TASK-SOLVING IN MATHEMATICS: THE SIGNIFICANCE OF VISUAL IMAGERY

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ABSTRACT – Mathematics education research argues that mathematical problem solving relies heavily on visualisation in its different forms and at different levels, far beyond the obvious field of geometry. Mathematics educators are thus encouraged and inspired to 'see' not only what comes 'within sight' but also what we are unable to see when reviewing their students' work. The qualitative case study described in this paper speaks to this research as it examines the use of visualisation processes (as called visual imagery) in word problem solving. In our study, 17 Grade 11 learners participated in one-on-one task-based interviews. They answered 10 word problems, which we compiled in a worksheet, whose aim was to analyse the evidence of visual imagery in the participants' solutions and problem solving strategies. To analyse this evidence, we developed a visual imagery analytical framework that facilitated the analysis of the participants' responses in the worksheet, their interview transcripts as well as gestures in video-recordings. The findings suggest that developing tasks that encouraged the use of visual imagery to solve mathematical problems help learners to improve their problem solving functionality. Thus, all the 17 participants managed to use some kind of visual imagery during task-based interviews, although at different levels and in many different ways.

Keywords: Problem solving; Word problems; Visualisation

INTRODUCTION

The study presented in this paper observed how selected Grade 11 learners were encouraged through task-based interviews to incorporate visualisation in their word problem solving. Although much neglected in actual classroom practices, the significance of visualisation as a strategic problem solving method in mathematics has attracted interest from mathematics education researchers over the years (Arcavi, 2003; Csíkos, Szitányi, & Kelemen, 2012; Rivera, 2014). Arcavi (2003, pp. 216–217) claims that as a human and cultural creation dealing with objects and entities different from physical phenomena, mathematics relies heavily on visualisation in its different forms and at different levels far beyond the obvious field of geometry and spatial visualisation. Further, Edens and Potter (2007) claim that "visualising objects and graphically representing numerical information are important mathematical tools that help students to solve problems and to understand [mathematical] concepts" (p. 285). Similarly, Csíkos, Szitányi, and Kelemen (2012) note that because "mathematical concepts and relations are often based on visual mental representations attached to verbal information, the ability to generate, retain and manipulate abstract images is obviously important in mathematical problem solving" (pp. 49–50). Visual strategies are thus necessary for mediating understanding of mathematical objects among problem solvers (Rivera, 2014). Hence, placing visualisation at the service of problem solving "may play a central role to inspire a whole solution, beyond the merely procedural" (Arcavi, 2003, p. 224).

This paper explores 1) the types of visual imagery evident in the learners' solutions to word problems and 2) an environment (i.e. types of tasks and classroom culture) that supports the use of visual methods and facilitates an improvement in problem solving performance. These objectives stem from a bigger visualisation study (Dongwi, 2018), which has shown how the participants' use of visual methods made word problems easier to solve. We ask the following question: What is the role of visual imagery in word problem solving of the selected learners?

VISUAL IMAGERY

When Presmeg (1986) defined a visual image as "a mental scheme depicting visual or spatial information" (p. 42), she claimed to have deliberately broadened the definition to ensure inclusion of all other kind imagery, which depict shape, pattern or form without conforming to the "picture in the mind" notion of imagery (Clements, 1982). This also included imagery which reflected the vividness and clarity of a picture (Presmeg, 1986). Visual imagery refers to the ability to formulate mental representations of the appearance of objects and to manipulate these representations in the mind (Hegarty & Kozhevnikov, 1999). To define and analyse visual imagery in our study, we adapted Presmeg's (1986) five categories of visual imagery. Presmeg (2014) acknowledges a considerable growth in research "on how visualisation is implicated in the teaching and learning of mathematics at all levels" (p. 151). She however noted that many of the questions that emanated from her early 1980s research were still unanswered and in need of investigation (Presmeg, 2014). It is against this background that we adapted her visual imagery framework, which we defined as follows:

Concrete pictorial imagery (CPI) – this refers to the concrete image(s) of an actual situation formulated in a person's mind – i.e., a picture in the mind, drawn on paper or described verbally.

Pattern imagery (PI) – this refers to the type of imagery in which concrete details are disregarded and pure relationships are depicted in a visual-spatial scheme.

Memory imagery (MI) – this refers to the ability to visualise an image that one has seen somewhere before. This too includes a history of recurrent occurrences.

Kinaesthetic imagery (KI) – this is the kind of imagery that involves muscular activity. A kinaesthetic visualiser wants to feel and touch.

Dynamic Imagery (DI) – this imagery involves processes of transforming shapes i.e. redrawing given or initially own-drawn figures with the aim of solving a problem.

METHODS AND PROCEDURES

All the participants in our study completed the Enacted Visualisation Geometric Reasoning Tasks Worksheet (EVGRT) as part of data collection. The EVGRT was a set of ten mathematical problems that we compiled into a worksheet that was administered to the selected learners during semi-structured one-on-one task-based interviews. Interviews were audio- and video recorded and responses, both verbal and nonverbal were transcribed. The interviews consisted of the EVGRT worksheet that individual participants completed in the presence of the first author of this paper.

What sets these tasks apart from ordinary daily mathematical problems was their unusual and interesting nature. They required some sort of visual imagery to solve, and inspired numerous possible solution strategies. The format of each task provoked the learners to discover their own methods and visual representations that also necessitated some sort of mathematical reasoning. In the bigger PhD study (Dongwi, 2018), we developed 10 items for the EVGRT worksheet with the appropriate rationale for each items. In this paper however, we only chose two out of those 10 tasks and discuss how three out of our 17 participants solved them. We opted for two tasks and three participants given the paper's space constraints.

FINDINGS

The data analysed for this paper presents and describes how each of the three selected participants used visual imagery to solve two EVGRT viz. the rectangle task and the dice (cube) task. These tasks were developed to encourage the participants to represent word problems visually when solving them. The nature of the tasks also created rich opportunities for the participants to articulate and reason their way through problem-solving process. We placed in parentheses the codes of the visual imagery framework (e.g. CPI, PI, MI, KI and DI) to show how the participants incorporated visual imagery in problem solving. A more detailed framework with observable indicators is in the bigger PhD study (Dongwi, 2018). Below is a synopsis of how each participant visually represented each of the given tasks in problem solving.

The rectangle task – word problem without pictorial representation

The longest side of a rectangle has a length of 63 cm and the diagonals have a length of 65 cm each. Find the width of the rectangle (in cm).

Jordan's problem solving strategy of the rectangle task

Jordan paused when he read 'the diagonals'. He claimed to have imagined drawing a triangle (CPI) during his long pause but he nevertheless sketched a rectangle (Figure 1a) that he later divided into triangles using the diagonals as sides for the triangles (DI). He marked with a question mark the width of the rectangle that he had to find.

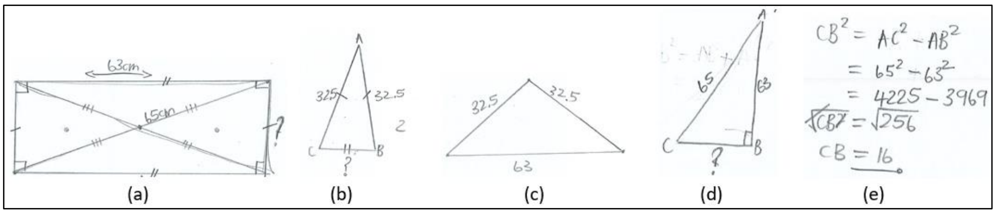


Figure 1: Jordan's visual representation of the rectangle task

Jordan extracted the first triangle (DI) (Figure 1b) from the complex rectangle (Figure 1a) to supposedly find side CB but then realised that he did not have sufficient information ("wait, it is going to be $AB^2 + AC^2 = BC^2$" (PI)). He also realised that the triangle was isosceles, which meant that he could not use the Pythagoras' theorem, so he revisited his original sketch. He quietly and thoughtfully traced triangles with his finger (KI) and then extracted the second triangle (DI) (Figure 1c) which he also realised that it could not generate the needed solution. With his head between his hands, he looked puzzled while staring at the sketched rectangle (CPI). He gestured lifting up each of the extracted tringles and placing it on top of its corresponding triangle in Figure 1 (a) (KI). He paused and then exclaimed, "Ooh…now I see!" (CPI). He noticed and extracted a right-angled triangle (DI), ABC (Figure 1d). He placed a question mark on the unknown side CB, and used the Pythagoras theorem to find the final answer (Figure 1e).

Ellena's problem solving strategy of the rectangle task

Slightly different from Jordan's method, Ellena solved the task in four steps. She straightaway sketched a rectangle after she read the problem for the first time and then the diagonal when she read it for the second time (CPI) (Figure 2a). She extracted a triangle in Figure (2b) in an attempt to find the width of the rectangle (DI). When she realised that this triangle could not generated what she wanted she wondered whether she had done anything wrong. She claimed to have been confused and could not figure out how she could find the width of the rectangle although she knew the lengths and the diagonals; she believed that she encountered a dead-end. This was partly because she did not yet at that stage notice the two right-angled triangles within the sketched rectangle (Figure 2a). Task-based interviews are helpful for such moments. The purpose of task-based interviews in our study was to prompt the participants to use visual imagery in problem solving. They were also encouraged to use both verbal and nonverbal modes to communicate their problem solving strategies (Dongwi, 2018). Cohen, Manion, and Morrison (2011) state that semi-structured interviews in particular enable researchers to motive their participants to discuss their thoughts, feelings and experiences. Hence, when Ellena was probed to unpack her complex rectangle in Figure 2(a), she identified different types of triangles that could be extracted from it (DI). She consequently extracted Figure (2c) and worked out her solution (Figure 2d) (PI, DI).

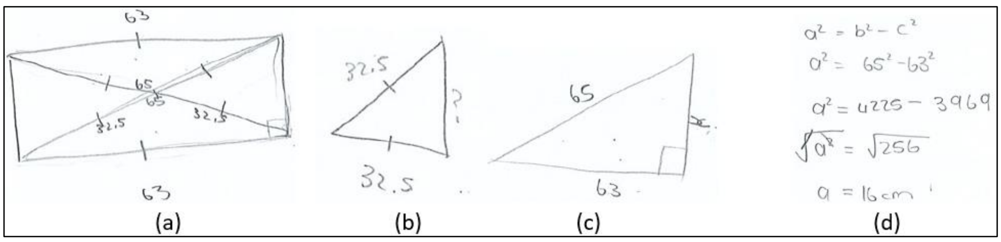


Figure 2: Ellena's visual representation of the rectangle task

Millie's problem solving strategy of the rectangle task

Millie claimed to have formulated mind pictures while reading the task (CPI). She took a long pause and then commented: "I'm just going to draw it". When asked what she was going to draw, she sketched a rectangle with a diagonal that she claimed represented her mental image (CPI) (Figure 3a). She worked privately on a calculator and then produced the solution in Figure 3 (b) (PI, CPI).

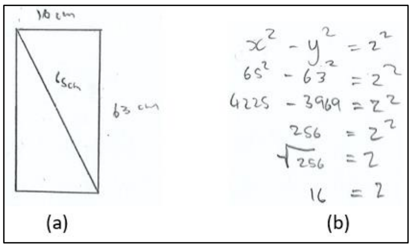
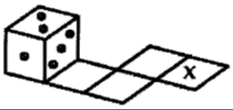


Figure 3: Millie's visual representation of the rectangle task

By sketching a rectangle with only one diagonal, it was easier for Millie to see the right-angled triangle that led to her using the Pythagoras' theorem to work out the solution much quicker than Jordan and Ellena whose sketches had both diagonals.

The dice (cube) task – word problem with an abstract diagram

On a dice, the numbers on opposite faces add up to 7. The dice in the diagram is rolled edge over edge along the path until it rests on the square labelled X. What is the number on top in that position?



By its nature, this task inspired the participants to use a combination of the 5VIs, ranging from moving pictures in the mind to actual dice rolling. The participants used hand gestures in this task more than they did in any other task in the worksheet. Below are summaries of how the three participants solved the dice task.

Jordan's problem solving strategy of the dice (cube) task

Jordan related the dice in the given diagram to a cube when he visualised the opposite sides of the dice (MI): "so if it is a cube, this side and this side will add up to 7, each one of the opposite faces" (Figure 4a) (KI). He silently used hand gestures as he proceeds through problem solving (CPI, KI). Asked to reveal what was in his mind he explained as follows:

> *See now this one is two* [circles the top of a dice with a pencil and writes 2 as he speaks (Figure 4c (KI))] *and the opposite will be 4…no… sorry 5 is equal to 7…we are looking for what will be the number when it fell on x* [circles the *x*]. *So it's gonna be 3 plus 4 is equal to 7, 1 plus 6 equals... When it's lying like that* [uses his hands to demonstrate the movement (KI) (Figure 4b)], *it will lie on 3* [writes 3, 2 and 6 on the square path to indicate the number that will land on each square as the dice is rolled (Figure 4c)]. …*the opposite number will be 4, so when it move to the side, 3, 2* [uses a pencil to visualise a rolling dice (KI, DI)], *6 and 1 then it will be… Okay* [holds his arms and rolling his head as if though moving a dice from one square to the next (CPI, KI, DI)] *here we have 3; x equals to 3*.

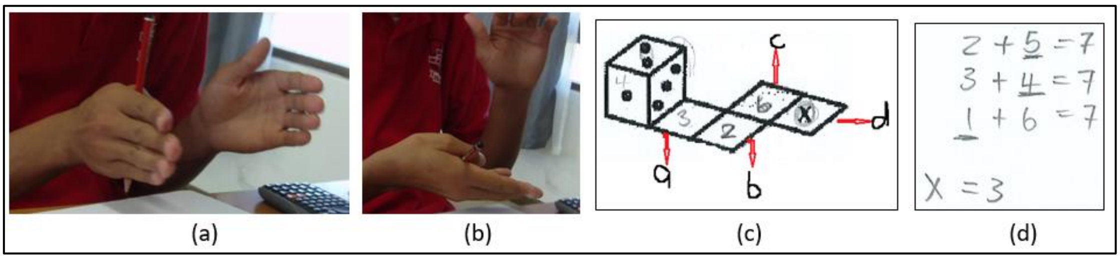


Figure 4: Jordan's visual representation of the dice (cube) task

Jordan visually demonstrated the actual rolling of the given dice by incorporating various 5VIs. Although he initially thought *x = 3* was the solution to question, he later rectified it as he concluded that 3 was the number on top when 4 landed on square X.

Ellena's problem solving strategy of the dice (cube) task

For this task, Ellena did not make any calculations or any drawings. She only used mind pictures combined with hand gestures for her problem-solving strategy.

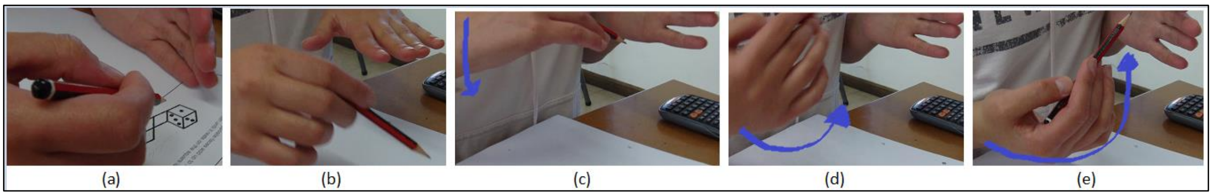


Figure 5: Ellena's visual representation of the dice (cube) task

In Figure 5 (a), Ellena's left hand represents the position (orientation) of the dice while her right hand shows the direction of the next move (CPI, KI, DI). She uttered, "*This one goes here*" (Figure 5b) before she became silent and used gestures in Figures 5(c) through 5(e) (CPI, KI, DI). The sequence of her movement through these figures was top-side-bottom (PI). After a few cycles of hand gestures and whispers, she requested for 'that box' (a designed box in a form of a cube that she had used in an earlier task) (MI, CPI). She calibrated it with numbers from 1 to 6, depicting the numbers on the faces of a dice that she literally rolled next to the given diagram (KI). She pronounced three as the answer to the question. Unlike Jordan and Millie there was no confusion whatsoever in Ellena's problem solving strategy for this task. She constantly reminded herself of what the main question was and she kept on repeating it (CPI) "*what is the number on top in that position?*" Hence, as she rolled the dice both in her mind and on paper (KI, DI), she always knew what she needed to find.

Millie's problem solving strategy of the dice (cube) task

When she turned to the task and before she read anything, Millie related to how she enjoyed working with diagrams and how such as the given one fascinated her (MI). She articulated shapes of the dice in each square even before she familiarised herself with the question (CPI). She even claimed to have visualised a rolling dice in her mind when she finally decided to read the question (CPI, DI). She demonstrated the movement of the dice by using her fingers and a pencil (KI, DI) (Figure 6a).

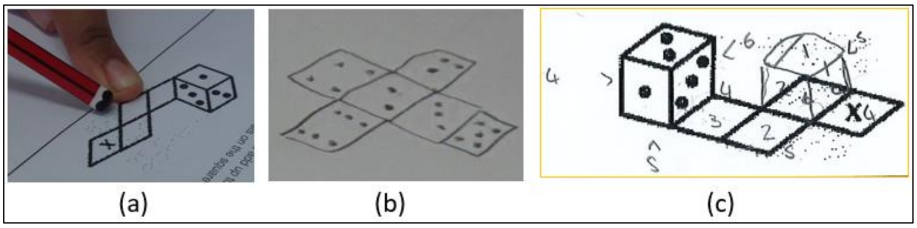


Figure 6: Millie's visual representation of the dice (cube) task

She paused a few times whenever she reached the third square with her imaginary rolling dice (CPI, KI, DI). She would then repeat the whole rolling process all over again (CPI, KI, DI). She admitted to have gotten a little confused by the dynamic pictures in her mind (DI). She sketched a net of the dice (CPI) (Figure 6b), turned the whole worksheet around to ensure a true representation of the given diagram (KI, DI). She worked from the net to sketch a dice on the third square (CPI, DI) (Figure 6c). Using the two sketches, she gestured with her fingers as she again visually moved her dice this time through the third square (KI, DI). These series of visual imageries enabled Millie to determine the number on square X and hence, to answer the question. She rolled an eraser in the square path to affirm the accuracy of her solution (KI, DI).

CONCLUSION

Visualisation processes used by the learners during problem-solving activities in our study proved both helpful and strategic. The participants used visual imagery in their minds, through gestures and when sketched on paper. When they solved the rectangle task, Jordan used trial and error sketching to determine the correct sketch that eventually helped him to solve the task. He mostly incorporated CPU, KI and DI through sketches and gestures during this task. Like Jordan, Ellena also used various sketches before she could determine the sketch that generated the correct method and solution to the rectangle task. When she somehow got confused, she employed dynamic imagery that enabled her to extract the necessary triangle that led her to solve the problem successfully. Unlike the other two participants, Millie's use of visual imagery to solve the rectangle task centred on concrete pictorial imagery. When she read the problem, she first internalised it and organised it in her mind before she uttered or sketched anything. This enabled her to conceptualise her thinking and effortlessly find the solution to the problem in relatively fewer steps in comparison to Jordan and Ellena. For the dice task, Jordan mostly used gestures with little writing on paper. Ellena only used gestures and Millie incorporated both gestures and sketches. All participants claimed to have experienced moving pictures in their minds when they imagined a rolling dice along the given path. They were able to see beyond only what came to mind. Their use of visual imagery in word problem solving offered them what Arcavi (2003) calls "a method of seeing the unseen" (p. 216), especially when they closed their eyes to see the numbers on the opposite faces of the dice while they imaginatively rolled it.

Lastly, from our observation, we have enough evidence to conclude that the use of visual imagery to solve word problems in our study facilitated the participants' problem-solving strategies and influenced their reasoning. Further, using diagrams helped the participants to see the hidden message in the linguistic information, which enabled them to better interpret and understand the questions. Van Garderen, Scheuermann and Poch, (2014) accentuate the complexity in the ability to use a diagram as a tool for solving word problems and warn that it should not be underestimated. Encoding information from a mathematical problem into a diagram requires an extensive knowledge base as it involves decoding the linguistic information and encoding it into visual information. This includes "knowledge related to the ability to select, produce and productively use a diagram as a problem-solving tool as well as the ability to critique and modify or generate a new diagram where needed within the context of a problem-solving situation" (p. 136).

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