VAN HIELE THEORY- BASED INSTRUCTION, GEOMETRIC PROOF COMPETENCE AND GRADE 11 STUDENTS' REFLECTIONS

by

ERIC MACHISI

Submitted in accordance with the requirements for

the degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS, SCIENCE AND TECHNOLOGY EDUCATION

in the subject

MATHEMATICS EDUCATION

at the

UNIVERSITY OF SOUTH AFRICA

SUPERVISOR: PROFESSOR N. N. FEZA

AUGUST 2020

DEDICATION

TO

MY LATE MOTHER

MODIE KANGARI MACHISI

AND

MY SONS HILLARY AND HARRY

THIS HUMBLE PIECE OF WORK IS A SIGN OF MY LOVE FOR YOU!

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ABBREVIATIONS

AICc	Bias-corrected Akaike Information Criterion
ANCOVA	Analysis of Covariance
ANOVA	Analysis of Variance
BC	Before Christ
CAPS	Curriculum and Assessment Policy Statement
CAQDAS	Computer Assisted Qualitative Data Analysis Software
CSV	Comma Separated Values
CVI	Content Validity Index
DBE	Department of Basic Education
EG	Euclidean Geometry
ESL	English as a Second Language
FET	Further Education and Training
FG	Focus Group
GCV	Generalized Cross Validation
GSP	Geometer’s Sketchpad
I-CVI	Item-level Content Validity Index
IT	Information Technology
LOESS	Locally Estimated Scatterplot Smoothing
LTSM	Learner and Teacher Support Materials
MANOVA	Multivariate Analysis of Variance
n. d	No Date
NAPTOSA	National Professional Teachers’ Organisation of South Africa
NCS	National Curriculum Statement

NNSSF	National Norms and Standards for School Funding
NSC	National Senior Certificate
NSNP	National School Nutrition Programme
OBE	Outcomes-Based Education
PDF	Portable Document Format
PEPP	People's Education for People's Power
qual	Qualitative
QUAN	QUANTITATIVE
RNCS	Revised National Curriculum Statement
RTF	Rich Text Format
SADTU	South African Democratic Teachers' Union
S-CVI	Scale-level Content Validity Index
SD	Standard Deviation
SES	Socio-Economic Status
Sm.	Smoothing Model
SPSS	Statistical Package for Social Sciences
StatsSA	Statistics South Africa
STD	Secondary Teacher's Diploma
STEM	Science, Technology, Engineering and Mathematics
TIMSS	Trends in International Mathematics and Science Study
UNCRC	United Nations Convention on the Rights of the Child

SUMMARY

This study sought to (a) investigate the effect of Van Hiele theory-based instruction on Grade 11 students' geometric proofs learning achievement, (b) explore students' views on their geometry learning experiences, and (c) develop a framework for better teaching and learning of Grade 11 Euclidean geometry theorems and non-routine geometric proofs. The study is based on Van Hiele's theory of geometric thinking. The research involved a convenience sample of 186 Grade 11 students from four matched secondary schools in the Capricorn district of Limpopo province, South Africa. The study employed a sequential explanatory mixed-methods design, which combined quantitative and qualitative data collection methods. In the quantitative phase, a non-equivalent groups quasi-experiment was conducted. A Geometry Proof Test was used to assess students' geometric proof construction abilities before and after the teaching experiment. Data analysis using non-parametric analysis of covariance revealed that students from the experimental group of schools performed significantly better than their counterparts from control group schools. In the qualitative phase, data were collected using focus group discussions and students' diary records. Results revealed that the experimental group students had positive views on their geometry learning experiences, whereas students from the control group of schools expressed negative views towards the teaching of Euclidean geometry and geometric proofs in their mathematics classes. Based on the quantitative and qualitative data findings, it was concluded that in addition to organizing instruction according to the Van Hiele theory, teachers should listen to students' voices and adjust their pedagogical practices to meet the expectations of a diverse group of students in the mathematics class. A framework for better teaching and learning of Grade 11 Euclidean geometry theorems and non-routine geometric proofs was thus developed, integrating students' views and Van Hiele theory-based instruction. The study recommends that teachers should adopt the modified Van Hiele theory-based framework to enhance students' mastery of non-routine geometric proofs in secondary schools.

Keywords: *Van Hiele theory, Van Hiele theory-based instruction, conventional teaching approaches, Euclidean geometry, non-routine geometric proof, learning achievement, students' views*

DECLARATION

I declare that the thesis entitled, “VAN HIELE THEORY- BASED INSTRUCTION, GEOMETRIC PROOF COMPETENCE AND GRADE 11 STUDENTS’ REFLECTIONS”, presents work carried out by myself and does not incorporate without acknowledgment any material previously submitted for a degree or diploma in any university; to the best of my knowledge it does not contain any materials previously published or written by another person except where due reference is made in the text; and all substantive contributions by others to the work presented, including jointly authored publications, are clearly acknowledged.

....

SIGNATURE DATE

(E MACHISI)

ACKNOWLEDGEMENTS

First, I would like to give special thanks to my supervisor, Professor Nosisi Nellie Feza. I could not have completed this journey without your support, your inspiration, and your vast experience. You were timeous in providing feedback, and you never gave up on me in times of hardships. What a great supervisor you are!

Apart from my supervisor, I would like to express my deep appreciation to the Institute for Science and Technology Education academic staff members: Professor David Mogari, Professor Jeanne Kriek, Professor Harrison Atagana, and Professor Keshnee Padayachee, for the encouragement, insightful comments and thoughtful criticisms that honed this thesis.

I am extremely grateful to the Research Ethics Committee of the University of South Africa, the Limpopo Department of Basic Education, in particular the Capricorn District Director, the Pietersburg and Mankweng Circuit Managers, and the Principals, School Governing Bodies, Teachers and Parents of the schools that participated in the study, for their approval.

To all the Grade 11 students who participated in this research, you really deserve a special place in my heart. Working with you was an exciting experience and your relationship with me is impossible to forget. This thesis would not have been accomplished without your participation.

Special thanks to Prince Neo Dumazi, Macha Molepo, Dimakatjo Mamabolo and Jacqueline Machubene, for lending a helping hand in the data collection process.

Finally, I would like to salute my close friends, Innocent Rice, Rufaro Felix Magara, Thoakge Johannes Mehlape, Godfrey Tembo, Sydney Matome Duba, Naphtali Thema and David Mathabatha, for their social, emotional, and moral support. Not forgetting my siblings, Stella, Square, Kennest, Betty, and Sweet, for their words of encouragement.

CHAPTER 1

INTRODUCTION

1.1 Background to the study

High school Euclidean geometry is the area of mathematics that offers students a natural place to learn mathematical proofs (Mwadzaangati, 2019). Other areas of mathematics taught in the South African secondary school mathematics curriculum, such as functions and algebra, number patterns, financial mathematics, calculus, probability and statistics, trigonometry, and analytical geometry, offer students limited opportunities to learn proofs (Shongwe, 2019).

Euclidean proofs offer students the opportunity to gain life skills such as visualization, deductive reasoning, logical argument, problem-solving, and critical thinking (Oflaz, Bulut, & Akcakin, 2016). Besides, proofs are used for verification, explanation, discovery, communication, systematization, aesthetic, and transfer purposes (Hemmi, 2010). Those who support the inclusion of Euclidean geometry in the secondary school mathematics curriculum argue that it helps to prepare students for careers in science, technology, engineering, and mathematics (STEM) that are undeniably important to economic growth (see Ndlovu & Mji, 2012).

However, despite numerous justifications for including Euclidean geometry and geometric proofs in secondary school mathematics curricula, the teaching and learning of this topic in South Africa has historically been problematic (see De Villiers & Heideman, 2014; Naidoo & Kapofu, 2020; Siyepu, 2014). Results from the Trends in International Mathematics and Science Study (TIMSS) of 2006 showed that geometry was the area of mathematics where the performance of South African students was dismal (Ndlovu & Mji, 2012). A follow-up study attributed the poor performance to poor teaching (Bowie, 2009). It was reported that educators had limited knowledge of Euclidean geometry content and the methodology of teaching it (Ntuli, 2014). This led politicians to suggest that Euclidean geometry should not be compulsory. As a result, the Revised National Curriculum Statement (RNCS) which came into effect in 2006 relegated Euclidean geometry to an optional paper – Mathematics Paper 3 (Naidoo, 2013).

Years later, researchers in South African universities reported that the exclusion of Euclidean geometry from the mainstream mathematics curriculum had increased the gap between secondary school and tertiary mathematics, for students enrolled in science and engineering programmes (Hlalele, 2020; Mouton, Louw & Strydom, 2012; Padayachee, Boshoff, Olivier & Harding, 2011; Wolmarans, Smit, Collier-Reed & Leather, 2010). A study carried out by Engelbrecht, Harding and Phiri (2010) found the 2009 cohort of first-year university mathematics students "weaker than their predecessors" (p. 3). This was the first group of students to write a Grade 12 mathematics examination that excluded Euclidean geometry. A similar study by the mathematics department of the University of the Witwatersrand showed a thirty-seven percent drop in the June mathematics pass rate for first-year students in 2009 (Blaine, 2009). These findings were consistent with trends observed at other universities in South Africa, namely the University of Cape Town, the University of Stellenbosch, the University of Pretoria, the North-West University, the University of KwaZulu-Natal, and the Nelson Mandela Metropolitan University (Blaine, 2009). It was argued that the lack of mathematical skills of the students was a consequence of Euclidean geometry which was no longer taught in all schools (Blaine, 2009).

In January 2012, the South African Department of Basic Education (DBE) introduced the Curriculum and Assessment Policy Statement (CAPS), which reintroduced Euclidean geometry and geometric proofs back into the mainstream mathematics curriculum (Alex & Mammen, 2014). Although the decision to make Euclidean geometry compulsory is commendable, the South African Democratic Teachers' Union (SADTU) and the National Professional Teachers' Organization of South Africa (NAPTOSA) shared the view that educators were not ready for the change, citing lack of adequate in-service teacher training prior to implementation (Ntuli, 2014). Evidence to support this position was found in surveys conducted by Olivier (2013; 2014) in two provinces in South Africa. In the 2013 survey, the teachers agreed that the CAPS training they received was inadequate for them to teach Euclidean geometry with confidence. In the follow-up survey, sixty-percent of the participants indicated that they were not comfortable with Euclidean geometry (Olivier, 2014).

South African educators wonder why geometry was brought back into the mainstream mathematics curriculum when the problems that led to its exclusion in the previous curriculum have not yet been fully addressed (Ndlovu, 2013). Even those with previous experience in teaching geometry still find it difficult to successfully teach most of their students (Ngirishi & Bansilal, 2019; Shange, 2016; Tachie, 2020). Govender (2014) maintains that the return of Euclidean geometry "has created a large amount of anxiety on the ground for both teachers and students" (p. 4). Some of the educators who are expected to teach Euclidean geometry under the CAPS have no previous contact with the topic (Tachie, 2020). The situation is aggravated by a lack of support from the subject advisors. Bradley and Scheiber (2010) noted that the subject advisors seem to lack adequate knowledge and skills needed to help teachers improve.

Given this orientation, one might reasonably wonder what the situation is in South African mathematics classrooms during Euclidean geometry lessons. Anecdotal evidence from discussions with fellow teachers during mathematics workshops in the province of Limpopo indicates that the main concern of the teachers is not about the content of Euclidean geometry. Instead, the biggest challenge is how to teach Euclidean geometry and geometric proofs in a way that ensures success for most students. This challenge was also reported in Malawi (see Mwadzaangati, 2019).

Mathematics educators may be willing to try out new ways of teaching Euclidean geometry and geometric proofs, but strong empirical evidence of successful teaching approaches is limited. It remains unclear what kind of teaching approaches could improve students' mathematical reasoning and proof skills in Euclidean geometry lessons (Jones, Fujita, & Kunimune, 2012; Miyazaki, et al., 2019). Teachers should therefore continue to look for ways to effectively teach Euclidean geometry proofs in a way that enhances the learning achievement of most students in the mathematics class.

1.2 Statement of the problem

Ideally, students are expected to graduate from secondary school with the ability to construct and write geometric proofs (Amidu & Nyarko, 2019; Dhlamini & De Villiers, 2013; Luneta, 2014; Salifu, Yakubu, & Ibrahim, 2018). This would ensure a smooth transition from high school to university mathematics. However, evidence from the National Senior Certificate (NSC) examination reports indicates that questions that require candidates to construct and write non-routine multi-step geometric proofs are poorly answered, and some candidates do not even try to answer the proof questions (see Department of Basic Education, 2015, 2016a, 2017, 2018, 2019, 2020; Hlalele, 2020). The weak performance of high school students on Euclidean geometry proofs is not unique to South Africa. Similar findings have been reported in Zimbabwe (Mukamba & Makamure, 2020), America (Oueini, 2019), Nigeria (Adeniji, Ameen, Dambatta, & Orilonise, 2018), Ghana (Armah, Cofie, & Okpoti, 2018), Namibia (Kanandjebo & Ngololo, 2017), Saudi Arabia (Al-Khateeb, 2016), Jordan (Tahani, 2016), Malawi (Mwadzaangati, 2015, 2019), Japan (Jones et al., 2012; Miyazaki, et al., 2019), and Turkey (Köğce, Aydın, & Yıldız, 2010).

It is evident from these reports that teaching Euclidean geometry proofs in secondary schools is problematic. If this situation remains unattended at the high school level, universities will have to put up with students who come to university without the necessary mathematical skills. This would mean that universities will have to continue to bear the burden of offering bridging courses and extended programmes to underprepared students. This has been found to put a strain on the already stretched financial and human resources of universities as students take longer to graduate (Atuahene & Russell, 2016; Council on Higher Education, 2013). As a result, the central concern of this study is: How can Grade 11 Mathematics teachers organize teaching and learning activities to enhance students' geometric proofs learning achievement?

Many studies have highlighted students' difficulties with mathematical proofs (see for example Harel & Fuller, 2009; Harel & Sowder, 2007; Selden & Selden, 2007; Stylianides & Stylianides, 2009). However, there has been limited research on how teachers can enhance students' geometric proofs learning achievement (Miyazaki, et al., 2019). As a result, most teachers lack the pedagogical knowledge to teach Euclidean geometry proofs effectively (Cirillo & Hummer, 2019; Mwadzaangati, 2015, 2019; Tachie, 2020). This is an area which still needs further investigation.

1.3 Research questions

The following questions will be addressed:

1) Does teaching and learning Euclidean geometry theorems and non-routine geometric proofs through Van Hiele theory-based instruction have any

statistically significant effect on Grade 11 students' geometric proofs learning achievement?

2) What are students' views on (a) the Van Hiele theory-based approach, and (b) conventional approach to teaching and learning Grade 11 Euclidean geometry theorems and non-routine geometric proofs?

1.4 Research objectives

The following objectives were set:

1) To implement Van Hiele theory-based instruction in the teaching of Grade 11 Euclidean geometry theorems and non-routine geometric proofs;
2) To test the effect of Van Hiele theory-based instruction on Grade 11 students' geometric proofs learning achievement;
3) To explore students' views on (a) the Van Hiele theory-based approach, and (b) conventional approach to teaching and learning Grade 11 Euclidean geometry theorems and non-routine geometric proofs;
4) To develop a framework for better teaching and learning of Grade 11 Euclidean geometry theorems and non-routine geometric proofs, integrating the views expressed by the students.

1.5 Significance of the study

It is anticipated that the findings of this study will benefit the following sectors of society:

1.5.1 Secondary school students

The direct beneficiaries of the findings of this research are students who have difficulties in understanding geometric proofs in secondary schools in South Africa. An improvement in strategies for teaching proofs paves the way for better student achievement, increases students' access to critical careers, such as mathematics, science, and technology, and ultimately their chances of survival and prosperity in the society.

1.5.2 Secondary school mathematics teachers

This study seeks to address the pedagogical concern of teachers about how to teach Euclidean geometry proofs in a way that ensures success for most of their students. In the absence of adequate pre- and in-service teacher training and development programmes, the results of this study may help teachers to discover how they can turn students' difficulties into opportunities to improve their pedagogical practices. The findings of this study may lead to changes in the approaches used by teachers to teach geometric proofs in countries where students do not perform well on the mathematical aspect of proof.

1.5.3 Curriculum advisers

Mathematics curriculum advisors will be guided on the kind of expertise they should acquire for them to be able to support teachers effectively and adequately in schools. Knowledge of mathematics content alone is not enough (Shulman, 1986).

1.5.4 Textbook publishers

Since textbooks are the only teaching resource available to teachers in many disadvantaged schools, the results of this study will provide suggestions for improving the sequencing and presentation of Euclidean geometry and geometry proof content and activities in textbooks for the benefit of the students.

1.6 Operational definitions of key terms

The following are definitions of terms as used in this study:

1.6.1 Van Hiele theory

Van Hiele theory is a model of geometric teaching and learning developed in the Netherlands by Pierre van Hiele and his wife Dina van Hiele-Geldof. The Van Hiele theory states that geometric thinking progresses through five hierarchical levels. To succeed at level (n), students must first master the geometric knowledge of level $(n-1)$. To facilitate movement between levels, the Van Hiele theory proposes that teaching and learning at each level should be sequenced as follows: information→guided orientation→explicitation→free orientation→integration (see the theoretical framework in Chapter 2 for further details).

1.6.2 Van Hiele theory-based instruction

Van Hiele theory-based instruction is a teaching approach developed by the researcher for purposes of teaching Euclidean geometry proofs. The approach is derived and adapted from some of the characteristics of the Van Hiele theory of geometrical thought. The Van Hiele theory states that students are less likely to succeed at level (n) if they have not mastered level $(n-1)$. In the context of the present study, this means that students cannot master Euclidean geometry proofs if they are unable to identify shapes, properties, relationships, and patterns. Implementing Van Hiele theory-based instruction would therefore require the teacher to first check whether students have mastered the prior knowledge of Euclidean geometry from lower grades (Grades 8-10) prior to introducing Grade 11 work. This is followed by remedial lessons designed to bridge any identified learning gaps. Another important aspect of the Van Hieles' theory is that students should engage in some guided exploration activities in which relationships and patterns are established before deductive proof is introduced. Teaching and learning activities are organized in accordance with the proposed sequence of the Van Hieles: information→guided exploration→explicitation→free orientation→integration.

1.6.3 Conventional teaching approaches

Conventional teaching approaches in the context of this study are teaching methods in which geometric knowledge is presented to students in the form of a lecture or by 'chalk and talk' or simply following the order of presentation set out in the textbook. The teacher holds the power and the responsibility over learning. Students who are slow to understand are left unattended. The coverage of the syllabus is more important than addressing students' needs. Teachers and textbooks are considered the only sources of knowledge. Students are perceived to have little or nothing to contribute, and student engagement is minimal. Students are left with no option except to memorize what they are taught and try to reproduce it in tests and examinations.

1.6.4 Euclidean geometry

Euclidean geometry is an aspect of mathematics that deals with properties and relationships of shapes, points, lines, angles, and positions, based on Euclid's definitions and assumptions.

1.6.5 Non-routine geometric proof

A non-routine geometric proof is a logical chain of reasoning that establishes the truth of a geometric statement using definitions, self-evident statements (axioms), and theorems.

1.6.6 Learning achievement

In this study, learning achievement refers to the level of student success in constructing and writing non-routine Euclidean geometry proofs, that is expressed in the form of scores obtained in a geometry proof test. In South Africa, students' learning achievement is ranked and reported according to the following criteria: 0−29 percent [Level 1−Not achieved], 30−39 percent [Level 2−Elementary achievement], 40−49 percent [Level 3−Moderate achievement], 50−59 percent [Level 4−Adequate achievement], 60−69 percent [Level 5−Substantial achievement], 70−79 percent [Level 6−Meritorious achievement], 80−100 percent [Level 7−Outstanding achievement] (Department of Basic Education, 2011).

It is important to note that an adequate level of learning achievement begins at 50%. Consequently, any student who scores below 50% is considered to have underperformed.

1.6.7 Student views

Student views as used in this study refer to what students say or report on their experience in teaching and learning Euclidean geometry theorems and non-routine geometric proofs.

1.7 The delimitation of the study

This study was confined to secondary schools in two townships (namely Mankweng and Seshego) in the Capricorn district of the province of Limpopo in South Africa. Four conveniently selected secondary schools (two in each township) participated in this study. A total of 186 Grade 11 students and two Grade 11 mathematics teachers participated in the study.

The study focused on the teaching of Grade 11 Euclidean geometry theorems and non-routine geometric proofs in selected schools. Therefore, the study does not cover all the learning concepts of Euclidean geometry. It also does not extend to all grades of the school system. The variables of interest were (a) students' geometric proofs learning achievement, and (b) students' views on teaching approaches used in their Euclidean geometry lessons. The purpose of the study was to evaluate the impact of Van Hiele theory-based instruction and conventional instruction on these variables.

The choice of methods was influenced by the theoretical viewpoint of pragmatism. A sequential explanatory mixed-methods design was adopted which combines quantitative and qualitative data collection methods in two different phases. The quantitative phase employed a non-equivalent groups quasi-experiment. The researcher implemented Van Hiele theory-based instruction in two secondary schools in Seshego township, while two Grade 11 mathematics teachers in two secondary schools in Mankweng township presented Euclidean geometry lessons in their usual way. The teaching experiment was completed in four weeks.

Quantitative data were obtained using a geometry proof test that was administered before and after the experiment, and data were analysed using non-parametric analysis of covariance (ANCOVA). During the qualitative phase, data were collected using diaries and focus group interviews, and analysed using computer-assisted qualitative data analysis software (CAQDAS) called MAXQDA. The quantitative and qualitative data were presented and analysed in two separate chapters. The main findings from the quantitative and qualitative data analysis are summarized in the discussion chapter.

1.8 Organization of the thesis

Chapter 1 briefly outlined the background to the study. The research problem and the importance of the study were presented. The research questions were stated and operational definitions of key terms were provided. The scope of the study was defined. Chapter 2 reviews literature on the teaching of geometric proofs from the Greek era to the twenty-first century. The knowledge gap is identified and a theoretical foundation for the study is proposed. Chapter 3 outlines the research methodology, from research design through sampling techniques to data collection and analysis. Chapter 4 presents the quantitative and qualitative data collected during the study. The findings are discussed in Chapter 5 and the research questions are addressed. The implications of the findings of the research are also explored in Chapter 5. Chapter 6 summarizes and concludes the study with recommendations and suggestions for future research. The constraints of the study are identified.

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.1 Introduction

This chapter is divided into two parts. Part One reviews literature on the following aspects: the evolution of geometric proofs; the importance of teaching and learning geometric proofs in high school; difficulties in learning and teaching geometric proofs; the teaching and learning of geometry and geometric proofs in South Africa; possible teaching strategies to improve students' geometric proofs learning achievement; emerging issues; and students' views. Part Two outlines the Van Hiele theory of geometric thinking, together with its implications for the teaching and learning of geometric proofs. A review of previous studies on Van Hiele theory-based instruction is presented to identify the research gap for the current study. A summary of the chapter is given in the final analysis.

PART ONE

2.2 The evolution of geometric proofs

Before the advent of classical Greece, mathematics was used in regions such as China, Mesopotamia, Egypt, and Southern India, primarily as a computational tool for addressing practical problems in surveying, accounting, and trade (Eves, 1990). Emphasis was placed on the results of the calculations, and there was no attempt to explain the validity of the results. In sharp contrast, ancient Greek mathematicians sought to demonstrate the truth of mathematical propositions through verbal explanations and constructions (Bramlet & Drake, 2013a). Knowing how something works was not enough for the Greeks. It was important to know why it worked. As a result, the Greek mathematicians transformed empirical mathematics into a demonstrative science based on deductive reasoning (Bramlet & Drake, 2013a). The Greeks emphasized that geometric facts should be determined by deduction, not by empirical methods (Stylianou et al., 2009). This was the beginning of the idea of proof.

It is unfortunate that most of the mathematical knowledge discovered in the Greek era has been lost due to impermanence of the media on which it was recorded (Shives, 2012). However, two of the early influential Greek mathematicians whose contributions to the evolution of geometric proofs are still found in modern records are: Thales of Miletus (624 – 546 BC), and Euclid of Alexandria (323 – 283 BC). Thales and Euclid are considered fathers of plane geometry (Finashin, 2015). Their individual contributions to the development of geometric proofs are set out in the following discussion.

2.2.1 Thales of Miletus (624 – 546 BC)

Thales of Miletus (624 – 546 BC), one of the seven great men of ancient Greece (Burton, 2007), was the first mathematician to use deductive reasoning in mathematics (Bramlet & Drake, 2013a). Thales used known geometrical facts to discover new geometric truths. This method is referred to as the deductive approach and was Thales' greatest contribution to the evolution of geometric proofs. Thales is praised for his discovery of five geometric propositions: (1) the diameter of a circle divides the circle into two equal segments, (2) angles in a triangle opposite two equal sides are equal, (3) vertically opposite angles are equal, (4) the angle subtended by a diameter in a circle is a right angle (Thales Theorem), and (5) triangles are congruent if they have two angles and one side in each that are respectively equal (Page, 2007). With these discoveries, the foundation for the learning of Euclidean proofs were developed for future mathematics students (Bramlet & Drake, 2013a).

The evolution of geometric proof in ancient Greece reached its peak with the work of another famous Greek mathematician: Euclid of Alexandria. The following section summarizes Euclid's contribution to geometry.

2.2.2 Euclid of Alexandria (323 – 283 BC)

Around the third century Before Christ (BC), Euclid of Alexandria produced a famous book called the Elements (Stylianou et al., 2009). In the Elements, Euclid contributed to the evolution of geometric proofs by organizing known geometrical knowledge on points, lines, and circles, into definitions, assumptions, and axioms (Bramlet & Drake, 2013a). As a result, Euclid's definitions, assumptions, and axioms gave rise to the need for improvements in the art of proving geometric propositions. Although the Greek mathematicians continued to justify their mathematical discoveries by construction and verbal explanations due to the lack of symbolic notation), every step of the proving process had to be justified using Euclid's definitions, assumptions, or axioms (Bramlet & Drake, 2013a). This set a new standard for geometry rigour (Harel & Sowder, 2007). As a result, the deductive approach characterized formal mathematics training in the post-Greek era (Stylianou et al., 2009). Plato's School of Philosophy was one of the first higher education institutions to emphasize training in deductive reasoning and proof for the Greek citizens. It is recorded in history that it was written on the entrance to Plato's School: "Let no one ignorant of geometry enter here!" (Anglin, 1994, p. 57). Thus, deductive reasoning skills were required for admission to Plato's School. Therefore, the call by modern universities to include geometry and proof in the secondary school mathematics curriculum is no surprise.

The next critical phase in the evolution of geometric proofs was the Renaissance period.

2.2.3 The Renaissance

The Renaissance was the period in European history that came shortly after the Middle Ages. This spanned the fourteenth to the seventeenth centuries and linked the Middle Ages with the Modern World (Palmer, Colton, & Kramer, 2013). This period was marked by a minor development in the theory of geometric proofs and an increase of interest in other areas of mathematics.

The inability of the Greek mathematicians to construct symbolic notation to promote the documentation of their mathematical discoveries led to the collapse of the Greek mathematics (Turchin, 1977). The Greek mathematicians focused on verbal proofs and proof by construction, which meant that their mathematics was limited to a small range of mathematical aspects (Bramlet & Drake, 2013a). The desire of European mathematicians to explore new mathematical concepts contributed to a change in focus from geometry to other areas of mathematics, such as calculus and algebra. As a result, deductive reasoning and proof became less important. According to Bramlet and Drake's (2013a) historical study, mathematicians of the Renaissance period (such as Johannes Kepler, Regiomontanus, Leonardo da Vinci, Nicolas Copernicus, Luca Pacioli, Thomas Harriot, and René Descartes) relied more on mathematical experience than on deductive reasoning. New mathematical findings were verified through empirical investigation and mathematical induction. However, such methods of justifying mathematical propositions were later found to be flawed and inadequate. For example, the fact that a statement is true in several cases does not automatically mean that it is universally valid. For this reason, much of the mathematical findings of the Renaissance era could not be trusted and had to be revisited (Bramlet & Drake, 2013a). This is because empirical claims are not recognized as proof (Stylianides & Stylianides, 2009). It was therefore important to validate the new developments in mathematics by means of robust proof.

2.2.4 The advent of symbolic notation

The nineteenth century recorded the biggest developments in mathematics since the beginning of the Greek period (Bramlet & Drake, 2013a). The quest for new mathematical knowledge drove European mathematicians to build a symbolic notation system to make mathematical computations simpler (Palmer et al., 2013). The invention of modern algebraic notation is attributed to Francois Viete, the French mathematician. He took the mathematics of ancient Greece and wrote it in a symbolic note (Palmer et al., 2013). This became an essential part in the evolution of deductive reasoning and geometric proof. Geometric proofs became easier to explain (Bramlet & Drake, 2013a). New interest in proofs led to a return of mathematics to the study of Euclid's axioms.

2.2.5 The beginning of teaching Euclidean geometry proofs in secondary school

The nineteenth century marked the beginning of formal teaching and learning of deductive proof in schools (Stylianou et al., 2009). In America, geometry skills were required for admission to universities (Barbin & Menghini, 2014) starting in 1844 (Furr, 1996). Learning proof became important in high school mathematics, as recommended by the Committee of Ten. This was a group of teachers and members of higher education institutions appointed to investigate the relationship between secondary school curriculum and university admission criteria (Stylianou et al., 2009). The Committee agreed that high school mathematics would train students in deductive reasoning. It was important not only to prepare students for university or college, but also to prepare them for their general well-being in society. The Committee defined geometry as the best place for students to learn reasoning and proof skills (Herbst, 2002). Since then, Euclidean proofs have been part of high school mathematics curricula in many countries.

Herbst (2002) identified three periods of geometric proof instruction since the time when high schools started teaching it. These are (1) the era of text, (2) the era of originals, and (3) the era of exercise.

2.2.5.1 The era of text

The era of text was the first period of geometric proof instruction in American high schools. Students were supposed to read, memorize, and reproduce long and complex paragraphs of geometric proofs in their textbooks (Adams, 2010). Teachers and textbooks did not give any detail on the method of proving. As a result, many students faced difficulties in memorizing the long and complex paragraphs of geometric proof texts (Herbst, 2002). It was noted that the replication of long paragraphs of geometric proofs without justification was of no benefit to the students. The Committee of Ten concluded that there was a need for pedagogical improvements to make geometric proof meaningful to students. This gave rise to the era of originals in which students began to learn how to construct proofs for geometric propositions.

2.2.5.2 The era of originals

In the era of originals, geometry instruction started to move away from the tendency to simply replicate proof texts (Herbst, 2002). In addition to the reproduction of textbook proofs, students were given the opportunity to construct their own proofs of ‘original’ geometric propositions (Subramanian, 2005). Geometry textbooks for this period included a long list of questions on originals. It was hoped that doing these exercises would develop students’ ability to reason for themselves and gain more geometric knowledge (Adams, 2010; Herbst, 2002; Subramanian, 2005). However, it was found that it was difficult for many students to prove the originals (Adams, 2010) and something different was needed.

Textbooks were revised to include more detailed visual aid diagrams and hints to help students construct proofs (Adams, 2010). In addition, a new practice of writing proofs emerged in which each statement had to be justified by a reason (Subramanian, 2005). Teachers were required to follow the same approach when presenting proofs to their students (Herbst, 2002). A mathematics conference was held to discuss possible instructional adaptations to enhance students' ability to construct proofs during the era of originals. It was recommended that informal geometry (also known as concrete geometry) be introduced at primary school level with the hope that, by the time the students went to high school, they would know the basic geometrical facts needed to construct proofs (Herbst, 2002).

Further changes to support students' performance in proofs were seen in the twentieth century, in the period characterized as the era of exercise (Herbst, 2002).

2.2.5.3 The era of exercise

The Committee of Ten recommended that textbooks should provide detailed information on methods and strategies for doing proofs (Adams, 2010; Herbst, 2002; Subramanian, 2005). In 1913, Arthur Schultze and Frank Sevenoak (Herbst, 2002) introduced a remarkable improvement in the format of writing geometric proofs. Schultze and Sevenoak invented the two columns format of writing geometric proofs, with statements on one side, and reasons on the other side (Herbst, 2002). Every statement in the proof had to be based on a definition, a theorem, an axiom, or a previously defined proposition (Schultze & Sevenoak, 1913). This made it easier for teachers to assess the work of their students (Subramanian, 2005). Teachers were expected to provide students with frequent drills until they 'understood' the theorems (Adams, 2010). This method of constructing geometric proofs by writing a logical sequence of statements justified with reasons is what came to be known as formal proof.

However, despite increasing efforts to support students' capacity to construct proofs, the lack of success on the part of students remained a matter of concern. It is still a matter of concern.

2.3 The teaching of Euclidean geometry and geometric proofs in South Africa

Formal mathematics education in South Africa started in the early 1950s, when the apartheid government came to power (Khuzwayo, 2005). The Bantu Education Act of 1953 implemented a colonial education system that restricted access to mathematics education for many Black South African children in South

Africa (Osayimwense, 2017). Schools for Whites were well-resourced with well-trained mathematics teachers, while schools for Black South Africans were under-resourced with poorly-trained mathematics teachers (McKeever, 2017). Most Black South African schools did not offer mathematics and science up to Grade 12 due to lack of qualified mathematics teachers (Gallo, 2020). This means that many Black South African students were denied access to Euclidean geometry and geometric proofs. As a result, many Black South African students did not have the opportunity to exercise deductive reasoning, creativity and critical thinking that characterize geometry at upper secondary school level.

The Bantu Education system prepared Black South African students for low-wage unskilled and semi-skilled labour, while their white counterparts were being groomed for high-salary careers (McKeever, 2017). Knowledge of mathematics was thus seen as unnecessary for Black South African children (Hayley, 2009). Most qualified high school mathematics teachers were Whites who taught mathematics in Afrikaans and English, making it difficult for the Black South African students to succeed in the subject because they had completed their primary education using their native languages (Gallo, 2020). Secondary school mathematics was therefore taught “as an abstract, meaningless subject, only to be memorized” (Khuzwayo, 2005, p. 310-311). Black South African students were taught to be recipients of mathematical ideas, and the active participation of students was not significant (Khuzwayo, 2005). The lack of academic success of Black South African students in mathematics was blamed on their race and culture, which were deemed inferior to that of their White peers (see Van den Berg, 1978).

To justify their apartheid mathematics education policies, Afrikaner academics could carry out mathematics education research on Black South African students, while African Black academics were prohibited from conducting similar studies on White students (Osayimwense, 2017). Groenewald’s (1976) research reported that Black South African students are far behind white students in terms of visual perception, geometric figure analysis and interpretation, and general arithmetic skills. This view is controversial, because it seems to suggest that Black South African students are naturally unable to learn geometry concepts. The results of the Groenewald (1976) study were subsequently questioned by the Mathematics Commission set up by the People’s Education for People's Power (PEPP) movement in the 1980s to examine the state of education in South Africa (Khuzwayo, 2005).

The Mathematics Commission argued that mathematics is a human invention and can therefore be manipulated by people to meet their needs at any time. The Mathematics Commission tried to develop new mathematics curricula to resolve the disparities in mathematics education, but this was rejected by the apartheid government (Khuzwayo, 2005). Unjust educational policies, such as the use of Afrikaans as a medium of instruction and unequal access to educational opportunities, culminated in the Soweto uprising, which saw the death of many young people as African Black students began mass demonstrations calling for better education (Gallo, 2020). The period between 1985 and 1990 was characterized by discussions on reforms, and the state of education in South Africa (Khuzwayo, 2005). The democratically elected South African government that came to power in 1994, embarked on a radical reform of the education system to address the inequalities of the past (Osayimwense, 2017).

In the post-apartheid era, all South African students, regardless of race, had unlimited access to learning mathematics (see Sehoole & Adeyemo, 2016). All students now had the opportunity to learn Euclidean geometry up to Grade 12. Other key post-apartheid education reforms included: the dissolution of 'white school' and 'black school' policies; the construction of new schools; the allocation of resources to mathematics and science in historically disadvantaged communities; and the development of a new school curriculum focused on a student-centred outcomes-based approach to education (Osayimwense, 2017).

According to the Department of Education (2003), the aim of Outcomes-Based Education (OBE) was to allow students to achieve their full potential through an activity-based student-centred approach to education. The approach was based on the principle of democracy. One of the crucial outcomes of mathematics education stated in the National Curriculum Statement (NCS) of 2003 was to enable students to "solve problems and make decisions using critical and creative thinking" (Department of Education, 2003, p. 2). This was opposed to passively learning mathematics, as was the case in the apartheid period.

However, although the theory behind OBE was strong, there were problems with its implementation. Many teachers did not understand what it meant to teach the OBE way due to lack of training (Ramoroka, 2006). As a result, mathematics teachers continued to teach Euclidean geometry and geometric proofs using direct instruction (Kutama, 2002). This comes as no surprise, given that the same poorly trained black teachers who taught mathematics in the apartheid education system, using the conventional approach, were expected to follow a radically different approach to mathematics education in the post-apartheid era.

In 2006, Euclidean geometry was removed from the mainstream mathematics curriculum, after a series of poor mathematics results (Bowie, 2009; Ntuli, 2014). It was noted that educators had limited content and pedagogical knowledge to effectively teach the topic. Kearsley (2010) concluded that South Africa erred by making Euclidean geometry non-compulsory because the skills gained through solving Euclidean geometry problems are not only essential in engineering and science, but also important in the lives of the citizens of the country.

A study by Engelbrecht et al. (2010) found that the 2009 group of university entrants (the first group to write a Grade 12 Mathematics Examination that excluded Euclidean geometry) were weaker than their predecessors in terms of their mathematical skills and knowledge. Another study at the University of Witwatersrand also reported a decrease in the performance of first-year students in the June 2009 mathematics pass rate (Blaine, 2009). Similar patterns were reported at other universities across South Africa. Mathematics education experts attributed the drop in the mathematics performance of first-year university students to the removal of Euclidean geometry from the secondary school mathematics curriculum (Mouton et al., 2012; Padayachee et al., 2011; Wolmarans et al., 2010).

In January 2012, South Africa launched a revised version of the NCS: The Curriculum and Assessment Policy Statement (CAPS). As part of its ongoing curriculum transformations and in keeping with international trends in mathematics education, South Africa reintroduced Euclidean geometry and proof into the mainstream mathematics curriculum. In the Mathematics CAPS for Further Education and Training (FET) (Grades 10-12), it is now compulsory for all South African mathematics students to learn proofs of Euclidean geometry theorems and riders. This move has been widely applauded by experts in mathematics education, as it is expected to bridge the skills gap between secondary school and university, particularly for students who plan to study STEM programmes.

However, the return of Euclidean geometry and proof in South Africa's mainstream mathematics "puts educators and teacher educators in a challenge similar to that of the past" (Ndlovu, 2013, p. 277). Some of the educators who are now supposed to teach Euclidean geometry in the CAPS did not do any Euclidean geometry during their years at high school, college or university (Govender, 2014). Others are victims of inadequate teacher training under the apartheid system. Apartheid teacher education did not have uniform requirements to provide guidance on what students in the different teacher training colleges had to learn (Diko, 2013). Given this orientation, it is clear that even the most experienced mathematics teachers face challenges in implementing the CAPS. Current developments in mathematics education require educators to shift from traditional teacher-centred approaches to new teaching strategies based on constructivist theories of teaching and learning, and inclusive education ideologies (Dube, 2016). It has been noted that many teachers are not fully equipped with the skills required to cope with these shifts (Dube, 2016). Educators need clear guidance and support to move from conventional instruction to modern research-based teaching approaches.

While South Africa's DBE has made substantial efforts to upgrade teachers' knowledge of Euclidean geometry content through in-service training, not all of the concerns of the teacher have been fully addressed (Ndlovu, 2013). To effectively teach Euclidean geometry and proof, teachers need not just the content knowledge, but also pedagogical knowledge for teaching the geometry content. A survey by Olivier (2013) involving 150 in-service mathematics teachers in two South African provinces, seeking their views on the Mathematics CAPS training they had received, found that: "teachers expressed uncertainty about how to implement the expected CAPS amendments in the classroom" (p. 20). In certain cases, the CAPS training facilitators themselves appeared to lack the expertise and skills required to help teachers improve.

Part of the amendments to the FET (Grades 10-12) mathematics curriculum is the reinstatement of proofs of Euclidean theorems and riders. Analysis of the 2014–2019 Grade 12 Mathematics Paper 2 examination results showed that the majority of students were unable to construct and write geometric proofs, many of whom did not even attempt these questions (Department of Basic Education, 2015, 2016a, 2017,2018,2019, 2020). This points to the view that finding ways to teach geometric proofs in a way that ensures success for the majority of the students is a big challenge to most teachers.

In South Africa, the teaching of Euclidean geometry has suffered in the past due to the teachers' lack of content and pedagogical knowledge (see Govender, 2014; Mosia, 2016; Tachie, 2020). Unless measures are taken to find better approaches to teaching it, there is danger of a return to the situation of the past. The Mathematics CAPS for Grades 10-12 only clarifies the order and pacing of topics but does not suggest how the topics should be taught (Bowie et al., 2014). In the CAPS document (see Department of Basic Education, 2011), Grade 11 Euclidean geometry is scheduled for Term 3, and is allocated a period of three weeks. Students are supposed to know seven theorems of the geometry of circles, and use the theorems and their converses to prove riders. In addition to listing the theorems to be learnt at the Grade 11 level, the Mathematics CAPS gives examples of the riders that students should be able to solve when they complete the chapter. The Mathematics CAPS does not, however, provide any guidelines to address teachers' pedagogical concerns. It is up to individual teachers to determine how they should teach the topic.

Inadequate support from the subject advisors aggravates the plight of the teachers. Dube (2016) concluded that training on pedagogical content knowledge is what South African mathematics teachers need to improve the quality of mathematics teaching in schools. The lack of pedagogical knowledge for teaching geometric proofs in the context of South Africa has led teachers to teach according to the textbook (McIntyre, 2007; Mthembu, 2007; Naidoo & Kapofu, 2020). Teachers continue to present proofs as ready-made mathematical ideas, and students are expected to memorize theorems and reproduce them in examinations without understanding (De Villiers & Heideman, 2014; Shongwe, 2019). Teachers seem to have no idea how to guide students to successfully construct and write Euclidean geometry proofs with understanding.

For many South African students who go to upper secondary school underprepared for formal geometry (Alex & Mammen, 2014), conventional teaching is unlikely to meet their needs (Abakpa & Iji, 2011). It only stifles understanding and alienates students from mathematics (Ndlovu & Mji, 2012). The findings of the 2014–2019 NSC Mathematics Paper 2 diagnostic reports suggest that conventional teaching approaches have failed to improve students' geometric proofs learning achievement (see Department of Basic Education, 2015, 2016a, 2017, 2018, 2019, 2020).

South African mathematics teachers may be willing to try out new approaches to teaching geometric proofs, but there is not enough empirical evidence of instructional practices that could enhance students' geometric proofs learning achievement (Stylianides & Stylianides, 2017). Developing teaching approaches that foster a solid understanding of Euclidean theorems and proofs rather than rote learning is therefore one of the biggest challenges facing mathematics teachers in the twenty-first century mathematics classroom. To devote their time to addressing this problem, South African mathematics teachers need to understand why knowledge of Euclidean geometry proofs is indispensable for secondary school students.

The next section emphasizes why geometric proofs should be part of secondary school mathematics curricula.

2.4 Reasons for teaching geometric proofs in secondary school

One of the reasons for teaching geometric proofs in secondary school is to prepare students for their tertiary studies (Adams, 2010). By studying geometric proofs students sharpen their abstract, logical reasoning, and spatial skills. These skills are required for admission into science-based careers such as civil and mechanical engineering, astronomy, construction, architecture, geology, masonry, cartography, and computer graphics (Abdullah & Zakaria, 2013; Alex & Mammen, 2014; Luneta, 2015; Sunzuma, Masocha, & Zezekwa, 2013). Enhancing students' geometric proofs learning achievement may in turn increase their chances of gaining entrance into science-based fields of study.

Learning proofs is important for all students to develop ordered thinking skills (Adams, 2010). The ability to criticize the work of others, to engage in debate, and to make logical arguments are skills used in everyday life. Through doing proofs students become critical thinkers who can build and validate their own knowledge and not rely on their teachers or textbooks (Bramlet & Drake, 2013b). In addition, students who cannot prove simply follow procedures and copy examples without reasoning (Bramlet & Drake, 2013b). Proof is therefore needed to make learning meaningful and prevent rote learning (Aylar & Sahiner, 2013).

Proof enhances students' general mathematical abilities (Stylianides, Stylianides, & Philippou, 2007; Thompson, Senk, & Johnson, 2012). In doing proof, students are equipped with tools, processes, and problem-solving techniques (Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012), and their understanding in other mathematical aspects is greatly influenced by their ability to prove (Gunhan, 2014). Students who have learnt proofs may use their proof skills to check whether their answers are correct – not only in geometry but also in other mathematical aspects such as trigonometry, calculus, and algebra. By doing so, students can notice and correct their own mistakes and thereby improve their overall mathematics performance.

Hemmi (2010) identified seven functions of proofs within mathematics: conviction, explanation, communication, systematization, intellectual challenge, transfer, and aesthetic. These functions of proofs also support the need to teach geometric proofs in high school. Proofs may be used to determine whether an assertion is true or false (Stylianides & Stylianides, 2009). This is the conviction function of proof and sets out the criteria for accepting or rejecting the mathematical claims. Mathematicians go beyond just establishing the truth of a mathematical statement by also explaining why the statement is valid. This, according to Cirillo (2009), is the main function of proof and helps to persuade the students rather than force the results upon them. Proofs are also used to communicate mathematical ideas (Zaslavsky et al., 2012) and provide a means to challenge students' intellectual capabilities. The skills and techniques that students acquire in doing proofs can be transferred to solving other mathematics problems. Mathematical proofs demonstrate the beauty of mathematics. Hemmi referred to this as the aesthetic function of proof.

According to Zaslavsky et al. (2012) Euclidean geometry is possibly the first place where students learn the skill of proving in the secondary school mathematics curriculum. In the twentieth century, attempts to introduce formal proofs outside geometry were unsuccessful (Stylianou et al., 2009). Based on the multiple roles of proof, many countries have placed more emphasis on the teaching and learning of Euclidean geometry and proof in schools (see for example Cirillo & Herbst, 2012; Stylianides et al., 2007; Weber & Alcock, 2009). However, this recommendation presents challenges to the teachers and the students in the mathematics classroom (Knuth, Choppin, & Bieda, 2009).

The next section discusses the difficulties faced by teachers and students in teaching and learning geometric proofs.

2.5 Difficulties with learning and teaching geometric proofs

The current emphasis on teaching and learning Euclidean geometry and proofs in secondary mathematics places greater demands on both the educators and the students. Available literature suggests that both teachers and students have difficulties in teaching and learning geometric proofs.

2.5.1 Students

Chief examiners' reports from across many parts of the world suggest that most secondary school students, including high performing ones, have difficulties with geometric proofs. In South Africa, the 2014-2019 examiners' reports on the performance of candidates per subject in the NSC examinations revealed that questions that required candidates to construct and write geometric proofs were not well answered (see Department of Basic Education, 2015, 2016a, 2017, 2018, 2019, 2020). In Malawi, Chief examiners' reports released by the Malawi National Examination Board for the period 2008-2013 concluded that "students fail mathematics mainly due to poor performance in geometric proof questions" (Mwadzaangati, 2015, p. 3308). Similar findings were reported in West African countries (see West African Examination Council, 2009, 2010, 2011).

Several empirical studies from America, England, Germany, Turkey, and Japan have also shown that most students in secondary schools do not know how to construct and write formal proofs (see for example Healy & Hoyles, 2000; Köğce et al., 2010; Recio & Godino, 2001; Reiss et al., 2001; Weber & Alcock, 2009). Thus, the problem of students' inability to construct geometric proofs is not unique to South African mathematics education. Researchers agree that constructing and writing a mathematical proof is a complicated skill for students (see for example Harel & Sowder, 2007; Heinze & Reiss, 2009; Stylianides & Stylianides, 2009; Thompson et al., 2012).

When developing strategies to improve students' geometric proofs learning achievement, it is important to consider the complexity of the challenges that students encounter before proposing potential ways to alleviate them (Selden & Selden, 2007). The main problems faced by students in the construction and writing of geometric proofs have been thoroughly investigated. These include (a) not knowing how to begin proof, (b) not being able to organize their reasoning in a logical sequence or to give reasons for their conclusions, (c) lack of prerequisite skills and conceptual understanding, (d) inability to use correct mathematical language and notation, (e) giving up too soon, (f) relying on empirical arguments rather than deductive reasoning, (g) failure to see the need for proof, and (h) low levels of cognitive development (see Fabiyi, 2017; Harel & Fuller, 2009; Harel & Sowder, 2007; Healy & Hoyles, 2000; Naidoo & Kapofu, 2020; Ngirishi & Bansilal, 2019; Selden & Selden, 2007; Stylianides & Stylianides, 2009). Although literature is replete with information on the kind of difficulties that students face in learning proof, little is known about potential ways to mitigate them.

The present study considers that the difficulties of students with geometric proofs indicate the existence of teaching practices that need to be improved. While there are various factors that contribute to students' mathematics achievement, it is widely accepted that what happens inside the classroom is the most important factor (Arnold & Bartlett, 2010; Barwell, Barton, & Setati, 2007). It is therefore important to explore the difficulties faced by teachers in teaching geometric proofs.

2.5.2 Teachers

Several reports from various countries around the world have identified the use of conventional teaching approaches as the main reason for students' difficulties with geometric proofs (see Bramlet & Drake, 2013b; Harel & Fuller, 2009; Mwadzaangati, 2015, 2019; Selden & Selden, 2007; Siyepu, 2014; West African Examination Council, 2009, 2010, 2011). Many teachers lack the pedagogical content knowledge required to teach the mathematical aspect of proof (Harel & Sowder, 2007; Blanton, Stylianou, & David, 2009; Ndlovu & Mji, 2012). Chief examiners' reports from South Africa, Malawi, Ghana, and Nigeria attribute the failure of students to construct and write geometric proofs to the teachers' lack of pedagogical knowledge (Department of Basic Education, 2015, 2016a, 2017, 2018, 2019, 2020; Mwadzaangati, 2015, 2019; West African Examination Council, 2009, 2010, 2011).

Current teaching of geometry concepts in many classroom settings is still oriented towards teacher-centred approaches (Abdullah & Zakaria, 2013; Mwadzaangati, 2019). Euclidean geometry lessons are still characterized by the traditional approach where teachers copy theorems and proofs from the textbook onto the chalkboard, and students in turn, copy the completed theorems and proofs into their notebooks (see Mthethwa, Bayaga, Bossé, & Williams, 2020; Tachie, 2020). Students are then expected to memorize the proofs and reproduce them in class exercises and tests without adequate comprehension (De Villiers & Heideman, 2014; Shongwe, 2019). This is how some of the teachers themselves have been taught during their high school years (Bramlet & Drake, 2013b; Gallo, 2020).

Future mathematicians, engineers, architects, and scientists are being lost in the school system due to the use of conventional methods of teaching that do not facilitate student engagement, creative and critical thinking, and deductive reasoning. There is therefore a great need to explore alternative approaches to teaching Euclidean theorems and proofs that would enhance the learning achievement of most students in Euclidean geometry.

The next section reviews the findings of available studies that looked at ways to promote the teaching and learning of geometric proofs in secondary schools.

2.6 Strategies to improve students' geometric proofs learning achievement

Numerous studies have explored students' difficulties with proofs (see Fabiyi, 2017; Harel & Sowder, 2007; Healy & Hoyles, 2000; Heinze & Reiss, 2009; Inglis & Alcock, 2012; Moore, 1994; Naidoo & Kapofu, 2020; Ngirishi & Bansilal, 2019; Stylianides & Stylianides, 2009), but only a few studies have concentrated on finding ways to improve students' geometric proofs learning achievement. Instructional strategies that were found to improve students' ability to construct geometric proofs include: the reading and colouring strategy (Cheng & Lin, 2006), the heuristic worked-out examples (Reiss, Heinze, Renkl & Groß, 2008), and the step-by-step unrolled strategy (Cheng & Lin, 2009).

2.6.1 Reading and colouring strategy: teaching experiment in Taiwan

Cheng and Lin (2006) investigated the effect of the reading and colouring strategy in helping incomplete provers to complete their geometric proofs. The incomplete provers were eight Grade 9 Taiwanese students identified from two classes. These students had learnt formal proofs in regular teaching but missed one necessary step in constructing multi-step geometric proofs.

A one-group pre-test/post-test quasi-experimental design was employed. The intervention involved the teacher demonstrating the reading and colouring strategy for about ten minutes. The strategy involved reading the question, representing given conditions on the given geometric figure using coloured pens, and writing the intermediary conclusions next to the diagram. Congruent elements were marked using the same colour. The students then completed proof tasks individually using the same items as in the pre-test. The proof tasks consisted of 22 items in which students' solutions were unacceptable in the pre-test. Results indicated that the reading and colouring strategy was effective as the students managed to construct correct proofs in 15 of 22 items. In a delayed post-test administered two months later, the incomplete provers' performance improved in 23 of 24 items.

It was concluded that the reading and colouring strategy helped the students to retrieve appropriate theorems and axioms for reasoning, and to reduce cognitive load when organizing the proof steps into a sequence (Cheng & Lin, 2006). Colouring kept "all information visible and operative" (Cheng & Lin, 2006, p. 295). However, it was observed that the reading and colouring strategy was not effective in cases where colouring caused visual disturbance. The study is also criticized for engaging a small number of participants. Future research could investigate the effectiveness of the reading and colouring strategy with larger samples of students.

Cheng and Lin (2007) conducted a follow-up study to their previous research. This time, they investigated the effectiveness and limitations of the reading and colouring strategy in whole-class regular teaching of multi-step geometric proofs. The study employed a two-groups pre-test/post-test quasi-experimental design. Sixty-four Grade 9 students from two classes in the same school were involved. Each class had 32 students. One class formed the treatment group and the other class constituted the control group. The two classes showed no significant difference in their pre-test average scores.The treatment group were taught using the reading and colouring strategy for a period of five weeks. This involved demonstration by the teacher and imitation by the students. The students in the control group were taught the conventional way. Four multi-step geometric proof questions were administered before and after the teaching experiment. Post-test results indicated that the students in the treatment group performed significantly better than the control group students, producing 60.6% of acceptable proofs compared to 30.3% for the control group (Cheng & Lin, 2007).

It was concluded that the reading and colouring strategy can be utilized in regular whole-class teaching. However, it was found to be "less effective to non-hypothetical bridging students", hence of little benefit to below-average students (Cheng & Lin, 2007, p. 113). Non-hypothetical bridging students are those that are not able to construct intermediary conclusions in a multi-step geometric proof. Approximately 40% of the students were unable to construct even one correct proof (Heinze, Cheng, Ufer, Lin, & Reiss, 2008).

2.6.2 Heuristic worked-out examples: teaching experiment in Germany

Reiss, Heinze, Renkl and Groß (2008) investigated the effect of using heuristic worked-out examples on students' geometric reasoning and proof competencies. Unlike the reading and colouring strategy which sought to enhance students' hypothetical bridging abilities, heuristic worked-out examples sought to give students a complete model of the proving process from the premise to the conclusion (Heinze et al., 2008). The heuristic worked-out examples approach was an extension of the classical worked-out examples approach, which consisted of a problem and the steps to its solution. Learning from heuristic worked-out examples meant that students had to understand how the proof was generated and why it worked.

The study involved 243 Grade 8 students from a secondary school in Germany. One hundred and fifty students from six classes constituted the treatment group and the remaining 93 students from four classrooms formed the control group. The two groups showed no significant difference in their pre-test performance. The teaching experiment began after all students had attended regular teaching and completed the geometry chapter. The treatment group worked with self-explaining heuristic worked-out examples for five lessons while control group students continued to have lessons on proof in the way they were usually taught by their teachers. Both groups wrote a post-test on geometric reasoning and proof.

The findings showed that students who were taught using heuristic worked-out examples performed substantially better than students who received regular instruction (Heinze et al., 2008). It was concluded that the heuristic worked-out examples approach was more effective than conventional teaching. Further analysis indicated that the heuristic worked-out examples favoured low and average achievers but did not have a major effect on high achievers. This was because the strategy emphasized aspects that the high achievers were already familiar with (Cheng & Lin, 2009). The strength of the heuristic worked-out examples approach lies in scaffolding learning by providing important geometry knowledge required by students. This allows students to focus more on the proving process than the recall of geometric facts (Heinze et al., 2008).

2.6.3 Step-by-step unrolled strategy: teaching experiment in Taiwan

Cheng and Lin (2009) conducted another study to test the effect of the step-by-step unrolled strategy on below-average students' performance on multi-step geometric proofs. A one-group pre-test/post-test quasi-experimental study was conducted. The study involved 11 students identified as below-average from five Grade 9 classes in Taiwan, based on their pre-test results. The students attended extra classes for a period of six weeks after regular lessons. The teaching experiment covered four geometry aspects: triangles, quadrilaterals, congruency, and parallel lines.

The step-by-step unrolled strategy divided the complex procedures of the proving process into small units of guided step-by-step reasoning activities (Cheng & Lin, 2009). The proof task was presented to the students in a 'covered' form. The first condition was unrolled, and students had to deduce what should be true based on the given condition. The second step was unrolled, and again students were asked to conclude what should be true based on the condition provided. The procedure was continued in the same way until the final step. It was hypothesized that the step-by-step unrolled strategy would help below-average students to develop hypothetical bridging skills. Both computational and narrative proof tasks were used.

A computational proof task is one that "asks students to find out the assigned measure(s) of configuration under given conditions" (Cheng & Lin, 2009, p. 125). The process of proving a computational proof task may look like number calculation. Figure 2.1 shows a typical example of a computational proof task:

Given that $\mathrm{AC} = \mathrm{CB}$ and $\mathrm{A\hat{C}D} = 80°$, show that $\mathrm{A\hat{B}C} = 40°$

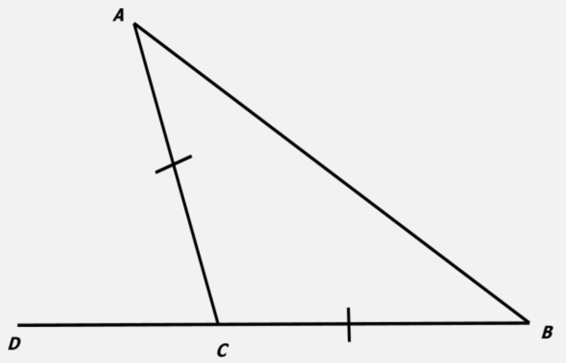


Figure 2.1: A typical example of a computational proof task

A narrative proof may take the form of a two-column proof or a descriptive format. A narrative proof differs from a computational proof in that it requires students to clearly express their reasoning from the premise to the conclusion, using appropriate theorems and axioms (Cheng & Lin, 2009).

Post-test results showed that the step-by-step unrolled strategy helped nine of the eleven students to successfully answer computational proof questions. However, it was found to be ineffective in solving multi-step narrative proofs.

2.7 Emerging issues

Experts in mathematics education agree that proof is an integral component of high school mathematics curricula. It is recommended that proof should be taught at all grade levels (see for example Cirillo & Herbst, 2012;

Stylianou et al., 2009; Weber & Alcock, 2009). However, the teaching and learning of Euclidean geometry proofs seems to be a challenge in many mathematics classrooms around the world. Many secondary school students perform poorly on geometric proof questions in tests and examinations (see Achor & Imoko, 2012; Department of Basic Education, 2015, 2016a, 2017, 2018, 2019, 2020; Mwadzaangati, 2015, 2019; West African Examination Council, 2009, 2010, 2011).

While students' difficulties with geometric proofs have been well researched (see for example Fabiyi, 2017; Harel & Sowder, 2007; Healy & Hoyles, 2000; Heinze & Reiss, 2009; Inglis & Alcock, 2012; Moore, 1994; Naidoo & Kapofu, 2020; Ngirishi & Bansilal, 2019; Recio & Godino, 2001; Stylianides & Stylianides, 2009), knowledge of the kind of instruction that would address students' geometric proof learning difficulties is still limited. Students' difficulties in learning Euclidean proofs have been attributed to poor quality of teaching in the mathematics classroom (Mwadzaangati, 2019). Conventional teaching approaches, predominantly dominated by teachers, have been found to be inadequate in meeting the learning needs of most students in Euclidean geometry. Empirical investigations undertaken to date, have not closed all the knowledge gaps.

The reading-and-colouring strategy developed by Cheng and Lin (2006) in Taiwan only benefited incomplete provers; those that were already able to construct intermediary steps but missed one necessary step in a multi-step geometric proof. The strategy had no positive effect on the performance of non-hypothetical bridging students; those who were unable to construct intermediary steps from the premise to the conclusion. The heuristic worked-out examples strategy developed by Reiss et al. (2008) in Germany benefited low and average achieving students and had no significant effect on high achievers. The step-by-step unrolled strategy developed by Cheng and Lin (2009) in Taiwan with below-average students was only effective in solving computational proofs. It was found to be ineffective in solving multi-step narrative proofs.

These strategies addressed the learning needs of certain students leaving out others. Contemporary theories of mathematics education advocate the development of teaching approaches that are inclusive in nature, with no child being left behind (see Department of Basic Education, 2011). To this end, teachers should develop teaching approaches that cater for students' diverse needs and at the same time address barriers to learning in regular classroom instruction (Dube, 2016). How to teach Euclidean geometry proofs in a way that is understood by most students, including those who lack the requisite geometry knowledge, is therefore subject to further investigation as the available studies have not fully addressed this challenge. To achieve success in this regard, the present study posits that, (1) geometry teaching and learning theories should be revisited, and (2) the students themselves should be allowed to give input on the teaching and learning process.

2.8 Students' views on their learning experiences

Efforts to find ways to improve students' geometric proofs learning achievement in secondary schools are less likely to succeed if the students' opinions are not considered. According to the United Nations Convention on the Rights of the Child (UNCRC), students should be consulted on matters that affect their lives (Abrahams & Matthews, 2011), and that includes their education. Arnot, McIntyre, Peddar, and Reay (2004) argue that students have the capacity to give "insightful and constructive" comments on their learning experiences in schools (p. 4). Students' perspectives on their learning experiences may be used to understand their attitudes towards the subject, and to structure future lessons in such a way as to maximize student academic, social, and emotional benefits (Borthwick, 2011). Capturing students' views on their teaching and learning experiences in mathematics should therefore be given priority in education.

The next section discusses the theory that influenced this research.

PART TWO

2.9 Theoretical framework of the study

One of the necessary conditions for successful teaching of mathematics is to understand the theoretical models that explain how students learn certain mathematical aspects. Educational researchers agree that theory informs practice (see Silver & Herbst, 2007; Skott, 2009). The most successful educational experiments are not random, but based on principles of multiple learning theories (Chung, 2001). With several studies documenting low achievement levels of most students in mathematics, it is critical that we revisit and reflect upon theories of learning mathematics “to identify what we may be missing” (Lundell & Higbee, 2001, p. 11).

The present study seeks to implement and test the effect of Van Hiele theory-based instruction on Grade 11 students’ geometric proofs learning achievement. The research further explores students’ views on the Van Hiele theory-based approach and the conventional approach to teaching Grade 11 Euclidean theorems and non-routine geometric proofs. The study draws its theoretical underpinnings from the geometry learning theory developed by the Van Hiele couple, which is one of the most detailed models for understanding students’ geometric thought (Abdullah & Zakaria, 2013). The Van Hiele theory provides a framework that could help teachers to design appropriate instruction to facilitate students’ learning of Euclidean geometry concepts. The theory was developed by Pierre Marie van Hiele and his wife Dina van Hiele-Geldof. It originated from the couple’s doctoral dissertations completed in 1957 at the University of Utrecht in the Netherlands (Fuys, Geddes, & Tischler, 1988).

Like the Van Hiele couple, many mathematics teachers in the twenty-first century are frustrated by the large number of students who find it difficult to understand Euclidean geometry concepts (Kutama, 2002). Pierre Marie van Hiele’s thesis explained why students experienced difficulties in learning geometry. Dina van Hiele-Geldof investigated the sequencing of the geometry content and learning activities for the development of students’ understanding (De Villiers, 2010). Revisiting the Van Hiele theory may help to explain why students have difficulties with geometric proofs and provide insight into how to organize learning and teaching activities to improve students’ geometric proofs learning achievement. If teachers would understand why students have problems with Euclidean geometry proofs, they would be able to suggest possible ways to intervene.

2.9.1 The Van Hiele theory

The Van Hieles were concerned that so many of their middle-grades students had difficulties in learning geometry (see Van Hiele, 1984; Van Hiele-Geldof, 1984). They concluded that high school geometry was too complicated for most of the students to fully comprehend (Malloy, 2002). The Van Hieles theorized that students had problems with secondary school geometry because they had not had enough previous geometry learning experiences at a lower level (Cirillo, 2009). This idea led them to investigate the prerequisite skills that students need to succeed in deductive reasoning.

On careful observation of their students' work, the Van Hieles concluded that students' geometric thinking seemed to progress through a sequence of five hierarchical levels (Van Hiele, 1984), each having its own unique characteristics which should be of interest to the mathematics teacher. Originally, the Van Hiele levels were numbered 0 to 4. It was Wirszup (1976) who changed the numbering so that Level 0 became Level 1 and Level 4 became Level 5. As a result, the Van Hiele levels are numbered and named differently by different researchers. The present study utilizes the original numbering of the Van Hiele levels, that is 0 to 4.

2.9.1.1 The Van Hiele levels of geometric thinking

2.9.1.1.1 Level 0:

This is the initial stage or basic level, labelled by some as visualization and others as recognition. Students who operate at this level can only identify geometric shapes (such as triangles, rectangles, and squares) by their appearance (Crowley, 1987). This is typical of students in pre-school up to Grade 2 (Malloy, 2002).

2.9.1.1.2 Level 1:

This is labelled by some as analysis and others as descriptive. At this level, students can now identify geometric shapes by their properties but cannot see how the shapes are interrelated and still cannot understand definitions (Crowley, 1987; Shaughnessy & Burger, 1985; Rahim, 2014). For example, the fact that a square is a rectangle is not yet understood. This is typical of students in Grades 2 to 5 (Malloy, 2002).

2.9.1.1.3 Level 2:

This is labelled by some as informal deduction and others as ordering. Students who have attained this level can now classify geometric shapes based on their properties. The concept of class inclusion is now understood and definitions become meaningful (Rahim, 2014). The square can now be recognized as a rectangle. However, formal reasoning is not yet understood (Van

Hiele, 1984). This is typical of students in Grades 5 to 8 (Malloy, 2002).

2.9.1.1.4 Level 3:

This is labelled as deduction (Crowley, 1987; Malloy, 2002). This is the level at which students can now construct proofs using the deductive approach since they now understand definitions, theorems, converses, and axioms (Crowley, 1987; Shaughnessy & Burger, 1985; Rahim, 2014). Students can now establish the connection among networks of theorems (Fuys et al., 1988). This is typical of students in upper secondary school (Malloy, 2002).

2.9.1.1.5 Level 4:

This stage is labelled as rigour. Students at this stage understand the relationships between different axiomatic systems, and can compare, analyse and create proofs in non-Euclidean geometries (Crowley, 1987; Rahim, 2014; Shaughnessy & Burger, 1985). This is typical of students at college or university.

2.9.1.2 Properties of the Van Hiele levels

According to the Van Hieles, students have to pass through all the levels without skipping any one of them. In order to succeed at level (n), students should first master level $(n-1)$. This is described as the property of fixed sequence (Usiskin, 1982). Each level is characterized by its own language and symbols. This is called the property of distinction (Fuys et al., 1988). For example, the fact that a square is a rectangle does not make sense to students at the visualization and analysis levels, but the same language makes sense to students at the informal deduction level. The property of adjacency implies that what was intrinsic at level(n) will become extrinsic at level $(n-1)$ (Van Hiele, 1984). For example, at Level 0 (visualization), students can only identify geometric shapes by looking at the physical appearance and they are not aware that those shapes possess properties. Students will start to recognize properties of shapes when they reach the analysis level (Level 1).

As Van Hiele (1984) put it: “Two people who reason at two different levels cannot understand each other” (p. 250). This is described as the property of separation and it sheds light on why the majority of secondary school students have difficulties with Euclidean geometry proofs. Some teachers present Euclidean geometry at a level higher than that of the student (Van Hiele-Geldof, 1984; Van Hiele, 1984). This is typical of what happens in upper secondary school (Grades 10-12) where teachers move straight into deductive reasoning and proof, assuming that students have mastered the geometry concepts of the lower grades. This results in a mismatch between the level of teaching and the students' levels of understanding. The lesson becomes a monologue, instead of a dialogue. To effectively teach geometry, there is need for teachers to align their teaching with students' current Van Hiele levels (Fuys et al., 1988). The Van Hieles cautioned against forcing students to a particular level when they are not ready, as this will result in students simply imitating the teacher's work without proper understanding (Van Hiele-Geldof, 1984). My experience of teaching Euclidean geometry affirms that students tend to memorize proofs of theorems, thus creating the impression that they have understood when, in fact, nothing has been learnt. For this reason, the focus of the present study is on the ability of the students to prove riders (non-routine geometric proofs) and not theorems, whose proofs can be memorized.

How teachers teach has more influence on students' achievement than students' biological maturation. This is described as the property of attainment (Usiskin, 1982). Some teaching methods can accelerate progress, while others can cause delays in learning development (Crowley, 1987). To facilitate movement between levels, the Van Hieles proposed that geometry instruction should be structured according to the learning phase framework, the details of which are outlined in the next section.

2.9.1.3 Van Hiele phases of teaching and learning

The Van Hieles proposed a sequence of five learning phases to help students attain a particular Van Hiele level. These are, inquiry→guided orientation→explicitation→free orientation→integration (Van Hiele, 1984). As summarized by Abdullah and Zakaria (2013), the inquiry phase involves teacher-student conversation to establish students' prior knowledge on the topic and to help students recognize the direction the lesson will take. In the guided orientation phase, students explore the topic and make discoveries through guided lesson activities. Explicitation offers students an opportunity to express and exchange ideas based on what they have observed in the second phase. The free orientation phase engages students in solving open-ended and more complex tasks, for example, multi-step geometry tasks that can be solved in more than one way. In the last phase (integration), students synthesize and summarize what they have learnt in order to develop a new network of relations. After going through all five of these phases, the student then attains a new level of geometric thinking (Van Hiele, 1984).

2.9.2 Implications of Van Hiele theory for teaching geometric proofs

If the Van Hiele theory is right, students going to upper secondary school should at least have achieved Level 2 (informal deduction) for them to be ready for Level 3 (deduction). This is the ideal situation. However, the situation prevailing in many mathematics classrooms is far from ideal. The method of geometry teaching that has been found to be prevalent in many classrooms is characterized by “checking homework, followed by teacher lecture and demonstration, followed in turn by student practice in a sequence of classroom instructional activities” (Sanni, 2007, p. 39). As a result of such teaching practices, most students leave secondary school with inadequate deductive reasoning and proof skills (Wang, 2009).

The Van Hiele theory gives insight into how teachers can effectively teach geometric proofs in classrooms where students’ difficulties with proof have been noticed. Knowledge of the Van Hiele theory is important to the teaching of geometric proofs (Cirillo, 2009). If the Van Hiele theory is valid, students will not understand geometric proofs if they have not mastered lower-level geometry concepts such as properties of shapes and definitions. Attempting to teach geometry proof to students who have not mastered the prior knowledge is likely to cause confusion between the teacher and the students. The implication for teaching geometric proof is that the mathematics teacher should first establish students’ current levels of geometric thinking to see if they are ready to learn proofs. If students are not ready, then the teacher should try to make up for the learning deficits to bring the students up to standard before introducing formal proofs. New geometry knowledge should be built upon students’ existing knowledge schema. Disregarding students’ current levels of geometric understanding may result in students memorizing facts and imitating the teacher without understanding.

If the Van Hieles’ property of attainment is valid, then the fact that most students come to secondary school not ready to learn geometric proofs should therefore not be used as an excuse when those students leave secondary school with weak deductive reasoning skills. It is how the teachers design and organize their teaching that determines whether the students will master deductive reasoning and proof skills. According to Noraini (2005), geometry cannot be taught like any other mathematics topic. For this reason, it is difficult for many mathematics teachers to plan activities that can enhance students' understanding of geometry concepts (Choi-Koh, 2000).

The Van Hiele teaching phases provide guidelines on how to design and organize instruction in a way that enhances students' understanding of geometry concepts at any level, which by implication include geometric proofs. The guided orientation phase suggests that teachers should allow students to explore and discover the properties of geometric shapes before they begin to solve complex tasks, such as proving theorems and riders. The implication here is that the teaching of geometric proofs should be preceded by investigative geometry in which the students reinvent geometry theorems and discover the facts by themselves rather than being told by the teacher. Educational psychologists have found that learning by discovery ensures higher levels of knowledge retention and promotes students' autonomy and independence (Bruner, 1960). This is also supported by Abdullah and Zakaria (2012), who concluded that geometry instruction should prioritize practical investigation, conjecturing, argumentation, and creative thinking.

The Van Hiele explicitation phase suggests that students should be given an opportunity to state the geometry theorems themselves using their own language, based on what they learnt during the exploration phase. The teacher is there to assist with the appropriate terminology. This is diametrically opposed to conventional teaching approaches, where the teacher writes theorems on the chalkboard, asks students to copy them into their notebooks, and to memorize the theorems.

The free orientation phase suggests that students should be given a chance to find their own ways to prove riders instead of limiting them to the techniques of the textbook or those known by the teacher. In sharp contrast, conventional teaching of geometric proofs is characterized by teacher demonstration, followed by students' practice. The danger here is that proving a geometric rider is not procedural, and hence the steps followed by the teacher in proving one rider on the chalkboard may not be applicable in proving the next rider. This can be frustrating to students who rely on their teacher for methods of solution. Proofs of geometric riders are unique and require students to fully understand the relevant geometric theorems, axioms, and definitions. The present study argues that the use of Van Hiele theory-based instruction ensures that students have complete acquisition of the relevant theorems and axioms during the guided exploration and explicitation phases, prior to the complex task of constructing and writing proofs.

The integration phase suggests that students should be allowed to share their proof strategies. In the process, students will discover that the process of proving riders can be done in multiple ways and this enriches their repertoire of solution strategies and problem-solving skills.

Several countries have carried out further research on the Van Hiele theory and subsequently realigned their geometry curriculum based on recommendations from their studies. The next section presents a review of the available studies on the Van Hiele theory.

2.9.3 Previous studies on the Van Hiele theory

Following the Van Hieles' findings, there has been a proliferation of research (a) to test the validity of the Van Hiele model and its assumptions, (b) to determine the Van Hiele levels of the students and teachers, and (c) to develop, implement and evaluate teaching experiments based on the Van Hiele model (Pusey, 2003).

2.9.3.1 Validating the Van Hiele theory

Usiskin (1982) conducted a study to find out if students' Van Hiele levels at the beginning of a one-year geometry course could be used to predict their end of year performance in geometry. The study involved 2699 Grade 10 students selected from thirteen high schools in America. A pre-test/post-test design was utilized and students' Van Hiele levels were assessed using a multiple-choice test. The results indicated a moderately strong correlation ($r = .64$) between students' scores at the beginning of the year and their scores at the end of the year. Most of the students operated at Level 0 (visualization) and Level

1 (analysis), and hence were not prepared for formal deductive geometry. These results confirmed the hierarchical levels of geometric thought of the Van Hieles. However, some students were found to oscillate between levels, making it difficult to categorize them.

Mayberry (1983) carried out a study to validate aspects of the Van Hiele theory. Nineteen student teachers from Georgia College in America were interviewed. The interview was based on seven common geometric concepts: parallel lines, right angles, squares, isosceles triangles, circles, congruence and similarity. An analysis of participants' responses showed that most of them were not ready for formal deductive geometry, operating at levels 0 or 1. Guttman's scalogram analysis of results confirmed the view that the Van Hiele levels form a hierarchy. Another interesting finding that emerged from Mayberry's study was that participants were found to operate at different levels on different geometry concepts (Mayberry, 1983). This result was confirmed by Gutiérrez, Jaime and Fortuny (1991) in a study involving 50 Spanish students, 41 of whom were student teachers in their third year of teacher training, and 9 were Grade 8 students. Based on their findings, Gutiérrez et al. (1991) concluded that it is possible for students to develop two consecutive Van Hiele levels at the same time. Thus, the view that the Van Hiele levels are discrete was refuted.

Senk (1989) investigated whether students' Van Hiele levels could predict their degree of success in proof writing. A sample of 241 American secondary school students were involved. The pre-test/post-test design was employed. Usiskin's (1982) multiple-choice test was used to assess the students' Van Hiele levels at the beginning of the geometry course. A proof test that consisted of 6 items was developed and administered as post-test. The results showed a moderately strong positive correlation between the students' pre-test and post-test scores. An analysis of variance (ANOVA) performed on the students' average post-test scores at different Van Hiele levels indicated a statistically significant difference $(p < 0.01)$, with students whose Van Hiele levels were at 2 or 3 scoring higher that those at levels 0 or 1. These findings support the Van Hieles' assertion that students cannot succeed at level (n) if they have not mastered level $(n-1)$. Senk (1989) concluded that there is a dire need for a strong geometry curriculum at lower school levels to ensure students success in high school geometry.

Mason (1995) reported results from a study involving 120 academically gifted students who had not yet started learning formal geometry. The students were selected from Grade 6-8 classes in 50 school different districts in America. The students completed a multiple-choice geometry test and 64 of them were interviewed using Mayberry's (1983) interview protocol. Analysis of students' responses confirmed that the Van Hiele levels are hierarchical. Interestingly, 35.8% of the gifted students skipped levels and the younger students attained significantly higher Van Hiele levels than the older ones (Mason, 1997). This corroborates the Van Hieles' assertion that achievement of higher Van Hiele levels does not dependent on age.

Based on the preceding literature, it can be concluded that the Van Hiele levels provide a valid way to categorize students' achievement in Euclidean geometry. The hierarchical nature of the Van Hiele levels was confirmed (Usiskin,1982; Senk,1989), whereas any discreteness of the Van Hiele levels was refuted (Gutiérrez et al., 1991).

2.9.3.2 Assessing students' Van Hiele levels

Several studies have assessed students' Van Hiele levels in many countries around the world. Feza and Webb (2005) investigated whether Grade 7 students met the requirements for geometry as stated in the RNCS. A sample of thirty Grade 7 students was selected from six previously disadvantaged primary schools in the Eastern Cape Province, South Africa. Data were collected using on-site observation of students, semi-structured interviews and video recordings. The results indicated that 10 students (33.33%) were at Level 0, 15 (50%) were between Levels 0 and 1, and only five students (16.67%) showed complete acquisition of Level 1, whereas the curriculum required them to be at Van Hiele Level 2 (informal deduction). Clearly, these students would go to secondary school underprepared. They are at risk of failing high school geometry unless secondary school teachers find pedagogical strategies to provide meaningful learning experiences that match the particular level of geometric thinking of such students.

Atebe (2008) investigated the Van Hiele levels of high school students from South Africa and Nigeria. The students were selected from the FET Band

(Grades 10-12). Initially, 144 students (72 from each country) were selected using purposive and stratified sampling techniques. However, five students (four from Nigeria and one from South Africa) withdrew their participation and only 139 students were involved in the final analysis. Data were collected using pen-and-paper tests, interviews and classroom video recordings. An analysis of results indicated that of the 68 students from Nigeria, 36 (53%) were at Level 0 (visualization), 15 (22%) were at Level 1 (analysis), 16 (24%) at Level 2 (informal deduction), and 1 (1%) at Level 3 (deduction). In the South African subsample 29 students (41%) were at Level 0, 16 (22%) operated at Level 1, 17 (24%) at Level 2, 2 (3%) at Level 3, 4 (6%) at Level 4, and 3 (4%) could not be classified. These results show that the greater number of the students were at Level 0, which means they were not ready to learn Euclidean geometry proofs (Atebe, 2008). The results are also consistent with previous findings by Feza and Webb (2005).

Alex and Mammen (2012) conducted a study to assess Grade 10 students' Van Hiele levels. The study involved 191 Grade 10 students from five secondary schools in Eastern Cape, South Africa. Purposive sampling was used to select the five schools that participated in the study. A multiple-choice test based on the properties of triangles and quadrilaterals was administered to the participants. The study found that 48% of the students operated at Level 0, 29% were at Level 1, 14% at Level 2, and 9% at Level 3. Thus, most of the students were not prepared for higher-grade Euclidean geometry, which involves proof and deductive reasoning. These findings were consistent with earlier studies by Usiskin (1982), Feza and Webb (2005), and Atebe (2008).

Van Putten, Howie and Gerrit (2010) investigated the Van Hiele levels of 32 third-year student teachers at the University of Pretoria in South Africa. The students' Van Hiele levels were assessed using multiple-choice test items as well as open-ended proof questions. The results of the study indicated that more than 50% of the students operated at Level 0. Even the geometry module offered to the students "did not bring about a sufficient improvement in their level of understanding for these students to be able to teach geometry adequately" (Van Putten, Howie & Gerrit, 2010, p. 22).

Luneta (2014) assessed the Van Hiele levels of a group of 128 first-year students at the University of Johannesburg in South Africa. Data were collected using questionnaires and interviews. The study revealed that most students had weak knowledge of basic geometry, with 78% of them efficiently functional at Level 1. To be effective and efficient in teaching geometry in the classroom, teachers should be at Level 4 (rigour). It is a pity that many teachers enter the teaching field ill-equipped (Van Putten et al., 2010). This is a matter of serious concern.

The preceding literature shows that students are trapped in a vicious circle of poor geometrical skills and understanding. Students leave primary school operating at lower Van Hiele levels than the standard set by their curriculum. They go to secondary school underprepared, which reduces their chances of success in higher-grade Euclidean geometry. The kind of teaching offered in many secondary schools appears not to meet the learning needs of these students. As a result, the students exit secondary school with deficiencies in their geometry knowledge. Studies involving pre-service mathematics teachers have shown that they, too, leave university with weak geometrical skills and understanding, and enter the profession ill-prepared. If the Van Hiele levels of the teachers are lower than the levels expected of their students, then the teachers will not be able to guide the students in the mathematics classroom (Van Putten et al., 2010). It is an incontestable fact that this situation is undesirable for any education system, and therefore requires immediate attention.

Reports indicating that the majority of students are operating at lower Van Hiele levels than the levels set by their mathematics curriculum are not unique to America, South Africa and Nigeria. Similar findings were reported in Lesotho (Evbuomwan, 2013), Malaysia (Abdullah & Zakaria, 2013; Meng & Idris, 2012), Turkey (Bal, 2014), Czech Republic (Haviger & Vojkůvková, 2015), Slovenia (Škrbec & Čadež, 2015), Ghana (Baffoe & Mereku, 2010), Yemen, Morocco, Kuwait, Tunisia, Georgia, Honduras, Oman, Saudi Arabia, Indonesia, Qatar and Botswana (Mullis, Martin, Foy, & Arora, 2012). There is no doubt that the problem appears to be in most countries around the world. It would be interesting to learn how some countries have tried to resolve the crisis. The next section highlights the findings from some of the key empirical studies on possible interventions and the extent to which the interventions have been effective.

2.9.3.3 Interventions based on the Van Hiele model

Several studies have tested the effect of interventions based on the Van Hiele model with students in different grades, and focusing on different concepts of Euclidean geometry.

Parsons, Stack and Breen (1998) investigated the effect of computer-based guided instruction using a software application called Windows TM Geometry. The study was carried out in America with a sample of eleven Grade 8 students. The objective of the study was to determine if Windows TM Geometry could improve the students' understanding of geometric concepts and definitions, and help them achieve Van Hiele Level 2 (informal deduction). The software (Windows TM Geometry) "is composed of online self-help tutorials to guide the geometry student through over 55 subject areas using over 700 practice problems" (Parsons et al., 1998, p. 82). The study employed a one-group pre-test/post-test quasi-experimental design. Participants completed three tests before and after the experiment. The tests were categorized as: Van Hiele Geometry Test, Entering Geometry Test, and Geometry Vocabulary Test. Pre-test results indicated that two students were below Level 0 (visualization), five were at Level 0, and four were at Level 1 (analysis). A one-tailed t-test for non-independent samples was performed to compare the students' pre-test/post-test scores on the Van Hiele Geometry Test. The results showed a statistically significant increase in students' scores. Most of the students had moved to Level 2 (informal deduction). It was concluded that the treatment (Windows TM Geometry) had a positive impact on students' levels of geometric thinking.

However, further analysis of students' pre-test/post-test scores on the Entering Geometry Test (which excluded proofs), and the Geometry Vocabulary Test using the two-tailed t-test showed a non-significant outcome. It was concluded that computer-assisted instruction was not effective in developing the terminology of geometry. These findings support the view that even though technology is widely known to enhance teaching and learning, it has its own limitations and therefore should be used to complement and not to replace the teacher. It is the teacher's responsibility to help students use appropriate geometric language (Howse & Howse, 2015; Van Hiele, 1984). Parsons et al.'s (1998) study has been criticized for using a small sample ($n = 11$). The fact that there was no control group also makes the results weak and unreliable. The study could therefore be replicated with a larger sample and a control group.

Liu (2005) tested the effectiveness of Van Hiele-based instruction in learning one Euclidean geometry theorem which states that: the angle subtended by an arc at the centre of a circle is twice the size of the angle subtended by the same arc at the circumference. The study employed a pre-test/post-test quasi-experimental design with a sample of 132 Form 3 (Grade 10) students from a Chinese school in Hong Kong. Sixty-five students from two classes made up the experimental group, and 67 students from the other two classes represented the control group. The Van Hiele Geometry Test (a multiple-choice test developed by Usiskin ,1982) was used as a pre-test.

The experimental group students were taught by the researcher. The treatment comprised four consecutive lessons, each lasting 50 minutes. Teaching materials in the form of worksheets on concepts related to the circle were developed using the first three levels of the Van Hiele model. Teaching was organized according to the Van Hiele teaching phases. In the initial stage, students were asked to sort circle diagrams based on similar characteristics. In the second stage, students used their own words to name angles in given circles. In stage 3, students were asked to measure two angles in a given diagram and establish the relationship between them. In the fourth stage, students gave feedback on their findings. The teacher assisted the students by introducing the relevant terminology. In stage 5, students worked on open-ended questions. In stage 6, the teacher used a proof method to consolidate students' understanding of the theorem. In the last stage (integration), students reviewed and summarized what they had learnt in previous stages.

The control group students were taught by another teacher using the traditional method. The teacher distributed worksheets on concepts relating to circles. The teacher then directly presented the theorem to the students, and told them to memorize it. The teacher then demonstrated the proof of the theorem from the textbook to help students see why it was true. A few examples from the textbook were given to demonstrate how the theorem is applied, and students copied the examples into their notebooks for future reference. Students were then assigned questions for practice in class and at home. Lessons were presented "according to the textbook without any teacher's personal and private instructional method” (Liu, 2005, p. 25).

The Van Hiele Geometry Test was used to assess students’ post-treatment levels of geometric thinking. This was coupled with a paper-and- pencil written quiz consisting of 11 questions marked out of 50. The written quiz comprised items that focused on the learnt theorem of the circle. Analysis of post-test scores showed a non-significant difference between the experimental and control groups’ Van Hiele levels. This came as no surprise because the treatments given to both groups were not based on the geometry concepts examined in the Van Hiele Geometry Test. Data from the written quiz showed that, on average, the treatment group scored higher than the control group. However, the fact that the participants were drawn from the same school is a threat to the validity of the findings due to the possibility of contamination. The study could therefore be replicated with the experimental and control groups from separate schools.

Meng (2009) investigated whether Van Hiele phase-based instruction changed students’ Van Hiele levels in learning Solid Geometry, focusing on cubes and cuboids. The study utilized the case study research design with a purposive sample of six Form One (Grade 8) students from a secondary school in Malaysia. Students were interviewed prior to treatment to assess their initial Van Hiele levels. The students’ initial Van Hiele levels ranged from Level 0 to Level 2. The students were then taught about the properties of the solid shapes using Van Hieles’ phase-based instruction together with the Geometer’s Sketchpad (GSP). The treatment involved seven sessions, each lasting 40 minutes. Fourteen GSP activities were developed and used in the lessons. A post-interview was conducted to assess change in the students’ Van Hiele levels after treatment. Results showed an increase in some students’ Van Hiele levels and no change in others. However, these findings cannot be reliable due to the small number of participants $(n = 6)$ and the absence of a control group. The study could therefore be replicated with a larger sample and a control group.

Abdullah and Zakaria (2013), conducted a quasi-experiment to assess the effect of Van Hiele phase-based learning on Form Two (Grade 9) students’ achievement in learning about the properties of quadrilaterals. The study involved two teachers and ninety-four students from a school in Malaysia. The students were split equally into two groups. One group constituted the experimental group and the other group formed the control group. The Van Hiele Geometry Test was administered to students in the two groups before and after the treatment. The treatment for the experimental group involved sequencing learning activities according to the Van Hiele phases of learning, and exploring the properties of quadrilaterals and their relationships using the GSP. The control group students were taught using traditional methods. An analysis of results using Wilcoxon t-test showed a significant difference between the two groups' post-treatment scores on the Van Hiele Geometry Test.

Interviews were conducted to further explore the nature of the differences in the Van Hiele levels of the students. An analysis of interview data revealed that before the treatment, both groups showed complete acquisition of Level 0 (visualization), low acquisition of Level 1 (analysis), and no acquisition of Level 2 (informal deduction) (Abdullah & Zakaria, 2013). After the treatment, most of the students in the control group only improved from Level 0 to Level 1. None of them acquired Level 2 (informal deduction). In the experimental group, only one student did not achieve the informal deduction level. The rest of the students showed complete mastery of Levels 0, 1 and 2. It was therefore concluded that Van Hiele's phase-based learning could be utilized to improve students' levels of geometric thinking.

Siew, Chong, and Abdullah (2013) implemented Van Hiele's phase-based learning with 221 Grade 3 students from a primary school in Malaysia. The study employed a one-group pre-test/post-test quasi-experimental design. The focus of the study was on learning two-dimensional shapes and symmetry. As part of the treatment, the students worked with tangrams for a period of three hours. The Chinese tangram is a puzzle that consist of seven pieces of geometric shapes: "a square, a parallelogram, two big right triangles, a medium sized right triangle and two small right triangles" (Siew et al., 2013, p. 102). A multiple-choice geometry test on two-dimensional shapes and symmetry was administered to the Grade 3 students before and after the experiment. Data were analysed using a paired samples t-test and results indicated that there was a statistically significant difference in the average pre-test and post-test scores. The students' Van Hiele levels improved from visualization (Level 0) to analysis (Level 1). Further analysis of pre-test and post-test average scores using multivariate analysis of variance (MANOVA) revealed that below-average students showed the greatest improvement compared to the average and above-average students. However, the study also suffers from internal validity issues. The absence of a control group means that maturation and history effects could possibly have influenced the outcome.

In South Africa, Alex and Mammen (2016) implemented van Hiele theory-based instruction in Grade 10, focusing on geometry concepts related to triangles and quadrilaterals. The study employed a pre-test/post-test quasi-experimental design with a sample of 359 Grade 10 students from five schools in Mthatha district, Eastern Cape Province. A total of 195 Grade 10 students formed the experimental group and the remaining 164 constituted the control group. Van Hiele theory-based instruction was implemented in the experimental group while conventional methods were being using in the control group. A multiple-choice test was administered to the experimental and control groups before and after five weeks of teaching. An analysis of results using the paired-samples *t*-test indicated a statistically significant difference in the mean scores of the two groups, in favour of the experimental group. There were more students in the experimental group than in the control group, who had moved from visualization (Level 0) to analysis (Level 1) and informal deduction (Level 2). It was concluded that van Hiele-based instruction was more effective than traditional methods in learning geometry concepts related to triangles and quadrilaterals.

Other studies that have implemented the Van Hiele phase-based instruction with control groups include those by Tay (2003) (Malaysia), Shi-Pui and Ka-Luen (2009) (China). Tay (2003) implemented the Van Hiele phase-based instruction in Form One, using manipulative materials. Shi-Pui and Ka-Luen (2009) used the Van Hiele-based instruction in the learning of Solid Geometry. In both studies, students who were taught using Van Hiele-based instruction performed better than those who received regular instruction.

Based on the preceding review, Van Hiele-based instruction integrated with the use of manipulative materials seems to give students better opportunities to learn geometry concepts than conventional instruction. The review of available literature shows that the effectiveness of Van Hiele-based instruction has been tested in the teaching and learning of the following geometry aspects: properties of triangles and quadrilaterals (Abdullah & Zakaria, 2013; Alex & Mammen, 2016), two-dimensional shapes and symmetry (Siew et al., 2013), solid geometry

(Meng, 2009; Shi-Pui & Ka-Luen, 2009), and one circle geometry theorem (Liu, 2005). Most of these studies sought to develop students' geometric thinking at primary and junior secondary school levels, except studies by Liu (2005), and Alex and Mammen (2016) that involved Grade 10 students. Thus, much attention has been given to the development of visualization (Level 0), analysis (Level 1) and informal deduction (Level 2).

The present study adds to previous findings by implementing Van Hiele theory-based instruction in the teaching and learning of deductive reasoning and proof at Grade 11 level, focusing on non-routine geometric proofs (riders).

2.10 Chapter summary and conclusion

There is consensus among mathematics education experts in the twenty-first century that the aspect of deductive reasoning and proof should remain part of the mathematics curriculum that is taught in secondary schools because of its multiple functions. However, knowledge of how to design and organize instruction to effectively teach geometric proofs in diverse learning environments with students of various cultural backgrounds is scarce. The history of the teaching of Euclidean geometry from the Greek Era to the twenty-first century indicates that teaching deductive reasoning and proof has been a daunting task for mathematics teachers. The nature of Euclidean geometry proof itself has developed from basic constructions and empirical demonstrations to writing a series of deductive reasoning steps justified by Euclidean theorems, converses, axioms, and definitions. How these theorems, converses, axioms, and definitions are presented to students is likely to determine their chances of success in writing the Euclidean proofs.

Teachers, both seasoned and inexperienced, have acknowledged that they do not have sufficient pedagogical expertise to improve the reasoning and proof skills of the students in Euclidean geometry (see Olivier, 2013, 2014). This has resulted in the continued use of conventional instruction in the teaching and learning of Euclidean theorems and proofs. Students are left with no choice but to memorize theorems, converses, axioms, and definitions, and reproduce these facts in tests and examinations without proper understanding. Many secondary school students have developed a negative attitude towards Euclidean geometry and proof because they do not understand the topic. An analysis of students' mathematics performance in school leaving examinations in various countries shows that many candidates are not even attempting questions on non-routine geometric proofs (see Department of Basic Education, 2015; Mwadzaangati, 2015, 2019; West African Examination Council, 2009, 2010, 2011).

There have been complaints from universities that students leave secondary school with weak reasoning and spatial skills, making it difficult for them to understand university mathematics. There is therefore a serious need for mathematics teachers at lower levels to do things differently. While it is important for primary school teachers to ensure that students acquire the basics of Euclidean geometry before they go to secondary schools, it is equally important for secondary school mathematics teachers to ensure that students go to college or university with adequate reasoning and deductive skills. Teachers at both primary and secondary school levels need to implement new teaching approaches to replace those that promote rote learning (Jones & Rodd, 2001).

Cheng and Lin (2006, 2009) developed two strategies to improve Grade 9 students' geometric proof competencies in Taiwan. The reading and colouring strategy was found to work best with incomplete provers, and the step-by-step unrolled strategy was effective in solving computational proofs. The heuristic worked-out examples developed by Reiss et al. (2008) improved the geometric proof competencies of below-average Grade 8 students in Germany. However, this does not necessarily mean that the same results will be obtained when these strategies are applied elsewhere due to differences in learning environments and cultural backgrounds. It is also not known whether these strategies work across all grade levels.

One of the limitations of the strategies developed by Cheng and Lin (2006, 2009), and Reiss et al. (2008), is that they used students from the same school and that makes their findings unreliable. In addition, the strategies they developed benefited some students, while others were left out. Contemporary perspectives on mathematics education support an inclusive approach to teaching that leaves no child behind. How to develop such teaching approaches is still open to further inquiry.

According to the Van Hiele model of geometric thought, students who go to Grade 11 are expected to have achieved the level of informal deduction for them to be ready for formal deduction (abstract reasoning and proof). Studies that have assessed students' Van Hiele levels have widely reported that students across all grade levels are operating at lower-than-expected Van Hiele levels (see Alex & Mammen, 2016; Abdullah & Zakaria, 2013; Atebe, 2008; De Villiers, 2010; Feza & Webb, 2005). That is, students go to Grade 11 not having mastered the informal deduction level, a prerequisite for success in learning proofs. This makes it difficult for the teachers to teach formal deduction. The traditional approach to teaching Euclidean theorems and proofs has been found to ineffective (Abakpa & Iji, 2011) and alternative teaching approaches are required.

Several researchers suggest that the teaching of proof should be preceded by classroom activities in which students explore, observe, formulate, and test conjectures (Blanton et al., 2009; Cassim, 2006; Harel & Fuller, 2009) to discover for themselves the origins of concepts and theorems (Kutama, 2002). This is in line with constructivist learning theories, which propose that students should be given the opportunity to develop their own knowledge by engaging with teaching and learning resources, their peers, and their teachers (Abdelfatah, 2010). However, available Grade 11 Mathematics textbooks and policy documents do not provide guidance to teachers on how to organize teaching and learning activities for better understanding of Euclidean geometry proofs by all the students in the mathematics class. Subject advisors, who should be the experts in the field, have been found to lack the necessary expertise to provide the guidance that the teachers require to improve their teaching. Even the geometry modules offered to pre-service teachers at university have been found to be insufficient in preparing the teachers for the challenges of the classroom (see Van Putten et al., 2010). The situation is frustrating to both the teachers and the students.

The present study seeks to resolve the challenge faced by Grade 11 mathematics teachers teaching Euclidean geometry proofs to students who operate at lower-than- expected Van Hiele levels. The next section presents the methods used in this research.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

The main objective of any research is to answer questions. The methodology of the study is therefore chosen based on the research questions (Plano Clark & Badiee, 2010). This chapter presents the methodology employed to address the research questions formulated in Chapter 1 (see section 1.3). The research design that was utilized in the study is described, and the justification for the chosen research design is given. The sampling procedures that were used in the study are outlined, and how data collection instruments were developed is explained. An account is given of how issues of the reliability and validity of data collection instruments were dealt with. Data collection and analysis procedures are discussed and ethical issues are addressed.

Prior to the discussion on the research design and methods used in the study, it is advisable that researchers should declare their philosophical assumptions (see Mackenzie & Knipe, 2006). The discussion that follows outlines the study's philosophical assumptions.

3.2 Research paradigm

In addition to the nature of the research questions, the choice of research methodology is also influenced by the individual researcher's ontological and epistemological assumptions (Lindsay, 2010). The term ontology refers to a person's view of the nature of knowledge, and epistemology is concerned with how knowledge is acquired (Scotland, 2012). The research paradigm defines the ontological and epistemological assumptions upon which a study is premised (Mertens, 2005; Neuman, 2006). It influences what we study, and how we study and interpret the research findings (see Mollard, 2014).

Human and social science research in the twentieth century was largely influenced by positivism, a research paradigm which underpins quantitative methodology (Tuli, 2010). Proponents of positivism contend that knowledge is that which can be objectively measured through empirical observations and manipulation of individual behaviour in an experiment. The purpose of research in the positivist paradigm is to test hypotheses in order to establish universal laws "to predict general patterns of human activity" (Tuli, 2010, p. 100). Human and social science researchers working from this perspective may collect data using quasi-experiments and cognitive tests (Mackenzie & Knipe, 2006). Admittedly, mathematics education requires rigorous inquiry to yield valid, reliable, and generalizable knowledge. However, research in the twenty-first century acknowledges that reality in social and human science research cannot be established using quantitative methodology alone (McGregor & Murnane, 2010).

Experiments involving human beings differ from laboratory-based experiments with chemicals and inanimate objects. Human beings have attitudes and feelings that cannot be quantitatively measured, yet are important in understanding social phenomena (Mertens, 2010). In their attempts to develop better approaches to teaching mathematics, reseachers should not ignore the student's voice. By paying attention to students' views, mathematics teachers could receive critical feedback to enhance students' achievement (Bansilal, James, & Naidoo, 2010). It is for this reason that the qualitative/constructivist methodology was incorporated into the present study to capture the voices and concerns of the students.

Both quantitative and qualitative research methodologies have strengths and weaknesses. Quantitative methods are well suited to measuring the magnitude of effect but cannot address the 'why' and 'how' questions (Buckley, 2015). Qualitative methods are helpful in explaining why and how certain factors in the teaching experiment or treatment administered to the participants contributed to the change in the observed phenomena (Buckley, 2015). However, findings from qualitative research cannot be generalized beyond the local participants since the results are normally based on small non-random samples (Johnson & Onwuegbuzie, 2004). Positivist and constructivist research methods are clearly complementary. For this reason, the present research adopted the pragmatist paradigm in a mixed-methods design to offset the limitations of mono-methods (see Ross & Onwuegbuzie (2012).

3.2.1 Pragmatism

Pragmatism is a philosophical movement that originated in the United States of America in the nineteenth century (West, 2012). It emerged as a third philosophical movement that sought to end the paradigm wars between positivists and constructivists.

The pragmatic approach to research advocates the use of any methods, techniques, and procedures from both positivist and constructivist methodologies in a single study, if it suits the research problem (Feilzer, 2010). The pragmatist knowledge claims are drawn from both the objective world of the positivists and the subjective world of constructivists. Thus, pragmatists' ontological assumptions are non-dualist. All that matters is what works best to address the purpose of the study (Tashakkori & Teddlie, 2009). Most importantly, the research questions are the driving force in the choice of research methods (Onwuegbuzie & Leech, 2005; Saunders, Lewis, & Thornhill, 2012).

The research questions of this research warranted the use of a sequential explanatory mixed-methods research design. The discussion that follows elaborates on this design.

3.3 The sequential explanatory mixed-methods research design

The sequential explanatory mixed-methods research design is a product of the pragmatist research philosophy. It integrates quantitative and qualitative methods at different phases of the study (Terrell, 2012). The sequential explanatory mixed-methods design employed in this research was made up of two phases. In the initial phase, quantitative data were collected using a quasi-experiment to objectively test the effect Van Hiele theory-based instruction on students' geometric proofs learning achievement. In the second phase, qualitative data were collected to explore students' views on the approaches used in the experimental and control groups. The results from the two phases were linked in the discussion chapter, which led to the development of a framework for better teaching and learning of Grade 11 Euclidean theorems and proofs.

Research in mathematics education has been criticized for piling up statistical data in the form of averages, standard deviations and *t*-tests, leaving many vital questions unanswered (Ross & Onwuegbuzie, 2012). It is therefore not surprising that there are increasing calls to incorporate qualitative data in mathematics education research to augment statistical analyses (Ross &

Onwuegbuzie, 2012). By utilizing the sequential explanatory mixed-methods design, this study used the qualitative data to further explore the findings from the quasi-experiment, as recommended by Smith (2012). The qualitative data helped to explain why the variables in the quasi-experiment were significant or non-significant predictors of students' achievement (Ivankova, Creswell, & Stick, 2006). It enhanced the validity of the findings obtained in the quasi-experiment (see Shah & Al-Bargi, 2013). It also helped to uncover additional information which would not be possible using quantitative methods alone (Creswell, 2009).

The dominant less-dominant QUANTITATIVE→qualitative (QUAN→qual) sequential explanatory mixed-methods design was utilized. A visual model of QUAN→qual sequential explanatory design is presented in Figure 3.1. The QUAN is capitalized to show that it was given more weight than the qual (Harwell, 2011):

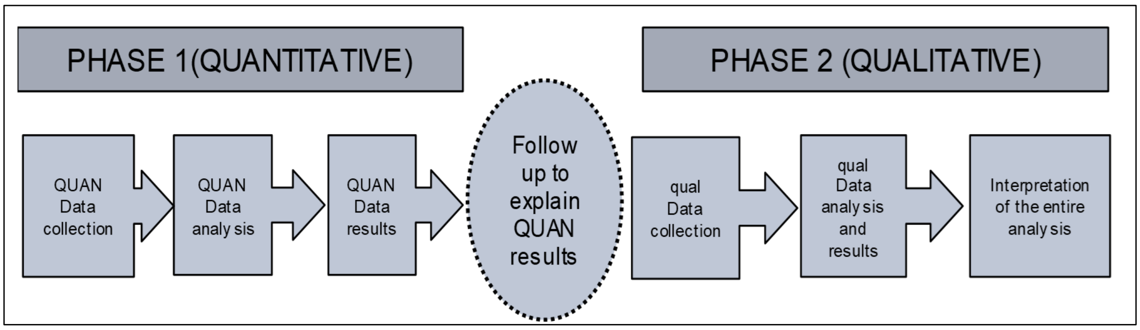


Figure 3.1: Sequential explanatory mixed methods design

Source: Creswell and Plano Clark (2007, p. 73)

3.3.1 Quantitative phase: non-equivalent groups quasi-experiment

It is unacceptable and unethical to randomly assign school children to experimental conditions (Fife-Schaw, 2012). For this reason, the study utilized the non-equivalent groups quasi-experimental design which utilizes intact groups of students. The quasi-experiment is not a true experiment in the sense that participants are not randomly assigned to the treatment and control conditions (Harris, et al., 2006). However, it is similar to a true experiment because it is used to determine if the treatment has an effect on the variables of interest. The term 'non-equivalent' simply means that the groups are likely to differ in some ways due to the presence of confounding variables. The visual model of the non-equivalent groups design is shown in Figure 3.2:

NR	(Experimental)	O_1	X	O_2
NR	(Control)	O_1		O_2

Notes. NR = Non-random assignment; O = Observation/measurement; X = treatment

Figure 3.2: Non-equivalent groups design

The symbol O_1 denotes the pre-test and O_2 denotes the post-test. The non-random assignment of participants into the treatment and control groups comes with other variables besides the treatment, which may influence the results of the experiment. Such variables are called confounders. The pre-test (O_1) was used to assess differences in performance between the treatment and control groups prior to administering treatment (X). A significant difference in the average pre-test scores between the two groups is a potential confounder (Gliner, Morgan, & Leech, 2009) which poses a threat to the internal validity of the research findings (Bell, 2010). The pre-test scores were therefore used as a covariate in the statistical analysis of results. Other measures that were taken to minimize the influence of confounding variables will be explained in section 3.4.4.

To address the first research question, the following hypotheses were tested at the 5% level of significance:

Null hypothesis ($\mathbf{H_0}$): Van Hiele theory-based instruction has no significant effect on Grade 11 students' geometric proofs learning achievement.

$$H_0: \mu_{van\ Hiele\ theory\text{-}based\ instruction} = \mu_{conventional\ instruction}$$

Alternative hypothesis $(\mathbf{H_1})$: Van Hiele theory-based instruction has a statistically significant effect on Grade 11 students' geometric proofs learning achievement.

$$H_1: \mu_{van\ Hiele\ theory\ \text{-}based\ instruction} \neq \mu_{conventional\ \ instruction}$$

3.3.2 Qualitative phase

The qualitative phase sought to gather data to answer research questions concerning students' views, attitudes, and feelings about (a) Van Hiele theory-based instruction and (b) conventional teaching. Diaries and focus groups were utilized. The qualitative data helped to build a holistic snapshot of the research topic.

3.3.2.1 Diaries

Diaries can be defined as documents created and maintained by individual people who record events, thoughts, feelings, attitudes and views on personal observations and experiences over time (Bytheway, 2012; Duke, 2012). Some researchers call them journals (Yi, 2008). While the researcher could have used lesson observation instead of diaries, it was not possible for the teacher/researcher to observe Euclidean geometry lessons in the control group schools and at the same time implement the proposed treatment in the experimental group. Besides, observing the same teachers daily for a longer period for purposes of research is burdensome and may cause discomfort to the teachers. It also has the potential to bias findings (Sheble & Wildemuth, 2009). For these reasons, students' diaries were found to be a suitable replacement for lengthy researcher observations (see Duke, 2012).

Data collected using the diary method is likely to be accurate because it is captured at or shortly after the occurrence of the event (Woll, 2013). Diaries have less recall errors compared to questionnaires and interviews that capture events long after they have occurred (Sheble & Wildemuth, 2009). In education, students' diaries are a source of valuable information that teachers may use to design effective lessons (Yi, 2008). However, diaries should not be used as the only data collection method in research (Woll, 2013). They should be combined with other data collection methods such as interviews in order to enrich the research findings or as a form of triangulation (Sheble & Wildemuth, 2009).

3.3.2.2 Focus groups

Focus groups are small groups of participants that are brought together by a trained facilitator or interviewer to discuss a topic (see Baral, Uprety, & Lamichhane, 2016; Bedford & Burgess, 2001; Wong, 2008). The environment in which the discussions are held should be "non-threatening" (Krueger & Casey, 2009, p. 2) to allow participants to express themselves freely without fear. There is variability in what researchers recommend as the optimum size for a focus group. The size of the group should therefore be decided by the individual researcher. What is important is to maximize participation and to reach data saturation. However, Adler and Clark (2008), recommend groups ranging from three to twelve participants.

Focus groups can be used to explore participants’ experiences, attitudes, perceptions, beliefs, opinions, and ideas on a given topic (Denscombe, 2007; Dilshad & Latif, 2013; Knight, 2012; Leung & Ratnapalan, 2009; Pearson & Vossler, 2016; Villard, 2003; Wong, 2008). In a mixed-methods design, focus groups help to shed more light on issues that emerged in the quantitative data analysis (Centers for Disease Control and Prevention, 2008; Freitas, Oliveira, & Jenkins, 1998). Focus group findings can also be used as supplementary data “to validate the findings of quantitative research” (Dilshad & Latif, 2013, p. 193). Knight (2012) concurs with Villard (2003) that focus groups are most productive when used to evaluate the success of teaching experiments. In education, focus group can help teachers to evaluate the effectiveness of their teaching. Such knowledge is key to the design, redesign and refinement of our pedagogical practices (see Office of Quality Improvement, 1999).

The advantages of the focus group technique are that: it is relatively inexpensive compared to individual interviews, and it allows the researcher to collect a large amount of information within a short timeframe (Baral et al., 2016; Krueger & Casey, 2009; Freitas et al., 1998). The focus group technique has high face validity (Pearson & Vossler, 2016) and participants are more likely to give honest responses (Leung & Ratnapalan, 2009). Unlike multiple personal interviews, focus groups provide opportunities for interactions among participants, which creates more valuable data. However, despite these various benefits, the focus group data collection technique has several noteworthy limitations.

The focus group discussion may be hijacked by outspoken individuals who have the potential to sway and supress important individual opinions (Health Promotion Unit, 2007; Leung & Ratnapalan, 2009; Wong, 2008). Some participants may talk over each other, thereby making it difficult to transcribe the data (Wilkinson, 2008). If a bigger number of participants are engaged, it may reduce participation opportunities for some members (Pearson & Vossler, 2016) and increase the danger of participants breaching confidentiality agreements (McParland & Flowers, 2012). In addition, focus group findings are not transferable to the larger population since participants are non-randomly selected (Leung & Ratnapalan, 2009).

Despite the disadvantages enumerated in the preceding discussion, focus groups were favoured ahead of other survey methods such as questionnaires and individual interviews. This is because focus groups can produce an appropriate amount of rich and valuable information using fewer resources (in terms of time and money) than multiple individual interviews and questionnaires (Office of Quality Improvement, 1999). Focus group interviews were used to explore students' views on Van Hiele theory-based instruction and conventional approaches to teaching and learning Grade 11 Euclidean geometry theorems and proofs. The intention was to supplement the findings obtained in the quantitative phase. It was assumed that high school students are old enough to engage in thoughtful analysis "especially on matters that clearly affect them" (Jackson & Davis, 2000, p. 145). Whitney (2005) adds that students "are experts about their schools and have definite opinions about what works well and what could be improved" (p. 3).

In the next section, the sampling procedures followed in this research are elaborated on.

3.4 Sampling

Since it is not always feasible for the researcher to include all units of the population of interest, it is recommended that the researcher should select some units of the larger population to participate in the study.

3.4.1 The target population

The larger group of people to which findings may be generalized is the target population (Fritz & Morgan, 2010). The targeted population in this research was the cohort of 2016 Grade 11 Mathematics students who were attending school in the townships of Limpopo province, South Africa. The researcher chose Grade 11 because this is the level at which most of the Euclidean geometry theorems and challenging riders are introduced, based on the South African CAPS (see Department of Basic Education, 2011). Limpopo was targeted because of having a consistent record of underperformance in the Grade 12 NSC examinations (see Department of Basic Education, 2016a). Township secondary schools were of interest to the researcher because this is where most students attend school. These schools have adequate classrooms, well-furnished computer laboratories and libraries, adequate teaching and learning resources, electricity, and water supply, but they still perform far below provincial and national targets (see Dhlamini, 2012).

3.4.2 The sampling frame

The sampling frame is part of the targeted group that contains the units the researcher can choose from (Luks & Bailey, 2011). The 2016 Grade 11 Mathematics students and teachers from Mankweng and Seshego township schools in the Capricorn district of Limpopo province in South Africa, constituted the sampling frame for this research. The townships of Mankweng and Seshego were targeted because of their proximity to the City of Polokwane (provincial capital of the province of Limpopo), which makes them easily accessible.

3.4.3 The study sample

The research sample consisted of 186 Grade 11 Mathematics students from four public secondary schools. Of these, 82 students from two secondary schools in Seshego township constituted the experimental group. The remaining 104 students from two secondary schools in Mankweng township formed the control group. A total of twenty-four students (6 per school) were recruited to keep diaries and participate in focus group discussions.

3.4.4 Sampling techniques

Schools were selected using the convenience sampling technique. Thus, selection was non-random and based on what was readily available (Fritz & Morgan, 2010). This is a common practice in educational research and other naturalistic studies (Dhlamini, 2012). The two schools from Mankweng township were matched with two similar schools from Seshego township using the Grade 12 Mathematics results for 2015 and the 2016 Schools' Master List data available on the South African Department of Basic Education Website (see Department of Basic Education, 2016b). Two Grade 11 mathematics students, one from Mankweng township and the other from Seshego township, were invited to a neutral venue to assign the schools to experimental and control groups. Using the lottery method, Mankweng schools were assigned to the control group while the Seshego schools were assigned to the experimental group.

The selected secondary schools were public no-fee schools in two townships of the same district. The four schools were categorized as Quintile 3 schools according to the government funding system for South African schools. The quintile funding system as described in the 2000 National Norms and Standards for School Funding (NNSSF) divides schools into five categories (quintiles), according to their socio-economic status (SES) or levels of poverty in the communities around them (Dieltiens & Motala, 2014). The quintile system ensures that the most disadvantaged schools receive the biggest share of the NNSSF. The communities' poverty scores are calculated using data compiled by Statistics South Africa (StatsSA). Quintiles 1 to 3 schools are in low SES areas characterized by low income, low education levels and a high unemployment rate and are regarded as the neediest schools. Such schools were "declared no-fee schools as of 2014" (Longueira, 2016, p. 48). Quintiles 4 and 5 schools are in high SES areas characterized by high income, high education levels and a low unemployment rate. These are fee-paying schools. Accordingly, schools in Quintiles 1 to 3 receive more NNSSF than their counterparts in Quintiles 4 and 5. Schools in the same quintile receive the same NNSSF allocation per student and thus are treated equally. The norms and standards funds are used by schools to buy learner and teacher support materials (LTSM) and pay for other utility bills.

In addition to being in similar low socio-economic environments and receiving the same NNSSF allocation per student, the selected schools were also part of the National School Nutrition Programme (NSNP), which targets Quintiles 1 to 3. The NSNP is an intervention programme that aims to reduce the impact of poverty on educational attainment (Rendall-Mkosi, Wenhold, & Sibanda, 2013). The schools had large student enrolments, ranging from 704 to 1265. All four selected schools had access to safe drinking water, electricity, a library, and a computer laboratory with a fixed projector.

An equally significant common characteristic of the selected schools was that the most experienced mathematics educators (college-trained) were responsible for teaching mathematics in the upper classes, while the newly qualified mathematics teachers (university-trained) were entrusted with lower grade mathematics. The schools from Mankweng were coded C1 and C2. These were the control group schools. In the same way, the schools from Seshego were coded E1 and E2. These were the experimental group schools. The townships are approximately one hundred and seventy-seven kilometres apart. Thus, chances of data contamination were reduced. Data contamination could have occurred if students from the two townships shared notes and learning experiences (see Hutchison & Styles, 2010). This could lead to a wrong conclusion in the testing of hypotheses (Keogh-Brown et al., 2007).

Teacher X at school C1 and Teacher Y at school C2 were both college-trained, with their highest teaching qualification being a Secondary Teacher's Diploma (STD). The teachers indicated that mathematics was one of their major subjects at college. The two teachers were permanently employed by the Limpopo Provincial Department of Basic Education. Both teachers had more than 15 years of experience teaching mathematics. Both teachers had taught Euclidean geometry in the old syllabus until it was scrapped out of the NCS in 2006. The two teachers participated in the Mathematics CAPS training workshops organized by the Limpopo Department of Basic Education in 2012. These teachers taught the 2014 and 2015 cohorts of Grade 12 students in their respective schools. Grade 12 students in 2014 and 2015 were the first two groups to write a Mathematics Paper 2 national examination which included Euclidean geometry in the CAPS. Their respective schools (School C1 and School C2) reported a Mathematics pass rate of less than 50 percent in the 2014 and 2015 NSC examinations (see section 4.2.2 of Chapter 4).

Self-selection, a type of convenience sampling method in which students volunteered to keep diaries and participate in focus group discussions, was used. It was assumed that self-selected participants would have a greater commitment and willingness to participate in the study than those recruited by persuasion. White (2006) argued that self-selected individuals "will be highly motivated and have strong opinions on the topic" (p. 188). The self-selection sampling technique also helped to avoid the potential risk of non-attendance and zero participation. Students at each school were informed about the topic of discussion, the time, and the venue for the focus group meetings (Office of Quality Improvement, 1999). Students were also told that they would be expected to keep diaries and record their learning experiences for the duration of the Euclidean geometry topic. Twenty-four students, six from each school, were recruited, resulting in four focus groups. Each focus group had three males and three females to ensure inclusiveness and gender balance. Two focus groups were made up of students from control group schools, and the other two groups were formed by students from experimental group schools. Thus, each school had its own focus group. Based on the advice given by Breen (2006), focus groups that are used to explore learning experiences should be made up of students who have had similar learning experiences.

3.5 Instrumentation

A geometry proof test, diaries, and focus group discussion guide were used to collect data in this research.

3.5.1 Geometry proof test

A geometry proof test which consisted of four long and open-ended proof questions was developed by the researcher to measure students' geometric proofs learning achievement before and after the teaching experiment. Each question was split into two, three or four parts to cater for the multiple intelligences of the students (see Appendix I). Selection of test items was informed by (1) the focus of the study, (2) Bloom's taxonomy, and (3) the South African Mathematics CAPS for the FET Band. The focus of this research was on proving riders and as such, all questions were proof questions. In keeping with Bloom's taxonomy, the proof questions set required students to recall geometric facts (Knowledge), demonstrate understanding (Comprehension), use problem-solving skills (Application), identify patterns, and organize ideas (Analysis), combine ideas (Synthesis), and make judgments (Evaluation). Questions were confined to the theorems and axioms prescribed for Grade 11 students in the CAPS. However, some questions required students to apply knowledge acquired in lower grades. The proof questions could be solved in multiple ways.

3.5.1.1 Validity and reliability of the geometry proof test

The validity of a test is the extent to which it measures exactly what it seeks to measure (see Heale & Twycross, 2015). A test is reliable or consistent if the same or similar results are obtainable when the test is re-administered to the same participants under the same conditions.

The initial draft of the geometry proof test had twelve proof items. The total mark allocation was 60 and the duration allowed was one hour. To ensure that the geometry proof test was valid, it was first developed using the guidelines outlined in section 3.5.1. The initial draft of the test was reviewed by the researcher's peers who were Grade 11 Mathematics teachers. The initial draft of the test was then revised based on the peer review comments. The revised test was emailed to a purposive sample of seven mathematics experts for validation. According to Zamanzadeh et al. (2015), at least five raters are recommended to avoid agreement due to chance. The sample consisted of four professors and three doctors of mathematics education from various universities in South Africa. These were identified by viewing their profiles from the universities' websites.

The seven mathematics experts were requested to judge the relevance and clarity of the proof items using a 4-point ordinal scale. The criteria that guided the experts in scoring the test items were adapted from Yaghmaie (2003) and Zamanzadeh et al. (2015) (see Table 3.1):

Table 3.1: Criteria for rating test items

Relevance
1= not relevant
2= item needs some revision
3= relevant but needs minor revision
4= very relevant
Clarity
1= not clear
2= item needs some revision
3= clear but needs minor revision
4= very clear

Note. Adapted from Yaghmaie (2003, p. 26) and Zamanzadeh et al. (2015, p. 168)

A test validation instrument developed by the researcher using the criteria in Table 3.1 was sent to the raters together with the geometry proof test (see Appendix J). The average rating score for each item was obtained by adding the two scores (relevance score + clarity score) and dividing the result by two. The resultant average scores per item per rater were recorded as shown in Table 3.2. The widely used technique for calculating item content validity index (I-CVI) using multi-rater agreement simply divides the number of raters who scored $3's$ and $4's$ for an item by the total number of raters in the panel (see Waltz & Bausell ,1983). This would give a CVI of 1.00 for all test items in Table 3.2 except 4.1 (0.86).

Table 3.2: Experts' final average rating scores per item

Item	Expert raters and ratings						
	1	2	3	4	5	6	7
1.0	4	4	4	4	4	4	4
2.1	4	4	4	4	4	4	4
2.2	4	4	4	4	4	4	4
2.3	4	4	4	4	3	4	3
2.4	4	4	4	4	4	4	4
3.1	4	4	3	4	4	4	4
3.2	4	4	3	4	4	4	4
3.3	4	4	3	4	4	4	4
4.1	4	1	4	4	4	4	4
4.2	4	4	4	4	4	4	4
4.3	4	3	4	4	4	4	4
4.4	3	4	4	4	4	4	4

Notes. 4 = very relevant and very clear; 3 = relevant and clear; 2 = item needs some revision; 1 = irrelevant

It is important to note, however, that techniques for calculating CVI by dichotomizing ratings are criticized for inflating the CVI values due to their failure to control for chance agreement (see Polit, Beck, & Owen, 2007). In view of this criticism, the researcher opted to use a modified kappa statistic (k^*) which adjusts each item content validity index for chance agreement. To obtain k^* for each test item, the probability of chance agreement (p_c) was calculated using the formula:

$$p_c = \left[\frac{N!}{A!\,(N-A)!} \right] (0{,}5)^N$$

where p_c: probability of chance agreement; N: number of raters; A: number of raters who gave the item a rating of either 3 or 4.

The modified kappa statistic (k^*) for each item was then computed using the formula:

$$k^* = \frac{[I\text{–}CVI] - p_c}{1 - p_c}$$

where k^*: modified kappa value; $I\text{–}CVI$: item content validity index;

p_c: probability of chance agreement

The item content validity indices $(I - CVI)$ and modified kappa values (k^*) obtained for each item are shown in Table 3.3. The overall content validity index of the test is the scale-level content validity index $(S - CVI)$. This was obtained by calculating the average of the item modified kappa values (Polit et al., 2007). The overall content validity index of the test instrument $(S - CVI)$ was 0.99 (see Table 3.3), which is greater than the least acceptable standard of 0.9 (see Waltz, Strickland & Lenz (2005).

Table 3.3: Item content validity indices and the modified kappa values

Item	Raters	Ratings (3's & 4's)	I-CVI	p_c	k^*
1.0	7	7	1.00	0.008	1.00
2.1	7	7	1.00	0.008	1.00
2.2	7	7	1.00	0.008	1.00
2.3	7	7	1.00	0.008	1.00
2.4	7	7	1.00	0.008	1.00
3.1	7	7	1.00	0.008	1.00
3.2	7	7	1.00	0.008	1.00
3.3	7	7	1.00	0.008	1.00
4.1	7	6	0.86	0.055	0.85
4.2	7	7	1.00	0.008	1.00
4.3	7	7	1.00	0.008	1.00
4.4	7	7	1.00	0.008	1.00
				$S - CVI/Ave$	0.99

Notes. $I - CVI$ = item level content validity index; p_c = probability of chance agreement; k^* = kappa value representing agreement on item relevance. $S - CVI/Ave$ = scale-level content validity index, averaging method

The results in Table 3.3 show that there was perfect agreement on item relevance in 11 out of 12 test items. It is important to note that adjustment for chance agreement had no effect on the modified kappa values in these cases (see Table 3.3). Adjustment for chance agreement lowered the validity index of item 4.1 by a margin of 0.01. One expert thought item 4.1 was irrelevant (see Figure 3.3). According to Waltz and Bausell (1983), a test item is accepted if its validity index is greater or equal to 0.79, otherwise it will be discarded (see also Polit et al., 2007; Zamanzadeh et al., 2015). Based on the validity indices in Table 3.3, all test items were therefore judged to be valid assessments of students' proof construction abilities. The validation form that was used by the experts to rate the items provided space for the raters to make suggestions for additions, deletions, and modifications of the test items to improve the instrument's face validity (see Appendix J). The raters' comments that necessitated further changes to the proof test are captured here:

Comment 1:

Proving for a cyclic quad is duplicated (2.1 & 4.4) thus it needs to be revised. All other items are Ok for Grade 11 Euclidean Geometry.

Comment 2:

… I think Question 4.1 is unnecessary – it does not need to be proved, since it is a direct corollary from a theorem. Learners need to implicitly use it in other questions. Question 4.3 can be 2 marks (not that difficult to prove).

Comment 3:

In the instructions for Question 3 the phrase "AB∥MP" is *so packed together and may affect the readability of your instructions. You may need to loosen up this phrase. We are not sure how this phrase could influence your participants' comprehension of the instructions and related diagram. One way to address this challenge could be to write the middle part* " ∥ " *in italics as "//* "and *also to insert spaces between the 3 components of the word/phrase, thus making it look like: "*AB *// MP ".*

Comment 4:

The marks that awarded for each of the questions and sub-questions were fair and realistic. The exception in my view is 3.2, which could be answered in just two steps. Seven (7) marks might perhaps be decreased to just 4/5 marks.

Figure 3.3: Mathematics experts' comments

Based on the comments in Figure 3.3, items 4.4 (allocated 5 marks) and 4.1 (allocated 1 mark) were deleted from the test. Mark allocation for Question 4.3 was maintained since it did not differ significantly from the 2 marks suggested by one of the experts. The remaining two items of Question 4 (4.2 & 4.3) were renumbered 4.1 and 4.2 respectively. Question 3 was modified by replacing '$\mathrm{AB} \parallel \mathrm{MP}$' with 'AB is parallel to MP'. Mark allocation for Question 3.2 was reduced from 7 marks to just 3 marks. The final version of the proof test now had 10 items, two less than the initial draft. All the remaining items had a validity index of 1.00, which represents perfect inter-rater agreement on relevance. Total mark allocation was now 50, ten less than the initial mark allocation. The time allocation of one hour was maintained.

The reliability of the revised proof test instrument was measured through the test-retest criterion. A conveniently selected sample of 27 Grade 11 students from a school outside the targeted research area wrote the same test twice. The second test was written two weeks after the first test. The reliability of the test was established by computing Pearson's correlation coefficient (r) in the Statistical Package for Social Sciences (SPSS) Version 24. Table 3.4 shows the SPSS output for Pearson's correlation coefficient (r), and its level of significance.

Table 3.4: Reliability statistics of the geometry proof test

Correlations			
		Time 1	Time 2
Time 1	Pearson Correlation	1	.824**
	Sig. (2- tailed)		.000
	N	27	27
Time 2	Pearson Correlation	.824**	1
	Sig. (2- tailed)	.000	
	N	27	27

Note. **Correlation is significant at the .01 level (2- tailed)

The results in Table 3.4 indicate that there was a statistically significant strong positive correlation $(r = .824, p = .000)$ between Time 1 and Time 2 scores on the geometry test. The recommended minimum acceptable value for test-retest reliability coefficient is .70 (Paiva, et al., 2014). The Pearson's correlation coefficient value $(r = .824)$ in Table 3.4 falls above this minimum reliability threshold. It was therefore concluded that the revised geometry proof test was reliable.

3.5.2 Focus group discussion guide

A focus group discussion guide (see Appendix M) was used to collect qualitative data to answer the second research question. The focus group discussion guide helps the moderator to facilitate the discussion in a standardized and structured way (Kuhn, 2016). It contains the key questions to be asked and their sequence. It helps to ensure that the focus group discussion stays on track and that all important areas of the research question(s) are addressed (Reid & Mash, 2014).

The researcher followed the recommendations by Krueger (2002) and Kuhn (2016) to design the focus group discussion guide used in this research. According to Krueger (2002) and Kuhn (2016), a typical focus group discussion guide should contain:

A Preliminary Section with labels for date, time, location, type of group, selection criteria used to recruit the participants, and number of participants present.

The Opening Section, which includes welcome and opening remarks; highlighting the purpose of the discussion; addressing issues of anonymity and confidentiality of responses; laying down the ground rules and expectations; announcing the estimated duration of the discussion; engaging in warm-up activity in which participants introduce themselves to the group.

The Question Section, which includes three categories of questions which are time-framed:

1) *Engagement questions* to get participants to talk to each other and to feel comfortable, and to build rapport.
2) *Exploration questions* which are questions focusing on the topic of discussion.
3) *Exit questions* which are follow-up questions to determine if there is anything else related to the topic that needs to be discussed.

The Closing Section, which includes wrapping up loose ends, giving participants an opportunity for final thoughts and comments, thanking participants for their input, and informing them of how the data will be used.

A key component of the focus group discussion guide is the *Question Section*. The quality of the data collected using focus group discussions depends on the quality of the questions asked by the facilitator (Center for Innovation in Research and Teaching, n.d.). As suggested by Lachapelle and Mastel (2017), focus group questions should be framed based on the following traits: behaviour, opinion, feelings, and sensory experiences. Questions on behaviour "focus on what a person has done or is doing" (Lachapelle & Mastel, 2017, p. 2). Exploring respondents' opinions involves asking about what they think on the issue being discussed. Questions about feelings seek to elicit respondents' emotional responses to the issue being discussed. Questions seeking information about what respondents have seen, touched and heard fall under sensory experience-type questions. Figure 3.4 shows the steps followed by the researcher to develop appropriate questions for the focus group discussion. These steps were adapted from National Oceanic and Atmospheric Administration (2015, p. 5-6):

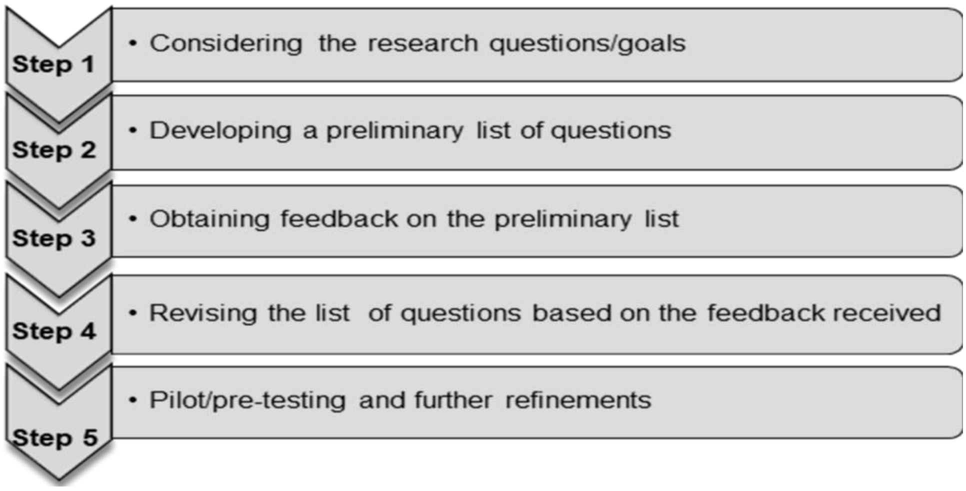


Figure 3.4: Steps followed when developing focus group questions

The reason for conducting focus group discussions was to explore students' experiences, perceptions, attitudes, beliefs, feelings and opinions on how Euclidean geometry theorems and proofs were taught in their mathematics classrooms. A consideration of the research goals was therefore essential to guide the researcher in developing relevant focus group questions. Generating a preliminary list was just a matter of brainstorming and writing all questions that came to mind, knowing that these questions would later be edited and reduced to a smaller number. Questions were developed under three categories: (1)

engagement questions, (2) exploration questions, and (3) exit questions. Ding (2014) clarifies what each of these question categories entails. Engagement questions are questions asked simply to get participants talking, relaxed and comfortable. They are sometimes referred to as ice-breakers (ETR, 2013). Exploration questions are questions which form the core or heart of the discussion. These are open-ended questions that seek to collect more specific data on the topic of discussion. Three to five questions under the exploration category are regarded as adequate (Ding, 2014). The exit questions are used to check if there is any key information that has been left out but that participants think is worth discussing.

The wording of the focus group questions was guided by several authorities. Good questions should be clear, open-ended, short, non-threatening, and one-dimensional (asking only about one clear idea) (Krueger & Casey, 2009). Open-ended questions do not constrain respondents to a limited range of options as is the case with closed questions. Based on the advice given by Krueger and Casey (2009), the following types of questions were avoided: dichotomous, leading, double-barrelled, value-laden, and 'why' questions.

The dichotomous type of questions require a simple 'yes' or 'no' response. These questions limit conversation and may lead to ambiguous responses (Canavor, 2006). Leading questions seem to give direction towards a particular response and hence may bias the results (Krosnick & Presser, 2009). Double-barrelled questions are questions that touch on two different issues. Such questions should be avoided because they may confuse respondents and also make responses hard to interpret (Krosnick & Presser, 2009). Double-barrelled questions are best separated into two parts. Value-laden questions are those that include emotionally charged words (for example blame, demand, unhelpful, force and unreasonable). Such questions indicate the interviewer's strong personal views on the issue being discussed and hence "can induce reactivity", which skews participants' reponses (Haslam & McGarty, 2014, p. 410). Lastly, 'why' questions were excluded because they "put participants on the spot, restrict the range of answers, and can inadvertently make someone feel defensive" (Canavor, 2006, p. 52).

Using the ideas in the preceding discussion, a preliminary list with ten questions (one engagement question, eight exploration questions, and one exit question) was developed by the researcher. Feedback on these potential questions was obtained from fellow postgraduate students and other experts (doctors and professors) in Mathematics Education. Based on their advice, three exploration questions were removed from the list as they were regarded as unnecessary. In addition, the wording in some questions was revised. The remaining seven questions were then entered into a focus group discussion guide draft. The developed focus group script was pre-tested by the selected facilitator on a group of Grade 11 students who were not part of this research. Various authorities have highlighted the value of pre-testing data collection instruments before a full-scale study. Pre-testing helps to notice weaknesses in the research instrument and to identify areas in need of further adjustments (Dikko, 2016). In the case of a focus group discussion, pre-testing serves to:

- highlight unclear and unnecessary questions (Calitz, 2005).
- determine whether the proposed duration of the discussion is acceptable (Dikko, 2016).
- give the facilitator an opportunity to improve questioning technique (Dikko, 2016).
- determine whether questions are enough to measure all the necessary concepts (Berg, 2012).
- improve quality, and add value and credibility to the study (Aitken, Gallagher, & Madronio, 2003; Van Wijk, 2013).

No further changes were made to the focus group discussion guide after the pre-testing exercise. All questions were clearly understood by the pilot group and met the requirements of the study. Based on the pretesting outcomes, it was estimated that the focus group discussion would take between one and a half to two hours.

3.5.3 Diary guide

A diary guide (see Appendix E) was developed by the researcher using guidelines from available literature. In the first part of the diary guide, the researcher clarified the purpose of the diary as suggested by Duke (2012) and Rausch (2014). Second, issues of anonymity and confidentiality were addressed to gain the trust of the participants (see section 3.8.1.2). Third, clear written instructions were given on the variables of interest that the diarists should write about (Bytheway, 2012; Rausch, 2014) and when the diary entries should be recorded. Providing information on the variables of interest was essential to relieve diarists of the burden of deciding what to include in the diary. On the part of the researcher, this was crucial to ensure that the research objectives would be addressed. Finally, an example of a completed diary entry (on a different topic from the one being investigated) was attached to the diary guide. This was important to guide diarists on the amount and type of data to be recorded (Duke, 2012).

While imposing the structure of the diary entry page by restricting entries to precategorized spaces makes it easier to complete the diary and analyse the data, it has the disadvantage that it limits the diarist to recording only that which can be slotted into the spaces provided. For this reason, there were no restrictions on the amount of information diarists could write per each variable of interest. Each diary was a small portable notebook made up of 192 pages. Daily entries were allowed to overflow to the next page when necessary.

Establishing a good rapport with participants is vital before data collection commences (Rausch, 2014). To this end, the researcher made multiple visits to the research sites prior to data collection and interacted with participants formally and informally to gain their trust. During this period, the researcher informed the Grade 11 students in the selected schools of the upcoming research activities.

In the next section, the data collection procedures employed in the study are explained.

3.6 Data collection

The data used in this research was collected through the administration of pre-tests and post-tests, students' diaries, and focus group discussions. Data collection commenced after the relevant ethical issues had been addressed (see section 3.8).

3.6.1 Pre-test administration

The geometry proof test developed in section 3.5.1 was administered to both the experimental and control groups in Term 3, just before Euclidean geometry was introduced. According to the South African Mathematics CAPS, Grade 11 Euclidean geometry should be taught in Term 3 (see Department of

Basic Education, 2011, p. 19). The choice to collect data during this period was therefore in accordance with policy. Four research assistants (2 males and 2 females) who were unemployed university graduates known to the researcher were hired to help administer the pre-test and post-test in participating schools. The research assistants were trained by the researcher for one day prior to the field work.

The teacher/researcher and research assistants visited the participating schools a week before the pre-test was administered to make prior arrangements with school principals, Grade 11 Mathematics teachers and their students. We asked for a list of Grade 11 Mathematics students at each school. This was used to generate codes to replace students' actual names to guarantee anonymity. The first student on the list of experimental group school E1 was coded E 1001, the second E 1002, and so on. Similarly, the first and second students on the list of experimental group school E2 were coded E 2001 and E 2002 respectively. In the same way, C 1001 and C 2001 represented the first student from control group schools C1 and C2 respectively. Pre-test answer sheets were coded in advance. Each research assistant was allocated a school to work with in administering the pre-test. The answer sheets and coding were verified by the teacher/researcher before packaging. The packaging of test papers and answer sheets was done by the researcher and the research assistants had no access to the test papers prior to the pre-test. The research assistants were trained on how to deal with irregularities and were also requested to be scrupulous in administering the pre-test.

The pre-test papers and answer sheets were delivered by the teacher/researcher to principals of participating schools a day before the set date. The school principals were requested to only release the test material to the research assistants on the set date and at the appropriate time. To ensure parity of test conditions, the pre-test was administered across the four school on the same day, starting and ending at the same time. Students' pre-test scripts and all test papers were collected by the research assistants and were submitted to the researcher. Some students refused to write the pre-test and that was respected without seeking reasons, as stipulated in their consent forms. The scripts were marked by a hired marker, with more than five years of experience in marking Grade 12 national examinations. The marker was not part of the research team that helped to administer the pre-test and post-test in participating schools.

3.6.2 Treatment

The teacher/researcher implemented Van Hiele theory-based instruction in the treatment schools while students in the control schools were taught by their teachers using their usual approaches. It was not possible for the researcher to teach both groups because the selected experimental and control schools were in separate areas, far from each other. However, the researcher and the two teachers who were responsible for the Grade 11 mathematics classes in the control schools were all guided by the same CAPS document and the same work schedules provided by the provincial DBE.

The CAPS document set out the Euclidean theorems that needed to be covered (see Appendix O) and the work schedules set out the time-frame in which the content of the topic should be covered (see Appendix P). Grade 11 Euclidean geometry is allocated three weeks in the CAPS (see Appendix O), but it was allocated four weeks in the work schedules sent to schools by the DBE (see Appendix P). We therefore agreed to cover the content in four weeks' time.

Mathematics teachers in the Capricorn district have been provided with ready-made lessons plans by the subject advisers to reduce the everyday burden of drawing up lessons plans. The lesson plans included full descriptions of teaching methods that teachers could use, and suggested activities for introduction, main body, and the closing of lessons (see Appendix Q). Although many teachers find these ready-made lessons to be convenient and simple to use due to their comprehensive nature, I found them rigid and insensitive to the needs of the students in the mathematics classroom. I used these lessons plans in 2015, and most of my students failed to understand the content of Euclidean geometry. I therefore decided to do things differently and try to implement a modified version of the Van Hiele theory-based approach to teaching Euclidean geometry theorems and proofs.

3.6.2.1 Van Hiele theory-based instruction

Figure 3.5 shows the geometry teaching and learning model designed by the researcher, incorporating the Van Hieles' assertions:

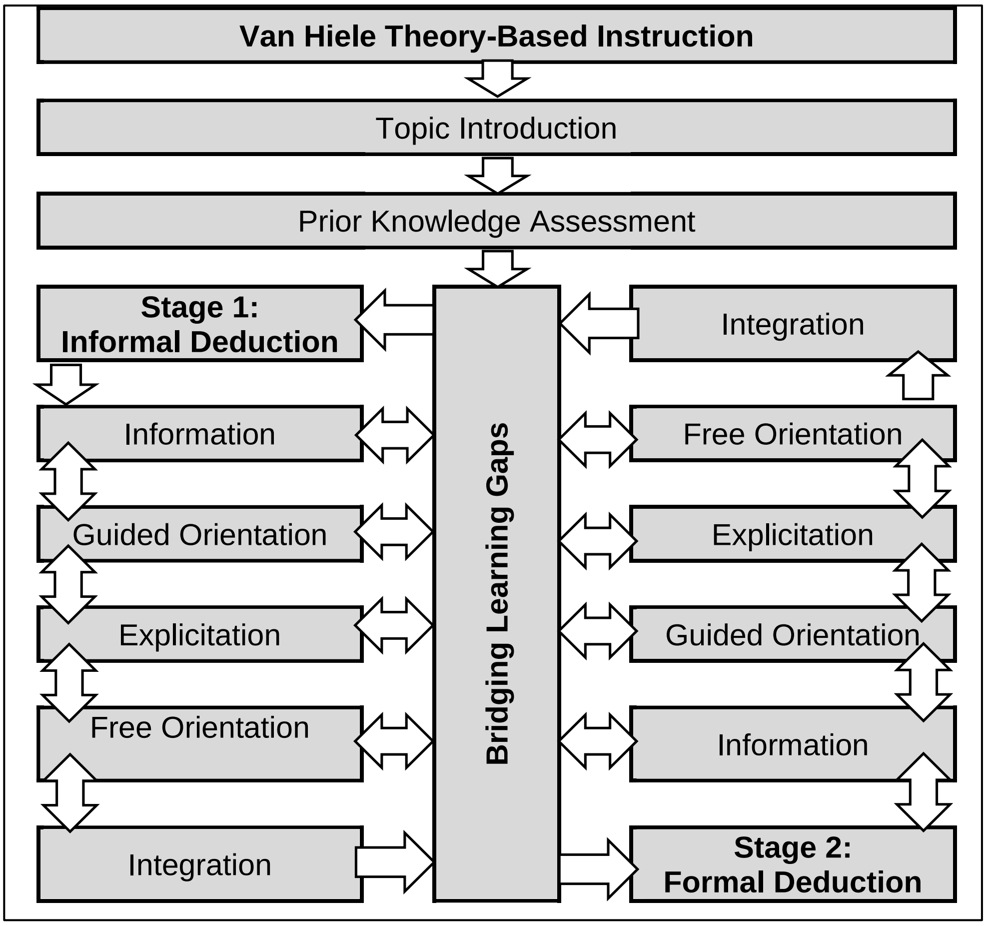


Figure 3.5: Proposed Van Hiele theory-based approach to teaching geometric proofs

The proposed Van Hiele theory-based approach to teaching Euclidean geometry proofs starts with informal deduction activities (Stage 1) before formal proofs (Stage 2). In the informal deduction stage, students engage in investigation activities using protractor, compass, ruler, paper-and-pencil or GSP with ready-made sketches to establish patterns and relationships in given geometric shapes. In other words, they 'reinvent' theorems and axioms. The GSP allows students to observe several examples of geometric shapes quickly without having to draw a separate figure each time as is the case with paper-and-pencil activities (Gray (2008). However, the use of GSP depends on the availability of computers and GSP software in classrooms, whereas paper-and-pencil investigation activities can be used in any school environment.

The South African Mathematics CAPS for the FET Band states that Grade 11 students should investigate before they prove theorems and riders (see Department of Basic Education, 2011, p. 14). This is consistent with the Van

Hiele theory which suggests that deductive reasoning (formal proof) should be preceded by informal deduction (investigative geometry). However, the CAPS document does not provide further details on what teachers and students should do as part of the investigation. It is left to the individual teachers to decide on the kind of investigation activities to do with their students.

The teacher/researcher examined the Grade 11 Mathematics textbooks commonly used in South African schools, namely, Classroom Mathematics, Platinum Mathematics, Study and Master, and Everything Mathematics (Siyavula). Only the Siyavula Grade 11 Mathematics textbook suggested paper-and-pencil investigation activities for four of the seven prescribed circle geometry theorems. The other theorems are just stated, proved, and applied without first being investigated. The paper-and-pencil investigation activities suggested in the Siyavula Grade 11 Mathematics textbook require thorough preparation and good time management on the part of the teacher. In South Africa, public schools administer common assessment tasks every quarter. Students in the same district write the same tests on set dates during the year. This pressurizes teachers to cover the prescribed syllabus content within the specified period. As a result, most teachers would skip the 'time-consuming' paper-and-pencil investigation activities and move straight to proving theorems and solving riders. To engage the experimental group students in investigation activities without consuming much time, the teacher/researcher replaced the traditional paper-and-pencil activities suggested in some of the Grade 11 Mathematics textbooks with similar activities in the GSP.

In both stages (Stage 1 and Stage 2) of the treatment, teaching and learning activities were sequenced according to the Van Hiele phases (see Figure 3.5). Bridging of learning gaps was done at every teaching and learning phase. The arrows in Figure 3.5 point either way, indicating that the movement from one phase/stage to the other is not rigid. That is, the model is flexible, allowing the teacher to go back to the previous phase/stage whenever it is necessary. The full details of how the proposed model was implemented are presented in the next sections.

3.6.2.1.1 Topic introduction [Lesson 1]

In Lesson 1, the topic was introduced by means of giving students a brief history of the origins of Euclidean geometry. This was done using a Power Point presentation. An old image of Euclid was displayed on screen and students were asked to guess whose image it was. It was amazing to hear some students saying: “Euclidean!”. The teacher/researcher then moved to the next slide where the names of the old man (Euclid) and his contributions to geometry were displayed. Students then noticed that the old man was named Euclid, not Euclidean. The teacher/researcher explained that the naming of the topic Euclidean geometry is in honour of Euclid and his contribution to geometry.

We then discussed the importance of studying Euclidean geometry and the role it plays in human life. The teacher/researcher displayed a list of careers in which knowledge of Euclidean geometry is critical such as architecture, aircraft designing, landscaping, automotive designing, cartography, engineering, and law. The teacher/researcher then explained why Euclidean geometry was brought back into South African mathematics education. Using physical structures in the classroom such as tables, chairs, roof trusses, cabinets, and windows, the teacher/researcher helped students to see that geometry is around us.

To conclude the introduction, the teacher/researcher displayed a bicycle on screen. Students had to identify the different shapes they saw in the bicycle structure (for example, triangles, quadrilaterals, and circles). The teacher/researcher explained that triangles were dealt with in Grades 8 and 9, quadrilaterals in Grade 10, and that Grade 11 Euclidean geometry deals with circles. The teacher/researcher conscientized students of the fact that for them to succeed in Grade 11 Euclidean geometry, they needed to recall work covered in lower grades. Students were informed that in the next lesson, they would write a revision task based on the Euclidean geometry concepts they learnt in the lower grades (Grades 8-10).

3.6.2.1.2 Assessing prior knowledge [Lesson 2]

A prior knowledge assessment test was administered to the experimental group students on Day 2 (see Appendix F) to identify areas of deficiency and to determine an appropriate level at which to start teaching. The test was also given to teachers in the control group. However, the researcher did not tell the teachers in the control group how to use it, as it would interfere with their conventional way of teaching Euclidean geometry. The assessment was only compulsory for students in the experimental group since it was part of the treatment procedures.

The Van Hieles highlighted that inadequate prior knowledge may impede current teaching and learning of Euclidean geometry if learning gaps are not addressed. This is because the teacher would teach at a level higher than the students' actual knowledge base. The Van Hieles referred to this as a mismatch between instruction and learning. Therefore, assessment of prior knowledge helped the teacher/researcher to adapt teaching to the level of the students, and to ensure that new knowledge was built on students' existing knowledge frameworks.

The prior knowledge assessment test comprised four questions on the Euclidean geometry concepts learnt in lower grades (Grades 8-10). These included, the geometry of straight lines, properties of two-dimensional shapes, proving congruency, and similarity. The test was written under strict examination conditions. Students' scripts were marked by the teacher/researcher, and areas of deficiency were identified by means of a test item analysis (see Appendix G). Test items with a high frequency of incorrect responses indicated areas where some students had serious deficiencies. The related geometry aspects together with the students concerned were identified for reteaching.

It is important to note here that the prior knowledge assessment test was completely different from the geometry proof test that was used to assess students' geometric proofs learning achievement before and after treatment.

3.6.2.1.3 Bridging learning gaps [Lesson 3]

The geometry aspects of co-interior angles, angles around a point and the exterior angle of a triangle (taught in Grades 8 and 9), had the highest frequency of incorrect responses (see Appendix G). Undoubtedly, these concepts are invaluable to proving riders. The fact that a greater number of Grade 11 students could not correctly answer some of the Euclidean geometry questions based on Grade 8 and 9 work is consistent with previous studies that found students to function below the expected levels of geometric thought (see section 2.9.3.2 in Chapter 2). Lesson 3 was devoted to giving students feedback on their test performance and to reteach areas of learning deficiency. However, not all learning deficiencies could be addressed in one day. It is for this reason that the bridging of learning gaps was incorporated into all phases of teaching and learning in the Van Hiele theory-based instruction. Where more than 50% of the students were found to have challenges with a geometry aspect, bridging lessons involved the whole class. Otherwise, only students at risk were targeted.

3.6.2.1.4 Stage 1: Informal deduction

At the level of informal deduction, students should be able to recognize properties of geometric shapes, and describe the relationships among them. To help students attain this level, the teacher/researcher organized lessons according to the Van Hieles' teaching phases: information↔guided orientation ↔ explicitation↔ free orientation ↔ integration. The arrows between the phases point either way to allow oscillation between phases when necessary.

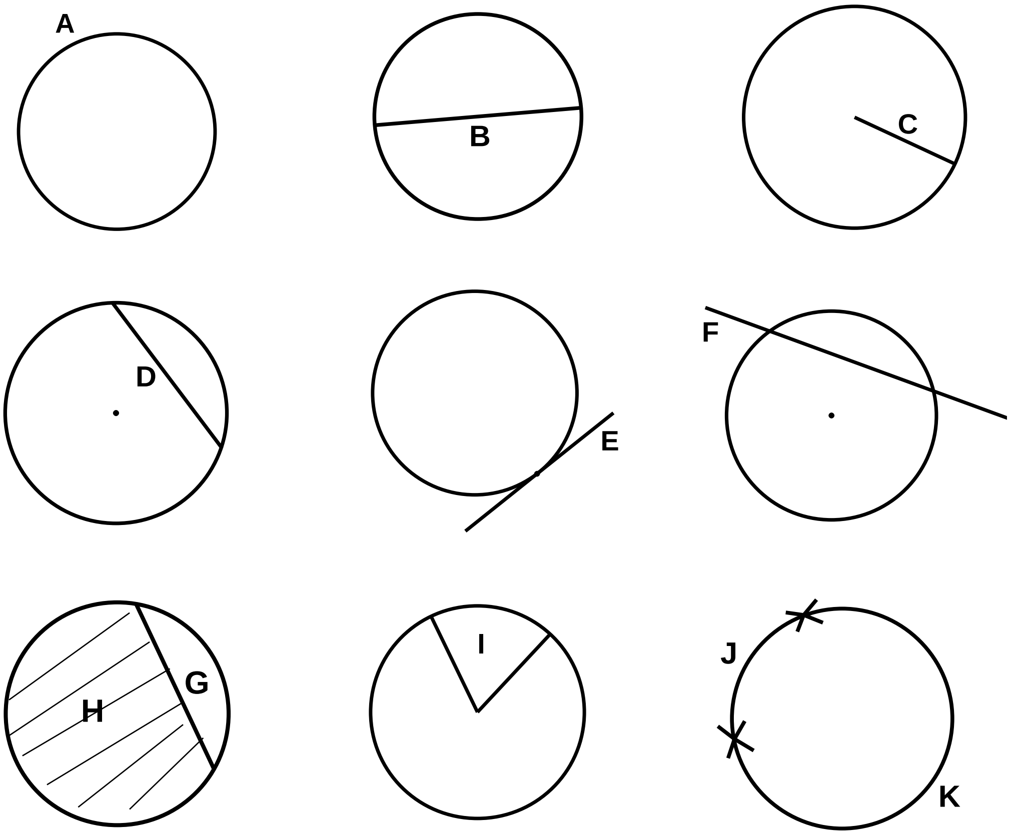
This section presents a full account of how the phases were implemented at the level of informal deduction.

3.6.2.1.4.1 Phase 1: Information [Lessons 4-5]

The Van Hieles' information phase is a two-way teacher-student interaction that seeks to give students an idea of the upcoming lessons. Ausubel (1960) contends that a preview of the upcoming content is essential when the new knowledge to be learnt is unfamiliar to the student. This serves to link new knowledge with the student's existing knowledge framework. It also helps teachers to discover what prior knowledge their students have about the topic. Lesson 4 and Lesson 5 were reserved for these purposes.

In Lesson 4 we discussed the circle and its component parts. Diagrams showing the different parts of a circle were projected onto a whiteboard (see Figure 3.6). Students were tasked to name the parts marked using letters of the alphabet, and to explain the given terms using their own words. The role of the teacher was simply to guide, correct, and add more details where necessary. Definitions of terms were negotiated and not imposed on the students. This is consistent with other contemporary views of mathematics education that put the student at the forefront of learning (see Dennick, 2012).

1. Name the parts labelled **A – K**



2. Explain the following terms using your own words:
- a) Circumference
- b) Radius
- c) Diameter
- d) Chord
- e) Segment
- f) Tangent
- g) Secant
- h) Arc

Figure 3.6: Parts of a Circle

In Lesson 5, we started with a recap of work done in the previous lesson on the circle and its component parts. The teacher/researcher then displayed fifteen diagrams related to the theorems and axioms students were going to explore in the next learning phase. The diagrams were projected onto a whiteboard one at a time using a Power Point presentation and students described what they saw in each case (see Figure 3.7). Feedback was given to students on the explanations that were expected in each of the diagrams in Figure 3.7:

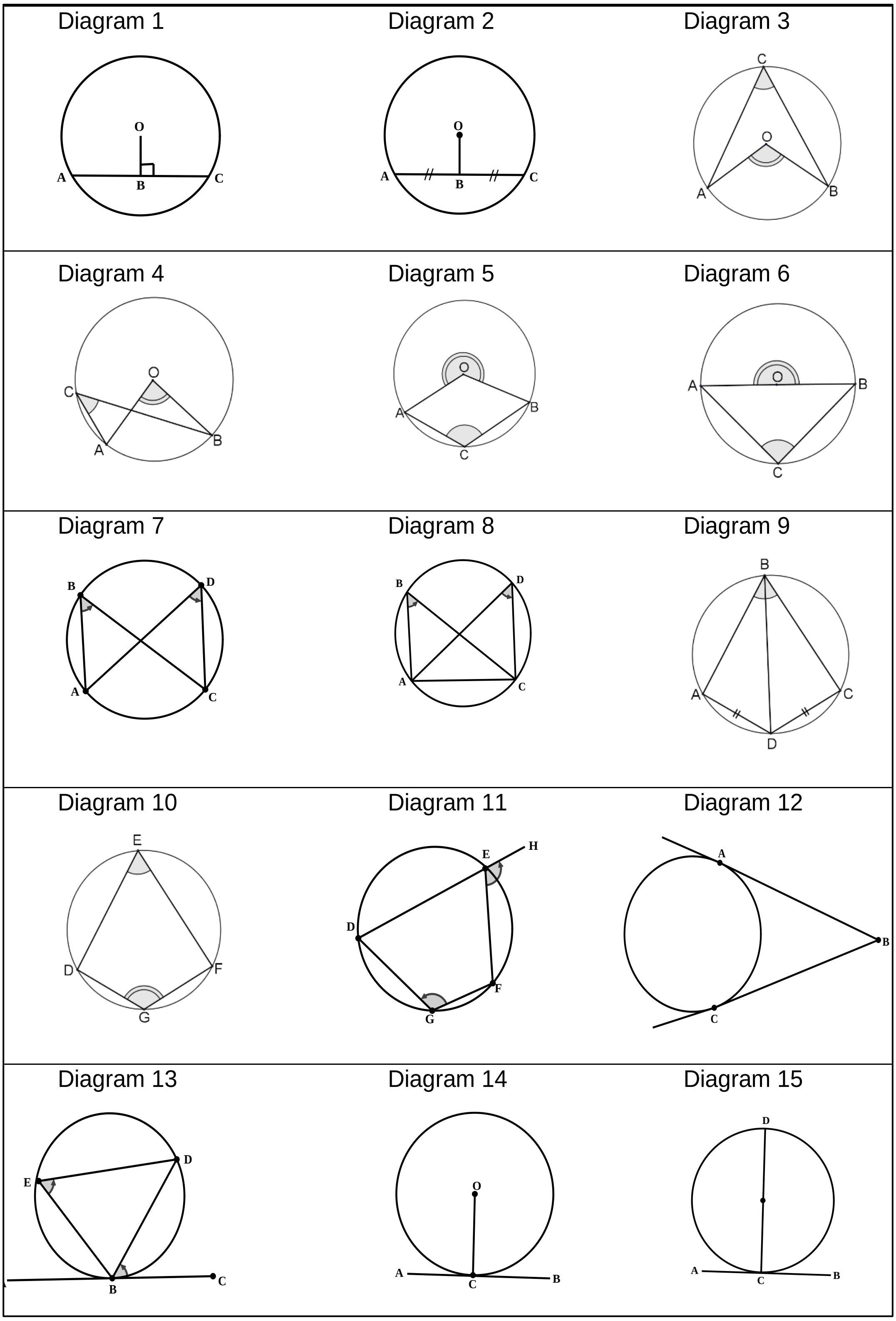


Figure 3.7: Diagrams on Grade 11 Euclidean geometry theorems and axioms

Notes:

Diagram 1:	Line OB is drawn from the centre of the circle perpendicular to chord AC.
Diagram 2:	Line OB is drawn from the centre of the circle to the midpoint of chord AC.
Diagram 3:	$A\hat{O}B$ lies at the centre of the circle. $A\hat{C}B$ lies at the circumference of the circle. Both $A\hat{O}C$ and $A\hat{B}C$ are subtended by the same arc AB.
Diagram 4:	$A\hat{O}B$ lies at the centre of the circle. $A\hat{C}B$ lies at the circumference of the circle. Both angles are subtended by the same arc AB.
Diagram 5:	$A\hat{O}B$ lies at the centre of the circle. $A\hat{C}B$ lies at the circumference of the circle.
Diagram 6:	Diameter AB subtends angle $A\hat{C}B$ at the circumference of the circle. The angle at the centre, that is $A\hat{O}B$, is a straight angle.
Diagram 7:	$A\hat{B}C$ and $A\hat{D}C$ are angles at the circumference of the circle. The two angles are subtended by the same arc AC.
Diagram 8:	$A\hat{B}C$ and $A\hat{D}C$ are subtended by the same chord AC and lie on the same side of the chord. They are in the same segment.
Diagram 9:	$A\hat{B}D$ and $C\hat{B}D$ are subtended by equal chords.
Diagram 10:	DEFG is a cyclic quadrilateral. All four vertices of the quadrilateral lie on the circumference of the circle. $\hat{E}$ and $\hat{G}$ are opposite angles of cyclic quadrilateral DEFG. The same holds true for $\hat{D}$ and $\hat{F}$.
Diagram 11:	$H\hat{E}F$ is the exterior angle of cyclic quadrilateral DEFG; $\hat{G}$ is the interior opposite angle.
Diagram 12:	AB and BC are two tangents drawn from the same point outside the circle.
Diagram 13:	$D\hat{B}C$ lies between tangent AC and chord DB. $B\hat{E}D$ lies in the alternate segment.
Diagram 14:	Tangent AB meets radius OC at point C.
Diagram 15:	Tangent AB meets diameter DC at point C.

One of the challenges that hinder students' progress in learning Euclidean geometry identified in literature is the inability to use appropriate geometry language. The Van Hiele theory points out that the teacher should help students to use the appropriate geometric terminology. To this end, the teacher had to supplement students' vocabulary with the following geometry terminology: angles subtended by the same arc; angles subtended by the same chord; angles in the same segment; angles subtended by equal chords; cyclic quadrilateral; interior opposite angle; and angle in the alternate segment. Students were exposed to the new terminology after they had used their own words to describe what they had observed in each diagram. This is in line with the long-standing educational practice of starting with what students know and progressing to the new knowledge. The geometry terminology that students acquired in this phase were needed to accurately report their findings in the next learning phase: the guided orientation phase.

3.6.2.1.4.2 Phase 2: Guided orientation [Lessons 6-12]

According to the South African Mathematics CAPS for Grades 10-12, Grade 11 students should investigate seven theorems of the geometry of circles (Department of Basic Education, 2011). In this research, investigation activities were carried out in the experimental schools' computer laboratories. The selected schools had at least twenty functional desktop computers. Permission was sought from the school principals to install the GSP in the schools' computer laboratories. The school principals had no idea of what GSP is all about and the researcher had to first demonstrate how it works to the schools' Information Technology (IT) committee members. After the demonstration exercise, the IT committee members in both schools approved the installation of the GSP in their computer laboratories.

A total of seven lessons were devoted to investigating Grade 11 circle geometry theorems using predesigned GSP sketches. In Lesson 6, students received training on how to use the GSP tools to measure angles and lengths, drag points, resize geometric shapes, animate, add text, and save their work. In Lesson 7 we did GSP activities 1a and 1b shown in Figures 3.8 and 3.9. Each activity had clear instructions guiding students on how to explore the given sketch diagram.

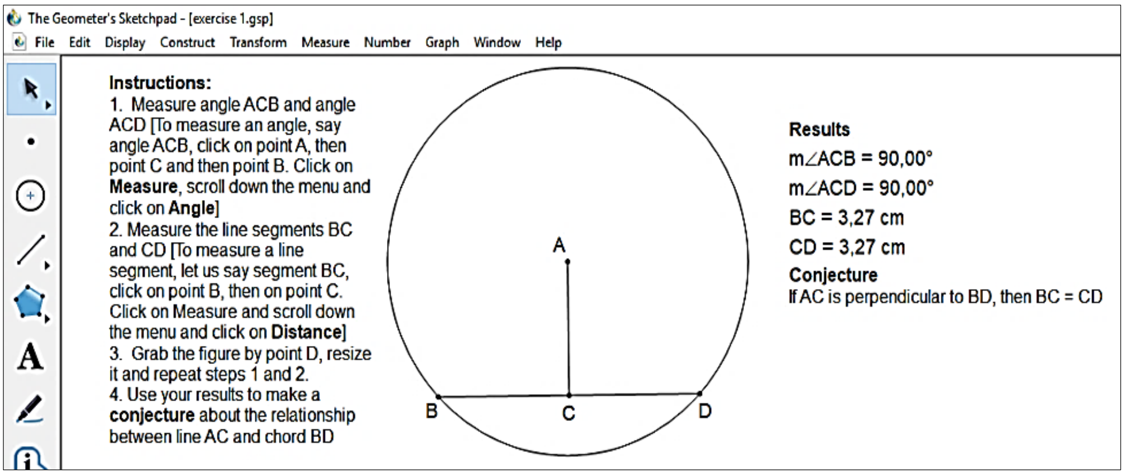


Figure 3.8: GSP Activity 1a: Line from centre perpendicular to chord

Activity 1a in Figure 3.8 helped students to discover that if the angle between line AC and chord BD is $90°$, the lengths of line segments BC and CD remain equal even when point D is dragged to a new position. The GSP results that were obtained in Activity 1a are displayed on the right side of the circle in

Figure 3.8. The conjecture was developed by the students themselves through observing their GSP results.

The teacher/researcher swapped instructions 1 and 2 in Figure 3.8 and asked students to redo the activity. Students discovered that if BC is equal to CD, then AC is perpendicular to BD. The teacher/researcher asked students to explain the difference between the following results:

Result 1: If $AC \perp BD$, then $BC = CD$.

Result 2: If $BC = CD$, then $AC \perp BD$.

Students were able to notice that Result 2 comes from reversing the order of statements in Result 1. The teacher/researcher used these findings to introduce the term *converse.*

Using another predesigned GSP sketch, students were asked to investigate what happens when point C changes position. They used the 'Animate Point' tool in GSP. Activity 1b in Figure 3.9 helped students to discover that if $\mathrm{BC} \neq \mathrm{CD}$, then the angle between line segment AC and chord BD is no longer a right angle; conversely, if the angle between line segment AC and chord BD is not a right angle, then $\mathrm{BC} \neq \mathrm{CD}$. The teacher/researcher emphasized that: if it is not given that $\mathrm{BC} = \mathrm{CD}$, students should not assume that $\mathrm{AC} \perp \mathrm{BD}$. Similarly, if it is not given that $\mathrm{AC} \perp \mathrm{BD}$, then we should not assume that $\mathrm{BC} = \mathrm{CD}$.

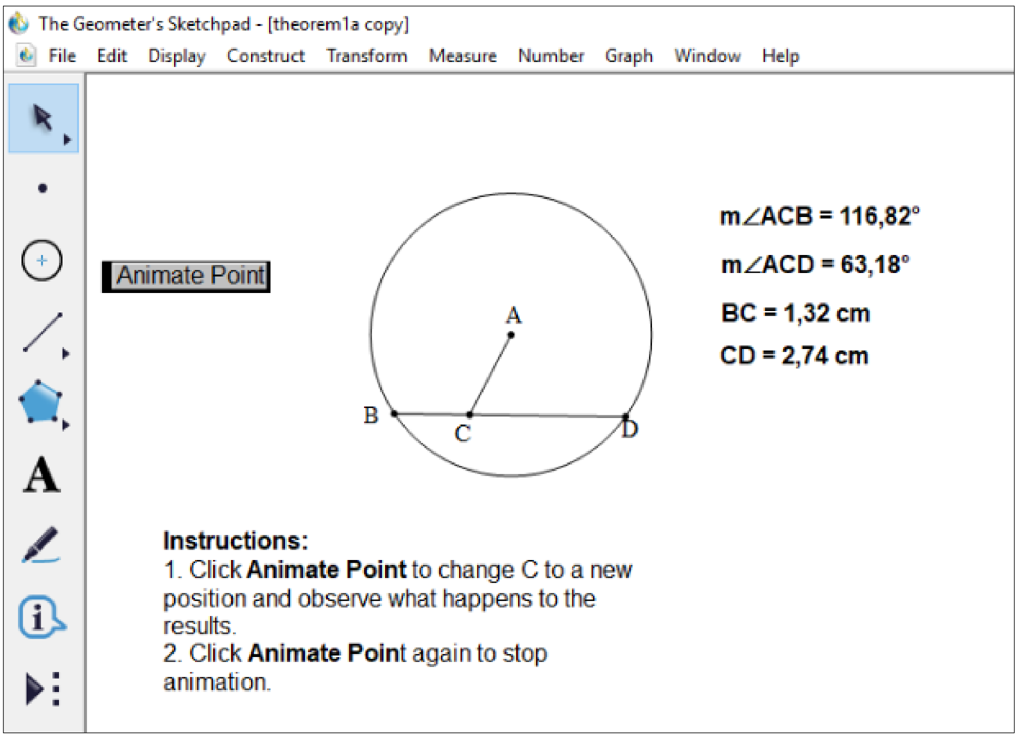


Figure 3.9: GSP Activity 1b: Line from centre to chord

In Lesson 8, students investigated the relationship between the angle subtended by an arc at the centre of the circle and the angle subtended by the same arc at the circumference of the circle. Figure 3.10 shows the GSP activity that was assigned to students and sample results:

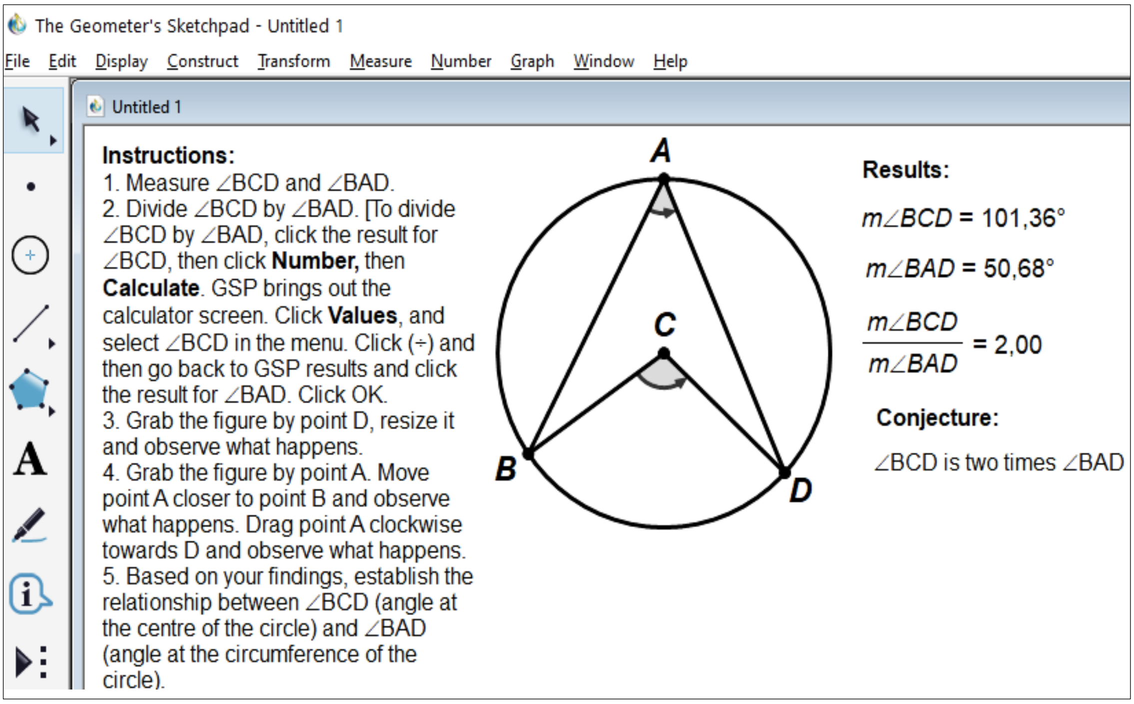


Figure 3.10: GSP Activity 2: Angle at the centre and angle at the circumference

Activity 2 helped students to discover that the measure of the angle subtended by an arc at the centre of the circle is twice the measure of the angle subtended by the same arc at the circumference of the circle. Thus, $B\hat{C}D = 2.B\hat{A}D$, which comes from manipulating the third statement under the GSP results displayed on the right side of the circle in Figure 3.10.

Students were informed that the relationship established in Figure 3.10 appears in three other versions. The teacher/researcher drew students' attention to the three other variations of the angle at the centre and angle at the circumference relationship. Using predesigned GSP sketches (see Figure 3.11), students were instructed to measure the size of $B\hat{C}D$ and $B\hat{A}D$ in each of the three given sketches. The results were consistent with what they discovered earlier on in Figure 3.10.

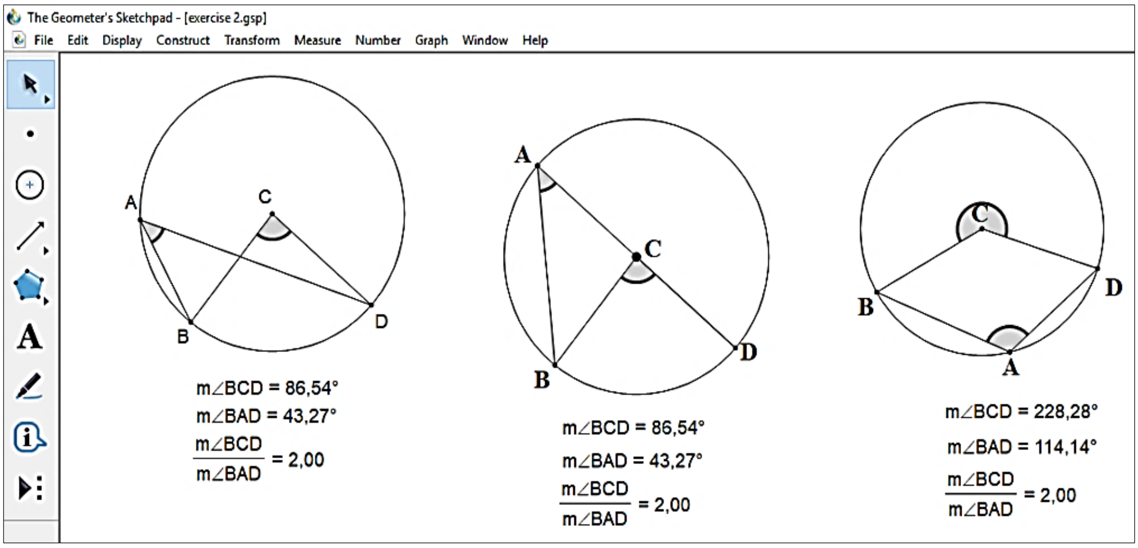


Figure 3.11: Variations of angle at the centre and angle at the circumference

In Lesson 9, students were assigned two GSP tasks. Activity 3 guided students towards discovering that the angle subtended by a diameter at the circumference of the circle measures $90°$ (see Figure 3.12).

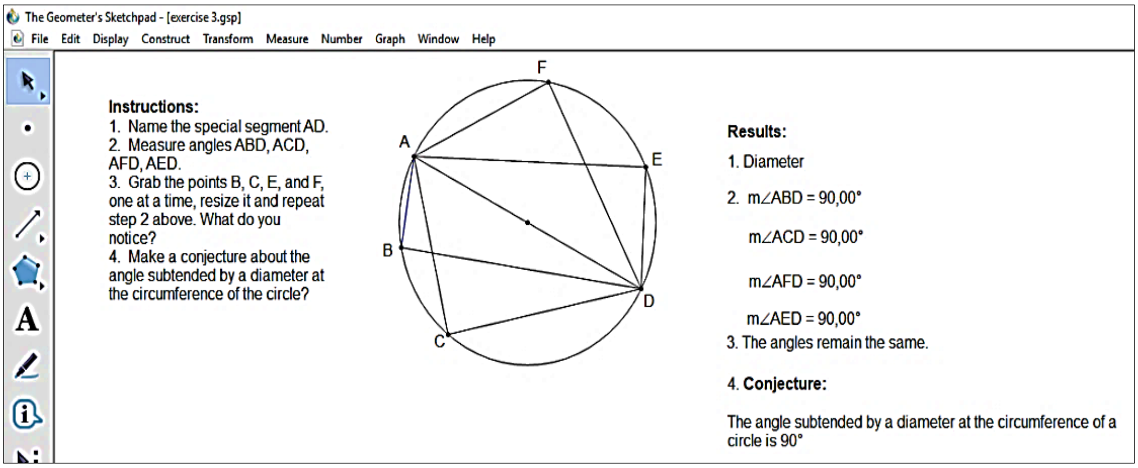


Figure 3.12: GSP Activity 3: Angle in a semi-circle

By dragging points B, C, F and E to new positions, students were able to explore multiple cases of angles subtended by a diameter. Of significance here is the fact that the angle measurements remain unchanged.

In Activity 4, students investigated what happens when we add opposite angles in a cyclic quadrilateral. Figure 3.13 shows the GSP sketch that students used and the results they obtained:

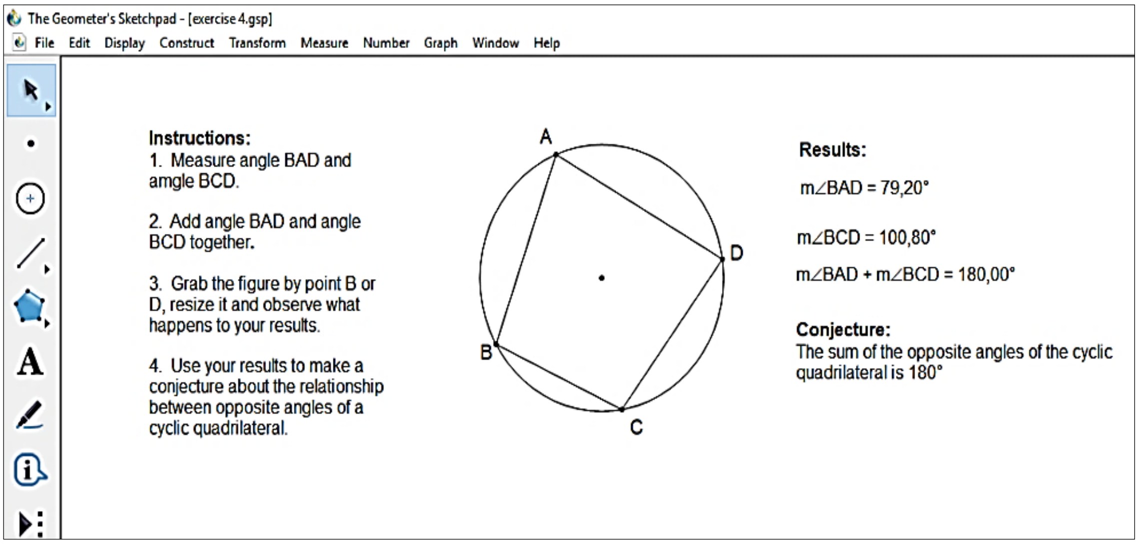


Figure 3.13: GSP Activity 4: Opposite angles of a cyclic quadrilateral

From the activity in Figure 3.13, students noticed that grabbing the figure by point B or D and resizing it resulted in the measurements of $B\hat{A}D$ and $B\hat{C}D$ changing, but the sum remained $180°$. The teacher/researcher emphasized that angles that add up to $180°$ are called supplementary angles. For consolidation purposes, students were requested to investigate if the results obtained were valid for $\hat{B}$ and $\hat{D}$.

In Lesson 10, students were given three GSP activities: Activity 5 and Activities 6a and 6b. Activity 5 guided students towards establishing the relationship between the exterior angle of a cyclic quadrilateral and the interior opposite angle. Figure 3.14 shows the GSP predesigned sketch that was used and a sample of results obtained.

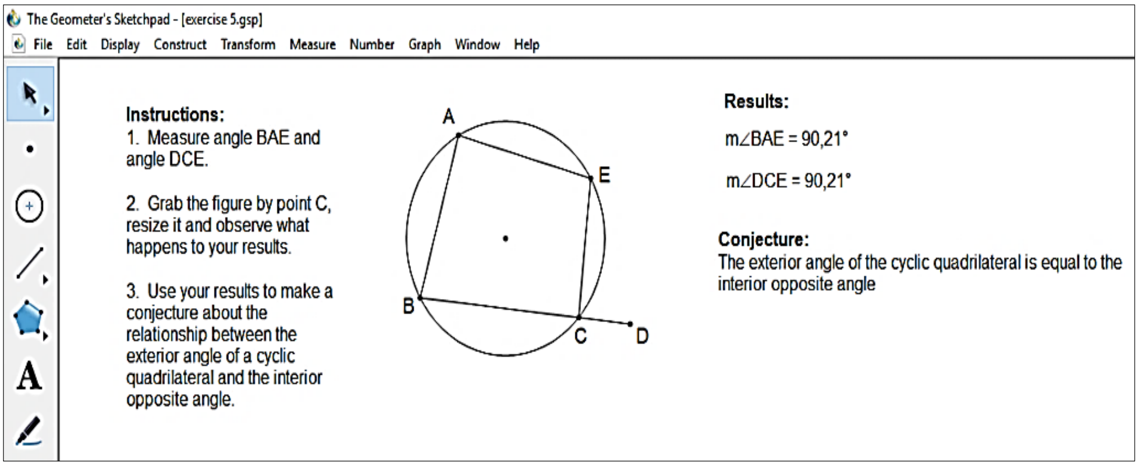


Figure 3.14: GSP Activity 5: Exterior angle of a cyclic quadrilateral

Students were further instructed to drag point B or E along the circumference of the figure and observe what happens to the results. Students noticed that the conjecture remained valid even when the vertices were shifted along the circumference. Activities 6a and 6b (see Figures 3.15 and 3.16) guided students to establish the relationship between angles subtended by the same arc/chord. Figure 3.15 shows angles subtended by the same arc whereas Figure 3.16 shows angles subtended by the same chord.

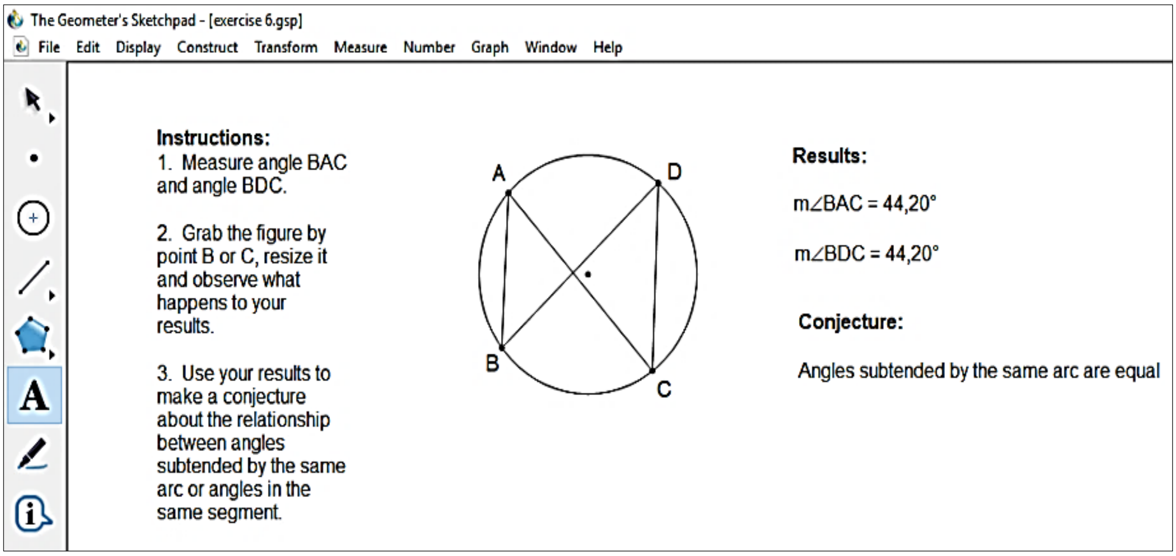


Figure 3.15: GSP Activity 6a: Angles subtended by the same arc

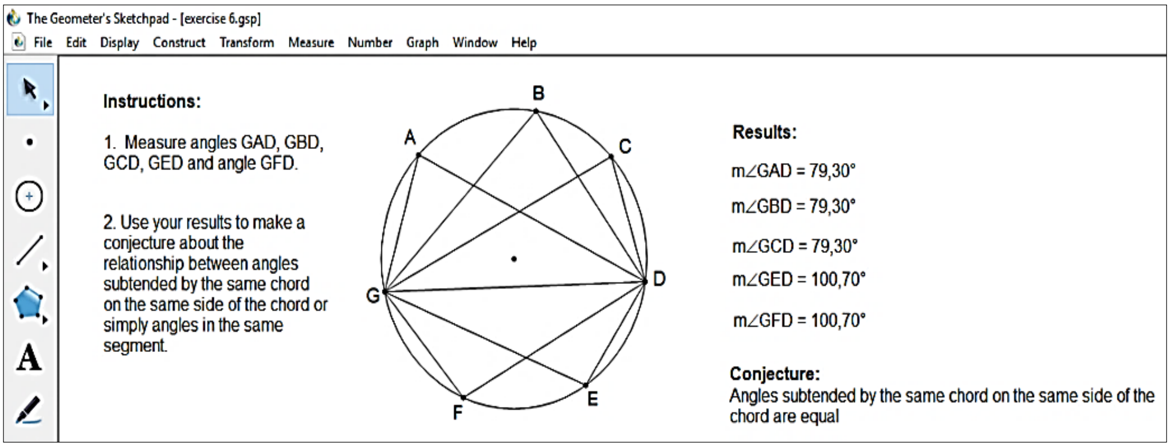


Figure 3.16: GSP Activity 6b: Angles subtended by the same chord

Figure 3.16 was an extended version of Figure 3.15. This was necessary to help students see that the angles are equal only if they lie on the same side of the chord. All five angles in Figure 3.16 are subtended by the same chord but only those that lie on the same side of the chord have the same value. It is therefore not enough to just say: 'Angles subtended by the same chord are equal'.

In Lesson 11, students were assigned two GSP activities: Activity 7 and Activity 8. In Activity 7, they investigated the relationship between the tangent and the radius. Figure 3.17 shows the instructions given and the results obtained:

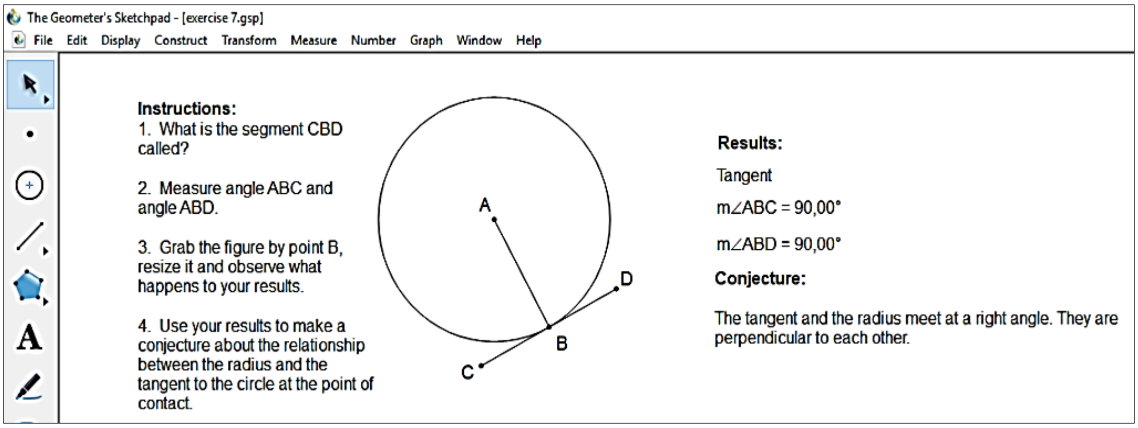


Figure 3.17: GSP Activity 7: Tangent and radius relationship

Activity 7 guided students to discover that the tangent and the radius meet at an angle of 90°; that is, they are perpendicular to each other. Dragging point B and resizing the figure had no effect on the results.

In Activity 8, students investigated the relationship between two tangents drawn from the same point outside a circle. Figure 3.18 shows the GSP predesigned sketch used together with the instructions that guided the students:

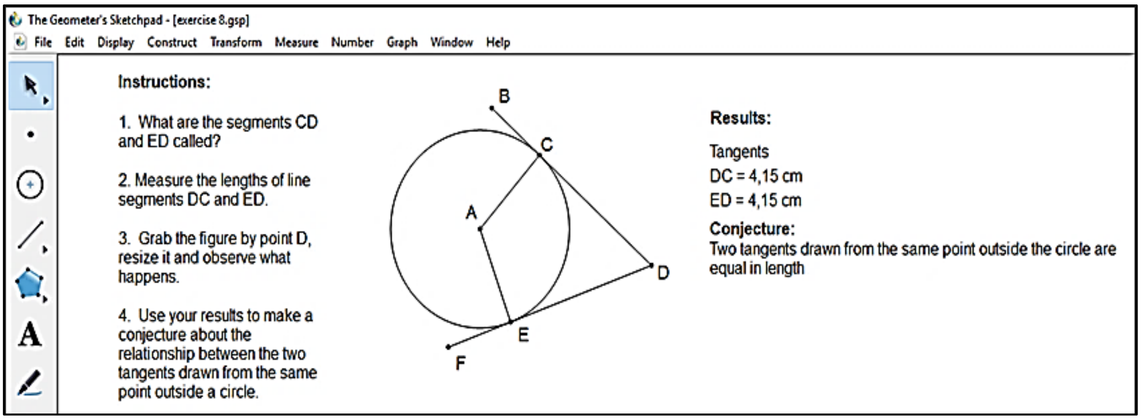


Figure 3.18: GSP Activity 8: Tangents from same point outside a circle

The results helped students to discover that two tangents drawn from the same point outside a circle are equal in length. Dragging the figure by point D and resizing it yielded similar results.

Lesson 12 marked the end of the guided orientation phase with two GSP activities: Activity 9 and Activity 10. Activity 9 guided students towards discovering that equal chords subtend equal angles. Figure 3.19 shows the GSP sketch used and the instructions given.

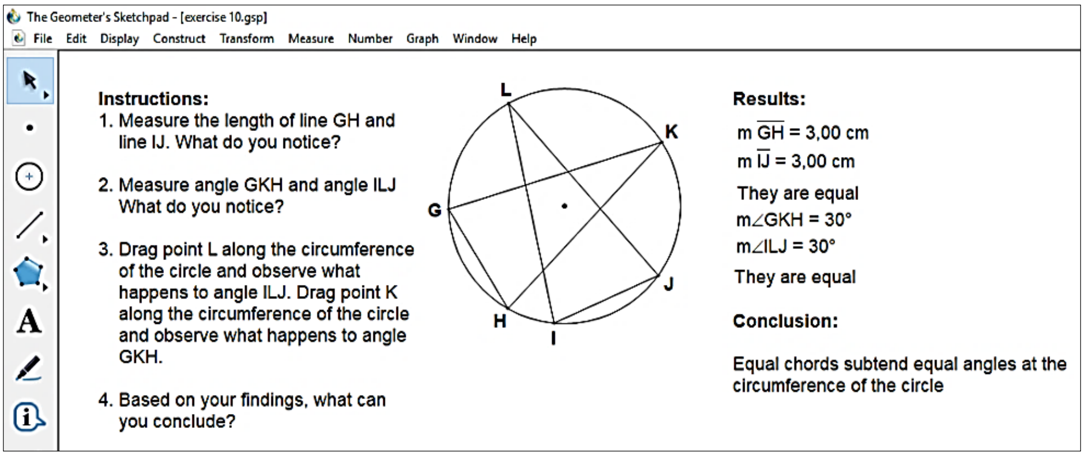


Figure 3.19: GSP Activity 9: Angles subtended by equal chords

In the last GSP activity, students investigated the relationship between the angle between a tangent and a chord at the point of contact and the angle subtended by the same chord in the alternate segment. Figure 3.20 shows the GSP sketch used and the results obtained.

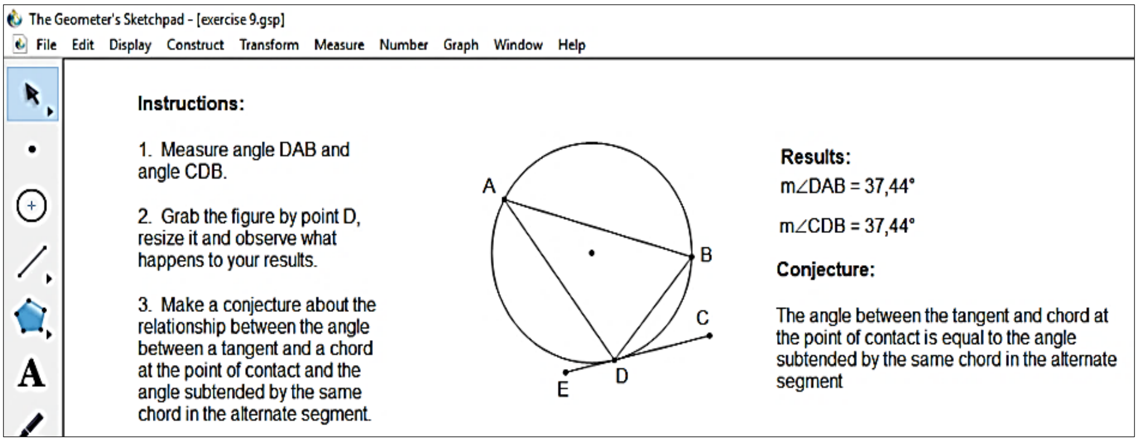


Figure 3.20: GSP Activity 10: Tangent and chord relationship

Transforming the figure by dragging point D towards point C or point A only changed the magnitude of the angle measurement but the initial observation remained unchanged.

Each of the GSP activities presented here was followed by Van Hiele's explicitation, free orientation and integration phases:

3.6.2.1.4.3 Phase 3: Explicitation [Lessons 7-12]

In each GSP activity, students explained in their own words what they had learnt about the given geometrical figure. The teacher/researcher acted merely as a facilitator, assisting students to use the relevant geometry terminology, and redirecting their thoughts when necessary.

3.6.2.1.4.4 Phase 4: Free orientation [Lessons 7-12]

In addition to the guided exploration activities, students were given ten minutes in each lesson to further explore similar GSP sketches without any given instructions and with no interference from the teacher.

3.6.2.1.4.5 Phase 5: Integration [Lessons 7-12]

In the integration phase, we discussed findings from the guided and free orientation activities to synthesize results and summarize the observed patterns and relationships of each of the investigated geometric figures. This marked the end of each lesson. The teacher/researcher highlighted the fact that observing the same pattern in several cases (a process called induction), does not guarantee that the observed pattern is valid in all cases. There could be a single case (known as a counter-example) among the cases not investigated in which the observed pattern would not be true. Therefore, there was need to validate results obtained from the GSP investigations through formal proofs.

Stage 1 (informal deduction) sought to give students an opportunity to establish patterns and relationships in geometric figures through practical investigations and the inductive process, before formal proofs. Based on the Van Hiele theory, this is a pivotal part of Euclidean geometry teaching and learning which provides the scaffolding needed for students to succeed in formal deduction. In Stage 2, the teacher/researcher introduced the idea of formal proofs. The sequence of instruction still followed the Van Hieles' teaching and learning phases. In Part 1, we dealt with proofs of theorems and in Part 2, we focused on the main aspect of this research: proving non-routine geometric proofs (riders).

3.6.2.1.5 Stage 2: Formal deduction Part 1-Proving theorems

In the first lesson of Stage 2 [Lesson 13], the teacher/researcher highlighted that empirical investigations are not proofs. Students were informed that the conjectures arrived at in Stage 1 through a series of observations may or may not always be true. Mathematical examples were given to emphasize the fact that the inductive process is vulnerable to counter-examples. For instance, the fact that, $1^2 - 1 + 41 = 41$, $2^2 - 2 + 41 = 43$, $3^2 - 3 + 41 = 47$, and $4^2 - 4 + 41 = 53$, are prime numbers does not necessarily prove that $n^2 - n + 41$ is a prime number for all $n \in \mathbb{N}$. This pattern is not true for $n = 41$. That is, $41^2 - 41 + 41 = 1681$ is not a prime number. Other real-life examples were given. The teacher/researcher further explained that mathematicians use the inductive process to develop mathematical ideas (hypotheses), which are then validated through formal deduction. We therefore needed to formally prove the conjectures obtained in Stage 1.

We defined a formal geometric proof as a logical argument or chain of reasoning that establishes the truth of a geometric statement using definitions, theorems, and axioms. The terms *axiom* and *theorem* were clarified. The importance of learning Euclidean geometry proofs was discussed.

We then devoted the next seven lessons to proving the conjectures we obtained in Stage 1.

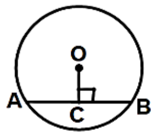
3.6.2.1.5.1 Information phase [Lesson 14]

The teacher/researcher emphasized that a conjecture becomes a theorem only if we prove that it is always valid, by making use of generally accepted statements, axioms, and theorems. Accordingly, we continued to refer to our GSP conclusions as conjectures and only changed this terminology after we had formally proved them. The teacher/researcher also stressed that the approach used to check if the conjectures are always true (deductive reasoning) differs from that which we employed previously to generate conjectures (induction).

Six GSP sketches were projected onto a whiteboard (see Figure 3.21). Students were asked to complete the statement of the conjecture and state what is given and what needed to be proved in each diagram. They did this activity in small groups. This was a form of prior knowledge assessment to gauge students' mastery of work done in the informal deduction stage. At the same time, it was intended to give students information about the upcoming geometry lessons. In keeping with the Van Hiele theory, the information phase was not a spoon-feeding exercise. Feedback on each diagram was only given after students had presented their group findings (see Figure 3.21).

Conjecture:
The line drawn from the centre of a circle perpendicular to a chord bisects chord

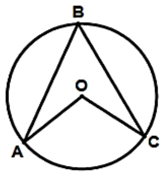
Given:
Circle with centre O and line OC ⊥ AB



Required to prove:
AC = CB

Conjecture: The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference

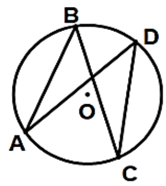
Given:
Circle with centre O, arc AC subtends ∠AOC at the centre, and ∠ABC at the circumference



Required to prove:
∠AOC = 2.∠ABC

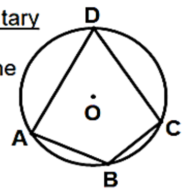
Conjecture: Subtended angles in the same segment of a circle are equal

Given:
Circle with centre O, and points B and D on the circumference of the circle. Arc AC subtends ∠ABC and ∠ADC in the same segment of the circle



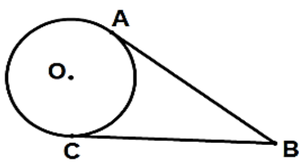
Required to prove:
∠ABC = ∠ADC

Conjecture: The opposite angles of a cyclic quadrilateral are supplementary
Given:
Circle with centre O, with points A, B, C and D on the circumference of the circle such that ABCD is a cyclic quadrilateral



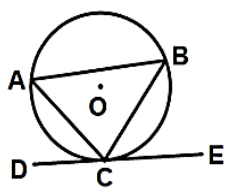
Required to prove:
∠ADC + ∠ABC = 180°

Conjecture:
Two tangents drawn to a circle from the same point outside the circle are equal in length
Given:
Circle with centre O and tangents AB and CB, where A and C are the respective points of contact of the two tangents with the circle



Required to prove:
AB = CB

Conjecture:
The angle between the tangent to a circle and a chord drawn at the point of contact is equal to the angle which the chord subtends in the alternate segment
Given:
Circle with centre O and tangent DE touching the circle at C. Chord AC subtends ∠ABC



Required to prove:
∠ACD = ∠ABC

Figure 3.21: GSP Sketches and related conjectures

3.6.2.1.5.2 Guided orientation phase [Lessons 15-20]

The Van Hiele theory does not specify how the phases of learning can be implemented in teaching proofs. It only indicates that for students to successfully achieve any level of geometric thought, learning activities should be organized according to the five learning phases. By demonstrating how the Van Hiele phases could be utilized in teaching geometric proofs, this research makes a significant contribution to existing knowledge on Van Hiele theory-based instruction.

In the guided orientation phase, the teacher/researcher employed Cheng and Lin’s (2009) step-by-step unrolled strategy to help students prove the conjectures established in Stage 1. Students were requested to sit in groups of three or four. Each group was given a diagram sheet with the step-by-step guiding questions (see Figure 3.22):

Conjecture:
The line drawn from the centre of a circle perpendicular to chord bisects chord
Given:
Circle with centre O and $OC \perp AB$
Required to prove:
AC = CB
Construction: Join OA and OB
(Use broken lines to show your constructions)
Hint: If we can show that ΔOCA and ΔOCB are congruent, then the sides CA and CB must be equal.
Guiding questions:
a) What can you conclude about $\angle C_1$ and $\angle C_2$? Give a reason for your conclusion.
b) If O is the centre of the circle and points A and B both lie on the circumference of the circle, what conclusion can be drawn about OA and OB? Give a reason for your answer
c) ΔOCA and ΔOCB are joined along OC. What conclusion can be drawn about OC?
d) Based on what you concluded in (a), (b) and (c), what can be concluded about ΔOCA and ΔOCB?

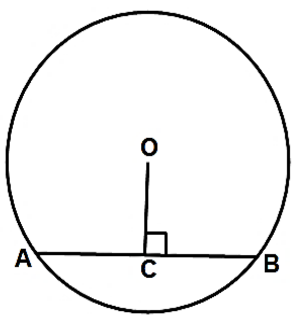


Figure 3.22: The step-by-step unrolled strategy

Similar guided orientation proof activities were prepared for the other conjectures.

3.6.2.1.5.3 Explicitation phase [Lessons 15-20]

In the explicitation phase, groups took turns to report their findings and justify their conclusions. Students were given time to criticize each other’s work.

This was intended to ensure that all conclusions made and the reasons given to support those conclusions are valid and generally accepted. The teacher/researcher explained that in the absence of further criticisms, the logical series of conclusions or statements students made would constitute a proof if they are supported by valid reasons. The final and unanimously agreed upon version of the proof of the conjecture was written on the chalkboard and the conjecture was restated as a theorem.

3.6.2.1.5.4 Free orientation phase [Lessons 15-20]

In the free orientation phase, students were assigned more complex geometry problems that required them to apply the learnt theorems. Students solved problems in groups, without receiving any guidance from the teacher.

3.6.2.1.5.5 Integration phase [Lessons 15-20]

During the integration phase, we discussed students' different solutions to problems assigned to them during the free orientation phase. The intention here was to reconcile students' solution methods into an integrated body of acceptable geometry solutions.

In Lesson 20, students were informed that we would prove riders in the next lessons. Some students wondered what proving a rider is all about. Instead of telling them what this involves, the teacher/researcher tasked the students to go and find out what proving geometric riders entails.

3.6.2.1.6 Stage 2: Formal deduction Part 2 – Proving riders

The focus of this research was on proving geometric riders. This is a more complex and more challenging activity than proving geometric theorems. Proofs of geometry theorems are procedural and routine and students can easily memorize the proofs and reproduce them in tests or examinations without understanding. Proofs of riders, by contrast, are non-procedural and non-routine. They require students to apply their reasoning, analytical, and problem-solving skills. Thus, proving geometric riders has more educational benefits than proving geometric theorems. For this reason, this research examined students' proof competencies on riders and not theorems.

The teacher/researcher hypothesized that two factors, (a) prior learning, particularly the learning of theorems, axioms, definitions, and properties of geometric shapes, and (b) organization of teaching and learning activities, would affect the students' progress in proving riders. This was informed by the Van Hiele theory. As a result, the teacher/researcher first had to take students through a variety of informal deduction activities before the learning of rider-proof.

The next section explains how the Van Hiele phases of geometry instruction were implemented in the teaching and learning of rider-proof.

3.6.2.1.6.1 Information phase [Lessons 21-24]

The purpose of the information phase was to establish what students knew about the topic and to give them an idea of what they were going to learn about in the coming lessons. In Lesson 21, students were requested to report back on the task assigned to them in the previous lesson. Some students responded by showing the teacher/researcher examples of riders in their mathematics textbook. Others mentioned that proving riders involves writing on one side, a series of statements that are supported by reasons written in short form (in brackets) on the other side. Some indicated that the process of proving riders appears to be difficult because they did not see any numbers to work with. It was encouraging to notice that the students wanted to know more about the process of proving riders. Students were commended for their efforts to get an idea of what a rider-proof entails.

The teacher/researcher explained that the process of proving riders differs from the approach used to establish conjectures during the GSP investigations. In the GSP investigations, we arrived at general conclusions based on observations of patterns and using numerical values in a few cases, in which we were not sure if our conclusions were true for all other cases. Students were informed that in the process of proving riders, we argue from the general to the particular case. The teacher/researcher further emphasized that proofs do not necessarily have to be written in a two-column format as reflected in the students' mathematics textbook. Students were informed that the idea of writing geometric proofs in two columns was developed by teachers to make the teaching, learning, and marking of geometric proofs easier. However, that is not the only way in which geometric proofs can be presented. Students were made aware that there is nothing wrong with writing a proof in paragraph form if all the necessary details are included. The teacher/researcher informed students that writing reasons in a short form is just a way to save time and is therefore not compulsory.

Students were informed that proving riders involves analysing the given information, drawing intermediary conclusions, and determining the step-by-step path that can be followed to arrive at the required conclusion. Each statement or claim that we make in the bridging process must be supported or justified using previously accepted statements which may be in the form of theorems and their converses, axioms, definitions, or properties of geometric figures. To mark the end of the lesson, students were tasked to go and write down all the theorems, converses, axioms, and properties of geometric figures that had been learnt so far, including those established in lower grades. In addition to listing theorems and their converses, axioms and properties of geometric figures, students were requested to classify the information under the following headings: lines, triangles, quadrilaterals, and circles.

In Lesson 22, the teacher/researcher divided the chalkboard into four parts, and wrote the headings: *Lines*, *Triangles*, *Circles*, and *Quadrilaterals*. Students took turns to write all their findings on the chalkboard, under the appropriate headings. We then discussed the students' findings as a class and mistakes were corrected. It was encouraging to note that students could write down most of the theorems and axioms about lines, triangles, quadrilaterals, and circles without the teacher's assistance. The teacher/researcher only assisted with the converses (where they existed) and a few other theorems and axioms which students had omitted. Students were then requested to copy the final list of theorems, converses, axioms, and properties of geometric figures into their notebooks. As part of their homework, students were tasked to go and write down the short versions of all the theorems, converses, axioms, and properties of lines, triangles, and quadrilaterals.

In Lesson 23, students took turns to write the short versions of the theorems, converses, axioms, and properties of geometric figures in the spaces provided on the chalkboard. The rest of the class were told to reserve their comments until the end of the activity. When all the items were completed, we then engaged in a class discussion to rectify mistakes and reinforce correct answers. To conclude the lesson, students were given a copy of acceptable reasons extracted from the Grade 12 Mathematics Examination Guideline to paste into the back of their notebooks. The teacher/researcher highlighted that the theorems, converses, axioms, and properties of geometric figures would be used to justify our statements/claims when proving riders. Students were informed that the next lesson would focus on forms of logic and properties of equality, which are essential in proving riders. Students were tasked to go and do some research on the transitive, substitution, addition, subtraction, reflexive, and symmetric properties of equality. The teacher/researcher advised students to use internet sources since some of this information may not be available in their mathematics textbooks.

In Lesson 24, the teacher/researcher divided the chalkboard into six parts with the following headings: *transitive property*, *substitution property*, *addition property*, *subtraction property*, *reflexive property*, and *symmetric property*, respectively. Students were given time to write down their findings in the spaces provided on the chalkboard. It was encouraging to notice that students filled all the spaces for the six properties. In writing down their findings, students used small letters. For instance, under the transitive property, they wrote: If $a = b$ and $b = c$, then $a = c$. Indeed, that is exactly what the transitive property says. The teacher/researcher commended students for their effort, and explained that properties of equality are useful not only in Euclidean geometry but also in algebra. Students were informed that there are more than six properties of equality. However, only those that were essential for proving riders were selected here. Other properties of equality include the multiplication property and the division property. These were mentioned in passing. Students were asked to go and learn more about these additional properties for enrichment purposes only.

Students were informed that in the coming lessons, they would be using the properties of equality to prove equality of angles and sides in given geometric figures. For this reason, the teacher/researcher suggested making amendments to the properties of equality that students had presented. For example, instead of writing: If $a = b$ and $b = c$, then $a = c$, we wrote: If $\hat{A} = \hat{B}$ and $\hat{B} = \hat{C}$, then $\hat{A} = \hat{C}$ and extended the result to equality of sides. That is: if $AB = CD$ and $CD = EF$, then $AB = EF$. Students were requested to copy the information in Figure 3.23 into their notebooks:

Transitive Property:	If $\widehat{A} = \widehat{B}$ and $\widehat{B} = \widehat{C}$, then $\widehat{A} = \widehat{C}$.
	If $AB = CD$ and $CD = EF$, then $AB = EF$.
Substitution Property:	If $\widehat{A} = \widehat{B}$ and $\widehat{A} = \widehat{C}$, then $\widehat{B} = \widehat{C}$.
	If $\widehat{A} + \widehat{B} = \widehat{C}$ and $\widehat{D} + \widehat{B} = \widehat{C}$,
	then $\widehat{A} + \widehat{B} = \widehat{D} + \widehat{B}$.
Addition Property:	If $\widehat{A} = \widehat{D}$, then $\widehat{A} + \widehat{B} = \widehat{D} + \widehat{B}$.
Subtraction Property:	If $\widehat{A} + \widehat{B} = \widehat{D} + \widehat{B}$, then $\widehat{A} = \widehat{D}$.
Reflexive Property:	$\widehat{A} = \widehat{A}$.
	$AB = AB$.
Symmetric property:	If $\widehat{A} = \widehat{B}$, then $\widehat{B} = \widehat{A}$.
	Similarly, if $AB = CD$, then $CD = AB$.

Figure 3.23: Properties of equality in Euclidean geometry

The teacher/researcher explained that the properties of equality are essential when combining statements in a rider-proof. The reflexive property is useful when proving congruence of triangles. The symmetric property simply reminds us that proving $\hat{A} = \hat{B}$ is the same as proving $\hat{B} = \hat{A}$. This informs us that we can work the proof from left to right or vice versa, which is an essential skill when proving riders. To conclude the information phase, students were given a few tips on how to prove riders more easily. These were developed from the suggestions by Ryan (2016):

Some Useful Tips to Solve Riders in Euclidean geometry:

- State exactly what must be proved.
- Write down all the given facts. Mark or indicate the given facts on the figure. If no diagram is provided, draw your own.
- Think what other facts can be drawn from the given information. Recall all the information (theorems, converses, axioms, definitions, and properties of geometric figures) that is related to the given facts.
- Try to apply the properties of equality to bridge your proof steps.
- If you get stuck, start from the other end of the proof, and work backwards (backward mapping).

- If you still cannot see the proof, then you need to read the given information again to make sure you have used all the givens. Examiners rarely include irrelevant information in a question.
- Remember key words such as tangent, diameter, cyclic quadrilateral, parallel, perpendicular, midpoint and bisector. Use the mnemonic DR-CPT (**D**OCTO**R** **C**A**P**E **T**OWN) to remind yourself of some of these key words:

  D-Diameter

  R-Radius

  C-Cyclic quadrilateral; Centre; Chord

  P-Parallel; Perpendicular

  T-Tangent

  These words suggest certain theorems and facts that could be useful in proving riders.
- Some questions may require you to make constructions to generate additional information.
- Look for congruent triangles and remember that congruent parts of congruent triangles are congruent. For example, if $\Delta ABC \equiv \Delta DEF$, then we can make any of the following conclusions:

  $AB = DE, BC = EF, AC = DF$ and $\hat{A} = \widehat{D}, \hat{B} = \widehat{E},\ \hat{C} = \widehat{F}$.
- Look for isosceles triangles and remember to use the 'if-sides-then-angles' or 'if-angles-then-sides' theorems.
- Look for parallel lines and if you find any, then think of the parallel-line theorems.
- Look for radii and remember that all radii of a circle are equal in length.
- Every single step in your chain of reasoning must be clearly expressed even if it appears to be obvious.
- Remember that diagrams are not necessarily drawn to scale. Therefore, you should not assume that two angles or two sides are equal just because they look equal.
- Never give up! Write down whatever you understand. Writing one step triggers another.

These tips were printed out and distributed to all students in the experimental group. Students were requested to paste the information into the back of their mathematics notebooks. Students were informed that in the next lesson they would start proving riders.

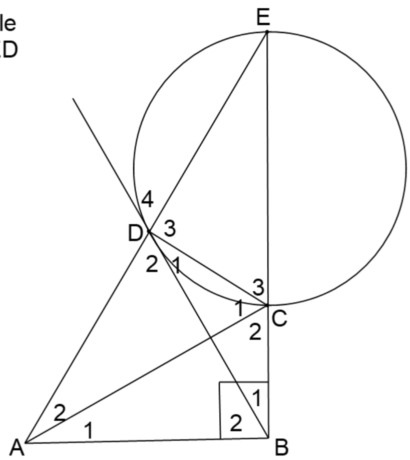
3.6.2.1.6.2 Guided orientation [Lessons 25-27]

Van Hiele's guided orientation phase involves students exploring the topic and making discoveries through guided lesson activities. The teacher/researcher prepared activities to guide students through diagram analysis, the labelling/ colouring strategy, and the proof construction process.

1) Diagram analysis

In Lesson 25, students were requested to sit in groups of three or four. The rider problem in Figure 3.24 was projected onto a whiteboard. Hard copies of the same rider were also distributed to each student. Students were given ten minutes to read the given information and analyse the given diagram. A list of guiding questions was handed out to each group to facilitate the analysis:

In the diagram alongside, EC is a diameter of circle DEC. EC is produced to B. BD is a tangent at D. ED is produced to A and AB $\perp$ BC.
Prove that:
(a) ABCD is a cyclic quadrilateral
(b) $\angle A_1 = \angle E$
(c) $\triangle$BDA is isosceles
(d) $\angle C_2 = \angle C_3$



Source: Phillips, Basson & Botha (2012, p. 241)

Figure 3.24: A typical rider for Grade 11 students

Guiding questions:

- Using '**D**OCTO**R** **C**A**P**E **T**OWN' (DR-CPT), identify key elements in the given information.
- If EC is a diameter of circle DEC, what can you conclude about $\widehat{D}_3$? Motivate your answer.
- If $\widehat{D}_3 = 90°$ and $\widehat{B} = 90°$, what conclusion can be drawn about quadrilateral ABCD? Give a reason for your answer.

- If ABCD is a cyclic quadrilateral, identify angles that are equal to $\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{B}_1$ and $\hat{A}_1$, giving reasons for your answers.
- If BD is a tangent to circle DEC at point D, then $\hat{D}_1 =$ ___ and $\hat{D}_4 =$ ____
- Lisa claims that $\hat{D}_2 = \hat{D}_4$. Is her claim valid? Explain.
- Groups were then given time to present their findings to the class. We then had a class discussion to rectify mistakes and consolidate correct responses.

2) The labelling/colouring strategy

Proving geometric riders is a complex task that places high demands on the student's working memory. Some students may be overwhelmed by the task and may give up in frustration. Labelling/colouring helps to reduce the amount of mental effort that students use to prove the rider. To conclude our diagram analysis lesson, students were requested to mark, label, or colour all equal angles in the same way using coloured pencils or markers. After students had attempted the task, the diagram in Figure 3.25 was displayed on screen for purposes of feedback and for students to see how the labelling could be done:

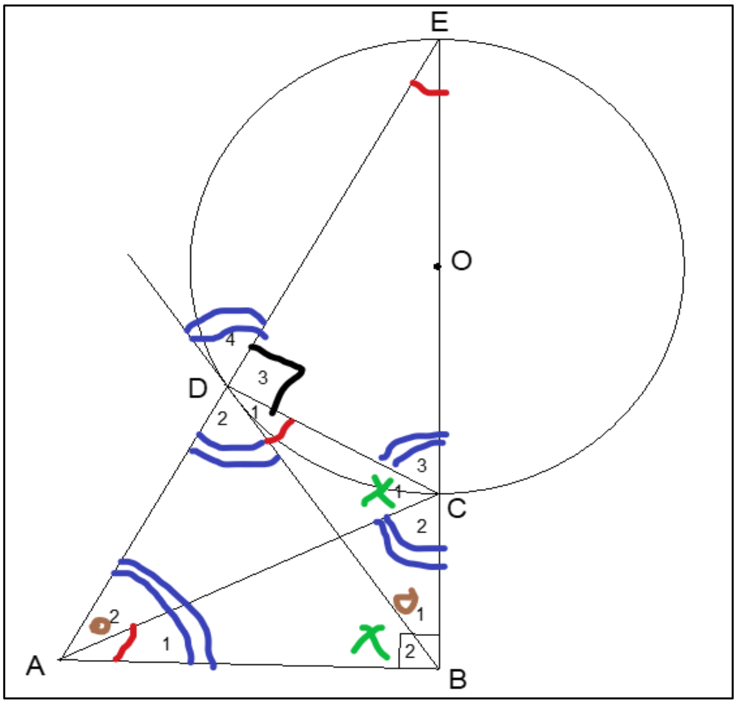


Figure 3.25: An example of the labelling or colouring strategy

Students were requested to keep their labelled diagrams safe for use in the next lesson.

3) Proof construction

In Lesson 26, the teacher/researcher reiterated that proving riders involves building a step-by-step argument using previously known facts in the form of theorems, axioms, or definitions, to arrive at the given conclusion. Students were reminded that they could write proofs using the two-column format or in the form of a paragraph. However, to avoid omitting crucial steps, they were encouraged to use the two-column method. To begin the proof construction process, students were given a task in which they constructed proofs of the riders in Figure 3.24 by filling in the missing statements and reasons (see Figure 3.26):

Proof Construction Activity 1:
Fill in the missing statements and reasons to complete the proofs:

(a) ***Required to prove****: ABCD is a cyclic quadrilateral*
$\widehat{D}_3 = 90°$ *(...)*
$\widehat{B}_1 + \widehat{B}_2 =$ .... *(Given)*
$\widehat{D}_3 = \widehat{B}_1 + \widehat{B}_2$ *(...)*
$\therefore ABCD$ *is a cyclic quadrilateral (..)*

(b) ***Required to prove****:* $\hat{A}_1 = \hat{E}$
ABCD is a cyclic quadrilateral (..)
$\therefore \hat{A}_1 =$ ....*(* $\angle s$ *in the same seg.)*
$\widehat{D}_1 = \hat{E}$ *(..)*
$\therefore \hat{A}_1 = \hat{E}$ *(...)*

(c) ***Required to prove****:* ΔBDA *is isosceles*
$\widehat{D}_2 = \widehat{D}_4$ *(...)*
$\widehat{D}_4 =$ .... *(tan-chord theorem)*
$\therefore \widehat{D}_2 =$ ....*(Both =* $\widehat{D}_4$*)*
$\hat{C}_3 = \hat{A}_1 + \hat{A}_2$ *(..)*
$\therefore \widehat{D}_2 = \hat{A}_1 + \hat{A}_2$ *(...)*
$\therefore \Delta BDA$ *is isosceles (...)*

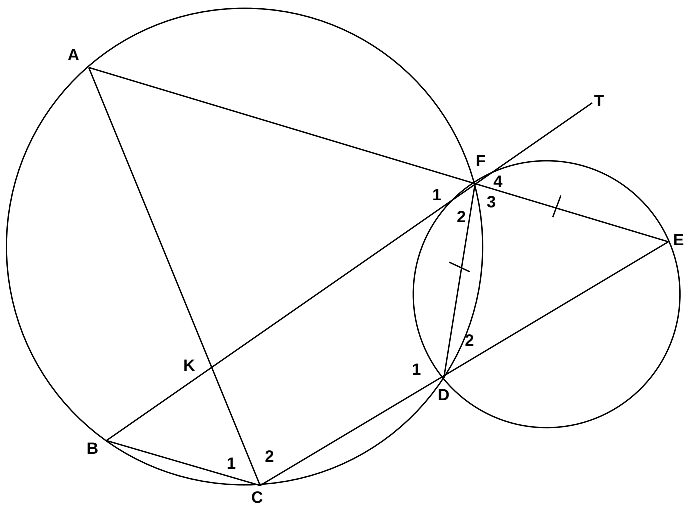
(d) ***Required to prove****:* $\hat{C}_2 = \hat{C}_3$
$\hat{C}_2 = \widehat{D}_2$ *(...)*
$\widehat{D}_2 =$ .... *(vert. opp* $\angle s$ *=)*
$\therefore \hat{C}_2 =$ .... *(Both =* $\widehat{D}_2$ *)*
$\widehat{D}_2 = \hat{C}_3$ *(..)*
$\therefore \hat{C}_2 = \hat{C}_3$ *(..)*

Figure 3.26: Proof construction task (1)

Students could refer to the properties of equality and list of theorems, axioms, and acceptable reasons recorded in their notebooks. This was a form of scaffolding learning. Students did this activity in groups and were given time to report back on their findings. We had a class discussion to iron out errors and misconceptions.

In Lesson 27, students were requested to sit in small groups. They were given another proof construction task which required them to sort given statements and reasons into meaningful proofs (see Figure 3.27):

In the accompanying figure, two circles intersect at F and D.



BT is a tangent to the smaller circle at F. Straight line AE is drawn such that $FD = FE$. CE is a straight line and chords AC and BF intersect at K. Prove that:

(a) $BT \parallel CE$

(b) $BCEF$ is a parallelogram

(c) $AC = BF$

(Department of Basic Education, 2011, p. 36)

Arrange the following statements and reasons into meaningful proofs

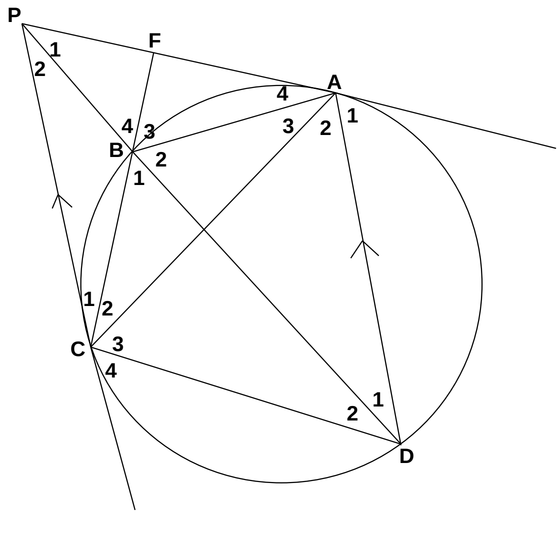
Statements	**Reasons**
(a) $\therefore$ BT **//** CE	[Both equal to $\hat{D}_2$]
$\hat{D}_2 = \hat{E}$	[tan chord theorem]
$\therefore\ \hat{F}_4 = \hat{E}$	[$\angle$s opp equal sides]
$\hat{F}_4 = \hat{D}_2$	[alt $\angle$s =]
(b) ∴ FE // CB	[proved]
BF//CE	[tan chord theorem]
BCEF is a parallelogram	[Both equal to $\hat{D}_2$]
$\hat{D}_2 = \hat{B}$	[opp sides of quad are //]
$\therefore \hat{F}_4 = \hat{B}$	[ext $\angle$ of a cyclic quad]
$\hat{F}_4 = \hat{D}_2$	[corresp $\angle$s =]
(c) AC = BF	[sides opp equal $\angle s$]
∴ AC = CE	[opp sides of a // m]
$\hat{D}_2 = \hat{A}$	[$\angle s$ opp equal sides]
CE = BF	[Both equal to $\hat{D}_2$]
$\therefore \hat{E} = \hat{A}$	[ext $\angle$ of a cyclic quad]
$\hat{E} = \hat{D}_2$	[Both = CE]

Figure 3.27: Proof construction task (2)

Students were given thirty minutes to complete the task. Diagram sheets were provided for students to practise the colouring/labelling strategy. Group leaders were given time to write their findings on the chalkboard. We then had a class discussion to rectify wrong proofs and reinforce the correct ones. The teacher/researcher emphasized that there is no single correct way to prove multi-step geometric riders. As part of their homework, students were tasked to go and try to find alternative ways to prove the riders in Figure 3.27. This was meant to help students see that the process of proving a rider does not follow a fixed sequence.

In Lesson 28, students were given time to report back on their homework activity. Mistakes were rectified and correct proofs were reinforced. We then proceeded to our last proof construction task in which students had to identify and correct errors and misconceptions in the given proofs. Students were asked to sit in small groups and the task in Figure 3.28 was distributed to all students.

PA and PC are tangents to the circle at A and C. $\mathrm{AD} \parallel \mathrm{PC}$, and PD cuts the circle at B. CB is produced to meet AP at F. AB, AC and DC are drawn.



Prove that:

(a) AC bisects $\mathrm{P\hat{A}D}$

(b) $\hat{\mathrm{B}}_1 = \hat{\mathrm{B}}_3$

(c) $\mathrm{A\hat{P}C} = \mathrm{A\hat{B}D}$

Source: (Phillips, Basson, & Botha, 2012, p. 241)

Figure 3.28: Proof construction task (3)

The following proof solutions contain numerous errors and misconceptions. Identify what is wrong in each case. Then write down the corrected proofs:

Proof attempt 1:

(a) Required to prove: AC bisects $\text{P}\widehat{\text{A}}\text{D}$

$\text{PA} = \text{PC}$ (Given) [Line 1]

$\widehat{\text{A}}_{3+4} = \widehat{\text{C}}_1 + \widehat{\text{C}}_2$ ($\angle$s opp equal sides) [Line 2]

$\widehat{\text{C}}_1 + \widehat{\text{C}}_2 = \widehat{\text{A}}_2$ (corresp. $\angle$s; AD $\parallel$ PC) [Line 3]

$\therefore \widehat{\text{A}}_{3+4} = \widehat{\text{A}}_2$ (Both $= \widehat{\text{C}}_1 + \widehat{\text{C}}_2$) [Line 4]

$\therefore$ AC bisects $\text{P}\widehat{\text{A}}\text{D}$ [Line 5]

Proof attempt 2:

(a) Required to prove: AC bisects $\text{P}\widehat{\text{A}}\text{D}$

$\widehat{\text{A}}_3 + \widehat{\text{A}}_4 = \widehat{\text{C}}_3$ (alt $\angle$s; AP $\parallel$ CD) [Line 1]

$\widehat{\text{C}}_3 = \widehat{\text{C}}_1 + \widehat{\text{C}}_2$ (AC bisects $\text{P}\widehat{\text{C}}\text{D}$) [Line 2]

$\therefore \widehat{\text{A}}_3 + \widehat{\text{A}}_4 = \widehat{\text{C}}_1 + \widehat{\text{C}}_2$ (Both $= \widehat{\text{C}}_3$) [Line 3]

$\widehat{\text{C}}_1 + \widehat{\text{C}}_2 = \widehat{\text{A}}_2$ (alt $\angle$s; PC $\parallel$AD) [Line 4]

$\therefore \widehat{\text{A}}_3 + \widehat{\text{A}}_4 = \widehat{\text{A}}_2$ (Both $= \widehat{\text{C}}_1 + \widehat{\text{C}}_2$) [Line 5]

$\therefore$ AC bisects $\text{P}\widehat{\text{A}}\text{D}$ [Line 6]

Proof attempt 3:

(a) Required to prove: AC bisects $\text{P}\widehat{\text{A}}\text{D}$

$\widehat{\text{A}}_2 = \widehat{\text{C}}_4$ (tan chord theorem) [Line 1]

$\widehat{\text{C}}_4 = \widehat{\text{C}}_1 + \widehat{\text{C}}_2$ (vert. opp $\angle$s) [Line 2]

$\therefore \widehat{\text{A}}_2 = \widehat{\text{C}}_1 + \widehat{\text{C}}_2$ (Both $= \widehat{\text{C}}_4$) [Line 3]

$\text{PA} = \text{PC}$ (tans from same pt) [Line 4]

$\widehat{\text{C}}_1 + \widehat{\text{C}}_2 = \widehat{\text{A}}_3 + \widehat{\text{A}}_4$ ($\angle$s opp equal sides) [Line 5]

$\therefore \widehat{\text{A}}_2 = \widehat{\text{A}}_3 + \widehat{\text{A}}_4$ (Both $= \widehat{\text{C}}_1 + \widehat{\text{C}}_2$) [Line 6]

$\therefore$ AC bisects $\text{P}\widehat{\text{A}}\text{D}$ [Line 7]

Proof attempt 1:

(b) Required to prove: $\hat{B}_1 = \hat{B}_3$

$\hat{B}_1 = \hat{A}_2$ (∠s in the same seg) [Line 1]

$\hat{A}_2 = \hat{D}_1 + \hat{D}_2$ (∠s opp equal sides) [Line 2]

$\therefore \hat{B}_1 = \hat{D}_1 + \hat{D}_2$(Both $= \hat{A}_2$) [Line 3]

$\hat{D}_1 + \hat{D}_2 = \hat{B}_3$ (tan − chord theorem) [Line 4]

$\therefore \hat{B}_1 = \hat{B}_3$ ($\hat{D}_1 + \hat{D}_2$) [Line 5]

Proof attempt 2:

(b) Required to prove: $\hat{B}_1 = \hat{B}_3$

$\hat{B}_1 = \hat{B}_4$ (vert. opp ∠s) [Line 1]

$\hat{B}_4 = \hat{B}_3$ ($\Delta BPF \equiv \Delta BAF$) [Line 2]

$\therefore \hat{B}_1 = \hat{B}_3$(Both $= \hat{B}_4$) [Line 3]

Proof attempt 3:

(b) Required to prove: $\hat{B}_1 = \hat{B}_3$

$\hat{B}_1 = \hat{B}_3$ (vert. opp ∠s) [Line 1]

Proof attempt 1:

(c) Required to prove: $A\hat{P}C = A\hat{B}D$

$A\hat{P}C = \hat{D}_1 + \hat{D}_2$(opp ∠s of a ||m) [Line 1]

$\hat{D}_1 + \hat{D}_2 = \hat{A}_1$(alt ∠s; AP ∥ CD) [Line 2]

$\therefore A\hat{P}C = \hat{A}_1$(Both $= \hat{D}_1 + \hat{D}_2$) [Line 3]

$\hat{A}_1 = \hat{B}_2$(tan − chord theorem) [Line 4]

$\therefore A\hat{P}C = \hat{B}_2 = A\hat{B}D$ (Both $= \hat{A}_1$) [Line 5]

Proof attempt 2:

(c) Required to prove: $A\hat{P}C = A\hat{B}D$

$A\hat{P}C = \hat{C}_4$ (Corresp. $\angle$s; AP $\parallel$ CD) [Line 1]

$\hat{C}_4 = \hat{B}_1$ (tan −chord theorem) [Line 2]

$\therefore A\hat{P}C = \hat{B}_1$(Both $= \hat{C}_4$) [Line 3]

$\hat{B}_1 = \hat{B}_2$ ($\Delta BCD \equiv \Delta BAD$) [Line 4]

$\therefore A\hat{P}C = \hat{B}_2 = A\hat{B}D$ (Both $= \hat{B}_1$) [Line 5]

Proof attempt 3:

(c) Required to prove: $A\hat{P}C = A\hat{B}D$

$A\hat{B}D = \hat{C}_3$ ($\angle$s in the same seg) [Line 1]

$\hat{C}_3 = \hat{D}_1 + \hat{D}_2$($\angle$s opp equal sides) [Line 2]

$\therefore A\hat{B}D = \hat{D}_1 + \hat{D}_2$ (Both $= \hat{C}_3$) [Line 3]

$\hat{D}_1 + \hat{D}_2 = A\hat{P}C$ (opp $\angle$s of a $\parallel$ m) [Line 4]

$\therefore A\hat{B}D = A\hat{P}C$ (Both $= \hat{D}_1 + \hat{D}_2$) [Line 5]

Each group was given time to report back on the errors and misconceptions they had found in each proof attempt. Students were also requested to write the corrected proofs on the chalkboard. The rest of the class could comment on each report to indicate whether they agreed or disagreed with it, or to add or subtract from what was presented. The teacher/researcher facilitated the discussion and finally concluded on the group findings.

3.6.2.1.6.3 Explicitation [Lesson 29]

In the explicitation phase, students were given the opportunity to verbally express and exchange their views about the proving process, based on what they had observed and learnt in the guided orientation phase. Students were then informed that in the coming lessons they would be proving riders without the teacher/researcher's guidance.

3.6.2.1.6.4 Free orientation [Lessons 30-33]

In the free orientation phase, students were given multi-step proof tasks to work on (see Worksheets 1-8 in Appendix H). Students worked independently of the teacher/researcher, hence the term 'free orientation'. They could work individually, in pairs, or in groups according to their preferences.

3.6.2.1.6.5 Integration [Lessons 30-33]

The integration phase was merged with the free orientation phase. Towards the end of each free orientation activity, we spared time to review the different approaches students had used to prove the given riders. Correct approaches were reinforced and wrong ones were corrected. The teacher/researcher presented alternative proofs to supplement what the students had presented in some cases. It was emphasized that geometric riders can be proved in multiple ways and that there is no fixed starting point in writing a rider-proof. What is important is to present a logical series of deductive statements justified by acceptable reasons. The teacher/researcher also stressed the essentiality of diagram analysis and the colouring/labelling technique before proving riders. Common errors and misconceptions were highlighted.

In the last few minutes of Lesson 33, the teacher/researcher announced the date for writing the post-test and the students were encouraged to prepare adequately for the test.

3.6.2.2 Conventional teaching

Students in the control group schools were taught by their mathematics teachers. A profile of the Euclidean geometry lessons delivered in the control schools is presented in Appendix Q. It is important to note that the same teaching methods (telling, explanation, question and answer, and illustration) are suggested in all Euclidean geometry lessons (see Appendix Q). Conventional teaching in the context of this research therefore refers to teaching by using the usual methods.

Based on peer observation and a review of the available literature, Euclidean geometry lessons in many classrooms are characterized by teachers copying theorems and proofs from the textbook onto the chalkboard, and students copying theorems and proofs into their notebooks. Teachers employing conventional methods in teaching Euclidean geometry move straight into proof and assume students have mastered the necessary prerequisites (such as definitions and properties of geometric figures) from lower grades. Students are not given an opportunity to investigate, observe and discover geometry theorems and axioms for themselves. Definitions, theorems, axioms, properties of geometric figures, and proofs are presented as ready-made ideas to be memorized by the students. The mathematics teacher and the mathematics textbook are regarded as the only sources of Euclidean geometry knowledge. Students who fail to understand the geometry presented by these two sources are considered unable to learn geometry.

Despite such teaching practices being widely criticized, their popularity remains high. The reasons why teachers continue to utilize traditional approaches in teaching Euclidean geometry were highlighted in Chapter 2 of this report.

3.6.3 Post-test administration

The post-test was written on a Friday of the fourth week in the third quarter of the year according to the South African school calendar. The teacher/researcher prepared the answer sheets for the post-test with the help of the research assistants. The coding system used in the pre-test was maintained. The only difference was that the answer sheets were labelled ‘post’. The answer sheets and coding were checked by the teacher/researcher before packaging. Packaging of test papers and answer sheets was done by the teacher/researcher. The research assistants had no access to the test papers prior to the date set for writing the post-test to prevent leakage of test papers and to protect the integrity and credibility of the post-test results. The post-test papers and answer sheets were delivered by the researcher to principals of participating schools the day before the date set for writing the test to prevent unnecessary delays on the day of writing the test. The research assistants were again reminded to invigilate scrupulously. The school principals were requested to only release the test material to the research assistants on the set date and at the appropriate time. To ensure equality of test conditions between the experimental and control group schools, the post-test was written on the same day at all four schools, also starting and ending at the same time. Students' post-test scripts and all test papers were collected, packed, and sealed by the research assistants and submitted to the teacher/researcher on the day the test was written.

The post-test scripts were marked by the same person who marked the pre-test scripts to ensure consistent marking. The recording of marks was done by the research assistants and verified by the teacher/researcher. The marker and the research assistants were remunerated for their services.

3.6.4 Diaries

The teacher/researcher met with the selected diarists during the first week of the third term (in the month of July of the year 2016), to discuss how the diary was to be completed. Each diarist was given a portable notebook to use as a diary. In addition, each diarist received a diary guide that outlined the purpose of the diary, variables of interest, issues of anonymity and confidentiality, and when the diary was to be completed. The teacher/researcher explained all the details of the diary guide and diarists could ask questions where they needed further clarity.

The teacher/researcher communicated with the diarists on a weekly basis to check on their progress and to encourage them to keep recording. Diaries were collected on the day that the students wrote the post-test.

3.6.5 Focus group discussions

Focus group discussions took place a week after post-test administration. Selected participants were informed in advance about the purpose, venue, date, and time of the focus group discussions. To avoid interfering with teaching and learning time, discussions were held after school hours at a local community hall that serves the township in which the schools are located. The teacher/researcher arranged transport to carry the participants from school to the venue. Food and refreshments were provided for the participants. A professional interviewer (with a Bachelor of Arts degree in English and Communication) was hired to facilitate the focus group discussions. The facilitator was first introduced to the students during the treatment period and had made several visits to the participating schools to create a good relationship with the students. The discussions lasted between one and half to two hours. The teacher/researcher stayed out of the discussions to avoid biased responses especially with the experimental group's students. The discussions were captured using a digital audio recorder. The teacher/researcher arranged transport to carry students to their respective homes after the discussions.

3.7 Data analysis

The research questions were answered by collecting and analysing both quantitative and qualitative data.

3.7.1 Quantitative data analysis

This study's quantitative phase explored the effect of Van Hiele theory-based instruction on the achievement of Grade 11 students in constructing non-routine geometric proofs. The study hypothesized that using Van Hiele theory-based instruction would have a statistically significant effect on the achievement of Grade 11 students. The hypothesis was tested using non-parametric analysis of covariance, taking pre-test score as a covariate. Initially, parametric analysis of covariance (ANCOVA) was identified as a suitable statistical tool for analysis of quantitative data in this study. ANCOVA assumes homogeneity of error variance and homogeneity of regression slopes across control and treatment groups. ANCOVA also assumes normality of the data. Due to violations of the assumption of normality and the assumption of equal error variances, non-parametric ANCOVA was used instead.

In non-parametric ANCOVA, non-parametric regression curves between the covariate and the dependent variable are fitted across control and treatment groups. A test for the difference in curves between control and treatment groups is performed. Non-parametric regression curves are plotted using two alternatives:

1) using smoothing models, and
2) using locally-weighted smoothing models.

A smoothing model based non-parametric regression curve is fitted using the "sm" package in R application. This package fits smoothing curves to both control and treatment groups using a smooth curve developed based on the smoothing parameter specified by alpha $= \frac{2r}{n}$, where r is the range of the data and n is the sample size (Bowman & Azzalini, 1997). Alternatively, locally-weighted smoothing model based non-parametric curves are fitted and tested using ‘fANCOVA’ package in R. The package ‘fANCOVA’ includes a set of R-functions to perform non-parametric ANCOVA for regression curves or surfaces.

In this study, non-parametric regression curves were fitted in R using both the smoothing model and the locally-weighted polynomial smoother. Three different methods for testing the equality or parallelism of non-parametric curves are available in ‘fANCOVA’: (1) based on an ANOVA-type statistic, (2) based on L-2 distance, and (3) based on variance estimators. The equality of the non-parametric curves was tested using an ANOVA-type statistic. If the p-value is below or equal to .05, the null hypothesis of no substantial difference in non-parametric curves between the control and treatment groups must be dismissed.

The testing of the significance of the null hypothesis alone is not sufficient and does little to advance scientific knowledge (Sun, Pan, & Wang, 2010). On the one hand, obtaining a statistically significant result does not automatically mean that findings are practically significant. On the other hand, obtaining a non-significant finding does not necessarily mean that results are not important. A statistically insignificant finding with a substantial effect size can be obtained (see for example Kirk, 1996). Concluding that findings are not practically meaningful based solely on lack of statistical significance could therefore be a big mistake. That is why it is highly recommended to measure the magnitude of the treatment effect for both significant and non-significant findings to help readers understand the practical significance of the results (Lakens, 2013; Lipsey et al., 2012; Sun et al., 2010).

In this research, partial eta-squared $\left(\eta_p^2\right)$ was used as an effect size measure. Partial eta-squared indicates the percentage of variance in the dependent variable that can be attributed to the independent variable while controlling for effects that are not accounted for by the model (such as individual differences and error). Partial η^2 is the commonly published estimation of the effect size in educational research for ANOVA-type studies (Hampton, 2012). This is so because it can easily be calculated from the information provided by SPSS. Partial eta-squared statistic is calculated as follows:

$$partial\ \eta^2 = \frac{SS_{treat}}{SS_{treat} + SS_{error}}$$

where: SS_{treat} = sum of squares for treatment

SS_{error} = sum of squares for error term associated with the treatment

As a rule of thumb, partial eta-squared effect size values are interpreted as small $(.01 \leq \eta_p^2 < .06)$, medium $(.06 \leq \eta_p^2 < .14)$, and large $(\eta_p^2 \geq .14)$ (Richardson, 2011).

3.7.2 Qualitative data analysis

Focus group discussions were conducted by the hired interviewer and were recorded using a digital audio recorder. The teacher/researcher transcribed the audio recordings of focus group discussions and the moderator audited them. Focus group data were coded using Computer Assisted Qualitative Data Analysis Software (CAQDAS), and diary information was coded through snapshots.

3.7.2.1 Transcribing focus group discussion audio recordings

The audio recordings were transferred from the digital recorder to the researcher's laptop. A folder with the name 'Focus group discussions' was created for the audio files. The audio files were named FG discussion C1, FG discussion C2, FG discussion E1 and FG discussion E2, to represent the participating schools, C1, C2, E1 and E2, respectively. Transcribing is a process of transforming audio data into textual data. Although there is no specific protocol for transcribing audio data, the present research followed guidelines suggested by McLellan, MacQueen, and Neidig (2003) to generate transcripts that are systematic and consistent. This is essential if the findings are to be credible.

The introductory and warm-up sections of the focus group discussions were excluded from the transcription because they were not needed for the data analysis. The process of transcribing started with the researcher listening to the audio several times before typing. The audio recordings were then transcribed verbatim (that is, exactly as said by the participants), including the filler words (for example, uhm, uh, like, eh), grammatical errors, mispronounced words, vernacular language, slang, word repetitions, and misused words. In cases where the researcher could not hear what was said by the speaker, the phrase 'inaudible segment' was typed in square brackets, together with a time stamp. Where two speakers spoke at the same time, making it impossible to decipher what was said by each speaker, the phrase 'cross talk' was placed in square brackets as suggested by McLellan et al. (2003). The participants actual names were replaced by pseudonyms.

A section break was inserted after each speaker's contribution to meet the requirements for qualitative data analysis with MAXQDA (see section 3.7.2.2). Each transcript was reviewed for accuracy by checking the transcript against the audio three times (McLellan et al., 2003). Transcription errors were corrected. The final scripts were saved in Rich Text Format (RTF), which makes it easier to import the documents into MAXQDA. The transcripts were coded FG C1, FG C2, FG E1 and FG E2, to represent focus group discussions with participants from schools C1, C2, E1 and E2, respectively. The files were saved in a folder named Focus group discussion transcripts.

3.7.2.2 Coding focus group discussion transcripts with MAXQDA

Coding is the process of assigning labels to the information that answers the research question(s) (Bazeley & Jackson, 2013). The coded data may be a single word, a phrase, a full sentence, a picture, or an entire page of text (Saldaña, 2013). Saldaña (2013) adds that there is no perfect way of coding qualitative data, because research questions are unique to context. It is a matter of choosing the right instrument for the right job, a characteristic of the pragmatist paradigm.

Coding of focus group discussion data was done using software known as MAXQDA, Version 2018. MAXQDA is a software package developed by a company called VERBI GmbH, based in Berlin, Germany. The program offers tools for importing documents, coding, categorizing text segments, and retrieving the coded segments. MAXQDA's user interface has four basic windows:

1) Document System window,
2) Code System window,
3) Document Browser window, and
4) Retrieved Segments window (see Figure 3.29):

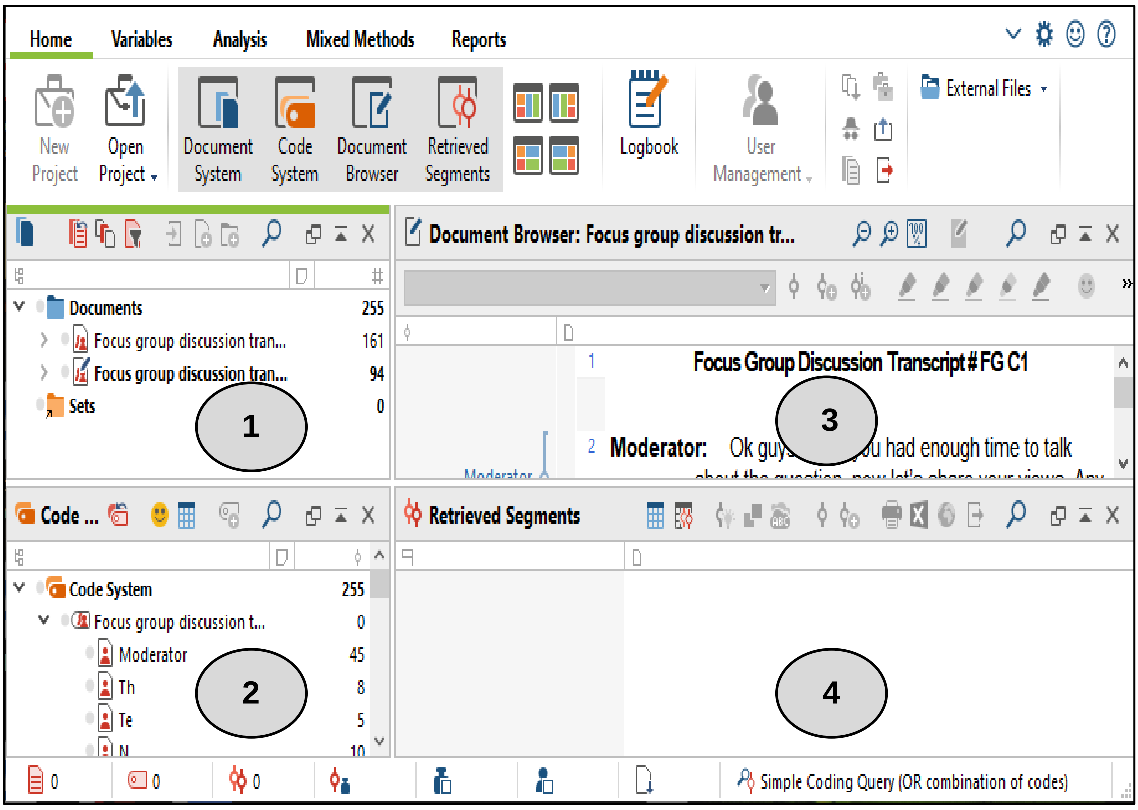


Figure 3.29: A snapshot of MAXQDA's user interface

The transcripts of the focus group discussions were imported into the Document System window using the Import feature of MAXQDA. Nodes or 'containers' for saving coded text segments were created and displayed in the Code System window. The labels for the experimental group data containers were: experimental group views and thoughts, experimental group feelings and emotions, experimental group attitudes, and experimental group likes and dislikes. Similarly, the labels for the control group data containers were: control group views and thoughts, control group feelings and emotions, control group attitudes, and control group likes and dislikes. These categories were based on the questions posed during discussions of the focus groups.

To open the transcript of the first focus group discussion, simply double-click it in the Document System where the file was imported and stored. The document is then shown in the Document Browser window where relevant information can be coded. MAXQDA automatically assigns paragraph position numbers to both the moderator's and participants' contributions. It is for this reason that a section break was inserted after every speaker's contribution during formatting of the focus group discussion transcripts. To code a word, phrase, single or multiple sentences, the researcher could simply highlight the segment of text to be coded in the Document Browser window or drag and drop it into the appropriate node or category created in the Code System window. The same procedure was followed to code relevant information in the four focus group transcripts. To retrieve coded text segments in a document, right-click respectively on the document and code names in the Document System and Code System. The coded segments will then be shown in the Retrieved Segments window to view. The retrieved segments were then exported to a word processor and saved as a document in the rich text format.

3.7.2.3 Coding diary records

Preparation of diary text for analysis started with numbering the pages in participants' diaries. The researcher then scanned all the pages of each participant's diary and saved each diary as a separate portable document format (PDF) file. The files were then saved in two separate folders labelled 'experimental group participant diaries' and 'control group participant diaries'. As was the case with focus group discussion transcripts, an a priori coding system was used to code the diary data. An a priori coding system uses pre-determined labels or categories to code the data (Saldaña, 2013). The researcher created a new Word document to save the coded segments of the participants' diary records.

Categories for coding relevant diary information were created in the Word document based on the guidelines given to diarists in their diary guide. Experimental group diary data were categorized as follows: experimental group diarists' views on lesson presentation, experimental group diarists' feelings and emotions, and experimental group diarists' reports of good and bad teaching practices. Similarly, control group diary data were coded as: control group diarists' views on lesson presentation, control group diarists' feelings and emotions, and control group diarists' reports of good and bad teaching practices. Coding of diary data was therefore a matter of taking a snapshot of the relevant text in each PDF diary document and pasting it under the appropriate category in the Word document file. The researcher then typed the identity of the diarist and the page number next to each coded segment.

3.7.2.4 Analysing coded segments

Data from focus group were categorized based on the main questions that were asked during the focus group discussions. Likewise, the diary data were categorized according to the guidelines provided to diarists in the diary guide. The categorized data were then compared to find similarities and differences in the views of the participants on the use of Van Hiele theory-based instruction and conventional approaches in the teaching and learning of Grade 11 Euclidean geometry and proof. Specific quotations from the focus group conversations and text section excerpts from the diaries of participants were presented, summarized, and discussed to answer the qualitative research questions.

3.7.2.5 Trustworthiness

In quantitative research, validity, reliability, and generalizability make up the scientific trinity that is used to evaluate the rigour of the study. However, these terms do not fit well with qualitative research (Noble & Smith, 2015). Trustworthiness is the term used in qualitative research to judge the rigour of the study (see Rolfe, 2006). Trustworthiness refers to the researcher's degree of confidence in the credibility, transferability, dependability, and confirmability of the qualitative research findings (Andrew & Halcomb, 2009). Maintaining trustworthiness in qualitative research is essential to enable readers to accept or refute the results.

Credibility refers to the veracity of the findings. Transferability/applicability is the extent to which results can be extended to other similar situations and environments, or with other classes (Ziyani, King, & Ehlers, 2004). Dependability/consistency is concerned with the stability of results over time (Bitsch, 2005). Results are consistent or dependable if, given the same raw data, other researchers would arrive at the same interpretations and conclusions. Confirmability/neutrality gives the reader the assurance that the qualitative data and its interpretation accurately reflect the views given by the participants and are not influenced by the personal interests, motivations, and perspectives of the researcher (Moon, Brewer, Januchowski-Hartley, Adams, & Blackman, 2016; Korstjens & Moser, 2018).

Several steps were taken to enhance the credibility of the data from focus group discussions. An outsider was hired to facilitate discussions. This was done to avoid the presence of the researcher biasing students' responses since the researcher was involved in administering treatment in the experimental group schools. The chosen facilitator entered the field a month before the focus group discussions were conducted to build a good relationship with the students. Prolonged engagement in the research site helps to gain the trust of the participants (Anney, 2014; Onwuegbuzie & Leech, 2007). As rapport increases, informants are more likely to disclose sensitive information and give honest responses (Krefting, 1991).

In addition, the credibility of focus group discussion data was enhanced by using triangulation. Triangulation refers to the use of multiple methods, researchers, and approaches to investigate the same phenomenon (Korstjens & Moser, 2018). Triangulation may also take the form of data collection at different times, places and people (Nokleby, 2011). In this research, data were gathered at different times and locations from four different focus groups. The data from focus group discussions were compared to what participants reported in their diaries. This was a methodological type of triangulation.

In a naturalistic inquiry, it is often not desirable to show that results are transferable to other contexts (Shenton, 2004). However, in the event that some readers and researchers may be interested in extrapolating the qualitative findings of this research to other contexts, a detailed description of the methodology, the context of the study, the research site, the sample and the sampling techniques used have been provided in earlier sections of this chapter.

The dependability of focus group discussions and diary data was enhanced by giving a concise and transparent overview of the qualitative research process from preparation, through the development of the focus group and diary guides to reporting results (see Noble & Smith, 2015). This is intended to make sure that an independent researcher can replicate the study and arrive at similar findings. The review of qualitative data findings by fellow postgraduate students and postgraduate supervisors at seminars and conferences also helped to ensure that the study complied with appropriate standards.

The findings of the qualitative research were made available to the students who participated in the qualitative study to confirm that their views were correctly expressed (see Holloway & Wheeler, 2010; Saldaña, 2013; Thomas & Magilvy, 2011). The researcher used quotations from the transcribed text and diaries of participants to show the link between the data and the findings (Elo, et al., 2014). However, care was taken to avoid overuse of the quotations as this has the potential to weaken the analysis (Graneheim & Lundman, 2004). The interview transcripts are attached to this report (see Appendix N). The audio recordings of focus group discussions, and the diaries of the participants were stored in a safe place.

3.8 Ethical issues

Research ethics refers to a system of morality that regulates the actions and decisions of the researcher during the study, starting from the conception of a research topic to the dissemination of research findings (Fouka & Mantzorou, 2011). Research ethics enable the researcher to behave in the correct and appropriate manner (Govil, 2013). To this end, the researcher took several steps to ensure this work complied with acceptable ethical standards.

The researcher made sure, in the initial phase, that this research was not a duplication of what was already done elsewhere by reading extensively around the research subject to define the research gap. The research proposal, together with the data collection tools, was then submitted to the Research Ethics Committee of the University of South Africa and ethical clearance was granted (see Appendix A1). Permission was also obtained from the Limpopo Department of Basic Education to engage public schools in this research. The authorization was granted (see Appendix A2).

Further ethical considerations that were made apply to the participants, fellow researchers, recipients of the results of educational research, and those who contributed directly and indirectly to this study. The following section provides the details.

3.8.1 Participants

Study participants are individuals or groups of individuals who engage directly or indirectly in the study process (Govil, 2013). This study involved school principals, Grade 11 students, mathematics teachers, parents, research assistants and mathematics education specialists. They were all entitled to four categories of rights: the right to informed consent, the right to anonymity and confidentiality, the right to benefit and not be harmed, and the right to privacy

(Lincoln, 2009; Ramrathan, Le Grange, & Shawa, 2017).

3.8.1.1 Informed consent

Informed consent means that the participants decide to participate willingly in the study and fully understand the demands and inconveniences associated with their participation. The researcher approached the sampled schools and obtained informed consent from the participants. The consent forms set out the research purpose, the associated procedures, the benefits of participation, and the demands involved (see Appendix B). Participants were told that they were free to refuse to participate without penalties. It was further clarified that they had the right to withdraw from the study at any time without giving any reasons. Those who opted out of the study were not forced to stay.

3.8.1.2 Anonymity and confidentiality

Participants were informed that their identities would not be disclosed when the research findings are reported. To guarantee anonymity, codes and pseudonyms were used to cover up real names. During the focus group discussions, the participants were not forced to answer the questions. Some of the participants decided not to return their diaries and their confidentiality was respected. Participants were informed that the raw data collected from them in this research would be kept in a secure place and treated with the utmost confidentiality.

3.8.1.3 Beneficence and nonmaleficence

Beneficence and nonmaleficence are concepts that are widely used in the health and nursing department but are also applicable to educational research. The concepts of beneficence and nonmaleficence recommend, respectively, that the researcher should optimize gains and reduce harm to the participants (Fouka & Mantzorou, 2011).

The research was conducted under normal teaching conditions in the natural school environment. Random selection was avoided because it would have caused discomfort to the students. Instead, the intact classes were used. Students in the control group schools were taught the usual way by their teachers. The teaching approach adopted by the researcher in the experimental group had no record of causing any harm or discomfort to the students. The topic taught at the time of the experiment was in accordance with the work schedule provided to schools by the Department of Basic Education.

Students who participated in focus group discussions were provided with food and the researcher organized transport to take them home after the group discussions. Students were also provided with diaries and pens and did not have to use their own resources for this research. The tools used to collect data were pre-tested on a small sample of participants prior to their use in the full-scale study. Mistakes and ambiguity in data collection tools were addressed.

3.8.1.4 Privacy

Protecting the right to privacy means respecting the autonomy of the participants to restrict access to their personal data and opinions (see Alderson & Morrow, 2011). Privacy is violated when the opinions and personal information of the participants are exchanged without their permission or consent (Fouka & Mantzorou, 2011).

Participants were fully informed about the investigation. The methods, sampling techniques, data collection tools, and procedures were discussed with the participants prior to their implementation. The researcher mentioned the people who would be allowed access to the raw data and explained the issues of data use, data storage and destruction (see The Norwegian National Research Ethics Committee, 2019). Participants were then asked to sign consent forms to indicate that they agreed to share their personal information and opinions with the researcher.

3.8.2 Fellow researchers

This study involved a review of the work of other researchers. Care was taken to ensure that all the work cited was properly referred to. The originality of this report was checked by Turnitin to ensure that the acceptable standards were met.

3.8.3 Users of educational research

The potential users of the findings of this research are mathematics teachers, textbook publishers, policy makers, subject advisors, pre-service teacher educators, and other educational researchers. The researcher is responsible for making these people aware of the findings of the research.

Following the guidance given by Govil (2013), this report is written in a clear manner, free from technical jargon to be understood by all interested users. The research findings are not prescriptive but suggestive. The report is written in a professional manner that does not harm others' feelings. The context of the study, the extent to which results can be generalized and the limitations of the study are clearly explained. The results of the study will eventually be published in an international academic journal. Copies of this report will be submitted to the Department of Basic Education of the Province of Limpopo and to the Library of the University of South Africa.

3.8.4 Contributors to the study

All persons who contributed directly or indirectly to this research were duly acknowledged in the preliminary pages of this report (see Acknowledgements).

3.9 Chapter summary and conclusion

This chapter dealt with the methodology of the study. The epistemological and ontological assumptions of the study were revealed. The design adopted in the study was explained and justified. Sampling procedures and data collection methods were outlined and issues relating to the reliability and validity of the geometry test instrument were addressed. The development of tools for collecting qualitative data was explained. In addition, the chapter outlined the treatment procedures for the experimental and control groups. The procedures for data collection and analysis were described and the trustworthiness of qualitative data was discussed. Finally, the ethical principles which regulated the conduct of the researcher were explained.

The results of the geometry tests, the diary entries and the focus group discussions are presented in the next chapter.

CHAPTER 4

QUANTITATIVE AND QUALITATIVE DATA FINDINGS

4.1 Introduction

As discussed in Chapter 3, the design of this study used quantitative and qualitative methods in two separate phases of the study. Phase One involved the collection and analysis of quantitative (QUAN) data to answer the first question in the study:

1) Does teaching and learning Euclidean geometry theorems and non-routine geometric proofs through Van Hiele theory-based instruction have any statistically significant effect on Grade 11 students' geometric proofs learning achievement?

Phase Two involved the collection and analysis of qualitative (qual) data to answer the second question in the study:

2) What are students' views on (a) the Van Hiele theory-based approach, and (b) the conventional approach to teaching and learning Grade 11 Euclidean geometry theorems and non-routine geometric proofs?

The data is therefore presented in this chapter following the QUAN-qual sequence. In other words, quantitative data is presented and analysed first, and qualitative data is presented and analysed second. The results of the quantitative and qualitative data analyses will be linked in Chapter 5 and their implications for classroom practice will be discussed.

4.2 Phase One: Quantitative data findings

In this phase, students' test scores are analysed using procedures outlined in Chapter 3. The analysis starts with testing the data for any violation of major assumptions for parametric statistics. These include: testing for homogeneity of regression slopes, testing for assumption of normality, and testing for homogeneity of error variances. Based on the results of these tests, an appropriate statistical test is then selected to determine whether there is a statistically significant difference in students' post-test scores due to treatment effects.

However, it is important to include data on the background characteristics of the study sample. This serves to provide readers with information on the type of participants involved in the study; to clarify to whom the study findings are applicable; to shed light on the generalizability of the findings and possible limitations; and to allow future replication of the study. To this end, a summary of the background characteristics of the participants is presented in the next section.

4.2.1 Background characteristics of the students in the study

Table 4.1 shows that 14 percent (26 out of 186) of the participants were repeating Grade 11. These students were being taught Grade 11 Euclidean geometry and geometric proofs for the second time. Nineteen (19) of the repeaters were in the control group and seven (7) in the experimental group. Fifty percent of the students reported living with both parents, while the remaining fifty percent lived with either a guardian or a single parent. Most of the students (103 out of 186) reported living with parents/guardians who had tertiary (college or university) qualifications. Fifty-one were in the control group, while the remaining 52 were in the experimental group. The impact of parents' education levels on children's academic achievement is a well-researched subject (see for example Chevalier, Harmon, Sullivan, & Walker, 2013; Khan, Iqbal, & Tasneem, 2016; Pufall et al., 2016). Fortunately, the experimental and control groups had about the same number of students whose parents had completed tertiary education. Thus, the possible impact of parental education levels on the achievement of students in the geometry test was probably spread evenly between the two groups.

One hundred and eleven (59.7%) of the 186 participants indicated that their parents or guardians were employed. The rest categorized their parents as either unemployed (28%) or self-employed (12.4%). Most of the parents (117 out of 186) were earning an average income and sixty-six (66) students classified their family income as low. This may explain why only 4.3% (8 out of 186) of the parents could afford to hire a private mathematics tutor for their children. One hundred and twenty-eight (128) of the 186 students had no access to a computer at home. Since most of the parents or guardians of the participants were employed, they would probably come home late and tired. As a result, few would have spared time helping their children with school work. It can therefore be concluded that most of the students in the sample depended solely on their school teachers for assistance in mathematics. Thus, what happens inside the classroom is a key determinant of student achievement in mathematics in these environments (see Arnold & Bartlett, 2010). The data in Table 4.1 was collected using the research tool in Appendix D:

Table 4.1: Background characteristics of student participants

Background Characteristic	**Description**	n	%
Grade Repetition	Repeaters	26	14.0
	Non-repeaters	160	86.0
		186	100
Parentage	No parents/Living with guardian	16	8.6
	Living with single parent	77	41.4
	Living with both parents	93	50.0
		186	100
Parent/Guardian's Highest Level of Education	Less than Grade 12	40	21.5
	Grade 12	43	23.1
	More than Grade 12	103	55.4
		186	100
Parent/Guardian's Employment Status	unemployed	52	28.0
	Self-employed	23	12.4
	Employed	111	59.6
		186	100
Family Income	Low income	66	35.5
	Average Income	117	62.9
	High Income	3	1.6
		186	100
Home facilities	Have no access to computer	128	68.8
	Have access to computer	58	31.2
		186	100
	Do not have private tutor	178	95.7
	Have a private tutor	8	4.3
		186	100

In addition to obtaining the background characteristics of the student participants, it was also considered necessary examine the performance record of the participating schools based on the 2015 Grade 12 Mathematics results. The next section is about this aspect.

4.2.2 Participating schools' 2015 Grade 12 Mathematics results

The common feature that made the selected schools suitable for participating in this study was that they performed below 50 percent in the 2015 Grade 12 Mathematics results (see Table 4.2). This was the second group of students to write the Mathematics Paper 2 examination, which included Euclidean geometry in the CAPS. The findings from the data obtained using the research tool in Appendix C are summarized in Table 4.2:

Table 4.2: Participating schools' 2015 Grade 12 Mathematics results

School Code	Year	Number Wrote	% Achieved at 30% and above
C1	2015	68	43
C2	2015	23	39
E1	2015	47	40
E2	2015	48	27

In 2013, before Euclidean geometry was made compulsory in the Grade 12 Mathematics examination, the Mathematics pass rates for C1, C2, E1 and E2 were 45.1%, 51.4%, 85.7% and 62.5% respectively. Comparing these results with the 2015 performance shows a significant drop in performance across all four schools, with E1 and E2 having sharper declines. Based on this analysis, it was assumed that the return of Euclidean geometry had contributed to the decline in the Mathematics pass rates of the four schools. Unless teachers try something different, the negative impact of Euclidean geometry on student mathematics outcomes could be seen for many years to come.

To address the first question in this study, a quasi-experiment was conducted with Grade 11 students in their natural school settings to test whether the proposed Van Hiele theory-based instruction had a statistically significant effect on students' geometric proofs learning achievement. Descriptive statistics for study variables are given in the following section.

4.2.3 Descriptive statistics

Table 4.3 shows that 55.9% (104 out of 186) of the participants were in the control group, while 44.1% were in the treatment group. The results show that

64.5% of the students (120 out of 186) were females. Their ages ranged from 16 to 21 with an average age of 17.19 years (SD = 1.116). Pre-test scores ranged between 0 and 36 with a mean of 3.30 (SD = 5.923). Post-test scores ranged from 0 to 100 with a mean of 29.99 (SD = 30.815). There was a huge gap between the pre-test and post-test average scores, showing the possible effects of the teaching approaches used. The standard deviation for the post-test scores was higher than the standard deviation for the pre-test scores. This indicates that post-test scores were more scattered than pre-test scores.

Table 4.3: Descriptive statistics

Variable		n	%	mean	SD
Group	Control	104	55.9		
	Treatment	82	44.1		
Gender	Female	120	64.5		
	Male	66	34.5		
Age		186	100	17.19	1.116
Pretest score		186	100	3.30	5.923
Post test score		186	100	29.99	30.815

4.2.4 Parametric analysis of covariance (ANCOVA)

In this study, randomization of students into control and experimental groups was not possible. Convenience sampling was therefore used. This brought with it some confounding variables that had the potential to skew results if not controlled. The measures outlined in section 3.4.4 of Chapter 3 were a form of experimental control of confounding factors. In this chapter, analysis of covariance is used as an additional measure to control for pre-treatment performance differences. This is a form of statistical control of potential confounders.

ANCOVA is an extension of the ANOVA to include a covariate. Like analysis of variance, ANCOVA is used to test whether there is a significant difference in group means between two or more independent groups on a dependent variable. The only difference is that analysis of covariance tests for differences in group means after adjusting for the covariate. A covariate is a third variable that is included in the statistical analysis because it is believed to have the potential to affect results. In this study, differences in students' performance prior to treatment were believed to be potential confounders in the outcomes of the post-test. For this reason, the pre-test score was used as a covariate.

Parametric ANCOVA assumes homogeneity of regression slopes, normality of data, and homogeneity of error variance. If these assumptions are not met, the non-parametric ANCOVA must be used. To test for these assumptions, data were entered in SPSS Version 24 with pre-test as *covariate*, post-test as *dependent variable*, and group as a *fixed factor*. The results of the analysis are presented in the next sections.

4.2.4.1 Testing the assumption of homogeneity of regression slopes

The test for homogeneity of regression slopes is carried out using test for significance of interaction term of pre-test and group. If the p-value for the interaction term is less than .05, we reject the null hypothesis of no significant difference and conclude that there is a significant interaction between the covariate (pre-test) and the independent variable (group). This would suggest that the researcher is unable to proceed with the parametric ANCOVA.

By default, SPSS does not include an interaction term between the covariate and independent variable in its general linear model. Therefore, the researcher had to request SPSS to include the *group*pre-test* interaction term in its model. The SPSS output for homogeneity of regression slopes is shown in the table for Tests of Between-Subjects Effects (see Table 4.4):

Table 4.4: SPSS output for homogeneity of regression slopes

Tests of Between-Subjects Effects					
Dependent Variable: Post-test					
Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	123971.906	3	41323.969	145.484	.000
Intercept	107958.452	1	107958.452	380.076	.000
Group	80524.233	1	80524.233	283.492	.000
Pre-test	12198.990	1	12198.990	42.947	.000
Group * Pre-test	**129.446**	**1**	**129.446**	**.456**	**.500**
Error	51696.072	182	284.044		
Total	342948.000	186			
Corrected Total	175667.978	185			

No significant effect of interaction term was reported, F (1,182) = .456, p = .500. This result supports assumption of homogeneity of regression slopes and suggests that parametric ANCOVA could be an appropriate statistic to analyse the main effects of the study. However, there are still other main assumptions to be tested.

4.2.4.2 Testing the assumption of normality

One of the main assumptions of parametric statistics is that sample means are normally distributed across independent samples. In the present study, the normality of post-test scores was checked by testing the normality of within-group residuals and the normality of standardized residuals of the overall model. A p-value less than .05 would mean that the assumption of normality is violated. To proceed with parametric ANCOVA in this case would require the dependent variable (post-test) to be transformed using one of the many options available to coax non-normal data into normality, such as the arithmetic, square root, inverse, box-cox, or log transformations. The main disadvantage of these data transformations is that the originality of the data is lost, which may lead to an incorrect interpretation of results. Therefore, a better choice would be to abandon parametric ANCOVA in favour of non-parametric ANCOVA, which does not assume data normality.

Tables 4.5 and 4.6 show the output of the SPSS for the normality of within-groups residuals and the normality of the residuals of the overall model respectively:

Table 4.5: SPSS output for normality of group residuals

Tests of Normality							
		Kolmogorov-Smirnov			Shapiro-Wilk		
	Group	Statistic	df	Sig.	Statistic	df	Sig.
Standardized Residual for Post-test	0	.222	104	.000	.729	104	**.000**
	1	.098	82	.048	.969	82	**.042**

Notes. 0 = Control group, 1 = Experimental group

Based on the Shapiro-Wilk's normality test, both groups have violated the assumption of normality of within-group residuals ($p < .05$).

Table 4.6: SPSS output for normality of the overall model

Tests of Normality						
	Kolmogorov-Smirnov			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
Standardized Residual for Post-test	.183	186	.000	.914	186	**.000**

Table 4.6 shows a severe deviation from normality in the standardized residuals for the overall model, as assessed by Shapiro-Wilk's test ($p < .001$). This suggests that parametric ANCOVA may not be an appropriate statistic for the analysis of the data in this study. However, there is still another key assumption that needs to be tested.

4.2.4.3 Testing for homogeneity of error variance: Levene's test

The Levene's test may be used when dealing with non-normal distributions to determine the equality of error variances across groups. The following hypotheses were tested:

H_0: Error variances of post-test scores are equal across the two groups

H_1: Error variances of post-test scores differ across the two groups

If the p-value in the Levene's test output is less than .05, the null hypothesis is rejected. This would be a violation of the assumption that error variances are homogeneous. Otherwise, we will not reject the null hypothesis and conclude that post-test score error variances are homogeneous across the two groups. Table 4.7 displays the outcome of the Levene's test:

Table 4.7: SPSS output for Levene's Test

Levene's Test of Equality of Error Variances			
Dependent Variable: Post-test			
F	df 1	df 2	Sig.
96.619	1	184	.000

The results of Levene's test indicate that the null hypothesis of equal error variances of post-test scores across the two groups must be rejected at .05 level of significance (F (1, 184) = 96.619, p =.000). Therefore, the assumption of equal error variances across groups is violated.

Field (2013) recommends that Levene's test results should be double-checked by calculating the variance ratio. This is because the Levene's test is not necessarily the best statistical measure to determine whether variances are uneven enough to cause serious problems (Field, 2013). The variance ratio is determined by dividing the larger variance by the smaller variance. If the calculated variance ratio is greater than 2, then the variances are heterogeneous. The variance ratio for the post-test data was 641.76/124.17= 5.17, which is more than twice the expected ratio. This confirms that the Levene's test outcome is valid.

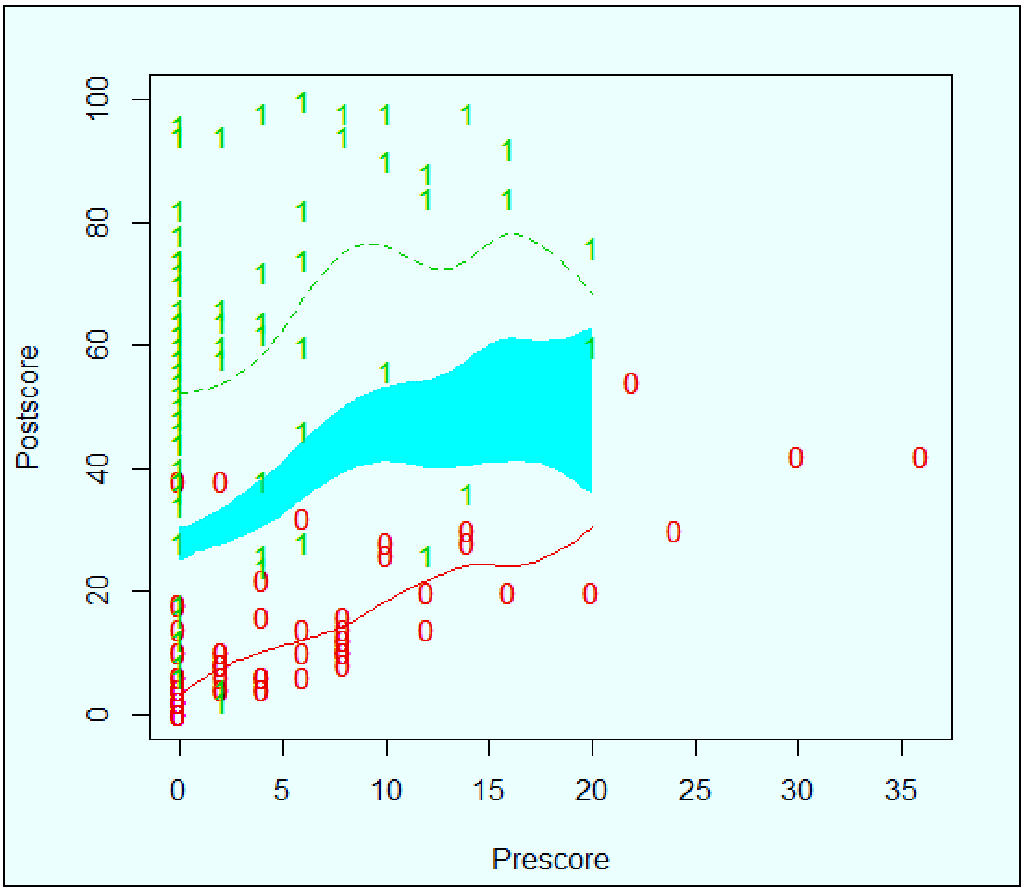
Based on these results, the researcher was unable to proceed with parametric ANCOVA. The non-parametric ANCOVA test statistic was used instead.

4.2.5 Non-parametric analysis of covariance

Parametric tests, on the one hand, assume that every continuous distribution follows a normal distribution and that the sample variances are homogeneous. Non-parametric tests, on the other hand, do not depend on any assumptions. While parametric tests have more statistical power than non-parametric tests, non-parametric tests are more robust than parametric tests. Robustness refers to the ability to withstand adverse statistical conditions such as the existence of outliers or violations of normality and homogeneity assumptions. In this study, the researcher had to compromise statistical power to gain robustness due to significant departures from normality and homogeneity of variance assumptions.

First, the collected quantitative data were prepared in Excel using the Comma-Separated Values (CSV) format. Accordingly, the prepared data file was saved as a ‘csv’ file. Non-parametric ANCOVA was performed on the data using the ‘sm’ and ‘fANCOVA’ packages built under R package version 3.4.4. The R codes used to do the analysis are found in Appendix K and the output is found in Appendix L. The acronym ‘sm’ refers to smoothing methods for non-parametric regression. The ‘sm. ancova’ function in R is used to fit a set of non-parametric regression curves with one or more covariates (see Bowman & Azzalini, 1997). The resulting curves are then compared graphically and statistically in a hypothesis test.

Figure 4.1 shows the smoothing model based non-parametric regression curves for the control and the treatment pre-test/post-test scores using the ‘sm. ancova’ function in R. The green coloured curve represents the fitted values at each observed covariate for the treatment group, while the red curve shows the fitted values at each observed covariate for the control group. The shaded region represents the band of separation between the fitted non-parametric curves for the control and treatment groups (see Figure 4.1).



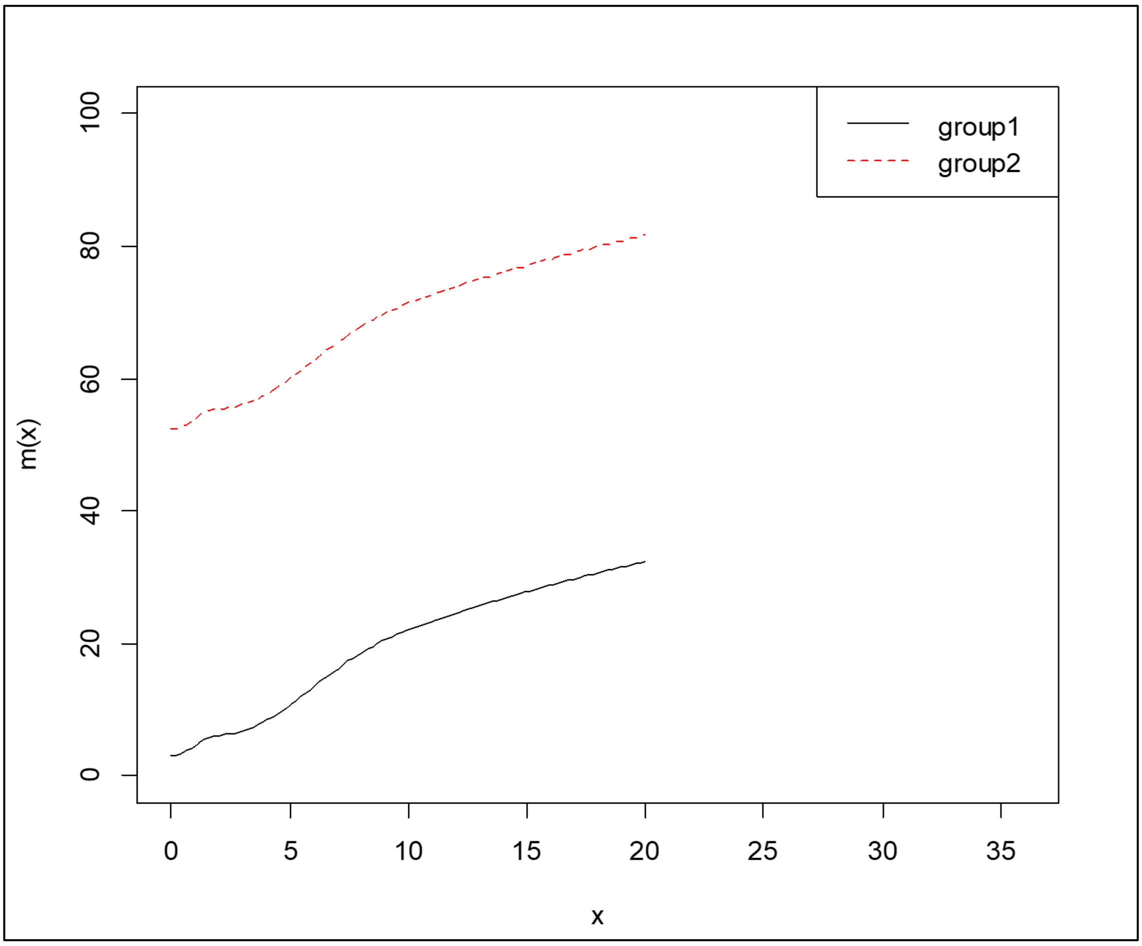
Note: 0 = Control group; 1 = Treatment group

Figure 4.1: Non-parametric smoothing curves for control and treatment groups

The ANCOVA test based on the smoothing model showed a significant difference in non-parametric regression curves between the control and the treatment groups $(h = 2.26, p = .000)$, at the 5% level of significance (see Appendix L for the output in R).

Further analysis of the data was done using the 'loess' function in the 'fANCOVA' package in R. LOESS stands for locally estimated scatterplot smoothing. It is the most flexible robust method of fitting a non-parametric model to the data because it is resistant to outliers and does not make any initial assumptions about the relationship among variables of interest (Cleveland, 1979). LOESS allows the relationship among variables to be determined by the data itself. Just like the 'sm. ancova' function, 'loess. ancova' fits smooth curves to the data using automatically selected local smoothing parameters. There are two methods for selecting smoothing parameters: AICc (bias-corrected Akaike Information Criterion) and GCV (Generalized Cross Validation) statistic. The

AICc smoothing parameter was automatically selected (see Appendix L). Figure 4.2 shows the LOESS curves fitted to the control and treatment group data. By visual inspection, the distribution of the LOESS curves indicates that the treatment group had higher scores than the control group.



Note. Group 1 = Control group; Group 2 = Treatment group

Figure 4.2: LOESS curves for treatment and control groups

Descriptive statistics produced by the 'loess. ancova' function in fANCOVA package had the estimate of intercept as 17.0987. This is the estimate of the median score for the control group. The estimated median score for the treatment group was 49.288 points higher than for the control group (see Appendix L for the output in R).

The ANCOVA test for significant difference in the two LOESS curves was carried out using function 'T.aov' in the fANCOVA package. As highlighted in Chapter 3, there are three methods available in fANCOVA to test the equality or parallelism of non-parametric curves. The function 'T.aov' is used to test the equality of non-parametric regression curves based on an ANOVA-type statistic. The results obtained from running 'T.aov' in R indicated that the null hypothesis of no significant difference in non-parametric curves between control and treatment groups must be rejected $(T = 595.9, p = .005, {\eta_p}^2 = .684)$, at the 5% level of significance (see Appendix L for the output in R).

Taken together, the non-parametric ANCOVA findings, based on both the smoothing model and the locally weighted polynomial smoothing model, showed that there was a significant difference in post-test scores between the control and the treatment groups in favour of the treatment group. Specifically, post-test results were substantially higher in the treatment group relative to the control group. It was therefore concluded that the Van Hiele theory-based instruction had a greater positive impact on the students' geometric proofs learning achievement than the conventional teaching approach.

To address the second question in this study, participants' diary records and focus group discussion data are analysed in the next section.

4.3 Phase Two: Qualitative data findings

Qualitative data were collected from the experimental and control group participants using focus group discussions and participant diaries. The aim was to explore more explanations that could add to the quantitative findings of Phase One of this study.

4.3.1 Focus group discussions

A total of four focus group discussions were held. Two of the group conversations were held with participants from the control schools, while the other two involved participants from the experimental schools. Initially, a total of 24 Grade 11 students, six from each school, were recruited to participate in the focus group discussions. However, eight of them decided to withdraw and only 16 participants participated. Each group consisted of members of both gender and had three to six participants. Table 4.8 shows the actual number of participants per focus group:

Table 4.8: Composition of student participants in each focus group

Category	Group Code	Composition of participants per group		Number of participants per group	Number of participants per category
		Males	Females		
Control	FG C1	2	1	3	9
	FG C2	3	3	6	
Experimental	FG E1	1	2	3	7
	FG E2	2	2	4	
Total		8	8		16

Table 4.8 indicates that of the 16 participants, nine were from the control group schools, and seven were from the experimental group schools. A possible reason for better interview attendance in the control group than in the experimental group could be that maybe students in the control group had more urgent concerns to raise than those in the experimental group. Differences in the number of participants across the four groups were not a matter of concern to the researcher since the purpose of the interviews was not to quantify responses but to examine the views of the participants on their Euclidean geometry and geometry proof learning experiences.

After introductions, the focus group discussions started with participants responding to warm-up and engagement questions, just to get them to speak and make them feel relaxed and familiar with the atmosphere (see Appendix M). During the focus group interviews, the main questions posed by the moderator were:

- What do you think about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?
- How do you feel about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?
- What do you like or dislike about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?

- Can you describe your attitude towards Euclidean geometry and geometric proofs?
- What did the teacher do that you think contributed to your attitude towards Euclidean geometry and geometric proofs?

Participants' responses to these questions were recorded, transcribed (see Appendix N) and coded using procedures outlined in Chapter 3. This section summarizes what the participants said in their responses. A few selected quotations are included to capture the essence of what was said and give readers an idea of how the participants responded.

4.3.1.1 Students' Euclidean geometry experiences verbalized

Focus group participants in the experimental group schools generally acknowledged that Euclidean geometry was well taught in their mathematics class:

> "I think they taught us in a good way. If I was going to rate, I would rate 10 over 10 because I understood everything about Euclidean geometry and geometric proofs. And now I have more knowledge, oh, yah"
> (**O**, FG E1, Position: 10 – 10)

> "… from my point of view, I think Euclidean geometry was taught very well in our mathematics class as we were able to solve the riders"
> (**Ha**, FG E2, Position: 12 – 12)

> "I think the way they taught us Euclidean geometry was very good and explicit because at one point they would give activities. They would leave us for like one hour … we will try to figure out how to come up with solutions, …that made us be a bit witty…because well they don't really give us answers to this question at first. They leave us then we will be able to discuss it with others, …"
> (**T**, FG E1, Position: 14 – 14)

Focus group participants from the control group schools, on the other hand, raised several concerns about how Euclidean geometry and geometric proofs were taught in their mathematics classes. Participants from school C1 shared the opinion that Euclidean geometry was not properly introduced. They frequently mentioned that key terms, such as chord and diameter, were not explained before theorems were introduced. One participant cited this as the reason students had difficulties with proofs:

"… in our school, when we were taught first time, our teacher didn't uh… *He didn't polish that* a chord is what? What is a diameter? *Where do we use it*? … He wanted to introduce Theorem 1 without introducing the first things of geometry. That's why geometry *gave us problems* when coming to the proofs"
(**Mp**, FG C1, Position: 13 – 13)

In agreement, participant **Bo** added:

"Eh Sir, the way our teacher introduced this geometry, he didn't explain what is this … inclusive [*Euclidean*] what what geometry? He didn't explain to us what kind of geometry is it and he didn't teach us how to prove it … and how some lines are called such as chord and what what is it a diameter, he just went straight to those theorems"
(**Bo**, FG C1, Position: 21 – 21)

"Eh, Sir, I think the teacher did some confusion at the first of this geometry..."
(**Bo**, FG C1, Position: 49 – 49)

Participant **Mp** articulated her view in the following way:

"… our teacher thought that because we started doing geometry … at those lowest grades, I think it's Grade 9 or Grade 10, so he thought maybe we know, what is chord, what is diameter, that's why he didn't think of touching those things …only to find that *even in the past we* didn't even understand"
(**Mp**, FG C1, Position: 23 – 23)

Focus group participants from school C2 mentioned that learning Euclidean geometry was challenging for them because their teacher rushed through the chapter and skipped certain sections:

"Some of us we find it difficult to understand because they are trying to cover the syllabus"
(**L**, FG C2, Position: 45 – 45)

"I remember there was this time Sir was going … somewhere else then he asked me to teach theorem 3, 4 and 5. So, he never came back to those theorems and show them to the whole class. I just took a book and then I write what's on the book and then I sat down"
(**N**, FG C2, Position: 27 – 27)

"They skipped other chapters [*sections*] of Euclidean geometry"
(**Ho**, FG C2, Position: 19 – 19)

"… just like the last theorems like theorem 6 and 7, … when we were doing geometry, we didn't do them"
(**Ho**, FG C2, Position: 23 – 23)

"… they did not teach us riders at all! They just teach us how the

> theorems (are) proved — proven, but riders they didn't even touch them" (**Th**, FG C2, Position: 25 – 25)

The main point emerging from these responses is that participants were dissatisfied with the way Euclidean geometry was taught in their mathematics class. Responses such as "they did not teach us riders at all" could explain why some control group participants got zero percent in the post-test. The view that the teacher in school C2 skipped riders is a possible topic for future study. It is worth knowing whether this was due to time constraints or lack of subject knowledge.

The next section describes how participants felt about the way Euclidean geometry and geometric proofs were taught in their mathematics class.

4.3.1.2 Students' feelings and emotions

Focus group participants in the experimental group schools indicated that they felt good about the way Euclidean geometry and geometric proofs were taught in their mathematics class:

> "I feel very good about it because eh, as they taught us, we were not only like listening to the teacher alone, we were giving our own thoughts, and our own like views from what we think about them… I feel good about it because we were able to do like things that I never thought I can do in my life… Firstly, when they introduced us to this topic *of* Euclidean geometry, I thought it was a difficult part but as I got to explore like *as they were teaching us about it* I was able to be free around my mates and then *I succeeded, even now* I am not like that perfect but I can do most of the things. Yah, I feel good because *it brought* a good experience … *in my life*"
> (**Mo**, FG E2, Position: 16 – 16)

> "I felt privileged to have been taught Euclidean geometry in this maths class because that GSP (Geometer's Sketchpad) theorems really works like, really helped me to be more interested in Euclidean geometry because those things I was doing them myself practically not just theoretically"
> (**Ch**, FG E2, Position: 18 – 18)

> "I feel good…because they teach us how to solve problems not only in the mathematics class but then in real life…"
> (**T**, FG E1, Position: 18 – 18)

Participant **Na** in focus group E1 explained how her feelings changed from 'bad at first' to 'good now':

> "I felt really bad at first because I had no idea what Euclidean

geometry was all about this year because we were doing something that we had never done before but then as time went on, I started feeling good because I was able to solve and come up with solutions. And it felt like I was being put on a test like as a challenge to test how far I can go or I can push myself or how I am willing to do things. So yah, I really feel good now…"
(**Na**, FG E1, Position: 16 – 16)

On the basis of these responses, it can be noted that focus group participants from experimental group schools derived positive feelings from the following aspects: active involvement in the learning process; expressing their own views and opinions about what is taught; exploring geometry concepts freely in the presence of their classmates; achieving what they thought they could not achieve; learning Euclidean geometry concepts practically, not just theoretically, and finally; seeing the relation between the concepts of Euclidean geometry and real life.

In contrast, many focus group participants from the control group schools shared negative feelings about how Euclidean geometry and geometric proofs were taught in their mathematics class. Participants from school C2 indicated that they felt bad about the way they were taught:

"Sir, I don't feel good because I don't know some of the theorems and there is a need whereby, I have to know especially riders. And riders have a lot of marks whereby when I can understand all of the theorems then I will be able to get the marks that are there"
(**Te**, FG C2, Position: 31 – 31)

"I feel bad because they did not teach us riders. Many question papers come with lots of riders. I can't write something that I don't know that's why we lose marks at geometry"
(**Th**, FG C2, Position: 41 – 41)

"I also feel bad because eh, some of us learners we prefer *that* eh, teachers should teach us and then that's where we get to understand the concepts and then when going home, we just revise and practise that"
(**N**, FG C2, Position: 43 – 43)

"It is heart-breaking when I look at the question paper, *I see a lot of marks but eish! I can't reach them because I don't have that* knowledge"
(**Te**, FG C2, Position: 65 – 65)

Focus group participants at school C1 spoke about feeling confused:

"I feel confused because when our teacher teaches us, we

> understand but when we get home, nothing! Like, we don't understand anything because the teacher is no more there"
> (**Mp**, FG C1, Position: 29 – 29)

> "I feel like this geometry is understandable but our teacher didn't be specific on that geometry, that's why we are a little bit confused"
> (**Bo**, FG C1, Position: 31 – 31)

Generally, it can be seen from the above statements that focus group participants in the control group schools were not satisfied with the teaching of Euclidean geometry and geometric proofs in their mathematics class. The participants were aware that geometry riders constitute a lot of marks in their test papers, but were disappointed that they did not have the skills required to successfully answer certain questions. For several of these participants, their mathematics teachers contributed to their negative feelings.

To gain more insight into participants' views and emotions, focus group members from both experimental and control group schools were asked to describe what they liked or did not like about how Euclidean geometry and geometric proofs were taught in their mathematics class. Participants discussed a range of teaching and learning experiences that they thought had the greatest impact on their perceptions and feelings. The next section describes the most striking responses that emerged from the group discussions.

4.3.1.3 Teaching strategies favoured and unfavoured by students

When prompted to discuss what they liked and did not like about how Euclidean geometry and geometric proofs were taught in their mathematics class, focus group participants from the experimental group schools mentioned several pedagogical practices they perceived to have had the greatest influence on their views and feelings. These included among other things: the use of Geometer's Sketchpad to investigate theorems; teaching at a slow pace; active engagement of all students in the class; and a free learning environment where making mistakes and giving wrong answers was part of the learning process. The responses of the participants representing these ideas included the following:

> "Eh, that part when we were taught in our maths class when we were using computers using the GSP software, I think when we were taught Euclidean geometry using that software was really good for us as learners because it wasn't like reading those theorems in a book. We were actually seeing them first-hand. We

were actually measuring those angles. In our books those things are not drawn to scale, you just read them and all you do is just memorise but that GSP software you can see them straight and you can measure those angles, the sides, you can see what exactly they are talking about"
(**Ch**, FG E2, Position: 14 – 14)

"What I like about the way we were taught is uh, our teacher was not in a hurry. He was patient and if a learner didn't understand he could explain more and give more examples"
(**O**, FG E1, Position: 24 – 24)

"…what I like was that everybody was able to participate in the lesson because Sir wrote statements on the chalkboard and everyone had a right or freedom to go there and fill the correct reason for that particular statement so the class was alive … we were jumping up and down, back and forth to the chalkboard…yah, I liked everything about how Euclidean geometry was taught"
(**Na**, FG E1, Position: 22 – 22)

Participant **T** reiterated:

"Well, what I like is the participation of everyone. That was on another level because well, we understood what Euclidean geometry was all about. In that way we were able to participate like all the time. We were even fighting over the chalk at times. That is what I liked"
(**T**, FG E1, Position: 26 – 26)

Reporting on the kind of learning environment that prevailed in their class during Euclidean geometry and geometry proof lessons, focus group participants stated:

"…the teacher made us to be free in class**.** He taught us in a way whereby like he was not that strict like all the time…he encouraged us to work in pairs so that we can help each other and he did not discourage us in any way or make me or make them feel uncomfortable in a way whereby we cannot even raise our hands …Even in the end we were fighting to write on the chalkboard…"
(**Mo**, FG E2, Position: 26 – 26)

"…what Sir did to make us feel comfortable was…telling us that no one is right and nobody is wrong. So, whenever you feel like answering you must do so even if you do not feel like your answer is right…"
(**Na**, FG E1, Position: 36 – 36)

Participant **T** added:

"He is always free with us... So, that is what I like about him. He's always a free man… most of us are not afraid to go towards him

and say this is the problem that I came across, so how can I try to solve this particular problem"
(**T**, FG E1, Position: 38 – 38)

While focus group participants from the experimental group schools liked several teaching and learning practices that had been implemented in their mathematics class, responses from focus group participants in the control group schools indicated that they did not like the way Euclidean geometry and geometric proofs were taught in their classes. Most participants from the C2 control group who contributed to the third discussion question responded in their vernacular language (see FG C2, Appendix N). However, only translated versions of their responses are presented here. The main issues posed by the participants included: teachers who teach at a fast pace to cover the syllabus; teachers who miss certain parts of Euclidean geometry; teachers who are impatient and insensitive to the needs of slow learners; and teachers who discourage learners. Participants commented as follows:

"I didn't like the way they taught us because ... they are fast and didn't think that we have slow learners"
(**Th**, FG C2, Position: 47 – 47)

"I don't like it because they summarize those chapters and when they summarize those chapters some of the things of Euclidean geometry…decrease our marks. When we go and say you did not teach us this, they say we must go and study and then we can't go and study for ourselves, it's them who are supposed to teach us those things"
(**Ho**, FG C2, Position: 51 – 51)

"Uhm, eish! Sometimes… when we approach him and explain that Sir here, we don't understand, he tells us that he has another class to attend"
(**Th**, FG C2, Position: 71 – 71)

"… when we tell him that we don't understand, then he says he has to finish the syllabus… so that when we write exams, we will not tell him that we didn't do this and that…he says he can't be stuck on Euclidean geometry forever. He has to move on to other chapters"
(**Co**, FG C2, Position: 73 – 73)

"…when we seek help from him, he shows us that attitude of saying 'I taught you this in class' … He is impatient with us"
(**Ho**, FG C2, Position: 75 – 75)

"They should stop using words of discouraging learners in class.

They love to discourage learners. … they tell us that I cannot pass. If they tell me that I cannot pass I will stop coming to school. Because I don't see the difference!"
(**Ho**, FG C2, Position: 89 – 89)

"And they should stop their habit of say maybe if you want to …ask a question, they say you did this last year and something that we did only once and we don't understand it. We need more knowledge to understand but they say you did it..."
(**L**, FG C2, Position: 95 – 95)

"The teachers are failing us…they forget that we are slow, that's why we ask but then the teachers are impatient with us"
(**Co**, FG C2, Position: 97 – 97)

Participants **Ko** and **Bo** from focus group C1 did not like the fact that the proving process seemed to be long and complicated when their teacher showed them how to prove the geometry riders:

"I dislike that geometry proofs…were long, they didn't shorten them, so they were difficult"
(**Ko**, FG C1, Position: 35 – 35)

"… what I didn't like is that the provings (*proofs*) of this geometry Sir were long when our teacher taught us how to prove them. That's why we were a little bit confused in the maths class"
(**Bo**, FG C1, Position: 37 – 37)

Participant **Mp**, also a member of the C1 focus group, stated that when the teacher taught other mathematics topics, she understood well but when the teacher taught Euclidean geometry, the teacher changed his attitude:

"…mostly when he teaches geometry, he changes his attitude but when he teaches other topics like Trigonometry, I understand very well"
(**Mp**, FG C1, Position: 51 – 51)

In another response to the same question, participant **Mp** stated:

"what I dislike is that,…*you may see* something that you don't understand on that circle, then you don't know how to ask a question, plus, it's in front of other learners, so you don't know if I am going to say it right or if Sir or Mam is going to understand what I am saying…So, this is one of the things that are killing us because we don't know how to express the questions or yah, or ask the questions"
(**Mp**, FG C1, Position: 33 – 33)

The above responses describe a learning environment where students were not free to express themselves. The student was afraid to ask questions about anything she did not understand, because she did not know how her peers and her teacher were going to respond. It appears that the learning environment at control school C1 inhibited the participation of all students.

During the final part of the focus group discussions, participants were asked to describe their attitude towards Euclidean geometry and geometric proofs. Participants were also asked to identify pedagogical practices that contributed to their attitude towards Euclidean geometry and geometric proofs. Several valuable and insightful responses were provided, the details of which are presented in the next section.

4.3.1.4 Students' attitudes

Many focus group participants from the experimental group schools reported that their attitude towards Euclidean geometry and geometric proofs had changed from negative to positive due to the influence of the treatment:

> "…my attitude was negative because I didn't know like (how) to solve Euclidean geometry (problems). I didn't know what Euclidean geometry is all about. So, when our teacher taught us, my attitude changed to being positive"
> (**O**, FG E1, Position: 30 – 30)

> "…at first, I was being negative about myself like how am I going to solve these things…then, as I got to explore…*solving* riders in many different ways… then that … just got me a positive attitude because now *I am able to do many things of* geometry"
> (**Mo**, FG E2, Position: 22 – 22)

> "Right now, my attitude is not the way it was before. It is more than positive"
> (**T**, FG E1, Position: 28 – 28)

> "My attitude at first was not good because I felt like Euclidean geometry was gonna defeat me because it's something I …never did before. But as time went on my attitude started to change… Then I started improving and started feeling better about myself…"
> (**Na**, FG E1, Position: 32 – 32)

Participant **Na** briefly described her post-treatment attitude towards Euclidean geometry in the following statement:

> "I can now tackle Euclidean geometry questions on my own and get them right… my skills have also improved. I am able to interpret

diagrams more accurately and apply the knowledge that I have acquired in previous days. Yes, so Euclidean geometry is not actually a difficult thing. It just needs a person to be determined and …to be focused all the time”
(**Na**, FG E1, Position: 4 – 4)

While the focus group participants from the experimental schools reported a positive change in their attitude towards Euclidean geometry and geometric proofs, the responses provided by the focus group participants from the control group schools were mostly negative. Dominant responses that emerged from discussions with control group participants included:

“…I have a bad attitude towards Euclidean geometry because I only understand few theorems, theorem 1, 2, maybe 3, but the rest — ai!”
(**N**, FG C2, Position: 55 – 55)

“…I have a bad attitude because when I try it at home, I find it very difficult…I give up!”
(**L**, FG C2, Position: 57 – 57)

“I have a bad attitude because I got some theorems but to prove that theorem 6 and 7 and riders, I don’t get it because is difficult”
(**Co**, FG C2, Position: 63 – 63)

“I have a bad attitude towards geometry because I find it difficult to understand what is being taught”
(**Mp**, FG C1, Position: 39 – 39)

When asked to shed light on the pedagogical practices they thought influenced their attitudes, participants in the focus groups reiterated points raised in previous sections. For experimental group participants, one of the factors that influenced their attitude towards geometry and geometric proofs was a learning atmosphere in which they were actively involved, relaxed and free to explore geometry concepts practically and not just theoretically, and were taught by a teacher who was not in a rush.

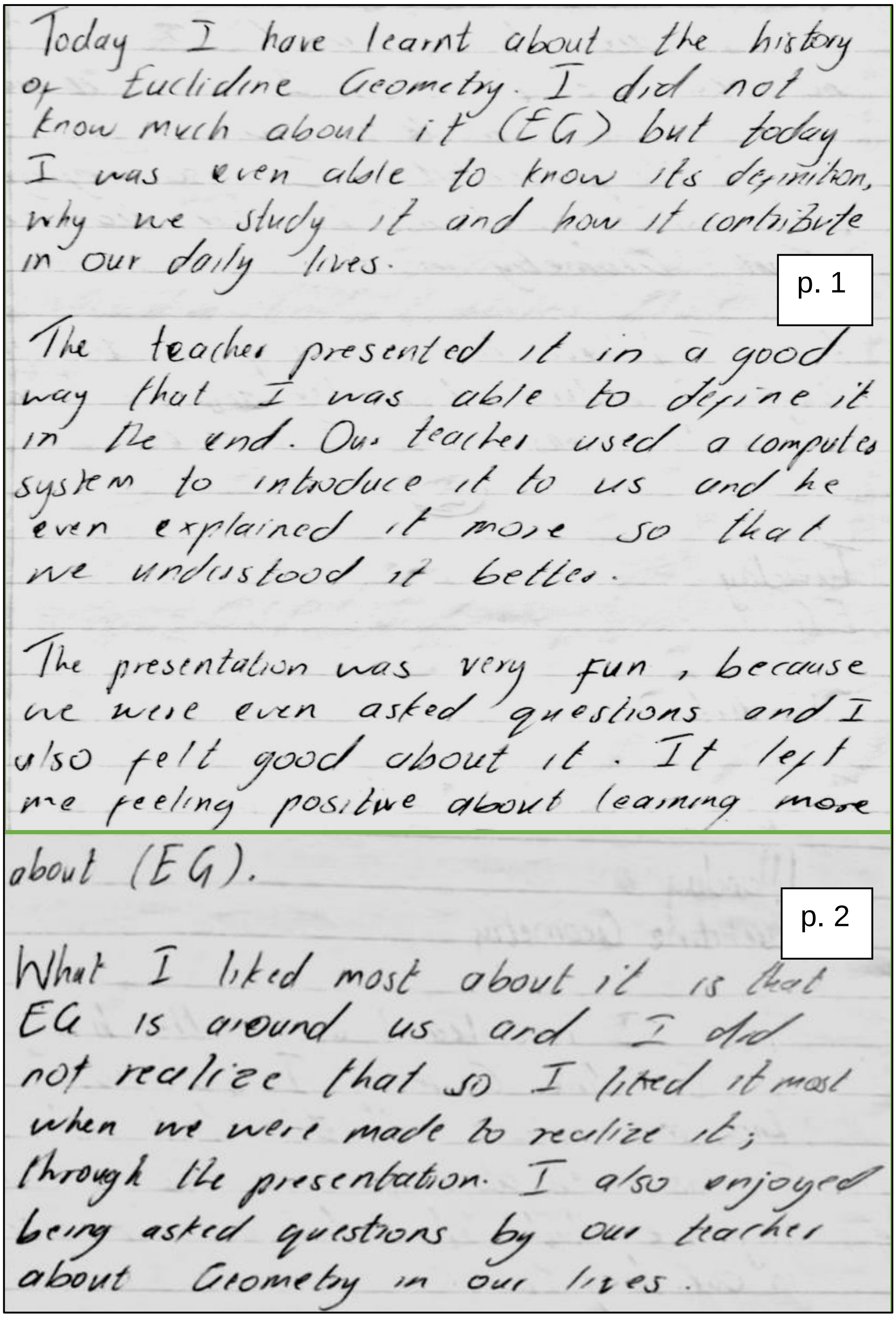
Contrary to these reports, focus group participants from the control group schools attributed their negative attitude towards Euclidean geometry and geometric proofs to teachers who did not introduce the topic properly, teachers who rushed through the topic to cover the syllabus, teachers who did not have time to address the needs of the students, and teachers who demoralized students through negative verbal comments – all of which led to the failure of the students to understand Euclidean geometry.

It is clear from the preceding presentation that a large amount of qualitative data was gathered through focus group discussions with the experimental and control group participants. Although the data provided in this section may be enough to answer the research questions, it is a good practice in research to use more than one approach to gather data on the same subject. This is intended to guarantee the validity of the study findings. To this end, participants' diary entries were also analysed to seek convergence with the findings of the focus group discussions.

4.3.2 Students' diary records

Of the 24 diaries issued to participants, a total of 10 diaries were completed and returned to the researcher. Five diaries were from the experimental group participants, and the other five came from the control group participants. At the beginning of the treatment, diarists were provided with a diary guide to help them record the necessary information based on their learning experience (see Appendix E). In completing their diaries, diarists were expected to include the following aspects: a brief description of how the lesson was presented, their thoughts and feelings about the presentation, what they liked or did not like about the presentation, and, finally, whether the lesson was understood. A lot of information was recorded in the diaries. However, not every piece of information is worthy of being cited and analysed here. Only segments containing the most important textual data will be extracted and analysed in this section.

As indicated in section 4.3.1.1, the focus group participants from the control group schools felt that one of the reasons why they had challenges with Euclidean geometry proofs was because the topic was not properly introduced. Participants from control group school C1 stated that the teacher went straight to the first theorem, without explaining the topic and its terminology (see section 4.3.1.1). It is therefore important to start the analysis of diary entries by looking at how Euclidean geometry was introduced in both experimental and control group schools to verify the students' claims. In the experimental group schools where Van Hiele theory-based instruction was implemented by the researcher, Day 1 of the teaching experiment was used to provide students with general information on the topic, its origin, its relevance to students and its contribution to human life. Figure 4.3 shows a diary entry by participant **Mo** reflecting on her learning experiences on Day 1:



Today I have learnt about the history of Euclidine Geometry. I did not know much about it (EG) but today I was even able to know its definition, why we study it and how it contribute in our daily lives.

p. 1

The teacher presented it in a good way that I was able to define it in the end. Our teacher used a computer system to introduce it to us and he even explained it more so that we understood it better.

The presentation was very fun, because we were even asked questions and I also felt good about it. It left me feeling positive about learning more about (EG).

p. 2

What I liked most about it is that EG is around us and I did not realize that so I liked it most when we were made to realize it; through the presentation. I also enjoyed being asked questions by our teacher about Geometry in our lives.

Figure 4.3: Day 1 diary entry by experimental group student Mo

According to participant **Mo**, the introduction to Euclidean geometry left her 'feeling positive' and the student wanted to learn more about the topic.

Another diarist from experimental group school E2 wrote:

• I have learned so much and taught that euclidean geometry is very useful in our everyday, which came to my surprise because I didn't know that it was useful.

• I loved every part of the lesson and very much enjoyed the presentation.

p. 1

I never knew that geometry was everywhere, and know that I know, I've always wanted to be like one of the greatest Scientists and mathematicians, so therefore

p. 1

I'm gonna practise the use of euclidean geometry in my everyday life to train my mind, think critically, reason logically, to understand and solve problems in the physical world and make a difference.

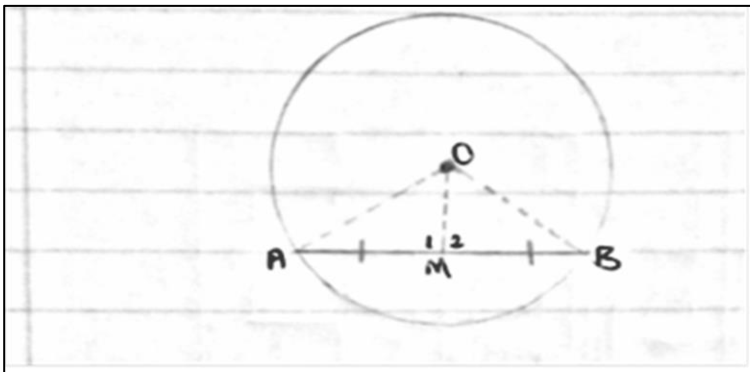
p. 2

Figure 4.4: Day 1 diary entry by experimental group student Kg

As part of her Day 1 learning experience, participant **Kg** said she was surprised to know that Euclidean geometry is useful in our everyday lives. She concluded her diary entry by stating that she would use Euclidean geometry knowledge in her life to understand and solve problems in the physical world.

It can be seen from the preceding diary record that providing students with a brief history of Euclidean geometry, showing them why they should study it, and how it relates to their everyday lives, is an important starting point for arousing students' interest in the topic.

On the other hand, an analysis of Day 1 entries by control group diarists supports what participants said in the focus group discussions, namely that their teacher(s) went straight to prove the first theorem. Figure 4.5 captures Day 1 diary entry by participant **Ko** from control group C1:

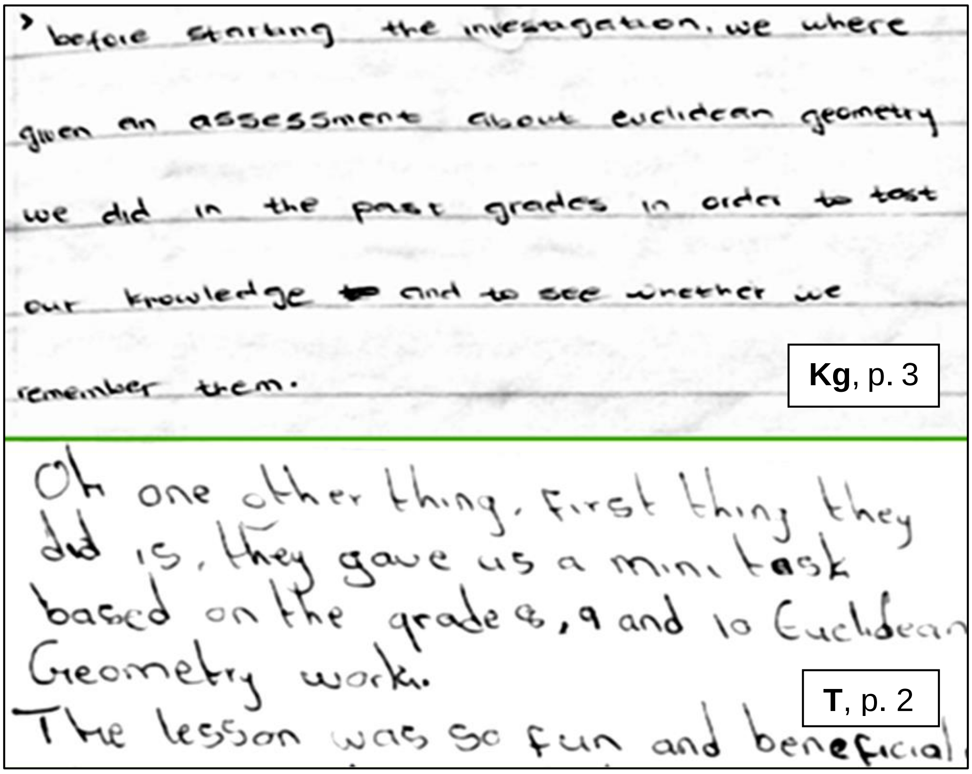


. In this theorem i think everything was great, easy and understandable
. The lesson was great the problem is that my teacher is a little fast. and
. The lesson was fast and the period was 30 minutes

Figure 4.5: Day 1 diary entry by control group student Ko

According to participant **Ko** from control group C1, Day 1's Euclidean geometry lesson was great except that the teacher was fast in presenting the lesson. Findings from the focus group discussions with control group participants revealed that some students (who identified themselves as being 'slow') were left behind by their teacher, who moved fast to cover the syllabus. However, it is worth noting that there are students who thrive under such conditions, particularly, those that are exceptionally gifted. It is therefore not surprising that in his Day 1 diary entry, participant **Ko** described the lesson as being 'great' although the teacher moved at a quick pace. As teachers teach at a fast pace, they meet the needs of the gifted students, but disadvantage the average and below-average students.

In the experimental group of schools, Day 2 was used to assess students on Grade 8-10 Euclidean geometry concepts to identify learning gaps that needed to be bridged. Figure 4.6 shows what some students wrote in their diaries:


before starting the investigation, we where given an assessment about euclidean geometry we did in the past grades in order to test our knowledge and to see whether we remember them.

Kg, p. 3

Oh one other thing. First thing they did is, they gave us a mini task based on the grade 8, 9 and 10 Euclidean Geometry work.
The lesson was so fun and beneficial

T, p. 2

Figure 4.6: Day 2 diary entries by experimental group students

Participant **T** acknowledged that the revision of Grade 8-10 work on Euclidean geometry was helpful, and participant **Kg** added that this was done to test whether they still understood previously learnt geometry concepts.

Despite differences in how Euclidean geometry was introduced to the experimental and control group students, it all seemed to set off in earnest. The next section contains descriptions of how the teachers' lesson presentations were judged by the students in subsequent lessons.

4.3.2.1 Experimental group students' diary reports on lesson presentation

Figure 4.7 summarizes lesson evaluations by the experimental group students on different days of the teaching experiment. An analysis of the students' diary reports shows results that are consistent with what was reported in the focus group discussions. Phrases such as '*presented wonderfully*', '*presented excellently*', '*very nice*' and '*very good*', were used by the students to evaluate their learning experiences in the experimental group schools. These words suggest that the experimental group participants had positive views on the proposed Van Hiele theory-based approach to teaching Euclidean geometry and geometric proofs.

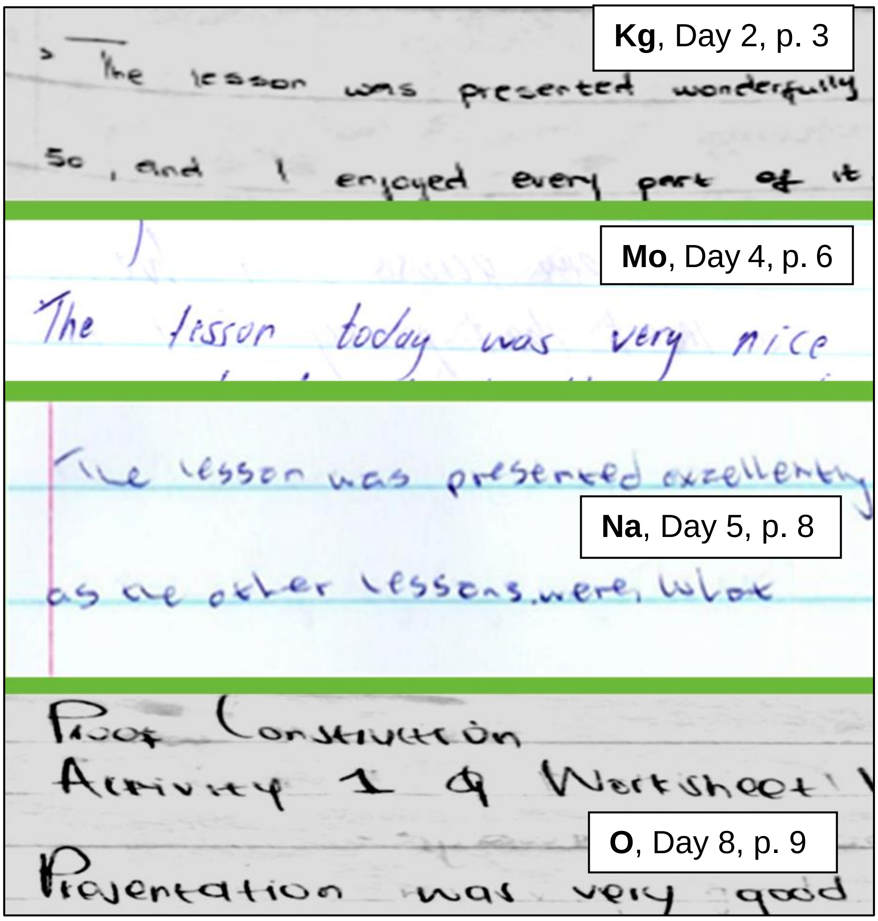


Figure 4.7: Experimental group students' views on lesson presentation

Although students in the experimental group schools wrote positively about their learning experiences, it is important to see how their peers in the control group schools evaluated their Euclidean geometry lessons.

4.3.2.2 Control group students' diary reports on lesson presentation

In focus group discussions, participants from the control group schools mentioned that their teachers were too fast, teaching to cover the syllabus, and skipping certain sections of geometry in the process. Students also complained that the teacher did not pay attention to them when they needed help. Figure 4.8 shows the text segments extracted from the students' diary entries:

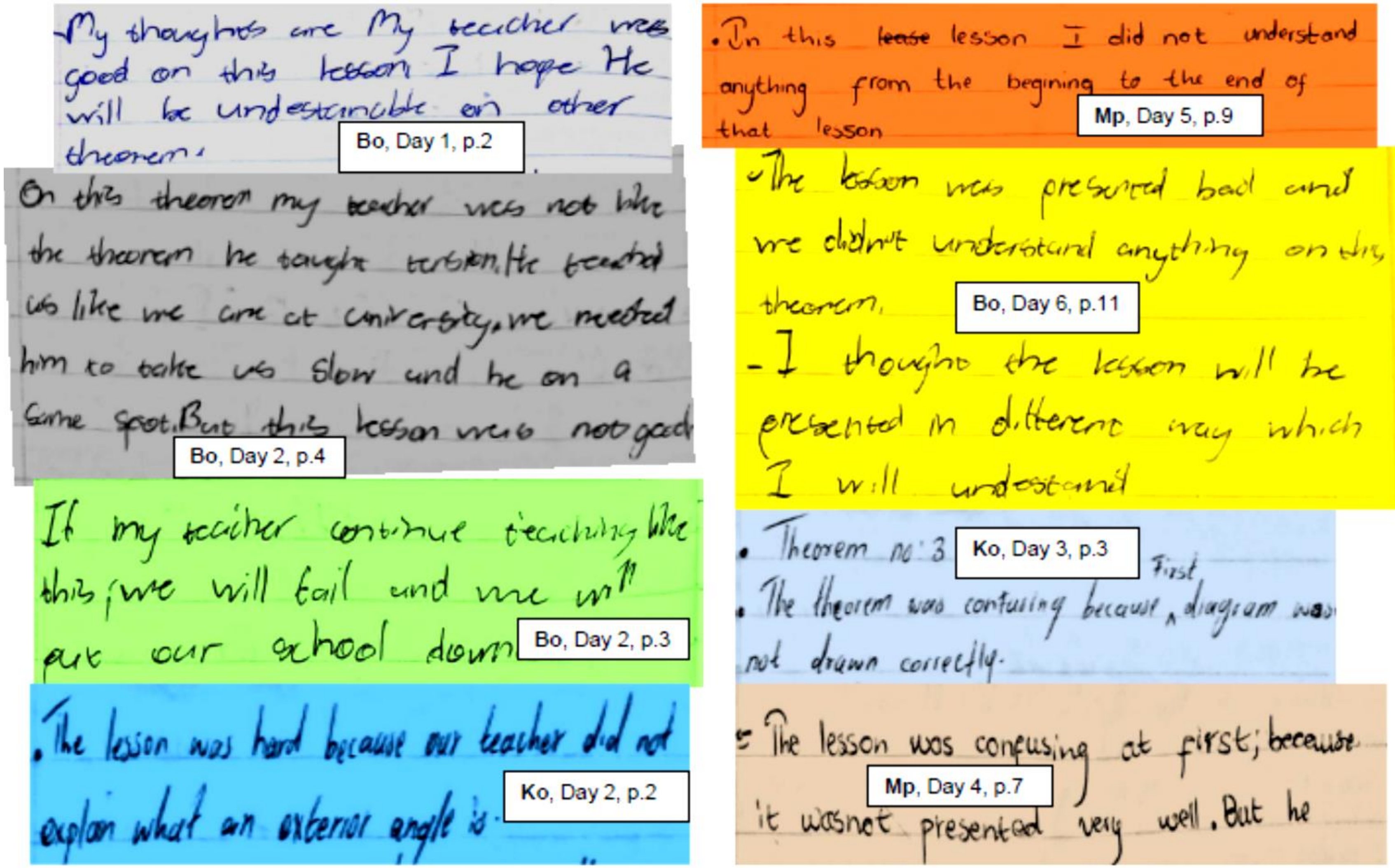


Figure 4.8: Control group students' diary reports on lesson presentations

Based on the control group participants' diary records, Day 1's lesson at school C1 was presented well and students looked forward to the next lessons. However, things turned bad starting from Day 2. Participant **Bo** wrote:

> "He *taught* us like we are at university. We needed him to take us slow…"
> (**Bo**, Day 2 diary entry, p. 4)

These words clearly indicate that the teacher was using the lecture method and moving at a fast pace. The issue of teachers teaching at a fast pace and leaving many students behind was also mentioned by control group participants in focus group discussions. Participant **Bo** goes on to record that if this kind of teaching continues, then students will fail mathematics and bring school results down.

On Day 2, Participant **Ko** found the lesson difficult to understand because the teacher did not explain what an exterior angle is. In focus group discussions with the control group participants, some students indicated that they struggled to understand Euclidean geometry concepts because their teacher did not explain the meaning of some key words. Thus, the views expressed by the control group participants in the focus group discussions are consistent with what they wrote in their diaries.

On Day 3, participant **Ko** noted that the presentation was confusing because the diagram used by the teacher was not drawn correctly. On Day 4, participant **Mp** also wrote that the lesson was confusing because it was not well presented. On Day 5 and Day 6 the lessons seemed to be worse than the previous presentations. This is evident from the quotations below:

> "I did not understand anything from the beginning to the end…"
> (**Mp**, Day 5 diary entry, p. 9)

> "The lesson was presented bad and we didn't understand anything… I thought the lesson will be presented in [a] different way which I will understand"
> (**Bo**, Day 6 diary entry, p. 11)

Although the first lesson was positively rated by control group students at school C1, subsequent lessons were negatively rated as the presentations did not meet the needs of the students. The statement by student **Bo**: "I thought the lesson will be presented in [a] different way which I will understand" is a call for a pedagogical shift in current teaching practices.

In addition to keeping a record of their views and thoughts on how Euclidean geometry lessons were presented in their mathematics classes, participants were also asked to record their feelings and emotions based on their learning experiences. The following section presents an analysis of the experimental and control group students' records of their feelings about the teaching of Euclidean geometry and geometric proofs in their mathematics classes.

4.3.2.3 Experimental and control group students' feelings and emotions on lesson presentations

In the focus group discussions, the experimental group participants expressed positive feelings about the Van Hiele theory-based approach to teaching Euclidean geometry and geometric proofs in their mathematics classes. The phrases in Figure 4.9 were extracted from the experimental group's diary records and are evidence of participants' positive feelings about their Euclidean geometry and geometry proof learning experiences:

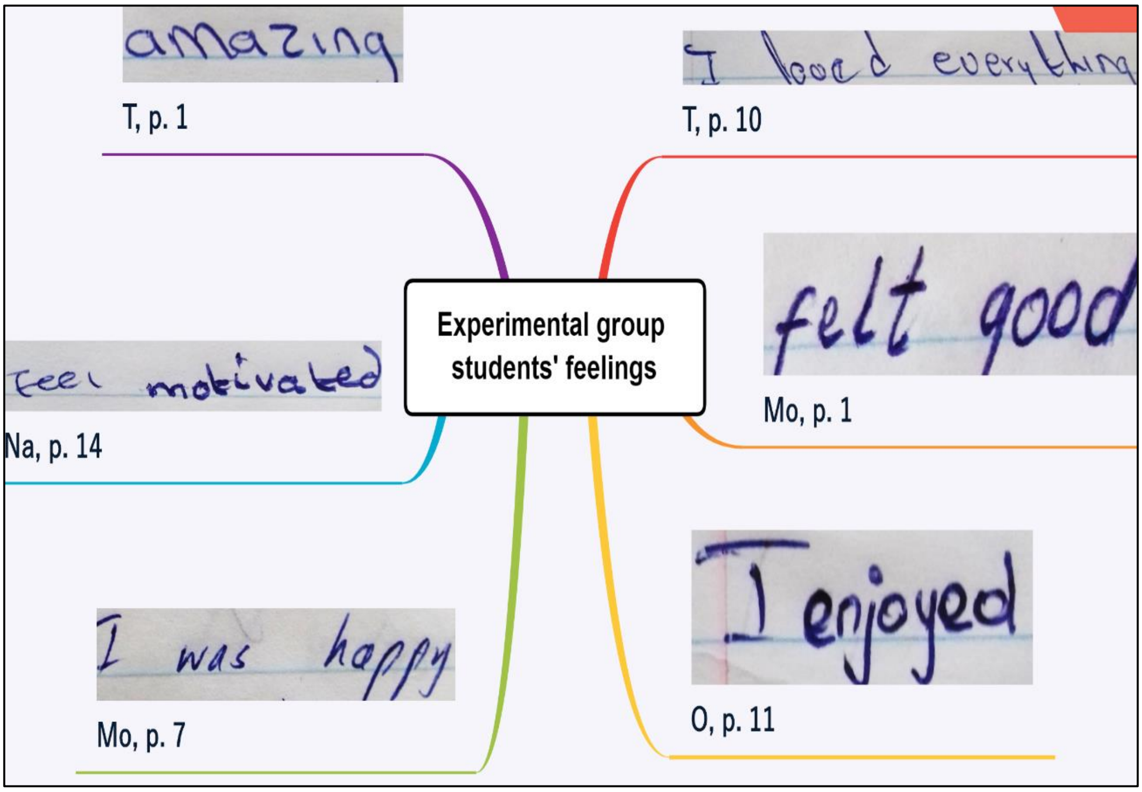


Figure 4.9: Experimental group students' records of their feelings and emotions on lesson presentations

Based on the words and phrases used by the experimental group participants to describe their feelings, it can be concluded that participants enjoyed learning Euclidean geometry and geometric proofs through the Van Hiele theory-based approach.

On the other hand, control group participants expressed negative feelings about how Euclidean geometry and geometric proofs were taught in their classes. Figure 4.10 shows the words and phrases taken from the students' diaries:

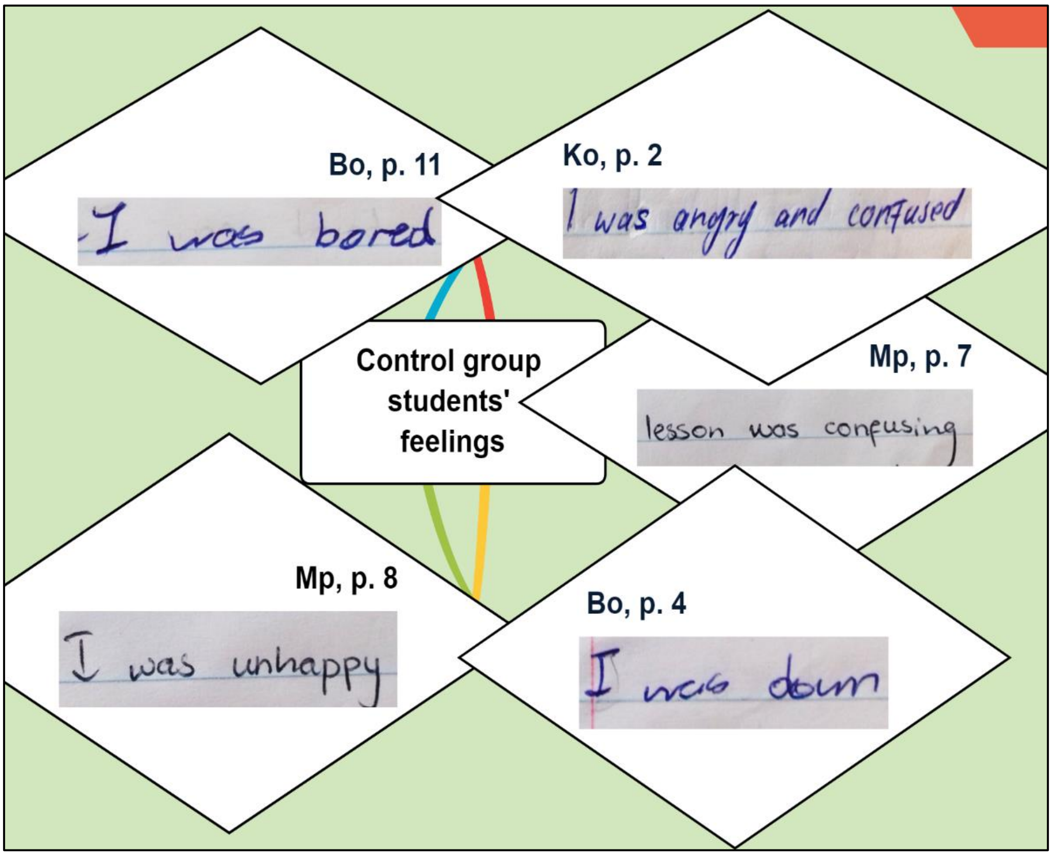


Figure 4.10: Control group students' records of their feelings and emotions on lesson presentations

The words '*bored', 'angry', 'confused', 'down'* and '*unhappy'* are reflective of participants' dissatisfaction with the way particular Euclidean geometry lessons were presented in the control group schools. These results are consistent with findings from the focus group discussions with the control group participants. If students are not happy with how mathematics teachers teach, then it is imperative that teachers try to adjust their teaching to meet the needs of the students.

In addition to recording their thoughts and feelings about their Euclidean geometry and geometry proof learning experiences, students were also asked to indicate what they liked or did not like about the presentation of each lesson. This information is of value to teachers as it helps them to know the kind of pedagogical practices that they should maintain or those that need to be changed. The next section presents the students' diary records of what they liked or did not like about their learning experiences.

4.3.2.4 Experimental and control group students' diary reports of good and bad teaching and learning practices

An analysis of the diary entries referred to in section 4.3.2.3 led to the conclusion that the experimental group participants were happy with the Van Hiele theory-based approach to teaching Euclidean geometry and geometric proofs implemented in their mathematics classes. On the other hand, control group participants expressed feelings of dissatisfaction with the way Euclidean geometry and geometric proofs were taught in their classes. It is worth exploring the aspects of the teaching approach used in the geometry class that led to positive and negative feelings among students. This kind of information helps to guide teachers in realigning their teaching practices to meet the needs of the students.

In the focus group discussions, the experimental group participants reported that they enjoyed the use of the Geometer's Sketchpad to practically investigate theorems. They mentioned that the teacher was not in a hurry and the learning environment was free and relaxed. Focus group participants also reported that they enjoyed the active participation of all students in the classroom and working in groups. However, these reports summarized the wide range of teaching and learning experiences they encountered during the teaching experiment. An analysis of students' records of their day-to-day learning experiences could provide more detail to validate and supplement what they said during the focus group discussions.

Figure 4.11 shows the text segments extracted from **Kg**'s diary. Teaching practices that had a positive impact on student **Kg** included: using a variety of teaching techniques, being calm and not in a hurry, showing students multiple ways to prove riders, and making students aware of the uses of Euclidean geometry in everyday life.

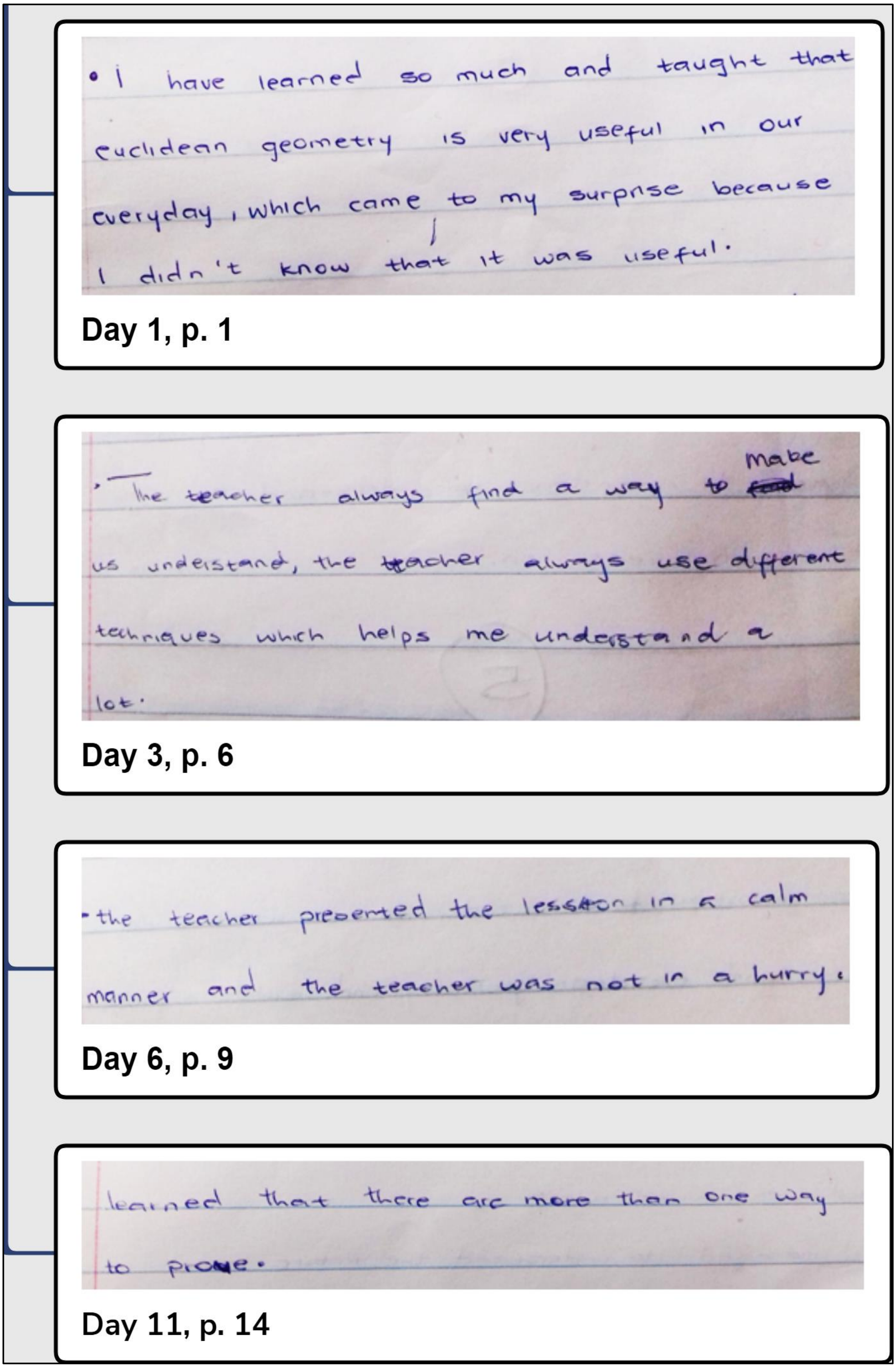


Figure 4.11: Student Kg's views on the teaching and learning process

Student **Na** (see Figure 4.12) from experimental group school E1 enjoyed being taught by a teacher who treated students fairly and allowed students to express themselves freely. The student wrote that using the GSP made geometry fun and easy. She enjoyed being taught by a patient teacher; one who ensured that all students moved at the same pace.

> as the other lessons were, What
> I love the most is that we are
> given freedom of expression and
> equal treatment. When you don't
> understand, Sir doesn't mind
> clarifying the problem. I think
> geometry just needs to be taught
> by someone with patience like Sir
> in order for us to grasp more informati
> I liked the fact that GSP makes
> geometry so easy, fun and
> uncomplicated. Everything just
> seems so easy with GSP. I also
> liked that Sir makes sure that
> we are all on the same page and
> moving at the same pace all the
> time.

p. 8

Sir really knows
how the students minds work and
his strategy and efforts really
work. He doesnt mind staying
behind and explaining what he had
taught again. We really need more
people like him in other departments.

p. 17

I liked how active we
were by running back and
forth to the board. It
was amazing!! I liked the
experience, as it made me
feel alive

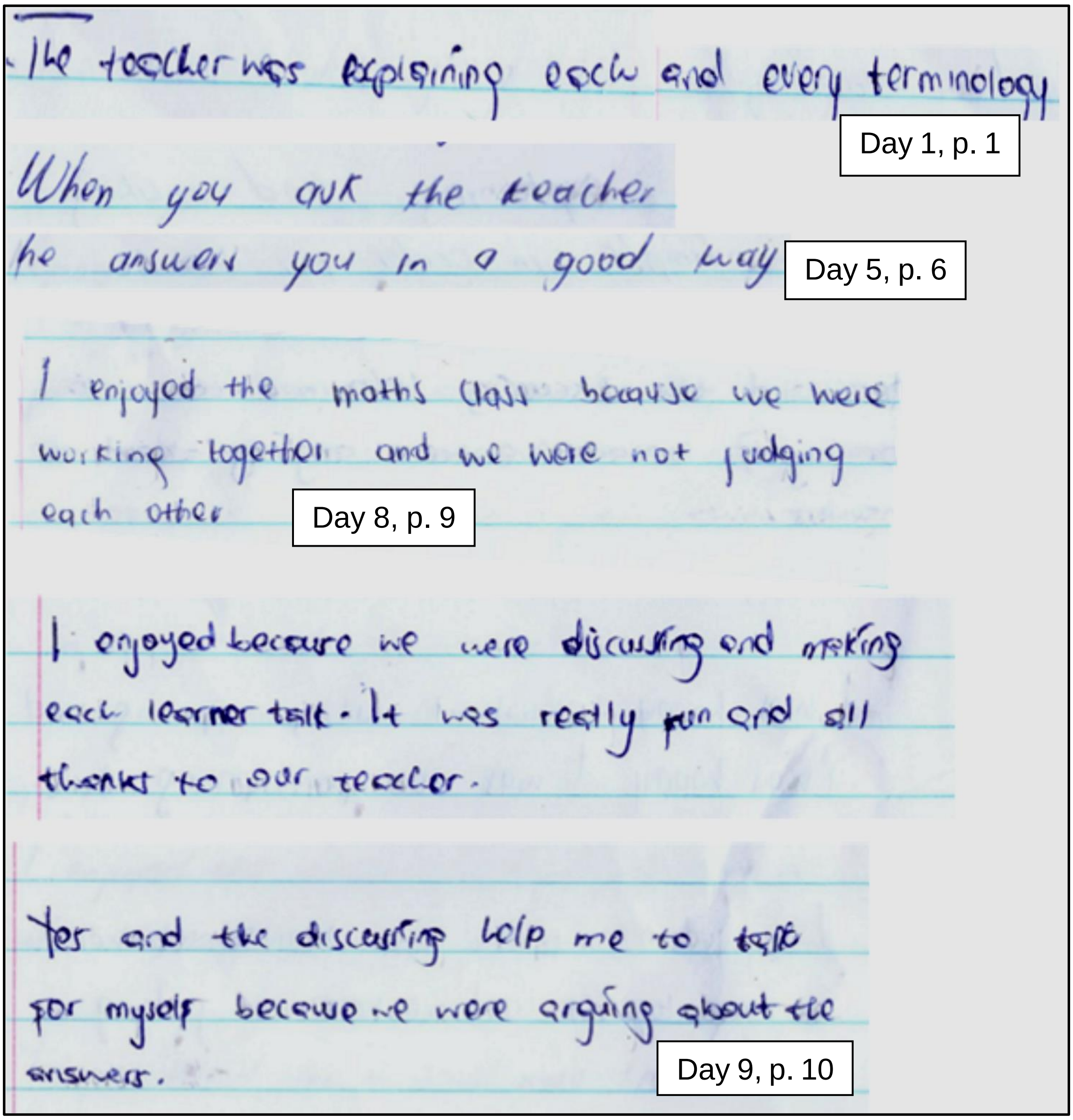
p. 10-11

Figure 4.12: Student Na's views on the teaching and learning process

Student **Na** liked being actively involved in the teaching and learning process. She acknowledged that the teacher did not mind staying behind to clarify and reteach some concepts. Based on how she experienced the teaching and learning process, student **Na** concluded that the teacher knew how students' minds work.

Student **O** from school E2 enjoyed working collaboratively with classmates, discussing and reasoning on the answers. She wrote in her diary that the teacher explained all the terminology of the topic. Student **O** added that the teacher responded to questions asked by the students 'in a good way'. Figure

4.13 shows student **O**’s diary reports on the kind of teaching and learning practices that inspired her most:


The teacher was explaining each and every terminology

Day 1, p. 1

When you ask the teacher he answers you in a good way

Day 5, p. 6

I enjoyed the maths class because we were working together and we were not judging each other

Day 8, p. 9

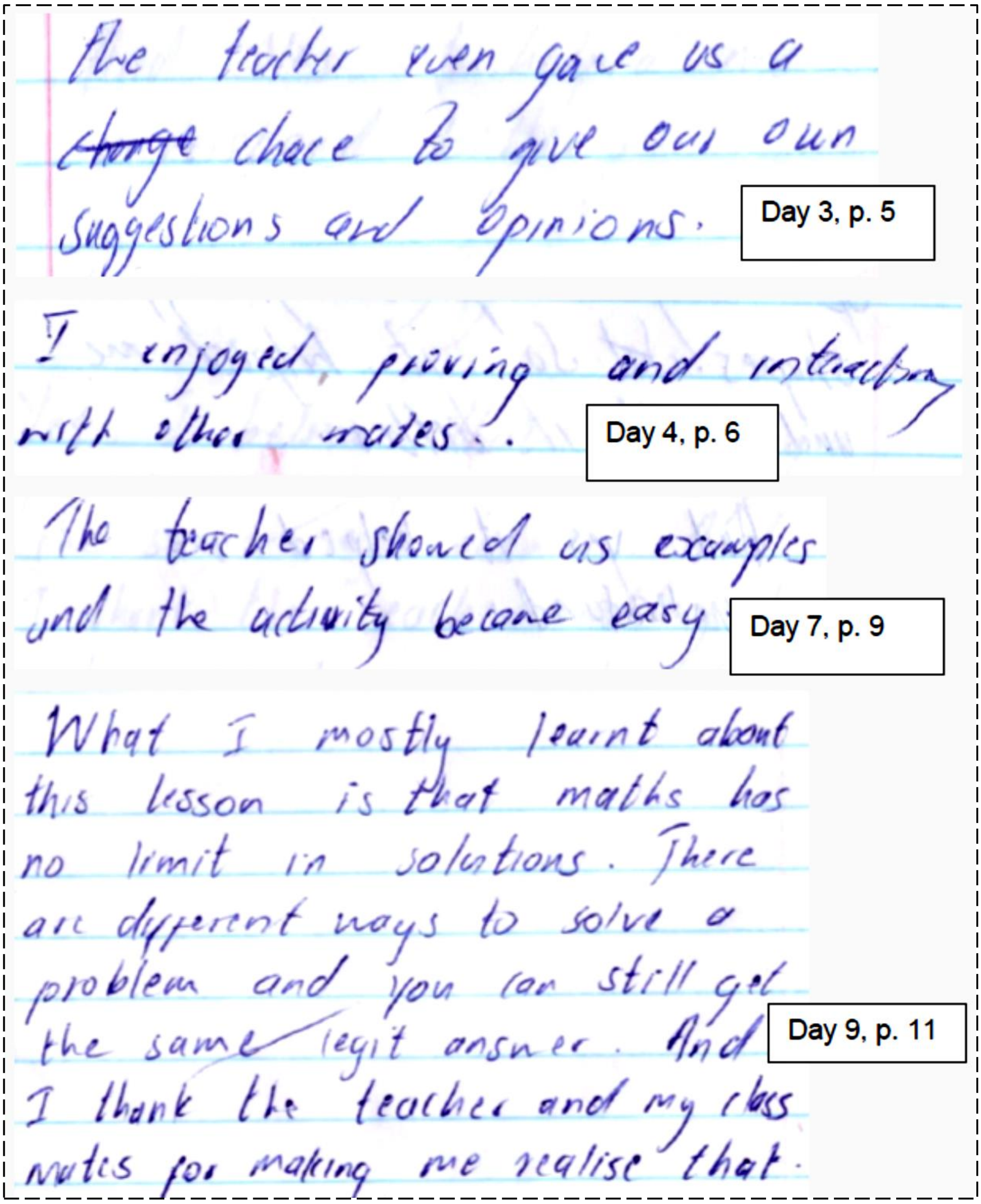
I enjoyed because we were discussing and marking each learner task. It was really fun and all thanks to our teacher.

Yes and the discussing help me to talk for myself because we were arguing about the answers.

Day 9, p. 10

Figure 4.13: Student O’s views on the teaching and learning process

Other teaching and learning practices that experimental group students liked included: the teacher giving them room to express their own opinions and suggestions on the solutions to the geometry problems; encouraging student-to-student interaction; using worked-out examples (modelling the proof process); and the teacher showing them multiple solution strategies. Figure 4.14 shows student **Mo**’s diary reports on her experience of the teaching and learning process at school E1:


The teacher even gave us a ~~charge~~ chace to give our own suggestions and opinions.

Day 3, p. 5

I enjoyed proving and interacting with other mates.

Day 4, p. 6

The teacher showed us examples and the activity became easy.

Day 7, p. 9

What I mostly learnt about this lesson is that maths has no limit in solutions. There are different ways to solve a problem and you can still get the same legit answer. And I thank the teacher and my class mates for making me realise that.

Day 9, p. 11

Figure 4.14: Student Mo's views on the teaching and learning process

While the experimental group students enjoyed their experience of the teaching and learning process during the teaching experiment, the same cannot be said for their counterparts in the control group. The reasons why control group students were not satisfied with the way Euclidean geometry and geometric proofs were taught in their classes were given in section 4.3.2.2. These included: teaching at a pace that is too fast for the students; using the lecture method; not explaining key terms; not varying teaching approaches; and diagrams not precisely drawn. Concerns such as teachers rushing through the topic and not explaining the key terms were also raised by control group students in the focus group discussions. Thus, the views recorded by the students in their diaries corresponded with what they reported in the focus group discussions. The review of students' diary entries was therefore essential not only for the purpose of triangulation, but also to seek additional views that might have been omitted in the discussions with participants. The diaries and the focus group discussions therefore complemented each other.

A summary of the chapter is provided in the following section.

4.4 Summary of the chapter

This chapter was divided into two phases: Phase One and Phase Two. In Phase One, the researcher presented and analysed numerical data to test whether the proposed Van Hiele theory-based instruction had a statistically significant effect on students' geometric proofs learning achievement. The results showed a statistically significant difference in the experimental and control group students' geometric proofs learning achievement.

In Phase Two, the researcher investigated the views of the students on the Van Hiele theory-based approach, and the conventional approach to teaching and learning Grade 11 Euclidean geometry theorems and proofs in their mathematics classrooms. Analysis of students' diary records and focus group transcripts revealed contrasting views about the approaches used to teach Euclidean geometry theorems and proofs in the experimental and control group schools. Experimental group students shared positive views about their learning experiences, while control groups students reported negative views on the same phenomena. The results of the qualitative analyses were consistent with the quantitative findings in the sense that students who shared negative views had attained lower test scores in the quasi-experiment while those who expressed positive views had obtained higher test scores.

The next chapter combines the quantitative and qualitative findings to develop a framework for better teaching and learning of Grade 11 Euclidean geometry theorems and proofs. The implications of the findings of the research for classroom practice will be highlighted.

CHAPTER 5

DISCUSSION OF FINDINGS

5.1 Introduction

This chapter provides a review and discussion of the findings of the study. The results of the analysis carried out in Chapter 4 are correlated and contrasted with previous studies and their contribution to existing knowledge is highlighted. The implications of the research results for instructional practice are discussed, and a framework for better teaching and learning of Euclidean geometry and geometric proofs is suggested. Finally, a summary of the chapter is given.

5.2 Key findings

The main findings from this study are:

- The Van Hiele theory-based instruction had a significant effect on Grade 11 students' geometric proofs learning achievement. Students' views on their geometry learning experiences led the teacher/researcher to discover that implementing Van Hiele theory-based instruction is not just a matter of sequencing learning activities according to the Van Hiele theory. There are additional human elements involved. Based on this finding, the initially proposed Van Hiele theory-based model is modified by the researcher into a comprehensive framework for better teaching and learning of Grade 11 Euclidean geometry theorems and proofs. This is the major contribution of the present study to existing knowledge. Figure 5.1 shows the constituents of the modified Van Hiele theory-based framework for better teaching and learning of Grade 11 Euclidean geometry theorems and proofs. The framework is made up of two arms: teacher support elements on the left arm, and the sequence of teaching and learning activities (Van Hiele theory-based instruction) on the right arm. Teacher support elements originated from the views shared by both the experimental and the control group of students who participated in the teaching experiment:

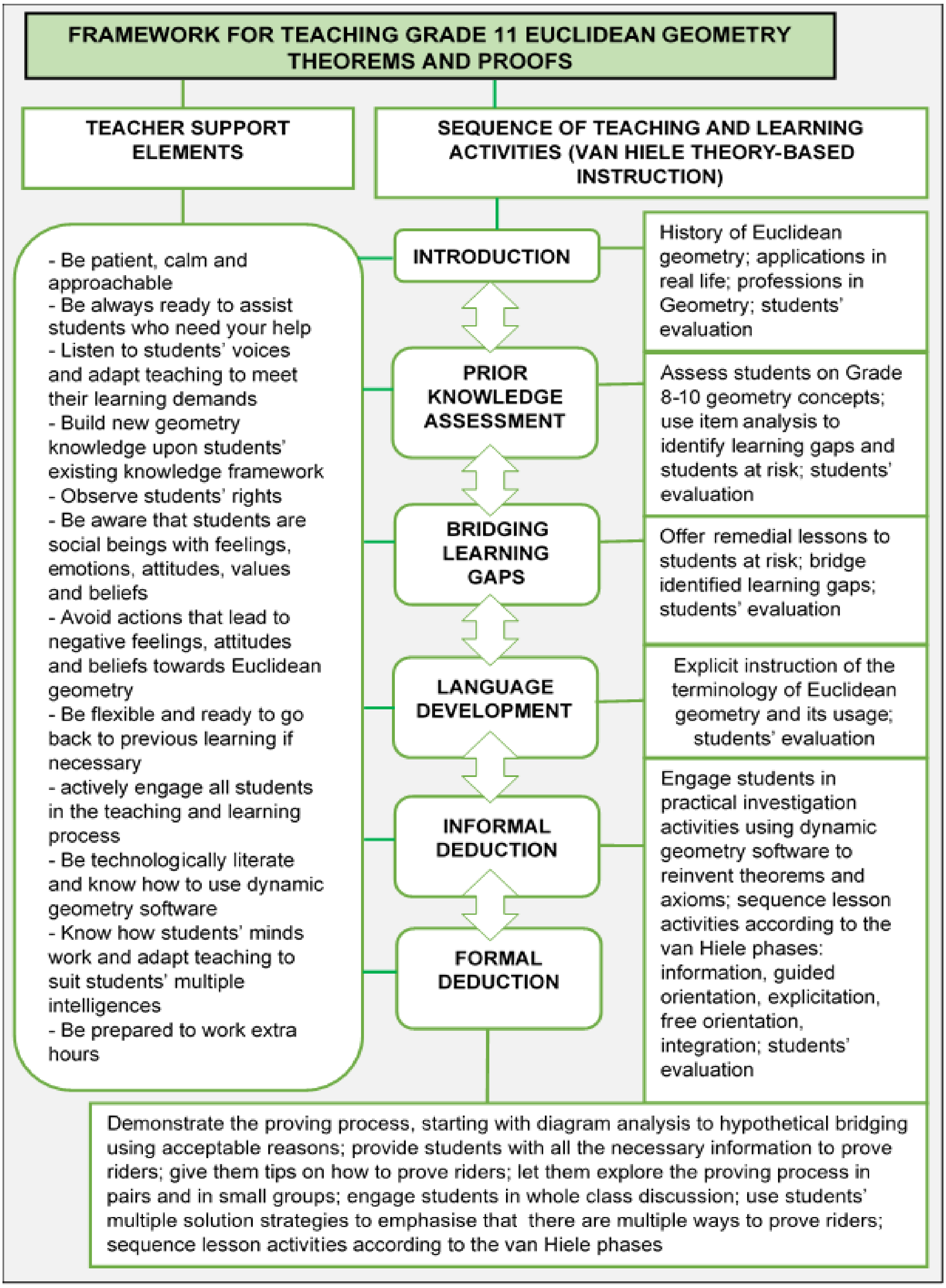


Figure 5.1: A modified Van Hiele theory-based framework for teaching and learning Grade 11 Euclidean theorems and proofs

The teacher support elements are tied to every learning stage in the sequence of teaching and learning activities to show that they are applicable to all levels. The teacher support elements act as the 'heart' and the sequencing of learning activities is the 'body'. If the 'heart' fails, then the 'body' is dead. The arrows in between the different levels in the sequence of learning activities point either way to indicate that the movement between levels is flexible. Thus, the teacher has the freedom to go back to previous learning activities if the situation demands such action. For example, if students are struggling with formal deduction because they missed some theorem or axiom during the informal deduction stage, the teacher should take students back to the practical investigation activities to review the theorem or axiom in question. This consumes a lot of time, of course, but the benefits are worth it. Also, Vygotsky's (1978) Zone of Proximal Development (ZPD) theory supports the idea of directing instruction at the student's current level of understanding.

- ANCOVA test of equality of non-parametric regression curves fitted for the experimental and control groups using the smoothing model indicated a statistically significant difference in the performance of the two groups $(h = 2.26, p = .000)$. Further analysis of post-test percentage scores using non-parametric ANCOVA based on the locally weighted polynomial smoothing model confirmed that indeed there was a statistically significant difference in students' performance between the experimental and control groups $(T = 595.9, p = .005, {\eta_p}^2 = .684)$. Visual inspection of smooth curves fitted for the experimental and control groups using the bias-corrected Akaike Information Criterion (AICc) revealed that the experimental group had higher post-test scores than the control group.
- An analysis of qualitative data from focus group discussions and students' diary records revealed that experimental group students had positive views on their geometry learning experiences:
  - Students reported that being informed about the history of Euclidean geometry, its role in human life, and how it relates to the physical world, inspired them to want to learn more about Euclidean geometry.
  - Students enjoyed the explicit instruction of the vocabulary of Euclidean geometry.
  - Students mentioned that practical investigation activities using the Geometer's Sketchpad helped them see the origins of the theorems and axioms for themselves, rather than memorizing from the textbook.

- Students were impressed by the fact that the teacher was always ready and able to help them when they needed his attention. They mentioned that the teacher did not mind staying behind to support students after normal teaching hours.
- The students acknowledged that the teacher knew how students' minds work, and varied teaching strategies to help students understand geometry concepts.
- Students appreciated being taught by a teacher who was calm, patient and not in a hurry. When they asked questions, the teacher responded in a positive way.
- Students mentioned that they could express themselves freely in class without fear of being judged by their peers or the teacher.
- Students enjoyed being actively involved in the learning process, taking turns to solve geometry riders on the chalkboard, in front of their classmates.
- Working in pairs and in groups, sharing multiple solution methods and correcting each other's mistakes in a constructive way, contributed to students' positive feelings.

- Students who were taught by their teachers in the regular (conventional) way revealed negative views on how Euclidean theorems and proofs were taught in their mathematics classrooms:
  - Students stated that the teacher was teaching to cover the syllabus instead of teaching to enhance students' learning achievement.
  - Students were not satisfied that the teacher(s) skipped certain sections of Euclidean geometry and proof.
  - Students were not happy that the teacher moved straight into formal proofs without first checking if students had mastered what one student referred to as the 'first things of geometry'.
  - Students complained that the teacher did not clarify the terminology of Euclidean geometry.
  - Students who were slow to grasp the content of Euclidean geometry did not receive support and extra help from their teacher when they needed it.

Taken together, these findings seem to suggest that the implementation of Van Hiele theory-based instruction offers a better service to students than traditional/conventional approaches to geometric proof instruction. However, there are specific characteristics of the teacher that are central to the effective implementation of the Van Hiele theory-based instruction. The identification of teacher-related characteristics that support the implementation of Van Hiele theory-based instruction is a key contribution of this study to previous research.

The findings of this research are explored in detail in the following sections.

5.2.1 Van Hiele theory-based instruction and students' geometric proofs learning achievement

A comparison of students' post-test scores on the Geometry Proof Test using non-parametric ANCOVA based on the locally-weighted polynomial smoothing model, showed a statistically significant difference in student performance between the experimental and control groups $(T = 595.9, p = .005, \eta_p{}^2 = .684)$. LOESS curves for the experimental and control groups showed that the experimental group had significantly higher post-test scores compared to the control group (see section 4.2.5 in Chapter 4). The estimated median score for the experimental group was 49.288 points greater than that of the control group. It was concluded that Van Hiele theory-based instruction had a statistically significant positive impact on students' geometric proofs learning achievement. The hypothesis of the study was therefore supported. These findings provide a response to the first research question, and are consistent with previous research on the impact of Van Hiele theory-based instruction on the levels of geometric thought among students.

Although several studies have tested the effect of Van Hiele theory-based instruction on students' understanding of geometry concepts, none of the studies found in literature have implemented Van Hiele theory-based instruction in teaching geometric proofs to students who go to upper secondary school underprepared. Abdullah and Zakaria (2013), and Alex and Mammen (2016), implemented Van Hiele theory-based instruction in Grades 9 and 10, focusing on the properties of triangles and quadrilaterals. Siew, Chong, and Abdullah (2013) applied Van Hiele phase-based instruction at Grade 3 level, focusing on the concept of symmetry of two-dimensional shapes. Meng (2009) and Shi-Pui and Ka-Luen (2009) implemented Van Hiele phase-based instruction in solid geometry. Liu (2005) implemented Van Hiele-based instruction in teaching one of the circle geometry theorems. These studies concentrated on developing students' geometric knowledge and skills at elementary and junior levels. Much attention has been directed towards developing students' visual, analytical, and informal deduction skills, and less attention has been paid to developing students' geometric proofs learning achievement.

A small number of studies that sought to address challenges with teaching geometric proofs were found in literature. These included: the reading and colouring strategy, a teaching experiment with Grade 9 students in Taiwan by Cheng and Lin (2006); the heuristic worked-out examples, a teaching experiment with Grade 8 students in Germany by Reiss, Heinze and Groß (2008); and, the step-by-step unrolled strategy, a teaching experiment with Grade 9 students in Taiwan by Cheng and Lin (2009) (see section 2.6 for details). However, none of these studies implemented Van Hiele theory-based instruction.

With several studies indicating that upper secondary school students cannot do geometric proofs because they do not have the requisite knowledge of geometry, the results of this study suggest that it is possible to support these students to catch up and master geometric proofs. Most students are victims of bad teaching in the past. As shown by the findings of this study, these students can still make significant progress within a short timeframe, provided they are given the right instruction. This is confirmed by Gutiérrez et al. (1991), who found that a student can master two Van Hiele levels simultaneously. The key point here is that students who go to a certain grade level with huge gaps in their geometry knowledge and skills should not be ignored. Mathematics teachers should view this as a challenge to improve their teaching skills.

Assessing the efficacy of teaching methods based on quantitative data analysis alone is a common weakness found in previous studies on Van Hiele theory-based instruction. This research supplements previous findings on Van Hiele theory-based instruction by asking students to provide feedback on the effectiveness of the method. The current study argued that verbal and written views by students on their geometry teaching and learning experiences could provide useful knowledge that could be used to restructure future Euclidean geometry lessons for better learning. In many countries, including South Africa, the practice of requesting students to evaluate teaching is only common at universities and colleges. Yet, students' ratings of their classroom learning experience have been found helpful even at primary school level (see for example Borthwick, 2011).

In Chapter 4, students' verbal and written views on their geometry learning experiences were presented. The main ideas emerging from students' views will be discussed in the next section.

5.2.2 Students' views on their geometry learning experiences

Qualitative data were collected from experimental and control group students through focus group discussions and diary records. During focus group discussions, the experimental group students frequently used the word *'good'* to rate how they were taught, and to describe their feelings about Euclidean geometry (see sections 4.3.1.1 & 4.3.1.2 in Chapter 4). Students also reported that their attitude towards Euclidean geometry and geometric proofs had changed from being negative to being positive because of their learning experiences (see section 4.3.1.4). An analysis of the experimental group of students' diary records revealed similar views to those expressed in the focus group discussions. In their diaries, the experimental group students reflected on how geometry lessons were presented in their mathematics class using phrases such as *'very nice'*, *'presented wonderfully'* and *'presented excellently'* (see section 4.3.2.1). In describing their feelings and emotions about how geometry lessons were presented in their classes, students wrote down words such as *'enjoyed'*, *'happy'*, and *'motivated'* (see section 4.3.2.3). Thus, in addition to increasing students' achievement scores, the Van Hiele theory-based instruction made students feel happy and positive about learning Euclidean geometry and proof.

In contrast, the word *'bad'* was frequently used by the control group students to describe their geometry learning experiences, feelings and attitudes towards Euclidean geometry and geometric proofs (see sections 4.3.1.1, 4.3.1.2 & 4.3.1.4). Diary entries by the control group students revealed that students were dissatisfied with the way Euclidean geometry and geometric proofs were taught in their mathematics classes. In their views on lesson presentation, students wrote words such as *‘bored’*, *‘unhappy’*, *‘angry’*, *‘down’* and *‘confused’* to describe their feelings and emotions during lessons (see section 4.3.2.3). Thus, views from the control group students indicate that conventional approaches to teaching Euclidean geometry and geometric proofs impact negatively on students’ feelings and attitudes towards the topic.

Evidence emerging from the field of neuroscience suggests that emotion and student achievement are inextricably connected. The emotions students feel due to their learning experiences may act as a rudder that guides future learning (Hinton, Fischer, & Glennon, 2012). Positive or good emotions make students want to be more involved in future learning activities, while negative or bad emotions may cause students to gravitate away from learning situations. To sum up, Hinton et al. (2012) concluded that it is common for people to want to be involved in situations that give rise to positive emotions and avoid conditions that lead to negative emotions.

However, simply knowing that Van Hiele theory-based instruction generates positive feelings and attitudes towards Euclidean geometry and geometric proofs is not enough to help teachers improve their teaching of the topic. One of the reasons teachers stick to old ways of teaching despite being increasingly called upon to try new teaching approaches is the lack of clarity on the new proposals. Teachers need to know what exactly causes Van Hiele theory-based instruction to work so well, and what exactly makes conventional instruction ineffective in teaching Euclidean geometry. The present study considers that it is the students who can provide an objective report on these issues. In marketing research, manufacturers ask consumers of their products whether they are satisfied with the product, and how they would like the product to be improved. Based on the consumers’ responses, manufacturers then know exactly what kind of product the consumers would want and can therefore incorporate the consumers’ views in the manufacturing of their new products. Similarly, in education, the students are the ‘consumers’, teachers are the ‘manufacturers’, and the way teachers teach is the ‘product’ that students are going to ‘consume’. The voices of the students are crucial to fully meet the needs of the students.

When the experimental group students rated the Van Hiele theory-based instruction, they specified exactly which learning experiences had a substantial influence on their views, feelings, and attitudes. On the other hand, the control group students also identified the learning experiences that accounted for their dissatisfaction. A critical review of the learning experiences that the experimental and control group students rated as 'good' and as 'bad' could be insightful in developing a framework for better teaching and learning of Euclidean geometry and geometric proofs in secondary schools. The most striking views that students expressed in focus group discussions and wrote down in their diaries are discussed in the next sections under the following headings:

1) Topic introduction;
2) Pace of teaching;
3) Terminology of Euclidean geometry;
4) Teacher support;
5) Pedagogical content knowledge and child psychology;
6) Collaborative learning;
7) Students' self-efficacy;
8) Practical investigation activities;
9) Student engagement and active participation; and
10) Equity and social justice.

5.2.2.1 Topic introduction

The experimental group of students reported that they enjoyed learning about the history of Euclidean geometry, why they should study it, and its practical use in their daily lives (see Figures 4.3 & 4.4). Student **Kg** from experimental school E2 recorded that she was surprised to discover that Euclidean geometry is useful in human life, and that it exists in the physical world (see Figure 4.4). She wrote in her diary that she had *'always wanted to be like one of the greatest scientists and mathematicians'*. She added that she was going to train her mind to *'think critically, reason logically, to understand and solve problems in the physical world and make a difference'* (see Figure 4.4). In another reflection on the introductory lesson, student **Mo** from the experimental group wrote: *'it left me feeling positive about learning more'*. It can be inferred from these findings that the way in which Euclidean geometry was introduced in the experimental group's geometry lessons stimulated the interest of the students in the topic. While there is no single best way to introduce a topic in mathematics, researchers agree that the introduction should capture students' interest, make them see the purpose of learning the content, and convince them that they are going to benefit (Fisher & Frey, 2011).

In the control group schools, students indicated that their teachers did not explain what Euclidean geometry is all about. Instead, they just went straight into proving the first theorem without explaining the terminology of the topic (see section 4.3.1.1). This is typical of conventional teaching practices. Student **Mp** from school C1 cited this kind of teaching as the reason behind most students' difficulties with geometric proofs. The control group students did not like the way Euclidean geometry was introduced in their mathematics classes. The only thing that inspired the control group of students to want to learn Euclidean geometry was the fact that it is allocated more marks in the question paper than any other mathematics topic in CAPS (see section 4.3.1.2).

5.2.2.2 Pace of teaching, time allocation and syllabus coverage

Students in the control group reported that they could not keep pace with their mathematics teachers, who moved fast to cover the syllabus before students wrote the common assessment tasks that are set at district level (see sections 4.3.1.1, 4.3.1.3 & 4.3.2.2). In one of the control group schools, students pointed out that the teacher skipped certain sections of Euclidean geometry in the process of rushing to finish the syllabus (see section 4.3.1.1). The students indicated that they expected their mathematics teachers to be slow and give them more time because they are *'slow learners'* (see section 4.3.1.3). In a study of the impact of instructional time on student performance, Cottaneo, Oggenfus and Wolfer (2016) concluded that the average and below-average students require more teaching time to achieve the same results as the above- average students.

Ramesh (2017) describes the implementation of a fast pace of teaching to cover the syllabus as an 'irregularity' that has been shown to have detrimental effects on student achievement (p. 14). Evidence in support of this position can be found in the post-test results of the control group students (see section 4.2.5). While syllabus coverage is important in view of the practice of common assessments in South Africa, teachers should remember that an ideal mathematics class is diverse, with a few students at the top, the majority being in the average and below-average categories. Against this background, a fast pace of teaching would serve the interests of a few students at the top only, and disadvantage most of the students in the lower categories. This creates inequality of learning opportunities for the students in the mathematics class, and the net effect is that most students would be left behind. Ramesh (2017) concluded that the real measure of students' learning outcomes "is not what teachers cover, it is about what students discover" (p. 17).

While students in the control group were frustrated at being left behind by their teachers, experimental group students enjoyed being taught by a teacher who *'was not in a hurry'* (see sections 4.3.1.3 & 4.3.2.4). Students in the experimental group indicated that they were given enough time to figure out solutions to geometry problems and discuss questions with their classmates (see section 4.3.1.1). In a diary entry, student **Na** from the experimental group liked the fact that the teacher made sure that students '*are on the same page and moving at the same pace all the time*' (see Figure 4.10). This result is in line with twenty-first century views on education which advocate a 'No Child Left Behind' kind of teaching approach (see United States Department of Education, Office of the Deputy Secretary, 2004).

5.2.2.3 The terminology of Euclidean geometry

Another pedagogical aspect that students mentioned in both the experimental and control groups relates to the terminology of Euclidean geometry. Student **O** from the experimental group stated that: *'The teacher was explaining each and every terminology'* of Euclidean geometry (see Figure 4.11). Students from control group C1, on the other hand, pointed out that the teacher did not explain the meaning of words such as chord and diameter, which are basic terms in Euclidean geometry (see section 4.3.1.1). Student **Mp** identified this as one of the reasons why they had difficulties with geometric proofs.

In South Africa and many other African countries, English is the language of teaching and learning. Yet, for most of the students, English is not their native language. As a result, many of the students are likely to encounter linguistic problems in mathematics (Meiers & Trevitt, 2010). Studies carried out by Ercikan, et al. (2015) in Australia, England, America and Canada, found a strong correlation between language mastery and student achievement in mathematics. Smith (2017) and Van der Walt (2009), agree with Ercikan et al. (2015) that knowledge of mathematical vocabulary influences students' mathematics attainment. Mastering mathematical vocabulary helps students to understand what is required to solve mathematics problems. The lack of understanding of the mathematical terminology, on the other hand, restricts students' access to mathematical ideas (Craig & Morgan, 2012; Prediger & Schüler-Meyer, 2017; Riccomini, Smith, Hughes, & Fries, 2015; Van der Walt, Maree, & Ellis, 2008).

In recognition of these views, researchers suggest that mathematical vocabulary should be explicitly taught (see for example Bay-Williams & Livers, 2009; Marzano, 2004; Sonbul & Schmitt, 2010), to help students gain mathematical proficiency (Riccomini et al., 2015). The explicit teaching of new words in mathematics takes away from the students the burden of guessing the meaning of foreign terms, so that they can concentrate more on application (Riccomini, et al., 2015). This reduces cognitive overload, particularly for the average and below-average students.

5.2.2.4 Teacher support

Students in the experimental group acknowledged the support they received from their teacher. They mentioned that the teacher *'was patient and if a learner didn't understand he could explain more and give more examples'* (**O**, FG E1, Position: 24-24). Student **T** from experimental group E1 reported that the teacher was '*always free'* to the extent that the students were *'not afraid to go towards him and say, this is the problem I came across, so how can I try to solve this particular problem'* (**T**, FG E1, Position: 38-38). A similar view was shared by student **Na** in her diary reports (see Figure 4.10): *'When you don't understand, Sir doesn't mind clarifying the problem.'* Student **Na** added that the teacher did not mind *'staying behind and explaining what he had taught again'*. Student **O** reported that the teacher responded to students' questions '*in a good way'* (see Figure 4.11).

On the contrary, statements made by students from the control group seem to suggest lack of teacher support in the control group mathematics classrooms. When control group students approached their teachers to seek help on what they had not understood in class, they received responses such as: *'I taught you this in class'*, and *'...you did this last year'* (see section 4.3.1.1). Student **Th** from control group C2 cited an instance when he approached the teacher for help and the teacher told them that he had another class to attend to (see section 4.3.1.1). Student **Co**, from the same control group also reported that when she asked for help on Euclidean geometry problems, the teacher told her that he had to move on to other chapters and could not stick to Euclidean geometry forever (see section 4.3.1.1).

Teacher support is defined as the degree to which students believe their teacher is willing to assist them in times of need (Patrick, Ryan, & Kaplan, 2007). A research conducted by Yu (2015) involving Grade 9 students found that teacher support had an indirect effect on student achievement in mathematics by improving their self-efficacy in mathematics. Martin and Dowson (2009) suggest that students who see their teachers as supportive and caring feel emotionally relaxed and motivated to take part in challenging classroom learning activities. In other related studies, teacher support was found to lead to increased class attendance (Klem & Connell, 2004), reduced disruptive behaviour, and improved student academic performance (Patrick et al., 2007). The findings of this research are consistent with these previous studies in the sense that students who found their teacher to be caring and supportive had higher test scores than those who found their teacher to be uncaring and unhelpful.

5.2.2.5 Pedagogical content knowledge and child psychology

In their diary reports, students in the experimental group expressed views related to the teacher's pedagogical content knowledge. Student **Kg** recorded that: *'The teacher always find a way to make us understand, the teacher always uses different techniques which helps me to understand a lot.'* Student **Na** added: *'Sir really knows how the students' minds work and his strategy and efforts really work... We really need more people like him in other departments.'* These views give credence to the philosophy of differentiated instruction (see for example Avgousti, 2017). An ideal mathematics class is made up of students of mixed ability. Therefore, a 'one-size-fits-all' approach would not meet the learning needs of some students. Through a variety of teaching methods, teachers can appeal to students of varying abilities.

Knowing how students' minds work is an aspect of child psychology that is part of the Van Hieles' theory. Knowledge of the Van Hiele theory helped the teacher/researcher to organize instruction to match students' current levels of geometric thought in the experimental school mathematics classroom. Effective teachers use students' thinking as a starting point for planning and "a resource for further learning" (Anthony & Walshaw, 2009, p. 11).

In the control group schools, where the geometry lessons seemed to be driven by the desire to finish the syllabus, students shared contrasting views to those expressed by the experimental group students. Statements such as *'I did not understand anything from the beginning to the end...'*, and *'I thought the lesson will be presented in [a] different way which I will understand'* in Figure 4.6 (see section 4.3.2.2) clearly indicate that conventional instruction did not meet the learning needs of some students.

5.2.2.6 Cooperative learning

When asked to indicate what they liked about the way Euclidean geometry and geometric proofs were taught in their mathematics class, students in the experimental group identified collaborative learning as one of the most striking features of their geometry learning experiences. This is evident in student **O**'s diary record (see Figure 4.11):

> 'I enjoyed the maths class because we were working together and we were not judging each other'
> 'I enjoyed because we were discussing and making each learner talk. It was really fun and all thanks to our teacher'
> '...discussing help[ed] me to talk for myself because we were arguing about the answers'

Collaborative learning is widely reported in literature and has been found to have significant benefits for mathematics students. A study by Nannyonjo (2007*)* found that students who worked collaboratively achieved better marks than those who worked individually. In another study, students who engaged in daily mathematics discussions were found to score higher marks in mathematics than those who had little or no discussion at all (Arends, Winaar, & Mosimege, 2017). As students learn a new mathematics topic, they need time to share solution strategies with their classmates, explain and defend their ideas or opinions, and consolidate their understanding (Lee, 2006). In addition, group work gives students the opportunity to make mistakes and be corrected by their peers (Fisher & Frey, 2011).

The findings here give weight to social constructivism, which claims that knowledge is socially constructed (see Westbrook et al., 2013).

5.2.2.7 Students' self-efficacies

As a result of their learning experience, students in the experimental group seemed to have high confidence in solving Euclidean geometry problems. This is evident in the statements below:

> 'I can now tackle Euclidean geometry questions on my own and get them right...my skills have also improved. I am able to interpret diagrams more accurately and apply the knowledge I have acquired in previous days. Yes, so Euclidean geometry is not actually a difficult thing.'
> (**Na**, FG E1, Position: 4-4)

> '...we were able to do...things that I never thought I can do in my life... I am not (like) perfect but I can do most of the things.'
> (**Mo**, FG E2, Position: 16- 16)

In contrast, students from the control group schools seemed to have low self-confidence. The following statements attest to this:

> '...when I look at the question paper, I see a lot of marks but I can't reach them because I don't have that knowledge.'
> (**Te**, FG C2, Position: 65 – 65)

> '...when I try it at home, I find it difficult, ... I give up!'
> (**L**, FG C2, Position: 57 – 57)

It can be seen from the above statements that students in the experimental group of schools believed they had the potential to solve geometry problems correctly, whereas their counterparts in the control group schools doubted their abilities. These findings suggest that Van Hiele theory-based instruction can be used to improve student confidence in Euclidean geometry. Students' levels of confidence on their ability to solve mathematical problems are referred to as their mathematics self-efficacies (see for example Zarch & Kadivar, 2006).

Research has since found a close connection between students' self efficacies and their mathematics performance. Students who believe that they can solve mathematics problems are highly motivated (Wang, 2013), work harder (Siegle & McCoach, 2007), and do not give up so easily when they face challenging mathematics problems (Bandura, 1977; Bouffard-Bouchard, Parent, & Larivee, 1991; Collins, 1982; Prabawanto, 2018). These attributes, in turn, lead to increased academic achievement (Bonnie & Lawes, 2016). On the other hand, students with low self-efficacy were found to have low levels of motivation and reduced commitment (Pajares, 1996; Zarch & Kadivar, 2006), leading to low academic achievement (Bonnie & Lawes, 2016).

Thus, the high sense of self-efficacy evident in the experimental group students' views correlates with the high scores that they obtained in the post-test. Similarly, the low self-efficacy reflected in statements made by the control group students corresponds with their low marks in the post-test. These findings authenticate previously established knowledge regarding the relationship between the self-efficacy levels of the students and their mathematics achievement. Studies have also indicated that students' beliefs about their ability to perform a mathematical task have an effect on their future decisions (Bandura, 1986). For instance, if students believe that they can prove Euclidean geometry riders, then they are likely to attempt such questions in their mathematics examination, whereas those who believe that they cannot prove riders will avoid such questions. However, an empirical study by Harlow, Burkholder, and Morrow (2002) established that students' beliefs about their mathematical abilities are malleable and can be influenced by using appropriate teaching and learning approaches. Van Hiele theory-based instruction appears to be one such teaching approach, because it led students who initially had low self-confidence to have a better sense of self-efficacy.

5.2.2.8 Practical investigation activities

Another aspect of Van Hiele theory-based instruction that students placed at the top of their list of the most influential learning experiences was the use of hands-on investigation activities using the GSP. This was succinctly captured by student **Ch** from experimental school E2 (see section 4.3.1.2):

> 'I felt privileged to have been taught Euclidean geometry in this maths class because that GSP...helped me to be more interested in Euclidean geometry because those things I was doing them myself practically, not just theoretically'
>
> 'I think when we were taught Euclidean geometry using that software was really good for us as learners because it wasn't like reading those theorems in a book. We were actually seeing them first-hand'

The views expressed by student **Ch** reinforce previously established knowledge on the role of practical work in mathematics education. Besides building students' interest in Euclidean geometry, practical activities offer students an opportunity to experiment, establish patterns, verify ideas, and reinvent theorems (Abdullah & Zakaria, 2012; Nath & Binny, 2018). Student **Ch**'s views support the theory of multiple intelligences, which notes that students embody various types of minds and thus learn, recall, act and understand differently (Gardner, 1991). Van Hiele theory-based instruction seems to be ideally suited for students with different intelligences than traditional instruction.

5.2.2.9 Student engagement and active participation

Student engagement was also identified to have positively influenced students' feelings and attitudes towards Euclidean geometry and geometric proofs in the experimental group of schools. The following statements corroborate this:

> "...what I like was that everyone was able to participate in the lesson...so the class was alive...we were jumping up and down, back and forth to the chalkboard..."
> (**Na**, FG E1, Position 22-22)

> "...we were able to participate like all the time. We were even fighting over the chalk at times. That is what I liked"
> (**T**, FG E1, Position 26-26)

> "I liked how active we were by running back and forth to the board. It was amazing! I liked the experience, as it made me feel alive"
> (Student **Na**'s diary report, see Figure 4.10)

The foregoing statements show that students in the experimental mathematics class enjoyed being actively involved and taking charge of the lessons. Thus, Van Hiele theory-based instruction is a student-centred teaching and learning approach. The views expressed by students here validate widespread calls for mathematics lessons to heighten student engagement. The benefits of engaging students in the learning process are commonly documented in literature. These include: increasing student satisfaction, reducing the feeling of being in isolation, motivating students to learn, and improving student performance (Martin & Bolliger, 2018). Toor and Mgombelo (2017) add that "engaging students in the learning process increases their attention and focus", which in turn helps to minimize disruptive behaviour in the classroom (p. 3005).

5.2.2.10 Equity and social justice

Another outstanding view that can be drawn from students' views relates to students' rights in the mathematics classroom. In a diary report, student **Na** from experimental school E1 loved the fact that students were '*given freedom of expression and equal treatment'* (Student **Na** diary report, see Figure 4.10). A similar view was expressed by student **Mo** in a focus group discussion when she mentioned that the teacher made students *'to be free in class'* (see section 4.3.1.3). Student **Mo** added: *'The teacher even gave us a chance to give our own suggestions and opinions'*. Every student enjoyed the right to participation. This is evident in the statement: *'... everyone had a right or freedom to go there and fill the correct reason for that particular statement ...'* (Student **Na**, FG E1, Position: 22-22, see section 4.3.1.3). Experimental group students were made aware that giving wrong answers is acceptable and is part of learning. The teacher told students that *'no one is right and nobody is wrong.... So, when you feel like answering, you must do so even if you do not feel like your answer is right...'* (Student **Na**, FG E1, Position: 36-36, see section 4.3.1.3). Another important aspect of the mathematics classroom culture in the experimental group was respect for each other. This was captured by student **O** in her diary report: '... *we were not judging each other'* (see Figure 4.11).

Twenty-first century mathematics education strives to foster equality in the mathematics classroom. Creating an equitable mathematics learning environment demands that the teacher observes the rights of the students. These include, among other things, the right to voice their opinions and to be heard (Kalinec-Craig, 2017); the right to make mistakes, share those mistakes with other students or the teacher, without being undermined (Steuer & Dresel, 2013); the right to be respected by other students and the teacher (Kazemi, 2018); the right to equal treatment; the right to ask questions and seek clarity where they do not understand (Davis, 2008); and the right to ask for extra help (Davis, 2008). The findings discussed in the preceding paragraph seem to align quite well with the proposed bill of rights for mathematics students. Van theory-based instruction thus offers all students in the mathematics classroom equal opportunities for learning.

The views expressed by students in the control group, by contrast, reflect a learning environment that is insensitive to the rights of students. The teachers were unwilling to attend to students' requests for help with concepts they did not understand in class (see section 4.3.1.1 & 4.3.1.3). Teachers gave petty reasons to dodge the students who needed extra help. These kinds of teaching and learning experiences have led many students in many secondary schools to disengage from mathematics (Wright, 2016). How then should mathematics teachers teach Grade 11 Euclidean geometry theorems and proofs to enhance students' achievement?

5.2.3 A framework for better teaching and learning of Grade 11 Euclidean theorems and proofs

In section 5.2.1, the statistical significance of the proposed Van Hiele theory-based model of instruction was discussed. The main idea emerging from the discussion is that Van Hiele theory-based instruction enhances students' geometric proofs learning achievement. This was found to be consistent with previous research on Van Hiele theory-based instruction. A review of the views of the experimental group of students on their experience in the geometry teaching and learning process clearly shows that the implementation of Van Hiele theory-based instruction is not just a matter of designing and presenting geometry lessons in accordance with the Van Hiele theory. There are additional elements of teacher characteristics that complement the Van Hiele teaching model (see Figure 5.1). This gave birth to a revised model for teaching Grade 11 Euclidean theorems and proofs. Figure 5.1 presented earlier (see section 5.2) showed the revised teaching framework that merges Van Hiele theory-based instruction with students' positive views into a comprehensive model. No other study was found in literature to have uncovered the elements of humanity that complement Van Hiele theory-based instruction. This is what makes the present study important and significant.

A key aspect of the modified Van Hiele theory-based framework for teaching Euclidean geometry theorems and proofs is the understanding that students are social beings whose opinions and input on their learning experiences should be listened to by those seeking ways to improve the academic achievement of the students. By deliberately pursuing students' expectations in the teaching and learning of Grade 11 Euclidean geometry and proof, the factors that hinder students' progress are exposed, and possible interventions can thus be developed based on context. This view is supported by the Professional Educator Standards Board (2009). The implications of the proposed framework for classroom practice will be discussed in more detail in section 5.3.1.

5.3 Implications of findings for educational practice, professional and curriculum development

The findings emerging from the preceding discussion have implications for classroom practice, teacher professional development, curriculum design, implementation, and evaluation.

5.3.1 Implications for teaching Euclidean theorems and proofs in secondary schools

Chief examiners' reports in many countries lament students' inability to construct non-routine multi-step geometric proofs in national mathematics examinations (Department of Education, 2015, 2016a, 2017, 2018, 2019, 2020; Mwadzangaati, 2015, 2019; West African Examination Council, 2009, 2010, 2011). The problem is attributed to teachers' lack of pedagogical knowledge for teaching this aspect of mathematics (see Mwadzangaati, 2015, 2019; West African Examination Council, 2009, 2010, 2011). Teachers in upper secondary school who are responsible for teaching Euclidean geometry proofs allege that students have difficulty with geometric proofs because they come to upper grades not adequately prepared for formal deduction. This observation is supported by numerous studies that have assessed students' Van Hiele levels at different grade levels and found that students are operating at much lower Van Hiele levels than expected (see Abdullah & Zakaria, 2013; Alex & Mammen, 2012, 2016; Atebe, 2008; De Villiers, 2010; Feza & Webb, 2005). Instead of facing the challenge, many high school mathematics teachers have left the problem unattended, and the spill-over effects have been noticed at universities and colleges (see for example Van Putten et al., 2010; Luneta, 2014).

While upper secondary school mathematics teachers cannot be blamed for the fact that students come to their classes with deficiencies in their geometry knowledge, teachers should accept the blame for students who leave their classes not having mastered the geometry knowledge and skills of that grade level. The findings of this study provide empirical evidence that, despite students going to upper secondary school underprepared, it is still possible to help these students achieve the expected levels of geometric thought, including formal deduction.

Mathematics teachers should acknowledge that teaching Euclidean geometry and geometric proofs is not like teaching any other mathematics topic. Geometry requires the teacher to have special pedagogical knowledge and skills. First, it is imperative for every geometry teacher to know about the Van Hiele theory, which explains how students' geometric thinking progresses from one level to the other. This has implications for the professional development of pre- and in-service mathematics teachers which will be discussed later in a separate section. The Van Hiele theory informs geometry teachers on how to organize and sequence teaching and learning activities within and between lessons to enhance students' understanding of geometry concepts.

Due to their lack of progress and based on their past learning experiences, many students go to upper secondary school with negative beliefs, feelings, and attitudes towards Euclidean geometry and geometric proofs. Moving straight into proving theorems and riders only serves to worsen the anxiety that these students already feel from their past learning experiences. The findings of this study indicate that the way the teacher introduces Euclidean geometry in the mathematics classroom matters. The study recommends that, in the introductory lesson, the teacher should give students a brief history of the origin of Euclidean geometry, explain why it is important for them to study the topic, and show them how geometry is connected to human life. This helps to arouse students' interest in learning more about Euclidean geometry.

To successfully teach geometric proofs in upper secondary school, mathematics teachers should embrace the fact that many students coming to their classes might not have acquired the prerequisite geometry knowledge and skills required to master formal proof. This could be due to poor teaching in the past, or simply because the students are slow to understand. Given this situation, upper secondary school teachers should avoid moving straight into proving geometry theorems and riders. The findings of this study suggest that upper secondary school geometry teachers should start by administering an informal test to assess students' understanding of the geometry knowledge and skills covered in lower grades. Test item analysis should be carried out to identify areas of deficiency and students who need regular support. The teacher should then reteach the geometry concepts that most students could not answer correctly in the test. Students at risk should be placed on a continuous remedial programme for the duration of the topic. This will demand that teachers increase their contact time with students. Mathematics teachers who want to see all their students succeed in learning geometry should be prepared to go the extra mile. To emphasize the importance of bridging learning gaps, the Van Hiele theory cautions teachers against forcing students to learn advanced geometry concepts when they are not ready, as this leads students to simply imitate the teacher without understanding (Van Hiele-Geldof, 1984). Enough time should therefore be spent on developing a proper foundation before formal deduction begins.

The terminology of Euclidean geometry should be explicitly taught. This includes key terms such as diameter, chord, tangent, secant, radius, cyclic quadrilateral, circumference, perpendicular, parallel, interior angle, and exterior angle. Proving geometry riders requires students to first read and understand the given information, which will facilitate their analysis of the given geometric figures. Therefore, mastery of the terminology of Euclidean geometry is key to accurate diagram analysis. If students do not understand the vocabulary of Euclidean geometry, certainly, they will face challenges with geometric proofs as was the case with control group students.

The Van Hiele theory states that students cannot achieve level (n) if they have not mastered level $(n-1)$. This means that students cannot master formal proofs if they have not achieved informal deduction skills. The South African mathematics syllabus for Grade 11 in the CAPS states that students should first investigate theorems before they start learning formal proofs (see Department of Basic Education, 2011). This is consistent with the Van Hiele theory. However, a review of literature on the implemented curriculum shows that many teachers do not engage students in investigation activities before they introduce proofs. Geometry lessons are still characterized by students copying theorems from the chalkboard or textbook into their notebooks without understanding. The teachers themselves seem to follow the order of activities presented in their mathematics textbooks. Knowledge of geometry theorems and axioms lays the foundation for proving geometry riders. If students do not have a clear understanding of the geometry theorems and axioms, then they would not be able to prove geometry riders.

This study suggests that engaging students in investigation activities before proof should not be a matter of choice but compulsory in the teaching of Euclidean geometry. The use of the Geometer's Sketchpad and ready-made GSP sketches to reinvent geometry theorems and axioms made geometry lessons more interesting, fun, and enjoyable for students in the experimental group. The GSP, through its click, drag and measure tools, allows students to explore numerous properties in geometric figures within a short space of time. The GSP also allows students to rotate and resize geometric figures to new positions, which enables students to see variations of the same theorem. The experimental group of students stated that they enjoyed learning geometry practically and seeing the results for themselves, as opposed to reading and memorizing theorems from the textbooks. Students also mentioned that they could remember most of the theorems and axioms without being reminded by the teacher. This provided the scaffolding that most students needed to have access to non-routine geometric proofs. It is also highly strongly recommended that mathematics teachers use technology and dynamic geometry applications (such as the GSP, GeoGebra and Dr Geo) to teach Euclidean geometry. The challenge here is that not every mathematics teacher is competent in the use of technology and dynamic geometry applications in the mathematics classroom. This has implications for the professional development of both pre-and in-service mathematics teachers.

There is no point in mathematics teachers to rush to cover the syllabus, leaving the students behind. Given that most students seem to have difficulty in understanding geometry concepts, a fast pace of teaching results in what the Van Hiele theory refers to as a mismatch between what is taught and the level of understanding of the students. The lesson becomes a monologue instead of it being a dialogue between the teacher and the students. The net result is that most students would not achieve the desired level of performance, as was the case with control group students in this study. That would be frustrating for both the mathematics teacher and the students. Findings from the present study have revealed that geometry students prefer to be taught by a teacher who is not in a hurry; one who is sensitive to the needs of the students. Students have a right to say they do not understand and teachers should listen, slow down the pace of teaching, and change their teaching approach if necessary. Teaching geometry is not about how much content the teacher covers in a specified timeframe; it is about how much geometry knowledge students gain from what is taught. Therefore, the pace of teaching should be regulated by students' understanding.

The process of proving geometry riders is a complex activity that should be explicitly taught. Mathematics teachers should not expect students to master the proving process on their own. Students in the control group lamented the lack of teacher guidance on how to prove geometry riders. Teachers should demonstrate the proving process, starting with diagram analysis, through hypothetical bridging steps, to the conclusion. This is a form of scaffolding to help students move from their current levels of performance to realizing their full potential through adult guidance. This is in line with Vygotsky (1978), who asserts that students learn by following adults' examples, and gradually become independent problem solvers. As the students gain experience in proving geometry riders, teacher assistance can gradually be withdrawn to allow students to freely explore solution methods without teacher interference. During the early stages of formal deduction activities, teachers should provide students with all the information they need to successfully prove geometry riders. This includes properties of equality (see Figure 3.23), a list of acceptable reasons as stipulated in the mathematics examination guideline, and tips to solve Euclidean geometry riders. This is important to reduce cognitive overload, particularly for below-average and average students.

As students explore their own solution methods, they should be allowed to discuss ideas with their classmates and their teacher. The teacher should create a learning environment in which students are able to share their opinions without being judged. It is their constitutional right to exercise freedom of expression. Teachers should make it known to students that incorrect responses are acceptable and form part of the learning process. Students should be encouraged to work collaboratively in pairs or in groups to correct each other's mistakes. This is consistent with the social constructivist learning theories. Working in groups offers students an opportunity to share their solution strategies and, in the process, students discover that geometry riders can be proved in multiple ways. Realizing that there are many ways to prove a rider boosted experimental group students' self-efficacy levels.

Finally, students should on a regular basis be given a chance to evaluate how teachers teach Euclidean geometry lessons. This should be done anonymously to ensure that students give honest and unbiased responses. Feedback from the students should then be used to guide lesson planning and presentation in subsequent lessons. A student's performance in Euclidean geometry and geometric proofs is not a product only of that student's cognitive abilities. There are other human elements that contribute significantly towards the student's academic development. The affective domain which deals with attitudes, feelings, emotions, values, and levels of appreciation, motivation, and enthusiasm, is a critical component of the geometry teaching and learning process. These attributes can only be assessed through listening to the student's voice. Many intervention programmes implemented in schools are imposed on the students from above, without incorporating the students' views. It is strongly recommended, based on the findings of this study, that students should have a voice in the design, implementation, and evaluation of the mathematics curriculum. Students are social beings who cannot be manipulated like objects in a laboratory experiment. The geometry teacher should therefore be patient, calm, approachable, helpful, and sensitive to students' perspectives.

The next section sets out the implications of the findings of the study for the professional development of teachers.

5.3.2 Implications for teacher professional development

In many countries, the underperformance of students in Euclidean geometry and geometric proofs has been attributed to the lack of pedagogical knowledge for teaching this topic (see Bramlet & Drake, 2013b; Mwadzaangati, 2015, 2019; Selden & Selden, 2007; West African Examination Council, 2009, 2010, 2011). Teaching is a dynamic art. The way teachers were trained to teach Euclidean geometry many years ago may be outdated in modern mathematics education. The findings of this study suggest that in-service mathematics teachers should receive fresh training on 'how to teach' Euclidean geometry and geometric proofs in a way that accommodates all students in the mathematics classroom. This should be facilitated by subject specialists with adequate content and pedagogical knowledge in Euclidean geometry. The training should include all mathematics teachers from primary to secondary school. To evaluate the effectiveness of the training programme, teachers should be assessed and certificates of competence should be issued at the end of the training programme.

Based on my experiences as a mathematics teacher in the context of the study, the existing training programmes do not include mechanisms for evaluating the effectiveness of the programme. Teachers just sign attendance registers and go back to their respective schools. To be effective teachers of Euclidean geometry, teachers should function at a higher Van Hiele level than the students they teach. Assessing the teachers' levels of competence after training is therefore important to monitor progress and identify those who need further support. In addition, the teachers' pedagogical knowledge for teaching Euclidean geometry should be continually updated to align with new research evidence. To this end, in-service teacher training should be a continuous and not just a one-off event.

While many geometry teachers may be aware of the Van Hiele theory and its application in teaching and learning Euclidean geometry, results of this study, coupled with evidence from the field of neuroscience (see Hinton et al.,2012), indicate that emotional support and teacher sensitivity to students' needs are indispensable partners in the implementation of Van Hiele theory-based instruction. The behaviour of the teachers in the control group schools led to the negative feelings and attitudes of the students towards Euclidean geometry and geometric proofs. The views expressed by the control group of students indicated that the teachers lacked the expertise to handle the emotions of the students. It is therefore recommended that geometry teachers be trained on how to manage the emotional domain of students to create a positive classroom climate that encourages geometry learning for all students regardless of their cognitive abilities. The Department of Basic Education should consider engaging neuroscientists to facilitate teacher training in managing the emotional aspects of the students.

Mathematics teachers themselves should not wait for the DBE to organize training for them to improve students' achievement in Euclidean geometry. It is the responsibility of every mathematics teacher to continue to engage in research to find new and innovative approaches to teaching Euclidean geometry and geometric proofs. It should be noted, however, that research is not a cheap exercise. The Department should therefore provide financial assistance to teachers who wish to engage in research targeted at enhancing the teaching of mathematics in schools. In addition, mathematics teachers should be provided with platforms to easily share their research findings.

One of the reasons teachers continue to use traditional teaching approaches is that research-based evidence of new teaching approaches does not reach them. A lot of research-based evidence that can guide teachers to effectively teach mathematics is available, but probably in places that are not easily accessible to many teachers. The Department of Basic Education should therefore provide sponsorship for mathematics teachers to publish their research-based evidence of effective teaching practices in journals and teacher magazines, which should then be distributed to all mathematics teachers in schools. Arranging teacher discussion forums and conferences would also go a long way towards helping to disseminate information that can guide mathematics teachers to improve their teaching.

There is no doubt that modern economies are driven by technology. To survive in the coming years, mathematics teachers (young and old, novice and experienced) should learn how to integrate technology not only in Euclidean geometry lessons, but also in the teaching of other mathematics topics. The findings of this study indicate that the use of dynamic geometry applications in geometry instruction has a beneficial impact on the emotional and cognitive domains of students. It is therefore crucial for every geometry teacher to learn how to integrate dynamic geometry applications into geometry lessons. Universities should integrate this into their pre-service mathematics teacher education programmes. Similar training programmes should also be organized for in-service mathematics teachers.

The mathematics teachers themselves should take teacher professional development seriously and positively. While experienced teachers are leaders in teaching practice, they should be willing to adopt new research-based teaching approaches. Some of the mathematics teaching practices used in the past are no longer effective and applicable to modern mathematics education. Therefore, mathematics teachers should be encouraged to upgrade their teaching qualifications. Efficient mathematics teachers are lifelong learners.

5.3.3 Implications for curriculum design, implementation, and evaluation

The opinions expressed by the students in the control groups indicate that the design of the geometry curriculum, as well as its implementation and evaluation, appear to be in a state of disharmony.

The geometry curriculum in the South African CAPS was imposed on teachers and students from above. The teachers and students were not involved in the design process. Studies conducted after CAPS training workshops for mathematics teachers revealed that many teachers are still not comfortable teaching Euclidean geometry (see Olivier, 2014). In one of the control group schools, the students reported that the teacher changed his attitude and behaved differently when teaching Euclidean geometry. In the other control group school, the students mentioned that certain sections of Euclidean geometry were skipped. This shows that the implementation of the geometry curriculum poses serious challenges for some teachers.

In addition, the practice of administering common tests during the year pressurizes teachers to rush through the syllabus, trying to cover all the prescribed geometry content before the dates set for the writing of the tests. Students in one of the control group schools told their teacher that the pace of teaching was too fast for them, but the teacher did not listen to their call to slow down. Instead, a negative response was given. At the end of it all, students may fail to answer geometry questions in the common tests and everyone (students, teachers, and curriculum designers) will be frustrated.

Students attribute their failure to understand Euclidean geometry and geometric proofs to poor teaching by their teachers. On the other hand, mathematics teachers defend themselves by saying that they covered the prescribed geometry content before the test was written. Teachers blame the students for not practising enough. At the end of each school term, mark schedules are submitted to the district, provincial and national government for analysis. Underperforming schools are identified and the principals of those schools are called to meetings with circuit managers, district directors, heads of departments and the Member of the Executive Council for education. There is nobody representing students’ voices in these meetings, yet the students are the principal stakeholders in the education system.

While curriculum design is primarily influenced by the needs of the economy, the views of the teachers and the students are vital. In other words, mathematics teachers and students should be involved in the design of the mathematics curriculum to ensure its smooth implementation. For example, students indicated that the time allocated to Euclidean geometry was not enough for them to master all the geometry concepts in their syllabus. This suggests that designers of the mathematics curriculum should consider increasing the time allocated to Euclidean geometry in the CAPS. In surveys conducted after Mathematics CAPS training workshops, some teachers revealed a low level of confidence in the teaching Euclidean geometry. This suggests to the curriculum designers that teachers should be thoroughly trained well in advance of the implementation of any new curriculum. In addition, curriculum design is not a one-off event. The designers of the mathematics curriculum should continuously adapt the curriculum to meet the needs of the teachers and the students. Unless the three parties realize that they need each other to survive, mathematics education in many countries is bound to fail.

Geometry teachers should be informed that the implementation of the geometry curriculum is not a matter of following the sequence of activities presented in the students' mathematics textbooks. Teachers are not supposed to be slaves to the textbook. Instead, they should be guided by their pedagogical knowledge of teaching geometry, recent research-based evidence, and the situation on the ground. To support teachers in the implementation of the mathematics curriculum, curriculum designers should provide guidance manuals for teachers on the various approaches that can be used to teach Euclidean geometry and other mathematics topics. These manuals should be updated continuously to keep pace with new research evidence.

Textbook publishers should revise textbook material to ensure that textbook content is consistent with new developments in mathematics education. The results of this study suggest that Grade 11 Mathematics textbooks should include investigation activities in which students can rediscover geometry theorems and axioms before they learn formal proofs. Mathematics textbooks should also guide teachers on how they can integrate technology in their geometry lessons. In addition, publishers may also include at the beginning of the chapter a brief history of Euclidean geometry, why students should study it, its role in human life, and a list of professions that use geometry knowledge and skills. This would help students to see the relevance of learning the topic. One student in the experimental group reported that she was surprised to learn that Euclidean geometry is useful in human life, and she became interested in learning more about the topic.

5.4 Summary of the chapter

This chapter discussed in more detail the results presented in Chapter 4. The main ideas that emerged from the discussion are that: Van Hiele theory-based instruction is more effective than conventional instruction in developing students' geometric proofs learning achievement. In addition, the implementation of Van Hiele theory-based instruction is not just about the organization of instruction according to the Van Hieles' proposals; the mathematics teacher should be responsive to the students' contextual needs.

The geometry teacher should be aware that students are social beings with feelings, emotions, attitudes, values, and beliefs, all of which have the potential to skew academic performance. The teacher's behaviour should therefore promote the development of positive feelings, attitudes, and beliefs about Euclidean geometry and geometric proofs. By listening to the students' voice, the geometry teacher should be able to adapt his or her teaching to meet the learning needs of a diverse group of students in the mathematics class. Students have the right to inform the teacher that they do not understand. They have the right to be actively engaged in the lesson, and not to be treated as empty containers. They have the right to tell the geometry teacher that the pace of teaching is too fast for them to understand what is being taught. They also have the right to evaluate the way they are taught and the geometry teacher should not feel offended by the students' feedback. Instead, the geometry teacher should observe all these students' rights and react positively.

Assessing students' prior knowledge and bridging learning gaps play a key role in developing students' understanding of geometry concepts. Providing students with information on the history of Euclidean geometry, its role in human life, its relationship with the physical world, and the various careers in which geometry knowledge and skills are applied, captures the attention of the students, and motivates them to want to learn more about the topic. Explaining the terminology of Euclidean geometry is also important, keeping in mind that most students learn geometry through the medium of English as a Second Language (ESL).

Practical investigation activities using dynamic geometry software not only motivate students, but also provide the necessary scaffolding that students need to master formal proofs. Geometry teachers should also learn how to integrate technology into their geometry lessons. Efficient geometry teachers do not rely solely on their experience, but always try new teaching approaches to enhance the academic achievement of students.

The in-service training of mathematics teachers should take place well in advance of the implementation of a new mathematics curriculum and should not be run concurrently with its implementation. The teachers and the students should also be involved in the process of curriculum development to ensure that their views are represented. This will go a long way towards closing the gaps between the intended curriculum and the implemented curriculum.

In the next chapter, the researcher gives readers a complete overview of the entire project. The limitations of the study will be highlighted and suggestions will be made for future research.

CHAPTER 6

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

6.1 Introduction

This study was prompted by several reports of secondary school students not performing well on the mathematical aspect of geometric proofs in national examinations across several countries. To address the problem, the following objectives were set in Chapter 1:

1) To implement Van Hiele theory-based instruction in the teaching of Grade 11 Euclidean geometry theorems and non-routine geometric proofs;
2) To test the effect of Van Hiele theory-based instruction on Grade 11 students' geometric proofs learning achievement;
3) To explore students' views on (a) the Van Hiele theory-based approach, and (b) conventional approach to teaching and learning Grade 11 Euclidean geometry theorems and non-routine geometric proofs;
4) To develop a framework for better teaching and learning of Grade 11 Euclidean geometry theorems and non-routine geometric proofs, integrating the views expressed by the students.

Chapter 2 presented a review of literature available on the evolution of Euclidean geometry proofs to understand the developments that have taken place in geometry instruction to date. The challenges faced by teachers and students in the teaching and learning of Euclidean geometry proofs were described. The Van Hiele theory and its implications for teaching and learning Euclidean geometry and proof were reviewed. The gap in knowledge that this research intended to fill was identified. Chapter 3 provided the details of how the Van Hiele theory-based instruction was implemented. Thus, the first objective was achieved. Chapter 4 summarized the quantitative and qualitative data that were obtained to address the second and the third objectives. In Chapter 5, the findings of the quantitative and qualitative data analyses were examined and discussed to address the fourth objective. The implications of the findings for classroom practice, teacher professional development, curriculum design, implementation, and evaluation were also outlined. This chapter presents a snapshot of the key points that emerged from Chapter 5 in response to the research questions. Finally, the shortcomings of the study are highlighted and suggestions for future research are proposed.

6.2 Summary of research findings

The following research questions were framed in Chapter 1:

1) Does teaching and learning Euclidean geometry theorems and non-routine geometric proofs through Van Hiele theory-based instruction have any statistically significant effect on Grade 11 students' geometric proofs learning achievement?
2) What are students' views on (a) the Van Hiele theory-based approach, and (b) conventional approach to teaching and learning Grade 11 Euclidean geometry theorems and non-routine geometric proofs?

In section 5.2.1 it was concluded that Van Hiele theory-based instruction had a statistically significant positive effect on students' geometric proofs learning achievement ($p < .05$). Thus, the first research question was answered. Section 5.2.2 discussed students' views on the teaching and learning of Euclidean geometry and geometric proofs in their mathematics classes. The discussion alluded to the view that the experimental group of students had positive views towards Van Hiele theory-based instruction (see section 5.2.2 for details). On the other hand, students who had received conventional instruction gave negative reports about their geometry learning experiences (see section 5.2.2 for details). Thus, the second research question was answered.

It was concluded that, in addition to organizing teaching and learning activities according to the Van Hieles' recommendations, teachers should pay attention to the students' voices and adjust their teaching accordingly. The human elements that are pivotal to the successful implementation of Van Hiele theory-based instruction were identified from the students' views. Based on the findings from section 5.2.1 and section 5.2.2, a framework for better teaching and learning of Grade 11 Euclidean geometry theorems and proofs was developed (see section 5.2.3 and Figure 5.1 for details).

6.3 Limitations of the study

Like any other research, the present study has its own limitations. Identifying the possible shortcomings of the research is important for contextualizing the findings and facilitate their interpretation by the reader.

The major limitation of this study was the non-random allocation of participants into treatment and control groups. Consequently, the findings cannot be extended beyond the geographical scope of study (see section 1.7). This research was limited to only four township secondary schools in the same district in the Limpopo province, South Africa, due to time and financial constraints. Therefore, the results of the study should be interpreted in this regard.

Also, only one focus group discussion was conducted per school due to time and financial resource restrictions. Engaging more than one focus group per school could have captured a bigger variety of responses that could have enriched the qualitative data findings. Besides, involving a larger sample of schools from different districts across the country could enhance the generalizability of findings and yield more definitive treatment effects.

While the teacher/researcher implemented Van Hiele theory-based instruction in both experimental schools, students in the control group schools were taught by different teachers, leading to variations in the way conventional instruction was implemented. This was not accounted for in the data analysis. Although, the teachers used the same lesson plans, it was not possible for the researcher to regulate teaching in the control group schools to make sure that teachers teach Euclidean geometry according to the lesson plans. Thus, what constituted conventional instruction in control group schools could be more complex than the definition presented in this study.

The study is limited to the teaching and learning of Euclidean geometry and geometric proofs at Grade 11 level in South Africa. The interpretation of the findings of the study should therefore be confined to the teaching and learning of Grade 11 Euclidean geometry and geometric proofs. The researcher believes, however, that the findings of the study may be relevant to the teaching of Grade 10 and Grade 12 Euclidean geometry and geometric proofs, although this is subject to investigation.

Finally, the teaching experiment was implemented in a period of four weeks. Given that students are going to upper secondary schools with a huge backlog in their geometry knowledge and skills, a period of four weeks may be inadequate to evaluate the effectiveness of the treatment. A longitudinal study may give a clearer picture of the treatment effects. However, that would require a bigger budget.

6.4 Recommendations for future research

Based on the limitations identified in the preceding section, it is recommended that future research should:

- Replicate the study with a larger sample of schools from different districts across the country. This would entail the training of teachers who would be able to implement the proposed treatment in experimental group schools, as it would be impractical for one teacher to implement the treatment in a number of schools every day.
- Implement the suggested framework for teaching and learning Grade 11 Euclidean geometry in a longitudinal study to achieve conclusive results.
- Extend the study to Grade 10 and 12 students.

REFERENCES

Abakpa, B. O., & Iji, C. O. (2011). Effect of Mastery Learning Approach on Senior Secondary School Students' Achievement in Geometry. *Journal of the Science Teachers Association of Nigeria, 46*(1), 165-177.

Abdelfatah, H. (2010). *Improving Attitudes towards Geometric Proof through A Suggested Story-Based Dynamic Geometry Approach* (Unpublished doctoral dissertation). University of Education Karlsruhe, Karlsruhe, Germany.

Abdullah, A. H., & Zakaria, E. (2012). The Activities Based on Van Hiele's Phase-Based Learning: Experts' and Preservice Teachers' Views. *Journal of Mathematics and Statistics, 8*(3), 385-395.

Abdullah, A. H., & Zakaria, E. (2013). Enhancing Students' Level of Geometric Thinking Through Van Hiele's Phase-based Learning. *Indian Journal of Science and Technology, 6*(5), 4432-4446.

Abrahams, K., & Matthews, T. (2011). *Child Rights Manual: Handbook for Parliamentarians.* Cape Town: Parliament of the Republic of South Africa.

Achor, E., & Imoko, B. (2012). Improving Some Nigeria Secondary Students' Achievement in Geometry: A Field Report on Team Teaching Approach. *New York Science Journal, 5*(1), 37-43.

Adams, C. (2010). *Proof of a Problem.* Pembroke, NC: University of North Carolina.

Adeniji, S. M., Ameen, S. K., Dambatta, B., & Orilonise, R. (2018). Effects of Mastery Learning Approach on Senior School Students' Academic Performance and Retention in Circle Geometry. *International Journal of Instruction, 11*(4), 951-962.

Adler, E. S., & Clark, R. (2008). *How it's done: An invitation to social research.* Belmont, CA: Thomson Wadsworth.

Aitken, L., Gallagher, R., & Madronio, C. (2003). Principles of recruitment and retention in clinical trials. *International Journal of Nursing Practice, 9*(6), 338-346.

Alderson, P. & Morrow, V. (2011). Respect for rights: privacy and confidentiality. In P. Alderson, & V. Morrow, *The ethics of research with children and young people: A practical handbook* (pp. 31-48). London: Sage.

Alex, J. K., & Mammen, K. J. (2012). A Survey of South African Grade 10 Learners' Geometric Thinking Levels in Terms of the Van Hiele Theory. *Anthropologist*, *14*(2), 123-129.

Alex, J. K., & Mammen, K. J. (2014). An Assessment of the Readiness of Grade 10 Learners for Geometry in the context of Curriculum and Assessment Policy Statement (CAPS) Expectation. *International Journal of Educational Sciences*, *7*(1), 29-39.

Alex, J. K., & Mammen, K. J. (2016). Lessons Learnt from Employing Van Hiele Theory Based Instruction in Senior Secondary School Geometry Classrooms. *Eurasia Journal of Mathematics, Science and Technology*, *12*(8), 2223-2236.

Al-Khateeb, M. A. (2016). The Extent of Mathematics Teachers' Awareness of Their Students' Misconceptions in Learning Geometrical Concepts in the Intermediate Education Stage. *European Scientific Journal , 12*(31), 357-372.

Amidu, B., & Nyarko, J. (2019). Van Hiele Geometric Thinking Levels of Junior High School Students of Atebubu Municipality in Ghana. *African Journal of Educational Studies in Mathematics and Sciences, 15*(1), 39-50.

Andrew, S., & Halcomb, E. J. (2009). Mixed Methods Research for Nursing and Health Sciences. London: Wiley-Blackwell.

Anglin, W. (1994). *Mathematics: A Concise History and Philosophy*. New York, NY: Springer.

Anney, V. N. (2014). Ensuring the Quality of the Findings of Qualitative Research: Looking at Trustworthiness Criteria. *Journal of Emerging Trends in Educational Research and Policy*, *5*(2), 272-281.

Anthony, G., & Walshaw, M. (2009). *Effective pedagogy of Education*. Brussels: International Academy of Education.

Arends, F., Winaar, L., & Mosimege, M. (2017, August). Teacher Classroom Practices and Mathematics Performance in South African Schools: A Reflection on TIMSS 2011. *South African Journal of Education*, *37*(3), Art.#1362, 11 pages. Retrieved August 26, 2019, from https://doi.org/10.15700/saje.v37n3a1362

Armah, R.B., Cofie, P.O., & Okpoti, C.A. (2018). Investigating the Effect of Van Hiele Phase-based Instruction on Pre-service Teachers' Geometric Thinking. *International Journal of Research in Education and Science, 4*(1), 314-330.

Arnold, C., & Bartlett, K. (2010). Improving Learning Achievement in Early Primary in Low-income Countries. Geneva: AGA Foundation.

Arnot, M., McIntyre, D., Peddar, D., & Reay, D. (2004). *Consultation in the Classroom.* Cambridge: Pearson.

Atebe, H. U. (2008). *Students' Van Hiele Levels of Geometric Thought and Conception in Plane Geometry: A Collective Case Study of Nigeria and South Africa* (Unpublished doctoral dissertation). Rhodes University, Grahamstown, South Africa.

Atuahene, F., & Russell, T. A. (2016). Mathematics Readiness of First-Year University Students. *Journal of Developmental Education, 39*(3), 12-32.

Ausubel, D. P. (1960). The use of advance organizers in the learning and retention of meaningful verbal materials. *Journal of Educational Pyschology, 51*(5), 267-272.

Avgousti, M. (2017). One Size Does Not Fit All: Understanding Differentiated

Instruction in Elementary Mixed-Ability Classrooms. Ontario Institute for Studies in Education of the University of Toronto, Department of Curriculum, Teaching and Learning. Ontario: University of Toronto.

Aylar, E., & Sahiner, Y. (2013). A Study on Teaching Proof to 7th Grade Students. *Procedia – Social and Behavioral Sciences, 116*(2014), 3427-3431.

Baffoe, E., & Mereku, D. K. (2010). The Van Hiele Levels of Understanding of Students Entering Senior High School in Ghana. *African Journal of Educational Studies in Mathematics and Sciences, 8*(2010), 51-61.

Bal, A. P. (2014). Predictor Variables for Primary School Students Related to Van Hiele Geometric Thinking. *Journal of Theory and Practice in Education, 10*(1), 259-278.

Bandura, A. (1977). Self-Efficacy: Towards a Unifying Theory of Behavioural Change. *Psychological Review*, *84*(2), 191-215.

Bandura, A. (1986). *Social Foundations of Thought and Action: A Social Cognitive Theory*. Englewood Cliffs, NJ: Prentice Hall.

Bansilal, S., James, A., & Naidoo, M. (2010). Whose voice matters? LEARNERS. *South African Journal of Education*, *30*(1), 153-165.

Baral, S., Uprety, S., & Lamichhane, B. (2016, March). Focus Group Discussion. Retrieved August 20, 2016, from Herd: http://www.herd.org.np/uploads/frontend/Publications/PublicationsAttachments1/1480653060-Focus%20Group%20Discussion_0.pdf

Barbin, E., & Menghini, M. (2014). History of Teaching Geometry. In A. Karp, & G. Schubring (Eds.), *Handbook on the History of Mathematics Education* (pp. 473-492). New York, NY: Springer.

Barwell, R., Barton, B., & Setati, M. (2007). Multilingual issues in mathematics education: Introduction. *Educational Studies in Mathematics, 64*(2), 113-119.

Bay-Williams, J. M., & Livers, S. D. (2009). Supporting Math Vocabulary Acquisition. *Teaching Children Mathematics, 16*(4), 238-245.

Bazeley, P., & Jackson, K. (2013). *Qualitative Data Analysis with NVIVO.* London: Sage.

Bedford, T., & Burgess, J. (2001). The Focus Group Experience. In M. Limb, C. Dwyer, M. Limb, & C. Dwyer (Eds.), *Qualitative Methodologies for Geographers: Issues and Debates* (pp. 121-135). New York, NY: Arnold.

Bell, B. A. (2010). Pretest–Posttest Design. In N. J. Salkind (Ed.), *Encyclopedia of Research Design* (pp. 1087-1092). Thousand Oaks, CA: Sage.

Berg, B. L. (2012). *Qualitative Research Methods for the Social Sciences* (7th ed.). Boston: Allyn and Bacon.

Bitsch, V. (2005). Qualitative Research: A Grounded Theory Example and Evaluation Criteria. *Journal of Agribusiness*, *23*(1), 75-91.

Blaine, S. (2009, June 26). Bring back the tough maths for matrics. Retrieved March 20, 2010, from Business Day, All Africa Global Media:

http://www.bdlive.co.za/articles/2009/06/26/bring-back-the-tough-maths-for-matrics

Blanton, M. L., Stylianou, D. A., & David, M. M. (2009). Understanding instructional scaffolding in classroom discourse on proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proof Across the Grades: A K-12 Perspective* (pp. 290-306). New York, NY: Routledge.

Bonnie, L., & Lawes, E. (2016). Assessing Students' Maths Self-Efficacy and Achievement. *Assessment News, 2(2016)*, 60-63.

Borthwick, A. (2011). Children's perceptions of, and attitudes towards, their mathematics lessons. In C. Smith (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, *31*(1), pp. 37-42.

Bouffard-Bouchard, T., Parent, S., & Larivee, S. (1991). Influence of Self-Efficacy on Self-Regulation and Performance Among Junior and Senior High-School Age Students. *International Journal of Behavioural Development, 14*, 153-164.

Bowie, L. (2009). What is Mathematics Paper 3 for? *Marang News, 5*.

Bowie, L., Davis, Z., Pillay, P., Nxumalo, H. B., Pleass, L. C., & Raju, M. G. (2014). *What's in the CAPS package? A Comparative Study of the National Curriculum Statement (NCS) and the Curriculum and Assessment Policy Statement (CAPS): Further Education and Training (FET) Mathematics.* Pretoria: Umalusi, Council for Quality Assurance in General and Further Education and Training.

Bowman, A. W., & Azzalini, A. (1997). *Applied Smoothing Techniques for Data Analysis: The Kernel Approach with S-Plus Illustrations.* Oxford: Oxford University Press.

Bradley, J., & Scheiber, J. (2010). Universalising Maths and Science Competence. In F. Antonie, K. Francis, & T. Kenny (Eds.), *Journal of the Helen Suzman Foundation, 59*(2010), 17-22.

Bramlet, D. C., & Drake, C. T. (2013a). A History of Mathematical Proof: Ancient Greece to the Computer Age. *Journal of Mathematical Sciences and Mathematics Education, 8*(2), 20-33.

Bramlet, D. C., & Drake, C. T. (2013b). The Role of Proof in the Curriculum. *Journal of Mathematical Sciences and Mathematics Education, 8*(1), 62-72.

Breen, R. L. (2006). A Practical Guide to Focus-Group Research. *Journal of Geography in Higher Education, 30*(3), 463-475.

Bruner, J. (1960). *The Process of Education.* Cambridge, MA: Harvard University Press.

Buckley, A. P. (2015). Using Sequential Mixed Methods in Enterprise Policy Evaluation: A pragmatic Design Choice? *The Electronic Journal of Business Research Methods, 13*(1), 16-26.

Burton, D. (2007). *The History of Mathematics: An Introduction* (6th ed.). New York, NY: McGraw-Hill.

Bytheway, B. (2012). The Use of Diaries in Qualitative Longitudinal Research. In B. Neale, & K. Henwood (Eds.), *Timescapes Methods Guides Series 2012 Guide No. 7.* Leeds: Economic and Social Research Council.

Calitz, M. G. (2005). *A Cultural Sensitive Therapeutic Approach to Enhance Educational Intelligence in Primary School Children.* Pretoria: University of South Africa.

Canavor, V. (2006). *From Proposal to Presentation: The Focus Group Process at NDI* (1st ed.). [ebook]: National Democratic Institute for International Affairs. Retrieved August 15, 2016, from https://www.ndi.org/sites/default/files/RESOURCE_From%20Proposal%20to%20Presentation%20The%20Focus%20Group%20Process_NDI.pdf

Cassim, I. (2006). *An exploratory study into grade 12 learners' understanding of theorems* (Unpublished master's dissertation). University of the Witwatersrand, Johannesburg.

Center for Innovation in Research and Teaching. (n.d.). *Effective Focus Group Questions.* Retrieved December 30, 2016, from Grand Canyon University CIRT: https://cirt.gcu.edu/research/developmentresources/research_ready/focus_groups/effective_questions

Centers for Disease Control and Prevention. (2008, July). Data Collection Methods for Program Evaluation: Focus Groups. Washington, DC: US Department of Health and Human Services. Retrieved August 16, 2016, from https://www.cdc.gov/healthyyouth/evaluation/pdf/brief13.pdf

Cheng, Y.-H., & Lin, F.-L. (2006). Using Reading and Coloring to Enhance Incomplete Provers' Performance in Geometry Proof. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education. 2*, pp. 289-296. Prague: PME.

Cheng, Y.-H., & Lin, F.-L. (2007). The Effectiveness and Limitations of Reading and Coloring Strategy in Learning Geometry. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education. 2*, pp. 113-120. Seoul: PME.

Cheng, Y.-H., & Lin, F.-L. (2009). Developing learning strategies for enhancing below average students' ability in constructing multi-steps geometry proofs. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education*, *1*, pp. 124-129. Taipei.

Chevalier, A., Harmon, C., Sullivan, V. O., & Walker, I. (2013, December 9). The impact of parental income and education on the schooling of their children. *IZA Journal of Labor Economics, 2*(8), 1-22.

Choi-Koh, S. S. (2000). The Activities Based on Van Hiele Model using Computer as a Tool. *Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education, 4*(2), 63-77.

Chung, C. J. (2001). Approaching Theory in Developmental Education. In D. B. Lundell, & J. L. Higbee (Eds.), *Theoretical Perspectives for Developmental Education* (pp. 19-25). Minneapolis, MN: Center for Research on Developmental Education and Urban Literacy.

Cirillo, M. (2009). Ten things to consider when teaching proof. *Mathematics Teacher, 103*(4), pp. 251-527.

Cirillo, M., & Herbst, P. (2012). Moving Toward More Authentic Proof Practices in Geometry. *The Mathematics Educator, 21*(2), 11-33.

Cirillo, M., & Hummer, J. (2019). Addressing Misconceptions in Secondary Geometry Proof. *Mathematics Teacher, 112*(6), 410-417.

Cleveland, W. S. (1979, December). Robust locally weighted regression and smoothing scatterplots, *Journal of American Statistical Association, 74*(368), 829-836.

Collins, J. L. (1982). Self-Efficacy and Ability in Achievement Behaviour. *Paper Presented at the Annual Meeting of the American Educational Research Association.* New York.

Cottaneo, M. A., Oggenfus, C., & Wolfer, S. C. (2016, March). *The More, the Better? The Impact of Instructional Time on Student Performance.* Germany: Institute for the Study of Labor.

Council on Higher Education. (2013). *A proposal for undergraduate curriculum reform in South Africa: The case for a flexible curriculum structure.* Pretoria: Council on Higher Education.

Craig, T., & Morgan, C. (2012). Language and Communication in Mathematics Education. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congresss on Mathematical Education* (pp. 529-533). Seoul, Korea: Springer.

Creswell, J. W. (2009). *Research Design: Qualitative, Quantitative, and Mixed Methods Approaches* (3rd ed.). Thousand Oaks, CA: Sage.

Creswell, J. W., & Plano Clark, V. L. (2007). *Designing and conducting mixed methods research.* Thousand Oaks, CA: Sage.

Crowley, M. L. (1987). The Van Hiele Model of the Development of Geometric Thought. In M. M. Lindquist, *Learning and Teaching Geometry, K-12, 1987 Yearbook of the National Council of Teachers of Mathematics* (pp. 1-16). Reston, VA: The National Council of Teachers of Mathematics.

Davis, S. L. (2008). Math Anxiety Bill of Rights. San-Marcos: Student Learning Assistance Center(SLAC). Retrieved September 12, 2019, from https://gato-docs.its.txstate.edu/slac/Subject/Math/Math-Study-Skills/Math-Anxiety-Bill-of-Rights/Math%20Anxiety%20Bill%20of%20Rights.pdf

Dennick, R. (2012). Twelve tips for incorporating educational theory into teaching practices. *Medical Teacher, 34*(8), 618-624.

Denscombe, M. (2007). *The Good Research Guide for Small-scale Social Research Projects* (3rd ed.). New York, NY: McGraw-Hill.

De Villiers, M. (2010). Some Reflections on the Van Hiele theory. *4th Congress of Teachers of Mathematics of the Croatian Mathematical Society.* Zagreb.

De Villiers, M., & Heideman, N. (2014). Conjecturing, Refuting and Proving within the context of Dynamic Geometry. *Learning and Teaching Mathematics, 2014*(17), 20-26.

Department of Basic Education. (2011). *Curriculum and Assessment Policy Statement (CAPS), Further Education and Training Phase Grades 10-12 Mathematics.* Pretoria: Department of Basic Education.

Department of Basic Education. (2013). National Senior Certicate Grade 11 Mathematics P2 Exemplar 2013. Republic of South Africa: Department of Basic Education. Retrieved June 20, 2016, from http://www.education.gov.za

Department of Basic Education. (2015). *National Senior Certificate Examinations: 2014 Diagnostic Report*. Pretoria: Department of Basic Education.

Department of Basic Education. (2016a). *2015 National Senior Certificate Examination Diagnostic Report.* Pretoria: Department of Basic Education.

Department of Basic Education. (2016b). *Schools Masterlist Data.* Retrieved July 20, 2016, from https://www.education.gov.za/Programmes/EMIS/EMISDownloads.aspx

Department of Basic Education. (2017). *National Senior Certificate Examination 2016 Diagnostic Report.* Pretoria: Department of Basic Education.

Department of Basic Education. (2018). *2017 National Senior Certificate Diagnostic Report Part 1.* Pretoria: Department of Basic Education.

Department of Basic Education. (2019). *National Senior Certificate 2018 Diagnostic Report Part 1.* Pretoria: Department of Basic Education.

Department of Basic Education. (2020). *2019 National Senior Certificate School Performance Report.* Pretoria: Department of Basic Education.

Department of Education. (2003). *National Curriculum Statement Grades 10-12 (General) Mathematics.* Pretoria: Department of Education.

Dhlamini, J. J. (2012). *Investigating the effect of implementing context-based problem solving instruction on learners' performance* (Unpublished doctoral dissertation)*.* University of South Africa, Pretoria, South Africa.

Dhlamini, S. S., & De Villiers, M. (2013). An investigation into grade 12 teachers' understanding of Euclidean geometry. *Global Research Journal on Mathematics Education, 2*(1), 97-118.

Dieltiens, V., & Motala, S. (2014). Quintile ranking system, South Africa. In M. Poisson, *Achieving transparency in pro-poor education incentives* (pp. 69-94). Paris, France: International Institute for Educational Planning.

Dikko, M. (2016). Establishing Construct Validity and Reliability: Pilot Testing of a Qualitative Interview for Research in Takaful (Islamic Insurance). *The Qualitative Report, 21*(3), 521-528.

Diko, N. (2013). Teacher Education. In V. Reddy, A. Juan, & T. Meyiwa (Eds.), *Education and Skills Development – Towards a 20 year review: Basic and Post School Education* (pp. 56-76). Pretoria: Human Sciences Research Council.

Dilshad, R. M., & Latif, M. I. (2013). Focus Group Interview as a Tool for Qualitative Research: An Analysis. *Pakistan Journal of Social Sciences, 33*(1), 191-198.

Ding, D. (2014). Determining environmental values using focus groups. In J. Appleton, & J. Appleton (Ed.), *Routledge Studies in Sustainable Development: Values in Sustainable Development* (pp. 58-67). New York, NY: Routledge.

Dube, N. V. (2016). *Teachers' Experiences of the Implementation of the Mathematics Curriculum and Assessment Policy Statement at the Further Education and Training (FET) Phase: Case Study.* University of Fort Hare, Faculty of Education. East London: University of Fort Hare.

Duke, J. (2012). Joining the Dots: Piloting the Work Diary as a Data Collection Tool. *Issues in Educational Research, 22*(2), 111-126.

Eadie, A., & Lampe, G. (2013). *Mathematics Grade 11 CAPS.* Claremont: The Answer.

Elo, S., Kääriäinen, M., Kanste, O., Pölkki, T., Utriainen, K., & Kyngäs, H. (2014). Qualitative Content Analysis: A Focus on Trustworthiness. *Sage Open*, *4*, 1-10.

Engelbrecht, J., Harding, A., & Phiri, P. (2010). Are OBE-Trained Learners Ready for University Mathematics? *Pythagoras, 72*, 3-13.

Ercikan, K., Chen, M. Y., Lyons-Thomas, J., Goodrich, S., Sandilands, D., Roth, W.-M., & Simon, M. (2015). Reading Proficiency and Comparability of Mathematics and Science Scores for Students from English and non-English Backgrounds: An International Perspective. *International Journal of Testing, 15*(2), 153-175.

ETR. (2013). Best Practices in Research and Evaluation: Focus Groups. *ETR Best Practices Guides*. Scotts Valley, CA: ETR. Retrieved August 15, 2016, from https://www.etr.org/ebi/assets/File/etr_best_practices_focus_groups.pdf

Evbuomwan, D. (2013). *An Investigation into the Difficulties faced by Form C Students in the Learning of Transformation Geometry in Lesotho Secondary Schools* (Unpublished master's dissertation). University of South Africa, Pretoria, South Africa.

Eves, H. (1990). *An Introduction to the History of Mathematics* (6th ed.). Philadelphia, PA: Saunders College Publishing.

Fabiyi, T. R. (2017). Geometry concepts in mathematics perceived difficult to learn by senior secondary school students in Ekiti State Nigeria. *IOSR*

Journal of Research & Method in Education, 7(1), 93-90.

Feilzer, M. Y. (2010). Doing Mixed Methods Research Pragmatically: Implications for the Rediscovery of Pragmatism as a Research Paradigm. *Journal of Mixed Methods Research, 4*(1), 6-16.

Feza, N., & Webb, P. (2005). Assessment Standards, Van Hiele levels, and grade seven learners' understanding of geometry. *Pythagoras, 62*, 36-47.

Field, A. P. (2013). *Discovering Statistics using IBM SPSS: And sex and drugs and rock 'n' roll* (4th ed.). London: Sage.

Fife-Schaw, C. (2012). Quasi-experimental Designs. In G. M. Breakwell, J. A. Smith, & D. B. Wright (Eds.), *Research Methods in Psychology* (4th ed., pp. 75-92). London: Sage.

Finashin, S. (2015). *History of Math Concepts: Essentials, Images and Memes.* Retrieved August 8, 2016, from http://metu.edu.tr/~serge/courses/303-2015/Book-essentials-A4.pdf

Fisher, D., & Frey, N. (2011). *The Purposeful Classroom: How to Structure Lessons with Learning Goals in Mind.* Alexandria, VA: ASCD.

Fouka, G., & Mantzorou, M. (2011). What are the Major Ethical Issues in Conducting Research? Is there a Conflict between the Research Ethics and the Nature of Nursing? *Health Science Journal, 5*(1), 3-14.

Freitas, H., Oliveira, M., & Jenkins, M. (1998, February). *The Focus Group, A Qualitative Research Method: Reviewing The Theory, and Providing Guidelines to its Planning.* University of Baltimore, Merrick School of Business. USA: Information Systems Research Group.

Fritz, A. E., & Morgan, G. A. (2010). Sampling. In N. J. Salkind (Ed.), *Encyclopedia of Research Design* (pp. 1303-1306). Thousand Oaks, CA: Sage.

Furr, J. (1996). A brief history of mathematics education in America (*Unpublished manuscript*). *University of Georgia, Athens, Georgia.* Retrieved October 16, 2017, from http://jwilson.coe.uga.edu/EMAT7050/HistoryWeggener.html.

Fuys, D., Geddes, D., & Tischler, R. (1988). The Van Hiele Model of Thinking in Geometry among Adolescents. *Journal for Research in Mathematics Education, 3*, 1-196.

Gallo, M. A. (2020). *Bantu Education, and Its Living Educational and Socioeconomic Legacy in Apartheid and Post-Apartheid South Africa* (Unpublished bachelor's dissertation). Fordham University, New York.

Gardner, H. (1991). *The Unschooled Mind: How Children Think and How Schools Should Teach.* New York, NY: Basic Books.

Gliner, J. A., Morgan, G. A., & Leech, N. L. (2009). *Research Methods in Applied Settings: An Integrated Approach to Design and Analysis* (2nd ed.). New York, NY: Routledge.

Govender, R. (2014). Rider strategies for solving school geometry problems. In M. Lebitso, & A. Maclean (Eds.), *Proceedings of the 20th Annual National Congress of the Association for Mathematics Education of South Africa* (*2*, pp. 4-5). Kimberley: AMESA.

Govil, P. (2013). Ethical Considerations in Educational Research. *International Journal of Advancement in Education and Social Science, 1*(2), 17-22.

Graneheim, U. H., & Lundman, B. (2004). Qualitative Content Analysis in Nursing Research: Concepts, Procedures and Measures to Achieve Trustworthiness. *Nurse Education Today, 24*(2), 105-112.

Gray, D. (2008, March 4). *Using the Geometer's Sketchpad in the Math Classroom to Improve Engagement, Transform the Teaching Environment, and Enhance Understanding.* Retrieved October 16, 2017, from Discoverarchive.vanderbilt.edu: http://discoverarchive.vanderbilt.edu/handle/1803/571

Groenewald, P. F. (1976). *Aspects in the traditional world of culture of the black child which hamper his intelligence: A cultural educational exploratory study.* Pretoria: Human Sciences Research Council.

Gunhan, B. (2014). A Case Study on the Investigation of Reasoning Skills in Geometry. *South African Journal of Education, 34*(2). Retrieved July 15, 2016, from http://www.sajournalofeducation.co.za

Gutiérrez, A., Jaime, A., & Fortuny, J. M. (1991). An alternative paradigm to evaluate the acquisition of the Van Hiele levels. *Journal for Research in Mathematics Education, 22*(3), 237-251.

Hampton, J. M. (2012). *The Reporting of Effect Size in Educational Psychology Literature.* Lancaster: Lancaster University.

Harel, G., & Fuller, E. (2009). Current Contributions Toward Comprehensive Perspectives on the Learning and Teahing of Proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth, *Teaching and Learning Proof Across the Grades: A K-12 Perspective* (pp. 355-370). New York, NY: Routledge.

Harel, G., & Sowder, L. (2007). Towards comprehensive perspectives on the teaching and learning of proof. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning.* Charlotte, NC: Information Age Publishing.

Harlow, L. L., Burkholder, G. J., & Morrow, J. A. (2002). Evaluating Attitudes, Skill, and Performance in a Learning-Enhanced Quantitative Methods Course: A Structural Modeling Approach. *Structural Equation Modeling: A Multidisciplinary Journal, 9*(3), 413-430.

Harris, A. D., McGregor, J. C., Perencevich, E. N., Furuno, J. P., Zhu, J., E, P. D., & Finkelstein, J. (2006). The Use and Interpretation of Quasi-Experimental Studies in Medical Informatics. *Journal of the American Medical Informatics Association, 13*(1), 16-23.

Harwell, M. R. (2011). Research Design in Qualitative/Quantitative/Mixed methods. In C. F. Conrad, & R. C. Serlin (Eds.), *The SAGE Handbook for Research in Education: Pursuing Ideas as the Keystone of Exemplary Inquiry* (pp. 147-165). Thousand Oaks, CA: Sage.

Haslam, A. S., & McGarty, C. (2014). *Research Methods and Statistics in Pyschology* (2nd ed.). Thousand Oaks, CA: Sage.

Haviger, J., & Vojkůvková, I. (2015). The Van Hiele Levels at Czech Secondary Schools. *Procedia-Social and Behavioural Sciences, 171*(2015), 912-918.

Hayley, P. (2009). *Educational reforms in South Africa: The Role of the Teacher in the Implementation of Educational Reforms in Post-Apartheid South Africa 1994-2008* (Unpublished master's thesis). University of Jyvaskyla, Finland.

Heale, R., & Twycross, A. (2015). Validity and reliability in quantitative studies. *Evidence Based Nursing, 18*(3), 66-67.

Health Promotion Unit. (2007). Focus Groups. Queensland Stay on Your Feet Community Good Practice Toolkit. Division of the Chief Health Officer, Queensland Health. Retrieved August 16, 2016, from https://www.health.qld.gov.au/__data/assets/pdf_file/0021/425730/33343.pdf

Healy, L., & Hoyles, C. (2000). A study of proof concepts in algebra. *Journal for Research in Mathematics Education, 31*(4), 183-190.

Heinze, A., & Reiss, K. (2009). Developing Argumentation and Proof Competencies in the Mathematics Classroom. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proof Across the Grades: A K-16 Perspective* (pp. 191-203). New York, NY: Routledge.

Heinze, A., Cheng, Y.-H., Ufer, S., Lin, F.-L., & Reiss, K. (2008). Strategies to foster students' competencies in constructing multi-steps geometric proofs: teaching experiments in Taiwan and Germany. *ZDM Mathematics Education, 40*, 443-453.

Hemmi, K. (2010). Three styles characterising mathematicians' pedagogical perspectives on proof. *Educational Studies in Mathematics, 75*(3), 271-291.

Herbst, P. G. (2002). Establishing a Custom for Proving in American School Geometry: Evolution of the Two-Column Proof in the Early Twentieth Century. *Educational Studies in Mathematics, 49*(3), 283-312.

Hinton, C., Fischer, K. W., & Glennon, C. (2012). *Mind, Brain, and Education: The Students at the Center Series.* Washington, DC: Jobs for the Future.

Hlalele, B. M. (2020). Developing the usage index for teaching with technology: A case study for Motheo district, South Africa. *The Journal for Transdiscliplinary Research in Southern Africa, 16*(1), 1-9.

Holloway, I., & Wheeler, S. (2010). *Qualitative Research in Nursing and Healthcare.* Oxford, UK: Blackwell.

Howse, T. D., & Howse, M. E. (2015, January). Linking the Van Hiele Theory to Instruction. *Teaching Children Mathematics, 21*(5), 305-313.

Hutchison, D., & Styles, B. (2010). *A Guide to Running Randomised Controlled Trials for Educational Researchers.* Slough: NFER.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*(4), 358-390.

Ivankova, N. V., Creswell, J. W., & Stick, S. L. (2006). Using Mixed-Methods Sequential Explanatory Design: From Theory to Practice. *Field Methods, 18*(1), 3-20.

Jackson, A. W., & Davis, G. A. (2000). *Turning Points 2000: Educating Adolescents in the 21st Century.* New York, NY: Teachers College Press.

Johnson, B. R., & Onwuegbuzie, A. J. (2004). Mixed methods research: A research paradigm whose time has come. *Educational Researcher, 33*(7), 14-26.

Jones, K., & Rodd, M. (2001). Geometry and Proof. In T. Rowland (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, *21*, pp. 95-100.

Jones, K., Fujita, T., & Kunimune, S. (2012). Promoting productive reasoning in the teaching of geometry in lower secondary school: Towards a future research agenda. *Pre-proceedings of ICME-12* (pp. 2387-2396). Seoul, Korea: COEX.

Kalinec-Craig, C. A. (2017). The Rights of the Learner: A Framework for Promoting Equity Through Formative Assessment in Mathematics Education. *Democracy and Education, 25*(2), Article 5. Retrieved September 12, 2019, from https://democracyeducationjournal.org/home/vol25/iss2/5/

Kanandjebo, L. N., & Ngololo, E. N. (2017). The Effects of 'Geometry Sketchpad' on Grade 12 Learners' Performance in Geometry. *International Electronic Journal of Mathematics Education, 12*(3), 735-747.

Kazemi, E. (2018). The Demands of the Rights of the Learners. *Democracy and Education, 26*(2), Article 6. Retrieved September 12, 2019, from https://democracyeducationjournal.org/home/vol26/iss2/7/

Kearsley, E. P. (2010). Effect of the National Senior Certificate and Outcomes-Based Education on the Civil Engineering Industry in South Africa. In D. J. Grayson (Ed.), *Proceedings of the Academy of Science of South Africa Forum. Critical Issues in School Mathematics and Science: Pathways to Progress* (pp. 14-19). Pretoria: Academy of Science of South Africa.

Keogh-Brown, M. R., Bachmann, M. O., Sheptsone, L., Hewitt, C., Howe, A., Ramsay, C. R., Campbell, M. J. (2007, October). Contamination in trials of educational interventions. *Health Technology Assessment, 11*(43), 1-107.

Khan, R. M., Iqbal, N., & Tasneem, S. (2016). The influence of Parents Educational level on Secondary School Students Academic achievements in District Rajanpur. *Journal of Education and Practice, 6*(16), 76-79.

Khuzwayo, H. B. (2005). A history of mathematics education research in South Africa: the apartheid years. In R. Vithal, J. Adler, & C. Keitel (Eds.), *Researching Mathematics Education In South Africa: Perspectives, Practices and Possibilities* (pp. 307-3027). Cape Town: Human Sciences Research Council.

Kirk, R. E. (1996). Practical Significance: A concept whose time has come. *Educational and Psychological Measurement, 56*(5), 746-759.

Klem, A. M., & Connell, J. P. (2004). Relationships Matter: Linking Teacher Support to Student Engagement and Achievement. *Journal of School Health, 74*(2), 262-273.

Knight, W. E. (2012). Using Focus Groups. In W. E. Knight, & R. A. Costomiris (Eds.), *Ball State University Assessment Workbook* (pp. 49-54). Muncie: Office of Institutional Effectiveness, Ball State University.

Knuth, E. J., Choppin, J. M., & Bieda, K. N. (2009). Middle School Students' Production of Mathematical Justifications. In D. A. Stylianou, M. L. Blaton, & E. J. Knuth, *Teaching and Learning Proof Across the Grades: A K-16 Perspective* (pp. 153-170). New York, NY: Routledge.

Kögce, D., Aydin, M., & Yildiz, C. (2010). The views of high school students about proof and their levels of proof: The case of Trabzon. *Procedia Social and Behavioral Sciences, 2*(2010), 2544-2549.

Korstjens, I., & Moser, A. (2018). Series: Practical guidance to qualitative research. Part 4: Trustworthiness and publishing. *European Journal of General Practice, 24*(1), 120-124.

Krefting, L. (1991). Rigour in Qualitative Research: The Assessment of Trustworthiness. *The American Journal of Occupational Therapy, 45*(3), 214-222.

Krosnick, J. A., & Presser, S. (2009). Question and Questionnaire Design. In J. D. Wright, & P. V. Marsden (Eds.), *Handbook of Survey Research* (2nd ed., pp. 263-313). San Diego, CA: Elsevier.

Krueger, R. A. (2002, October). *Designing and Conducting Focus Group Interviews.* St. Paul, MN: University of Minnesota. Retrieved August 25, 2016, from https://www.eiu.edu/ihec/Krueger-FocusGroupInterviews.pdf

Krueger, R. A., & Casey, M. A. (2009). *Focus Groups: A Practical Guide for Applied Research* (4th ed.). San Francisco, CA: Sage.

Kuhn, G. (2016, November 16). *What is a Moderator's Guide? Focus Groups Upstate NY*. Retrieved December 5, 2016, from Drive Research: https://www.driveresearch.com/single-post/2016/10/30/What-is-a-Moderators-Guide-Focus-

Kutama, M. E. (2002). *An investigation into process-based instruction in the teaching of Grade 8 and 9 Euclidean geometry* (Unpublished master’s dissertation). University of South Africa, Pretoria, South Africa.

Lachapelle, P., & Mastel, T. (2017). Using Focus Groups for Community Development. Bozeman, MT: Montana State University Extension.

Lakens, D. (2013, November 26). Calculating and reporting effect sizes to facilitate cumulative science: a practical primer for t-tests and ANOVAs. *Frontiers in Psychology Cognition, 4*(863), 1-12.

Lee, C. (2006). *Language for Learning Mathematics: Assessment for Learning in Practice.* Berkshire, UK: Open University Press.

Leung, F.-H., & Ratnapalan, S. (2009, February). Spotlight on Focus Groups. *Canadian Familyt Physician, 55*(2), 218-219.

Limpopo Department of Education. (2013). National Senior Certificate (CAPS) Grade 11 Mathematics P2 November 2013. Republic of South Africa: Department of Basic Education.

Limpopo Department of Education. (2014). National Senior Certificate Grade 11 Mathematics P2 November 2014. Republic of South Africa: Department of Basic Education.

Lincoln, Y. S. (2009). Ethical practices in qualitative research. In D. M. Mertens, & P. E. Ginsberg (Eds.), *The handbook of social research ethics* (pp. 150-169). Thousand Oaks, CA: Sage

Lindsay, M. (2010). The Philosophical Underpinnings of Educational Research. *Polyglossia, 19*(2010), 5-11.

Lipsey, M. W., Puzio, K., Yun, C., Hebert, M. A., Steinka-Fry, K., Cole, M. W., . . . Busick, M. D. (2012). *Translating the Statistical Representation of the Effects of Educational Interventions Into More Readily Interpretable Forms (NCSER 2013-3000).* Washington, DC: National Center for Special Education Research.

Liu, K.-w. (2005). *The Effectiveness of Van Hiele-Based Instruction.* Unpublished MA Thesis, University of Hong Kong, Pokfulam, Hong Kong SAR.

Longueira, R. (2016). *Exploring The Functionality of The South African Education Quintile Funding System.* University of Pretoria, Faculty of Education. Pretoria: University of Pretoria.

Luks, S., & Bailey, D. (2011). Sampling, Random and Nonrandom. In B. Badie, D. Berg-Schlosser, & L. Morlino (Eds.), *International Encyclopedia of Political Science* (pp. 2347-2352). Thousand Oaks, CA: Sage.

Lundell, D. B., & Higbee, J. L. (2001). *Theoretical Perspectives for Developmental Education.* Minneapolis, MN: Center for Research on Developmental Education and Urban Literacy.

Luneta, K. (2014). Foundation Phase Teachers' (Limited) Knowledge of Geometry. *South African Journal of Childhood Education, 4*(3), 71-86.

Luneta, K. (2015). Understanding students' misconceptions: An analysis of final Grade 12 examination questions in geometry. *Pythagoras, 36*(1), 1-11.

Mackenzie, N., & Knipe, S. (2006). Research dilemmas: Paradigms, methods and methodology. *Issues in Educational Research, 16*(2), 193-205.

Malloy, C. E. (2002). The Van Hiele Framework. In D. K. Pugalee, J. Frykholm, H. Slovin, A. Johnson, & C. E. Malloy (Eds.), *Navigating Through Geometry Grades 6-8.* Reston, VA: National Council of Teachers of

Mathematics.

Martin, A. J., & Dowson, M. (2009). Interpersonal Relationships, Motivation, Engagement, and Achievement: Yields for Theory, Current Issues, and Educational Practice. *Review of Educational Research, 79*(1), 327-365.

Martin, F., & Bolliger, D. U. (2018). Engagement Matters: Student perceptions on the importance of engagement strategies in the online learning environment. *Online Learning Journal, 22*(1), 205-222.

Marzano, R. J. (2004). *Building Background Knowledge for Academic Achievement.* Alexandria, VA: Association for Supervision and Curriculum Development .

Mason, M. M. (1995). Geometric Understanding in Gifted Students Prior to a Formal Course in Geometry. *Paper presented at the17th Annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, October 21-24.* Columbus, Ohio.

Mason, M. M. (1997). The Van Hiele model of geometric understanding and mathematically talented students. *Journal for the Education of the Gifted, 21*(1), 39-53.

Mayberry, J. (1983). The Van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education, 14*(1), 58-69.

McGregor, S. L., & Murnane, J. A. (2010). Paradigm methodology and method: Intellectual integrity in consumer scholarship. *International Journal of Consumer Studies, 34*(4), 419-427.

McIntyre, R. N. I. (2007). *Analysing geometry in the Classroom Mathematics and Mind Action Series mathematics textbooks using the Van Hiele levels* (Unpublished master's thesis). University of the Witwatersrand, Johannesburg, South Africa.

McKeever, M. (2017). Educational Inequality in Apartheid South Africa. *American Behavioral Scientist, 16*(1), 114-131.

McLellan, E., MacQueen, K. M., & Neidig, J. L. (2003, February). Beyond the Qualitative Interview: Data Preparation and Transcription. *Field Methods, 15*(1), 63-84.

McParland, J. L., & Flowers, P. (2012). Nine lessons and recommendations from the conduct of focus group research in chronic pain samples. *British Journal of Health Psychology, 17*(3), 492-504.

Meiers, M., & Trevitt, J. (2010). Language in the mathematics classroom. *The Digest, NSWIT, 2010*(2), 1-16.

Meng, C. C. (2009). Enhancing Students' Geometric Thinking Through Phase-Based Instruction Using Geometer's Sketchpad: A Case Study. *J. Pendidik dan Pendidikan, Jil, 24*, 89-107.

Meng, C. C., & Idris, N. (2012). Enhancing Students' Geometric Thinking and Achievement in Solid Geometry. *Journal of Mathematics Education, 5*(1), 15-

Mertens, D. M. (2005). *Research methods in education and psychology: Integrating diversity with quantitative and qualitative approaches* (2nd ed.). Thousand Oaks, CA: Sage.

Mertens, D. M. (2010). *Research and Evaluation in Education and Psychology* (3rd ed.). Thousand Oaks, CA: Sage.

Miyazaki, M., Nagata, J., Chino, K., Sasa, H., Fujita, T., Komatsu, K., & Shimizu, S. (2019). Curriculum Development for Explorative Proving in Lower Secondary School Geometry: Focusing on the Levels of Planning and Constructing Proof. *Frontiers in Education, 4*(31), 1-9.

Mollard, E. (2014). Exploring Paradigms in Postpartum Depression Research: The Need for Feminist Pragmatism. *Health Care for Women International, 0*, 1-14.

Moon, K., Brewer, T. D., Januchowski-Hartley, S. R., Adams, V. M., & Blackman, D. A. (2016). A Guideline to Improve Qualitative Social Science Publishing in Ecology and Conservation Journals. *Ecology and Society, 21*(3), 17[Online]. Retrieved November 29, 2016, from http://dx.doi.org/10.5751/ES-08663-210317

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*, 249-266.

Mosia, M. S. (2016). *Improving Teachers' Technological Pedagogical Content Knowledge for Teaching Euclidean Geometry Using Integrated Information and Communication Technologies Software* (Unpublished doctoral thesis). University of the Free State, Bloemfontein.

Mouton, N., Louw, G., & Strydom, G. (2012). A Historical Analysis of the Post-Apartheid Dispensation Education in South Africa (1994-2011). *International Business and Economics Research Journal, 11*(11), 1211-1222.

Mthembu, S. G. (2007). *Instructional approaches in the teaching of Euclidean geometry in Grade 11* (Unpublished master's thesis). University of KwaZulu-Natal, Durban, South Africa.

Mthethwa, M., Bayaga, A., Bossé, M. J., & Williams, D. (2020). GeoGebra for learning and teaching: A parallel investigation. *South African Journal of Education, 40*(2), 1-12.

Mudaly, V. (2007). Proof and proving in secondary school. *Pythagoras, 66*, 64-75.

Mukamba, E., & Makamure, C. (2020). Integration of GeoGebra in Teaching and Learning Geometric Transformations at Ordinary Level in Zimbabwe. *Contemporary Mathematics and Science Education, 1*(1), 1-8.

Mullis, I. V., Martin, M. O., Foy, P., & Arora, A. (2012). *TIMSS 2011 International Results in Mathematics.* Boston College, Lynch School of Education. Chestnut Hill, MA, USA: TIMSS & PIRLS International Study Center.

Mwadzaangati, L. (2015). Mathematical Knowledge for Teaching Geometric Proof: Learning from Teachers' Practices. *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 3308-3308). Prague, Czech Republic: CERME.

Mwadzaangati, L. (2019). Comparison of geometric proof development tasks as set up in the textbook and as implemented by teachers in the classroom. *Pythagoras, 40*(1), 1-14.

Naidoo, J. (2013). Does social class influence learner reasoning in geometry? *Global Journal of Medical Research, 13*(3), 27-34.

Naidoo, J., & Kapofu, W. (2020). Exploring female learners' perceptions of learning geometry in mathematics. *South African Journal of Education, 40*(1), 1-11.

Nannyonjo, H. (2007). *Education inputs in Uganda: An Analysis of Factors Influencing Learning Achievement in Grade Six. World Bank Working Paper No.98.* Washington, DC: World Bank.

Nath, B. K., & Binny, M. I. (2018). *Significance of Mathematics Laboratory in Schools.* Malappuram: University of Calicut.

National Oceanic and Atmospheric Administration. (2015). Social Science Tools for Coastal Programs: Introduction to Focus Groups. Office for Coastal Management. Retrieved August 15, 2016, from https://coast.noaa.gov/data/digitalcoast/pdf/focus-groups.pdf

Ndlovu, M. (2013). The learning of geometry as moving from one thinking level to the next: revising Van Hiele. In Z. Davies, & S. Jaffer (Eds.), *Proceedings of the 19th Annual Congress of the Association for Mathematics Education of South Africa. 1*, pp. 277-279. Cape Town: AMESA.

Ndlovu, M., & Mji, A. (2012). Pedagogical implications of students' misconceptions about deductive geometric proof. *Acta Academia, 44*(3), 175-205.

Neuman, W. (2006). *Social Research Methods: Qualitative and quantitative approaches* (6th ed.). Boston, MA: Pearson.

Ngirishi, H., & Bansilal, S. (2019). An exploration of high school learners' understanding of geometric concepts. *Problems of Education in the 21st Century, 77*(1), 82-96.

Noble, H., & Smith, J. (2015). Issues of Validity and Reliability in Qualitative Research. *Evidence-Based Nursing, 18*(2), 34-35.

Nokleby, H. (2011). Triangulation with Diverse Intentions. *Karlstads Universitets Pedagogiska Tidskrift, 7*(1), 140-156.

Noraini, I. (2005). *Pedagogy in Mathematics Education* (2nd ed.). Kuala Lumpur: Utusan Publication.

Ntuli, N. (2014, July 23). *Fears over teacher preparedness.* Retrieved November 21, 2014, from Independent Online News: http://www.iol.co.za/news/south-africa/kwazulu-natal/fears-over-teacher-preparedness

Office of Quality Improvement. (1999). Focus Groups: When and Why to Use. Madison: University of Wisconsin. Retrieved August 20, 2016, from http://www.virginia.edu/processsimplification/resources/WiscFocusGroups.pdf

Oflaz, G., Bulut, N., & Akcakin, V. (2016). Pre-service classroom teachers' proof schemes in geometry: a case study of three pre-service teachers. *Eurasian Journal of Educational Research, 16*(63), 133-152.

Olivier, W. (2013, May). *Reflection on the training of teachers for the CAPS mathematics curriculum-a brief report.* Retrieved November 21, 2014, from SAMF: http://www.samf.ac.za/documents/CAPS_mathematics_curriculum.pdf

Olivier, W. (2014, April). *Reflections on the Implementation of CAPS Mathematics in the Classroom.* Retrieved November 21, 2014, from SAMF: http://www.samf.ac.za/documents/Reflection_on_implementation_of_CAPS_Mathematics_in_the_Classroom.pdf

Onwuegbuzie, A. J., & Leech, N. L. (2005). On becoming a pragmatic researcher: The importance of combining quantitative and qualitative research methodologies. *International Journal of Social Research Methodology, 8*(5), 375-387.

Onwuegbuzie, A. J., & Leech, N. L. (2007). Validity and Qualitative Research: An Oxymoron? *Quality and Quantity, 41*(2), 233-249.

Osayimwense, O. (2017). *History of mathematics education in South Africa* (Unpublished manuscript). Univerity of Benin, Benin City, Nigeria.

Oueini, S. (2019). *The Impact of Intelligent Tutoring Software on Geometry Students* (Unpublished doctoral thesis). University of South Carolina, California, Columbia.

Padayachee, P., Boshoff, H., Olivier, W., & Harding, A. (2011). A blended learning Grade 12 intervention using DVD technology to enhance the teaching and learning of mathematics. *Pythagoras, 32*(1), 1-8.

Page, J. (2007, October 29). *Thales of Miletus - A Short Biography.* Retrieved August 8, 2016, from OpenStax-CNX: https://cnx.org/exports/aaf9c2f8-d18f-401c-801d-adf0166a24c9@1.pdf/thales-of-miletus---a-short-biography-1.pdf

Paiva, C. E., Barroso, E. M., Carneseca, E. C., Souza, C. d., Santos, F. T., López, R. V., & Paiva, B. S. (2014). A critical analysis of test-retest reliability in instrument validation studies of cancer patients under palliative care: a systematic review. *BMC Medical Research Methodology, 14*(1), 8.

Pajares, F. (1996). Self-Efficacy Beliefs in Academic Settings. *Review of Educational Research, 66*(4), 542-578.

Palmer, R. R., Colton, J., & Kramer, L. S. (2013). *A History of Europe in the Modern World* (11th ed.). New York, NY: McGraw-Hill.

Parsons, R. R., Stack, R., & Breen, J. (1998). Writing and Computers: Increasing Geometric Content Based Knowledge Using the Van Hiele Model. *Sixth Annual Curriculum and Instruction Research Symposium.* Vermillion, SD: The University of South Dakota.

Patrick, H., Ryan, A. M., & Kaplan, A. (2007). Early Adolescents' Perceptions of the Classroom Social Environment, Motivational Beliefs, and Engagement. *Journal of Educational Psychology, 99*(1), 83-98.

Pearson, D., & Vossler, A. (2016). Methodological Issues in Focus Group Research. *Counselling Psychology Review, 31*(1), 8-16.

Phillips, M. D., Basson, J., & Botha, C. (2012). *MInd Action Series Mathematics 11 Textbook* (1st ed.). Sanlamhof: All Copy Publishers.

Plano Clark, V. L., & Badiee, M. (2010). Research Questions in Mixed Methods Research. In A. Tashakkori, & C. Teddlie (Eds.), *SAGE Handbook of Mixed Methods in Social & Behavioral Research* (pp. 275-305). Thousand Oaks, CA: Sage.

Polit, D. F., Beck, C. T., & Owen, S. V. (2007). Is the CVI an Acceptable Indicator of Content Validity? Appraisal and Recommendations. *Research in Nursing and Health, 30*, 459-467.

Prabawanto, S. (2018). The Enhancement of Students' Mathematical Self-Efficacy Through Teaching with Metacognitive Scaffolding Approach. *Journal of Physics: Conference Series 1013*, 1-7.

Prediger, S., & Schüler-Meyer, A. (2017). Fostering the Mathematics Learning of Language Learners: Introduction to Trends and Issues in Research and Professional Development. *Eurasia Journal of Mathematics, Science and Technology Education, 13*(7b), 4049-4056.

Professional Educator Standards Board. (2009). Defining Student Voice. Washington: Washington Professional Educator Standards Board (PESB). Retrieved September 13, 2019, from http://www.waproteach.org/rsc/pdf/WAProTeachStudentVoice.pdf

Pufall, E., Eaton, J. W., Nyamukapa, C., Schur, N., Takaruza, A., & Gregson, S. (2016). The relationship between parental education and children’s schooling in a time of economic turmoil: The case of East Zimbabwe, 2001 to 2011. *International Journal of Educational Development, 51*(2016), 125-134.

Pusey, E. L. (2003). *The Van Hiele Model of Reasoning in Geometry: A Literature Review.* Raleigh, NC: North Carolina State University.

Rahim, M. H. (2014). Research implications for teaching and learning strategies in undergraduate mathematics. *Proceedings of the Frontiers in Mathematics and Science Education Research Conference 1-3 May 2014* (pp. 122-130). Famagusta, North Cyprus: FISER.

Ramesh, S. (2017, March). Syllabus Completion. *Mentor Magazine, 10*(10), 14-17.

Ramoroka, N. J. (2006). *Educators' understanding of the premises underpinning outcomes-based education and its impact on their* *classroom assessment practices* (Unpublished master's dissertation). University of Pretoria, Pretoria.

Ramrathan, L., Le Grange, L., & Shawa, L. B. (2017). Ethics in educational research. In L. Ramrathan, L. Le Grange, & P. Higgs (Eds.), *Education Studies for Initial Teacher Development* (1st ed., pp. 432-443). Cape Town: JUTA.

Rausch, A. (2014). Using Diaries in Research on Work and Learning. In C. Harteis, A. Rausch, J. Seifried, C. Harteis, A. Rausch, & J. Seifried (Eds.), *Discourses on Professional Learning: On the Boundary Between Learning and Working* (pp. 341-366). Dordrecht: Springer.

Recio, A. M., & Godino, J. D. (2001). Institutional and personal meanings of proof. *Educational Studies in Mathematics, 48*(1), 83-99.

Reid, S., & Mash, B. (2014). African Primary Care Research: Qualitative Interviewing in Primary Care. *African Journal of Primary Health Care and Family Medicine, 6*(1), Art.#632, 6 Pages. Retrieved July 10, 2016, from http://dx.doi.org/10.4102/phcfm.v6i1.632

Reiss, K. M., Heinze, A., Renkl, A., & Groß, C. (2008). Reasoning and proof in geometry: effects of learning environment based on heuristic worked-out examples. *ZDM Mathematics Education, 40*, 455-467.

Reiss, K., Klieme, E., & Heinze, A. (2001). Prerequisites for the understanding of proofs in the geometry classroom. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (pp. 97-104). Utrecht: Utrecht University.

Rendall-Mkosi, K., Wenhold, F., & Sibanda, N. B. (2013). *Case Study of the National School Nutrition Programme in South Africa.* School of Health Systems and Pubic Health, University of Pretoria. Pretoria: University of Pretoria.

Riccomini, P. J., Smith, G. W., Hughes, E. M., & Fries, K. M. (2015, May 8). The Language of Mathematics: The Importance of Teaching and Learning Mathematical Vocabulary. *Reading and Writing Quarterly, 31*(3), 235-252.

Richardson, J. T. (2011). Eta squared and partial eta squared as measures of effect size in educationa research. *Educational Research Review, 6*(2), 135-147.

Rolfe, G. (2006). Validity, trustworthiness and rigour: quality and the idea of qualitative research. *Journal of Advanced Nursing, 53*(3), 304-310.

Ross, A., & Onwuegbuzie, A. J. (2012). Prevalence of Mixed Methods Research in Mathematics Education. *The Mathematics Educator, 22*(1), 84-113.

Ryan, M. (2016). *Geometry for Dummies* (3rd ed.). New Jersey: John Wiley and Sons Inc.

Saldaña, J. (2013). *The Coding Manual for Qualitative Researchers* (2nd ed.). London: Sage.

Salifu, A. S., Yakubu, A.-R., & Ibrahim, F. I. (2018). Van Hiele Geometric Thinking Levels of Pre-Service Teachers of E.P. College of Education, Bimbilla-Ghana. *Journal of Education Practice, 9*(23), 108-119.

Sanni, R. (2007). Teaching geometry in schools: An investigative rather than instructive process. *Pythagoras, 65*, 39-44.

Saunders, M. K., Lewis, P., & Thornhill, A. (2012). *Research methods for business students* (5th ed.). London: Pearson.

Schultze, A., & Sevenoak, F. (1913). *Plane Geometry.* New York: MacMillan.

Scotland, J. (2012). Exploring the Philosophical Underpinnings of Research: Relating Ontology and Epistemology to the Methodology and Methods of the Scientific, Interpretive, and Critical Research Paradigms. *English Language Teaching, 5*(9), 9-16.

Sehoole, C., & Adeyemo, K. S. (2016). Access to, and Success in, Higher Education in Post-apartheid South Africa: Social Justice Analysis. *Journal of Higher Education in Africa, 14*(1), 1-18.

Selden, A., & Selden, J. (2007). *Overcoming Students' Difficulties in Learning to Understand and Construct Proofs.* Tennessee Technological University, Department of Mathematics. Cookeville, TN: Tennessee Technological University.

Senk, S. L. (1989). Van Hiele levels and achievement in writing proofs. *Journal for Research in Mathematics Education, 20*(3), 309-321.

Shah, S. R., & Al-Bargi, A. (2013). Research Paradigms: Researchers' Worldviews, Theoretical Frameworks and Study Designs. *AWEJ: Arab World English Journal, 4*(4), 252-264.

Shange, D. S. (2016). *An exploration of teachers' experiences in teaching Grade 10 geometry within the curriculum and assessment policy statement (CAPS) in Umbumbulu circuit* (Unpublished master's dissertation). University of KwaZulu-Natal, Durban, South Africa.

Shaughnessy, M. J., & Burger, W. F. (1985). Spadework Prior to Deduction in Geometry. *Mathematics Teacher, 78*(6), 419-428.

Sheble, L., & Wildemuth, B. (2009). Research Diaries. In B. Wildemuth (Ed.), *Applications of Social Research Methods to Questions in Information and Library Science* (pp. 211-221). Santa Barbara, CA: Libraries Unlimited.

Shenton, A. K. (2004). Strategies for Ensuring Trustworthiness in Qualitative Research Projects. *Education for Information, 22*(2), 63-75.

Shi-Pui, K., & Ka-Luen, C. (2009). The Van Hiele Phases of Learning in Studying Cube Dissection. Retrieved February 15, 2017, from http://math.unipa.it/~grim/21_project/Kwan358-363.pdf

Shives, M. S. (2012, April 30). *Paradigms and Mathematics: A Creative Perspective.* Retrieved August 08, 2016, from Docs-Engine.Com: http://www.maa.org/sites/default/files/images/upload_library/46/HOMSIGMAA/shives_HOM_winner_2013.pdf

Shongwe, B. (2019). *Exploring Grade 11 learners' functional understanding of proof in relation to argumentation in selected high schools* (Unpublished doctoral thesis).University of KwaZulu-Natal, Durban, South Africa.

Shulman, L. S. (1986, February). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher, 15*(2), 4-14.

Siegle, D., & McCoach, B. D. (2007). Increasing Student Mathematics Self-Efficacy Through Teacher Training. *Journal of Advanced Academics, 18*(2), 278-312.

Siew, N. M., Chong, C. L., & Abdullah, M. R. (2013). Facilitating Students' Geometric Thinking Through Van Hiele's Phase-Based Learning Using Tangram. *Journal of Social Sciences, 9*(3), 101-111.

Silver, E., & Herbst, P. (2007). Theory in mathematics education scholarship. In F. Lester (Ed.), *Second Handbook of Research on Mathematics and Learning* (pp. 39-56). New York, NY: Information Age.

Siyepu, S. (2014). Analysis of Van Hiele’s Theory In Circle Geometry: A Focus in FET Level. In M. Lebitso, & A. Maclean (Eds.), *Proceedings of the 20th Annual National Congress of the Association for Mathematics Education of South Africa* (pp. 61-63). Kimberley: AMESA.

Skott, J. (2009). Theoretical loops in mathematics education: theory-practice relationships revisited. *Asian Journal of Educational Research and Synergy, 1*(2), 74-88.

Škrbec, M., & Čadež, T. H. (2015, June). Identifying and Fostering Higher Levels of Geometric Thinking. *Eur Educationasia Journal of Mathematics, Science and Technology, 11*(3), 601-617.

Smith, O. S. (2017). *The Influence of Language on the Teaching and Learning of Mathematics.* Walden University, College of Education. Minneapolis, MN: Walden University.

Smith, R. L. (2012). Mixed Methods Research Designs: A Recommended Paradigm for the Counselling Profession. *Ideas and Research You Can Use, 48*(1), 1-6.

Sonbul, S., & Schmitt, N. (2010). Direct teaching of vocabulary after reading: Is it worth the effort? *English Language Teachers' Journal, 64*(3), 253-260.

Steuer, G., & Dresel, M. (2013). A Constructive Error Climate an Element of Effective Learning Environments. *Management Learning, 57*(2), 262-275.

Stylianides, A. (2011). Towards a comprehensive knowledge package for teaching proof: A focus on the misconception that empirical arguments are proofs. *Pythagoras, 32*(1), 1-10.

Stylianides, G. J., & Silver, E. A. (2009). Reasoning-and-Proving in School Mathematics: The Case of of Pattern Identification. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proof Across the Grades: A K-16 Perspective* (pp. 235-249). New York, NY: Routledge.

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal of Research in Mathematics Education, 40*(3), 314-352.

Stylianides, G. J., Stylianides, A. J., & Philippou, G. N. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematical Teacher Education, 10*(3), 145-166.

Stylianides, G. S., & Stylianides, A. J. (2017). Research-based interventions in the area of proof: the past, the present, and the future. *Educational Studies in Mathematics, 96*(2), 119-127.

Stylianou, D. A., Blanton, M. L., & Knuth, E. J. (2009). *Teaching and Learning Proof Across the Grades: A K-16 Perspective.* New York, NY: Routledge.

Subramanian, L. (2005). *An Investigation of High School Geometry Students' Proving and Logical Thinking Abilities and the Impact of Dynamic Geometry Software on Student Performance* (Unpublished doctoral dissertation). University of Central Florida, Orlando.

Sun, S., Pan, W., & Wang, L. L. (2010). A Comprehensive Review of Effect Size Reporting and Interpreting Practices in Academic Journals in Education and Psychology. *Journal of Educational Psychology, 102*(4), 989-1004.

Sunzuma, G., Masocha, M., & Zezekwa, N. (2013). Secondary School Students' Attitudes Towards their Learning of Geometry: A Survey of Bindura Urban Secondary Schools. *Greener Journal of Educational Research, 3*(8), 402-410.

Tachie, S. A. (2020). The challenges of South African teachers in teaching euclidean geometry. *International Journal of Learning, Teaching and Educational Research, 19*(8), 297-312.

Tahani, A. (2016). Effects of Van Hiele Model in geometric Concepts Acquisition: The Attitudes towards Geometry and Learning Transfer Effect of the First Three Grades Students in Jordan. *International Education Studies*, *9*(4), 87-98.

Tashakkori, T., & Teddlie, C. (2009). *Foundations of mixed methods research.* Thousand Oaks, CA: Sage.

Tay, B. (2003). *A Van Hiele-Based Instruction and its Impact on the Geometry Achievement of Form One Students* (Unpublished master’s dissertation). University of Malaya, Kuala Lumpur.

Terrell, S. R. (2012, January). Mixed-Methods Research Methodologies. *The Qualitative Report, 17*(1), 254-280.

The Norwegian National Research Ethics Committee. (2019). *Guidelines for Research Ethics in the Social Sciences, Humanities, Law and Theology.* Oslo, AS: The National Committee for Research Ethics in the Social Sciences and Humanities.

Thomas, E., & Magilvy, J. K. (2011). Qualitative Rigour or Research Validity in Qualitative Research. *Journal for Specialists in Pediatric Nursing, 16*(2), 151-155.

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education, 43*(3), 253-295.

Toor, A., & Mgombelo, J. (2017). Maths Centers: A Pedagogical Tool for Student Engagement in International Maths Class. *Proceedings of CERME10* (pp. 3004-3010). Dublin: HAL.

Tuli, F. (2010, September). The Basis of Distinction Between Qualitative and Quantitative Research in Social Science: Reflection on Ontological, Epistemological and Methodological Perspectives. *Ethiopian Journal of Education and Sciences, 6*(1), 97-108.

Turchin, V. F. (1977). *The Phenomenon of Science.* New York, NY: Columbia University Press.

United States Department of Education, Office of the Deputy Secretary. (2004). *No Child Left Behind: A ToolKit for Teachers.* Washington, D.C: US Department of Education.

Usiskin, Z. (1982). *Van Hiele Levels and Achievement in Secondary School Geometry.* Chicago, IL: University of Chicago.

Van den Berg, D. J. (1978). *A pedagogical study of the black man's mathematical ability.* Pretoria: South African Human Sciences Research Council.

Van der Walt, M. (2009). Study Orientation and Basic Vocabulary in Mathematics in Primary School. *South African Journal for Science and Technology, 28*(4), 378-392.

Van der Walt, M., Maree, K., & Ellis, S. (2008). A Mathematics Vocabulary Questionnaire for Use in the Intermediate Phase. *South African Journal of Education, 28*(4), 489-504.

Van Hiele, P. M. (1984). A Child's Thought and Geometry. In D. Fuys, D. Geddes, & R. Tischler (Eds.), *English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre M. van Hiele* (pp. 247-256). Brooklyn, NY: Brooklyn College.

Van Hiele-Geldof, D. (1984). The Didactic of Geometry in the Lowest Class of Secondary School. In D. Fuys, D. Geddes, & R. Tischler (Eds.), *English Translation of Selected Writings of Dina van Hiele-Geldoff and Pierre M. van Hiele* (pp. 10-222). Brooklyn, NY: Brooklyn College.

Van Putten, S., Howie, S., & Gerrit, S. (2010). Making geometry compulsory: Are we prepared? *Perspectives in Education, 28*(4), 22-31.

Van Wijk, E. H. (2013). Managing Ethical Problems in Qualitative Research Involving Vulnerable Populations, Using a Pilot Study. *International Journal of Qualitative Methods, 12*(1), 570-586.

Villard, J. A. (2003, July 16). Use of Focus Groups: An Effective Tool for Involving People in Measuring Quality and Impact. US Department of Education. Retrieved August 16, 2016, from https://files.eric.ed.gov/fulltext/ED482279.pdf

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes.* Cambridge, MA: Harvard University Press.

Waltz, C. F., & Bausell, B. R. (1983). *Nursing research: Design, statistics and computer analysis* (2nd ed.). Philadelphia, PA: F. A. Davis Company.

Waltz, C. F., Strickland, O. L., & Lenz, E. R. (2005). *Measurement in Nuring and Health Research* (3rd ed.). New York, NY: Springer.

Wang, H. (2009). Teaching and learning how to write proofs in Concepts of Geometry. *US-China Education Review, 6*(12), 74-80.

Wang, X. (2013). Why Students Choose STEM Majors: Motivation, High School Learning and Post-Secondary Context of Support. *American Educational Research Journal, 50*(5), 1081-1121.

Weber, K., & Alcock, L. (2009). Proof in Advanced Mathematics Classes. In D. Stylianou, D. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 323-338). New York, NY: Routledge.

West African Examination Council. (2009). *Chief Examiner's Report on May/June School Certificate Examination.* Lagos: Lucky Star.

West African Examination Council. (2010). *Chief Examiner's Report.* Lagos: West African Examination Council.

West African Examination Council. (2011). *Chief Examiner's Report.* Lagos: West African Examination Council.

West, C. P. (2012). *A mixed methods sequential explanatory study of the impact of chronic pain on family resilience.* Queensland, Australia: James Cook University.

Westbrook, J., Durrani, N., Brown, R., Orr, D., Pryor, J., Boddy, J., & Salvi, F. (2013). *Pedagogy, Curriculum, Teaching Practices and Teacher Education in Developing Countries: Final Report, Education Rigorous Literature Review.* University of Sussex, Department for International Development. University of Sussex.

White, K. (2006). Self-Selected Sample. In K. White (Ed.), *The Sage Dictionary of Health and Society* (p. 188). London: Sage.

Whitney, L. (2005). Middle Level Teachers Using Focus Group Research. *Education, 126*(1), 3-8.

Wilkinson, S. (2008). Focus Groups. In J. A. Smith (Ed.), *Qualitative Psychology: A Practical Guide to Research Methods* (2nd ed., pp. 186-206). London: Sage.

Wirszup, I. (1976). Breakthroughs in the Psychology of Learning and Teaching Geometry. In J. Martin, & D. Bradbard (Eds.), *Space and Geometry: Papers from a Research Workshop* (pp. 75-97). Athens, GA: University of Georgia.

Woll, H. (2013). Process Diary as Methodological Approach in Longitudinal Phenomenological Research. *Indo-Pacific Journal of Phenomenology, 13*(2), 1-11.

Wolmarans, N., Smit, R., Collier-Reed, B., & Leather, H. (2010). Addressing concerns with the NSC: An analysis of first-year student performance in Mathematics and Physics. *Proceedings of the 18 th Conference of the Southern African Association for Research in Mathematics, Science and Technology Education* (pp. 274-284). KwaZulu-Natal.

Wong, L. P. (2008). Focus Group Discussion: A Tool for Health and Medical Research. *Singapore Medical Journal, 49*(3), 256-261.

Wright, P. (2016). Social Justice in the Mathematics Classroom. *London Review of Education, 14*(2), 104-118.

Yaghmaie, F. (2003). Content validity and its estimation. *Journal of Medical Education, 3*(1), 25-27.

Yi, J.-Y. (2008). The Use of Diaries as a Qualitative Research Method to Investigate Teachers' Perception and Use of Rating Schemes. *Journal of Pan-Pacific Association of Applied Linguistics, 12*(1), 1-10.

Yu, R. (2015). *Teacher Classroon Practices, Student Motivation, and Mathematics Achievement in High School: Evidence from HSLS:09 Data.* Blacksburg, VA: Virginia Polytechnic Institute and State University.

Zamanzadeh, V., Ghahramanian, A., Rassouli, M., Abbaszadeh, A., Alavi-Majd, H., & Nikanfar, A.-R. (2015). Design and Implementation Content Validity Study: Development of an Instrument for Measuring Patient-Centered Communication. *Journal of Caring Sciences, 4*(2), 165-178.

Zarch, M. K., & Kadivar, P. (2006). The Role of Mathematics Self-Efficacy and Mathematics Ability in the Structural Model of Mathematics Performance. *Proceedings of the 9th WSEAS International Conference on Applied Mathematics, May 27-29*, (pp. 242-249). Istanbul, Turkey.

Zaslavsky, O., Nickerson, S. D., Stylianides, A. J., Kidron, I., & Winicki-Landman, G. (2012). The Need for Proof and Proving: Mathematical and Pedagogical Perspectives. In G. Hanna, & M. de Villiers (Eds.), *Proof and Proving in Mathematics Education* (pp. 215-229). New York, NY: Springer.

Ziyani, I., King, L. J., & Ehlers, V. J. (2004). Using Triangulation of Research Methods to Investigate Fanily Planning Practice in Swaziland. *Africa Journal of Nursing and Midwifery, 6*(1), 12-17.

APPENDICES

APPENDIX A: APPROVAL LETTERS

A 1: ETHICAL CLEARANCE CERTIFICATE

UNISA college of science, engineering and technology

Dear Eric Machisi (47021136)

Date: 2016-01-26

Application number:
2015_CGS/ISTE_012

REQUEST FOR ETHICAL CLEARANCE: (The effects of van Hiele theory-based instruction on grade 11 learners achievement in constructing geometric proofs)

The College of Science, Engineering and Technology's (CSET) Research and Ethics Committee has considered the relevant parts of the studies relating to the abovementioned research project and research methodology and is pleased to inform you that ethical clearance is granted for your research study as set out in your proposal and application for ethical clearance.

Therefore, involved parties may also consider ethics approval as granted. However, the permission granted must not be misconstrued as constituting an instruction from the CSET Executive or the CSET CRIC that sampled interviewees (if applicable) are compelled to take part in the research project. All interviewees retain their individual right to decide whether to participate or not.

We trust that the research will be undertaken in a manner that is respectful of the rights and integrity of those who volunteer to participate, as stipulated in the UNISA Research Ethics policy. The policy can be found at the following URL:
http://cm.unisa.ac.za/contents/departments/res_policies/docs/ResearchEthicsPolicy_apprvCounc_21Sept07.pdf

Please note that the ethical clearance is granted for the duration of this project and if you subsequently do a follow-up study that requires the use of a different research instrument, you will have to submit an addendum to this application, explaining the purpose of the follow-up study and attach the new instrument along with a comprehensive information document and consent form.

Yours sincerely

Prof Ernest Mnkandla
Chair: College of Science, Engineering and Technology Ethics Sub-Committee

Prof IOG Moche
Executive Dean: College of Science, Engineering and Technology

RECEIVED
2016 -01- 26
OFFICE OF THE EXECUTIVE DEAN
College of Science, Engineering and Technology

University of South Africa
College of Science, Engineering and Technology
The Science Campus
C/o Christiaan de Wet Road and Pioneer Avenue,
Florida Park, Roodepoort
Private Bag X6, Florida, 1710
www.unisa.ac.za/cset

UNISA college of science, engineering and technology

A 2: LIMPOPO DEPARTMENT OF EDUCATION APPROVAL

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

Private Bag X 9711
POLOKWANE
0700
Tel: 015 285 7300
Fax: 015 285 7499

DEPARTMENT OF
EDUCATION

CAPRICORN POLOKWANE DISTRICT

Enq : 2/1/R
Enq : Mphaphuli AJ
Tel No.: 015 285 7410
Email : MphaphuliAJ@edu.limpopo.gov.za

To : Mr Machisi E
1034 Zone 8
SESHEGO
0699

SUBJECT: PERMISSION TO CONDUCT RESEARCH IN SOME SCHOOLS IN POLOKWANE DISTRICT

1. The above matter refers.
2. The Department wishes to inform you that your request to conduct a research has been approved.
3. The following conditions should be considered.
   3.1. The research should not have any financial implication for Limpopo Department of Education.
   3.2. Arrangements should be made with both the Circuit Offices and schools concerned.
   3.3. The conduct of research should not anyhow disrupt the Academic Programs at Schools.
   3.4. The research should not be conducted during the examinations especially the fourth term.
   3.5. During the study, the research ethics should be practiced, in particular the principle voluntary participation (the people involved should be respected).

Cnr Blaauwberg & Yster Street, Ladanna

"We Belong, We Care, We Serve"

3.6. Upon completion of research study, the researcher shall share the final product of the research with Department.

4. Furthermore you are expected to produce this letter at Schools/ Offices where you intend to conduct your research as evidence that you are permitted to conduct the research.

5. The Department appreciates the contribution that you wish to make and wish you success in your research.

Best wishes

MR MOTHEMANE KD
ACTING DISTRICT DIRECTOR

04/07/2016
DATE

LIMPOPO PROVINCE
DEPARTMENT OF EDUCATION
POLOKWANE DISTRICT OFFICE
DSM OFFICE
2016 -07- 04
PRIVATE BAG X9711
POLOKWANE, 0700
KGORO YA THUTO LIMPOPO PROVINCE

APPENDIX B: LETTERS OF PERMISSION AND CONSENT

B 1: LETTER TO THE DISTRICT SENIOR MANAGER

Enquiries : Mr Eric Machisi
Cell : 0721474618
Work : 015 223 6592
E-mail : 47021136@mylife.unisa.ac.za

1034 Zone 8
Seshego
0699

4 July 2016

The District Senior Manager
Limpopo Department of Education
Capricorn Polokwane District
Private Bag X 9711
Polokwane
0700

Dear Sir/Madam

REQUEST FOR PERMISSION TO CONDUCT RESEARCH IN SCHOOLS

My name is Eric Machisi. I am a Mathematics Education student at the University of South Africa (UNISA). The research I wish to conduct for my doctoral thesis involves exploring the effects of van Hiele theory-based instruction on Grade 11 learners' achievement in constructing geometric proofs. This project will be conducted under the supervision of Professor Nosisi Nellie Feza of the Institute for Science and Technology Education (ISTE) (UNISA).

I am hereby seeking your permission to approach a number of township secondary schools in the Capricorn Polokwane District to provide participants for this project.

Attached herewith is a copy of the University of South Africa ethical clearance certificate, the project information statement together with copies of the consent and assent forms to be used in the study.

Upon completion of the study, I undertake to provide the Department of Basic Education with a bound copy of the full research report. For any further information, please feel free to contact me on 072 147 4618 or e-mail at 47021136@mylife.unisa.ac.za

Thank you for your time and consideration in this matter. Hoping to hear from you soon

Yours faithfully

Eric Machisi
University of South Africa Student

For the attention of the District Senior Manager:

PROJECT INFORMATION STATEMENT

PROJECT TITLE:

THE EFFECT OF VAN HIELE THEORY-BASED INSTRUCTION ON GRADE 11 LEARNERS' ACHIEVEMENT IN CONSTRUCTING GEOMETRIC PROOFS

The objectives of the study are:

- To design and implement Van Hiele theory-based instruction in the teaching of geometric proofs in township secondary schools;
- To measure the impact of Van Hiele theory-based instruction on learners' achievement and compare it with that of conventional instruction in the teaching of geometric proofs;
- To investigate learners' views on the implementation of Van Hiele theory-based instruction in the teaching of geometry and proofs;
- To investigate learners' views on the use of conventional approaches in the teaching of geometry and proofs.

Significance of the study

The study is significant in the following ways:

- It seeks to find ways to obviate learners' difficulties with geometric proofs, and hence enhance learners' overall mathematics achievement.
- It addresses educators' pedagogical concern of how to teach geometric proofs in a manner that guarantees success for the majority if not all their learners.
- It makes a call for a pedagogical shift in current approaches to teaching mathematics, particularly the teaching of geometric proofs in the Curriculum and Assessment Policy Statement (CAPS).
- It provides valuable first-hand information on real matters of the classroom and forms a basis for making recommendations to the Department of Basic Education (DBE) on the kind of teacher development and support programmes they should consider implementing in schools.

Benefits of the research to participating schools

- The study will help debunk the perception among many educators that most learners cannot prove geometric riders.
- The study is likely to change learners' perception that proving geometric riders is a difficult mathematical aspect.
- The study acts as a remedial programme for learners who have difficulty in understanding geometric proofs.
- The study may help educators discover how they can turn learners' difficulties into opportunities to improve the quality of teaching.

The research plan and method

A convenience sample of four secondary schools from two townships in Capricorn district will participate in the study. Two schools from one township will constitute the experimental group whereas the other two schools from another township will form the control group. The researcher will implement Van Hiele theory-based instruction in the experimental group schools while learners in the control group schools will be taught by their educators as usual. The programme is expected to run for a period of four weeks during the third quarter of the year 2016. Data will be collected through administering pre-tests and post-tests in both experimental and control group schools. A few selected learners from both townships will participate in focus group discussions to elicit their views on the methods of instruction used in their classes during the teaching and learning of geometry and proofs. Permission will be sought from the learners and their parents prior to their participation in the research. Only those who consent and whose parents consent will participate. Mathematics educators and subject advisers will be requested to validate the geometry achievement test instrument before implementation. Their participation will also be based on informed consent. All information collected will be treated in the strictest confidence and will be used only for purposes of the study. Neither the school nor individual learners will be identifiable in any reports that are written. Participants may withdraw from the study at any time with no penalty. The role of the school is voluntary and the school principal may decide to withdraw the school's participation at any time. There are no known risks to participation in this study. Recording devices will be used only in recording focus group discussions and no identifying information will be collected. If a learner requires support because of their participation in this research, steps will be taken to accommodate this.

Schools' involvement

Once I have received permission to approach learners to participate in the study, I will:

- Obtain informed consent from participants.
- Arrange for informed consent to be obtained from participants' parents.
- Arrange time with participants for data collection

Thank you for taking your time to read this information.

Eric Machisi
Primary Researcher
University of South Africa

Professor Nosisi Nellie Feza
Supervisor
University of South Africa
418 Robert Sobukhwe Building
Nana Sita Street
Pretoria
Tel: 012 337 6168

B 2: LETTER TO PRINCIPALS OF PARTICIPATING SCHOOLS

Enquiries : Mr. Eric Machisi
Cell : 0721474618
Work : 015 223 6592
E-mail : 47021136@mylife.unisa.ac.za

1034 Zone 8
SESHEGO
0742

5 July 2016

Dear **Principal**

REQUEST FOR PERMISSION TO CONDUCT RESEARCH IN YOUR SCHOOL

My name is Eric Machisi. I am a Mathematics Education student at the University of South Africa (UNISA). The research I wish to conduct for my doctoral thesis involves exploring the effects of van Hiele theory-based instruction on Grade 11 learners' achievement in constructing geometric proofs. This project will be conducted under the supervision of Professor Nosisi Nellie Feza of the Institute for Science and Technology Education (ISTE) (UNISA).

I am hereby seeking permission to use your school as a research site for the study which involves working with Grade 11 mathematics learners and their educators. I would be grateful to receive your support in this regard.

I have sought and gained permission from the District Senior Manager to involve Grade 11 mathematics learners and educators in my research. I guarantee total confidentiality of all information collected in my research. Neither the school nor the individual learners and educators will be identifiable in any reports that will be written. I will only report information that is in the public domain and within the law.

Please find attached herewith this letter, a copy of the project information statement outlining the details of the study, the School Principal Consent form, the District Senior Manager approval letter and the University of South Africa Ethical Clearance Certificate. Please also note that the participation of your school is voluntary and that you are free to withdraw from the study at any stage.

For any further information, please feel free to contact me on 072 147 4618 or e-mail at 47021136@mylife.unisa.ac.za

Thank you for your time and consideration in this matter. Hoping to hear from you soon

Yours faithfully

Eric Machisi
University of South Africa Student

For the attention of the School Principal:

PROJECT INFORMATION STATEMENT

PROJECT TITLE:

THE EFFECT OF VAN HIELE THEORY-BASED INSTRUCTION ON GRADE 11 LEARNERS' ACHIEVEMENT IN CONSTRUCTING GEOMETRIC PROOFS

Objectives of the research

The objectives of the study are:

- To design and implement Van Hiele theory-based instruction in the teaching of geometric proofs in township secondary schools;
- To measure the impact of Van Hiele theory-based instruction on learners' achievement and compare it with that of conventional instruction in the teaching of geometric proofs;
- To investigate learners' views on the implementation of Van Hiele theory-based instruction in the teaching of geometry and proofs;
- To investigate learners' views on the use of conventional approaches in the teaching of geometry and proofs.

Significance of the study

The study is significant in the following ways:

- It seeks to find ways to obviate learners' difficulties with geometric proofs and hence, enhance learners' overall mathematics achievement.
- It addresses educators' pedagogical concern of how to teach geometric proofs in a manner that guarantees success for the majority if not all their learners.
- It makes a call for a pedagogical shift in current approaches to teaching mathematics, particularly the teaching of geometric proofs in the Curriculum and Assessment Policy Statement (CAPS).
- It provides valuable first-hand information on real matters of the classroom and forms a basis for making recommendations to the Department of Basic Education (DBE) on the kind of teacher development and support programmes they should consider implementing in schools.

Benefits of the research to participating schools

- The study will help debunk the perception among many educators that most learners cannot prove geometric riders.
- The study is likely to change learners' perception that proving geometric riders is a difficult mathematical aspect.
- The study acts as a remedial programme for learners who have difficulty in understanding geometric proofs.
- The study may help educators discover how they can turn learners' difficulties into opportunities to improve the quality of teaching.

The research plan and method

A convenience sample of four secondary schools from two townships in Capricorn district will participate in the study. Two schools from one township will constitute the experimental group whereas the other two schools from another township will form the control group. The researcher will implement Van Hiele theory-based instruction in the experimental group schools while learners in the control group schools will be taught by their educators as usual. The programme is expected to run for a period of four weeks during the third quarter of the year 2016. Data will be collected through administering pre-tests and post-tests in both experimental and control group schools. A few selected learners from both townships will participate in focus group discussions to elicit their views on the methods of instruction used in their classes during the teaching and learning of geometry and proofs. Permission will be sought from the learners and their parents prior to their participation in the research. Only those who consent and whose parents consent will participate. Mathematics educators and subject advisers will be requested to validate the geometry achievement test instrument before implementation. Their participation is also based on informed consent. All information collected will be treated in the strictest confidence and will be used only for purposes of the study. Neither the school nor individual learners will be identifiable in any reports that are written. Participants may withdraw from the study at any time with no penalty. The role of the school is voluntary and the school principal may decide to withdraw the school's participation at any time. There are no known risks to participation in this study. Recording devices will be used only in recording focus group discussions and no identifying information will be collected. If a learner requires support because of their participation in this research, steps will be taken to accommodate this.

Schools' involvement

Once I have received permission to approach learners to participate in the study, I will:

- Obtain informed consent from participants.
- Arrange for informed consent to be obtained from participants' parents.
- Arrange time with participants for data collection.

Thank you for taking your time to read this information.

Eric Machisi	**Professor Nosisi Nellie Feza**
Primary Researcher	Supervisor
University of South Africa	University of South Africa

School Principal Consent Form

I give permission to Eric Machisi to invite Grade 11 mathematics learners and educators in this school to participate in investigating the effect of van Hiele theory-based instruction on grade 11 learners' achievement in constructing geometric proofs.

I have read the **Project Information Statement** explaining the purpose of the research and understand that:

- The role of the school is voluntary.
- I may decide to withdraw the school's participation at any time.
- Grade 11 mathematics learners and educators will be invited to participate and that permission will be sought from them and also from learners' parents.
- Only learners who consent and whose parents consent will participate in this research.
- All information obtained will be treated in strictest confidence.
- The school, learners' and educators' names will not be used and individual learners and educators will not be identifiable in any reports about the study.
- There are no known risks to participation in this study.
- The school will not be identifiable in any reports about the study.
- Participants may withdraw from the study at any time without penalty.
- A report of findings will be made available to the school.
- I may seek further information on the project from the researcher on 072 147 4618 or e-mail at 47021136@mylife.unisa.ac.za

____________________________ ________________________

School Principal's Signature **Date**

B 3: LETTER TO PARENTS/GUARDIANS

Enquiries : Mr Eric Machisi
Cell : 0721474618
Work : 015 223 6592
E-mail : 47021136@mylife.unisa.ac.za

1034 Zone 8
SESHEGO
0742

18 July 2016

Dear Parent/Guardian

REQUEST FOR YOUR CHILD TO PARTICIPATE IN A RESEARCH PROJECT

My name is Eric Machisi. I am a Mathematics Education student at the University of South Africa (UNISA). I am delighted to take this opportunity to seek your permission to involve your child in my research project entitled "*The effects of van Hiele theory-based instruction on grade 11 learners' achievement in constructing geometric proofs*". I am undertaking this study as part of my doctoral research at the University of South Africa. The purpose of the study is to find ways that can enhance learners' achievement in constructing geometric proofs.

If you allow your child to participate, I shall request your child to attend geometry lessons and write a pre-and post-test to check progress. The study will take place during regular school activities. The tests results will only be used for research purposes and will not count towards your child's term mark. There is also a possibility that your child might be interviewed at the end of the project. The purpose of the interview will be to investigate learners' perceptions and emotions on the method of instruction used in their classes in the teaching and learning of geometry and proofs. The project is expected to last for a period of four weeks. The data generated in this project will help to find ways to provide better mathematics education to your child.

All information that is collected in this study will be treated with utmost confidentiality and will be used for research purposes only. No identifying information will be used throughout the study, that is, your child's name and the name of his/her school will not be disclosed in any written report on this study. There are no foreseeable risks to your child by participating in this study.

Please note that your child's participation in this study is voluntary. You are free to refuse permission for your child to take part in this project and I guarantee that your refusal will not affect your child in any way. Your child will still have all the benefits that would be otherwise available to learners at the school. Your child may stop participating at any time they wish, for any or no reason without losing any of their rights. Participation in this study will involve no costs to your child and your child will not be paid for participating in this study.

In addition to your permission, your child will also be requested to agree or refuse to participate in the study by signing an assent form. If your child does not wish to participate in the study, he or she will not be included and there will be no penalty. The information gathered from your child's participation will be stored safely in a lockable room and on a password locked computer for five years after the study. Thereafter, the records will be destroyed.

Please sign the consent form on the next page, indicating whether I may or may not involve your child in this project. If you have any questions or issues for clarity, please do not hesitate to contact me or my study supervisor, Professor Nosisi Nellie Feza, Institute for Science and Technology Education (ISTE), University of South Africa (UNISA). My contact number is 072 147 4618 and my e-mail is 47021136@mylife.unisa.ac.za. The e-mail of my supervisor is fezann@unisa.ac.za.

Thank you for taking your time to read this letter.

Yours faithfully

Eric Machisi
University of South Africa Student

Parental Consent form

I, the parent/legal guardian of ...., acknowledge that I have read and understood the information provided above. The nature and purpose of the study has been explained to me and I have been given an opportunity to ask questions and my questions have been adequately answered. If I have additional questions, I know the person I should contact. I will receive a copy of this parental consent form after I sign it.

Please tick (✓) the appropriate category. Then sign and have your child return the slip.

Thank you in advance!

☐ **Yes**, you may involve my child in your research.

☐ **No,** please do not involve my child in your research.

-- ---------------

Parent / Legal Guardian's Name & Signature **Date**

B 4: LETTER REQUESTING WRITTEN CONSENT/ASSENT FROM LEARNERS TO PARTICIPATE IN THE STUDY

Enquiries : Mr Eric Machisi
Cell : 0721474618
Work : 015 223 6592
E-mail : 47021136@mylife.unisa.ac.za

1034 Zone 8
SESHEGO
0742

19 July 2016

Dear Learner

My name is Eric Machisi. I am a doing a research on the teaching and learning of geometric proofs in secondary schools as part of my studies at the University of South Africa (UNISA). Your principal has given me permission to conduct this study at your school. I am delighted to invite you to participate in my study. I am doing this study to find ways that your teachers may use to help you understand geometric proofs better. This will help you and many other learners of your age in different schools.

If you decide to participate in this study, I will ask you to write a pre-and post-test on geometric proofs learnt in Grade 11. Your names will not appear on the answer sheets and the marks obtained will not count or contribute towards your marks at school. The results will be used for the purpose of research only and will be withheld until the study is over. I will not share the test results with your educators or parents. At the end of the program I might request you to attend a focus group discussion that will take about one - and - half to two hours. The discussion will be tape recorded and the researcher may wish to quote from the discussion in reporting the study's results. Your name will not be revealed in any publications resulting from this study.

You may discuss anything in this letter with your parents, friends or anyone else you feel comfortable talking to before you decide whether or not you want to participate in the study. You do not have to decide immediately. If there are any words or issues that you may want me to explain more about, I will be readily available at any time. Please note that you do not have to be in this research if you do want to be involved. The choice to participate is yours. You do not have to decide immediately. Give yourself time to think about it. If you choose to participate, you may stop taking part at any time and I guarantee that nothing undesirable will happen to you.

This study is considered safe and free from any harm to participants. If anything unusual happens to you in the course of the study, I would need to know. Feel free to contact me anytime with your questions or concerns. You will not be paid for taking part in this study. I will not tell people that you are in this research and I will not share any information about the study with anyone except my supervisor, Professor Nosisi Nellie Feza. Information collected from this study will be kept confidential. Throughout the study, participants will be identified by codes instead of names. The results of the study will be presented to the University of South Africa for academic purposes and later published in order that interested people may learn from the research. When the research is done, I will let you know what I have discovered and learnt from the study by making available a written report about the research results.

If you have any questions, you may ask them now or later, even when the study has started. If you wish to ask questions later, you may talk to me or have your parents or another adult to call me at 072 147 4618 or e-mail at: 47021136@mylife.unisa.ac.za.

Please sign the attached consent/assent form to indicate whether or not you agree to participate in the study. Do not sign the form until you have all your questions answered and have understood the contents of this letter.

Thank you for taking your time to read this letter.

Eric Machisi

University of South Africa Student

Consent / Assent form

I have accurately read and understood this letter which asks me to participate in a study at our school. I have had the opportunity to ask questions and I am happy with the answers I have been given. I know that I can ask questions later if I have them.

I understand that taking part in this research is voluntary (my choice) and that I may withdraw from the study at any time for any or no reason. I understand that if I withdraw from the study at any time, this will not affect me in any way.

I understand that my participation in this study is confidential and that no material that could identify me will be used in any reports on this study. I had time to consider whether or not I should take part in this study and I know who to contact if I have questions about the study.

I **agree / do not agree** [****strike out one****] to take part in this study.

-------------------------------	----------------------------	-----------------
Learner's name (print)	**Learner's signature**	**Date**

--------------------------------	---------------------------	-----------------
Witness's name (print)	**Witness' signature**	**Date**

*Witness must be over 18 years and present when signed

--------------------------------	---------------------------	------------------------
Parent/Legal guardian's name	**Parent/Legal guardian's signature**	**Date**

Interview Consent / Assent and confidentiality Agreement

I, ………………………………………………….., grant consent/assent that information I share during the interview discussions may be used by the researcher, Eric Machisi, for research purposes. I am aware that the interview discussion will be digitally recorded and grant consent/assent for these recordings, provided that my privacy will be protected. I undertake not to divulge any information that is shared in the interview discussions with the researcher to any other person in order to maintain confidentiality

Participant's Name (Please Print) : ……………………………………………

Participant's signature : ………………………………….............

Researcher's Name (Please Print) : …………………………………………...

Researcher's Signature : ……………………………………………

Date : …………………………………………...

B 5: LETTER TO MATHEMATICS EXPERTS

Enquiries : Mr Eric Machisi
Cell : 0721474618
Work : 015 223 6592
E-mail : 47021136@mylife.unisa.ac.za

1034 Zone 8
Seshego
0742

8 July 2016

Dear Esteemed Mathematics Expert

REQUEST FOR ASSISTANCE IN VALIDATING A GRADE 11 GEOMETRIC PROOF TEST

My name is Eric Machisi. I am a Mathematics education student at the University of South Africa (UNISA). The research I wish to conduct for my doctoral thesis involves exploring the effects of van Hiele theory-based instruction on learners' achievement in constructing geometric proofs. The study involves collecting data from learners through administering a geometry test to grade 11 learners. It is a requirement that the test instrument must be validated before it is administered to participants. I am therefore requesting you to assist in validating the test items based on relevance and clarity.

Attached to this letter is a copy of the geometry test and the validation form you may use if you are willing to take part in the study. Please note that participation is voluntary and hence you are free to choose not to take part should you wish to do so. I guarantee total confidentiality of all information collected in my research and no names or identifiable information will be used in any reports that will be written.

For any further information, please feel free to contact me on 072 147 4618 or e-mail at 47021136@mylife.unisa.ac.za

Thank you for your time and consideration in this matter.

Yours faithfully

Eric Machisi
University of South Africa Student

APPENDIX C: SCHOOL AND TEACHER PROFILE

SCHOOL AND TEACHER PROFILE FORM

INSTRUCTIONS:

- You are kindly requested to complete both section A and section B of this form
- If possible, please respond to all items
- The information collected here will constitute the data for the present study
- Your responses will be treated with utmost confidentiality and anonymity is guaranteed

SECTION A: SCHOOL PROFILE

TYPE OF SCHOOL: *(Indicate with X)*	Public	
	Independent	
FEE OR NO FEE SCHOOL: *(Indicate with X)*	Fee receiving school	
	No fee receiving school	
LOCATION OF THE SCHOOL: *(Indicate with X)*	Township	
	Rural	

SCHOOL FACILITIES: *(Indicate with X)*		**YES**	**NO**
	Computer laboratory/laptops		
	Overhead data projector		
	Interactive geometry software		
	School library		
	Science laboratory		

If your school has the above facilities, are they functional? *(Indicate with X)*

		YES	**NO**
	Computer laboratory		
	Overhead projector		
	Interactive geometry software		
	School library		
	Science laboratory		

GRADE 12 MATHEMATICS RESULTS FOR THE PAST TWO YEARS

		Number Wrote	% Achieved
	2013		
	2014		
	2015		

2016 SCHOOL ENROLMENT

Overall School Enrolment	
Number of Grade 11 learners doing Mathematics	

<table>
<tr><th colspan="10">SECTION B: EDUCATOR PROFILE</th></tr>
<tr><td colspan="6"></td><td colspan="2">AGE</td><td colspan="2"></td></tr>
<tr><td colspan="6">GENDER: (Mark with “X”)</td><td colspan="2">Male</td><td colspan="2"></td></tr>
<tr><td colspan="6"></td><td colspan="2">Female</td><td colspan="2"></td></tr>
<tr><td colspan="10">POPULATION GROUP: (Mark with “X”)</td></tr>
<tr><td>Black</td><td></td><td>Coloured</td><td></td><td>Indian</td><td></td><td>White</td><td></td><td>Other</td><td></td></tr>
<tr><td colspan="10">HIGHEST PROFESSIONAL QUALIFICATION: (Mark with “X”)</td></tr>
<tr><td>Certificate</td><td></td><td>Diploma</td><td></td><td>Degree</td><td></td><td>Honours</td><td></td><td colspan="2"></td></tr>
<tr><td>Masters</td><td></td><td>Doctorate</td><td></td><td>Other</td><td></td><td>Specify</td><td colspan="3"></td></tr>
<tr><td colspan="10">TEACHING EXPERIENCE IN EDUCATION: (Mark with “X”)</td></tr>
<tr><td colspan="2" rowspan="5"></td><td colspan="4">0 -1 year</td><td colspan="2"></td><td colspan="2" rowspan="5"></td></tr>
<tr><td colspan="4">1-5 years</td><td colspan="2"></td></tr>
<tr><td colspan="4">5 -10 years</td><td colspan="2"></td></tr>
<tr><td colspan="4">10-15 years</td><td colspan="2"></td></tr>
<tr><td colspan="4">More than 15 years</td><td colspan="2"></td></tr>
<tr><td colspan="10">EMPLOYMENT STATUS: (Mark with “X”)</td></tr>
<tr><td colspan="6" rowspan="2"></td><td colspan="2">Temporary</td><td colspan="2"></td></tr>
<tr><td colspan="2">Permanent</td><td colspan="2"></td></tr>
<tr><td colspan="10">EMPLOYING BODY: (Mark with “X”)</td></tr>
<tr><td colspan="2" rowspan="2"></td><td colspan="6">Provincial Department of Education</td><td colspan="2"></td></tr>
<tr><td colspan="6">School Governing Body</td><td colspan="2"></td></tr>
<tr><td colspan="10">WHICH GRADES ARE YOU CURRENTLY TEACHING AT SCHOOL?
(Mark with “X”)</td></tr>
<tr><td colspan="6" rowspan="6"></td><td colspan="2">GRADES</td><td colspan="2"></td></tr>
<tr><td colspan="2">8</td><td colspan="2"></td></tr>
<tr><td colspan="2">9</td><td colspan="2"></td></tr>
<tr><td colspan="2">10</td><td colspan="2"></td></tr>
<tr><td colspan="2">11</td><td colspan="2"></td></tr>
<tr><td colspan="2">12</td><td colspan="2"></td></tr>
<tr><td colspan="10">ARE YOU CURRENTLY TEACHING THE SUBJECT(S) IN WHICH YOU SPECIALISED IN YOUR PROFESSIONAL QUALIFICATIONS? (Mark with “X”)</td></tr>
<tr><td colspan="6" rowspan="2"></td><td colspan="2">YES</td><td colspan="2"></td></tr>
<tr><td colspan="2">NO</td><td colspan="2"></td></tr>
<tr><td colspan="10">If your answer is “No”, please indicate the reason(s) from the list below:</td></tr>
<tr><td colspan="8">There was no other teacher to teach this subject</td><td colspan="2"></td></tr>
<tr><td colspan="8">Redeployment and Rationalisation</td><td colspan="2"></td></tr>
<tr><td colspan="8">Left the teaching profession for some time and re-entered at a later stage</td><td colspan="2"></td></tr>
<tr><td colspan="8">This was the only subject left at the school</td><td colspan="2"></td></tr>
<tr><td colspan="8">Phasing out of other subjects</td><td colspan="2"></td></tr>
<tr><td colspan="8">Other (please specify below)</td><td colspan="2"></td></tr>
</table>

APPENDIX D: LEARNER PROFILE

LEARNER'S BACKGROUND CHARACTERISTICS								
						LEARNER CODE		
AGE				**GENDER:** (Mark with "X")				
							Male	
							Female	
GRADE REPETITION: (Mark with "X")								
Repeater					Non-repeater			
HOME LANGUAGE: (Mark with "X")								
Afrikaans		English		Sepedi		Sotho		
Venda		Tswana		Tsonga		Zulu		
Xhosa		Ndebele		Swati		Other		
LOCATION OF RESIDENCE: (Mark with "X")								
Village		Informal settlement			Township			
PARENTAGE: (Mark with "X")								
Living with both parents				Living with single parent				
No parents/living with guardian or siblings								
PARENT/ GUARDIAN HIGHEST LEVEL OF EDUCATION: (Mark with "X")								
Mother	Less than grade 12		Grade 12		More than grade 12			
Father	Less than grade 12		Grade 12		More than grade 12			
Guardian	Less than grade 12		Grade 12		More than grade 12			
EMPLOYMENT STATUS OF PARENT(S)/GUARDIAN: (Mark with "X")								
Parent(s)/guardian employed								
Parent(s)/guardian self-employed								
Parent(s)/guardian unemployed								
FAMILY INCOME STATUS: (Mark with "X")								
Low			Average			High		
HOME FACILITIES: (Mark with "X")								
Have access to a computer		Do not have access to a computer						
Have a private mathematics tutor		Do not have a private mathematics tutor						

APPENDIX E: DIARY GUIDE

Purpose of the research

This study explores the effect(s) of teaching approaches used in the mathematics classroom on Grade 11 students' learning achievement. The study also explores students' views on their Euclidean geometry learning experiences.

Purpose of the diary

Your diary will provide me with important information about your day-to-day learning experiences during Euclidean geometry lessons and how your experiences affected your attitudes, views, and emotions about the topic. This information will help me to develop questions for group discussions with you at a later stage of the research.

Privacy and confidentiality

Please **do not** write your names, the name of your school or mathematics teacher in your diary. The information collected from your diaries will be used for academic purposes only. Your name, school and mathematics teacher's names **will not** be used in reporting the findings of the study. Your diaries will be kept in a secure place and treated with utmost confidentiality.

Guidelines for diary completion

Thank you for agreeing to keep a diary of your day-to-day teaching and learning experiences for the period that Euclidean geometry will be taught at your school. It would be helpful if you could make entries into your diary daily. However, I do not want this to be a tiresome task. Please try to make entries into the diary every evening. If you feel that you do not have enough time to make your diary entry on the day that the lesson was taught, it is still fine if you do it a day after. I have tried to make the diary as easy as possible to complete and please feel free to contact me on 072 147 4618 or email at 47021136@mylife.unisa.ac.za for assistance with any issues that may arise in completing your diaries.

In completing your diary, please try to include the following:

- the date
- lesson topic
- a description of how the lesson was presented by the teacher
- your thoughts and feelings/emotions about the way the lesson was presented [Try to evaluate or judge the lesson presentation]
- what you liked or disliked, enjoyed or did not enjoy about the presentation
- Do you believe the way the teacher taught the lesson helped you to understand the topic?

If there are any other experiences that you would like to write about which are not indicated here, please feel free to include them in your diary. You are encouraged to write your diary in English and please do not worry about grammar or spelling errors. You and your diary entries will remain anonymous. Your diary consists of 192 A-5 pages and therefore there are no restrictions on the amount of information you can record. Daily diary entries can overflow to the next page when necessary.

Thank you so much for taking your time and effort to complete the diary. Please do not hesitate to contact me for any assistance you may require to complete your diary.

Eric Machisi

[Researcher and University of South Africa Student]

APPENDIX F: PRIOR KNOWLEDGE ASSESSMENT TASK AND MARKING GUIDE

GRADE 11 EUCLIDEAN GEOMETRY READINESS TEST

(Informal Assessment Based on Grade 8 -10 Work)

LEARNER'S CODE: ________________

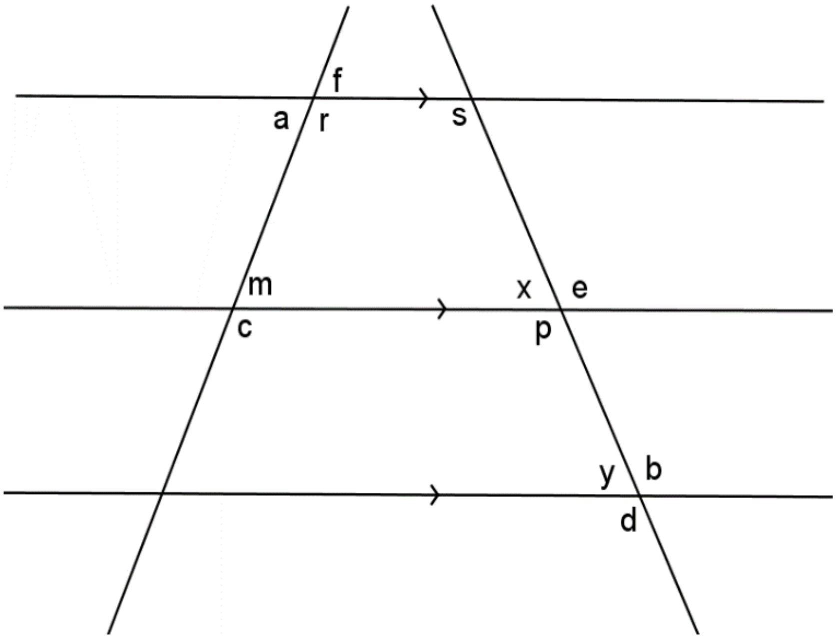
MARKS: **40** TIME: **1 HOUR**

INSTRUCTIONS AND INFORMATION:

Read the instructions carefully before answering the questions:

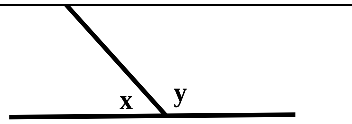
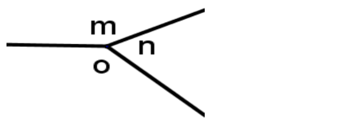
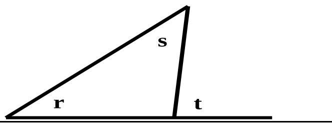
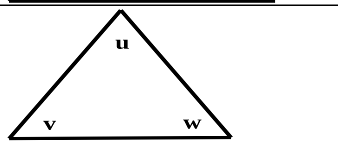
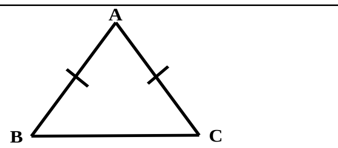
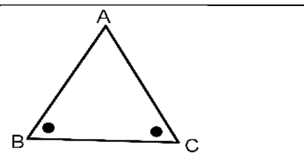
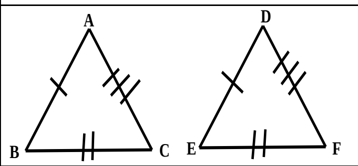
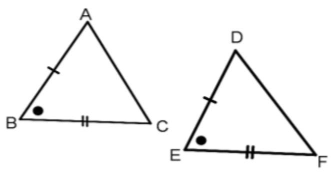
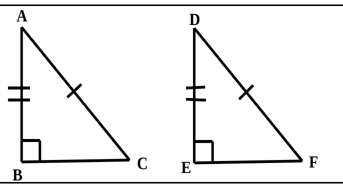
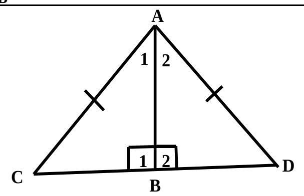
1. This question paper consists of 3 long questions.
2. Answer ALL questions.
3. Write your answers in the spaces provided.
4. Write neatly and legibly.
5. Diagrams are NOT necessarily drawn to scale.

1. Study the diagram below and answer the questions that follow:



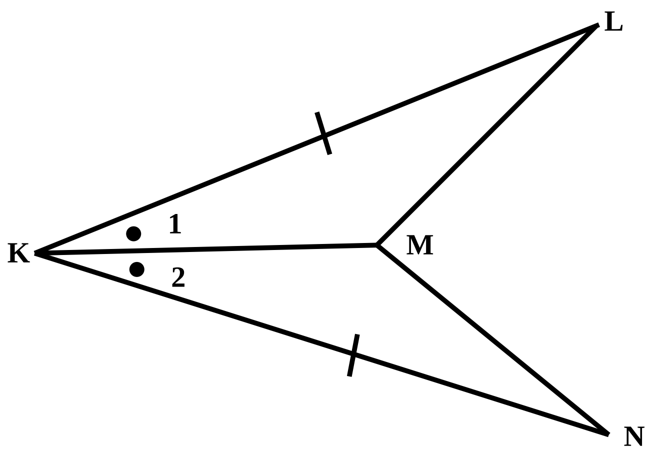
	Name an angle that is:	Answers
(a)	Vertically opposite to ***e***	(1)
(b)	Vertically opposite to ***f***	(1)
(c)	Alternate to ***b***	(1)
(d)	Alternate to ***a***	(1)
(e)	Corresponding to ***y***	(1)
(f)	Corresponding to ***c***	(1)
(g)	Co-interior to ***r***	(1)
(h)	Co-interior to ***p***	(1)

2. Fill in the missing information

(a)		Statement: $x + y =$ ____________________ Reason: _________________________________ (2)
(b)		Statement: $m + n + o =$ ________________ Reason: ______________________________ (2)
(c)		Statement: $t =$ _____________________ Reason: ___________________________ (2)
(d)		Statement: $v + u + w =$ _________________ Reason: ______________________________ (2)
(e)		If $AB = AC$, then _________________________ Reason: ______________________________ (2)
(f)		If $\hat{B} = \hat{C}$, then _______________________________ Reason: ________________________________ (2)
(g)		Statement: $\Delta ABC \equiv \Delta DEF$ Reason: _____________________________ (1)
(h)		Statement: $\Delta ABC \equiv \Delta DEF$ Reason: _____________________________ (1)
(i)		Statement: $\Delta ABC \equiv \Delta DEF$ Reason: _____________________________ (1)
(j)		Statement: $\widehat{A}_1 = \widehat{A}_2$ Reason: _____________________________ (1)

3. Study the following sketches

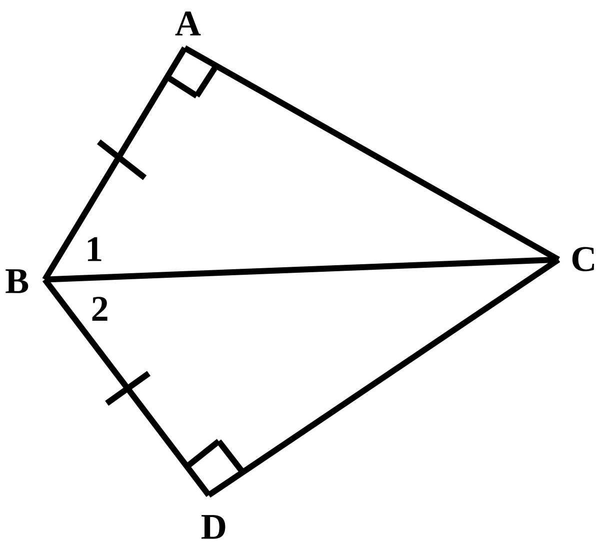
(a)



Prove that: $\Delta KLM \equiv \Delta KNM$ (4)

Statement	Reason

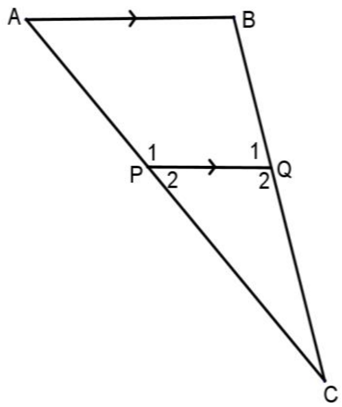
(b)



Prove that: $\hat{B}_1 = \hat{B}_2$ (4)

Statement	Reason

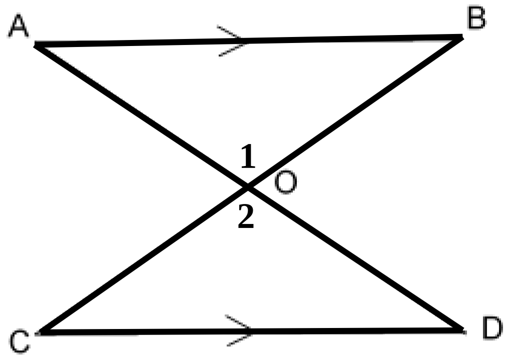
(c) In the sketch, $AB//PQ$



Show that: $\Delta ABC \;///\Delta PQC$ (4)

Statement	Reason

(d) In the sketch, AB // CD.



Prove that: ΔABO /// ΔDCO (4)

Statement	Reason

GRAND TOTAL: **40**

MARKING GUIDE

1(a)	p		✓ answer
(b)	a		✓ answer
(c)	p		✓ answer
(d)	M		✓ answer
(e)	x		✓ answer
(f)	r		✓ answer
(g)	m		✓ answer
(h)	y		✓ answer
2(a)	$x + y = 180°$ ($\angle$s on a str. line)		✓S ✓R
(b)	$m + n + o = 360°$ ($\angle$s round a pt **OR** $\angle$s in a rev)		✓S ✓R
(c)	$t = r + s$ (ext $\angle$ of a Δ)		✓S ✓R
(d)	$v + u + w = 180°$ (sum of $\angle$s in Δ)		✓S ✓R
(e)	$\hat{B} = \hat{C}$ ($\angle$s opp equal sides)		✓S ✓R
(f)	$AB = AC$ (sides opp equal $\angle$s)		✓S ✓R
(g)	SSS		✓R
(h)	SAS		✓R
(i)	RHS		✓R
(j)	$\Delta ABC \equiv \Delta ABD$ **OR** $\equiv \Delta$s		✓R
3(a)	**Statement**	**Reason**	
	$KL = KN$	Given	✓S & R
	$\hat{K}_1 = \hat{K}_2$	Given	✓S & R
	$KM = KM$	Common	✓S & R
	$\therefore \Delta KLM \equiv \Delta KNM$	SAS	✓R
(b)	**Statement**	**Reason**	
	$\hat{A} = \hat{D}$	Both $= 90°$	✓S & R
	BC = BC	Common	✓S & R
	BA = BD	Given	✓S & R
	$\therefore \Delta BAC \equiv \Delta BDC$	RHS	✓S & R
	$\therefore \hat{B}_1 = \hat{B}_2$	$\Delta BAC \equiv \Delta BDC$ or $\equiv \Delta$s	
(c)	**Statement**	**Reason**	
	$\hat{A} = \hat{P}_2$	Corresp $\angle s$; AB // PQ	✓S & R
	$\hat{B} = \hat{Q}_2$	Corresp $\angle s$; AB // PQ	✓S & R
	$\hat{C} = \hat{C}$	Common	✓S & R
	$\Delta ABC /// \Delta PQC$	AAA / $\angle\angle\angle$	✓S & R
(d)	**Statement**	**Reason**	
	$\hat{A} = \hat{D}$	alt $\angle s$; AB // CD	✓S & R
	$\hat{B} = \hat{C}$	alt $\angle s$; AB // CD	✓S & R
	$\hat{O}_1 = \hat{O}_2$	Vert opp $\angle s =$	✓S & R
	$\Delta ABO /// \Delta DCO$	AAA / $\angle\angle\angle$	✓S & R

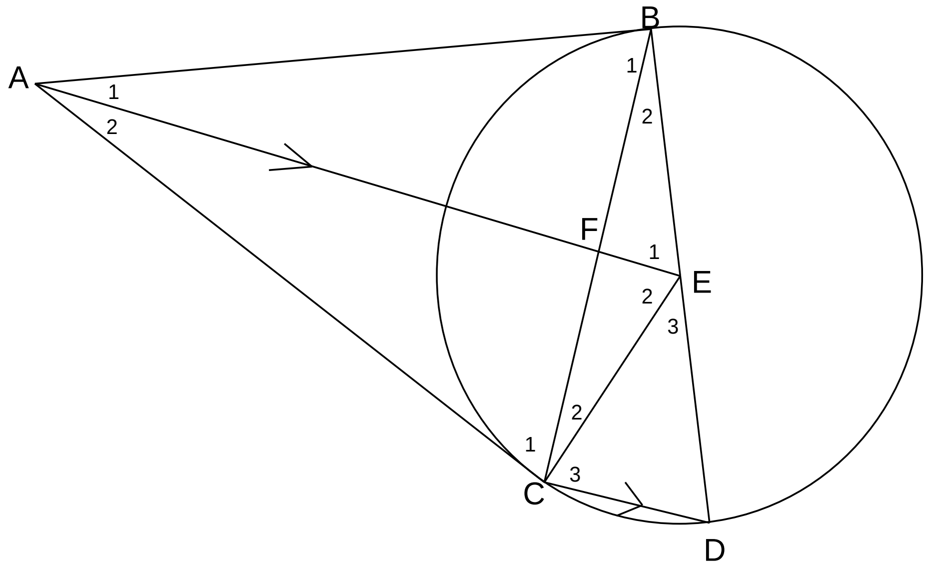
APPENDIX G: TEST ITEM ANALYSIS

Item	Aspect	Number of incorrect responses
1 (a)	Vertically opposite angles	
1 (b)	Vertically opposite angles	
1 (c)	Alternating angles	
1 (d)	Alternating angles	
1 (e)	Corresponding angles	
1 (f)	Corresponding angles	
1 (g)	Co-interior angles	
1 (h)	Co-interior angles	
2 (a)	Angles on a straight line	
2 (b)	Angles around a point	
2 (c)	Exterior angle of a triangle	
2 (d)	Angles of a triangle	
2 (e)	Properties of an isosceles triangle	
2 (f)	Properties of an isosceles triangle	
2 (g)	Congruency	
2 (h)	Congruency	
2 (i)	Congruency	
3 (a)	Proof (congruency)	
3 (b)	Proof (congruency)	
3 (c)	Proof (similarity)	
3 (d)	Proof (similarity)	

APPENDIX H: WORKSHEETS

Worksheet 1 [Classwork]

In the accompanying figure, BD is a diameter of the circle. E is the centre of the circle. AB and AC are tangents to the circle. $AE \parallel CD$. AE intersects BC at F and CE is drawn.



Prove that:

(a) EBAC is a cyclic quadrilateral (6)

(b) AE bisects $B\hat{E}C$ (7)

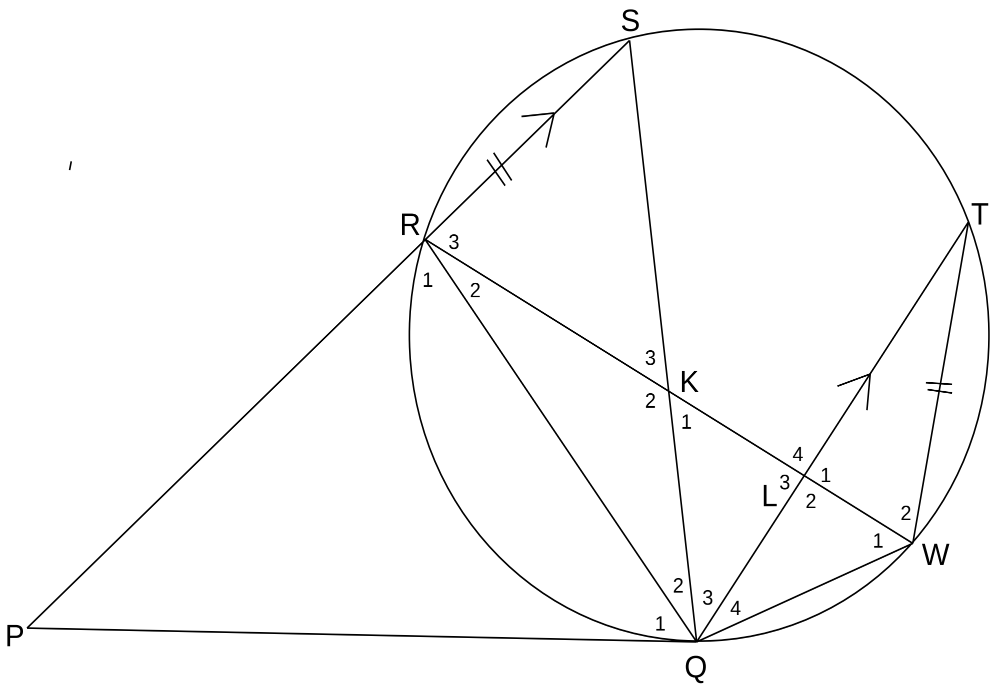
(c) EB is a tangent to circle AFB (6)

[19 marks]

(Eadie & Lampe, 2013, p. 9.26)

Worksheet 2 [Homework]

In the diagram below, PQ is a tangent to the circle at Q. PRS is a secant of circle RSQWT. RW cuts SQ at K and QT at L. $PS \parallel QT$. $RS = TW$.



Prove that:

(a) KQ is a tangent to circle LQW (6)

(b) $\hat{R}_1 = \hat{L}_3$ (7)

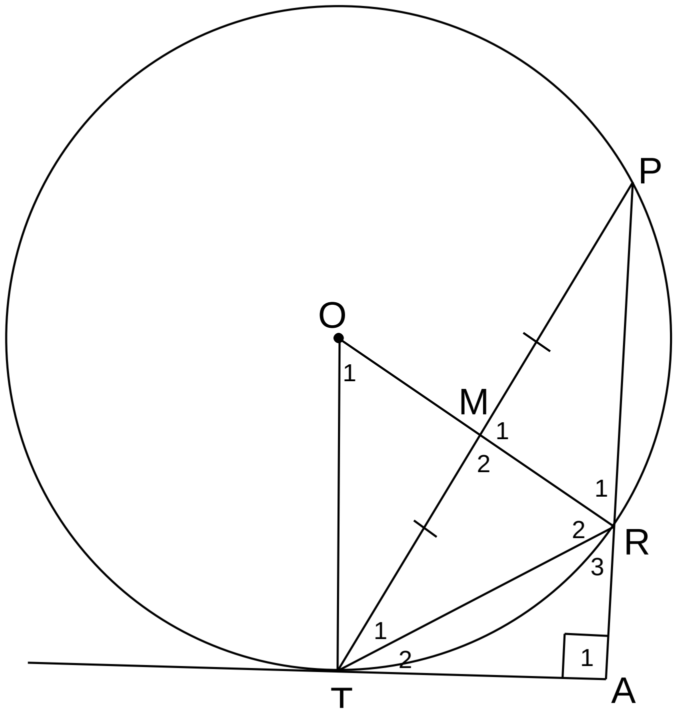
(c) PRKQ is a cyclic quadrilateral (10)

[23 marks]

(Eadie & Lampe, 2013, p. 9.26)

Worksheet 3 [Classwork]

TA is a tangent to the circle PRT. M is the midpoint of chord PT. O is the centre of the circle. PR is produced to intersect with TA at A and $TA \perp PA$. T and R are joined. OR and OT are radii.



Prove that:

(a) MTAR is a cyclic quadrilateral (5)

(b) $PR = RT$ (6)

(c) TR bisects $P\hat{T}A$ (6)

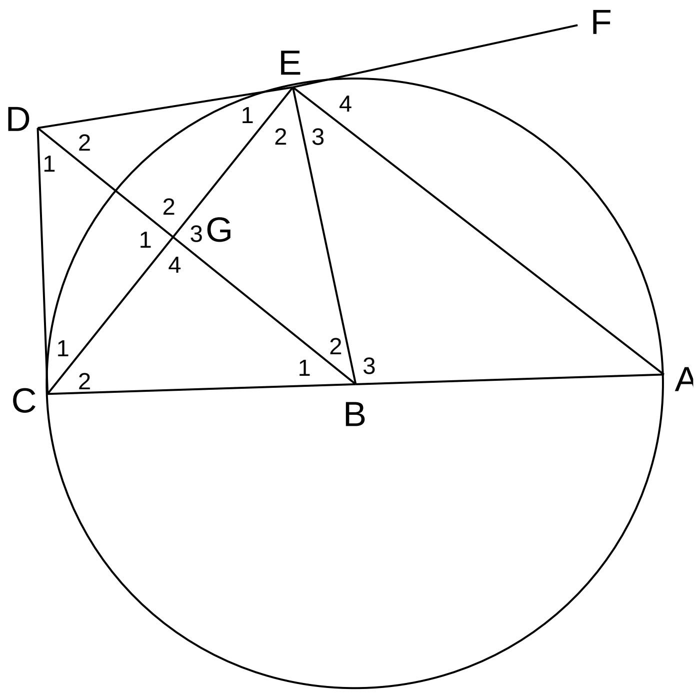
(d) $\hat{T}_2 = \frac{1}{2}\hat{O}_1$ (5)

[22 marks]

(Phillips, Basson, & Botha, 2012, p.241)

Worksheet 4 [Homework]

AC is a diameter of the circle centre B. FED is a tangent to the circle at E. $BG \perp EC$. BG produced meets FE produced at D. DC is drawn.



Prove that:

(a) $BG \parallel AE$ (5)

(b) BCDE is a cyclic quadrilateral (6)

(c) DC is a tangent to circle EAC (10)

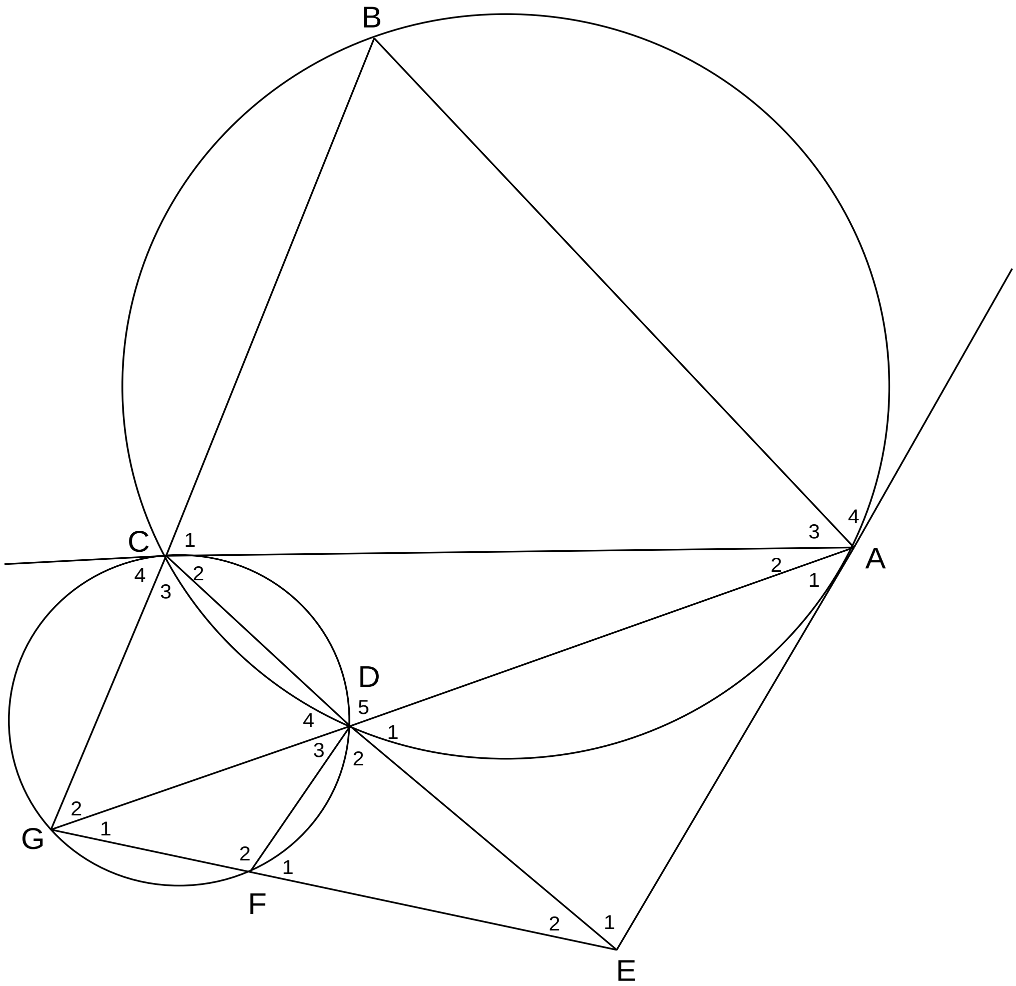
(d) DC is a tangent to circle BCG (10)

[31 marks]

(Phillips, Basson, & Botha, 2012, p.244)

<u>Worksheet 5</u> [Classwork]

In the diagram, EA is a tangent to circle ABCD at A. AC is a tangent to circle CDFG at C. CE and AG intersect at D.



Prove that:

(a) $BG \parallel AE$ (6)

(b) AE is a tangent to circle FED (6)

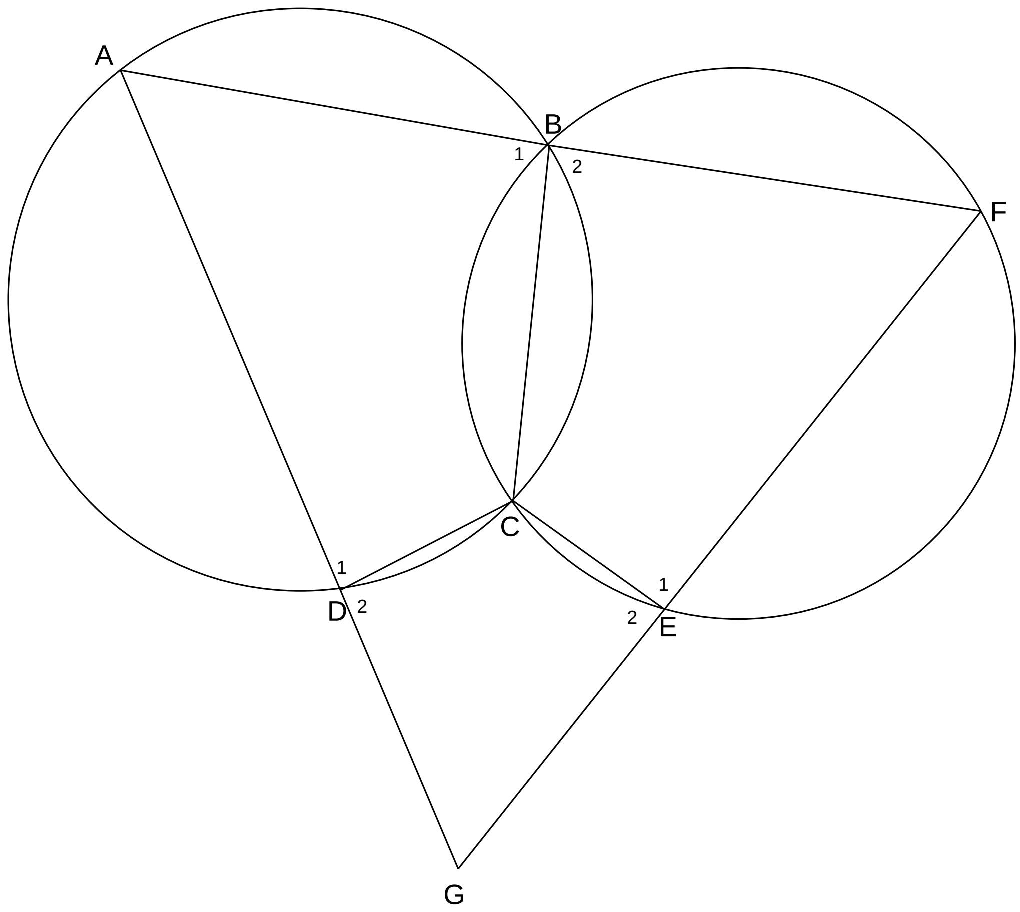
(c) $AB = AC$ (6)

[18 marks]

(Department of Basic Education, 2013, p. 12)

Worksheet 6 [Homework]

Two circles intersect at B and C. ADG, FEG and ABF are straight lines.

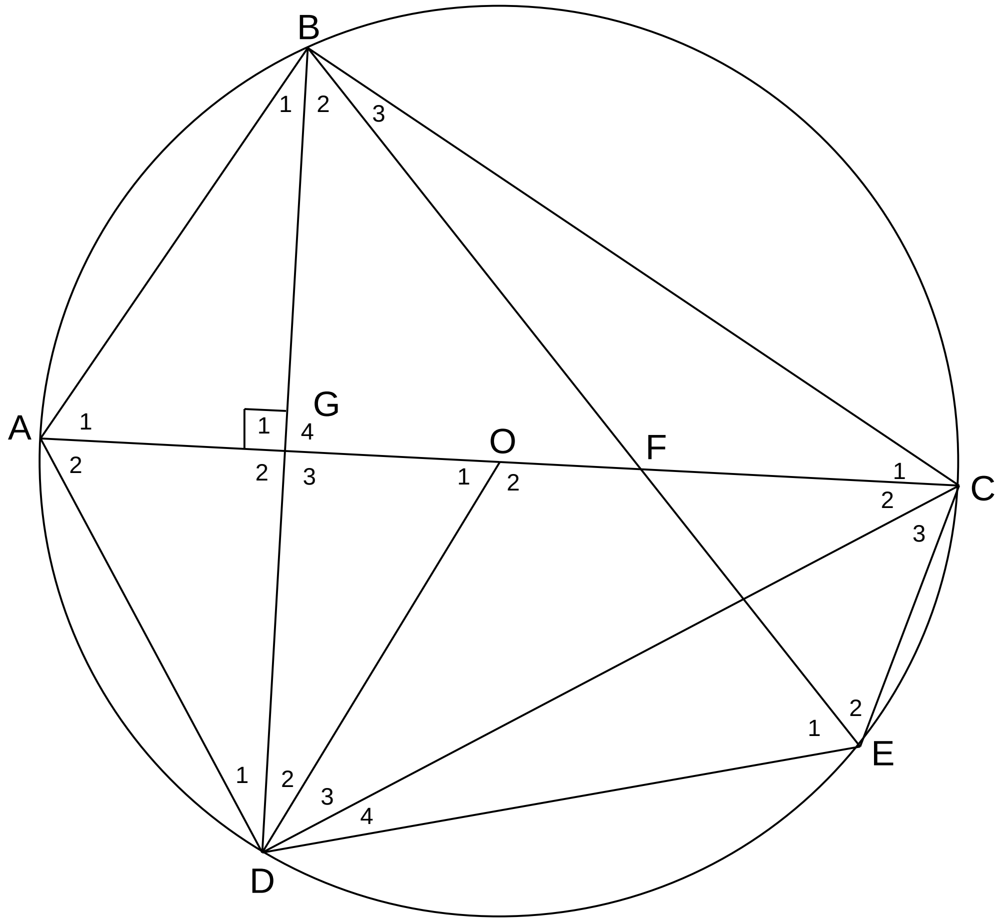


Prove that DCEG is a cyclic quadrilateral (6 marks)

(Limpopo Department of Education, 2014, p. 11)

Worksheet 7 [Classwork]

Given circle centre O with diameter AOD and chord $CD \perp AB$.



Prove that:

(a) $B\hat{C}O = D\hat{C}O$ (6)

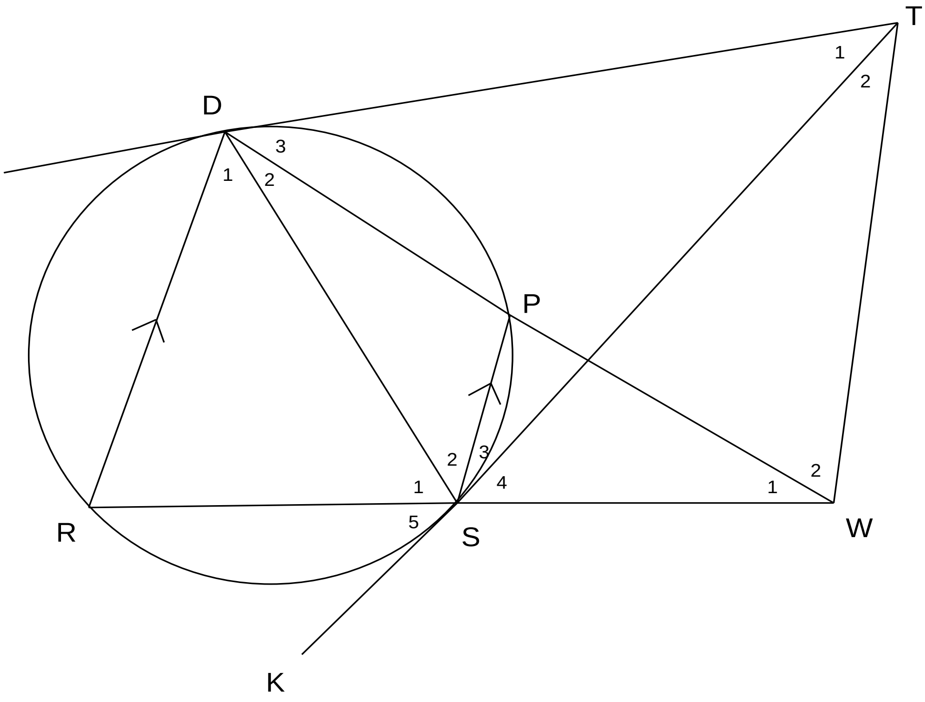
(b) $B\hat{E}D = 2A\hat{D}G$ (6)

(c) ODEF is a cyclic quadrilateral (9)

[21 marks]

Worksheet 8 [Homework]

TD is a tangent to circle RSPD. RS and DP produced meet at W. KST is a straight line. $\hat{S}_4 = \hat{S}_2$. $DR \parallel PS$.



Prove that:

(a) SWTD is a cyclic quadrilateral (5)

(b) TS is a tangent to circle RSPD (7)

(c) $TW \parallel PS$ (6)

[14 marks]

(Limpopo Department of Education, 2013, p. 8)

APPENDIX I: GEOMETRY PROOF TEST AND MARKING GUIDE

GRADE 11 EUCLIDEAN GEOMETRY TEST

(PROVING RIDERS)

MARKS: 50 TIME ALLOWED: 1 HOUR

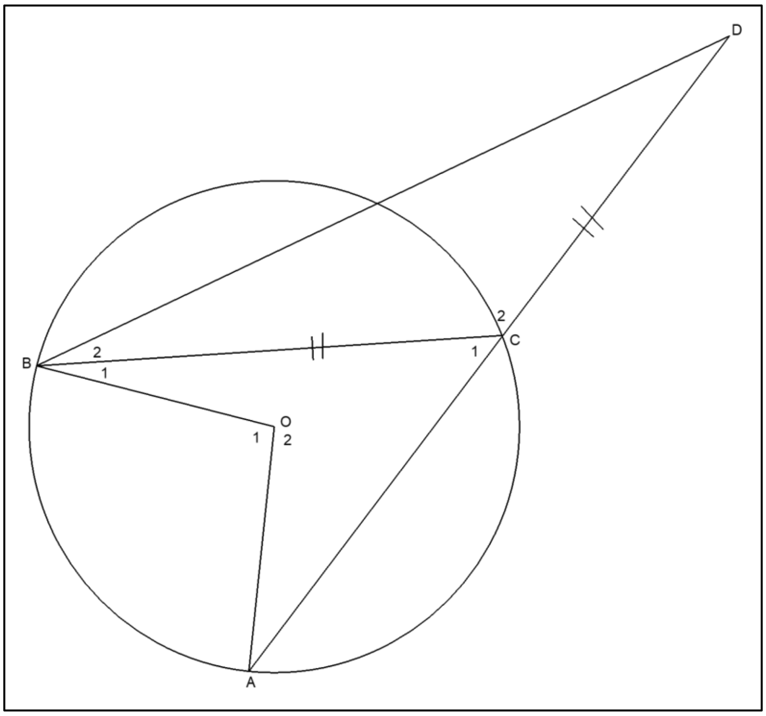
INSTRUCTIONS AND INFORMATION

This question paper consists of 4 long questions

1. Answer ALL questions
2. Write neatly and legibly
3. Diagrams are NOT necessarily drawn to scale
4. Number your answers correctly according to the numbering system used in this question paper.

QUESTION 1

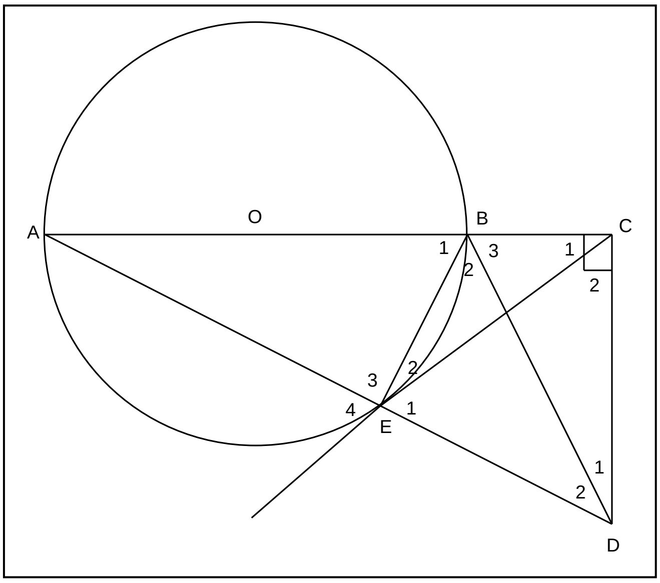
O is the Centre of the circle, AC is produced to D, and $CD = CB$.



Prove that $\hat{O}_1 = 4\hat{D}$ **[8 marks]**

QUESTION 2

Diameter AB is produced to C. CE is a tangent to the circle at E. AE is produced to D, and $DC \perp AC$.



Prove that:

(2.1) BEDC is a cyclic quadrilateral (4)

(2.2) $\hat{\mathrm{D}}_1 = \hat{\mathrm{A}}$ (5)

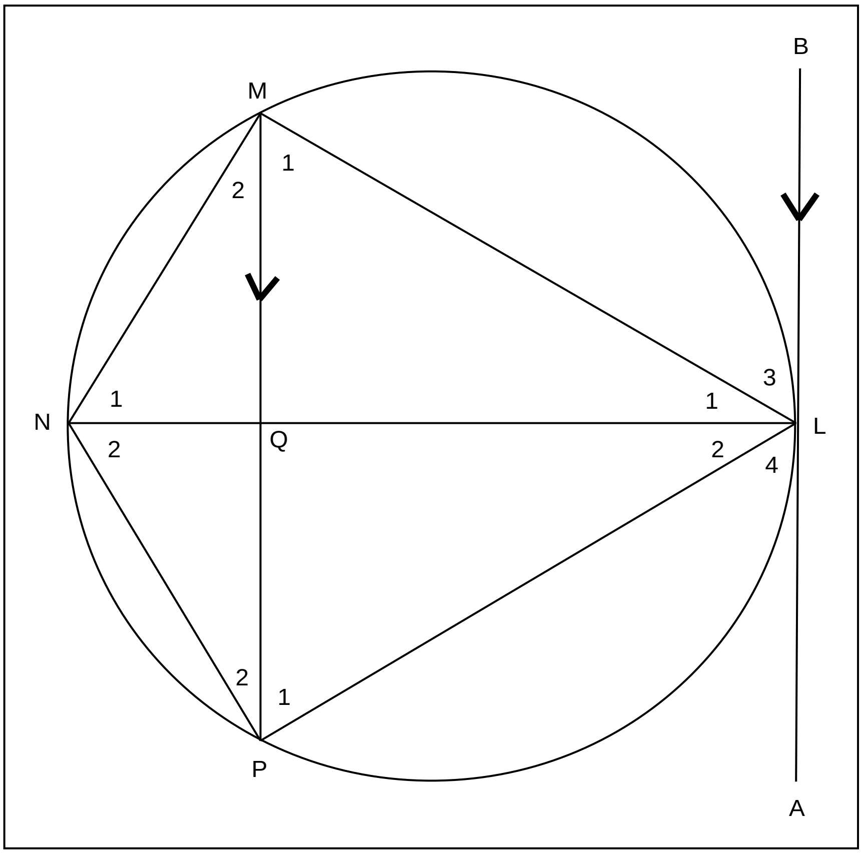
(2.3) $\mathrm{CE} = \mathrm{CD}$ (7)

(2.4) $\hat{\mathrm{B}}_1 = \hat{\mathrm{B}}_3$ (6)

[22 marks]

QUESTION 3

AB is a tangent to a circle LMNP. AB is parallel to MP.



Prove that:

(3.1) $\mathrm{LM} = \mathrm{LP}$ (6)

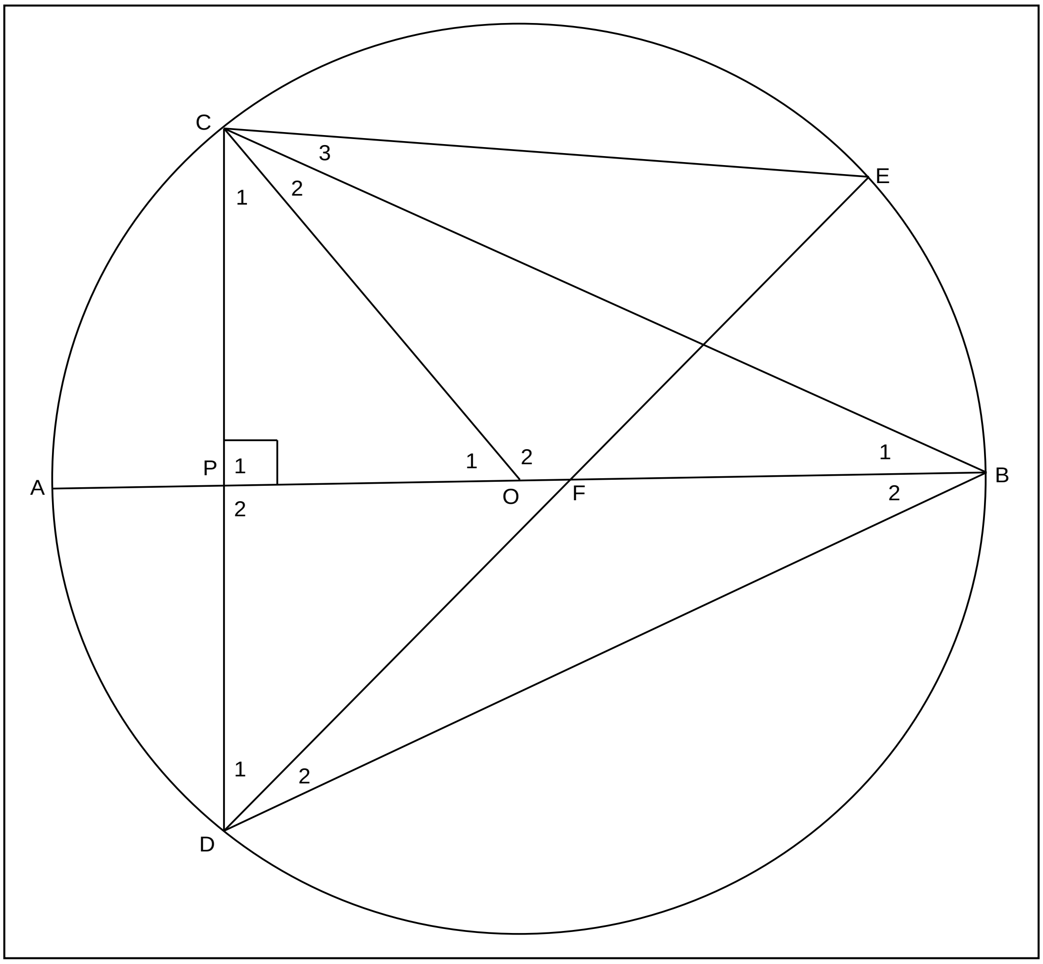
(3.2) LN bisects $\mathrm{M\hat{N}P}$ (3)

(3.3) LM is a tangent to circle MNQ (4)

[13 marks]

QUESTION 4

In the diagram, O is the centre of the circle. $CD \perp AB$ at P



Prove that:

(4.1) $\hat{B}_1 = \hat{B}_2$ (4)

(4.2) $\hat{E} = 2\hat{B}_1$ (3)

[7 marks]

GRAND TOTAL: 50

Thank you for your participation!

MARKING GUIDE

1.	$CD = CB$ (given)	
	$\therefore \hat{D} = \hat{B}_2$ (∠s opp equal sides)	✓ S ✓ R
	$\hat{C}_1 = \hat{B}_2 + \hat{D}$ (ext. ∠ of a Δ)	✓ S ✓ R
	$\therefore \hat{C}_1 = 2\hat{D}$ $(\hat{B}_2 = \hat{D})$	✓ S
	$\hat{O}_1 = 2\hat{C}_1$ (∠ at centre = 2.∠ at cirumference)	✓ S ✓ R
	$\therefore \hat{O}_1 = 2(2\hat{D})$ $= 4\hat{D}$	✓ S
		(8)
2.1	$\hat{E}_3 = 90°$ (∠ in semi − circle)	✓S ✓R
	$\hat{C} = 90°$ (given)	
	$\therefore \hat{E}_3 = \hat{C}$ (Both $= 90°$)	✓S
	∴ BEDC is a cyclic quadrilateral	
	(ext ∠ = int opp ∠) OR (converse ext ∠ of a cyclic quad)	✓R
		(4)
2.2	$\hat{D}_1 = \hat{E}_2$ (∠s in the same seg. OR ∠s subtended by the same chord)	✓S ✓R
	$\hat{E}_2 = \hat{A}$ (tan chord theorem)	✓S ✓ R
	$\therefore \hat{D}_1 = \hat{A}$ (Both $= \hat{E}_2$)	✓R
		(5)
2.3	$\hat{D} = \hat{B}_1$ (ext ∠ of a cyclic quad)	✓S ✓ R
	$\hat{B}_1 = \hat{E}_4$ (tan chord theorem)	✓S ✓R
	$\hat{E}_4 = \hat{E}_1$ (vert opp ∠s)	✓S ✓ R
	$\therefore \hat{D} = \hat{E}_1$	
	∴ CE = CD (sides opp. equal ∠s)	✓R
		(7)
2.4	$\hat{B}_3 = \hat{E}_1$ (∠s in the same seg)	✓S ✓R
	$\hat{E}_1 = \hat{E}_4$ (vert opp ∠s)/proved	✓ S/R
	$\therefore \hat{B}_3 = \hat{E}_4$ (both $= \hat{E}_1$)	✓ S/R
	But $\hat{E}_4 = \hat{B}_1$ (tan chord theorem/proved)	✓S/R
	$\therefore \hat{B}_3 = \hat{B}_1$ (both $= \hat{E}_4$)	✓R
	OR	**(6)**
	$\hat{B}_1 = \hat{D}$ (ext.∠of a cyclic quad)	✓S ✓R
	$\hat{D} = \hat{E}_1$ (∠s opp. equal sides; CE = CD)	✓S ✓R
	$\therefore \hat{B}_1 = \hat{E}_1$ (both $= \hat{D}$)	
	$\hat{E}_1 = \hat{B}_3$ (∠s in the same seg)	✓S ✓R
	$\therefore \hat{B}_1 = \hat{B}_3$ (both $= \hat{E}_1$)	
		(22)
3.1	$\hat{L}_3 = \hat{M}_1$ (alt ∠s; AB ∥ MP)	✓S ✓R
	$\hat{L}_3 = \hat{P}_1$ (tan chord theorem)	✓S ✓R
	$\hat{M}_1 = \hat{P}_1$ (both $= \hat{L}_3$)	✓S/R
	$\therefore \boldsymbol{LM = LP}$ (sides opp. equal sides)	✓R
		(6)
	OR	
	$\hat{L}_4 = \hat{P}_1$ (alt ∠s; AB ∥ MP)	✓S ✓R
	$\hat{L}_4 = \hat{M}_1$ (tan chord theorem)	✓S ✓R
	$\therefore \hat{P}_1 = \hat{M}_1$ (both $= \hat{L}_4$)	✓S/R
	∴ LM = LP (sides opp. equal angles)	✓R
		(6)

3.2	$LM = LP$ (proved)	✓S/R
	$\therefore \hat{N}_1 = \hat{N}_2$ (Equal chords; equal $\angle$s)	✓S ✓R
	OR	
	$\hat{N}_1 = \hat{P}_1$ ($\angle$s in the same seg)	✓S/R
	$\hat{P}_1 = \hat{M}_1$ (proved)	
	$\therefore \hat{N}_1 = \hat{M}_1$ (both $= \hat{P}_1$)	
	But $\hat{M}_1 = \hat{N}_2$ ($\angle$s in the same seg)	✓S/R
	$\therefore \hat{N}_1 = \hat{N}_2$ (both $= \hat{M}_1$)	✓S/R
	$\therefore$ LN bisects $M\hat{N}P$	
		(3)
	OR	
	$\hat{L}_3 = \hat{N}_1$ (tan chord theorem)	
	$\hat{L}_3 = \hat{M}_1$ (alt $\angle$s; $AB \parallel MP$)	
	$\therefore \hat{N}_1 = \hat{M}_1$ (both $= \hat{L}_3$)	✓S/R
	But $\hat{M}_1 = \hat{N}_2$ ($\angle$s in the same seg)	✓S/R
	$\therefore \hat{N}_1 = \hat{N}_2$ (both $= \hat{M}_1$)	✓S/R
	$\therefore$ LN bisects $M\hat{N}P$	**(3)**
	OR	
	$\hat{L}_4 = \hat{N}_2$ (tan chord theorem)	
	$\hat{L}_4 = \hat{P}_1$ (alt $\angle$s; $AB \parallel MP$)	
	$\therefore \hat{N}_2 = \hat{P}_1$(both $= \hat{L}_4$)	✓S/R
	But $\hat{P}_1 = \hat{N}_1$($\angle$s in the same seg)	✓S/R
	$\therefore \hat{N}_1 = \hat{N}_2$ (both $= \hat{P}_1$)	✓S/R
	$\Longrightarrow$ LN bisects $M\hat{N}P$	
		(3)
3.3	$\hat{M}_1 = \hat{P}_1$ (proved)	✓ S/R
	$\therefore \hat{N}_1 = \hat{P}_1$ (proved)	✓ S/R
	$\therefore \hat{M}_1 = \hat{N}_1$ (both $= \hat{P}_1$)	✓ S/R
	LM is a tangent to circle MNQ ($\angle$ between line and chord)**OR** (converse tan chord theorem)	✓ R
		(4)
	OR	
	$\hat{L}_3 = \hat{M}_1$ (alt $\angle$s; $AB \parallel MP$)	✓ S/R
	$\hat{L}_3 = \hat{N}_1$ (tan chord theorem)	✓ S/R
	$\therefore \hat{M}_1 = \hat{N}_1$ (both $= \hat{L}_3$)	✓ S/R
	$\Longrightarrow$ LM is a tangent to circle MNQ	
	($\angle$ between line and chord)OR (converse tan chord theorem)	✓ R
		(4)
4.1	$CP = DP$($\perp$ line from centre to chord bisects chord)	✓ S/R
	$\hat{P}_1 = \hat{P}_2$ (both $= 90°$)	✓ S/R
	BP is common	✓ S
	$\therefore \Delta BCP \equiv \Delta BDP$ (SAS)	✓ S/R
	$\therefore \hat{B}_1 = \hat{B}_3$ $(\Delta BCP \equiv \Delta BDP)$	
		(4)
4.2	$\hat{E} = \hat{B}_1 + \hat{B}_2$ ($\angle$s in the same seg)	✓S ✓R
	But $\hat{B}_1 = \hat{B}_2$ (proved)	✓S/R
	$\therefore \hat{E} = \hat{B}_1 + \hat{B}_1 = 2\hat{B}_1$	
		(3)

APPENDIX J: TEST VALIDATION FORM

You are kindly requested to provide feedback on the validity of each test item by inserting a cross (**X**) in the appropriate spaces. Your feedback is highly valued and greatly appreciated.

Item / Criteria	1	2.1	2.2	2.3	2.4	3.1	3.2	3.3	4.1	4.2	4.3	4.4
Relevance												
1 = not relevant												
2 = item needs some revision												
3 = relevant but needs minor revision												
4 = very relevant												
Clarity												
1 = not clear												
2 = item needs some revision												
3 = clear but needs minor revision												
4 = very clear												
Total Rating Score Per Item												
Average Rating Score Per Item												

You may provide **comments and **suggestions** for improvements here:

………

………

………

………

………

………

………

………

………

………

Signature: ______________________ **Date:** ______________________

APPENDIX K: ‘sm’ ANCOVA AND fANCOVA CODES IN ‘R’ PACKAGE

```
> rm(list=ls())
> dat=read.csv(file.choose(),header=T)
> library(sm)
> library(fANCOVA)
> attach(dat)
> names(dat)
> with(dat,ancova.np<-sm.ancova(Prescore,Postscore,Group,model="equal"))
> sm.ancova(x=Prescore,y=Postscore,group=Group,model="equal")
> loess.ancova(Prescore,Postscore,Group,degree=2,criterion=c("aicc","gcv"),
+ family=c("gaussian","symmetric"),method=c("Speckman","Backfitting"),
+ iter=10,tol=0.01,user.span=NULL,plot=T)
> T.aov(Prescore,Postscore,Group,B=200,degree=1,criterion=c("aicc","gcv"),
+ family=c("gaussian","symmetric"),tstat=c("DN","YB"),
+ user.span=NULL)
```

APPENDIX L: 'sm' ANCOVA AND fANCOVA OUTPUT IN 'R' PACKAGE

```
> rm(list=ls())
> dat=read.csv(file.choose(),header=T)
> library(sm)
Package 'sm', version 2.2-5.6: type help(sm) for summary information
Warning message:
package 'sm' was built under R version 3.4.4
> library(fANCOVA)
fANCOVA 0.5-1 loaded
Warning message:
package 'fANCOVA' was built under R version 3.4.4
> attach(dat)
> names(dat)
[1] "Student.ID" "Group"      "Age"        "Gender"     "Prescore"
[6] "Postscore"  "X"          "X.1"
> with(dat,ancova.np<-sm.ancova(Prescore,Postscore,Group,model="equal"))
Test of equality :  h =  2.26096    p-value =  0
> sm.ancova(x=Prescore,y=Postscore,group=Group,model="equal")
```

Test of equality: h = 2.26096 p-value = 0

```
> loess.ancova(Prescore,Postscore,Group,degree=2,criterion=c("aicc","gcv"),
+ family=c("gaussian","symmetric"),method=c("Speckman","Backfitting"),
+ iter=10,tol=0.01,user.span=NULL,plot=T)
$linear.fit
            [,1]
```

(Intercept) 17.09871

group1 49.28838

```
$smooth.fit
Call:
loess(formula = lm.res ~ x, span = span1, degree = degree, family = family)
Number of Observations: 186
Equivalent Number of Parameters: 4.86
Residual Standard Error: 16.85
There were 50 or more warnings (use warnings() to see the first 50)
> T.aov(Prescore,Postscore,Group,B=200,degree=1,criterion=c("aicc","gcv"),
+ family=c("gaussian","symmetric"),tstat=c("DN","YB"),
+ user.span=NULL)
Test the equality of curves based on an ANOVA-type statistic
Comparing 2 nonparametric regression curves
Local polynomial regression with automatic smoothing parameter selection via AICC is used for curve fitting.
Wide-bootstrap algorithm is applied to obtain the null distribution.
Null hypothesis: there is no difference between the 2 curves.
```

T = 595.9 p-value = 0.004975

```
There were 50 or more warnings (use warnings() to see the first 50)
>
```

APPENDIX M: FOCUS GROUP DISCUSSION GUIDE

Preliminary Section [For internal use only]

Date	
Time	
Location	
Type of group	
Selection criteria	
Number of participants present	
Number of male participants present	
Number of female participants present	

Focus Group Script:

Opening Section

Introduction:

Hello everybody! **Welcome** and thank you for volunteering to participate in this focus group discussion. We know that you have your own business to do and we greatly appreciate that you have sacrificed your time to be with us today. My name is **[insert moderator's name here]** and assisting me is [**insert note-taker's name here**]. We are conducting discussion groups with Grade 11 learners like yourselves from different secondary schools in Capricorn district, on behalf of Mr Eric Machisi, who is a student with the University of South Africa. The **purpose** of the discussion is to get your **views** on the way Euclidean geometry was taught in your mathematics classrooms. Your feedback is very important to us as it will guide researchers in developing ways to improve the quality of teaching and learning of Euclidean geometry in schools.

My role as a facilitator will be to guide the discussion by asking you several open questions that each one of who can respond to. [**Insert note-taker's name here]** will observe, take notes, and record an audio of the conversation. We are recording the conversation because we do not want to miss any of your comments. This is only for purpose of the research. The recorded information will be transcribed, summarized, and combined with information recorded in focus group discussions conducted elsewhere. I would like to assure you that whatever you say in this discussion will be **anonymous**. This means that **no names or personal information** will be used in our final report. The final report will be published by the University of South Africa for interested parties to read.

Before we start, I want everyone to know that **there are no right or wrong answers** to the questions asked in this discussion, only differing views. **Both positive and negative views are important** to us. So, please **feel free to be honest** and to share all your views with us even if they differ from what others have said. You **do not have to agree with the views of other participants** in the group. We encourage everyone to participate and you do not have to speak in any order. However, the most important rule we should observe is that **only one person speaks at a time**. We may be tempted to interrupt when someone is talking but please let us wait until they have finished. Please be reminded that information provided in this room **must be kept confidential**. This means that you should not tell anyone what was said by others here today. We would greatly appreciate it if members **respect each other's privacy** by not discussing the comments of other group members when you leave this room. Remember the golden rule: Treat others in the same way you would want them to treat you. Do you have any questions before we get started? [answers]. Please, let us **switch off our cell phones** or simply put them on silent mode to avoid disturbances when we get started. If you must respond to a call, please do so as quietly as possible and re-join us as quickly as possible. Once again, thank you very much for your cooperation. Our discussion will take no more than two hours. Without further delays, let us get started.

Warm-up:

Let us start by getting to know each other. Please tell us: (1) your first name; and (2) an activity you like to do in your spare time (*Point to someone to start; randomly select people to demonstrate that people do not talk in sequence*).

Question Section

(a) Engagement: *Ask a general question to get participants talking to each other, to make them feel comfortable, and to build rapport*

- When you think of Euclidean geometry, what comes to your mind? Please talk to each other. You have five minutes to do that.
- Ok, our five minutes has elapsed. I would love to hear your different views. Anyone of you can be first to tell us his or her response (*Give all participants who want to say something time to speak, and remember to say 'thank you' after each speaker)*

Thank you very much for all your contributions. It was quite interesting to hear your different views. Now, let us proceed to our next set of questions.

(b) Exploration: *Ask specific questions focusing on the topic of discussion*

- What do you think about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?
- How do you feel about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?
- What do you like or dislike about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?
- Can you describe your attitude towards Euclidean geometry and geometric proofs?
- What did the teacher do that you think contributed to your attitude towards Euclidean geometry and geometric proofs?

(c) Exit: *Ask a follow-up question to determine if there is anything else related to the topic that needs to be discussed*

Before we end the discussion, is there anything you wanted to add that you did not get a chance to bring up earlier? (*Give participants time to speak*).

Closure

Thank you so much for your time and sharing your opinions and emotions with us. Your feedback will be valuable to our research and this has been a very successful discussion. We hope you found this discussion interesting. If there is anything you are unhappy with or wish to complain about, please feel free to talk to me at the following number: 072 147 4618. I see our time is up and we have come to the end of our discussions. Once again, thank you very much for your participation. As you walk out, please collect your food and gift from the people seated next to the exit.
I wish you all a safe journey on your way back home.
Good bye!

APPENDIX N: FOCUS GROUP DISCUSSION TRANSCRIPTS

N 1: EXPERIMENTAL GROUP TRANSCRIPT – FG E1

Moderator: **When you think of Euclidean geometry, what comes to your mind?** Please talk to each other. You have five minutes to do that (*Pause*).

Moderator: Alright, eh, thank you so much for your multiple contributions as you were discussing but now, I would love to hear your different views in terms of whenever you think of Euclidean geometry, what comes to your mind. I want to hear your views personally. Let's start with eh **Na**!

Na: Eh, so when I heard of Euclidean geometry, I thought of quadrilaterals but in turned out that Euclidean geometry was all about all shapes, including circles, and other quadrilaterals. So, what came to my mind when I saw that we are going to solve Euclidean geometry about circles I thought eh it was difficult because I have never done anything like that before. So I didn't believe myself at first and I had already gave up saying I will never get this right but then as Sir continued to teach us and as he unpacked the whole topic, then it became a lot more easier for me to understand it and I am quite happy to say that I have improved and I can now tackle Euclidean geometry questions on my own and get them right. And also, my skills have also improved. I am able to interpret diagrams more accurately and apply the knowledge that I have acquired in previous days. Yes, so Euclidean geometry is not actually a difficult thing. It just needs a person to be determined and to — yes, to be focused all the time.

Moderator: Thank you **Na**. Uhm, **T**!

T: Ok, when I think of Euclidean geometry right, uhm, I have always loved this part of Euclidean geometry in Mathematics. Like in Mathematics as a whole, I have always loved Euclidean geometry. Uhm, what I like about Euclidean geometry or what I have been in love about it is because they give you things and then they ask you questions based on that thing. So, if you are able to interpret it then it won't be a very tough situation for you to come up with solutions. So, whenever I think of Euclidean geometry, or whenever I hear of Euclidean geometry, I have always become happy you know, because this is the part of mathematics that I love the most and I am very good at it. So, it is not really a barrier to me to solve Euclidean geometry problems. To come up with solutions is not really hard to me.

Moderator: Thank you so much, **T**. Uhm, **O**!

O: When I think of Euclidean geometry, firstly, I didn't know how to solve theorems (riders) and it was difficult for me. But since our teacher taught us how to prove and solve, so, I started liking how to solve theorems (riders). And when I think of Euclidean geometry, I become happy because I was not working alone. We were working in pairs, and that made us know more or have more knowledge about Euclidean geometry.

Moderator: Thank you **O**. Eh, thank you so much all of you for your contributions. It was quite interesting to hear your different views. Let us now proceed to our next set of questions. We are going to explore the first question: **What do you think about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom? O**!

O: I think they taught us in a good way. If I was going to rate, I would rate 10 over 10 because I understood everything about Euclidean geometry and geometric proofs. And now I have more knowledge, oh, yah.

Moderator: Thank you **O**!

Na: Ok, I think it was taught exceptionally well because we were doing each theorem individually every day and then after doing the theorem, we were given an activity to do. So, uhm, that made us like gain more knowledge and have experience on how to solve certain riders. So, yah we became very familiar with the whole topic. So, I think yah Euclidean geometry was taught very, very well.

Moderator: Thank you **Na**!

T: Eh, I think the way they taught us Euclidean geometry was very good and explicit because at one point they would give activities. They would leave us for like one hour thirty minutes or so. So, we will try to figure out how to come up with solutions, how to solve this problem, and then that made us be a little bit witty than before because well they don't really give us answers to this question at first. They leave us then we will be able to discuss it with others, then, yah that is how it was done.

Moderator: Thank you **T** for your view. Eh, let's move to the next one! **How do you feel about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?** How do you feel? Uhm, **Na**!

Na: Uhm, I felt really bad at first because I had no idea what Euclidean geometry was all about this year because we were doing something that we had never done before but then as time went on, I started feeling good because I was able to solve and come up with solutions. And it felt like I was being put on a test like as a challenge to test how far I can go or I can push myself or how I am willing to do things. So yah, I really feel good now about Euclidean geometry.

Moderator: Thank you very much **Na**. Anyone else who wants to — Uhm, **T**!

T: Uhm, I feel good about Euclidean geometry because they teach us how to solve problems not only in the mathematics class but then in real life because you will be able to solve problems in different perspectives. Then, that is what is happening in real life because we come across many problems in our daily lives. With Euclidean geometry we are now able to come up with solutions to solve this and that.

Moderator: Thank you **T**. **O**, do you have something?

O: Yes! What I like about Euclidean geometry is that you can solve many problems with many solutions and the other thing is uhm working with our teacher made us know more about theorems. That's what I like and the last thing I like is we were working in pairs and we showed each other, which one is right and which one is wrong. Then after that our Sir came and showed us which is wrong and what is right. Yah.

Moderator: Thank you **O**! Let's move on to our third question: **What do you like or dislike about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?** Uhm, **Na**!

Na: Ok, uhm, what I like was that everybody was able to participate in the lesson because sir wrote statements on the chalkboard and everyone had a right or freedom to go there and fill the correct reason for that particular statement so the class was alive so yah we were jumping up and down, back and forth to the chalkboard just to — yah, I liked everything about how Euclidean geometry was taught.

Moderator: Thank you **Na**. Uhm, **O**!

O: What I like about the way we were taught is uh, our teacher was not in a hurry. He was patient and if a learner didn't understand he could explain more and give more examples. So that's what I like about the way we were taught.

Moderator: Thank you **O**. Uhm, **T**!

T: Well, what I like is the participation of everyone. That was on another level because well, we understood what Euclidean geometry was all about. In that way we were able to participate like all the time. We were even fighting over the chalk at times. That is what I liked.

Moderator: Thank you **T**. Eh, let's quickly move on to our fourth question. **Can you describe your attitude towards Euclidean geometry and geometric proofs?** Uhm, **T**!

T: Ok, my attitude has always been positive towards Euclidean geometry. But now I think it grew remarkably on another level. Right now, my attitude is not the way it was before. It is more than positive you know.

Moderator: Thank you **T**. Uhm, **O**!

O: Firstly, I didn't like, uhm, my attitude was negative because I didn't know like (how) to solve Euclidean geometry. I didn't know what Euclidean geometry is all about. So, when our teacher taught us, my attitude changed to being positive. So, now I know more about solving problems and Euclidean geometry. So, I would say, and since my attitude changed, uhm, I think I would have more knowledge or work more in order to have better attitude.

Moderator: Thank you **O**! Uhm, **Na**!

Na: My attitude at first was not good because I felt like Euclidean geometry was *gonna* defeat me because it's something I have never did (done) before. But as time went on my attitude started to change because I told myself that I would not be defeated by a bunch of diagrams with complicated lines. Then I started improving and started feeling better about myself and now I view Euclidean geometry as something that eh, I can take as a — like —.

Moderator: Thank you **Na**! This leads us to our last question. **What did your teacher do that you think contributed to your attitude towards Euclidean geometry and geometric proofs?** Uhm, **O**!

O: Our teacher made me love the way we solve and he taught us and explained each and every theorem, not being in a hurry. And the other thing is he made us comfortable to talk to him in order to solve, and — yes.

T: Uhm, one thing I like about Sir is that he doesn't really tell you that this answer is wrong because he knows that if he do (does) so he will take your confidence down. So, he is free. He always free with us. You will be free to talk to him even it doesn't involve mathematics things. So, that is what I like about him. He's always a free man. You don't, like most of us are not afraid to go towards him and say this is the problem that I came across, so how can I try to solve this particular problem. So, you are always free to go to Sir and that is what I like him.

Moderator: Thank you **T**. Before we end the discussion, is there anything you wanted to add that you did not get a chance to bring it up earlier on? (Pause) Alright, thank you so much for your time and sharing your opinions and emotions with us. Your feedback would be a valuable asset to our research and this has been a very successful discussion. We hope you found this discussion interesting. If there is anything you are unhappy with or wish to complain about please feel free to talk to me at the following number, 072 147 4618. I see your time is up and we have to come to the end of this discussion. Once again thank you so much for your participation. I wish you all a safe journey back home. Goodbye!

N 2: EXPERIMENTAL GROUP DISCUSSION TRANSCRIPT– FG E2

Moderator: **When you think of Euclidean geometry what comes to your mind?** Please talk to each other. You have five minutes to do this (*Pause*).

Moderator: Ok, your five minutes has elapsed. I would love to hear your different views. Anyone can first tell us what his/her response or views on what you were discussing.

Mo: From what we were talking about mostly we talked about circles and quads, tangents and chords. So, from my view like Euclidean geometry *ye e dirang ke rena* (the one that we are doing) is mostly about circles, yah.

Moderator: Ok, thank you very much for all your contributions. Anyone else who wants to voice out?

Kg: What does the question say?

Moderator: The question is, uhm, when you think of Euclidean geometry, what comes in your mind?

Kg: Solving problems. Ah, well Euclidean geometry needs someone who can think like critically so because solving riders is hard, like you have to think to solve it.

Moderator: Ok, thank you **Kg** for input. Do we anyone else? (*pause*) Uhm, it is quite interesting to hear your different views. Now, let us proceed to our next set of questions. We are going to exploration. **What do you think about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom? Mo**!

Mo: Uhm, from what I think like, firstly I didn't know how to solve like to prove using a laptop or computer. But as — when we went into our classroom and Sir taught us about it, then I was so impressed and got more like interested on knowing how to solve these problems. And I think the way that they teach and mostly like *be ba re dumelela re rena* like *re fa di* views *tša rena* (they allowed us like to give our own views). And, it's good, yah!

Moderator: Ok, thank you **Mo**! Anyone else who wants to — yes, **Ha**!

Ha: Uhm, from my point of view I think Euclidean geometry was taught very well in our mathematics class as we were able to solve the riders and how to prove our shapes. Then we were able to know how to solve these types of questions so that when we know that these types are going to appear on question papers then we know how to answer them. So, yah I think Euclidean geometry was taught very well as we were able to understand how Euclidean geometry was able to — be confined (??).

Moderator: Thank you **Ha**! Another one who is interested in the question? Yes, **Ch**!

Ch: Eh, that part when we were taught in our maths class when we were using computers using the GSP software, I think when we were taught Euclidean geometry using that software was really good for us as learners because it wasn't like reading those theorems in a book. We were actually seeing them first-hand. We were actually measuring those angles. In our books those things are not drawn to scale, you just read them and all you do is just memorise but that GSP software you can see them straight and you can measure those angles, the sides, you can see what exactly they are talking about.

Moderator: Ok, thank you **Ch**! Uhm, **how do you feel about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom? Mo**!

Mo: I feel very good about it because eh, as they taught us, we were not only like listening to the teacher alone, we were giving our own thoughts, and our own like views from what we think about them. And then I feel good, yah, I feel good about it because we were able to do like things that I never thought I can do in my life. Like, I never thought, *sa mathomo* (firstly), eish! Firstly, when they introduced us to this topic *ya* (of) Euclidean geometry, I thought it was a difficult part but as I got to explore like *ge ba re ruta ka tšona* (as they were teaching us about it) I was able to be free around my mates and then *ka kgona*, *le gona jwatše* (I was able, even now) I am not like that perfect but I can do most of the things. Yah, I feel good because *e tlišise* (it brought) a good experience like *mo bophelong ba ka* (in my life).

Moderator: Thank you **Mo**! Anyone else? Yes, **Ch**!

Ch: I felt privileged to have been taught Euclidean geometry in this maths class because that GSP theorems (software) really works like, really helped me to be more interested in Euclidean geometry because those things I was doing them myself practically not just theoretically.

Moderator: Thank you **Ch**! **What do you like or dislike about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?** Uhm, **Kg**!

Kg: Actually, I love everything that was taught because it helped me to train my mind, and to think critically, and to reason logically. It helped me to understand and solve problems in the physical world and it made me to gain life skills like being able to explain, being able to convince, being able to verify, communicate and to prove.

Moderator: Thank you **Kg**! Anyone who wants to add? Ok, **can you describe your attitude towards Euclidean geometry and geometric proofs?** Uhm, **Mo**!

Mo: My attitude towards Euclidean geometry and geometric proofs like at first, I was being negative about myself like how am I going to solve these things, they are so difficult. And then, as I got to explore and then *gwa ba* *le di* (there were these) different parts *tša go solver di* (of solving) riders in many different ways, like eish, from what they taught us, they said that mathematics you can solve things like in many ways and then that thing just got me a positive attitude because now *ke kgona go dira dilo tše dintši tša* (I am able to do many things of) geometry.

Moderator: Thank you **Mo**! Uhm, let's move on to our last question. **What did the teacher do that you think contributed to your attitude towards Euclidean geometry and geometric proofs? Ha**!

Ha: The teacher made these types of geometry to make them more easier because the way he proved them on the board, made it look so easy that it had to make us make it look so easy. So, that's why the teacher had to make everything easier for us to not get anything less unspeakable (??).

Moderator: Thank you **Ha**. **Mo**!

Mo: Eh, the teacher made us to be free in class**.** He taught us in a way whereby like he was not that strict like all the time. He made things look easier like our theorem statements, he called it a *bible* so when I think of solving and coming up with reasons I just think of Ok, in the *bible* there is this reason, and then I can solve. Like he didn't deny any of our answers. He let us be free and he even taught us like he encouraged us to work in pairs so that we can help each other and he did not discourage us in any way or make me or make them feel uncomfortable in a way whereby we cannot even raise our hands being afraid to say that the answer is wrong or is right. In our last part when we were no longer working with GSP and computer, he allowed us to write on the chalkboard. Even in the end we were fighting to write on the chalkboard and being able to be enlightened and free and making jokes, yes, yah.

Moderator: Thank you **Mo**. Before we end the discussion, is there anything you wanted to add that you did not get a chance to bring it up earlier on? (Pause) Ok, seems like we brought forth all the relevant information. Uhm, in closure, thank you so much for your time and sharing your opinions and emotions with us. Your feedback will be a valuable asset to our research and this has been a very successful discussion. We hope you found this discussion interesting. If there is anything you are unhappy with or wish to complain about please feel free to talk to me at the following number 072 147 4618. I see our time is up and we have to come to the end of the discussion. Once again thank very much for your participation. I wish you all a safe journey on your way back home. Goodbye!

N 3: CONTROL GROUP DISCUSSION TRANSCRIPT – FG C1

Moderator: Ok guys I think you had enough time to talk about the question, now let's share your views. Any person can start first.

Mp: Ok, uhm, please repeat the question

Moderator: The question says: **When you think of Euclidean geometry what comes to your mind?**

Mp: Ok, I think it is a circle, a circle that has a point at the centre. So that's what I think of Euclidean geometry

Moderator: Ok, let's hear from others if you have anything else to say.

Ko: When I think of Euclidean geometry I think of a circle with a centre, and a circle which has lines on it.

Moderator: Ok, that's **Ko**'s contribution. **Bo** what do you have to say?

Bo: Eh, on my view sir about this geometry, I think it's a circle which has angles inside it and which includes some chords and diameters.

Moderator: Ok, thank you very much for your contribution let's move on to the next question. The next question is like this: **What do you think about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom at your school?** Any person can speak first. You can give yourself time to think if you feel like you need to think about the question. Remember there are no right or wrong answers. Whatever you say is acceptable. It is your own view and that's what we are interested in.

Mp: Uhm I think uh, how the proofs were introduced, am I right?

Moderator: Yes.

Mp: Ok, and then, uh, in our school, when we were taught first time, our teacher didn't uh - didn't uh — what can I say? *Ase a pholiše* (didn't polish*), ase a pholiše gore* (didn't polish that) a chord is what? What is a diameter? *Re e-user ko kae?* (Where do we use it?) These opposite angles and what what interior angles, — so he didn't even uh, what can I say? He wasn't so specific on that. He just, *ne a, ke tla reng?* (he was, what can I say?) He wanted to introduce Theorem 1 without introducing the first things of geometry. That's why geometry *ere file bothata* (gave us problems) when coming to the proofs.

Moderator: Thank you very much for your contribution **Mp**. Anything else that you have to say from the other members of the panel?

Ko: Eh, I think that [*cross talk*] I think that Euclidean geometry before the teacher teaches us — he or she should explain some of the words that cannot be understandable.

Moderator: Ok, you can proceed.

Ko: But geometry was great, he introduced it very well — and it's understandable.

Moderator: Ok, **Bo** do you have anything to say?

Bo: Eh, when we talking about this geometry sir, I think the teacher should have some discussion with other teachers so that they can bring their views and share those views on how they will teach the student about this geometry so that the learners can understand that geometry.

Moderator: So, when you look at how the teacher presented the geometry and the proofs at your school, what do you think about the way that it was presented?

Bo: Eh Sir, the way our teacher introduced this geometry, he didn't explain what is this inclu —what what, is it *inclusive* [Euclidean] what what geometry? He didn't explain to us what kind of geometry is it and he didn't teach us how to prove it and how some lines are called such as chord and what what…is it a diameter, he just went straight to those theorems.

Moderator: Thank you very much for your contribution guys, **Mp** do you have something else to say?

Mp: Yes, I think the reason why geometry it is so difficult at first, it is because our teacher thought that because we started doing geometry at grade— at those lowest grades, I think it's grade 9 or grade 10, so he thought maybe we know, what is chord, what is diameter, that's why he didn't think of touching those things like — *kudu* (much) — [*cross talk*] and only to find that *le gona ko morago* (even in the past) we didn't even understand.

Moderator: That's interesting. Thank you very much. Let's move on to our next question. **How do you feel about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?** Remember any one of us can speak first.

Ko: The question?

Moderator: How do you feel about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?

Ko: Uhm, I feel like some of the proofs were difficult but when we go through them, the teacher teaches us how to prove them, he made them easier.

Moderator: Ok, let's hear from others. How do you feel about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?

Mp: I feel confused because when our teacher teaches us, we understand but when we get home, nothing! Like, we don't understand anything because the teacher is no more there.

Moderator: Interesting contribution! **Bo**, do you have anything to say?

Bo: Yes, on my feeling sir, eh, I feel like this geometry is understandable but our teacher didn't be specific on that geometry, that's why we are a little bit confused.

Moderator: Ok, thank you very much for your contribution to this question. Let's move on to our next question. Our next question says: **What do you like or dislike about the way geometry and geometric proofs were taught in your mathematics classroom?**

Mp: Uhm, what I dislike is that, uhm, you can, I mean like *o kano bona* (you may see) something that you don't understand on that circle, then you don't know how to ask a question, plus, it's in front of other learners, so you don't know if I am going to say it right or if sir or mam is going to understand what I am saying because I don't understand and I am trying to be understandable. So, I am not sure if sir or mam will understand. So, this is one of the things that are killing us because we don't know how to express the questions or yah, or ask the questions.

Moderator: Ok, thank you for your contribution, let me repeat the question before we hear views from other members of our group. The question says: What do you like or dislike about the way that geometry or geometric proofs were taught in your classroom?

Ko: uhm, I dislike that geometric proofs like they were long, they didn't shorten them, so they were difficult.

Moderator: Ok, **Bo** do you have anything to say?

Bo: Yes, what I like about this geometry sir is that some of those theorems are just simple and what I didn't like is that the provings of this geometry sir were long when our teacher taught us how to prove them. That's why we were a little bit confused in the maths class.

Moderator: Ok, thank you very much guys for your contributions on this question. Let us move on to the next question, which is the second last question. It says: **Can you describe your attitude towards geometry and geometric proofs?** Any one of us can speak first.

Mp: I have a bad attitude towards geometry because I find it difficult to understand what is being taught.

Moderator: Thank you very much **Mp** for your contribution, let's hear from the other members of the panel. What do you have to say on this one?

Ko: I had a bad attitude before understanding Euclidean geometry but now I understand it better so my attitude is good on it.

Moderator: Thank you very much for your contribution. **Bo**, do you have anything to say?

Bo: Yes, eh, my attitude was bad at the first of the introduction of this geometry but at least our teacher tried to explain how to prove and how to —eh— to do what, eh, ah!

Moderator: You can use your mother tongue if you want to express yourself clearly.

Bo: Eh, how to express those equations sir! And, now it's better that we understand that geometry and my attitude is very well.

Moderator: Ok, thank you very much for your contributions now let's come to the last question. The last question says: **What did the teacher do that you think contributed to your attitude towards geometry and geometric proofs?**

Ko: Uhm, I think at the first time he didn't introduce the Euclidean geometry well, so it gave me a bad attitude but when times goes on [*cross talk*] — my attitude changed.

Moderator: Ok, let's hear from the other members of the panel.

Bo: Eh, Sir, I think the teacher did some confusion at the first of this geometry but when we were busy with eh, [*cross talk*] with this topic sir, he tried to explain to us what is this geometry all about and at least my attitude was good than at the first of this topic sir.

Moderator: Ok, thank you very much. **Mp**, do you have anything to contribute?

Mp: Yes, I don't really blame the teacher. I blame myself for not concentrating at first because I knew that I didn't understand geometry very well but I didn't pay attention to that. So, yes, I know that I don't understand geometry very well. So, my teacher didn't do anything but mostly when he teaches geometry, he changes his attitude but when he teaches other topics like trigonometry I understand very well and—yes.

Moderator: That's interesting! Thank you very much for your contributions to these questions. Maybe before we end the discussion, is there anything else that you wanted to add which you did not have a chance to bring forward earlier on with regards to the teaching of geometry at your school?

Mp: Sir, I think that we must have enough time to focus on geometry since geometry is a problem to many students. I think that we are not the only ones that have a problem with geometry. Almost half a school we have a problem with geometry so I think they must focus a lot maybe we can have maybe studies after school to focus on geometry because geometry has more marks.

Moderator: Yah it's true [*cross talk*].

Mp: He must make sure that maybe at least when he knocks off— maybe we understood something and he is sure that we did understand that — maybe giving us a task *nyana* (small task) or a test or something just to prove that we did understand.

Moderator: Yes, thank you very much for your contribution. Guys do you have anything else to add to our discussion which you think we did not talk about. You have nothing to add. Ok, thank you very much for your time and sharing your opinions and emotions with me and then I think your feedback is going to be valuable in my research and I am happy that our discussion was very successful and interesting. If there is anything else that you are unhappy with or wish to complain about you can contact me at my number. And then our time I think is up and we have come to the end of our discussion and once again thank you very much for your participation. I wish you all a safe journey on your way back.

N 4: CONTROL GROUP DISCUSSION TRANSCRIPT – FG C2

Moderator: Ok, my first question is: **When you think of Euclidean geometry, what comes to your mind?** And please when you want to say something just indicate by raising your hand so that we don't have two people speaking at the same time. **Th**!

Th: Uhm, I'm thinking, uhm — theorems!

Moderator: Ok, yah just say whatever you think.

Th: I'm thinking about theorems. Yah, I'm thinking about theorems.

Moderator: Thank you. **Te**!

Te: I am thinking about shapes.

Moderator: Ok, thank you. Anybody else? **N**!

N: I think of theorems which shall be proven either wrong or right.

Moderator: Ok, thank you. Anybody else who has anything to say? **Ho**!

Ho: [*Inaudible segment, 22 seconds of interview missing, 01:10 — 01:32*]

Moderator: Ok, I am encouraging you to speak a bit louder so that your voice can be audibly recorded. Now, let's move on to our next question. My next question is like this, it says: **What do you think about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?** Yes, **C**!

Co: It was just difficult.

Moderator: Ok, **N**!

N: Eh, I think, oh, I know, the teacher was a good teacher, uhm — and if a learner, one or two, he or she doesn't understand, uhm, it was a bit difficult for the learner to go and approach the teacher. Eh — *O fela pelo nyana* (S/he is a bit impatient)

Moderator: Ok, thank you very much for your contribution. Anybody else with anything to say? **Th**!

Th: Eh, Sometimes the teachers were, eh, when it comes to teaching all the — eh, *go teacher'a di chapter ka moka* (teaching all the chapters), they didn't do that. They skipped others.

Moderator: Ok, anybody else with anything to add? **Ho**!

Ho: Uhm, I agree that eh — the two speakers were right that eh, Euclidean geometry is hard, yeah, it's hard. They skipped other chapters of Euclidean geometry [*Inaudible segment, 16 seconds of interview missing, 04:14 — 04:30*]

Moderator: Ok are you able to give examples of the information that you think was skipped by the teacher?

Ho: Eh, examples?

Moderator: I mean, can you just elaborate when you say the teacher was skipping some of the things in Euclidean geometry? Specify what kind of concepts did the teacher skip?

Ho: Eh, just like the last theorems like theorem 6 and 7, sometimes in question papers they set them but when we were doing geometry, we didn't do them.

Moderator: Ok, **Th**!

Th: Uhm, eh, they did not teach us riders at all! They just teach us how the theorems (are) proved — proven but riders they didn't even touch them.

Moderator: Ok, thank you very much for your contribution. **N**!

N: I remember there was this time sir was going to — where was he going? Somewhere else then he asked me to teach theorem 3,4 and 5. So, he never came back to those theorems and show them to the whole class. I just took a book and then I write what's on the book and then I sat down.

Moderator: Ok, and when the teacher came back, the teacher did not explain [*cross talk*]

N: No! He said I wrote the theorems on the board so everyone should go and study them.

Moderator: Ok, thank you so much for your contributions to that question, let's move on to our next question. The next question says: **How do you feel about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom?** Yes, **Te**!

Te: Uhm, Sir, I don't feel good because I don't know some of the theorems and there is a need whereby I have to know especially riders and riders have a lot of marks whereby when I can understand all of the theorems then I will be able to get the marks that are there.

Moderator: Yes, **Ho**! Do you have anything to say?

Ho: Yes, I feel good because I write my notes at home. When I come to school on Monday, I get to understand [*Inaudible segment, 2 seconds of interview missing, 07:35—07:37*]

Moderator: Ok, **Co**!

Co: I feel bad because some of us we don't write notes *ko gae* (at home). We just copy what the teachers teach us then we can go home and study.

Moderator: Ok, **L**!

L: I feel bad because there are some theorems neh, uhm — Sir, *nka adder'a Sepedi nyana*? (can I add a bit of Sepedi?)

Moderator: Ok. You are allowed to do that.

L: Like you should know some theorems neh, in order to do *tsela tsa* (those ones of) riders. Yah, there are some I don't know. So, and then *ka moka dilo tsela ko nale tse dingwe a ke di fihleleli* (Everything put together, there are certain things I cannot reach).

Moderator: Ok, **Th!**

Th: I feel bad because they did not teach us riders. Many question papers come with lots of riders. I can't write something that I don't know that's why we lose marks at geometry.

Moderator: Yes, I agree with you**. N**!

N: I also feel bad because eh, some of us learners we prefer *gore* (that) eh, teachers should teach us and then that's where we get to understand the concepts and then when going home, we just revise and practise that.

Moderator: Ok, **L**, you want to add something?

L: Yah, eh [*Inaudible segment*] *go nale, nka reng* syllabus, so they are trying *gore ba tsamaiše* syllabus. So, there are some things they need in order *gore ba phuše* syllabus. *Go swanetše gore* syllabus *ya*, I mean chapter *ya di* theorems *ebe le nako e ntši* because for some of us we find it difficult *gore re understande* because *ba phuša* syllabus.
(Yah, eh [Inaudible segment] there is, what can I say, a syllabus, so they are trying to cover the syllabus. So, there are some things they need in order to cover the syllabus. There is need for the syllabus, I mean chapter of theorems to be given a lot of time because for some of us we find it difficult to understand because they are trying to cover the syllabus)

Moderator: OK, thank you very much for your contribution to that question. Let's move on to the next question. The next question says: **What do you like or dislike about the way Euclidean geometry and geometric proofs were taught in your mathematics classroom? Th**!

Th: I didn't like the way they taught us because of they are fast and didn't think that we have slow learners. They can't catch all the things that the teacher says because of fast [*Inaudible segment, 4 seconds of interview missing, 10:36 — 10:40*] so that they want to finish the chapter.

Moderator: Ok, thank you very much for your contribution. Anybody else with anything to add? **N**!

N: I feel good because uhm, Sir *a re rutang,* like *ge a ruta, wa kwagala, wa kwisisega and then nna, ge ke sa e kwisisi botse ke taba ya gore o busy o kitimisa di chapter*
(Sir who teaches us, like when he teaches, he is understandable and then what I don't understand is why he is busy chasing after the chapters).

Moderator: Ok, **Ho**!

Ho: I don't like it because they summarize those chapters and when they summarize those chapters some of the things of Euclidean geometry [inaudible segment] they decrease our marks. When we go and say you did not teach us this, they say we must go and study and then we can't go and study for ourselves, it's them who are supposed to teach us those things.

Moderator: Yes, that's a very important point. Anything else that you want to add? Te!

Te: Euclidean geometry I like it because *e nale* a lot of marks *tše eleng gore di ka go thuša gore ophase maths* and *le gona gape*, I don't like it *ge mathitšhere ba sa re direle gore re be* good *ka yona* because *ge re kaba* good *ka yona kemo retlo kgona go phasa* maths *botse* because *etshwere di maraka tše dintši ka gare ga* question paper.
(Euclidean geometry I like it because it has a lot of marks that can help you to pass maths. And also, I don't like it when teachers do not make us to be good at it because if we can be good at it then we will be able to pass maths well because it has a lot of marks in the question paper)

Moderator: Ok, thank you very much for your contribution. Do you have anything else to add? Ok, let's move on to our next question. The next question says: **Can you describe your attitude towards Euclidean geometry and geometric proofs? N**!

N: I could say that I have a bad attitude towards Euclidean geometry because I only understand few theorems: theorem 1, 2, maybe 3, but the rest — ai! [Laughter] [*cross talk*] [*Inaudible segment, 12 seconds of interview missing, 13:17 — 13:29*]

Moderator: Ok, **L**! Do you want to say something?

L: Yah, Le nna I have a bad attitude because when I try it at home, I find it very difficult, that I am trying to concentrate, like — I give up! Yah (laughs)

Moderator: Ok, **Ho**!

Ho: Nna, I have a good attitude because now I understand geometry. Much of it I understand so I have a good attitude.

Moderator: Ok, Th!

Th: I have both good and bad. Let me start with good. I know all the theorems and then I can't prove riders, yah.

Moderator: **Co**!

Co: I have a bad attitude because I got some theorems but to prove that theorem 6 and 7 and riders, I don't get it because is difficult.

Moderator: **Th**!

Te: *Nna* sir, attitude *yaka e* bad*. Ebolaiša pelo ge ke lebeletse mo* question paper ka o re *ke bona di maraka tše dintši mara eish! Ake kgone go di fihlelela ka ore akena* knowledge *yela*

(Sir, my attitude is bad. It's heart breaking, when I look at the question paper, I see a lot of marks, but eish! I cannot reach them because I don't have the required knowledge)

Moderator: That's interesting! **L** do you want to add some more?

L: Yes, I do have a bad attitude neh but I really love Euclidean geometry it's because like those things you can feel like you see them. Like the answers are on the question paper but you can't prove them — and then you lose marks.

Moderator: Yah, I understand you. Alright, thank you very much for your contributions to that question. Let's move on to our last question and the last question says: **What did the teacher do that you think contributed to the attitude you have towards Euclidean geometry and geometric proofs?**

N: As I have indicated *gore o fela pelo,* so *go boima, bothata gore o mmobotse gore* sir *ke kgopela o nthuse ka* this and that. *Otla go botsa gore tsamaya kantle, otla bona gore otšwa jwang.*

Moderator: Uhm, that's interesting! **Th**!

Th: *Uhm, eish! go nale nako ye ngwe akere … re kgona gore a ruta a le busy, ge re molata re mohlalusetša problem gore sir kamo a re kwešiši a re botse gore yena o nale class ye aswanetše gore a e attende [Inaudible segment, 2 seconds of interview missing, 16:35 — 16:37].*

Moderator: Ok, I get your point. **Co**!

Co: *Eh, go boima because ge re mmotsa gore a re kwešiši then o re botsa gore oswanetše afetše syllabus [Inaudible segment] gore ye re tlo ngwala re seke ra mmotsa gore ase re dire eng nyana, eng nyana because o re yena aka se stucke mo Euclidean geometry forever. O swanetse a fetele go di chapter tse dingwe.*

Moderator: I get your point! Anything else that you want to add? **Ho**!

Ho: *Eh, go boima* because *ge re nyaka* help *mo yena go nale nako ye ngwe o re fa* attitude *yela ya gore o re rutile yona ka classeng* [Inaudible segment, 17:31 — 17:39]. *O re felela di pelo.*

Moderator: Ok, thank you very much for your contributions to our last question and then is there anything else that maybe you need to add to our discussion which I have not asked about? **Ho**!

Ho: *Ke nale* suggestion, *bona ba go romela di* schedules *ba swanetše ba fe* Euclidean geometry *nako e* enough *gore re kgone go di tshwara ka moka* [Inaudible segment] *ga ba sa lebelela gore nako ye bare file ke e nyane gore bare rute* geometry *ka yona.*

Moderator: That's an interesting contribution! **Th** your hand was up!

Th: *Nna ke nagana gore* eh sometimes *le dikolo* they must eh maybe increase mathematics time. Just like geometry needs more time*, ga e nyake nako nyana*.

Moderator: Ok, I get your point. Te!

Te: *Lenna* Sir *ke kwana le bona ka* gore geometry *ge re sa efe nako,* failing rate *ya* Maths *ka classeng e ba yegolo, e ba entši.*

Moderator: That's a very important contribution. Anything else that you want to add from the girls? **L**!

L: *Lenna* I want to repeat *taba ye kgale a e bolela*. Like *ba swanetše ba e fe nako* Euclidean geometry because *redira* theorem 1, then tomorrow *re dira* theorem 2, *ke nako e nyane*. *A kere re swanetše re be le nako ya gore re kgone* go practiser re *bone gore* this theorem *re a e understander* so we can go to another one. So *bona ba re bea* pressure. *Re ka se dire dilo tše pedi ka nako e tee.*

Moderator: Ok, that's a very important contribution. **N**!

N: *Nna ke suggester'a gore, tše tša goswana le* maths *le* physics *a di swanela go ba* after break. Because after break, *re boa re khutshe ba bangwe ba robala*. So *ge ba ka di bea mathomong tše tša go swana le geometry* — let's come to the problem! Because *re tlabe re le* fresh. [Inaudible segment, 8 seconds of interview missing, 20:30—20:38] *Ba feteša di* chapter *banna! Di* theorem *tše di re bolaile banna!*

Moderator: Ok, **L**!

L: Like they should teach us slow because *re nale* some learners *ba* slow. There are some of those learners *ba leka go ditshwara pele* then *ke mo ba tlo kgona go di kwešiša*. So, uhm, the teachers *ge re fihla go* geometry, *ba be* patient *le rena ba seke ba re 'Ai mara ba e tseba monna!' Ba e kwešiša?*

Moderator: Yes, I understand you. **Ho**!

Ho: *Ba swanetše ba tlogele mantšu a bona ka mo classeng a go discourager bana ba sekolo. Ba rata go discourager bana ba sekolo.* [Inaudible segment] *ba re botsa gore nna nkase phase. Ge ba re botša gore nna nkase phase, nna nkase tle sekolong. Go no tshwana!*

Moderator: I understand you. **N**!

N: I also want to add onto **L**'s point. Even though the teachers *ba sa re rute* slow but then if *wena o okwa gore ase o kgotsofale* that's when *o ka emelela wa ya go teacher'a wa mmotsa* and *a go fe nako ya gore o kwešiše* because *go ngwala rena* at the end of the day.

Moderator: Ok, **Th**!

Th: *Nna, ke ema le **H**. Nto ye ba swanetše ba e dire ke go re ba seke bare ge o fihla go meneer o mmotšisa gore maneer mo ake kwešiši ano fihla o wena a re tsamaya o nyaka lentšu la gore eng end, otla kgona o boa go nna. Otla bona gore eh! Ya re bolaya nto yeo!*

Moderator: I understand you. Ke a go kwešiša. L!

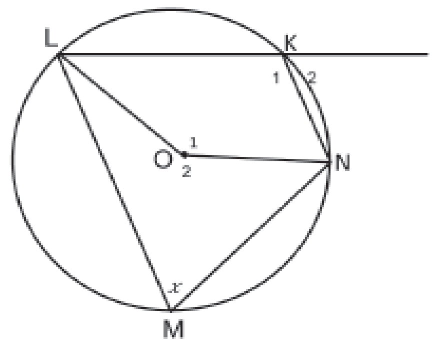
L: **A**nd *bastope nto ya bona ya gore* maybe if you want to ask*, obotšiše* question neh, *ba re o e dirile* last year and *nto yela ya gore o e dirile ka tee fela* and we don't understand it. We need more knowledge to understand but they say you did it **L**.

Moderator: I get your point. **Co**!

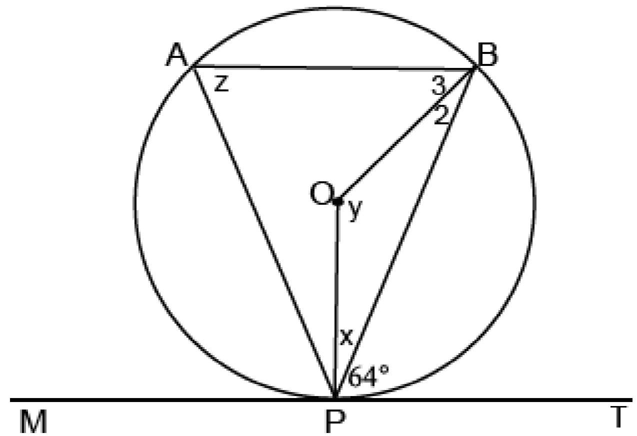
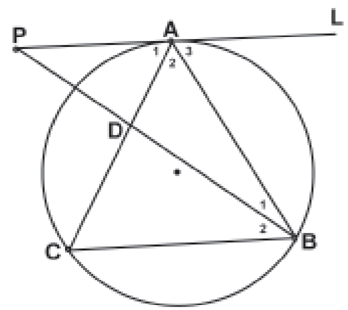
Co: *Mathitshere ba re flopisa* because most of the time *ge ngwana wa sekolo a sa kwešiši ko klaseng*, like if *ge a re ba repeater* and *then ge a botšisetsa gore o e dirile* last year, what's the *use ya gore a botšiše gape* because *nto yela o e dirile* last year *ba sa diri selo*? Like *ba rebala gore re tshwara* slow, that's *why re botšiša* but then *mathitshere ale a re felela pelo.*

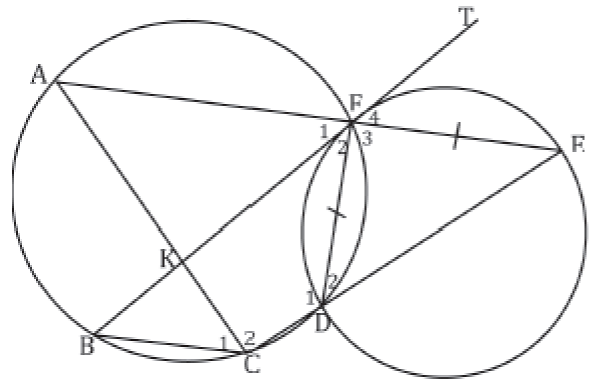
Moderator: Guys**, t**hank you very much for your contributions. Is there anyone else who wants to add something?

APPENDIX O: MATHEMATICS GRADE 11 CURRICULUM AND ASSESSMENT POLICY STATEMENT

<table>
<tr><th colspan="4">GRADE 11: TERM 3</th></tr>
<tr><th>No. of weeks</th><th>Topic</th><th>Curriculum Statement</th><th>Clarification</th></tr>
<tr><td>1</td><td>Measurement</td><td>1. Revise the Grade 10 work.</td><td></td></tr>
<tr><td>3</td><td>Euclidean Geometry</td><td>Accept results established in earlier grades as axioms and also that a tangent to a circle is perpendicular to the radius, drawn to the point of contact.
Then investigate and prove the theorems of the geometry of circles:
• The line drawn from the centre of a circle perpendicular to a chord bisects the chord;
• The perpendicular bisector of a chord passes through the centre of the circle;
• The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre);
• Angles subtended by a chord of the circle, on the same side of the chord, are equal;
• The opposite angles of a cyclic quadrilateral are supplementary;
• Two tangents drawn to a circle from the same point outside the circle are equal in length;
• The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.
Use the above theorems and their converses, where they exist, to solve riders.</td><td>Comments:
Proofs of theorems can be asked in examinations, but their converses (wherever they hold) cannot be asked.
Example:
1. AB and CD are two chords of a circle with centre O. M is on AB and N is on CD such that OM ⊥ AB and ON ⊥ CD. Also, AB = 50mm, OM = 40mm and ON = 20mm. Determine the radius of the circle and the length of CD. (C)
2. O is the centre of the circle below and $\hat{O}_1 = 2x$.


2.1. Determine $\hat{O}_2$ and $\hat{M}$ in terms of x. (R)
2.2. Determine $\hat{K}_1$ and $\hat{K}_2$ in terms of x. (R)
2.3. Determine $\hat{K}_1 + \hat{M}$. What do you notice? (R)
2.4. Write down your observation regarding the measures of $\hat{K}_2$ and $\hat{M}$. (R)
p. 34</td></tr>
</table>

GRADE 11: TERM 3			
No. of weeks	**Topic**	**Curriculum Statement**	**Clarification**
			3. *O* is the centre of the circle above and MPT is a tangent. Also, $OP \perp MT$. Determine, with reasons, x, y and z. (C)  4. Given: $AB = AC$, $AP \parallel BC$ and $\hat{A}_2 = \hat{B}_2$. .  p. 35 Prove that: 4.1 *PAL* is a tangent to circle *ABC*; (P) 4.2 *AB* is a tangent to circle *ADP*. (P)

GRADE 11: TERM 3			
No. of weeks	Topic	Curriculum Statement	Clarification
			5. In the accompanying figure, two circles intersect at F and D.  p. 36 *BFT* is a tangent to the smaller circle at *F*. Straight line *AFE* is drawn such that *FD* = *FE*. *CDE* is a straight line and chord *AC* and *BF* cut at K. Prove that: *5.1 BT // CE* (C) *5.2 BCEF* is a parallelogram (P) *5.3 AC = BF* (P)

(Department of Basic Education, 2011, p. 34-36)

APPENDIX P: 2016 GRADE 11 MATHEMATICS WORK SCHEDULE

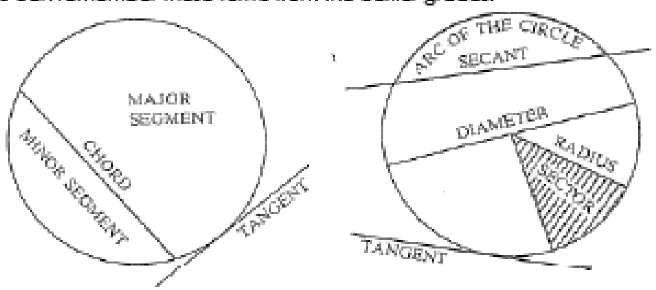
TERM THREE			
TIME/DATE	**MAIN TOPIC**	**CONTENT**	**ASSESSMENT**
Week 1 – 4 **18 July – 12 Aug 2016**	**Euclidean Geometry**	Accept results established in earlier grades as axioms and also that a tangent to a circle is perpendicular to the radius, drawn to the point of contact. Then investigate and prove the theorems of the geometry of circles: • The line drawn from the centre of a circle perpendicular to a chord bisects the chord; • The perpendicular bisector of a chord passes through the centre of the circle; • The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre); • Angles subtended by a chord of the circle, on the same side of the chord, are equal; • The opposite angles of a cyclic quadrilateral are supplementary; • Two tangents drawn to a circle from the same point outside the circle are equal in length; • The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment. Use the above theorems and their converses, where they exist, to solve riders.	Exercises Short test

APPENDIX Q: PROFILE OF EUCLIDEAN GEOMETRY LESSONS TAUGHT IN THE CONTROL SCHOOLS

GRADE	11	SUBJECT	Mathematics	WEEK	2	TOPIC	Line from Centre of Circle: Time 50 mins	Lesson	1

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	

LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • Describing the parts of a circle. • The line segment drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

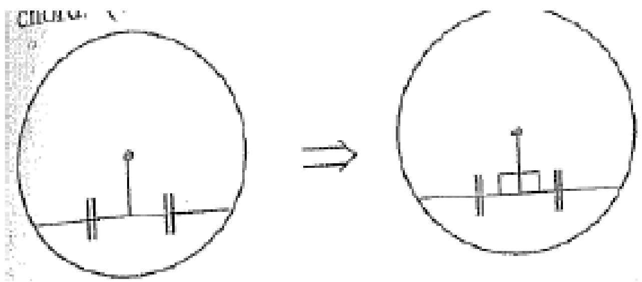
TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. **2. Lesson Development** **2.1. <u>Introduction</u>** • Questions for a baseline assessment: o *Ask the learners questions based on congruency of triangles and Pythagoras' Theorem.* • As the lesson progresses: o *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.* **2.2. <u>Main Body (Lesson Presentation)</u>** o Before we begin to look at the theorems you must know the following for Grade 11 Euclidian Geometry. Let us familiarize ourselves with the terminology used in circles. o See if you can remember these terms from the earlier grades:  o The geometry we shall deal with in Grade 11 involves circle theorems and their proofs. o We shall state a theorem and then do the proof thereof.	1. **<u>Baseline</u>** 2. **<u>Informal Assessment</u>** 3. **<u>Homework Exercise</u>** 4. **<u>Answers to Activities</u>**	Baseline: Lesson & Activities: 45 min Corrections:	• **Lesson references:** o Classroom Mathematics, Standard 9; pp. 309 – 312. o Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160. • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers); • **Other useful text references:** o Teaching Mathematics Std. 9; pp. 111 – 113. o New Mathematics; Std. 9; pp. 382 – 389.

- What follows thereafter will be examples of how the particular theorem is applied in the solution of riders.
- You will then be expected to do exercises, which will graduate from numerical type to those requiring proofs.
- **To solve the riders in these exercises, a number of theorems will, most probably, have to be used in an integrated way.**
- The first three theorems we shall deal with can be referred to as "chord and centre of circle theorems".
- These theorems have three interrelated factors, namely:

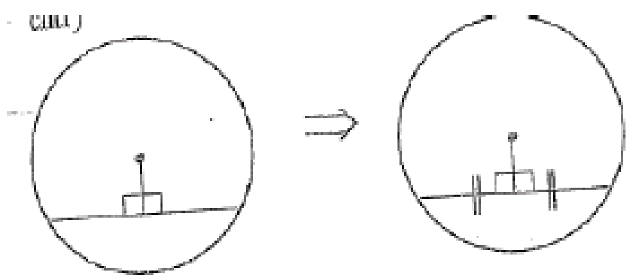
1. The **centre** of the circle;
2. The **midpoint** of the chord; and
3. The **perpendicular** drawn from the centre.

- Given any two, the third factor will be true.
- Let us illustrate the statement above using diagrams...

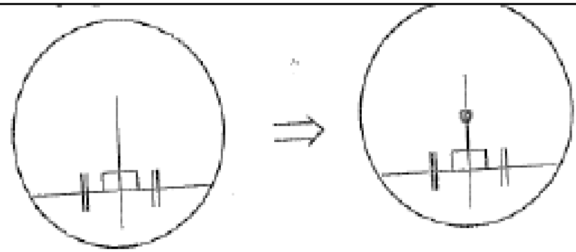
1. The line from the **centre** of a circle to the **midpoint** of the chord is **perpendicular** to the chord.



2. The line from the **centre perpendicular** to the chord will **bisect** the chord.



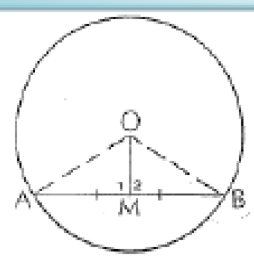
3. The **perpendicular bisector** of the chord passes through the **centre** of the circle.



- Now, let us take a look at the first theorem and how it can be proved.

THEOREM:

The line segment joining the centre of a circle to the midpoint of a chord is perpendicular to the chord. (Line from centre)



Given:	Any circle with centre O and chord AB. OM is drawn with M as the midpoint of AB.	
Required To Prove:	$OM \perp AB$	
Construction:	Join OA and OB	
Proof:	**Statement:**	**Reason:**
	In ΔOMA and ΔOMB	
	$OA = OB$	Radii
	$OM = OM$	Common side
	$AM = MB$	Given
	$\therefore \Delta OMA \equiv \Delta OMB$	S, S, S

$\therefore \hat{M}_1 = \hat{M}_2$ Corresponding angles of congruent triangles $but\ \hat{M}_1 + \hat{M}_2 = 180°$ AMB is a straight line $\therefore \hat{M}_1 = \hat{M}_2 = 90°$ $\therefore OM \perp AB$ **HINTS ON SOLVING RIDERS!!!** 1. Read the question ONE statement at a time and INTERPRET. For example, given AB//CD $\rightarrow$ find any equal alternate or corresponding angles. 2. Fill in the interpreted facts on the rider. 3. Look for any other facts that you might have missed, e.g. equal subtended angles, exterior angle of cyclic quadrilaterals, etc. 4. Make sure that the theorem that is given has been applied. 5. Then take a look at what you are required to prove and interpret it, e.g. prove that ABCD is a cyclic quadrilateral $\rightarrow$ prove 2 subtended angles equal; OR prove that the exterior angle is equal to the opposite interior angle; OR prove opposite angles supplementary. 6. The answer should be there, if not, go back to step 1 and see of you missed any fact. 7. If all else fails, write down all the interpreted facts and continue with the question assuming the first fact was proved. 2.3. Conclusion • Revise key elements of the lesson. • Make the learners aware that the proof of the second theorem will be dealt with in the next lesson. Only then will we also begin to do exercises based on these theorems. • Provide a chalkboard summary of the key concepts of the lesson.		Conclusion: ±5 min	

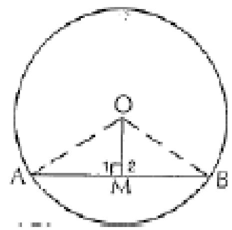
GRADE	11	SUBJECT	Mathematics	TERM 3	WEEK 2	TOPIC	Perpendicular from Centre Time: 55 min	Lesson	2

LESSON SUMMARY FOR: DATE STARTED:		**DATE COMPLETED:**	
LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • The line segment drawn from the centre of a circle perpendicular to a chord bisects the chord.		

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. **2. Lesson Development** **2.1. Introduction** • Questions for a baseline assessment: o *Ask the learners questions based on congruency of triangles and Pythagoras' Theorem.* • As the lesson progresses: o *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.* **2.2. Main Body (Lesson Presentation)** o In the previous lesson, we introduced you to the terminology you are likely to encounter when dealing with circle theorems and riders. o We also proved the theorem (line through centre and midpoint of chord). o This lesson looks at another theorem on chords in the circle. o When you have done the proof of this theorem, you should be able to start solving riders which involve the two theorems. o **Note that the 2nd theorem is often referred to as the converse of the first theorem.**	1. **Baseline**	Baseline: Lesson & Activities: 35 min Corrections: 15 min	• **Lesson references:** o Classroom Mathematics, Standard 9; pp. 309 – 312. o Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160. o Maths Tutor for the Matric Examinations; Grades 11 & 12; Higher Grade; Paper 2; pp. 65 – 98. • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers); o Scientific Calculators.

THEOREM:

The perpendicular drawn from the centre of a circle to a chord bisects the chord. (Perp. from centre to chord)



Given: Any circle with centre O and $OM \perp AB$.

Required To Prove: $AM = MB$

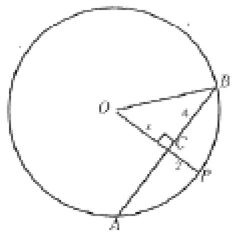
Construction: Join OA and OB

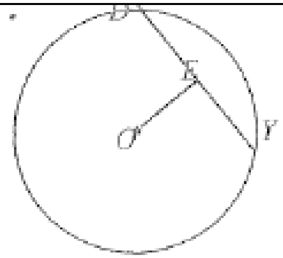
Proof:	**Statement:**	**Reason:**
	In ΔOMA and ΔOMB	
	$OA = OB$	Radii
	$OM = OM$	Common side
	$\therefore \hat{M}_1 = \hat{M}_2 = 90°$	Given that $OM \perp AB$
	$\therefore \Delta OMA \equiv \Delta OMB$	90°, H, S
	$AM = MB$	Corresponding sides of congruent triangles

- Now, let's take a look at how the two theorems we have learned are applied in solving riders.

NOTE TO THE TEACHER: From the textbook also do the formal proof of: The perpendicular bisector of a chord passes through the centre of the circle

- **Other useful text references:**
  - Teaching Mathematics Std. 9; pp. 111 – 113.
  - New Mathematics; Std. 9; pp. 382 – 389.

Example 1 Chord AB of circle O measures 8cm and radius $OP \perp AB$ and meets AB at C. If CP = 2cm, calculate the diameter of the circle.  **Solution:** $AC = BC = 4cm$...............$[OC \perp AB]$ $OP = (x+2)cm = OB$.......$[radii]$ $In\ \Delta OCB:$ $OB^2 = OC^2 + BC^2$...........$[Pythagoras]$ $\therefore (x+2)^2 = x^2 + 4^2$ $\therefore x^2 + 4x + 4 = x^2 + 16$ $\therefore 4x = 12$ $\therefore x = 3$ $\therefore radius = (3+2)cm = 5cm$ $\Rightarrow diameter = 2(5cm) = 10cm$ **Example 2** In the figure below, circle centre O has a chord DY of length 18cm and a radius of length 12cm. Calculate the perpendicular distance from O to DY, correct to two decimal places.			



Solution:

$In\ \Delta ODE:$

$DE = \frac{1}{2}DY$.................$[perp.\ from\ centre\ ;\ OE \perp DY]$

$\therefore DE = 9\ cm$

$OD^2 = OE^2 + DE^2$.....$[pythagoras]$

$\therefore OE^2 = 12^2 - 9^2 = 63$

$\therefore OE = \sqrt{63} = 7{,}94cm$

ACTIVITIES:

- Do a selection of these problems with your learners in class (informal assessment);
  - The rest of the activities could be given as homework.

1. Refer to the sketch below:

2. <u>Informal Assessment</u>

3. <u>Homework Exercise</u>

Please note that due to space constraints, all exercises, informal and homework are provided in the 1st column.

4. <u>Answers to Activities</u>

1. $r = 50\ mm\ ;\ CD = 80\ mm$

2.1. $r\sqrt{3}$

2.2. $0{,}27m$

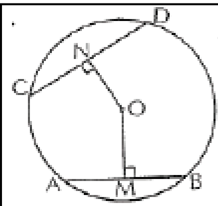
2.3. $17{,}84m$

2.4. $4{,}25cm$

3.1. $10\ mm$

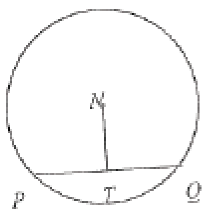
3.2. $70mm$

O is the centre. AB = 60mm; OM = 40mm. Calculate the radius of the circle and the length of CD if ON = 30mm.

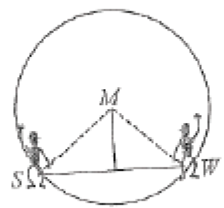


2. All answers correct to two decimal places where necessary:

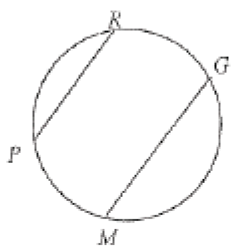
2.1. Find the length of chord PQ in terms of r if $NT \perp PQ$ and NT is half the radius in a circle of radius r units.



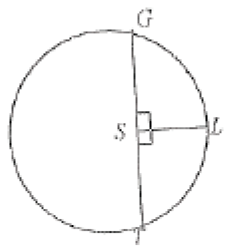
2.2. A swing of length 2m is suspended by two wires MS and MW of length 2m each. How far must the centre of the swing be above the ground to ensure that the swing does not hit the ground?

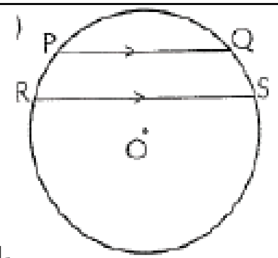
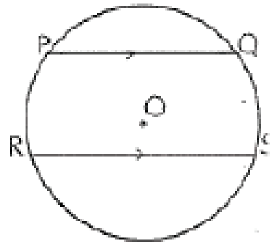


2.3. PR//MG lie in a circle of radius 10m. If PR = 8cm and MG = 10m, calculate how far apart the lines are.



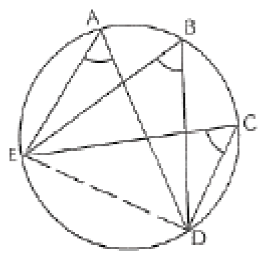
2.4. If SL = 2cm and chord GT = 7,5cm, find the length of the radius.



3. Find the distance between the chords in both cases. O is the centre in both figures. PQ = 60mm; RS = 80mm and radius = 50mm.  3.1.  3.2. **2.3. Conclusion** • Revise key elements of the lesson. • Provide a chalkboard summary of the key concepts of the lesson.		Conclusion: ±5 min	

GRADE	11	SUBJECT	Mathematics	TERM 3	WEEK 2	TOPIC	Angle at Centre Time: 55 min	Lesson	3

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	
LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • The angle which an arc of a circle subtends at the centre of the circle is twice the angle it subtends at any point on the circle.		

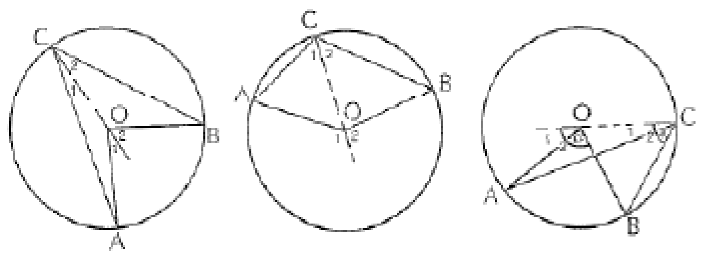
TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. **2. Lesson Development** **2.1. Introduction** • Make the learners take note that there are three facts regarding angles subtended in a circle that they should know, namely: o *The angle subtended at the centre of the circle is twice the angle subtended by the same arc or chord in the same segment;* o *Angles subtended by the same chord in the same segment are equal; and* o *Equal chords subtend equal angles at the centre or the circumference.* • You might also want to illustrate what is meant by the word "subtends". In the diagram below, chord or arc ED subtends equal angles A, B and C at the circumference.  • As the lesson progresses: o *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.*	1. **Baseline**	Baseline: Lesson & Activities: 35 min Corrections: 15 min	• **Lesson references:** o Classroom Mathematics, Standard 9; pp. 312 – 313. o Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160. • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers); o Scientific Calculators. • **Other useful text references:** o Teaching Mathematics Std. 9; pp. 114 – 115.

2.2. Main Body (Lesson Presentation)

- In this lesson we will prove the first fact regarding angles subtended in a circle as stated above.
- We will then move on to looking at how the theorem is used in solving riders.

THEOREM:

> **The angle which an arc of a circle subtends at the centre of the circle is twice the angle it subtends at any point on the circle. (Angle at centre)**



(i) (ii) (iii)

Given: Any circle with centre O and arc AB subtending $A\hat{O}B$ at the centre and $A\hat{C}B$ at the circle.

Required To Prove: $A\hat{O}B = 2A\hat{C}B$

Construction: Draw CO and produce (extend)

Proof:	Statement:	Reason:
	In ΔCAO	
	$\hat{O}_1 = \hat{C}_1 + \hat{A}$	Exterior angle of a triangle = sum of interior opposite angles
	$but\ \hat{C}_1 = \hat{A}$	OC = OA ; radii
	$\therefore \hat{O}_1 = 2\hat{C}_1$	
	Similarly, in ΔCBO	

- New Mathematics; Std. 9; pp. 390 – 395.

$\hat{O}_2 = 2\hat{C}_2$

$Reference$: $Figures\ (i)\ and\ (ii)$

$\therefore \hat{O}_1 + \hat{O}_2 = 2\hat{C}_1 + 2\hat{C}_2$

$\therefore \hat{O}_1 + \hat{O}_2 = 2(\hat{C}_1 + \hat{C}_2)$

$\therefore A\hat{O}B = 2A\hat{C}B$

$Reference$: $Figure\ (iii)$

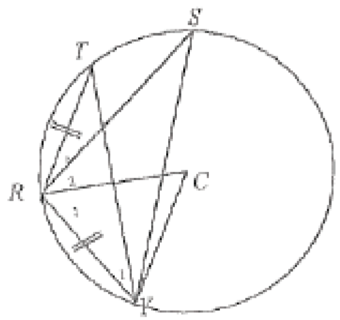
$\therefore \hat{O}_2 - \hat{O}_1 = 2\hat{C}_2 - 2\hat{C}_1 = 2(\hat{C}_2 - \hat{C}_1)$

$\therefore A\hat{O}B = 2A\hat{C}B$

- Now, let us take a look at a few examples of how the theorem is applied in solving riders...

Example 1

In the circle centre C, chord RT = RY. $\hat{Y}_1 = 23°$. Calculate, with reasons:

1.1. $\hat{T}$. 1.2. $\hat{C}$. 1.3. $\hat{R}_1$	

Solutions:

1.1. $\hat{T} = \hat{Y}_1$..........$[RT = RY]$ or $\angle's\ opp\ equal\ sides$

$\therefore \hat{T} = 23°$

1.2. $\hat{C} = 2\hat{T}$.........$[\angle\ at\ centre = 2 \times \angle\ at\,circ]$

$\therefore \hat{C} = 46°$

1.3. In ΔCRY : $CR = CY$........*[radii]*

$\therefore \hat{R}_1 = R\hat{Y}C$......................*[$\angle$s opp = sides]*

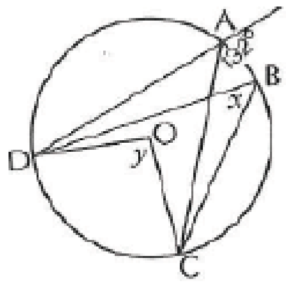
$R\hat{Y}C + \hat{R}_1 + \hat{C} = 180°$

$R\hat{Y}C + \hat{R}_1 = 180° - 46° = 134°$

$\therefore R\hat{Y}C = 67°$

Example 2

In the diagram below, find the values of x and y.



Solutions:

$D\hat{A}C = 180° - 132°$..........*[$\angle$s on a str. line]*

$\therefore D\hat{A}C = 48°$

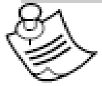
but $D\hat{A}C = \frac{1}{2}y$..................*[$\angle$ at centre]*

$\therefore y = 96°$

$x = \frac{1}{2}y$...............................*[$\angle$ at centre]*

$\therefore x = 48°$

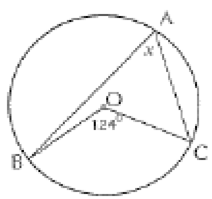
ACTIVITIES:



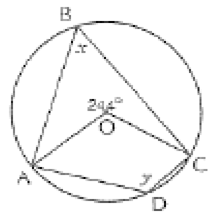
- **Do a selection of these problems with your learners in class (informal assessment);**
- **The rest of the activities could be given as homework.**

1. O is the centre of the circle in each case. Find the sizes of the required angles, giving reasons:

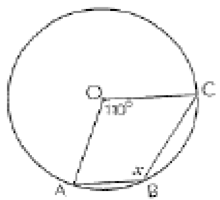
a)



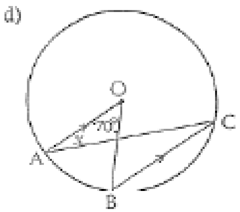
b)



c)

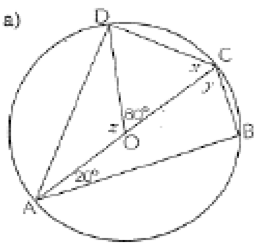


d)

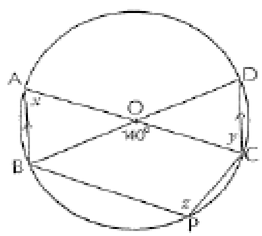


2. Find the values of x, y and z. (O is the centre)

a)



b)



2. <u>**Informal Assessment**</u>

3. <u>**Homework Exercise**</u>

Please note that due to space constraints, all exercises, informal and homework are provided in the 1st column.

4. <u>**Answers to Activities**</u>

1. Activity 1

a. $x = 62°$

b. $x = 58°$; $y = 122°$

c. $x = 125°$

d. $x = 35°$

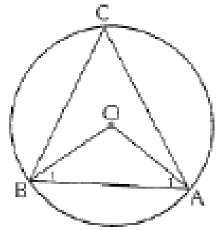
2. Activity 2

a. $x = 60°; y = 70°; z = 120°$

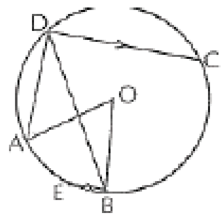
b. $x = 70°; y = 70°; z = 110°$

3. $\hat{C} = 50°$

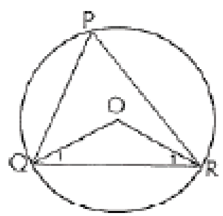
3. If $\hat{B}_1 = 40°$, find $\hat{C}$.



4. If $A\hat{O}B = D\hat{B}E = 60°$, and DC//EB, prove AC to be a diameter of the circle.



5. Prove that $\hat{Q}_1 + \hat{P} = 90°$.



Conclusion: ±5 min

2.3. Conclusion

- Revise key elements of the lesson.
- Provide a chalkboard summary of the key concepts of the lesson.

GRADE	11	SUBJECT	Mathematics	TERM 3	WEEK 2	TOPIC	Consolidation of Concepts Time: 55 min	Lesson	4

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	
LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • The line segment drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord. • The line segment drawn from the centre of a circle perpendicular to a chord bisects the chord. • The angle which an arc of a circle subtends at the centre of the circle is twice the angle it subtends at any point on the circle.		

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. **2. Lesson Development** **2.1. Introduction** • Inform the learners that they will need to complete the worksheet by the end of the period. • As the lesson progresses: o *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.* **2.2. Main Body (Lesson Presentation)** o This lesson consolidates the concepts learned thus far. o Complete the worksheet given below. **2.3. Conclusion** • Revise key elements of the lesson. • Provide a chalkboard summary of the key concepts of the lesson.	**1. Baseline** **2. Informal Assessment** **3. Homework Exercise** **4. Answers to Activities**	Baseline: Lesson & Activities: 35 min Corrections: 15 min Conclusion: ±5 min	• **Lesson references:** o Classroom Mathematics, Standard 9; pp. 309 – 319. o Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160. o DoE/Feb – March 2009; Mathematics/P3 • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers); o Worksheet. o Scientific Calculators.

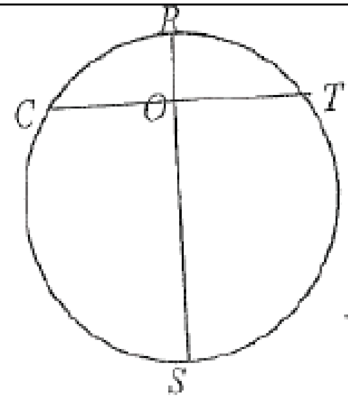
WORKSHEET:

QUESTION 1

A cross CRTS is to be placed in a circular window of radius 1,7m. The centre of the cross-bar, CT, must be 0,5m from the top of the cross. Let M be the centre of the circle.

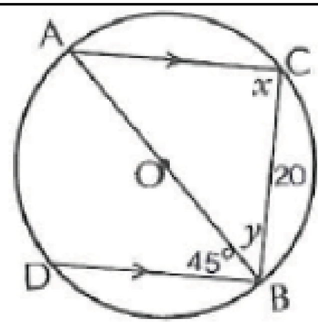
1.1. Calculate, with reasons, the length of CT, correct to one decimal place.

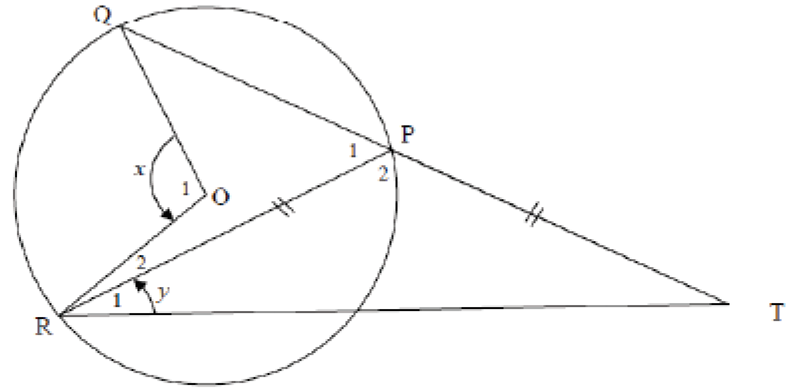
1.2. What would be the length of the metal required to construct the cross?



QUESTION 2

Find x, y and the lengths of AC and OB.



QUESTION 3	
In the figure alongside, O is the centre of the circle and PT = PR. Let $\hat{R} = y$ and $\hat{O}_1 = x$. 3.1. Express x in terms of y. 3.2. If TQ = TR and x = 120⁰, calculate the measure of: 3.2.1. y. 3.2.2. $\hat{R}_2$. (Hint: Draw QR)	

GRADE	11	SUBJECT	Mathematics	TERM 3	WEEK 3	TOPIC	Extension of Angle at Centre Time: 55 min	Lesson	1

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	
LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • The angle subtended at the circle by a diameter is a right angle. • Angles in the same segment of a circle are equal.		

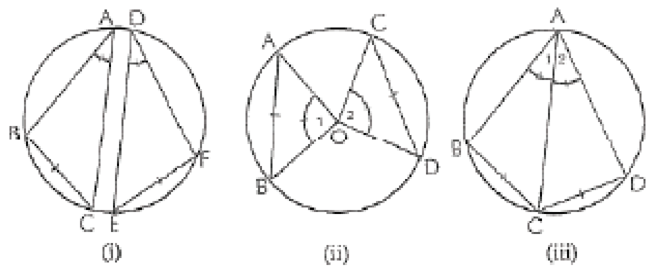
<table>
<tr><th>TEACHER ACTIVITIES</th><th>LEARNER ACTIVITIES</th><th>TIMING</th><th>RESOURCES NEEDED</th></tr>
<tr>
<td>

1. **Teaching Methods**
   - Telling; Explanation; Question and Answer; Illustration; etc.

2. **Lesson Development**

2.1. **Introduction**

- The theorem we proved in the previous lesson has some corollaries you need to know:
  - *Equal chords of a circle subtend equal angles at points on the circle;*
  - *Equal chords of a circle subtend equal angles at the centre of the circle; and*
  - *Chords of a circle are equal if they subtend equal angles at points on the circle.*



If $BC = EF$, then $\hat{A} = \hat{D}$ If $AB = CD$, then $\hat{O}_1 = \hat{O}_2$ If $\hat{A}_1 = \hat{A}_2$, then $BC = CD$

- As the lesson progresses:
  - *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.*

</td>
<td>

1. **Baseline**

</td>
<td>

Baseline:

Lesson & Activities: 35 min

Corrections: 15 min

</td>
<td>

- **Lesson references:**
  - Classroom Mathematics, Standard 9; pp. 313 – 319.
  - Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160.
- **Materials needed:**
  - Chalkboard (or any other appropriate form of board);
  - Chalks (or markers);
  - Scientific Calculators.
- **Other useful text references:**
  - Teaching Mathematics Std. 9; pp. 116 – 120.

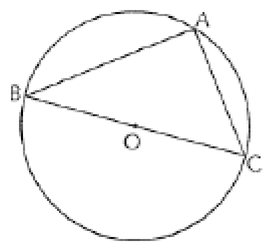
</td>
</tr>
</table>

2.2. Main Body (Lesson Presentation)

- In this lesson we will look at two more theorems which are basically extensions of the "angle at centre" theorem.

THEOREM:

The angle subtended at the circle by a diameter is a right angle. (Angle in semi-circle)



NOT AS FORMAL PROOF:

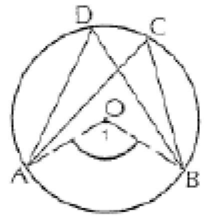
Given:	Circle with centre O and diameter BOC. A is any point on the circumference.	
Required To Prove:	$B\hat{A}C = 90°$	
Construction:	Draw AB and AC.	
Proof:	**Statement:**	**Reason:**
	$B\hat{O}C = 2B\hat{A}C$	Angle at centre = 2 x angle at circle
	$but\ B\hat{O}C = 180°$	Straight angle
	$\therefore B\hat{A}C = \frac{1}{2}(180°) = 90°$	

- The converse of the above theorem is useful in solving many riders. That is, if a chord of a circle subtends a right angle on the circumference of a circle then the chord is a diameter of the circle.
- Let us look at another theorem which is also an extension of the angle at centre theorem.

- New Mathematics; Std. 9; pp. 395 – 405.

THEOREM:

Angles in the same segment of a circle are equal. (Angles in same segment)

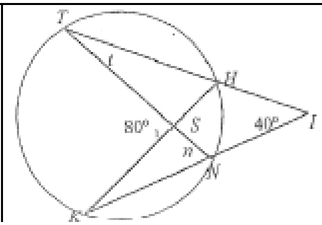


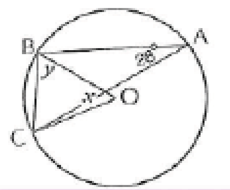
Given:	Circle with centre O, with arc AB subtending $A\hat{D}B$ and $A\hat{C}B$ in the same segment.	
Required To Prove:	$A\hat{D}B = A\hat{C}B$	
Construction:	Join AO and BO.	
Proof:	**Statement:**	**Reason:**
	$\hat{O}_1 = 2A\hat{D}B$	Angle at centre
	$\hat{O}_1 = 2A\hat{C}B$	Angle at centre
	$\therefore 2A\hat{D}B = 2A\hat{C}B$	Both $= \hat{O}_1$
	$\therefore A\hat{D}B = A\hat{C}B$	

Example 1

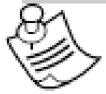
Given $\hat{S}_1 = 80°$ and $\hat{I} = 40°$.

Calculate the values of t and n.



Solutions: $In\ \Delta TNI : n = t + 40°$................[$ext.\angle\ of\ a\ \Delta$]...........(1) $In\ \Delta KNS : \hat{K} = t$........................[$\angle s\ in\ same\ segment$] $and\ \ n = 80° - t$..........................[$ext.\angle\ of\ a\ \Delta$]...........(2) $\therefore t + 40° = 80° - t$ $\therefore 2t = 40°$ $\therefore t = 20°$ $\Rightarrow n = 20° + 40° = 60°$ **Example 2** Find the numerical values of x and y.  **Solutions:** $x = 2\hat{A}$................[$\angle\ at\ centre$] $\therefore x = 2(28°) = 56°$ $In\ \Delta OBC : OB = OC$............[$radii$] $\therefore O\hat{B}C = O\hat{C}B = y$..............[$\angle s\ opp. = sides$] $\therefore y + y + x = 180°$..............[$\angle s\ of\ a\ \Delta$] $\therefore 2y = 180° - 56° = 124°$ $\therefore y = 62°$			

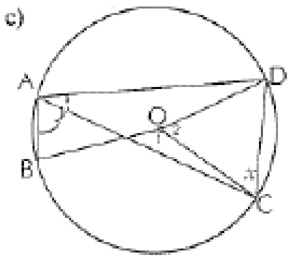
ACTIVITIES:



- Do a selection of these problems with your learners in class (informal assessment);
  - The rest of the activities could be given as homework.

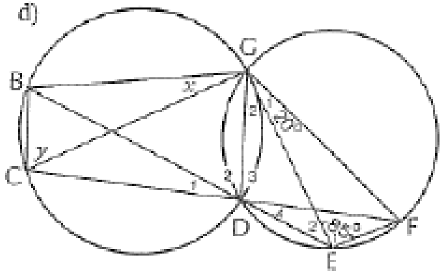
1. Find the values of x and y.

c)

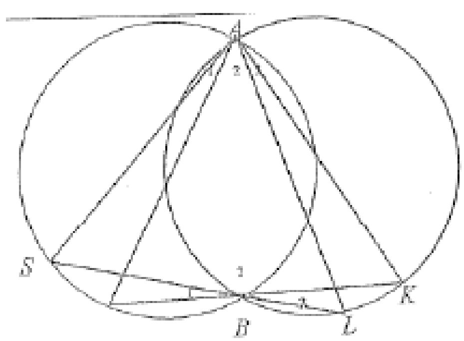


$B\hat{O}C = 130°$; $D\hat{O}C = 60°$

d)



2. Two circles intersect at A and B. PBK and SBL are straight lines. Prove that: $\hat{A}_1 = \hat{A}_3$.



2. <u>**Informal Assessment**</u>

3. <u>**Homework Exercise**</u>

Please note that due to space constraints, all exercises, informal and homework are provided in the 1st column.

4. <u>**Answers to Activities**</u>

1. Activity 1

c. $x = 60°$; $y = 95°$

d. $x = 20°$; $y = 75°$

3.1. $x = 30°; y = 60°; z = 60°$

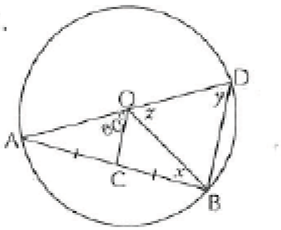
3.2. $x = 35°; y = 35°; z = 220°$
$w = 55°$

3.3. $x = 52°; y = 52°; z = 38°$
$w = 26°$

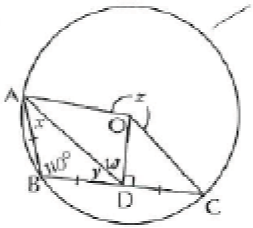
3.4. $x = 44°; y = 22°; z = 68°$

3. Calculate the unknown angles (marked) in the following diagrams. O is the centre of the circles. Give full reasons for your answers.

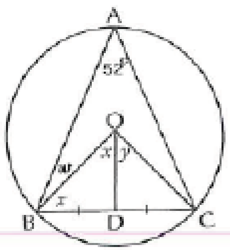
1.



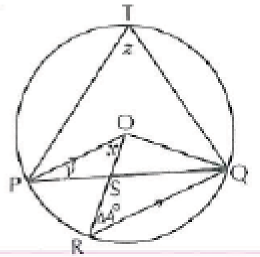
2.



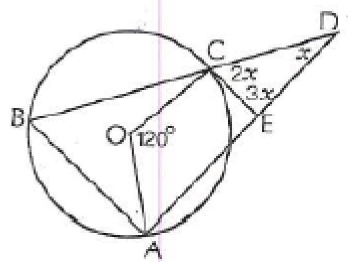
3. Given: AB = AC



4.



4. Refer to the diagram below:

Prove that AB//EC.	

2.3. Conclusion • Revise key elements of the lesson. • Provide a chalkboard summary of the key concepts of the lesson.		Conclusion: ±5 min	

GRADE	11	SUBJECT	Mathematics	TERM 3	WEEK 3	TOPIC	Consolidation Exercise Time: 55 min	Lesson	2

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	

LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • The line segment drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord. • The line segment drawn from the centre of a circle perpendicular to a chord bisects the chord. • The angle which an arc of a circle subtends at the centre of the circle is twice the angle it subtends at any point on the circle. • The angle subtended at the circle by a diameter is a right angle. • Angles in the same segment of a circle are equal.

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. **2. Lesson Development** **2.1. Introduction** • Inform the learners that they will need to complete the worksheet by the end of the period. • As the lesson progresses: o *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.* **2.2. Main Body (Lesson Presentation)** o This lesson consolidates the concepts learned thus far. o Complete the worksheet given below. **2.3. Conclusion** • Revise key elements of the lesson. • Provide a chalkboard summary of the key concepts of the lesson.	1. **Baseline** 2. **Informal Assessment** 3. **Homework Exercise** 4. **Answers to Activities**	Baseline: Lesson & Activities: 35 min Corrections: 15 min Conclusion: ±5 min	• **Lesson references:** o Classroom Mathematics, Standard 9; pp. 309 – 319. o Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160. • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers); o Worksheet.

WORKSHEET:

QUESTION 1

In the figure, $\hat{T}_1 = 32°$; $\hat{L}_1 = 58°$ and $E\hat{T}P = 78°$.

1.1. Calculate with reasons:

1.1.1. $\hat{E}_2$.

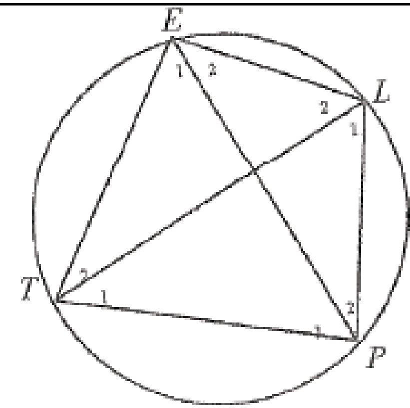
1.1.2. $\hat{E}_1$.

1.1.3. $\hat{T}_2$.

1.1.4. $\hat{P}_2$.

1.1.5. $T\hat{L}P$

1.2. What can be deduced about TL?



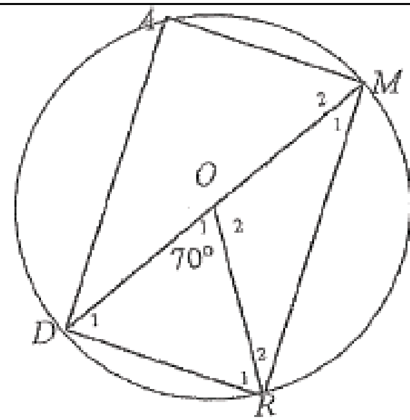
QUESTION 2

In the figure O is the centre of the circle. Find, with reasons:

2.1. $\hat{A}$.

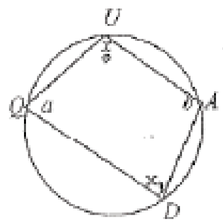
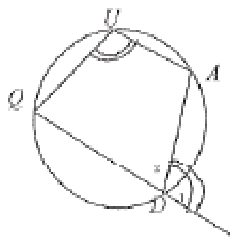
2.2. $\hat{M}_1$.

2.3. $\hat{D}_1$



GRADE	11	SUBJECT	Mathematics	TERM 3	WEEK 3	TOPIC	Cyclic Quadrilaterals 1 Time: 55 min	Lesson	3

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	
LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • The opposite angles of a cyclic quadrilateral are supplementary.		

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. **2. Lesson Development** **2.1. <u>Introduction</u>** • There are two factors that must be known in cyclic quadrilaterals: o *The opposite angles of a cyclic quadrilateral are supplementary; and*  $x + y = 180°$ $a + b = 180°$ o *The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.*  • As the lesson progresses:	1. **<u>Baseline</u>**	Baseline: Lesson & Activities: 35 min Corrections: 15 min	• **Lesson references:** o Classroom Mathematics, Standard 9; pp. 319 – 324. o Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160. o Maths Tutor for the Matric Examinations; Grades 11 & 12; Higher Grade; Paper 2; pp. 65 – 98. • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers)

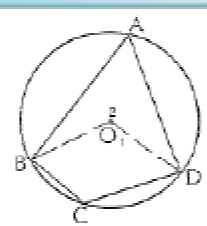
- *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.*

2.2. **Main Body (Lesson Presentation)**

- **A cyclic quadrilateral is a quadrilateral whose four vertices lie on the circumference of a circle.** That means, its sides are chords of the circle.
- In this lesson we will look at the first theorem on cyclic quads...

THEOREM:

The opposite angles of a cyclic quadrilateral are supplementary. (Opp. angles of cyclic quad.)



Given:	Circle with centre O containing cyclic quad. ABCD.	
Required To Prove:	$\hat{A}+\hat{C}=180^\circ$ *and* $\hat{B}+\hat{D}=180^\circ$	
Construction:	Join BO and DO.	
Proof:	**Statement:**	**Reason:**
	$\hat{O}_1 = 2\hat{A}$	Angle at centre = 2 x angle at circle
	$\hat{O}_2 = 2\hat{C}$	Angle at centre = 2 x angle at circle
	$\therefore \hat{O}_1+\hat{O}_2 = 2\hat{A}+2\hat{C}$	
	$But\ \hat{O}_1+\hat{O}_2 = 360^\circ$	Angles around a point

- **Other useful text references:**
  - Teaching Mathematics Std. 9; pp. 214 – 220.
  - New Mathematics; Std. 9; pp. 406 – 416.

$\therefore \hat{A}+\hat{C}=180^\circ$

Similarly, by joining AO and CO, we could prove that

$\hat{B}+\hat{D}=180^\circ$

HOW TO PROVE THAT A QUADRILATERAL IS CYCLIC

1. **If a line segment joining two points subtends equal angles at two other points on the same side of the line segment, then these four points are concyclic.**
2. **If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.**
3. **If an exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is a cyclic quadrilateral.**

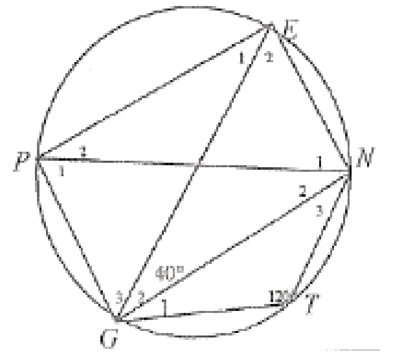
Example 1

In the given figure, $\hat{G}_2=40^\circ$ and $\hat{T}=120^\circ$. Find, with reasons:

1.1. $\hat{P}_1$

1.2. $\hat{P}_2$

1.3. $G\hat{N}E$



Solutions:

1.1. $\hat{P}_1+\hat{T}=180^\circ$...........$[opp.\angle s\ of\ a\ cyclic\ quad.]$

$\therefore \hat{P}_1=180^\circ-120^\circ$

$=60^\circ$

1.2. $\hat{P}_2 = \hat{G}_2$....................[*$\angle$s in same segment*]

$\therefore \hat{P}_2 = 40°$

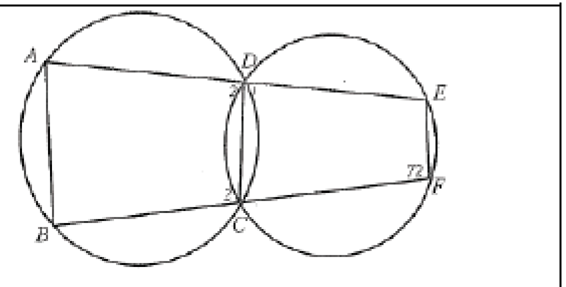
1.3. $(\hat{P}_1 + \hat{P}_2) + G\hat{N}E = 180°$........[*opp.$\angle$s of a cyclic quad.*]

$\therefore G\hat{N}E = 180° - (60° + 40°)$

$= 80°$

Example 2

The two circles intersect at D and C. ADE and BCF are double chords. AB and EF are drawn. If $\hat{F} = 72°$, calculate $\hat{B}$.



Solutions:

$\hat{D}_1 + \hat{F} = 180°$................[*opp.$\angle$s of a cyclic quad.*]

$\therefore \hat{D}_1 = 180° - 72° = 108°$

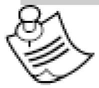
$\hat{D}_1 + \hat{D}_2 = 180°$..............[*adj.$\angle$s on a str. line*]

$\therefore \hat{D}_2 = 180° - 108° = 72°$

but $\hat{D}_2 + \hat{B} = 180°$..........[*opp.$\angle$s of a cyclic quad.*]

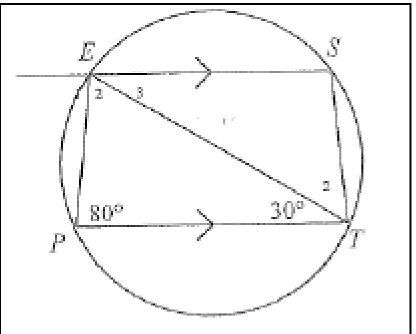
$\therefore \hat{B} = 180° - 72° = 108°$

ACTIVITIES:

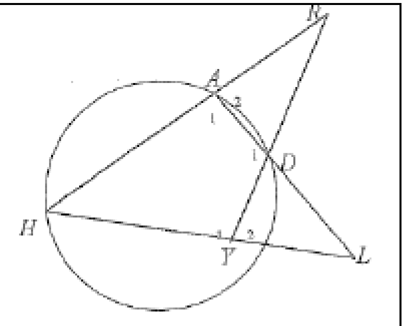


- Do a selection of these problems with your learners in class (informal assessment);
- The rest of the activities could be given as homework.

1. Refer to diagram below:

In circle PEST, ES//PT. Given that $\hat{P} = 80°$ and $\hat{T}_1 = 30°$, determine, with reasons, the size of: 1.1. $\hat{T}_2$ 1.2. $\hat{S}$	

2. Refer to the diagram below:

Calculate all the angles of the cyclic quadrilateral if $\hat{R} = 40°$ and $\hat{L} = 70°$.	

2. <u>Informal Assessment</u>

3. <u>Homework Exercise</u>

Please note that due to space constraints, all exercises, informal and homework are provided in the 1st column.

4. <u>Answers to Activities</u>

1.1. $\hat{T}_2 = 50°$

1.2. $\hat{S} = 100°$

2. Activity 2

$\hat{Y}_1 = 105°$

$\hat{A}_1 = 75°$

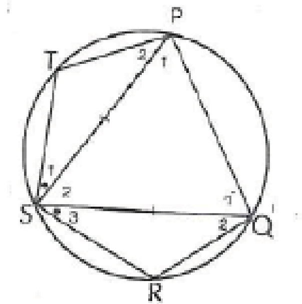
$\hat{H} = 35°$

$\hat{D} = 145°$

<table>
<tr>
<td>
3. Refer to the diagram below:
<table>
<tr>
<td>PS = QS and $\hat{S}_1 = \hat{S}_3$.

Prove that:

3.1. $\hat{T} = \hat{R}$

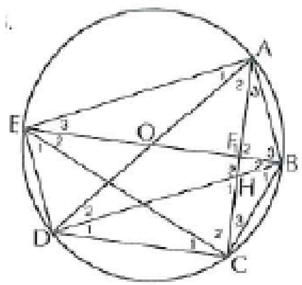
3.2. $TS = RS$</td>
<td>
</td>
</tr>
</table>
4. Refer to the diagram below:
<table>
<tr>
<td>AOD and EOB are diameters. $AF \perp EB$.

Prove that:

4.1. EFHD is cyclic.

4.2. $B\hat{A}D = D\hat{H}C$.

4.3. $\hat{C}_3 = \hat{A}_1$.

4.4. EB bisects $A\hat{B}C$.</td>
<td>
</td>
</tr>
</table>

2.3. Conclusion

- Revise key elements of the lesson.
- Provide a chalkboard summary of the key concepts of the lesson.
</td>
<td></td>
<td>Conclusion: ±5 min</td>
<td></td>
</tr>
</table>

GRADE	11	SUBJECT	Mathematics	TERM 3	WEEK 3	TOPIC	Cyclic Quadrilaterals 2 Time: 55 min	Lesson	4

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	
LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.		

<table>
<tr><th>TEACHER ACTIVITIES</th><th>LEARNER ACTIVITIES</th><th>TIMING</th><th>RESOURCES NEEDED</th></tr>
<tr><td>

1. **Teaching Methods**
   - Telling; Explanation; Question and Answer; Illustration; etc.

2. **Lesson Development**

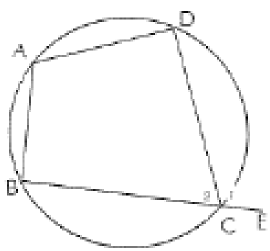
2.1. <u>Introduction</u>

- This lesson deals with the second factor in cyclic quadrilaterals which we illustrated in the previous lesson.
- As the lesson progresses:
  - *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.*

2.2. <u>Main Body (Lesson Presentation)</u>

THEOREM:

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. (Ext. angle of cyclic quad)



</td><td>

1. <u>**Baseline**</u>

</td><td>

Baseline:

Lesson & Activities: 35 min

Corrections: 15 min

</td><td>

- **Lesson references:**
  - Classroom Mathematics, Standard 9; pp. 319 – 324.
  - DoE/November 2009; Mathematics/P3
  - Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160.
  - Maths Tutor for the Matric Examinations; Grades 11 & 12; Higher Grade; Paper 2; pp. 65 – 98.
- **Materials needed:**
  - Chalkboard (or any other appropriate form of board);
  - Chalks (or markers).

</td></tr>
</table>

NOT AS FORMAL PROOF:

Given:	Cyclic quad. ABCD with BC produced to E, creating an exterior angle $\hat{C}_1$.	
Required To Prove:	$\hat{C}_1 = \hat{A}$	
Proof:	**Statement:**	**Reason:**
	$\hat{C}_1 + \hat{C}_2 = 180°$	$B\hat{C}E$ is a straight angle
	$\hat{A} + \hat{C}_2 = 180°$	Opposite angles of a cyclic quad.
	$\therefore \hat{C}_1 + \hat{C}_2 = \hat{A} + \hat{C}_2$	
	$\therefore \hat{C}_1 = \hat{A}$	

- Now, let us look at how this theorem is used in conjunction with the previous one in the solution of riders.

Example 1

ER is the diameter of circle EXTR. XE = XT.

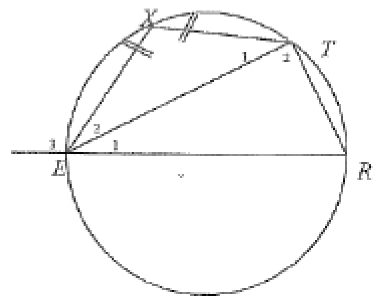
If $\hat{E}_3 = 105°$, calculate the following, giving reasons:

1.1. $\hat{T}_1$

1.2. $\hat{X}$

1.3. $\hat{R}$

1.4. $\hat{E}_1$



Solution:

1.1. $\hat{T}_2 = 90°$....................[$\angle$ *in a semi – circle*]

$\hat{E}_3 = \hat{T}_1 + \hat{T}_2$..............[*ext.* $\angle$ *of a cyclic quad.*]

$\therefore 105° = \hat{T}_1 + 90°$

$\therefore \hat{T}_1 = 15°$

- **Other useful text references:**
  - Teaching Mathematics Std. 9; pp. 214 – 220.
  - New Mathematics; Std. 9; pp. 406 – 416.

1.2. *In* ΔXET :

$\hat{T}_1 = \hat{E}_2$.......................[$\angle s$ *opp.* = *sides* ; $XE = XT$]

$\hat{X} + \hat{E}_2 + T_1 = 180°$....[$\angle s$ *of a* Δ]

$\therefore \hat{X} = 180° - (15° + 15°)$

$= 150°$

1.3. $\hat{X} + \hat{R} = 180°$.............[*opp.*$\angle s$ *of a cyclic quad*]

$\therefore \hat{R} = 180° - 150°$

$= 30°$

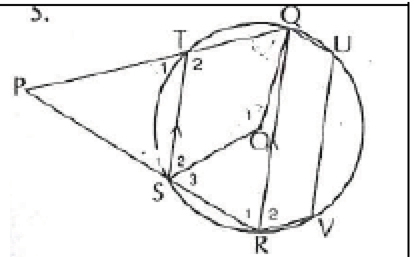
1.4. *In* ΔTER :

$\hat{E}_1 + \hat{T}_2 + \hat{R} = 180°$........[$\angle s$ *of a* Δ]

$\therefore \hat{E}_1 = 180° - (90° + 30°)$

$= 60°$

Example 2

TS//QR and O is the centre of the circle. Prove that: 2.1. PQ = PR 2.2. QOSP is a cyclic quadrilateral	5. 

Solutions:

2.1. $\hat{S}_1 = \hat{R}_1$.......................[*corresp.*$\angle s$; $ST \,//\, RQ$]

but $\hat{S}_1 = P\hat{Q}R$...........[*ext.*$\angle$ *of a cyclic quad.*]

$\hat{R}_1 = P\hat{Q}R$

$\Rightarrow PQ = PR$..............[*sides opp* = $\angle s$]

2.2. $\hat{O}_1 = 2\hat{R}_1$....................[$\angle$ *at centre*]

but $\hat{R}_1 = \hat{S}_1$................[*corresp.* $\angle s$; $ST \parallel RQ$]

$\therefore \hat{O}_1 = 2\hat{S}_1$

$\hat{P} = 180° - (\hat{S}_1 + \hat{T}_1)$

$= 180° - 2\hat{S}_1$

$= 180° - \hat{O}_1$

$\Rightarrow$ *QOSP is a cyclic quad*.......[*opp.* $\angle s$ *of quad are supplementary*]

CHARACTERISTICS OF A CYCLIC QUAD:

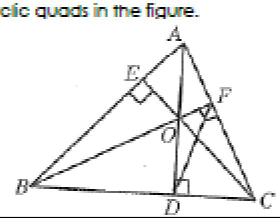
At this point you could summarize the facts learned about cyclic quads as follows...

- **THE FOUR VERTICES LIE ON THE CIRCUMFERENCE OF A CIRCLE.**
- **EACH PAIR OF OPPOSITE ANGLES IS SUPPLEMENTARY.**
- **THE EXTERIOR ANGLE IS EQUAL TO THE INTERIOR OPPOSITE ANGLE.**

ACTIVITIES:

- Do a selection of these problems with your learners in class (informal assessment);
- The rest of the activities could be given as homework.

1. Name, with reasons, 4 cyclic quads in the figure.

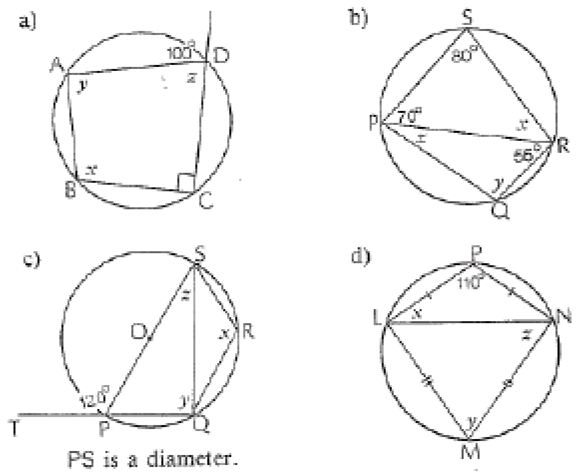


2. Informal Assessment

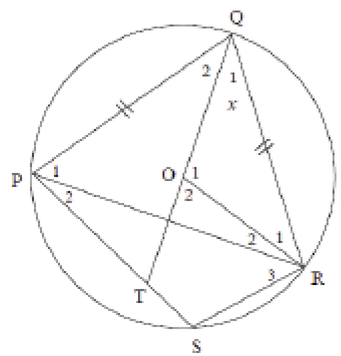
3. Homework Exercise

Please note that due to space constraints, all exercises, informal and homework are provided in the 1st column.

2. Find the size of each of the unknown angles, giving reasons.



3. In the diagram below, O is the centre of the circle. P, Q, R and S are points on the circumference of the circle. TOQ is a straight line such that T lies on PS. PQ = QR and $\hat{Q}_1 = x$.



4. Answers to Activities

1.1. $AFOE$
1.2. $BEOD$
1.3. $FODC$
1.4. $ABDF$

2. Activity 2
a. $x = 100°; y = 90°; z = 80°$
b. $x = 30°; y = 100°; z = 25°$
c. $x = 120°; y = 90°; z = 30°$
d. $x = 35°; y = 70°; z = 55°$

6.1. $\hat{R}_1 = 50°$
6.2. $\hat{R}_2 = 30°$
6.3. $\hat{S}_1 = 30°$
6.4. $\hat{P}_1 = 60°$
6.5. $\hat{P}_2 = 40°$

$\therefore \hat{C} = 46°$

1.3. In ΔCRY : $CR = CY$........[radii]

$\therefore \hat{R}_1 = R\hat{Y}C$.....................[$\angle s$ opp = sides]

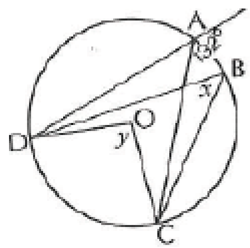
$R\hat{Y}C + \hat{R}_1 + \hat{C} = 180°$

$R\hat{Y}C + \hat{R}_1 = 180° - 46° = 134°$

$\therefore R\hat{Y}C = 67°$

Example 2

In the diagram below, find the values of x and y.



Solutions:

$D\hat{A}C = 180° - 132°$..........[$\angle s$ on a str. line]

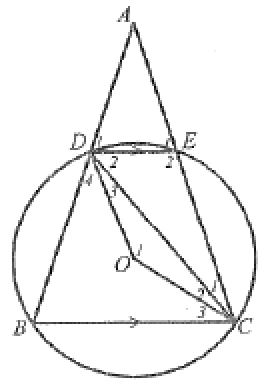
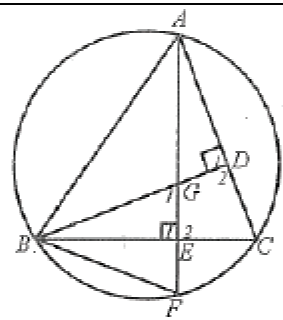
$\therefore D\hat{A}C = 48°$

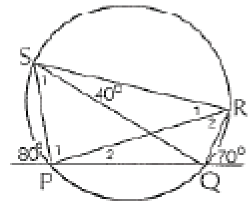
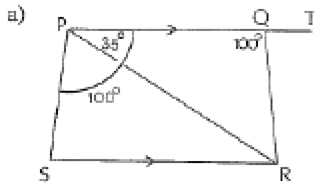
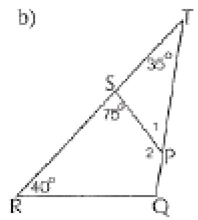
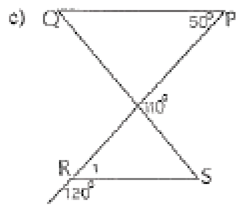
but $D\hat{A}C = \frac{1}{2}y$.................[$\angle$ at centre]

$\therefore y = 96°$

$x = \frac{1}{2}y$..............................[$\angle$ at centre]

$\therefore x = 48°$

3.1. Calculate, with reasons, $\hat{P}_1$ in terms of x. 3.2. Show that TQ bisects $P\hat{Q}R$. 3.3. Show that STOR is a cyclic quadrilateral. 4. Refer to the diagram below: Chords BD and CE of circle O are produced to meet at A. If DE//BC, prove that ADOC is cyclic.  5. Refer to the diagram below: In ΔABC, $AE \perp BC$ and $BD \perp AC$. AE is produced to meet the circumcircle of ΔABC at F. 			

6. Refer to the diagram below: Find $\hat{R}_1$; $\hat{R}_2$; $\hat{S}_1$; $\hat{P}_1$ and $\hat{P}_2$  7. Verify that PQRS is a cyclic quad in each of the following examples. (Give reasons)    **2.3. Conclusion** • Revise key elements of the lesson. • Provide a chalkboard summary of the key concepts of the lesson.		Conclusion: $\pm$5 min	

GRADE	11	SUBJECT	Mathematics	TERM 3	Week 4	TOPIC	Proving Riders 2 Time: 55 min	Lesson	4

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	

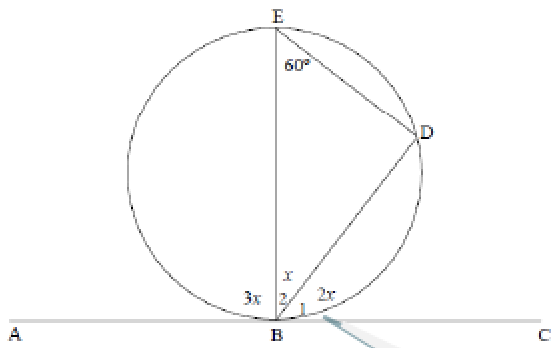
LESSON OBJECTIVES	The learners will consolidate their knowledge of the following concepts: • The line segment drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord. • The line segment drawn from the centre of a circle perpendicular to a chord bisects the chord. • The angle which an arc of a circle subtends at the centre of the circle is twice the angle it subtends at any point on the circle. • The angle subtended at the circle by a diameter is a right angle. • Angles in the same segment of a circle are equal. • The opposite angles of a cyclic quadrilateral are supplementary. • An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. • Two tangents drawn to a circle from the same point outside the circle are equal in length. • The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. **2. Lesson Development** **2.1. Introduction** • We continue working through riders from previous examination papers. • The idea is to expose the learners to typical exam questions and what it takes to prove these riders. • As the lesson progresses: o *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.* **2.2. Main Body (Lesson Presentation)** o In the previous lesson, we dealt with riders that involve mostly cyclic quadrilaterals. o We will now attempt to deal with riders that involve the tan-chord theorem and intersecting circles. o As we advised in the previous lesson, keep track of the given information in each rider and deduce facts related to the theorems you have learned.	**1. Baseline** **2. Informal Assessment** **3. Homework Exercise** **4. Answers to Activities**	Baseline: Lesson & Activities: 35 min	• **Lesson references:** o DoE/Exemplar 2008; Mathematics/P3 o DoE/May-June/2008; Mathematics/HG/P2 • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers). • **Other useful text references:** o Guidelines Study Aids; Mathematics Paper 2; Geometry &

<table>
<tr>
<td>

Example 1

In the figure below, $\hat{E} = 60°$. ABC is a straight line.



1.1. Calculate, with reasons, the value of x.

- **We can use the fact that ABC is a straight line…**

$3x + x + 2x = 180°$....[$\angle s$ *on a str.line*]
$6x = 180°$
$\therefore x = 30°$

1.2. Prove that AC is a tangent to the circle.

- **We have to show that an angle between AC and a chord of the circle is equal to another angle in the alternate segment.**

$\hat{B}_1 = 2x = 60°$...........[*proven in* 1.1. *above*]
$\hat{E} = 60$......................[*given*]
Now, $\hat{E} = \hat{B}_1$
$\therefore$ *AC is a tangent*....[$\angle$ *btwn line & chord* $= \angle$ *in alt. segment*]

- **Now we can claim that EB is a diameter and $\hat{D} = 90°$ as an angle in a semi-circle.**

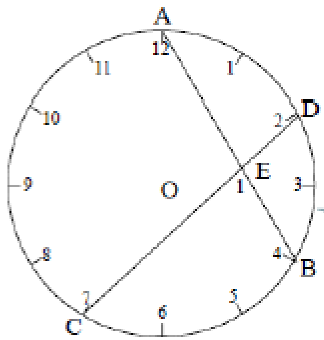
</td>
<td>

At this point we cannot assume that $3x = 90°$ because we don't know yet that EB is a diameter.

</td>
<td>Corrections:
15 min</td>
<td>Trigonometry; Grade 11 Higher Grade; pp. 133 – 160.
o Classroom Mathematics, Standard 9; pp. 309 – 365.
o New Mathematics; Std. 9; pp. 382 – 427.</td>
</tr>
</table>

Example 2

The diagram below shows a clock-face. 12 is joined to the 4 and 2 is joined to the 7 with straight lines AB and CD respectively. O is the centre of the clock-face.



Try to recognize diagrams associated with the theorems you have learned by looking at typical angles formed, adding on the given information, etc.

2.1. If AO and DO are joined, show that $A\hat{O}D = 60°$.

- A clock has 12 sectors, each say α.

$\therefore 12\alpha = 360°$

$\therefore \alpha = 30°$

$\therefore AOD = 60°$......[$\angle$ *at centre*]

2.2. If CO and BO are joined, calculate $C\hat{O}B$.

$From\ 2.1.,\ C\hat{O}B = 3\alpha$

$\therefore C\hat{O}B = 3(30°) = 90°$

2.3. Hence, calculate $\hat{E}_1$.

$C\hat{A}B = \frac{1}{2}(90°) = 45°$........[$\angle$ *at centre*]

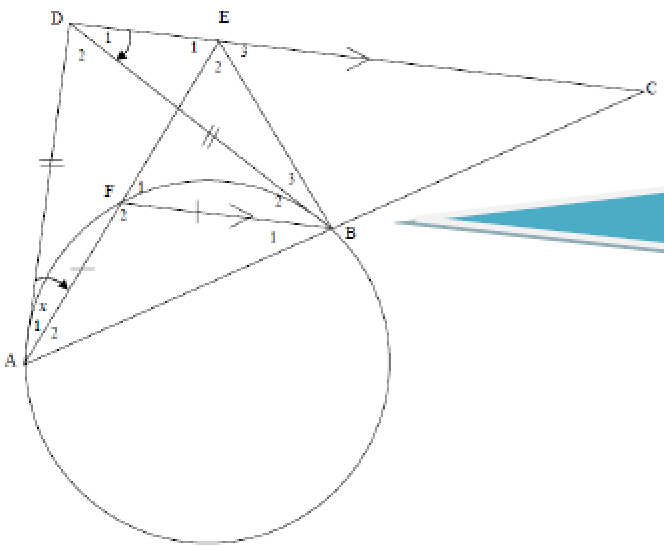
$A\hat{C}D = \frac{1}{2}(60°) = 30°$.......[$\angle$ *at centre*]

$Now,\ \hat{E}_1 = C\hat{A}B + A\hat{C}D$

$= 45° + 30° = 75°$

Example 3

In the sketch below, DA and DB are tangents to the circle at A and B. AF = FB. AB produced cuts the line through D, which is parallel to FB, at C. AF produced meets DC at E and $D\hat{A}E = x$.



The information given includes tangents, so be on the lookout for angles which satisfy the **tan-chord theorem**. We also have angles which resemble those in the theorem on **angles in the same segment**. Mention is also made of parallel lines, so look out for alternate $\angle s$, corresponding $\angle s$, etc.

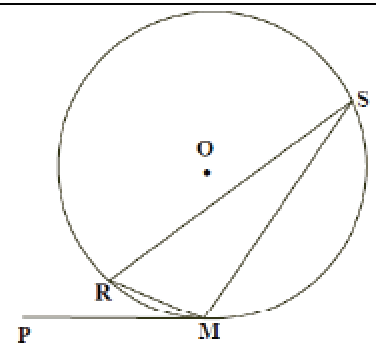
3.1. Find, with reasons, 5 angles each equal to x.

- $\hat{B}_1 = x$.................[$tan - chord\ theorem$]
- $A_2 = x$.................[$FA = FB$]
- $\hat{B}_2 = x$.................[$D\hat{A}B = D\hat{B}A = 2x$; $tan - chord\ theorem$]
- $\hat{D}_1 = \hat{B}_2 = x$........[$alt.\angle s$; $DC \,//\, FB$]
- $C = B_1 = x$.........[$corresp.\angle s$; $DC \,//\, FB$]

3.2. Prove that ABED is a cyclic quadrilateral. $\hat{A}_2 = \hat{D}_1 = x$..................[*from* 3.1. *above*] *but these are angles subt. by BE* $\therefore$ *ABED is cyclic.* 3.3. Prove that $A\hat{B}E = 3D\hat{A}E$. $\hat{B}_3 = \hat{A}_1 = x$..................[$\angle$*s in the same segment*] *Now* $A\hat{B}E = \hat{B}_1 + \hat{B}_2 + \hat{B}_3$ $\therefore A\hat{B}E = 3x$ $= 3D\hat{A}E$ 3.4. Prove that AD = BC. $\hat{D}_1 = \hat{C} = x$ $\therefore BD = BC$..................[*isosceles* Δ] *but* $BD = AD$...............[tans. *from common pt*] $\therefore AD = BC$ **2.3. Conclusion** • Revise key elements of the lesson. • Provide a chalkboard summary of the key concepts of the lesson. • You may also provide the learners with more riders from previous exam papers for them to practice with.		Conclusion: ±5 min	

WORKSHEET:

QUESTION 1

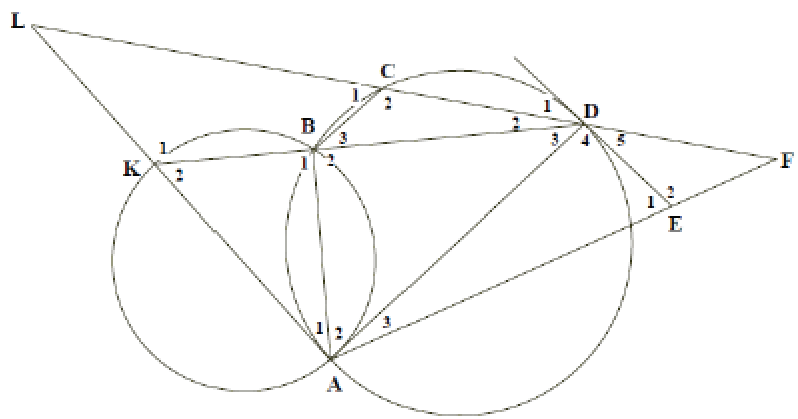
1.1. In the diagram alongside, O is the centre of circle RSM. Use the diagram to prove the theorem which states that: ***If PM is a tangent to the circle, then*** $P\hat{M}R = \hat{S}$. **(6)**	

1.2. In the diagram below, two circles intersect each other at A and B. ED is a tangent to circle ABCD. DA is a tangent to circle AKB. DBK is a straight line. AK and DC are produced to meet at L. LCD and AE are produced to meet at F. CD = DF.

1.2.1. LKBC is a cyclic quadrilateral. **(5)**

1.2.2. $\hat{B}_2 = L\hat{A}D$. **(3)**

1.2.3. $DE // LA$. **(5)**



[19]

QUESTION 2

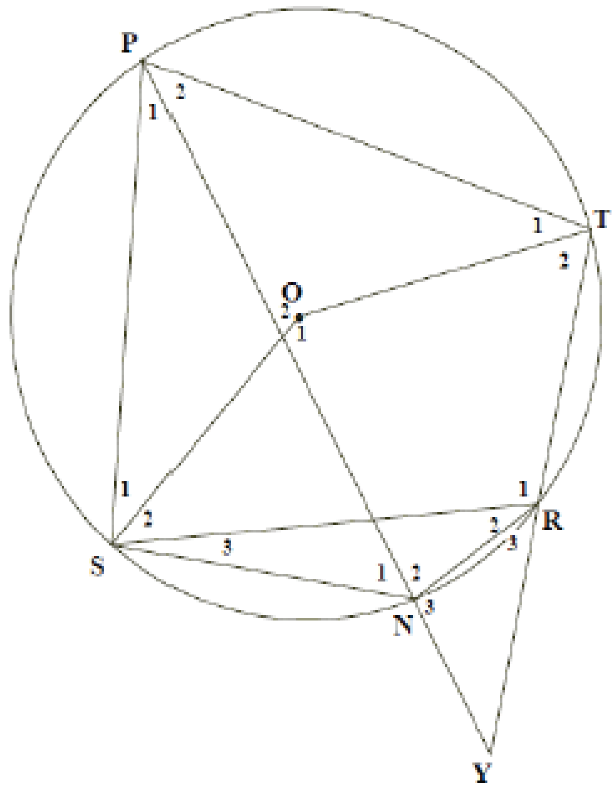
In the diagram below, circle PTRNS, with centre O, is drawn. NP bisects $S\hat{P}T$. PN and TR are produced to meet at Y. NR = NY. Let $\hat{O}_1 = 4x$.

2.1. Determine, with reasons, the following in terms of x.

2.1.1. $\hat{R}_3$ **(5)**

2.1.2. $P\hat{S}R$ **(4)**

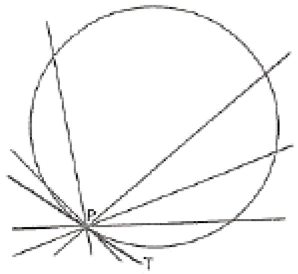
2.2. Hence, prove that OSYR is NOT a cyclic quadrilateral. **(3)**



[12]

GRADE	11	SUBJECT	Mathematics	TERM 3	Week 4	TOPIC	Tangents to a Circle 1 Time: 55 min	Lesson	1

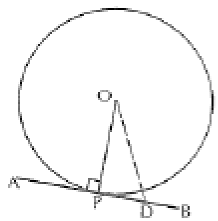
LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	
LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • The angle between a tangent and the radius at the point of contact is 90⁰. • Two tangents drawn to a circle from the same point outside the circle are equal in length. • The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.		

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. **Teaching Methods** • Telling; Explanation; Question and Answer; Illustration; etc. 2. **Lesson Development** 2.1. <u>Introduction</u> • Start off by **defining and illustrating what a tangent is.**  • All other points on line PT lie outside the circle, except for point P, the only point of contact. • Thereafter, make the learners aware that there are three facts about tangents that they need to know. These are: o *The angle between a tangent and the radius at the point of contact is 90⁰;* o *Tangents drawn from the same point outside the circle to the circle are equal in length; and* o *The angle between a tangent and a chord is equal to the angle the chord subtends in the alternate segment.*	1. <u>Baseline</u>	Baseline: ±5 min Lesson & Activities: 35 min Corrections: 10 min	• **Lesson references:** o Classroom Mathematics, Standard 9; pp. 324 – 330. o Teaching Mathematics Std. 9; pp. 240 – 246. o DoE/November 2008; Mathematics/P3 o DoE/Preparatory Examination 2008; Mathematics/P3 • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers). • **Other useful text references:** o Guidelines Study

- As the lesson progresses:
  - *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.*

2.2. Main Body (Lesson Presentation)

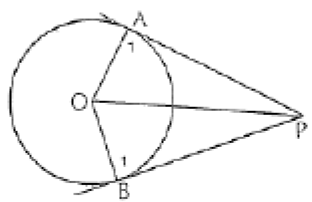
- The first fact refers to a tangent being perpendicular to a radius, which we can illustrate as:



- The second one refers to two tangents drawn from the same point outside the circle. We need to prove this theorem.

THEOREM:

Two tangents drawn to a circle from the same point outside the circle are equal in length. (Tan. from same point)



Given: Circle with centre O and tangents PA and PB touching the circle at A and B respectively.

Required To Prove: $PA = PB$

Construction: Join OA, OB and OP.

Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160.

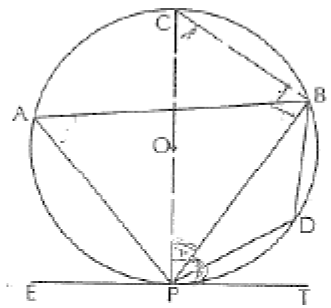
- Maths Tutor for the Matric Examinations; Grades 11 & 12; Higher Grade; Paper 2; pp. 65 – 98.
- New Mathematics; Std. 9; pp. 417 – 427.

Proof:	Statement:	Reason:
	$In\ \Delta PAO\ and\ \Delta PBO$	
	$P\hat{A}O = P\hat{B}O = 90^\circ$	$\tan \perp rad$
	$PO = PO$	Common side
	$AO = BO$	Radii
	$\therefore \Delta PAO \equiv \Delta PBO$	90°, H, S
	$\therefore PA = PB$	

- The third fact is another theorem on tangents which is often referred to as the "tan-chord" theorem.

THEOREM:

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment. (Tan-chord theorem)



Given: Circle with centre O, with tangent EPT and chord PB. Point A is on major arc PB and point D is on minor arc PB.

Required To Prove:
$(a)\ B\hat{P}T = \hat{A}$
$(b)\ E\hat{P}B = \hat{D}$

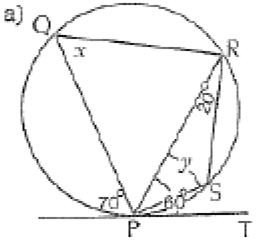
Construction: Draw diameter CP and join C to B.

Proof:	**Statement:**	**Reason:**
	$(a)\ \hat{P}_1 + \hat{P}_2 = 90°$	$\tan \perp$ *diameter*
	but $C\hat{B}P = 90°$	Angle in semi-circle
	$\therefore \hat{P}_1 + \hat{C} = 90°$	Angles of a $\Delta = 180°$
	$\therefore \hat{P}_2 = \hat{C}$	
	but $\hat{C} = \hat{A}$	Angles in the same segment
	$\therefore \hat{A} = \hat{P}_2 \left(B\hat{P}T\right)$	
	$(b)\ E\hat{P}B + \hat{P}_2 = 180°$	PQS is a straight line
	and $\hat{A} + \hat{D} = 180°$	Opp. angles of cyclic quad.
	$\therefore E\hat{P}B + \hat{P}_2 = \hat{A} + \hat{D}$	
	but $\hat{P}_2 = \hat{A}$	Proved in (a) above
	$\therefore E\hat{P}B = \hat{D}$	

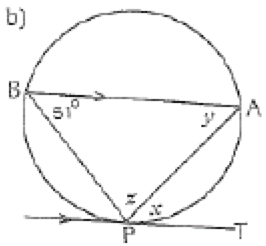
Example1

In the diagrams below, PT is a tangent. Determine the magnitude of the angles marked x, y and z.

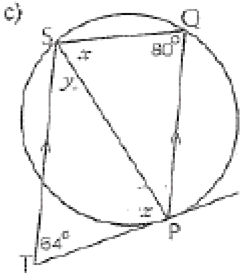
a)



b)



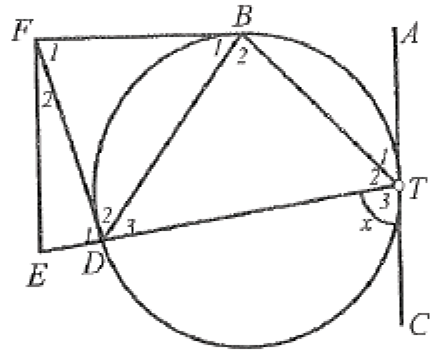
c)



Solutions: 1. <u>**Diagram (a)**</u> $x = 60°$..............................[tan$-$ *chord theorem*] $x + \hat{S} = 180°$......................[*opp.*$\angle$*s of a cyclic quad.*] $\therefore \hat{S} = 180° - 60° = 120°$ $y + 120° + 20° = 180°$.......[$\angle$*s of a* Δ] $\therefore y = 180° - 140° = 40°$ 2. <u>**Diagram (b)**</u> $x = 51°$..............................[tan$-$ *chord theorem*] $x = y$..............................[*alt.*$\angle$*s*; $BA // PT$] $\therefore y = 51°$ $z + y + 51° = 180°$.............[$\angle$*s of a* Δ] $\therefore z = 180° - 102° = 78°$ 3. <u>**Diagram (c)**</u> $z = 80°$..............................[tan$-$ *chord theorem*] $y + z + 64° = 180°$............[$\angle$*s of a* Δ] $\therefore y = 180° - 144° = 36°$ $(x + y) + 80° = 180°$.........[*co*$-$int $\angle$*s*; $ST // QP$] $\therefore x + y = 120°$ $\Rightarrow x = 120° - 36° = 84°$ ➢ Make the learners take note that the examples above are direct applications of the tan-chord theorem in relation to other theorems they have learned in the earlier grades. They involve finding numerical values of angles. ➢ The example which follows is a little bit complex (riders). A rider like this one often involves "proving something" about angles, sides or entire quadrilaterals.			

Example 2

In the diagram below, FD, FB and ATC are tangents to the circle. EF is parallel to CTA and meets TD produced at E. Let $E\hat{T}C = x$.



2.1. Find, with reasons, two other angles each equal to x.

$\hat{B}_2 = x$............[$tan - chord\ theorem$]

$\hat{E} = x$..............[$alt.\angle s$; $EF \,//\, CTA$]

2.2. Prove that $\hat{B}_2 = \hat{D}_1$.

$\hat{D}_1 = 180° - (\hat{D}_2 + \hat{D}_3)$.........[$\angle s$ on a str. line]

$= 180° - \left(\hat{T}_2 + \hat{D}_3\right)$..........[$tan - chord$; FD & BD]

$= 180° - \left(\hat{T}_2 + \hat{T}_1\right)$............[tan$-$ $chord$; TA & TB]

$= x$...................................[$A\hat{T}C$ is a straight angle]

$= \hat{B}_2$..................................[proven in 2.1. above]

2.3. Prove that $FE = FB$.

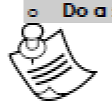
$\hat{E} = \hat{D}_1 = x$

$\therefore FE = FD$.............................[*sides opp.* = *angles*]

but $FD = FB$.........................[*tangents from same point*]

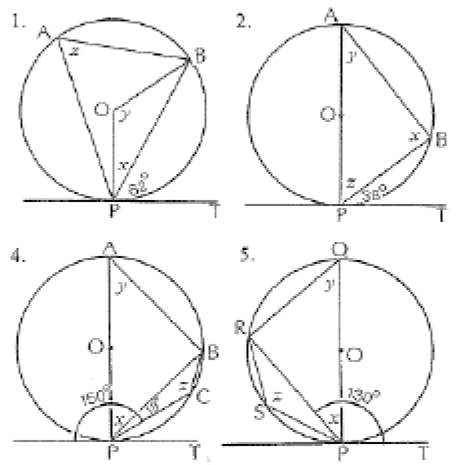
$\therefore FE = FB$

ACTIVITIES:



- **Do a selection of these problems with your learners in class (informal assessment);**
  - **The rest of the activities could be given as homework.**

1. Find the size of x, y and z in each case. O is the centre and PT is a tangent in each case. Reasons must be given.



N.B. $B\hat{P}C = 18°$

2. In the figure below, RDS is a tangent to circle O at D. If BC = DC and $C\hat{D}S = 40°$, calculate, with reasons, the measures of:

2. <u>Informal Assessment</u>

3. <u>Homework Exercise</u>

Please note that due to space constraints, all exercises, informal and homework are provided in the 1st column.

4. <u>Answers to Activities</u>

1.1. $x = 28°$; $y = 124°$

$z = 62°$

1.2. $x = 90°$; $y = 38°$

$z = 52°$

1.3. $x = 42°$; $y = 42°$

$z = 48°$

1.4. $x = 42°$; $y = 48°$

$z = 132°$

1.5. $x = 40°$; $y = 50°$

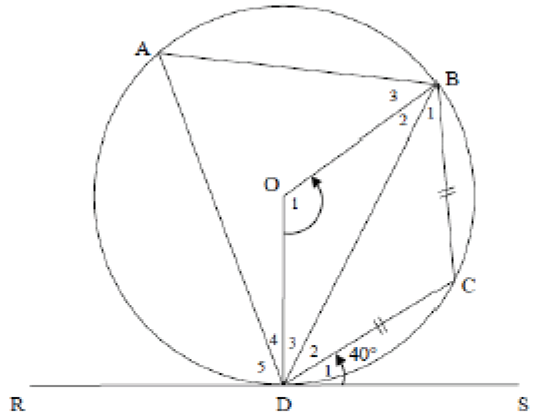
$z = 130°$

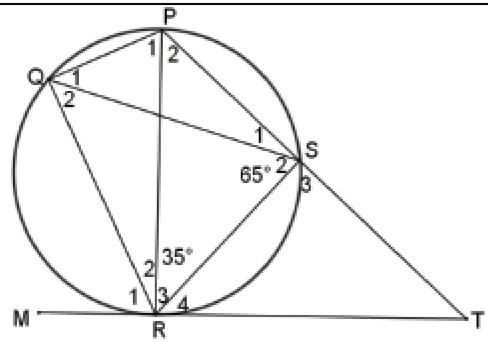
2.1. $B\hat{D}C = 40°$

2.2. $\hat{C} = 100°$

2.3. $\hat{A} = 80°$

2.4. $\hat{O}_1 = 160°$

 2.1. $B\hat{D}C$ 2.2. $\hat{C}$ 2.3. $\hat{A}$ 2.4. $\hat{O}_1$ 3. In the sketch below P, Q, R and S are points on the circle. PR is the diameter of the circle and MRT is a tangent at R. $\hat{S}_2 = 65°$ and $\hat{R}_3 = 35°$. Determine, with reasons, the sizes of: 3.1. $\hat{R}_1$ 3.2. $\hat{R}_4$ 3.3. $\hat{S}_3$	3.1. $\hat{R}_1 = 65°$ 3.2. $\hat{R}_4 = 55°$ $\hat{S}_3 = 90°$		

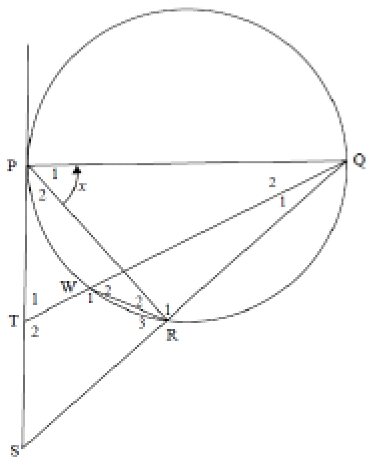


4. In the figure below, PQ is a diameter to circle PWRQ. SP is a tangent to the circle at P. Let $\hat{P}_1 = x$.

4.1. Why is $PRQ = 90^\circ$?

4.2. Prove that $\hat{P}_1 = \hat{S}$.

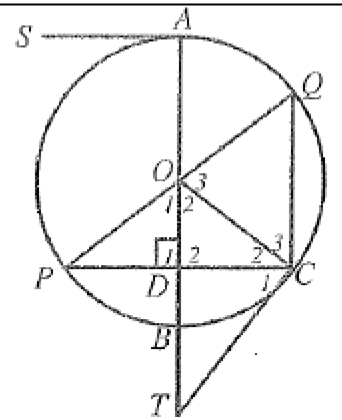
4.3. Prove that SRWT is a cyclic quadrilateral.



GRADE	11	SUBJECT	Mathematics	TERM 3	Week 4	TOPIC	Tangents to the Circle 2 Time: 55 min	Lesson	2

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	
LESSON OBJECTIVES	The learners will be taught and learn the following concepts: • The angle between a tangent and the radius at the point of contact is 90^0. • Two tangents drawn to a circle from the same point outside the circle are equal in length. • The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.		

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. **2. Lesson Development** **2.1. Introduction** • You might want to begin with a quick round-up of the two theorems dealt with in the previous lesson because the learners will need the facts stated in those theorems to solve the riders which follow in this lesson • As the lesson progresses: o *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.* **2.2. Main Body (Lesson Presentation)** o This lesson is a continuation of the previous one on tangents. o We are going to solve riders which involve mainly the theorems on tangents. **Example 1** In the diagram below, SA and TC are tangents to the circle at A and C. AT is the perpendicular bisector of chord PC at D. QC//AD and AD cuts PQ at O. O and C are joined. 1.1. Prove that SA//PC. 1.2. Prove that OP = OC. 1.3. Prove that TC is a tangent to circle ODC.	1. Baseline	Baseline: Lesson & Activities: 35 min Corrections: 15 min	• **Lesson references:** o Classroom Mathematics, Standard 9; pp. 324 – 330. o Maths Tutor for the Matric Examinations; Grades 11 & 12; Higher Grade; Paper 2; pp. 65 – 98. • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers). • **Other useful text references:** o Guidelines Study Aids; Mathematics Paper 2; Geometry &



Solutions:

1.1. Prove that SA//PC.

$AB \text{ is a } \perp \text{ bisector of chord } PC$
$\therefore AB \text{ is a diameter}$
$S\hat{A}D = 90°$..................[tan – chord theorem]
$\hat{D}_2 = 90°$......................[$AD \perp PC$]
$\therefore SA \,//\, PC$......................[alt.∠s are equal]

Note that we could have used co-interior angles here by showing that $S\hat{A}D = \hat{D}_1 = 90°$

1.2. Prove that OP = OC.

$\Delta OPD \equiv \Delta OCD$..............[S,∠,S ; line from centre theorem]
$\therefore OP = OC$

1.3. Prove that TC is a tangent to circle ODC.

$\hat{C}_1 = \hat{Q}$............................[tan – chord theorem]
but $\hat{Q} = \hat{O}_1$......................[corresp.∠s ; $QC \,//\, AD$]

Trigonometry; Grade 11 Higher Grade; pp. 133 – 160.

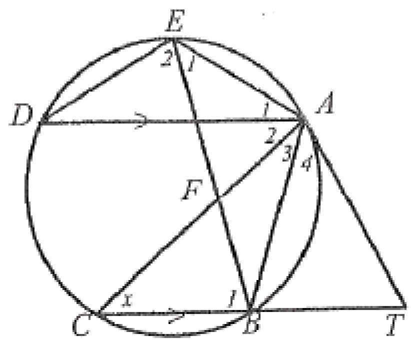
- New Mathematics; Std. 9; pp. 417 – 427.
- Teaching Mathematics Std. 9; pp. 240 – 246.

$and\ \hat{O}_1 = \hat{O}_2$....................[$\Delta OPD \equiv \Delta OCD$]

$\therefore TC$ *is a tangent to circle ODC*.......[$\angle$ *btwn line & chord* = $\angle$ *in alt. segment*]

Example 2

T A is a tangent and TBC is a secant to the circle. Chords DA and CB are parallel, and DE = CB. Let $\hat{C} = x$.



2.1. Name 3 other angles, giving reasons, each equal to x.

$\hat{A}_2 = x$.................[*alt* $\angle s$; $DA // CT$]

$\hat{E}_1 = x$.................[$\angle s$ *in same segment*]

$\hat{A}_4 = x$.................[*tan – chord theorem*]

2.2. Prove that $E\hat{A}T$ is bisected by CA.

$\hat{A}_1 = \hat{A}_3$.................[*equal chords subt. equal* $\angle s$; $DE = CB$]

$\therefore \hat{A}_1 + \hat{A}_2 = \hat{A}_3 + \hat{A}_4$........[$\hat{A}_2 = \hat{A}_4 = x$]

$\therefore C\hat{A}E = C\hat{A}T$

$\Rightarrow CA$ *bisects* $E\hat{A}T$

2.3. Prove that TAFB is a cyclic quadrilateral.

$\hat{B}_1 = \hat{A}_1 + \hat{A}_2$..........[$\angle$s in same segment]

$\therefore \hat{B}_1 = C\hat{A}T$............[CA bisects $E\hat{A}T$]

but $\hat{B}_1$ is an ext.$\angle$ of quad. TAFB]

and $C\hat{A}T$ is an int. opp.$\angle$

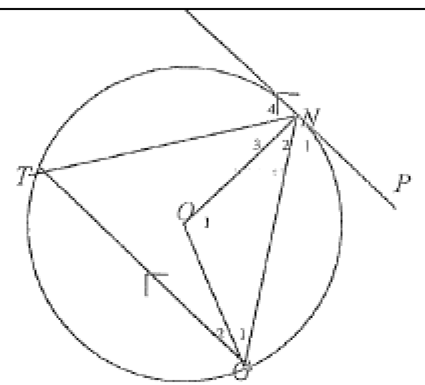
$\Rightarrow$ TAFB is a cyclic quad.

ACTIVITIES:

- Do a selection of these problems with your learners in class (informal assessment);
- The rest of the activities could be given as homework.

1. In the figure NP is a tangent circle centre O. NP//TG and $\hat{O}_1 = 120°$. Calculate with reasons, the size of:

1.1. $\hat{N}_2$
1.2. $\hat{N}_1$
1.3. $\hat{G}_2$
1.4. $\hat{N}_4$



2. <u>Informal Assessment</u>

3. <u>Homework Exercise</u>

Please note that due to space constraints, all exercises, informal and homework are provided in the 1st column.

4. <u>Answers to Activities</u>

1.1. $\hat{N}_2 = 30°$
1.2. $\hat{N}_1 = 60°$
1.3. $\hat{G}_2 = 30°$
1.4. $\hat{N}_4 = 60°$

4.1. $\hat{R}_1 = 80°$
4.2. $\hat{Q}_4 = 80°$
4.3. $\hat{Q}_2 = 40°$
4.4. $\hat{Q}_1 = 50°$
4.5. $\hat{P}_2 = 10°$

2. In the figure below, TP and TQ are tangents to circle O and TCS is a secant. Chord PR//CS, and CS cuts RQ at A and PQ at B.

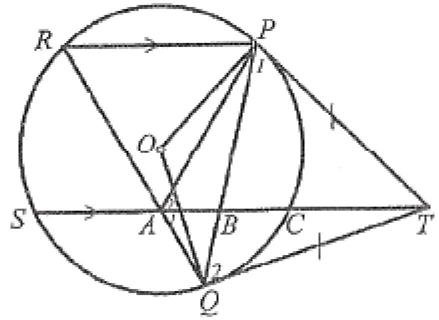
2.1. POQT is a cyclic quadrilateral.

2.2. $\hat{A}_1(T\hat{A}Q) = \hat{P}_1(T\hat{P}Q)$.

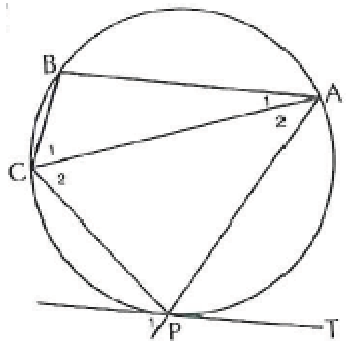
2.3. PAQT is cyclic quadrilateral.

2.4. TP is a tangent to circle PAB.

2.5. Circles PAB and O touch at P.

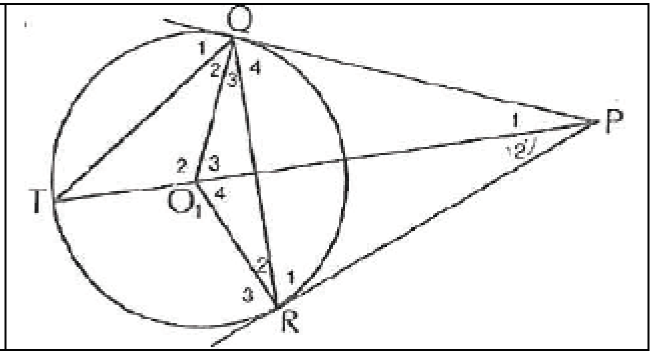


3. In the sketch below, $\hat{C}_1 = \hat{C}_2$; $B\hat{C}P = 120°$ and $\hat{P}_1 = 60°$. Prove that PT is a tangent to the circle.



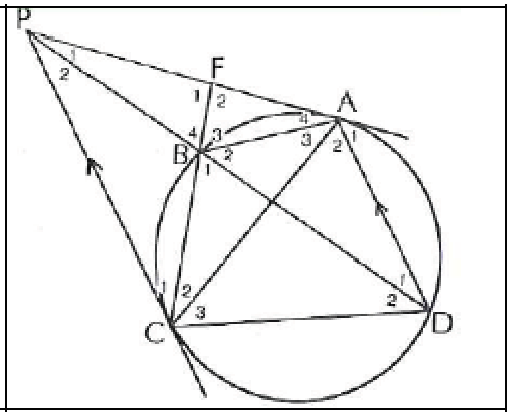
4. In the figure below, PQ and PR are tangents. O is the centre of the circle. POT is straight. $\hat{O}_1 = 100°$ and $\hat{R}_2 = 10°$. Determine the size of:

4.1. $\hat{R}_1$
4.2. $\hat{Q}_4$
4.3. $\hat{Q}_2$
4.4. $\hat{Q}_1$
4.5. $\hat{P}_2$



5. PA and PC are tangents to the circle at A and C. AD//PC and PD cuts the circle at B. CB is produced to meet AP at F. AB, AC and DC are joined.
Prove that:

5.1. AC is the bisector of $P\hat{A}D$.
5.2. $\hat{B}_1 = \hat{B}_3$.
5.3. $AP = PC$.
5.4. $A\hat{P}C = A\hat{B}D$.
5.5. $\hat{A}_4 = \hat{P}_2$



GRADE	11	SUBJECT	Mathematics	TERM 3	Week 4	TOPIC	Proving Riders 1 Time: 55 min	Lesson	3

LESSON SUMMARY FOR: DATE STARTED:		DATE COMPLETED:	

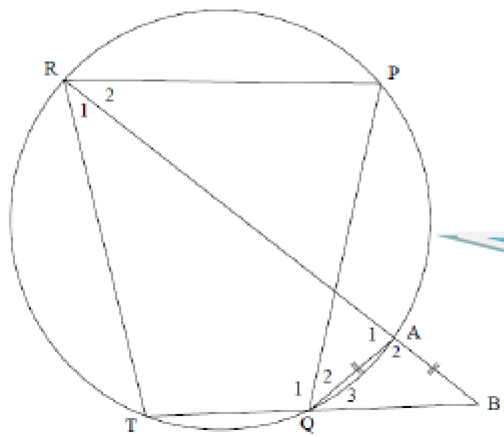
LESSON OBJECTIVES	The learners will consolidate their knowledge of the following concepts: • The line segment drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord. • The line segment drawn from the centre of a circle perpendicular to a chord bisects the chord. • The angle which an arc of a circle subtends at the centre of the circle is twice the angle it subtends at any point on the circle. • The angle subtended at the circle by a diameter is a right angle. • Angles in the same segment of a circle are equal. • The opposite angles of a cyclic quadrilateral are supplementary. • An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. • Two tangents drawn to a circle from the same point outside the circle are equal in length. • The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching Methods • Telling; Explanation; Question and Answer; Illustration; etc. 2. Lesson Development 2.1. Introduction • You need to make the learners aware that this lesson, and the lesson which follows, is aimed at consolidating all the facts learned in the theorems dealt with. • The lessons will involve solving riders from previous examinations which cover the range of the theorems stated above. • As the lesson progresses: o *You need to do corrections, together with the learners, on the board; explain and clarify misconceptions.* 2.2. Main Body (Lesson Presentation) o As stated in the introduction, the aim of this lesson is to put together all the facts and knowledge you have gained from the lessons in Euclidean Geometry thus far and use that to prove (solve) riders. o Let us look at a few riders from previous examination papers and work through them.	1. Baseline 2. Informal Assessment 3. Homework Exercise 4. Answers to Activities	Baseline: Lesson & Activities: 35 min	• **Lesson references:** o DoE/November 2008; Mathematics/P3 o DoE/Preparatory Examination 2008; Mathematics/P3 o DoE/May-June/2008; Mathematics/SG/P2 • **Materials needed:** o Chalkboard (or any other appropriate form of board); o Chalks (or markers). • **Other useful text references:**

<table>
<tr>
<td>

Example 1

In the diagram below, points R, P, A, Q and T lie on a circle. RA bisects $\hat{R}$ and AB = AQ. RA and TQ produced meet at B.



Prove that:

1.1. $AQ \text{ bisects } P\hat{Q}B$.

$\hat{Q}_3 = \hat{R}_1 = \hat{R}_2 = x$.......[ext.$\angle$ of a cyclic quad; RA bisects $\hat{R}$]
$\hat{R}_2 = \hat{Q}_2 = x$...............[$\angle$s in same segment]
Now $\hat{Q}_2 = \hat{Q}_3$

OR

$\hat{Q}_2 + \hat{Q}_3 = \hat{R}_1 + \hat{R}_2$.......[ext.$\angle$ of a cyclic quad; RA bisects $\hat{R}$]
but $\hat{Q}_2 = \hat{R}_2 = \hat{R}_1$.........[$\angle$s in same segment; RA bisects $\hat{R}$]
$\therefore \hat{Q}_3 = \hat{Q}_2$

</td>
<td>

After reading through the given information, try to mark out angles that are equal. Also see if you can recognize any theorems that the rider is based on.

</td>
<td>

Corrections:

15 min

</td>
<td>

- Guidelines Study Aids; Mathematics Paper 2; Geometry & Trigonometry; Grade 11 Higher Grade; pp. 133 – 160.
- Classroom Mathematics, Standard 9; pp. 309 – 365.
- New Mathematics; Std. 9; pp. 382 – 427.

</td>
</tr>
</table>

1.2. $TR = TB$.

$\hat{Q}_3 = \hat{B} = x$.................[$\angle$s opp. equal sides; $AQ = AB$]
$\hat{R}_1 = \hat{B} = x$.................[from 1.1.]
$\therefore TR = TB$.................[sides opp. equal angles]

If TR = TB, that would make ΔTRB to be isosceles. Therefore, we would have to prove that two of its angles (those opposite the sides in question) are equal.

1.3. $\hat{P} = T\hat{R}P$

$\hat{P} = \hat{A}_1$.......................[$\angle$s in same segment]
$\hat{A} = \hat{Q}_3 + \hat{B}$...............[ext.$\angle$ of ΔABC]
$\hat{Q}_3 + \hat{B} = 2\hat{Q}_3$.............[$\hat{Q}_3 = \hat{B}$; $\angle$s opp. = sides]
$2\hat{Q}_3 = 2\hat{R}_1$..................[from 1.1.]
$2\hat{R}_1 = P\hat{R}T$

OR

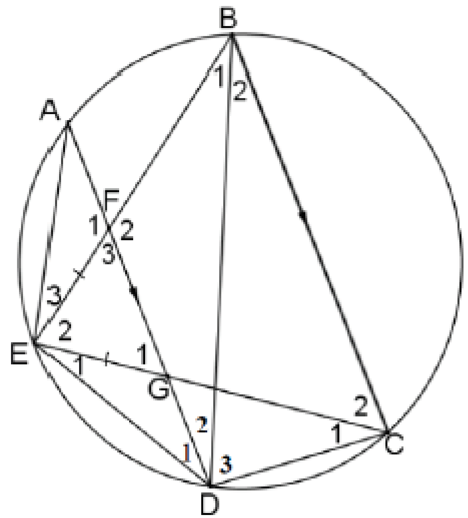
$T\hat{R}P = 2x$
$\hat{A}_1 = \hat{Q}_3 + \hat{B} = 2x$.........[ext.$\angle$ of a Δ]
$\therefore \hat{P} = \hat{A}_1 = 2x$..............[$\angle$s in same segment]
$= T\hat{R}P$

Example 2

In the sketch below, $AD // BC$ with A, B, C, D and E points on the circumference of the circle. EB, EC and BD are straight lines and EF = EG.

Prove that:

2.1. $\hat{B}_2 = \hat{D}_2$.
2.2. $\hat{E}_1 = \hat{E}_3$.
2.3. FBCG is a cyclic quadrilateral.



2.1. $\hat{B}_2 = \hat{D}_2$.

$B_2 = D_2$...............$[alt.\angle s;\ AD \parallel BC]$

2.2. $\hat{E}_1 = \hat{E}_3$.

$\hat{E}_1 = \hat{B}_2$..............$[\angle s\ in\ same\ segment]$
$\therefore \hat{E}_1 = \hat{D}_2$
$but\ D_2 = E_3$.......$[\angle s\ in\ same\ segment]$
$\therefore \hat{E}_1 = \hat{E}_3$

2.3. FBCG is a cyclic quadrilateral.

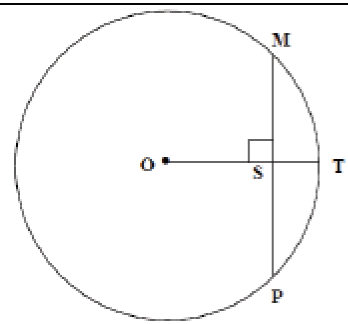
$EF = EG$...................[$given$] $\hat{F}_3 = \hat{G}_1$.......................[$\angle s\ opp. = sides$] $\hat{G}_1 = \hat{C}_2$.......................[$corresp.\angle s;\ AD // BC$] $\therefore \hat{F}_3 = \hat{C}_2$ $\therefore FBCG\ is\ a\ cyclic\ quad$........[$ext.\angle = int.\ opp\angle$] **2.3. <u>Conclusion</u>** • Revise key elements of the lesson. • Provide a chalkboard summary of the key concepts of the lesson.		Conclusion: ±5 min	

WORKSHEET:

QUESTION 1

In the diagram alongside, O is the centre of the circle. Radius OT and chord PM intersect at S. $OT \perp PM$. PM = 16 units; ST = 4 units and OS = x units.

Calculate, giving reasons, the numerical value of x.

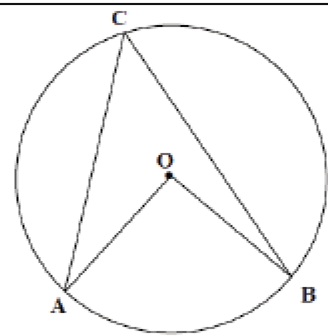


[6]

QUESTION 2

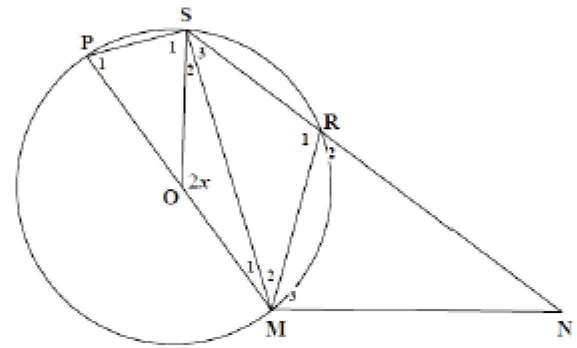
2.1.

In the diagram alongside, A, B and C are points on the circle. Use the diagram to prove the theorem which states that: ***If O is the centre of the circle, then*** $A\hat{O}B = 2\hat{C}$. **(6)**



2.2.

In the diagram alongside, PSRM is a circle with centre O. SR is produced to N and MN is drawn. Let $M\hat{O}S = 2x$.



2.2.1. Name, with reasons, THREE other angles each equal to x. **(5)**

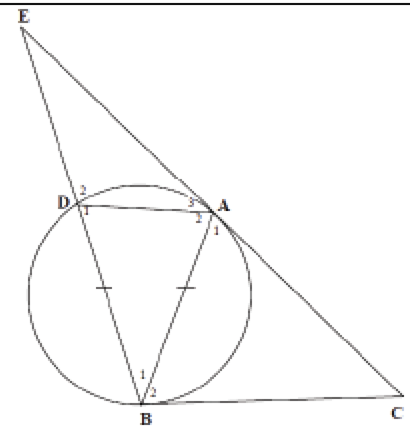
2.2.2. Express $\hat{S}_2$ in terms of x. **(3)**

2.2.3. Prove, with reasons, whether SRMO is a cyclic quadrilateral. **(3)**

[17]

QUESTION 3

In the diagram below, BC and CAE are tangents to circle DAB. BDE is a straight line. BD = BA. Prove that:



3.1. $\hat{D}_2 = \hat{A}_2 + \hat{A}_3$ **(3)**

3.2. $DA // BC$ **(4)**

[7]