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Guest Editorial

WOFACS '92: Interdisciplinarity and Collaboration

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This edition of SACJ is devoted to the Proceedings of WOFACS '92: the Workshop on Formal Aspects of Computer Science. The event was hosted at the University of Cape Town by FACCS-Lab, the Laboratory for Formal Aspects and Complexity in Computer Science, in July 1992.

The goal of WOFACS '92 was to bring together in a structured learning environment those Southern African computer scientists and mathematicians (academics and graduate students) interested in theoretical computer science. For this event FACCS-Lab imported four researchers eminent in their field, each to give a course of 10 lectures over a two-week period. Topics were carefully chosen so as to appeal to both mathematicians and computer scientists, and to reflect current work in the area of Formal Aspects of Computer Science. Each course was offered at beginning MSc level, and each could be taken by graduate students for credit at their respective home institutions. The guest lecturers and their topics were:

- Prof Roger Maddux (Iowa State University), *Predicate Transformer Semantics and Boolean Algebras with Operators*;
- Prof Austin Melton (Kansas State University), *Domains, Powerdomains and Power Structures*;
- Dr Hans Jürgen Ohlbach (Max Planck Institut für Informatik), *Deduction Systems in Artificial Intelligence*; and
- Prof Jeffery Zucker (McMaster University), *Theory of Computation.*

In addition, a number of Southern African researchers each gave an invited one-hour overview of the research done by themselves and their collaborators at their respective institutions. They were:

- Prof Ian Alderton (UNISA), *Cartesian Closedness*;
- Prof Guillaume Brümmer (University of Cape Town), *Completions in Topology and Elsewhere*;
- Prof Willem Fouché (University of Pretoria), *Identifying randomness given by high descriptive complexity*;
- Prof Johannes Heidema (UNISA), *Some Logics of Semantic Information*;
- Prof Stef Postma (University of the Western Cape), *Octolisp: a set of solutions; a source of problems*;
- Prof Yuri Velinov (University of Zimbabwe), *Categories, Nets and Parallel Computation*;
- Prof Basie von Solms (Randse Afrikaanse Universiteit), *Formal Languages and Automata as the Basis of a Number of Research Projects at the Rand Afrikaans University*; and
- Prof Peter Wood (University of Cape Town), *Detecting Bounded Recursion in Datalog Programs.*

WOFACS '92 was attended by about 75 participants from across the country, roughly half of whom were academics, the remainder being graduate students. Grants were made available to some graduate students who could not obtain funding from their home institutions. Participants who required accommodation were housed in a University Residence, and there was sufficient opportunity to enjoy the beauty of the Cape. Apart from travel and accommodation costs WOFACS '92 was a free service to the community: no fees of any kind were levied.

Simply put, WOFACS '92 was a *developmental* endeavour. The organisers considered Formal Aspects of Computer Science to be an increasingly important field of study, the pursuit of which should be encouraged on a broad front in the Southern African environment. To create a sense of community it was important to bring all interested parties together. On the other hand, we felt that it would be premature to issue a Call for Papers and organise a conference. Thus arose the idea of a Workshop, where people come to learn, and to share information on research projects.

It happens that the WOFACS endeavour fits in with two points of view recently put forward in the editorial pages of the *SA Computer Journal*. In a guest editorial in SACJ 6 (March 1992) Ed Coffman, the FRD-sponsored guest at the 1991 SA Computer Research Conference, gives his impressions of Computer Science research in South Africa. His overriding impression is that South Africa is weak in the area of Formal Aspects (or, as Coffman says, computer and computation theory). Coffman strongly recommends a concerted development in this area. He allows that Mathematics Departments could play an important role in this development – provided they make a major commitment and do not regard the area as a mere service discipline.

Coffman mentions further the practical advantages in a financially constrained environment of inter-departmental and inter-institutional cooperation. At a more fundamental level this point has also been addressed by SACJ Editor

Derrick Kourie. In his Editor's Notes to SACJ 8 (November 1992) Kourie discusses the dual forces of competition and collaboration at play in the Southern African research environment, concludes that there is too much of the former, and makes a strong plea for the latter. In Kourie's view, it is in the nature of scientific research that it will flourish where there is a healthy spirit of collaboration. Moreover, Kourie contends, the benefits of such collaboration should not remain confined to single Departments, but must be extended to an inter-departmental and inter-institutional level.

The foregoing ideas fit well with the basic FACCS-Lab philosophy of interdisciplinarity. We believe that computer scientists and mathematicians can and should collaborate, and that if they do it will be to their mutual benefit. To quote from the FACCS-Lab 1992 Report:

> FACCS-Lab aims to bring Formal Methods and Complexity Theory to bear on problems of Computer Science, in a structured interdisciplinary research programme intended to contribute to the development of research manpower in South Africa. In a developing country such as South Africa it is essential that the more applied sciences (such as Computer Science) should retain a good research base. It is also important that the theoretical sciences (such as Mathematics) should retain links with real-world developmental issues. FACCS-Lab aims to provide a bridge between Mathematics and Computer Science in South Africa, to further its overall developmental aim.

Activities such as WOFACS '92 are intended to contribute to this overall developmental aim. No doubt we can still improve our efforts, and feedback would be welcome.

It is the pleasure and privilege of any organiser to express thanks to those who contribute to the success of an event. I would like to do so, conveying my sincere thanks and that of FACCS-Lab to:

- Roger Maddux, Austin Melton, Hans Jürgen Ohlbach and Jeff Zucker, for coming, for presenting a course, evaluating students, and contributing to this SACJ edition;
- The local speakers, for helping to clarify the picture of what is being done in Southern Africa;
- Janet Goslett, the WOFACS secretary, and Maureen le Sar, my personal secretary, for keeping the show on the road;
- Laurette Pretorius, Hardy Hulley and Janet Goslett, for being actively involved in writing material for this edition;
- The Foundation for Research Development, and the University of Cape Town Research Committee, for funding WOFACS '92; and
- Cliff Moran, Dean of Science at UCT, for finding the money to publish this edition of SACJ.

Editor's Notes

This issue of SACJ is an archive of material presented at a workshop on formal aspects of computer science. The workshop – known as WOFACS '92 – was organised by the Laboratory for Formal Aspects and Complexity in Computer Science (FACCS-Lab) in the Department of Mathematics at the University of Cape Town. As one of the Research Leaders of FACCS-Lab, Prof Chris Brink was a prime mover in getting the event off the ground. As guest editor of this SACJ edition, he has collated the material and, most importantly, organised the funding for this issue.

Consequently, SACJ subscribers are able to reap the benefit of having access to WOFACS material without affecting the production of other editions of the journal. (The next edition is scheduled to appear in the near future.) On behalf of readers, I would like to thank Prof Brink for his initiatives, as well as the four contributors for making their work available. SACJ's production editor, Riël Smit – who has handled the final typesetting in his usual efficient, competent and uncomplaining fashion – also deserves our sincere thanks.

I hope and trust that this archival material will be of lasting value to those who teach and research in the area of formal aspects of computing.

Derrick Kourie
Editor

Production notes

I hope readers are not too perturbed by the fact that half of the articles in this issue is set in single column rather than the usual two column style. This was necessitated by the many wide formulae used in these papers.

Production Editor

A Working Relational Model: The derivation of the Dijkstra-Scholten predicate transformer semantics from Tarski's axioms for the Peirce-Schröder calculus of relations

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Abstract

The definitions for Dijkstra's predicate transformer semantics can be justified by considering the meaning of a program as a binary relation on states (which holds between two states if there is a terminating computation connecting them) together with a set of states (those which initiate eternal computations). In fact, all of the definitions can be proved as theorems, using Tarski's ten equational postulates for the calculus of relations. These postulates define relation algebras, so we take the meaning of a program to be a pair of elements of a complete relation algebra, one corresponding to the binary relation determined by the terminating computations, the other corresponding to the set of states initiating eternal computations. The predicate transformers for a program are defined from these two elements. In this abstract setting we derive all the significant definitions and theorems of Dijkstra and Scholten's "Predicate Calculus and Program Semantics".
Keywords: *relation algebras, semantics, relational, axiomatic, predicate transformers, preconditions.*
Computing Review Categories: *F.3.2, F.3.1, F.4.1*

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1 Introduction

This paper is a revised and more complete version of some notes on the book "Predicate Calculus and Program Semantics" by Edsger W. Dijkstra and Carel S. Scholten [DS90]. The notes were used in conjunction with a course of ten lectures, called "Predicate Transformer Semantics and Boolean Algebras with Operators", given at the Workshop on Formal Aspects of Computer Science (WOFACS'92), University of Cape Town, 6–17 July 1992. The suggested text for this course was [DS90]. Chapters 5, 6, and 8 of [DS90] include material on Boolean algebras, unary operators on Boolean algebras, and part of Tarski's Fixed Point Theorem for monotone functions on a complete lattice. This material serves as preparation for the predicate transformer semantics presented in Chapters 7 and 9, according to which the meaning of each program is determined by its two associated predicate transformers. Predicate transformers are unary operators on a Boolean algebra. Predicate transformer semantics are explained and justified in Chapters 7 and 10 by relating them to "operational semantics", which, in this case, means the analysis of a program according to the connection it establishes between the states of an abstract machine through its terminating computations and the set of states which initiate its eternal computations.

As it happens, such operational semantics are absolutely perfectly suited for a formal presentation within the century-old calculus of relations, invented by Augustus De Morgan, Charles S. Peirce, and Ernst Schröder. In this approach, a program's meaning is determined by a binary relation on states (which holds between two states if there is a terminating computation connecting them) and a set of states (those which initiate eternal computations).

But even more is possible. The purely abstract "postulational method" of [DS90 pp.124–5] can be used. Indeed, all of the theorems can be proved from the set of ten equational postulates proposed by Tarski [T41] as an axiomatization for a portion of the calculus of relations. These postulates serve as the definition of relation algebras. In this abstract relational semantics, the meaning of a program is determined by two elements of a relation algebra, one corresponding to the binary relation determined by the terminating computations, the other corresponding to the set of states initiating eternal computations. The predicate transformers for a program are defined from these two elements.

The operational semantics for each compound program are determined by the operational semantics for its constituent parts. The exact nature of the connection depends on the way in which the parts are combined. In every case it is possible to translate this operational connection directly into the calculus of relations, thus providing a definition by which the two elements associated with a compound program can be computed from the the elements associated with its constituent parts. The connections between the predicate transformers of compound programs and the predicate transformers of their constituent parts, which are presented in [DS90] as

definitions, thereby become theorems. Because of the applicability of relation algebras to program semantics, the material for the course at WOFACS'92 was expanded to include a section on relation algebras, which, incidentally, happen to be a special kind of Boolean algebras with operators, and many of the principal definitions and theorems from Chapters 7–10 of [DS90] were proved from Tarski's ten axioms, such as the Main Repetition Theorem and the characterizations of the predicate transformers for **do** $B \rightarrow S$ **od** as least and greatest fixed points.

The paper is organized as follows. Section 2 contains an explanation of the predicate transformer semantics, based largely on quotations from [DS90]. Sections 3–6 consist entirely of background material, none of which is new (except possibly one specialized theorem about relation algebras). An attempt has been made to make these sections complete, that is, they start from axioms and build up all the results needed for proving the theorems on semantics in later sections. Almost all the results in these sections have been provided with detailed proofs, but, in order to make the sections more convenient for reference, all the proofs have been relegated to an Appendix. Reconstructing all of these proofs without consulting the Appendix is a very challenging exercise. The main part of the paper begins in Section 7, where arbitrary interpretations are studied, those in which each program receives its meaning as a pair of elements in a relation algebra, but absolutely no connection is assumed to hold between the elements assigned to a compound program and the elements assigned to its constituent parts. In spite of this generality, much can be proved, including a "healthiness" condition for all programs and characterizations of determinacy. Section 8 introduces "correct" interpretations, those which "correctly" (that is, according to the dictates of [DS90]) connect the elements assigned to the parts of a complex program with those assigned to the program itself. After some motivation for these connections based on quotations from [DS90], the definition of correct interpretation is given, followed by a string of theorems relating the predicate transformers of complex programs to the predicate transformers of their parts. In Section 9 it is shown that if the basic commands of the language satisfy the "law of the excluded miracle", then so do all the compound programs. Some results concerning determinism are presented in Section 10, and the "Main Repetition Theorem" of [DS90] is given in Section 11. A formula in [DS90] is referred to by a notation of the form "(n,m)", which means "formula (m) of Chapter n". This device is adopted from [DS90].

2 The Semantics of a Program

Quotations are taken from [DS90]. Page and line numbers are given for each quotation. For example, "121_{1-5}" refers to lines 1 through 5 from the bottom of page 121, while "124^{10-11}" refers to lines 10 and 11 from the top of page 124.

"The semantics of a programming language defines for each program written in that language 'what that program means'. ... The semantics refers to the execution of the programs, i.e., what would happen each time such a program is fed into an appropriate computer." (121_{1-5}) "In the current case, the question is: what are we going to ignore and what are we going to take into account when referring to 'what would happen each time such a program is fed into an appropriate computer'? To begin with —and not surprisingly so— we are going to ignore all physical characteristics of the computer:" (122_{8-13}) ... "it is the abstract machine that matters, because that is the one we can think about." (122_{1-2})

"To simplify matters further, we ignore all devices for input and output of information: the value of the input absorbed by a computation is deemed to be captured by the initial state of the machine —i.e., the state in which the machine starts the computation— , the value of the output is similarly deemed to be captured by the final state of the machine —i.e., the state in which the machine is left upon completion of the computation— ." (123_{8-13}) Various "observations strongly support our decision not to give input and output a special status in our considerations." (124^{10-11}) "Accordingly, we shall no longer define the semantics via the detour of the nett [*sic*] effect of the computations a program may evoke; instead we shall define more directly how, for any program in the language, initial and final states are connected." (124_{8-11})

"The only things of interest that remain are the initial state, the final state, and how a program defines a connection between them. Since computations no longer enter the picture, we can forget about machines and computational models. The postulational method allows us to treat programs as uninterpreted formulae, i.e., mathematical objects in their own right, that we can deal with while ignoring the fact that they are interpretable as executable code."(125^{1-6})

"To begin with, we look for a helpful classification of computations. A major dichotomy is into terminating and eternal computations: a terminating computation has an initial state and a final state, whereas an eternal computation has no final state." (126_{7-10})

"The general way of distinguishing points in state space is the introduction of some predicate X; a predicate X defines a dichotomy of the state space in the sense that each state satisfies either X or $\neg X$." (126_{1-4})

"In summary, we propose to partition the computations into the following —indeed mutually exclusive— classes:

'eternal' —all computations that fail to terminate

'finally X' —all computations terminating in a final state satisfying X

'finally $\neg X$' —all computations terminating in a final state satisfying $\neg X$." (127^{15-22})

"We now take the view that we know everything that is to be known about the semantics of a program S if we know, for any predicate X and for any initial state, computations from which of the three classes are possible when S is started in that initial state." (127_{13-16})

"For given S and X, we propose the following three predicates .... The first of three predicates is independent of X.

$wp.S.true$: holds precisely in those initial states for which no computation under control of S belongs to the class 'eternal'

$wlp.S.X$: holds precisely in those initial states for which no computation under control of S belongs to the class 'finally $\neg X$'

$wlp.S.(\neg X)$: holds precisely in those initial states for which no computation under control of S belongs to the class 'finally X'." (127_{1-9})

"That is, $wp.S$ and $wlp.S$, being functions from predicates to predicates, emerge as predicate transformers. (The first one has, so far, only been applied to the argument $true$, but that will be remedied shortly.)" (128^{19-21})

"As we said, we consider the semantics of a program S fully characterized by the knowledge of the predicates $wp.S.true$ and $wlp.S.X$ for all X, i.e., the knowledge of the predicate $wp.S.true$ and the predicate transformer $wlp.S$." (128_{13-15})

"And now the time has come to introduce, in terms of the predicate $wp.S.true$ and the predicate transformer $wlp.S$, the second predicate transformer we associate with program S. It is the predicate transformer $wp.S$ given by

(2) $$[wp.S.X \equiv wp.S.true \wedge wlp.S.X] \quad \text{for all } X.$$ (129^{7-11})

"From the interpretation of the conjuncts in the right-hand side of (2) we derive the interpretation of $wp.S.X$:

$wp.S.X$: holds precisely in those initial states for which each computation under control of S belongs to the class 'finally X'.

The names wp and wlp are derived from 'weakest precondition' and 'weakest liberal precondition', respectively: $wp.S.X$ is 'the weakest precondition under which S is guaranteed to establish the postcondition X', $wlp.S.X$ is 'the weakest precondition under which S is guaranteed to establish the postcondition X if computation terminates'. In the jargon: $wlp.S$ is concerned with 'the partial correctness of S' (i.e., apart from possible failure to terminate), whereas $wp.S$ is concerned with 'the total correctness of S' (i.e., termination included)." (129^{15-26})

We now introduce two binary relations between states, r_S and e_S, with the following interpretations.

r_S : holds precisely for those pairs of states for which there is a terminating computation under control of S whose initial state is the first state of the pair, and whose final state is the second state.

e_S : holds precisely for those pairs of states for which there is an eternal computation under control of S whose initial state is the first state of the pair.

The first relation r_S could be called the "terminating-computation relation of S", because it contains all pairs of states that are connected by terminating computations of S. The second relation e_S is an example of what

is called here a "domain relation", that is, one which is entirely determined by its domain because it happens to be the direct product of its domain with the whole universe of states. The sets of states are obviously in one-to-one correspondence with the domain relations on states, and every predicate X has its corresponding domain relation. We incorporate sets in the calculus of relations here in the guise of domain relations. (Many other ways are possible, but this one is most convenient for our purposes.) The domain of e_S is the set of states which initiate eternal computations of S, so perhaps e_S could be called the "eternal-computation relation of S".

The predicate transformers *wlp.S* and *wp.S* send predicates to predicates. Accordingly, we will define operations wlp_S and wp_S, in terms of r_S and e_S, which send domain relations to domain relations. To do so, we introduce three operations on binary relations between states, namely, relative multiplication (also called composition), intersection, and complementation, as follows. Let x and y be binary relations on a set of states U. For any states $u, v \in U$, let $\langle u, v\rangle$ be the ordered pair whose first state is u and second state is v. Then

$$\begin{aligned}
\overline{x} &= \{\langle u, v\rangle : \langle u, v\rangle \notin x \text{ and } u, v \in U\} \\
x\,;y &= \{\langle u, w\rangle : \text{there is some } v \in U \text{ such that } \langle u, v\rangle \in x \text{ and } \langle v, w\rangle \in y\} \\
x \cdot y &= \{\langle u, v\rangle : \langle u, v\rangle \in x \text{ and } \langle u, v\rangle \in y\}
\end{aligned}$$

Now let x be the domain relation consisting of those pairs whose first state belongs to the class "finally X". Then we define

$$\begin{aligned}
\mathrm{wlp}_S\,(x) &= \overline{r_S\,;\overline{x}} \\
\mathrm{wp}_S\,(x) &= \overline{r_S\,;\overline{x}} \cdot \overline{e_S}
\end{aligned}$$

It is now a very simple, but extremely important, exercise to verify that these definitions agree with the interpretations from [DS90], that is, $\mathrm{wlp}_S\,(x)$ is the domain relation whose domain consists of all those states satisfying the predicate *wlp.S.X*, and similarly for $\mathrm{wp}_S\,(x)$ and *wp.S.X*. Toward that end, here is a fairly literal reading of the definition of $\mathrm{wlp}_S\,(x)$. The pair of states $\langle u, w\rangle$ is in $\mathrm{wlp}_S\,(x)$ if and only if it is not in $r_S\,;\overline{x}$, that is, there does not exist a third state v which can be reached from u by a terminating computation under control of S starting at u with final state v such that $\langle v, w\rangle$ is not in the domain relation x. Now $\langle v, w\rangle$ is not in x if and only if v does not satisfy X. Therefore $\langle u, w\rangle$ is in $\mathrm{wlp}_S\,(x)$ if and only if every terminating computation of S starting at u must end at a state satisfying X, that is, every terminating computation of S starting at u belongs to the class "finally X". Of course, if $\langle u, w\rangle$ is in $\mathrm{wlp}_S\,(x)$, then there may be eternal computations starting at u, but no terminating computation from u can belong to the class "finally $\neg X$". Therefore $\langle u, w\rangle$ is in $\mathrm{wlp}_S\,(x)$ if and only if no computation of S from u belongs to the class "finally $\neg X$", in agreement with the interpretation of *wlp.S.X* from [DS90].

This much has been observed before, and the formula defining $\mathrm{wlp}_S\,(x)$ has appeared elsewhere. A relational definition of $\mathrm{wp}_S\,(x)$ must deal with nontermination. Some authors have used a formula for $\mathrm{wp}_S\,(x)$ which is simply incorrect, that is, one which does not accord with the interpretation of *wp.S.X* given above from [DS90]. For example, if 1 denotes the universal relation between states, *i.e.*, all pairs of states belong to 1, then $\overline{r_S\,;\overline{x}} \cdot r_S\,;1$ is the domain relation for the predicate which holds at precisely those states at which *wlp.S.X* holds and a terminating computation of S is possible. Defining $\mathrm{wp}_S\,(x)$ to be $\overline{r_S\,;\overline{x}} \cdot r_S\,;1$ is the same as identifying guaranteed termination with the existence of some terminating computation, *i.e.*, mistakenly assuming $\overline{e_S} = r_S\,;1$. Another frequently used strategy is to add a fictional state, "—usually called 'bottom'— that can be interpreted as 'stuck in an eternal computation'." ($125_{2\text{–}3}$) This causes difficulties. See [R91] for a survey of such attempts. "In this context, experience with the relational calculus has not been too favourable. The transition to the relational calculus does almost suffice for the elimination of 'bottom', but not quite! Moreover the relational calculus, which treats the two arguments of a relation on the same footing, does not, by itself, reflect the asymmetry between initial and final state". ($126^{17\text{–}22}$) The trick used here to avoid "bottom" and reflect asymmetry is quite simple: translate directly from [DS90] into the relational calculus. Since nontermination is handled in [DS90] by the predicate *wp.S.true*, we translate *wp.S.true* into a domain relation, obtaining $\overline{e_S}$, according to the interpretations of e_S and *wp.S.true* given above, and then define $\mathrm{wp}_S\,(x)$ by imitating (2). Since "bottom" is not used in [DS90] it is, naturally, not used here either. The interpretation for $\mathrm{wp}_S\,(x)$ thus obtained is: $\langle u, w\rangle$ is in $\mathrm{wp}_S\,(x)$ if and only if every terminating computation of S starting at u belongs to the class "finally X" and no eternal computation of S begins at u, *i.e.*, every computation starting at u belongs to the class "finally X".

The definitions of $\mathrm{wlp}_S\,(x)$ and $\mathrm{wp}_S\,(x)$ are taken as a starting point in Section 7, and results from [DS90] can be successfully proved from the ten equational axioms for relation algebras (given in Sections 3 and 5). In view of this success, it is interesting that, immediately following their comments concerning "bottom" and asymmetry, Dijkstra and Scholten say, "These are admittedly only tentative explanations for the not-too-fortunate experience with the relational calculus. Other possible explanations are that no one trying to apply the relational calculus in this area mastered it well enough, or that the relational calculus needs a few notational revisions before it can be considered a workable tool." ($126^{22\text{–}26}$) Perhaps they are right.

One final observation parallels the end of the Preface in [DS90]. "Honesty compels to add to this wish that there is one possible —and, alas, likely— 'improvement' we are not waiting for, viz., the translation of our theory into set-theoretical terminology —by interpreting predicates as characteristic functions of subsets of states— so as to make it all the more familiar. Little is so regrettable as to see one's work 'improved upon' by the introduction of traditional complications one has been very careful to avoid. ... the existence of individual machine states enters the picture only when our theory is applied to program semantics, ... the theory itself does not need a postulate of the existence of individual states and, therefore, should not be cluttered by their introduction." (viii$_{1-4}$ and ix$^{1-11}$) Rest assured that such cluttering has not occurred here. This Introduction notwithstanding, the theory presented below is purely postulational, starting from an arbitrary complete relation algebra whose elements need not be binary relations on states (although such algebras are covered by the theory). Indeed, since there are relation algebras which are not isomorphic to any algebra of binary relations over a set, the theory has a strictly wider application than any theory dealing only with relations between states. Machine states, sets of them, and relations between them are surely important for explaining the inspiration, motivation, and applications for the theory, as has been done here and in Chapters 7 and 10 of [DS90], but the theory itself, both in [DS90] and here in Sections 7–11, has no mention of states and "should not be cluttered by their introduction". Concerning their intended audience, Dijkstra and Scholten say, "We most sincerely hope to reach them, to thrill them, and to inspire them to improve their own work and ours". (viii$_{4-5}$) This paper is, if not an improvement, then, at least, inspired by their work.

3 Orderings, Lattices, and Boolean Algebras

This and the next three sections contain only background material. With few exceptions, the coverage has been limited to those definitions and results which are actually used in the sections on semantics.

Parentheses are omitted from terms according to the convention that a repeated binary operation is computed from left to right, *e.g.*, $x + y + z = (x + y) + z$, the operation ; takes precedence over $\cdot$, and $\cdot$ takes precedence over $+$. The scope of joins and meets is always as small as possible.

Sometimes we use "iff" as an abbreviation for "if and only if".

Definition 1. *Let B be an arbitrary set and let $\leq$ be a binary relation on B.*

(i) *The relation $\leq$ is called a* **partial ordering of** *B if for all $x, y \in B$ we have*
  (1) *$x \leq x$ ($\leq$ is reflexive over B),*
  (2) *if $x \leq y$ and $y \leq z$ then $x \leq z$ ($\leq$ is transitive),*
  (3) *if $x \leq y$ and $y \leq x$ then $x = y$ ($\leq$ is antisymmetric).*

(ii) *Let I be an arbitrary set and suppose there is some $x_i \in B$ for every $i \in I$. An element $y \in B$ is an* **upper bound of** *$\{x_i : i \in I\}$ if $x_i \leq y$ for every $i \in I$.*

(iii) *If y is an upper bound of $\{x_i : i \in I\}$ and $y \leq z$ for every upper bound z of $\{x_i : i \in I\}$, then y is called the* **least upper bound of** *$\{x_i : i \in I\}$ or, simply, the* **join of** *$\{x_i : i \in I\}$ and is denoted $\sum\{x_i : i \in I\}$ or $\sum_{i \in I} x_i$.*

(iv) *An element $y \in B$ is a* **lower bound of** *$\{x_i : i \in I\}$ if $y \leq x_i$ for every $i \in I$.*

(v) *If y is a lower bound of $\{x_i : i \in I\}$ and $z \leq y$ for every lower bound z of $\{x_i : i \in I\}$, then y is called the* **greatest lower bound of** *$\{x_i : i \in I\}$ or the* **meet of** *$\{x_i : i \in I\}$) and is denoted $\prod\{x_i : i \in I\}$ or $\prod_{i \in I} x_i$.*

(vi) *If $\leq$ is a partial ordering of B and the join and meet of $\{x_i : i \in I\}$ both exist whenever I is a two-element set, then $\langle B, \leq \rangle$ is called a* **lattice.**

(vii) *A lattice is* **complete** *if the join and meet of $\{x_i : i \in I\}$ always exist, regardless of the cardinality of I.*

We frequently use the notation "$\sum\{x : \varphi(x)\}$" or "$\sum_{\varphi(x)} x$", where $\varphi(x)$ is some condition on x. The meaning of this notation is simply $\sum_{i \in I} y_i$, where $I = \{x : \varphi(x)\}$ and $y_i = i$ for every $i \in I$. A similar explanation applies to "$\prod\{x : \varphi(x)\}$" and "$\prod_{\varphi(x)} x$".

Definition 2. *A* **Boolean algebra** *is an algebraic structure of the form $\mathfrak{B} = \langle B, +, {}^- \rangle$, where B is a nonempty set, $+$ is a binary operation on B, and ${}^-$ is a unary operation on B, such that the following axioms are satisfied for all $x, y, z \in B$:*

(Ba$_1$) $x + y + z = x + (y + z)$,
(Ba$_2$) $x + y = y + x$,
(Ba$_3$) $x = \overline{\overline{x} + \overline{y}} + \overline{\overline{x} + y}$.

An additional binary operation $\cdot$ on B is defined by

(Ba_4) $x \cdot y = \overline{\overline{x} + \overline{y}}$.

The axiomatization (Ba_1)–(Ba_3) is due to E. V. Huntington ([Hu33], [Hu33a]). There is a fascinating open problem connected with this axiomatization (Problem 1.1 of [HMT71]). The "dual" of (Ba_3) is

(Ba_3') $x = \overline{\overline{x + \overline{y}} + \overline{x + y}}$.

If an algebra $\mathfrak{B} = \langle B, +, {}^- \rangle$ satisfies (Ba_1), (Ba_2), and (Ba_3'), must it be a Boolean algebra? Probably the answer is "no". It is interesting to note if $\mathfrak{B}$ is a *finite* algebra satisfying (Ba_1), (Ba_2), and (Ba_3') then $\mathfrak{B}$ is, in fact, a Boolean algebra. The reason for this is that every finite algebra which satisfies (Ba_3') must also satisfy (Ba_3). To see this, suppose that $\mathfrak{B}$ is a finite algebra which satisfies (Ba_3'). From the form of (Ba_3') it is clear that the operation $^-$ is onto. Since $\mathfrak{B}$ is finite, the operation $^-$ must also be one-to-one. Substitute $\overline{x}$ for x in (Ba_3') to get $\overline{x} = \overline{\overline{x} + \overline{\overline{y}} + \overline{\overline{x} + y}}$. Since $^-$ is one-to-one, this entails $x = \overline{\overline{x} + \overline{y}} + \overline{\overline{x} + y}$. Thus ($\mathrm{Ba}_3$) holds in $\mathfrak{B}$.

Theorem 3. *The following identities are satisfied in every Boolean algebra.*

(i) $x + \overline{x} = y + \overline{y}$.
(ii) $\overline{\overline{x}} = x$.
(iii) $\overline{x} = \overline{x + \overline{y}} + \overline{x + y}$.
(iv) $x + (y + \overline{y}) = z + \overline{z}$.
(v) $x + x = x + \overline{y + \overline{y}}$.
(vi) $x + x = x$.
(vii) $x \cdot x = x$.
(viii) $x \cdot y = y \cdot x$.
(ix) $x \cdot y \cdot z = x \cdot (y \cdot z)$.
(x) $(x + y) \cdot x = x$.
(xi) $x = x \cdot y + x \cdot \overline{y}$.
(xii) $x = (x + \overline{y}) \cdot (x + y)$.
(xiii) $(x + y) \cdot \overline{x} = y \cdot \overline{x}$.
(xiv) $x + x \cdot y = x$.
(xv) $x \cdot (y + z) = x \cdot y + x \cdot z$.
(xvi) $\overline{x + y} = \overline{x} \cdot \overline{y}$.
(xvii) $\overline{x \cdot y} = \overline{x} + \overline{y}$.
(xviii) $x + y \cdot z = (x + y) \cdot (x + z)$.
(xix) $\overline{x} \cdot y + x \cdot z = (x + y) \cdot (\overline{x} + z)$.

From 3(i) it follows that, in every Boolean algebra $\mathfrak{B}$, the sets $\{x + \overline{x} : x \in B\}$ and $\{\overline{x + \overline{x}} : x \in B\}$ each contain exactly one element.

Definition 4. *For every Boolean algebra $\mathfrak{B}$, the unique element in $\{x + \overline{x} : x \in B\}$ is denoted by 1, and the unique element in $\{\overline{x + \overline{x}} : x \in B\}$ is denoted by 0.*

Theorem 5. *The following identities are satisfied in every Boolean algebra.*

(i) $1 = x + \overline{x}$.
(ii) $0 = x \cdot \overline{x}$.
(iii) $\overline{1} = 0$.
(iv) $\overline{0} = 1$.
(v) $x + 1 = 1$.
(vi) $x \cdot 0 = 0$.
(vii) $x + 0 = x$.
(viii) $x \cdot 1 = x$.

Definition 6. *For every Boolean algebra $\mathfrak{B}$, define a binary relation $\leq$ on B as follows: $x \leq y$ iff $x + y = y$.*

Theorem 7. *Let $\mathfrak{B}$ be a Boolean algebra.*

(i) *The relation $\leq$ is a partial ordering of B.*
(ii) *For every $x \in B$, $0 \leq x$ and $x \leq 1$.*
(iii) *For all $x, y, z \in B$, if $x \leq y$ then $x + z \leq y + z$ and $x \cdot z \leq y \cdot z$.*
(iv) *$\langle B, \leq \rangle$ is a lattice in which the join of $\{x, y\} \subseteq B$ is $x + y$ and the meet of $\{x, y\}$ is $x \cdot y$.*
(v) *The following statements are equivalent for all $x, y \in B$:*
  (1) $x \leq y$,
  (2) $\overline{y} \leq \overline{x}$,
  (3) $x + y = y$,

(4) $x \cdot y = x$,
(5) $\overline{x} + y = 1$,
(6) $x \cdot \overline{y} = 0$.

Theorem 8. *Let $\mathfrak{B}$ be a Boolean algebra. Let I be an arbitrary set. Suppose $x_i \in B$ for every $i \in I$, and $y \in B$.*

(i) *If $I = \emptyset$ then $\sum_{i \in I} x_i = 0$ and $\prod_{i \in I} x_i = 1$.*
(ii) *If $\sum_{i \in I} x_i$ exists then $\prod_{i \in I} \overline{x_i}$ also exists and $\overline{\sum_{i \in I} x_i} = \prod_{i \in I} \overline{x_i}$.*
(iii) *If $\prod_{i \in I} x_i$ exists then $\sum_{i \in I} \overline{x_i}$ also exists and $\overline{\prod_{i \in I} x_i} = \sum_{i \in I} \overline{x_i}$.*

Theorem 9. *Let $\mathfrak{B}$ be a Boolean algebra. Let I be an arbitrary set and suppose $x_i, y_i \in B$ for every $i \in I$.*

(i) *If $\sum_{i \in I} x_i$ and $\sum_{i \in I} y_i$ exist, then $\sum_{i \in I}(x_i + y_i)$ also exists and $\sum_{i \in I}(x_i + y_i) = \sum_{i \in I} x_i + \sum_{i \in I} y_i$.*
(ii) *If $\prod_{i \in I} x_i$ and $\prod_{i \in I} y_i$ exist, then $\prod_{i \in I}(x_i \cdot y_i)$ also exists and $\prod_{i \in I}(x_i \cdot y_i) = \prod_{i \in I} x_i \cdot \prod_{i \in I} y_i$.*
(iii) *Suppose $x_i \leq y_i$ for every $i \in I$. If $\sum_{i \in I} x_i$ and $\sum_{i \in I} y_i$ exist, then $\sum_{i \in I} x_i \leq \sum_{i \in I} y_i$. If $\prod_{i \in I} x_i$ and $\prod_{i \in I} y_i$ exist, then $\prod_{i \in I} x_i \leq \prod_{i \in I} y_i$.*

Theorem 10. *Let $\mathfrak{B}$ be a Boolean algebra. Let I and J be arbitrary sets. Suppose $x_i \in B$ for every $i \in I \cup J$.*

(i) *If $\sum_{i \in I} x_i$ and $\sum_{i \in J} x_i$ exist, then $\sum_{i \in I \cup J} x_i$ also exists and $\sum_{i \in I \cup J} x_i = \sum_{i \in I} x_i + \sum_{i \in J} x_i$,*
(ii) *If $\prod_{i \in I} x_i$ and $\prod_{i \in J} x_i$ exist, then $\prod_{i \in I \cup J} x_i$ also exists, and $\prod_{i \in I \cup J} x_i = \prod_{i \in I} x_i \cdot \prod_{i \in J} x_i$.*
(iii) *If $\sum_{i \in I} x_i$ and $\sum_{j \in J} x_j$ exist and $I \subseteq J$, then $\sum_{i \in I} x_i \leq \sum_{j \in J} x_j$.*
(iv) *If $\prod_{i \in I} x_i$ and $\prod_{i \in J} x_i$ exist and $I \subseteq J$, then $\prod_{i \in J} x_i \leq \prod_{i \in I} x_i$.*

4 Operators on Boolean Algebras

This section is a brief exposition of some of the theory of unary operators on Boolean algebras. Except for residuals, the material is taken from [JT51] and [DS90]. The terminology used here is a mixture of the terminology from those two sources.

Definition 11. *For every Boolean algebra $\mathfrak{B}$ and every function f mapping B to B, f^* is the* **dual of** *f, where f^* is defined for every $x \in B$ by $f^*(x) = \overline{f(\overline{x})}$.*

Theorem 12. [DS90 (6,3) p.83] *Every function f on a Boolean algebra $\mathfrak{B}$ is the dual of its dual, i.e., $f^{**} = f$.*

Definition 13. *Let f and g be functions on a Boolean algebra $\mathfrak{B}$.*

(i) *g* **is a conjugate of** *f just in case for all $x, y \in B$, $f(x) \cdot y = 0$ iff $x \cdot g(y) = 0$.*
(ii) *g* **is a residual of** *f just in case for all $x, y \in B$, $f(x) \leq y$ iff $x \leq g(y)$.*

Theorem 14. *Let f, g, and h be functions on a Boolean algebra $\mathfrak{B}$.*

(i) *If g and h are conjugates of f, then $g = h$.*
(ii) *If g and h are residuals of f, then $g = h$.*
(iii) *The following statements are equivalent:*
  (1) *g is a conjugate of f,*
  (2) *f is a conjugate of g,*
  (3) *g^* is a residual of f,*
  (4) *f^* is a residual of g.*
(iv) *The following statements are also equivalent:*
  (1) *g is a residual of f,*
  (2) *f^* is a residual of g^*,*
  (3) *g^* is a conjugate of f,*
  (4) *f is a conjugate of g^*.*

In view of the symmetry expressed by the equivalence of **14**(iii)(1) and **14**(iii)(2), we shall say "f and g are conjugate" instead of "f is a conjugate of g".

Definition 15. *A function f on a Boolean algebra $\mathfrak{B}$ is*

(i) **normal** *if* $f(0) = 0$,
(ii) **monotone** *(or* **increasing***) if* $x \leq y$ *implies* $f(x) \leq f(y)$ *for all* $x, y \in B$,
(iii) **additive** *(or* **finitely disjunctive***) if* $f(x + y) = f(x) + f(y)$ *for all* $x, y \in B$,
(iv) **multiplicative** *(or* **finitely conjunctive***) if* $f(x \cdot y) = f(x) \cdot f(y)$ *for all* $x, y \in B$,
(v) **completely additive** *(or* **positively disjunctive***) if, for every indexed set* $\{x_i : i \in I\} \subseteq B$, *if* $\sum_{i \in I} x_i$ *exists and* I *is not empty then* $\sum_{i \in I} f(x_i)$ *also exists and* $f(\sum_{i \in I} x_i) = \sum_{i \in I} f(x_i)$,
(vi) **universally disjunctive** *if it is both normal and completely additive,*
(vii) **completely multiplicative** *(or* **positively conjunctive***) if, for every indexed set* $\{x_i : i \in I\} \subseteq B$, *if* $\prod_{i \in I} x_i$ *exists and* I *is not empty then* $\prod_{i \in I} f(x_i)$ *also exists and* $f(\prod_{i \in I} x_i) = \prod_{i \in I} f(x_i)$,
(viii) **universally conjunctive** *if* f *is positively conjunctive and* $f(1) = 1$.

Theorem 16. [JT51 1.2] *For every Boolean algebra $\mathfrak{B}$ and every $x \in B$, the function $x \cdot (\text{-})$ is universally disjunctive and positively conjunctive, and the function $x + (\text{-})$ is universally conjunctive and positively disjunctive.*

Theorem 17.

(i) [JT51 p.898] *Every completely additive function on a Boolean algebra is additive and every additive function is monotone.*
(ii) *Every completely multiplicative function on a Boolean algebra is multiplicative and every multiplicative function is monotone.*

Theorem 18. [DS90 (6,9) p.84] *A function on a Boolean algebra is universally, positively, or finitely disjunctive if and only if its dual is universally, positively, or finitely conjunctive, respectively.*

Theorem 19. [JT51 1.13 and 1.14] [DS90 (11,1) p.202]

(i) *The conjugate g of a function f on a Boolean algebra $\mathfrak{B}$, if it exists, is given by*

$$g(y) = \prod_{f(x) \cdot y = 0} \overline{x}$$

for every $y \in B$.
(ii) *The function f has a conjugate just in case the following conditions are satisfied:*
  (1) *f is universally disjunctive,*
  (2) $\sum_{f(x) \leq y} x$ *exists for every* $y \in B$.

Theorem 20. [JT51 1.15] *Let f and g be functions on a Boolean algebra $\mathfrak{B}$. The following statements are equivalent:*

(1) *f and g are conjugate.*
(2) *For all* $x, y \in B$,
  (a) $f(x \cdot \overline{g(y)}) \leq f(x) \cdot \overline{y}$,
  (b) $g(y \cdot \overline{f(x)}) \leq g(y) \cdot \overline{x}$.
(3) $f(0) = 0$, $g(0) = 0$, *and, for all* $y, z \in B$,
  (a) $f(y) \cdot z \leq f(y \cdot g(z))$,
  (b) $g(z) \cdot y \leq g(z \cdot f(y))$.

Definition 21. *For any function f and any $i \in \omega$, f^i is the result of composing f with itself i times. More precisely, $f^0(x) = x$ and $f^{i+1} = f(f^i(x))$ for every x and every $i \in \omega$.*

The following theorem is designed for use in the proof of Theorem **30**

Theorem 22. *Let f and g be conjugate functions on a Boolean algebra $\mathfrak{B}$. Let $b \in B$ and assume $\sum_{i \in \omega} f^i(b)$ exists. Let $c \in B$ and define $h(x) = c + f(x)$ for every $x \in B$. For every $y \in B$, if $y \leq b + h(y)$ and $z = y \cdot \overline{\sum_{i \in \omega} f^i(b)}$ then $z \leq h(z) = c + f(z)$.*

5 Relation Algebras

Charles S. Peirce ([P1870], [P1880], and especially [P1883]) combined the work of George Boole ([B1847]) and Augustus De Morgan ([D1856], [D1864]) to create a calculus of relations which was extensively developed by Ernst Schröder ([S1895]). A fragment of this calculus was axiomatized by Alfred Tarski ([T41]). His axiomatization was the basis for the definition of relation algebras ([JT48], [CT51], [JT52]). For further introductory and historical material on relation algebras see [CT51], [J82], [J91], [JT52], [M91], [M91a], and [TG87]. This section contains just enough basic definitions and results for the applications given later. Everything in the section except Theorem **30** can be found in [CT51] or [JT52].

Definition 23. ***A relation algebra*** *is an algebraic structure of the form*

$$\mathfrak{A} = \langle A, +, {}^{-}, ;, {}^{\smile}, 1' \rangle ,$$

where $\langle A, +, {}^{-} \rangle$ *is a Boolean algebra,* $;$ *is a binary operation on* A, ${}^{\smile}$ *is a unary operation on* A, *and* $1'$ *is an element of* A, *such that the following axioms are satisfied for all* $x, y, z \in A$:

(Ra_1) $x;y;z = x;(y;z)$,
(Ra_2) $(x+y);z = x;z + y;z$,
(Ra_3) $x;1' = x$,
(Ra_4) $\breve{\breve{x}} = x$,
(Ra_5) $(x+y)^{\smile} = \breve{x} + \breve{y}$,
(Ra_6) $(x;y)^{\smile} = \breve{y};\breve{x}$,
(Ra_7) $\breve{x};\overline{x;y} + \overline{y} = \overline{y}$.

An additional binary operation $\dagger$ *on* A *is defined by*

(Ra_8) $x \dagger y = \overline{\overline{x};\overline{y}}$.

Note that (Ra_7) is equivalent to $\breve{x};\overline{x;y} \leq \overline{y}$ and to $\breve{x};\overline{x;y} \cdot y = 0$.

Examples. Let U be an arbitrary set, called "the universe". Let $\mathrm{Re}(U) = \{x : x \subseteq U \times U\}$, that is, $\mathrm{Re}(U)$ is the set of all binary relations on the universe U. We can obtain a relation algebra of the form

$$\mathfrak{Re}(U) = \langle \mathrm{Re}(U), +, {}^{-}, ;, {}^{\smile}, 1' \rangle ,$$

by defining $1'$ and the operations $+$, ${}^{-}$, $;$, and ${}^{\smile}$ in the following ways. First, let $1'$ be the identity relation on U, that is,

(1) $1' = \{\langle u, u\rangle : u \in U\}$.

Next, for any binary relations $x, y \in \mathrm{Re}(U)$, let

(2) $x + y = \{\langle u, v\rangle : \langle u, v\rangle \in x \text{ or } \langle u, v\rangle \in y\}$,
(3) $\overline{x} = \{\langle u, v\rangle : \langle u, v\rangle \in U \text{ and } \langle u, v\rangle \notin x\}$,
(4) $x;y = \{\langle u, w\rangle : \text{there is some } v \in U \text{ such that } \langle u, v\rangle \in x \text{ and } \langle v, w\rangle \in y\}$,
(5) $\breve{x} = \{\langle v, u\rangle : \langle u, v\rangle \in U\}$.

Thus $x + y$ is the union of x and y, $\overline{x}$ is the complement of x with respect to $U \times U$, $x;y$ is the relative product of x and y, and $\breve{x}$ is the converse of x. It is a straightforward exercise to verify that $\mathfrak{Re}(U)$ satisfies axioms (Ba_1)-(Ba_3) and (Ra_1)-(Ra_7). Therefore $\mathfrak{Re}(U)$ is a relation algebra. By definition (Ba_4), $x \cdot y$ is the intersection of x and y. The relation $x \dagger y$, defined by (Ra_8), is called the relative sum of x and y.

If U is empty, then $\mathfrak{Re}(U)$ is an algebra with just one element in it. If U contains exactly one element, then $\mathfrak{Re}(U)$ is Boolean, that is, it satisfies the identity $1' = 1$. On the other hand, if $\mathfrak{Re}(U)$ satisfies $1' = 1$, then U is either empty or has exactly one element. If U is finite, so is $\mathfrak{Re}(U)$. If U is a countable infinite set, then $\mathfrak{Re}(U)$ is an uncountable algebra.

Theorem 24. *The following statements hold in every relation algebra.*

(i) $x \leq y$ *iff* $\breve{x} \leq \breve{y}$.
(ii) $\breve{0} = 0$.
(iii) $\breve{1} = 1$.
(iv) $\overline{\breve{x}} = \breve{\overline{x}}$.
(v) $(x \cdot y)^{\smile} = \breve{x} \cdot \breve{y}$.
(vi) $0 = \breve{x} \cdot y$ *iff* $0 = x \cdot \breve{y}$.
(vii) *The function* ${}^{\smile}$ *is universally disjunctive and universally conjunctive.*
(viii) $\breve{1}' = 1'$.
(ix) $x;(y+z) = x;y + x;z$.
(x) *If* $x \leq y$ *then* $z;x \leq z;y$ *and* $x;z \leq y;z$.
(xi) *The functions* $x;(\text{-})$ *and* $\breve{x};(\text{-})$ *are conjugate.*

(xii) $\overline{y;x};\breve{x} \leq \overline{y}$.
(xiii) *The functions* $(\text{-});x$ *and* $(\text{-});\breve{x}$ *are conjugate.*
(xiv) $x;y \cdot z = 0$ *iff* $\breve{x};z \cdot y = 0$ *iff* $z;\breve{y} \cdot x = 0$.
(xv) *The functions* $x;(\text{-})$ *and* $(\text{-});x$ *are universally disjunctive.*
(xvi) $x;y \cdot z \leq x;(y \cdot \breve{x};z)$.
(xvii) $y;x \cdot z \leq (y \cdot z;\breve{x});x$.
(xviii) $x;0 = 0 = 0;x$.
(xix) $x;1' = x = 1';x$.
(xx) $x \leq x;1$.
(xxi) $x \leq 1;x$.
(xxii) $1;1 = 1$.
(xxiii) *If* $x;1 = x$ *then* $\overline{x};1 = \overline{x}$.
(xxiv) *If* $x;1 = x$ *then* $(x \cdot y);z = x \cdot y;z$.
(xxv) *If* $x;1 = x$ *and* $y;1 = y$ *then* $(x \cdot y);1 = x \cdot y$.
(xxvi) *If* $x;1 = x$ *then* $(x \cdot 1');y = x \cdot y$.
(xxvii) *If* $x;1 = x$ *then* $(y \cdot \breve{x});z = y;(x \cdot z) = (y \cdot \breve{x});(x \cdot z)$.

Definition 25. *An element* x *of a relation algebra* $\mathfrak{A}$ *is a* **domain element** *if* $x;1 = x$.

For every set U, the domain elements of $\mathfrak{Re}(U)$ are the domain relations on U, that is, the relations of the form $V \times U$ where $V \subseteq U$.

Theorem 26. *Let* $\mathfrak{A}$ *be a relation algebra and let* D *be the set of domain elements of* $\mathfrak{A}$. *Then*

(i) D *is closed under* $^-$, $+$, *and* $\cdot$.
(ii) D *is closed under* $x;(\text{-})$ *for every* $x \in A$.
(iii) *If* $\{x_i : i \in I\} \subseteq D$ *and* $\sum_{i \in I} x_i$ *exists, then* $\sum_{i \in I} x_i \in D$.
(iv) *If* $\{x_i : i \in I\} \subseteq D$ *and* $\prod_{i \in I} x_i$ *exists, then* $\prod_{i \in I} x_i \in D$.

Definition 27. *An element* x *of a relation algebra* $\mathfrak{A}$ *is a* **functional element** *if* $\breve{x};x \leq 1'$.

The functional elements of $\mathfrak{Re}(U)$ are the partial functions from U to U.

Theorem 28. *Let* $\mathfrak{A}$ *be a relation algebra.*

(i) [CT51 3.39] *If* x *and* y *are functional elements of* $\mathfrak{A}$, *then so is* $x;y$.
(ii) [CT51 4.2] *If* x *is a functional element of* $\mathfrak{A}$, *then* $x;(y \cdot z) = x;y \cdot x;z$ *for all* $y, z \in A$.
(iii) [CT51 4.2] *An element* x *of* $\mathfrak{A}$ *is functional if and only if* $x;y \cdot x;\overline{y} = 0$ *for every element* $y \in A$.

Definition 29. *For every element* x *of a relation algebra let* $x^0 = 1'$, *and, for every* $i \in \omega$, *let* $x^{i+1} = x;x^i$. *Let* $x^\omega = \sum\{x^i : i \in \omega\}$ *whenever the join exists.*

The following theorem is designed for use in the proof of Theorem **53(ii)(iv)**

Theorem 30. *Assume* y, r, b, *and* c *are elements of a relation algebra* $\mathfrak{A}$ *and* $y \leq b + c + r;y$. *Assume* $\sum_{i \in \omega}(r^i;b)$ *exists. Let* $z = y \cdot \overline{\sum_{i \in \omega}(r^i;b)}$. *Then* $z \leq c + r;z$.

6 Tarski's Fixed Point Theorem

In 1927 Knaster and Tarski proved that every function, mapping subsets of a set U to subsets of U, which is increasing (with respect to set-theoretical inclusion) has at least one fixed point. In 1939 Tarski proved a lattice-theoretical generalization of this theorem. The generalization was published (along with many applications) in 1955 [T55, Theorem 1] and is presented in this section.

Theorem 31. (Tarski [T55]) *Assume that* $\langle A, \leq \rangle$ *is a complete lattice and that* f *is a monotone function from* A *to* A, *that is,*

$$\text{if } x, y \in A \text{ and } x \leq y \text{ then } f(x) \leq f(y).$$

Let I, D, *and* F *be the subsets of* A *which are increased, decreased, and fixed by* f, *respectively, i.e.,*

$$I = \{x : x \leq f(x)\}, \quad D = \{x : f(x) \leq x\}, \quad F = \{x : x = f(x)\}.$$

Then

(i) $\langle F, \leq \rangle$ *is a nonempty complete lattice.*
(ii) $\sum I = \sum F \in F$.
(iii) $\prod D = \prod F \in F$.

It may happen that the least upper bound of $X \subseteq F$ in the lattice $\langle F, \leq \rangle$ of fixed points may differ from the least upper bound of X in $\langle A, \leq \rangle$. For an example, let A be the set of all subsets of $\{a, b, c\}$, ordered by inclusion. For each $S \subseteq \{a, b, c\}$, let $f(S) = \{a, b, c\}$ if S has two or more elements and $f(S) = S$ if S has fewer than two elements. Let F be the fixed points of f, and let $X = \{\{a\}, \{b\}\} \subseteq F$. Then $\sum X = \{a, b\}$ in $\langle A, \leq \rangle$ but $\sum X = \{a, b, c\}$ in $\langle F, \leq \rangle$.

7 Relational Semantics: Arbitrary Interpretations

First we describe a class of programming languages which we call *suitable*. A suitable language $\mathcal{L}$ contains two disjoint classes of objects, called *predicates* and *statements*. (A better choice would be *commands*, since commands are to be obeyed while statements are either true or false.) Let $\mathcal{L}_S$ be the set of commands (or statements) of $\mathcal{L}$, and let $\mathcal{L}_P$ be the set of predicates of $\mathcal{L}$. Roughly speaking, commands are intended to denote actions which should be performed, while predicates are intended to denote conditions which may or may not hold in a given state or situation. The commands are divided into two classes, basic and compound. A suitable language must contain three specific basic commands, namely, *havoc*, *abort*, and *skip*, but may contain some unspecified number of other possibilities, such as assignment commands. (Assignment commands will not be explicitly treated here.) The compound commands of a suitable language are built up from basic commands using various means of combination. Specifically, if S_0 and S_1 are commands then so is $S_0; S_1$. If S is any command and B is any predicate, then **do** $B \rightarrow S$ **od** is a command. If S_i is a command correlated with each $i \in I$, where I is an arbitrary index set, then **if** $i : B_i \rightarrow S_i$ **fi** is also a command. If $\{B_i : i \in I\} = \{B\}$ and $\{S_i : i \in I\} = \{S\}$ we use the notation **if** $B \rightarrow S$ **fi** in place of **if** $i : B_i \rightarrow S_i$ **fi**. Every compound command can be obtained in exactly one of these three ways. We need no special assumptions concerning the set of predicates $\mathcal{L}_P$. This set could be a full first-order language, or one of higher type. The part of the theory presented here is independent of these possibilities.

Definition 32. *An* **interpretation** $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ *of a suitable programming language $\mathcal{L}$ is a complete relation algebra $\mathfrak{A}$ together with three maps*

$$\mathrm{r} : \mathcal{L}_S \to A, \quad \mathrm{e} : \mathcal{L}_S \to A, \quad \text{and} \quad \mathrm{d} : \mathcal{L}_P \to A,$$

such that

(i) $\mathrm{e}_S ; 1 = \mathrm{e}_S$ *for every command* $S \in \mathcal{L}_S$,
(ii) $\mathrm{d}_B ; 1 = \mathrm{d}_B$ *for every predicate* $B \in \mathcal{L}_P$.

To get a examples of interpretations, imagine that U is a set of machine states, that each command S has a corresponding terminating-computation relation r_S, that each command S has an eternal-computation relation e_S of the form $E \times U$, where E is the set of states initiating eta] eternal computations of S, and that each predicate B has a corresponding domain relation d_B of the form $X \times U$, where X is the set of states satisfying B. Then e_S and d_B are domain elements of $\mathfrak{Re}(U)$, as required by **32**(i)(ii), so $\langle \mathfrak{Re}(U), \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation.

If $\mathfrak{A}$ is any relation algebra whatsoever, an interpretation will be obtained by setting $\mathrm{r}_S = \mathrm{e}_S = \mathrm{d}_B = 0$ for every S and B, and another interpretation is obtained by setting $\mathrm{r}_S = \mathrm{e}_S = \mathrm{d}_B = 1$. A slightly more interesting interpretation results from setting $\mathrm{r}_S = 1'$, $\mathrm{e}_S = 0$, and $\mathrm{r}_B = 1$. This is tantamount to saying that every command always terminates and does nothing (leaves each state unchanged), and every predicate holds at every state.

In the remainder of this section we prove several general results applicable to any suitable programming language $\mathcal{L}$ and any interpretation $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ of $\mathcal{L}$. We will form various meets and joins without explicit mention of the fact that we can do so by the hypothesis that $\mathfrak{A}$ is complete.

Definition 33. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every command $S \in \mathcal{L}_S$ and every $x \in A$, define* $\mathrm{wlp}_S(x)$ *and* $\mathrm{wp}_S(x)$ *as follows:*

(i) $\mathrm{wlp}_S(x) = \overline{\mathrm{r}_S ; \bar{x}}$,
(ii) $\mathrm{wp}_S(x) = \overline{\mathrm{r}_S ; \bar{x}} \cdot \overline{\mathrm{e}_S}$.

In case x is a domain element, $\mathrm{wlp}_S(x)$ is called the "weakest liberal precondition guaranteeing x", and $\mathrm{wp}_S(x)$ is called the "weakest precondition guaranteeing x". We will usually apply the functions $\mathrm{wlp}_S(\text{-})$ and $\mathrm{wp}_S(\text{-})$ only to domain elements, although they are defined for all elements of the relation algebra. It is interesting that the extended definition allows the recovery of r_S from $\mathrm{wlp}_S(\text{-})$, since $\mathrm{r}_S = \mathrm{wlp}^*_S(1')$. Note that 1' is not a domain element unless $\mathfrak{A}$ is Boolean, by **24**(xix).

Theorem 34. *For every command $S \in \mathcal{L}_S$ and every $x \in A$, if x is a domain element, then $\mathrm{wlp}_S(x)$ and $\mathrm{wp}_S(x)$ are also domain elements.*

Proof. This theorem is an immediate consequence of **26**(i)(ii) and **33**.

Theorem 35. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every command $S \in \mathcal{L}_S$,*

(i) $\mathrm{wp}_S(x) = \mathrm{wlp}_S(x) \cdot \overline{\mathrm{e}_S}$,
(ii) [DS90 (7,1) p.129] $\mathrm{wlp}_S(1) = 1$,
(iii) $\mathrm{wp}_S(1) = \overline{\mathrm{e}_S}$,
(iv) [DS90 (7,2) p.129] $\mathrm{wp}_S(x) = \mathrm{wlp}_S(x) \cdot \mathrm{wp}_S(1)$.

Proof. **35**(i): This part follows immediately from **33**(i)(ii).
35(ii): $\mathrm{wlp}_S(1) = \overline{\mathrm{r}_S;\overline{1}} = \overline{\mathrm{r}_S;0} = \overline{0} = 1$ by **33**(i), **5**(iii), **24**(xviii), and **5**(iv), respectively.
35(iii): $\mathrm{wp}_S(1) = \mathrm{wlp}_S(1) \cdot \overline{\mathrm{e}_S} = 1 \cdot \overline{\mathrm{e}_S} = \overline{\mathrm{e}_S}$ by **35**(i)(ii), **3**(viii), and **5**(viii).
35(iv): This part follows from **35**(i)(iii).

Theorem 36. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every $S \in \mathcal{L}_S$ and every $x \in A$,*

(i) $\mathrm{wlp}^*_S(x) = \mathrm{r}_S;x$,
(ii) $\mathrm{wp}^*_S(x) = \mathrm{r}_S;x + \mathrm{e}_S$.

Proof. (i): Using **11**, **33**(i), and **3**(ii), we get $\mathrm{wlp}^*_S(x) = \overline{\mathrm{wlp}_S(\overline{x})} = \overline{\mathrm{r}_S;\overline{\overline{x}}} = \mathrm{r}_S;x$.
(ii): $\mathrm{wp}^*_S(x) = \overline{\mathrm{wp}_S(\overline{x})} = \overline{\mathrm{r}_S;\overline{\overline{x}} \cdot \overline{\mathrm{e}_S}} = \mathrm{r}_S;x + \mathrm{e}_S$ by **33**(ii) and **3**(ii)(xvii).

Theorem 37. [DS90 (7,0) p.129, and R0 p.132] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every $S \in \mathcal{L}_S$, $\mathrm{wlp}^*_S(\text{-})$ is universally disjunctive and $\mathrm{wlp}_S(\text{-})$ is universally conjunctive.*

Proof. Since $\mathrm{r}_S;(\text{-})$ is universally disjunctive by **24**(xv), it follows from **36**(i) that $\mathrm{wlp}^*_S(\text{-})$ is also universally disjunctive. We conclude that $\mathrm{wlp}_S(\text{-})$ is universally conjunctive by **18**.

Dijkstra and Scholten say "for every *wlp.S* we define, we shall honour the obligation to show that it meets requirement R0: *wlp.S* is universally conjunctive" (132^{10-12}). There is no such obligation here, because a consequence of **37** is that all predicate transformers arising from an arbitrary interpretation must automatically satisfy the healthiness condition R0.

Theorem 38. [DS90 (7,8) p.132] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every command $S \in \mathcal{L}_S$, $\mathrm{wp}_S(\text{-})$ is positively conjunctive and $\mathrm{wp}^*_S(\text{-})$ is positively disjunctive.*

Proof. For every nonempty indexed set $\{x_i : i \in I\} \subseteq A$, we have

$$\begin{aligned}
\mathrm{wp}_S\Big(\prod_{i\in I} x_i\Big) &= \mathrm{wlp}_S\Big(\prod_{i\in I} x_i\Big) \cdot \overline{\mathrm{e}_S} && \textbf{35}(\mathrm{i}) \\
&= \Big(\prod_{i\in I} \mathrm{wlp}_S(x_i)\Big) \cdot \overline{\mathrm{e}_S} && \textbf{37} \\
&= \prod_{i\in I} \Big(\mathrm{wlp}_S(x_i) \cdot \overline{\mathrm{e}_S}\Big) && I \neq \emptyset,\ \textbf{16} \\
&= \prod_{i\in I} \mathrm{wp}_S(x_i) && \textbf{35}(\mathrm{i})
\end{aligned}$$

Thus $\mathrm{wp}_S(\text{-})$ is positively conjunctive. It follows by **18** that its dual, $\mathrm{wp}^*_S(\text{-})$, is positively disjunctive.

Corollary 39. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every $S \in \mathcal{L}_S$ and all $x, y \in A$,*

(i) $\mathrm{wlp}_S(x) \cdot \mathrm{wlp}_S(y) = \mathrm{wlp}_S(x \cdot y)$,
(ii) $\mathrm{wp}_S(x) \cdot \mathrm{wp}_S(y) = \mathrm{wp}_S(x \cdot y)$.

Proof. This theorem follows from **38** by **17**(i).

Theorem 40. [DS90 (7,6) p.131] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every command $S \in \mathcal{L}_S$ and all $x, y \in A$,*
$\mathrm{wp}_S(x) \cdot \mathrm{wlp}_S(y) = \mathrm{wp}_S(x \cdot y)$.

Proof.

$$\begin{aligned} \mathrm{wp}_S(x) \cdot \mathrm{wlp}_S(y) &= \mathrm{wlp}_S(x) \cdot \overline{\mathrm{e}_S} \cdot \mathrm{wlp}_S(y) && \textbf{35}(\mathrm{i}) \\ &= \mathrm{wlp}_S(x) \cdot \mathrm{wlp}_S(y) \cdot \overline{\mathrm{e}_S} && \textbf{3}(\mathrm{viii})(\mathrm{ix}) \\ &= \mathrm{wlp}_S(x \cdot y) \cdot \overline{\mathrm{e}_S} && \textbf{39}(\mathrm{i}) \\ &= \mathrm{wp}_S(x \cdot y) && \textbf{35}(\mathrm{i}) \end{aligned}$$

Theorem 41. [DS90 (7,5) p.131] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every command $S \in \mathcal{L}_S$ and every $x \in A$, if $\mathrm{wp}_S(0) = 0$ then $\mathrm{wp}_S(x) \leq \mathrm{wlp}^*_S(x)$.*

Proof. We have

$$\begin{aligned} 0 &= \mathrm{wp}_S(0) && \text{hypothesis} \\ &= \mathrm{wp}_S(x \cdot \overline{x}) && \textbf{5}(\mathrm{ii}) \\ &= \mathrm{wp}_S(x) \cdot \mathrm{wlp}_S(\overline{x}) && \textbf{40} \end{aligned}$$

so $\mathrm{wp}_S(x) \leq \overline{\mathrm{wlp}_S(\overline{x})} = \mathrm{wlp}^*_S(x)$ by **3**(ii), **7**(v), **11**.

Definition 42. [DS90 p.131[7–9]] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. A command $S \in \mathcal{L}_S$ is* **deterministic** *if*
$\mathrm{wlp}^*_S(x) \leq \mathrm{wp}_S(x)$ *for all $x \in A$.*

The following corollary is called a definition in [DS90], but, because it contains the extra hypothesis $\mathrm{wp}_S(0) = 0$ (called "R1" in [DS90]), it has been split here into theorem **41** and definition **42**.

Corollary 43. [DS90 (7,7) p.131] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. For every $S \in \mathcal{L}_S$, if $\mathrm{wp}_S(0) = 0$, then S is deterministic iff $\mathrm{wp}_S(x) = \mathrm{wlp}^*_S(x)$ for all $x \in A$.*

Proof. Combine **41** and **42**.

The next theorem characterizes determinism even in the absence of hypothesis R1. In the concrete case, it says that S is deterministic iff no two terminating computations start at the same state and end at different states, *i.e.*, r_S is a partial function, and no state initiates both a terminating and an eternal computation of S, *i.e.*, r_S and e_S have disjoint domains. Note that a deterministic S can still have eternal computations.

Theorem 44. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is an interpretation of $\mathcal{L}$. A command $S \in \mathcal{L}_S$ is deterministic iff $\breve{\mathrm{r}}_S ; \mathrm{r}_S \leq 1'$ and $\mathrm{r}_S \cdot \mathrm{e}_S = 0$.*

Proof. The following statements are equivalent.

S is deterministic	
For all x, $\mathrm{wlp}^*_S(x) \leq \mathrm{wp}_S(x)$	**42**
For all x, $\mathrm{r}_S ; x \leq \overline{\mathrm{r}_S ; \overline{x}} \cdot \overline{\mathrm{e}_S}$	**36**(i), **33**(ii)
For all x, $\mathrm{r}_S ; x \cdot (\mathrm{r}_S ; \overline{x} + \mathrm{e}_S) = 0$	**7**(v), **3**(xvii)(ii)
For all x, $\mathrm{r}_S ; x \cdot \mathrm{r}_S ; \overline{x} + \mathrm{r}_S ; x \cdot \mathrm{e}_S = 0$	**3**(xv)
For all x, $\mathrm{r}_S ; x \cdot \mathrm{r}_S ; \overline{x} = 0$ and for all x, $\mathrm{r}_S ; x \cdot \mathrm{e}_S = 0$	**7**(ii)(iv)
r_S is functional and $\mathrm{r}_S \cdot \mathrm{e}_S = 0$	(see next paragraph)

The equivalence of the last two statements comes from the following observations. By **28**(iii), r_S is functional iff for all x, $\mathrm{r}_S ; x \cdot \mathrm{r}_S ; \overline{x} = 0$. If for all $x \in A$, $\mathrm{r}_S ; x \cdot \mathrm{e}_S = 0$, then, taking $x = 1$, we have $\mathrm{r}_S \cdot \mathrm{e}_S \leq \mathrm{r}_S ; 1 \cdot \mathrm{e}_S = 0$ by **24**(xx) and **7**(iii). Conversely, if $\mathrm{r}_S \cdot \mathrm{e}_S = 0$, then for every $x \in A$,

$$\begin{aligned} \mathrm{r}_S ; x \cdot \mathrm{e}_S &= (\mathrm{r}_S \cdot \mathrm{e}_S) ; x && \textbf{3}(\mathrm{viii}), \textbf{32}(\mathrm{i}), \textbf{24}(\mathrm{xxiv}) \\ &= 0 ; x && \text{hypothesis} \\ &= 0 && \textbf{24}(\mathrm{xviii}) \end{aligned}$$

8 Relational Semantics: Correct Interpretations

The definition of a "correct" interpretation for a suitable language is given below. The remarks that follow are justification for the definition. We follow [DS90] closely and quote from it often.

"Operationally interpreted, the only thing we know about an execution of *havoc* is that it terminates; upon its termination the machine may be in any state, i.e., all variables spanning the state space may have been set to unpredictable, unrelated values" (134_{1-2} and 135^{1-3}) Thus every state is connected to every other state by a terminating computation of *havoc*, and *havoc* has no eternal computations. We therefore want a correct interpretation to satisfy the conditions $r_{havoc} = 1$ and $e_{havoc} = 0$.

"The operational interpretation of *abort* is that for all initial states its execution fails to terminate" (135_{1-2}), that is, every state initiates an eternal computation of *abort*, and *abort* has no terminating computations. Hence we require $r_{abort} = 0$ and $e_{abort} = 1$.

"The operational interpretation of *skip* is that its execution, which is guaranteed to terminate, leaves the values of all variables unchanged" (136^{11-12}), that is, there are no eternal computations, and every computation has the same final state as initial state. Hence we want $r_{skip} = 1'$ and $e_{skip} = 0$.

The operational interpretation of $S_0;S_1$ is "first execute S_0, then execute S_1". We therefore identify " the state at which the execution of S_0 terminates with the state in which the execution of S_1 is initiated. (Accordingly, in a computation in which the execution of S_0 fails to terminate, the execution of S_1 is not initiated at all.) This is the standard technique of implementing the semicolon: termination of its left-hand operand starts its right-hand operand." (190_{1-2} and 191^{1-4}) Thus a terminating computation of $S_0;S_1$ starts at a state that begins a terminating computation of S_0 that ends at a state that begins a terminating computation of S_1 that ends at the final state of the computation of $S_0;S_1$. Naturally, the equation which asserts this is $r_{S_0;S_1} = r_{S_0};r_{S_1}$. A state initiates an eternal computation of $S_0;S_1$ if it either initiates an eternal computation of S_0, or else initiates a terminating computation of S_0 that ends at a state that begins an eternal computation of S_1. This is expressed equationally by $e_{S_0;S_1} = e_{S_0} + r_{S_0};e_{S_1}$.

"In those states in which no guard is satisfied, **if** $i : B_i \to S_i$ **fi** is semantically equivalent to *abort*. If one guard is satisfied, then S_i is executed for some value of i such that B_i is satisfied by the initial state" (144_{1-4}). Therefore $r_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}} = \sum_{i\in I}(d_{B_i}\cdot r_{S_i})$. A computation is a terminating computation of **if** $i : B_i \to S_i$ **fi** if, for some $i \in I$, it is a terminating computation of S_i whose initial state satisfies B_i. The states initiating eternal computations of **if** $i : B_i \to S_i$ **fi** are those in which no B_i is satisfied, together with those which, for some $i \in I$, satisfy B_i and initiate an eternal computation of S_i. Therefore $e_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}} = \prod_{i\in I}\overline{d_{B_i}} + \sum_{i\in I}(d_{B_i}\cdot e_{S_i})$.

Note that if $d_{B_i} = 0$ for every $i \in I$, then $r_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}} = \sum_{i\in I} 0 = 0 = r_{abort}$ and $e_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}} = \prod_{i\in I} 1 + \sum_{i\in I} 0 = 1 = e_{abort}$. So if no B_i is satisfied, then **if** $i : B_i \to S_i$ **fi** is indeed "semantically equivalent to *abort*".

A DO command is written **do** $B \to S$ **od**, where B is a predicate and S is a command. A computation for **do** $B \to S$ **od** is "a finite number (possibly zero) of terminating executions of **if** $B \to S$ **fi**, followed by a terminating execution of **if** $\neg B \to$ *skip* **fi**." (194^{9-14}) Therefore $r_{\mathbf{do}\, B\to S\, \mathbf{od}} = \sum_{i\in\omega}\left((r_{\mathbf{if}\, B\to S\, \mathbf{fi}})^i ; (r_{\mathbf{if}\, \neg B\to \mathit{skip}\, \mathbf{fi}})\right) = \sum_{i\in\omega}\left((d_B\cdot r_S)^i ; (\overline{d_B}\cdot 1')\right)$.

To motivate the operational definition of the eternal computations of **do** $B \to S$ **od**, Dijkstra and Scholten divide the computations of **do** $B \to S$ **od** into two sets, the "inner eternal" and "outer eternal" computations. The domain relation for the set of "inner eternal" computations is $\sum_{i\in\omega}\left((d_B\cdot r_S)^i ; (d_B\cdot e_S)\right)$, because "a finite sequence of executions of **if** $B \to S$ **fi** gives rise to an eternal computation if and only if the execution of some **if** $B \to S$ **fi** —obviously the last one in the sequence— gives rise to an eternal computation under control of S." (194_{1-7}) The "outer eternal" computations are "all eternal computations under control of **do** $B \to S$ **od** that consist of an infinite sequence of executions of **if** $B \to S$ **fi**." (194_{14-15}) Let C be the predicate that holds at every initial state from which an outer eternal computation is possible, and let c be the corresponding domain relation. For each state satisfying C a terminating computation of **if** $B \to S$ **fi** is possible, hence B holds in that state, which is therefore in the domain of d_B. Thus

(10,3) p.195 $$c \le d_B.$$

At every state where C holds "it should be possible that, after the first execution of S, C holds again" (195_{5-6}), that is, there should be a terminating computation of **if** $B \to S$ **fi** whose final state is again in the domain of c. Hence

(10,4) p.195 $$c \le r_S;c.$$

Combining these two observations, we get

$$c \le d_B \cdot r_S;c.$$

Thus c is a solution of the equation

(10,5) p.195 $$y \leq d_B \cdot r_S ; y.$$

"This equation has in general many solutions. We shall now show that C is its weakest solution". (196^{1-2})
Claim: c is the weakest solution of (10,5).

Proof. (See 196^{3-11}). Suppose (10,5) holds. We wish to show that $y \leq c$. Suppose we have a state in the domain of y. By (10,5), this state satisfies B, and it is also in the domain of $r_S ; y$. Hence there is a terminating computation of S whose final state is in the domain of y. We may now repeat the argument to get a second terminating computation which ends at a state in the domain of y, and then do it again, and so on. This yields an infinite sequence of terminating computations of S, showing that the first state initiates an eternal computation of **do** $B \rightarrow S$ **od**, as desired.

It follows by the Claim that $c = \sum\{y : y \leq d_B \cdot r_S ; y\}$. (Incidentally, since the function $d_B \cdot r_S ; (-)$ is monotone, it follows by Tarski's Theorem that $c = \sum\{y : y = d_B \cdot r_S ; y\}$ and $c = d_B \cdot r_S ; c$.) Combining the formulas for both inner and outer eternal computations gives us one possible definition, namely

$$e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}} = \sum_{i \in \omega} \left((d_B \cdot r_S)^i ; (d_B \cdot e_S) \right) + \sum\{y : y \leq d_B \cdot r_S ; y\}.$$

However, a shorter definition can be obtained by *not* dividing the types of eternal computations of **do** $B \rightarrow S$ **od** into two classes. Consider a state in the domain of $e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}}$. First, B must hold at this state, since otherwise the execution of **do** $B \rightarrow S$ **od** would terminate immediately. The state is therefore in the domain of d_B. Since B holds, S is executed. This either leads to an eternal computation of S, that is, the state is in the domain of e_S, or else there is no such eternal computation. The state must therefore initiate a terminating computation of S, for if not, we would have a state satisfying B from which no computation of S is possible, contradicting our assumption that the state does initiate a computation of **do** $B \rightarrow S$ **od**. The state initiates no eternal computations of S, but does initiate an eternal computation of **do** $B \rightarrow S$ **od**, so at least one of the terminating computations of S must end in a state from which an eternal computation of **do** $B \rightarrow S$ **od** is possible. This conclusion is equivalent to asserting that the state is in the domain of $r_S ; e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}}$. Putting these inclusions together, we conclude that any state in the domain of $e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}}$ must be in the domain of $d_B \cdot e_S + r_S ; e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}})$, that is, $e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}} \leq d_B \cdot (e_S + r_S ; e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}})$. Conversely, we can argue that if $y \leq d_B \cdot (e_S + r_S ; y)$ then $y \leq e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}}$. Indeed, a state in the domain of y must satisfy B, and either an eternal computation of S is possible from that state, in which case it initiates an eternal computation of **do** $B \rightarrow S$ **od**, or else it initiates a terminating computation of S that ends in a state which is again in the domain of y. Either this new state initiates an eternal computation of S or a terminating computation of S that ends at a state in the domain of y, and so on. We either eventually get into an eternal computation of S, or else create an infinite sequence of terminating computations of S. Either way we get an eternal computation of **do** $B \rightarrow S$ **od**, so $y \leq e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}}$. This shows that $e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}}$ is, in fact, the weakest solution of $y \leq d_B \cdot (e_S + r_S ; y)$, *i.e.*,

$$e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}} = \sum\{y : y \leq d_B \cdot (e_S + r_S ; y)\}.$$

Definition 45. *An interpretation* $\langle \mathfrak{A}, r, e, d \rangle$ *of s suitable language* $\mathcal{L}$ *is* **correct** *if the following conditions hold.*

(i) $r_{havoc} = 1$ *and* $e_{havoc} = 0$.
(ii) $r_{abort} = 0$ *and* $e_{abort} = 1$.
(iii) $r_{skip} = 1'$ *and* $e_{skip} = 0$.
(iv) *For all* $S_0, S_1 \in \mathcal{L}_S$, $r_{S_0 ; S_1} = r_{S_0} ; r_{S_1}$ *and* $e_{S_0 ; S_1} = e_{S_0} + r_{S_0} ; e_{S_1}$.
(v) *For every index set* I, *every* $\{B_i : i \in I\} \subseteq \mathcal{L}_P$, *and every* $\{S_i : i \in I\} \subseteq \mathcal{L}_S$,

$$r_{\mathbf{if}\, i : B_i \rightarrow S_i\, \mathbf{fi}} = \sum_{i \in I} (d_{B_i} \cdot r_{S_i}),$$

$$e_{\mathbf{if}\, i : B_i \rightarrow S_i\, \mathbf{fi}} = \prod_{i \in I} \overline{d_{B_i}} + \sum_{i \in I} (d_{B_i} \cdot e_{S_i}).$$

In particular, $r_{\mathbf{if}\, B \rightarrow S\, \mathbf{fi}} = d_B \cdot r_S$ *and* $e_{\mathbf{if}\, B \rightarrow S\, \mathbf{fi}} = \overline{d_B} + d_B \cdot e_S$.
(vi) *For every* $B \in \mathcal{L}_P$ *and every* $S \in \mathcal{L}_S$,

$$r_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}} = \sum_{i \in \omega} \left((d_B \cdot r_S)^i ; (\overline{d_B} \cdot 1') \right),$$

$$e_{\mathbf{do}\, B \rightarrow S\, \mathbf{od}} = \sum\{y : y \leq d_B \cdot (e_S + r_S ; y)\}.$$

Definition 45 is only concerned with correctness for *havoc*, *abort*, *skip*, composition ;, **if**, and **do**, since these are the only language features that are treated in this paper. Clearly, for more complex features the definition of correctness should be extended. For example, if the predicates in $\mathcal{L}_P$ are built up from variables by standard connectives of propositional calculus, then would be natural to add the following conditions to the definition of correctness.

$$\begin{aligned} \mathrm{d}_{\mathbf{true}} &= 1 \\ \mathrm{d}_{\mathbf{false}} &= 0 \\ \mathrm{d}_{\neg B} &= \overline{\mathrm{d}_B} \\ \mathrm{d}_{B \wedge C} &= \mathrm{d}_B \cdot \mathrm{d}_C \\ \mathrm{d}_{B \vee C} &= \mathrm{d}_B + \mathrm{d}_C \\ \mathrm{d}_{B \to C} &= \overline{\mathrm{d}_B} + \mathrm{d}_C \\ \mathrm{d}_{B \leftrightarrow C} &= \mathrm{d}_B \cdot \mathrm{d}_C + \overline{\mathrm{d}_B} \cdot \overline{\mathrm{d}_C} \end{aligned}$$

The next theorem shows that correct interpretations are extremely abundant.

Theorem 46. *For every suitable programming language $\mathcal{L}$ and every complete relation algebra $\mathfrak{A}$ there exist maps* r, e, *and* d *such that $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation. In fact, if we assume*

(1) d *is any map from predicates to domain elements of $\mathfrak{A}$,*
(2) *r' is map from basic commands to elements of $\mathfrak{A}$ such that $r'(havoc) = 1$, $r'(abort) = 0$, and $r'(skip) = 1'$,*
(3) *e' is a map from basic commands to domain elements of $\mathfrak{A}$ such that $e'(havoc) = 0$, $e'(abort) = 1$, and $e'(skip) = 0$,*

then r' and e' can be extended in a unique way to maps r *and* e *such that $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation.*

Proof. The proof of the existence of unique extensions r and e proceeds by a standard induction on the complexity of commands. For basic commands, r and e must agree with r' and e'. Any command S which is not basic is built from simpler commands in exactly one way. This determines the appropriate formula from **45**(iv)–(vi) to use in defining r_S and e_S. To show that $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation, we must verify that **32**(i)(ii) hold. We already have **32**(ii) by hypothesis (1). Note that 0 and 1 are domain elements by **24**(xviii)(xxii), so hypothesis (3) is not inconsistent with **32**(i). For basic commands S, e_S is a domain element by (3). What remains is to prove that the expressions for e_S in **45**(iv)–(vi) yield domain elements when their inputs are assumed to be domain elements.

For any commands S_0 and S_1, if e_{S_0} and e_{S_1} are domain elements, then so is $\mathrm{e}_{S_0} + \mathrm{r}_{S_0};\mathrm{e}_{S_1}$ by **26**(i)(ii).

Let $\{B_i : i \in I\} \subseteq \mathcal{L}_P$ and $\{S_i : i \in I\} \subseteq \mathcal{L}_S$. Suppose e_{S_i} is a domain element for every $i \in I$. By (1) we know that d_{B_i} is a domain element for every $i \in I$, and hence $\prod_{i \in I} \overline{\mathrm{d}_{B_i}}$ is a domain element by **26**(i)(iv). We also know that $\mathrm{d}_{B_i} \cdot \mathrm{e}_{S_i}$ is a domain element for every $i \in I$ by **26**(i), so $\sum_{i \in I} (\mathrm{d}_{B_i} \cdot \mathrm{e}_{S_i})$ is a domain element by **26**(iii). It follows that $\prod_{i \in I} \overline{\mathrm{d}_{B_i}} + \sum_{i \in I} (\mathrm{d}_{B_i} \cdot \mathrm{e}_{S_i})$ is a domain element by **26**(i).

Suppose e_S is a domain element. We will show that

(4) $$\{y : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\} \text{ is closed under } (\text{-});1.$$

Suppose $y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)$. Then

$$\begin{aligned} y;1 &\leq (\mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y));1 && \text{hypothesis, } \mathbf{24}(\mathrm{x}) \\ &= \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y);1 && (1), \mathbf{25}, \mathbf{24}(\mathrm{xxiv}) \\ &= \mathrm{d}_B \cdot (\mathrm{e}_S;1 + \mathrm{r}_S;y;1) && (\mathrm{Ra}_2) \\ &= \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y;1) && \text{hypothesis} \\ &= \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;(y;1)) && (\mathrm{Ra}_1) \end{aligned}$$

It follows from (4) that $\{y;1 : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\} \subseteq \{y : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\}$, so, by **10**(iii),

(5) $$\sum\{y;1 : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\} \subseteq \sum\{y : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\}.$$

But $\sum\{y : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\} \leq \left(\sum\{y : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\}\right);1 = \sum\{y;1 : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\}$ by **24**(xx)(xv), so, together with (5), this gives us $\sum\{y;1 : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\} = \sum\{y : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\}$.

Thus $\sum\{y : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S ; y)\}$ is a join of elements of the form $y;1$, each of which is a domain element because $y;1;1 = y;(1;1) = y;1$ by (Ra_1) and **24**(xxii). Therefore $\sum\{y : y \leq \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S ; y)\}$ is a domain element by **26**(iii). This completes the proof of **46**.

In the remainder of this section and in Sections 9–11 we prove results for an arbitrary suitable programming language $\mathcal{L}$ with a correct interpretation $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ of $\mathcal{L}$.

Theorem 47. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation of $\mathcal{L}$. For every $x \in A$,*

(i) [DS90 (7,10) p.133] $\mathrm{wlp}_{havoc}(x) = 0 \dagger x$.
(ii) [DS90 (7,12) p.133] $\mathrm{wp}_{havoc}(x) = 0 \dagger x$.
(iii) [DS90 (7,11) p.133] $\mathrm{wp}_{havoc}(1) = 1$.
(iv) [DS90 (7,13) p.135] $\mathrm{wlp}_{abort}(x) = 1$.
(v) [DS90 (7,15) p.135] $\mathrm{wp}_{abort}(x) = 0$.
(vi) [DS90 (7,14) p.135] $\mathrm{wp}_{abort}(1) = 0$.
(vii) [DS90 (7,16) p.136] $\mathrm{wlp}_{skip}(x) = x$.
(viii) [DS90 (7,18) p.136] $\mathrm{wp}_{skip}(x) = x$.
(ix) [DS90 (7,17) p.136] $\mathrm{wp}_{skip}(1) = 1$.

Proof.

47(i): $\mathrm{wlp}_{havoc}(x) = \overline{\mathrm{r}_{havoc} ; \overline{x}}$ **33**(i)
$= \overline{1 ; \overline{x}}$ **45**(i)
$= 0 \dagger x$ **5**(iv), (Ra_8)

47(ii): $\mathrm{wp}_{havoc}(x) = \mathrm{wlp}_{havoc}(x) \cdot \overline{\mathrm{e}_{havoc}}$ **35**(i)
$= 0 \dagger x \cdot \overline{0}$ **47**(i), **45**(i)
$= 0 \dagger x$ **5**(iv)(viii)

47(iii): $\mathrm{wp}_{havoc}(1) = 0 \dagger 1$ **47**(ii)
$= \overline{\overline{0} ; \overline{1}}$ (Ra_8)
$= \overline{\overline{0} ; 0}$ **5**(iii)
$= \overline{0}$ **24**(xviii)
$= 1$ **5**(iv)

47(iv): $\mathrm{wlp}_{abort}(x) = \overline{\mathrm{r}_{abort} ; \overline{x}} = \overline{0 ; \overline{x}} = \overline{0} = 1$.
47(v): $\mathrm{wp}_{abort}(x) = \mathrm{wlp}_{abort}(x) \cdot \overline{\mathrm{e}_{abort}} = 1 \cdot \overline{1} = 0$.
47(vi): By **47**(v), $\mathrm{wp}_{abort}(1) = 0$.
47(vii): $\mathrm{wlp}_{skip}(x) = \overline{\mathrm{r}_{skip} ; \overline{x}} = \overline{1' ; \overline{x}} = \overline{\overline{x}} = x$.
47(viii): $\mathrm{wp}_{skip}(x) = \mathrm{wlp}_{skip}(x) \cdot \overline{\mathrm{e}_{skip}} = x \cdot \overline{0} = x$.
47(ix): By **47**(viii), $\mathrm{wp}_{skip}(1) = 1$.

In connection with **47**(i)(ii), it should be noted that $0 \dagger x$ is the appropriate counterpart of what is denoted by "$[x]$" in [DS90].

According to [DS90], the two equations in the next theorem define the predicate transformer semantics for $S_0 ; S_1$. Through them, "the predicate transformers characterizing the semantics of $S_0 ; S_1$ are expressed in terms of the predicate transformers characterizing the semantics of S_0 and S_1" $(137_{13\text{–}16})$.

Theorem 48. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation of $\mathcal{L}$. For all commands $S_0, S_1 \in \mathcal{L}_S$ and every $x \in A$,*

(i) [DS90 (7,23) p.137] $\mathrm{wlp}_{S_0 ; S_1}(x) = \mathrm{wlp}_{S_0}(\mathrm{wlp}_{S_1}(x))$,
(ii) [DS90 (7,25) p.137] $\mathrm{wp}_{S_0 ; S_1}(x) = \mathrm{wp}_{S_0}(\mathrm{wp}_{S_1}(x))$,
(iii) [DS90 (7,24) p.137] $\mathrm{wp}_{S_0 ; S_1}(1) = \mathrm{wp}_{S_0}(\mathrm{wp}_{S_1}(1))$.

Proof.

$$
\begin{aligned}
\textbf{48(i)}:\quad \mathrm{wlp}_{S_0;S_1}(x) &= \overline{\mathrm{r}_{S_0;S_1};\overline{x}} && \textbf{33(i)}\\
&= \overline{(\mathrm{r}_{S_0};\mathrm{r}_{S_1});\overline{x}} && \textbf{45(iv)}\\
&= \overline{\mathrm{r}_{S_0};(\mathrm{r}_{S_1};\overline{x})} && (\mathrm{Ra}_1)\\
&= \overline{\mathrm{r}_{S_0};\overline{\mathrm{r}_{S_1};\overline{x}}} && \textbf{3(ii)}\\
&= \overline{\mathrm{r}_{S_0};\overline{\mathrm{wlp}_{S_1}(x)}} && \textbf{33(i)}\\
&= \mathrm{wlp}_{S_0}(\mathrm{wlp}_{S_1}(x)) && \textbf{33(i)}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{48(ii)}:\quad \mathrm{wp}_{S_0;S_1}(x) &= \mathrm{wlp}_{S_0;S_1}(x)\cdot\overline{\mathrm{e}_{S_0;S_1}} && \textbf{35(i)}\\
&= \mathrm{wlp}_{S_0}(\mathrm{wlp}_{S_1}(x))\cdot\overline{\mathrm{e}_{S_0;S_1}} && \textbf{48(i)}\\
&= \mathrm{wlp}_{S_0}(\mathrm{wlp}_{S_1}(x))\cdot\overline{\mathrm{e}_{S_0}+\mathrm{r}_{S_0};\mathrm{e}_{S_1}} && \textbf{45(iv)}\\
&= \mathrm{wlp}_{S_0}(\mathrm{wlp}_{S_1}(x))\cdot\overline{\mathrm{e}_{S_0}}\cdot\overline{\mathrm{r}_{S_0};\mathrm{e}_{S_1}} && \textbf{3(xvi)}\\
&= \mathrm{wlp}_{S_0}(\mathrm{wlp}_{S_1}(x))\cdot\overline{\mathrm{r}_{S_0};\overline{\overline{\mathrm{e}_{S_1}}}}\cdot\overline{\mathrm{e}_{S_0}} && \textbf{3(ii)(viii)(ix)}\\
&= \mathrm{wlp}_{S_0}(\mathrm{wlp}_{S_1}(x))\cdot\mathrm{wlp}_{S_0}(\overline{\mathrm{e}_{S_1}})\cdot\overline{\mathrm{e}_{S_0}} && \textbf{33(i)}\\
&= \mathrm{wlp}_{S_0}(\mathrm{wlp}_{S_1}(x)\cdot\overline{\mathrm{e}_{S_1}})\cdot\overline{\mathrm{e}_{S_0}} && \textbf{39(i)}\\
&= \mathrm{wp}_{S_0}(\mathrm{wp}_{S_1}(x)) && \textbf{35(i)}
\end{aligned}
$$

48(iii): Set $x = 1$ in **48**(ii).

According to [DS90], "the semantics of *IF* satisfies" (139_4) the three equations in the next theorem.

Theorem 49. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d}\rangle$ is a correct interpretation of $\mathcal{L}$. For every index set I, every $\{B_i : i \in I\} \subseteq \mathcal{L}_P$, every $\{S_i : i \in I\} \subseteq \mathcal{L}_S$, and every $x \in A$,*

(i) [DS90 (7,27) p.139] $\mathrm{wlp}_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}}(x) = \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \mathrm{wlp}_{S_i}(x))$,

(ii) [DS90 (7,29) p.139] $\mathrm{wp}_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}}(x) = \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \mathrm{wp}_{S_i}(x))\cdot\sum_{i\in I}\mathrm{d}_{B_i}$,

(iii) [DS90 (7,28) p.139] $\mathrm{wp}_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}}(1) = \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \mathrm{wp}_{S_i}(1))\cdot\sum_{i\in I}\mathrm{d}_{B_i}$.

In particular, $\mathrm{wlp}_{\mathbf{if}\, B\to S\, \mathbf{fi}}(x) = \mathrm{d}_B + \mathrm{wlp}_S(x)$ *and* $\mathrm{wp}_{\mathbf{if}\, B\to S\, \mathbf{fi}}(x) = \mathrm{d}_B\cdot\mathrm{wlp}_S(x)$.

Proof.

$$
\begin{aligned}
\textbf{49(i)}:\quad \mathrm{wlp}_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}}(x) &= \overline{\mathrm{r}_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}};\overline{x}} && \textbf{33(i)}\\
&= \overline{\left(\sum_{i\in I}(\mathrm{d}_{B_i}\cdot\mathrm{r}_{S_i})\right);\overline{x}} && \textbf{45(v)}\\
&= \overline{\sum_{i\in I}((\mathrm{d}_{B_i}\cdot\mathrm{r}_{S_i});\overline{x})} && \textbf{24(xv)}\\
&= \overline{\sum_{i\in I}(\mathrm{d}_{B_i}\cdot\mathrm{r}_{S_i};\overline{x})} && \textbf{32(ii), 24(xxiv)}\\
&= \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \overline{\mathrm{r}_{S_i};\overline{x}}) && \textbf{8(ii), 3(xvii)}\\
&= \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \mathrm{wlp}_{S_i}(x)) && \textbf{33(i)}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{49(ii)}:\quad & \mathrm{wp}_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}}(x)\\
&= \mathrm{wlp}_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}}(x)\cdot\overline{\mathrm{e}_{\mathbf{if}\, i:B_i\to S_i\, \mathbf{fi}}} && \textbf{35(i)}\\
&= \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \mathrm{wlp}_{S_i}(x))\cdot\overline{\prod_{i\in I}\overline{\mathrm{d}_{B_i}} + \sum_{i\in I}(\mathrm{d}_{B_i}\cdot\mathrm{e}_{S_i})} && \textbf{49(i), 45(v)}\\
&= \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \mathrm{wlp}_{S_i}(x))\cdot\left(\sum_{i\in I}\mathrm{d}_{B_i}\cdot\prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \overline{\mathrm{e}_{S_i}})\right) && \textbf{3(ii)(xvi)(xvii), 8(ii)(iii)}\\
&= \prod_{i\in I}\left((\overline{\mathrm{d}_{B_i}} + \mathrm{wlp}_{S_i}(x))\cdot(\overline{\mathrm{d}_{B_i}} + \overline{\mathrm{e}_{S_i}})\right)\cdot\sum_{i\in I}\mathrm{d}_{B_i} && \textbf{3(viii)(ix), 9(ii)}\\
&= \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \mathrm{wlp}_{S_i}(x)\cdot\overline{\mathrm{e}_{S_i}})\cdot\sum_{i\in I}\mathrm{d}_{B_i} && \textbf{3(xviii)}\\
&= \prod_{i\in I}(\overline{\mathrm{d}_{B_i}} + \mathrm{wp}_{S_i}(x))\cdot\sum_{i\in I}\mathrm{d}_{B_i} && \textbf{35(i)}
\end{aligned}
$$

49(iii): Set $x = 1$ in **49**(ii).

Next we derive a useful formula for $\mathrm{wlp}_{\mathbf{do}\, B\to S\, \mathbf{od}}(x)$.

Theorem 50. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d}\rangle$ is a correct interpretation of $\mathcal{L}$. For every predicate $B \in \mathcal{L}_P$, every command $S \in \mathcal{L}_S$, and every element $x \in A$,*

$$\mathrm{wlp}_{\mathbf{do}\, B\to S\, \mathbf{od}}(x) = \overline{\sum_{i\in\omega} \left((\mathrm{d}_B \cdot \mathrm{r}_S)^i \,;\, (\overline{\mathrm{d}_B} \cdot \overline{x}) \right)}.$$

Proof.

$$\begin{aligned}
\mathrm{wlp}_{\mathbf{do}\, B\to S\, \mathbf{od}}(x) &= \overline{\mathrm{r}_{\mathbf{do}\, B\to S\, \mathbf{od}}\,;\overline{x}} && \textbf{33}(\mathrm{i})\\
&= \overline{\left(\sum_{i\in\omega} \left((\mathrm{d}_B \cdot \mathrm{r}_S)^i \,;\, (\overline{\mathrm{d}_B} \cdot 1') \right)\right);\overline{x}} && \textbf{45}(\mathrm{vi})\\
&= \overline{\sum_{i\in\omega} \left((\mathrm{d}_B \cdot \mathrm{r}_S)^i \,;\, (\overline{\mathrm{d}_B} \cdot 1') \,;\overline{x} \right)} && \textbf{24}(\mathrm{xv})\\
&= \overline{\sum_{i\in\omega} \left((\mathrm{d}_B \cdot \mathrm{r}_S)^i \,;\, \left((\overline{\mathrm{d}_B} \cdot 1') \,;\overline{x}\right) \right)} && (\mathrm{Ra}_1)\\
&= \overline{\sum_{i\in\omega} \left((\mathrm{d}_B \cdot \mathrm{r}_S)^i \,;\, (\overline{\mathrm{d}_B} \cdot \overline{x}) \right)} && \textbf{32}(\mathrm{ii}),\ \textbf{24}(\mathrm{xxiii})(\mathrm{xxvi})
\end{aligned}$$

Theorem 51. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d}\rangle$ is a correct interpretation of $\mathcal{L}$. For every $S \in \mathcal{L}_S$, $x \in A$, and $i \in \omega$, $(\mathrm{wlp}_S)^i(x) = \overline{\mathrm{r}_S^i\,;\overline{x}}$.*

Proof. We prove this by induction. First note that $(\mathrm{wlp}_S)^0(x) = x = \overline{1';\overline{x}} = \overline{\mathrm{r}_S^0;\overline{x}}$ by **3**(ii), **24**(xix), and **29**. Next, assume that $(\mathrm{wlp}_S)^i(x) = \overline{\mathrm{r}_S^i;\overline{x}}$. Then we have

$$\begin{aligned}
(\mathrm{wlp}_S)^{i+1}(x) &= \mathrm{wlp}_S\big((\mathrm{wlp}_S)^i(x)\big)\\
&= \mathrm{wlp}_S\big(\overline{\mathrm{r}_S^i;\overline{x}}\big) && \text{hypothesis}\\
&= \overline{\mathrm{r}_S;\mathrm{r}_S^i;\overline{x}} && \textbf{33}(\mathrm{i})\\
&= \overline{\mathrm{r}_S;\big(\mathrm{r}_S^i;\overline{x}\big)} && \textbf{3}(\mathrm{ii})\\
&= \overline{\mathrm{r}_S;\mathrm{r}_S^i;\overline{x}} && (\mathrm{Ra}_1)\\
&= \overline{\mathrm{r}_S^{i+1};\overline{x}} && \textbf{29}
\end{aligned}$$

"The Main Repetition Theorem involves well-founded sets because it deals with *wp.DO*, which captures guaranteed termination of the repetition. Since *wlp.DO* is not concerned with guaranteed termination, we may expect *wlp.DO* to be simpler to deal with than *wp.DO*. This expectation is confirmed by" (185^{5-9}) the following theorem.

Theorem 52. [DS90 (9,41) p.185] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d}\rangle$ is a correct interpretation of $\mathcal{L}$. For every $B \in \mathcal{L}_P$, every $S \in \mathcal{L}_S$, and every $x \in A$,*

$$\mathrm{wlp}_{\mathbf{do}\, B\to S\, \mathbf{od}}(x) = \prod_{i\in\omega} (\mathrm{wlp}_{\mathbf{if}\, B\to S\, \mathbf{fi}})^i (\mathrm{d}_B + x).$$

Proof.

$$\begin{aligned}
\mathrm{wlp}_{\mathbf{do}\, B\to S\, \mathbf{od}}(x) &= \overline{\sum_{i\in\omega} \left((\mathrm{d}_B \cdot \mathrm{r}_S)^i \,;\, (\overline{\mathrm{d}_B} \cdot \overline{x}) \right)} && \textbf{50}\\
&= \prod_{i\in\omega} \overline{(\mathrm{d}_B \cdot \mathrm{r}_S)^i \,;\overline{\mathrm{d}_B + x}} && \textbf{8}(\mathrm{ii}),\ \textbf{3}(\mathrm{xvi})\\
&= \prod_{i\in\omega} \overline{(\mathrm{r}_{\mathbf{if}\, B\to S\, \mathbf{fi}})^i \,;\overline{\mathrm{d}_B + x}} && \textbf{45}(\mathrm{v})\\
&= \prod_{i\in\omega} (\mathrm{wlp}_{\mathbf{if}\, B\to S\, \mathbf{fi}})^i (\mathrm{d}_B + x) && \textbf{51}
\end{aligned}$$

The next theorem contains the predicate transformer semantics for $\mathbf{do}\, B\to S\, \mathbf{od}$, which are introduced in [DS90] as definitions, along with this advice. "We suggest that in this chapter the reader just accept these definitions as such, without wondering from where they come or what inspired them. Such background information will be provided in the next chapter." The only justification given here for these "definitions" is a proof.

Theorem 53. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation of $\mathcal{L}$. For every predicate $B \in \mathcal{L}_P$, every command $S \in \mathcal{L}_S$, and every element $x \in A$, the following statements hold.*

(i) [DS90 (9,1) p.171] $\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x)$ *is the greatest (weakest) solution y of*

$$(\mathrm{d}_B + x) \cdot (\overline{\mathrm{d}_B} + \mathrm{wlp}_S(y)) = y.$$

(ii) [DS90 (9,2) p.171] $\mathrm{wp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x)$ *is the least (strongest) solution y of*

$$(\mathrm{d}_B + x) \cdot (\overline{\mathrm{d}_B} + \mathrm{wp}_S(y)) = y.$$

(iii) $\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x) = \sum\{y : (\mathrm{d}_B + x) \cdot (\overline{\mathrm{d}_B} + \mathrm{wlp}_S(y)) = y\}$.

(iv) $\mathrm{wp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x) = \prod\{y : (\mathrm{d}_B + x) \cdot (\overline{\mathrm{d}_B} + \mathrm{wp}_S(y)) = y\}$.

Proof. **53**(i) and **53**(ii) immediately imply **53**(iii) and **53**(iv), respectively. In fact, **53**(i) and **53**(ii) happen to follow from **53**(iii) and **53**(iv) by Tarski's Theorem. To prove **53**(i), first note that

$$\begin{aligned}
&\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x) \\
&= \overline{\sum_{i \in \omega} ((\mathrm{d}_B \cdot \mathrm{r}_S)^i ; (\overline{\mathrm{d}_B} \cdot \overline{x}))} && \textbf{50} \\
&= \overline{(\mathrm{d}_B \cdot \mathrm{r}_S)^0 ; (\overline{\mathrm{d}_B} \cdot \overline{x}) + \sum_{i \in \omega} ((\mathrm{d}_B \cdot \mathrm{r}_S)^{i+1} ; (\overline{\mathrm{d}_B} \cdot \overline{x}))} && \textbf{10}(\mathrm{i}) \\
&= \overline{\overline{\mathrm{d}_B} \cdot \overline{x} + \sum_{i \in \omega} ((\mathrm{d}_B \cdot \mathrm{r}_S)^{i+1} ; (\overline{\mathrm{d}_B} \cdot \overline{x}))} && \textbf{29}, \textbf{24}(\mathrm{xix}) \\
&= \overline{\overline{\mathrm{d}_B} \cdot \overline{x} + \sum_{i \in \omega} ((\mathrm{d}_B \cdot \mathrm{r}_S) ; (\mathrm{d}_B \cdot \mathrm{r}_S)^i ; (\overline{\mathrm{d}_B} \cdot \overline{x}))} && \textbf{29} \\
&= \overline{\overline{\mathrm{d}_B} \cdot \overline{x} + \sum_{i \in \omega} ((\mathrm{d}_B \cdot \mathrm{r}_S) ; ((\mathrm{d}_B \cdot \mathrm{r}_S)^i ; (\overline{\mathrm{d}_B} \cdot \overline{x})))} && (\mathrm{Ra}_1) \\
&= \overline{\overline{\mathrm{d}_B} \cdot \overline{x} + (\mathrm{d}_B \cdot \mathrm{r}_S) ; \sum_{i \in \omega} ((\mathrm{d}_B \cdot \mathrm{r}_S)^i ; (\overline{\mathrm{d}_B} \cdot \overline{x}))} && \textbf{24}(\mathrm{xv}) \\
&= \overline{\overline{\mathrm{d}_B} \cdot \overline{x} + (\mathrm{d}_B \cdot \mathrm{r}_S) ; \overline{\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x)}} && \textbf{50}, \textbf{3}(\mathrm{ii}) \\
&= \overline{\overline{\mathrm{d}_B} \cdot \overline{x} + \mathrm{d}_B \cdot \mathrm{r}_S ; \overline{\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x)}} && \textbf{32}(\mathrm{ii}), \textbf{24}(\mathrm{xxiv}) \\
&= (\mathrm{d}_B + x) \cdot (\overline{\mathrm{d}_B} + \overline{\mathrm{r}_S ; \overline{\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x)}}) && \textbf{3}(\mathrm{ii})(\mathrm{xvi})(\mathrm{xvii}) \\
&= (\mathrm{d}_B + x) \cdot (\overline{\mathrm{d}_B} + \mathrm{wlp}_S(\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x))) && \textbf{33}(\mathrm{i})
\end{aligned}$$

Thus $\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x)$ satisfies the equation $y = (\mathrm{d}_B + x) \cdot (\overline{\mathrm{d}_B} + \mathrm{wlp}_S(y))$. Next we show that every solution of this equation is included in $\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(x)$. Assume that y is a solution, *i.e.*,

$$y = (\mathrm{d}_B + x) \cdot (\overline{\mathrm{d}_B} + \mathrm{wlp}_S(y)).$$

Rewriting this equation by **3**(ii)(xvi)(xvii) and **33**(i) gives

$$\overline{y} = \overline{\mathrm{d}_B} \cdot \overline{x} + \mathrm{d}_B \cdot \mathrm{r}_S ; \overline{y}$$

so by **7**(iv), **32**(ii), and **24**(xxiv) we have

$$\overline{\mathrm{d}_B} \cdot \overline{x} \leq \overline{y}, \tag{1}$$

$$(\mathrm{d}_B \cdot \mathrm{r}_S) ; \overline{y} \leq \overline{y}. \tag{2}$$

Note that $(\mathrm{d}_B \cdot \mathrm{r}_S)^0 ; (\overline{\mathrm{d}_B} \cdot \overline{x}) = 1' ; (\overline{\mathrm{d}_B} \cdot \overline{x}) = \overline{\mathrm{d}_B} \cdot \overline{x} \leq \overline{y}$ by **29**, **24**(xix), and (1). Also, if $(\mathrm{d}_B \cdot \mathrm{r}_S)^i ; (\overline{\mathrm{d}_B} \cdot \overline{x}) \leq \overline{y}$, then,

$$\begin{aligned}
(\mathrm{d}_B \cdot \mathrm{r}_S)^{i+1} ; (\overline{\mathrm{d}_B} \cdot \overline{x}) &= (\mathrm{d}_B \cdot \mathrm{r}_S) ; (\mathrm{d}_B \cdot \mathrm{r}_S)^i ; (\overline{\mathrm{d}_B} \cdot \overline{x}) && \textbf{29} \\
&= (\mathrm{d}_B \cdot \mathrm{r}_S) ; ((\mathrm{d}_B \cdot \mathrm{r}_S)^i ; (\overline{\mathrm{d}_B} \cdot \overline{x})) && (\mathrm{Ra}_1) \\
&\leq (\mathrm{d}_B \cdot \mathrm{r}_S) ; \overline{y} && \text{hypothesis, } \textbf{24}(\mathrm{x}) \\
&\leq \overline{y} && (2)
\end{aligned}$$

We therefore conclude by induction that

$$\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x})\right)\leq \overline{y}.$$

By **7**(v) this is equivalent to

$$y\leq \overline{\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x})\right)}.$$

But $\overline{\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x})\right)}=\mathrm{wlp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)$ by **50**, so $y\leq \mathrm{wlp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)$. Therefore $\mathrm{wlp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)$ is a solution of $(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}_S(y))=y$ that also contains all solutions. We conclude that

$$\mathrm{wlp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)=\sum\{y:(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}_S(y))=y\},$$

which completes the proof of **53**(i) and **53**(iii). For **53**(ii) and **53**(iv), we show that $\mathrm{wp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)$ is a solution of the equation

$$(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wp}_S(y))=y,$$

and that $\mathrm{wp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)$ is contained in every solution. First note that $\mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;(\text{-}))$ is monotone by **24**(x), **7**(iii), and **3**(viii). By Tarski's Theorem **31**(ii), $\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\}$ is therefore a (in fact, the largest) solution of the equation $y=\mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)$, so

$$\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\}=\mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;\textstyle\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\}) \tag{3}$$

Then

$$\begin{aligned}
&\overline{\mathrm{wp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)}\\
&=\overline{\mathrm{wlp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)}+\mathrm{e}_{\mathbf{do}\,B\to S\,\mathbf{od}} && \textbf{35}(\mathrm{i}),\ \textbf{3}(\mathrm{ii})(\mathrm{xvii})\\
&=\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x})\right)\\
&\quad+\textstyle\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\} && \textbf{50},\ \textbf{3}(\mathrm{ii}),\ \textbf{45}(\mathrm{vi})\\
&=\overline{\mathrm{d}_B}\cdot\overline{x}+\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^{i+1};(\overline{\mathrm{d}_B}\cdot\overline{x})\right)\\
&\quad+\textstyle\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\} && \textbf{10}(\mathrm{i}),\ \textbf{29},\ \textbf{24}(\mathrm{xix})\\
&=\overline{\mathrm{d}_B}\cdot\overline{x}+\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^{i+1};(\overline{\mathrm{d}_B}\cdot\overline{x})\right)\\
&\quad+\mathrm{d}_B\cdot\mathrm{e}_S+\mathrm{d}_B\cdot\mathrm{r}_S;\textstyle\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\} && (3),\ \textbf{3}(\mathrm{xv})\\
&=\overline{\mathrm{d}_B}\cdot\overline{x}+(\mathrm{d}_B\cdot\mathrm{r}_S);\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x})\right)\\
&\quad+\mathrm{d}_B\cdot\mathrm{e}_S+\mathrm{d}_B\cdot\mathrm{r}_S;\textstyle\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\} && \textbf{29},\ (\mathrm{Ra}_1),\ \textbf{24}(\mathrm{xv})\\
&=\overline{\mathrm{d}_B}\cdot\overline{x}+\mathrm{d}_B\cdot\mathrm{e}_S\\
&\quad+\mathrm{d}_B\cdot\mathrm{r}_S;\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x})\right)\\
&\quad+\mathrm{d}_B\cdot\mathrm{r}_S;\textstyle\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\} && (\mathrm{Ba}_1),\ (\mathrm{Ba}_2),\ \textbf{32}(\mathrm{ii}),\ \textbf{24}(\mathrm{xxiv})\\
&=\overline{\mathrm{d}_B}\cdot\overline{x}+\mathrm{d}_B\cdot\mathrm{e}_S\\
&\quad+\mathrm{d}_B\cdot\mathrm{r}_S;\Big(\sum_{i\in\omega}\left((\mathrm{d}_B\cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x})\right)\\
&\qquad+\textstyle\sum\{y:y\leq \mathrm{d}_B\cdot(\mathrm{e}_S+\mathrm{r}_S;y)\}\Big) && (\mathrm{Ba}_1),\ \textbf{3}(\mathrm{xv}),\ \textbf{24}(\mathrm{ix})\\
&=\overline{\mathrm{d}_B}\cdot\overline{x}+\mathrm{d}_B\cdot\mathrm{e}_S\\
&\quad+\mathrm{d}_B\cdot\mathrm{r}_S;\left(\overline{\mathrm{wlp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)}+\mathrm{e}_{\mathbf{do}\,B\to S\,\mathbf{od}}\right) && \textbf{50},\ \textbf{3}(\mathrm{ii}),\ \textbf{45}(\mathrm{vi})\\
&=\overline{\mathrm{d}_B}\cdot\overline{x}+\mathrm{d}_B\cdot\mathrm{e}_S+\mathrm{d}_B\cdot\mathrm{r}_S;\overline{\mathrm{wp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(x)} && \textbf{32}(\mathrm{ii}),\ \textbf{24}(\mathrm{xxiv})
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x)\\
&= \overline{\overline{\mathrm{d}_B}\cdot\overline{x} + \mathrm{d}_B\cdot\mathrm{e}_S + \mathrm{d}_B\cdot\mathrm{r}_S;\overline{\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x)}} && \mathbf{3}\text{(ii)}\\
&= (\mathrm{d}_B + x)\cdot(\overline{\mathrm{d}_B} + \overline{\mathrm{e}_S})\cdot(\overline{\mathrm{d}_B} + \overline{\mathrm{r}_S;\overline{\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x)}}) && \mathbf{3}\text{(ii)(xvi)(xvii)}\\
&= (\mathrm{d}_B + x)\cdot(\overline{\mathrm{d}_B} + \overline{\mathrm{e}_S})\cdot(\overline{\mathrm{d}_B} + \mathrm{wlp}_S(\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x))) && \mathbf{33}\text{(i)}\\
&= (\mathrm{d}_B + x)\cdot(\overline{\mathrm{d}_B} + \overline{\mathrm{e}_S}\cdot\mathrm{wlp}_S(\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x))) && \mathbf{3}\text{(xviii)}\\
&= (\mathrm{d}_B + x)\cdot(\overline{\mathrm{d}_B} + \mathrm{wp}_S(\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x))). && \mathbf{3}\text{(viii), }\mathbf{35}\text{(i)}
\end{aligned}
$$

Thus $\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x)$ is a solution of the equation $(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wp}_S(y)) = y$. Next we show that $\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x)$ is contained in every solution. Assume that y is a solution, *i.e.*,

$$y = (\mathrm{d}_B + x)\cdot(\overline{\mathrm{d}_B} + \mathrm{wp}_S(y)).$$

Then

$$
\begin{aligned}
\overline{y} &= \overline{\mathrm{d}_B}\cdot\overline{x} + \mathrm{d}_B\cdot\overline{\mathrm{wp}_S(y)} && \mathbf{3}\text{(ii)(xvi)(xvii)}\\
&= \overline{\mathrm{d}_B}\cdot\overline{x} + \mathrm{d}_B\cdot(\mathrm{r}_S;\overline{y} + \mathrm{e}_S) && \mathbf{33}\text{(ii), }\mathbf{3}\text{(ii)(xvii)}\\
&= \overline{\mathrm{d}_B}\cdot\overline{x} + \mathrm{d}_B\cdot\mathrm{e}_S + \mathrm{d}_B\cdot\mathrm{r}_S;\overline{y} && \mathbf{3}\text{(xv)}, (\mathrm{Ba}_1), (\mathrm{Ba}_2)\\
&= \overline{\mathrm{d}_B}\cdot\overline{x} + \mathrm{d}_B\cdot\mathrm{e}_S + (\mathrm{d}_B\cdot\mathrm{r}_S);\overline{y} && \mathbf{32}\text{(ii), }\mathbf{24}\text{(xxiv)}
\end{aligned}
$$

Let $b = \overline{\mathrm{d}_B}\cdot\overline{x}$, $c = \mathrm{d}_B\cdot\mathrm{e}_S$, and $u = \mathrm{d}_B\cdot\mathrm{r}_S$. Then $\overline{y} = b + c + u;\overline{y}$. Let $z = \overline{y}\cdot\overline{\sum_{i\in\omega}(u^i;b)}$. By **30** we have $z \le c + u;z$, so

$$\overline{y}\cdot\overline{\sum_{i\in\omega}(u^i;b)} = z \le \sum\{z : z \le c + u;z\}. \tag{4}$$

Then we get

$$
\begin{aligned}
\overline{y} &= \overline{y}\cdot\sum_{i\in\omega}(u^i;b) + \overline{y}\cdot\overline{\sum_{i\in\omega}(u^i;b)} && \mathbf{3}\text{(ii)(xi)}\\
&\le \sum_{i\in\omega}(u^i;b) + \sum\{z : z \le c + u;z\} && (4), \mathbf{7}\text{(iii)(iv)}\\
&= \sum_{i\in\omega}((\mathrm{d}_B\cdot\mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x}))\\
&\qquad + \sum\{z : z \le \mathrm{d}_B\cdot\mathrm{e}_S + (\mathrm{d}_B\cdot\mathrm{r}_S);z\} && \text{definitions of } b, c, u\\
&= \sum_{i\in\omega}((\mathrm{d}_B\cdot\mathrm{r}_S)^i;(\overline{\mathrm{d}_B}\cdot\overline{x}))\\
&\qquad + \sum\{z : z \le \mathrm{d}_B\cdot(\mathrm{e}_S + \mathrm{r}_S;z)\} && \mathbf{32}\text{(ii), }\mathbf{24}\text{(xxiv), }\mathbf{3}\text{(xv)}\\
&= \overline{\mathrm{wlp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x)} + \mathrm{e}_{\mathbf{do}\ B\to S\ \mathbf{od}} && \mathbf{50}, \mathbf{3}\text{(ii)}, \mathbf{45}\text{(vi)}\\
&= \overline{\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x)}, && \mathbf{35}\text{(i)}, \mathbf{3}\text{(ii)(xvii)}
\end{aligned}
$$

so $\mathrm{wp}_{\mathbf{do}\ B\to S\ \mathbf{od}}(x) \le y$ by **7**(v), which completes the proof of **53**(ii)(iv).

9 The "Law of The Excluded Miracle"

Note that $\mathrm{wp}_S(0) = \overline{\mathrm{r}_S;\overline{0}}\cdot\overline{\mathrm{e}_S} = \overline{\mathrm{r}_S;1}\cdot\overline{\mathrm{e}_S}$, so $\mathrm{wp}_S(0)$ is the domain element for the set of states which are not the initial state of any computation of S. Dijkstra and Scholten wish to exclude such states, and they therefore consider the following "axiom" [DS90 (7,4) p.130, and R1 p.132].

R1 $$0 = \mathrm{wp}_S(0) = \overline{\mathrm{r}_S;1}\cdot\overline{\mathrm{e}_S}.$$

R1 is also called the "law of the excluded miracle". Dijkstra and Scholten say "for every *wp.S* we define, we shall honour the obligation to show that it meets requirement R1: $[wp.S.false \equiv false]$" ($132^{13-15}$). This is done by the next theorem, which says that ***havoc***, ***abort***, and ***skip*** satisfy R1, and compound commands satisfy R1 if they are built from simpler ones that satisy R1. All parts of the next theorem do fail in some (necessarily incorrect) interpretation.

Theorem 54. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation of $\mathcal{L}$.*

(i) $\mathrm{wp}_S(0) = 0$ *for every* $S \in \{havoc, abort, skip\}$.
(ii) *For all* $S_0, S_1 \in \mathcal{L}_S$, *if* $\mathrm{wp}_{S_0}(0) = 0$ *and* $\mathrm{wp}_{S_1}(0) = 0$ *then* $\mathrm{wp}_{S_0;S_1}(0) = 0$.
(iii) *For every index set* I, *every* $\{B_i : i \in I\} \subseteq \mathcal{L}_P$, *and every* $\{S_i : i \in I\} \subseteq \mathcal{L}_S$, *if* $\mathrm{wp}_{S_i}(0) = 0$ *for every* $i \in I$, *then* $\mathrm{wp}_{\mathbf{if}\, i:B_i \to S_i\, \mathbf{fi}}(0) = 0$.
(iv) *For every* $S \in \mathcal{L}_S$ *and every* $B \in \mathcal{L}_P$, *if* $\mathrm{wp}_S(0) = 0$ *then* $\mathrm{wp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(0) = 0$.
(v) *If* $\mathrm{wp}_S(0) = 0$ *for every basic command* S, *then* $\mathrm{wp}_S(0) = 0$ *for every* $S \in \mathcal{L}_S$.

Proof.

54(i): We have $\mathrm{wp}_{havoc}(0) = 0 \dagger 0 = \overline{\overline{0}; \overline{0}} = \overline{1;1} = \overline{1} = 0$ by **47**(ii), (Ra_8), **5**(iii)(iv), and **24**(xxii), $\mathrm{wp}_{abort}(0) = 0$ by **47**(v), and $\mathrm{wp}_{skip}(0) = 0$ by **47**(viii).

54(ii): If $\mathrm{wp}_{S_0}(0) = 0$ and $\mathrm{wp}_{S_1}(0) = 0$ then $\mathrm{wp}_{S_0;S_1}(0) = \mathrm{wp}_{S_0}(\mathrm{wp}_{S_1}(0)) = \mathrm{wp}_{S_0}(0) = 0$ by **48**(ii).

54(iii): Assume $\mathrm{wp}_{S_i}(0) = 0$ for every $i \in I$. Then

$$\begin{aligned} \mathrm{wp}_{\mathbf{if}\, i:B_i \to S_i\, \mathbf{fi}}(0) &= \prod_{i \in I} \left(\overline{\mathrm{d}_{B_i}} + \mathrm{wp}_{S_i}(0)\right) \cdot \sum_{i \in I} \mathrm{d}_{B_i} && \textbf{49}(\mathrm{ii}) \\ &= \prod_{i \in I} \overline{\mathrm{d}_{B_i}} \cdot \sum_{i \in I} \mathrm{d}_{B_i} && \text{hypothesis, } \textbf{5}(\mathrm{vii}) \\ &= 0 && \textbf{8}(\mathrm{ii}), \textbf{5}(\mathrm{ii}), \textbf{3}(\mathrm{viii}) \end{aligned}$$

54(iv): Assume $\mathrm{wp}_S(0) = 0$. Note first that

$$\begin{aligned} \mathrm{wp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(0) &= \prod\{y : (\mathrm{d}_B + 0) \cdot (\overline{\mathrm{d}_B} + \mathrm{wp}_S(y)) = y\} && \textbf{53}(\mathrm{iv}) \\ &= \prod\{y : \mathrm{d}_B \cdot \mathrm{wp}_S(y) = y\} && \textbf{5}(\mathrm{ii})(\mathrm{vii}), \textbf{3}(\mathrm{xv}), (\mathrm{Ba}_2) \end{aligned}$$

But $0 \in \{y : \mathrm{d}_B \cdot \mathrm{wp}_S(y) = y\}$ because $\mathrm{d}_B \cdot \mathrm{wp}_S(0) = 0$ by assumption. It follows that $\prod\{y : \mathrm{d}_B \cdot \mathrm{wp}_S(y) = y\} \leq 0$, hence $\mathrm{wp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(0) = 0$.

54(v): This follows from **54**(i)–(iv) by induction on the complexity of commands.

10 Determinism for Correct Interpretations

From their operational interpretation it is natural to expect that *skip* and *abort* should be deterministic. It is also natural to say that "*havoc* is not deterministic. In fact, *havoc* is almost as nondeterministic as possible". ($134_{2\text{–}3}$) However, even under the operational interpretation there is one case in which *havoc is* deterministic, namely, when there is only one machine state.

Theorem 55. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation of $\mathcal{L}$.*

(i) *skip and abort are deterministic,*
(ii) *havoc is deterministic if and only if $\mathfrak{A}$ is Boolean, i.e.,* $1' = 1$.

Proof. **55**(i): By **44** and **45**(ii)(iii), *skip* is deterministic because 0 is functional and $0 \cdot 1 = 0$, while *abort* is deterministic because $1'$ is functional and $1' \cdot 0 = 0$.

55(ii): By **44** and **45**(i), *havoc* is deterministic iff 1 is functional, *i.e.*, iff $\breve{1};1 \leq 1'$. But $\breve{1};1 = 1$, so *havoc* is deterministic iff $1' = 1$.

An obvious sufficient (but not necessary) condition for $S_0;S_1$ to be deterministic is that S_0 and S_1 are deterministic.

Theorem 56. *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d} \rangle$ is a correct interpretation of $\mathcal{L}$. For all $S_0, S_1 \in \mathcal{L}_S$, if S_0 and S_1 are deterministic, then so is $S_0;S_1$.*

Proof. Assume S_0 and S_1 are deterministic. By **44** we get

(1) r_{S_0} and r_{S_1} are functional,
(2) $\mathrm{r}_{S_0} \cdot \mathrm{e}_{S_0} = 0$ and $\mathrm{r}_{S_1} \cdot \mathrm{e}_{S_1} = 0$.

From (1) it follows that $r_{S_0;S_1}$ is functional by **28**(i) and **45**(iv). Also,

$$\begin{aligned}
r_{S_0;S_1} \cdot e_{S_0;S_1} &= r_{S_0};r_{S_1} \cdot (e_{S_0} + r_{S_0};e_{S_1}) && \textbf{45}(\text{iv}) \\
&= r_{S_0};r_{S_1} \cdot e_{S_0} + r_{S_0};r_{S_1} \cdot r_{S_0};e_{S_1}) && \textbf{3}(\text{xv}) \\
&= (e_{S_0} \cdot r_{S_0});r_{S_1} + r_{S_0};r_{S_1} \cdot r_{S_0};e_{S_1}) && \textbf{3}(\text{viii}), \textbf{32}(\text{i}), \textbf{24}(\text{xxiv}) \\
&= (e_{S_0} \cdot r_{S_0});r_{S_1} + r_{S_0};(r_{S_1} \cdot e_{S_1}) && (1), \textbf{28}(\text{ii}) \\
&= 0;r_{S_1} + r_{S_0};0 && (2), \textbf{3}(\text{viii}) \\
&= 0 && \textbf{24}(\text{xviii})
\end{aligned}$$

Therefore $r_{S_0;S_1}$ is deterministic by **44**.

Theorem 57. [DS90 (7,39) p.144] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, r, e, d \rangle$ is a correct interpretation of $\mathcal{L}$. Assume S_i is deterministic for every $i \in I$ and $d_{B_i} \cdot d_{B_j} = 0$ whenever $i \neq j$ and $i, j \in I$. Then* $\mathbf{if}\, i : B_i \rightarrow S_i\, \mathbf{fi}$ *is deterministic.*

Proof. From the assumptions we get, by **44**,

(1) $d_{B_i} \cdot d_{B_j} = 0$ whenever $i \neq j$ and $i, j \in I$,
(2) r_{S_i} is functional and $r_{S_i} \cdot e_{S_i} = 0$ for every $i \in I$.

Then

$$\begin{aligned}
&(r_{\mathbf{if}\, i:B_i \rightarrow S_i\, \mathbf{fi}})^{\smile};r_{\mathbf{if}\, i:B_i \rightarrow S_i\, \mathbf{fi}} \\
&= \Big(\sum_{i \in I} d_{B_i} \cdot r_{S_i}\Big)^{\smile};\sum_{j \in I}(d_{B_j} \cdot r_{S_j}) && \textbf{45}(\text{v}) \\
&= \sum_{i \in I}(d^{\smile}_{B_i} \cdot r^{\smile}_{S_i});\sum_{j \in I}(d_{B_j} \cdot r_{S_j}) && \textbf{24}(\text{vii}) \\
&= \sum_{i,j \in I}((d^{\smile}_{B_i} \cdot r^{\smile}_{S_i});(d_{B_j} \cdot r_{S_j})) && \textbf{24}(\text{xv}) \\
&= \sum_{i,j \in I}(r^{\smile}_{S_i};(d_{B_i} \cdot d_{B_j} \cdot r_{S_j})) && \textbf{32}(\text{ii}), \textbf{24}(\text{xxvii}), \textbf{3}(\text{ix}) \\
&= \sum_{i,j \in I,\ i=j}(r^{\smile}_{S_i};(d_{B_i} \cdot d_{B_j} \cdot r_{S_j})) \\
&\quad + \sum_{i,j \in I,\ i \neq j}(r^{\smile}_{S_i};(d_{B_i} \cdot d_{B_j} \cdot r_{S_j})) && \textbf{10}(\text{i}) \\
&= \sum_{i \in I}(r^{\smile}_{S_i};(d_{B_i} \cdot d_{B_i} \cdot r_{S_i})) \\
&\quad + \sum_{i,j \in I,\ i \neq j}(r^{\smile}_{S_i};(0 \cdot r_{S_j})) && (1) \\
&= \sum_{i \in I}(r^{\smile}_{S_i};(d_{B_i} \cdot r_{S_i})) && \textbf{5}(\text{vi})(\text{vii}), \textbf{3}(\text{viii}), \textbf{24}(\text{xviii}) \\
&\leq \sum_{i \in I}(r^{\smile}_{S_i};r_{S_i}) && \textbf{24}(\text{x}), \textbf{7}(\text{iv}), \textbf{9}(\text{iii}) \\
&\leq \sum_{i \in I} 1' && (2), \textbf{9}(\text{iii}) \\
&= 1'
\end{aligned}$$

Therefore $r_{\mathbf{if}\, i:B_i \rightarrow S_i\, \mathbf{fi}}$ is functional. From

$$\begin{aligned}
(3) \qquad \sum_{i \in I}(d_{B_i} \cdot r_{S_i}) \cdot \prod_{i \in I} \overline{d_{B_i}} &\leq \sum_{i \in I} d_{B_i} \cdot \prod_{i \in I} \overline{d_{B_i}} && \textbf{7}(\text{iv}), \textbf{9}(\text{iii}) \\
&= \sum_{i \in I} d_{B_i} \cdot \overline{\sum_{i \in I} d_{B_i}} && \textbf{8}(\text{ii}) \\
&= 0 && \textbf{5}(\text{ii})
\end{aligned}$$

and

(4)
$$\begin{aligned}
&\sum_{i\in I}(\mathrm{d}_{B_i}\cdot \mathrm{r}_{S_i})\cdot\sum_{j\in I}(\mathrm{d}_{B_j}\cdot \mathrm{e}_{S_j})\\
&=\sum_{i,j\in I}(\mathrm{d}_{B_i}\cdot \mathrm{r}_{S_i}\cdot \mathrm{d}_{B_j}\cdot \mathrm{e}_{S_j}) && \textbf{16}, \textbf{3}(\text{viii})\\
&=\sum_{i,j\in I,\,i=j}(\mathrm{d}_{B_i}\cdot \mathrm{r}_{S_i}\cdot \mathrm{d}_{B_j}\cdot \mathrm{e}_{S_j})+\sum_{i,j\in I,\,i\neq j}(\mathrm{d}_{B_i}\cdot \mathrm{r}_{S_i}\cdot \mathrm{d}_{B_j}\cdot \mathrm{e}_{S_j}) && \textbf{10}(\text{i})\\
&=\sum_{i\in I}(\mathrm{r}_{S_i}\cdot \mathrm{e}_{S_i}\cdot \mathrm{d}_{B_i})+\sum_{i,j\in I,\,i\neq j}(\mathrm{d}_{B_i}\cdot \mathrm{d}_{B_j}\cdot \mathrm{r}_{S_i}\cdot \mathrm{e}_{S_j}) && \textbf{3}(\text{viii})(\text{ix})\\
&=\sum_{i,j\in I}(0\cdot \mathrm{d}_{B_i})+\sum_{i,j\in I,\,i\neq j}(0\cdot \mathrm{r}_{S_i}\cdot \mathrm{e}_{S_j}) && (1),\,(2)\\
&=0
\end{aligned}$$

we get

$$\begin{aligned}
&\mathrm{r}_{\textbf{if}\,i:B_i\to S_i\,\textbf{fi}}\cdot \mathrm{e}_{\textbf{if}\,i:B_i\to S_i\,\textbf{fi}}\\
&=\sum_{i\in I}(\mathrm{d}_{B_i}\cdot \mathrm{r}_{S_i})\cdot\Big(\prod_{i\in I}\overline{\mathrm{d}_{B_i}}+\sum_{j\in I}(\mathrm{d}_{B_j}\cdot \mathrm{e}_{S_j})\Big) && \textbf{45}(\text{v})\\
&=\sum_{i\in I}(\mathrm{d}_{B_i}\cdot \mathrm{r}_{S_i})\cdot\prod_{i\in I}\overline{\mathrm{d}_{B_i}}+\sum_{i\in I}(\mathrm{d}_{B_i}\cdot \mathrm{r}_{S_i})\cdot\sum_{j\in I}(\mathrm{d}_{B_j}\cdot \mathrm{e}_{S_j}) && \textbf{3}(\text{xv})\\
&=0 && (3),\,(4)
\end{aligned}$$

so $\textbf{if}\,i : B_i\to S_i\,\textbf{fi}$ is deterministic by **44**.

Theorem 58. [DS90 (9,14) p.173] *Assume $\mathcal{L}$ is suitable and $\langle\mathfrak{A},\mathrm{r},\mathrm{e},\mathrm{d}\rangle$ is a correct interpretation of $\mathcal{L}$. For every $S\in\mathcal{L}_S$ and every $B\in\mathcal{L}_P$, if S is deterministic, then so is* $\textbf{do}\,B\to S\,\textbf{od}$.

Proof. This proof follows the one in [DS90 p.173]. Assume S is deterministic. By **42** we know that

(1)
$$\mathrm{wlp}^*_S(x)\leq \mathrm{wp}_S(x)\text{ for all }x\in A,$$

and we need to show $\mathrm{wlp}^*_{\textbf{do}\,B\to S\,\textbf{od}}(x)\leq \mathrm{wp}_{\textbf{do}\,B\to S\,\textbf{od}}(x)$ for every x.

$$\begin{aligned}
&\mathrm{wlp}^*_{\textbf{do}\,B\to S\,\textbf{od}}(x)\\
&=\overline{\mathrm{wlp}_{\textbf{do}\,B\to S\,\textbf{od}}(\overline{x})} && \textbf{11}\\
&=\overline{\textstyle\sum\{y:(\mathrm{d}_B+\overline{x})\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}_S(y))=y\}} && \textbf{53}(\text{iii})\\
&=\textstyle\prod\{\overline{y}:(\mathrm{d}_B+\overline{x})\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}_S(y))=y\} && \textbf{8}(\text{ii})\\
&=\textstyle\prod\{\overline{y}:\overline{\mathrm{d}_B}\cdot x+\mathrm{d}_B\cdot\overline{\mathrm{wlp}_S(y)}=\overline{y}\} && \textbf{3}(\text{ii})(\text{xvi})(\text{xvii})\\
&=\textstyle\prod\{y:\overline{\mathrm{d}_B}\cdot x+\mathrm{d}_B\cdot\overline{\mathrm{wlp}_S(\overline{y})}=y\} && \textbf{3}(\text{ii})\\
&=\textstyle\prod\{y:\overline{\mathrm{d}_B}\cdot x+\mathrm{d}_B\cdot\mathrm{wlp}^*_S(y)=y\} && \textbf{11}\\
&=\textstyle\prod\{y:(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}^*_S(y))=y\} && \textbf{3}(\text{xix})\\
&=\textstyle\prod\{y:(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}^*_S(y))\leq y\} && \text{Tarski's Theorem}\\
&\leq\textstyle\prod\{y:(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wp}_S(y))\leq y\} && (1),\,\textbf{10}(\text{iv})\\
&=\textstyle\prod\{y:(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wp}_S(y))=y\} && \text{Tarski's Theorem}\\
&=\mathrm{wp}_{\textbf{do}\,B\to S\,\textbf{od}}(y) && \textbf{53}(\text{iv})
\end{aligned}$$

For the equalities above which are justified by a reference to Tarski's Theorem, we need to know that the functions $(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}^*_S(\text{-}))$ and $(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wp}_S(\text{-}))$ are monotone, but this is a consequence of **37**, **38**, **16**, and **17**(i)(ii). For the inequality justified by (1) and **10**(iv), we note that if $(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wp}_S(y))\leq y$ then $(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}^*_S(y))\leq y$ since $(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}^*_S(y))\leq(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wp}_S(y))$ by (1), so $\{y:(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wp}_S(y))\leq y\}\subseteq\{y:(\mathrm{d}_B+x)\cdot(\overline{\mathrm{d}_B}+\mathrm{wlp}^*_S(y))\leq y\}$, to which we apply **10**(iv).

11 The Main Repetition Theorem

Theorem **59**, presented in this section, is a generalization of what is called "the Main Repetition Theorem" in [DS90]. An informal statement of this result runs as follows. Assume

(1) P is a predicate,
(2) if P and B hold at a state then no eternal computation of S is possible from that state,
(3) if P and B hold at the initial state of a computation of S, then P holds at the final state, and the initial state is in the relation G to (is "greater than") the final state,
(4) there is no infinite sequence of states such that P and B hold at every state in the sequence, and each state is in relation G to the next state.

It follows from these assumptions that $\mathrm{wp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(P)$ holds where P does, that is, P is a sufficient (but usually not necessary) condition for the guaranteed termination of $\mathbf{do}\,B\to S\,\mathbf{od}$ at a state satisfying P. Theorem **59** generalizes the Main Repetition Theorem in two ways. First, it does not include the assumption that G is transitive, a possibility noted in [DS90 pp.174–5]. Second, as is the case for all the results in this paper, it applies to arbitrary relation algebras, not just the ones built from true binary relations on a set.

Theorem 59. [DS90 (9,26) p.180 *The Main Repetition Theorem*] *Assume $\mathcal{L}$ is suitable and $\langle \mathfrak{A}, \mathrm{r}, \mathrm{e}, \mathrm{d}\rangle$ is a correct interpretation of $\mathcal{L}$. For every command $S \in \mathcal{L}_S$, every predicate $B \in \mathcal{L}_P$, and all $p, g \in A$, if*

(1) $p;1 = p$,
(2) $p \cdot \mathrm{d}_B \le \overline{\mathrm{e}_S}$,
(3) $p \cdot \mathrm{d}_B \cdot \mathrm{r}_S \le g \cdot \breve{p}$,
(4) $\sum\{z : z \le (g \cdot p \cdot \mathrm{d}_B \cdot \breve{p} \cdot \breve{\mathrm{d}_B});z\} = 0$,

then $p \le \mathrm{wp}_{\mathbf{do}\,B\to S\,\mathbf{od}}(p)$.

Proof. First we prove that

(5) for every $y \in A$, if $y \le \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)$ then $p \cdot y = 0$.

Assume $y \le \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)$. Then

$$\begin{aligned}
p \cdot y &\le p \cdot \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y) && \text{hypothesis, } \mathbf{7}\text{(iii)}, \mathbf{3}\text{(viii)(ix)}\\
&= p \cdot \mathrm{d}_B \cdot \mathrm{e}_S + p \cdot \mathrm{d}_B \cdot \mathrm{r}_S;y && \mathbf{3}\text{(xv)}\\
&= p \cdot \mathrm{d}_B \cdot \mathrm{r}_S;y && (2), \mathbf{7}\text{(v)}, \mathbf{3}\text{(ii)}, (\mathrm{Ba}_2), \mathbf{5}\text{(vii)}\\
&= (p \cdot \mathrm{d}_B \cdot \mathrm{r}_S);y && (1), \mathbf{32}\text{(ii)}, \mathbf{26}\text{(i)}, \mathbf{24}\text{(xxiv)}\\
&\le (g \cdot p \cdot \mathrm{d}_B \cdot \breve{p});y && (3), \mathbf{7}\text{(iv)}, \mathbf{24}\text{(x)}\\
&= (g \cdot p \cdot \mathrm{d}_B \cdot \breve{p});(\mathrm{d}_B \cdot y) && \text{hypothesis, } \mathbf{7}\text{(iv)}, \mathbf{24}\text{(x)}\\
&= (g \cdot p \cdot \mathrm{d}_B \cdot \breve{p} \cdot \breve{\mathrm{d}_B});y && \mathbf{32}\text{(ii)}, \mathbf{24}\text{(xxvii)}\\
&= (g \cdot p \cdot \mathrm{d}_B \cdot \breve{p} \cdot \breve{\mathrm{d}_B});(p \cdot y) && (1), \mathbf{3}\text{(viii)(ix)}, \mathbf{24}\text{(xxvii)}
\end{aligned}$$

Since $p \cdot y$ belongs to a set whose join is 0 by (4), we conclude that $p \cdot y = 0$, finishing the proof of (5). Next,

$$\begin{aligned}
p \cdot \mathrm{e}_{\mathbf{do}\,B\to S\,\mathbf{od}} &= p \cdot \textstyle\sum\{y : y \le \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\} && \mathbf{45}\text{(vi)}\\
&= \textstyle\sum\{p \cdot y : y \le \mathrm{d}_B \cdot (\mathrm{e}_S + \mathrm{r}_S;y)\} && \mathbf{16}\\
&= \textstyle\sum\{0\} && (5)\\
&= 0.
\end{aligned}$$

It follows that

(6) $$p \le \overline{\mathrm{e}_{\mathbf{do}\,B\to S\,\mathbf{od}}}$$

by **3**(ii) and **7**(v). Next we prove

(7) $$p \cdot (\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p}) = 0 \text{ for every } i \in \omega$$

by induction on i. For $i = 0$ we have $p \cdot (\mathrm{d}_B \cdot \mathrm{r}_S)^0;(\overline{\mathrm{d}_B} \cdot \overline{p}) = p \cdot 1';(\overline{\mathrm{d}_B} \cdot \overline{p}) = p \cdot \overline{\mathrm{d}_B} \cdot \overline{p} = 0$ by **29**, **24**(xix), **3**(viii)(ix), and **5**(ii)(vi). For the inductive step, assume $p \cdot (\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p}) = 0$. Then

$$
\begin{aligned}
p \cdot (\mathrm{d}_B \cdot \mathrm{r}_S)^{i+1};(\overline{\mathrm{d}_B} \cdot \overline{p}) &= p \cdot (\mathrm{d}_B \cdot \mathrm{r}_S);(\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p}) && \textbf{29} \\
&= p \cdot (\mathrm{d}_B \cdot \mathrm{r}_S);((\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p})) && (\mathrm{Ra}_1) \\
&= (p \cdot \mathrm{d}_B \cdot \mathrm{r}_S);((\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p})) && (1), \textbf{24}(\mathrm{xxiv}), \textbf{3}(\mathrm{ix}) \\
&\leq \breve{p};((\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p})) && (3), \textbf{7}(\mathrm{iv}), \textbf{24}(\mathrm{x}) \\
&= \breve{p};(p \cdot (\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p})) && (1), \textbf{3}(\mathrm{vii}), \textbf{24}(\mathrm{xxvii}) \\
&= \breve{p};0 && \text{inductive hypothesis} \\
&= 0 && \textbf{24}(\mathrm{xviii})
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
p \cdot \overline{\mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(p)} &= p \cdot \sum_{i \in \omega} ((\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p})) && \textbf{50}, \textbf{3}(\mathrm{ii}) \\
&= \sum_{i \in \omega} (p \cdot (\mathrm{d}_B \cdot \mathrm{r}_S)^i;(\overline{\mathrm{d}_B} \cdot \overline{p})) && \textbf{16} \\
&= \sum_{i \in \omega} 0 && (7) \\
&= 0
\end{aligned}
$$

so

$$
p \leq \mathrm{wlp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(p) \tag{8}
$$

by **7**(v). From (6) and (8) we conclude by **35**(i) and **7**(iv) that $p \leq \mathrm{wp}_{\mathbf{do}\, B \to S\, \mathbf{od}}(p)$, as desired.

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Appendix A. Proofs for Sections 3–6

Proof of **3**.

$$\begin{aligned}
\textbf{3(i)}:\quad x+\overline{x} &= \left(\overline{\overline{x}+\overline{\overline{y}}+\overline{\overline{x}+\overline{y}}}\right)+\left(\overline{\overline{\overline{x}+\overline{\overline{y}}}+\overline{\overline{\overline{x}}+\overline{y}}}\right) && (\mathrm{Ba}_3)\\
&= \left(\overline{\overline{\overline{y}}+\overline{x}+\overline{\overline{y}+\overline{x}}}\right)+\left(\overline{\overline{\overline{y}+\overline{\overline{x}}}+\overline{\overline{y}+\overline{\overline{x}}}}\right) && (\mathrm{Ba}_2)\\
&= \left(\overline{\overline{y}+\overline{\overline{x}}+\overline{\overline{y}+\overline{x}}}\right)+\left(\overline{\overline{\overline{y}+\overline{\overline{x}}}+\overline{\overline{y}+\overline{x}}}\right) && (\mathrm{Ba}_1),\ (\mathrm{Ba}_2)\\
&= y+\overline{y} && (\mathrm{Ba}_3)
\end{aligned}$$

$$\begin{aligned}
\textbf{3(ii)}:\quad \overline{\overline{x}} &= \overline{\overline{\overline{x}}+\overline{\overline{x}}+\overline{\overline{\overline{x}}+\overline{x}}} && (\mathrm{Ba}_3)\\
&= \overline{\overline{x}+\overline{\overline{x}}+\overline{\overline{x}+\overline{\overline{x}}}} && (\mathrm{Ba}_2)\\
&= \overline{\overline{x}+\overline{\overline{x}}+\overline{x+\overline{x}}} && \textbf{3(i)}\\
&= x && (\mathrm{Ba}_3)
\end{aligned}$$

$$\begin{aligned}
\textbf{3(iii)}:\quad \overline{x} &= \overline{\overline{\overline{x}+\overline{y}}+\overline{\overline{x}+y}} && (\mathrm{Ba}_3)\\
&= \overline{x+\overline{y}+\overline{x+y}} && \textbf{3(ii)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(iv)}:\quad x+(y+\overline{y}) &= x+(x+\overline{x}) && \textbf{3(i)}\\
&= x+x+\overline{x} && (\mathrm{Ba}_1)\\
&= x+x+(\overline{x+x}+\overline{x+\overline{x}}) && \textbf{3(iii)},\ (\mathrm{Ba}_2)\\
&= x+x+\overline{x+x}+\overline{x+\overline{x}} && (\mathrm{Ba}_1)\\
&= x+\overline{x}+\overline{x+\overline{x}} && \textbf{3(i)}\\
&= z+\overline{z} && \textbf{3(i)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(v)}:\quad x + x &= \overline{\overline{x+x} + \overline{\overline{x+\overline{x}}}} + \overline{\overline{x+x} + (x+\overline{x})} && (\mathrm{Ba}_3)\\
&= \overline{\overline{x+x} + \overline{x+\overline{x}}} + \overline{y+\overline{y}} && \textbf{3(iv)}\\
&= \overline{\overline{x}} + \overline{y+\overline{y}} && \textbf{3(iii)}\\
&= x + \overline{y+\overline{y}} && \textbf{3(ii)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(vi)}:\quad x + x &= \overline{\overline{x+x} + \overline{\overline{x+\overline{x}} + \overline{x}}} + \overline{\overline{x+x} + (\overline{x+\overline{x}} + \overline{x})} && (\mathrm{Ba}_3)\\
&= \overline{\overline{x+x} + \overline{\overline{x+\overline{x}} + \overline{x}}} + \overline{\overline{x+x} + \overline{x+\overline{x}} + \overline{x}} && (\mathrm{Ba}_1)\\
&= \overline{\overline{x+x} + \overline{\overline{x+\overline{x}} + \overline{x}}} + \overline{\overline{x}+\overline{x}} && \textbf{3(iii)}\\
&= \overline{\overline{x+x+\overline{x}} + \overline{\overline{x+\overline{x}} + \overline{x}}} + \overline{\overline{x}+\overline{x}} && \textbf{3(v)}\\
&= \overline{\overline{x}+\overline{x}} + \overline{\overline{\overline{x+x}+\overline{x}} + \overline{\overline{x}+x}+x} && (\mathrm{Ba}_2)\\
&= \overline{\overline{x}+\overline{x}} + \overline{\overline{x}+x} && (\mathrm{Ba}_3)\\
&= x && (\mathrm{Ba}_3)
\end{aligned}$$

$$\begin{aligned}
\textbf{3(vii)}:\quad x \cdot x &= \overline{\overline{x}+\overline{x}} && (\mathrm{Ba}_4)\\
&= \overline{\overline{x}} && \textbf{3(vi)}\\
&= x && \textbf{3(ii)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(viii)}:\quad x \cdot y &= \overline{\overline{x}+\overline{y}} && (\mathrm{Ba}_4)\\
&= \overline{\overline{y}+\overline{x}} && (\mathrm{Ba}_2)\\
&= y \cdot x && (\mathrm{Ba}_4)
\end{aligned}$$

$$\begin{aligned}
\textbf{3(ix)}:\quad x \cdot y \cdot z &= \overline{\overline{\overline{x}+\overline{y}}+\overline{z}} && (\mathrm{Ba}_4)\\
&= \overline{\overline{x}+\overline{y}+\overline{z}} && \textbf{3(ii)}\\
&= \overline{\overline{x}+(\overline{y}+\overline{z})} && (\mathrm{Ba}_1)\\
&= \overline{\overline{x}+\overline{\overline{y}+\overline{z}}} && \textbf{3(ii)}\\
&= x \cdot (y \cdot z) && (\mathrm{Ba}_4)
\end{aligned}$$

$$\begin{aligned}
\textbf{3(x)}:\quad (x+y)\cdot x &= \overline{\overline{\overline{x+y}+\overline{x}}} && (\mathrm{Ba}_4)\\
&= \overline{\overline{x+y}+(\overline{x+y}+\overline{x+\overline{y}})} && \textbf{3(iii)}\\
&= \overline{\overline{x+y}+\overline{x+y}+\overline{x+\overline{y}}} && (\mathrm{Ba}_1)\\
&= \overline{\overline{x+y}+\overline{x+\overline{y}}} && \textbf{3(vi)}\\
&= \overline{\overline{x}} && \textbf{3(iii)}, (\mathrm{Ba}_2)\\
&= x && \textbf{3(ii)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xi)}:\quad x &= \overline{\overline{x}+\overline{y}}+\overline{\overline{x}+y} && (\mathrm{Ba}_3)\\
&= \overline{\overline{x}+\overline{y}}+\overline{\overline{x}+\overline{\overline{y}}} && \textbf{3(ii)}\\
&= x\cdot y + x\cdot\overline{y} && (\mathrm{Ba}_4)
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xii)}:\quad x &= \overline{\overline{x}} && \textbf{3(ii)}\\
&= \overline{\overline{x+\overline{y}}+\overline{x+y}} && \textbf{3(iii)}\\
&= (x+\overline{y})\cdot(x+y) && (\mathrm{Ba}_4)
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xiii)}:\quad (x+y)\cdot\overline{x} &= (x+y)\cdot((\overline{x}+y)\cdot\overline{x}) && \textbf{3(x)}\\
&= (x+y)\cdot(\overline{x}+y)\cdot\overline{x} && \textbf{3(ix)}\\
&= (y+\overline{x})\cdot(y+x)\cdot\overline{x} && \textbf{3(viii)}, (\mathrm{Ba}_2)\\
&= y\cdot\overline{x} && \textbf{3(xii)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xiv)}:\quad x+x\cdot y &= x\cdot y+x\cdot\overline{y}+x\cdot y && \textbf{3(xi)}\\
&= x\cdot y+x\cdot y+x\cdot\overline{y} && (\mathrm{Ba}_1), (\mathrm{Ba}_2)\\
&= x\cdot y+x\cdot\overline{y} && \textbf{3(vi)}\\
&= x && \textbf{3(xi)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xv)}:\quad x\cdot(y+z) &= x\cdot(y+z)\cdot y+x\cdot(y+z)\cdot\overline{y} && \textbf{3(xi)}\\
&= x\cdot((y+z)\cdot y)+x\cdot((y+z)\cdot\overline{y}) && \textbf{3(ix)}\\
&= x\cdot y+x\cdot((y+z)\cdot\overline{y}) && \textbf{3(x)}\\
&= x\cdot y+x\cdot(z\cdot\overline{y}) && \textbf{3(xiii)}\\
&= x\cdot y+x\cdot y\cdot z+x\cdot(z\cdot\overline{y}) && \textbf{3(xiv)}\\
&= x\cdot y+(x\cdot z\cdot y+x\cdot z\cdot\overline{y}) && (\mathrm{Ba}_1), \textbf{3(viii)(ix)}\\
&= x\cdot y+x\cdot z && \textbf{3(xi)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xvi)}:\quad \overline{x}\cdot\overline{y} &= \overline{\overline{\overline{x}}+\overline{\overline{y}}} && (\mathrm{Ba}_4)\\
&= \overline{x+y} && \textbf{3(ii)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xvii)}:\quad \overline{x\cdot y} &= \overline{\overline{\overline{x}}+\overline{\overline{y}}} && (\mathrm{Ba}_4)\\
&= \overline{x}+\overline{y} && \textbf{3(ii)}
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xviii)}:\quad x+y\cdot z &= x+\overline{\overline{y}+\overline{z}} && (\mathrm{Ba}_4)\\
&= \overline{\overline{x}}+\overline{\overline{y}+\overline{z}} && \textbf{3(ii)}\\
&= \overline{\overline{x}\cdot(\overline{y}+\overline{z})} && \textbf{3(xvii)}\\
&= \overline{\overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{z}} && \textbf{3(xv)}\\
&= \overline{\overline{x+y}+\overline{x+z}} && \textbf{3(xvi)}\\
&= (x+y)\cdot(x+z) && (\mathrm{Ba}_4)
\end{aligned}$$

$$\begin{aligned}
\textbf{3(xix)}:\quad (x+y)\cdot(\overline{x}+z) &= (x+y)\cdot\overline{x}+(x+y)\cdot z && \textbf{3(xv)}\\
&= y\cdot\overline{x}+(x+y)\cdot z && \textbf{3(xiii)}\\
&= y\cdot\overline{x}+x\cdot z+y\cdot z && (\mathrm{Ba}_2), \textbf{3(xv)}, (\mathrm{Ba}_1)\\
&= y\cdot\overline{x}+x\cdot z+y\cdot z\cdot x+y\cdot z\cdot\overline{x} && \textbf{3(xi)}, (\mathrm{Ba}_1)\\
&= \overline{x}\cdot y+\overline{x}\cdot y\cdot z+(x\cdot z+x\cdot z\cdot y) && \textbf{3(viii)(ix)}, (\mathrm{Ba}_1), (\mathrm{Ba}_2)\\
&= \overline{x}\cdot y+x\cdot z && \textbf{3(xiv)}
\end{aligned}$$

Proof of **5**. Using **4** and (Ba_4), we convert (i)–(viii) into equivalent identities, and indicate why each of them holds.

(i′) $y+\overline{y}=x+\overline{x}$ holds by **3**(i).
(ii′) $\overline{y+\overline{y}}=\overline{x}+\overline{\overline{x}}$ holds by **3**(i).

(iii′) $\overline{x+\overline{x}} = \overline{y+\overline{y}}$ holds by **3**(i).
(iv′) $\overline{\overline{x+\overline{x}}} = y+\overline{y}$ holds by **3**(i)(ii).
(v′) $x+(y+\overline{y}) = z+\overline{z}$ holds by **3**(iv).
(vi′) $\overline{x}+\overline{\overline{y+\overline{y}}} = \overline{z+\overline{z}}$ holds by **3**(ii)(iv).
(vii′) $x+\overline{y+\overline{y}} = x$ holds by **3**(v)(vi).
(viii′) $\overline{x}+\overline{\overline{y+\overline{y}}} = x$ holds by **3**(ii)(v)(vii).

Proof of **7**.

7(i): Let $x, y, z \in B$. Then $x \leq x$ since $x+x=x$ by **3**(vi). If $x \leq y$ and $y \leq z$, then $x+y=y$ and $y+z=z$, so, using these equations and (Ba_1), we have $x+z = x+(y+z) = x+y+z = y+z = z$, *i.e.*, $x \leq z$. Finally, if $x \leq y$ and $y \leq x$, then $x+y=y$ and $y+x=x$, so $x=y$ by (Ba_2).

We have $x \cdot y + x = x$ by **3**(xiv) and (Ba_2), so $x \cdot y \leq x$, and $x \cdot y + y = y$ by **3**(xiv), so $x \cdot y \leq y$. Thus $x \cdot y$ is a lower bound of $\{x, y\}$. If $z \leq x$ and $z \leq y$, then $z+x=x$ and $z+y=y$, so, by **3**(xviii), $z + x \cdot y = (z+x) \cdot (z+y) = x \cdot y$, and therefore $z \leq x \cdot y$. Thus $x \cdot y$ is the greatest lower bound of $\{x, y\}$.

7(ii): We obtain $0 \leq x$ from **5**(vii), (Ba_2), and **6**, while $x \leq 1$ follows from **5**(v) and **6**.

7(iii): Suppose $x \leq y$, *i.e.*, $x+y=y$. Then

$$\begin{aligned} x+z+(y+z) &= x+y+(z+z) && (\mathrm{Ba}_1), (\mathrm{Ba}_2) \\ &= x+y+z && \textbf{3}(\mathrm{vi}) \\ &= y+z && x+y=y \end{aligned}$$

so $x+z \leq y+z$. Also,

$$\begin{aligned} x \cdot z + y \cdot z &= (x+y) \cdot z && \textbf{3}(\mathrm{viii})(\mathrm{xv}) \\ &= y \cdot z && x+y=y \end{aligned}$$

so $x \cdot z \leq y \cdot z$.

7(iv): We have

$$\begin{aligned} x+(x+y) &= x+x+y && (\mathrm{Ba}_1) \\ &= x+y && \textbf{3}(\mathrm{vi}) \end{aligned}$$

so $x \leq x+y$, and

$$\begin{aligned} y+(x+y) &= x+y+y && (\mathrm{Ba}_2) \\ &= x+(y+y) && (\mathrm{Ba}_1) \\ &= x+y && \textbf{3}(\mathrm{vi}) \end{aligned}$$

so $y \leq x+y$. Thus $x+y$ is an upper bound of $\{x, y\}$. Suppose z is an upper bound of $\{x, y\}$. Then $x \leq z$ and $y \leq z$, *i.e.*, $x+z=z$ and $y+z=z$, so $x+y+z = x+(y+z) = x+z = z$ by (Ba_1). Thus $x+y \leq z$.

7(v): (1) and (3) are equivalent by **6**. Assume (3) holds. Then

$$\begin{aligned} 1 &= y+1 && \textbf{5}(\mathrm{v}) \\ &= y+(x+\overline{x}) && \textbf{5}(\mathrm{i}) \\ &= \overline{x}+(x+y) && (\mathrm{Ba}_1), (\mathrm{Ba}_2) \\ &= \overline{x}+y && (3) \end{aligned}$$

so (5) holds. Using (Ba_4) and **5**(iii), we see that (6) is equivalent to $\overline{x+\overline{y}} = \overline{1}$, but the latter statement is equivalent to (5) by **3**(ii). Assume (5) holds. Then

$$\begin{aligned} \overline{x} &= (\overline{x}+\overline{y}) \cdot (\overline{x}+y) && \textbf{3}(\mathrm{xii}) \\ &= (\overline{x}+\overline{y}) \cdot 1 && (5) \\ &= \overline{x}+\overline{y} && \textbf{5}(\mathrm{viii}) \end{aligned}$$

so (2) holds by **6**. Assume (2) holds, *i.e.*, $\overline{y} + \overline{x} = \overline{y}$ by **6**. Then

$$\begin{aligned} x \cdot y &= \overline{\overline{x} + \overline{y}} && (\mathrm{Ba}_4) \\ &= \overline{\overline{y} + \overline{x}} && (\mathrm{Ba}_2) \\ &= \overline{\overline{x}} && (2) \\ &= x && \mathbf{3}(\mathrm{ii}) \end{aligned}$$

so (4) holds. Assume (4) holds. Then

$$\begin{aligned} y &= y + y \cdot x && \mathbf{3}(\mathrm{xiv}) \\ &= x \cdot y + x && (\mathrm{Ba}_2),\ \mathbf{3}(\mathrm{viii}) \\ &= x + y && (4) \end{aligned}$$

Thus (4) implies (3).

Proof of **8**.

8(i): Assume $I = \emptyset$. Then $\{x_i : i \in I\} = \emptyset$. Every element of $\mathfrak{B}$ is an upper bound of $\emptyset$, so the least upper bound of $\emptyset$ is the least element of $\mathfrak{B}$, namely, by **7**(ii), 0. Similarly, every element of $\mathfrak{B}$ is a lower bound of $\emptyset$, so the greatest upper bound of $\emptyset$ is the greatest element of $\mathfrak{B}$, namely 1.

8(ii): Assume $\sum_{i \in I} x_i$ exists. We need to show that $\overline{\sum_{i \in I} x_i}$ is the greatest lower bound of $\{\overline{x_i} : i \in I\}$. For every $i \in I$, $x_i \leq \sum_{i \in I} x_i$, hence $\overline{\sum_{i \in I} x_i} \leq \overline{x_i}$ by **7**(v). Thus $\overline{\sum_{i \in I} x_i}$ is a lower bound of $\{\overline{x_i} : i \in I\}$. Let z be a lower bound of $\{\overline{x_i} : i \in I\}$. Then for every $i \in I$, we have $z \leq \overline{x_i}$, hence $x_i \leq \overline{z}$ by **7**(v) and **3**(ii). It follows that $\sum_{i \in I} x_i \leq \overline{z}$, so $z \leq \overline{\sum_{i \in I} x_i}$.

Proof of **9**.

9(i): Suppose $\sum_{i \in I} x_i$ and $\sum_{i \in I} y_i$ exist. For every $i \in I$, we have $x_i \leq \sum_{i \in I} x_i$ and $y_i \leq \sum_{i \in I} y_i$, hence also $x_i + y_i \leq \sum_{i \in I} x_i + \sum_{i \in I} y_i$ by **7**(iii). Thus $\sum_{i \in I} x_i + \sum_{i \in I} y_i$ is an upper bound of $\{x_i + y_i : i \in I\}$. If z is an upper bound of $\{x_i + y_i : i \in I\}$, then z is an upper bound of $\{x_i : i \in I\}$, since $x_i \leq x_i + y_i \leq z$ for every $i \in I$, so $\sum_{i \in I} x_i \leq z$, and, similarly, $\sum_{i \in I} y_i \leq z$. It follows that $\sum_{i \in I} x_i + \sum_{i \in I} y_i \leq z$ by **7**(iii). This shows that $\sum_{i \in I} x_i + \sum_{i \in I} y_i$ is the least upper bound of $\{x_i + y_i : i \in I\}$, so the desired equation holds.

9(iii): Suppose $\sum_{i \in I} x_i$ and $\sum_{i \in I} y_i$ exist, and $x_i \leq y_i$ for every $i \in I$. Then $x_i + y_i = y_i$ and $x_i \cdot y_i = x_i$ for every $i \in I$, so, by **9**(i), $\sum_{i \in I} x_i + \sum_{i \in I} y_i = \sum_{i \in I} (x_i + y_i) = \sum_{i \in I} y_i$, and, by **9**(ii), $\prod_{i \in I} x_i + \prod_{i \in I} y_i = \prod_{i \in I} (x_i + y_i) = \prod_{i \in I} x_i$. Thus, by **6**, $\sum_{i \in I} x_i \leq \sum_{i \in I} y_i$ and $\prod_{i \in I} x_i \leq \prod_{i \in I} y_i$.

Proof of **10**.

10(i): Suppose $\sum_{i \in I} x_i$ and $\sum_{i \in J} x_i$ exist. Then $\sum_{i \in I} x_i + \sum_{i \in J} x_i$ is an upper bound of $\{x_j : j \in I \cup J\}$, for if $j \in I \cup J$, then either $j \in I$ or $j \in J$. In case $j \in I$, we have $x_j \leq \sum_{i \in I} x_i \leq \sum_{i \in I} x_i + \sum_{i \in J} x_i$ by **7**(iv), and similarly, $x_j \leq \sum_{i \in I} x_i + \sum_{i \in J} x_i$ in case $j \in J$. If z is an upper bound of $\{x_j : j \in I \cup J\}$, then z is both an upper bound of $\{x_j : j \in I\}$ and an upper bound of $\{x_j : j \in J\}$. Hence $\sum_{i \in I} x_i \leq z$ and $\sum_{i \in J} x_i \leq z$, so $\sum_{i \in I} x_i + \sum_{i \in J} x_i \leq z$ by (Ba_1) and **6**. We have shown that $\sum_{i \in I} x_i + \sum_{i \in J} x_i$ is the least upper bound of $\{x_j : j \in I \cup J\}$, so the desired equation holds.

10(iii): Assume $\sum_{i \in I} x_i$ and $\sum_{j \in J} x_j$ exist and $I \subseteq J$. If $i \in I$, then $i \in J$ since $I \subseteq J$, so $x_j \leq \sum_{j \in J} x_j$. Thus $\sum_{j \in J} x_j$ is an upper bound of $\{x_i : i \in I\}$. But $\sum_{i \in I} x_i$ is the least upper bound of $\{x_i : i \in I\}$, so $\sum_{i \in I} x_i \leq \sum_{j \in J} x_j$.

Proof of **12**. By **3**(ii) and **11**, $f^{**}(x) = \overline{f^*(\overline{x})} = \overline{\overline{f(\overline{x})}} = f(x)$ for every $x \in B$, so $f = f^{**}$.

Proof of **14**.

14(i): Assume g and h are conjugates of f. Then, for all $x, y \in B$,

$f(x) \cdot y = 0$ iff $x \cdot g(y) = 0$
$f(x) \cdot y = 0$ iff $x \cdot h(y) = 0$

so

(1) $x \cdot g(y) = 0$ iff $x \cdot h(y) = 0$.

Consider a fixed y. Set $x = \overline{g(y)}$. Then $x \cdot g(y) = \overline{g(y)} \cdot g(y) = 0$ by **5**(ii) and **3**(viii). By (1), $\overline{g(y)} \cdot h(y) = 0$, so $h(y) \leq g(y)$ by **3**(viii) and **7**(v). Similarly, set $x = \overline{h(y)}$ and get $g(y) \leq h(y)$ from (1). By **7**(i), $g(y) = h(y)$.

Proof of **19**.

19(i): Assume g is a conjugate of f. Then

$$\overline{g(y)} = \sum_{x \leq \overline{g(y)}} x = \sum_{x \cdot g(y) = 0} x = \sum_{f(x) \cdot y = 0} x, \tag{3}$$

so, taking complements, we have $g(y) = \prod_{f(x) \cdot y = 0} \overline{x}$ by **3**(ii) and **8**(ii).

19(ii): First assume f has a conjugate function, say g. Since $0 = 0 \cdot g(1)$, it follows that $0 = f(0) \cdot 1 = f(0)$, so f is normal. Let $\{x_i : i \in I\} \subseteq B$ with $I \neq \emptyset$. Assume $\sum_{i \in i} x_i$ exists. Since g^* is a residual of f, the following statements are equivalent for every $y \in B$:

(4) $f(\sum_{i \in I} x_i) \leq y$
$\sum_{i \in I} x_i \leq g^*(y)$
$x_i \leq g^*(y)$ for every $i \in I$
$f(x_i) \leq y$ for every $i \in I$
(5) y is an upper bound of $\{f(x_i) : i \in I\}$

If $y = f(\sum_{i \in I} x_i)$ then (4) is true, so $f(\sum_{i \in I} x_i)$ is an upper bound of $\{f(x_i) : i \in I\}$. On the other hand, (5) implies (4), so if y is an upper bound of $\{f(x_i) : i \in I\}$ then $f(\sum_{i \in I} x_i) \leq y$. Thus $f(\sum_{i \in I} x_i)$ is the least upper bound of $\{f(x_i) : i \in I\}$, *i.e.*, $f(\sum_{i \in I} x_i) = \sum_{i \in I} f(x_i)$. This shows that f is completely additive. Since f is also normal, f is universally disjunctive. Finally, $\sum_{f(x) \leq y} x$ exists for every $y \in B$ since, by (3),

$$g^*(y) = \overline{g(\overline{y})} = \sum_{f(x) \cdot \overline{y} = 0} x = \sum_{f(x) \leq y} x.$$

To complete the proof of (ii), we assume (1) and (2), and prove that f has a conjugate. It follows from (2) that we may define a function g by setting

$$g(y) = \overline{\sum_{f(x) \leq \overline{y}} x} = \prod_{f(x) \cdot y = 0} \overline{x}$$

for every $y \in B$. If $f(z) \cdot y = 0$, then $\prod_{f(x) \cdot y = 0} \overline{x} \leq \overline{z}$, so $z \cdot g(y) = 0$. To prove the converse, note that, by (1), f is monotone and

$$f(\overline{g(y)}) = f(\sum_{f(x) \cdot y = 0} x) = f(\sum_{f(x) \leq \overline{y}} x) = \sum_{f(x) \leq \overline{y}} f(x) \leq \overline{y} \tag{6}$$

Then

1.	$z \cdot g(y) = 0$	hypothesis
2.	$z \leq \overline{g(y)}$	1
3.	$f(z) \leq f(\overline{g(y)})$	2, f is monotone
4.	$f(z) \leq \overline{y}$	3, (6)
5.	$f(z) \cdot y = 0$	4

This completes the proof that $f(z) \cdot y = 0$ iff $z \cdot g(y) = 0$, so g is a conjugate of f.

Proof of **20**.

(1) iff (2): Suppose f and g are conjugate. Then f is monotone by **17**(i) and **19**(ii), so $f(x \cdot \overline{g(y)}) \leq f(x)$. Furthermore, from $x \cdot \overline{g(y)} \cdot g(y) = 0$ we get $f(x \cdot \overline{g(y)}) \cdot y = 0$ since f and g are conjugate, so $f(x \cdot \overline{g(y)}) \leq \overline{y}$. Thus

$f(x \cdot \overline{g(y)}) \leq f(x) \cdot \overline{y}$. By symmetry, we also have $g(x \cdot \overline{f(x)}) \leq g(x) \cdot \overline{y}$. Thus (1) implies (2). For the converse, assume (2). If $x \cdot g(y) = 0$, then $x \cdot \overline{g(y)} = x$, hence $f(x) = f(x \cdot \overline{g(y)}) \leq f(x) \cdot \overline{y} \leq \overline{y}$ by (2)(a), so $f(x) \cdot y = 0$. Conversely, if $f(x) \cdot y = 0$, then $y = y \cdot \overline{f(x)}$, so $g(y) = g(y \cdot \overline{f(x)}) \leq \overline{x}$ by (2)(b), hence $x \cdot g(y) = 0$. Thus f and g are conjugate.

(1) iff (3): Suppose f and g are conjugate. Then f and g are normal by **19**(ii), *i.e.*, $f(0) = 0 = g(0)$. To show that (3)(a) holds we first observe that f is additive by **19**(ii) and that (2)(a) holds by the first part of the proof. Then

$$\begin{aligned}
f(y) \cdot z &= f\big(y \cdot g(z) + y \cdot \overline{g(z)}\big) \cdot z && \textbf{3}(\text{xi}) \\
&= \big(f(y \cdot g(z)) + f(y \cdot \overline{g(z)})\big) \cdot z && f \text{ is additive} \\
&\leq \big(f(y \cdot g(z)) + f(y) \cdot \overline{z}\big) \cdot z && (2)(\text{a}),\ \textbf{7}(\text{iii}) \\
&\leq f(y \cdot g(z)) && \textbf{3}(\text{viii})(\text{ix})(\text{xv}),\ \textbf{5}(\text{ii})(\text{vi})(\text{vii}),\ \textbf{7}(\text{iv})
\end{aligned}$$

The proof of (3)(b) is similar. Thus (2) implies (3). For the converse, assume (3). If $f(y) \cdot z = 0$, then, since g is normal and (3)(b) holds, $g(z) \cdot y \leq g(z \cdot f(y)) = g(0) = 0$. Conversely, if $g(z) \cdot y = 0$ then $f(y) \cdot z = 0$ by the normality of g and (3)(a). Thus f and g are conjugate.

Proof of **22**. First of all, f and g are monotone by **19**(ii), **15**(vi), and **17**(i). For every $j \in \omega$, we have

$$\begin{aligned}
g\big(\prod_{i\in\omega} \overline{f^i(b)}\big) &\leq g\big(\overline{f^{j+1}(b)}\big) && g \text{ is monotone} \\
&= g\big(\overline{f(f^j(b))}\big) && \textbf{21} \\
&\leq \overline{f^j(b)} && f \text{ and } g \text{ are conjugate},\ \textbf{20}(2)(\text{b}),\ \textbf{3}(\text{viii}),\ \textbf{5}(\text{vii}),\ \textbf{7}(\text{iv})
\end{aligned}$$

so

$$g\big(\prod_{i\in\omega} \overline{f^i(b)}\big) \leq \prod_{j\in\omega} \overline{f^j(b)}. \tag{1}$$

Next,

$$b \cdot \prod_{i\in\omega} \overline{f^i(b)} \leq b \cdot \overline{f^0(b)} = b \cdot \overline{b} = 0 \tag{2}$$

by **21** and **5**(ii). Finally,

$$\begin{aligned}
z &= y \cdot \overline{\sum_{i\in\omega} f^i(b)} && \text{definition of } z \\
&= y \cdot \prod_{i\in\omega} \overline{f^i(b)} && \textbf{8}(\text{ii}) \\
&\leq (b + h(y)) \cdot \prod_{i\in\omega} \overline{f^i(b)} && \text{hypothesis},\ \textbf{7}(\text{iii}) \\
&= b \cdot \prod_{i\in\omega} \overline{f^i(b)} + h(y) \cdot \prod_{i\in\omega} \overline{f^i(b)} && \textbf{3}(\text{viii})(\text{xv}) \\
&= h(y) \cdot \prod_{i\in\omega} \overline{f^i(b)} && (2),\ \textbf{5}(\text{vii}),\ (\text{Ba}_2) \\
&= (c + f(y)) \cdot \prod_{i\in\omega} \overline{f^i(b)} && \text{definition of } h \\
&\leq c + f(y) \cdot \prod_{i\in\omega} \overline{f^i(b)} && \textbf{3}(\text{viii})(\text{xv}),\ \textbf{7}(\text{iii})(\text{iv}) \\
&\leq c + f\big(y \cdot g\big(\prod_{i\in\omega} \overline{f^i(b)}\big)\big) && f \text{ and } g \text{ are conjugate},\ \textbf{20}(3)(\text{a}),\ \textbf{7}(\text{iii}),\ (\text{Ba}_2) \\
&\leq c + f\big(y \cdot \prod_{j\in\omega} \overline{f^j(b)}\big) && (1),\ \textbf{7}(\text{iii}),\ \textbf{3}(\text{viii}),\ f \text{ is monotone},\ (\text{Ba}_2) \\
&= c + f(z) && \textbf{8}(\text{ii}),\ \text{definition of } z
\end{aligned}$$

Proof of **24**.

24(i): Suppose $x \leq y$, *i.e.*, $x + y = y$. Then $\breve{y} = (x + y)^{\smile} = \breve{x} + \breve{y}$ by (Ra_5), so $\breve{x} \leq \breve{y}$. Conversely, if $\breve{x} \leq \breve{y}$, then $\breve{x} + \breve{y} = \breve{y}$, so, by (Ra_4) and (Ra_5), $x + y = \breve{\breve{x}} + \breve{\breve{y}} = (\breve{x} + \breve{y})^{\smile} = (\breve{y})^{\smile} = y$, hence $\breve{x} \leq \breve{y}$.

$$\begin{aligned}
\textbf{24}(\text{ii}):\quad \breve{0} &= 0 + \breve{0} && \textbf{5}(\text{vii}),\ (\mathrm{Ba}_2)\\
&= \breve{\breve{0}} + \breve{0} && (\mathrm{Ra}_4)\\
&= (\breve{0} + 0)^{\smile} && (\mathrm{Ra}_5)\\
&= (\breve{0})^{\smile} && \textbf{5}(\text{vii})\\
&= 0 && (\mathrm{Ra}_4)
\end{aligned}$$

$$\begin{aligned}
\textbf{24}(\text{iii}):\quad \breve{1} &= (1 + \breve{1})^{\smile} && \textbf{5}(\text{v}),\ (\mathrm{Ba}_2)\\
&= \breve{1} + \breve{\breve{1}} && (\mathrm{Ra}_5)\\
&= \breve{1} + 1 && (\mathrm{Ra}_4)\\
&= 1 && \textbf{5}(\text{v})
\end{aligned}$$

24(iv): For every y, $\breve{y} + \breve{\overline{y}} = (y + \overline{y})^{\smile} = \breve{1} = 1$ by (Ra_5), **5**(i), and **24**(iii), so

$$\breve{\overline{y}} + \breve{\breve{y}} = \breve{\overline{y}} \tag{1}$$

by **2**(ii) and **7**(v). Then

$$\begin{aligned}
\breve{\overline{x}} &= \overline{\breve{x}} + \breve{\overline{x}} && (1)\\
&= \breve{\breve{\overline{x}}} + \breve{\overline{\breve{x}}} && (\mathrm{Ra}_4)\\
&= (\breve{\overline{x}} + \overline{\breve{x}})^{\smile} && (\mathrm{Ra}_5)\\
&= (\breve{\overline{x}})^{\smile} && (\mathrm{Ba}_2),\ (1)\\
&= \overline{\breve{x}} && (\mathrm{Ra}_4)
\end{aligned}$$

$$\begin{aligned}
\textbf{24}(\text{v}):\quad (x \cdot y)^{\smile} &= (\overline{\overline{x} + \overline{y}})^{\smile} && (\mathrm{Ba}_4)\\
&= \overline{(\overline{x} + \overline{y})^{\smile}} && \textbf{24}(\text{iv})\\
&= \overline{\breve{\overline{x}} + \breve{\overline{y}}} && (\mathrm{Ra}_5)\\
&= \overline{\overline{\breve{x}} + \overline{\breve{y}}} && \textbf{24}(\text{iv})\\
&= \breve{x} \cdot \breve{y} && (\mathrm{Ba}_4)
\end{aligned}$$

24(vi): If $0 = \breve{x} \cdot y$, then

$$\begin{aligned}
0 &= \breve{0} && \textbf{24}(\text{ii})\\
&= (\breve{x} \cdot y)^{\smile} && \text{hypothesis}\\
&= \breve{\breve{x}} \cdot \breve{y} && \textbf{24}(\text{v})\\
&= x \cdot \breve{y} && (\mathrm{Ra}_4)
\end{aligned}$$

and the proof of the converse is similar.

24(vii): By **24**(vi), $^{\smile}$ is a conjugate of $^{\smile}$. It follows that $^{\smile}$ is universally disjunctive by **19**(ii). By **24**(iv) and **3**(ii), $^{\smile}$ is the dual of $^{\smile}$, so $^{\smile}$ is also universally conjunctive by **18**.

$$\begin{aligned}
\textbf{24}(\text{viii}):\quad \breve{1}' &= \breve{1}';1' && (\mathrm{Ra}_3)\\
&= \breve{1}';\breve{\breve{1}}' && (\mathrm{Ra}_4)\\
&= (\breve{1}';1')^{\smile} && (\mathrm{Ra}_6)\\
&= (\breve{1}')^{\smile} && (\mathrm{Ra}_3)\\
&= 1' && (\mathrm{Ra}_4)
\end{aligned}$$

$$\begin{aligned}
\textbf{24}(\text{ix}):\quad x;(y+z) &= \breve{\breve{x}};(\breve{\breve{y}}+\breve{\breve{z}}) && (\text{Ra}_4)\\
&= \breve{\breve{x}};(\breve{y}+\breve{z})^{\breve{}} && (\text{Ra}_5)\\
&= ((\breve{y}+\breve{z});\breve{x})^{\breve{}} && (\text{Ra}_6)\\
&= (\breve{y};\breve{x}+\breve{z};\breve{x})^{\breve{}} && (\text{Ra}_2)\\
&= (\breve{y};\breve{x})^{\breve{}}+(\breve{z};\breve{x})^{\breve{}} && (\text{Ra}_5)\\
&= \breve{\breve{x}};\breve{\breve{y}}+\breve{\breve{x}};\breve{\breve{z}} && (\text{Ra}_6)\\
&= x;y+x;z && (\text{Ra}_4)
\end{aligned}$$

24(x): Suppose $x \leq y$, *i.e.*, $x+y=y$. Then, by (ix), $z;x+z;y=z;(x+y)=z;y$, so $z;x \leq z;y$, and $x;z+y;z=(x+y);z=y;z$ by (Ra_2), so $x;z \leq y;z$ as well.

24(xi): Let x be fixed. Define functions f and g by $f(y)=x;y$ and $g(y)=\breve{x};y$ for every y. By **24**(x), f and g are monotone. Then

$$\begin{aligned}
g(y\cdot\overline{f(z)}) &\leq g(y)\cdot g(\overline{f(z)}) && g \text{ is monotone, } \textbf{7}(\text{iv})\\
&= g(y)\cdot\breve{x};\overline{x;y}\\
&\leq g(y)\cdot\overline{z} && (\text{Ra}_7),\ \textbf{3}(\text{viii}),\ \textbf{7}(\text{iii})
\end{aligned}$$

so **20**(2)(b) holds, and

$$\begin{aligned}
f(z\cdot\overline{g(y)}) &\leq f(z)\cdot f(\overline{g(y)}) && f \text{ is monotone, } \textbf{7}(\text{iv})\\
&= f(z)\cdot x;\overline{\breve{x};y}\\
&= f(z)\cdot\breve{\breve{x}};\overline{\breve{x};y} && (\text{Ra}_4)\\
&\leq f(z)\cdot\overline{y} && (\text{Ra}_7),\ \textbf{3}(\text{viii}),\ \textbf{7}(\text{iii})
\end{aligned}$$

so **20**(2)(a) holds as well. It follows by **20** that f and g are conjugate, as desired.

$$\begin{aligned}
\textbf{24}(\text{xii}):\quad \overline{y;x};\breve{x} &= (\overline{y;x};\breve{x})^{\breve{}\breve{}} && (\text{Ra}_4)\\
&= (\breve{\breve{x}};(\overline{y;x})^{\breve{}})^{\breve{}} && (\text{Ra}_6)\\
&= (\breve{\breve{x}};\overline{(y;x)^{\breve{}}})^{\breve{}} && \textbf{24}(\text{iv})\\
&= (\breve{\breve{x}};\overline{\breve{x};\breve{y}})^{\breve{}} && (\text{Ra}_6)\\
&\leq (\overline{\breve{y}})^{\breve{}} && (\text{Ra}_7),\ \textbf{24}(\text{i})\\
&= (\overline{\breve{y}})^{\breve{}} && \textbf{24}(\text{iv})\\
&= \overline{y} && (\text{Ra}_4)
\end{aligned}$$

24(xiii): Let x be fixed. Define functions f and g by $f(y)=y;x$ and $g(y)=y;\breve{x}$ for every y. By **24**(x), f and g are monotone. Then

$$\begin{aligned}
g(y\cdot\overline{f(z)}) &\leq g(y)\cdot g(\overline{f(z)}) && g \text{ is monotone, } \textbf{7}(\text{iv})\\
&\leq g(y)\cdot\overline{z} && \textbf{24}(\text{xii}),\ \textbf{3}(\text{viii}),\ \textbf{7}(\text{iii})
\end{aligned}$$

so **20**(2)(b) holds, and

$$\begin{aligned}
f(z\cdot\overline{g(y)}) &\leq f(z)\cdot f(\overline{g(y)}) && f \text{ is monotone, } \textbf{7}(\text{iv})\\
&\leq f(z)\cdot\overline{y} && (\text{Ra}_4),\ \textbf{24}(\text{xii}),\ \textbf{3}(\text{viii}),\ \textbf{7}(\text{iii})
\end{aligned}$$

so **20**(2)(a) holds as well. Therefore f and g are conjugate by **20**.

24(xiv): This part follows immediately from **24**(xi) and **24**(xiii). In fact, **24**(xiv) is equivalent to the conjunction of **24**(xi) and **24**(xiii).

24(xv): This part follows immediately from **24**(xi) and **24**(xiii) by **19**(ii).

24(xvi): The desired equation is identical to equation **20**(3)(a) with $f = x;(\text{-})$ and $g = \breve{x};(\text{-})$, and it therefore holds by **24**(xi) and **20**.

24(xvii): The desired equation is identical to equation **20**(3)(a) with $f = (\text{-});x$ and $g = (\text{-});\breve{x}$, and it therefore holds by **24**(xiii) and **20**.

24(xviii): By **24**(xi)(xiii) and either **19**(ii) or **20**, $x;(\text{-})$ and $(\text{-});x$ are normal, *i.e.*, $x;0 = 0 = 0;x$.

24(xix):
$$\begin{aligned} x;1' &= x && (\mathrm{Ra}_3)\\ &= \breve{\breve{x}} && (\mathrm{Ra}_4)\\ &= (\breve{x};1')\breve{} && (\mathrm{Ra}_3)\\ &= \breve{1}';\breve{\breve{x}} && (\mathrm{Ra}_6)\\ &= \breve{1}';x && (\mathrm{Ra}_4)\\ &= 1';x && \textbf{24}(\mathrm{viii}) \end{aligned}$$

24(xx):
$$\begin{aligned} x &= x;1' && (\mathrm{Ra}_3)\\ &\leq x;1 && \textbf{24}(\mathrm{x}) \end{aligned}$$

24(xxi):
$$\begin{aligned} x &= 1';x && \textbf{24}(\mathrm{xix})\\ &\leq 1;x && \textbf{24}(\mathrm{x}) \end{aligned}$$

24(xxii):
$$\begin{aligned} 1 &\leq 1;1 && \textbf{24}(\mathrm{xx})\\ &\leq 1 && \textbf{7}(\mathrm{ii}) \end{aligned}$$

24(xxiii): Assume $x;1 = x$. Then

$$\begin{aligned} \overline{x};1 &= \overline{x;1};1 && \text{hypothesis}\\ &= \overline{x;1};\breve{1} && \textbf{24}(\mathrm{iii})\\ &\leq \overline{x} && \textbf{24}(\mathrm{xii})\\ &\leq \overline{x};1 && \textbf{24}(\mathrm{xx}) \end{aligned}$$

24(xxiv): If $x;1 = x$ then

$$\begin{aligned} (x \cdot y);z &\leq x;1 \cdot y;z && \textbf{24}(\mathrm{x})\\ &= x \cdot y;z && \text{hypothesis}\\ &= (y \cdot x;\breve{z});z && \textbf{24}(\mathrm{xvii})\\ &\leq (y \cdot x;1);z && \textbf{24}(\mathrm{x})\\ &= (y \cdot x);z && \text{hypothesis} \end{aligned}$$

24(xxv): If $x;1 = x$ and $y;1 = y$ then $(x \cdot y);1 = x \cdot y;1 = x \cdot y$ by **24**(xxiv).

24(xxvi): If $x;1 = x$ then $(x \cdot 1');y = x \cdot 1';y = x \cdot y$ by **24**(xxiv)(xix).

24(xxvii): Assume $x;1 = x$. We get $(y \cdot \breve{x});(x \cdot z) \leq (y \cdot \breve{x});z$ and $(y \cdot \breve{x});(x \cdot z) \leq y;(x \cdot z)$ by **24**(x). For the opposite inclusions, we argue as follows.

$$\begin{aligned} (y \cdot \breve{x});z &= (y \cdot \breve{x});z \cdot 1\\ &\leq (y \cdot \breve{x});(z \cdot (y \cdot \breve{x})\breve{};1) && \textbf{24}(\mathrm{xvi})\\ &= (y \cdot \breve{x});(z \cdot (\breve{y} \cdot x);1) && \textbf{24}(\mathrm{v}),\ (\mathrm{Ra}_4)\\ &\leq (y \cdot \breve{x});(z \cdot x;1) && \textbf{24}(\mathrm{x})\\ &= (y \cdot \breve{x});(z \cdot x) && \text{hypothesis} \end{aligned}$$

$$\begin{aligned} y;(x\cdot z) &= y;(x\cdot z)\cdot 1 \\ &\le (y\cdot 1;(x\cdot z)^{\smile});(x\cdot z) && \textbf{24}(\text{xvii}) \\ &\le (y\cdot 1;\breve{x});(x\cdot z) && \textbf{24}(\text{i})(\text{x}) \\ &= (y\cdot (x;1)^{\smile});(x\cdot z) && \textbf{24}(\text{iii})(\text{v}) \\ &= (y\cdot x^{\smile});(x\cdot z) && \text{hypothesis} \end{aligned}$$

Proof of **26**.

26(i): Assume $x \in D$, *i.e.*, $x;1 = x$. Then $\overline{x};1 = \overline{x}$ by **24**(xxiii), so $\overline{x} \in D$. If $x, y \in D$ then $(x+y);1 = x;1 + y;1 = x + y$ by (Ra_2), so $x + y \in D$, and $x \cdot y \in D$ by **24**(xxv).

26(ii): Let $x \in A$ and $y \in D$. Then $y;1 = y$, so, by (Ra_1), $x;y;1 = x;(y;1) = x;y$, and hence $x;y \in D$.

26(iii): Assume $\{x_i : i \in I\} \subseteq D$ and $\sum_{i\in I} x_i$ exists. Then $\sum_{i\in I} x_i = \sum_{i\in I}(x_i;1) = \big(\sum_{i\in I} x_i\big);1$ by **24**(xv), so $\sum_{i\in I} x_i \in D$.

26(iv): This follows from **26**(i)(iii) by **8**(iii) and **3**(ii).

Proof of **28**. **28**(i): Suppose x and y are functional elements of $\mathfrak{A}$. Then

$$\begin{aligned} (x;y)^{\smile};(x;y) &= \breve{y};\breve{x};(x;y) && (\text{Ra}_6) \\ &= \breve{y};(\breve{x};x);y && (\text{Ra}_1) \\ &\le \breve{y};1';y && x \text{ is functional, } \textbf{24}(\text{x}) \\ &= \breve{y};y && (\text{Ra}_3) \\ &\le 1' && y \text{ is functional} \end{aligned}$$

so $x;y$ is functional as well.

$$\begin{aligned} \textbf{28}(\text{ii}):\quad x;y\cdot x;z &\le x;(z\cdot \breve{x};(x;y)) && \textbf{24}(\text{xvi}) \\ &= x;(z\cdot \breve{x};x;y) && (\text{Ra}_1) \\ &\le x;(z\cdot 1';y) && x \text{ is functional, } \textbf{24}(\text{x}),\ \textbf{7}(\text{iii}) \\ &= x;(z\cdot y) && \textbf{24}(\text{xix}) \\ &\le x;z\cdot x;y && \textbf{24}(\text{x}),\ \textbf{7}(\text{iv}) \end{aligned}$$

28(iii): Assume first that x is functional. Let $y \in A$. Then $x;y\cdot x;\overline{y} = x;(y\cdot\overline{y}) = x;0 = 0$ by **28**(ii), **5**(ii), **24**(xviii). For the converse, suppose $x;y\cdot x;\overline{y} = 0$ for every element y. Take $y = 1'$ and get $0 = x;1'\cdot x;\overline{1'} = x\cdot x;\overline{1'}$ by (Ra_3). Then $0 = \overline{1'}\cdot\breve{x};x$ by **24**(xiv), so $\breve{x};x \le 1'$. Thus x is functional.

Proof of **30**. Let $f(x) = r;x$ and $g(x) = \breve{r};x$ for every $x \in A$. Then f and g are conjugate by **24**(xi). Apply **22**.

Proof of **31**. This proof follows Tarski's proof [T55]. First we prove

(1) $$\text{if } X \subseteq I \text{ then } x \le f(\textstyle\sum X) \text{ for every } x \in X.$$

Assume $X \subseteq I$. Then $\sum X$ exists since $\langle A, \le\rangle$ is complete. We prove the conclusion of (1) as follows.

1.	$x \in X$	hypothesis
2.	$x \le f(x)$	1, $X \subseteq I$, definition of I
3.	$x \le \sum X$	1, definition of $\sum$
4.	$f(x) \le f(\sum X)$	3, f is monotone
5.	$x \le f(\sum X)$	2, 4, $\le$ is transitive

According to (1), $f(\sum X)$ is a upper bound of X whenever $X \subseteq I$, so

(2) $$\textstyle\sum X \le f(\sum X) \text{ for every } X \subseteq I.$$

From (2) and the definition of I we have

(3) $$\sum X \in I \text{ for every } X \subseteq I.$$

Next we prove that

(4) $$I \text{ is closed under } f$$

as follows.

1.	$x \in I$	hypothesis
2.	$x \leq f(x)$	1, definition of I
3.	$f(x) \leq f(f(x))$	2, f is monotone
4.	$f(x) \in I$	3, definition of I

From (3) and (4) we see that

(5) $$f(\sum I) \in I.$$

It follows from (5) that

(6) $$f(\sum I) \leq \sum I.$$

Combining (2) (with $X = I$) and (6), we get

(7) $$f(\sum I) = \sum I.$$

By (7) and the definition of F,

(8) $$\sum I \in F,$$

so F is not empty. From (8) we have

(9) $$\sum I \leq \sum F.$$

Note that $F = I \cap D \subseteq I$, hence $\sum F \leq \sum I$. Together with (9), this gives us

(10) $$\sum I = \sum F.$$

In view of (8) and (10), we have completed the proof of **31**(ii). The proof of **31**(iii) is similar, and will be omitted.

31(i): We have seen that F is nonempty, so what remains to show that $\langle F, \leq \rangle$ is a complete lattice. For this it suffices to show that every $X \subseteq F$ has a join and meet in $\langle F, \leq \rangle$. Let $X \subseteq F$. Consider the complete sublattice $\langle \{x : \sum X \leq x\}, \leq \rangle$ of $\langle A, \leq \rangle$. We prove that

(11) $$\{x : \sum X \leq x\} \text{ is closed under } f$$

as follows.

1.	$\sum X \leq x$	TABhypothesis
2.	$f(\sum X) \leq f(x)$	1, f is monotone
3.	$\sum X \leq f(\sum X)$	(2), $X \subseteq F \subseteq I$
4.	$\sum X \leq f(x)$	2, 3, $\leq$ is transitive

Let f' be the restriction of f to $\{x : \sum X \leq x\}$. Let $F' = \{x : \sum X \leq x = f(x)\}$. Thus F' is the set of fixed points of f which are upper bounds of X. When **31**(ii) is applied to $\langle \{x : \sum X \leq x\}, \leq \rangle$ and f', the conclusion is that $\sum F' \in F'$. Hence $\sum F' \in F$ since $F' \subseteq F$, and $\sum F'$ is an upper bound of X since $\sum X \leq \sum F'$. $\sum F'$ is the least upper bound of the set of fixed points of f which are upper bounds of X, so $\sum F'$ is the least upper bound of X in $\langle F, \leq \rangle$. Similarly, the greatest lower bound of X in $\langle F, \leq \rangle$ exists, so $\langle F, \leq \rangle$ is a complete lattice.

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Contents

GUEST EDITORIAL

WOFACS '92: Interdisciplinarity and Collaboration
C Brink 1

Editor's Notes 2

SPECIAL CONTRIBUTIONS

Introduction to Computability Theory
J Zucker and L Pretorius 3

Denotational Semantics and Domain Theory
J Goslett, H Hulley and A Melton 31

Deduction Systems Based on Resolution
N Eisinger and HJ Ohlbach 44

A Working Relational Model: The derivation of the Dijkstra-Scholten predicate transformer semantics from Tarski's axioms for the Peirce-Schröder calculus of relations
RD Maddux 92

Pretoria Drukkers