



ERRORS DISPLAYED BY LEARNERS IN THE LEARNING OF GRADE 11 GEOMETRY

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Abstract - Geometry is one of the topics that a number of studies have identified as problematic to learners despite the fact that it is also the most practical useable form of mathematics at all grades. The aim of this study was to establish the kind of errors learners display when responding to specific Euclidean Geometry questions. It used the van Hiele levels of geometrical thought as a theoretical framework to understand and gauge learners' knowledge of Geometry. This research was conducted in grade 11 at a high school in Bushbuckridge (Bohlabela district, Manyeleti Circuit) in Mpumalanga Province, in South Africa. The study used data from 30 randomly selected learners out of a total of 265. A Geometry grade 11 test served as an instrument. Content analysis of learners written responses was used to extract data from learners' scripts. The data was then coded with predetermined categories. This study found that learners misapplied rules, demonstrated signs of weak conceptual knowledge in Geometry and had very weak problem-solving skills. This weakness spilled into their inability to solve proof problems.

1. INTRODUCTION AND BACKGROUND

This study was conceived as an investigation into the kind of errors learners display when responding to specific grade 11 Euclidean Geometry questions. This is because Geometry has been identified as one of the topics where most learners have most errors (Roux, 2003). Studies point to the fact that most learners at secondary schools in South Africa have weak knowledge of Geometry resulting in them exhibiting serious errors in Geometry questions in the school leaving examinations (Oberdorf & Taylor-Cox, 1999; Bowie, 2009; Roux, 2003; Kearsley, in ASSAF, 2009, Van der Sandt, 2007). It is hoped that teachers will be able use the outcomes of this study to intervene meaningfully in the learning of Geometry thereby assisting learners in achieving excellent mathematics learning and practices

In 2008 Geometry was made a voluntary section (paper3) of mathematics in South Africa and it almost went out of the curriculum because most schools were not teaching it (Department of Education, 2009; Van Putten et al, 2010). Many schools that did not have the required capacity to teach geometry decided to opt out. Learners did not take up Mathematics Paper 3 because of the added load and in order to improve their chances of passing mathematics with higher marks (Bowie, 2009). This disadvantaged mathematics learners because geometry is the basis for learners' success with further studies at tertiary level in the mathematical, engineering and health sciences (Kearsley, in ASSAF, 2009). However, Geometry has been re-introduced in 2012 as compulsory. The Department of Education's (DBE) Curriculum and Assessment Policy (CAPS) indicates that from 2012 in grade 10, 2013 in Grade 11 and 2014 in grade 12, Geometry became a compulsory examinable section of mathematics which is written as a section of Paper 2. Geometry forms 20% of the paper in Grade 10 and 26.7% in both Grades 11 and 12 (DBE, 2011).

The basic research question of the study is: What are the most common types of errors that grade 11 learners display when responding to Euclidian geometry tasks? The study goes on to also establish the sources of such errors and why learners do them. In the process of collating the errors learners make,



van Hiele's theory of geometric thought is then used to identify the learners' levels of understanding in relation to Geometry.

2. LITERATURE PERSPECTIVES

2.1 Theoretical framework

There is a long history of error analysis in mathematics education (Radatz, 1979) and teachers and researchers have long recognized its value. When analysing and diagnosing errors displayed by learners, an identification of the root cause of errors and how best they can be corrected in order to benefit both learners and teachers is made. According to Howell, Fox and Morehead (1993) error analysis is an assessment approach that allows the teacher to determine whether learners are making consistent mistakes when performing basic computations. By pinpointing the pattern of an individual learner's errors, the teacher can then directly teach the correct procedure for solving the problem (van der Sandt, 2007). Error analysis involves the evaluation of learners' errors to determine their cause (Luneta, 2013). When errors are diagnosed one establishes the learner's areas of weaknesses and identifies specific errors the learner is frequently making and ascertains why these errors are being made. It is important to know why learners are making certain errors and how to capitalize on the error to facilitate learning. It is critical to identify the cause of the errors because it points towards the difficulties learners are encountering during a learning experience (Luneta, 2008). Teaching becomes irrelevant if it does not address learners' errors.

2.2 Errors

Harper (2010) defines an error as a "deviation from accuracy or correctness" while Luneta (2008, p. 386) defines it as a simple symptom of the difficulties a student is encountering during a learning experience. Elbrink (2008) defines errors as "mistakes learners make when solving problems that may be caused by carelessness, misinterpretation of symbols or text; lack relevant experience or knowledge related to that mathematical topic, learning objective, concept; lack of awareness or inability to check the answer given; or the results of misconceptions". Synonyms of this word are blunder, slip, oversight, mistake, fault, transgression, trespass, and misdeed.

2.3 Sources of errors

Different sources of errors have been identified by various researchers in the learning of geometry, namely: faulty reasoning; prior knowledge; procedural and conceptual knowledge; faulty schema; educators; and content knowledge. The following subsections elaborate on such

2.3.1 Faulty reasoning

According to Michael (2001) errors are a result of conceptual or reasoning difficulties that hinder learners' mastery of any discipline. This is supported by Brodie (2010) who contends that it is a big problem for learners to reason and communicate mathematically when given word problems and construction of proofs

2.3.2 Prior knowledge

In mathematics classes, research shows that learners can enter the classroom holding misconceptions that have the strong potential to derail new learning (Chiu & Liu, 2004). Expressing similar views is Mullis et al in Luneta (2008) who argue that the issue of the "learner's background and the context within which learning takes place can be sources of misconception. According to Kilpatrick, Swafford & Findell (2001), errors usually originate in prior instruction as learners incorrectly generalize prior knowledge to grapple with new tasks.

2.3.3 Procedural and conceptual knowledge

Luneta (2013) in his book asserts that learners' errors in geometry are as a result of the procedural way that mathematics is taught. Luneta and Makonye (2010) affirm when they argue that errors result if learners fail to build procedures from conceptual knowledge. According to Marek, Cowan and Cavallo (2011), this results in the rote learning of mathematics where algorithms are learnt without connecting them to the underlying semantic information. This may be a source of misconception. The implication here is that learners' errors in geometry are as a result of both their learning as well as the way learners were taught.

2.3.4 Faulty schema

According to Olivier (1989) learning basically involves the interaction between a child's schemas and new ideas. Sometimes some new ideas may be so difficult to any available schema, that it is impossible to link it to any existing schema. This will make assimilation or accommodation impossible. Olivier (1989) then argued that in this case "the learner creates a new box and tries to memorize the idea".

2.3.5 Educators

According to Luneta (2013), the main cause of learners' errors and misconceptions are mathematics teachers. He identifies teaching habits and ineffective teaching approaches such as persistently teacher-centred approaches as some of the sources of learners' errors. Wrong questioning techniques, such as incomplete, ambiguous, or unnecessarily difficult questions can cause learners to make mistakes.

2.3.6 Content knowledge

Hill, Ball and Schilling (2008) define pedagogic content knowledge as knowledge about the purposes for teaching a given subject matter, knowledge about the order in which subject matter should be presented and knowledge about the instructional strategies useful for teaching content.

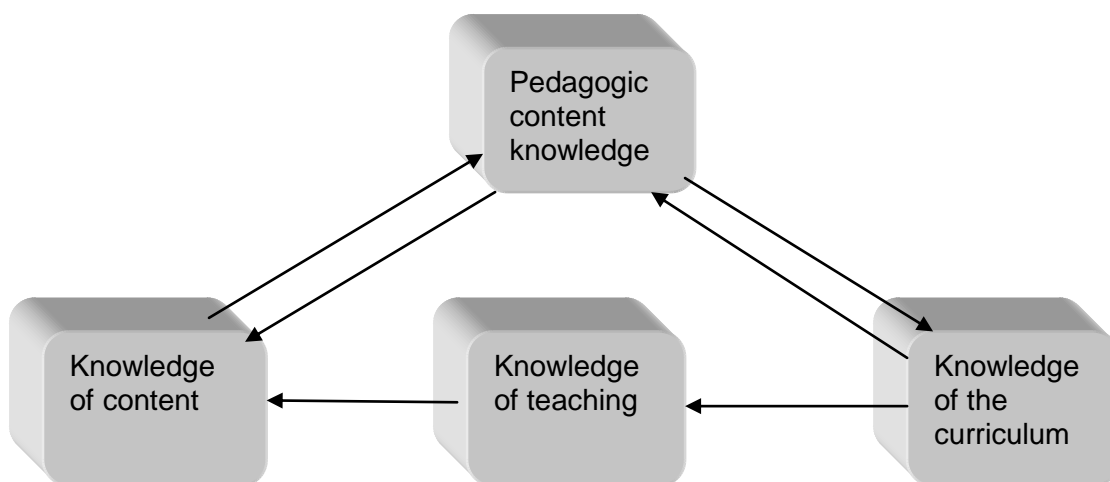


Figure 1: The network of pedagogical content knowledge (An, Kulm, & Wu, 2004)



In South Africa, the 2012 presentation by Linda Chisholm to the minister of Basic Education highlights inadequate content and pedagogical content knowledge (DBE, 2012). Teachers' lack of content knowledge results in teachers not explicitly explaining the concepts to the satisfaction of the learners.

2.4 The role played by errors

Several studies have been done that support the notion and point to the many positive roles errors can play in learners' conceptual understanding (Donovan & Bransford, 2005). According to Borasi (1994), mistakes made in the class are actually catalysts for the learning that took place and describes such errors as springboards for inquiry. Errors are seen as value sources of information about the learning process, providing clues that educators should take advantage of in order to uncover current learners' knowledge and how they come to construct such knowledge (Keith & Frese, 2008). Teaching should therefore change the perspective from condemning errors into one that embraces them. Fang (2010) states that errors are "not to be understood simply as a failure of learners, but as the symptoms of the nature of the conceptions which underlie learners' mathematical activity". When teachers teach learners that errors are the beginning of the learning process, that errors should be used as a stepping stones leading to success, teachers are much more likely to empower learners in error correction strategies (Brodie, Shalem, Sapire & Manson, 2010).

2.5 Handling of errors and implications for teaching

Teachers and learners should have the ability and skills to handle errors. The most important issue is how to engage with errors without blaming learners. According to Swan (2005), the best strategy of handling errors is to embrace errors as a point of contact with learners' thinking and as points of conversation, which can generate discussions about mathematical ideas. In this way learners' thinking and mathematical knowledge are brought into contact with each other. Teachers should therefore understand learners' errors, contemplate their causes and methodologically correct them. Making errors should be regarded as part of the process of learning. Errors are considered inevitable and should be used as an important source of information about the learning process. Teachers should regard errors as a clue for uncovering what learners already know and how they have constructed such knowledge. Teaching to avoid learners developing errors appears to be unhelpful and could result in errors being hidden from the educator and from learners themselves. With the widely recognized conceptual change framework, errors initially conceptualised negatively are now seen as a natural stage in knowledge construction and thus inevitable (Makonye, 2011).

2.6 The Role of van Hiele's Theory of Geometrical thought

This theory has five levels of geometrical thought (Musser, Burger & Peterson, 2011). Students progress and develop their knowledge of geometry through these levels (van Hiele, 1986, 1999). The 1st level (level 0) is called visualisation: here learners can identify a shape but are not able to provide its properties. The 2nd level (level 1) is called analysis: here learners are able to identify particular properties of shapes but not in logical order. The 3rd level (level 2) is called informal deduction: here learners can combine the shapes and its properties to provide a precise definition as well as relate the shape to other shapes (van de Sandt, 2007). The 4th level (level 3) is called deduction: here learners apply formal deductive arguments such as in proofs. The 5th level (level 4) is called rigor: it is characterized by "formal reasoning about mathematical systems" (Yee, 2006).

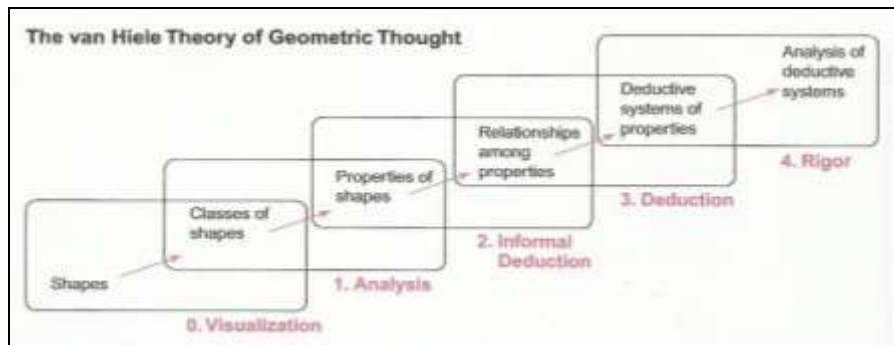


Figure 2: The van Hiele theory of geometric thought, (van De Walle, 2001)

As alluded earlier, this study then ropes in van Hiele’s theory of Geometrical thought. That is, upon the identifying of the errors learners make. Van Hiele’s theory then contributes to the understanding of why learners make certain errors by pointing out their level of Geometry understanding. Teachers and or facilitators can then intervene in the learning process using the identified geometric levels of understanding to design and appropriately select materials to correct learners’ errors and misconceptions.

3. RESEARCH METHODOLOGY

3.1 Research Design

This research study utilized a mixed-methods approach to investigate the errors displayed by learners in the learning of Grade 11 Geometry. Johnson and Onwuegbuzie (2004) define mixed methods research as “the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts, or language into a single study”. Mixed method research provides a better understanding of research problems than either approach alone.

3.3 Participants, sampling and data instruments

A random sampling technique was used to select 30 grade 11 mathematics learners from a population of 265 learners (Creswell, 2008). A Geometry Grade 11 test served as an instrument in this study together with interviews. We specifically selected scripts of learners who had challenges in geometry. These learners’ test scripts were then gleaned to identify errors displayed by learners. We then randomly selected the final 30 learners from this group. Interviews were also conducted to help identify the sources of these errors and to consolidate the data collected from learners’ scripts.

3.4 Data Collection

The writing of the Geometry test was followed by a content analysis of learners written responses in order to extract data from learners’ scripts. Interviews then followed with selected learners where further clarity was sought

3.5 Reliability and validity of data collected

The data used in this research is reliable and valid because the test that was used was a formal Geometry grade 11 paper 2 district common paper. This test was used for continuous assessment (CASS). This test also met examination guidelines as stipulated in the mathematics policy document and all the taxonomy levels were covered.

3.6 Ethical considerations

Participation was voluntary, consent was obtained and the anonymity of the participants was protected.

4. DATA ANALYSIS

For this study the research applied a sequential explanatory design. The sequential explanatory design incorporates two phases of data collection and analysis. It is conducted in a quantitative, then qualitative sequence (Creswell, 2008). In the first phase of the study, quantitative data were collected and analysed to provide a general understanding of the research problem (learners' test scripts). In the second phase of the study, qualitative data were collected and analysed to provide further explanation of the findings identified in the initial quantitative phase (interviews). The sequential flow of the quantitative and qualitative phases is shown in the diagram below:

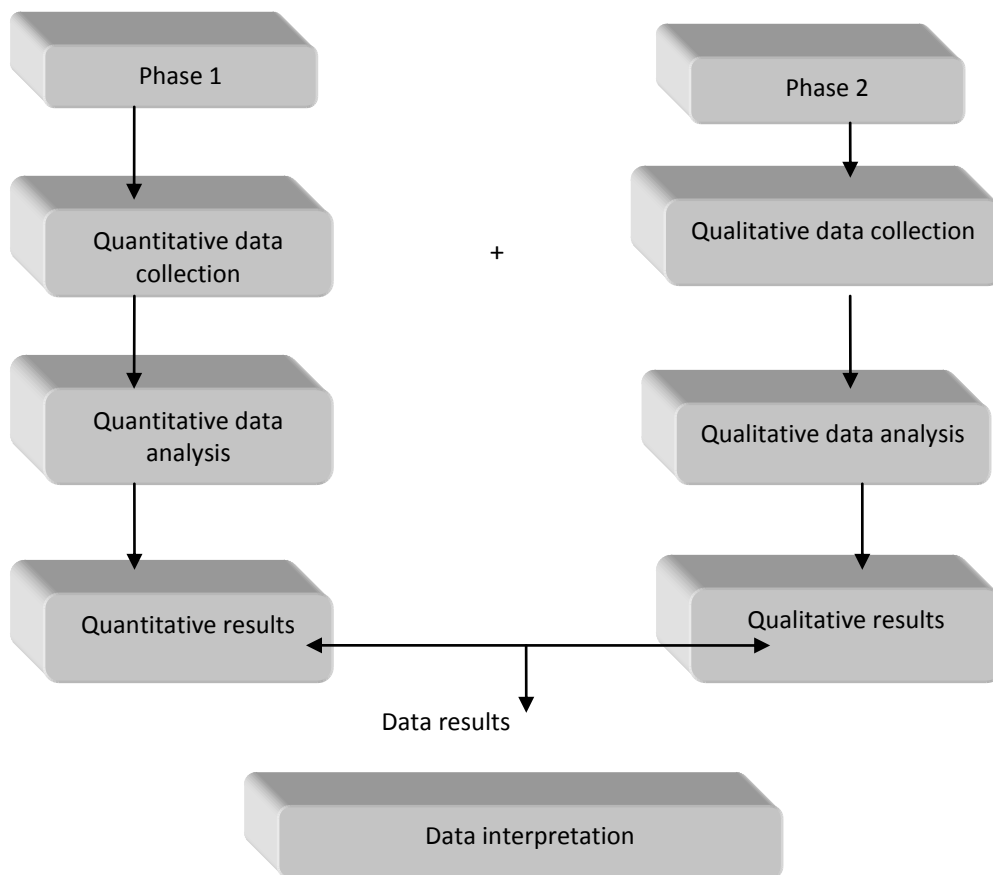


Figure 3: Data collection and analysis procedure

This design played an important role in the triangulation of data.

A geometry test was used as a first data collection instrument. This test was consisted of 5 questions. The 30 scripts of the sampled learners were marked by their teacher. The researchers then gleaned the scripts to identify the type of errors made by learners. These errors were coded in terms of themes and the van Hiele levels (see Tables 1, 2 and 3). The errors were then analysed inductively and deductively. Furthermore, these errors were coded with predetermined categories. Learners were coded from L1 to

L30. The L refers to a learner and the numbers 1 to 30 refers to the number of learners who participated in this research. Learners' test scripts were marked and their responses were categorized as C1 to C4 in terms of the developed statement and reasons.

Table 1: Classification of learners answers from their test scripts

| Category Number | Explanation |
|-----------------|--|
| C1 | Correct statement, correct reason. |
| C2 | Incorrect statement, incorrect answer. |
| C3 | Correct statement, incorrect reason |
| C4 | Incorrect statement, correct reason. |

Learners' responses were also pre-categorized in terms of the type of errors made, namely slips, conceptual errors and procedural errors.

Table 2: Classification of learners' errors as described in the literature

| Category | Type of error | Explanation |
|----------|------------------|--|
| Err1 | Slip | Blunder, minor error committed because the learner was in a hurry. |
| Err 2 | Conceptual error | Lack of knowledge of the concept caused by an insufficient mastery of basic facts, concepts and skills. |
| Err3 | Procedural error | Learners know the concept and properties of figures but cannot apply it to the problem. Blindly apply procedures without really knowing what's going on. |

Lastly, learners' responses were then categorized (deductively) in terms of van Hiele levels of Geometric thought (van Hiele, 1986).

Table 3: Van Hiele levels of geometric thought as described in this study

| Category number (Van Hiele Levels) | Explanation |
|------------------------------------|---|
| VHL0 | Learners can name and identify common geometric shapes. |
| VHL1 | learners can recognize a geometric shape based on its properties, but cannot recognize relationships between classes of figures |
| VHL2 | Learners identify class inclusion of shapes. |
| VHL3 | Learners can construct geometric proofs. |
| VHL4 | Learners understand the relations between geometrical concepts. |

Errors were then summarized per question per learner in a tabular form in terms of the above categories. For example, if L1 is coded C2, Err2, and VHL1 for his/her test, it would mean the learner has given an incorrect statement and an incorrect answer, has made a conceptual error (lack of the knowledge of a concept) and is showing a sign of operating at van Hiele 1.

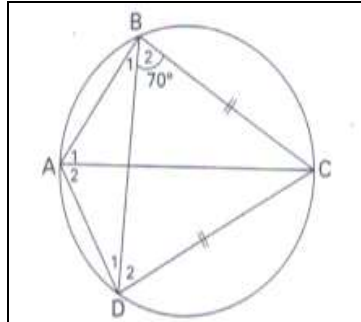
5. FINDINGS AND DISCUSSION

5.1 Question by question analysis to highlight errors made by some learners

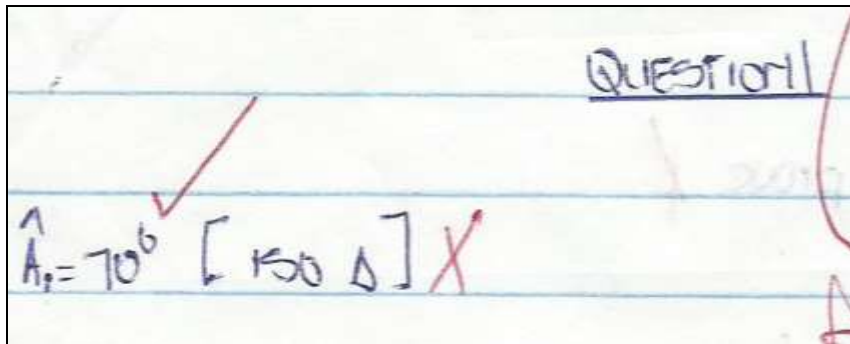
The test was based on cognitive levels 3 and 4 in terms of mathematics CAPS document (DBE, 2011).

QUESTION 1

Given: A, B, C and D are four points on the circumference of a circle, $BC=DC$ and $\angle B_2 = 70^\circ$.
Learners were required to prove that $\angle A_1 = \angle A_2$.



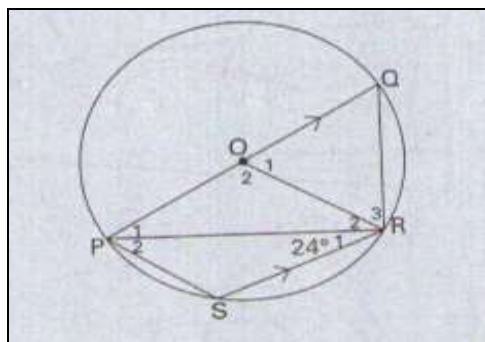
Two-third of the learners provided an incorrect proof giving wrong reasons as witnessed by L4 answer below:



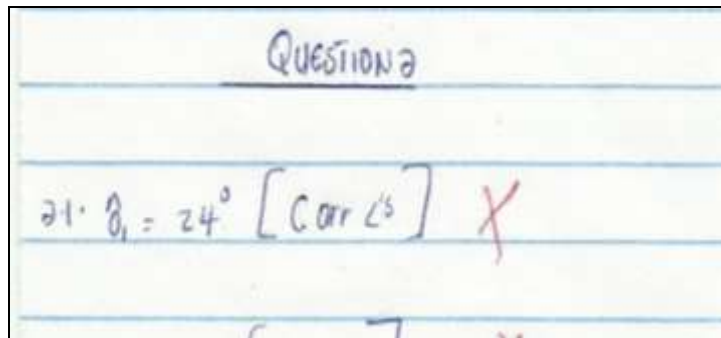
This clearly shows incompetence and lack of logic in terms of proof questions. In the figure, only $\angle B_2$ and $\angle D_2$ are isosceles angles.

Question 2

In the diagram, POQ is a diameter of the circle O. Chord SR is drawn parallel to PQ. OR and PR are drawn. $\angle R_1 = 24^\circ$. Learners were required to calculate the sizes of $\angle O_1$, $\angle Q$ and $\angle S$



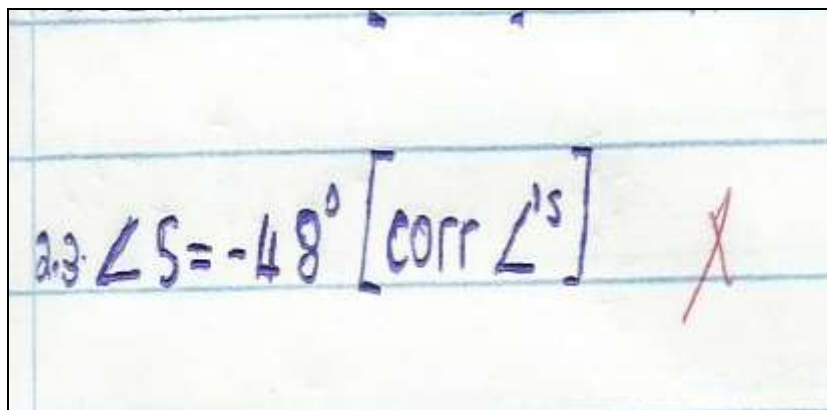
Responses for $\angle O_1$: 13 learners provided an incorrect proof. Most of them just attempted to give the value of $\angle O_1$, and not the required proof as evidenced by L7 response below:



This is a sign of learner showing lack of logic in solving geometry questions. The learner was guessing the answer and the reason after realizing that there are parallel lines. There is no corresponding relationship between $\angle O_1$ and $\angle R_1$, the given angle equal to 24° .

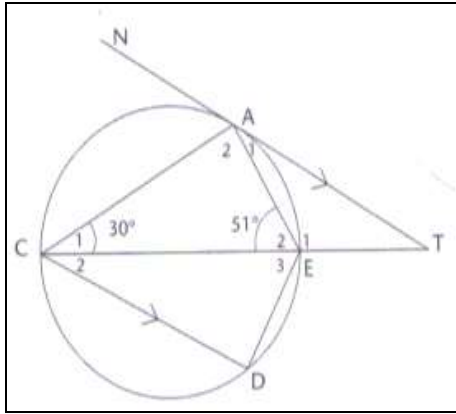
Responses to $\angle Q$: Half of the learners provided incorrect solutions. These learners made unjustified conclusions based on wrong reasons, and it seems they were guessing the final answer and reasons. These learners show lack of deductive reasoning in solving geometry questions.

Responses to $\angle S$: One third of the learners provided an incorrect solution. Learners who made mistakes gave reasons such as radii, corr \angle s, ext \angle s of a cyclic quad, which are all wrong because there is no angle related to $\angle S$ with these reasons.



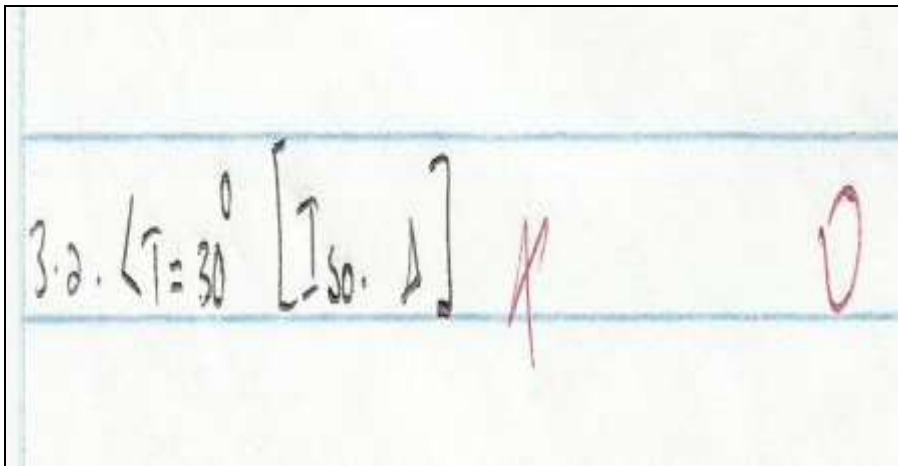
Question 3

Given TAN is a tangent to the circle at A and TN is \parallel to CD. CE is produced to meet the tangent at T. $\angle C_1 = 30^\circ$ and $\angle E_2 = 51^\circ$. Learners were required to calculate the sizes of $\angle A_1$, $\angle T$, $\angle C_2$ and $\angle D$.



Responses to $\angle A_1$: 28 learners provided the correct solution with correct reasons. Learners' performance on this question was therefore excellent.

Response to $\angle T$: 14 learners provided incorrect solutions and reasons. Amongst the incorrect reasons which were not related to the question were given, namely Iso triangle, Alt \angle s, ext triangle, ext angles of a triangle, co-int \angle s, were given.



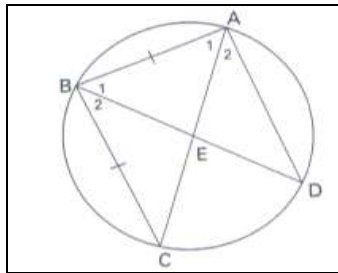
This is a sign of learners who were just guessing when giving reasons. They showed an inability of using properties in correct contexts.

Response to $\angle C_2$: Learners found this question challenging as evidenced by the number of learners who scored no marks. The main reason was caused by the fact that in order to get the correct answer, learners should have calculated the value of $\angle T$ correctly (using the sum of interior \angle s of $\triangle AET$). While 16 learners correctly calculated the value of $\angle T$ correctly, they failed to see its connection with $\angle C_2$. Some learners thought $\angle C_2$ and $\angle C_1$ are angles on the same segment. This is an error since the two angles are subtended by different arcs. They are also not equal because they are subtended by unequal chords.

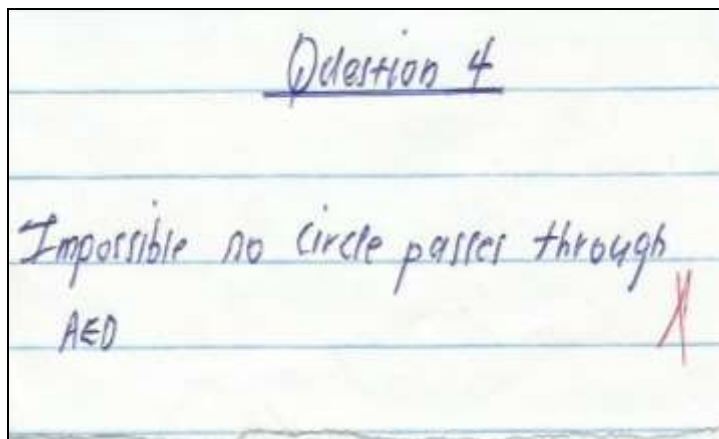
Response to $\angle D$: 9 learners provided full correct answer, 3 learners provided part of the answer whereas 17 learners provided incorrect answers.

Question 4

Given: $AB = BC$. Prove that AB is a tangent at A to the circle. Learners were required to prove that AB is a tangent at A to the circle passing through A , E and D .

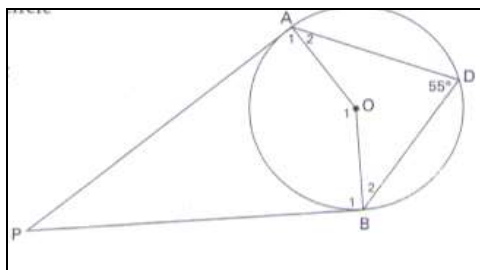


Response to question 4: All learners except L30 failed to provide the required proof. This is a clear sign that learners experience challenges in terms of proof questions. A third of candidates scored no marks for this question writing that it was impossible (see L4 answer below).

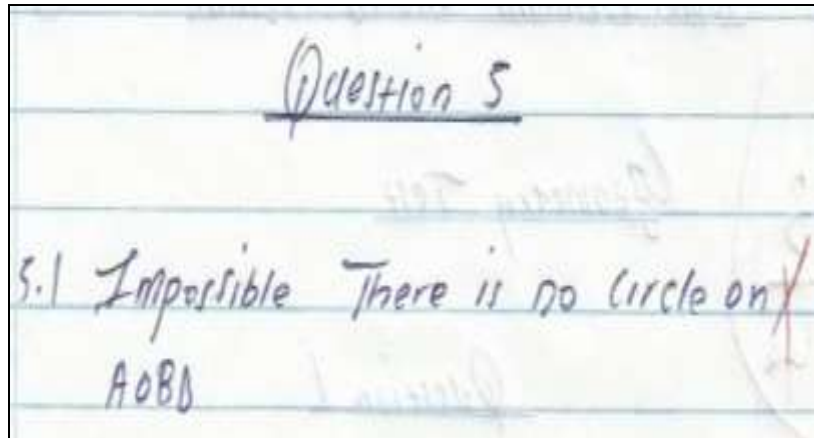


Question 5

PA and PB are tangents to the circle centre O . $\angle ADB = 55^\circ$. In Question 5.1, learners were required to prove that $AOBP$ is a cyclic quadrilateral. In 5.2 learners were required to calculate the value of $\angle P$



All learners except L30, who provided only one step of the solution, provided incorrect and irrelevant responses (refer to L6 incorrect response below).



5.2 Learners' test responses per question in terms of the outlined codes and categories

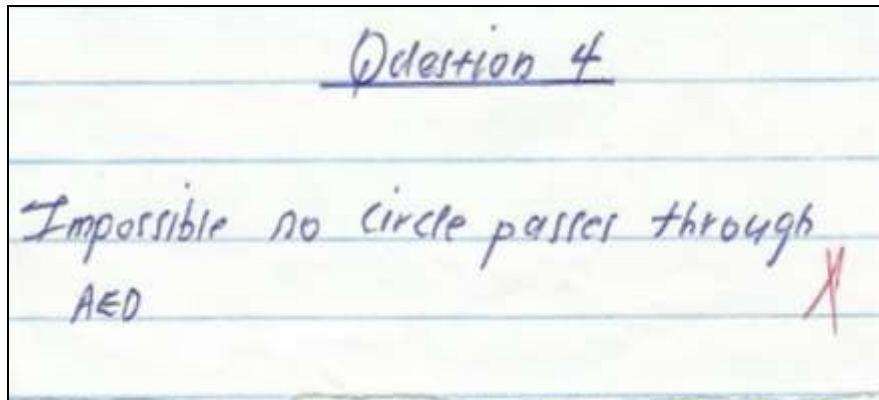
Table 4: Learners, their errors and van Hiele levels

| Learners | Question 1 | Question 2 | Question 3 | Question 4 | Question 5 | Van Hiele level |
|----------|------------|------------|------------|------------|------------|-----------------|
| L1 | C2,Err2 | C2,Err2 | C4,Err3 | C2,Err3 | C2,Err3 | 0 |
| L2 | C2,Err2 | C2,Err2 | C2,Err2 | C2,Err3 | C2,Err3 | 0 |
| L3 | C2,Err3 | C2,Err2 | C3,Err2 | C2,Err3 | C2,Err3 | 0 |
| L4 | C2,Err3 | C2,Err2 | C4,Err3 | C2,Err3 | C2,Err3 | 0 |
| L5 | C2,Err2 | C2,Err2 | C3,Err2 | C2,Err3 | C2,Err3 | 0 |
| L6 | C2,Err2 | C2,Err2 | C4,Err3 | C2,Err3 | C2,Err3 | 0 |
| L7 | C2,Err3 | C2,Err2 | C2,Err2 | C2,Err3 | C2,Err3 | 0 |
| L8 | C2,Err2 | C2,Err2 | C3,Err2 | C2,Err3 | C5,Err3 | 0 |
| L9 | C2,Err2 | C3,Err1 | C2,Err2 | C2,Err3 | C2,Err3 | 0 |
| L10 | C2,Err1 | C2,Err2 | C2,Err2 | C2,Err3 | C2,Err3 | 0 |
| L11 | C4,Err1 | C3,Err2 | C2,Err2 | C2,Err3 | C2,Err3 | 0 |
| L12 | C2,Err3 | C3,Err2 | C2,Err2 | C2,Err3 | C2,Err3 | 0 |
| L13 | C3,Err1 | C1 | C4,Err3 | C2,Err3 | C2,Err3 | 0 |
| L14 | C3,Err1 | C1 | C2,Err1 | C2,Err3 | C2,Err3 | 0 |
| L15 | C1 | C2,Err2 | C4,Err3 | C2,Err3 | C2,Err3 | 0 |
| L16 | C1 | C1 | C4,Err3 | C2,Err3 | C2,Err3 | 0 |
| L17 | C2 | C1 | C2 | C2,Err3 | C2,Err3 | 0 |
| L18 | C1 | C1 | C1 | C2,Err3 | C5,Err3 | 0 |
| L19 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 0 |
| L20 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 0 |
| L21 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 0 |
| L22 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |
| L23 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |
| L24 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |
| L25 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |
| L26 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |
| L27 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |
| L28 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |
| L29 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |
| L30 | C1 | C1 | C1 | C2,Err3 | C2,Err3 | 1 |

The following major errors were identified:

❖ **There should always be a visible circle for a line to be a tangent.**

This was an error made by learners when answering test item question 4. Learners were tested on their knowledge and competency of how to prove a tangent. Some of the following reasons were given as a response: cannot prove because AED is a triangle, impossible the tangent does not pass through the circle. It seems learners were taught that a straight line is considered a tangent only if it touches an outside part of a visible circle at one point only. To them there is no visible circle on the diagram.



Learners' responses and poor performance in this question is consistent with the research by Siyepu (2005) who found that learning to write proofs in geometry is one of the most difficult topics learners. The problem that learners are presenting is: they were not taught to make additional construction so as to help them visualise the problem. They should have drawn a circle passing through points A, E & D. As a solution, learners need to be taught all the properties of a tangent, namely the angle between a tangent and a chord is equal to the angle in the alternate segment, and a tangent is perpendicular to a radius. They should be taught that for a line to be a tangent, it must satisfy these properties, even if there is no visible circle. For example, in Figure 4 below, AB may be a tangent if we can prove that $\angle B_1 = \angle D$. The reason will be that $\angle B_1$ will be the angle between tangent AB and chord BC while $\angle D$ will be the angle in the alternate segment.

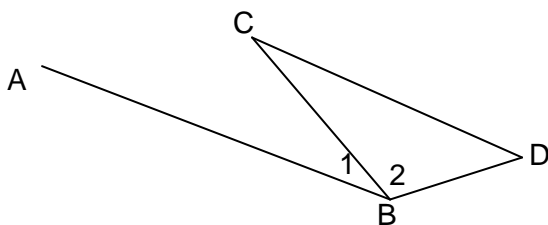


Figure 4: Angle between tangent and chord

❖ **There should always be a visible circle for a quadrilateral to be cyclic**

This was an error displayed by learners when answering question 5.1. Learners were required to prove that AOBP is a cyclic quadrilateral. In the diagram, there was no circle passing through AOBP. To make matters worse, P was drawn outside the circle. Some of the incorrect reasons given were: impossible, there is no circle on AOBP, not true P outside the circle to mention but a few. It seems learners believe a quadrilateral will be cyclic only if there is a visible circle. As a solution, learners should be taught that a quadrilateral is cyclic if the opposite angles are supplementary, if the exterior angle is equal to the

interior opposite angle and if the angles on the same segment are equal even if there is no visible circle. They should also be advised to draw the circle where it's not visible so that it will be easy for them to recall theorems of a cyclic quadrilateral.

For example, in the below figure, ABCD will be a cyclic quadrilateral if it can be proved that $\angle B + \angle D_1 = 180^\circ$ [opp. \angle 's of a cyclic quad] or if $\angle B = \angle D_2$ [ext $\angle =$ int. opp. \angle].



Figure 5: Quadrilateral

Learners should also be taught to draw the circle where it's not visible so that it will be easy for them to recall theorems of a cyclic quadrilateral.

❖ **Application of concepts in incorrect contexts to solve geometric problems.**

Question by question analysis reveal that learners have weak conceptual knowledge (see Table 5 below). Learners committed many C2 errors, which were incorrect statements and reasons. This is an indication that they did not understand the concepts taught and subsequently applied them in wrong contexts.

Table 5: Number of errors made by learners in terms of categories as explained in Table 1

| CATEGORY | QUESTION NUMBER | | | | | TOTAL |
|----------|-----------------|----|----|----|----|-------|
| | Q1 | Q2 | Q3 | Q4 | Q5 | |
| C1 | 15 | 17 | 12 | 0 | 0 | 44 |
| C2 | 12 | 9 | 8 | 30 | 30 | 89 |
| C3 | 2 | 3 | 4 | 0 | 0 | 9 |
| C4 | 1 | 1 | 6 | 0 | 0 | 8 |

Most of the wrong answers and statement were therefore due to conceptual errors. These were as a result of learners' inability to derive meaning from the definitions or explanations provided, and therefore, did not adequately translate the concepts described and apply them to the geometry problems. Learners could not apply properties of geometric concepts in correct contexts to simplify geometry problems. When facing a geometry problem, learners usually gave a reason unrelated to a problem. This is consistent with the studies conducted by Cunningham & Roberts, (2010) and Kabaca et al., (2011) who all confirmed that learners have difficulties in understanding geometry concepts. This is also alluded to by Siyepu (2005), whose study also reveals learners' display errors as a result of their inability to understand geometric concepts. According to Oberdorf & Taylor-Cox (1999) "lack of exposure to proper vocabulary" is one of the reasons for learners' errors in geometry. For a success in learning Geometry the understanding of geometrical concepts is essential.

❖ **Learners have problems with proof questions**

The analysis of learners' work reveals poor performance in terms of questions which involves proof. This became clear when learners answered questions 4 and 5. All the learners participating in the

study, with the exception of one learner scored 0. Learners cannot reason deductively and present a logical, coherent argument. Researchers like Ohtani (in De Villiers, 1998) identified the traditional method of providing learners with ready-made definitions as a teaching practice in geometry which inhibits learners' competency in proof. Educators should design assessment activities that promote higher order thinking skills which will result in deductive and logical reasoning, thereby enhances learners' competency in proof questions.

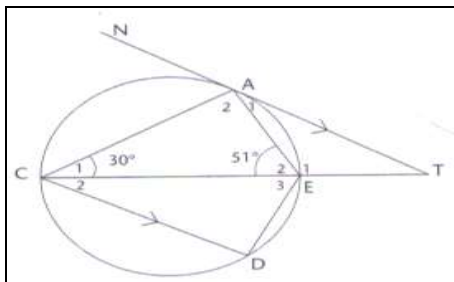
❖ **Slips, conceptual, procedural and order of operations.**

Table shows frequency of errors made by learners in terms of slips, conceptual, and procedural:

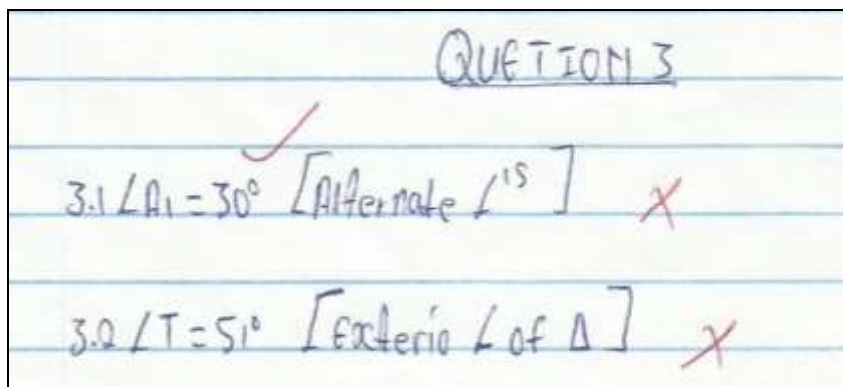
Table 6: Number of errors made by learners in terms of slips, conceptual and procedural errors

| CATEGORY | QUESTION NUMBERS | | | | | TOTAL |
|----------|------------------|----|----|----|----|-------|
| | Q1 | Q2 | Q3 | Q4 | Q5 | |
| ERR1 | 4 | 6 | 8 | 0 | 0 | 18 |
| ERR2 | 7 | 25 | 27 | 0 | 0 | 59 |
| ERR3 | 4 | 16 | 39 | 30 | 30 | 96 |

Table 6 above shows that fewer slips were made question 1 and 2. Slips (Err1) were the lowest performed errors e.g. $x + 10^\circ$ [alternate angles] instead of $x = 10^\circ$. According to Luneta and Makonye (2010), learners sometimes make these types of errors through lapses in concentration, hasty concentration or reasoning. Lots of procedural and conceptual errors (ERR2 and ERR3 errors) were made in question 3, 4 and 5. Procedural errors (Err3) were committed as a result of not understanding concepts and therefore applying them in wrong contexts, which resulted in poor deductive reasoning and poor logic skills. For example, when answering question 3.1 and 3.2, one of the learners gave wrong reasons. The learner made conceptual errors because she applied incorrect concepts in an incorrect context.



A learner's response



This is wrong because firstly line NT is not parallel to line CT. Secondly, while $\angle T_2$ an exterior angle of triangle AET, the exterior angle is equal to two interior opposite angles and not just one angle.

❖ **Van Hiele levels**

The data analysis of this research reveals that most learners are operating at levels 0 and 1 (see Figure 5):

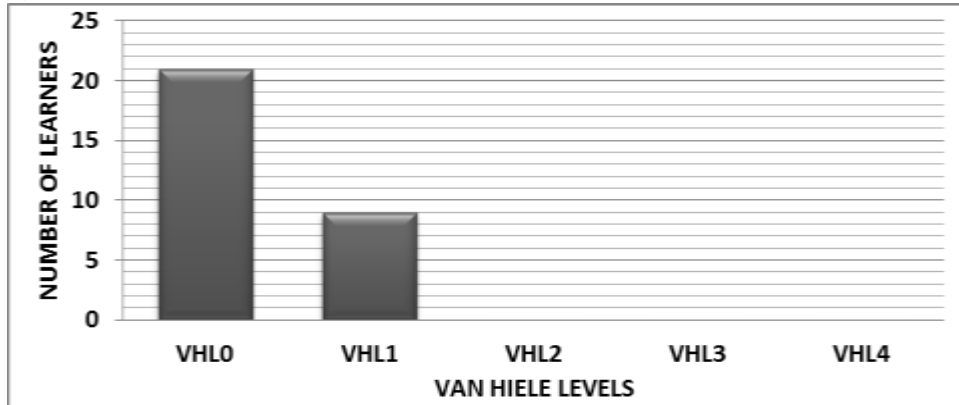


Figure 5: The results of the van Hiele level

The above figure clearly shows that achieving the van Hiele levels 2, 3 and 4 remains problematic for learners as evident in the marks that learners got in questions that involved proof or more than one procedural step. The majority of the learners find it easier to calculate the value of unknown angles than proving equal angles, tangents; and cyclic quadrilateral. These learners are simply not ready for the study of deductive Geometry problems due to lack of rudimental concepts of geometry.

6. CONCLUSION, RECOMMENDATIONS AND IMPLICATIONS FOR MATHEMATICS EDUCATION.

This study embraces theory below of engaging errors: This model encourages the acceptance of errors as part of the learning process. The acceptance of errors should be followed by the engagement of errors which will eventually result in the correction of such errors.

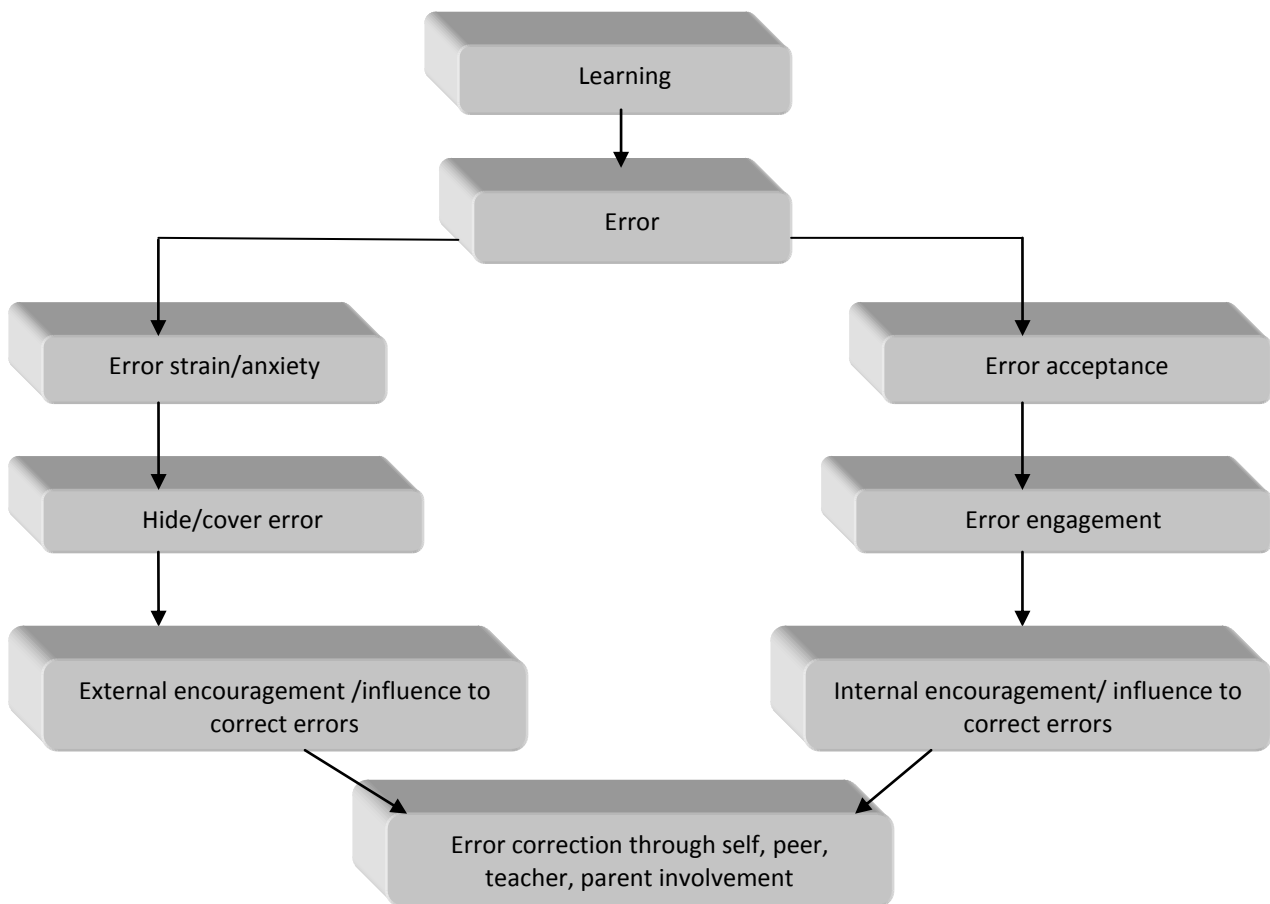


Figure 6: The error engagement model

This theory promotes the correction of errors through the self, peer, teacher and parental involvement. No matter how big an error is, both educators and learners should have the guts and motivation of correcting such an error. The recognition of errors is an effective and motivating starting point from which to plan lessons designed to fill the gaps in learners' knowledge. Errors should therefore not be viewed from negative perspective. Fang (2010) suggests errors should be seen as value sources of information about the learning process, providing clues that educators should take advantage of in order to uncover current learners' knowledge and how they come to construct such knowledge. Educators should therefore understand learners' errors, contemplate their causes and methodologically correct them (Luneta, 2013). This may help educators to develop alternative ways of instruction and classroom organization that provide the better fit teaching styles and the learning strategies for overcoming these errors. Regarding the arguments and findings advanced in this report, and in a quest to improve the poor performance of learners in geometry, it is suggested that educators should be empowered in terms of error identification and handling techniques. Learners should be exposed to goal-oriented activities which promote higher order thinking because this report found that no learner was operating at van Hiele levels 2, 3 and 4. This will help improve learners' proof skills and conceptual understanding.

The educational implication therefore is that teachers should understand learners' errors, contemplate their causes and methodologically correct them. This research supports Musser, Burger and Peterson(2011)'s recommendations that teachers should know their learners' mathematical thinking to be able to structure their teaching of new ideas to work with or correct those ways of thinking, thereby preventing learners from making errors. Teachers need to be able to analyze errors and evaluate alternative ideas, anticipate learners' errors and common misconceptions and be able to



interpret students' incomplete thinking. Scholars have indicated that errors cannot be prevented completely and that too heavy a reliance on error prevention can have detrimental effects (Van Dyck et al., 2005). For these reasons, a shift from an exclusive error prevention approach towards an error management strategy is proposed by this research.

In view of the findings, the study recommends that error analysis be incorporated in all mathematics Continuous Professional Teacher Development (CPTD) programmes if teachers are to develop the capacity to interpret and respond to their learners' errors in productive ways.

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