FIRST YEAR STUDENTS IN THE FOUNDATION PHASE PROGRAMME'S CAPACITY TO WORK WITH FRACTIONS – WHAT DO THEY DISPLAY?

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ABSTRACT– This paper reported on the capacity of the first year students in the Foundation Phase programme to work with fractions. These students have completed their school career and most have graduated from the new dispensation of choosing between mathematics and mathematics literacy as part of the curriculum. All the students (117) admitted into the Foundation Phase teacher training programme came from six of the 9 provinces of South Africa. The report is investigating how much knowledge of fractions they have retained. Conceptual and procedural knowledge was assessed in a test with 67 multiple choice and step-by-step solutions seeking items to respond to. Only the correct final answer was rewarded. The test items comprised of multiple choice, short-answer and open calculations to show their own understanding and procedural steps towards a solution. A combination of the test items were constructed and included to help the students get feedback and confirmation of the retained knowledge as they go through the tasks. Everyone took the pen and paper test in the second week of the 2015 classes. After the scripts were evaluated the results in 11 categories range from highest (63%:37%) in "Understanding of fractions concepts" to lowest (15%:85%) "Manipulating fractions symbols: Addition" where (Correct%:Incorrect%) format refers to attempted tasks excluding no responses. In conclusion the average performance of the students in working with fractions was 39% correct to 61% incorrect respectively, corresponding to 45% as minimum entry mark to Faculty of Education for Mathematics. Errors like adding both denominators and numerators as whole numbers and consistently doing it in subtraction revealing grave misconceptions were displayed and isolated and more attention will focus on them during the training period.

Keywords: Fractions; Errors; misconceptions, Mathematics Education, South Africa.

1. INTRODUCTION

The report is investigating how much conceptual and procedural knowledge of fractions they have retained after they have completed their schooling career. The aim of this investigation was to collect the participants' uninfluenced and raw knowledge of the concepts fractions, decimals and percentages as early as possible in the year. It is assumed that they have accumulated knowledge concepts fractions, decimals and percentages during their school time of about 12 years. Though I have been teaching and learning of Mathematics has a long history of achievements, under-achievements as well as failures both on the side of teacher and the side of learners. The teacher would put all the efforts to come through to the learners by preparing a lesson with all the relevant teaching aids hoping for success. The learner, on the other hand, would try all he/she can to understand what the teacher was presenting on the day and hope to have understood everything. Unfortunately since the beginning of learning, Mathematics included, this gap between what the teacher has taught and what the learner has learnt in class, errors have been noticed because "there will never be a one-to-one relationship teaching and learning" asserted Nunan (1995, p133) in language teaching. This gap manifests as errors in their responses in assessments tasks. These errors are classified according to the nature and frequency. Models and reforms in mathematics teaching and learning have been suggested, proposed and implemented to try to close the gap and influence the understanding and achievement of both teachers and learners positively (Balfanz & Byrnes, 2006). South Africa is no different in the reforms and changes in the curriculum and models to improve the teaching and learning including that of mathematics.

In Mathematics the section fractions under number development is a concept taught and learnt in schools universally. Dubinsky (1991), Gray and Tall (2007) and Sfard (1991) argued and agreed that learners acquire new mathematical ideas in inconstant ways, mainly procedurally/operationally or conceptually/structurally as well as keep these two “lanes” of development of their mathematical knowledge separate. The students invariably form conceptual schemes of what they managed to pull and put together from the learning opportunity. This “saved” scheme could be the same as provided in the learning opportunity or very different as well depending on how the previous related conceptual schemes have accommodated the new situation or concept. This seems to apply in fractions-decimals-percentage as well. This section is almost covered in a similar way and similar pace per group of grades in many countries but differ in delivery and depth from country to country (Mullis & Martin, 2014).

2. OVERVIEW OF THE INVESTIGATION

The learning and teaching of fractions is an area of mathematics that both learners and teachers find particularly challenging (Moss & Case, 1999; Pearn & Stephens, 2004, 2007; Clarke & Roche, 2009; Harvey, 2011). Zhou (2011) asserts that difficulties learners have with fractions are conceptual in nature. Many of them have instrumental understanding (operational or procedural) of fractions without knowing why the procedures are used which Zhou (2011) calls relational understanding. In many cases students do not understand the part/whole relationships described in fraction notation. This results in unconnected learning as this type of learning is reinforced by the assessment methods that sometimes expect fragmented and memorized learning (Zhou, 2011). This kind of assessment method encourages both teachers and learners to see mathematics as a collection of a large number of facts and procedures.

There is a link between the concepts fractions, decimals and percentages. Some of the procedures in common fractions do facilitate the transition to concepts decimals and percentages and vice versa. Fractions, Decimals and Percentages are different ways of showing the same number value. A half is written as (a fraction); 0.5 (a decimal) and 50% (a percentage). Mathematics teachers need to be conversant with these conceptual connections to facilitate smoother transitions as well as to seek appropriate teaching approaches to minimize the unintended misconceptions that might arise. Teacher education has both the duty to facilitate the isolation and closure of gaps in content knowledge of the teacher in the making as well as exposing them to possible options of teaching approaches. Though Dembele and Miaro (2003) argued that the availability of resources is a crucial dimension and factor for school effectiveness, a threshold level of school resources is desirable to lay a necessary platform to develop and flourish for effective teaching and learning. A competent Mathematics teacher is one of the critical resources in the learning of mathematics.

Teacher education depends on the quality of students admitted into the course to produce teachers who qualify to go and teach mathematics in a manner that exceeds their own experience. It is a teacher-learner-teacher conundrum of inefficient mathematics teaching that needs a stop. Wits University education Professor Mary Metcalfe told City Press that "Many teachers need more support with both mathematics and the specific challenges of teaching concepts," and it is them who produce some of the students that get trained as teachers (City Press Newspaper, 2012). One expectation is that the teacher training institutions should attempt to narrow the gap of limited conceptual and content knowledge of teachers in service as evidenced by factual errors during lessons contributing to low levels of learner achievements (Shepherd, 2013). It is possibly expected of these institutions to make up for years of deficient mathematics learning in the period of initial teacher training programme. Teaching first year students is the most difficult because of the diverse knowledge base levels and background as they begin their university studies that include mathematics learning (Swedosh, 1996). Working with fractions in the first year mathematics class; the researcher want to know how wide the conceptual knowledge gap in our Foundation Phase first year students is in order to make an attempt at breaking the cycle of supplying the education system with teachers with deficient pedagogical content knowledge of fractions.

Every learner forms a concept image of fractions to create cognitive structure associated with the concept under discussion. This personal concept image includes all the mental pictures and associated properties and processes (Tall & Vinner, 1981). Any learner's initial knowledge of fractions concept is typically superficial and fragile but with time the network of relationships needs to deepen and become richer as they go through their school career (Star, 2005). When a personal concept image begins to diverge from formal concept definition a fault-line is displayed as an error. When this error is consistently displayed in similar tasks, the learner is said to have formed a misconception. Researchers like Engelhardt (1977), Swedosh (1996) and Ashlock (2010) report that learners, when dealing with fractions, tend to hold misconceptions and make numerous errors in their attempts to work solutions out. Beside careless errors, these researchers cite common basic error types such as the inability to arrange them in ascending or descending order, grouping errors, basic fact errors and incorrect operations, to more complex problems such as the inability to apply the fractions skills in solving word problems. Two previous cohorts of first year students in this programme did display the aforementioned issues when dealing with fractions, decimals and percentage in the mathematics module that the researcher taught. So the researcher felt that it was important to worry about the types of misconceptions and their associated errors on fractions, how to teach as well as how learners learn fractions and formalized this investigation around this specific cohort of students to inform the interventions and strategies in the module delivery. Anyway, students' errors displayed on written work may also reveal deep-rooted misconceptions that will always be repeated unless some intervention is applied.

The debate on conceptual and procedural knowledge and which is better for optimal understanding of mathematical concepts goes on. Baroody, Feil, and Johnson, (2007) argue that the conceptual approach focuses on the meaningful memorisation of skills. This approach is based on the assumption that mathematics constitutes a network of skills and concepts of packages access as schemes (Saenz-Ludlow, 1995). Star (2005), on the other hand, argues that flexibility in the use and application of these skills must be a measure of how deep the procedural and conceptual knowledge the learner understands the concept. An optimal balance between these two approaches is for teachers to help children to acquire needed facts, rules, formulas, and procedures in a meaningful way (i.e. with comprehension). The role of the teacher is to guide the learner towards understanding the concepts and mastering the associated procedural skills. I agree with Star (2005) when he argues that the conceptual and procedural knowledge are critical components of students' level of mathematical proficiency and thus merit carefully crafted study. This ideal balance is even more profound in the learning of fractions and the related concept to prepare learners in the transition from arithmetic to algebra (Brown & Quinn, 2006).

3. THE RESEARCH DESIGN AND METHODS

This exploratory investigation utilized a quantitative collection method through a questionnaire of fractions-decimals-percentages test items to establish and have an overview of how much of these concepts this cohort of students have retained. All these students have completed and passed their matric already. Participants in this investigation were the whole group of 2015 first year students in the Foundation Phase Programme who wrote the same test at the same time. All first year (freshman) students do a mathematics course on fractions-decimals-percentages in the first year of the programme and were part of the study out of convenience. The participants were given 80 minutes to complete the test. The aim was to collect their uninfluenced and raw knowledge of the concepts as early as possible in the year. The group sat the exploratory Pre-Test in the second week of February. These students were unaware that there would be a test and therefore did not attempt to prepare for it which would potentially affect the results thereof. One hundred and seventeen (117) students participated in the pen-and-pencil test taking activity. The majority (80%) were females and 20% males constituted the group. The task had sixty-seven (67) questions with ninety-three (93) test items to respond to.

3.1 Design of questionnaire task items

The test included a majority of multiple-choice items together with short-answer and extended-response (free-response) items on fractions-decimals-percentage (TIMMS, 2012). These questions were demanding the students to work from total basics of the section of work even though they had completed their schooling careers. The test consisted of 93 items and was specifically designed to measure procedural as well as general attributes of fraction knowledge in areas related to: concept, addition, subtraction, multiplication, division, equivalence, order or compare, and transfer or convert between common fraction to either decimal or percentage form and vice versa. The format of the final instrument was made up of multiple-choice questions (MCQs) designed to test a concept, terminology, notation, or key formulae and open-ended questions (OEQs). The test questions were designed so that if a student had a particular misconception (e.g. Questions 10, 11, 13, 14, 20), this fact would be apparent when the response of the student to that question was considered. Students were supposed to answer in their own way, words, and steps (e.g. Questions 1 – 24; 26 - 42). Secondly, the questions and style of questioning would help the student remember or confirm their recollected knowledge where applicable (e.g. Questions 26 – 31; 43 - 67). The test attempts to demand the students to demonstrate all these four cognitive skills (TIMSS 2015). The cognitive skills are to knowing facts and procedures, using concepts appropriately, solving routine problems as well as reasoning in the process of solving a problem.

The questions were spread over the main concepts on fractions, decimals and percentage as well as the relationships between the three concepts. The questions ranged from definitions, to operations, to conversion between them. The pen-and-pencil test task on fractions covered all work on types, relationship, recognition, classification and the associated operations on fractions. The open-ended questions gave participants an opportunity to show steps towards expected solutions the way they remember and know. The participants were supposed to choose their own approach, expression and skills to respond to the tasks in the extended-response (free-response) sections of the tasks. These types of questions included the four operations; addition, subtraction, multiplication and division, comparisons of fractions were included as well. The students were supposed to remember and have known as fractions are covered from beginning of school. This group of students had completed their schooling career and are expected to know the section very well after so many repeat learning and reinforcements experiences over the years.

Examples of some of the questions in the task are provided in Table 1.

Addition	**Subtraction**	**Multiplication**	**Division**
C.= $2\frac{2}{3} + 6\frac{2}{7}$	**C.=** $5\frac{2}{3} - 2\frac{2}{4}$	**A.=** $\frac{8}{5} \times \frac{2}{6}$	**C.=** $5\frac{2}{3} \div 2\frac{2}{4}$
Order from **greatest to least**: 0.9, $\frac{4}{5}$, 81%, $1\frac{1}{5}$			
Conversion	**Fractions**	**Decimal**	**Percent**
	$\frac{2}{5}$	0.4	40 %

Table 1: Examples of questions in the questionnaire

After the students wrote the test, every script was assessed and marks recorded in a spreadsheet. Every response or lack of it (no response) was recorded and the initial analyses of the scripts are discussed.

3.2 Overview analysis of collected data

The pre-test task has revealed a set of common misconceptions and associated errors displayed by the students. Students were given 80 minutes to complete the test. Every test item was examined and a record was kept in the form of:

- how many students attempted each question item,
- how many students answered correctly,
- how many students had chosen a wrong answer (where applicable), as well as
- what was the performance in the particular categories/group

The 67 questions with 93 test items were put together and demanded the students to respond to these five categories. These categories comprised of question items from two to fourteen spread all over the questionnaire. These categories helped facilitate a clearer, simpler and deeper analysis of the students' responses to the items. The groupings could still be made up of different combinations because of the cognitive demands that run across many of the question items. The selections of items to be included in particular groupings were informed by the glaring and obvious appearance and instructions to be followed. Some of them might have more demand on the students than others and thus would be better grouped under a different category. Some of the descriptions of categories came from the cognitive demands like compare, order, add, convert, etc.

3.3 Brief discussion of responses

A brief discussion of the responses to the test across five categories from general conceptual attributes and procedural knowledge. Each category has a summary of the breakdown of the responses or lack thereof with correct (**Cor**), incorrect (**Incor**) as well as no response (**NR**). The overall average of the students responses was only **34%** in the correct column; **53%** in the incorrect column with **13%** of the test items left with no response.

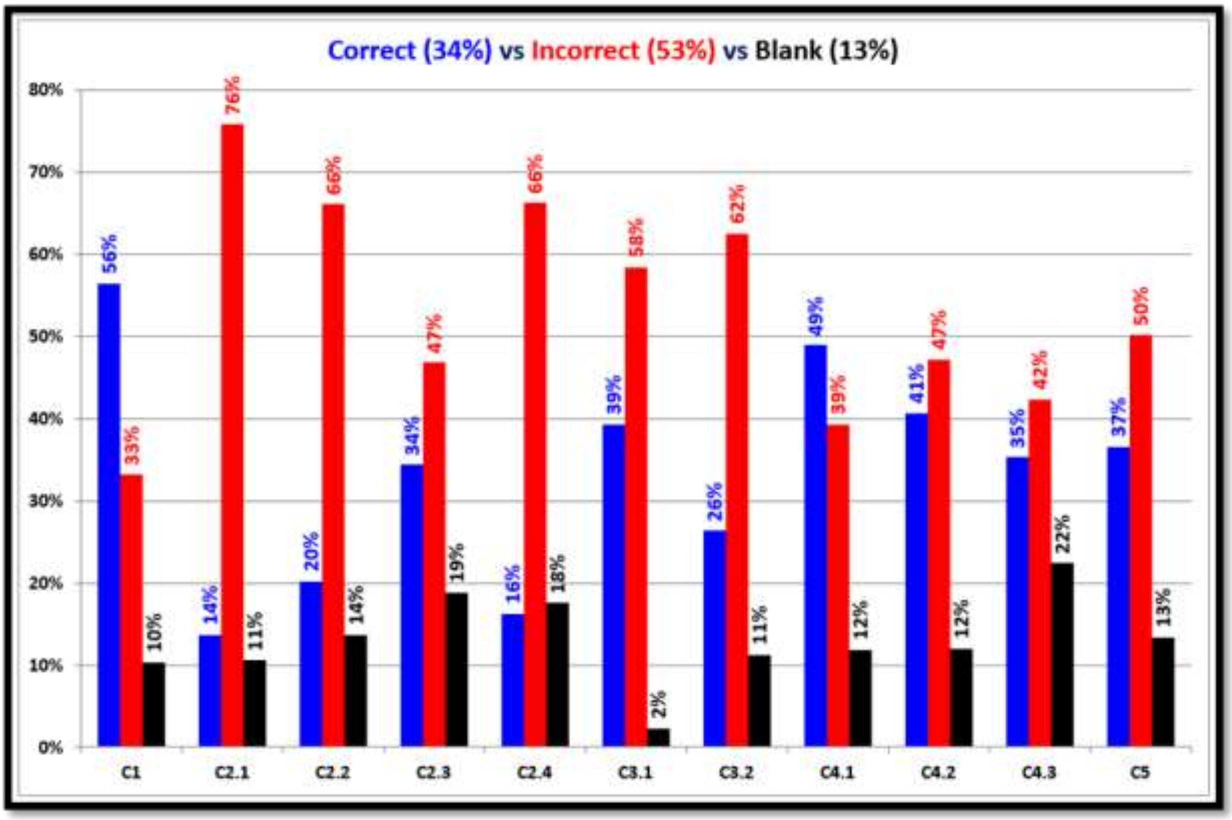

Figure 1: Summary of categories, collection of questions and performance.

Figure 1 displays the categories and subcategories, the applicable question items, with the students' performances per category. After the scripts were evaluated the results in 5 categories range from highest (**56%**:**33%:10%**) in "Understanding of fractions concepts" to lowest (**14%:76%:11%**) "Manipulating fractions symbols: Addition" where (**Correct%:Incorrect%:NoResponse%**) format refers to attempted tasks including not done. The section of manipulating fractions to add, subtract, multiply or divide proved the most difficult. The section (category 3) of comparing and sequencing fractions saw an increase in correct responses **44% and 25%** with very few no responses as well. The next section (category 4) of converting from one to another saw more balanced performance with **48%, 41%** and **38%**. The

Each category has a group of questions items that fit as much as possible. A question item was fitted in only one group to highlight just one of the cognitive skill to be demonstrated. Some of the categories has many question items whilst others have as few as two items (C2.4). The biggest group (C1) has fourteen items. The overview analysis is presented in the abovementioned five categories. The following is a category-by-category analysis from the fraction test.

Category 1: Understanding of fractions concepts: e.g. definitions and conversions (56%:33%:10%)
This category dealt with terminology of fractions. The questions sought to help them dispel or confirm their thoughts or memory retrieval from the pool of many accumulated schemes of knowledge on fractions, decimals and percentages. They see the position of the numerator + denominator as top or bottom in relation to the division sign in a/b.
The students were able to respond to the set of question items as almost sixty percent (56%) were correct responses to the questions. Only 10 percent left these items unattended to when about a third (33%) supplied totally unacceptable responses.

Category 2: Manipulating fractions symbols: addition, subtraction, multiplication, division

The second category dealt with addition, subtraction, multiplication, division in how they expressed themselves. These tasks were examples of the open-ended responses to the items. The students were supposed to display how their own procedural ways lead to the solutions. In this category the three questions were given from easy to complex fractions. Those who could not do the tasks and get the correct solution displayed this common way of solution.

Category 2.1: Manipulating fractions symbols: addition (14%:76%:11%)
The worst performance of the group appears to be where the student was supposed to show own steps in addition (C2.1). A big majority (**76%**) of the students did not get the correct answer. Though 11% did not respond to the questions in this category, 14% of them were able to solve all three problems correctly. A closer look at the individual test items in this category revealed that the item that proved most difficult was 13C.

The question item 13C was the addition of mixed fractions and many students were not able to deal with. For 13A, 13B and 13C students got 18%, 15% and 8%, whilst those who did not respond to the question items were 8%, 9% and 15%, respectively. This particular set of questions revealed the misconceptions some students had about addition or subtraction in fractions. They treated the numerator and denominator as two whole numbers. The “whole number bias” was evident in their responses (Ni & Zhou, 2005). Their errors and misconceptions were well displayed in these examples:

In both cases this student adds/subtracts the numerators together, then adds/subtracts the denominators together. In the addition tasks the sums are added further to get 13 or 15.

In the items 14A and 14B the same is done but introduces a gap of conceptual knowledge of dividing by zero.

In both sets of tasks, 13C and 14C are left blank because apparently the method seems to have hit a strong wall.

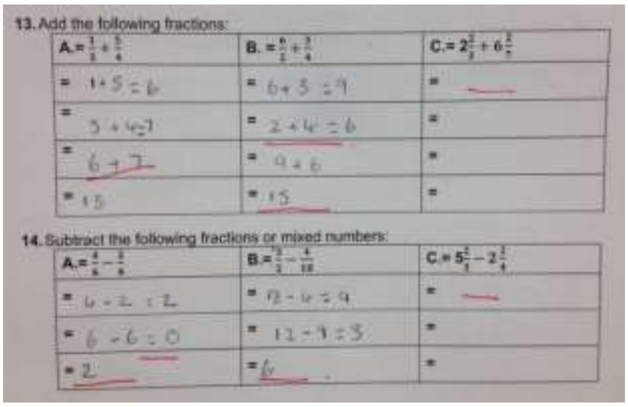

Figure 2: Misconceptions in addition and subtraction

Category 2.2: Manipulating fractions symbols: subtraction (20%:66%:14%)
The students were supposed to show own solution steps in this set of questions. Two thirds (**66%**) of the students were not able to solve the problems. There was an increase (14%) of no response to the questions in this category, with a 6% increase (14%) of them able to solve all three problems correctly. A closer look at the individual test items in this category revealed that the item that proved most difficult was 14C again. A common error in this set of question was the same as in the addition (see fig 3) of whole number bias (Ni & Zhou, 2005).

Category 2.3: Manipulating fractions symbols: multiplication (34%:47%:19%)
The multiplication category had 9 question items (Question 10 and 11) for the students to respond to including 16A. The students were multiplying and simplifying (reducing the fraction to its simplest form). In each case they were working it out instead of choosing from given options. 34% of them were able to get the correct answer in all the involved items. 19% on the other hand did not respond to any of these questions. About fifty percent (47%) of these students managed to work out the correct solution for each question item.

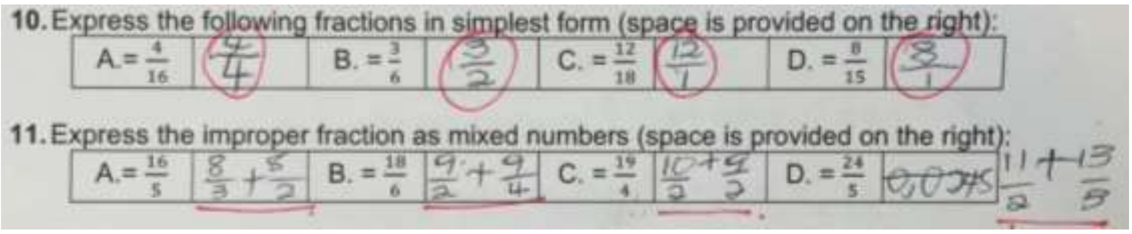

Figure 3: Misconceptions in manipulating fractions symbols

In question 10 the whole number bias seems to have influenced the response. It appears that the student left the numerator as it and divided the denominator by the numerator for each question. In 10C, for an example, 12 goes 18 once and repeated that in 10D.

In question 11 this student was simply splitting the "whole number" in the numerator and denominator. Working in 11A the student split 16/5 into 8/2 and 8/3 as these numbers are treated as whole numbers on their own. The student seems to be reversing what others were doing in addition and subtraction questions 13 and 14.

Category 2.4: Manipulating fractions symbols: division (16%:66%:18%)

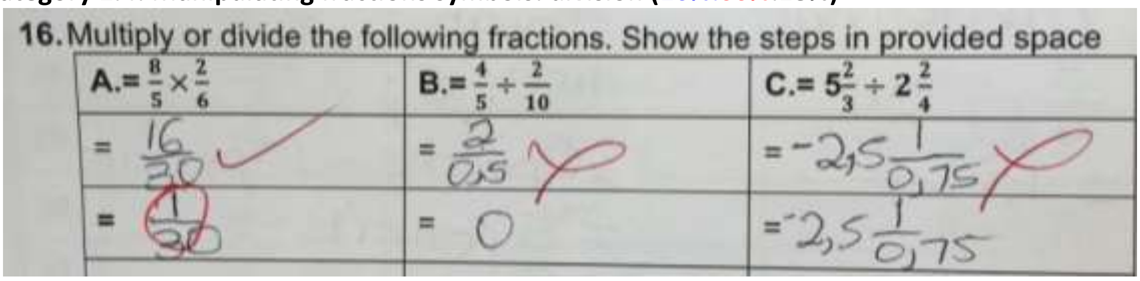

Figure 4: Misconceptions in manipulating fractions symbols in fractions

This category had two items (16B and 16C) that the students had to respond to. Only 16% of the students got the correct answer. More students (18%) did not respond to these two questions at all. 66% of the 117 students worked on the solutions and got the correct answers. The whole number bias is displayed in the given image. In the question item 16B 4 was divided by 2 and 5 was divided by 10 to yield 2 and 0.5 respectively. It is difficult to explain the final answer zero (0) at the end. This misconception was repeated in 16C consistently.

Category 3.1: Comparing and sequencing fractions: same forms of fractions (39%:58%:2%)

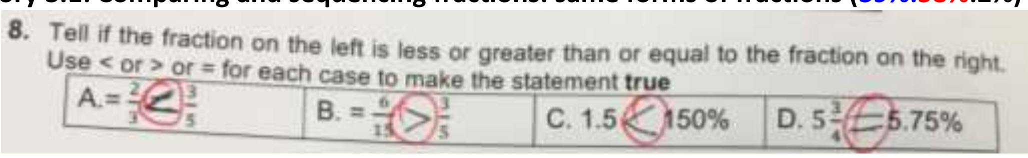

Figure 5: Comparing and sequencing fractions: same forms of fractions (8A and 8B)

This category had three items including 8A and 8B that the students had to respond to. Almost all the students (2%) responded to these questions. Thirty nine percent (39%) of the 117 students provided the correct answers. The rest (58%) made incorrect choices of the sign to make the each statement true in the comparisons. There is very little one can make out of the entries until a face-to-face interview is done where this particular student would share some light. It is possible, though, to suppose that the whole number bias was at play again because in both cases the numerators and denominators were "bigger" to make their responses correct for whole numbers.

Category 3.2: Comparing and sequencing fractions: mixed forms of fractions (26%:62%:11%)

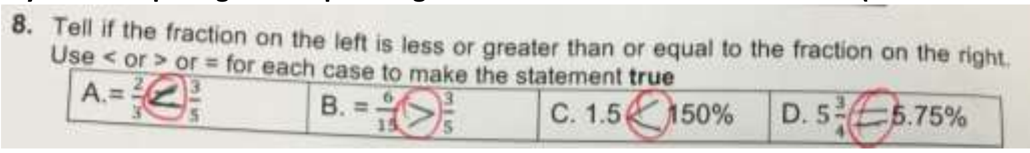

Figure 5: Comparing and sequencing fractions: mixed forms of fractions (8C and 8D)

This category had twelve items including 8C and 8D that the students had to respond to. Only 26% of the students got the correct answer. Eleven percent (11%) of these students responded to these questions. The rest (62%) provided incorrect responses in these mixed forms fractions comparisons. This category demanded that the student retrieved a conceptual scheme of converting to one form to another (fraction, decimal or percentage) before a correct comparison is made. When one of the conceptual schemes is faulty or insufficiently developed or totally not retrieved, an error is due to happen. The numbers utilised in forming the question items wanted the student to go to basics and avoid guessing a response. 8D is a typical example of two values so similar until the percentage sign is observed. This particular student in this example provided the exact solicited response.

Category 4.1: Alternative forms of fractions: convert to decimals (49%:39%:12%)

This category had twelve items that the students had to respond to. Close to half (49%) of the students were able to convert and got the correct answer. 12% students did not respond to these twelve questions at all. 39% of the 117 students responded but got incorrect answers. This particular group of question items had a combination of own answer and choose from the provided options. Most students got the correct answer from the section of items with provided options to choose from.

Category 4.2: Alternative forms of fractions: convert to percentage (41%:47%:12%)

This category had eleven items that the students had to convert to percentage form. Close to half (47%) of the students were unable to convert to the correct percentage form answer. 12% students did not respond to these eleven question items at all. On the other hand, 41% of the 117 students worked on the solutions and got the correct answers. Again this particular group of question items had a combination of own answer and choose from the provided options. Most students got the correct answer from the section of question items with provided options to choose from.

Category 4.3: Alternative forms of fractions: convert to common fraction (35%:42%:22%)

This category had twelve items that the students had to respond to. Close to a third (35%) of the students were able to convert from either decimal or percentage forms and got the correct answer. It was noticeable that almost for every 5 students 1 (22%) student did not respond to these twelve questions at all. Of the 117 students 42% were unable to convert to the correct common fraction form. The design of the test had both own answer as well as choose from the provided options kind of question items. Most students got the correct answer from the section of items with provided options to choose from. The lowest percentage of correct response in this group of question items was question 30B which had only 4% of success. The students were supposed to convert 1.005 to percentage form and 42% left it blank whilst 54% got it wrong.

Category 5: Solving mathematical word problems with fraction elements (37%:50%:13%)

This category had twelve task items that the students had to work out. Exactly half (50%) of the students were unable to solve the given mathematical word problems with fraction elements. Questions "How many halves are there in two-fifths? and What is half of one-fourth?" are the two questions that had the lowest percentage of correct response in this group. They had only 2% and 3% of success respectively. The conceptual demand of these two questions was apparently very difficult to understand for those who attempted to solve them. 13% students did not respond to these twelve question items at all. On the other hand, 37% of the 117 students worked on the solutions and got the correct answers.

The demand on the capacity of the student's retrieval of conceptual schemes developed over the years of schooling where they were taught fractions, decimals and percentage.

4. CONCLUSION

In conclusion the average performance of the students in working with fractions-decimals-percentage from the attempted questions items was **39%** correct to **61%** incorrect respectively. The displayed misconceptions were presented as examples of how some of them fit the past researchers findings. These misconceptions and very interesting responses to where the students were offering own steps in the solution versus making a choice from a list of possible answers. The "whole number bias" was seen to have been the main source of the errors and misconceptions in the incorrect responses (Ni & Zhou, 2005). Students use the single-unit counting scheme as applied to whole numbers operations to interpret instructional data on fractions. The examples provided a window to the types of consistent errors that learners often displayed. They were convinced that their methods were correctly recalled for the correct question items they applied them in. It is interesting to observe that students have integrity in the answers they provided to these questions as it displayed the misconceptions in many cases of incorrect ones (Green, Piel, and Flowers, 2008). Some of the explanations of what the student was doing can only be through interviews. One will possibly understand the thought process as well possible source of the erroneous thought process.

Brown and Quinn (2006, p38) summarise the main possible gap in the teaching and working with fractions

> "Capable high school students often complain that they cannot do fractions; fluency with rational numbers means doing with understanding. Regrettably, many students are taught algorithms before they have had the time to develop the fundamental concept. Their only alternative when confronted with fraction operations is to match what is being presented with one of the disconnected, previously-memorised algorithms from earlier mathematics experiences. If the situation being presented is novel or is not in a recognisable form, then a student's best effort is no more than a good guess. The errors that were made repeatedly demonstrate that a good guess is not sufficient."

The error analysis revealed a large number of misconceptions that students have related to the subject of fractions, decimals and percentages. These misconceptions will be addressed in the Foundation Phase class in the second semester combined with an intensive tutor programme. In addition Brown and Quinn (2006) are emphatic that an individual must generalise a concept if the concept is not understood yet. They admit that it is a genuine problem in mathematics education because mathematics is cumulative in nature and needs a philosophical change to promote depth learning as opposed to breadth learning to discourage memorizing facts and algorithms (Brown & Quinn, 2006).

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