MISCONCEPTIONS OF THE LIMIT CONCEPT IN A MATHEMATICS COURSE FOR ENGINEERING STUDENTS

by

TERTIA JORDAAN

submitted in part fulfilment of the requirements for the degree of

MASTER OF EDUCATION – WITH SPECIALISATION IN MATHEMATICS EDUCATION

at the

UNIVERSITY OF SOUTH AFRICA

SUPERVISOR: Ms R PAULSEN

28 FEBRUARY 2005

“I declare that ‘Misconceptions of the limit concept in a mathematics course for engineering students’ is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.”

T Jordaan

I want to dedicate this research project to the following people:

- My husband, Kobus for his support and understanding during my time of study;
- My only child, Jaco who did not understand much of what was happening and had to bear with me behind the computer or books during his last pre-school year;
- My loyal and dedicated domestic worker for 13 years, Agnes who lost her battle against her disease on 31 December 2004.

ACKNOWLEDGEMENTS

With the completion of this research report I want to express my sincere thanks to:

- My Creator for the strength and wisdom to complete the research project.
  Philippians 4:13: ‘I can do everything through Him who strengthens me’.

- Prof Dirk Wessels for his initial help in planning the research project.

- Ms Ronél Paulsen, my supervisor for her guidance and assistance throughout the duration of the project. A special word of thanks for the formatting of the final electronic script and help with the sketching of the graphs.

- The library staff for their help with the literature search.

- My family and friends who believed in me and supported me.

T JORDAAN

White River
February 2005

Summary:

In this investigation an attempt was made to determine the misconceptions that engineering students have of the idea of a limit. A comprehensive literature study showed that there are a number of common misconceptions that students normally form. The empirical investigation was done in two phases. A questionnaire on the idea of a limit was given to the students during the first phase. During the second phase six interviews were conducted. The findings were grouped according to the nature of a limit and students' views on the relationship between the continuity of a function at a point and the limit at that point. An analysis of these findings led to the identification of the misconceptions that these students have of the idea of a limit.

Key words: Limit; Mathematics Education; Calculus; Misconceptions

Opsomming:

In hierdie ondersoek is gepoog om die wanbegrippe wat ingenieursstudente van die limietbegrip vorm, bloot te stel. ‘n Omvattende literatuurstudie het ‘n aantal algemene wanbegrippe aan die lig gebring. Die empiriese ondersoek het in twee fases plaasgevind. Tydens die eerste fase is ‘n vraelys aan die studente gegee in ‘n poging om meer te wete te kom van hulle begrip van ‘n limiet. Die vraelys is opgevolg deur ses onderhoude. Die responsies is gegroepeer in terme van die aard van ‘n limiet en studente se sienings van die kontinuïteit van ‘n funksie by ‘n punt en die limiet by daardie punt. Die analisering van hierdie responsies het die identifisering van ‘n aantal wanbegrippe by hierdie groep studente moontlik gemaak.

Sleutelwoorde: Limiet; Wiskunde-onderwys; Calculus; Wanbegrippe

TABLE OF CONTENTS

FIGURES

TABLES

CHAPTER 1

BACKGROUND AND OVERVIEW OF THE STUDY

1.1 Introduction

The researcher became aware of students' problems with the idea of a limit when she started lecturing at tertiary level two years ago. The first topic in the second-year syllabus was limits. This included the limit of a function at a point, a limit as x approaches infinity and the relationship between limits and continuity. Little time, about four hours, was allocated to these aspects related to limits. This was just a brief preparation for the derivative and differentiation which took up a large part of the syllabus and the lecturing time.

Due to the inexperience of the researcher lecturing at tertiary level, the teaching was strictly according to the textbook. At the end of the unit, the students could calculate limits in a variety of problems, but they found it very difficult to explain the idea of a limit in their own words. The researcher became curious and wanted to investigate the idea of limits further. She wanted to determine why students find this idea so complicated; whether the complexity is inherent to the nature of the idea itself; whether it is due to the way in which it is taught or whether it is due to any other reasons.

A possible explanation for the identified problem is that the idea of a limit is often taught in isolation. Students see the idea as a concept on its own without any relationship to other Calculus ideas. Students do not realize the important role that limits play in the study of Calculus at tertiary level in Differential Calculus, Integral Calculus and continuity.

They cannot explain why the idea of a limit is fundamental to Calculus. They are unable to link limits to the broader field of Calculus. The motivation for this investigation thus was to determine the conceptions that students have of the idea of a limit and to try and reveal the specific misconceptions concerning limits.

1.2 Background of the study

White & Mitchelmore (1996:79) mention that there is much concern about the large numbers of students taking Calculus and the rote, manipulative learning that takes place. Research into the understanding of Calculus has shown a whole spectrum of concepts that cause problems for students. In particular, student difficulties with the abstract concepts of rate of change, limit, tangent and function are well documented. These concepts involve mathematical objects or processes specific to Calculus. The general tendency now is for less emphasis on skills and greater emphasis on the understanding of the underlying concepts.

In an article, *The History of Limits,* the authors state that the idea of "limit" is the most fundamental concept of Calculus. Every major concept of Calculus namely derivative, continuity, integral, convergence or divergence is defined in terms of limits. In fact "limit" is what distinguishes at the most basic level what we call Calculus, as the mathematics of change, variation, related rates and limits, from the other branches of mathematics (algebra, geometry and trigonometry). (http://occawlonline.pearsoned.com/bookbind/pubbooks/thomas_awl/ chapter1/medialib/cu).

Cornu (1991:153) agrees that the mathematical concept of limit holds a central position which permeates the whole of mathematical analysis – as a foundation of the theory of approximation, of continuity and of differential calculus and integral calculus. He is of the opinion that this mathematical concept is a particularly difficult idea, typical of the kind of thought required in advanced mathematics.

Cornu further mentions that one of the greatest difficulties in teaching and learning the idea of a limit does not only lie in its richness and complexities, but also in the extent to which the cognitive aspects can not be generated purely from the mathematical definition. The distinction between the definition and the concept itself is didactically very important. Remembering the definition of a limit is one thing, but acquiring the fundamental conception is another.

The limit concept has long been considered fundamental to an understanding of Calculus and Real Analysis. Recent studies have confirmed that a complete understanding of the limit concept among students is comparatively rare (Williams 1991:219). There is general agreement in the literature that students have trouble with the idea of limit, whether it is in the context of functions and continuity or of sequences and series (Davis & Vinner, 1986; Tall, 1992; Monaghan, 1991; Bezuidenhout, 2001; Williams, 1991). Moreover, many of the difficulties encountered by students in dealing with other concepts (continuity, differentiability, integration) are related to their difficulties with limits. The reason for this may be due to inappropriate and weak mental links between knowledge of limits and knowledge of other calculus concepts such as continuity, derivative and integral. Well-constructed mental representations of the network of relationships among calculus concepts are essential for a thorough understanding of the conceptual underpinnings of the calculus (Bezuidenhout 2001:487).

Williams (1991:219) states that conceptions of limit are often confounded by issues of whether:

- a function can reach its limit,
- a limit is actually a boundary,
- limits are dynamic processes or static objects
- limits are inherently tied to motion concepts.

These issues give rise to incomplete or alternative conceptions of limit. Williams is of the opinion that these alternative conceptions of limit relate closely to the view of limiting processes held by the mathematical community prior to Cauchy's rigorous delta-epsilon definition of limit.

Williams (1991:219) mentions Ervynck's view that most students completing a course in calculus have a pre-rigorous understanding of limit and very few ever achieve full understanding of the rigorous definition. Cottrill et.al (1996:167) confirm students' difficulties when they say that most students have little success in understanding this important mathematical idea. Although for many students such pre-rigorous understanding may suffice, Tall (1980:173) suggested that such informal models of limit could lead to more serious misunderstandings and interfere with future learning.

There are unavoidable sources of naive misconceptions inherent to the idea of a limit (Davis & Vinner 1986:298-300; Tall 1992:501). One is the influence of *language or words*, in which certain terms remind us of ideas that intrude into students' attempts to represent mathematical concepts. These terms are phrases such as 'tends to', 'approaches' or 'gets close to'. When these phrases are used in relation to a sequence *approaching* a limit, they invariably carry the implication that the terms of the sequence *cannot equal the limit.*

For example, in the sequence 1; ½; ; ¼; ...., the $\lim_{n \to \infty} \frac{1}{n} = 0$, although the last term will never be equal to zero.

In addition to the words, there are the *ideas* that these words conjure up, which have their origins in earlier experiences. These ideas should not be part of a mental representation of the mathematical concept. Another source of misconceptions is the sheer complexity of the ideas. These ideas cannot appear “instantaneously in complete and mature form”, so that “some parts of the idea will get adequate representations before other parts will”. It would therefore be necessary for each student to gradually build mental representations of the limit concept.

1.3 The problem statement

The following key problems or questions related to the idea of a limit need to be investigated. They are:

- How do students understand the idea of a limit?
- What kind of misconceptions do they form?
- How do they relate the continuity/discontinuity of a function at a point to the existence of a limit at that point?

1.4 Aim of the research

The aim of this investigation is to identify the specific misconceptions that engineering students have of the limit concept. The focus will be on the limit of a function. The researcher wants to determine whether these misconceptions are different to or in accordance with those found in the literature.

In order to achieve this aim, the following objectives are formulated:

- To gather data of the students' understanding of the limit concept by means of a questionnaire and interviews,
- To analyse the gathered data in order to determine misconceptions that students have.

1.5 The research design

This research project was conducted at the Nelspruit campus of the Tshwane University of Technology. The researcher lectures Mathematics at this institution. The students are enrolled for a Diploma in Electrical Engineering. Mathematics is a compulsory subject in this course. Many of the students have to follow a bridging course before they can commence with Mathematics 1. There are two reasons for this practice. The first reason being their low marks in Mathematics in the Grade 12 examination and the second, that they can be better prepared for the Mathematics 1 course. Mathematics 1, 2 and 3 are semester courses and students can complete their Mathematics courses required for this diploma within a year and a half.

The content of the Mathematics 2 course corresponds more or less to a first-year Calculus course. The main topics are differentiation and integration including their applications in real-life, engineering situations. The focus of the course is on the application of students' mathematical knowledge and skills in an engineering context. They are taught the mathematics that they need so that they can use and apply this knowledge in other engineering subjects such as Control Systems and Digital Systems. It is a multi-cultural classroom and the students are taught through the medium of English, which is their second language.

Initially, the second year students were chosen to be the subjects in the study. The original plan was to interview six of them to determine their initial understanding of a limit prior to the discussion in class. Then limits were to be taught in class by making use of discussions, problem solving tasks, group work and other written tasks. These tasks were to be assessed in order to determine possible misconceptions. The idea then was to interview the same six students for a second time to determine in which way the teaching has altered their views on limits, if at all.

Due to the fact that limits were omitted from the second-year course at the beginning of the second semester of the year 2004, the planned strategy had to be adapted. The decision was made to use the third-year students in the study because they dealt with limits during the first semester of the same year. The approach was strictly according to the textbook due to the researcher's inexperience at that time. The second-years were to take part in the pilot study of the planned questionnaire as well as in the interviews. A pilot study of the questionnaire was completed first in order to check whether it needed any changes. The responses were recorded and analysed in order to identify possible misconceptions. Thereafter two students were interviewed in order to determine whether the interview schedule needed to be changed.

1.5.1 Literature study

A library search was done with the following descriptors: limit, misconceptions, Calculus and Mathematics Education. The aim of the literature study was to identify relevant literature and research projects that were conducted on the teaching and learning of limits to shed more light on the research questions.

The following related aspects were investigated in the literature:

- The historical development of limits;
- Epistemological obstacles in the historical development of limits;
- The nature of the idea of a limit; informal and rigorous definitions;
- Models of limit held by students;
- Misconceptions of the idea of a limit;
- Prerequisites for the understanding of limits - functions and continuity and
- Possible strategies for the teaching and learning of the idea of a limit.

1.5.2 Empirical study

This investigation took place in two phases in the form of a quantitative-descriptive (survey) design (de Vos 2002:142-143). The aim was to obtain insight into the students' understanding of a limit in order to identify their misconceptions.

In the first phase a questionnaire on the idea of a limit was given to the whole group of 47 third-year mathematics students. The items in the questionnaire focused on the concept itself and not on calculating limits or the manipulative aspects thereof (See Appendix A). The data in the questionnaire were recorded and thereafter analysed in order to find out more about these students' understanding.

The second phase consisted of interviews with six of the students. Students who gave exceptional responses or answers different to the usual ones in the questionnaire were selected for interviews. The interview schedule consisted of six questions with the aim once again to determine their misconceptions (See Appendix B). The interviews were tape-recorded and later transcribed. The responses were analysed according to the nature of a limit, the relationship between the continuity of functions and limits and other conceptions that might be identified.

1.6 Value of the research

The important role that limits play in Calculus is acknowledged by most educators and educationists, but it is a fact that the inherent nature of limits is quite complex. Most students find it difficult to understand this idea. For many years teachers and lecturers have been seeking for answers to why students have misconceptions and what they can do to help them develop mathematically correct conceptions.

The identification of the students' misconceptions ought to be of value in the following ways:

1. It might lead to a better understanding of the students' thought processes and the quality of learning that takes place.
2. Knowledge of misconceptions can be employed in planning more effective teaching strategies and methods.
3. It can also be used to present richer learning experiences to the students.
4. The researcher can assist other lecturers in teaching limits in a more effective way.
5. The identification of misconceptions can create worthwhile opportunities to enhance learning.
6. Removing misconceptions and providing better understanding of the idea of a limit.

1.7 Terminology

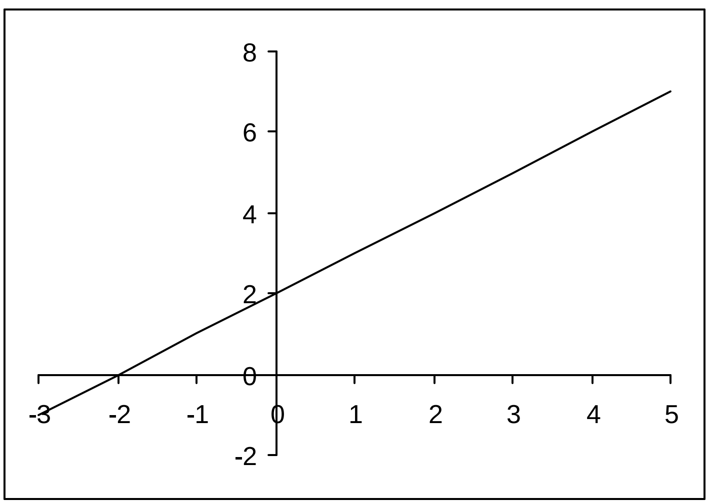
1.7.1 The limit concept

Limits occur in many different mathematical contexts. They are the limit of a sequence, a limit of a series and a limit of a function [$f(x)$ as $x \to a$ or $f(x)$ as $x \to \infty$]. Limits also occur in the notion of continuity, differentiability and integration (Tall 1992:501). The focus of this study is on the limit of functions only (continuous as well as discontinuous) and two examples are given:

Examples of the limit of a function

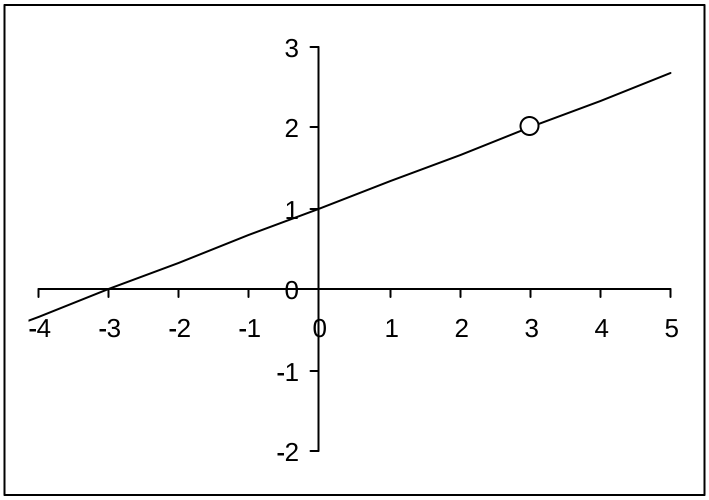
1 For the function $f(x) = x + 2$, the $\lim_{x \to 1} f(x) = 3$

Fig 1:



2 If $g(x)=\dfrac{x^2-9}{3x-9}$, then $\lim_{x\to 3} g(x)=2$

Fig 2:



The limit concept has an informal (intuitive) definition as well as a formal (rigorous) epsilon-delta definition.

An informal definition of the limit is:

Let f be a function which is defined at all x near a. If, as x tends to *a* from both the left and the right, $f(x)$ tends to some number b, then we say that the limit of $f(x)$, as x tends to a, is b $\lim_{x\to a} f(x)=b$

The formal epsilon-delta definition is as follows:

A function $f(x)$ is said to approach a limit b as x approaches the value of a if, given any small positive å, it is possible to find a positive number δ such that | $f(x)$ – a| < å for all x satisfying 0 < |x – a| < δ .

1.7.2 Misconceptions

Olivier (1989:11) and Bezuidenhout (1998:397) agree that a lecturer/teacher cannot transmit knowledge ready-made and intact to the students/learners. They see errors and misconceptions as the natural result of people's efforts to construct their own knowledge. These misconceptions are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge. Misconceptions, therefore, cannot be avoided. They may however, hinder the development of more sophisticated understandings of fundamental mathematical concepts to a great extent.

Such errors and misconceptions should not be treated as terrible things to be uprooted. This may confuse the learner and shake his/her confidence in his/her previous knowledge. Instead, making errors is best regarded as part of the process of learning.

This suggests that lecturers become aware of their students' possible misconceptions. A classroom atmosphere that is tolerant of errors and misconceptions ought to be created. Misconceptions should be taken into account when teaching strategies are developed. The identification of misconceptions can create worthwhile opportunities to enhance learning.

1.7.3 Misconceptions related to limits

David Tall's earliest research began with calculus and limits. This led him to the discovery of differences between mathematical theories and cognitive beliefs in many individuals. For example, the limit 'nought point nine recurring' has mathematical limit equal to one. Cognitively there is a tendency to view the concept as getting closer and closer to one, without actually reaching it. (*Limits, Infinitesimals and Infinities* : http://www.warwick.ac.uk/staff/David.Tall/themes/limits-infinity.html).

He gives the following reason for this tendency. The primitive brain notices movement. Therefore the mental idea of a sequence of points tending to a limit is more likely to focus on the moving points than on the limit point. The idea of a limit is thus first conceived as a process and then as a concept. In the case of the limit, the process of tending to a limit is a potential process that may never reach its limit (it may not even have an explicit finite procedure to carry out the limit process). This gives rise to cognitive conflict in terms of cognitive images that conflict with the formal definition.

Tall & Vinner (1981:152) use the term concept image to describe the total cognitive structure (in an individual's mind) that is associated with a specific mathematical concept. This includes all the mental pictures and associated properties and processes. The concept image is built up over time through all kinds of experiences and changes as the individual meets new stimuli. Every individual has his own, unique concept image of a mathematical idea.

The concept definition, on the other hand is quite a different matter. Tall & Vinner (1981:152) regard the concept definition to be a form of words that specify that concept. Such a concept definition may be formal and given to the individual as part of a mathematical theory. It may be personal, invented by the individual to (partially) describe his concept image. A personal concept definition may be considered as part of the concept image, but the formal concept definition may or may not be (Tall 1980:172).

In most countries, the limit is introduced in an informal manner first, with a formal definition coming much later, if at all. This means that a concept image is built up long before any formal concept definition is given. The concept images of limit are therefore likely to contain factors which conflict with the formal/concept definition (Tall 1980:174).

For example, if the initial teaching tends to emphasize the process of approaching a limit, rather than the concept of limit itself, then the concept imagery contains many factors which conflict with the formal definition (‘approaches but cannot reach’, ‘cannot pass’, ‘tends to’). The students develop images of limits which relate to misconceptions concerning the process of ‘getting close’ or ‘growing large’ or ‘going on forever’ (Cornu 1991:156).

In the literature the following types of misconceptions related to limits are described:

- confusion over whether a limit is actually reached,
- confusion regarding the static characteristic of a limit and
- uncertainty about whether limits are dynamic processes or static objects (Cottrill et.al 1996:173 ; Williams 1991:219 ; Tall 1980:175 ; Thabane 1998:23 ; Davis & Vinner 1986:294).

1.8 Layout of the investigation

The way in which the rest of this research project is discussed is as follows:

Chapter 2 gives an overview of the literature study that was done. The historical development of limits is sketched. The nature of limits as well as the epistemological obstacles inherent to limits is described. Students’ conceptions and misconceptions of limits are described. Limit-related aspects such as functions and continuity also receive attention. Important aspects regarding the teaching and learning of limits are also included.

In Chapter 3 the empirical investigation is described. This includes a description of the research design. The instruments used to gather data namely the questionnaire and interview are highlighted. Reasons are given for the inclusion of each item in the specific measuring instrument. The ways in which the gathered data were analysed, also receive attention.

Chapter 4 consists of a representation of the data that were gathered during the empirical investigation. An analysis of the responses in the questionnaire as well as the interviews is given. The analysed data are interpreted in terms of the findings from the literature.

In the final chapter, Chapter 5, summaries are given of the findings from the literature review, the research approach and the findings in the empirical investigation. Research findings are put in perspective with what is known about misconceptions in the literature and what was found during the empirical part of the research. The limitations of this research project are described as experienced by the researcher. A conclusion is drawn from all the different aspects and the chapter is concluded with recommendations for further research on limits.

CHAPTER 2

THEORETICAL BACKGROUND TO THE STUDY OF LIMITS

2.1 Introduction

The purpose of this chapter is to present a literature review of the research that has been done on limits. The historical development of the idea of a limit is described briefly. The views of various authors and researchers on the nature of limits are then discussed and evaluated. The models that students have of the idea of a limit are described. This includes students' understanding of the concept as well as their misconceptions. The problem-centred approach to learning is discussed as possible teaching method. Thereafter, the role of technology in the teaching and learning of limits is discussed.

2.2 The historical development of limits

For many centuries, the ideas of a limit were confused with vague and sometimes philosophical ideas of infinity i.e. infinitely large and infinitely small numbers and other mathematical entities. The idea of a limit was also confused with subjective and undefined geometric intuitions (*The History of Limits,* http://occawlonline.pearsoned.com/bookbind/pubbooks/thomas_awl/chapter1/medialib/cu).

Cornu (1991:159) is of the opinion that the idea of a limit was introduced to resolve three types of difficulty:

- Geometric problems, for example, the calculations of area, 'exhaustion' and consideration of the nature of geometric lengths;
- The problem of the sum and rate of convergence of a series;

- The problems of differentiation that come from the relationship between two quantities that simultaneously tends to zero.

The term *limit* in our modern sense is a product of the late 18th and early 19th century Enlightenment in Europe. It was developed as a means of putting the differential and integral calculus on a rigorous foundation. Until this time, there were only rare instances in which the idea of a limit was used rigorously and correctly. Our modern definition is less than 150 years old. The idea of a limit remains one of the most important concepts (and sometimes one of the most difficult) for students of mathematics to understand (*The History of Limits*, http://occawlonline.pearsoned.com/bookbind/pubbooks/thomas_awl/chapter1/medialib/cu).

In the beginning, when Newton and Leibniz were developing calculus, their work was based on the idea of ratios and products of arbitrarily small quantities or numbers. Newton called them fluxions and Leibniz referred to them as differentials. These ideas were built on the work of Wallis, Fermat and Decartes who had found a need in their work for some concept of the 'infinitely small' (*The Platonic Realms,* http://www.mathacademy.com/platonic_realms/encyclopedia/encyhome.html).

The root of these issues poses the following question: "How close can two numbers be without being the same number?" Another question is: "How small can a number be without being zero?" The effective answer given by Newton and others was that there are infinitesimals. They are thought of as positive quantities that are smaller than any non-zero real number. Such a concept seemed necessary, because the differential calculus relied crucially on the consideration of ratios, both of whose terms were vanishing simultaneously to zero. Despite the success of

Newton's and Leibniz's methods as well as the enthusiasm of mathematicians such as Lagrange and Euler, the idea of infinitesimal became more and more difficult to support during the 18th century. The main complaint was that it is impossible to imagine in any concrete way an object that is infinitely small. Without a firm theoretical basis for infinitesimals, mathematicians could not be completely confident in their methods. Such a theoretical basis did not seem forthcoming, despite nearly two centuries of effort by the mathematical community of Europe.

A new way of thinking about ratios of vanishing quantities was being introduced by the French mathematician, D'Alembert, namely the method of limits. His formulation was nearly identical to that in use today, although it relied heavily on geometrical intuitions. For example, D'Alembert saw the tangent to a curve as a limit of secant lines, as the end points of the secants converged on the point of tangency and became identical with it 'in the limit'. This is exactly how the derivative is motivated by calculus courses world-wide today. However, considered purely as a geometrical argument, without a numerical or functional foundation, this idea of limiting secant lines is subject to age-old objections of the sort exhibited by Zeno's paradoxes (*The Platonic Realms,*
http://www.mathacademy.com/platonic_realms/encyclopedia/encyhome.html).

Another great French mathematician, Cauchy, provided the rigorous formulation of the limit concept that would meet all objections. Cauchy's definitions of the derivative and the integral as limits of functions transformed our understanding of the calculus. It opened the door to a rich period of growth and innovation in mathematics (*The Platonic Realms,* http://www.mathacademy.com/platonic_realms/encyclopedia/encyhome.html).

2.3 Epistemological obstacles in the historical development

It is useful to study the history of the limit concept to locate periods of slow development and the difficulties which arose during its development. This may indicate the presence of epistemological obstacles.

Brousseau in Cornu (1991:159) defines an epistemological obstacle as knowledge which functions in a certain domain of activity and becomes well-established. Then it fails to work satisfactorily in another context where it malfunctions and leads to contradictions. It then becomes necessary to destroy the original insufficient knowledge and to replace it with a new concept that operates satisfactorily in the new domain.

Epistemological obstacles occur both in the historical development of scientific thought and in educational practice. They have two essential characteristics:

- They are unavoidable and essential constituents of knowledge to be acquired,
- They are found, at least in part, in the historical development of the concept (Cornu 1991:158).

There are four major epistemological obstacles in the history of the limit concept according to Cornu (1991:159-162), these are:

- The failure to link geometry to numbers,
- The idea of the infinitely large and infinitely small,
- The metaphysical aspect of the idea of limit and
- Is the limit attained or not?

The idea of a limit is difficult to introduce in mathematics because it seems to have more to do with metaphysics or philosophy. This metaphysical aspect of limit is one of the principal obstacles for today's students. This obstacle makes the comprehension of the limit extremely difficult, particularly because it cannot be calculated directly using familiar methods of algebra and arithmetic. The question whether the limit is actually reached or not, is still alive in the minds of today's students. This is another obstacle to the idea of a limit.

2.4 The nature of the idea of a limit

The idea of a limit is first defined informally and then formally (rigorously) before different views on its nature are discussed. As mentioned in Chapter 1, the focus of this investigation is on the limits of functions.

An informal definition of limit

If the values of a function $f(x)$ approach the value L as x approaches c, we say that f has limit L as x approaches c and write $\lim_{x \to c} f(x) = L$

The formal definition

The limit of $f(x)$ as x approaches c is the number L if the following criterion holds: Given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about c such that for all x, $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

The idea of a limit signifies a progression to a higher level of mathematical thinking which Tall (1992:495) calls, advanced mathematical thinking. This progression involves a difficult transition. It begins where concepts have an intuitive basis founded on experience to one where they are specified by formal definitions and their properties reconstructed through logical deductions. During this transition (and long after), earlier experiences and their properties as well as the growing body of deductive knowledge, exist simultaneously in the mind. This produces a wide variety of cognitive conflict that can act as an obstacle to learning.

Tall (1991:501) quotes Cornu who sees the idea of a limit as the first mathematical concept that students meet where one does not find a definite answer by a straightforward mathematical calculation. Abstract ideas such as limit, could be conceived operationally as processes or structurally as objects. The dual character of mathematical concepts that have both a procedural and a structural aspect was investigated by many researchers. Sfard (1991:20) used the word reification to describe the gradual development of a process becoming an object. Dubinsky (1991:101) postulated a theory of how concepts start as processes which are encapsulated as mental objects that are then available for higher-level abstract thought. Gray & Tall mentioned in Kidron & Zehavi (2002:206) formulated the idea of a *procept*, as a combination of process and concept, called up by a single symbol. In the case of the limit, the same symbol $\lim_{n \to \infty} a_n$ may be simply a process, the process of tending to a limit. If it can be seen as a new entity that is detached from the process that produced it, then it becomes an object. Processes can then be performed on this object and a new kind of static construct obtained (Kidron & Zehavi 2002: 205-206).

There is general agreement in the literature that process or operational conceptions must precede the development of structural object notions (Cottrill et.al 1996:173; Tall 1992:508). Various authors have the tendency to set up a dichotomy between dynamic or process conceptions of limit and static or formal conceptions. The latter is normally identified with the formal ε - δ definition (Cottrill et al 1996:174).

Kidron & Zehavi (2002:205) use Cornu's explanation that the definition of limit is formulated in terms of an unencapsulated process 'give me an $\varepsilon > 0$ and I will find an N such that .... ', rather than a concept, in the form 'there exists a function $N(\varepsilon)$ such that .....' as in the case of a limit of a sequence.

There is general agreement that process or operational conceptions must precede the development of structural or object notions (Sfard 1991:1). There are, however, two different views of the relationship students have to a process conception of limit, also called a dynamic conception. Some authors seem to indicate that a dynamic conception is easy and natural for students to develop (Tall 1992; Williams 1991; Tall & Vinner 1981). According to this view, the main difficulty is for students to pass from a dynamic conception to a formal understanding of limits. There is even a suggestion that students' dynamic ideas hinder their movement toward developing a formal idea.

Other authors seem to feel that developing a strong dynamic conception is necessary for a formal understanding. A formal understanding must build on the student's dynamic conception. In this view, the difficulty comes in constructing the dynamic idea and this difficulty is the obstacle to understanding (Davis & Vinner 1986 ; Cottrill et al 1996).

The nature of the idea of a limit is relatively complex. Epistemological obstacles occur because of the nature of the concept itself. The models that students have of the idea of a limit are discussed in greater detail in the next section. These models can have certain misconceptions which are the focal point of this investigation.

2.5 Models of limit held by students

2.5.1 Spontaneous conceptions of limits

The teaching of most mathematical ideas does not begin on virgin territory. In the case of limits, the student already has a certain number of ideas, images and knowledge, from daily experience, before any teaching on this subject commences. Cornu (1991:154) describes these conceptions of an idea that occur prior to formal teaching, as spontaneous conceptions. During a mathematics lesson these spontaneous ideas do not disappear, but they mix with newly acquired knowledge, are modified and adapted to form the student's personal ideas.

In the case of the idea of a limit, the words 'tends to' and 'limit', have a significance for the students before any lesson begins. The students continue to rely on these meanings after they have been given a formal definition. Investigations have shown many different meanings for the expression 'tends towards'. Some of them are: 'to approach' (eventually staying away from it), 'to approach ... without reaching it', 'to approach ... just reaching it and to resemble' (without any variation).

The word limit itself can have many different meanings to different individuals at different times. Most often it is considered as an 'impassable limit'. It can also be: an impassable limit that is reachable, a point that one approaches, without reaching it (or reaching it), the end or finish and an interval (Cornu 1991:154-155 ; Monaghan 1991:21-22).

2.5.2 Misconceptions of the idea of limit

Many authors distinguish between concept images and concept definitions (Vinner 1991; Tall 1992; Davis & Vinner 1986; Cornu 1991).

The term concept image is used to describe the total cognitive structure that is associated with a concept. This includes all the mental pictures and associated properties and processes. This concept image is built up over the years through all kinds of experiences and is subject to change. The concept definition, on the other hand, is seen as a form of words used to specify that concept (Tall & Vinner 1981:152).

For each individual, a concept definition generates its own concept image. There is no single idea of limit in the minds of students. The concept images of limit contain factors that conflict with the formal concept definition. This gives rise to several kinds of misconceptions. Cornu (1991) mentions the following example: “It is clear that the initial teaching of limits tends to emphasize the process of approaching a limit, rather than the concept of the limit itself. The concept imagery associated with this process, contains many factors that conflict with the formal definition (‘approaches but cannot reach’, ‘cannot pass’, ‘tends to’, etc.) Students develop images of limits and infinity which relate to misconceptions concerning the process of ‘getting close’ or ‘growing large’ or ‘going on forever’” (p 156).

The following types of misconceptions are described in literature (Williams 1991:219; Thabane 1998:65; Laridon 1992:398; Cottrill et. al 1996:173).

- Confusion over whether a function can reach its limit,
- Confusion over whether a limit is actually a boundary,
- Confusion whether limits are dynamic processes or static objects and
- Whether limits are inherently tied to motion concepts.

Thabane (1998:65) and Laridon (1992:398) mention more common misconceptions. Students think that 'limits' simply entail substituting the value at which the limit is to be found, into the expression. They often think that limits are only encountered when trying to ascribe a value to a function at a point where the function is undefined. Students often think that function values and limits are the same. Students talk of a limit not being defined at a point, when it is the function that is not defined at the point. Students think only about the manipulative aspects and do not focus on the idea of the limit. Bezuidenhout (2001:489) argued that formula might make calculations easier, but did not promote understanding. He advises that the development of students' conceptual understanding should go hand in hand with the development of their manipulative skills.

The conclusion can be drawn that the idea of a limit is complex. Students have difficulty in understanding this important idea. Cottrill et al (1996:172) mention that they have not found any reports of success in helping students overcome their difficulties. Some authors report that even using technology has not been successful in doing so.

2.5.3 Prerequisites for the understanding of limits

In order to understand limits and other calculus concepts, students ought to have a global interpretation of the behaviour of functions and their graphs (Lauten, Graham, & Ferrini-Mundy 1994:227). This includes the idea of the continuity of functions. Students also need to be familiar with graphs i.e. the interpretation of given graphs as well as the sketching of a graph when its algebraic formula is given. Another important related aspect is the idea of infinity.

2.5.3.1 Global understanding of functions

The function concept is one of the most fundamental in all of mathematics and one for which students seldom develop a satisfactory understanding. Reasons students have difficulty with the concept seem to centre round its complexity and its generality. The concept has many facets and associated sub-concepts which can be stated at different levels of abstraction (Dreyfus & Eisenberg 1983:119).

Ferrini-Mundy & Lauten (1993:157) mention that various researchers in mathematics education have been interested in students' understanding of functions. These researchers have studied the sub-concepts of function (domain, range, representation and correspondence), representations used for function (graphs, rules, tables and arrow diagrams) and ways in which students use and conceptualise functions.

This research has shown that the working definition of function held by most students is that of function as a rule of correspondence, represented by a formula. Students are strongly committed to this view and believe that a function must have the same rule of correspondence over its entire domain. Functions defined differently on different parts of the domain (piecewise functions) present great difficulty. A number of these researchers have noted that students interpret functions in a point-by-point, or local, way rather than globally. Tasks that encourage students to interpret graphs qualitatively may help build a global interpretation.

In this discussion, the focus is primarily on the 'limit of a function' and the 'continuity of a function at a point'.

2.5.3.2 The continuity of functions

The issue of continuity has become one of practical as well as theoretical importance. Continuous functions are the functions we normally use in the equations that describe numerical relations in the world around us. They are the functions we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. It is important to know when continuity is called for, what it entails and how to test for it (Finney & Thomas 1993:91).

A function $y = f(x)$ that can be graphed throughout its domain with one continuous motion of the pen (that is without lifting the pen) is an example of a continuous function. A function is continuous if it is continuous at each point of its domain. The continuity test can be used to test for continuity at a point.

The continuity test

A function $y = f(x)$ is continuous at $x = c$ if and only if it meets all three of the following conditions:

1 $f(c)$ exists (f is defined at c)

2 $\lim\limits_{x \to c} f(x)$ exists (f has a limit as $x \to c$)

3 $\lim\limits_{x \to c} f(x) = f(c)$ (the limit equals the function value)

To test for continuity at endpoints, the appropriate one-sided limit is used (Finney & Thomas 1993:92).

For any function $y = f(x)$ it is important to distinguish between continuity at $x = c$ and having a limit as $x \to c$. The limit is where the function is headed as $x \to c$. Continuity is the property of arriving at the point where $f(x)$ has been heading when x actually gets to c (Finney & Thomas 1993:98).

2.6 The teaching and learning of the idea of a limit

The teaching and learning of this important, but complex idea ought to be planned and implemented with great care. The problem-centred approach based on a constructivist perspective seems to be an effective way to teach and learn the idea of a limit.

2.6.1 A constructivist perspective on teaching and learning

The central idea of the constructivist theory is that mathematical knowledge cannot be transferred ready-made from one person to another. It ought to be constructed by every individual learner. This theory maintains that students are active meaning-makers who continually construct their own meanings for ideas communicated to them. This is done in terms of their own, existing knowledge base. This suggests that a student finds a new mathematical idea meaningful to the extent that connections are established between the new idea and his/her existing knowledge (Bezuidenhout 1998:390).

Mathematical learning is a constructive process that requires individual cognitive activity. This individual cognitive activity cannot be controlled and manipulated by a lecturer as if it is a physical phenomenon. This view of learning is in contrast to the view that a student is a passive receiver of knowledge and that misconceptions can be deleted from memory by taking in clear explanations from the person giving the explanations. Due to this constructive process, the mathematical knowledge (contained in the explanations) can be modified during the process of transmission (Bezuidenhout 1998:390).

Learning a mathematical idea, such as limit, involves a construction process. This implies that students build on and modify their existing concept images. The concept image consists of all the mental pictures together with the set of properties that an individual associates with a given concept. The concept images which an individual constructs through his own activities may differ in various respects to the formal mathematical concepts. This leads to the formation of alternative conceptions or misconceptions (Bezuidenhout 1998:391).

2.6.2 A problem-centred learning approach to teaching

A problem-centred approach to mathematics teaching is based on the acceptance that students construct their own knowledge. This approach attempts to establish individual and social procedures to monitor and improve the nature and quality of those constructions. The construction of mathematical knowledge is firstly an individual and secondly a social activity (Murray et. al 1998:30).

In the problem-centred approach, problem solving plays a dual role according to Human (1993:22). The development of problem solving skills plays a central role on the one hand and problem solving is the dominant learning type on the other hand. Students learn mathematics through solving problems. They do not learn new mathematics when they solve problems by purely utilising existing knowledge.

To learn through the solution of problems, students need to be exposed to problems which are new to them. The knowledge that develops through this process should not have been available beforehand. This, however, does not neglect the fact that there is still some mathematical knowledge that students cannot discover by themselves, namely the physical knowledge and the social knowledge.

The role of the teacher is no longer that of transmitter of knowledge to the students, but rather a facilitator of their learning. This facilitating role that the teacher plays puts a lot of pressure on his/her organisational skills as well as subject didactical knowledge.

Social interaction plays an important role in the problem-centred approach. Murray et al (1998:31) mention the following three purposes of social interaction in the classroom:

- Social interaction creates the opportunity for students to talk about their thinking and encourages reflection;
- Students learn not only through their own constructions but also from one another and
- Through classroom social interaction the students also interact with the teacher.

The teacher therefore plays an important role in selecting appropriate problems that have to be solved. The teacher also has to create a classroom culture that is conducive to learning. Murray et al (1998:31) stress the fact that the amount and quality of learning that takes place, depend on the classroom culture and on the expectations of both the teacher and the students.

Laridon (1992:399) warns that most students need to be eased into the difficult concepts of calculus, especially the idea of a limit. He advocates methods that are generally intuitive rather than rigorous, especially in the beginning. He states that nothing is guaranteed to build up an antipathy to calculus more than a ‘long, formal, dry treatment’ of the idea of a limit as step one. If in addition, the ε - δ definition is used, the limit notation will more than likely trigger a mental block whenever it is encountered in future. When teaching first courses in calculus, it must be kept in mind that it took the most brilliant minds of the mathematical community centuries to arrive at this rigour (Laridon 1992:400).

2.6.3 The role of technology in teaching and learning limits

The research on the role of technology in teaching and learning limits is very limited. Tall (1992:503) reports that teaching the idea of a limit using the computer has, on the whole, fared badly.

Cornu (1991:165) is of the opinion that the computer may very well play a significant role in providing an environment where the student may gain appropriate experiences to construct the limit concept. However, such approaches are very likely to contain their own peculiar epistemological obstacles. It is necessary to reflect on student experiences in the new environment to see precisely what is learnt and in what form the knowledge is stored in the memory.

Cornu (1991:166) further states that the interaction with the computer may involve programming. The individual can construct computer processes that may permit the acquisition and mastery of the corresponding mathematical ideas. It may involve pre-prepared software to enable the student to experience carefully selected environments that model the idea of a limit. It is also possible to imagine a kind of computer ‘toolbox’ for the learning of the mathematical idea: a computer environment that will permit students to manipulate objects and to construct knowledge.

Various other approaches are possible. In a context such as that of studying limits, it is vital that the computer software is designed within a teaching strategy based on the careful analysis of the idea that must be acquired.

Lauten et.al (1994:227) mention that an increasing number of students is using graphic calculators as they become more available and inexpensive. Claims are widespread that:

- the use of graphic calculators will allow for more conceptual approaches;
- students will understand the relationships between symbolic algebra and graphical representations more readily;
- students will now be able to solve problems that were previously inaccessible to them because of the formal mathematics required for their solution.

It is evident that more research is needed to determine the specific role that technology plays in learning about limits.

2.7 Conclusion

There is ample evidence in literature and research results that the idea of a limit is complex. There were certain epistemological obstacles in the historical development thereof and there still are in educational practice today.

Research on student understanding of limits is less extensive. Williams (1991:219) mentions that recent studies have confirmed that a complete understanding of limits among students is comparatively rare. Although students can calculate limits and deal with problems involving limits, they do not have a satisfactory understanding of the nature of the idea itself.

Students seem to experience conflict between formal, precise definitions and the informal, natural language interpretations used conveniently in discourse. Students commonly believe that a limit cannot be reached. They are uneasy about the mismatch between their intuitions and the answers they produce through mathematical manipulation (Lauten et.al 1994:227). It is quite discouraging to read that Cottrill et.al (1996:172) have found no reports of success in helping students overcome these difficulties because lecturers want students to have correct and complete conceptual understanding.

It is clear that the last word on the teaching and learning of limits has not been spoken. Further research is needed to shed more light on the following issues:

- Which teaching strategy is likely to bring about the desired understanding of limits in students?
- How can technology be used to improve student understanding of limits?
- In what way can knowledge of students' misconceptions be used to improve teaching strategies and the quality of their learning?

With this theoretical background in mind, the aim of this investigation is to identify the misconceptions that engineering students have of limits. This investigation is an attempt to add valuable and interesting findings and/or viewpoints.

CHAPTER 3

METHOD OF RESEARCH

3.1 Introduction

The aim of this investigation was to determine engineering students' understanding of the idea of a limit in order to identify possible misconceptions. In this chapter the various procedures used to achieve these goals are discussed. The following aspects regarding the investigation are described:

- The research design;
- The methods used to collect the data;
- A description of the items as well as a motivation for including them in the questionnaire and the interview schedule;
- The sampling procedures used for the questionnaire as well as the interviews;
- The pilot testing of the questionnaire and interviews and
- The procedures used for analysing the obtained data.

A quantitative-descriptive design, also known as a survey, was used in this research project. The investigation was done in two phases. In the first phase a questionnaire was used to gather data on students' understanding of limits. The responses were recorded and analysed.

During the second phase interviews were conducted with six students. These students were selected according to their responses in the questionnaire. Students whose responses indicated the presence of misconceptions were chosen to be interviewed.

3.2 The survey design

In survey research the investigator selects a sample of respondents and administers a questionnaire and/or conducts interviews to collect information on the variables of interest. The data that are gathered are used to describe characteristics of a certain population. Surveys are generally used to find out more about people's beliefs, attitudes, values, understandings as well as other types of information. They are frequently used in education, politics, business, government and psychology because accurate information can be obtained for large numbers of people with a small sample. Most surveys describe the incidence, frequency and distribution of the characteristics of an identified population (Mc Millan & Schumacher 2001:304-309 & de Vos 2002:142-143).

In this survey, students' understanding of the idea of a limit was to be determined in order to identify their misconceptions. The survey as research design lends itself best to answering the research questions asked in this investigation. These questions are:

- How do students understand the idea of a limit?
- What kinds of misconceptions do they have?
- How do students relate the continuity/discontinuity of a function at a point to the existence of a limit at that point? (compare 1.2 & 5.4)

The survey also allows that the information can be collected by means of a questionnaire at a relatively low cost. The questionnaire can be distributed to the whole group and collected immediately after completion. All respondents have to answer the same questions and the anonymity of each participant can be ensured.

Heid (http://www.maa.org/saum/maanotes49/109.html) stresses the fact that students' answers on tests do not always show their true level of understanding. Sometimes they understand more than their answers indicate. Other times, despite using the correct words, they do not understand what they write. The use of in-depth interviews is of great value in this regard. This content-based type of interview is not just an oral test or quiz but rather a way to dig more deeply into the complexities of students' mathematical understandings.

3.3 Data collection methods

Data were collected by means of a questionnaire followed by six one-to-one interviews. The aspects related to the questionnaire are discussed first.

3.3.1 The questionnaire

The purpose of the questionnaire was to determine how the students understand the idea of a limit of a function. The focus was on the informal definition only, since the epsilon-delta definition was not discussed in this specific mathematics course. The researcher developed her own questionnaire with the following objectives in mind:

- To determine the student's understanding of a limit per se;
- To determine the student's understanding of the continuity or discontinuity of a function at a point;
- To find out how the students understand the existence or non-existence of a limit at a point of continuity/discontinuity;
- To determine how the students deal with the graphing of a function with a discontinuity at a point.

The questionnaire consisted of questions and statements. Open questions as well as closed questions were included. Students were also asked to sketch the graph of a function with a discontinuity at a point. (See Appendix A)

3.3.1.1 Discussion of the items

The aim with Question 1 was to determine how students understand the nature of a limit. These aspects are described in the literature as common conceptions of students regarding this idea.

The researcher also wanted to determine whether the students are able to verbalise the idea of a limit when no graph or equation of a function is given as is asked in Question 2.

Question 3 was included to see if students are able to interpret the symbolic expression of a limit. Another reason was to find out more about students' understanding of the relationship between limits and function values and limits and continuity.

The aim with Question 4 was to determine the students' understanding of a discontinuity of a function and how they deal with the limit at that point. Another reason was to find out how they understand right-hand limits, left-hand limits and a limit and the function value.

The difference between Question 5 and Question 4 is that no graph of the function was presented here. The most important reason why this question was included, was to see if students are able to sketch the graph being undefined at $x = 3$. Their answers should also reveal their understanding of continuity and discontinuity and the existence of a limit at an undefined point.

3.3.1.2 Pilot testing

A pilot test of the questionnaire was conducted with the Mathematics 2 students. They were 24 students and have dealt with limits in Grade 12 as well as very briefly in the Mathematics 1 course. The aim of the pilot test was to get feedback about the questionnaire before it was used in the study. The researcher wanted to determine the time needed for completion, whether the directions and items are clear and whether the data obtained from the questionnaire reflect the real understanding of the students. The respondents were asked to give feedback on individual items as well as the questionnaire as a whole.

Each student's responses to all the items were recorded in a table. They were then analysed in terms of the following categories:

- Students' understanding of the nature of the idea of a limit;
- Students understanding of limits and continuity;
- Students' understanding of limits and function values and
- Any other findings of interest.

Their comments on the different items and the total questionnaire were also read and interpreted. This feedback was used to alter and improve the questionnaire before it was administered to the sample in the study.

3.3.1.3 Sampling procedures

The Mathematics III course had 47 students during the second semester (2004). This group of students took part in the empirical study. These students were completing the last of the required Mathematics courses for the Diploma in Electrical Engineering. They were a multicultural group to whom English is a second language.

The idea of a limit was taught during the first semester of 2004 as part of the Mathematics II course. Since then, the students have not done any work on this concept at all. They might not remember as much as was hoped, because most of them rely on their short-term memory. They do not spend enough time working with the idea in order to make it part of their long-term memory.

The questionnaire was administered to the group as a whole during one of their lectures. More or less 35 minutes were needed to complete the items. The students were not allowed to communicate with each other- each student's own work and thoughts were required.

3.3.2 In-depth interviews

Interviews have particular strengths. They are a useful way of getting large amounts of data quickly. They are an especially effective way of obtaining depth in data as needed in this investigation. However, interviews also have limitations. They involve personal interaction and cooperation is therefore essential. The primary disadvantage of the interview is its potential for subjectivity and bias. Great care must therefore be taken to remain objective and open-minded during the interview (de Vos 2002:305 & Mc Millan & Schumacher 2001:268).

The aim of these content-based interviews was to find out more about each student's personal understanding of limits in a way that the questionnaire did not necessarily reveal. In other words, to determine the nature of each student's concept image of a limit. It was a way of accessing what he/she thought in the absence of additional purposeful teaching. An interview is not a tutoring session, because the goal is to understand what the interviewee is thinking and understanding. It is also to assess the breadth and depth of his/her mathematical understandings in general.

The objectives for the interviews were very similar to those mentioned in the questionnaire. They were:

- To assess the student's graphical understanding of a limit,
- To assess the student's symbolic understanding of a limit,
- To determine the student's understanding of the continuity or discontinuity of a function at a point,
- To determine how the latter influences the students' understanding of the existence or non-existence of a limit at that point.

3.3.2.1 The interview schedule

Each student's understanding is unique and this understanding is best revealed through open-ended questions and related probes. The interview focuses on the content, what the participant is saying as well as the process, reading between the lines focusing on how he/she talks and behaves. The interview schedule consisted of a core set of six questions on limits. The purpose was to find out how the student made sense of the mathematics he/she was using (See Appendix B).

The aim with Question 1 was to see if students understand the symbolic expression of limits. Another reason was to find out how they understand the idea of infinity and how they link a limit to the behaviour of the function when x becomes very big.

In Question 2 the researcher wanted to determine how the students think about approximation and if this understanding is linked to the conception of a limit as an approximation and/or a limit as unreachable.

The idea of a limit is found in another context in Question 3 being the definition of a derivative in Differential Calculus. The researcher wanted to determine the students' understanding of this very important definition especially how they understand the limit in this formula.

Question 4 was aimed at determining whether the students understand this symbolic expression of a limit. Furthermore, to find out how they understand the relationship between a function being defined or undefined at a point and the existence of a limit. It was also included to find out whether students link limits to function values at the point in question.

The students' understanding of the graphical interpretation of a limit was investigated in Question 5. It could also give an indication whether they view the limit and the function value as being one and the same thing.

The students' responses to Question 6 were thought to reveal more about their ability to deal with an undefined point of a function when sketching its graph. It could also give an indication of their ability to sketch a graph of a function and their thoughts on the existence or non-existence of a limit at that undefined point.

3.3.2.2 Pilot testing

The interview schedule needed to be piloted and revised based on the success of those pilots in achieving the goals of the interview. Two pilot interviews were tape-recorded and later transcribed. The following questions needed to be answered objectively:

- Were the questions understood as intended?
- Were the questions adequate catalysts in determining the students' mathematical understanding?
- Were there other questions that ought to be included?

- Was the sequence of questions appropriate for the purpose of the interview?
- Did the responses reveal the student's true understanding of limits?

3.3.2.3 Sampling procedures

As mentioned earlier, there were 47 students in the Mathematics 3 course. Six of these students were selected for an interview. They were chosen according to their responses in the questionnaire. Students with unusual, interesting or incomplete responses were chosen. A high-achiever as well as a low-achiever was also included. The reason for this was to explore possible differences between good and poor students' understanding of a limit.

These interviews were tape-recorded. Attention was also given to the non-verbal expressions of the interviewees. The interviews were later transcribed and each response was analysed in depth to assess the student's understanding of a limit. The aim was to identify possible misconceptions in their understanding.

3.3.2.4 Analysis of transcripts

The data were analyzed in order to identify the misconceptions that these students have of the idea of a limit. The transcripts were analysed by making use of open coding. Open coding is the part of analysis that pertains specifically to the naming and categorising of phenomena through the close examination of data. During open coding, the data are broken down into discrete parts, closely examined and compared for similarities and differences. Questions are asked about the phenomena as reflected in the data.

Conceptualising the data becomes the first step in analysis. It involves taking apart an observation or response and giving each discrete incident or idea a name. This name then stands for, or represents a phenomenon. This is done by comparing incident with incident so that similar phenomenon can be given the same name.

Once particular phenomena have been identified in the data, the concepts are grouped around them. The process of grouping concepts that seem to pertain to the same phenomenon is called categorising. The categories are then named. The categories are further developed in terms of their properties (de Vos 2002:346-347).

The data were grouped according to the following aspects or categories:

- Students' understanding of the nature of a limit;
- Students' understanding of limits and continuity;
- Students' understanding of limits and function values and
- Any other interesting findings.

3.4 Conclusion

The procedures that are implemented during an investigation such as this play an important role in determining the ultimate success thereof, as well as the validity and reliability of the end results. This chapter focused on the research design used in this research project namely the survey design. The methods used to gather the necessary data were described. These are the questionnaire and interviews. A motivation or reason for the inclusion of each item in the questionnaire as well as the interview schedule was also given.

The sampling procedures used to determine the respondents in the questionnaire and the interviewees were discussed. Another important aspect of the investigation namely the pilot testing of the chosen measuring instruments was also discussed. Finally, the planned procedure for analysing the data was described. In the following chapter the data are represented, analysed and interpreted in order to reveal the research findings.

CHAPTER 4

DATA ANALYSIS AND INTERPRETATION

4.1 Introduction

The purpose of this chapter is to present, analyse and interpret the data that were gathered in the empirical investigation. The aim of the data collection was to determine the students' understanding of the idea of a limit in order to identify possible misconceptions. The focus was on the idea of the limit of a function as described by the informal definition and not the epsilon-delta definition.

The data collected in the questionnaire are represented first. Thereafter, data capturing essential and relevant parts of students' responses in the transcribed interviews are presented. The data are then analysed and interpreted in order to reveal the misconceptions that the students have.

4.2 Representation and analysis of the data in the questionnaire

The questionnaire was administered to 42 students during the last 40 minutes of a two-hour mathematics lecture. On that specific day, five students were absent and never completed the questionnaire at all. The students were informed about the questionnaire the week before, but not on the nature of its content. The reason for this was that the researcher did not want them to revise what they had done earlier. They had to rely on what they could remember. Some students commented that they could not remember much about limits because they had not worked with the idea for about eight months.

4.2.1 Question 1

The number of true and false responses to the five questions in Question 1 is shown below.

Table 1:

Item number	True	False	No response
1.1	21	18*	3
1.2	37	4 *	1
1.3	30 *	10	2
1.4	24	17 *	1
1.5	22 *	17	3

* Correct answer of the item

Some common mistakes were:

- Twenty-one of the students (50%) indicated that a limit is a number past which a function cannot go. This indicates that they view a limit as a boundary (item 1.1)
- Eighty eight percent of the students ($\frac{37}{42}$) thought that a limit describes how the value of a function moves as the value of x moves towards a certain point (item 1.2). These students probably associate a limit with the function value for a specific value of x.
- In item 1.3, ten of the students (24%) disagreed with the statement that a limit is a number that the function value gets closer to but never reaches. The other 76% of the students viewed the limit as unreachable.

- Twenty-four students (57%) were incorrect by saying that a limit is an approximation that can be made as accurate as you wish (item 1.4)
- Forty percent (17) of the students did not agree that the limit of a function can fail to exist at a point as asked in item 1.5. These students are of the opinion that there must be a limit at a certain point on a function.

4.2.2 Question 2

In this question students were asked to give their own explanation of the limit of a function. Students experienced difficulty in answering this question as indicated by the low response rate of 52, 4 %. Some of the incorrect responses are as follows:

- Student 27: "*When x approaches s, then L gets very big". This student probably associates a limit with the idea of infinity.*
- Student 20: "*When s approaches the number L, x approaches zero*"
- Student 1: "*The function is continuous at x = L*"
- Student 4: "*It means that either it converges or diverges*"
- Student 11: "*The limit shows how the function is going to be*"
- Student 25: "*A function gives a certain value for every value for x*"
- Student 35: "*Approaches infinity*"

Three students (numbers 9, 30 and 41) tried to answer the question by simply writing the wording of the question down. One of the more sensible answers was "When x approaches s, it will produce the limit, L."

4.2.3 Question 3

In this question the students were asked to circle the number of the statement(s) that must be true. The responses of the 42 students to this question are represented in tabular form below:

Table 2:

No	3.1	3.2	3.3	3.4		No	3.1	3.2	3.3	3.4
1		*				22	*		*	
2			*			23			*	
3				*		24	*			
4	*					25			*	
5				*		26	*			
6		*				27			*	
7	*		*			28		*		
8			*			29				*
9	*					30	*			
10	*					31		*		
11				*		32			*	
12	*					33			*	
13		*	*			34			*	
14				*		35				*
15	*					36			*	
16			*			37	*		*	
17			*			38			*	
18			*			39			*	
19			*			40	*			
20			*			41	*			
21	*					42	*	*	*	

- Only six students (14%) chose 3.4, which is the correct answer. They realized that none of the other options must be true.
- Fifteen (36%) of the students encircled item number 3.1. This revealed that they view a limit as a substitution process.

- Six students chose 3.2 as an answer (14%). They were of the opinion that a function must be continuous at a point to have a limit at that point.
- Twenty-one of the students (50%) encircled 3.3 thus indicating that a function must be defined at a point to have a limit at that point.
- Three students (no. 13, 22 and 37) encircled two items and student 42 encircled three items. They were under the impression that a function must be defined and continuous at a point to have a limit and that the limit is equal to the function value at the point.

4.2.4 Question 4

Summary of the 42 answers given in each item of this question:

Table 3:

Item number	Correct	Incorrect	No response
4.1	2	26	14
4.2	10	15	17
4.3	13	15	14
4.4	9	17	16
4.5	13 *	9	20
4.6	9	12	21
4.7	5	9	28

* The correct answer to item 4.5 is – . Only two students had this answer but 11 of them had . The latter answer was also considered to be correct which gives a total of 13.

Common wrong answers are:

- Zero, -2, 3 and -3 in item 4.1

- In item 4.2, fourteen (33%) of the students had 3 or –3 as answers
- Nine students thought that the limit of $F(x)$ at $x = 3$ equals the function value of F at $x = 3$ (item 4.7)

The students experienced difficulty with the discontinuity at $x = 3$ as is clear from the small number of correct responses. They did not realize that the function value of $F(3) = 3$ and not two and that the limit of $F(x)$ as x approaches three is actually two.

4.2.5 Question 5

In this question students had to sketch the graph of a function and thereafter answer questions on the graph. The results are indicated in the table:

Table 4:

Item number	Correct	Incorrect	No response
Graph	3	22	17
5.1	6	19	17
5.2	17	9	16
5.3	14	15	13
5.4	3	11	28
5.5	3	18	21

Typical incorrect answers that occurred:

- In item 5.1, eight students (19%) said that $f(x)$ becomes zero at the point $x = 3$ instead of being undefined at the point.
- Nine (21%) of the students thought that the graph is continuous at the point $x = 3$ (item 5.2).
- Fifteen students were of the opinion that $f(x)$ does not have a limit at $x = 3$ (item 5.3)
- Eleven students wrote 0, 2 or 3 in item 5.4.

- In item 5.5, seventeen students answered 0, 2 or 3.

The students experienced difficulty in sketching the graph of $f(x)$. Seventeen of the students (40%) did not even try to sketch the graph and consequently could not answer the questions on the graph with ease. Some of the students had zero as the y-intercept instead of one. Others had the gradient of the graph incorrect. Some of them just ignored the point $x = 3$ because they did not know how to deal with the discontinuity at that point. In order to try something, some of them made $y = 0$ which resulted in an x-intercept at $x = 3$.

4.3 Representation and analysis of the data from the interviews

Two weeks after the completion of the questionnaire, the interviews were conducted. The second semester lectures were almost coming to an end and the test week was due to start the following week. Due to a time constraint, students were asked to volunteer for the interviews. At first they were reluctant as they thought that they could not remember much about limits. As mentioned earlier, they had not worked with limits for about eight months. This could be seen as a reasonable excuse. In the end there were six volunteers and they were interviewed one after the other on the same day. Each interview was tape recorded and lasted about 10-15 minutes. The numbers one to six were allocated to the students according to the sequence in which they were interviewed.

The students were not asked to calculate limits algebraically, because they were able to find most limits in that way. The questions were aimed at finding out how they understand the idea of a limit, how they think about limits and function values and how they understand the effect of a discontinuity at a point on the existence of the limit at the same point. (See Appendix B for the interview schedule)

4.3.1 Excerpts from interviews on question 1

In question 1, students were asked to comment on the symbolic expression $\lim_{x \to \infty} \frac{1}{x} = 0$. A rough sketch of the graph of the function was available for use by the students. Not one of the six students could remember that the function is called a hyperbola. Some of the responses to the question: “What happens to the function when x approaches infinity?” were:

S_1: *When x gets very big, it becomes smaller.*

S_2: *f(x) gets closer to zero. It can’t be zero, it just reaches zero*.

S_5: *The graph approaches the x-axes, but never touches it.*

S_6: *If x approaches infinity, f(x) also approaches infinity*.

The students realized that if x becomes very big, the function value will reach zero, although it will never be zero itself. They all used the value of $f(x)$ and did not think that the limit will actually be zero. The explanation was that if you divide one by numbers that get bigger and bigger, the answer will eventually reach zero. They all mentioned that the limit approaches zero, but never reaches zero. This shows that they viewed a limit as a dynamic process and not as a static object similarly to what Bezuidenhout (2001:491) and Williams (1991:219) concluded from their research on limits.

On the question whether we can write $f(\ )$ = 0, five of the six students answered yes. Two of the responses were as follows:

S_2: *No, well umm, it depends on the formula. Infinity is a symbol representing numbers that are very big, numbers that you can’t really write.*

S_6: *Yes. If you divide one by numbers getting bigger and bigger,* $\frac{1}{x}$ *becomes smaller and smaller and will eventually reach zero.*

This indicated that they see the symbol, , as a number representing big numbers and not really as a symbol that represents infinity. Student S_2 named a symbol, but he gave the impression that you could find f( ) and that its value depends on the formula of a specific function. In this specific case, the function reaches zero when x increases.

4.3.2 Excerpts from the interviews on question 2

In this question, students were asked what the value of $0{,}\bar{9}$ is. None of them could remember that it is called a recurring decimal number. They all said that the value is one, but explained it in interesting ways.

S_2: *It is equal to one, as when you round off numbers, although it is not right. When you use your calculator, it won't give you one.*

S_1: *If you round off this thing, it is close to one.*

S_3: *It means that there are still more nines. We can say it is almost equal to one, but not really one*.

S_4: *If I have to round it off, I can say it is equal to one.*

S_5: *It is close to one.*

S_6: *We can try to make it one*.

These responses, with the exception of S_4, indicated that they were thinking of the process of rounding off numbers and that the answer is close to one. This way of thinking is very similar to that in question 1 where a limit was seen as a number that a function is approaching or getting closer to, but never reaches.

4.3.3 Students' explanations in question 3

This specific question dealt with the idea of a limit in a different context namely that of $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. They were asked what the meaning of the limit in this symbolic expression is as well as what this expression is usually used for. The students experienced great difficulty with this question as indicated by their responses:

S_1: *We use the formula for the gradient. If h becomes zero, it is going to be x.*

S_2: *We use it to calculate the limit. In this formula we do not really care about h. It just helps us to calculate everything. If we divide by a zero, it will not be possible we can not use a value at this point. We cancel out the h.*

S_3: *In Calculus,…in a sequence? I do not know, maybe to find f(x).*

S_4: *I think it is one or two equations and then you have to substitute. If x approaches nought, no, when h approaches nought, no, we can't divide by nought. Ach, I don't know. If you divide by zero, it is undefined.*

S_5: *They give you f(x) and then you substitute f(x) into all the x's. I do not really know. If h approaches zero, it will be zero.*

S_6: *I would say if h approaches zero, then f(x) will approach infinity. Ja. I'm not sure where we use it.*

From these explanations it is clear that the students did not have a clear concept image of the idea of a limit in this context. They can calculate a derivative by using this formula if they are asked to do so, but they cannot describe the concept correctly in their own words.

4.3.4 Excerpts from the responses to question 4

This question dealt with another type of symbolic expression namely $\lim_{x \to 1} f(x) = 3$. The students were asked to describe the meaning of the expression in their own words. The focus was on whether the function has to be defined at a point to have a limit at that point and what the relationship between the function value and the limit at that point is. Some of the responses were as follows:

S_2: *It says where x is one, our limit is three so as x approaches a one, our f(x) will approach three. It just approaches the one or the three, it never reaches the one or the three.*

S_3: *It means that the limit does exist at three.*

S_4: *Ok, it means that if x approaches one, then the graph goes to three.*

S_5: *As x approaches one, the function approaches three.*

Five of the six students were adamant that the function must be defined at $x = 1$ to have a limit of three. If the function is undefined at that point, there will be no limit. Student S_2 was the only one who thought that the function needn't have to be defined at the point. If it is undefined, it can still have a limit. Further, the other five students were all very confident that $f(1)=3$. Student S_2 motivated his answer in this way:

The limit is found by substitution, it depends on the formula you have.

4.3.5 Responses to question 5

In this question, the students were given the graph of a function, $f(x)$. The formula of the function was not given. They had to show where the limit of $f(x)$ as x 2 is as well as what they think $\lim_{x \to 2} f(x)$ is. Every student pointed to the point three on the y-axes. Each student motivated his/her answer by saying that if $x = 2$, the value of $f(x)$ is equal to 3.

They did not use the word limit, but rather the value of $f(x)$ at x = 3. All of them said that $\lim_{x \to 2} f(x) = 3$ the same reason as above.

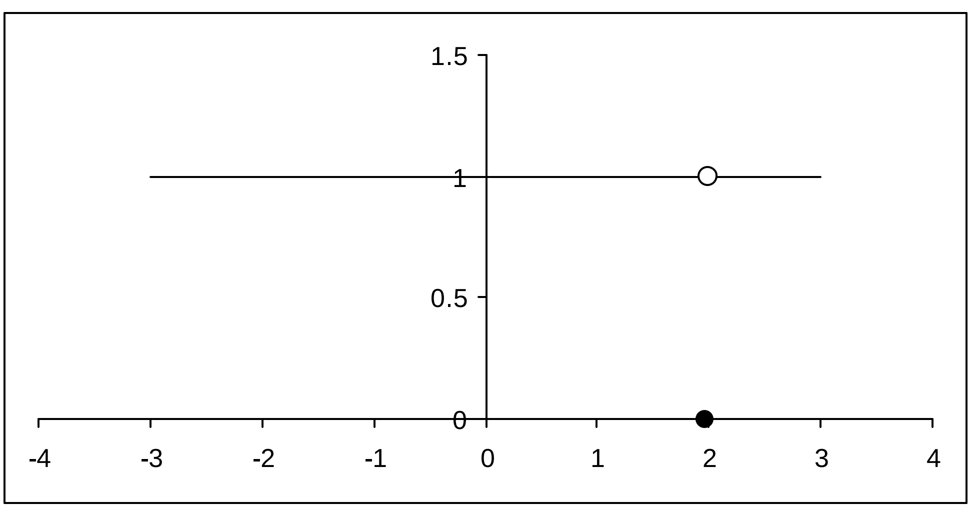
These responses indicated that the limit and the function value were seen as being the same, i.e. $\lim_{x \to 2} f(x) = f(2)$. Bezuidenhout (2001:495) is of the opinion that this misconception may be mainly due to the use of a method of substitution to find limits algebraically. This method was also used when the idea of a limit was taught to the students at the beginning of the year.

4.3.6 Excerpts from interviews on question 6

In the sixth and final question of the interview, students were asked to sketch the graph of the function, $f(x) = \dfrac{x-2}{x-2}$ which has a discontinuity at the point x = 2. They had to say whether the function has a limit at x = 2 as well as what the value of $f(2)$ is.

All the students attempted the graph by choosing values for x and calculating the corresponding $f(x)$ values. At the point x = 2, they had some difficulty. Students S_1, S_3 and S_4 had the following graph:

Fig 3



On the question, what happens when $x = 2$, five of them answered that $f(x)$ becomes zero. At first, they did not think of a discontinuity at that point. On the question, what $\frac{0}{0}$ is, three of them answered that it was zero. The other three interviewees were correct in saying that it was undefined, although $f(2)$ remained equal to zero for some of them.

Student S_2, who normally gets an A-symbol in mathematics, was the only one who said that the function was undefined at $x = 2$ (although he was one of those who said that $f(x)=0$). He elaborated on this by saying that the function is not continuous at the particular point and that there is no limit at $x = 2$. On the question whether the limit could be one, his response was no, because the function does not approach one. He was under the impression that there was no limit at the point because the function was undefined at the point.

Due to a hint that the function could be discontinuous at $x = 2$, the students realized that they had to make an open circle at $x = 2$ because the function was undefined at the point. Three of the students could then say that the limit at $x = 2$ is equal to one, although they still believed that the value of $f(x)$ at $x = 2$ is equal to zero.

4.4 Students' conceptions of the idea of a limit

During the interviews, students were given the opportunity to express their ideas and thoughts not only on limits, but also on infinity and continuity. As mentioned earlier, the students had not been doing any work on limits for more than eight months and had to rely on their memories to provide answers to the set questions. One might expect that this could reveal their true understanding even better in comparison to having done the work only recently. At times, some of them seemed a little embarrassed although they tried their very best throughout the interviews.

4.4.1 Summary of the identified conceptions on limits

The following conceptions on the nature of a limit were identified:

- Students see a limit as a boundary (item 1.1)
- Students see a limit as unreachable (item 1.3)
- Students see a limit as an approximation (item 1.4)
- Students are under the impression that a function will always have a limit at a point (item 1.5)
- Students view a limit as a dynamic process and not as a static object (item 1.2 and interview questions 1 & 6)

Conceptions regarding the relationship between a continuous function and a limit were:

- Students think that a function has to be defined at a point to have a limit at that point. A function that is undefined at a certain point does not have a limit.
- Students think that when a function has a limit, then it has to be continuous at that point. (Item 3 of the questionnaire and Question 4 & 6 of the interview)

Other misconceptions were:

- The limit is equal to the function value at a point, i.e. a limit can be found by a method of substitution (Questions 3, 4 & 5 of the questionnaire).
- When one divides zero by zero, the answer is zero (Interview question 6). Most of the students know that any other number divided by zero is undefined.

- When one rounds a number off, the number gets closer to another number, but does not equal it, ex. 0,999… 1 (Interview question 2).

4.4.2 Interpretation of the findings

The identification of these conceptions points to the fact that most of these students have very limited conceptions about limits and continuity. This could be ascribed to the fact that little time (more or less four hours) was allocated to the teaching of limits and continuity as an introduction to the derivative and differentiation of various functions. The time delay from working with limits and encountering them again in the questionnaire and interviews could also play a role in these limited conceptions. At the beginning of the second semester of 2004, limits were completely eliminated from the Mathematics 2 syllabus. The course now starts with derivatives and the focus is largely on the application of the rules used in finding derivatives. This might change again at the beginning of the next academic year (2005).

Several researchers have experienced the difficulty that students have with limits and have investigated the conceptions that they have. The conceptions mentioned above, are in accordance with those that these researchers have found. Williams (1991:219) and Tall (1992:501) concluded that conceptions of limit are often confounded by issues of whether a function can reach its limit, whether a limit is actually a boundary, whether limits are dynamic processes or static objects and whether limits are inherently tied to motion concepts.

Thabane (1998:66) and Bezuidenhout (2001:493) investigated first year students' understanding of limits and found that students have in mind that the limit is about mechanical substitution. They also found that students only think about the manipulative aspects and do not focus on the concept of a limit. According to their findings, students talk of a limit not being defined at a point when it is the function that is not defined at that point. Laridon (1992:398) confirms the fact that students think that limits simply entail substituting the value at which the limit is to be found, into the expression. He also found that students think that limits and function values are the same. Furthermore, limits are only encountered when trying to ascribe a value to a function at a point where it is undefined (Example: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$).

Lauten, Graham & Ferrini-Mundy (1994:227) go a step further by saying that students commonly believe that a limit cannot be reached and are uneasy about the mismatch between their intuitions and the answers they produce through mathematical manipulation. They seem to experience conflict between formal, precise definitions and the informal, natural language interpretations used commonly in the discourse. Tall (1992:502) is of the opinion that the formal definition is also fraught with cognitive problems. In this course, the students only dealt with the informal definition, the formal definition was not discussed.

Tall in *Limits, Infinitesimals and Infinities* (http://www.warwick.ac.uk/staff/David.Tall/themes/limits-infinity.html) explains that the primitive brain notices movement. The mental notion of the limit of a function is more likely to focus on the moving point than on the limit point. The limit concept is conceived first as a process and then as a concept or object. Anna Sfard (1991:1) calls this the dual nature of mathematical conceptions. She is of the opinion that abstract notions such as function or limit, can be conceived in two fundamentally different ways: structurally as objects, and operationally as processes. These two approaches, although ostensibly incompatible, are in fact complementary. The processes of learning and of problem-solving consist in an intricate interplay between operational and structural conceptions of the same notions. The operational conception is, for most people the first step in the acquisition of new mathematical notions. The transition from computational operations to abstract objects is a long and inherently difficult process. This could explain why most students see a limit as a process and not as an object- not enough time is allowed to make this transition.

4.5 Conclusion

The data analysis and data interpretation shed some light on the ways in which students think about limits. The identified misconceptions are very similar to the ones identified by other researchers. The findings of this study show that many students' knowledge and understanding rest largely on isolated facts and procedures and that their conceptual understanding of limits, continuity and infinity is deficient. Lecturers ought to become aware of their students' understanding and possible misconceptions. Diagnosing the nature of students' conceptual problems enables lecturers to develop specific teaching strategies to address such problems and to enhance conceptual understanding.

CHAPTER 5

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

The idea of a limit is viewed by many authors as the most fundamental concept of calculus. Every major concept of calculus namely derivative, continuity, integral, convergence or divergence is defined in terms of limits. The idea of a limit is thus a very important concept, but one of the difficult ones for calculus students to understand (compare 1.1). This concept has a very long history which could reflect its complexity. It took mathematicians centuries to develop the idea of a limit, although the modern definition is less than 150 years old (compare 2.2).

Different investigations done on limits, show that the majority of students do not master the idea of a limit, even at an advanced stage of their studies. This does not prevent them from working out exercises, solving problems and passing their examinations. A possible explanation for this could be that this concept is often taught in isolation - students encounter the idea for a period of time and hardly ever again. Students do not understand why this idea is fundamental to calculus and cannot link it to other aspects of calculus (compare 1.1).

There is much concern about the large numbers of students taking calculus and the rote, manipulative learning that takes place. Most of the time, the approach in textbooks as well as the teaching strategies focus on the manipulative aspects of limits and not on understanding the limit per se. These practices do not promote conceptual understanding and could result in misconceptions (compare 1.2 & 2.7).

Students need time to develop the correct understanding of a limit - a concept image that is in accordance with the concept definition (compare 2.5.2). As a result of this, the general tendency now is for less emphasis on skills and greater emphasis on the understanding of the underlying concepts of calculus such as limits.

5.2 Summary of the findings

5.2.1 Summary of the literature review on the nature of limits

Tall (1992:501) quotes Cornu who views the idea of a limit as the first mathematical idea that students encounter where one does not find a definite answer by a straightforward mathematical calculation. This is quite a different (and unexpected) situation to deal with in comparison to the students' previous experiences with mathematical ideas. Tall (1992:495) confirms this when he mentions that the idea of a limit signifies a progression to a higher level of mathematical thinking which he calls advanced mathematical thinking. This progression involves a difficult transition. It starts off where concepts have an intuitive basis founded on experience to one where they are specified by formal definitions and their properties reconstructed through logical deductions. During this transition (and long after), earlier experiences and their properties as well as the growing body of deductive knowledge, exist simultaneously in the mind. This produces a wide variety of cognitive conflict that can act as an obstacle to learning (compare 2.4.2).

Furthermore, abstract ideas such as limit could be conceived operationally as processes and structurally as objects. The dual character of mathematical ideas that have both a procedural and a structural aspect was investigated by many researchers. Anna Sfard (1991:20) used the word reification to describe the gradual development of a process becoming an object. Dubinsky (1991:101) postulated a theory of how concepts start as processes which are encapsulated as mental objects that are then available for higher-level abstract thought. Gray & Tall mentioned in Kidron & Zehavi (2002:206) formulated the notion of 'procept' as a combination of process and concept evoked by a single symbol. In the case of a limit, the same symbol $\lim_{n\to\infty} a_n$ may be simply a process (the process of tending to a limit), but if it can be seen as a new entity that is detached from the process that produced it, then it becomes an object. Other processes can then be performed on this object to obtain a new kind of static construct (Kidron & Zehavi 2002:205-206).

5.2.2 Summary of students' conceptions of limits

In the light of what was said above, one can understand that most students find it difficult to come to terms with this complex idea. The following types of misconceptions are described in the literature (Williams 1991:219; Thabane 1998:65; Laridon 1992:398; Cottrill et. al 1996:173).

- Confusion over whether a function can reach its limit,
- Whether limits are inherently tied to motion concepts,
- Confusion over whether a limit is actually a bound and
- Confusion whether limits are dynamic processes or static objects.

Thabane (1998:65) and Laridon (1992:398) found that students think that 'limits' simply entail substituting the value at which the limit is to be found into the expression. They often think that limits are only encountered when trying to ascribe a value to a function at a point where the function is undefined. Students often think that limits and function values are the same. They talk of a limit not being defined at a point when it is the function that is not defined at a point. Students think only about the manipulative aspects and do not focus on the idea of a limit. One explanation for this could be the way in which the concept is taught to students. Another reason could be the approach used in most calculus text books namely the use of formulas and calculations of limits. Bezuidenhout (2001:489) argued that formula might make calculations easier, but do not promote understanding. In this way the source and nature of misconceptions may go undetected (compare 2.5.2).

5.2.3 Summary of the research approach

This research project was undertaken to determine engineering students' understanding of the limit concept in order to identify possible misconceptions.

This investigation was in the form of a quantitative-descriptive (survey) design and was done in two phases. In the first phase a questionnaire on the idea of a limit was given to a group of 42 third-year mathematics students. The questions focused on the concept itself and not on calculating limits or the manipulative aspects thereof (Appendix A).

The students had not done any work on limits for a period of about eight months. At that time, limits were discussed according to their prescribed textbook, due to the inexperience of the researcher. It was just touched on as an introduction to the slope of a tangent to a curve and the derivative which covered a large part of the syllabus. At first, tables were used to find limits where the value of *x,* where the limit was to be found, was omitted. Later limits were calculated algebraically. The continuity of functions was also discussed briefly. The data in the questionnaires were recorded and thereafter analysed in order to find out more about these students' understanding.

The second phase consisted of interviews with six of the students. The interview schedule consisted of six questions with the aim once again to find out more about their understanding (Appendix B). The interviews were tape-recorded and later transcribed. The responses were analysed according to the nature of a limit, the relationship between continuity and limits and other conceptions for example rounding off and division by zero. A summary of these findings are given in 5.2.4.

5.2.4 Summary of the findings in the empirical investigation

The findings in the empirical investigation are very similar to the descriptions that are found in the literature. The findings are as follows (compare Chapter 4):

Findings regarding students' views on the nature of a limit:

- Some students see a limit as a boundary past which a function cannot go,
- Some of them see a limit as unreachable i.e. the limit comes closer and closer to a point but never reaches the specific value,
- Many students see limits as a dynamic process and not as static objects which is related to the above-mentioned view,
- Some students view a limit as an approximation meaning that you can make the limit as accurate as you wish.

Other misconceptions related to limits are:

- A function must be defined at a point to have a limit at that point - discontinuous functions can not have a limit at the point of discontinuity,
- A function has to be continuous at a point to have a limit at that point,
- The value of the limit at a point is equal to the function value at that point i.e. $\lim_{x \to 3} f(x) = f(3)$. Students view finding limits and the substitution process as exactly the same thing.

Other interesting findings:

- In the case of an approximation like 0,999… the students see it as less than one and not equal to one,
- Zero divided by zero is equal to zero although most of them know that any number divided by zero is undefined (compare Chapter 4).

5.3 Limitations of the study

The original aim of this research project was to identify specific misconceptions that the second year engineering students have of the idea of a limit after having discussed it in class. The plan was to teach this idea by using alternative strategies to lectures such as group work, problem solving tasks and various tasks on their computer program called MATLAB. In the past this specific course started with limits which were followed by derivatives and the whole topic of differentiation.

At the beginning of the second semester of 2004, limits were completely omitted from the syllabus. Due to this, the research project had to be adapted. There was no time in the semester plan to include this topic. The decision was then taken to use the present third year students as subjects, although they had not worked with limits for a period of about eight months. The second year students were used in the pilot study to test the content of the questionnaire and two students were interviewed. These students did limits in Grade 12 and just revised it briefly in their Mathematics I course.

Two limiting factors have been identified. They are:

- The time span (of eight months) between when limits were discussed with the third year students and when the real answering of the questionnaire took place and
- The fact that the topic was taught and learnt in a traditional way strictly according to the prescribed textbook.

5.4 Conclusion

The project was undertaken to find answers to the following research questions:

1 How do students understand the idea of a limit?
2 What kind of misconceptions do they have?
3 How do they relate the continuity/discontinuity of a function at a point to the existence of a limit at that point? (compare 1.3)

These research questions can now be answered as follows:
The outstanding observation was that students see a limit as unreachable. This could be due to the language used in many textbooks

to describe limits for example ‘tends to’ and ‘approaches’. These words are verbs or action words and as Monaghan (1991:23) describes, the action in this mathematical setting is ‘getting to a limit’ sets up a dynamic interpretation of a limit. These words represent a movement towards a point without ever getting there. This explains the dynamic view of the majority of students that a limit is a process and not an object. Another view of a limit that the students have is that a limit is a boundary point. This could be because of their experience with speed limits, although that could always be exceeded.

Many students think that a function must be defined at a point to have a limit at that point. If there is a point of discontinuity there could not be a limit at that point. They also think that a limit is equal to the function value at that point. This is true for continuous functions but not for piecewise functions or discontinuous functions.

5.5 Recommendations for further research

The following recommendations are made as a result of the findings in this investigation:

- To investigate those cognitive processes which are assumed to be involved in the student’s acquisition of the idea of a limit;
- To investigate the role of a problem-centred approach on the learning of limits and students subsequent understanding;
- To investigate the effect that the use of technology, for example computer programs and graphing calculators might have on students’ understanding of limits;
- To develop a teaching strategy that will lead to the most accurate understanding of limits.

The first reason for these recommendations is that the researcher was curious to find out more about these issues, but they could not be addressed due to the changes in the syllabus for the specific mathematics course. There was no time at all in the semester plan to include limits. Another reason is the acknowledgement of the fact that students construct their own meaning of mathematical ideas and that knowledge should not be transmitted to them in a ready-made manner. The researcher wanted to move away from the old-fashioned talk-and-chalk method to a more contemporary approach of allowing students to construct their own meaning and developing their own understanding of limits by solving problems and working in groups.

The investigation of these four aspects related to the teaching and learning of this important idea in calculus, ought to lead to more effective teaching strategies. The use of better teaching strategies ought to result in more precise understandings of the students and fewer misconceptions. This is the aspiration of every mathematics educator worldwide.

BIBLIOGRAPHY

Bezuidenhout, J. 1998. First-year University Students' Understanding of Rate of Change In *International Journal of Mathematical Education in Science and Technology 29(3): 389-399.*

Bezuidenhout, J. 2001. Limits and Continuity: Some Conceptions of First-year Students In *International Journal of Mathematical Education in Science and Technology 32(4): 487-500.*

Cornu, B. 1991. Limits In *Advanced Mathematical Thinking,* edited by Tall, D. ME Library: Kluwer Academic Press: 153-166.

Cottrill, J; Dubinsky, E; Nichols, D; Schwingendorf, K; Thomas, K & Vidakovic, D. 1996. Understanding the Limit Concept: Beginning with a Coordinated Process Scheme in *Journal of Mathematical Behaviour* 15:167-192.

Davis, R.B. & Vinner, S. 1986. The Notion of Limit: Some Seemingly Unavoidable Misconception Stages In *Journal of Mathematical Behaviour 5: 281-303.*

de Vos, AS (ed). 2002. *Research at Grass Roots for the Social Sciences and Human Service Professions 2nd edition.* Van Schaik: Pretoria.

Dreyfus, T & Eisenberg, T. 1983. The Function Concept in College Students: Linearity, Smoothness and Periodicity in *Focus on Learning Problems in Mathematics* 5: 119-132.

Dubinsky, E 1991. Reflective Abstraction in Advanced Mathematical Thinking in Tall, E (ed) In *Advanced Mathematical Thinking.* ML Library: Kluwer Academic Press: 95-123.

Ferrini-Mundy, J & Lauten, D. 1993. Teaching and Learning Calculus in Wilson, PS (ed) *Research Ideas for the Classroom: High School Mathematics.* New York: MacMillan NCTM: 155-173.

Finney, RL & Thomas, GB. 1993. *Calculus 2nd Ed.* Addison-Wesley Publishing Company: New York.

Heid, MK. In-depth Interviews to Understand Student Understanding (http://www.maa.org/saum/maanotes49/109.html).

History of Limits (http://occawlonline.pearsoned.com/bookbind/pubbooks/thomas awl/ chapter1/medialib/cu).

Human, PG. 1993. Die Rasionaal vir Probleemgesentreerde Wiskunde-onderwys in Wessels, DCJ 2000 *Reader for MEDMA.* Unisa: Pretoria: 22-23.

Kidron, I & Zehavi, N. 2002. The Role of Animation in Teaching the Limit Concept in *International Journal of Computer Algebra in Mathematics Education* 9(3):205-227.

Laridon, PE. 1992. Teaching Calculus In Moodley, M, Njisane, RA & Presmeg, NC (eds) *Mathematics education for Inservice and Pre-service teachers.* Pietermaritzburg: Shuter & Shooter: 381-401.

Lauten, AD, Graham, K & Ferrini-Mundy, J. 1994. Student Understanding of Basic Calculus Concepts: Interaction with the Graphics Calculator In *Journal of Mathematical Behaviour* 13:225-237.

Math Academy Online, 1997-2004. Limits in *Platonic Realms Interactive Mathematics Encyclopedia.* (http://www.mathacademy.com/platonic_realms/encyclopedia/encyhome.html

McMillan, JH & Schumacher, S. 2001. *Research in Education (5th Ed) A Conceptual Introduction* Addison Wesley Longman: New York.

Monaghan, J. 1991. Problems with the Language of Limits in *For the Learning of Mathematics* 11(3):20-24*.*

Murray, H; Olivier, AI & Human, PG. 1998. Learning through Problem Solving. Malati project: IWWOUS. In Wessels, DCJ. 2000. *Reader for MEDMA.* Unisa: Pretoria: 30-38.

Olivier, A. 1989. Handling Pupils' Misconceptions In *Presidential address delivered at the Thirteenth National Convention on Mathematics, Physical Science and Biology Education, 3-7 July:* 1-12.

Sfard, A. 1991. On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin In *Educational Studies in Mathematics* 22: 1-36.

Tall, DO. 1980. Mathematical Intuition, with Special Reference to Limiting Processes, *Proceedings of the Fourth International Congress on Mathematical Education :* Berkeley: 170-181.

Tall, DO & Vinner, S. 1981. Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics 12: 151-169.*

Tall, D. (ed) 1991. *Advanced Mathematical Thinking.* ME Library : Kluwer Academic Press.

Tall, D. 1992. The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity and Proof In Grouws, DA (ed) *Handbook of Research on Mathematic Teaching and Learning.* New York: McMillan: 495-511.

Tall, D. 2004. Limits, Infinitesimals and Infinities. (http://www.warwick.ac.uk/staff/David.Tall/themes/limits-infinity.html).

Tall, DO & Vinner, S. 1981. Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics 12: 151-169.*

Thabane, JL. 1998. *Students' Understanding of the Limit Concept in a First- year Calculus Course.* MSc Dissertation: Wits University.

Vinner, S. 1991. The Role of Definitions in the Teaching and Learning of Mathematics In Tall, D (ed) *Advanced Mathematical Thinking.* ML Library: Kluwer Academic Press: 65-81.

White, P & Mitchelmore, M. 1996. Conceptual Knowledge in Introductory Calculus In *Journal for Research in Mathematics* 27(1):79-95.

Williams, SR. 1991. Models of Limit held by College Calculus Students in *Journal for Research in Mathematics Education* 22(3):219-236.

Appendix A

THE IDEA OF A LIMIT

QUESTIONNAIRE

The purpose of this questionnaire is to determine how you understand the idea of a limit. Please answer these questions as best as you can. The results will be used in a research project currently in progress at this institution. You will remain anonymous throughout the investigation. Thank you for your cooperation.

Name: .... Student no: ....

Mark the statements about limits as true or false. Circle the T if you think that the statement is true or F if you think that it is false (not true).

1.1 T F A limit is a number past which a function cannot go.

1.2 T F A limit describes how the value of a function moves as the value of x moves towards a certain point.

1.3 T F A limit is a number that the function value gets closer to but never reaches.

1.4 T F A limit is an approximation that can be made as accurate as you wish.

1.5 T F The limit of a function can fail to exist at a certain point.

2 Describe in your own words what it means to say that the limit of a function f as x $\rightarrow$ s is some number L.

3 Which statement(s) in 3.1 to 3.3 below *must be true* if f is a function for which $\lim_{x\to2} f(x)=3$? Encircle the number in front of your choice(s). Encircle 3.4 if you think that none of them are true.

3.1 $f(2) = 3$

3.2 f is continuous at the point $x = 2$

3.3 f(x) is defined at $x = 2$

3.4 None

For the function g graphed below, find

4.1 $\lim_{x\to3^-} g(x)=$ ……..

4.2 $\lim_{lix\to3^+} g(x)=$ …….

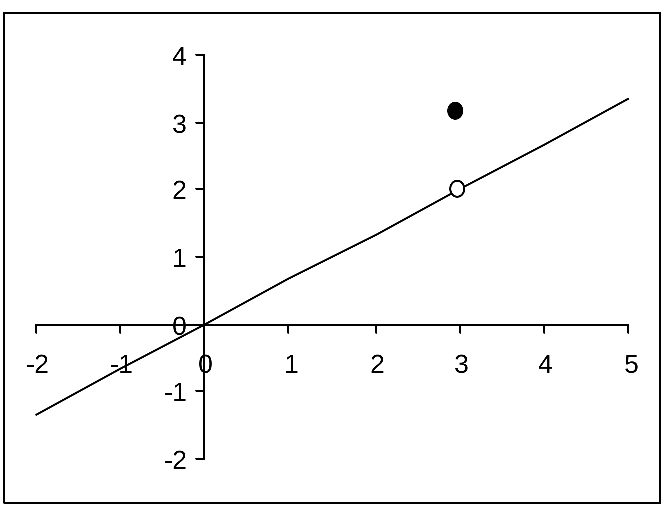
4.3 $\lim_{x\to3} g(x)=$ ……..

4.4 g(3) = ……

4.5 $\lim_{x\to-\infty} g(x)=$ ………

4.6 $\lim_{x\to\infty} g(x)=$ ……

4.7 Complete: The limit of g as x 3 is (equal/not equal) to the function value of g at x = 3. Underline your choice.



5. Sketch the graph of the function $f(x)=\dfrac{x^2-9}{3x-9}$

Answer the following questions:

5.1 What happens at the point where $x = 3$?

5.2 The function is (continuous / discontinuous) at the point $x = 3$ (Underline your choice).

5.3 Does the limit of $f(x)$ exist at $x = 3$? (YES / NO)

5.4 If you have answered yes in 5.3, what is the limit of $f(x)$ at $x = 3$?

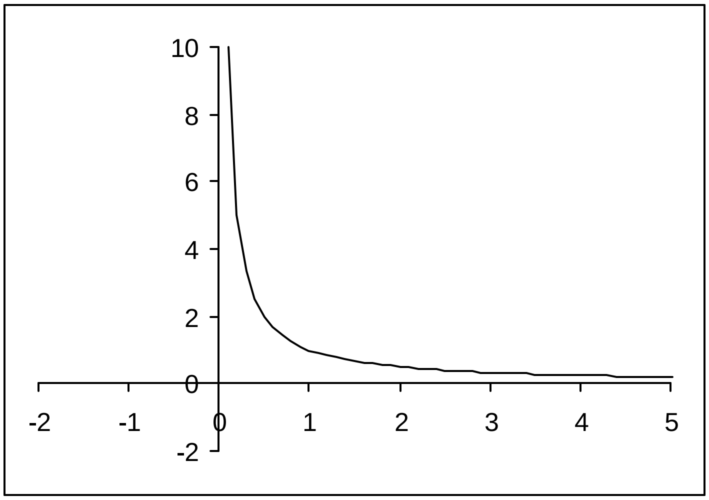
5.5 What is the value of the function at $x = 3$, i.e. $f(3)$?

Appendix B

The interview schedule

QUESTION 1

Let $f(x) = \frac{1}{x}$



1.1 What do you understand by $\lim_{x \to \infty} \frac{1}{x} = 0$?

1.2 Can we write $f(\infty) = 0$?

QUESTION 2

What is the value of 0,999?

QUESTION 3

3.1 What does the following expression mean?

$$\lim_{h \to \infty} \frac{f(x+h) - f(x)}{h}$$

3.2 What do we use this formula for?

QUESTION 4

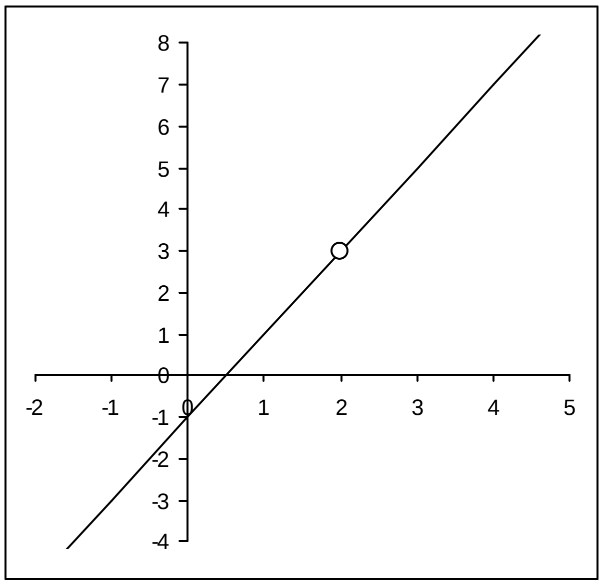
4.1 Explain the following: $\lim_{x \to 1} f(x) = 3$

4.2 Must the function be defined at x = 1 to have a limit?

4.3 Must $f(1)$ = 3 or not?

QUESTION 5

Look at the graph:



5.1 What is the limit of $f(x)$ as x 2?

5.2 What is $\lim_{x \to 2} f(x)$?

QUESTION 6

Let $f(x) = \dfrac{x-2}{x-2}$

6.1 Sketch the graph of the function.

6.2 What is the value of $f(2)$?

6.3 Does $f(x)$ have a limit at x = 2?

6.4 If yes, what is the limit?