AN APOS EXPLORATION OF THE CONCEPTUAL UNDERSTANDING OF ALGEBRAIC EXPRESSIONS

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ABSTRACT– Algebra systematizes the construction and analysis of formulas, equations, algebraic expressions, and functions that make an important part of mathematics and its applications. This paper explores the conceptual difficulties and understandings of algebraic expressions and related concepts that a sample of grade 8 learners have, through APOS theory lens. APOS pre-supposes what it means to learn and understand a concept by constructing and reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the application of those concepts. Qualitative design was employed in which 74 grade 8 learners from two of fifteen purposely selected secondary schools in a district of the Eastern Cape in South Africa were taught algebraic expressions following a prescribed genetic decomposition. Subsequently, a test was written thereafter six learners were interviewed on the bases of their responses. Results indicated that about 93% of the learners operated in the action stage where they could not perform mental manipulation that transform objects by reacting to external cues that give precise details on what steps to take. Only 2% showed evidence of a mental construction as an action that happens in the mind, which could be interiorized as the process on understanding algebraic expressions.

Keywords: Algebraic expressions; APOS theory, conceptual understanding, genetic decomposition.

Introduction

Dubinsky and McDonald (2001) define APOS as a theory of learning mathematics can help us understand the learning process by providing explanations of phenomena that we can observe in *learners* who are trying to construct their understandings of mathematical concepts and by suggesting directions for pedagogy that can help in this learning process. The APOS theory is a theory about how learners construct knowledge. It arose out of an attempt to understand the mechanism of reflective abstraction as the key to constructing logico-mathematical knowledge as described by Piaget (Meel, 2003 pg.150). The APOS theory is an extension of Piaget's idea to include advanced mathematical concepts. The APOS theory offers insight into learners' conceptual difficulties and understanding of algebraic expressions and related concepts.

The trust and focus of this paper lies with an APOS exploration of the conceptual difficulties and understanding of algebraic expressions with related concepts that a sample of grade 8 learners displayed, through the lens of the APOS theory. Why a focus on algebraic expressions? Algebraic expressions form part of the senior phase CAPS curriculum in South Africa. A substantial amount of time is allocated to the section on evaluating expressions and simplification of algebraic expressions in grades 7 to 9.

The discussion proceeds as follows:

- Literature review, on the possible conceptions of algebra that may be promoted by the school curriculum as well as the conceptual difficulties learners face when learning algebra are explored.
- A brief synopsis of the APOS theory is given, focusing on the elements of the theory that are relevant to the study.
- The methodology followed and the findings are discussed followed by the conclusion.

Literature review: school algebra

Algebra systematizes the construction and analysis of formulas, equations, algebraic expressions, and functions that make an important part of mathematics and its applications. According to Kieran (1992, p. 391), algebra is conceived as a branch of mathematics that deals with symbolizing and generalizing numerical relationships and mathematical structures, and operating within those structures. On the other hand, for example, French (2002, p.3) holds the view that algebra develops thinking skills and it makes possible for insightful explanations of a wide range of phenomena in the world to be presented. From these viewpoints, we contend that learning algebra is an important milestone in learners' mathematical development.

Existing research on school algebra served as a frame of reference for identifying issues concerning the teaching and learning of algebra in high school. The literature review has helped us to formulate the following themes that may impact on how algebra is taught and learned in school. The two major themes identified for the purposes of this report fall into two major categories: 1) Conceptions of school algebra and 2) Conceptual difficulties learners have in respect of algebra.

Conceptions of school algebra

Algebra is often referred to as 'generalized arithmetic', a process of exploring a given numeric situation for patterns and relationships. This may be understood to mean that the development from arithmetic thinking into algebraic thinking occurs naturally. Demana & Leitzel (1988) have demonstrated that exposing learners to numerical computations and problem solving first makes it possible for them to understand basic concepts of algebra. The National Council of Teachers of Mathematics [NCTM] (2001) asserts that "Both arithmetic and algebra are useful for describing important relationships in the world. Although arithmetic is effective in describing a static picture of the world, algebra is dynamic and a necessary vehicle for describing changing world" (p. 1) . The NCTM brings an important element, when learners are encouraged to describe and represent quantities in different ways, learners "develop notions of the idea and usefulness of variables" (p. 3). Hence, the use of symbols to communicate ideas expands; as a result, algebraic expressions in line with mathematical properties emerge. However, French (2002) cautions that: "a proper understanding of algebraic process is inevitably very dependent on a corresponding understanding and facility with arithmetical operations" (p. 47). Even though French's statement may be disputed, it highlights the fact that it is not a smooth transition from arithmetic to algebraic mode of operation, more especially if the arithmetical structure is not well understood.

Conceptual difficulties learners have in algebra

There are documented obstacles that learners encounter in their study of algebra. In the next passage, we briefly highlight some of the challenges as reported by various researchers. Much research has been done around learners' understanding of algebraic notation (MacGregor and Stacey, 1997) and learners' understanding of literal terms and expressions (Kieran, 1989; Küchemann, 1978), and on systematic mistakes that learners make in simplifying algebraic expressions (Booth, 1984; Linchevski & Herscovics, 1996). Moreover, learners experience serious problems in grouping or combining like terms, unlike in arithmetic where operations yield other numbers; in algebra, operations may yield an algebraic term or an algebraic expression. Küchemann (1981), established that it is difficult for learners to understand letter symbols as denoting a variable particularly in a formula defining a function, but less problematic when used as an unknown in a simple equation. Linchevski and Livneh (1999) observed that learners who make errors in manipulating algebraic expressions repeat some of those errors when dealing with arithmetic expressions. This implies that learners' understanding of arithmetic and algebraic expressions are interconnected. Therefore, it seems that most difficulties in learning algebra may predominantly be due to learners' poor understanding of the variable and algebraic expression.

In their study, Lee and Wheeler (1987) report the learners' inability to read and use algebraic symbols in a meaningful way. Viewing algebra as a language of mathematics, then this requires that learners in general should be fluent in the language. Hence understanding the concept of a variable and algebraic expression and the meaning of the solutions indicate that the language is grasped. In other words, comprehending the algebra language implies the ability to read, write, and manipulate both numbers and symbolic representations in formulas, expressions and equations. Related to this is the ability to understand the meaning and structure of expressions and formulas, often referred to as possessing symbol sense. Arcavi (1994) makes a distinction between some features of symbol sense, such as the skill to set up a formula, the ability to recognize equivalent expressions and to see how they may 'tell a different story'. Arcavi claims that success in algebra and mathematics in general, rest upon symbol sense: "Symbol sense is the algebraic component of a broader theme: sense-making in mathematics" (Arcavi, 1994, p. 32).

Therefore, given this background, we confine the interpretation of symbol sense in this research to the understanding of the meaning and structure of algebraic expressions. Interpreting symbol sense in this way enabled us to understand the concept of structure sense more deeply. Structure sense focuses on abilities such as using equivalent structures of an expression flexibly and creatively (Hoch, 2003; Linchevski & Livneh, 1999). We suspect that within the current curriculum and teaching practices, most learners, to the (often limited) extent that they do at all recognize the letter symbols as placeholders for numbers (variables), make sense of algebraic expressions purely as prescriptions to execute the given computations (Sfard & Linchevsky, 1994, p 207). Given such challenges experienced by learners at different levels this paper set out to identify and expose the difficulties learners experience in understanding equivalent algebraic expressions in grade 8 mathematics.

Theoretical Framework

Construction of knowledge and understanding of new ideas require that learners be actively thinking about it. They need to make connections between old and new knowledge, to engage in reflective thinking; as well as going through existing ideas to find those that seem most useful in giving meaning to new concepts being learnt. It is therefore essential that before learners are exposed to the learning of algebraic expressions, they are first equipped with necessary background knowledge upon which new knowledge can be built. The background knowledge in this paper constitutes the genetic decomposition of the concept. A genetic decomposition is a set of mental constructs which might describe how the concept can develop in the mind of an individual (Dubinsky, 1991). These constructs proposed in this paper are:

- Number sense and operations
- Algebraic properties
- Concept of a variable
- Algebraic terms and expressions (the symbolic language of mathematics)
- Manipulation of algebraic expressions

Lack of knowledge on any of the above mentioned concepts may lead to serious challenges in the learners' ability to understand and wisely use the idea of equivalent expressions. Moreover, we believe that when learners are knowledgeable about properties governing algebraic manipulations, including the order of operations, they are less likely to encounter problems in transforming algebraic expressions to obtain their equivalent forms.

The researcher employed Action–Process–Object–Schema (APOS) theory advocated by Dubinsky (1991) to explore leaner's conceptual understanding of algebraic expressions. According to the APOS theory the individual develops understanding by first manipulating previously constructed mental or physical objects to form actions. The actions are then interiorized to form processes which are then encapsulated to form objects. Once the object is formed, it can be de-encapsulated back to the process that gave birth to it. The actions, processes and objects can be organized in schemas (Asiala et al., 1996, p.8).

APOS theory involves four stages, an action, process, object, and schema. The description of action, process, object and schema are provided below are consistent with those given by Maharaj (2010). The stages are hierarchical in nature, meaning that in a particular mathematical situation, a learner operating at a schema level must have a totality of understanding gathered through action, process and object levels.

Action:
In the APOS theory action is the building block of understanding. Dubinsky (1990) thinks of action as "a transformation of objects perceived by an individual as essentially external and as requiring, either explicitly or from memory, step-by-step instruction on how to perform the operation". At the action level a learner is able to carry out rote procedures, in other words, learners need to act on the given algebraic expression with an intention of achieving a simplified algebraic expression, that is, equivalent algebraic expression. For example, a learner acting mathematically (simplifying) on an algebraic expression like $2x\ (6x + 5) - 3x\ (4x - 5)$ is expected to interpret and assign meaning to algebraic expressions as descriptors of sequences of computations, e.g. to interpret $2(x + 3)$ as "add 3 to the given number and multiply the answer by 2". At this stage a learner is aware of and uses basic properties of operations, especially the distributive property.
Process:
When an action is repeated and an individual reflects upon it, he or she can make an internal mental construction called a process which the individual can think of as transforming the same kind of action, but no longer with the need of external stimuli. The learner is able to see the process as a whole, can use multiple representations, can reverse the process and compose with other processes. At this stage, a learner is confident that algebraic manipulation is a process that delivers equivalent expressions and subsequently, a learner becomes confident in own ability to produce equivalent expressions, that is, a learner trust his or her own manipulative ability.
Object:
An object is constructed from a process when the learner becomes aware of the process as a totality and realizes that transformations can act on it and can actually compose such transformations, then we say the individual has condensed the process into a cognitive object (Maharaj,2010, p. 43). For example if a student is in a position to notice appropriate actions and processes and can apply procedures as a whole and understands that transformations can be performed on it, then a student is at an object level of the APOS theory. For example, in this study, a learner is at an object level when the learner has acquired basic knowledge of equivalence that includes the following components (the list may be extended):

- to become aware of equivalence, and to understand what it means
- to come to appreciate how equivalence can be utilized, and specifically
- to become able to recognize replacements in oral or written expositions of mathematical work
- to become able to recognize situations in which it may be useful to replace an expression with an equivalent expression
- to become able to generate equivalent expressions

Schema:
According to Dubinsky (1994) the schema is a collection of processes and objects that a learner uses to organize understand and make sense of mathematical concepts. A schema is an organizing structure that a learner invokes to deal with new and unfamiliar mathematical situations. For this study, a learner is at a schema level when they decide on their own to apply a schema when presented with a particular mathematical situation (Maharaj, 2013). For example, at a schema level, learners develop a

more flexible method of handling notation, are able to see algebraic expressions either as a process to calculate a needed result or as an object to be manipulated as part of a more complex piece of symbolism. As a result, the learner decides on her or his own volition that to evaluate $3x + 12$ for different values of x is much less work than to evaluate $3x(6x + 10) – 9x(2x + 3) + 12$.

A genetic decomposition for equivalent of algebraic expressions

The usage of APOS theory when conducting research on learners' construction of specific mathematical knowledge requires the development of a genetic decomposition. The genetic decomposition of a specific mathematical concept is said to be a well-defined set of mental constructs that may explain the way a learner formulates the mathematical concept in the mind (Jojo, 2015). A genetic decomposition of the concept of equivalence consists of a detailed description of possible actions, mathematical behaviors and reactions that would be invoked in the mind of a learner who has developed the concept of equivalence. According to the APOS theory a learner can begin the development of the concept of equivalence out of different actions that result in different understandings of the concept (DeVries and Armon, 2004).

Liebenberg, Sassman and Olivier (1999) propose two dimensions of understanding algebraic equivalence. The first dimension of understanding is that two algebraic expressions are equivalent if the expressions have equal values for all the values of the variable. The second dimension of understanding concerns the usefulness of the idea of algebraic equivalence so that the transformation of one algebraic expression into another becomes meaningful for the learners. What this means is that as learners simplify an expression they need to develop an understanding that:

1. the transformation of one algebraic expression into another results in two expressions that are equivalent,
2. the consequence of having equivalent expressions is that one expression can be replaced with another expression that makes the task at hand easier or convenient.

Below is a summary of the genetic decomposition for utilizing the idea of equivalent expressions

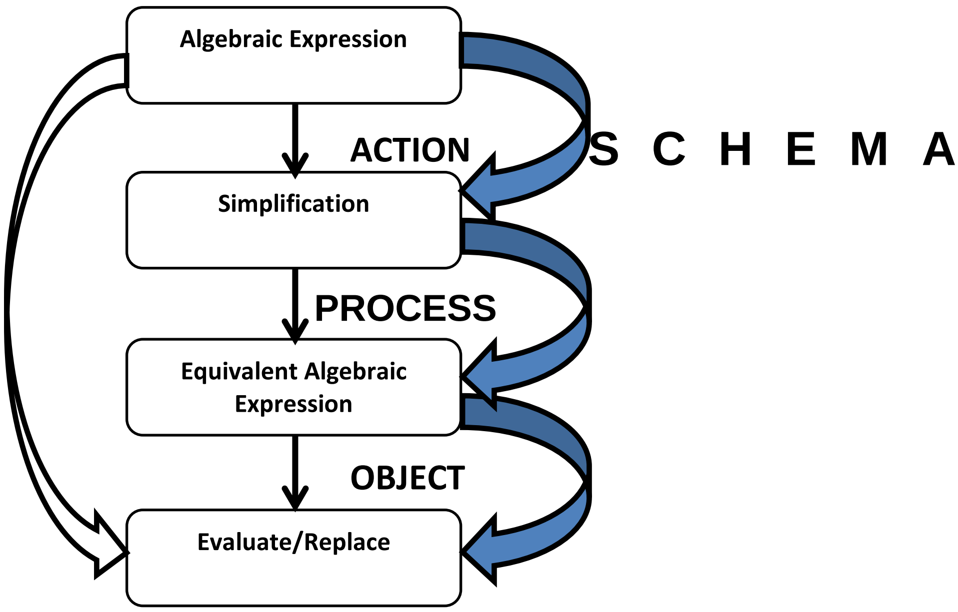

Figure 3: A summary to utilize equivalent algebraic expressions

For a learner to have a schema for equivalent expressions, he/she should be able to simplify algebraic expressions, manipulate operations and resolution of brackets appropriately, and be able to identify and apply equivalence in other higher order expressions.

We conjecture that certain conceptual prerequisites would need to be in place before learners could be expected to start to use equivalence of algebraic expressions out of their own volition:

- Learners need to interpret (assign meaning to) algebraic expressions as descriptors of sequences of computations, e.g. to interpret $2(x + 3)$ as "add 3 to the given number and multiply the answer by 2".
- More fundamentally, learners need to have some concept of quantitative variables, and need to interpret letter symbols in algebraic expressions as placeholders for unknowns and as variables.
- Learners need to be aware that an algebraic expression may assume many different numerical values, depending on the numerical values assigned to the independent variable(s).
- Learners need to be aware of the mathematical phenomenon that (apparently) different algebraic expressions such as $2x\ (6x + 5) - 3x(4x - 5)$ and $25x$ may consistently produce the same answers.

The study reported here focused on grade 8 learners' conceptual understanding of algebraic expressions. In our view conceptual understanding of algebraic expressions entails:

- Being able to simplify an algebraic expression in order to create an equivalent expression
- Appreciating what is achieved by simplifying an expression
- Appreciating that once you have created an equivalent expression you can replace with the other
- Being able to evaluate an expression by substituting values into it

Fundamentally as a prelude to understanding or appreciating the equivalence of expressions (algebraic or arithmetic), we can conjecture that learners must have a sound understanding of the purpose of simplification. As a result of complexity that is created by the presence of literal symbols particularly in an algebraic expression, the ability to simplify algebraic expression of various sorts, whenever it is possible, becomes the important algebraic (arithmetic) skill. Simplification is something that is done a lot in mathematics in general, and depending on the features of the expression (algebraic or arithmetic) with which you are dealing with, then we can say: one expression is simpler than another if it has fewer terms, or if its parts have fewer terms (generalizing, fractions in particular). With these in learners' minds we believe that the attainment of the habit of replacing autonomously will become practical and reachable.

Methodology of the reported research

The research reported is a single qualitative case study. A qualitative approach was used with the intention of understanding the contexts in which the participants operated (Creswell, 2007). This was a case study because of its particularity and complexity as a single case, used to understand its activity within important circumstances. The study aimed to explore learners' conceptual understanding of algebraic expressions. The research question addressed was:

- How do learners' construct their understanding of manipulation of algebraic expressions?

The research participants were 74 grade 8 learners from two of fifteen purposely selected secondary schools in a district of the Eastern Cape in South Africa. Learners wrote a preliminary test in a form of an activity sheet on the understanding of algebraic expressions for thirty minutes after they were taught algebraic expressions following the proposed genetic decomposition. Semi-structured interviews were then conducted with six participants on the basis of their responses. Figure 2 shows the activities that learners wrote for data collection.

Name & Surname:____

School:____

Answer the following questions on the answer sheet. DO NOT USE A CALCULATOR.

1. Simplify the expression $2x(5x+6)-5x(2x+1)+3x$
2. What is the value of $2x(5x+6)-5x(2x+1)+3x$ if$x=24{,}3$?
3. What is the value of $2x(5x+6)-5x(2x+1)+3x$ if $x=7{,}6$?
4. For what value of *x* is $2x(5x+6)-5x(2x+1)+3x$ equal to 18,5?
5. For what value of *x* is $2x(5x+6)-5x(2x+1)+3x$ equal to 60?

Figure 4: Learner activity

Findings and discussions

The researcher did not mark learners' written responses for right or wrong answers, rather mental constructions relevant to the simplification of algebraic expressions to determine the level of conceptual understanding at which the learners operated according to the given responses were studied. As stated in the proposed genetic decomposition of the conceptual understanding of algebraic expressions, the learners were expected to show evidence of building the following mental constructions:

1. Simplify the expression: $2x(5x+6)-5x(2x+1)+3x$. Learners were expected to know that by simplifying an algebraic expression, "rules" of mathematics are used to draw together different parts of an expression. An algebraic expression is in simplest form when it has no like terms.

$$
\begin{aligned}
& 2x(5x+6)-5x(2x+1)+3x \\
= \quad & 10x^2+12x-10x^2-5x+3x \\
= \quad & 10x
\end{aligned}
$$

2. Expect the learner to replace $2x(5x+6)-5x(2x+1)+3x$ with $10x$ and compute $10\times 24{,}3=243$

3. Expect the learner to replace $2x(5x+6)-5x(2x+1)+3x$ with $10x$ and compute $10\times 7{,}6=76$

4. Expect the learner to replace $2x(5x+6)-5x(2x+1)+3x$ with $10x$ and solve $10x=18{,}5$ to find $x=1{,}85$

5. Expect the learner to replace $2x(5x+6)-5x(2x+1)+3x$ with $10x$ and solve $10x=60$ to find $x=6$

Some of these attributes were not prevalently reflected in the learners' responses. For example, learner 1's response revealed that the learner could be functioning at action level, but this learner provided two responses as shown in figure 3.

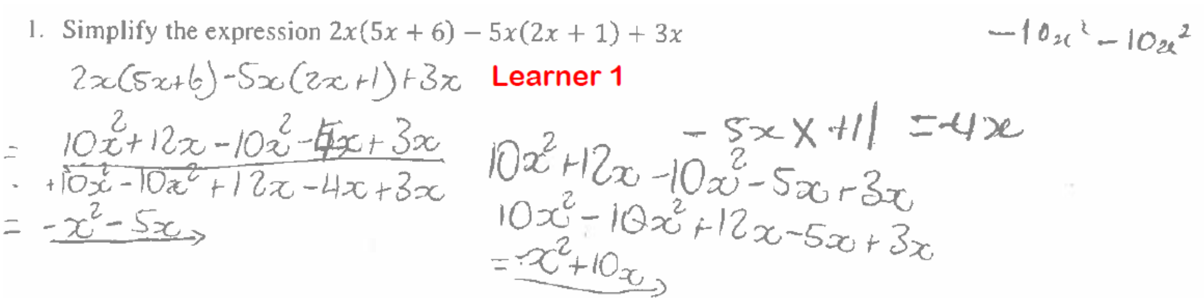

Figure 5: Example of a learner response

During interviews, when Learner 1 was asked on why she gave two responses, she said:
Learner 1: *bendingekho sure, phaya ecaleni* (I was not sure, that is why I did a calculation by the other side*), I multiplied (-5x + 1) and I got 4x*
Researcher: How did you do that?
*Learner 1: Jaa, ngoku ndabona uba I made a mistake, then ndaphinda ndabala kwakhona (*Yes, now I see that I made a mistake, I then tried to redo my calculations). *But kuba bendingekho sure (*but because I was not sure*) ndaziyeka zombini (*I submitted both versions*)*
This reflects the uncertainty of the relevant mental constructions made by the learner. Initially the learner multiplied incorrectly -5x×1=-4x. In the second attempt this has been corrected to yield -5x. This implies that even though actions on multiplication of terms and removal of brackets was done externally out of learner's mind, transformations through processing of the information were engaged in, hence the second response. This indicates that the manipulations necessary for processing the given algebraic expression went beyond the action level and were encapsulated to the process level.

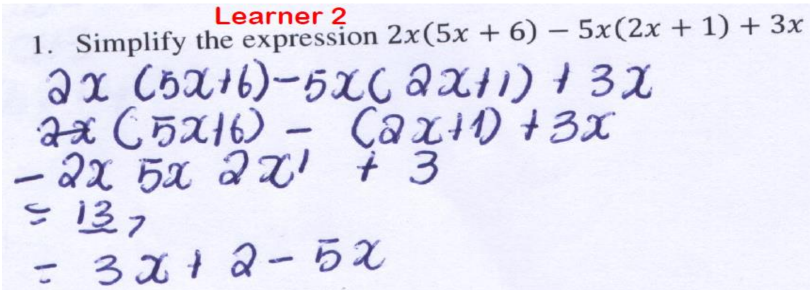

Figure 6: Example of a learner response

Learner 2's response revealed that the learner could be functioning at action level. But the response as shown in figure 4 clearly indicates the learner's inability to simplify the given algebraic expression. This implies that the learner did not make relevant mental constructions; as a result, no proper transformations took place. The calculations presented do not reflect any construction of mental processes but the response is as if the learner does not know what to do with the given problem. No mental constructions were formed. The learner is therefore operating at a level even below action stage.
During interviews, Learner 2 revealed the fascination he had when he saw the problem.
Researcher: Why did you cancel the 2x outside the bracket in step 2?
Learner 2: *Bendingayazi uba itheni ingekho phakathi kwi brackets.* (I did not know why it was not inside the brackets.
Researcher: What about the 5x, and all the other operations in the other steps?
Learner 2: *Mh………, bendingazi uba mandithini* (I did not know what to do)
This clearly shows that the learner had no clue on how to tackle the problem, nor was he aware of the role played by a bracket in an algebraic expression.

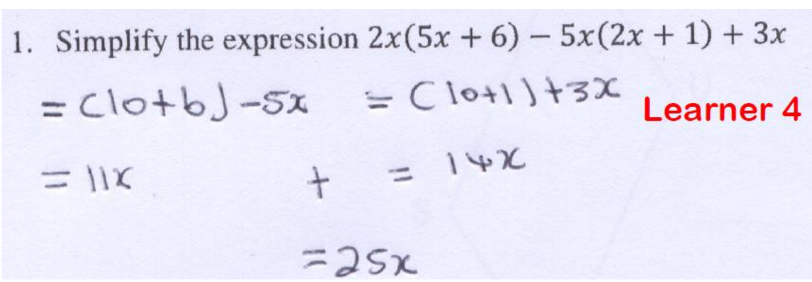

Figure 7: Example of leaner response

The same applies to the response displayed in figure 5.
When learner 4 was asked why he left out the variable x in the second step, he said:
Learner 4: *Ngaphandle ko x, nditsho ndikwazi uku add a 10 + 6 ndifumane u 16, xa ndi minus a u 5, I get 11 and then ndibuyise u x. (*Without x, the problem becomes easier for me, I add 10 and 6 to get 16, then subtract 5 to get 11, after which I bring back x, then it becomes 11x
Researcher: Did you do the same on the right hand side?
Learner 4: *Yes, 5 x 2 gives me 10 and then add 3x*
Researcher: What about the +1 in the bracket?
Learner 4: *Hayi ke…………, u 5 is next to 2, akukho number next to +1.* U 1 ukude (Oh no, the 5 is next to the 2 in the bracket and there is no number next to 1. 1 is far away. This indicates clearly that this learner also has no clue on how brackets should be removed in algebraic expressions.

Conclusions

Results revealed that about 93% of the learners could not even perform actions in the given expression. It looks like they were fascinated by the given expression, did not know what to do with it, they could not perform mental manipulation that transform objects by reacting to external cues that give precise details on what steps to take. They failed to perform the first operations with removing the brackets. This was clearly revealed during interviews. Evidence of the understanding of the associative and distributive properties was absent. There was no way they could move to a level of understanding equivalence in algebraic fractions. In relation to the proposed genetic decomposition of equivalent algebraic expressions, the learners only displayed knowledge of number sense and operations such that when they were required to work with algebra, they reduced given expressions to just numbers.

Only 2% showed evidence of a mental construction as an action that takes place entirely in the mind, which could be interiorised to a process on understanding algebraic expressions. These students went through the simplifications of the expressions through removal of brackets. However, these students could not interpret equivalence of fractions. During interviews they could not explain the meaning of the equal sign relationship in the given expressions.

It was evident from this study that most learners' struggle with manipulations to get solutions of algebraic expressions was from their lack of basic mathematical operations like (i) the meaning of brackets in an expression, (ii) order of operations, (iii) identification and manipulations of like and unlike terms and (iv) substitution in general. These are the skills that should be nurtured for them to build mental constructs relevant to understanding algebraic expressions. It is recommended that teachers provide basic lessons on introduction of algebra such that learners can be able to build and connect their arithmetic knowledge, develop it and apply to algebraic manipulations

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