

EXPLORING LEARNERS' DIFFICULTIES IN SOLVING GRADE 12 DIFFERENTIAL CALCULUS: A CASE STUDY OF ONE SECONDARY SCHOOL IN POLOKWANE DISTRICT

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ABSTRACT - Grade 12 learners have difficulty in solving differential calculus questions using first principles and rules of differentiation. This study investigated how Grade 12 learners perform this task. A convenient sample of thirty-seven ($n=37$) learners of a secondary school in the Polokwane District of the Limpopo province, South Africa, participated in this study. An interpretive paradigm was used for this study, which followed an explanatory sequential mixed-methods design. Data were collected from learners using diagnostic test and semi-structured interview schedules with some learners and their teacher. The results revealed that 52% of the learners performed better in finding the derivative of functions by using the rules of differentiation, compared with 23% of the learners who performed better on the questions by using first principles. The interviews showed that learners experienced difficulties when substituting $(x + h)$ in the formula if the function comprised more than one term, and also when dividing by h in the formula. Using the rules of differentiation, learners experienced difficulties when given functions involving fractions. The study therefore recommends that learners should be taught how to differentiate functions using first principles and rules of differentiation by working to correct the types of mistake learners display in using differentiation.

Keywords: differential calculus, rules of differentiation, first principles, conceptual knowledge, procedural knowledge

1. INTRODUCTION

Differential calculus is a branch of mathematics that is taught to secondary school learners in various countries (Lam, 2009). Rohde, Jain, Poddar and Ghosh (2012) describe differential calculus as a subject which can be applied to anything that moves or changes or has a shape. Tall (2009: 481) argues that "calculus begins with the desire to quantify how things change (the function concept), the rate at which they change (the derivative), the way in which they accumulate (the integral), and the relationship between the two (i.e. the fundamental theorem of calculus and the solutions of differential equation)". It is often used by mathematicians, physicists, economists, engineers and other experts to solve real-life problems (Laridon, Jawurek, Kitto, Pike, Myburgh, Rhodes-Houghton, Sasman, Scheiber, Sigabi, & Van Rooyen, 2007). For example, it can be applied to the study of machinery of all kinds, electric lighting and wireless, optics and thermodynamics (Rohde et al., 2012). In economics and commerce, differential calculus helps to solve problems in finding maximum profit or minimum cost and the like (Berresford & Rockett, 2015). Differential calculus is an important topic; as such, if its basics are not well understood, then real-life problems cannot be solved without difficulty (Rohde et al., 2012).

In South Africa, differential calculus (i.e. differentiation and application of differential calculus) has been taught in Grade 12 for many years. However, according to the Department of Basic Education (DBE), the learning of differential calculus still poses difficulties for South African Grade 12 mathematics learners, since a number of them continue to find it difficult to solve problems based on first principles and rules of differentiation; cubic function; and applications in optimisation (DBE 2011; 2012; 2013; 2014; 2015). One of the reasons for the difficulties encountered in teaching and learning calculus in secondary schools is that it is generally perceived to be abstract and involving complex ideas (Zachiarides, Pamfilos, Christou, Maleev, & Jones, 2007); moreover, learners do not comprehend the

key concepts of calculus (Artique, Batanero, & Kent, 2007). The focus in this study is on how learners solve problems involving first principles and rules of differentiation.

Research studies attribute some of learners' difficulties with calculus problems to weak understanding of functions and other related graphs (e.g DBE, 2014; DBE, 2015). Some learners' difficulties are attributed simply to the procedures used when practising the routine steps followed when solving calculus problems. Zachiarides et al. (2007) argue that focusing on procedural understanding rather than conceptual understanding when teaching calculus contributes towards learners' difficulties in dealing with calculus problems. Despite all the theories about learners' difficulties in solving calculus problems, there appears to be little attention given to finding the derivatives of functions using first principles and rules of differentiation.

2. PURPOSE OF THE STUDY

The purpose of this study is to report on the Grade 12 learners' academic achievement and difficulties when finding the derivatives of functions using first principles and rules of differentiation. Despite this investigation being a case study based on one school in the Polokwane District of Limpopo, its findings may still inform and assist Grade 12 mathematics teachers about the learners' difficulties in this area. The study set out to answer the following research questions:

- (i) To what extent do Grade 12 learners understand the derivatives of functions using first principles and rules of differentiation?
- (ii) What difficulties do Grade 12 learners encounter in finding the derivatives of functions using first principles and rules of differentiation?

3. THEORETICAL PERSPECTIVES

Mathematical knowledge is often difficult for learners to acquire (Kaminski, Sloutski, & Heckler, 2009). Different approaches are used by teachers to transfer knowledge during classroom teaching. Some of the approaches used in classroom teaching enable learners to be proficient in mathematics. Mathematical proficiencies are discussed by Kilpatrick, Swafford, and Findell (2001), who identified five strands of mathematical proficiencies, which are not independent but represent different aspects of a whole. The five strands are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. The strands identified are interwoven and interdependent (NCTM, 2001). They provide a framework for discussing the knowledge, skills, abilities, and beliefs that constitute mathematical proficiency.

Although this study intended to use Kilpatrick et al. (2001) mathematical proficiency, for the purpose of this study conceptual understanding and procedural fluency are used to underpin the study. Conceptual understanding is when a learner is able to comprehend mathematical concepts, operations, and relations (Kilpatrick et al., 2001; Watson & Sullivan, 2008). Skemp (1976) argues that it is not enough for a learner to understand *how* to perform mathematical procedure, but also to know *why* a particular task is performed. In the context of this study, learners should know how to differentiate using first principles: $\frac{dy}{dx}$ or $f'(x)$ or $y' = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ and rules of differentiation $\frac{dy}{dx}$ or $f'(x)$ or $y' = nx^{n-1}$ to find the derivatives of given functions.

Procedural understanding/fluency is knowledge of procedures: knowledge of how and when to use them appropriately, and the skills in performing them flexibly, accurately, and efficiently (Kilpatrick et al., 2001; Watson & Sullivan, 2008). Procedural knowledge is built on conceptual understanding (NCTM, 2000, 2014). The two strands should work hand in hand. Grade 12 learners should be in a position to know the procedures of finding the derivative of functions using first principles and rules of differentiation. However, the challenge may not be the procedures used by learners. These types of

task require higher-order thinking skills; they need conceptual knowledge and procedural knowledge complementing each other.

4. LITERATURE REVIEW

Studies show that often poor academic achievement in calculus is caused by a deficiency in the teacher's content knowledge of the subject (Lam, 2009). Lam studied the content knowledge of calculus of 27 in-service mathematics teachers by using a questionnaire dealing with images and definitions of various calculus concepts. The results revealed a lack of knowledge of various differential calculus concepts; furthermore, the teachers tended to focus on procedural knowledge of calculus (Lam, 2009). Muzangwa and Chifamba (2012) argue that the lack of conceptual understanding in calculus limits its usefulness in related science applications. Axtell (2006) agrees, concluding that the calculus curriculum needs to be improved by focusing on conceptual understanding rather than procedural understanding only.

Coe (2007) and Ubuz (2007) researched learners' fundamental ideas of rate of change in learning calculus and concluded that learners lack the understanding of fundamental ideas of *change* and *rate*, which in calculus provide an important underpinning of the derivatives. Anton, Biven and Davis (2005) postulate that calculus is introduced by involving limits and differentiations where learners are presented with the formal abstracts of definition of limits and limit laws:

$\frac{dy}{dx}$ or $f'(x)$ or $y' = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ and $\frac{dy}{dx}$ or $f'(x)$ or $y' = nx^{n-1}$ for differentiations.

Muzangwa and Chifamba (2012) explored the errors and misconceptions about the rate of change in numeric, graphic and symbolic representations with constants and variable rates. The average rate was explained using learners' understanding of linear equations and constant rate. The instantaneous rate was eventually renamed derivative and symbolic manipulations used to find the derivative from first principles $\frac{dy}{dx}$ or $f'(x)$ or $y' = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$, noting the fraction as the average rate. They concluded that calculus teaching was affected by the lecturers' lack of conceptual knowledge of calculus.

Herbert (2011) researched first-year tertiary students taking introductory calculus, rate and hence the derivative of functions. She used two-sample t-tests to measure the examination results of the introductory calculus limits. The results revealed that the first-year students displayed a lack of conceptual understanding of introductory calculus. Herbert further showed that students lacked knowledge of rate in calculus as compared with area and integration. Other difficulties in understanding of rate included confusion relating to symbols and their use of variables (White & Mitchelmore, 1996), lack of awareness between slope, rate and first derivative (Porzio, 1997), and misunderstandings related to the average and instantaneous rate (Hassan & Mitchelmore, 2006). The rate is a troublesome concept which hampers conceptual understanding of the derivative (Herbert, 2011). Herbert suggests that when teaching introductory calculus, the teacher should start with area and integration, rather than rate and derivative, which is inconsistent with secondary school calculus. The integration of calculus is not part of many secondary school mathematics syllabi.

Muzangwa and Chifamba (2012) conducted a study on errors and misconceptions in undergraduate student calculus by administering two exercises on Calculus 1 and 2. The results revealed that most of the students' errors were caused by knowledge gaps in basic algebra. Furthermore, Muzangwa and Chifamba showed that errors and misconceptions possessed by students were generated by lack of advanced mathematics thinking, which could have been caused by lecturers during teaching and learning of calculus.

The DBE releases an annual report on errors and misconceptions Grade 12 learners display when writing the final Paper 1 of the mathematics examination. These reports cover calculus, among other topics of mathematics. One hundred scripts for Paper 1 of mathematics are randomly selected each

year at each province across all 123 marking centres countrywide during the marking of the examination. These scripts are randomly selected from a number of districts, to cover low, medium and high scores. Throughout the marking process, markers, senior markers and chief markers note learners' responses to questions, paying particular attention to common errors and misconceptions. The analysis is done per question on the random sample of 100 scripts. This entails recording the marks obtained by learners from the 100 scripts on a per question basis.

From the analysis, a detailed explanation was provided per question/sub question in the following two main parts: Section 1: general overview of learner performance in the question paper as a whole. Section 2: comments on candidates' performance on individual questions. This included the following: (1) general comments on the performance of learners in the specific questions, stating whether a question was well answered or poorly answered; (2) why the question was poorly answered, including common errors and misconceptions; and (3) suggestions for improvement in relation to teaching and learning. Subject advisory support and provision of the reports from all nine provinces for each paper per subject were consolidated and the information summarised in the report. The following are the findings reported by the DBE from 2011 to 2015.

In 2011, the DBE reported that learner performance was better in the question of determining $f'(x)$ from first principles if $f(x) = -4x^2$. However, the report indicated that there were still issues that needed attention by teachers so that learners could lose as few marks as possible. Learners commonly made notational errors in the differentiation by first principles and consequently lost marks (DBE, 2011). It was further directed that teachers needed to perform drill and practice exercises where notation was specifically emphasised, and place more emphasis on multiplying binomials by binomials in the GET phase; many learners are still multiplying binomials by binomials incorrectly, but manipulate the answer for the first principles' differentiation to be $f'(x) = 8x$.

Furthermore, it was reported that when learners were asked to evaluate $\frac{dy}{dx}$ if $y = \frac{3}{2x} - \frac{x^2}{2}$ many of them struggled to correctly identify the coefficients, variables and exponents in the expression of y before differentiating. A general error was to write $\frac{3}{2x}$ as $-6x^{-1}$. This showed how a lack of mastery of the work on exponents learnt in grades 9 and 10 prevents accurate answers, even though grade 12 differentiation methods are understood. It was suggested that for learners to improve their performance in differential calculus they needed to pay attention to notation, so that the function and the derivative are distinguished (DBE, 2011).

In 2012, the DBE reported that notation was still a problem to learners. There were many basic algebraic manipulation errors, such as:

- (1) the limit in the incorrect place $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$;
- (2) leaving out the bracket in $\lim_{h \rightarrow 0} (4x + 2h)$ and writing $\lim_{h \rightarrow 0} 4x + 2h$;
- (3) substituting 0 into $\lim_{h \rightarrow 0} (4x + 2h)$.

In the year 2012, learners were given $g(x) = \frac{x^2+x-2}{x-1}$ simply to test their understanding of differentiation. The question was, however, poorly answered because many learners did not first simplify $g(x)$ to $x + 2$ by factorising the numerator and then dividing by the denominator. Moreover, many of those that did this were unable to determine that $g'(x) = 1$ (DBE, 2012). It was further reported that a common error was determining the derivative of the numerator and the denominator separately. It was suggested that for improvement in solving calculus questions teachers needed to do the following: (1) to stress the importance of notation in finding the derivative from first principles; (2) drill and practise the rules of differentiation; and (3) that the original function must be in power form so that they can correctly identify the coefficient, variable and exponent before the rules of differentiation can be applied (DBE, 2012).

In 2015, the learners were given this question: "If $(x) = x^2 - 3x$, determine $f'(x)$ from first principles" and the question was well answered, but the question Determine $\frac{dy}{dx}$ if $y = (x^2 - \frac{1}{x^2})^2$ and the question Determine $D_x \left(\frac{x^3-1}{x-1} \right)$ were poorly answered (DBE, 2015). It was reported that although first principles were taught and reiterated in class, learners still had a problem with notation in the last question. It was reported that the following errors kept recurring:

- (1) Algebraic errors such as incorrect expansion, adding unlike terms and changing of signs in multiplication;
- (2) Incorrect use of the formula as well as notational mistakes;
- (3) Failure to replace x with $(x + h)$ into the $-3x$ part of the equation; and
- (4) Omitting the factorisation part to derivative;
- (5) Learners found difficulty in squaring the binomial containing a rational term in the question: Determine $\frac{dy}{dx}$ if $y = (x^2 - \frac{1}{x^2})^2$; and that the learners had great difficulty in factorising the difference of two cubes. In addition to what had been suggested in 2011, 2012 and 2013 to improve learners' performance in solving differential calculus it was further suggested that the basic algebraic manipulation in earlier grades should be taught properly and revised on an ongoing basis (DBE, 2015).

5. Methodology

This study followed an explanatory sequential mixed-methods design in which quantitative data informed qualitative data (Creswell, 2014). The quantitative data was collected first, followed by qualitative data. An interpretive paradigm was used in this study. A diagnostic test was administered to a convenient sample ($n=37$) of Grade 12 learners, with the aim of measuring the level of achievement between the learners finding the derivatives of functions using first principles and those using rules of differentiation. The diagnostic test consisted of six test items to assess Grade 12 learners finding the derivatives of functions using first principles and rules of differentiation, in which four items were allocated for each rule. Qualitative data were generated from the semi-structured interviews with a Grade 12 mathematics teacher and seven randomly selected learners who were chosen to participate in the interviews to make sense of the difficulties learners encountered in this process. The semi-structured interviews were constructed following the learners' responses in the diagnostic test.

6. Significance of the Paper

The study played a fundamental role in informing the teacher about her learners' academic achievement when finding the derivatives of functions using first principles and rules of differentiation. The teacher also knew the approaches and the type of support that can be used in teaching calculus, and understood that her learners possessed procedural knowledge instead of conceptual knowledge. Again, learners realised the mistakes they had made and why they had committed those mistakes.

7. Research Findings

7.1 Table 1 presents the results using Didis and Erbas' (2015) categories of correct responses (CR), incorrect responses (IR), incomplete responses (InR), and blank responses (BR). In the context of this study the CR category describes the learners who obtained full marks, displaying both conceptual and procedural understanding: the IR category were learners who obtained no mark, displaying lack of conceptual understanding: the InR category indicated learners who obtained some marks but lacked procedural knowledge, and the BR category also having no mark and lacking both conceptual and procedural knowledge.

7.2 Presentation of Diagnostic Test Results

Table 1 depicts the analysis of learners' performance in the diagnostic test.

Table 1: Derivatives using First Principles' Test Items' Results

First principles	CR	IR	InR	BR
1.1 $f(x) = 3x^2$	35% (13)	46.0% (17)	10.8% (4)	8.1% (3)
1.2 $f(x) = 2x^2 - x$	24.3% (9)	13.5% (5)	10.8% (4)	51.4% (19)
1.3 $f(x) = \frac{1}{x}$	18.9% (7)	5.4% (2)	13.5% (5)	62.2% (23)

**** Table1 used percentages and absolute numbers (frequencies) to represent the results using the categories**

Table 1 shows that the results showed high percentages for test items 1.2 and 1.3 in the BR category: 51.4% and 62.2% respectively, revealing that learners lacked both conceptual and procedural understanding of finding the derivatives using first principles (Kilpatrick et al., 2001; Watson & Sullivan, 2008). Learners lacked conceptual understanding in the IR category for test 1.1 with 46.0%, while 1.2 and 1.3 test items showed lower percentages (13.5% and 5.4%) respectively. The IR category revealed that learners gave wrong formulas and wrong calculations to determine the derivatives of functions using first principles. However, for test item 1.1, 35.5% of the learners were found in the CR category, showing that they could perform operations such as using the correct formula, correct substitutions and correct computations in finding the derivatives using first principles.

Table 2: Derivatives using Rules of Differentiation Test Items' Results

Rules for differentiations	CR	IR	InC	BR
2.1 $f(x) = x^3 - 3x^2 + 9x$	64.9% (24)	8.1% (3)	27.0% (10)	0.0% (0)
2.2 $f(x) = \frac{4}{\sqrt{x}} - \frac{x^3}{9}$	43.2% (16)	32.4% (12)	8.1% (3)	21.6% (8)
2.3 $f(x) = x^2 + \frac{1}{2x^3}$	56.8% (21)	10.8% (4)	24.3% (9)	8.1% (3)

**** Table 2 used percentages and absolute numbers (frequencies) to represent the results using the categories**

The results in Table 2 above show that the higher percentages in the CR category for test items 2.1, 2.2, and 2.3 (64.9%, 43.2%, and 56.8% respectively) suggest learners' comprehension of operations (Kilpatrick et al., 2001; Watson & Sullivan, 2008) in finding the derivatives of these functions using rules of differentiation. The IRs category for test item 2.2 reveals that 32.4% of learners struggled to differentiate \sqrt{x} , as learners did not make sense at all by changing the function to $f(x) = 4\sqrt{x} - 9x^2$. The results suggested learners' lack of conceptual and procedural knowledge (Kilpatrick et al., 2001) when finding of the derivatives of the given functions. The other challenge learners faced was found in test item 2.3, with InR results of 24.3%, which suggested learners lacked the correct knowledge of when and how to use the procedures appropriately (Kilpatrick et al., 2001) to find the derivatives of the given functions using the rules of differentiation to change the denominator x^3 to x^{-3} . These learners simply found the derivatives of the denominator x^3 without converting it to the numerator.

7.3 Teacher Semi-structured Interview

The teacher interview was conducted once in the selected school that participated in this study. The interview session took about 15 minutes and was recorded on a laptop. The rationale for this interview was to understand in depth the difficulties these learners encountered in finding the derivatives of functions in the diagnostic test using first principles and rules of differentiation. The results of the interview showed that learners were able to find the derivatives of simple functions, but could not do so when faced with challenging ones. The results of the interview also revealed that learners did not know the formula for finding the derivatives of functions using first principles; some of the learners used the positive sign instead of the negative sign in

$\frac{dy}{dx}$ or $f'(x)$ or $y' = \lim_{h \rightarrow 0} \frac{f(x+h)+f(x)}{h}$ and some just wrote the formula $\frac{dy}{dx}$ or $f'(x)$ or $y' = \frac{\lim_{h \rightarrow 0} f(x+h)-f(x)}{h}$, which was the wrong formula for first principles. The extract below shows the teacher's response in the interview. The texts in the extract are reported verbatim.

Extract 1: Teacher Semi-structured Interview

- Researcher: What challenges do your learners face in finding the derivatives using first principles and rules for differentiations?
- Teacher: These learners have a serious problem in calculus, mostly in finding the derivatives using the first principle. Some give wrong formula, some do incorrect substitution and wrong calculations. Most of the learners cannot find the derivatives of functions involving fractions.
What do you think can be the cause of the problem?
- Researcher: These learners lack background of algebra and maybe even the way we teach them can be cause of the problem. *Le gona ba tswafa, ge o ba file mmereko ba dira forogo* [Again, these learners are lazy; when given an activity to do at home they do it in the morning].

Extract 1 above shows the learners' lack of conceptual knowledge as shown in following incorrect procedures to find the derivatives using first principles. The teacher uses the word 'some' in explanation, referring to the learners who cannot write the correct formula and incorrect substitution of terms in the formula, which suggests a lack of procedural knowledge (Watson & Sullivan, 2008). The teacher also uses the word 'most' referring to a large number of learners who cannot differentiate functions that include variables. Furthermore, learners appear to have more problems when finding the derivatives of functions that involved fractions of the variable x . The lack of a 'background of algebra' suggests a lack of conceptual knowledge which seemed to have hampered the learning of calculus (Muzangwa & Chifamba, 2012). The lack of commitment of learners to their school work was found to be one of the factors that contributed to the poor performance in this connection.

7.3 Learner Semi-structured Interviews

The semi-structured interviews were conducted with seven learners, who revealed that the calculus concept was difficult for them to understand. All the learners said that they did not even know the meaning of the derivatives and were taught only the calculations, which suggests that learners lacked conceptual knowledge of the rate of change in calculus (Herbert, 2011). Furthermore it was found that learners were faced with many formulas to memorise, which is why they often gave the wrong formula for first principles in finding the derivatives. Learners 1, 2, 4 and 5 indicated that the approaches used by the teacher to teach derivatives of functions using first principles led to learners' lack of content knowledge (Lam, 2009), while the time allocated for teaching the topic of derivatives was insufficient. Learner 4's response revealed that she encountered challenges when finding the derivatives of functions using the formula; it confused her. All seven of the learners had difficulty in finding the derivatives of functions that included fractions of the variable x . Extract 2 below records the interview between the researcher and Learner 7 about the difficulties in finding the derivatives of functions using first principles and rules of differentiation.

Extract 2: Learner 7 Semi-structured interview

- Researcher: What difficulties do you face in finding the derivatives of functions?
- Learner 7: We have a problem in using the first principle in finding the

- derivatives even though we have a problem in using the rules for differentiations but it is better.
What do you think can be the cause of the problem?
- Researcher: The way they teach us, they just show us how to calculate
Learner: without explaining to us what the derivatives mean. The teacher also teach in a fast pace and we ultimately not understand.
Can you just explain what was the real problem when finding the derivatives of functions?
- Researcher: The main problem is the formula and the functions that have
Learner: fractions. We are not able to substitute even when give a straight function without a fraction.

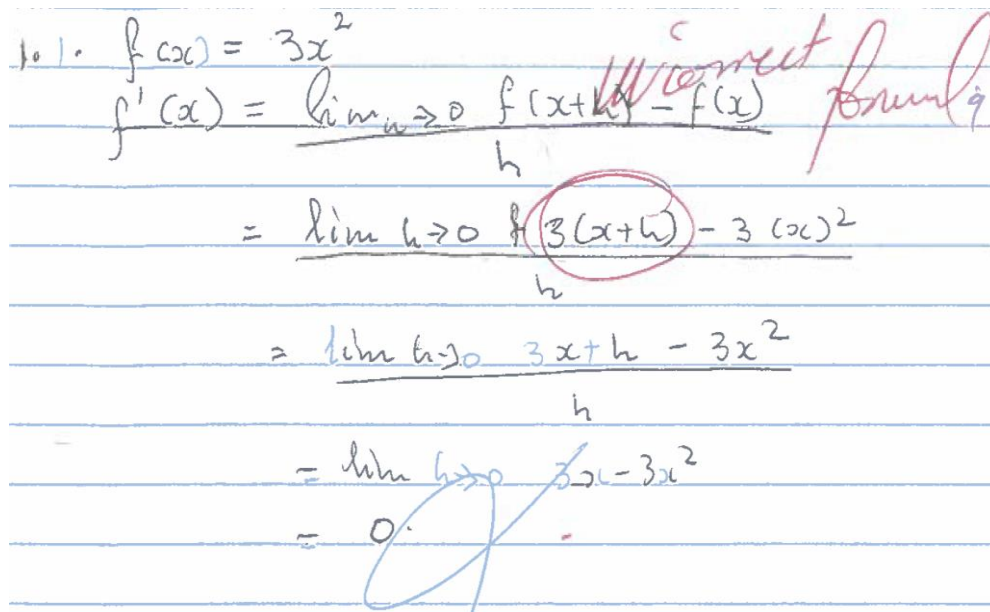
Muzangwa and Chifamba (2012) postulate that that the lack of conceptual knowledge of calculus is a challenge and can limit learners when learning other related science applications. Learner 7 highlighted some of the causes of difficulties in learning the derivatives: ‘the way they teach us’, which suggests the approach used by the teachers focuses on procedural knowledge (Lam, 2009). The response also indicated that the teacher focused only on calculations and did not explain what the derivatives are (cf. Coe (2007) and Ubuz (2007)). The results revealed that learners lacked knowledge of using the correct formula and functions, including fractions, which suggests a lack of mathematical thinking which could have been the result of the method of teaching and learning of calculus (Muzangwa & Chifamba, 2012).

8. Discussion of Results

Kilpatrick et al. (2001) suggest that both conceptual and procedural knowledge are needed for mathematical proficiency. The diagnostic test results revealed that learners performed better in finding the derivatives of the functions using the rules of differentiation than using first principles. The results suggested that learners displayed conceptual knowledge when they were able to find the derivatives of functions 2.1. to 2.3 using the rules of differentiation. A high percentage for the rules of differentiation when finding the derivatives ranged between 43.2% and 64.9%, while a lower percentage was found on the first principle ones, which ranged between 18.9% and 35.1% for the CR category. This showed that learners performed more poorly in finding the derivatives of functions using first principles than when using the rules of differentiation. This was revealed in the BR category, when learners demonstrated a lack of procedural and conceptual knowledge by failing to attempt to find the derivatives of the functions using first principles. For the BR category, the percentage using first principles when finding the derivatives of functions ranged between 8.1% and 62.2% and that of the rules of differentiation ranged between 2.1% and 8.1. Herbert (2011) concurs that most of the first-year students in tertiary institutions have difficulties in learning introductory calculus.

The lack of conceptual knowledge in finding the derivatives of functions using first principles was supported by both the teacher and learner interviews. The three examples below represent what was found in the learners’ diagnostic test results as regards the formula, substitution, and computations.

Example of incorrect first principles formula

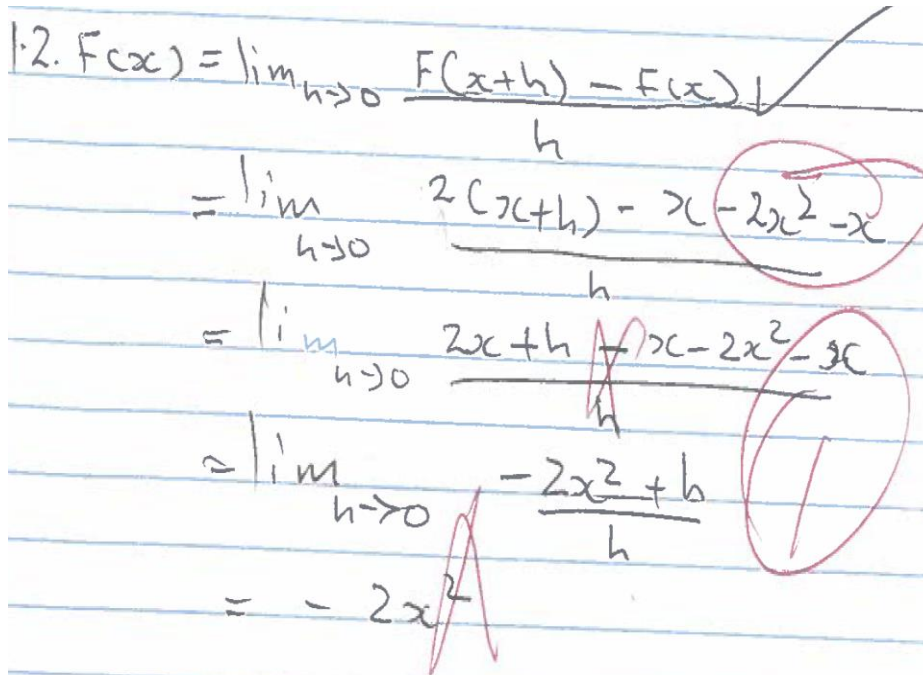


$$\begin{aligned}
 10.1. \quad f(x) &= 3x^2 \\
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{Incorrect formula} \\
 &= \lim_{h \rightarrow 0} \frac{3(x+h) - 3x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x+h - 3x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x - 3x^2}{h} \\
 &= 0
 \end{aligned}$$

Vignette 1

This is the first principles formula given by one of the learners in the diagnostic test: $f'(x) = \frac{\lim_{h \rightarrow 0} f(x+h) - f(x)}{h}$, which was the wrong formula to find the derivatives of functions using first principles. The NCTM (2014) argues that procedural knowledge is built on conceptual knowledge. The results suggest that learners lack both the procedural and conceptual knowledge of calculus in finding the derivatives of the functions using first principles. Learners themselves claim that some of the difficulties are caused by the teacher's approach of just wanting them to memorise formulas without explaining the meaning of the formulas. These results are supported by Axtell (2006) and Muzangwa and Chifamba (2012), claiming that teaching approaches used by teachers of calculus focused on procedural understanding only rather than starting with conceptual knowledge.

Example of incorrect substitution



$$\begin{aligned}
 1.2. f(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x+h) - x - 2x^2 - x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2x+h - x - 2x^2 - x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2x^2 + h}{h} \\
 &= -2x^2
 \end{aligned}$$

Vignette 2

This is an example of incorrect substitution into the function $f(x) = 2x^2 - x$ when finding its derivatives: $f(x) = \lim_{h \rightarrow 0} \frac{2(x+h) - x - 2x^2 - x}{h}$ in which the learner did not consider squaring the bracket in the first term. Also, the second term should have been $(x+h)$ because of the negative sign between the first and second terms, and lastly the last term should have been $(2x^2 - x)$ instead of $-2x^2 - x$.

White and Mitchelmore (1996) argue that learners experience difficulties in relating symbols and the use of the variables in finding the derivatives of functions. This substitution shows lack of advanced mathematics thinking, also due to the teaching approach (Muzangwa & Chifamba, 2012).

Examples of incorrect computations

2.3. $f(x) = x^2 + \frac{1}{2x^3}$

$f'(x) = x^2 + 2x^3$

~~$= 2x + 6x^2$~~

Vignette 3

The example provided here represents one of the learners' responses in test item 2.3: $f(x) = x^2 + \frac{1}{2x^3}$, which required learners to find the derivatives using the rules of differentiation. This is an example of some of the responses: $f(x) = x^2 + 2x^3$ and most of them found the answer as $f'(x) = 2x^2 + 6x^2$, suggesting a lack of conceptual knowledge of differential calculus (Herbert, 2011). The response of this learner shows that since the learners had to use rules of differentiation $f'(x) = nx^{n-1}$, she just decided to change the fraction into a whole number without considering the sign of the exponent of the variable in the denominator. The learners' deficiencies in mathematical operations hamper their learning of calculus.

NTCM (2014) argues that procedural knowledge should build on conceptual knowledge. The findings reveal that learners lacked both, which contributed to their difficulties when solving differential calculus problems.

Scholars agree that learners' failure to understand the fundamental ideas of change and rate can hamper learning of calculus (Coe, 2007; Ubuz, 2007). Learners' interviews revealed that teachers did not help them to understand the concepts but to memorise the derivatives of the functions by either using first principles or rules of differentiation. Learners' procedural knowledge seemed better than their conceptual knowledge, as they were able to respond to simple functions better than complex ones.

Muzangwa and Chifamba (2012) argued that their study of undergraduate students focusing on errors and misconceptions in calculus showed that the difficulties in calculus are caused by knowledge gaps of basic algebra. The teacher in the participating school concurred with this. Their poor background in basic algebra was revealed when learners failed to substitute the terms in the formula, or find the derivative of functions that included fractions. Learners lacked knowledge of converting the variable x in the denominator to x^{-1} before they could find its derivative using either first principles or rules of differentiation. An example was test item 2.3: $f(x) = x^2 + \frac{1}{2x^3}$ in which learners were supposed to

write the fraction as $\frac{x^{-5}}{2}$ before it could be differentiated. It seems that learners did not comprehend calculus concepts, which led them to use the wrong procedures.

Teaching approaches also appear to have affected learners' academic achievement. Commentators agree that the lack of conceptual knowledge of learners in calculus can be caused by lecturers (Muzangwa & Chifamba, 2012; Coe, 2007; Ubuz, 2007). In this study, learners 1, 2, 4, 5, and 7 felt that the approaches used by the teacher were a challenge to the learners; learner 7 concurred that their teacher just taught them the procedures without explaining their meanings. Axtell (2006) argues that the calculus curriculum needs to be revised to improve teaching and learning and focus on conceptual understanding rather than procedural understanding only.

In her study, Herbert (2011) revealed that first-year tertiary students lacked conceptual knowledge of introductory calculus. The teacher in our study reflected on her own approach and that of her colleagues to teaching differential calculus, conceding that it could have played a part in the learners' misunderstanding of calculus. Learner 7 felt that the time factor contributed towards the teacher's approaches as she was teaching at a fast pace to complete the syllabus.

9. CONCLUSION

The study focused on learners' difficulties in finding the derivative of functions using first principles and rules of differentiation. The findings revealed that learners performed more poorly in finding the derivatives using first principles than using the rules of differentiation. The poor performance was possibly due to the teachers' teaching approaches and learners' lack of procedural and conceptual knowledge. The study therefore recommends that learners should be taught how to differentiate functions using first principles and rules of differentiation by working to correct the types of mistake learners display in this area.

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