MISCONCEPTIONS AND RELATED ERRORS DISPLAYED BY PRE-SERVICE FOUNDATION PHASE TEACHERS IN TRANSFORMATION GEOMETRY

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Abstract – This paper reports on a bigger study aimed at identifying pre-service teachers' misconceptions with transformation geometry and addressing these using the van Hiele phases of learning. I report here only on the misconceptions identified. An action research approach was conducted with 82 BEd Foundation Phase student teachers and data was collected through tests and semi-structured interviews, used to complement the test results. Content analysis of participants' responses to written tasks and interviews were used to determine errors, leading to the formulation of misconceptions. The findings revealed that the errors displayed ranged from non-systemic errors due to students misreading information or forgetting some piece of information, to systemic errors or misconceptions which may be attributable to various reasons. Misconceptions included problems with solution strategies and rules previously learned; insufficient skill in working with practical measurements resulting in faulty procedures when measuring lengths of lines and sizes of angles; the tendency to consider geometrical figures as material objects, resulting in participants relying on their visual perceptions rather than reasoning based on the properties of the figures. The findings also seem to support literature that claims that most pre-service teachers have not developed beyond van Hiele's Level 2 of geometric reasoning.

Keywords: Misconceptions; Transformation geometry; van Hiele theory.

1. INTRODUCTION AND BACKGROUND

Several objectives for teaching mathematics in the Foundation Phase in South Africa have been identified. Among those objectives is the "deep conceptual understanding in order to make sense of mathematics" (Department of Basic Education, 2011, p. 8). Thus Foundation Phase learners are expected to master key mathematical concepts from their early years of learning the subject. Primary elementary mathematics is regarded as "fundamental, as it contains the rudiments of many key concepts in more advanced branches of the discipline" (Ma, 2010, p. 116). Unfortunately, evidence points to the fact that school learners often lack this deep conceptual understanding in mathematics (Atebe, 2008; Makonye, 2011).

In particular, learners at all levels of schooling have challenges with the learning of geometry (van der Sandt, 2007). For example, in Luneta's (2014) study, in which BEd Foundation Phase students' basic knowledge of geometry was investigated, he found out that although the student teachers had passed Grade 12, their knowledge of basic geometry was found wanting. It is therefore not surprising that teachers are also part of the challenge with the learning of geometry because they provide teaching which is often of poor quality, "with teachers not able to answer questions in the curriculum they teach" (Bernstein, 2013:1). This suggests that the content knowledge student teachers obtain from their training courses is not adequate and relevant for the learners they are supposed to teach.

Furthermore, in geometry, information might be communicated at different levels of reasoning of the teacher and the student, which becomes a major cause of misconception (Luneta 2015). Thus there is a possibility of errors occurring because students have difficulties in understanding the instructional approaches used by the teacher. Research needs to be done to determine the nature of students' (including pre-service teachers) difficulties with learning geometry, as well as come up with possible solutions. An important aspect of this research direction would be the identification of students' mistakes (Panaoura & Gagatsis, 2009). This paper reports on a study that sought to identify and address pre-service Foundation Phase teachers' misconceptions with transformation geometry.

Transformational geometry has been described as an aspect of geometry which deals with the way geometrical shapes or objects are changed into their various images under reflections, translations, rotations, glide reflections and magnifications on a plane (Evbuomwan, 2013). Within the broader context of geometry, students have been found to be struggling with the learning of transformation geometry (Bansilas & Naidoo, 2012; Luneta, 2015). Bansilas and Naidoo (2012) discovered that misconceptions involving the concept of transformational geometry included the incompetent use of algebraic manipulations. This would happen when, for example, students, in response to "a translation of 3 units to the right", write "x – 3 = -2 -3 = 5". Also, the use of rules rather than visualisation has been found to create problems because students couldn't always remember the rules involved. According to a study by Ilaslan (2013), teachers do not feel confident enough to implement transformational geometry especially rotation since they lack adequate training and support. It is possible that teachers may not teach mathematics topics that they find to be very difficult, or "they may treat topics whose conceptual demand they do not appreciate, perfunctorily" (Chick, 2002: 180). Hence it is important that pre-service teachers who will be laying the foundation for further learning of geometry in the near future are prepared adequately in their studies involving transformation geometry. A study such as this one could help build the confidence needed by these teachers to teach the topic.

With the reintroduction of transformation geometry into the curriculum in the primary school level in South Africa (Department of Basic Education, 2011), it is crucial that prospective primary school teachers are well grounded in the topic. This coupled with the fact that a number of the student teachers that enroll for the BEd Foundation Phase program, with whom this study was carried out, did not learn transformation geometry as part of their high school curriculum. Thus these students have a gap of knowledge on transformation geometry. Yet they are expected to master the topic as part of the curriculum in the BEd Foundation Phase programme for which they are enrolled.

This study aimed to identify BEd students' misconceptions with transformation geometry. The research questions were:
What errors are displayed by BEd (Foundation Phase) students when working with transformation geometry?
What misconceptions are associated with the errors displayed?

2. LITERATURE PERSPECTIVES

Theoretical framework

Error analysis, also referred to as error pattern analysis, is the study of errors in learners' work with a view to finding explanations for these reasoning errors (Herholdt & Sapire, 2014). In their approach to error analysis, Rach, Ufer & Heinze (2013:23) distinguish between a "pragmatic, outcome- oriented, and an analytic, process-oriented path of action" (p. 23), as depicted in the figure below:

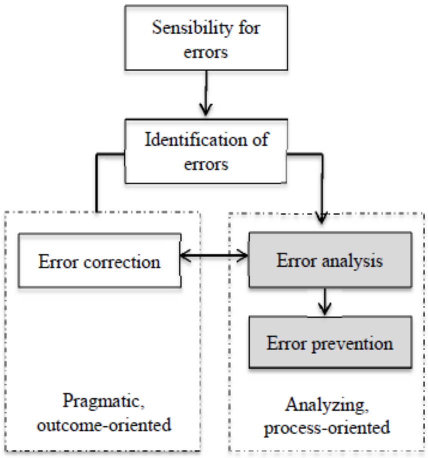


Figure 1: Process model for learning in error situations (Rach, Ufer & Heinze, 2013)

While the former approach depicted on the left in the above figure, proceeds directly from error recognition to error rectification, the latter route includes a closer analysis of the error and the generation of error prevention strategies. With respect to this study, the latter approach was applicable since, after detecting the students' errors and misconceptions that are responsible for such errors, these were analysed and an intervention program was carried out to address them with the hope that in the long run they would be prevented or minimized in future.

Misconceptions and associated errors

One important context in which teachers need to play a vital role in trying to improve student performance in mathematics, particularly geometry, is identifying and addressing student errors and misconceptions. According to Panaoura & Gagatsis (2009), errors were seen for the first time in a positive way in the work of Piaget, whereby they allowed the tracing of the reasoning mechanisms adopted by students. This means that making errors in computation, for example, is not completely bad because it becomes an important part of the learning process if these errors are dealt with diagnostically.

Errors and misconceptions are related, even though they are different in many aspects. "An error is a mistake, slip, blunder or inaccuracy" (Luneta & Makonye 2010: 35). A basic premise in differentiating between an error and a misconception is that errors are easily detected in learners' work such as written text or speech, while misconceptions are often hidden from mere observation without scrutiny. If errors and misconceptions were to be put on a continuum, one would have non-systematic errors on one end, and the more serious systematic errors, which are deeply rooted in misconceptions, on the opposite end (Makonye, 2011: 11). The aim of this study was therefore to discover the "hidden" misconceptions.

Classification of errors

Various authors have put forward different ways in which errors can be classified. Most of these classifications fall under two broad categories, namely, non-systematic and systematic errors.

Non-systematic errors

Non-systematic errors are normally referred to as slips and might exist due to students' carelessness, misreading information or forgetting some piece of information, unintentionally. Normally, students will easily correct such errors by themselves because there are no fundamental and faulty conceptual structures associated with them. However, teachers are warned that "errors in the learning of mathematics are not simply the absence of correct answers or the result of unfortunate accidents. They are the consequence of definite processes whose nature must be discovered" (Wiens, 2007: 5). Even if these errors were simple slip ups, they should not be taken lightly because they can demoralize learners and impede their performance when committed, thus becoming “a serious inhibitor to learning as mathematics builds on itself” (Schnepper & McCoy, 2013: 1). These types of errors were detected in this study, as discussed under section 4 below.

Systematic errors

VanLehn (1990) indicated that systematic errors mean consistent application of faulty methods, algorithms or rules. The cause of systematic errors may relate to students’ procedural knowledge, conceptual knowledge, or links between these two types of knowledge (Xiaobao, 2006: 4). Examples of systematic errors have been suggested, such as errors due to deficiencies of mastery prerequisite skills, facts and concepts and errors due to the application of irrelevant rules or strategies (Egodawatte 2011, p. 36). Further examples suggested include making logically invalid inferences, applying an improper version of a definitions or theorem and having the right solution to the wrong question, or making a mistake in a basic skill (Schnepper & McCoy, 2013: 1). Some of the examples of systematic errors suggested here are applicable in this study, as discussed under section 4 below.

The van Hiele theory

Problems experienced by learners (and pre-service teachers) when learning geometry have been investigated in various studies, many of which used the van Hiele theory (Mayberry, 1983; Feza & Webb, 2005; Wu & Ma, 2006; Luneta, 2014). In studies that used the van Hiele theory to identify students’ misconceptions (Siyepu, 2005; Atebe, 2008; Alex & Mammen, 2014; Luneta, 2015), it proved to be a useful framework for extracting, measuring, understanding and addressing students’ difficulties with school geometry. In the context of this study, the van Hiele theory played an important role in that it helped diagnose some of the causes of the misconceptions displayed by the students in that questions were set to cater for particular levels of development of students’ reasoning in geometry. Also, as part of the bigger study of which this paper is a report on only a portion of, the van Hiele theory was used to address the misconceptions identified.

According to the van Hiele theory, learners advance through five hierarchical levels of reasoning when they learn geometric concepts. These levels are recognition, analysis, ordering, deduction and rigour. For this study it was further critical that the pre-service teachers were assisted in attaining a level of geometric thinking in transformation geometry that would possibly, in turn, enable them to provide assistance to learners in their own classrooms when they teach the concept of transformation geometry in future. Therefore, as part of determining their misconceptions with transformation geometry it was important to get an idea of the level of geometric reasoning at which they were, based on the van Hiele theory.

Closely linked to the van Hiele theory is Soon’s levels of geometric thinking on transformation geometry. Soon (1989) developed these levels by identifying the kind of competences that students need to master at each of the van Hiele levels (see table on section 3.3). In discussing the findings of this study, reference will be made to Soon’s and van Hiele levels.

RESEARCH DESIGN AND METHODOLOGY

Research paradigm, approach and method

This study is based on the interpretivist paradigm, which seeks to understand the situation from the perspective of the participant (Ary, Jacobs & Razavieh, 2002). The study was interpretive in nature because its aim was to understand the misconceptions displayed by students and then attempt to get answers from the perspective of the students through face to face interactions during classwork activities and detailed written work.

The study qualitative in nature in that its aim was to understand a social phenomenon, from the perspective of the human participants in the study, data is collected in natural setting and research aims at generating theory rather than testing theory (Ary, Jacobs, & Razavieh, 2002; Creswell, 2010).

The action research approach was used to conduct this study. Winter and Munn-Giddings (2001: 8)) describe action research as "a study of a social situation carried out by those involved in that situation in order to improve both their practice and the quality of their understanding". The aim of the study was to improve the researcher's practice through the implementation of an intervention program, with the hope of an increase in the quality of students' understanding of transformation geometry.

Sampling

Purposeful sampling (McMillan and Schumacher, 2001) was done, with participants being a group of 82 second year BEd Foundation Phase students at a university in the Mpumalanga province in South Africa. This group of students had to learn transformation geometry as part of their BEd (year 2) curriculum, thus being the relevant group for the investigation of the phenomenon of interest, namely, misconceptions displayed in transformation geometry. The whole group of second year students enrolled for the BEd programme was selected because this was an action research study where an inquiry was done while executing an intervention aimed at improvement (Waterman, Tillen, Dickson & de Koning 2001: 4) of the researcher's practice as well as the level of understanding of all the students.

Data collection and analysis

Data was collected by means of tests and interviews. First, a test was administered to participants before any instruction on the topic of transformation geometry was done. The purpose of the test was to identify students' errors in working with transformation geometry problems. Questions in the test covered the topic as stipulated in the curriculum of the BEd programme, namely, concepts on translation, reflection and rotation. The test items were based on and adapted from the general frameworks for investigating learners' van Hiele levels of development, as suggested by Mayberry (1981) and Usiskin (1982).

The test items were formulated in such a way that they tested students' understanding of transformation geometry according to the van Hiele levels. The test questions were based mainly on Soon (1989), who determined van Hiele-like levels for learning transformation geometry, as well as on Burger and Shaughnessy (1986), who placed the study of isometric transformations in the van Hiele framework.

Soon's (1989) levels are depicted in the table below. For the purpose of this study, only levels 1 through 4 were considered. In discussing the findings, an association will be made between the errors made by students in the study and these levels.

Table 1: Soon's (1989) levels

Levels	Characteristics: The student ...
Level 1	Identifies transformation by the changes in the figure; (a) in simple drawings of figures and images; and (b) in pictures of everyday applications. Identifies transformation by performing actual motion; names, discriminates the transformation. Names or labels transformations using standard and/or non-standard names and labels appropriately. Solves problems by operating on changes of figures or motion rather than using properties of the changes.
Level 2	Uses the properties of changes to draw the pre-image or image of a given transformation. Discovers properties of changes to figures resulting from specific transformation. Uses appropriate vocabulary for the properties and transformation. Is able to locate axis of reflection, centre of rotation, translation vector and centre of enlargement. Relates transformations using coordinates. Solves problems using known properties of transformations.
Level 3	Performs composition of simple transformations. Describes changes to states (pre-image, image) after composite transformations. Represents transformations using coordinates and matrices. Inter-relates the properties of changes to a figure resulting from transformations. Given initial and final states, can name a single transformation. Given initial and final states, can decompose and recombine a transformation as a composition of simple transformations.
Level 4	Gives geometric proofs using transformational approach. Gives proofs using the coordinates and matrices. Thinks through multi-step problems and gives reasons for problems.
Level 5	Understands – associative, commutative, inverse, identity with respect to a composite transformation operation. Identifies groups of transformations. Proves or disproves subsets of transformation from group structures.

Interviews were then conducted with selected students (21 in all) based on their responses in the test, to determine the sources of the errors they displayed in the test scripts.

Inductive analysis of data was done, with the aim of uncovering patterns and regularities in data as well as general explanations, in order to generate theory (Hatch, 2002). Content analysis of the scripts was done. The responses to the written pre-test and the interviews were carefully analysed to determine common patterns of difficulties and errors, leading to the formulation of misconceptions that students had.

Validity, reliability and ethical considerations

The test was first piloted with different groups of BEd students who were not participants in the study, to determine whether they elicited the intended responses and interactions. The piloting process proved the instruments to be reliable.

Written consent was obtained from all participants and confidentiality was maintained. Students were assured of inclusivity since the follow-up intervention was supposed to benefit everybody.

FINDINGS AND DISCUSSION

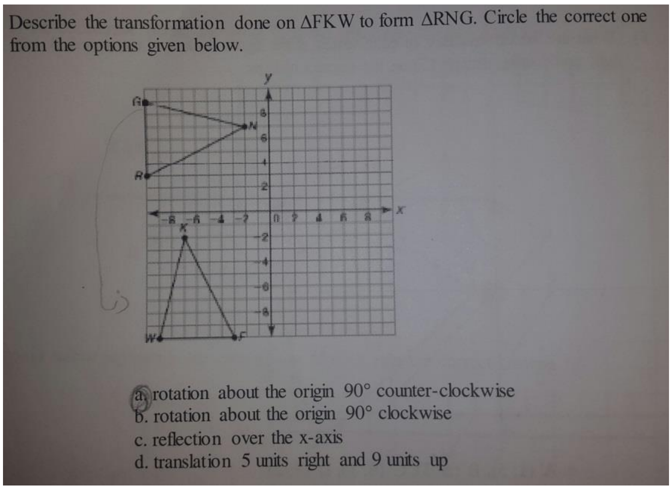
Analysis of the student test scripts as well as insight from follow-up interviews confirmed that the students have various misconceptions and associated errors with the topic of transformation geometry. These will be discussed under two broad categories, namely, non-systematic errors and systematics errors.

Non-systematic errors

As mentioned before, these are unintentional slips or errors due to carelessness.

Misreading information

An error due to misreading which figure is the original and which one is the image, was displayed, leading to an incorrect answer. For example in the multiple choice question below, 26 students (32%) chose option ‘a’ as the correct answer in the question below. Later on some acknowledged during interviews that they did not really “read carefully” on which triangle the transformation was performed.

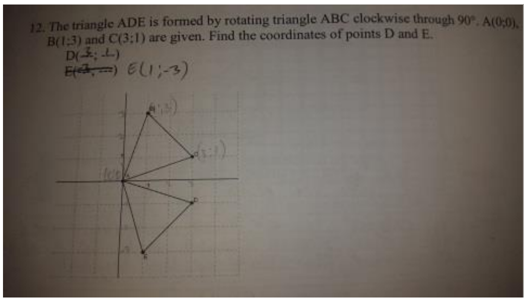

Describe the transformation done on ΔFKW to form ΔRNG. Circle the correct one from the options given below.

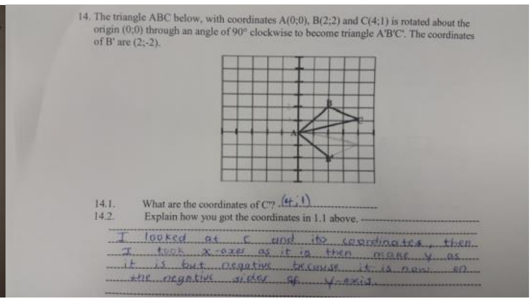
a. rotation about the origin 90° counter-clockwise
b. rotation about the origin 90° clockwise
c. reflection over the x-axis
d. translation 5 units right and 9 units up

In some instances, students carelessly read “anti-clockwise” as “clockwise”, leading to incorrect identification of figures and images (Soon level 1). However, this error could also be associated with language problems, in cases where, for example, students mentioned that they “always confused clockwise and anti-clockwise direction”.

Missing information

Errors were identified, that involved writing the incorrect ordered pair, by omitting the negative sign, even though the points are correctly labelled or identified on the system of axes. This is the situation in both cases in the examples below. It should be noted that in the second example, question 14, the student made a double error: the missing sign as well as the fact that her explanation is correct for a reflection and not for a rotation. Thus this example also concurs with Schnepper’s (2013) idea of an error involving having the right solution to the wrong question.



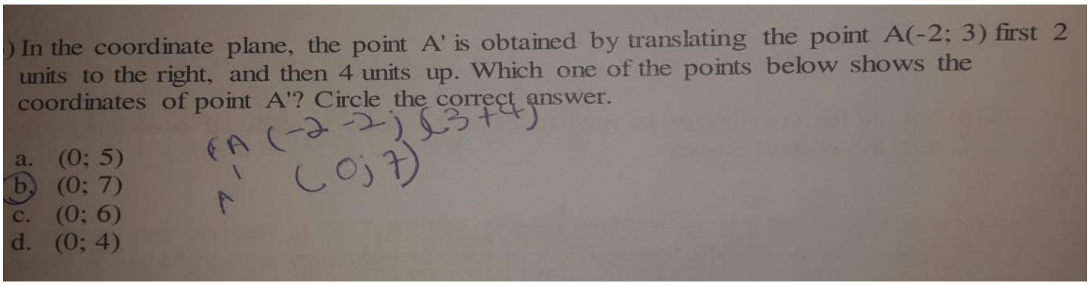


Systematic errors

Students displayed quite a number of errors that relate to their procedural knowledge and conceptual knowledge. These are what we referred to earlier on as misconceptions. A discussion of these errors follows below.

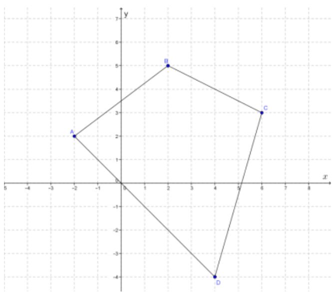
Errors involving basic operations

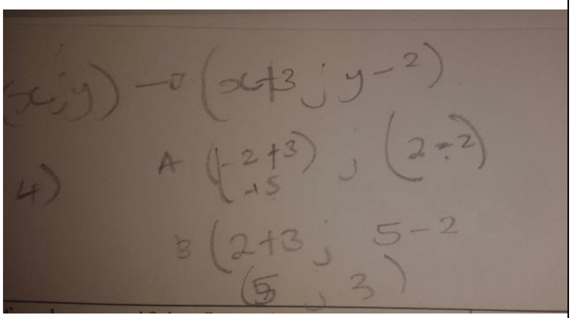
Errors like the ones below were common, where the student made double error by first interpreting "2 units to the right' as "- 2" and secondly by working out -2-2 = 0.



In the case below, a student worked out -2 + 3 = 5.

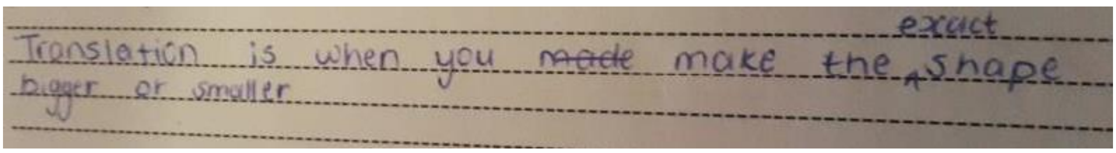
What are the coordinates of each vertex if the figure below is translated 3 units to the right and 2 units down? Circle the correct answer.



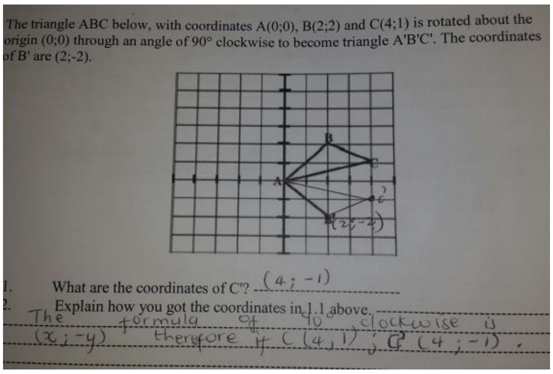


Application of irrelevant rules, strategies or incorrect/incomplete/incomprehensible descriptions of transformations

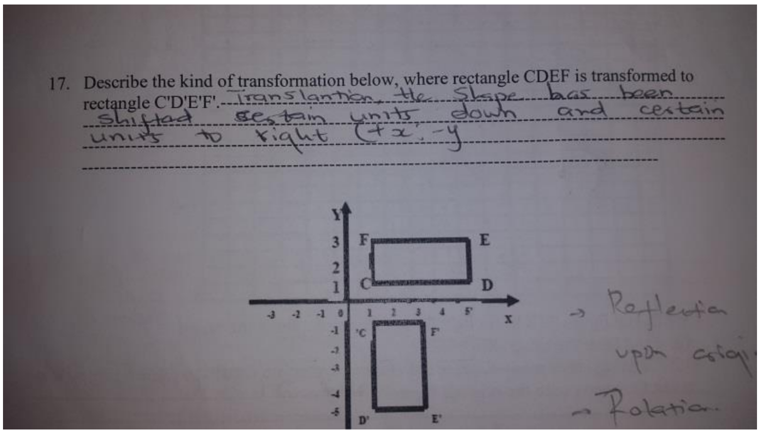
Below is an error where a student gave a description that indicates that he is confusing translation with enlargement.



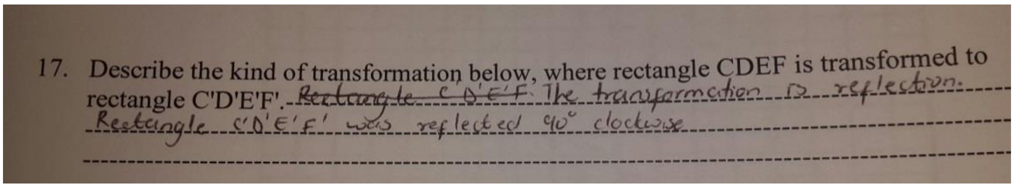
In answering the following question, 53 students (65%) either applied the rule for, or a description of reflection even though the question involved a rotation, such as in the example below.



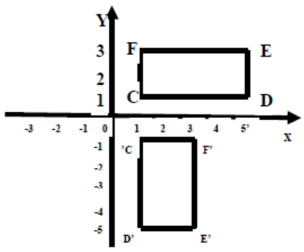
In the following example, the student describes a shift/translation but gives the rule for a reflection.



For the same question as the one above, the student describes the transformation as a reflection but uses terminology that is pertinent to a rotation, such as “90° clockwise”

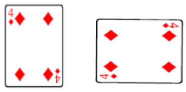


Errors involving incomplete descriptions of transformations, especially rotation, were prevalent, such as in the two examples below. Statements such as “Rotation of 90° clockwise” or “90° rotation” were used to describe the transformation of figure CFED to C'F'E'D. 60% of the students who correctly identified the transformation below as a rotation failed to describe it completely, thus they were unable to “name the transformation, given initial and final states” (Soon level 3).



Incomprehensible explanations or descriptions were not uncommon, such as
“Translation is when you move a shape to other positions, using the given numbers”.

“Card has turned 1 unit” or has “rotated one more time”. That is, failing to use angles to describe the amount of turn. Turning one unit/one more time in this case is taken as meaning “turning once to the left/right/down/up” as in the following diagram:

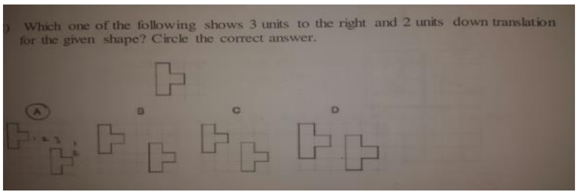


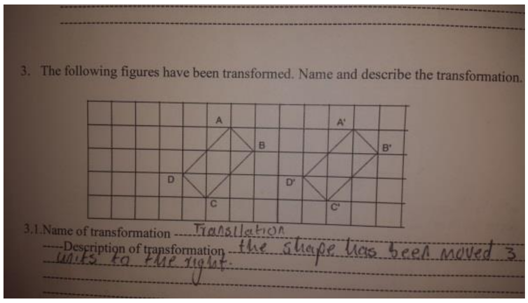
“The position of this card is like it is still on process for rotating…”  or simply “rotation of 90°”, suggesting that some students think that rotation can only be through angles of 90°, 180°, 270° and 360°.

The above errors suggest that the students could not achieve level 2, as depicted in Soon’s levels.

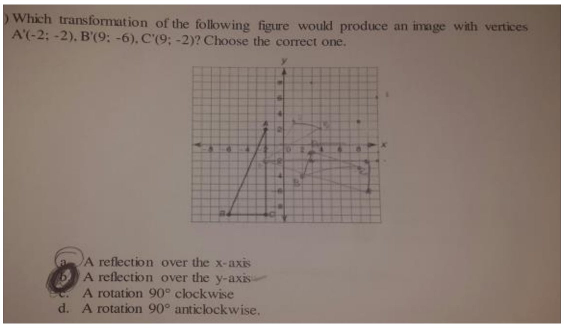
Errors involving inability to discover properties of changes to figures resulting from specific transformation.

In translating a shape, a point at one “furthest” end of the shape is chosen and then only this one particular point is used to determine the given translation, that is, to find out how far the shape has been shifted. In the example on the left below, for example, 18% of the students chose option A as the correct answer. Another example is given below on the right.



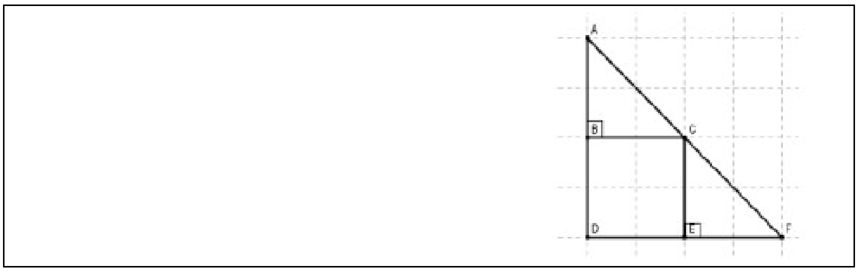


Possibly due to lack of visualisation skills, some students could not “visualise” (even though they plotted correct points and drew correct image) that in the question below, the transformation performed cannot be a reflection on any of the axes. These students could therefore not “identify transformation by the changes in the figure, that is, Soon’s level 1.

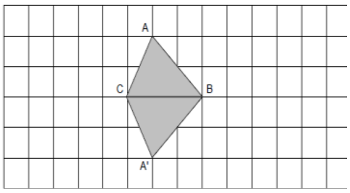


Errors involving the introduction of the system of axes where it does not exist

In trying to describe what translation is: “…figure shifts along the axis (y-axis and x-axis) to new coordinate either left or right, or up and down”, as if figure can’t shift diagonally. Hence the thinking can be limited to vertical or horizontal shifts only, resulting in errors when describing translation. For example, in describing the situation in the diagram below, a student stated: “Triangle ABC in my understanding has been translated vertically to form triangle CEF”. It seems as if this student could not recognise that the triangle has shifted ‘diagonally’ (Soon’s level 1).

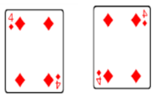


In describing the transformation below, 93% of the students who correctly identified this as a reflection referred to it as reflection "across the x-axis".



For these learners the horizontal line CB (and possibly the grid lines) meant that the diagram was drawn on a system of axes.

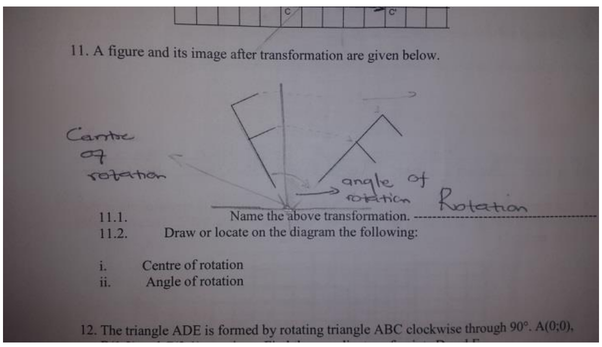
The assumption of the existence of the system of axis was prevalent with other transformations involving reflection, even when there were no grid lines. For example, regarding the transformation below, there were statements like: "The card is reflected on the x- axis (or y- axis) because x-axis is like horizontal and y-axis is like vertical"



Again, this kind of thinking limits students' reasoning to vertical or horizontal reflection only. This can be evidenced in a question where students were asked to reflect a triangle on any of its sides to create a quadrilateral.

Despite the fact that 18 out of the 82 students did not attempt this question (set at Soon's level 2), all students (43) who successfully drew the correct quadrilateral (kite) reflected the triangle on the horizontal side, rather than on the other slanted/diagonal sides, which would also create some kites. Even though this is not an error, it does, however, limit the thinking and reasoning of the students, especially because all those who tried reflecting on the other slanted sides (13 students) could not perform the reflection correctly, while a few others totally did not follow the instruction for the question.

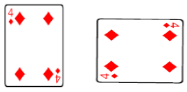
The system of axes was drawn in order to determine the centre of rotation and angle of rotation. Seemingly the assumption is that the centre of rotation has to be the point of origin (0; 0), as displayed in the diagram below:



Errors involving language/terminology issues

Language related errors were eminent and cutting across all types of transformation problems or questions students had to work on. Some of these errors have already been implied within other errors discussed above.

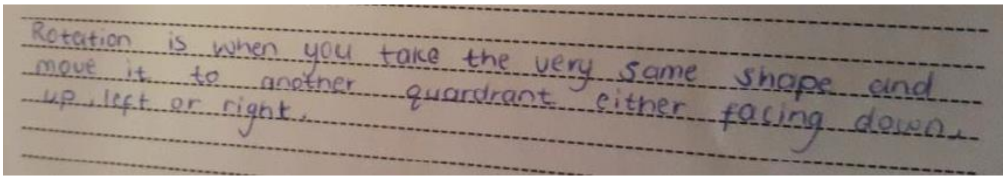
The use of imprecise terminology was prevalent. For example:

"…card has shifted from vertical to horizontal"  , creating misleading notions because in transformation geometry, a "shift" is associated with a translation rather than the dictionary meaning of 'move', 'budge', 'upside down'.

"…rotation of counter anti-clockwise", meaning anti-clockwise or counter-clockwise.

"Translation is an act of uniform movement", for lack of better or appropriate words to use.

In the statement below, direction words such as down, up, left, right are used to imply a "turn" in this context.

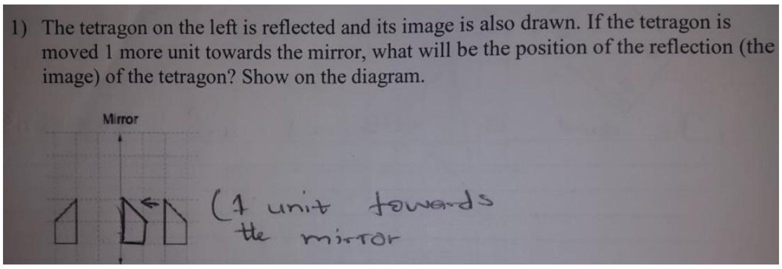


Errors involving misinterpretation of instructions were displayed. For example:

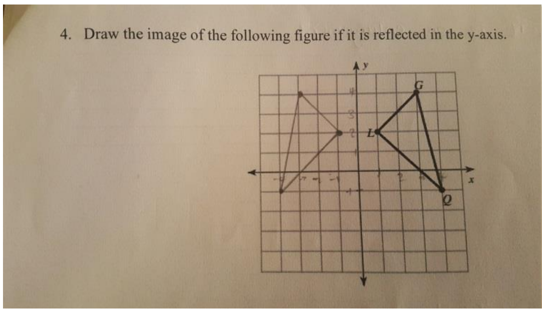
"Draw/locate angle of rotation on the diagram" was interpreted as "draw or write the size of the angle on the diagram"

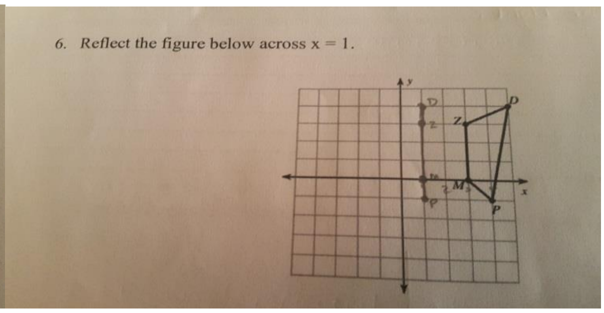
"Describe" is reduced to mean "name". For example, in a question that required students to describe two different transformations that would move a figure onto its image, 54% of the students only mentioned the type of transformation without describing it.

If a figure has to be "moved 1 more unit" this can be interpreted as "skip one unit and then place it" as seen in the example below. Although not many students (11) made this error, it is worth mentioning because it affected their ability to correctly perform the required reflection.



The instruction: "Draw the image when the figure is translated/reflected/rotated ..." is taken to mean something different from "Translate/reflect/rotate the figure ...". For example, the student whose work appears below was able to correctly follow the instruction on question 4 below but not the one on question 6, even though in both cases the line of reflection was correctly identified:





During interviews the student explained, even though not convincingly enough, that when asked "to reflect the figure" she thought she was supposed to "show how it would look like in the mirror", not to draw its image. This gives some insight into why 27 students never attempted or gave no response to question 6 above, 42 got it wrong (most of them simply reflected the figure on the y-axis - perhaps also because of the misconception about reflection on the axes only, as discussed earlier on) and only 13 students gave the correct answer.
Problems with inappropriate use of vocabulary in relations to transformations, suggest that these students could not achieve Soon's level 2.

CONCLUSION

Based on the findings of this study, BEd Foundation Phase students grapple with the learning of the topic of transformation geometry and display a range of misconceptions which need to be addressed in order to give them a better change at teaching geometry to their own learners in the future. The findings suggest that these students lack both conceptual and procedural knowledge that would enable them to fully comprehend and competently teach the topic. For example, their inability to discover properties of changes to figures resulting from specific transformations, or to work appropriately with correct and relevant rules and descriptions is an indication of lack of understanding of the concepts involved in transformation geometry. These findings concur with the suggestion that students have "deficiencies of mastery prerequisite skills, facts and concepts and errors due to the application of irrelevant rules or strategies" (Egodawatte 2011: 36).

Students' failure to visualise certain transformations that are performed suggests that they did not get enough opportunity to practise visualisation skills during their schooling years. This was acknowledged by students during interviews, when they mentioned that their teachers at high school insisted on them using the rules for transformation geometry because then "once you 'keep the rules in your head' you will never go wrong". This concurs with the notion that teachers may teach topics whose conceptual demand they do not appreciate, perfunctorily (Chick, 2002). Students' lack of visualisation skills implies that some students do not possess level 1 characteristics as depicted in Soon's (1989) van Hiele like levels of achievement in transformation geometry. Therefore it is not surprising that none of the students in the study possessed characteristics that are required for level 4 reasoning.

As recommendation towards addressing the misconceptions identified in this study, teachers should provide students with opportunities to experience hands – on manipulatives that will give them plenty of practice to manipulate in order to get visual perspectives of how the shapes move as they are translated, reflected or rotated. This should be achieved before any rules are learned or emphasized. Thereafter, students should work on step by step instructions on how to relate the transformed shapes with the properties of both objects and images so that they can discover and master the rules themselves. The bigger study from which this paper is based deals with an intervention programme that aims to address the misconceptions identified, by conducting a series of lessons based on the van Hiele phases of learning. These phases entail a step by step approach that helps students in gradually advancing from one level of geometric thinking to the next higher level during classroom teaching.

Lastly, teachers and learners could both benefit if teachers could make effort to familiarize themselves with their students' errors and misconceptions and work on strategies to improve them. This is especially crucial in the South African context where error analysis and diagnosis are not part of courses offered in pre-service teacher education programmes (Luneta, 2008). To this effect, I would recommend the analytic, process-oriented approach to error analysis as suggested by Rach, Ufer & Heinze (2013).

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