \cdots \vee h(\operatorname{coz}(\delta_n)). \end{aligned}$$

Since each $\delta_i \in \boldsymbol{O}_{\mathfrak{c}_M(a)}$, we have $\operatorname{coz}(\delta_i) \prec a$, and hence $\operatorname{coz}(\delta_i) \prec\!\!\prec a$ because M is coz-interpolative. So, putting $d = \operatorname{coz}(\delta_1) \vee \cdots \vee \operatorname{coz}(\delta_n)$, we have that $d \prec\!\!\prec a$ and $y \leq h(d)$. By Lemma 3.3.2, this proves that h is an N-homomorphism, and hence f is an N-map.

($\Leftarrow$): Suppose that f is an N-map, and let B be a closed sublocale of M. As before, we write h in place of f^*. We know from Theorem 3.3.1(c) that $\langle(\mathcal{R}h)[\boldsymbol{O}_B]\rangle \subseteq \boldsymbol{O}_{f_{-1}[B]}$. To prove the reverse inclusion, pick $a \in M$ such that $B = \mathfrak{c}_M(a)$, and let $\alpha \in \boldsymbol{O}_{f_{-1}[B]} = \boldsymbol{O}_{\mathfrak{c}_L(h(a))}$. Then $\operatorname{coz}\alpha \prec h(a)$, and hence $\operatorname{coz}\alpha \prec\!\!\prec h(a)$ because M is coz-interpolative. Since h is an N-homomorphism, we can find $d \in \operatorname{Coz} M$ such that $d \prec\!\!\prec a$ and $\operatorname{coz}\alpha \prec\!\!\prec h(d)$. Take $\delta \in \mathcal{R}M$ with $\operatorname{coz}\delta = d$. Now, $\operatorname{coz}\alpha \prec\!\!\prec h(d)$ implies $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}((\mathcal{R}h)(\delta))$, and so, by [14, Lemma 4.4], α is a multiple of $(\mathcal{R}h)(\delta)$. Since $\operatorname{coz}\delta \prec\!\!\prec a$, we have that $\delta \in \boldsymbol{O}_{\mathfrak{c}_M(a)}$, and hence

$$(\mathcal{R}h)(\delta) \;\in\; (\mathcal{R}h)[\boldsymbol{O}_B] \;\subseteq\; \langle(\mathcal{R}h)[\boldsymbol{O}_B]\rangle,$$

which then implies $\alpha \in \langle(\mathcal{R}h)[\boldsymbol{O}_B]\rangle$. Therefore $\boldsymbol{O}_{f_{-1}[B]} \subseteq \langle(\mathcal{R}h)[\boldsymbol{O}_B]\rangle$, and equality follows. □

Regarding the $\boldsymbol{M}$-ideals in this context of pushing forward, the types of calculations that we have seen a number of times now yield the following.

Proposition 3.3.4. *Let $f\colon L \to M$ be a localic map. Then:*

(a) $\langle (\mathcal{R}f^*)[\boldsymbol{M}_A] \rangle \subseteq \boldsymbol{M}_{f_{-1}[A]}$ *for every sublocale A of M.*

(b) $\langle (\mathcal{R}f^*)[\boldsymbol{M}^T] \rangle \subseteq \boldsymbol{M}^{(\beta f)_{-1}[T]}$ *for every sublocale T of βM iff f is a WN-map.*

3.4 What happens in $C(X)$?

All the results in Sections 3.2 and 3.3 hold for $C(X)$, *mutatis mutandis.* We shall not state all of them. Instead, we shall set up the tools for proving them and, as an illustration of how to use the tools, prove just one deduced from each section.

In Chapter 2 we showed how to relate the $\boldsymbol{O}$- and $\boldsymbol{M}$-ideals of $C(X)$ to those of $\mathcal{R}(\Omega(X))$. That will be used in Chapter 4. For recurrent purposes it is convenient to relate the two types of ideals using the machinery developed in Diagram (3.1.3) in Section 3.1.

Using the notation of that diagram, we have that since $\delta_X\colon \Omega(\beta X) \to \beta(\Omega(X))$ is a frame isomorphism, it is also a localic isomorphism. The proofs of the next lemma and the two corollaries following it are immediate from what we proved in Chapter 2 and the fact we have just mentioned about δ_X.

Lemma 3.4.1. *For any $p \in \beta X$, $\varphi_X[\boldsymbol{M}^p] = \boldsymbol{M}^{\{\delta_X(\tilde{p}),1\}}$.*

Corollary 3.4.2. *For any $p \in \beta X$, $\varphi_X[\boldsymbol{O}^p] = \boldsymbol{O}^{\{\delta_X(\tilde{p}),1\}}$.*

Corollary 3.4.3. *For any $A \subseteq \beta X$, $\varphi_X[\boldsymbol{M}^A] = \boldsymbol{M}^{\delta_X[\tilde{A}]}$ and $\varphi_X[\boldsymbol{O}^A] = \boldsymbol{O}^{\delta_X[\tilde{A}]}$.*

The following lemma will be needed below. Let $f\colon X \to Y$ be a continuous map and $A \subseteq X$. As can be deduced from [35, Proposition VI.1.3.1],

$$\mathrm{Pt}(\tilde{A}) = \{\tilde{x} \mid x \in A\} \qquad \text{hence} \qquad \mathrm{Pt}\left(\widetilde{f[A]}\right) = \{\widetilde{f(x)} \mid x \in A\}.$$

As observed in [35, II.2.4], if $f\colon X \to Y$ is a continuous map, then $(\Omega(f))_*(\tilde{x}) = \widetilde{f(x)}$ for every $x \in X$.

Lemma 3.4.4. *If $f\colon X \to Y$ is a continuous function and $A \subseteq X$, then* $\widetilde{f[A]} = (\Omega(f))_*[\tilde{A}]$.

Proof. Since both $\widetilde{f[A]}$ and $(\Omega(f))_*[\tilde{A}]$ are spatial sublocales of $\Omega(Y)$, to show that they coincide we need only show that they contain exactly the same points. But this is easy to deduce from the little discussion immediately preceding the statement of the lemma. □

Here is one more tool that we shall use. Let $f\colon X \to Y$ be a continuous map. Since $\Omega(f) = f^{-1}$ and since $(g \circ k)^{-1} = k^{-1} \circ g^{-1}$ for any two composable functions, the square

$$\begin{array}{ccc} C(Y) & \xrightarrow{C(f)} & C(X) \\ \downarrow{\scriptstyle \varphi_Y} & & \downarrow{\scriptstyle \varphi_X} \\ \mathcal{R}(\Omega(Y)) & \xrightarrow{\mathcal{R}(\Omega(f))} & \mathcal{R}(\Omega(X)) \end{array}$$

commutes in the category of rings, so that

$$\varphi_X \circ C(f) = \mathcal{R}(\Omega(f)) \circ \varphi_Y \qquad \text{and hence} \qquad C(f)^{-1} \circ \varphi_X^{-1} = \varphi_Y^{-1} \circ (\mathcal{R}(\Omega(f)))^{-1}.$$

Since φ_X is an isomorphism, we therefore have

$$C(f)^{-1} = \varphi_Y^{-1} \circ \mathcal{R}(\Omega(f))^{-1} \circ \varphi_X. \tag{3.4.1}$$

Now here are the $C(X)$ versions of Theorem 3.2.1(b) and Theorem 3.2.2(a).

Corollary 3.4.5. *If $f\colon X \to Y$ is a continuous function, then:*

(a) $C(f)^{-1}[\boldsymbol{M}^A] \subseteq \boldsymbol{M}^{(\beta f)[A]}$ *for every* $A \subseteq \beta X$.

(b) $\boldsymbol{O}^{(\beta f)[A]} \subseteq C(f)^{-1}[\boldsymbol{O}^A]$ *for every* $A \subseteq \beta X$.

Proof. (a) Consider the frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ and the sublocale $\delta_X[\widetilde{A}]$ of $\beta(\Omega(X))$. Since $\beta(\Omega(f)) \circ \delta_Y = \delta_X \circ \Omega(\beta f)$ from Diagram (3.1.3), we have $(\beta(\Omega(f)))_* \circ \delta_X = \delta_Y \circ (\Omega(\beta f))_*$, upon taking right adjoints and recalling that δ_X and δ_Y are isomorphisms. Consequently, in light of Lemma 3.4.4,

$$\beta(\Omega(f))_*[\delta_X[\tilde{A}]] = \delta_Y[\Omega(\beta f)_*[\tilde{A}]] = \delta_Y\left[\widetilde{\beta f[A]}\right].$$

We know from Theorem 3.2.1(b) that $\mathcal{R}(\Omega(f))^{-1}[\boldsymbol{M}^{\delta_X[\tilde{A}]}] \subseteq \boldsymbol{M}^{(\beta(\Omega(f)))_*[\delta_X[\tilde{A}]]}$. So, computing $C(f)^{-1}[\boldsymbol{M}^A]$ via the equality in (3.4.2), we obtain

$$\begin{aligned}
C(f)^{-1}[\boldsymbol{M}^A] &= \varphi_Y^{-1}\left[\mathcal{R}(\Omega(f))^{-1}\left[\varphi_X\left[\boldsymbol{M}^A\right]\right]\right] \\
&= \varphi_Y^{-1}\left[\mathcal{R}(\Omega(f))^{-1}\left[\boldsymbol{M}^{\delta_X[\tilde{A}]}\right]\right] && \text{by Corollary 3.4.3} \\
&\subseteq \varphi_Y^{-1}\left[\boldsymbol{M}^{(\beta(\Omega(f)))_*\left[\delta_X[\tilde{A}]\right]}\right] \\
&= \varphi_Y^{-1}\left[\boldsymbol{M}^{\delta_Y\left[\widetilde{\beta f[A]}\right]}\right] \\
&= \varphi_Y^{-1}\left[\varphi_Y\left[\boldsymbol{M}^{(\beta f)[A]}\right]\right] && \text{by Corollary 3.4.3} \\
&= \boldsymbol{M}^{(\beta f)[A]},
\end{aligned}$$

which proves the result.

(b) Similar to that of part (a), except that we must invoke Theorem 3.2.2(a) in this case. □

Next, we prove the $C(X)$ version of Theorem 3.3.1. It is not hard to show that if K is a closed subspace of X, then $\widetilde{K}$ is a closed sublocale of $\Omega(X)$, and, in fact, $\widetilde{K} = \mathfrak{c}_{\Omega(X)}(X \smallsetminus K)$. In consequence,

> *the closed sublocales of* $\Omega(X)$ *are precisely the sublocales* $\widetilde{K}$*, for* K *a closed subspace of* X*.*

We shall need the following lemma. Since $f_{-1}[\mathfrak{c}_M(m)] = f^{-1}[\mathfrak{c}_M(m)]$ for any localic map $f\colon L \to M$ and $m \in M$, we shall write the localic inverse image of closed sublocales as in the latter case. This is to avoid $((\Omega(f))_*)_{-1}[-]$ in favour of $(\Omega(f))_*^{-1}[-]$.

Lemma 3.4.6. *If* $f\colon X \to Y$ *is a continuous map, then* $(\Omega(f))_*^{-1}[\widetilde{K}] = \widetilde{f^{-1}[K]}$ *for every closed subset* K *of* Y*.*

Proof. As in the case of images, it suffices to show that these two sublocales have the same points. Observe that

$$(\Omega(f))_*^{-1}[\widetilde{K}] = (\Omega(f))_*^{-1}\left[\mathfrak{c}_{\Omega(Y)}(Y \smallsetminus K)\right] = \mathfrak{c}_{\Omega(X)}\big(f^{-1}[Y \smallsetminus K]\big) = \mathfrak{c}_{\Omega(X)}\big(X \smallsetminus f^{-1}[K]\big).$$

Now, for any $x \in X$,

$$\begin{aligned}
\tilde{x} \in \mathfrak{c}_{\Omega(X)}\big(X \smallsetminus f^{-1}[K]\big) \quad &\text{iff} \quad X \smallsetminus \{x\} \in \mathfrak{c}_{\Omega(X)}\big(X \smallsetminus f^{-1}[K]\big) \\
&\text{iff} \quad X \smallsetminus f^{-1}[K] \subseteq X \smallsetminus \{x\} \\
&\text{iff} \quad x \in f^{-1}[K] \\
&\text{iff} \quad \tilde{x} \in \widetilde{f^{-1}[K]},
\end{aligned}$$

which proves that $\mathrm{Pt}\left((\Omega(f))_*^{-1}[\widetilde{K}]\right) = \mathrm{Pt}\left(\widetilde{f^{-1}[K]}\right)$, whence the result follows by spatiality. $\square$

Note that if $f\colon L \to M$ is a localic isomorphism, then $f_{-1}[-]$ is exactly the set-theoretic $f^{-1}[-]$.

Corollary 3.4.7. *If $f\colon X \to Y$ is a continuous map and K is a closed subset of βY, then $\langle C(f)[\boldsymbol{O}^K]\rangle = \boldsymbol{O}^{(\beta f)^{-1}[K]}$.*

Proof. Consider the frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ and the closed sublocale $\delta_Y[\widetilde{K}]$ of $\beta(\Omega(Y))$, and apply Theorem 3.3.1(b) to this data to obtain the equality

$$\left\langle \mathcal{R}(\Omega(f)) \left[\boldsymbol{O}^{\delta_Y[\widetilde{K}]}\right] \right\rangle = \boldsymbol{O}^{(\beta(\Omega(f)))_*^{-1}[\delta_Y[\widetilde{K}]]}.$$

The commutativity of the diagram above that led to equation (3.4.2) gives the equality $\mathcal{R}(\Omega(f)) = \varphi_X \circ C(f) \circ \varphi_Y^{-1}$, and so, taking into account the fact that $\varphi_Y^{-1}\left[\boldsymbol{O}^{\delta_Y[\widetilde{K}]}\right] = \boldsymbol{O}^K$, by Corollary 3.4.3,

$$\left\langle \mathcal{R}(\Omega(f)) \left[\boldsymbol{O}^{\delta_Y[\widetilde{K}]}\right] \right\rangle = \left\langle \varphi_X \left[C(f)\left[\boldsymbol{O}^K\right]\right] \right\rangle = \varphi_X \left[\left\langle C(f)\left[\boldsymbol{O}^K\right]\right\rangle\right];$$

the latter because φ_X is an isomorphism (actually, being onto suffices). On the other hand, from Diagram (3.1.3) we have $\beta(\Omega(f)) = \delta_X \circ \Omega(\beta f) \circ \delta_Y^{-1}$, so that, upon taking right adjoints and then localic inverse images (each of which reverses the order of composition), we have

$$(\beta(\Omega(f)))_*^{-1}\left[\delta_Y[\widetilde{K}]\right] = \delta_X\left[(\Omega(\beta f))_*^{-1}\left[\delta_Y^{-1}\left[\delta_Y[\widetilde{K}]\right]\right]\right] = \delta_X\left[(\Omega(\beta f))_*^{-1}[\widetilde{K}]\right] = \delta_X\left[\widetilde{(\beta f)^{-1}[K]}\right];$$

the last equality emanating from Lemma 3.4.6. Therefore, in light of Corollary 3.4.3,

$$\boldsymbol{O}^{(\beta(\Omega(f)))_*^{-1}[\delta_Y[\widetilde{K}]]} = \boldsymbol{O}^{\delta_X\left[\widetilde{(\beta f)^{-1}[K]}\right]} = \varphi_X\left[\boldsymbol{O}^{(\beta f)^{-1}[K]}\right].$$

Since φ_X is an isomorphism, it therefore follows that $\langle C(f)[\boldsymbol{O}^K]\rangle = \boldsymbol{O}^{(\beta f)^{-1}[K]}$. $\square$

Theorem 3.4.8. *If $f\colon X \to Y$ is a continuous function, then:*

(a) $C(f)^{-1}[\boldsymbol{M}^S] \subseteq \boldsymbol{M}^{(\beta f)[S]}$ *for every* $S \subseteq \beta X$.

(b) $\boldsymbol{O}^{(\beta f)[S]} \subseteq C(f)^{-1}[\boldsymbol{O}^S$ *for every* $S \subseteq \beta X$.

Proof. (a) Since $f^*(U) = f^{-1}[V]$ for every $V \in \Omega(Y)$, and since $(g \circ k)^{-1} = k^{-1} \circ g^{-1}$ for any two composable functions, the square

$$\begin{array}{ccc} C(Y) & \xrightarrow{\;C(f)\;} & C(X) \\ {\scriptstyle \varphi_Y}\downarrow & & \downarrow{\scriptstyle \varphi_X} \\ \mathcal{R}(\Omega(Y)) & \xrightarrow{\;\mathcal{R}f^*\;} & \mathcal{R}(\Omega(X)) \end{array}$$

commutes in the category of rings, so that

$$\varphi_X \circ C(f) = \mathcal{R}f^* \circ \varphi_Y \qquad \text{and hence} \qquad C(f)^{-1} \circ \varphi_X^{-1} = \varphi_Y^{-1} \circ (\mathcal{R}f^*)^{-1}.$$

Since φ_X is an isomorphism, we therefore have

$$C(f)^{-1} = \varphi_Y^{-1} \circ (\mathcal{R}f^*)^{-1} \circ \varphi_X. \tag{3.4.2}$$

Now consider the localic map $f_*\colon \Omega(X) \to \Omega(Y)$ and the sublocale $\widetilde{S}$ of $\Omega(\beta X)$. We know from Theorem 3.2.1(b) that $(\mathcal{R}f^*)^{-1}[\boldsymbol{M}^{\widetilde{S}}] \subseteq \boldsymbol{M}^{(\beta f_*)[\widetilde{S}]}$. So, computing $C(f)^{-1}[\boldsymbol{M}^S]$ via the equality in (3.4.2), we obtain

$$\begin{aligned} C(f)^{-1}[\boldsymbol{M}^S] &= \varphi_Y^{-1}[(\mathcal{R}f^*)^{-1}[\varphi_M[\boldsymbol{M}^S]] \\ &= \varphi_Y^{-1}[(\mathcal{R}f^*)^{-1}[\boldsymbol{M}^{\widetilde{S}}]] \qquad \text{in light of Corollary 3.4.3} \\ &\subseteq \varphi_Y^{-1}[\boldsymbol{M}^{(\beta f_*)[\widetilde{S}]}] \\ &= \varphi_Y^{-1}[\boldsymbol{M}^{(\beta f)_*[\widetilde{S}]}] \\ &= \varphi_Y^{-1}[\boldsymbol{M}^{\widetilde{(\beta f)[S]}}] \qquad \text{by Lemma 3.4.4} \\ &= \varphi_Y^{-1}[\varphi_Y[\boldsymbol{M}^{(\beta f)[S]}]] \qquad \text{by Corollary 3.4.3} \\ &= \boldsymbol{M}^{(\beta f)[S]}, \end{aligned}$$

which proves the result.

(b) The proof of this part is similar, except that we must invoke Theorem 3.2.2(a) in this case. □

We now want to obtain some $C(X)$-analogues of the other results from the previous sections. For that we need to know how closed sublocales transfer from spaces to induced locales.

Lemma 3.4.9. *Let K be a closed subspace of X. Then $\widetilde{K}$ is a closed sublocale of $\Omega(X)$. In fact, $\widetilde{K} = \mathfrak{c}_{\Omega(X)}(X \smallsetminus K)$.*

Proof. Denote by $h\colon \Omega(X) \to \Omega(K)$ the induced frame homomorphism $U \mapsto U \cap K$. If $U \in \Omega(X)$, let $U^{\ltimes}$ denote the largest open set in X with $U^{\ltimes} \cap K = U \cap K$. With this notation,

$$\widetilde{K} = \{U^{\ltimes} \mid U \in \Omega(X)\}.$$

The bottom element of $\widetilde{K}$ is $h_*(0_{\Omega(K)}) = \emptyset^{\ltimes}$. Since K is closed, $X \smallsetminus K$ is open, and is the largest open subset of X disjoint from K. Therefore $0_{\widetilde{K}} = X \smallsetminus U$. Consequently, $\widetilde{K} \subseteq \mathfrak{c}_{\Omega(X)}(X \smallsetminus K)$. For the reverse inclusion, let $U \in \mathfrak{c}_{\Omega(X)}(X \smallsetminus K)$. We argue that $U = U^{\ltimes}$. Since $U \subseteq U^{\ltimes}$, we show that $U^{\ltimes} \subseteq U$. Let $x \in U^{\ltimes}$. If $x \in X \smallsetminus K$, then $x \in U$. If $x \in K$, then $x \in U^{\ltimes} \cap K = U \cap K$; so $x \in U$. Thus, in either of the two exhaustive possibilities, we have $x \in U$. Therefore $U = U^{\ltimes}$, showing that $\mathfrak{c}_{\Omega(X)}(X \smallsetminus K) \subseteq \widetilde{K}$, and hence $\widetilde{K} = \mathfrak{c}_{\Omega(X)}(X \smallsetminus K)$. $\square$

We deduce immediately from this lemma that

> *the closed sublocales of $\Omega(X)$ are precisely the sublocales $\widetilde{K}$, for K a closed subspace of X.*

We can now apply this to obtain the $C(X)$-version of Theorem 3.2.1(c). First though, given a continuous function $f\colon X \to Y$, let us express $(\mathcal{R}f^*)^{-1}$ as a suitable composite as we did $C(f)^{-1}$ in equation (3.4.2) in the proof of Theorem 3.4.8. From the equality $C(f)^{-1} \circ \varphi_X^{-1} = \varphi_Y^{-1} \circ (\mathcal{R}f^*)^{-1}$ we get

$$(\mathcal{R}f^*)^{-1} = \varphi_Y \circ C(f)^{-1} \circ \varphi_X^{-1}.$$

Theorem 3.4.10. *The following are equivalent for a continuous function $f\colon X \to Y$.*

(1) *$C(f)^{-1}[\boldsymbol{M}^S] = \boldsymbol{M}^{(\beta f)[S]}$, for every subset S of βX.*

(2) *f is a WN-map.*

Proof. We show first that $C(f)^{-1}[\boldsymbol{M}^K] = \boldsymbol{M}^{(\beta f)[K]}$ for every closed subset K of βX if and only if $(\mathcal{R}f^*)^{-1}[\boldsymbol{M}^F] = \boldsymbol{M}^{(\beta f_*)[F]}$ for every closed sublocale F of $\Omega(\beta X)$.

Suppose, first, that $(\mathcal{R}f^*)^{-1}[\boldsymbol{M}^F] = \boldsymbol{M}^{(\beta f_*)[F]}$ for every closed sublocale F of $\Omega(\beta X)$. Let K be a closed subset of βX. Then, by Lemma 3.4.9, $\widetilde{K}$ is a closed sublocale of $\Omega(\beta X)$. So, by hypothesis, and applying Corollary 3.4.3 and Lemma 3.4.4, we get

$$(\mathcal{R}f^*)^{-1}[\varphi_X[\boldsymbol{M}^K]] = (\mathcal{R}f^*)^{-1}[\boldsymbol{M}^{\widetilde{K}}] = \boldsymbol{M}^{(\beta f_*)[\widetilde{K}]} = \boldsymbol{M}^{\widetilde{(\beta f)[K]}} = \varphi_Y[\boldsymbol{M}^{(\beta f)[K]}],$$

which implies

$$\varphi_Y^{-1}[(\mathcal{R}f^*)^{-1}[\varphi_X[\boldsymbol{M}^K]]] = \boldsymbol{M}^{(\beta f)[K]},$$

and hence $C(f)^{-1}[\boldsymbol{M}^K] = \boldsymbol{M}^{(\beta f)[K]}$, in light of (3.4.2).

For the other way round, suppose that $C(f)^{-1}[\boldsymbol{M}^K] = \boldsymbol{M}^{(\beta f)[K]}$, for every closed subset K of βX. Let S be a closed sublocale of $\Omega(\beta X)$. Then, as observed above, $S = \widetilde{K}$, for some closed subset K of βX. Then, by hypothesis and the equality in (3.4.2),

$$\varphi_Y^{-1}[(\mathcal{R}f^*)^{-1}[\varphi_X[\boldsymbol{M}^K]]] = \boldsymbol{M}^{(\beta f)[K]}.$$

Since $\varphi_X[\boldsymbol{M}^K] = \boldsymbol{M}^{\widetilde{K}} = \boldsymbol{M}^S$, the foregoing equality implies

$$(\mathcal{R}f^*)^{-1}[\boldsymbol{M}^S] = \varphi_Y[\boldsymbol{M}^{(\beta f)[K]}] = \boldsymbol{M}^{\widetilde{(\beta f)[K]}} = \boldsymbol{M}^{(\beta f_*)[\widetilde{K}]} = \boldsymbol{M}^{(\beta f_*)[S]},$$

which establishes the claim. $\square$

Chapter 4

Annihilator ideals and the socle of $\mathcal{R}L$

In [15], the author characterizes the socle (we will recall the definition shortly) of the ring $\mathcal{R}L$ in terms of atoms of the frame L. In this chapter we propose to show that it is an $\boldsymbol{O}$-ideal associated with some rather special sublocale of βL. We will then characterize (again in terms of sublocales) when it has certain algebraic properties. Towards that end, and also for other purposes, we shall need to express annihilator ideals also as $\boldsymbol{O}$-ideals. We shall then see that for certain sublocales, the annihilator of an $\boldsymbol{O}$-ideal associated with a sublocale is the $\boldsymbol{O}$-ideal associated with the supplement of that sublocale. A similar phenomenon occurs (and even more frequently) for the $\boldsymbol{M}$-ideals.

4.1 Annihilator ideals

Recall that the *annihilator* of a subset S of a ring A, denoted $\mathrm{Ann}(S)$, is the ideal

$$\mathrm{Ann}(S) = \{a \in A \mid as = 0 \text{ for every } s \in S\}.$$

In [15, Lemma 3.1], it is shown that for any set $S \subseteq \mathcal{R}L$, the annihilator of S is, in our present notation,

$$\mathrm{Ann}(S) = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(a^*))},$$

where $a = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in S\}$. In the same lemma it is shown that, in fact, the set of annihilator ideals of $\mathcal{R}L$ is the collection

$$\{\boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(b^*))} \mid b \in L\}.$$

We demonstrate that these ideals are precisely the ideals $\boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(b))}$, for $b \in L$. Indeed, since $\boldsymbol{O}^U = \boldsymbol{M}^U$ for any open sublocale U of βL, and since $\boldsymbol{M}^A = \boldsymbol{M}^{\bar{A}}$ for any sublocale A of βL, we have

$$\boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(b^*))} = \boldsymbol{M}^{\overline{\mathfrak{o}_{\beta L}(r_L(b))}} = \boldsymbol{M}^{\mathfrak{o}_{\beta L}(r_L(b))} = \boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(b))}.$$

Remark 4.1.1. In [32], the authors study rings (which they call AIP-*rings*) in which every annihilator ideal is pure. Based on Theorem 2.4.1 and the foregoing discussion in this section, we have that $\mathcal{R}L$ is an AIP-ring if and only if L is basically disconnected. Hence, also, $C(X)$ is an AIP-ring if and only if X is basically disconnected.

Now, for any sublocale A of βL, $A \vee A^{\#} = \beta L$. This implies $\boldsymbol{O}^A \cap \boldsymbol{O}^{A^{\#}} = \boldsymbol{O}^{\beta L} = \{\mathbf{0}\}$, and similarly for $\boldsymbol{M}^A$. In consequence, we have that for any sublocale A of βL,

$$\boldsymbol{O}^{A^{\#}} \subseteq \operatorname{Ann}(\boldsymbol{O}^A) \qquad \text{and} \qquad \boldsymbol{M}^{A^{\#}} \subseteq \operatorname{Ann}(\boldsymbol{M}^A).$$

Naturally, one wonders if these containments are actually not equalities. We will show that for closed sublocales they are, but for open sublocales they generally are not. A sublocale is *regular-open* if it equals the interior of its closure. Regular-open sublocales of L are exactly the sublocales $\mathfrak{o}_L(a)$, for $a \in \mathfrak{B}L$.

Theorem 4.1.2. *Let L be a completely regular frame.*

(a) *For any closed sublocale A of βL,* $\operatorname{Ann}(\boldsymbol{M}^A) = \boldsymbol{M}^{A^{\#}}$ *and* $\operatorname{Ann}(\boldsymbol{O}^A) = \boldsymbol{O}^{A^{\#}}$.

(b) *If U is an open sublocale of βL, then* $\operatorname{Ann}(\boldsymbol{M}^U) = \boldsymbol{M}^{U^{\#}}$ *iff U is regular-open.*

(c) *If U is an open sublocale of βL, then* $\operatorname{Ann}(\boldsymbol{O}^U) = \boldsymbol{O}^{U^{\#}}$ *only if U is regular-open.*

Proof. (a) Let $A = \mathfrak{c}_{\beta L}(I)$, for some $I \in \beta L$. As shown in [13, Lemma 4.4],

$$\bigvee\Big\{ \operatorname{coz}\alpha \mid \alpha \in \boldsymbol{M}^{\mathfrak{c}_{\beta L}(I)} \Big\} = \bigvee\Big\{ \operatorname{coz}\alpha \mid \alpha \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} \Big\} = \bigvee I;$$

and so

$$\operatorname{Ann}(\boldsymbol{M}^A) = \operatorname{Ann}(\boldsymbol{O}^A) = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(\bigvee I)^*)} = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(I^*)}.$$

On the other hand,

$$\boldsymbol{M}^{\mathfrak{o}_{\beta L}(I)} = \boldsymbol{M}^{\overline{\mathfrak{o}_{\beta L}(I)}} = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(I^*)},$$

which then proves that $\operatorname{Ann}(\boldsymbol{M}^A) = \boldsymbol{M}^{A^\#}$ because $A^\# = \mathfrak{o}_{\beta L}(I)$.

For the other equality, since $\boldsymbol{O}^{A^\#} \subseteq \operatorname{Ann}(\boldsymbol{O}^A)$, we need only show the reverse containment. So let $\alpha \in \operatorname{Ann}(\boldsymbol{O}^A) = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(I^*)}$. Then $\mathfrak{c}_{\beta L}(I^*) \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha))$, which, on taking interiors, yields

$$\mathfrak{o}_{\beta L}(I) \subseteq \mathfrak{o}_{\beta L}(I^{**}) \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*),$$

thus showing that $\alpha \in \boldsymbol{O}^{\mathfrak{o}_{\beta L}(I)}$, that is, $\alpha \in \boldsymbol{O}^{A^\#}$. Therefore $\operatorname{Ann}(\boldsymbol{O}^A) \subseteq \boldsymbol{O}^{A^\#}$, and we have the desired equality.

(b) Pick $I \in \beta L$ such that $U = \mathfrak{o}_{\beta L}(I)$. Then, using the result in part (a) and the fact that $\boldsymbol{M}^S = \boldsymbol{M}^{\overline{S}}$ for each sublocale S of βL, we obtain

$$\operatorname{Ann}(\boldsymbol{M}^{\mathfrak{o}_{\beta L}(I)}) = \operatorname{Ann}(\boldsymbol{M}^{\mathfrak{c}_{\beta L}(I^*)}) = \boldsymbol{M}^{\mathfrak{o}_{\beta L}(I^*)} = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(I^{**})}.$$

Consequently, $\operatorname{Ann}(\boldsymbol{M}^U) = \boldsymbol{M}^{U^\#}$ if and only if $\boldsymbol{M}^{\mathfrak{c}_{\beta L}(I^{**})} = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(I)}$, which holds if and only if $\mathfrak{c}_{\beta L}(I^{**}) = \mathfrak{c}_{\beta L}(I)$, which, in turn, holds if and only if $I = I^{**}$. This is so if and only if U is regular-open.

(c) Choose $I \in \beta L$ such that $U = \mathfrak{o}_{\beta L}(I)$. Then

$$\operatorname{Ann}(\boldsymbol{O}^U) = \operatorname{Ann}(\boldsymbol{M}^U) = \operatorname{Ann}(\boldsymbol{M}^{\mathfrak{o}_{\beta L}(I)}) = \boldsymbol{M}^{\mathfrak{o}_{\beta L}(I^*)} = \boldsymbol{O}^{\mathfrak{o}_{\beta L}(I^*)}.$$

Now, if U is not regular open, then $I < I^{**}$ as elements of βL. So there is an $\alpha \in \mathcal{R}L$ such that $\operatorname{coz}\alpha \in I^{**}$ and $\operatorname{coz}\alpha \notin I$. The latter implies $\alpha \notin \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)}$, that is, $\alpha \notin \boldsymbol{O}^{U^\#}$. On the other hand though, $\operatorname{coz}\alpha \in I^{**}$ implies that $r_L(\operatorname{coz}\alpha) \leq I^{**}$, so that $I^* \leq r_L(\operatorname{coz}\alpha)^*$, and consequently $\mathfrak{o}_{\beta L}(I^*) \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)$, whence $\alpha \in \boldsymbol{O}^{\mathfrak{o}_{\beta L}(I^*)} = \operatorname{Ann}(\boldsymbol{O}^U)$. This proves that if $\operatorname{Ann}(\boldsymbol{O}^U) = \boldsymbol{O}^{U^\#}$, then U is regular-open. □

The condition that U be regular-open is not sufficient for the annihilator of $\boldsymbol{O}^U$ to coincide with $\boldsymbol{O}^{U^\#}$. Here is an example showing this.

Example 4.1.3. Let $L = \Omega(\mathbb{R})$, and put $a = (0, 1)$. Then $a = a^{**}$, and so the open sublocale $U = \mathfrak{o}_{\beta L}(r_L(a))$ of βL is regular-open. Since every element of L is a cozero element, there exists some $\alpha \in \mathcal{R}L$ such that $a = \operatorname{coz}\alpha$. Now, as shown in the course of the proof of item (c) in the theorem above, $\operatorname{Ann}(\boldsymbol{O}^U) = \boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(a)^*)}$, which then shows that $\alpha \in \operatorname{Ann}(\boldsymbol{O}^U)$. On the other hand though, $\alpha \notin \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))}$, otherwise we would have $\operatorname{coz}\alpha \in r_L(a)$, which would imply $a \prec\!\!\prec a$, which is false. This shows that $\operatorname{Ann}(\boldsymbol{O}^U) \neq \boldsymbol{O}^{U^\#}$.

4.2 The socle

Recall that the *socle* of a ring A, denoted $\mathrm{Soc}(A)$, is the ideal of A generated by its minimal ideals. If A has no minimal ideal, then $\mathrm{Soc}(A)$ is the zero ideal. The socle is also expressible as an intersection of certain types of ideals. Recall that an ideal of a ring A is *essential* if it has non-zero intersection with every non-zero ideal of A. If A has no non-zero nilpotent element (for instance, if $A = \mathcal{R}L$), then an ideal I of A is essential if and only if $\mathrm{Ann}(I) = \{0\}$. It is well known that

$$\mathrm{Soc}(A) = \bigcap\{E \subseteq A \mid E \text{ is an essential ideal of } A\}.$$

We mentioned at the beginning of the Chapter that one of our goals is to express the socle of $\mathcal{R}L$ as an $\boldsymbol{O}$-ideal. In preparation thereof, let us recall that a sublocale of a frame is said to be *nowhere dense* [37] if it misses the smallest dense sublocale of the frame. Nowhere dense sublocales have nowhere dense closure [37]. Observe that if A is nowhere dense, then $A^{\#}$ is dense; and conversely if A is complemented. Thus, a closed sublocale is nowhere dense if and only if it is of the form $\mathfrak{c}(a)$ for some dense element a.

Let us now introduce a sublocale that will play a crucial role in describing the socle. For any frame M, denote by $\mathrm{Nd}(M)$ the sublocale

$$\mathrm{Nd}(M) = \bigvee\{S \in \mathcal{S}(M) \mid S \text{ is nowhere dense}\}.$$

Since a sublocale is nowhere dense precisely when it misses the smallest dense sublocale, and since the closure of a nowhere dense sublocale is nowhere dense, it is clear that

$$\mathrm{Nd}(M) = M \smallsetminus \mathfrak{B}M = \bigvee\{\mathfrak{c}_M(x) \mid x \text{ is a dense element of } M\}.$$

Now, recalling how joins of sublocales are computed, and keeping in mind that an element above a dense one is dense, one checks quickly that, in terms of elements,

$$\mathrm{Nd}(M) = \{a \in M \mid a \text{ is a meet of dense elements}\}.$$

We shall need the following notion which was introduced by Plewe [37]. A frame is *dense in itself* if each of its Boolean sublocales has a dense supplement. As Plewe observed, a sober space is dense in itself (in the usual topological sense of having no isolated point) precisely when the frame of its open subsets is dense in itself. For our purposes, we need some characterizations that are established in [37]. We recite them in the upcoming proposition, and add new ones, including one in terms of elements.

Proposition 4.2.1. [Plewe's criteria] *The following are equivalent for any frame M.*

(1) *M is dense in itself.*

(2) *M is covered by its nowhere dense sublocales. That is,* $\mathrm{Nd}(M) = M$.

(3) *There exists a family of nowhere dense sublocales whose join is a dense sublocale of M.*

(4) *Every non-void open sublocale of M meets some nowhere dense sublocale of M.*

(5) *There exists a family $\{a_i \mid i \in I\} \subseteq M$ consisting of dense elements such that $\bigwedge_i a_i = 0$.*

Proof. The equivalence of the first three statements is part of [37, Proposition 5]. Statements (2) and (5) are equivalent because (i) nowhere dense sublocales have nowhere dense closures, (ii) a closed sublocale $\mathfrak{c}_M(a)$ is nowhere dense if and only if a is a dense element, and (iii) for any $\{a_i \mid i \in I\} \subseteq M$, $\bigvee_i \mathfrak{c}_M(a_i) = M$ if and only if $\bigwedge_i a_i = 0$.

(3) $\Leftrightarrow$ (4): A sublocale of a frame is dense if and only if it meets every non-void open sublocale of the frame [21, Lemma 9.2]. Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be the set of all nowhere dense sublocales of M. For any open sublocale U of M we have

$$U \cap \mathrm{Nd}(M) = U \cap \bigvee_\lambda A_\lambda = \bigvee_\lambda (U \cap A_\lambda),$$

since complemented sublocales are linear. The equivalence under consideration follows from this because $\mathrm{Nd}(M)$ is the join of all nowhere dense sublocales of M. $\square$

We shall be interested in characterizing when the socle of $\mathcal{R}L$ is the zero ideal, and when it is an essential ideal. Because the socle will turn out to be an $\boldsymbol{O}$-ideal, we first present criteria, in terms of sublocales, for determining when an $\boldsymbol{O}$-ideal is the zero ideal, and when it is an essential ideal. These will actually be needed even in describing the socle as an $\boldsymbol{O}$-ideal.

Lemma 4.2.2. *The following are equivalent for any sublocale A of βL.*

(1) *$\boldsymbol{O}^A$ is the zero ideal.*

(2) *A is dense in βL.*

(3) *$\boldsymbol{M}^A$ is the zero ideal.*

Proof. (1) $\Rightarrow$ (2): Assume that $\boldsymbol{O}^A$ is the zero ideal. Then $\boldsymbol{O}^{\bar{A}}$ is the zero ideal, which implies $\mathrm{Ann}(\boldsymbol{O}^{\bar{A}}) = \mathcal{R}L$. Since $\bar{A}$ is a closed sublocale of βL, Theorem 4.1.2(a) tells us that $\boldsymbol{O}^{\beta L \smallsetminus \bar{A}} = \mathcal{R}L$. Thus, $\mathbf{1} \in \boldsymbol{O}^{\beta L \smallsetminus \bar{A}}$, which implies

$$\beta L \smallsetminus \bar{A} \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\mathbf{1})^*) = \mathfrak{o}_{\beta L}(0_{\beta L}) = \mathsf{O}.$$

It follows from this that $\bar{A} = \beta L$, and so A is dense in βL.

(2) $\Rightarrow$ (3): If A is dense in βL, then $\boldsymbol{M}^A = \boldsymbol{M}^{\bar{A}} = \boldsymbol{M}^{\beta L}$, which is the zero ideal.

(3) $\Rightarrow$ (1): This follows from the fact that $\boldsymbol{O}^A \subseteq \boldsymbol{M}^A$. □

Before we move to the characterization of essential $\boldsymbol{O}$-ideals, let us use this corollary to address a natural question regarding the containment $\boldsymbol{O}^A \subseteq \boldsymbol{O}^{A^{\#\#}}$, which always holds because $A^{\#\#} \subseteq A$. If A is complemented, then this containment is actually an equality because then $A = A^{\#\#}$. There are however instances when the containment is strict.

Example 4.2.3. Let X be any Tychonoff space which is dense in itself. Then βX is dense in itself. Put $L = \Omega(X)$. Since $\beta L \cong \Omega(\beta X)$, βL is dense in itself, and so $\mathfrak{B}(\beta L)^{\#} = \beta L$ by one of Plewe's criteria. Therefore $\mathfrak{B}(\beta L)^{\#\#} = \mathsf{O}$, and consequently $\boldsymbol{O}^{\mathfrak{B}(\beta L)} = \{\mathbf{0}\}$ by Lemma 4.2.2. On the other hand though, $\boldsymbol{O}^{\mathfrak{B}(\beta L)^{\#\#}} = \mathcal{R}L$.

Lemma 4.2.4. *The following are equivalent for a sublocale A of βL.*

(1) *$\boldsymbol{O}^A$ is essential.*

(2) *$\boldsymbol{M}^A$ is essential.*

(3) *A is nowhere dense.*

Proof. (1) $\Rightarrow$ (2): This is so because $\boldsymbol{O}^A \subseteq \boldsymbol{M}^A$.

(2) $\Rightarrow$ (3): Assume that $\boldsymbol{M}^A$ is essential. Then $\boldsymbol{M}^{\bar{A}}$ is essential (as the two ideals are equal). Since $\bar{A}$ is a closed sublocale, Theorem 4.1.2(a) gives $\mathrm{Ann}(\boldsymbol{M}^{\bar{A}}) = \boldsymbol{M}^{\beta L \smallsetminus \bar{A}}$. The essentiality

of $\boldsymbol{M}^{\bar{A}}$ then says $\boldsymbol{M}^{\beta L \smallsetminus \bar{A}}$ is the zero ideal, which, by Lemma 4.2.2, implies $\beta L \smallsetminus \bar{A}$ is dense, whence $\bar{A}$ is nowhere dense, and hence A is nowhere dense.

(3) $\Rightarrow$ (1): Assume that A is nowhere dense. Then $\bar{A}$ is nowhere dense, and so $\beta L \smallsetminus \bar{A}$ is dense. From Theorem 4.1.2(a), we have $\operatorname{Ann}(\boldsymbol{O}^{\bar{A}}) = \boldsymbol{O}^{\beta L \smallsetminus \bar{A}}$. Since $\beta L \smallsetminus \bar{A}$ is dense, Lemma 4.2.2 tells us that $\boldsymbol{O}^{\beta L \smallsetminus \bar{A}}$ is the zero ideal, which then implies $\boldsymbol{O}^{\bar{A}}$ is essential. Therefore $\boldsymbol{O}^{A}$ is essential because $\boldsymbol{O}^{\bar{A}} \subseteq \boldsymbol{O}^{A}$. □

Remark 4.2.5. In [22], the authors prove for $C(X)$ a result almost similar to Lemma 4.2.4, but restricted to *closed* subspaces of βX. Our result is thus certainly much sharper, even when restricted to $C(X)$. Another comment is that their proof does not use annihilators, as ours does, but instead uses a result attributed to McKnight in [9].

Let us digress slightly to compare the two previous lemmas. For complemented sublocales, nowhere denseness is the antithesis of denseness, because the first concept says "interior is void", whilst the second says "closure is the whole thing". In rings, there is a notion which is the antithesis of essentiality of ideals. Namely, an ideal I of a ring A is said to be *small* if for any ideal J of A, the equality $I + J = A$ implies $J = A$. Compare with essentiality which says the equality $I \cap J = \{0\}$ implies $J = \{0\}$.

Now the two previous lemmas say "nowhere denseness is to essentiality what denseness is to being zero". Considering what we have said in the preceding paragraph, perhaps one could have expected nowhere denseness to be to essentiality what denseness is to smallness. Actually that is exactly what we have because, as we show below, being zero in $\mathcal{R}L$ is precisely being small. Thus, the two lemmas harmonize with the antitheses mentioned above.

That the only small ideal of $\mathcal{R}L$ is the zero ideal follows from the fact that in any ring an ideal is small if and only if it is contained in the Jacobson radical of the ring, and the Jacobson radical of $\mathcal{R}L$ is the zero ideal, as was shown by Ighedo in her PhD thesis [26, Remark 2.1.1]. Since her proof of this fact requires knowledge of maximal ideals of $\mathcal{R}L$, we proffer the following direct proof that "small $\equiv$ zero" in $\mathcal{R}L$.

Proposition 4.2.6. *The only small ideal of $\mathcal{R}L$ is the zero ideal.*

Proof. Let I be a small ideal of $\mathcal{R}L$. Let $\alpha \in I$, and consider any $\gamma \in \mathcal{R}L$ with $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\alpha$.

Find $\sigma \in \mathcal{R}L$ such that

$$\operatorname{coz}\gamma \wedge \operatorname{coz}\sigma = 0 \quad \text{and} \quad \operatorname{coz}\sigma \vee \operatorname{coz}\alpha = 1.$$

Then, by the properties of the cozero map, $\gamma\sigma = \mathbf{0}$ and $\sigma^2 + \alpha^2$ is invertible. Therefore $I + \langle\sigma\rangle = \mathcal{R}L$. Since I is small, $\langle\sigma\rangle = \mathcal{R}L$, which implies $\operatorname{coz}\sigma = 1$, and hence $\gamma = \mathbf{0}$. Since $\operatorname{coz}\alpha = \bigvee\{\operatorname{coz}\tau \mid \operatorname{coz}\tau \prec\!\!\prec \operatorname{coz}\gamma\}$, by complete regularity, it follows that $\operatorname{coz}\alpha = 0$, and hence $\alpha = \mathbf{0}$. Therefore I is the zero ideal. □

This ends the digression, and we pick up the discussion on when the ideals associated with sublocales of L are the zero ideal or essential ideals.

In Lemma 4.2.2 we saw that each of the ideals $\boldsymbol{O}^A$ and $\boldsymbol{M}^A$ is the zero ideal if and only if A is a dense sublocale of βL. In Lemma 4.2.4 we saw that the ideals $\boldsymbol{O}^A$ and $\boldsymbol{M}^A$ are essential precisely when A is a nowhere dense sublocale of A. Since, for any sublocale S of L, $\boldsymbol{O}_S = \boldsymbol{O}^{r_L[S]}$ and $\boldsymbol{M}_S = \boldsymbol{M}^{r_L[S]}$, it follows that:

- each of the ideals $\boldsymbol{O}_S$ and $\boldsymbol{M}_S$ is the zero ideal if and only if $r_L[S]$ is a dense sublocale of βL; and
- each of the ideals $\boldsymbol{O}_S$ and $\boldsymbol{M}_S$ is essential if and only if $r_L[S]$ is a nowhere dense sublocale of βL.

We wish express these characterizations within L without invoking βL. For that we need the following lemma, which we prove more generally than is needed for current purposes. Recall that if $h\colon M \to L$ is a dense onto frame homomorphism, then $h_*(b^*) = h_*(b)^*$ for every $b \in L$. In **Loc**, this says if $f\colon L \to M$ is a dense one-one localic map, then $f(a^*) = f(a)^*$ for every $a \in L$.

Lemma 4.2.7. *For any dense one-one localic map $f\colon L \to M$, we have the following:*

(a) *S is dense in L iff $f[S]$ is dense in M.*

(b) *S is nowhere dense in L iff $f[S]$ is nowhere dense in βL.*

Proof. (a) If S is dense in L, then $0_L \in S$, and so $f(0_L) \in f[S]$. But $f(0_L) = 0_M$ since f is dense, so $f[S]$ is dense in M. Conversely, if $f[S]$ is dense in M, then $f(0_L)$, which is the bottom of $f[S]$ since f is dense, belongs to $f[S]$, so that $f(0_L) = f(s)$, for some $s \in S$. Since f is one-one, this implies $s = 0_L$, showing that S is dense in L.

(b) Suppose that S is nowhere dense in L. Consider any $m \in f[A] \cap \mathfrak{B}M$, so that $m = f(a)$, for some $s \in S$, and $m = b^*$ for some $b \in M$. Thus $f(a) = b^*$, and hence (in light of f being injective and f^* commuting with pseudocomplementation),

$$s = f^*(f(a)) = f^*(b^*) = (f^*(b))^*$$

which implies $s \in S \cap \mathfrak{B}L$, hence $s = 1$ since S is nowhere dense in L . Therefore $m = f(s) = 1$, which proves that $f[S]$ is nowhere dense in M.

Conversely, suppose $f[A]$ is nowhere dense in M. Consider any $s \in S \cap \mathfrak{B}L$. Then $s = x^*$ for some $x \in L$, which implies $f(s) = f(x^*) = f(x)^*$. Thus, $f(s) \in f[S] \cap \mathfrak{B}M = \mathsf{O}$, which implies $f(s) = 1_M$, and therefore $s = 1_L$ because f is one-one and $f(1_L) = 1_M$. Therefore S is nowhere dense in L. □

Corollary 4.2.8. *For any sublocale S of L, we have the following,*

(a) $\boldsymbol{O}_S$ *is the zero ideal iff* $\boldsymbol{M}_S$ *is the zero ideal iff S is dense in dense L.*

(b) $\boldsymbol{O}_S$ *is essential iff* $\boldsymbol{M}_S$ *is essential iff S is nowhere dense in dense L.*

We can now describe the socle of $\mathcal{R}L$ in the desired manner.

Theorem 4.2.9. $\operatorname{Soc}(\mathcal{R}L) = \boldsymbol{O}^{\operatorname{Nd}(\beta L)}$.

Proof. Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be the set of all nowhere dense sublocales of βL, so that $\operatorname{Nd}(\beta L) = \bigvee_\lambda A_\lambda$. Lemma 4.2.4 tells us that, for each λ, $\boldsymbol{O}^{A_\lambda}$ is an essential ideal, and hence, in light of the socle being the intersection of all essential ideals,

$$\operatorname{Soc}(\mathcal{R}L) \subseteq \bigcap_\lambda \boldsymbol{O}^{A_\lambda} = \boldsymbol{O}^{\bigvee_\lambda A_\lambda} = \boldsymbol{O}^{\operatorname{Nd}(\beta L)}.$$

Now consider any essential ideal of I of $\mathcal{R}L$, and denote by $\Delta(I)$ the closed sublocale of βL given by

$$\Delta(I) = \mathfrak{c}_{\beta L}\Big(\bigvee\{r_L(\operatorname{coz}\alpha) \mid \alpha \in I\}\Big).$$

As shown in the proof of [14, Proposition 5.2], $\boldsymbol{O}^{\Delta(I)} \subseteq I \subseteq \boldsymbol{M}^{\Delta(I)}$. Therefore $\boldsymbol{M}^{\Delta(I)}$ is an essential ideal, implying that $\operatorname{Ann}(\boldsymbol{M}^{\Delta(I)})$ is the zero ideal, and hence $\boldsymbol{O}^{\Delta(I)}$ is an essential ideal because $\operatorname{Ann}(\boldsymbol{O}^{\Delta(I)}) = \operatorname{Ann}(\boldsymbol{M}^{\Delta(I)})$, as shown in the proof of Theorem 4.1.2. Thus, by Lemma 4.2.4, there is an index λ_0 such that $\boldsymbol{O}^{A_{\lambda_0}} \subseteq I$. Since the socle is the intersection of all essential ideals, it follows from this that $\bigcap_\lambda \boldsymbol{O}^{A_\lambda} \subseteq \operatorname{Soc}(\mathcal{R}L)$, and so we have the claimed equality. □

Since $\operatorname{Nd}(\beta L)$ is a join of closed sublocales, each of which is nowhere dense, the following corollary follows from Theorem 2.4.1 and Lemmas 2.3.1 and 4.2.4.

Corollary 4.2.10. $\operatorname{Soc}(\mathcal{R}L)$ *is the intersection of all the pure essential ideals of* $\mathcal{R}L$. *If* L *is basically disconnected, then* $\operatorname{Soc}(\mathcal{R}L)$ *is pure.*

Let us now address the question of when the socle of $\mathcal{R}L$ is zero. Recall that one of Plewe's criteria says a frame M is dense in itself if and only if there is a family $\{A_i\}$ of nowhere dense sublocales of M such that $\bigvee_i A_i$ is a dense sublocale of M. The following result therefore follows from Lemma 4.2.2 and Theorem 4.2.9.

Corollary 4.2.11. $\operatorname{Soc}(\mathcal{R}L)$ *is zero iff* βL *is dense in itself.*

Let us pause for a moment and interpret this result in $C(X)$. This is with the view to showing some stark differences between the vanishing of the socle in classical function rings and its vanishing in pointfree function rings. Recall that $C(X) \cong \mathcal{R}(\Omega(X))$, $\beta(\Omega(X)) \cong \Omega(\beta X)$, and X is dense in itself if and only if βX is dense in itself. Since a sober space is dense in itself if and only if the frame of its open sets is dense in itself [37], it follows that

$\operatorname{Soc}(C(X))$ *is zero iff* βX *is dense in itself iff* X *is dense in itself.*

Corollary 4.2.11 tells us that, in frames, we do have the localic version of the first of these equivalences. We shall see that one implication in the localic version of the other equivalence fails.

If L is a dense sublocale of M, then $\mathfrak{B}L = \mathfrak{B}M$. Therefore, if $A \subseteq L$ is a nowhere dense sublocale of L, then A is a nowhere dense sublocale of M. Consequently, if L is dense in itself, so that L is covered by its nowhere dense sublocales, then M has nowhere dense sublocales with dense join, and so, by one of Plewe's criteria, M is dense in itself. In particular,

if L *is dense in itself, then* βL *is dense in itself,* and we can thus deduce from Corollary 4.2.11 the following result.

Corollary 4.2.12. *If L is dense in itself, then* $\mathrm{Soc}(\mathcal{R}L)$ *is zero. The converse fails.*

Here is an example avouching that (unlike in classical function rings) if $\mathrm{Soc}(\mathcal{R}L)$ is zero, it does not follow that L is dense in itself. Recall that an element p of a frame L is called *prime* if $p < 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. The set of prime elements of L is denoted $\mathrm{Pt}(L)$. In regular frames the primes are exactly the elements that are maximal strictly below the top.

Example 4.2.13. Let L be a Boolean frame with no primes (such as $\mathfrak{B}(\Omega(\mathbb{R}))$). Then of course L is not dense in itself. We claim that βL is dense in itself. If βL were not dense in itself, then (being spatial) we would have a Tychonoff space X with an isolated point such that $\beta L \cong \Omega(X)$. Then there would exist $p \in \mathrm{Pt}(\beta L)$ with $p \vee p^* = 1$. We cannot have $j_L(p) = 1$, as that would imply $j_L(p^*) = 0$, whence we would have $p^* = 0$ as j_L is dense, leading to $p = 1$. Therefore $j_L(p) < 1$, and since primes in regular frames are precisely the maximal elements, a simple calculation would imply that $j_L(p) \in \mathrm{Pt}(L)$, which is a contradiction as L has no primes. Thus, by Corollary 4.2.11, $\mathrm{Soc}(\mathcal{R}L)$ is zero even though L is not dense in itself.

To close the discussion on the vanishing socle, let us briefly say a word on the discrepancy between the behavior of dense subspaces vis-à-vis that of dense sublocales with regard to inheritance of the dense-in-itself property. We have seen that a frame with a dense sublocale that is dense in itself is itself dense in itself; however, a dense sublocale of a dense in itself frame (even a spatial one, at that) is not necessarily dense in itself. We show that if we restrict to smooth sublocales then the dense ones among them inherit the property of being dense in oneself. We do not assume any separation axiom.

Proposition 4.2.14. *A smooth dense sublocale of a dense-in-itself frame is dense in itself.*

Proof. Let $L \subseteq M$ be a smooth dense sublocale of a dense-in-itself frame M. Let $(C_i \mid i \in I)$ be a collection of complemented sublocales of M with $L = \bigvee_{i\in I} C_i$, and let $(N_j \mid j \in J)$ be the collection of all nowhere dense sublocales of M. Then $M = \bigvee_{j\in J} N_j$ since M is dense in itself. Since L is dense in M, $\mathfrak{B}L = \mathfrak{B}M$, and so for any nowhere dense sublocale N of M, $L \cap N$ is a nowhere dense sublocale of L. For each $i \in I$, $C_i \subseteq \bigvee_{j\in J} N_j$, and so, by linearity of complemented sublocales,

$$C_i = C_i \cap \bigvee_{j\in J} N_j = \bigvee_{j\in J} (C_i \cap N_j).$$

Since each C_i is contained in L, the collection $\{C_i \cap N_j \mid (i,j) \in I \times J\}$ is a family of nowhere dense sublocales of L, covering L. Therefore L is nowhere dense. $\square$

Corollary 4.2.15. *If L is a smooth sublocale of βL, then* $\mathrm{Soc}(\mathcal{R}L)$ *is zero iff L is dense in itself.*

Now we turn to characterizing when the socle of $\mathcal{R}L$ is essential. Recall from [37] that Plewe calls a frame *scattered* if every non-void closed sublocale contains a non-void open Boolean sublocale. He observes that a sober space X is scattered if and only if the frame $\Omega(X)$ is scattered.

Corollary 4.2.16. *The following statements about* $\mathrm{Soc}(\mathcal{R}L)$ *are equivalent.*

(1) $\mathrm{Soc}(\mathcal{R}L)$ *is essential.*

(2) $\mathrm{Nd}(\beta L)$ *is nowhere dense.*

(3) $\mathfrak{B}(\beta L)$ *is complemented.*

(4) βL *has a largest nowhere dense sublocale.*

(5) βL *has a smallest dense open sublocale.*

Proof. By Lemma 4.2.4 and the description of the socle in Theorem 4.2.9, $\mathrm{Soc}(\mathcal{R}L)$ is essential if and only if $\mathrm{Nd}(\beta L)$ is nowhere dense; which proves the equivalence of statements (1) and (2). Since, for any frame M, $\mathrm{Nd}(M) = M \smallsetminus \mathfrak{B}M$, we have

$$\begin{aligned} \mathrm{Nd}(M) \text{ is nowhere dense} \quad & \text{iff} \quad \mathrm{Nd}(M) \cap \mathfrak{B}M = \mathsf{O} \\ & \text{iff} \quad (M \smallsetminus \mathfrak{B}M) \cap \mathfrak{B}M = \mathsf{O} \\ & \text{iff} \quad \mathfrak{B}M \text{ is complemented.} \end{aligned}$$

Therefore statements (2) and (3) are equivalent.

Since $\mathrm{Nd}(\beta L)$ is the join of all nowhere dense sublocales of βL, it is clear that $\mathrm{Nd}(\beta L)$ is nowhere dense if and only if it is the largest nowhere dense sublocale of βL. This proves the equivalence of statements (2) and (4).

Since the closure of any nowhere dense sublocale is nowhere dense, if $\mathrm{Nd}(\beta L)$ is nowhere dense, then it is, in fact, a closed sublocale. But clearly, a frame has a largest (closed) nowhere dense sublocale if and only if it has a smallest dense open sublocale. Thus, statements (4) and (5) are equivalent. □

Remark 4.2.17. We feel compelled to mention that if a frame has a smallest dense open sublocale, it does not mean that its smallest dense sublocale is open. Incidentally, Banaschewski and Pultr prove in [8] that, for any frame L, $\mathfrak{B}L$ is open if and only if L has an open Boolean dense sublocale. Their proof is frame-theoretic. We offer the following localic one, which is much shorter. If $\mathfrak{B}L$ is open, then of course L has an open Boolean dense sublocale. Conversely, suppose U is an open dense Boolean sublocale of L. Denote pseudocomplementation in U by $(-)^{\neg}$. The density of U implies that, for any $u \in U$, $u^{\neg} = u \to 0_U = u \to 0_L = u^*$. Since U is Boolean, $u = u^{\neg\neg} = u^{**}$, which says $U \subseteq \mathfrak{B}L$, and hence $U = \mathfrak{B}L$, implying that the smallest dense sublocale is open.

In [37], Plewe proves that a frame is scattered if and only if every sublocale has a largest nowhere dense sublocale. We therefore have the following corollary.

Corollary 4.2.18. *If βL is scattered, then* $\mathrm{Soc}(\mathcal{R}L)$ *is an essential ideal. The converse fails.*

Here is an example showing that the converse of the corollary does not hold.

Example 4.2.19. Let $L = \beta(\Omega(\mathbb{N}))$. Since $\mathbb{N}$ is locally compact, $\Omega(\mathbb{N})$ is an open sublocale of L. Since $\Omega(\mathbb{N})$ is a dense sublocale of L, $\mathfrak{B}L = \mathfrak{B}(\Omega(\mathbb{N})) = \Omega(\mathbb{N})$, and so $\mathfrak{B}L$ is a complemented sublocale of L. Since $\beta L \cong L$, it follows that $\mathfrak{B}(\beta L)$ is a complemented sublocale of βL, and so, by Corollary 4.2.16, $\mathrm{Soc}(\mathcal{R}L)$ is an essential ideal of $\mathcal{R}L$. But of course βL is not scattered because $\beta L \cong \Omega(\beta\mathbb{N})$, and $\beta\mathbb{N}$ is not scattered as $\beta\mathbb{N} \smallsetminus \mathbb{N}$ is a closed subspace with no isolated point.

A few comments about the ideal $\boldsymbol{O}_{\mathrm{Nd}(L)}$ are in order. Emanating from Corollary 4.2.16 is the natural question whether it is necessary and sufficient that the sublocale $\mathrm{Nd}(L)$ of L be nowhere dense for $\mathrm{Soc}(\mathcal{R}L)$ to be essential. We show that it is necessary. We need a lemma, which we state more generally than is really needed for our purposes.

We recalled earlier that if $f\colon L \to M$ is a one-one dense localic map, then $f(a^*) = f(a)^*$ for all $a \in L$.

Lemma 4.2.20. *Let $f\colon L \to M$ be a dense surjective localic map.*

(a) $\mathrm{Nd}(L) \subseteq f_{-1}[\mathrm{Nd}(M)]$.

(b) *If M is scattered, then $f_{-1}[\mathrm{Nd}(M)] = \mathrm{Nd}(L)$.*

(c) *If $A \subseteq M$ is a nowhere dense sublocale, then $f_{-1}[A]$ is nowhere dense in L.*

(d) *If M is scattered and A is a sublocale of M such that $f_{-1}[A]$ is nowhere dense in L, then A is nowhere dense in M.*

Proof. (a) Let $a \in \mathrm{Nd}(L)$. Then, as observed earlier, $a = \bigwedge_i a_i$, for some dense elements $a_i \in L$. Therefore $f(a) = \bigwedge_i f(a_i)$, which implies $f(a) \in \mathrm{Nd}(M)$ because (being dense and surjective) f maps dense elements to dense elements. Thus, $f[\mathrm{Nd}(L)] \subseteq \mathrm{Nd}(M)$, which implies $\mathrm{Nd}(L) \subseteq f_{-1}[\mathrm{Nd}(M)]$.

(b) Let us recall that, as was observed by Plewe [38, p. 315], pullback along any localic map with scattered codomain preserves all joins. Therefore, if M is scattered,

$$\begin{aligned} f_{-1}[\mathrm{Nd}(M)] &= f_{-1}\left[\bigvee\{\mathfrak{c}_M(m) \mid m \text{ is dense in } M\}\right] \\ &= \bigvee\{f_{-1}[\mathfrak{c}_M(m)] \mid m \text{ is dense in } M\} \\ &= \bigvee\{\mathfrak{c}_L(f^*(m)) \mid m \text{ is dense in } M\} \\ &\subseteq \bigvee\{\mathfrak{c}_L(a) \mid a \text{ is dense in } L\} \qquad \text{since } f^* \text{ preserves density} \\ &= \mathrm{Nd}(L); \end{aligned}$$

so that we have $f_{-1}[\mathrm{Nd}(M)] = \mathrm{Nd}(L)$, by part (a).

(c) Let A be a nowhere dense sublocale of M. Then $A \cap \mathfrak{B}M = \mathsf{O}$, and consequently $f_{-1}[A] \cap f_{-1}[\mathfrak{B}M] = \mathsf{O}$. Observe that $f[\mathfrak{B}L] \subseteq \mathfrak{B}M$ because if $a \in \mathfrak{B}L$, then $a = a^{**}$, which implies $f(a) = f(a^{**}) = f(a)^{**}$ because f is dense and injective. Thus, $\mathfrak{B}L \subseteq f_{-1}[\mathfrak{B}M]$, and hence

$$f_{-1}[A] \cap \mathfrak{B}L \;\subseteq\; f_{-1}[A] \cap f_{-1}[\mathfrak{B}M] \;=\; \mathsf{O},$$

which says $f_{-1}[A]$ is nowhere dense.

(d) If M is scattered, then every sublocale of M is complemented [37]. Since localic preimage functions preserve complements, and since A is complemented in M, $f_{-1}[A]$ is complemented in

L, with $(f_{-1}[A])^{\#} = f_{-1}[A^{\#}]$. Therefore the hypothesis that $f_{-1}[A]$ is nowhere dense implies that $f_{-1}[A^{\#}]$ is dense in L, and therefore $0 \in f_{-1}[A^{\#}]$, which, in light of f being dense, implies $0 = f(0) \in A^{\#}$, thus showing that $A^{\#}$ is a dense sublocale, and therefore A is nowhere dense since A is complemented. □

Now, the localic map $r_L \colon L \to \beta L$ is dense and surjective, so this lemma applies to it. Applying it, we obtain the following results.

Proposition 4.2.21. *Let L be a completely regular frame.*

(a) $\mathrm{Soc}(\mathcal{R}L) \subseteq \boldsymbol{O}_{\mathrm{Nd}(L)}$.

(b) *If* $\mathrm{Soc}(\mathcal{R}L)$ *is essential, then* $\mathrm{Nd}(L)$ *is nowhere dense in L.*

(c) *If βL is scattered, then* $\mathrm{Soc}(\mathcal{R}L)$ *is essential iff* $\mathrm{Nd}(L)$ *is nowhere dense iff L has a largest nowhere dense sublocale.*

Proof. (a) From the containment $\mathrm{Nd}(L) \subseteq (r_L)_{-1}[\mathrm{Nd}(\beta L)]$, as per Lemma 4.2.20(a), we deduce that $r_L[\mathrm{Nd}(L)] \subseteq \mathrm{Nd}(\beta L)$, and therefore

$$\mathrm{Soc}(\mathcal{R}L) = \boldsymbol{O}^{\mathrm{Nd}(\beta L)} \subseteq \boldsymbol{O}^{r_L[\mathrm{Nd}(L)]} = \boldsymbol{O}_{\mathrm{Nd}(L)}.$$

(b) If $\mathrm{Soc}(\mathcal{R}L)$ is essential, then $\mathrm{Nd}(\beta L)$ is nowhere dense in βL by Corollary 4.2.16. By Lemma 4.2.20(c), $(r_L)_{-1}[\mathrm{Nd}(\beta L)]$ is nowhere dense, and hence by Lemma 4.2.20(a), $\mathrm{Nd}(L)$ is nowhere dense.

(c) Clearly, we need only prove the right-to-left implication in the first equivalence. So assume that $\mathrm{Nd}(L)$ is nowhere dense. By Lemma 4.2.20(b), $(r_L)_{-1}[\mathrm{Nd}(\beta L)] = \mathrm{Nd}(L)$, and so by Lemma 4.2.20(d), $\mathrm{Nd}(\beta L)$ is nowhere dense, and so $\mathrm{Soc}(\mathcal{R}L)$ is essential by Corollary 4.2.16. □

We conclude with following comments.

(a) It should be clear that if L is compact, then $\boldsymbol{O}^{\mathrm{Nd}(\beta L)} = \boldsymbol{O}_{\mathrm{Nd}(L)}$. The converse fails. Indeed, for the frame $L = \Omega(\mathbb{R})$ we have $\mathrm{Nd}(L) = L$ and $\mathrm{Nd}(\beta L) = \beta L$ since L and βL are dense in themselves, and so $\boldsymbol{O}^{\mathrm{Nd}(\beta L)} = \boldsymbol{O}_{\mathrm{Nd}(L)} = \{\mathbf{0}\}$.

(b) The containment $\boldsymbol{O}^{\mathrm{Nd}(\beta L)} \subseteq \boldsymbol{O}_{\mathrm{Nd}(L)}$ can be proper. To see this, observe that, for any frame M, $\mathrm{Nd}(M) = \mathsf{O}$ if and only if M is Boolean. In particular, since βM is compact, $\mathrm{Nd}(\beta M) = \mathsf{O}$

precisely when M is a finite Boolean algebra. Now let L be the power set of any infinite set. Then L is a Boolean frame, but βL is not Boolean, and so $\mathrm{Nd}(\beta L) \neq \mathsf{O}$. Since for any frame M and $A \in \mathcal{S}(\beta M)$, $\boldsymbol{O}^A = \mathcal{R}M$ if and only if $A = \mathsf{O}$, we have $\boldsymbol{O}^{\mathrm{Nd}(\beta L)} \neq \mathcal{R}L$. Since L is Boolean, $\mathrm{Nd}(L) = \mathsf{O}$, and so $\boldsymbol{O}_{\mathrm{Nd}(L)} = \boldsymbol{O}^{\mathsf{O}} = \mathcal{R}L$. Therefore $\boldsymbol{O}^{\mathrm{Nd}(\beta L)} \subset \boldsymbol{O}_{\mathrm{Nd}(L)}$.

Chapter 5

Mapping ideals to sublocales

In this chapter we introduce a mapping from the lattice of ideals of $\mathcal{R}L$ into the lattice of sublocales of L. We shall then examine some properties of this mapping. Its ancestry goes back to the 1954 paper of Gillman, Henriksen and Jerison [23] in which they present a proof of the theorem of Gelfand and Kolmogoroff that is about the bijection between the sets of maximal ideals of the rings $C(X)$ and $C^*(X)$.

To recall, Gillman, Henriksen and Jerison introduced the notation $\Delta(I)$ as a shorthand for associating an ideal I of $C(X)$ with the closed subset of βX given by

$$\Delta(I) = \bigcap\{\mathrm{cl}_{\beta X}\, Z(f) \mid f \in I\},$$

where $Z(f)$ denotes the zero-set of f. Since its introduction, wherever $\Delta(I)$ has appeared (sometimes written as $\theta(I)$), it has just been a notation of convenience. In this chapter our aim is to make Δ a homomorphism of the algebraic structures that are known as quantales that were introduced by Mulvey [34].

5.1 Making Δ a quantale homomorphism

A *quantale* is a complete lattice Q with an associative binary operation $\cdot$ such that

$$a \cdot \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \cdot b_i) \qquad \text{and} \qquad \Big(\bigvee_{i\in I} b_i\Big) \cdot a = \bigvee_{i\in I}(b_i \cdot a)$$

for every $a \in Q$ and every family $(b_i \mid i \in I)$ of elements of Q. The quantale is *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in Q$. Every frame is a commutative quantale if we take $\cdot$ to be $\wedge$. A *quantale homomorphism* is a mapping between quantales that preserves all joins and the binary operation. As mentioned above, quantales were introduced by Mulvey [34] and are studied in detail in [40].

Let us remind the reader that by "ring" we mean a commutative ring with identity. The lattice $\mathrm{Idl}(A)$ of ideals of a ring A, partially ordered by inclusion, is complete, with sum for join. We view it as a quantale with the binary operation given by the usual ideal product. That is, for any ideals I and J of A,

$$I \cdot J = \left\{ \sum_{i=1}^{n} u_i v_i \mid n \in \mathbb{N}, u_i \in I, v_i \in J \right\}.$$

As usual, we simply write IJ for this product.

Definition 5.1.1. We define the map $\Delta_L \colon \mathrm{Idl}(\mathcal{R}L) \to \mathcal{S}(\beta L)^{\mathrm{op}}$ by the equation

$$\Delta_L(I) = \bigcap_{\alpha \in I} \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha)) = \mathfrak{c}_{\beta L}\Big(\bigvee_{\alpha \in I} r_L(\operatorname{coz}\alpha)\Big) = \mathfrak{c}_{\beta L}\Big(\bigcup_{\alpha \in I} r_L(\operatorname{coz}\alpha)\Big).$$

Observe that the join in the definition of $\Delta_L(I)$ is a union because it is directed. When we are dealing with one frame, we shall suppress the subscript. We remark (for later use) that Δ is surjective on closed sublocales. Indeed, if A is a closed sublocale of βL, say $A = \mathfrak{c}_{\beta L}(J)$ for some $J \in \beta L$, then the set $Q = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \in J\}$ is easily checked to be an ideal of $\mathcal{R}L$ with $\Delta(Q) = A$. We shall see in the following proposition that Δ need not be injective, but is always injective on what Johnstone [29] calls "neat" ideals. Let us recall what they are.

An ideal I of a ring A is said to be *neat* if $mI = I$. Neat ideals are also called "pure". In $\mathcal{R}L$, $\alpha \in mI$ if and only if $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\gamma$, for some $\gamma \in I$. See [11, Lemma 3.4] for details.

In a number of instances we shall use [14, Lemma 4.4], which states that if γ and δ belong to $\mathcal{R}L$ and $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\delta$, then γ is a multiple of δ.

Proposition 5.1.2. *Let I and J be ideals of $\mathcal{R}L$. Then:*

(a) $\Delta(I) = \Delta(mI)$.

(b) $\Delta(I) \subseteq \Delta(J)$ *implies* $mJ \subseteq mI$.

(c) $\Delta(I) = \Delta(J)$ *iff* $mI = mJ$.

(d) Δ *is injective on neat ideals.*

Proof. (a) In view of the definition, it suffices to show that

$$\bigcup_{\alpha \in I} r_L(\operatorname{coz}\alpha) = \bigcup_{\alpha \in mI} r_L(\operatorname{coz}\alpha).$$

The one containment is trivial because $mI \subseteq I$. If $c \in \bigcup_{\alpha \in I} r_L(\operatorname{coz}\alpha)$, then there is an $\alpha_0 \in I$ such that $c \in r_L(\operatorname{coz}\alpha_0)$, which says $c \prec\!\!\prec \operatorname{coz}\alpha_0$. Pick $\gamma \in \mathcal{R}L$ with $c \prec\!\!\prec \operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\alpha_0$. Then γ is a multiple of α_0, and so $\gamma \in I$. From $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\alpha_0$, we have that $\gamma \in mI$, which then implies $c \in \bigcup_{\alpha \in mI} r_L(\operatorname{coz}\alpha)$; establishing the other containment. Therefore $\Delta(I) = \Delta(mI)$.

(b) Suppose that $\Delta(I) \subseteq \Delta(J)$. Let $\gamma \in mJ$ and pick $\tau \in J$ such that $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\tau$. Therefore $\operatorname{coz}\gamma \in \bigcup_{\alpha \in J} r_L(\operatorname{coz}\alpha)$. From the containment $\Delta(I) \subseteq \Delta(J)$ we deduce that

$$\bigcup_{\alpha \in J} r_L(\operatorname{coz}\alpha) \subseteq \bigcup_{\alpha \in I} r_L(\operatorname{coz}\alpha).$$

Thus, there exists $\rho \in I$ such that $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\rho$, which implies $\gamma \in mI$. Therefore $mJ \subseteq mI$.

(c) The forward implication follows from (b), and the other follows from (a).

(d) This follows from (c) because an ideal Q of $\mathcal{R}L$ is neat if and only if $Q = mQ$. $\square$

It will be convenient to give the map $A \mapsto \boldsymbol{O}^A$ a name. So, let us do so.

Definition 5.1.3. We define the map $\Psi_L \colon \mathcal{S}(\beta L)^{\mathrm{op}} \to \mathrm{Idl}(\mathcal{R}L)$ by $\Psi_L(A) = \boldsymbol{O}^A$. When dealing with one frame, we shall drop the subscript.

Recall that a *P-frame* is a completely regular frame in which every cozero element is complemented. We are aiming for the first result announced in the abstract, which will culminate in showing that we have an adjunction

$$\mathrm{Idl}(\mathcal{R}L) \underset{\Psi}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{S}(\beta L)^{\mathrm{op}}$$

precisely when L is a P-frame. To recall, if X and Y are posets, two monotone functions $f \colon X \to Y$ and $g \colon Y \to X$ are said to be in a *Galois connection*, with f on the left and g on the right, written $f \dashv g$, if

$$\forall x \in X, \;\; \forall y \in Y, \;\; f(x) \leq y \iff x \leq g(y).$$

A useful characterization is that

$$f \dashv g \quad \iff \quad f \circ g \leq \mathrm{id}_Y \;\text{ and }\; \mathrm{id}_X \leq g \circ f.$$

En route to the result we are aiming for, we establish some preliminary ones, including others that are not really germane to the task at hand, but which we find to be noteworthy nevertheless. One of the latter kind generalizes [9, Lemma 1.6], and significantly sharpens it because it also mentions an instance (not observed in [9]) of when the converse holds.

Recall that a completely regular frame L is called an *almost P-frame* [4] if $c = c^{**}$ for every $c \in \operatorname{Coz} L$.

Proposition 5.1.4. *Let L be a completely regular frame.*

(a) *For any sublocale A of βL, $\Delta(\boldsymbol{O}^A) = \Delta(\boldsymbol{O}^{\overline{A}}) = \Delta(\boldsymbol{M}^{\overline{A}}) = \overline{A}$.*

(b) *If A and B are sublocales of βL with $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$, then $\overline{B} \subseteq \overline{A}$. The converse holds if L is an almost P-frame.*

(c) *For any ideal I of $\mathcal{R}L$, $\boldsymbol{O}^{\Delta(I)} = mI$.*

Proof. (a) Since $\overline{A} \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha))$ for every $\alpha \in \boldsymbol{M}^{\bar{A}}$, and since $\boldsymbol{O}^{\bar{A}} \subseteq \boldsymbol{O}^A \subseteq \boldsymbol{M}^A$, we have

$$\overline{A} \subseteq \Delta(\boldsymbol{M}^{\bar{A}}) = \Delta(\boldsymbol{M}^A) \subseteq \Delta(\boldsymbol{O}^A) \subseteq \Delta(\boldsymbol{O}^{\overline{A}}).$$

Consequently, we need only show that $\Delta(\boldsymbol{O}^{\overline{A}}) \subseteq \overline{A}$. Put $H = \bigwedge A$, so that $\overline{A} = \mathfrak{c}_{\beta L}(H)$. Now, in light of the fact that

$$\Delta(\boldsymbol{O}^{\overline{A}}) = \mathfrak{c}_{\beta L}\left(\bigcup_{\alpha \in \boldsymbol{O}^{\overline{A}}} r_L(\operatorname{coz}\alpha)\right),$$

it suffices to show that $\bigcup\{r_L(\operatorname{coz}\alpha) \mid \alpha \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(H)}\} = H$. But this is indeed so because, for any $\gamma \in \mathcal{R}L$, $\gamma \in \boldsymbol{O}^{\mathfrak{c}_{\beta L}(H)}$ if and only if $\operatorname{coz}\gamma \in H$, and $H = \bigcup\{r_L(\operatorname{coz}\alpha) \mid \operatorname{coz}\alpha \in H\}$.

(b) If $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$, then, in light of the result in (a),

$$\overline{B} = \Delta(\boldsymbol{M}^{\overline{B}}) = \Delta(\boldsymbol{M}^B) \subseteq \Delta(\boldsymbol{O}^A) = \overline{A};$$

which proves the first part of (b). Next, assume that L is an almost P-frame and A and B are sublocales of βL with $\overline{B} \subseteq \overline{A}$. We must show that $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$. Let $\alpha \in \boldsymbol{O}^A$. By definition,

$A \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)$, which, on taking closures and noting that $r_L(\operatorname{coz}\alpha)^{**} = r_L(\operatorname{coz}\alpha)$ since L is an almost P-frame, yields

$$\overline{B} \subseteq \overline{A} \subseteq \overline{\mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)} = \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha)^{**}) = \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha)),$$

which implies $\alpha \in \boldsymbol{M}^{\overline{B}} = \boldsymbol{M}^B$. Therefore $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$.

(c) Since $\Delta(I) = \mathfrak{c}_{\beta L}\big(\bigcup_{\alpha\in I} r_L(\operatorname{coz}\alpha)\big)$, for any $\gamma \in \mathcal{R}L$ we have

$$\begin{aligned} \gamma \in \boldsymbol{O}^{\Delta(I)} \quad &\iff \quad \operatorname{coz}\gamma \in \bigcup\{r_L(\operatorname{coz}\alpha) \mid \alpha \in I\} \\ &\iff \quad \operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\alpha, \quad \text{for some } \alpha \in I \\ &\iff \quad \gamma \in mI, \end{aligned}$$

which then proves the claim. □

We have the following application to $C(X)$. Recall that a Tychonoff space is called an *almost P-space* if each of its G_δ-sets has dense interior. These spaces were studied in detail by Levy [30]. A space X is an almost P-space if and only if $\Omega(X)$ is an almost P-frame. In [9, Lemma 1.6], Dietrich shows that if, for subsets A and B of βX, $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$, then $\overline{B} \subseteq \overline{A}$. We show that the converse holds if X is an almost P-space. For that, we need a lemma.

Recall that complemented (and hence closed) sublocales of a spatial frame are spatial. Let K be a closed subset of a Tychonoff space X. Since

$$\operatorname{Pt}(\widetilde{K}) = \{\widetilde{w} \mid w \in K\} = \big\{X \smallsetminus \{w\} \mid w \in K\big\} = \operatorname{Pt}\big(\mathfrak{c}_{\Omega(X)}(X \smallsetminus K)\big),$$

the latter by a simple calculation, it follows that $\widetilde{K} = \mathfrak{c}_{\Omega(X)}(X \smallsetminus K)$ because both these sublocales are spatial.

In what follows we use the overline for both the closure in spaces and locales. There will be no danger of confusion. Observe from [35, Proposition VI.1.3.1] that if A and B are subsets of a T_D-space (and hence of a Tychonoff space), then $A \subseteq B$ if and only if $\widetilde{A} \subseteq \widetilde{B}$.

Lemma 5.1.5. *If S is a subset of a Tychonoff space X, then $\overline{\widetilde{S}} = \widetilde{\overline{S}}$.*

Proof. By what we have just observed, it suffices to show that $\overline{\widetilde{S}} = \mathfrak{c}_{\Omega(X)}\big(X \smallsetminus \overline{S}\big)$. Since $\overline{\widetilde{S}}$ is a closed sublocale of $\Omega(X)$, there exists $U \in \Omega(X)$ such that $\overline{\widetilde{S}} = \mathfrak{c}_{\Omega(X)}(U)$. So we must show that $U = X \smallsetminus \overline{S}$. Since $S \subseteq \overline{S}$, $\widetilde{S} \subseteq \widetilde{\overline{S}}$, and since $\widetilde{\overline{S}}$ is a closed sublocale, we have $\overline{\widetilde{S}} \subseteq \widetilde{\overline{S}}$. This says $\mathfrak{c}_{\Omega(X)}(U) \subseteq \mathfrak{c}_{\Omega(X)}(X \smallsetminus \overline{S})$, which implies $X \smallsetminus \overline{S} \subseteq U$. On the other hand, the containment $\widetilde{S} \subseteq \overline{\widetilde{S}} = \widetilde{X \smallsetminus U}$ implies $S \subseteq X \smallsetminus U$, so that $U \cap S = \emptyset$, and hence $U \cap \overline{S} = \emptyset$ because an open set misses a set if and only if it misses the closure of that set. Therefore $U \subseteq X \smallsetminus \overline{S}$, hence we have the desired equality, whence the lemma follows. □

Corollary 5.1.6. *Let X be an almost P-space and A and B subsets of βX such that $\overline{B} \subseteq \overline{A}$. Then $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$.*

Proof. By hypothesis, $\Omega(X)$ is an almost P-frame and, for the sublocales $\widetilde{\overline{A}}$ and $\widetilde{\overline{B}}$ of $\Omega(\beta X)$ induced by $\overline{A}$ and $\overline{B}$, we have $\widetilde{\overline{B}} \subseteq \widetilde{\overline{A}}$. Thus, by Lemma 5.1.5, the sublocales $\widetilde{A}$ and $\widetilde{B}$ of $\Omega(\beta X)$ satisfy the containment $\overline{\widetilde{B}} \subseteq \overline{\widetilde{A}}$. Proposition 5.1.4(b) implies $\boldsymbol{O}^{\widetilde{B}} \subseteq \boldsymbol{M}^{\widetilde{A}}$. From Corollary 2.1.3 we deduce that $\varphi_X[\boldsymbol{O}^B] \subseteq \varphi_X[\boldsymbol{M}^A]$, which implies $\boldsymbol{O}^B \subseteq \boldsymbol{M}^A$ because φ_X is a ring isomorphism. □

We are now ready to present the first main result in the chapter.

Theorem 5.1.7. *Regarding the maps $\Delta\colon \mathrm{Idl}(\mathcal{R}L) \to \mathcal{S}(\beta L)^{\mathrm{op}}$ and $\Psi\colon \mathcal{S}(\beta L)^{\mathrm{op}} \to \mathrm{Idl}(\mathcal{R}L)$, we have the following results.*

(a) *Δ is a quantale homomorphism and Ψ preserves meets.*

(b) *Δ and Ψ are in a Galois connection, with Δ on the left, iff L is a P-frame.*

Proof. (a) Let us show first that Δ preserves joins. Let $\{I_k \mid k \in K\} \subseteq \mathrm{Idl}(\mathcal{R}L)$. We claim that

$$\bigvee_{\alpha \in \sum_k I_k} r_L(\operatorname{coz}\alpha) = \bigvee_{k \in K}\Big(\bigvee_{\alpha \in I_k} r_L(\operatorname{coz}\alpha)\Big). \tag{5.1.1}$$

For a fixed $k_0 \in K$,

$$\bigvee_{\alpha \in I_{k_0}} r_L(\operatorname{coz}\alpha) \le \bigvee_{\alpha \in \sum_k I_k} r_L(\operatorname{coz}\alpha),$$

which yields the inequality $\geq$ in (5.1.1). For the opposite inequality, let us keep in mind that

$$\bigvee_{\alpha \in \sum_k I_k} r_L(\operatorname{coz}\alpha) = \bigcup_{\alpha \in \sum_k I_k} r_L(\operatorname{coz}\alpha).$$

If a cozero element c belongs to this union, there is an α_0 in $\sum_k I_k$ such that $c \in r_L(\operatorname{coz}\alpha_0)$. Therefore there are finitely many indices $k_1, \ldots, k_n$ in K and elements $\alpha_{k_i} \in I_{k_i}$, for $i = 1, \ldots, n$, such that $\alpha_0 = \alpha_{k_1} + \cdots + \alpha_{k_n}$. Since r_L preserves finite joins of cozero elements, and since $\operatorname{coz}\alpha_0 \leq \operatorname{coz}(\alpha_{k_1}) \vee \cdots \vee \operatorname{coz}(\alpha_{k_n})$, we have

$$r_L(\operatorname{coz}\alpha_0) \leq r_L(\operatorname{coz}(\alpha_{k_1})) \vee \cdots \vee r_L(\operatorname{coz}(\alpha_{k_n})) \leq \bigvee_{\alpha\in\sum_k I_k} r_L(\operatorname{coz}\alpha),$$

which proves the desired inequality, and hence establishes the claimed equality. We argue from this that Δ preserves joins. We need to keep in mind that joins in $\mathcal{S}(\beta L)^{\mathrm{op}}$ are intersections. Let $\{I_k \mid k \in K\} \subseteq \operatorname{Idl}(\mathcal{R}L)$. Then.

$$\begin{aligned}
\Delta\left(\bigvee_{k\in K}^{\operatorname{Idl}(\mathcal{R}L)} I_k\right) = \Delta\Big(\sum_k I_k\Big) &= \mathfrak{c}_{\beta L}\Big(\bigvee_{\alpha\in\sum_k I_k} r_L(\operatorname{coz}\alpha)\Big)\\
&= \mathfrak{c}_{\beta L}\Big(\bigvee_{k\in K}\Big(\bigvee_{\alpha\in I_k} r_L(\operatorname{coz}\alpha)\Big)\Big)\\
&= \bigcap_{k\in K}\mathfrak{c}_{\beta L}\Big(\bigvee_{\alpha\in I_k} r_L(\operatorname{coz}\alpha)\Big)\\
&= \bigcap_{k\in K}\Delta(I_k)\\
&= \bigvee_{k\in K}^{\mathcal{S}(\beta L)^{\mathrm{op}}} \Delta(I_k),
\end{aligned}$$

which proves that Δ preserves joins.

Now we show that Δ preserves the binary product. Let I and J be ideals of $\mathcal{R}L$. The product $\Delta(I)\cdot\Delta(J)$ is the meet of these two sublocales taken in $\mathcal{S}(\beta L)^{\mathrm{op}}$, which is their join calculated in $\mathcal{S}(\beta L)$. Therefore,

$$\begin{aligned}
\Delta(I)\cdot\Delta(J) &= \mathfrak{c}_{\beta L}\Big(\bigvee_{\alpha\in I} r_L(\operatorname{coz}\alpha)\Big) \vee \mathfrak{c}_{\beta L}\Big(\bigvee_{\gamma\in J} r_L(\operatorname{coz}\gamma)\Big)\\
&= \mathfrak{c}_{\beta L}\left(\bigvee_{\alpha\in I} r_L(\operatorname{coz}\alpha) \wedge \bigvee_{\gamma\in J} r_L(\operatorname{coz}\gamma)\right)\\
&= \mathfrak{c}_{\beta L}\left(\bigvee_{(\alpha,\gamma)\in I\times J}\big(r_L(\operatorname{coz}\alpha)\wedge r_L(\operatorname{coz}\gamma)\big)\right) \qquad \text{by the frame law}\\
&= \mathfrak{c}_{\beta L}\left(\bigvee_{(\alpha,\gamma)\in I\times J} r_L(\operatorname{coz}(\alpha\gamma))\right).
\end{aligned}$$

Now, if $\rho \in IJ$, then $\rho = \alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n$, for some finitely many elements $\alpha_i \in I$ and $\gamma_i \in J$, which then, by the properties of the cozero map and the fact that r_L preserves finite joins of cozero elements, implies that

$$r_L(\operatorname{coz}\rho) \;\le\; r_L\big(\operatorname{coz}(\alpha_1\gamma_1)\big) \vee \cdots \vee r_L\big(\operatorname{coz}(\alpha_n\gamma_n)\big) \;\le\; \bigvee_{(\alpha,\gamma)\in I\times J} r_L(\operatorname{coz}(\alpha\gamma)).$$

Consequently,

$$\bigvee_{\tau\in IJ} r_L(\operatorname{coz}\tau) \;\le\; \bigvee_{(\alpha,\gamma)\in I\times J} r_L(\operatorname{coz}(\alpha\gamma)) \;\le\; \bigvee_{\tau\in IJ} r_L(\operatorname{coz}\tau),$$

from which we deduce that $\Delta(IJ) = \Delta(I)\cdot\Delta(J)$. In all then, Δ is a quantale homomorphism.

Next, we show that Ψ preserves meets. Undecorated joins of sublocales are understood to be taken in $\mathcal{S}(\beta L)$. If $\{A_k \mid k\in K\} \subseteq \mathcal{S}(\beta L)^{\mathrm{op}}$, then

$$\Psi\left(\bigwedge_{k\in K}^{\mathcal{S}(\beta L)^{\mathrm{op}}} A_k\right) = \boldsymbol{O}^{\bigvee_{k\in K}A_k} = \bigcap_{k\in K}\boldsymbol{O}^{A_k} = \bigcap_{k\in K}\Psi(A_k) = \bigwedge_{k\in K}^{\mathrm{Idl}(\mathcal{R}L)}\Psi(A_k),$$

which shows that Ψ preserves meets.

(b) We comment first that the results in (a) tell us that both these maps are monotone, so it does make sense to talk about them possibly being in a Galois connection. Proposition 5.1.4(a) says $\Delta(\Psi(A)) = \bar{A}$ for every $A\in\mathcal{S}(\beta L)^{\mathrm{op}}$. Since $\bar{A}\le A$ in $\mathcal{S}(\beta L)^{\mathrm{op}}$, we therefore have $\Delta\circ\Psi \le \mathrm{id}_{\mathcal{S}(\beta L)^{\mathrm{op}}}$. Now recall from [11, Corollary 3.10]) that L is a P-frame if and only if every ideal of $\mathcal{R}L$ is neat. Since an ideal I of $\mathcal{R}L$ is neat if and only if $I = mI$, we have

$$\begin{aligned} L \text{ is a } P\text{-frame} \quad &\Longleftrightarrow \quad I \subseteq mI \text{ for every } I\in\mathcal{R}L \\ &\Longleftrightarrow \quad \mathrm{id}_{\mathrm{Idl}(\mathcal{R}L)} \le \Psi\circ\Delta. \end{aligned}$$

It follows therefore that Δ is left adjoint to Ψ if and only L is a P-frame. □

We now wish to interpret this in $C(X)$. For that we need some background, sourced mainly from [36]. Recall that for any frame L, the lattice

$$\mathcal{S}_{\mathfrak{c}}(L) = \{S\in\mathcal{S}(L) \mid S \text{ is a join of closed sublocales}\}$$

is a frame, with partial order $\subseteq$ and joins as in $\mathcal{S}(L)$. If L is subfit, then $\mathcal{S}_{\mathfrak{c}}(L)$ is a Boolean frame, with complements equal to supplements calculated in $\mathcal{S}(L)$. Furthermore, $\mathfrak{B}(\mathcal{S}(L)^{\mathrm{op}})$, the Booleanization of $\mathcal{S}(L)^{\mathrm{op}}$, is the Boolean frame $\mathcal{S}_{\mathfrak{c}}(L)^{\mathrm{op}}$. If Y is a T_1-space and $\mathfrak{P}(Y)$ denotes the powerset of Y, then the map

$$\tau_Y\colon \mathfrak{P}(Y)^{\mathrm{op}} \to \mathcal{S}_{\mathfrak{c}}(\Omega(Y))^{\mathrm{op}} \qquad \text{given by} \qquad T\mapsto \widetilde{T}$$

is an isomorphism of Boolean algebras, and hence a frame isomorphism.

Next, any ring isomorphism $\phi\colon A \to B$ induces a quantale isomorphism $\widehat{\phi}\colon \mathrm{Idl}(A) \to \mathrm{Idl}(B)$ given by $I \mapsto \phi[I]$. Recall the ring isomorphism $\varphi_X\colon C(X) \to \mathcal{R}(\Omega(X))$ for any Tychonoff space X, and, as in Section 2.1, view the Stone-Čech compactification of $\Omega(X)$ as being given by the dense-onto frame homomorphism $\Omega(i_X)\colon \Omega(\beta X) \to \Omega(X)$. We consequently have the map

$$\mathsf{F}_X\colon \mathrm{Idl}(C(X)) \to \mathrm{Idl}(\mathcal{R}(\Omega(X))) \to \mathcal{S}(\Omega(\beta X))^{\mathrm{op}} \to \mathcal{S}_{\mathfrak{c}}(\Omega(\beta X))^{\mathrm{op}} \to \mathfrak{P}(\beta X)^{\mathrm{op}}$$

given by the composite

$$\mathsf{F}_X = \tau_{\beta X}^{-1} \circ \mathsf{b}_{\mathcal{S}(\Omega(\beta X))^{\mathrm{op}}} \circ \Delta_{\Omega(X)} \circ \widehat{\varphi_X}.$$

Being a composite of quantale homomorphisms, the map $\mathsf{F}_X\colon \mathrm{Idl}(C(X)) \to \mathfrak{P}(\beta X)^{\mathrm{op}}$ is itself a quantale homomorphism. We show that it is precisely the map of Gillman, Henriksen and Jerison.

Proposition 5.1.8. $\mathsf{F}_X(I) = \bigcap\{\mathrm{cl}_{\beta X} Z(f) \mid f \in I\}$, *for every ideal* I *of* $C(X)$.

Proof. Since $\big(\Delta_{\Omega(X)} \circ \widehat{\varphi_X}\big)(I)$ is a closed sublocale of $\Omega(\beta X)$ for any ideal I of $C(X)$, and since the Booleanization map $\mathsf{b}_{\mathcal{S}(\Omega(\beta X))^{\mathrm{op}}}$ sends every closed (actually, every complemented) sublocale to itself, it suffices to show that

$$\big(\tau_{\beta X} \circ \mathsf{F}_X\big)(I) = \big(\Delta_{\Omega(X)} \circ \widehat{\varphi_X}\big)(I).$$

To compute the sublocale on the left, recall that if K is a closed subset of Y then the induced closed sublocale of $\Omega(Y)$ is $\mathfrak{c}_{\Omega(Y)}(Y \smallsetminus K)$. Recall also from the discussion preceding Lemma 2.1.1 that, for any $f \in C(X)$,

$$\beta X \smallsetminus \mathrm{cl}_{\beta X} Z(f) = \Omega(i_X)_*\big(\mathrm{coz}(\varphi(f))\big) = r_{\Omega(X)}\big(\mathrm{coz}(\varphi_X(f))\big).$$

Thus,

$$\begin{aligned}
\big(\tau_{\beta X} \circ \mathsf{F}_X\big)(I) = \widetilde{\mathsf{F}_X(I)} &= \mathfrak{c}_{\Omega(\beta X)}\Big(\beta X \smallsetminus \bigcap_{f\in I} \mathrm{cl}_{\beta X} Z(f)\Big) \\
&= \mathfrak{c}_{\Omega(\beta X)}\Big(\bigcup_{f\in I}\big(\beta X \smallsetminus \mathrm{cl}_{\beta X} Z(f)\big)\Big) \\
&= \mathfrak{c}_{\Omega(\beta X)}\left(\bigcup_{f\in I} r_{\Omega(X)}\big(\operatorname{coz}(\varphi_X(f))\big)\right) \\
&= \mathfrak{c}_{\Omega(\beta X)}\left(\bigcup_{\alpha\in\varphi_X[I]} r_{\Omega(X)}(\operatorname{coz}\alpha)\right) \\
&= \Delta_{\Omega(X)}(\varphi_X[I]) \\
&= \big(\Delta_{\Omega(X)} \circ \widehat{\varphi_X}\big)(I),
\end{aligned}$$

whence the result follows. □

Given a Tychonoff space X, we have the maps

$$\mathsf{F}_X\colon \operatorname{Idl}(C(X)) \to \mathfrak{P}(\beta X)^{\mathrm{op}} \qquad \text{and} \qquad \mathsf{G}_X\colon \mathfrak{P}(\beta X)^{\mathrm{op}} \to \operatorname{Idl}(C(X))$$

given by $\mathsf{F}_X(I) = \bigcap\{\mathrm{cl}_{\beta X} Z(f) \mid f \in I\}$ and $\mathsf{G}_X(A) = \boldsymbol{O}^A$. It is clear that G_X preserves meets because they are set-theoretic unions in $\mathfrak{P}(\beta X)^{\mathrm{op}}$ and intersections in $\operatorname{Idl}(C(X))$.

The first part in the next corollary follows from Proposition 5.1.8 and the discussion preceding it. We will deduce the second part from Theorem 5.1.7(b).

Corollary 5.1.9. *For any Tychonoff space X, F_X is a quantale homomorphism and G_X preserves meets. Furthermore, $\mathsf{F}_X \dashv \mathsf{G}_X$ iff X is a P-space.*

Proof. We first argue that $\mathsf{F}_X \dashv \mathsf{G}_X$ if and only if $\Delta_{\Omega(X)} \dashv \Psi_{\Omega(X)}$. As observed in the proof of Theorem 5.1.7(b), $\Delta_{\Omega(X)} \circ \Psi_{\Omega(X)} \leq \mathrm{id}_{\mathcal{S}(\Omega(\beta X))^{\mathrm{op}}}$, and so

$$\Delta_{\Omega(X)} \dashv \Psi_{\Omega(X)} \quad \text{iff} \quad I \subseteq \boldsymbol{O}^{\Delta_{\Omega(X)}(I)} \ \text{ for every } I \in \operatorname{Idl}(\mathcal{R}(\Omega(X))).$$

In our notation, [9, Lemma 1.6] says $\mathsf{F}_X(\mathsf{G}_X(A)) = \overline{A}$ for every $A \subseteq \beta X$, so that $\mathsf{F}_X \circ \mathsf{G}_X \leq \mathrm{id}_{\mathfrak{P}(\beta X)^{\mathrm{op}}}$. Thus,

$$\mathsf{F}_X \dashv \mathsf{G}_X \quad \text{iff} \quad J \subseteq \boldsymbol{O}^{\mathsf{F}_X(J)} \ \text{ for every } J \in \operatorname{Idl}(C(X)).$$

Now suppose that $\Delta_{\Omega(X)} \dashv \Psi_{\Omega(X)}$, and let J be an ideal of $C(X)$. Then $\varphi_X[J]$ is an ideal of $\mathcal{R}(\Omega(X))$, and therefore $\varphi_X[J] \subseteq \boldsymbol{O}^{\Delta_{\Omega(X)}(\varphi_X[J])}$, which, in light of Corollary 2.1.3 and the equality $\tau_{\beta X} \circ \mathsf{F}_X = \Delta_{\Omega(X)} \circ \widehat{\varphi_X}$ established in the proof of Proposition 5.1.8, implies

$$\varphi_X[J] \subseteq \boldsymbol{O}^{\tau_{\beta X}(\mathsf{F}_X(J))} = \boldsymbol{O}^{\widetilde{\mathsf{F}_X(J)}} = \varphi_X\left[\boldsymbol{O}^{\mathsf{F}_X(J)}\right],$$

whence we get $J \subseteq \boldsymbol{O}^{\mathsf{F}_X(J)}$, thus proving that $\mathsf{F}_X \dashv \mathsf{G}_X$. Conversely, suppose that $\mathsf{F}_X \dashv \mathsf{G}_X$, and let I be an ideal of $\mathcal{R}(\Omega(X))$. Since φ_X is a ring isomorphism, there exists an ideal J of $C(X)$ such that $I = \varphi_X(J)$. Then (as $\mathsf{F}_X \dashv \mathsf{G}_X$), $J \subseteq \boldsymbol{O}^{\mathsf{F}_X(J)}$, which implies

$$I = \varphi_X(J) \subseteq \varphi_X\left[\boldsymbol{O}^{\mathsf{F}_X(J)}\right] = \boldsymbol{O}^{\widetilde{\mathsf{F}_X(J)}} = \boldsymbol{O}^{\tau_{\beta X}(\mathsf{F}_X(J))} = \boldsymbol{O}^{\Delta_{\Omega(X)}(\varphi_X[J])} = \boldsymbol{O}^{\Delta_{\Omega(X)}(I)},$$

whence $\Delta_{\Omega(X)} \dashv \Psi_{\Omega(X)}$.

Now, since X is a P-space precisely when $\Omega(X)$ is a P-frame, it follows from Theorem 5.1.7(b) that $\mathsf{F}_X \dashv \mathsf{G}_X$ if and only if X is a P-space. □

If $f\colon L \to M$ is a join-preserving map between complete lattices and f_* denotes its right adjoint, then for every $b \in M$ for which $f^{-1}(b) \neq \emptyset$, the equality $f(f_*(b)) = b$ always holds. Now, since Δ is surjective on closed sublocales, we know from Theorem 5.1.7(b) that if L is a P-frame, then $\Delta_*(A) = \boldsymbol{O}^A$ for every closed sublocale A of βL. Below we produce an explicit example (necessarily in some frame L which is not a P-frame) of a closed sublocale A of βL for which $\Delta_*(A) \neq \boldsymbol{O}^A$.

Example 5.1.10. Let $L = \Omega(\mathbb{R})$, and put $a = (0, 1)$. Since $\mathbb{R}$ is metrizable, every element of L is a cozero element. Pick $\gamma \in \mathcal{R}L$ with $\operatorname{coz}\gamma = a$. Let A be the closed sublocale $A = \mathfrak{c}_{\beta L}(r_L(a))$ of βL. Then $\Delta(\boldsymbol{O}^A) = A$ by Proposition 5.1.4(a), and so, $\boldsymbol{O}^A \leq \Delta_*(A)$. We now produce an ideal J of $\mathcal{R}L$ with $\boldsymbol{O}^A \subset J$ (proper containment) and $\Delta(J) = A$. This will show that $\Delta_*(A) \neq \boldsymbol{O}^A$. Let $J = \langle\gamma\rangle$, the ideal generated by γ. Since, for any $\tau \in \mathcal{R}L$, $\operatorname{coz}\tau \prec\!\!\prec \operatorname{coz}\gamma$ implies $\tau \in \langle\gamma\rangle$, it is easy to see that $\bigvee_{\alpha\in J} r_L(\operatorname{coz}\alpha) = r_L(a)$. Also, from what we observed about the $\boldsymbol{O}$-ideals associated with closed sublocales, we see that $\boldsymbol{O}^A = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \prec\!\!\prec a\}$, whence we deduce that $\gamma \notin \boldsymbol{O}^A$ because a is not complemented.

Naturally, one wonders if we can identify some class of closed sublocales A for which $\Delta_*(A) = \boldsymbol{O}^A$. Rather unexpectedly, sublocales which are generalizations of P-sets are of this type. Taking a cue from spaces, the author of [18] calls a closed sublocale a *P-sublocale* if it is interior to every zero-sublocale containing it. He then shows that a closed sublocale $\mathfrak{c}_L(a)$ of a frame L is a P-sublocale if and only if every cozero element of L which is below a is actually rather below a.

In the proof of the next theorem we will use the fact that if $I \prec J$ in βL, then $\bigvee I \in J$. For a proof see the paragraph preceding [14, Example 4.11].

Theorem 5.1.11. *If A is a join of P-sublocales of βL, then $\Delta_*(A) = \boldsymbol{O}^A$.*

Proof. We prove this first for P-sublocales. So let $B = \mathfrak{c}_{\beta L}(I)$ be a P-sublocale of βL. We are going to show that for any $c \in \operatorname{Coz} L$, the containment $r_L(c) \subseteq I$ implies $c \in I$. [Caution: $r_L(c)$ is not necessarily a cozero element of βL]. Find a sequence (c_n) in $\operatorname{Coz} L$ with $c_n \prec\!\!\prec c_{n+1}$ for each n and $\bigvee_{n\in\mathbb{N}} c_n = c$. Since $r_L(c_n) \prec\!\!\prec r_L(c_{n+1})$ for each n, $\bigvee_{n\in\mathbb{N}} r_L(c_n)$ is a cozero element of βL, and, furthermore, it is below I, and so $\bigvee_{n\in\mathbb{N}} r_L(c_n) \prec I$ because $\mathfrak{c}_{\beta L}(I)$ is a P-sublocale. Therefore $\left(\bigvee_{n\in\mathbb{N}} r_L(c_n)\right)^* \vee I = 1_{\beta L}$. But now

$$\Big(\bigvee_{n\in\mathbb{N}} r_L(c_n)\Big)^* = \bigwedge_{n\in\mathbb{N}} r_L(c_n)^* = \bigwedge_{n\in\mathbb{N}} r_L(c_n^*) = r_L\Big(\bigwedge_{n\in\mathbb{N}} c_n^*\Big) = r_L\Big(\Big(\bigvee_{n\in\mathbb{N}} c_n\Big)^*\Big) = r_L(c^*) = r_L(c)^*,$$

which then implies $r_L(c) \prec I$, and hence $c \in I$. Since Δ is surjective on closed sublocales,

$$\Delta_*(B) = \sum\{J \in \operatorname{Idl}(\mathcal{R}L) \mid \Delta(J) = B\}.$$

Since $\Delta(\boldsymbol{O}^B) = \overline{B} = B$, $\boldsymbol{O}^B \leq \Delta_*(B)$. Consider any ideal J of $\mathcal{R}L$ with $\Delta(J) = B$. Then, from the definition, $\bigvee_{\alpha\in J} r_L(\operatorname{coz}\alpha) = I$. Thus, if $\alpha \in J$, then $r_L(\operatorname{coz}\alpha) \subseteq I$, and thus by what we proved above, $\operatorname{coz}\alpha \in I$, which implies $\alpha \in \boldsymbol{O}^B$. Therefore $\boldsymbol{O}^B$ is the largest ideal of $\mathcal{R}L$ mapped to B by Δ, hence $\Delta_*(B) = \boldsymbol{O}^B$.

Now suppose that $A = \bigvee_{k\in K} A_k$, for some P-sublocales A_k of βL. Then

$$\Delta_*\Big(\bigvee_{k\in K} A_k\Big) = \Delta_*\left(\bigwedge_{k\in K}^{\mathcal{S}(\beta L)^{\mathrm{op}}} A_k\right) = \bigwedge_{k\in K}^{\operatorname{Idl}(\mathcal{R}L)} \Delta_*(A_k) = \bigcap_{k\in K} \boldsymbol{O}^{A_k} = \boldsymbol{O}^{\bigvee_{k\in K} A_k} = \boldsymbol{O}^A,$$

which proves the result. □

5.2 Characterizing Woods' WN-maps

Our goal in this section is, among other things, to use the material in the previous section to characterize the WN-homomorphisms that we discussed in Chapter 3. Recall from [40, p. 25] that if $\phi\colon A \to B$ is a ring homomorphism, then the map $\mathrm{Idl}(A) \to \mathrm{Idl}(B)$ given by $I \mapsto \langle \phi[I] \rangle$ is a quantale homomorphism, where $\langle - \rangle$ denotes ideal-generation. Therefore the square in the following theorem resides in the category of quantales.

Theorem 5.2.1. *A frame homomorphism $h\colon M \to L$ is a WN-homomorphism iff the square*

$$\begin{array}{ccc} \mathrm{Idl}(\mathcal{R}M) & \xrightarrow{\langle (\mathcal{R}h)[-] \rangle} & \mathrm{Idl}(\mathcal{R}L) \\ \Big\downarrow{\scriptstyle \Delta_M} & & \Big\downarrow{\scriptstyle \Delta_L} \\ \mathcal{S}(\beta M)^{\mathrm{op}} & \xrightarrow{((\beta h)_*)_{-1}[-]} & \mathcal{S}(\beta L)^{\mathrm{op}} \end{array}$$

commutes.

Proof. Let I be an ideal of $\mathcal{R}M$. Since for any ideal Q of $\mathcal{R}L$, the join of the form $\bigvee_{\alpha \in Q} r_L(\operatorname{coz} \alpha)$ can be taken over any generating subset of Q, we have

$$\Delta_L(\langle (\mathcal{R}h[I]) \rangle) = \mathfrak{c}_{\beta L} \left(\bigvee_{\tau \in (\mathcal{R}h)[I]} r_L(\operatorname{coz} \tau) \right).$$

On the other hand,

$$\begin{aligned} ((\beta h)_*)_{-1}[\Delta_M(I)] &= ((\beta h)_*)_{-1} \left[\mathfrak{c}_{\beta M} \Big(\bigvee_{\alpha \in I} r_L(\operatorname{coz} \alpha) \Big) \right] \\ &= \mathfrak{c}_{\beta L} \left((\beta h) \Big(\bigvee_{\alpha \in I} r_L(\operatorname{coz} \alpha) \Big) \right) \\ &= \mathfrak{c}_{\beta L} \Big(\bigvee_{\alpha \in I} (\beta h) \big(r_L(\operatorname{coz} \alpha) \big) \Big). \end{aligned}$$

Therefore the square commutes if and only if

$$\bigvee_{\alpha \in I} (\beta h) \big(r_M(\operatorname{coz} \alpha) \big) = \bigvee_{\tau \in (\mathcal{R}h)[I]} r_L(\operatorname{coz} \tau) \tag{5.2.1}$$

for every ideal I of $\mathcal{R}M$.

Now we suppose that the square commutes, and show that h is a WN-homomorphism. Let $c \in \operatorname{Coz} M$, and take $\gamma \in \mathcal{R}M$ such that $c = \operatorname{coz} \gamma$. For $I = \langle \gamma \rangle$, the left side of equation (5.2.1) is $(\beta h)(r_M(c))$. Since $\mathcal{R}h$ is a ring homomorphism, the ideal of $\mathcal{R}L$ generated by $(\mathcal{R}h)[\langle \gamma \rangle]$

is the principal ideal $\langle(\mathcal{R}h)(\gamma)\rangle$. Since $\operatorname{coz}((\mathcal{R}h)(\gamma)) = h(\operatorname{coz}\gamma)$, the right side of (5.2.1) with $I = \langle\gamma\rangle$ is $r_L(h(c))$, which then shows that h is a WN-homomorphism.

Conversely, suppose that h is a WN-homomorphism. Let I be an ideal of $\mathcal{R}M$, and take any $\alpha \in I$. Then $(\beta h)(r_M(\operatorname{coz}\alpha)) = r_L(h(\operatorname{coz}\alpha))$. But $h(\operatorname{coz}\alpha) = \operatorname{coz}((\mathcal{R}h)(\alpha))$, and $(\mathcal{R}h)(\alpha) \in (\mathcal{R}h)[I]$; so we deduce from this that the inequality $\leq$ in equation (5.2.1) holds. On the other hand, let $\tau \in (\mathcal{R}h)[I]$. Then there exists $\alpha \in I$ such that $\tau = (\mathcal{R}h)(\alpha)$, hence $\operatorname{coz}\tau = h(\operatorname{coz}\alpha)$, whence

$$r_L(\operatorname{coz}\tau) = r_L(h(\operatorname{coz}\alpha)) = (\beta h)(r_M(\operatorname{coz}\alpha)),$$

since h is a WN-homomorphism. It follows from this that the other inequality also holds, and so the square commutes. $\square$

The upper morphism in the square in Theorem 5.2.1 is extension of ideals, and the lower morphism is localic inverse image. We can form a "dual" square, with contraction of ideals replacing extension, and direct image replacing localic inverse image. We show below that the resulting square always commutes, regardless of the homomorphism. In preparation for that, we need to recall two concepts.

A frame homomorphism is said to be *perfect* if its right adjoint preserves directed joins. It is well known that a frame homomorphism into a compact regular frame is perfect. Recall that a localic map $f\colon L \to M$ is said to be *closed* if the induced direct-image map sends closed sublocales to closed sublocales. This is so precisely when $f[\mathfrak{c}_L(a)] = \mathfrak{c}_M(f(a))$ for every $a \in L$. Any localic map with a compact regular domain is closed.

For use below, observe that since $(\beta h)(r_M(a)) \leq r_L(h(a))$ for any frame homomorphism $h\colon M \to L$ and $a \in M$, we have $r_M(a) \leq (\beta h)_*(r_L(h(a)))$.

Theorem 5.2.2. *For any frame homomorphism $h\colon M \to L$, the square*

$$\begin{array}{ccc} \operatorname{Idl}(\mathcal{R}L) & \xrightarrow{(\mathcal{R}h)^{-1}[-]} & \operatorname{Idl}(\mathcal{R}M) \\ {\scriptstyle \Delta_L}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_M} \\ \mathcal{S}(\beta L)^{\mathrm{op}} & \xrightarrow{(\beta h)_*[-]} & \mathcal{S}(\beta M)^{\mathrm{op}} \end{array} \qquad (5.2.2)$$

commutes.

Proof. For any $I \in \mathrm{Idl}(\mathcal{R}L)$,

$$\Delta_M((\mathcal{R}h)^{-1}[I]) = \mathfrak{c}_{\beta M}\left(\bigvee_{\tau\in(\mathcal{R}h)^{-1}[I]} r_M(\operatorname{coz}\tau)\right),$$

and

$$\begin{aligned}(\beta h_*)[\Delta_L(I)] &= (\beta h)_*\left[\mathfrak{c}_{\beta L}\Big(\bigvee_{\alpha\in I} r_L(\operatorname{coz}\alpha)\Big)\right] \\ &= \mathfrak{c}_{\beta M}\left((\beta h)_*\Big(\bigvee_{\alpha\in I} r_L(\operatorname{coz}\alpha)\Big)\right) \qquad \text{since } (\beta h)_* \text{ is a closed map} \\ &= \mathfrak{c}_{\beta M}\Big(\bigvee_{\alpha\in I}(\beta h)_*\big(r_L(\operatorname{coz}\alpha)\big)\Big) \qquad \text{since } \beta h \text{ is perfect .}\end{aligned}$$

Consequently, we shall be done if we can prove that, for any ideal I of $\mathcal{R}L$,

$$\bigvee_{\tau\in(\mathcal{R}h)^{-1}[I]} r_M(\operatorname{coz}\tau) = \bigvee_{\alpha\in I}(\beta h)_*\big(r_L(\operatorname{coz}\alpha)\big).$$

But now if $\tau \in (\mathcal{R}h)^{-1}[I]$, then $(\mathcal{R}h)(\tau) \in I$ and

$$(\beta h)(r_M(\operatorname{coz}\tau)) \le r_L(h(\operatorname{coz}\tau)) = r_L(\operatorname{coz}(\mathcal{R}h)(\tau)),$$

so that if we set $\alpha = (\mathcal{R}h)(\tau)$, we have $\alpha \in I$ and $r_M(\operatorname{coz}\tau) \le (\beta h)_*(r_L(\operatorname{coz}\alpha))$. We therefore have the inequality

$$\bigvee_{\tau\in(\mathcal{R}h)^{-1}[I]} r_M(\operatorname{coz}\tau) \;\le\; \bigvee_{\alpha\in I}(\beta h)_*\big(r_L(\operatorname{coz}\alpha)\big).$$

To establish the opposite inequality, let us note that from the equality $(\beta h)_* \circ r_L = r_M \circ h_*$, which always holds,

$$\bigvee_{\alpha\in I}(\beta h)_*\big(r_L(\operatorname{coz}\alpha)\big) = \bigvee_{\alpha\in I} r_M(h_*(\operatorname{coz}\alpha)).$$

Given $\alpha \in I$, let $c \in r_M(h_*(\operatorname{coz}\alpha))$, and pick $\delta \in \mathcal{R}M$ such that $c \lll \operatorname{coz}\delta \lll h_*(\operatorname{coz}\alpha)$. Then $h(\operatorname{coz}\delta) \lll h(h_*(\operatorname{coz}\alpha)) \le \operatorname{coz}\alpha$. This says $\operatorname{coz}\big((\mathcal{R}h)(\delta)\big) \lll \operatorname{coz}\alpha$, which implies $(\mathcal{R}h)(\delta)$ is a multiple of α, and hence $\delta \in (\mathcal{R}h)^{-1}[I]$. Since $c \in r_M(\operatorname{coz}\delta)$, we deduce that

$$r_M(h_*(\operatorname{coz}\alpha)) \;\le\; \bigvee_{\tau\in(\mathcal{R}h)^{-1}[I]} r_M(\operatorname{coz}\tau),$$

and, upon taking joins over all $\alpha \in I$, we get the desired inequality. This completes the proof. $\square$

5.3 The Lindelöf analogue

The subcategory of **CRFrm** consisting of the compact objects is coreflective, with the coreflector $\beta\colon$ **CRFrm** $\to$ **KCRFrm**. The map Δ associates with each ideal of $\mathcal{R}L$ a sublocale of βL defined in terms of the right adjoint of the homomorphism $j_L\colon \beta L \to L$. In the opposite direction, Ψ associates with each sublocale of βL an ideal of $\mathcal{R}L$ defined in terms of how it relates to a certain open sublocale of βL.

Now, the subcategory of **CRFrm** consisting of the Lindelöf objects is also coreflective, with the coreflector $\lambda\colon$ **CRFrm** $\to$ **KCRFrm** which we will describe shortly. See [31] for details. We wish to consider a map which associates with each ideal of $\mathcal{R}L$ a sublocale of λL defined analogously to the compact case, but using the right adjoint of coreflection map $\lambda_L\colon \lambda L \to L$ to L from Lindelöf objects. In the opposite direction, we will associate with each sublocale of λL an ideal of $\mathcal{R}L$ defined in terms of how it relates to a certain open sublocale of λL.

Turning to some background, the frame of σ-ideals of $\operatorname{Coz} L$ is denoted by λL. It is a Lindelöf completely regular frame. The map $\lambda_L\colon \lambda L \to L$ that sends a σ-ideal to its join in L is a dense surjective frame homomorphism, and it is the coreflection map to L from Lindelöf completely regular frames. We denote its right adjoint by ϱ_L, and recall that, for any $a \in L$,

$$\varrho_L(a) = \{c \in \operatorname{Coz} L \mid c \leq a\}.$$

Thus, if $a \in \operatorname{Coz} L$, then $\varrho_L(a)$ is the principal ideal of $\operatorname{Coz} L$ generated by a. Comparing ϱ_L to r_L, we observe the parallelism:

> $r_{L|\operatorname{Coz} L}\colon \operatorname{Coz} L \to \beta L$ *is a lattice homomorphism, and* $\varrho_{L|\operatorname{Coz} L}\colon \operatorname{Coz} L \to \lambda L$ *is a* σ*-frame homomorphism. In fact,* $\varrho_{L|\operatorname{Coz} L}\colon \operatorname{Coz} L \to \operatorname{Coz}(\lambda L)$ *is a* σ*-frame isomorphism.*

For any $I \in \beta L$, let $\langle I\rangle_\sigma$ denote the σ-ideal of $\operatorname{Coz} L$ generated by I. Explicitly, for any $c \in \operatorname{Coz} L$,

> $c \in \langle I\rangle_\sigma$ *iff there is a sequence* (c_n) *in* I *such that* $c = \bigvee_n c_n$.

We denote by k_L the dense surjective frame homomorphism

$$k_L\colon \beta L \to \lambda L \qquad \text{given by} \qquad k_L(I) = \langle I\rangle_\sigma.$$

It is not difficult to show that the composite

$$\beta L \xrightarrow{k_L} \lambda L \xrightarrow{\lambda_L} L$$

is the homomorphism $j_L\colon \beta L \to L$, so that $r_L = (k_L)_* \circ \varrho_L$, and hence $k_L \circ r_L = \varrho_L$ since k_L is surjective. We will use this observation below.

The map Ψ in Section 5.1 is defined in terms of $\boldsymbol{O}$-ideals, which themselves are defined in terms of sublocales of βL. In order to have an analogy in terms of λL, we first define the following ideals.

For any sublocale A of λL, the ideal $\boldsymbol{N}^A$ of $\mathcal{R}L$ is defined by

$$\boldsymbol{N}^A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{o}_{\lambda L}(\varrho_L((\operatorname{coz}\alpha)^*))\} = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{o}_{\lambda L}(\varrho_L(\operatorname{coz}\alpha)^*)\}.$$

One checks routinely that $\boldsymbol{N}^A$ is indeed an ideal of $\mathcal{R}L$. Actually, these $\boldsymbol{N}$-ideals are some special cases of $\boldsymbol{O}$-ideals, as we show below.

Lemma 5.3.1. *If $h\colon M \to L$ is a surjective frame homomorphism, A is a sublocale of L, and $a \in L$, then $A \subseteq \mathfrak{o}_L(a)$ iff $h_*[A] \subseteq \mathfrak{o}_M(h_*(a))$.*

Proof. It suffices to show that $A \cap \mathfrak{c}_L(a) = \mathsf{O}$ if and only if $h_*[A] \cap \mathfrak{c}_L(h_*(a)) = \mathsf{O}$. But this follows easily from the fact that h_* is injective (as h is surjective) and $h_*(1) = 1$. □

Proposition 5.3.2. *For any sublocale A of λL, $\boldsymbol{N}^A = \boldsymbol{O}^{(k_L)_*[A]}$.*

Proof. For any $\alpha \in \mathcal{R}L$,

$$\begin{aligned}
\alpha \in \boldsymbol{N}^A \quad &\Longleftrightarrow \quad A \subseteq \mathfrak{o}_{\lambda L}(\varrho_L(\operatorname{coz}\alpha)^*) \\
&\Longleftrightarrow \quad (k_L)_*[A] \subseteq \mathfrak{o}_{\beta L}\big((k_L)_*(\varrho_L(\operatorname{coz}\alpha)^*)\big) \qquad \text{by Lemma 5.3.1} \\
&\Longleftrightarrow \quad (k_L)_*[A] \subseteq \mathfrak{o}_{\beta L}\big(r_L(\operatorname{coz}\alpha)^*\big) \qquad \text{since } (k_L)_* \circ \varrho_L = r_L \\
&\Longleftrightarrow \quad \alpha \in \boldsymbol{O}^{(k_L)_*[A]},
\end{aligned}$$

which proves the proposition. □

This proposition says if we view λL as a sublocale of βL (which we can do by identifying λL with its isomorphic copy $(k_L)_*[\lambda L]$), then the $\boldsymbol{N}$-ideals are exactly the $\boldsymbol{O}$-ideals associated with sublocales of λL. Since the map $A \mapsto \boldsymbol{O}^A$ is not injective (indeed, for any dense sublocale D of βL, $\boldsymbol{O}^D$ is the zero ideal, as one checks readily), there is nothing a priori that says the sets of $\boldsymbol{O}$-ideals and $\boldsymbol{N}$-ideals do not coincide. They however generally do not; and below we identify the frames for which they do coincide.

Recall that a frame L is called *pseudocompact* if every element of $\mathcal{R}L$ is bounded, in the sense of f-rings. There are several characterizations, such as L is pseudocompact if and only if λL is compact [6, Proposition 2] if and only if $\beta L \cong \lambda L$. The latter makes one implication in Theorem 5.3.3 below unsurprising.

In the proof of the upcoming result we shall use the notion of codenseness. A frame homomorphism is called *codense* if the top of its domain is the only elements it maps to the top. In the category of regular frames, codense morphisms are exactly the injective ones. We shall also use the fact (see, for instance, [17, Corollary 3.5]) that the neat ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{O}^A$ for A a closed sublocale of βL. The notation used in [17] is different though. Note, further, that if A and B are closed sublocales of βL with $\boldsymbol{O}^A = \boldsymbol{O}^B$, then $A = B$.

Theorem 5.3.3. *The $\boldsymbol{O}$-ideals of $\mathcal{R}L$ are exactly the $\boldsymbol{N}$-ideals iff L is pseudocompact.*

Proof. If L is pseudocompact, then $k_L \colon \beta L \to \lambda L$ is an isomorphism, as can be deduced from the characterization of βL in [4, Corollary 8.2.7]. It then follows from Proposition 5.3.2 that the sets of $\boldsymbol{O}$-ideals and $\boldsymbol{N}$-ideals coincide.

Conversely, suppose that the sets of $\boldsymbol{O}$-ideals and $\boldsymbol{N}$-ideals coincide. We prove that λL is compact by showing that $k_L \colon \beta L \to \lambda L$ is codense. Consider then any $I \in \beta L$ with $k_L(I) = 1_{\lambda L}$. By hypothesis, there is a sublocale A of λL such that $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \boldsymbol{N}^A$. Thus, by Proposition 5.3.2, $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \boldsymbol{O}^{(k_L)_*[A]}$, which makes $\boldsymbol{O}^{(k_L)_*[A]}$ a neat ideal of $\mathcal{R}L$, and so by [19, Lemma 2.9], $\boldsymbol{O}^{(k_L)_*[A]} = \boldsymbol{O}^{\mathrm{cl}_{\beta L}(k_L)_*[A]}$, which then implies $\mathrm{cl}_{\beta L}(k_L)_*[A] = \mathfrak{c}_{\beta L}(I)$.

Now,

$$\mathrm{cl}_{\beta L}(k_L)_*[A] = \mathfrak{c}_{\beta L}\Big(\bigwedge (k_L)_*[A]\Big) = \mathfrak{c}_{\beta L}\Big((k_L)_*\Big(\bigwedge A\Big)\Big),$$

which then implies $I = (k_L)_*(\bigwedge A)$, and hence, in light of k_L being surjective,

$$\bigwedge A = k_L\Big((k_L)_*\Big(\bigwedge A\Big)\Big) = k_L(I) = 1_{\lambda L}.$$

This implies that $I = 1_{\beta L}$, as desired. So k_L is injective and hence is an isomorphism, making λL compact, and hence L pseudocompact. $\square$

Now we define two maps $\Gamma_L \colon \mathrm{Idl}(\mathcal{R}L) \to \mathcal{S}(\lambda L)^{\mathrm{op}}$ and $\Phi_L \colon \mathcal{S}(\lambda L)^{\mathrm{op}} \to \mathrm{Idl}(\mathcal{R}L)$ by adapting the definitions of Δ and Ψ. Namely, for any $I \in \mathcal{R}L$ and $A \in \mathcal{S}(\lambda L)^{\mathrm{op}}$,

$$\Gamma_L(I) = \mathfrak{c}_{\lambda L}\left(\bigvee_{\alpha\in I} \varrho_L(\operatorname{coz}\alpha)\right) \qquad \text{and} \qquad \Phi_L(A) = \boldsymbol{N}^A.$$

As usual, we will forget the subscripts when such selective amnesia leads to no harm.

Remark 5.3.4. A quick remark here may not come amiss. Unlike in the compact case, the join in the definition of $\Gamma_L(I)$ cannot be replaced with a union even though it is directed. Indeed, if $c \in \operatorname{Coz} L$ is not complemented, then, for the ideal $I = \{\gamma \in \mathcal{R}L \mid \operatorname{coz}\gamma \prec\!\!\prec c\}$, we have $c \in \bigvee_{\alpha\in I}\varrho_L(\operatorname{coz}\alpha)$ but $c \notin \bigcup_{\alpha\in I}\varrho_L(\operatorname{coz}\alpha)$. Indeed, if the latter were false, there would exist $\gamma \in \mathcal{R}L$ such that $c \leq \operatorname{coz}\gamma \prec\!\!\prec c$, making c complemented. To see the former, find a sequence (c_n) of cozero elements of L with $c_n \prec\!\!\prec c_{n+1}$ for each n and $c = \bigvee_n c_n$. Then choose, for each n, $\gamma_n \in \mathcal{R}L$ with $c_n = \operatorname{coz}(\gamma_n)$. Then each γ_n belongs to I, whence the claimed membership holds.

The maps Γ and Δ are connected through a frame homomorphism as follows.

Proposition 5.3.5. *The triangle*

$$\begin{array}{ccc} & \mathrm{Idl}(\mathcal{R}L) & \\ {}^{\Delta}\swarrow & & \searrow^{\Gamma} \\ \mathcal{S}(\beta L)^{\mathrm{op}} & \xrightarrow{((k_L)_*)_{-1}[-]} & \mathcal{S}(\lambda L)^{\mathrm{op}} \end{array}$$

commutes. That is, $\Gamma = ((k_L)_*)_{-1}[-] \circ \Delta$.

Proof. For any $I \in \mathrm{Idl}(\mathcal{R}L)$,

$$\begin{aligned}
\Big(((k_L)_*)_{-1}[-] \circ \Delta\Big)(I) &= ((k_L)_*)_{-1}\left[\mathfrak{c}_{\beta L}\left(\bigvee_{\alpha\in I} r_L(\operatorname{coz}\alpha)\right)\right] \\
&= \mathfrak{c}_{\lambda L}\left(k_L\left(\bigvee_{\alpha\in I} r_L(\operatorname{coz}\alpha)\right)\right) \\
&= \mathfrak{c}_{\lambda L}\left(\bigvee_{\alpha\in I} k_L\left(r_L(\operatorname{coz}\alpha)\right)\right) \\
&= \mathfrak{c}_{\lambda L}\left(\bigvee_{\alpha\in I} \varrho_L(\operatorname{coz}\alpha)\right) \\
&= \Gamma(I),
\end{aligned}$$

which proves the result. $\square$

This proposition yields a corollary that is a perfect analogue of Proposition 5.1.4(a). Recall that in the β-case we have $\Delta(\boldsymbol{O}^A) = \mathrm{cl}_{\beta L} A$ for any sublocale A of βL. The λ-case analogue of this is precisely the following result.

Corollary 5.3.6. *For any sublocale A of λL, $\Gamma(\boldsymbol{N}^A) = \mathrm{cl}_{\lambda L} A$.*

Proof. For brevity, we write κ for the localic map $(k_L)_*$. Now,

$$
\begin{aligned}
\Gamma(\boldsymbol{N}^A) &= \kappa_{-1}\big[\Delta\big(\boldsymbol{N}^A\big)\big] && \text{by Proposition 5.3.5} \\
&= \kappa_{-1}\Big[\Delta\big(\boldsymbol{O}^{\kappa[A]}\big)\Big] && \text{by Proposition 5.3.2} \\
&= \kappa_{-1}\big[\,\mathrm{cl}_{\beta L}\,\kappa[A]\big] && \text{by Proposition 5.1.4(a)} \\
&= \kappa_{-1}\Big[\mathfrak{c}_{\beta L}\Big(\bigwedge \kappa[A]\Big)\Big] \\
&= \kappa_{-1}\Big[\mathfrak{c}_{\beta L}\Big(\kappa\Big(\bigwedge A\Big)\Big)\Big] && \text{since } \kappa \text{ is a localic map} \\
&= \mathfrak{c}_{\lambda L}\Big(k_L\Big(\kappa\Big(\bigwedge A\Big)\Big)\Big) \\
&= \mathfrak{c}_{\lambda L}\Big(\bigwedge A\Big) && \text{since } k_L \circ (k_L)_* = \mathrm{id}_{\lambda L} \text{ as } k_L \text{ is onto} \\
&= \mathrm{cl}_{\lambda L}\, A,
\end{aligned}
$$

which proves the result. $\square$

We saw in Proposition 5.1.4(c) that, for any ideal I of $\mathcal{R}L$, $\boldsymbol{O}^{\Delta(I)} = mI$; an ideal related to L by the characterization

$$\gamma \in mI \quad \Longleftrightarrow \quad (\exists\, \alpha \in I)(\mathrm{coz}\,\gamma \prec\!\!\prec \mathrm{coz}\,\alpha).$$

We shall see that we have an analogous situation in the λ-case. Towards that end, we introduce the following definition.

Definition 5.3.7. Given an ideal I of $\mathcal{R}L$, we define the ideal sI of $\mathcal{R}L$ by

$$sI = \Big\{\gamma \in \mathcal{R}L \mid \mathrm{coz}\,\gamma \prec\!\!\prec \bigvee_{n=1}^{\infty} \mathrm{coz}(\alpha_n) \text{ for some sequence } (\alpha_n) \text{ in } I\Big\}.$$

It is routine to check that sI is indeed an ideal of $\mathcal{R}L$. Furthermore, the condition defining sI is a "countable" version of the condition characterizing mI because, by the properties of the cozero map, $\gamma \in mI$ if and only if $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}(\alpha_1) \vee \cdots \vee \operatorname{coz}(\alpha_n)$ for some finite set $\{\alpha_1, \ldots, \alpha_n\} \subseteq I$.

Here are some quick observations about the ideal sI.

Observation 5.3.8. For any ideal I of $\mathcal{R}L$:

(a) sI is neat;

(b) $mI \subseteq sI$; and

(c) $mI = sI$ if and only if $sI \subseteq I$, since mI is the largest neat ideal of $\mathcal{R}L$ contained in I.

For use in the proof of the next lemma, let us recall that ϱ_L induces a σ-frame isomorphism $\varrho_{L|\operatorname{Coz} L}\colon \operatorname{Coz} L \to \operatorname{Coz}(\lambda L)$, as a consequence of which we have that, for any $c, d \in \operatorname{Coz} L$, $c \prec\!\!\prec d$ if and only if $\varrho_L(c) \prec\!\!\prec \varrho_L(d)$.

Lemma 5.3.9. *For any ideal I of $\mathcal{R}L$, $\boldsymbol{N}^{\Gamma(I)} = sI$. That is, $\Phi(\Gamma(I)) = sI$.*

Proof. Given $\gamma \in \mathcal{R}L$, we have

$$\begin{aligned} \gamma \in \boldsymbol{N}^{\Gamma(I)} \quad & \text{iff} \quad \Gamma(I) \subseteq \mathfrak{o}_{\lambda L}(\varrho_L(\operatorname{coz}\gamma)^*) \\ & \text{iff} \quad \mathfrak{c}_{\lambda L}\left(\bigvee_{\alpha \in I} \varrho_L(\operatorname{coz}\alpha)\right) \subseteq \mathfrak{o}_{\lambda L}(\varrho_L(\operatorname{coz}\gamma)^*) \\ & \text{iff} \quad \varrho_L(\operatorname{coz}\gamma)^* \vee \bigvee_{\alpha \in I} \varrho_L(\operatorname{coz}\alpha) = 1_{\lambda L}. \end{aligned}$$

Since λL is Lindelöf, this last statement holds if and only if there is a sequence (α_n) in I such that

$$\varrho_L(\operatorname{coz}\gamma)^* \vee \bigvee_{n=1}^{\infty} \varrho_L(\operatorname{coz}(\alpha_n)) = 1_{\lambda L}.$$

Now, since the restriction of ϱ_L to $\operatorname{Coz} L$ is a σ-frame homomorphism, since the rather below relation coincides with the completely below relation in normal frames, and since λL is normal (being a regular Lindelöf frame), we have that $\gamma \in \boldsymbol{N}^{\Gamma(I)}$ if and only if there is a sequence (α_n) in I such that

$$\varrho_L(\operatorname{coz}\gamma) \prec\!\!\prec \varrho_L\left(\bigvee_{n=1}^{\infty} \operatorname{coz}(\alpha_n)\right),$$

which, in turn, holds if and only if $\operatorname{coz}\gamma \prec\!\!\prec \bigvee_n \operatorname{coz}(\alpha_n)$. This proves the proposition. $\square$

Now, since Δ is a quantale homomorphism, and since Γ is a composite of quantale homomorphisms (Proposition 5.3.5), it follows that Γ is a quantale homomorphism. A calculation similar to that which showed that Ψ preserves meets shows that Γ preserves meets. We now have the following analogue Theorem 5.1.7(b).

Theorem 5.3.10. *The following are equivalent for a completely regular frame L.*

(1) $\Gamma \dashv \Psi$.

(2) *L is a P-frame.*

(3) *$I \subseteq sI$ for every ideal I of $\mathcal{R}L$.*

Proof. (1) $\Leftrightarrow$ (3): Since $\Gamma(\Psi(A)) = \mathrm{cl}_{\lambda L} A$ for every sublocale A of λL (Corollary 5.3.6), so that we always have $\Gamma \circ \Psi \leq \mathrm{id}_{\mathcal{S}(\lambda L)^{\mathrm{op}}}$, $\Gamma \dashv \Psi$ if and only if $I \subseteq \Psi(\Gamma(I))$ for every ideal I of $\mathcal{R}L$, that is, if and only if $I \subseteq sI$ for every I in light of Lemma 5.3.9. Therefore (1) and (3) are equivalent.

(3) $\Rightarrow$ (2): Let $c \in \mathrm{Coz}\, L$, and pick $\gamma \in \mathcal{R}L$ such that $c = \mathrm{coz}\,\gamma$. Let I be the principal ideal $\langle\gamma\rangle$. Since $\gamma \in I$, (3) says $\gamma \in sI$, so there is a sequence (γ_n) in I such that $\mathrm{coz}\,\gamma \prec\!\!\prec \bigvee_n \mathrm{coz}(\gamma_n)$. Since each γ_n is a multiple of γ, $\mathrm{coz}(\gamma_n) \leq \mathrm{coz}\,\gamma$, and so $\mathrm{coz}\,\gamma \prec\!\!\prec \mathrm{coz}\,\gamma$, which implies that c is complemented. Therefore L is a P-frame.

(2) $\Rightarrow$ (3): If L is a P-frame, then every ideal of $\mathcal{R}L$ is neat. Hence, for any ideal I of $\mathcal{R}L$, $I = mI \subseteq sI$. □

Recall from [29, Lemma V 2.8] that an ideal I of a ring A is neat if and only if $IJ = I \cap J$, for every ideal J of A. Thus, if L is a P-frame, then the quantale $\left(\mathrm{Idl}(\mathcal{R}L), \cdot, \sum\right)$ is exactly the frame $\left(\mathrm{Idl}(\mathcal{R}L), \cap, \sum\right)$, which then makes the map $\Delta\colon \mathrm{Idl}(\mathcal{R}L) \to \mathcal{S}(\beta L)^{\mathrm{op}}$ a frame homomorphism, and hence $\Gamma\colon \mathrm{Idl}(\mathcal{R}L) \to \mathcal{S}(\lambda L)^{\mathrm{op}}$ is a frame homomorphism by Proposition 5.3.5. Thus, $sI = \Gamma_*(\Gamma(I))$, for every $I \in \mathrm{Idl}(\mathcal{R}L)$, which makes the mapping $I \mapsto sI$ a nucleus on $\mathrm{Idl}(\mathcal{R}L)$. We therefore have the following corollary to Theorem 5.3.10.

Corollary 5.3.11. *If L is P-frame, then the set $\{sI \mid I \in \mathrm{Idl}(\mathcal{R}L)\}$ is a sublocale of* $\mathrm{Idl}(\mathcal{R}L)$. *Furthermore, it is dense.*

Remark 5.3.12. It vexes us that we are unable to characterize the ideals I for which $sI \subseteq I$. We should point out though that strongly divisible ideals, as defined by Azarpanah [2], are of this kind. To recall, an ideal I in a ring is called *strongly divisible* if for every sequence (u_n) in I, there is an element $u \in I$ such that each u_n is a multiple of u. One checks easily that if I is a strongly divisible ideal of $\mathcal{R}L$, then $sI \subseteq I$. In $C(X)$, they include the maximal ideals M such that $C(X)/M \cong \mathbb{R}$ [2, Corollary 4.3].

We close by presenting analogues of Theorems 5.2.1 and 5.2.2. Each is about commutativity of a certain square. In contrast with the β-case, we will show that the λ-version of the square in Theorem 5.2.1 always commutes.

Proposition 5.3.13. *For any frame homomorphism $h\colon M \to L$, the square*

$$\begin{array}{ccc} \mathrm{Idl}(\mathcal{R}M) & \xrightarrow{\langle(\mathcal{R}h)[-]\rangle} & \mathrm{Idl}(\mathcal{R}L) \\ \Big\downarrow{\scriptstyle \Gamma_M} & & \Big\downarrow{\scriptstyle \Gamma_L} \\ \mathcal{S}(\lambda M)^{\mathrm{op}} & \xrightarrow{((\lambda h)_*)_{-1}[-]} & \mathcal{S}(\lambda L)^{\mathrm{op}} \end{array}$$

is commutative.

Proof. A calculation analogous to that in the proof of Theorem 5.2.1 shows that the square commutes if and only if for any ideal I of $\mathcal{R}M$,

$$\bigvee_{\alpha \in I} (\lambda h)\big(\varrho_M(\operatorname{coz}\alpha)\big) = \bigvee_{\tau \in (\mathcal{R}h)[I]} \varrho_L(\operatorname{coz}\tau).$$

Since $(\lambda h)\big(\varrho_L(c)\big) = \varrho_L(h(c))$ for every $c \in \operatorname{Coz} M$, it follows that the square in question commutes if and only if

$$\bigvee_{\alpha \in I} \varrho_L(h(\operatorname{coz}\alpha)) = \bigvee_{\tau \in (\mathcal{R}h)[I]} \varrho_L(\operatorname{coz}\tau).$$

But this last equation always holds because for any $\alpha \in \mathcal{R}L$, the element $\tau = (\mathcal{R}h)(\alpha)$ belongs to $(\mathcal{R}h)[I]$ and $h(\operatorname{coz}\alpha) = \operatorname{coz}\tau$. Therefore the square above always commutes. $\square$

The discrepancy between the two results can be explained as follows. A WN-homomorphism $h\colon M \to L$ is defined by requiring the containment $(\beta h)(r_M(c)) \subseteq r_L(h(c))$ to be an equality for every $c \in \operatorname{Coz} M$. On the other hand, the corresponding containment $(\lambda h)(\varrho_M(c)) \subseteq \varrho_L(h(c))$ is always an equality. Viewed differently, as remarked earlier, ϱ_L restricts to a σ-frame isomorphism

$\operatorname{Coz} L \to \operatorname{Coz}(\lambda L)$, whereas r_L does not necessarily restrict to a σ-frame isomorphism onto $\operatorname{Coz}(\beta L)$.

Turning to the analogue of Theorem 5.2.2, let us write $\mathcal{S}\colon \mathbf{Frm} \to \mathbf{Frm}$ for the functor that sends L to $\mathcal{S}(L)^{\mathrm{op}}$ and a morphism $h\colon M \to L$ to the morphism $(h_*)_{-1}[-]\colon \mathcal{S}(M)^{\mathrm{op}} \to \mathcal{S}(L)^{\mathrm{op}}$, which we name $\mathcal{S}(h)$.

Proposition 5.3.14. *For any frame homomorphism $h\colon M \to L$, the square*

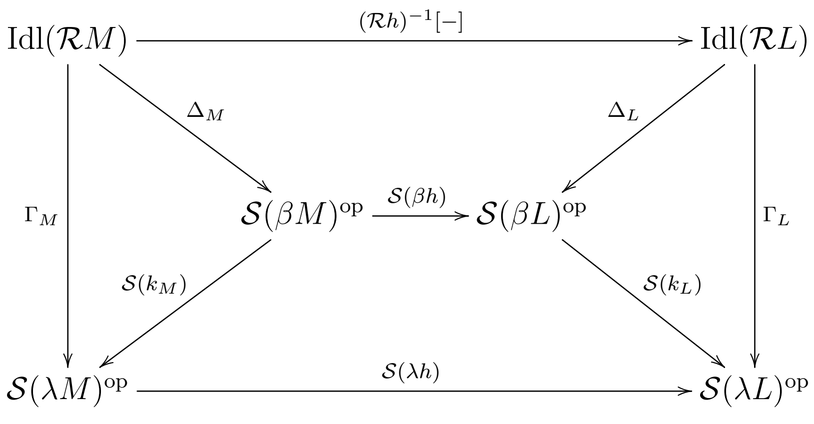
$$\begin{CD} \mathrm{Idl}(\mathcal{R}L) @>{(\mathcal{R}h)^{-1}[-]}>> \mathrm{Idl}(\mathcal{R}M) \\ @V{\Gamma_L}VV @VV{\Gamma_M}V \\ \mathcal{S}(\lambda L)^{\mathrm{op}} @>{(\lambda h)_*[-]}>> \mathcal{S}(\lambda M)^{\mathrm{op}} \end{CD}$$

is commutative.

Proof. The square on the left of the diagrams

$$\begin{CD} \beta M @>{\beta h}>> \beta L \\ @V{k_M}VV @VV{k_L}V \\ \lambda M @>{\lambda h}>> \lambda L \end{CD} \qquad \begin{CD} \mathcal{S}(\beta M)^{\mathrm{op}} @>{\mathcal{S}(\beta h)}>> \mathcal{S}(\beta L)^{\mathrm{op}} \\ @V{\mathcal{S}(k_M)}VV @VV{\mathcal{S}(k_L)}V \\ \mathcal{S}(\lambda M)^{\mathrm{op}} @>{\mathcal{S}(\lambda h)}>> \mathcal{S}(\lambda L)^{\mathrm{op}} \end{CD}$$

is known to commute, and hence the one on the right also commutes. Now, in the diagram



the triangles commute by Proposition 5.3.5, the upper trapezoid commutes by Theorem 5.2.2, and the lower trapezoid commutes, as just noted. So it follows that the outer square commutes, which is precisely what we are supposed to prove. $\square$

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