MATHEMATICS BEGINS WITH DIRECT HUMAN EXPERIENCE. AN APOS APPROACH TO CONCEPTUAL UNDERSTANDING OF A MATHEMATICAL CONCEPT

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Abstract

This paper reports on the APOS (Actions, Processes, Objects and Schema) approach to difficulties experienced by first year engineering students at a University of Technology in constructing the concept of the chain rule in differential calculus. A proposed initial genetic decomposition (IGD), describing the mental constructs which students are supposed to make in order to understand the chain rule was suggested. Instructional treatment followed the activities, classroom discussions and exercises (ACE) model proposed by Dubinsky (1991). This paper in particular presents the discussions on interviews with group representatives seeking clarity on responses to four exercises on differentiation of trigonometric functions done collaboratively in class on the understanding of the chain rule. In a class of 78, students worked collaboratively in 12 groups of about six participants each. The interviews in this study were conducted with selected individuals from the different groups for clarity and explanations on written responses. This was done to get feedback on how the students perceived the chain rule and to fulfil the verification purpose where the group response was clarified. These interviews were semi-structured and questions were open-ended. They followed a guide designed to elicit the students' understanding of the chain rule based on the tasks given. Analysis of results revealed to a greater extent a process understanding of the chain rule concept and when using the Triad (intra-, inter- and trans-) mechanism to explain the interview discussions, it was revealed that most students operated on inter- stage. Differentiation as a process was complete with most groups but they struggled with basic algebraic manipulations, understanding of composition and decomposition of functions.

APOS, genetic decomposition, trigonometric functions, chain rule, calculus and Triad mechanism

Introduction

The key mechanism for an individual to obtain new mathematical meaning is for him/her to construct mental representations of direct experiences relevant to that concept. A structured set of mental constructs which might describe how the concept can develop in the mind of an individual is called the **genetic decomposition** of that particular concept. Dubinsky (1991) proposed that the mental constructions that the learner might make include Actions, Processes, Objects and Schemas (APOS). The initial genetic decomposition (IGD) of the concept of the chain rule suggested below guided the researcher's teaching instruction in class and the construction of the interview and discussion tasks. APOS ascertains that to understand a mathematical concept begins with manipulating previously constructed mental or physical objects in the learner's mind to form **actions**; actions would then be interiorised to form

processes which are then encapsulated to form **objects** (Dubinsky, 1991). These objects could be de-encapsulated back to the processes from which they are formed, which would be finally organized in **schemas**. Understanding the chain rule was explored in relation to the schema relevant to it. For an elaboration of these concepts refer to Maharaj (2010, p43).

The chain rule states that, if a function *g*, is differentiable at *a* and *f* is differentiable at *g(a)*, then $h = fog$ is a differentiable function at *a*, so that $h^{'}(a) = (fog)'(a) = f^{'}(g(a)).g^{'}(a)$. Hassani (1998) examined students' understanding on graphical, numerical (tabular) and algebraic/ symbolic presentations of composition of functions and the chain rule. His study revealed that first-year undergraduate calculus students have a very meagre understanding of the composition of functions and their ability to explain or apply the chain rule is significantly related to their algebraic manipulative skills and their general knowledge of function concepts and function composition. Also Swanson (2006) asserted that the complexity of the chain rule deserves exploration because students struggle to understand it and because of its importance in the calculus curriculum. Despite the importance of the chain rule in calculus and its difficulty for students, the chain rule has been scarcely studied in mathematics educational research (Clark et al, 1997; Gordon, 2005; Uygur & Ozdas, 2007; Webster, 1978). These students' difficulties include the inability to apply the chain rule to functions and also with composing and decomposing functions (Clark et al, 1997; Cottrill, 1999; Hassani, 1998).

The researcher's observations on students' performance in calculus using the chain rule has also revealed a difficulty in understanding and applying the concept as compared to other sections (exponential and logarithmic functions, trigonometry and complex numbers) in first year engineering mathematics. This led to the researcher's interest of how learners conceptualized the chain rule and how they could learn this concept effectively. The article therefore reports on how students can learn and apply the chain rule and thus inform the teaching of the concept.

Students were provided with group activities in class that were designed to induce them to make the suitable mental constructions as suggested by the initial genetic decomposition (IGD). The tasks used in this study helped students gain experience in constructing actions corresponding to the chain rule.

Lastly, the students were then provided with complicated activities where they needed to organize a variety of previously constructed objects, like functions and derivatives of composition of functions, into a schema that could be applied to chain rule problem situations. More specifically the researcher examined students' attempts to answer the tasks given in class, their tests, and exercises with regard to their understanding of functions, composition of functions and the chain rule.

In the studies done on APOS using computers by (Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., St. John, D., Tolias, T., & Vidakovic, D., 1997), it was shown that the students were in a better position to make mental constructions using computers when finding graphically the limits of certain functions. The researcher did not use computers and was not sure about certain mental constructions not being constructed. Written responses were examined for errors made and interviews were conducted with selected candidates in relation to interesting responses displayed in the questionnaires. It is hoped that this engagement would fill in doubtful gaps of such a study.

Theoretical Analysis

Piaget's work concentrated on the development of mathematics knowledge with young children, and rarely going beyond adolescence. Also, pedagogical strategies are almost absent from Piaget's work (Asiala, 2004). The researcher felt that the emphasis on exploring the use of chain rule with trigonometric functions and verbal representations of calculus concepts can be fostered through reflective abstractions following Dubinsky's (1991b) model of conceptual understanding. Dubinsky believes that the concept of reflective abstraction that was introduced by Piaget (Berth & Piaget, 1966) can be a powerful tool to describe the development of advanced mathematical thinking and that it could be used to analyse any mathematical knowledge applicable to higher education. Therefore, the researcher adopted the APOS approach (Dubinky, 1991a), based on intuitive appeal as there has been little empirical research done documenting the use of it on students' conceptual understanding of the chain rule in the African continent. APOS has been used in research focusing on understanding of various mathematical concepts, (Pascual, 2004; Sfard, 1991; Tall, 1994; Dubinsky, 1991a; De Vries, 2001; Gray & Tall, 2002; Clark et al, 1997).

This study was conducted according to a specific framework for research and curriculum development in advanced mathematics education, which guided the systematic enquiry of how students acquire mathematical knowledge and what instructional interventions contribute to student learning. The framework consists of three components: **theoretical analysis, instructional treatment,** and **observations and assessment** of student learning as proposed by Asiala et al (2004).

It is in the theoretical analysis of the chain rule where the researcher presented an initial genetic decomposition (IGD) describing specific mental constructions which a learner should make in order to develop his/her conceptual understanding of the chain rule. These are modelled in the following diagram (Jojo, 2012).

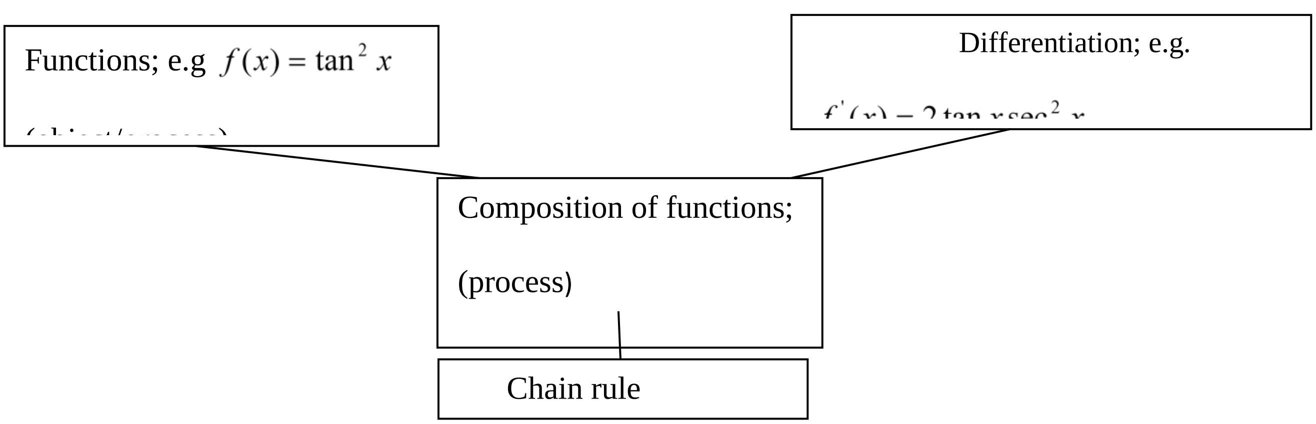


Figure 1: Initial genetic decomposition of the chain rule

The IGD for understanding the chain rule proposed that understanding could begin with: (i) students understanding the composition of two or more functions, (ii) transformation of the functions to one composite function, (iii) understanding the derivative concept of the composite functions, and (iv) coordination of the two processes to obtain this derivative which would then be encapsulated to using the chain rule for differentiation.

Construction of knowledge in this study was analysed through reflective abstraction at the heart of which is APOS (Dubinsky, 1991b) which then incorporates Piaget's **Triad** mechanism. Reflective abstraction has two components namely a projection of existing knowledge onto a higher plane of thought and the reorganization of existing knowledge structures (Dubinsky, 1991a). Reflective abstraction is therefore a process of construction of knowledge, and Dubinsky outlines five kinds of construction in reflective abstraction: *Interiorisation*, *Co-ordination*, *Encapsulation*, *Generalisation* and *Reversal*. For full explanation of these concepts in chain rule concept knowledge construction, see Jojo , Brijlall and Maharaj, (2013).

The Triad mechanism occurring in three stages explained other constructions in the mind implicating mental representations and transformations in the analysis of schema formations. These stages are the Intra- stage which focuses on 'a single entity', followed by the Inter- stage which is 'study of transformations between objects' and the Trans- stage noted as 'schema development connecting actions, processes and objects' (Jojo et al.; (2013)). Tall (1996) argues that there is no transformation that takes place when a child compares the sizes of objects. Zingiswa, Brijlall and Maharaj, (2005) assert that the idea of transformation is a tool the teacher can use, a tool that does not dehumanise learning because it corresponds to one aspect of the voluntary activities of the student's mind. Jojo (2011) used the flow diagram (see Figure 2) to explain the activities involved in construction of the chain rule concept and to illustrate APOS extended.

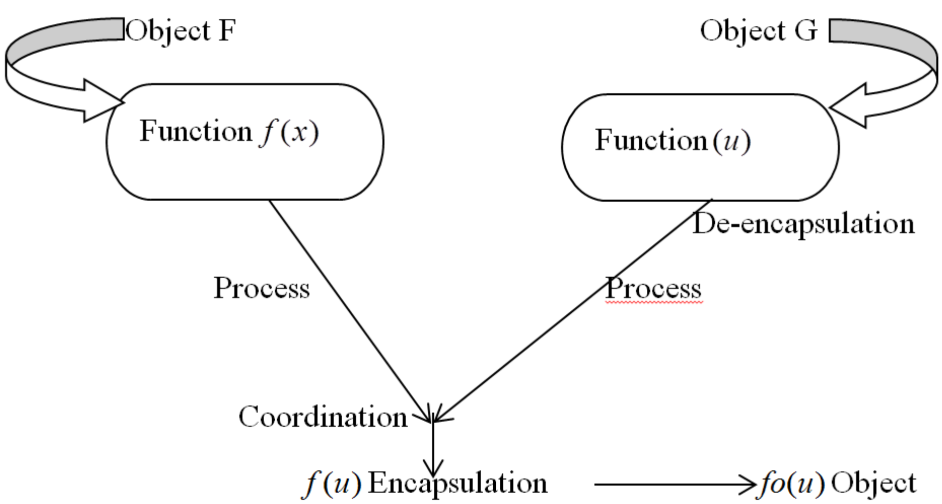


Figure 2: Illustration of the composition of functions

The overview of the model is that one begins with two functions F and G and transforms them into a single function, F o *u*. The transformation begins by de-encapsulation of F and *u* back to the process F(*x*) and *u* ie u from which it came. The two processes are then coordinated to obtain the process *x* on F(*u*), which is then encapsulated to the object F o *u*.

The tricky part is in identifying the function u as a single entity in $f(x)$. The *x* in $f(x)$ is not *x* but another function in *x*. Thus the derivative $u^{'}$ has to be evaluated first and then multiplied by the derivative, $f'(u)$ to get $f'(u).u'$. To avoid the confusion of the two functions in *x*, one may represent the function *g(x)* as *u*. This will then enable one to find separately the derivatives $u^{'}$ and $f'(u)$. Result would then be found when one multiplies $u' \times f'(u)$. This would safe guard against two errors which the students usually make,

(1) Finding $f'(x) \times u'$, for example the derivative, $\cos(3x^2+4x-5) \neq -\sin x.(6x+4)$ but is $-\sin(3x^2+4x-5).(6x+4)$ and (2) Finding $f'(u')$ where one derivative is plugged onto the other one. For example the derivative of $\tan(3x^4) \neq \sec^2 12x^3$ but is $\sec^2(3x^4).12x^3$. Clear connections to the chain rule are made as the student collects all the objects formed as derivatives and multiplies the results to find the derivative of the composite function. The correct actions in total, applied and reversed with different equations, indicates acquisition of the constructed **schema** of the chain rule by the student modelled in Figure 3.

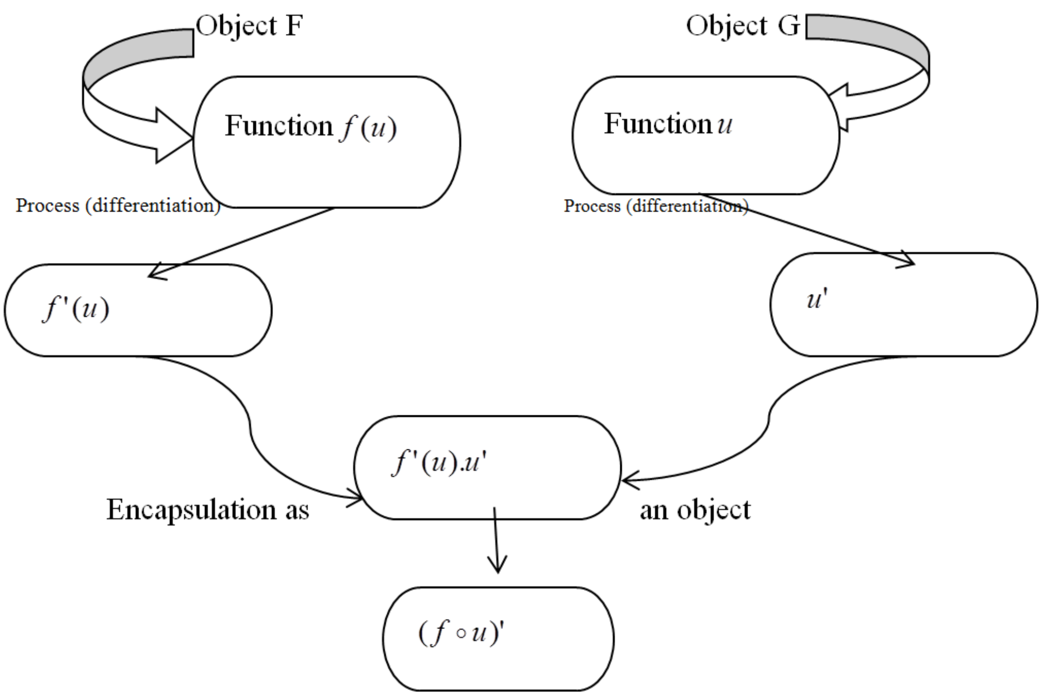


Figure 2: A model of the chain rule

Students were provided with activities in class that were designed to induce them to make the suitable mental constructions suggested in an initial genetic decomposition.

The students interiorised their actions by discussing their actions with others collaboratively. They also had to write verbal descriptions of their actions using their own words. Mental constructions that a learner makes include actions, processes, objects and schema of the chain rule. Within the APOS context, **actions** made externally in identifying a composition of functions should first be interiorised and reflected upon identifying functions within functions. A student should then **process** and transform these functions such that they are encapsulated in totality to form **objects** through differentiation.

Aims and objectives

The study aims at exploring how students:

(1) develop a process or object conception of a function
(2) develop a process or object conception of a composition of functions
(3) Can use the previously constructed schemas of functions, composition of functions and derivative to define the chain rule

Literature Review

Duffin and Simpson (2000) identified and named three components of understanding as (1) the building, (2) the having, and (3) the enacting. They defined 'building understanding' as the formation of connections between internal mental structures. 'Having understanding' is said to be the state of these connections at any particular time and 'enacting understanding' as the use of the connections available at a particular moment to solve a problem or construct a response to a question. Thus this is the type of understanding that may be visible from students' work when responding to mathematical tasks. Duffin and Simpson also talked about the breadth and depth of understanding. They described the breadth of understanding to be determined by the number of different possible starting points that the learner may have in solving a problem. The depth may be evidenced by the way the learners could unpack each stage of their solution in more detail by referring to more concepts.

Imagine a situation where one is given a function say, $f(x) = (x^3 + 5x)^2$ to differentiate. A learner may identify this function as being represented structurally as $(x^3 + 5x)(x^3 + 5x)$, and may expand the expression before differentiating. This learner might then use the product rule to differentiate the resulting expression. Another learner who recognizes the function in the form of $f(u) = (x^3 + 5x)^2$ where $f(x) = x^2$ and $u = x^3 + 5x$, may use the chain rule to differentiate $f(x)$. A learner who sees this function to be represented structurally in one form only lacks breadth of understanding. For example, a learner could indicate the stages at which the power rule, the product rule or the chain rule have been applied, that is, alongside the work shown in performing the mathematical task. This demonstrates a deeper understanding of solving the task than in a case where the structures will be manipulated by applying a rule with no explanations at all. A learner who instrumentally carries out manipulations is likely to be unaware of the mistakes he or she has made.

On the contrary, the understanding of a mathematical concept is explained in this article with the help and adoption of APOS. It can be also agreed that mathematical ideas begin with human activity and then proceed to be abstract concepts (Dubinsky & McDonald, 2001). It is therefore important to understand how the construction of concepts in the mind, lead to abstraction of mathematical knowledge. This interpretation of the relevant knowledge construction processes is essential since it points to the contributions we get from APOS analysis. These include understanding the importance of human thought, (mental constructions relating to the concept), and pointing to effective pedagogy for a particular concept.
This article *reports on research that was conducted into students' construction of an underlying structure of the chain rule.* This focus was accomplished by means of:

(1) a discussion of the types of structures constructed by students when learning the chain rule with the view to clarifying their understanding of: (i) the composition of function and (ii) the derivative;

(2) finding out how the lack or availability of these structures hamper or assist students' understanding of the chain rule, and
(3) determining the students' actual engagement with tasks and how these tasks link with the expected outcomes highlighted in the initial genetic decomposition.

Participants, Instructional design Methodology

The study was qualitative in nature, targeting 78 first-year electrical engineering students at a university of technology students in KwaZulu-Natal in 2011. Data was collected in the form of **A**ctivities, **C**lassroom discussions and **E**xercises done out of class, (ACE). These were conducted in the form of tutorial tests on tasks pre-supposed to test students' understanding of (1) composition and decomposition of functions, (2) differentiation, and (3) use and applications of chain rule in differentiation of loaded trigonometric functions. These were administered during tutorial sessions held one Friday afternoon in a one hour slot as a pedagogical approach based on APOS. These tutorial lessons were held to reinforce learning and teaching done during the instruction of the calculus concepts in normal lectures, thus sampling was convenient. Students in these sessions were divided into six groups of +/- 30 groups each supervised by different tutors simultaneously in different classroom venues. It was not important whether the responses were correct or wrong, of more significance, was the procedure students used and the interactions they had with each other before they agreed on a particular response. Interviews were then conducted with five participants chosen on the basis of their responses to the written instruments.

The next section uses tables, written extracts, and interviews to describe the comprehension of students with regard to the use and application of the chain rule concept.

Analysis and Discussion

The worksheets were analysed for meaning which is one of the mechanisms necessary for understanding a concept. These included detecting the connections made by students to other concepts, calculations made using the chain rule, the chain rule technique used, and mental images on which the chain rule was based, and composition and decomposition of functions.

Discussion on task 1

Differentiate:	$y = \sin^3(4x)$

Table 1 summarises the question analysis of task 1 using the responses presented by the groups in this task.

Table 1: Analysis of task 1

	Incorrect responses	Partially correct responses	Completely correct responses	Chain rule preference	Connection to other concepts
No of groups	6	4	2	12	8

Interviews with the student whose work had been displayed on the extracts revealed their deeper understanding of the technique they used in differentiating. The other group that presented an incorrect response, left out the square sign after differentiating with respect to the power of the sin function. Their actions were not interiorized with regard to the derivative concept and this had an impact on applying the chain rule in the given task. Their mental images could not be related to the string of symbols forming the expression, since they could not interpret both the symbols and or manipulations. Since calculations reflect the active part of mental constructions, the differentiation rules for these students were not perceived as entities on which actions could be made. Dubinsky (2010) asserts that in such cases the difficulty does not depend on the nature of the formal expressions, but rather on the loss of the connections between the expressions and the situation instructions.

Discussion on Task 2

Differentiate:	$y = \ln \frac{e^{4x} \sin x}{x \tan x}$

Table 2 summarises the question analysis of task 2 using the responses presented by the groups in this task.

Table 2: Analysis of task 2

	Incorrect responses	Partially correct responses	Completely correct responses	Chain rule preference	Connection to other concepts
No of groups	6	3	3	12	8

Generally, one of two strategies were employed by students. The first form technique called for a specific connection between the applications of natural logarithms and differentiation. A coherent collection of the logarithmic rules and differentiation was displayed by16,7% of the groups. Those groups were operating in the Trans- stage since they reflected on the explicit structure of the chain rule and were also able to operate on the mental constructions which made up their collection. Those students presented responses showing internal processes for manipulating logarithmic objects. Their schema enabled them to understand, organize, deal with and make sense out of application of the product rule, quotient, logarithmic rules and the chain rule. The other three groups could not apply logarithmic rules correctly and as such could not process the differentiation of the given task. The interpretation of logarithms was then incorrect since a bracket was left out in step three of the response. Thus the function differentiated was not the originally given one. Those students did not recognize the relationships between application of natural logarithms and algebraic manipulations resulting in multiplications when they were due and subtractions where appropriate. They perceived differentiation as a separate entity and even the rules applied were not remembered correctly. These were operating in the Intra- stage of the Triad.

Discussion on task 3

Differentiate: $y = \ln(x^2 \cos x)$

Table 3 summarizes the question analysis of task 3 using the responses presented by the groups in this task.

Table 3: Analysis of task 3

	Incorrect responses	Partially correct responses	Completely correct responses	Chain rule preference	Connection to other concepts
No of groups	2	2	8	12	12

Some students misrepresented the derivative of $\cos x$ *as* $\sin x$, they left out the negative sign. Those students just differentiated as an action not taking care and without constructing a meaning into it. For them it was just a matter of using rules and knowing that the derivative of this function is just that. No processes were coordinated. Those students were operating in the action stage since they saw the given function as a formula and the errors of signs left out where they should have been, meant that those actions were not interiorized to processes. According to the Triad, they were operating in the Intra- stage since the students had a collection of rules for finding derivatives of functions in various situations, but did not recognise the relationships between them.

Discussion on Task 4

Differentiate: $y = \cot^3(x^5 + e^{\sqrt{x^2+3}})$

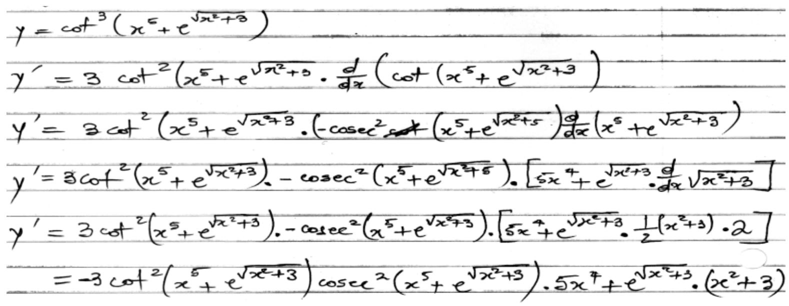
Table 2 summarizes the question analysis of task 4 using the responses presented by the groups in this task.

Table 2: Analysis of task 4

	Incorrect responses	Partially correct responses	Completely correct responses	Chain rule preference in differentiation	Connection to other concepts
No of groups	6	4	2	12	12

All the groups applied the chain rule to the first task incorrectly differentiating in a straight form from left to right. It was only 16,7% of the groups that presented completely correct responses. One of the groups who left out the bracket then went on to detach the derivative $5x^4$ of x^5 from the + sign. These mistakes were not detected by any of the other members of the same group. Those students struggled with the connection of previously learnt algebraic skills such as the use of brackets where appropriate and the manipulation of algebraic terms in a function. The calculations presented after differentiating using the chain rule successfully were therefore not correct for 83,3% responses received. When one representative was interviewed and asked to state the chain rule, he wrote: ‘I just differentiate from left to right and multiply.’

The actions explained cannot guarantee the correct application of the chain rule especially when the task calls for use of logarithms. Such students are said to operate in the action stage of APOS with their actions on algebraic manipulations not interiorised and as such could not be applied with other concepts.



$y = \cot^3 (x^5 + e^{\sqrt{x^2+3}})$

$y' = 3 \cot^2 (x^5 + e^{\sqrt{x^2+3}} \cdot \frac{d}{dx} (\cot (x^5 + e^{\sqrt{x^2+3}})$

$y' = 3 \cot^2 (x^5 + e^{\sqrt{x^2+3}} \cdot (-\mathrm{cosec}^2 (x^5 + e^{\sqrt{x^2+5}}) \frac{d}{dx} (x^5 + e^{\sqrt{x^2+3}})$

$y' = 3 \cot^2 (x^5 + e^{\sqrt{x^2+3}}) \cdot -\mathrm{cosec}^2 (x^5 + e^{\sqrt{x^2+5}}) \cdot [5x^4 + e^{\sqrt{x^2+3}} \cdot \frac{d}{dx} \sqrt{x^2+3}]$

$y' = 3 \cot^2 (x^5 + e^{\sqrt{x^2+3}}) \cdot -\mathrm{cosec}^2 (x^5 + e^{\sqrt{x^2+3}}) \cdot [5x^4 + e^{\sqrt{x^2+3}} \cdot \frac{1}{2} (x^2+3) \cdot 2]$

$= -3 \cot^2 (x^5 + e^{\sqrt{x^2+3}}) \mathrm{cosec}^2 (x^5 + e^{\sqrt{x^2+3}}) \cdot 5x^4 + e^{\sqrt{x^2+3}} \cdot (x^2+3)$

Extract 1 Group response to task 4

In step two of their solution, they left out the second bracket but recaptured it back in the following steps. This was a common error made by other students regarding change in operations after differentiation. Also they presented the derivative of $\sqrt{x^2+3}$ as $\frac{1}{2}(x^2+3)$ instead of $\frac{1}{2}(x^2+3)^{-\frac{1}{2}}$. The latter error led them to multiply $\frac{1}{2} \times 2$ and got (x^2+3). Their solution is a guided notation using a form technique of chain rule application and was used with caution initially. They demonstrated mental construction of the chain rule as an object. At this point in his (group representative) development he displayed the ability to reverse certain processes. He could trace back the steps of which functions were already and still to be differentiated. According to the Triad, they operated in the Inter- stage with regard to the chain rule application since they displayed construction of the underlying structure of the chain rule as an object through reflection on relationships between various processes from previous stages. This was done by putting $\frac{d}{dx}$ before functions still to be differentiated. The two errors are not associated with chain rule applications but indicate that they did not verify their response. They seemed excited and sure about chain rule application.

Conclusion

The initial genetic decomposition illustrated in figure 1, was suggested as a way of organising hypotheses about how learning the chain rule concept could take place. Instructional design was aimed at assuring that the students make the desired mental constructions. The structures constructed by students when learning the chain rule in the tasks given to the groups indicated that students had schema of the derivative even though they had not developed understanding of the composition of functions. It was observed that in some groups, they would first work the problems as individuals. They would then compare their responses and arguments would ensue as each one defends his/her response whilst the others convince each other on how they presented their solutions. This argument would subside as they reach agreements of different ways of approach of the same task displaying acceptable responses. In this process students learn from each other, explain procedures followed in doing the tasks and an agreed upon response was then presented as a response to the task for the group. Working collaboratively, they taught, strengthened, learnt, and reinforced different skills in learning the chain rule concept.

Although they displayed little understanding of the composition of functions, they still showed evidence of operating in the Inter- stage regarding the derivative. This disputes findings presented by Cotrill, (1999). He found that understanding composition and decomposition of functions is a pre-requisite to understanding the chain rule. The extracts chosen reveal that students using the Leibniz technique in eighty per cent degree presented correct responses to each of the tasks. Students interviewed indicated that when they use this technique, it always show the remaining functions still to be differentiated until finding derivatives for all functions represented in the composite function. The error indicated in extract 2 indicates that students have to have a schema understanding of the derivative to be able to do and apply the chain rule. They might operate on the Intra- level where the student had a collection of rules for finding derivatives of functions in various situations, but has no recognition of the relationships between them. They saw the various rules for differentiation as not related. These were students who were skilled at algebraic manipulations, easily able to assimilate rules and procedures in a cognitive structure that consisted of a list of unconnected actions, processes and objects to produce correct answers.

I suggest a modification of the mental constructions proposed in the IGD. The modified genetic decomposition should include over and above what is noted in the IGD, the knowledge of basic algebraic manipulations. These would then caution the students on the use and application of the chain rule. Further research on exploration of APOS on mental constructions formed when working with integrals. APOS explores the mental constructions made in learning many other concepts in the form of proposing genetic decomposition, nonetheless, the chain rule is still the major tool used in differential and integral calculus. It must therefore be taught to and with full understanding.

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