THE CORRELATION BETWEEN GRADE 11 LEARNERS' FUNCTIONAL UNDERSTANDING OF PROOF AND ARGUMENTATION

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ABSTRACT – The purpose of this study was to determine the association between argumentation ability and multiple indicators of functional understanding of proof (verification; explanation; communication; discovery; and, systematization) after controlling for gender and sociodemographic variables. Guided by the sociocultural theory, data was drawn from the administration of two survey questionnaires to 135 Grade 11 Dinaledi high school learners from ethnically and socioeconomically diverse communities in the eThekwini metropolitan area, South Africa. Logistic regression was used to identify relationships between argumentation ability and functional understanding of proof. Argumentation ability was positively but tenuously associated with verification, explanation, communication, discovery, and systematization after controlling for gender. Whereas the explanatory function of proof exerted the greatest and statistically significant influence on learners' argumentation ability, the communication function of proof exerted the smallest and statistically insignificant influence on argumentation ability. Findings suggested that functional understanding of proof may enhance learners' argumentation ability. Explicit instruction on the functions that proof performs in mathematics is recommended.

Keywords: Functional Understanding of Proof; Argumentation; Multiple Regression; Explanatory Function; Gender

INTRODUCTION

Attempts to teach proof to high school learners (frequently during short periods of time) have been unsuccessful (Clements & Battista, 1992; Hadas, Hershkowitz, & Schwarz, 2000; Pedemonte, 2007). Given that the 'failure to teach proofs seems to be universal' (Hadas, Hershkowitz, & Schwarz, 2000, p. 128), functional understanding of proof and argumentation, activities Edwards (1997) refers to as the "territory before proof", need to be part of the mathematical activities that precede and support the development of proofs. Along this line, Marrades and Gutiérrez (2000) argue that it is vitally important for both teachers and researchers in the area of proof to know learners' understanding of functions of mathematical proof in order to understand their attempts to solve proof problems. The general motivation for this study came from the need to measure learners' understanding of the functions of proof in mathematics and argumentation quality since lack thereof contributes to difficulties with learning proofs meaningfully (e.g., de Villiers, 1990, Healy & Hoyles, 1998). The researcher joins Hsieh, Horng, and Shy (2012) in considering proof in the classroom more broadly 'as the product of a spectrum of activities starting with exploration, and progressing to the stages of conjecturing, informal explanation, and justification' (p. 288).

The phrase "functional understanding of proof" is used to refer to the function that proof performs in mathematics. De Villiers' (1990) model describes five functions that proof performs in mathematics: verification, explanation, communication, discovery, and systematisation. Thus, proving in the mathematical classroom includes not only cognitive functions (explanation and discovery) but also social (verification and communication,) and epistemological ones (systematization). According to Hanna (2000), the explanatory function of proving helps to make mathematics meaningful and understandable. This "enlightening" or illumination function brings argumentation into the arena. Support for this view comes from Hanna's (2007) statement that '[a]n argument presented with sufficient rigor will enlighten and convince more students, who in turn may convince their peers' (p. 22).

Of course, it is necessary to be clear about what is meant by argumentation. Although Pedemonte (2007) correctly argues that there is no common definition for the concept of argumentation in the field of mathematics education, the researcher adopted van Eemeren, Grootendorst, Johnson, Plantin, and Willard's (2013) definition of argumentation that is compatible with classroom contexts, '[a]rgumentation is a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint' (p. 5).

Toulmin's (2003) model decomposes an argument into six constitutive elements and describes the relationships between them: claim, data, warrant, backing, rebuttals, and qualifiers. It was useful in analysing learners' quality of argumentation. Guided by Vygotsky's (1978) sociocultural theory of learning, this research question was posed: What is the relationship between learners' functional understanding of proof and their on argumentation ability?

Geometry in South African high schools The importance of Euclidean geometry education as an integral component of mathematics curriculum was confirmed when it was made compulsory once again in South African high schools in 2011 (Bleeker, Stols, & Van Putten, 2013; Department of Basic Education [DBE], 2011). This reintroduction of proof into the CAPS mathematics curriculum reflected the notion that there is an appreciation of proof as the basis of mathematical knowledge. This notion finds support in Hersh's (1997) claim that proof is an essential tool for promoting mathematical understanding. However, for many learners, proof is just a ritual withoutmeaning (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). This perspective is reinforced when learners are required to write proofs according to a certain scheme or solely with symbols. In South Africa, as in most countries, the geometry curriculum includes Euclidean proof and analytical geometry. Whereas Euclidean geometry focuses on space and shape using a system of logical deductions, analytical geometry focuses on space and shape using algebra and a Cartesian coordinate system (Department of Basic Education [DBE], 2011; Uploaders, 2013).

In this study geometry has been taken to be the mathematics of shape and space, which traditionally incorporates but is not limited to Euclidean geometry. This study focused exclusively on Euclidean geometry on the basis that learner performance in this area has been consistently poor compared to the other geometries just mentioned. The South African high school mathematics curriculum, Euclidean geometry is the place where learners should engage in formal deductive reasoning as they do proofs. As previously mentioned, functional understanding of proof, one of the Specific Aims advocated in CAPS for mathematics, is based on van Hiele's (1986) broad theory of geometric thinking. Specifically, Euclidean proof (formal deduction) starts in Grade 10. In this grade, learners are expected to investigate, make conjectures, and prove the properties of the sides, angles, diagonals and areas of quadrilaterals; namely, kite, parallelogram, rectangle, rhombus, square, and trapezium (Department of Basic Education [DBE], 2011). In addition, they are required not only to know that a single counterexample can disprove a conjecture, but also that numerous specific examples supporting a conjecture do not constitute a general proof. Accordingly, very few will contest the notion that Grade 10 instruction is assumed to have had an impact on learners' functional understanding of proof in mathematics. Hence, this study investigated this understanding in Grade 11 learners. However, the weakness in CAPS is that there appears to be a lack of explicit content on the functions of proof as well as the historical aspects of proof.

As the researcher argued earlier, it is precisely this absence of instruction on functional understanding of proof that seem to inhibit learners' ability to construct proofs. By making the functions explicit, the intended curriculum can be realised. Support for this insistence arose out of Idris' (2006) assertion that since functional understanding of proof is a largely conventional concept, its learning cannot take place without explicit instruction. Needless to say, this is not a suggestion that ability to prove is secondary but an attempt to underscore functional understanding as a prerequisite aspect of constructing Euclidean proof.

THEORETICAL BACKGROUND

Mathematics education research has shown that most learners have serious difficulties with constructing proofs (Conner, 2007; de Villiers & Heideman, 2014). Harel and Sowder (1998) locate the cause of learners' difficulty in the logical aspect of proof construction. Thompson, Senk, and Johnson (2012) argue that some of the most persistent proof-related difficulties identified among learners in secondary school are a consequence of the confusion about the functions of proof in mathematics. Most research studies have focused on proof and proving as content of the curriculum to be learnt and taught. For instance, Knuth (2002) investigated teachers' conceptions of proof, Wu (2006) and Chin and Lin (2009) focused on learning how to read and write proofs, Hanna and Barbreau (2008) investigated ways to learn proof, and Harel and Sowder (2007) investigated the teaching of proof. A relatively small number of studies has discussed the functions of proof in mathematics (for example, de Villiers, 1990; Bell, 1976, & Hanna, 2000).

As far as the researcher could ascertain, only Healy and Hoyles' (1998) attempts to capture learners' functional understanding of proof. They conducted a nationwide (England and Wales) survey of 2 459 Grade 10 learners' functional understanding of proof in mathematics and how those learners constructed logical arguments (proof) in algebra and geometry. In particular, they used an open-ended survey questionnaire on which learners were to write about everything they knew of proof and its functions in mathematics. Further, they investigated the influence of statutory instruction on the nature of proof following suggestions that such instruction could contribute to deeper understanding of the notion of proof itself and thus improve its didactic treatment in the classroom. They found that the function of proof as a means to verify was prevalent. Hanna (1995) posit that learning about the functions of proof in mathematics is of primary importance to mathematicians. In the same vein, the researcher contends that the value of understanding the functions of proof in mathematics needs to be reflected in the mathematics classroom itself.

Very few can readily disagree with the contention that no single explanation accounts for the low scholastic achievement in Euclidean geometry. However, there is scarcity of empirical evidence on the influence of learners' functional understanding of proof on the quality of argumentation. Knipping (2003) recommends that it would be interesting if the relationship between functions of proof and argumentation structures were examined. Alibert and Thomas (1991) discuss the relationship between functional understanding of proof largely from a theoretical basis rather than conducting a systematic investigation. They believe that learners' distorted understanding of the functions of proof is a direct consequence of instruction that presents proof as a finished product; an approach that deprives learners of opportunities to be partners in mathematical knowledge construction.

Vygotsky's (1978) sociocultural theory of learning underpins the study reported herein. In particular, focus in this study was on the theory's notion of "zone of proximal development (ZPD)" which he describes as the guidance provided to a learner by a more knowledgeable other (teacher, parent, sibling, or peer) towards the attainment of new knowledge. According to Wersch (1985), interpreter of Russian sociocultural psychology, the notion ZPD was introduced as an attempt 'to deal with two practical problems: the assessment of children's intellectual abilities and the evaluation of instructional practice'. The rationale for the choice of this notion of ZPD lies in the fact that the three concepts, "learning", "argumentation", and "functional understanding of proof" not only have a social character but also that embedded in them is the overall aim of understanding what learners can do to construct proofs as a result of instructional practices that focus on the "territory before proof". Throughout this study, the term "learning" is used in a broad sense to encompass not only cognitive but also affective (attitudes and beliefs) notions of learners' mathematical experience (Stylianides & Stylianides, 2018). From a cognitive perspective, the term denotes the social process of appreciating the centrality of proof in mathematics and knowing how to make mathematically acceptable claims and justify them rather than to mean providing answers designed to reflect rehearsed application of procedures and algorithms only. This definition is consistent with Vygotsky's (1978) sociocultural theory which considers learning as involving scaffolding of a learner by "more knowledgeable others" such as a teachers, parents, older siblings or even peers.

The present study, therefore, expanded on previous research by disaggregating quality of argumentation and functional understanding of proof into their association with a variety of indicators. The hypothesis was that learners who appreciated the functions of proof engage in high quality argumentation. This relationship was tested controlling for gender.

METHODS

Sample design

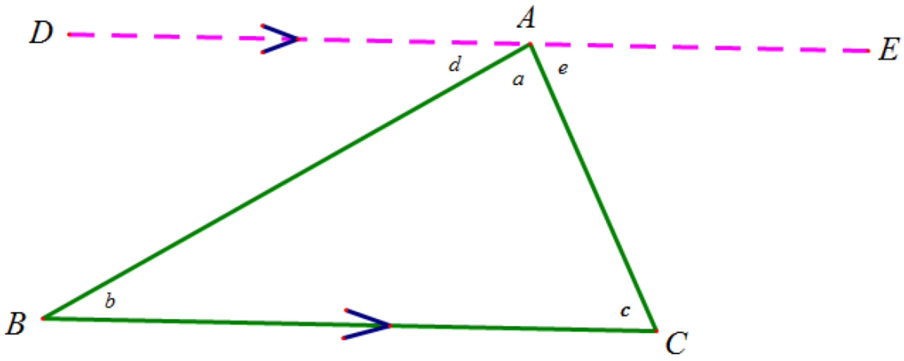
Data were drawn, at a single point in time, from three schools randomly selected sample of ten Dinaledi schools in ethnically and socioeconomically diverse communities in the urban district of the Ethekwini metropolitan area in September of 2017. The rationale for selecting Dinaledi schools for the investigation was that these schools were monitored by a team that included senior education department officials and individuals with an interest in educational research. Sample characteristics are in presented in Table1. Additional sociodemographic variables were also assessed. School type was defined as “fee paying” and “no fee”. Race was assessed with a single item: “Do you think of yourself as (1) African, (2) White (3) Indian, (4)or (4) Coloured. Respondents were grouped as African or nonAfrican for multivariateanalysis. Three levels of socioeconomic status (SES) were based on the highest educationallevel completed by either parent/guardian of respondents. Where this information was missing(n=13), the school type attended by the respondent was used to infer their SES.

Table 1: Characteristics of the sample: Sociodemographic factors

		Female	Male	Total
		78	57	135
Characteristic		(54.1%)	(45.9%)	100%
School type				
	Fee-paying	28	24	52
	No-fee	44	39	83
Race				
	White	3	5	8
	African	44	39	83
	Indian	18	16	34
	Coloured	7	3	10
Socioeconomic status				
	Low	46	39	85
	Middle	23	19	42
	High	3	5	8

Measures

The independent variables that were hypothesized as influencing (predicting) the dependent variable were: verification, explanation, communication, discovery, and systematization. Functional understanding of proof was assessed with the Learners’ Functional understanding of Proof (LFUP) scale. Quality of argumentation was assessed with the Argumentation Frame in Euclidean Geometry (AFEG) using the mathematical statement that *The interior angles of a triangle sum up to* 180^0. The duration of the questionnaires was 30 minutes. It consisted of prompts as shown in Figure 1.



Please, make ANY statement or claim from the diagram and justify it. Please, think carefully as you argue your points using the guide provided below.

My statement is that ……………………………………………...…………………………………………

My reason for making this statement is that ……………………………..

Arguments against my idea might be that ……....……………………………………………………………

Figure 1. The AFEG questionnaire

Two researchers determined the reliability of the tool using Cohen's (1968) kappa coefficient (κ). In addition, this coefficient was appropriate to use on the basis that we adopted a multicategory rubric comprising an ordinal scale in which responses were classified into 1 of 5 types of categories. Cohen's interrater agreements (κ) were: content = .95 and argumentation = .97. As Altman (1991) suggests, these values indicated very good agreement between raters.

Analysis

The LFUP questionnaire has 25 Likert scale items that range from 1 ("Strongly disagree") to 5 ("Strongly agree"). The scores on the LFUP scale were treated as interval level scale which was amenable to parametric statistical analyses. A five-tiered grading scale was used to assess learners' functional understanding of proof. Mean responses were interpreted according to the following categories: 0–<1.5 (unencultured); 1.5–<2.5 (poorly encultured); 2.5–< 3.5 (hybrid); 3.5–<4.5 (moderately encultured); 4.5–≤5 (extremely encultured). Learners' quality of argumentation was the dichotomous (binary, i.e., low or high) dependent variable whose values were to be predicted and therefore only contained data coded as 0 or 1.Table 2 describes how the quality of argumentation was assessed. The analysis wasperformed with the assistance of SPSS v.24 (Statistical Package for the Social Sciences forWindows (SPSS), 2017).

Table 2: Definition and coding of argument components

Argument	Definition	Code description	Quality
My statement is that …	A claim (C) is a conclusion put forward publicly for general acceptance (Toulmin, 2003).	No reply; uncodifiable.	Low
My statement is that …		C (Claim; conclusion)	Low
My reason is that …	A warrant is ground (G) provided in justifying the claim.	C+G (Providing reason for claim)	Low
Arguments against my idea might be that …	A rebuttal (R) meant statement that sought to diminish the strength of a conclusion (Pollock, 2001).	C+G+R (Refutation of claim/ground)	High

RESULTS AND DISCUSSION

Attempts to interpret the correlation between functional understanding of proof and argumentation quality were hampered by the possible existence of a third variable that may influence the relationship between the two variables. The researcher used partial correlations technique to statistically control or nullify the effects of gender (Wilson & MacLean, 2011), as the third or secondary variable, on the relationship between the primary variables; namely, functional understanding of proof and argumentation quality. The partialling out of gender was informed by research (e.g., Geary, 1999; Healy & Hoyles, 2000) which suggests that learner performance in mathematics tends to be a function of gender.

Since the zero-order correlations have already been analysed above, the researcher considered the section with the partial correlations in Table 3. In the previous section, the significant relationship between functional understanding of proof and gender seemed to suggest that gender has influence in explaining the understanding-argumentation association. However, the partial correlations section shows that controlling for gender further weakens the strength of the significant relationship between functional understanding of proof and argumentation ability (r = .214, p = .013). Clearly, controlling for gender was justified given that it was, as shown in Table 3, one secondary variable that seemed to influence the relationship between the two primary variables.

Table 3: Assessing the influence of functional understanding of proof on argumentation; controlling for gender

Control Variables			LFUP score	AFEG score	Gender
-none-[a]	LFUP score	Correlation	1.000		
		Significance (2-tailed)			
		df	0		
	AFEG score	Correlation	.225	1.000	
		Significance (2-tailed)	.009	.	
		df	133	0	
	Gender	Correlation	.171	.089	1.000
		Significance (2-tailed)	.047	.302	
		df	133	133	0
Gender	LFUP score	Correlation	1.000		
		Significance (2-tailed)			
		df	0		
	AFEG score	Correlation	.214	1.000	
		Significance (2-tailed)	.013	.	
		df	132	0	

a. Cells contain zero-order (Pearson) correlations.

The multiple correlation coefficient between argumentation scores and covariates combined, R, was computed. Then, the coefficient of determination ($R2$) which is the square of the Pearson product moment correlation coefficient, was used to express the proportion of variability in argumentation that can be accounted for by particular fucntional understanding of proof. According to Muijs' (2004) criteria, this model is of poor fit as it meant that only as low as 6.3 % of the variance in the argumentations scores were explained by the covariates (Table 4).

Table 4: A summary of the R, R square and adjusted R square in analysis of LFUP and AFEG

					Change Statistics				
Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	R Square Change	F Change	df1	df2	Sig. F Change
1	.252[a]	.063	.056	.87517	.063	9.011	1	133	.003

a. Predictors: (Constant), T15

b. Dependent Variable: AFEG score

Multiple regression was run to tease out which of the functional understanding of proof variables were most closely associated with argumentation quality (Table 5). The beta (*β*) values in Table 5 provide interesting information about some of these factors with regard to their relative effects on argumentation. First, whereas knowing that proof explains had the strongest positive and statistically significant effect on argumentation where *β* = .502 and the level of significance*, p* = .006, knowing both that proof is a means to verify and discover had nonsignificant impact on argumentation. Second, whereas knowing that proof is a means to systematize and communicate mathematical ideas yielded nonsignificant results, the former had a weakest negative effect (*β* = -.074) and the latter the strongest negative effect (*β* = –.327). Third, only knowing that proof systematises had a statistically nonsignificant result at .174 (*p* > .005) effect on argumentation. The interesting conclusion here was that only having an understanding that proof as a means to explain can be used to predict learners' argumentation ability.

Table 5: The beta coefficient in regression analysis

Model		Unstandardised Coefficients B	Unstandardised Coefficients Std. Error	Standardised Coefficients Beta	t	Sig.
1	(Constant)	.001	.569		.001	.999
	Verification	.140	.182	.083	.770	.443
	Explanation	.524	.186	.502	2.814	.006
	Communication	-.073	.228	-.074	-.318	.751
	Discovery	.164	.187	.119	.875	.383
	Systematization	-.266	.195	-.327	-1.368	.174

a. Dependent Variable: AFEG score

CONCLUSION

The purpose of this study was to investigate the relationship (if any) between learners' functional understanding of proof and their argumentation ability. Guided by the sociocultural theory, the basis for the hypothesis was formulated and appropriate data collection and analysis methods were selected. The hypothesis was supported by the empirical evidence in that the correlation between functional understanding of proof and quality of argumentation was, although tenuous, positive and statistically significant. Whereas, relative to each other, the explanatory function of proof exerted the greatest and statistically significant influence on learners' argumentation ability, the communication function of proof exerted the smallest and statistically insignificant influence on argumentation ability. In other words, the explanatory function of proof was found to be the factor which best predicted learners' success in argumentation ability. One limitation of this study is that the findings cannot be generalized to the population of Grade 11 learners.

The recommendation that Euclidean geometry curriculum needs to be revamped for the purpose of making functional understanding of proof and argumentation explicit and assessable content has implications for two constituencies. Instructional practices in high schools and methods modules at teacher training institutions need to include these exploratory activities (functional understanding of proof and argumentation) prior to engaging in the final step of formal proof construction. The limitation of the study is that learners' responses to the questionnaires were not probed. Future research initiatives need to blend close-ended items with open-ended questions to enhance insights into learners' functional understanding of proof because the results have shown that functional understanding of proof is a significant predictor of the quality of argumentation. Overall, the results of this study are offered as a contribution to the field's growing understanding of learners' activities prior to constructing proofs.

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