# A Lie Symmetry Analysis of the Black-Scholes Merton Finance Model through modified Local one-parameter transformations

**by**

**Tshidiso Phanuel Masebe**

Submitted in accordance with the requirements for the degree of

**Doctor of Philosophy** in the subject **Applied Mathematics** at

the University of South Africa

September 2014

Supervisor : Dr J.M. Manale

# Contents

# List of Figures

# Abstract

The thesis presents a new method of Symmetry Analysis of the Black-Scholes Merton Finance Model through modified Local one-parameter transformations. We determine the symmetries of both the one-dimensional and two-dimensional Black-Scholes equations through a method that involves the limit of infinitesimal $\omega$ as it approaches zero. The method is dealt with extensively in [23]. We further determine an invariant solution using one of the symmetries in each case. We determine the transformation of the Black-Scholes equation to heat equation through Lie equivalence transformations.

Further applications where the method is successfully applied include working out symmetries of both a Gaussian type partial differential equation and that of a differential equation model of epidemiology of HIV and AIDS. We use the new method to determine the symmetries and calculate invariant solutions for operators providing them.

Keywords: Black-Scholes equation; Partial differential equation; Lie Point Symmetry; Lie equivalence transformation; Invariant solution.

# Declaration

Student number: 46640193

I declare that **A Lie Symmetry Analysis of the Black-Scholes Merton Finance Model through modified Local one-parameter transformations** is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of a complete list of references.

*SIGNATURE* *DATE*

# Dedication

I would like to dedicate this thesis to my father Philip Masebe, my late mother, Dolly Daisy Masebe who has always been my pillar of strength and an encouragement me to work hard in my studies. I further would like to thank my son, Reotshepile Phillip Masebe who has always been a source of inspiration and motivation for me to hold steadfast.

# Acknowledgement


- I am indebted to my supervisor, Dr J.M. Manale for the guidance, patience and professional assistance he has offered throughout this thesis. It is with gratitude that I have tapped from his vast wisdom and knowledge.

- Special words of thank you go to Prof M. T. Kambule for having introduced me to Symmetry Analysis. I also acknowledge the teaching I received from Prof N.H. Ibragimov in my Masters degree programme. It has rounded my understanding of Lie Symmetry Analysis. I also thank Dr M.P. Mulaudzi for the lessons he offered me on Mathematics of Finance.

- Words of thank you go to all my friends, former colleagues and current colleagues at Tshwane University of Technology , and to Staff members in the Mathematics Department at University of South Africa. Family members, brothers and a sister with whom we have a formidable bond I say thanks. Thanks to a special lady for her words of encouragement, and expressing confidence in my ability to put together an undertaking of this nature.

- I acknowledge the financial support of my institution Tshwane University of Technology and the financial aid at University of South Africa.

- With God everything is possible.

# Introduction

The Lie group analysis of differential equations is the area of mathematics pioneered by Sophus Lie in the 19th century (1849-1899). Sophus Lie made a profound and far-reaching discovery that all *ad-hoc* techniques designed to solve ordinary differential equations (e.g separable variables, homogeneous, exact) could be explained and deduced simply by his theory. These techniques were in fact special cases of a general integration procedure of classifying ordinary differential equations in terms of their symmetry groups. This discovery led to Lie identifying a full set of equations which could be integrated or reduced to lower order equations by his method [6],[14].

The Lie Symmetry method is analytic and highly algorithmic. The method systematically unifies and extends *ad-hoc* techniques to construct explicit solutions for differential equations. The emphasis is on explicit computational algorithms to discover symmetries admitted by differential equations and to construct invariant solutions resulting from the symmetries [6],[13].

Lie group analysis established itself to be an effective method of solving non-linear differential equations analytically. In fact the first general solution of the problem of classification was given by Sophus Lie for an extensive class of partial differential equations [14]. Since then many researchers have done work on various families of differential equations. The results of their work have been captured in several outstanding literary works by amongst others Ibragimov (1999), Hydon(2000), Bluman and Anco(2002), etc [6], [9], [13].

The present project titled **A Lie Symmetry Analysis of the Black-Scholes Merton Finance Model through modified Local one-parameter transformations** seeks to explore the analysis of the widely used one-dimensional model of Black-Scholes partial differential equation through determining the new symmetries and invariant solutions of some of these symmetries. The analysis is through a new method developed in [23], [28]. Further exploration of the method can be found in [25], [26],[27]. Throughout the project we use Lie point symmetries.

The Black-Scholes equation is a partial differential equation that governs the value of financial derivatives. Determining the value of derivatives had been a problem in finance for almost 70 years since 1900. In the early 70s, Black and Scholes made a pioneering contribution to finance by developing a Black-Scholes equation under restrictive assumptions and the option valuation formula. Scholes obtained a Nobel Prize for economics in 1997 for his contribution (Black had passed on in 1995 and could not receive the prize personally) [32]. Furthermore, information on the derivation of and the restrictive assumptions in the development of the Black-Scholes equation can be obtained in the texts [4], [8], [10], [11], [12], [21], [22] and [30].

The thesis outline is as follows:

Chapter 1 presents the concept of Local One-parameter Point Symmetries. We introduce concepts of Local One-parameter Point transformations, generator, prolongation formulas, determining equation and Lie algebras. These concepts serve as tools in the analysis of the Black-Scholes partial differential equation.

Chapter 2 presents the Symmetry Analysis of equation (2.1) as outlined in [9]. The chapter presents the symmetry of (2.1),finite symmetry transformations and the transformation of the equation (2.1) to the heat equation.

Chapter 3 is the core of the thesis. It introduces the Symmetry Analysis of equation (2.1) by using the method developed through contributions in ([23]) and ([28]). The chapter details the symmetries of one-dimensional Black-Scholes equation (2.1) as well as the two-dimensional case through modified Local one-parameter transformations.

Chapter 4 involves the transformation of the transformed Black-Scholes equation to heat through Lie equivalence transformations. We calculate this transformation by using a method similar but with slight modification to that developed in [17].

Chapter 5 presents other areas where the method was successfully applied. The application includes determining the symmetries and invariant solutions of the Gaussian type differential equation and symmetries in the epidemiology of HIV and AIDS.

## Contributions to the study

The thesis is based on peer reviewed contributions to the study as outlined in the publications ([1]) and ([28]). Several texts for example ([6]), ([13]),([14]) etc. present symmetries of differential equations in the way initially introduced by Sophus Lie. However, in this publications we introduce an alternative way to determine these symmetries, and the method present new additional symmetries.

# Chapter 1

# Local One-parameter Point Symmetries

The chapter presents the underlying theory of Lie Symmetry Analysis and the tools we will use in subsequent chapters. The most common of all symmetries in practice are Local One-parameter Point symmetries.

## 1.1 Local One-parameter Point transformations

To begin, we consider the following definition.

**Definition 1** Let

$$\bar{\mathbf{x}} = \mathbf{G}\left(\mathbf{x}; \epsilon\right) \tag{1.1}$$

be a family of one-parameter $\epsilon \in R$ invertible transformations, of points $\mathbf{x} = (x^1, \cdots, x^N) \in \mathbf{R}^N$ into points $\bar{\mathbf{x}} = (\bar{x}^1, \cdots, \bar{x}^N) \in \mathbf{R}^{\mathbf{N}}$. These are known as one-parameter transformations, and subject to the conditions

$$\bar{\mathbf{x}}|_{\epsilon=0} = \mathbf{x}. \tag{1.2}$$

That is,

$$\mathbf{G}\left(\mathbf{x}; \epsilon\right)\Big|_{\epsilon=0} = \mathbf{x}. \tag{1.3}$$

Expanding (1.1) about $\epsilon = 0$, in some neighborhood of $\epsilon = 0$, we get

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \left( \left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0} \right) + \frac{\epsilon^2}{2} \left( \left. \frac{\partial^2 \mathbf{G}}{\partial \epsilon^2} \right|_{\epsilon=0} \right) + \cdots = \mathbf{x} + \epsilon \left( \left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0} \right) + O\left(\epsilon^2\right). \quad (1.4)$$

Letting

$$\xi\left(\mathbf{x}\right) = \left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0}, \quad (1.5)$$

reduces the expansion to

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \xi\left(\mathbf{x}\right) + O\left(\epsilon^2\right). \quad (1.6)$$

**Definition 2** The expression

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \xi\left(\mathbf{x}\right), \quad (1.7)$$

is called a Local One-parameter Point transformation.

The components of $\xi\left(\mathbf{x}\right)$ are called the infinitesimals of (1.1) [6].

## 1.2 Local One-parameter Point transformation groups

The set $G$ of transformations

$$\bar{\mathbf{x}}_{\epsilon_i} = \mathbf{x} + \epsilon_i \left( \left. \frac{\partial \mathbf{G}}{\partial \epsilon_i} \right|_{\epsilon_i=0} \right) + \frac{\epsilon_i^2}{2} \left( \left. \frac{\partial^2 \mathbf{G}}{\partial \epsilon_i^2} \right|_{\epsilon_i=0} \right) + \cdots, \quad i = 1, 2, 3, \cdots, \quad (1.8)$$

becomes a group only when truncated at $O\left(\epsilon^2\right)$.

That is, $G$ is a group since the following properties hold under binary operation $+$:

1. **Closure**. If $\bar{\mathbf{x}}_{\epsilon_1}, \bar{\mathbf{x}}_{\epsilon_2} \in G$ and $\epsilon_1, \epsilon_2 \in R$, then

$$\bar{\mathbf{x}}_{\epsilon_1} + \bar{\mathbf{x}}_{\epsilon_2} \;=\; (\epsilon_1 + \epsilon_2)\xi\left(\mathbf{x}\right) \;=\; \bar{\mathbf{x}}_{\epsilon_3} \in G, \quad \text{and} \quad \epsilon_3 = \epsilon_1 + \epsilon_2 \in R.$$

2. **Identity**. If $\bar{\mathbf{x}}_0 \equiv I \in G$ such that for any $\epsilon \in R$

$$\bar{\mathbf{x}}_0 + \bar{\mathbf{x}}_\epsilon \;=\; \bar{\mathbf{x}}_\epsilon \;=\; \bar{\mathbf{x}}_\epsilon + \bar{\mathbf{x}}_0,$$

then $\bar{\mathbf{x}}_0$ is an identity in $G$.

3. **Inverses**. For $\bar{\mathbf{x}}_\epsilon \in G$, $\epsilon \in R$, there exists $\bar{\mathbf{x}}_\epsilon^{-1} \in G$, such that

$$\bar{\mathbf{x}}_\epsilon^{-1} + \bar{\mathbf{x}}_\epsilon \;=\; \bar{\mathbf{x}}_\epsilon + \bar{\mathbf{x}}_\epsilon^{-1}, \quad \bar{\mathbf{x}}_\epsilon^{-1} \;=\; \bar{\mathbf{x}}_{\epsilon^{-1}},$$

and $\epsilon^{-1} = -\epsilon \in D$, where $+$ is a binary composition of transformations and it is understood that $\bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon - \mathbf{x}$. Associativity follows from the closure property.

**Example 1** :

Group of Rotations in the Plane

$$\bar{x}_1 \;=\; x_1 \cos\epsilon + x_2 \sin\epsilon,$$

$$\bar{x}_2 \;=\; x_2 \cos\epsilon - x_1 \sin\epsilon.$$

That is,

$$\bar{x}_1 \;=\; x_1 + x_2\epsilon,$$

$$\bar{x}_2 \;=\; x_2 - x_1\epsilon.$$

**Example 2** : Group of Translations in the Plane

$$\bar{x}_i \;=\; x_i + \epsilon,$$

$$\bar{x}_j \;=\; x_j.$$

**Example 3** :

Group of Scalings in the Plane

$$\bar{x}_i = (1+\epsilon)x_i,$$

$$\bar{x}_j = (1+\epsilon)^2 x_j. \qquad [6],[15].$$

## 1.3 The group generator

The Local One-parameter Point transformations in (1.7) can be rewritten in the form

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon\xi\,(\mathbf{x}) \cdot \nabla\ \mathbf{x}, \tag{1.9}$$

so that

$$\bar{\mathbf{x}} = (1 + \epsilon\xi\,(\mathbf{x}) \cdot \nabla)\,\mathbf{x}. \tag{1.10}$$

An operator,

$$G = \xi\,(\mathbf{x}) \cdot \nabla, \tag{1.11}$$

can then be introduced, so that (1.9) assumes the form

$$\bar{\mathbf{x}} = (1 + \epsilon G)\,\mathbf{x}. \tag{1.12}$$

The operator (1.11) has the expanded form

$$G = \sum_{k=1}^{N} \xi^k \frac{\partial}{\partial x^k}, \tag{1.13}$$

or simply

$$G = \xi^k \frac{\partial}{\partial x^k}. \qquad [6],[16],[18]. \tag{1.14}$$

## 1.4 Prolongations formulas

It often happens that the invariant function $F$ does not only depend on the point $\mathbf{x}$ alone, but also on the derivatives. When that is the case then we have to use the prolonged form of the operator $G$.

### 1.4.1 The case $N = 2$, with $x^1 = x$ and $x^2 = y$

The case $N = 2$, with $x^1 = x$ and $x^2 = y$ reduces (1.13) to

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{1.15}$$

In determining the prolongations, it is convenient to use the operator of total differentiation

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots, \tag{1.16}$$

where

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \quad \cdots. \tag{1.17}$$

The derivatives of the transformed point is then

$$\bar{y}' = \frac{d\bar{y}}{d\bar{x}}. \tag{1.18}$$

Since

$$\bar{x} = x + \epsilon\xi \quad \text{and} \quad \bar{y} = y + \epsilon\eta, \tag{1.19}$$

then

$$\bar{y}' = \frac{dy + \epsilon d\eta}{dx + \epsilon d\xi}. \tag{1.20}$$

That is,

$$\bar{y}' = \frac{dy/dx + \epsilon d\eta/dx}{dx/dx + \epsilon d\xi/dx}. \tag{1.21}$$

Now introducing the operator $D$:

$$\bar{y}' = \frac{y' + \epsilon D(\eta)}{1 + \epsilon D(\xi)} = \frac{(y' + \epsilon D(\eta))(1 - \epsilon D(\xi))}{1 - \epsilon^2(D(\xi))^2}. \tag{1.22}$$

Hence

$$\bar{y}' = \frac{y' - \epsilon(D(\eta) - y'D(\xi)) - \epsilon^2(D(\xi))}{1 - \epsilon^2(D(\xi))^2}. \tag{1.23}$$

That is,

$$\bar{y}' = y' + \epsilon(D(\eta) - y'D(\xi)), \tag{1.24}$$

or

$$\bar{y}' = y' + \epsilon\zeta^1, \tag{1.25}$$

with

$$\zeta^1 = D(\eta) - y'D(\xi). \tag{1.26}$$

It expands into

$$\zeta^1 = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y. \tag{1.27}$$

The first prolongation of $G$ is then

$$G^{[1]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'}. \tag{1.28}$$

For the second prolongation, we have

$$\bar{y}'' = \frac{y'' + \epsilon D(\zeta^1)}{1 + \epsilon D(\xi)} \approx y'' + \epsilon\zeta^2, \tag{1.29}$$

with

$$\zeta^2 = D(\zeta^1) - y''D(\xi). \tag{1.30}$$

This expands into

$$\begin{aligned} \zeta^2 &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\ &- y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y''. \end{aligned} \tag{1.31}$$

The second prolongation of $G$ is then

$$G^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'} + \zeta^2\frac{\partial}{\partial y''}. \tag{1.32}$$

Most applications involve up to second order derivatives. It is reasonable then to pause here, for this case [14] .

### 1.4.2 Invariant functions in $R^2$

**Theorem 1** A function $F(x,y)$ is an invariant of the group of transformations (1.13) if for each point $(x,y)$ it is constant along the trajectory determined by the totality of transformed points $(\bar{x},\bar{y})$:

$$F(\bar{x},\bar{y}) = F(x,y). \tag{1.33}$$

This requires that

$$GF = 0, \tag{1.34}$$

leading to the characteristic system

$$\frac{dx}{\xi} = \frac{dy}{\eta}. \tag{1.35}$$

**Proof.** Consider the Taylor series expansion of $F(\mathbf{\bar{x}})$ with respect to $\epsilon$:

$$F(\bar{x},\bar{y}) = F(\bar{x},\bar{y})\Big|_{\epsilon=0} + \epsilon\frac{\partial\bar{F}}{\partial\epsilon}\Big|_{\epsilon=0} + \cdots . \tag{1.36}$$

This can be written in the form

$$F(\bar{x},\bar{y}) = F(\bar{x},\bar{y})\Big|_{\epsilon=0} + \epsilon\left(\frac{\partial\bar{x}}{\partial\epsilon}\frac{\partial\bar{F}}{\partial\bar{x}} + \frac{\partial\partial\bar{y}}{\partial\epsilon}\frac{\partial\bar{F}}{\bar{y}}\right)\Big|_{\epsilon=0} + \cdots . \tag{1.37}$$

That is,

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon\left(\xi\frac{\partial\bar{F}}{\partial\bar{x}} + \eta\frac{\partial\bar{F}}{\bar{y}}\right)\Big|_{\epsilon=0} + \cdots , \tag{1.38}$$

or

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\bar{F} + \cdots . \tag{1.39}$$

Hence

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon G\bar{F}, \tag{1.40}$$

with

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}. \tag{1.41}$$

For $\epsilon = 0$ then we get

$$F(\bar{x}, \bar{y}) = F(x, y), \tag{1.42}$$

thus proving the theorem [16].

### 1.4.3 Multi-dimensional cases

In dealing with the multi-dimensional cases, we may recast the generator (1.13) as

$$G = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{1.43}$$

We consider the kth-order partial differential equation

$$F(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}) \quad \text{where} \quad x = (x_1 \dots x_n), \quad u_{(1)} = \frac{\partial u}{\partial x}. \tag{1.44}$$

By definition of symmetry, the transformations (1.1) form a symmetry group $G$ of the system (1.44) if the function $\bar{u} = \bar{u}(\bar{x})$ satisfies (1.43) whenever the function $u = u(x)$ satisfies (1.44). The transformed derivatives $\bar{u}_{(1)}, \dots, \bar{u}_{(k)}$ are found from (1.4) by using the formulae of change of variables in the derivatives, $D_i = D_i(f^j)\bar{D}_j$. [6]

Here

$$D_i = \frac{\partial}{\partial x^i} + u_i^a(\frac{\partial}{\partial u^a}) + u_{ij}^a(\frac{\partial}{\partial u_j^a} + \cdots) \tag{1.45}$$

*is the total derivative operator w.r.t.* $x^i$ and $\bar{D}_j$ is given in terms of the transformed variables. The transformations (1.13) together with the transformations on $\bar{u}_{(1)}$ form a group, $G^{[1]}$, which is the first prolonged group which acts in the space $(x, u, \bar{u}_{(1)})$. Likewise, we obtain the prolonged groups $G^{[2]}$ and so on up to $G^{[k]}$.

The infinitesimal transformations of the prolonged groups are:

$$\begin{gathered}
\bar{u}_i^a \approx u_i^a + a\zeta_i^a(x, u, u_{(1)})\,, \\
\bar{u}_{ij}^a \approx u_{ij}^a + a\zeta_{ij}^a(x, u, u_{(1)}, u_{(2)})\,, \\
\vdots \\
\bar{u}_{i_1 \dots i_k}^a \approx u_{i_1 \dots i_k}^a + a\zeta_{i_1 \dots i_k}^a(x, u, u_{(1)}, \dots, u_{(k)})\,.
\end{gathered} \tag{1.46}$$

The functions $\zeta_i^a(x,u,u_{(1)})$, $\zeta_{ij}^a(x,u,u_{(1)},u_{(2)})$ and $\zeta_{i_1\dots i_k}^a(x,u,u_{(1)},\dots,u_{(k)})$ are given, recursively, by the *prolongation formulas*:

$$\begin{aligned}
\zeta_i^a &= D_i(\eta^a) - u_j^a D_i(\xi^j)\,, \\
\zeta_{ij}^a &= D_j(\zeta_i^a) - u_{il} D_j(\xi^l)\,, \\
&\vdots \\
\zeta_{i_1\dots i_k}^a &= D_{i_k}(\zeta_{i_1\dots i_{k-1}}^a) - u_{li_1\dots i_{k-1}} D_{i_k}(\xi^l)\,.
\end{aligned} \tag{1.47}$$

The generator of the prolonged groups are:

$$\begin{aligned}
G^{[1]} &= \xi^i(x,u)\tfrac{\partial}{\partial x^i} + \eta^a(x,u)\tfrac{\partial}{\partial u^a} + \zeta_i^a(x,u,u_{(1)})\tfrac{\partial}{\partial u_i^a}\,, \\
&\vdots
\end{aligned} \tag{1.48}$$

$$\begin{aligned}
G^{[k]} = {}& \xi^i(x,u)\tfrac{\partial}{\partial x^i} + \eta^a(x,u)\tfrac{\partial}{\partial u^a} + \zeta_i^a(x,u,u_{(1)})\tfrac{\partial}{\partial u_i^a} \\
&+ \dots + \zeta_{i_1\dots i_k}^a(x,u,\dots,u_{(k)})\tfrac{\partial}{\partial u_{i_1\dots i_k}^a}\,.
\end{aligned} \tag{1.49}$$

[6], [15].

**Definition 3** A differential function $F(x,u,\dots,u_{(p)})$, $p \geq 0$, is a $p$th-order differential invariant of a group $G$ if

$$F(x,u,\dots,u_{(p)}) = F(\bar{x},\bar{u},\dots,\bar{u}_{(p)}), \tag{1.50}$$

i.e. if $F$ is invariant under the prolonged group $G^{[p]}$, where for $p=0$, $u_{(0)} \equiv u$ and $G^{[0]} \equiv G$.

**Theorem 2** A differential function $F(x,u,\dots,u_{(p)})$, $p \geq 0$, is a $p$th-order differential invariant of a group $G$ if

$$G^{[p]}F = 0\,, \tag{1.51}$$

where $G^{[p]}$ is the $p$th prolongation of $G$ and for $p=0$, $G^{[0]} \equiv G$.

The substitution of (1.49) and (1.50) into (1.5) gives rise to

$$E^\sigma(\bar{x},\bar{u},\bar{u}_{(1)},\dots,\bar{u}_{(k)}) \approx E^\sigma(x,u,u_{(1)},\dots,u_{(k)}) + a(G^{[k]}E^\sigma)\,, \tag{1.52}$$

$$\sigma = 1,\dots,\tilde{m}\,.$$

Thus, we have

$$G^{[k]}E^{\sigma}(x,u,u_{(1)},\dots,u_{(k)})=0, \qquad \sigma=1,\dots,\tilde{m}, \tag{1.53}$$

whenever (1.44) is satisfied. The converse also applies.

### 1.4.4 Invariant functions in $R^N$

**Theorem 3** A function $F(\mathbf{x})$ is an invariant of the group of transformations (1.13) if for each point $\mathbf{x}$ it is constant along the trajectory determined by the totality of transformed points $\bar{\mathbf{x}}$:

$$F(\bar{\mathbf{x}})=F(\mathbf{x}). \tag{1.54}$$

This requires that

$$GF=0, \tag{1.55}$$

leading to the characteristic system

$$\frac{dx^1}{\xi^1}=\cdots=\frac{dx^N}{\xi^N}. \tag{1.56}$$

**Proof.** Consider the Taylor series expansion of $F(\bar{\mathbf{x}})$ with respect to $\epsilon$:

$$F(\bar{\mathbf{x}})=F(\mathbf{x})\Big|_{\epsilon=0}+\epsilon\frac{\partial\bar{\mathbf{F}}}{\partial\epsilon}\Big|_{\epsilon=0}+\cdots. \tag{1.57}$$

This can be written in the form

$$F(\bar{\mathbf{x}})=F(\bar{\mathbf{x}})\Big|_{\epsilon=0}+\epsilon\frac{\partial\bar{\mathbf{x}}}{\partial\epsilon}\cdot\bigtriangledown\bar{F}\Big|_{\epsilon=0}+\cdots. \tag{1.58}$$

That is,

$$F(\bar{\mathbf{x}})=F(\bar{\mathbf{x}})\Big|_{\epsilon=0}+\epsilon\xi\cdot\bigtriangledown\bar{F}\Big|_{\epsilon=0}+\cdots. \tag{1.59}$$

For $\epsilon=0$ then we get

$$F(\bar{\mathbf{x}})=F(\mathbf{x})), \tag{1.60}$$

thus proving the theorem [16].

## 1.5 Determining equations

Equations (1.53) are called the *determining equations*. They are written compactly as

$$G^{[k]}E^{\sigma}(x,u,u_{(1)},\ldots,u_{(k)})\,\big|_{(1)}=0, \qquad \sigma=1,\ldots,\tilde{m},$$

where $\big|_{(1)}$ means evaluated on the surface (1.44).

The determining equations are linear homogeneous partial differential equations of order $k$ for the unknown functions $\xi^i(x,u)$ and $\eta^a(x,u)$. These are consequences of the prolongation formulae (1.47). Equations (1.5) also involve the derivatives $u_{(1)},\ldots,u_{(k)}$ some of which are eliminated by the system (1.44). We then equates the coefficients of the remaining unconstrained partial derivatives of $u$ to zero. In general, (1.5) decomposes into an overdetermined system of equations, that is, there are more equations than the $n+m$ unknowns $\xi^i$ and $\eta^a$. There are computer algebra programs that can perform the task of solving determining equations [3].

Since the determining equations are linear homogeneous, their solutions form a *vector space* $L$ [6].

## 1.6 Lie algebras

There is another important property of the determining equations, viz. if the generators

$$G_1=\xi_1^i(x,u)\frac{\partial}{\partial x^i}+\eta_1^a(x,u)\frac{\partial}{\partial u^a}$$

and

$$G_2=\xi_2^i(x,u)\frac{\partial}{\partial x^i}+\eta_2^a(x,u)\frac{\partial}{\partial u^a}$$

satisfy the determining equations, so do their *commutator* $[G_1,G_2]=G_1G_2-G_2G_1$

$$[G_1,G_2]=\left(G_1(\xi_2^i)-G_2(\xi_1^i)\right)\frac{\partial}{\partial x^i}+\left(G_1(\eta_2^a)-G_2(\eta_1^a)\right)\frac{\partial}{\partial u^a}$$

which obeys the properties of bilinearity, skew-symmetry and Jacobi's identity, viz.

1. **Bilinearity**. If $G_1, G_2, G_3 \in L$, then

$$[\alpha G_1 + \beta G_2, G_3] \;=\; \alpha [G_1, G_3] + \beta [G_2, G_3], \quad \alpha, \beta \text{ are scalars}$$

2. **Skew symmetry**. If $G_1, G_2 \in L$, then

$$[G_1, G_2] \;=\; -[G_2, G_1].$$

3. **Jacobi Identity**. If $G_1, G_2, G_3 \in L$, then

$$[[G_1, G_2], G_3] + [[G_2, G_3], G_1] + [[G_3, G_1], G_2] = 0.$$

The vector space $L$ of all solutions of the determining equations forms a *Lie algebra* which generates a multi-parameter group admitted by (1.44) [15],[16].

## 1.7 Solvable Lie algebras

In this section, we will show that if $r = 1$, then the order of an ODE can be reduced constructively by one. If $n > 2$ and $r = 2$, the order can be reduced constructively by two. But if $n > 2$ and $r > 2$, it will not necessarily follow that the order can be reduced by more than one. However, if the $r$-dimensional Lie algebra of infinitesimal generators of an admitted r-parameter group has a q-dimensional solvable subalgebra, then the order of the ODE can be reduced constructively by $q$.

**Definition 4** A subalgebra $A$ of the Lie algebra $L^r$ with dimension $r$, is called an ideal or normal subalgebra of $L^r$ if $[G_\alpha, G_\beta] \in A$, for all $G_\alpha \in A$ and $G_\beta \in L^r$.

**Definition 5** The Lie algebra $L^r$, with dimension $r$, is called an $r$-dimensional solvable Lie algebra if there exists a chain of subalgebras

$$A^1 \subset A^2 \subset \cdots \subset L^r,$$

with $A^{k-1}$ being an ideal of $A^k$, and $k \leq r$.

**Definition 6** A Lie algebra $A$ is Abelian if $[G_\alpha, G_\beta] = 0$, if both $G_\alpha$ and $G_\beta$ are in $A$. [16]

**Theorem 4** An abelian algebra is solvable [16].

**Theorem 5** A two-dimensional algebra is solvable.

**Proof** Let $L$ be a two-dimensional Lie Algebra with infinitesimal generators $X_1$ and $X_2$ as basis vectors. Suppose that

$$[X_1, X_2] = aX_1 + bX_2 = Y$$

If $C_1X_1 + C_2X_2 \in L$, then

$$\begin{aligned} [Y, C_1X_1 + C_2X_2] &= C_1[Y, X_1] + C_2[Y, X_2] \\ &= C_1b[X_2, X_1] + C_2a[X_1, X_2] \\ &= (C_2a - C_1b)Y \end{aligned}$$

Therefore $Y$ is a one-dimensional ideal of $L$. Hence the proof [5].

## 1.8 Lie equations

One-parameter groups are obtained by their generators by means of *Lie's theorem*:

**Theorem 6** Given the infinitesimal transformations $\bar{x^i} = x^i + \epsilon\xi^i(x)$, $\bar{u^\alpha} = u^\alpha + \epsilon\eta^\alpha(x)$ or its symbol $G$, the corresponding one-parameter group $G$ is obtained by solution of the *Lie equations*

$$\frac{d\bar{x}^i}{d\epsilon} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{d\epsilon} = \eta^\alpha(\bar{x}, \bar{u}),$$

subject to the initial conditions

$$\bar{x}^i|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha|_{\epsilon=0} = u^\alpha. \qquad [18] \tag{1.61}$$

## 1.9 Canonical Parameter

If in the group property 1., discussed above, the expression $\varphi(\epsilon_1, \epsilon_2)$ can be written as

$$\varphi(\epsilon_1, \epsilon_2) = \epsilon_1 + \epsilon_2,$$

then the parameter $a$ is said to be *canonical*. In general, a canonical parameter exists whenever $\varphi$ exists. That is, one has the following theorem:

**Theorem 7** : For any $\varphi(a, b)$, there exists the canonical parameter

$$\tilde{a} = \int_0^a \frac{da'}{A(a')},$$

where

$$A(a) = \frac{\varphi(a, b)}{b}|_{b=0}. \qquad [18]$$

This system, with $a$ as the canonical parameter below, transforms form-invariantly in variables $t, x, y, z, u, v, w, p, \mu$ (see [**?**]) under

$$\bar{t} = t \exp\left[\int_0^a \frac{da'}{\bar{\mu} F'(\mu)}\right], \; \bar{x} = x \exp\left[\int_0^a \frac{da'}{\bar{\mu} F'(\mu)}\right], \; \bar{y} = y \exp\left[\int_0^a \frac{da'}{\bar{\mu} F'(\mu)}\right],$$
$$\bar{z} = z \exp\left[\int_0^a \frac{da'}{\bar{\mu} F'(\mu)}\right], \; \bar{u} = u \exp\left[\int_0^a \frac{da'}{\bar{\mu} F'(\mu)}\right], \; \bar{v} = v \exp\left[\int_0^a \frac{da'}{\bar{\mu} F'(\mu)}\right],$$
$$\bar{w} = w \exp\left[\int_0^a \frac{da'}{\bar{\mu} F'(\mu)}\right], \; \bar{p} = p \; \exp\left[\int_0^a \frac{da'}{\bar{\mu} F'(\mu)}\right], \; F(\bar{\mu}) = a + F(\mu),$$

where

$$F(\mu) = \frac{1}{\mu F'(\mu)}, [16]$$

## 1.10 Canonical variables

**Theorem 8** : Every one-parameter group of transformations ( $\bar{x} = f(x, y, \epsilon)$, $\bar{y} = g(x, y, \epsilon)$ reduces to a group of translations $\bar{t} = t + \epsilon$, $\bar{u} = u$ with the generator

$$X = \frac{\partial}{\partial t}$$

by a suitable change of variables

$$t = t(x, y), \quad u = u(x, y).$$

The variables $t, u$ are called canonical variables.

**Proof**: Under change of variables the differential operator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

transforms according to the formula

$$X = X(t)\frac{\partial}{\partial t} + X(u)\frac{\partial}{\partial u}. \tag{1.62}$$

Therefore, canonical variables are found from the linear partial differential equation of the first order:

$$\begin{aligned} X(t) &\equiv \xi(x, y)\frac{\partial t(x, y)}{\partial x} + \eta(x, y)\frac{\partial t(x, y)}{\partial y} = 1 \\ X(u) &\equiv \xi(x, y)\frac{\partial u(x, y)}{\partial x} + \eta(x, y)\frac{\partial u(x, y)}{\partial y} = 0. \end{aligned} \tag{1.63}$$

Hence the proof [15].

**Theorem 9** : By a suitable choice of the basis $G_1, G_2$, any two-dimensional Lie algebra can be reduced to one of the four different types, which are determined by the following canonical structural relations:

I.

$$[G_1, G_2] = 0, \quad G_1 \vee G_2 \neq 0; \tag{1.64}$$

II.

$$[G_1, G_2] = 0, \quad G_1 \vee G_2 = 0; \tag{1.65}$$

III.

$$[G_1, G_2] \neq 0, \quad G_1 \vee G_2 \neq 0; \tag{1.66}$$

IV.

$$[G_1, G_2] \neq 0, \quad G_1 \vee G_2 = 0, \tag{1.67}$$

where

$$G_1 \vee G_2 = \xi_1\eta_2 - \eta_1\xi_2,$$

and

$$G_1 = \xi_1\frac{\partial}{\partial x} + \eta_1\frac{\partial}{\partial y}, \quad G_1 = \xi_2\frac{\partial}{\partial x} + \eta_2\frac{\partial}{\partial y}. \qquad [15].$$

**Type I.**

$$[G_1, G_2] = 0, \quad G_1 \vee G_2 \neq 0.$$

This condition reduces

$$y'' = f(y'),$$

to

$$\int \frac{dy'}{f(y')} = x + C_1,$$

with $C_1$ being the integration constant.

**Type II.**

$$[G_1, G_2] = 0, \quad G_1 \vee G_2 = 0;$$

This condition reduces

$$y'' = f(x),$$

to

$$y = \int \left( \int f(x)dx \right) dx + C_1x + C_2.$$

with $C_1$ and $C_2$ being the integration constants.

**Type III.**

$$[G_1, G_2] \neq 0, \quad G_1 \vee G_2 \neq 0;$$

This condition reduces

$$y'' = \frac{1}{x} f(y'),$$

to

$$\int \frac{dy'}{f(y')} = \ln(x) + C_1,$$

with $C_1$ being the integration constant.

**Type IV.**

$$[G_1, G_2] \neq 0, \quad G_1 \vee G_2 = 0,$$

This condition reduces

$$y'' = y' f(x),$$

to

$$y = C_1 \int e^{\int f(x) dx} dx + C_2.$$

**Theorem 10** : The basis of an algebra $L_r$ can be reduced by a suitable change of variable to one of the following forms:

I.

$$G_1 = \frac{\partial}{\partial x}, \quad G_2 = \frac{\partial}{\partial y};$$

II.

$$G_1 = \frac{\partial}{\partial y}, \quad G_2 = x \frac{\partial}{\partial y};$$

III.

$$G_1 = \frac{\partial}{\partial y}, \quad G_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y};$$

IV.

$$G_1 = \frac{\partial}{\partial y}, \quad G_2 = y\frac{\partial}{\partial y}.$$

The variables $x$ and $y$ are called canonical variables.

## 1.11 One Dependent and Two Independent Variables.

We consider the equations

$$u_t = u_{xx}, \tag{1.68}$$

. In order to generate point symmetries for equation (1.68), we first consider a change of variables from $t, x$ and $u$ to $t^*, x^*$ and $u^*$ involving an infinitesimal parameter $\epsilon$. A Taylor's series expansion in $\epsilon$ near $\epsilon = 0$ yields

$$\left.\begin{array}{rcl} \bar{t} & \approx & t + \epsilon T(t,x,u) \\ \bar{x} & \approx & x + \epsilon \xi(t,x,u) \\ \bar{u} & \approx & u + \epsilon \zeta(t,x,u) \end{array}\right\} \tag{1.69}$$

where

$$\left.\begin{array}{rcl} \frac{\partial \bar{t}}{\partial \epsilon}|_{\epsilon=0} & = & T(t,x,u) \\ \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0} & = & \xi(t,x,u) \\ \frac{\partial \bar{u}}{\partial \epsilon}|_{\epsilon=0} & = & \zeta(t,x,u) \end{array}\right\}. \tag{1.70}$$

The tangent vector field (1.12) is associated with an operator

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u}, \tag{1.71}$$

called a symmetry generator. This in turn leads to the invariance condition

$$G^{[2]}\left[F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})\right]|_{\{F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})=0\}} = 0, \tag{1.72}$$

where $G^{[2]}$ is the second prolongation of $G$. It is obtained from the formulas:

$$\begin{aligned} G^{[2]} &= G + \zeta_t^1 \frac{\partial}{\partial u_t} + \zeta_x^1 \frac{\partial}{\partial u_x} + \zeta_{tt}^2 \frac{\partial}{\partial u_{tt}} \\ &\quad + \zeta_{tx}^2 \frac{\partial}{\partial u_{tx}} + \zeta_{xx}^2 \frac{\partial}{\partial u_{xx}}, \end{aligned}$$

where

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right] u_t - \frac{\partial \xi}{\partial x} u_x, \tag{1.73}$$

$$\zeta_x^1 = \frac{\partial g}{\partial x} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right] u_x - \frac{\partial T}{\partial t} u_t, \tag{1.74}$$

$$\begin{aligned} \zeta_{tt}^2 &= \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right] u_t - \frac{\partial^2 \xi}{\partial t^2} u_x \\ &\quad + \left[f - 2\frac{\partial T}{\partial t}\right] u_{tt} - 2\frac{\partial \xi}{\partial t} u_{tx}, \end{aligned}$$

$$\begin{aligned} \zeta_{xx}^2 &= \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right] u_x - \frac{\partial^2 T}{\partial x^2} u_t \\ &\quad + \left[f - 2\frac{\partial T}{\partial x}\right] u_{xx} - 2\frac{\partial T}{\partial x} u_{tx}, \end{aligned}$$

and

$$\begin{aligned} \zeta_{tx}^2 &= \frac{\partial^2 g}{\partial t \partial x} + u\frac{\partial^2 f}{\partial t \partial x} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t \partial x}\right] u_t \\ &\quad + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t \partial x}\right] u_x - \left[f - \frac{\partial T}{\partial t} - \frac{\partial \xi}{\partial x}\right] u_{tx} \\ &\quad - \frac{\partial T}{\partial x} u_{tt} - \frac{\partial \xi}{\partial t} u_{xx}. \end{aligned}$$

## 1.12 One Dependent and Three Independent Variables.

In order to generate point symmetries for equation (1.68), we first consider a change of variables from $t, x$ and $u$ to $t^*, x^*$ and $u^*$ involving an infinitesimal parameter $\epsilon$. A Taylor's series expansion in $\epsilon$ near $\epsilon = 0$ yields

$$\left.\begin{aligned} \bar{t} &\approx t + \epsilon T(t,x,u) \\ \bar{x} &\approx x + \epsilon \xi(t,x,u) \\ \bar{u} &\approx u + \epsilon \zeta(t,x,u) \end{aligned}\right\} \tag{1.75}$$

where

$$\left.\begin{aligned} \frac{\partial \bar{t}}{\partial \epsilon}\Big|_{\epsilon=0} &= T(t,x,u) \\ \frac{\partial \bar{x}}{\partial \epsilon}\Big|_{\epsilon=0} &= \xi(t,x,u) \\ \frac{\partial \bar{u}}{\partial \epsilon}\Big|_{\epsilon=0} &= \zeta(t,x,u) \end{aligned}\right\}. \tag{1.76}$$

The tangent vector field (1.12) is associated with an operator

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u}, \tag{1.77}$$

called a symmetry generator. This in turn leads to the invariance condition

$$G^{[2]}\left[F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})\right]|_{\{F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})=0\}} = 0, \tag{1.78}$$

where $G^{[2]}$ is the second prolongation of $G$. It is obtained from the formulas:

$$\begin{aligned} G^{[2]} &= G + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} \\ &\quad + \zeta_{tx}^2\frac{\partial}{\partial u_{tx}} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}}, \end{aligned}$$

where

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x. \tag{1.79}$$

$$\zeta_x^1 = \frac{\partial g}{\partial x} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t. \tag{1.80}$$

$$\begin{aligned} \zeta_{tt}^2 &= \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x \\ &\quad + \left[f - 2\frac{\partial T}{\partial t}\right]u_{tt} - 2\frac{\partial \xi}{\partial t}u_{tx}. \end{aligned}$$

$$\begin{aligned} \zeta_{xx}^2 &= \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 T}{\partial x^2}u_t \\ &\quad + \left[f - 2\frac{\partial T}{\partial x}\right]u_{xx} - 2\frac{\partial T}{\partial x}u_{tx}. \end{aligned}$$

and

$$\begin{aligned} \zeta_{tx}^2 &= \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 f}{\partial t\partial x} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t\partial x}\right]u_t \\ &\quad + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right]u_x - \left[f - \frac{\partial T}{\partial t} - \frac{\partial \xi}{\partial x}\right]u_{tx} \\ &\quad - \frac{\partial T}{\partial x}u_{tt} - \frac{\partial \xi}{\partial t}u_{xx}. \end{aligned}$$

We now look at two dimensional and three dimensional heat equation given respectively by

$$u_t = u_{xx} + u'_{yy} \tag{1.81}$$

and

$$u_t = u_{xx} + u_{yy} + u_{zz}. \tag{1.82}$$

In order to generate point symmetries for equation (1.81), we first consider a change of variables from $t, x, y$ and $u$ to $t^*, x^*, y^*$ and $u^*$ involving an infinitesimal parameter $\epsilon$. A Taylor's series expansion in $\epsilon$ near $\epsilon = 0$ yields

$$\left.\begin{array}{rcl} \bar{t} & \approx & t + \epsilon T(t,x,y,u), \\ \bar{x} & \approx & x + \epsilon \xi(t,x,y,u), \\ \bar{y} & \approx & y + \epsilon \varphi(t,x,y,u), \\ \bar{u} & \approx & u + \epsilon \zeta(t,x,y,u). \end{array}\right\} \tag{1.83}$$

where

$$\left.\begin{array}{rcl} \frac{\partial \bar{t}}{\partial \epsilon}|_{\epsilon=0} & = & T(t,x,y,u), \\ \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0} & = & \xi(t,x,y,u), \\ \frac{\partial \bar{y}}{\partial \epsilon}|_{\epsilon=0} & = & \varphi(t,x,y,u), \\ \frac{\partial \bar{u}}{\partial \epsilon}|_{\epsilon=0} & = & \zeta(t,x,y,u). \end{array}\right\}. \tag{1.84}$$

The tangent vector field (1.84) is associated with an operator

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial u}, \tag{1.85}$$

called a symmetry generator. This in turn leads to the invariance condition

$$G^{[2]}\left[F(t,x,u_t,u_x,u_y,u_{tx},u_{ty},u_{tt},u_{xx},u_{xy},u_{yy})\right]|_{\{F(t,x,u_t,u_x,u_y,u_{tx},u_{ty},u_{tt},u_{xx},u_{xy},u_{yy})=0\}} = 0, \tag{1.86}$$

where $G^{[2]}$ is the second prolongation of $G$. It is obtained from the formulas:

$$\begin{array}{rcl} G^{[2]} & = & G + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_y^1\frac{\partial}{\partial u_y} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} \\ & & + \zeta_{tx}^2\frac{\partial}{\partial u_{tx}} + \zeta_{ty}^2\frac{\partial}{\partial u_{ty}} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}} + \zeta_{xy}^2\frac{\partial}{\partial u_{xy}} + \zeta_{yy}^2\frac{\partial}{\partial u_{yy}}, \end{array}$$

where

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x - u_y\frac{\partial \varphi}{\partial t}, \tag{1.87}$$

$$\zeta_x^1 = \frac{\partial g}{\partial x} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t - u_y\frac{\partial \varphi}{\partial x}, \tag{1.88}$$

$$\zeta_y^1 = \frac{\partial g}{\partial y} + u\frac{\partial f}{\partial y} + \left[f - \frac{\partial \varphi}{\partial x}\right]u_y - \frac{\partial \varphi}{\partial t}u_t - u_x\frac{\partial \xi}{\partial y} - u_t\frac{\partial T}{\partial y}, \tag{1.89}$$

$$\begin{array}{rcl} \zeta_{tt}^2 & = & \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x - \frac{\partial^2 \xi}{\partial t^2}u_y \\ & & + \left[f - 2\frac{\partial T}{\partial t}\right]u_{tt} - 2\frac{\partial \xi}{\partial t}u_{tx} - 2\frac{\partial \varphi}{\partial t}u_{yt}, \end{array}$$

$$\begin{aligned}
\zeta^2_{tx} &= \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 f}{\partial t\partial x} + \left[\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t\partial x}\right]u_t \\
&+ \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right]u_x - u_y\frac{\partial^2 \varphi}{\partial t\partial x} - \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial x}\right]u_{tx} \\
&- \left[\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\right]u_{xx} - \left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right]u_{xy}.
\end{aligned}$$

$$\begin{aligned}
\zeta^2_{ty} &= \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 f}{\partial t\partial x} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t\partial x}\right]u_t \\
&+ \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right]u_x - \left[f - \frac{\partial T}{\partial t} - \frac{\partial \xi}{\partial x}\right]u_{tx} \\
&- \frac{\partial T}{\partial x}u_{tt} - \frac{\partial \xi}{\partial t}u_{xx}.
\end{aligned}$$

$$\begin{aligned}
\zeta^2_{xx} &= \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 \varphi}{\partial x^2}u_y - \frac{\partial^2 T}{\partial x^2}u_t \\
&+ \left[f - 2\frac{\partial \xi}{\partial x}\right]u_{xx} - 2\frac{\partial \varphi}{\partial x}u_{xy}, -2\frac{\partial T}{\partial x}u_{tx},
\end{aligned}$$

$$\begin{aligned}
\zeta^2_{ty} &= \frac{\partial^2 g}{\partial t\partial y} + u\frac{\partial^2 f}{\partial t\partial y} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial t\partial y}\right]u_y - \frac{\partial^2 \xi}{\partial t\partial y}u_x - \frac{\partial^2 T}{\partial y^2}u_t \\
&- \left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\right]u_{yy} - 2\left[\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial y}\right]u_{yt},
\end{aligned}$$

$$\begin{aligned}
\zeta^2_{yy} &= \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 f}{\partial^2 y} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right]u_y - \frac{\partial^2 \xi}{\partial y^2}u_x - \frac{\partial^2 T}{\partial y^2}u_t \\
&- \left[f - 2\frac{\partial \varphi}{\partial y}\right]u_{yy} - 2\left[\frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial y}\right]u_{yt},
\end{aligned}$$

## 1.13 One Dependent and Four Independent Variables.

For the equation (1.82) the tangent vector is given by

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial y} + \beta\frac{\partial}{\partial z} + \zeta\frac{\partial}{\partial u}, \tag{1.90}$$

and

$$\begin{aligned}
G^{[2]} &= G + \zeta^1_t\frac{\partial}{\partial u_t} + \zeta^1_x\frac{\partial}{\partial u_x} + \zeta^1_y\frac{\partial}{\partial u_y} + \zeta^1_z\frac{\partial}{\partial u_z} + \zeta^2_{tt}\frac{\partial}{\partial u_{tt}} \\
&+ \zeta^2_{tx}\frac{\partial}{\partial u_{tx}} + \zeta^2_{ty}\frac{\partial}{\partial u_{ty}} + \zeta^2_{xx}\frac{\partial}{\partial u_{xx}} + \zeta^2_{xy}\frac{\partial}{\partial u_{xy}} + \zeta^2_{xz}\frac{\partial}{\partial u_{xz}} + \zeta^2_{yy}\frac{\partial}{\partial u_{yy}} + \zeta^2_{yz}\frac{\partial}{\partial u_{yz}} \\
&+ \zeta^2_{zz}\frac{\partial}{\partial u_{zz}} + \zeta^2_{zt}\frac{\partial}{\partial u_{zt}}
\end{aligned}$$

$$\begin{aligned}
\zeta^2_{zz} &= \frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 f}{\partial y^2}u + u_z(2\frac{\partial f}{\partial z} - \frac{\partial^2 \beta}{\partial z^2}) - u_x\frac{\partial^2 \xi}{\partial z^2} - u_y\frac{\partial^2 \varphi}{\partial y^2} - u_t\frac{\partial^2 \tau}{\partial z^2} + u_{zz}(f - 2\frac{\partial \beta}{\partial z}) \\
&-2u_{zx}\frac{\partial \xi}{\partial z} - 2u_{zy}\frac{\partial \varphi}{\partial z} - 2u_{zt}\frac{\partial \tau}{\partial z},
\end{aligned}$$

$$\zeta^2_{nn} = \frac{\partial^2 g}{\partial n^2} + \frac{\partial^2 f}{\partial (n-1)^2}u + u_n(2\frac{\partial f}{\partial n} - \frac{\partial^2 \theta}{\partial n^2} + \cdots \tag{1.91}$$

## 1.14 One Dependent and $n$ Independent Variables.

The Local One-parameter Point transformations

$$\begin{aligned} \bar{x} &= X_i(x,u,\epsilon) = x_i + \epsilon\xi(x,u) + 0\epsilon^2 & (1.92)\\ \bar{u} &= U(x,u,\epsilon) = u + \epsilon\eta(x,u) + 0\epsilon^2 \quad i = 1,2,\ldots,n & (1.93) \end{aligned}$$

acting on $(x,u)$ - space has generator

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}$$

The $k$th extended *infinitesimals* are given by

$$\xi(x,u), \eta(x,u), \eta^{(1)}(x,u,\partial u), \ldots, \eta^{(1)}(x,u,\partial u,\ldots,\partial u^{(1)}), \tag{1.94}$$

and the corresponding $k$th extended generator is

$$X^{(k)} = X_i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial u} + \zeta_i^1\frac{\partial}{\partial u_i} + \ldots + \zeta_i^k\frac{\partial}{\partial u_{i_1 i_2 \ldots i_l}} \quad i = 1,2,\ldots,n \;\; l = 1,2,\ldots,k \;\; k \geq 1[6]. \tag{1.95}$$

**Theorem 11** The extended infinitesimals satisfy the recursive relations

$$\begin{aligned} \eta_i^{(1)} &= D_i\eta - (D_i\xi_j)u_j, \quad i = 1,2,\ldots,n & (1.96)\\ \zeta^k_{i_1 i_2 \ldots i_k} &= D_{i_k}\zeta^{k-1}_{i_1 i_2 \ldots i_{k-1}} - (D_i\xi_j)u_{i_1 i_2 \ldots i_{k-1}}j & (1.97) \end{aligned}$$

$i = 1,2,\ldots,n$ for $l = 1,2,\ldots,k$ with $k \geq 2$

**Proof.** Let $A$ be an $n \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} D_1X_1 & \cdots & D_1X_n \\ \vdots & \cdots & \vdots \\ D_nX_1 & \cdots & D_nX_n \end{bmatrix}$$

and assume that $A^{-1}$ exists. From equation (1.92) and the matrix $A$ we have that

$$\mathbf{A} = \begin{bmatrix} D_1(x_1+\epsilon\xi_1) & D_1(x_2+\epsilon\xi_2) & \cdots & D_1(x_n+\epsilon\xi_n) \\ D_2(x_1+\epsilon\xi_1) & D_2(x_2+\epsilon\xi_2) & \cdots & D_2(x_2+\epsilon\xi_2) \\ \vdots & \cdots & \cdots & \vdots \\ D_n(x_1+\epsilon\xi_1) & D_n(x_2+\epsilon\xi_2) & \cdots & D_n(x_n+\epsilon\xi_n) \end{bmatrix} + 0(\epsilon^2) = I + \epsilon B + 0(\epsilon^2)$$

where $I$ is the identity matrix and

$$\mathbf{B} = \begin{bmatrix} D_1\xi_1 & D_1\xi_2 & \cdots & D_1\xi_n \\ D_2\xi_1 & D_2\xi_2 & \cdots & D_2\xi_n \\ \vdots & \vdots & \cdots & \vdots \\ D_n\xi_1 & D_n\xi_2 & \cdots & D_n\xi_n \end{bmatrix}$$

Then $A^{-1} = I - \epsilon B + 0(\epsilon^2)$ Using some transformations we arrive at that

$$\begin{bmatrix} \zeta^k_{i_1 i_2 \ldots i_k} 1 \\ \zeta^k_{i_1 i_2 \ldots i_k} 2 \\ \vdots \\ \zeta^k_{i_1 i_2 \ldots i_k} n \end{bmatrix} = \begin{bmatrix} D_1 \zeta^{k-1}_{i_1 i_2 \ldots} \\ D_2 \zeta^{k-1}_{i_1 i_2 \ldots} \\ \vdots \\ D_n \zeta^{k-1}_{i_1 i_2 \ldots} \end{bmatrix} - B \begin{bmatrix} u_{i_1 i_2 \ldots i_{k-1}} 1 \\ u_{i_1 i_2 \ldots i_{k-1}} 2 \\ \vdots \\ u_{i_1 i_2 \ldots i_{k-1}} n \end{bmatrix}$$

$i = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k \geq 2$ and this leads to (1.97). Hence the proof. The details of the proof are contained in [6].

## 1.15 $m$ Dependent and $n$ Independent Variables.

We consider the case of $n$ independent variables $x = (x^1 \ldots . x^n)$ and $m$ dependent variables $u(x) = u^1(x) \ldots u^n(x)$. Partial derivatives are denoted by $u^\mu_i = \frac{\partial u^\mu}{\partial x^i}$. The notation

$$\partial u \equiv \partial^1 u = u^1_1(x) \ldots u^1_n(x) \ldots u^m_1(x) \ldots u^m_n(x)$$

denotes the set of all first-order partial derivatives

$$\begin{aligned} \partial^p u &= \{u^p_{i_1 \ldots i_p} | \mu = 1 \ldots m : i_1 \ldots i_p = 1 \ldots n\} \\ &= \{\frac{\partial^p u^p(x)}{\partial x^{i_1} \ldots \partial x^{i_p}} | \mu = 1 \ldots m : i_1 \ldots i_p = 1 \ldots n\} \end{aligned}$$

denotes the set of all partial derivatives of order $p$. Point transformations of the form

$$\bar{x} = f(x, u) \tag{1.98}$$

$$\bar{u} = g(x, u) \tag{1.99}$$

acting on the $n+m$ dimensional space $(x,u)$ has as its pth extended transformation

$$(\bar{x})^i = f^i(x,u) \tag{1.100}$$

$$(\bar{u^\mu}) = g^\mu(x,u) \tag{1.101}$$

$$(\bar{u^\mu_i}) = h^\mu_i(x,u,\partial u) \tag{1.102}$$

$$\vdots \tag{1.103}$$

$$(\bar{u^\mu_{i_1\ldots i_p}}) = h^\mu_{i_1\ldots i_p}(x,u,\partial u\ldots\partial^p u) \tag{1.104}$$

with $i, i_1, \ldots, i_p = 1, \ldots, n; \quad \mu = 1\ldots m; \quad \frac{\partial(\bar{u^\mu})}{\partial((\bar{x})i}$. The transformed components of the first-order derivatives are determined by

$$\begin{bmatrix} (\bar{u})^\mu_1 \\ (\bar{u})^\mu_2 \\ \vdots \\ (\bar{u})^\mu_n \end{bmatrix} = \begin{bmatrix} h^\mu_1 \\ h^\mu_2 \\ \vdots \\ h^\mu_n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 g^\mu \\ D_2 g^\mu \\ \vdots \\ D_n g^\mu \end{bmatrix}$$

where $A^{-1}$ is the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} D_1 f^1 & \cdots & D_1 f^n \\ \vdots & \cdots & \vdots \\ D_n f^1 & \cdots & D_n f^n \end{bmatrix}$$

in terms of the total derivative operators

$$D_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ii_1} \frac{\partial}{\partial u^\mu_{i_1}} + \ldots,$$

$i = 1, \ldots, n$ [7]. The transformed components of the higher-order derivatives are determined by

$$\begin{bmatrix} (\bar{u})^\mu_{i_1\ldots i_p}1 \\ (\bar{u})^\mu_{i_1\ldots i_p}2 \\ \vdots \\ (\bar{u})^\mu_{i_1\ldots i_p}n \end{bmatrix} = \begin{bmatrix} h^\mu_{i_1\ldots i_p}1 \\ h^\mu_{i_1\ldots i_p}2 \\ \vdots \\ h^\mu_{i_1\ldots i_p}n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 h^\mu_{i_1\ldots i_{p-1}}1 \\ D_2 h^\mu_{i_1\ldots i_{p-1}}2 \\ \vdots \\ D_n h^\mu_{i_1\ldots i_{p-1}}n \end{bmatrix}$$

The situation where the point transformation (5.154,5.155) is a one-parameter group of transformation given by

$$\bar{x}^i = f^i(x,u,\epsilon) = x^i + \epsilon\xi^i(x,u) + 0(\epsilon^2), \quad i = 1, \ldots.n \tag{1.105}$$

$$\bar{u}^\mu = g^\mu(x,u,\epsilon) = u^\mu + \epsilon\xi^i(x,u) + 0(\epsilon^2), \quad \mu = 1, \ldots.m \tag{1.106}$$

will have the corresponding generator given by

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu} \qquad [7] \qquad (1.107)$$

# Chapter 2

# Symmetry Analysis of Black-Scholes Equation

In this chapter we state the Symmetry Analysis as presented in the paper by Ibragimov and Gazizov ([9]). We present the symmetries of one-dimensional Black-scholes equation, finite symmetry transformations of determined operators, transformation to heat equation and invariant solutions from some operators. The one-dimensional Black-scholes model is given by the partial differential equation

$$u_t + \frac{1}{2}A^2 x^2 u_{xx} + Bxu_x - Cu = 0 \tag{2.1}$$

with constant coefficients $A, B$ and $C$, where $A \neq 0$, and define $D \equiv B - \frac{A^2}{2}$. The Symmetry Analysis of the equation (2.1) as outlined in [9] follows in the subsequent sections.

## 2.1 Symmetries

For the Black-Scholes model (2.1), $n = 1, x^1 = x$ the generator or symbol of infinitesimal symmetry is given as

$$X = \xi^1(t,x)\frac{\partial}{\partial t} + \xi^2(t,x)\frac{\partial}{\partial x} + \eta(t,x)\frac{\partial}{\partial u} \tag{2.2}$$

Its extension up to the second prolongation is given by

$$X^{(2)} = X + \eta_t^{(1)} \frac{\partial}{\partial u_t} + \eta_x^{(1)} \frac{\partial}{\partial u_x} + \eta_{xx}^{(2)} \frac{\partial}{\partial u_{xx}} \quad (2.3)$$

where $X$ is defined by equation (2.2) and the functions $\zeta^0, \zeta^1$and $\zeta^{11}$ are given by

$$\begin{aligned}\zeta^0 &= D_t(\eta) - u_t D_t(\xi^0) - u_x^i D_t(\xi^1) \\ &= \eta_t + u_t \eta_u - u_t \xi_t^0 - u_t^2 \xi_u^0 - u_x \xi_t^1 - u_t u_x \xi_u^1\end{aligned}$$

$$\zeta^i = D_i(\eta) - u_t D_i(\xi^0) - u_x^i D_i(\xi^1)$$

$$\zeta^1 = \eta_x + u_x \eta_u - u_t \xi_x^0 - u_t u_x \xi_u^0 - u_x \xi_x^1 - u_x^2 \xi_u^1$$

$$\begin{aligned}\zeta^{ij} &= D_j(\zeta^i) - u_{tx^i} D_j(\xi^0) - u_{x^i x^k} D_j(\xi^k) \\ \zeta^{11} &= D_x(\zeta^1) - u_{tx} D_x(\xi^0) - u_{xx} D_x(\xi^1) \\ &= \eta_{xx} + 2u_x \ eta_{xu} + u_{xx}\eta_u + u_x^2 \eta_{uu} - 2u_{tx}\xi_x^0 \\ &- u_t \xi_{xx}^0 - 2u_t u_x \xi_{xu}^0 - (u_t u_{xx} + 2u_x u_{tx})\xi_u^0 - u_t u_x^2 \xi_{uu}^0 \\ &- 2u_{xx}\xi_x^1 - u_x \xi_{xx}^1 - 2u_x^2 \xi_{xu}^1 - 3u_x u_{xx} \xi_u^1 - u_x^3 \xi_{uu}^1\end{aligned}$$

where $D_x$ and $D_t$ are total derivatives with respect to the variables $x$ and $t$ respectively and are defined by

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots .$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots . \quad [9]$$

The determining equation is given by

$$\{\zeta^0 + \frac{1}{2} + A^2 x u_{xx} \xi^1 + \frac{1}{2} A^2 x^2 \zeta^{11} + B u_x \xi^1 + B x \zeta^1 - C\eta\}|_{u_t = -\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu} = 0 \quad (2.4)$$

The substitutions of $\zeta_t^0, \zeta^1$ and $\zeta^{11}$ in the determining equation yields that

$$\begin{aligned}
&\eta_t + (-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)(\eta_u - \xi_t^0) - (-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)^2\xi_u^0 - \\
&\quad u_x\xi_t^1 - (-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)u_x\xi_u^1 + A^2xu_{xx}\xi^1 + \frac{1}{2}A^2x^2\eta_{xx} + A^2x^2u_x\eta_{xu} \\
&\quad + \frac{1}{2}A^2x^2u_{xx}(\eta_u - 2\xi_x^1 - 3u_x\xi_u^1) + \frac{1}{2}A^2x^2u_x^2(\eta_{uu} - 2\xi_{ux}^1) - A^2x^2u_{tx}\xi_x^0 \\
&\quad - \frac{1}{2}A^2x^2(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_{xx}^0 - A^2x^2(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_{xu}^0 \\
&\quad - \frac{1}{2}A^2x^2u_{xx}(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_u^0 - \frac{1}{2}A^2x^2(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_{uu}^0 \\
&\quad - \frac{1}{2}A^2x^2u_x^3\xi_{uu}^1 + Bv_x\xi^1 - A^2x^2u_{tx}u_x\xi_u^0 + Bxu_x(\eta_u - \xi_x^1) \\
&\quad - Bx(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_x^0 + Bx\eta_x - Bxu_x(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_u^0 \\
&\quad - Bxu_x^2\xi_u^1 - C\eta = 0.
\end{aligned} \tag{2.5}$$

The solution of determining equation (2.5) provides an infinite dimensional vector space of infinite symmetries of the equation (2.1) spanned by the operators

$$\begin{aligned}
X_1 &= 2A^2t^2\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x} + \{(\ln x - Dt)^2 + 2A^2Ct^2 - A^2t\}u\frac{\partial}{\partial u} \\
X_2 &= 2t\frac{\partial}{\partial t} + x(\ln x + Dt)\frac{\partial}{\partial x} + (Ct)u\frac{\partial}{\partial u} \\
X_3 &= \frac{\partial}{\partial t} \\
X_4 &= A^2xt\frac{\partial}{\partial x} + \{\ln x - Dt\}u\frac{\partial}{\partial u} \\
X_5 &= x\frac{\partial}{\partial x}
\end{aligned} \tag{2.6}$$

and

$$X_6 = u\frac{\partial}{\partial u}, \quad X_\infty = g(t,x)\frac{\partial}{\partial u} \tag{2.7}$$

The function $g(t,x)$ in (2.7) is an arbitrary solution of equation (2.1) and $X_\infty$ is an infinite symmetry [9].

### 2.1.1 Finite symmetry transformations

The finite symmetry transformations

$$\bar{t} = f(t,x,u,\epsilon), \quad \bar{x} = g(t,x,u,\epsilon), \quad \bar{u} = h(t,x,u,\epsilon),$$

corresponding to the generators (2.6) and (2.7) are found by solving the *Lie equations* (1.8). These are given by:

$$
\begin{aligned}
X_1 &: \ \bar{t} = \frac{t}{1-2A^2\epsilon_1 t}, \quad \bar{x} = x^{\frac{t}{1-2A^2\epsilon_1 t}}, \quad \bar{u} = u\sqrt{1-2A^2\epsilon_1 t}\, e^{\frac{[(\ln x - Dt)^2 + 2A^2Ct^2]\epsilon_1}{1-2A^2\epsilon_1 t}} \\
X_2 &: \ \bar{t} = t\epsilon_2^2, \quad \bar{x} = x\epsilon_2 e^{D(\epsilon_2^2-\epsilon_2)t}, \quad \bar{u} = ue^{C(\epsilon_2^2-\epsilon_2)t} \quad \epsilon_2 \neq 0 \\
X_3 &: \ \bar{t} = t+\epsilon_3, \quad \bar{x} = x, \quad \bar{u} = u \\
X_4 &: \ \bar{t} = t, \quad \bar{x} = xe^{A^2\epsilon_4 t}, \quad \bar{u} = ux^{\epsilon_4}e^{(\frac{1}{2}A^2\epsilon_4^2 - D\epsilon_4)t} \\
X_5 &: \ \bar{t} = t, \quad \bar{x} = x\epsilon_5, \quad \bar{u} = u \quad \epsilon_5 \neq 0
\end{aligned}
$$

and

$$
\begin{aligned}
X_6 &: \ \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u\epsilon_6 \quad \epsilon_6 \neq 0 \\
X_\phi &: \ \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u + g(t,x)
\end{aligned}
$$

The functions $\epsilon_1, \ldots, \epsilon_6$ are the parameters of the one-parameter groups generated by $X_1, \ldots X_6$, and $g(t,x)$ is an arbitrary solution of (2.1). The operators $X_1, \ldots X_6$ generate a six-parameter group and $X_\phi$ generates an infinite group.

## 2.2 Transformation to heat equation

In the paper [9] as well the text [17] Gazizov and Ibragimov inform that equation (2.1) is reducible to heat equation

$$v_\tau = v_{yy} \tag{2.8}$$

by Lie equivalence transformation

$$\tau = \beta(t), \quad y = \alpha(t,x), \quad v = \gamma(t,x)u, \quad \alpha_x \neq 0, \quad \beta_t \neq 0. \tag{2.9}$$

With change of variables (2.9) the equation (2.8) becomes

$$u_{xx} + \Big[\frac{2\gamma_x}{\gamma} + \frac{\alpha_x\alpha_t}{\beta_t} - \frac{\alpha_{xx}}{\alpha_x}\Big]u_x - \frac{\alpha_x^2}{\beta_t}u_t + \Big[\frac{\gamma_{xx}}{\gamma} + \frac{\alpha_x\alpha_t\gamma_x}{\beta_t\gamma} - \frac{\alpha_x^2\gamma_t}{\beta_t\gamma} - \frac{\alpha_{xx}\gamma_x}{\alpha_x\gamma}\Big]u = 0 \tag{2.10}$$

Equation (2.10) compared to (2.1) written in the form

$$u_{xx} + \frac{2B}{A^2x}u_x + \frac{2}{A^2x^2}u_t - \frac{2C}{A^2x^2}u = 0$$

results in the system

$$\begin{aligned} \frac{\alpha_x^2}{\beta_t} &= -\frac{2}{A^2x^2} && (2.11)\\ \frac{2\gamma_x}{\gamma} + \frac{\alpha_x\alpha_t}{\beta_t} - \frac{\alpha_{xx}}{\alpha_x} &= \frac{2B}{A^2x} && (2.12)\\ \frac{\gamma_{xx}}{\gamma} + \frac{\alpha_x\alpha_t\gamma_x}{\beta_t\gamma} - \frac{\alpha_x^2\gamma_t}{\beta_t\gamma} - \frac{\alpha_{xx}\gamma_x}{\alpha_x\gamma} &= -\frac{2C}{A^2x^2} && (2.13) \end{aligned}$$

Solving for $\alpha$ in (2.11) results in that

$$\alpha(t,x) = \frac{\varphi(t)}{A}\ln x + \psi(t), \quad \beta_t = -\frac{1}{2}\varphi(t)^2 \tag{2.14}$$

with $\varphi(t)$ and $\psi(t)$ as arbitrary functions. The substitution of (2.14) in (2.12) results in

$$\gamma(t,x) = \nu(t)x^{\frac{B}{A^2}-\frac{1}{2}+\frac{\psi_t}{A\varphi}+\frac{\psi_t}{2A^2\varphi}\ln x} \tag{2.15}$$

with an arbitrary function $\nu(t)$. The solution of (2.13) results in two possibilities, either

$$\varphi = \frac{1}{L-Kt}, \quad \psi = \frac{M}{L-Kt} + N, \quad K \neq 0,$$

and the function $\nu(t)$

$$\frac{\nu_t}{\nu} = \frac{M^2K^2}{2(L-Kt)^2} - \frac{K}{2(L-Kt)} - \frac{A^2}{8} + \frac{B}{2} - \frac{B^2}{2A^2} - C$$

or

$$\varphi = L, \quad \psi = Mt + N, \quad L \neq 0,$$

and the function $\nu(t)$

$$\frac{\nu_t}{\nu} = \frac{M^2}{L^2} - \frac{K}{2(L-Kt)} - \frac{A^2}{8} + \frac{B}{2} - \frac{B^2}{2A^2} - C$$

with arbitrary constants K,L,M and N. This results in two different transformations that associate (2.1) and (2.8). The first transformation is given by

$$\begin{aligned} y &= \frac{\ln x}{A(L-Kt)} + \frac{M}{L-Kt} + N, \quad \tau = -\frac{1}{2K(L-Kt)} + P, \quad K \neq 0\\ v &= E\sqrt{L-Kt}e^{[\frac{MK^2}{2(L-Kt)}-\frac{1}{2}(\frac{B}{A}-\frac{A}{B})^2-C]t-Ct}x^{\frac{B}{A^2}-\frac{1}{2}+\frac{MK}{A(L-Kt)}+\frac{K\ln x}{2A^2(L-Kt)}}u && (2.16) \end{aligned}$$

while the second transformation is

$$\begin{aligned} y &= \frac{L}{A}\ln x + Mt + N, \quad \tau = -\frac{L^2}{2}t + P, \quad L \neq 0\\ v &= Ee^{\frac{M^2}{2L^2}-\frac{1}{2}(\frac{B}{A}-\frac{A}{B})^2t}x^{\frac{B}{A^2}-\frac{1}{2}+\frac{M}{AL}}u && (2.17) \end{aligned}$$

The calculation presents two different transformations that associate (2.1) and (2.8) ([9]).

## 2.3 Invariant Solutions

One of the important components of classification of symmetry analysis is the construction of invariant solutions. The paper ([9]) illustrates the calculation of invariant solution by considering a subgroup and the procedure is applied to each operator and the results are as follows: The one-parameter subgroup with the generator

$$X = X_1 + X_2 + X_3 = \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

provides invariants given by

$$I_1 = t - \ln x, \qquad I_2 = \frac{u}{x}.$$

The invariant solution is of the form

$$I_2 = \phi(I_1) \quad \text{or} \quad u = x\phi(z), \quad z = t - \ln x.$$

The substitution of these invariants in equation (2.1) results in the equation with constant coefficients given by

$$\frac{A}{2}\phi'' + (1 - B - \frac{A}{2})\phi' + (B - C)\phi = 0$$

which is solvable. Applying the procedure to individual operators the results are as follows:

$$X_1 : u = \frac{1}{\sqrt{t}}e^{(\frac{(\ln x - Dt)^2}{2A^2 t} + Ct)\phi(\frac{\ln x}{t})}, \quad \phi'' = 0$$

hence

$$u = (K_1\frac{\ln x}{t^{\frac{3}{2}}} + \frac{K_2}{\sqrt{t}})e^{(\frac{(\ln x - Dt)^2}{2A^2 t} + Ct)}$$

$$X_2 : u = e^{Ct}\phi(\frac{\ln x}{\sqrt{t}} - D\sqrt{t}), \quad A^2\phi'' - z\phi' = 0, \quad z = \frac{\ln x}{\sqrt{t}} - D\sqrt{t}$$

whence

$$\phi(z) = K_1\int_0^z e^{\frac{\mu^2}{2A^2}}d\mu + K_2$$

$$X_3 : u = \phi(x), \quad \frac{1}{2}A^2x^2\phi'' + Bx\phi' - C\phi = 0.$$

The equation reduces to linear coefficients when $z = \ln x$ is introduced.

$$X_4 : u = e^{(\frac{(\ln x - Dt)^2}{2A^2t})}\phi(t), \quad \phi' + (\frac{1}{2t} - C)\phi = 0$$

whence

$$\phi = \frac{K}{\sqrt{t}}e^{Ct}$$

and hence

$$u = \frac{K}{\sqrt{t}}e^{(\frac{(\ln x - Dt)^2}{2A^2t} + Ct)}$$

$$X_5 : u = \phi(t), \quad \phi' - C\phi = 0 \tag{2.18}$$

whence

$$u = Ke^{Ct} \tag{2.19}$$

$K, K_1, K_2$ are constants of integration, and

$$D = B - \frac{1}{2}A^2.$$

Operators $X_6, X_\infty$ do not provide invariant solutions.

# Chapter 3

# Introduction of new method

This chapter presents the Symmetry Analysis of transformed one-dimensional and two-dimensional Black-Scholes equations. The method employs a formula that we shall henceforth be refer to as the Manale's formula. The rationale and justification of Manale's formula is presented in Appendix A . We determine the symmetries from a modified transformation with an infinitesimal $\omega \to 0$. We compute the Commutator Table for operators in one-dimensional and determine invariant solutions for one operator in each case. Graphical solutions are presented for the calculated solutions.

## 3.1 The transformed one-dimensional Black-Scholes equation

We present a new method to Symmetry Analysis of (2.1) where we start with transforming the equation in a different independent variable, $r$. The one-dimensional Black-Scholes equation (2.1) is transformed using the following change of variables.

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} \\ x u_x &= \frac{\partial u}{\frac{\partial x}{x}} = \frac{\partial u}{\partial \ln x} \end{aligned} \tag{3.1}$$

Let

$$
\begin{aligned}
& r = \ln x, \quad \text{then} \\
& \frac{\partial r}{\partial x} = \frac{1}{x} \\
& \frac{\partial x}{\partial r} = x
\end{aligned}
\tag{3.2}
$$

We therefore express

$$
x u_x = \frac{\partial u}{\partial r} \tag{3.3}
$$

Also,

$$
\begin{aligned}
& u_{xx} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) \\
& x^2 u_{xx} = x^2 \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}\Big) \\
& = x\Big\{\frac{\partial}{\frac{\partial x}{x}}\Big(\frac{\partial u}{\partial x}\Big)\Big\} \\
& = x\Big\{\frac{\partial}{\partial \ln x}\Big(\frac{\partial u}{\partial x}\Big)\Big\} \\
& = x\Big\{\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial x}\Big)\Big\} \\
& = x\Big\{\frac{\partial}{\partial r}\Big(\frac{1}{x}\frac{\partial u}{\partial \ln x}\Big)\Big\} = x\Big\{\frac{\partial}{\partial r}\Big(\frac{1}{x}\frac{\partial u}{\partial r}\Big)\Big\} \\
& = x\Big\{-\frac{1}{x^2}\frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{1}{x}\frac{\partial^2 u}{\partial r^2}\Big\} \\
& = x\Big\{-\frac{1}{x}\frac{\partial u}{\partial r} + \frac{1}{x}\frac{\partial^2 u}{\partial r^2}\Big\} \\
& = -\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}
\end{aligned}
\tag{3.4}
$$

Therefore

$$
\begin{aligned}
& x u_x = u_r \\
& x^2 u_{xx} = u_{rr} - u_r
\end{aligned}
\tag{3.5}
$$

where $r$ is given by equation (3.2). We substitute equation (3.5) in equation (2.1) and define

$$
D = B - \frac{A^2}{2},
$$

then the Black-Scholes one dimensional equation transforms to

$$
u_t + \frac{1}{2}A^2 u_{rr} + D u_r - C u = 0. \tag{3.6}
$$

## 3.2 Solution of determining equation for (3.6)

The infinitesimal generator for point symmetry admitted by equation (3.6) is of the form

$$X = \xi^1(t,r)\frac{\partial}{\partial t} + \xi^2(t,r)\frac{\partial}{\partial r} + \eta(t,r)\frac{\partial}{\partial u} \tag{3.7}$$

The extension of the equation up to the second prolongation is given by

$$X^{(2)} = X + \eta_t^{(1)}\frac{\partial}{\partial u_t} + \eta_r^{(1)}\frac{\partial}{\partial u_r} + \eta_{rr}^{(2)}\frac{\partial}{\partial u_{rr}} \tag{3.8}$$

where $X$ is defined by equation (3.7).

The determining equation is given by

$$\eta_t^{(1)} + \frac{1}{2}A^2\eta_{rr}^{(2)} + D\eta_r^{(1)} - C\eta = 0 \tag{3.9}$$

when

$$u_{rr} = (-\frac{2}{A^2})[u_t + Du_r - Cu] \tag{3.10}$$

where we define the following from ([6],[15])

$$\begin{aligned}
&\eta = fu + g \\
&\eta_t^{(1)} = g_t + f_t u + [f - \xi_t^1]u_t - \xi_t^2 u_r \\
&\eta_r^{(1)} = g_r + f_r u + [f - \xi_r^2]u_r - \xi_r^1 u_t \\
&\eta_{rr}^{(2)} = g_{rr} + f_{rr}u + [2f_r - \xi_{rr}^2]u_r - \xi_{rr}^1 u_t \\
&+ [f - 2\xi_r^2]u_{rr} - 2\xi_r^1 u_{tr}
\end{aligned} \tag{3.11}$$

The substitutions of $\eta_t^{(1)}, \eta_r^{(1)}$ and $\eta_{rr}^{(2)}$ in the determining equation yields that

$$\begin{aligned}
&g_t + f_t u + [f - \xi_t^1]u_t - \xi_t^2 u_r + (\frac{1}{2}A^2)\{g_{rr} + f_{rr}u + [2f_r - \xi_{rr}^2]u_r \\
&- \xi_{rr}^1 u_t + [f - 2\xi_r^2](-\frac{2}{A^2}[u_t + Du_r - Cu]) - 2\xi_r^1 u_{tr}\} \\
&+ (D)[g_r + f_r u + [f - \xi_r^2]u_r - \xi_r^1 u_t] - Cfu - Cg = 0
\end{aligned} \tag{3.12}$$

We set the coefficients of $u_r, u_{tr}, u_t$ and those free of these variables to zero. We thus have the following monomials which we termed defining equations

$$u_{tr} \quad : \quad \xi^1_r = 0, \tag{3.13}$$

$$u_t \quad : \quad -\xi^1_t + 2\xi^2_r = 0 \tag{3.14}$$

$$u_r \quad : \quad -\xi^2_t + A^2 f_r + D\xi^2_r - \frac{1}{2}A^2\xi^2_{rr} = 0, \tag{3.15}$$

$$u^0_r \quad : \quad g_t + \frac{1}{2}A^2 g_{rr} + Dg_r - Cg = 0, \tag{3.16}$$

$$u \quad : \quad f_t + \frac{1}{2}A^2 f_{rr} + Df_r - 2C\xi^2_r = 0 \tag{3.17}$$

From defining equations (3.13) and (3.14) we have that

$$\xi^2_{rr} = 0 \tag{3.18}$$

Thus

$$\xi^2 = ar + b \tag{3.19}$$

which can be expressed using Manale's formula with infinitesimal $\omega$ as

$$\xi^2 = \frac{a \ \sin(\frac{\omega r}{i}) + b\phi \ \cos(\frac{\omega r}{i})}{-i\omega}, \quad \text{where} \quad \phi = \sin(\frac{\omega}{i}), \tag{3.20}$$

and $a$ and $b$ are arbitrary functions of $t$. We differentiate equation (3.20) with respect to $r$ and $t$ and obtain

the following equations

$$\xi^2_r = a \ \cos(\frac{\omega r}{i}) - b\phi \ \sin(\frac{\omega r}{i}), \tag{3.21}$$

$$\xi^2_{rr} = \frac{-\omega}{i} \ a \ \sin(\frac{\omega r}{i}) - \frac{\omega}{i} \ b \ \phi \ \cos(\frac{\omega r}{i}), \tag{3.22}$$

$$\xi^2_t = \frac{\dot{a} \ \sin(\frac{\omega r}{i}) + \dot{b}\phi \ \cos(\frac{\omega r}{i})}{-i\omega} \tag{3.23}$$

and from defining equation (3.14) we have $\xi^1_t = 2\xi^2_r$ which implies that

$$\xi^1 = 2at \ \cos(\frac{\omega r}{i}) - 2bt\phi \ \sin(\frac{\omega r}{i}) + C. \tag{3.24}$$

We substitute equations (3.21), (3.22) and (3.23)

in the defining equation (3.15) to get the expression for $f_r$ given by

$$f_r = \cos(\frac{\omega r}{i})\Big\{ -\frac{\omega b\phi}{2i} - \frac{\dot{b}\phi}{A^2 i\omega} - \frac{Da}{A^2}\Big\} + \sin(\frac{\omega r}{i})\Big\{ -\frac{\omega a}{2i} - \frac{\dot{a}}{A^2 i\omega} + \frac{Db\phi}{A^2}\Big\} \tag{3.25}$$

Integrating equation (3.25) with respect to $r$ gives the expression for $f$

$$f = \sin(\frac{\omega r}{i})\Big\{-\frac{b\phi}{2} - \frac{\dot{b}\phi}{A^2\omega^2} - \frac{Dia}{A^2\omega}\Big\} + \cos(\frac{\omega r}{i})\Big\{\frac{a}{2} + \frac{\dot{a}}{A^2\omega^2} - \frac{Dib\phi}{A^2\omega}\Big\} + k(t) \quad (3.26)$$

We use equations (3.25) and (3.26) to get expressions for $f_{rr}$ and $f_t$ given by

$$f_{rr} = \sin(\frac{\omega r}{i})\Big\{-\frac{\omega^2 b\phi}{2} - \frac{\dot{b}\phi}{A^2} + \frac{Da\omega}{A^2 i}\Big\} + \cos(\frac{\omega r}{i})\Big\{\frac{\omega^2 a}{2} + \frac{\dot{a}}{A^2} + \frac{Db\omega\phi}{A^2 i}\Big\} \quad (3.27)$$

and

$$f_t = \sin(\frac{\omega r}{i})\Big\{-\frac{\dot{b}\phi}{2} - \frac{\ddot{b}\phi}{A^2\omega^2} - \frac{Di\dot{a}}{A^2\omega}\Big\} + \cos(\frac{\omega r}{i})\Big\{\frac{\dot{a}}{2} + \frac{\ddot{a}}{A^2\omega^2} - \frac{Di\dot{a}\phi}{A^2\omega}\Big\} + k'(t) \quad (3.28)$$

We substitute equations (3.21), (3.25), (3.27) and (3.28) into the defining equation (3.17) and solve the equation

$$\begin{aligned}&\sin(\frac{\omega r}{i})\Big\{-\frac{\dot{b}\phi}{2} - \frac{\ddot{b}\phi}{A^2\omega^2} - \frac{Di\dot{a}}{A^2\omega} + 2Cb\phi\Big\} + \cos(\frac{\omega r}{i})\Big\{\frac{\dot{a}}{2} + \frac{\ddot{a}}{A^2\omega^2} - \frac{Di\dot{b}\phi}{A^2\omega} - 2Ca\Big\}\\ &+ k'(t) + \sin(\frac{\omega r}{i})\Big\{-\frac{bA^2\omega^2\phi}{4} - \frac{\dot{b}\phi}{2} + \frac{Da\omega}{2i}\Big\} + \cos(\frac{\omega r}{i})\Big\{\frac{aA^2\omega^2}{4} + \frac{\dot{a}}{2} + \frac{Db\omega\phi}{2i}\Big\} +\\ &\cos(\frac{\omega r}{i})\Big\{-\frac{Db\omega\phi}{2i} - \frac{D\dot{b}\phi}{A^2 i\omega} - \frac{D^2 a}{A^2}\Big\} + \sin(\frac{\omega r}{i})\Big\{-\frac{Da\omega}{2i} - \frac{D\dot{a}}{A^2 i\omega} + \frac{D^2 b\phi}{A^2}\Big\} = 0\end{aligned} \quad (3.29)$$

We collect all the coefficients of sine function together and equate them to zero. A similar step is taken with the cosine function.
For the coefficients of sine function we have:

$$\begin{aligned}&-\frac{\dot{b}\phi}{2} - \frac{\ddot{b}\phi}{A^2\omega^2} - \frac{Di\dot{a}}{A^2\omega} - \frac{bA^2\omega^2\phi}{4} - \frac{\dot{b}\phi}{2} - \frac{Da\omega}{2i}\\ &+ \frac{Da\omega}{2i} - \frac{D\dot{a}}{A^2\omega i} - \frac{D^2 b\phi}{A^2} + 2Cb\phi = 0\end{aligned} \quad (3.30)$$

which simplifies to a second-order ordinary linear differential equation

$$\ddot{b} + \dot{b}A^2\omega^2 + \frac{bA^4\omega^4}{4} - \frac{D^2 b}{A^2} + 2CA^2\omega^2 b = 0 \quad (3.31)$$

Solving equation (3.31) we proceed as follows. Let

$$\beta = \frac{A^2\omega^2}{2}, \quad \text{and} \quad k_1 = -\frac{D^2}{A^2} - 2C \quad (3.32)$$

We also set

$$\alpha_1 = b\beta^2 - k, \quad \text{then} \quad \dot{\alpha_1} = \dot{b}\beta^2, \ddot{\alpha_1} = \ddot{b}\beta^2 \quad (3.33)$$

Equation (3.31) transforms to

$$\ddot{\alpha_1} + 2\dot{\alpha_1} + \alpha_1\beta^2 = 0. \tag{3.34}$$

To find the solution of equation (3.34) we proceed as follows. We set

$$\alpha_1 = cz \tag{3.35}$$

where $c = c(t), z = z(t)$. Then

$$\alpha_1' = c'z + cz' \tag{3.36}$$

$$\alpha_1'' = c''z + 2c'z' + cz'' \tag{3.37}$$

We substitute equations (3.35), (3.36) and (3.37) into equation (3.34) and after rearranging we solve the equation

$$cz'' + (2c + 2c')z + (c'' + 2c' + \beta^2 c)z = 0 \tag{3.38}$$

The choice for $c$ is such that

$$2c + 2c' = 0, \tag{3.39}$$

whence

$$c = e^{-t}. \tag{3.40}$$

The equation (3.38) simplifies to

$$z'' + (\beta^2 - 1)z = 0 \tag{3.41}$$

The solution for equation (3.34) is now written

$$\alpha_1 = e^{-t}\Big(C_1\frac{\sin\bar{\omega}\cos\bar{\omega}t}{\bar{\omega}}\Big) + C_2 e^{-t}\frac{\sin\bar{\omega}t}{\bar{\omega}} \tag{3.42}$$

so that when $\beta = \pm 1$ or $\omega \to 0$ the solution for $z$ is linear, and we define

$$\bar{\omega} = \sqrt{\beta^2 - 1} \tag{3.43}$$

We substitute for $b$ in equation (3.33) to obtain that

$$b = \frac{e^{-t}}{\beta^2}\Big\{\Big(C_1\frac{\sin\bar{\omega}\cos\bar{\omega}t}{\bar{\omega}}\Big) + C_2 e^{-t}\frac{\sin\bar{\omega}t}{\bar{\omega}}\Big\} + \frac{D^2}{\beta^2 A^2} + \frac{4C}{\beta} \tag{3.44}$$

Similarly for the coefficients of the cosine function we have

$$\begin{aligned}&\frac{\dot{a}}{2}+\frac{\ddot{a}\phi}{A^2\omega^2}-\frac{Di\dot{b}\phi}{A^2\omega}+\frac{aA^2\omega^2\phi}{4}+\frac{\dot{a}}{2}+\frac{Db\omega\phi}{2i}-\frac{Db\omega\phi}{2i}\\&-\frac{D\dot{b}\phi}{A^2\omega i}-\frac{D^2a}{A^2}+2aC=0\end{aligned} \tag{3.45}$$

which simplifies to a second-order ordinary linear differential equation

$$\ddot{a}+\dot{a}A^2\omega^2+\frac{aA^4\omega^4}{4}-\frac{aD^2}{A^2}-2aA^2\omega^2C=0 \tag{3.46}$$

Solving equation (3.46) we find the solution for $a$ to be

$$a=\frac{e^{-t}}{\beta^2}\Big\{\Big(C_3\frac{\sin\bar{\omega}\cos\bar{\omega}t}{\bar{\omega}}\Big)+C_4\frac{\sin\bar{\omega}t}{\bar{\omega}}\Big\}+\frac{D^2}{\beta^2A^2}-\frac{4C}{\beta} \tag{3.47}$$

and we also have that

$$k'(t)=0 \quad\Rightarrow\quad k(t)=C_5 \tag{3.48}$$

We differentiate equations (3.44) and (3.47) to obtain expressions for $\dot{a}$ and $\dot{b}$

$$\dot{a}=-\frac{e^{-t}}{\beta^2}\Big(C_3\frac{\sin\bar{\omega}\cos\bar{\omega}t}{\bar{\omega}}+C_4\frac{\sin\bar{\omega}t}{\bar{\omega}}\Big)+\frac{e^{-t}}{\beta^2}\Big(-C_3\sin\bar{\omega}\sin\bar{\omega}t+C_4\cos\bar{\omega}t\Big) \tag{3.49}$$

Similarly

$$\dot{b}=-\frac{e^{-t}}{\beta^2}\Big(C_1\frac{\sin\bar{\omega}\cos\bar{\omega}t}{\bar{\omega}}+C_2\frac{\sin\bar{\omega}t}{\bar{\omega}}\Big)+\frac{e^{-t}}{\beta^2}\Big(-C_1\sin\bar{\omega}\sin\bar{\omega}t+C_2\cos\bar{\omega}t\Big) \tag{3.50}$$

We substitute equations (3.32),(3.44),(3.47),(3.48),(3.49) and (3.50) into equation (3.26) and get the expression for $f$ given as

$$\begin{aligned}f=\sin(\frac{\omega r}{i})\Big\{&-\frac{C_1\phi e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^2\bar{\omega}}-\frac{C_2\phi e^{-t}\sin\bar{\omega}t}{2\beta^2\bar{\omega}}-\frac{D^2\phi}{2\beta^3A^2}-\frac{4C\phi}{\beta}\\&+\frac{C_1\phi e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3\bar{\omega}}+\frac{C_2\phi e^{-t}\sin\bar{\omega}t}{2\beta^3\bar{\omega}}+\frac{C_1\phi e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3}\\&-\frac{C_2\phi e^{-t}\cos\bar{\omega}t}{2\beta^3}-\frac{C_3Di\omega e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3}-\frac{C_4Di\omega e^{-t}\sin\bar{\omega}t}{2\beta^3}-\frac{D^3i\omega}{2\beta^3A^2}\Big\}\\&+\cos(\frac{\omega r}{i})\Big\{\frac{C_3e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^2\bar{\omega}}+\frac{C_4e^{-t}\sin\bar{\omega}t}{2\beta^2\bar{\omega}}+\frac{D^2}{2\beta^3A^2}-\frac{4C\phi}{\beta}\\&-\frac{C_3e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3\bar{\omega}}-\frac{C_4e^{-t}\sin\bar{\omega}t}{2\beta^3\bar{\omega}}-\frac{C_3e^{-t}\sin\bar{\omega}\sin\bar{\omega}t}{2\beta^3}\\&-\frac{C_4e^{-t}\cos\bar{\omega}t}{2\beta^3}-\frac{C_1\phi Di\omega e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3}\\&-\frac{C_2\phi Di\omega e^{-t}\sin\bar{\omega}t}{2\beta^3}-\frac{D^3\phi i\omega}{2\beta^3A^2}\Big\}+C_5\end{aligned}$$

### 3.2.1 Infinitesimals for equation (3.6)

The linearly independent solutions of the defining equations (3.12) lead to the infinitesimals

$$\begin{aligned}\xi^1 = &\cos\left(\frac{\omega r}{i}\right)\Big\{\frac{2te^{-t}}{\beta^2\bar{\omega}}\Big\{\Big(C_3\sin\bar{\omega}\cos\bar{\omega}t\Big) + C_4\sin\bar{\omega}t\Big\} + \frac{2tD^2}{\beta^2A^2}\Big\}\\ &- \sin\left(\frac{\omega r}{i}\right)\Big\{\frac{2t\phi e^{-t}}{\beta^2\bar{\omega}}\Big\{\Big(C_1\sin\bar{\omega}\cos\bar{\omega}t\Big) + C_2\sin\bar{\omega}t\Big\}\\ &+ \frac{2t\phi D^2}{\beta^2A^2}\Big\} - \frac{8Ct}{\beta}\cos\left(\frac{\omega r}{i}\right) + \frac{8Ct\phi}{\beta}\sin\left(\frac{\omega r}{i}\right) + C_6\end{aligned} \tag{3.51}$$

$$\begin{aligned}\xi^2 = &\sin\left(\frac{\omega r}{i}\right)\Big\{\frac{e^{-t}}{-\beta^2\omega\bar{\omega}i}\Big(C_3\sin\bar{\omega}\cos\bar{\omega}t + C_4\sin\bar{\omega}t\Big) - \frac{D^2}{i\omega\beta^2A^2}\Big\}\\ &+ \cos\left(\frac{\omega r}{i}\right)\Big\{\frac{e^{-t}\phi}{-\beta^2\omega\bar{\omega}i}\Big(C_1\sin\bar{\omega}\cos\bar{\omega}t - C_2\sin\bar{\omega}t\Big) - \frac{D^2\phi}{i\omega\beta^2A^2}\Big\}\\ &+ \frac{4Ci}{\omega\beta}\sin\left(\frac{\omega r}{i}\right) + \frac{4Ci\phi}{\omega\beta}\cos\left(\frac{\omega r}{i}\right)\end{aligned} \tag{3.52}$$

$$\begin{aligned}f = &\sin(\frac{\omega r}{i})\Big\{ - \frac{C_1\phi e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^2\bar{\omega}} - \frac{C_2\phi e^{-t}\sin\bar{\omega}t}{2\beta^2\bar{\omega}} - \frac{D^2\phi}{2\beta^3A^2} - \frac{4C\phi}{\beta}\\ &+ \frac{C_1\phi e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3\bar{\omega}} + \frac{C_2\phi e^{-t}\sin\bar{\omega}t}{2\beta^3\bar{\omega}} + \frac{C_1\phi e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3}\\ &- \frac{C_2\phi e^{-t}\cos\bar{\omega}t}{2\beta^3} - \frac{C_3Di\omega e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3} - \frac{C_4Di\omega e^{-t}\sin\bar{\omega}t}{2\beta^3} - \frac{D^3i\omega}{2\beta^3A^2}\Big\}\\ &+ \cos(\frac{\omega r}{i})\Big\{\frac{C_3e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^2\bar{\omega}} + \frac{C_4e^{-t}\sin\bar{\omega}t}{2\beta^2\bar{\omega}} + \frac{D^2}{2\beta^3A^2} - \frac{4C\phi}{\beta}\\ &- \frac{C_3e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3\bar{\omega}} - \frac{C_4e^{-t}\sin\bar{\omega}t}{2\beta^3\bar{\omega}} - \frac{C_3e^{-t}\sin\bar{\omega}\sin\bar{\omega}t}{2\beta^3}\\ &- \frac{C_4e^{-t}\cos\bar{\omega}t}{2\beta^3} - \frac{C_1\phi Di\omega e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3}\\ &- \frac{C_2\phi Di\omega e^{-t}\sin\bar{\omega}t}{2\beta^3} - \frac{D^3\phi i\omega}{2\beta^3A^2}\Big\} + C_5\end{aligned} \tag{3.53}$$

### 3.2.2 The symmetries for equation (3.6)

According to (3.12), the infinitesimals: (3.53), (3.51) and (3.52), lead to the generators

$$\begin{aligned}X_1 = &\Big( -\frac{2te^{-t}\phi}{\beta^2\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial t} + \Big(\frac{e^{-t}i\phi}{\beta^2\bar{\omega}\omega}\sin\bar{\omega}\cos\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial r}\\ &+ \Big\{ -\frac{e^{-t}\phi}{2\beta^2\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big) + \frac{e^{-t}\phi}{2\beta^3\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big)\\ &+ \frac{e^{-t}\phi}{2\beta^3}\sin\bar{\omega}\cos\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big) - \frac{Di\phi\omega e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3}\cos\Big(\frac{\omega r}{i}\Big)\Big\}u\frac{\partial}{\partial u}\end{aligned} \tag{3.54}$$

$$\begin{aligned}X_2 = &\Big( -\frac{2te^{-t}\phi}{\beta^2\bar{\omega}}\sin\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial t} - \Big(\frac{e^{-t}i\phi}{\beta^2\bar{\omega}\omega}\sin\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial r}\\ &+ \Big\{ -\frac{e^{-t}\phi}{2\beta^2\bar{\omega}}\sin\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big) + \frac{e^{-t}\phi}{2\beta^3\bar{\omega}}\sin\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big)\\ &- \frac{e^{-t}\phi}{2\beta^3}\cos\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big) - \frac{Di\phi\omega e^{-t}\sin\bar{\omega}t}{2\beta^3}\cos\Big(\frac{\omega r}{i}\Big)\Big\}u\frac{\partial}{\partial u}\end{aligned} \tag{3.55}$$

$$\begin{aligned}X_3 = &\Big(\frac{2te^{-t}}{\beta^2\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial t} + \Big(\frac{e^{-t}i}{\beta^2\bar{\omega}\omega}\sin\bar{\omega}\cos\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial r}\\ &+ \Big\{\frac{e^{-t}}{2\beta^2\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big) - \frac{e^{-t}}{2\beta^3\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big)\\ &- \frac{e^{-t}}{2\beta^3}\sin\bar{\omega}\sin\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big) - \frac{Di\omega e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3}\sin\Big(\frac{\omega r}{i}\Big)\Big\}u\frac{\partial}{\partial u}\end{aligned} \tag{3.56}$$

$$\begin{aligned}X_4 = &\Big(\frac{2te^{-t}\phi}{\beta^2\bar{\omega}}\sin\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial t} + \Big(\frac{e^{-t}i\phi}{\beta^2\bar{\omega}\omega}\sin\bar{\omega}t\sin\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial r}\\ &+ \Big\{\frac{e^{-t}}{2\beta^2\bar{\omega}}\sin\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big) - \frac{e^{-t}}{2\beta^3\bar{\omega}}\sin\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big)\\ &+ \frac{e^{-t}}{2\beta^3}\cos\bar{\omega}t\cos\Big(\frac{\omega r}{i}\Big) - \frac{Di\omega e^{-t}\sin\bar{\omega}t}{2\beta^3}\sin\Big(\frac{\omega r}{i}\Big)\Big\}u\frac{\partial}{\partial u}\end{aligned} \tag{3.57}$$

$$\begin{aligned}X_5 = &\frac{2tD^2}{\beta^2A^2}\Big(\cos\Big(\frac{\omega r}{i}\Big) - \phi\sin\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial t} + \frac{D^2}{\omega\beta^2A^2}\Big(\phi\cos\Big(\frac{\omega r}{i}\Big)\\ &+ \sin\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial r} - \frac{D^2i\omega}{2\beta^3A^2}\Big(\sin\Big(\frac{\omega r}{i}\Big) + \phi\cos\Big(\frac{\omega r}{i}\Big)\\ &+ D\sin\Big(\frac{\omega r}{i}\Big) + D\phi\cos\Big(\frac{\omega r}{i}\Big)\Big)u\frac{\partial}{\partial u}\end{aligned} \tag{3.58}$$

$$X_6 = u\frac{\partial}{\partial u} \tag{3.59}$$

$$X_7 = \frac{\partial}{\partial t} \tag{3.60}$$

$$X_8 = -\frac{8Ct}{\beta}\Big(\cos\Big(\frac{\omega r}{i}\Big) + \phi\sin\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial t} + \frac{4Ct}{\beta\omega}\Big(\phi\cos\Big(\frac{\omega r}{i}\Big) + \sin\Big(\frac{\omega r}{i}\Big)\Big)\frac{\partial}{\partial r}$$
$$+ \frac{4C}{\beta}\Big(\phi\sin\Big(\frac{\omega r}{i}\Big) - \cos\Big(\frac{\omega r}{i}\Big)\Big)u\frac{\partial}{\partial u} \tag{3.61}$$

The defining equation (3.16) gives an infinite symmetry

$$X_\infty = g(t,r)\frac{\partial}{\partial u} \tag{3.62}$$

### 3.2.3 Table of Commutators

The determined set of operators (3.54) to (3.61) form a Lie Algebra if their commutator is bilinear, anti symmetric and satisfy the Jacobi identity as stated in (1.6). The commutator for the pair of operators (3.54) and (3.55) is determined as follows:

$$\begin{aligned}
[X_1, X_2] \;=\;& \big\{X_1\big(-\frac{2te^{-t}\phi}{\beta^2\bar\omega}\sin\bar\omega t\sin\big(\frac{\omega r}{i}\big)\big) - X_2\big(-\frac{2te^{-t}\phi}{\beta^2\bar\omega}\sin\bar\omega\cos\bar\omega t\sin\big(\frac{\omega r}{i}\big)\big)\big\}\frac{\partial}{\partial t}\\
+\;& \big\{X_1\big(\frac{e^{-t}i\phi}{\beta^2\bar\omega\omega}\sin\bar\omega t\cos\big(\frac{\omega r}{i}\big)\big) - X_2\big(\frac{e^{-t}i\phi}{\beta^2\bar\omega\omega}\sin\bar\omega\cos\bar\omega t\cos\big(\frac{\omega r}{i}\big)\big)\big\}\frac{\partial}{\partial r}\\
+\;& \big\{X_1\big(-\frac{e^{-t}\phi}{2\beta^2\bar\omega}\sin\bar\omega t\sin\big(\frac{\omega r}{i}\big) + \frac{e^{-t}\phi}{2\beta^3\bar\omega}\sin\bar\omega t\sin\big(\frac{\omega r}{i}\big)\\
+\;& -\frac{e^{-t}\phi}{2\beta^3}\cos\bar\omega t\sin\big(\frac{\omega r}{i}\big) - \frac{Di\phi\omega e^{-t}\sin\bar\omega t}{2\beta^3}\cos\big(\frac{\omega r}{i}\big)\big)\\
-\;& X_2\big(-\frac{e^{-t}\phi}{2\beta^2\bar\omega}\sin\bar\omega\cos\bar\omega t\sin\big(\frac{\omega r}{i}\big) + \frac{e^{-t}\phi}{2\beta^3\bar\omega}\sin\bar\omega\cos\bar\omega t\sin\big(\frac{\omega r}{i}\big)\\
+\;& \frac{e^{-t}\phi}{2\beta^3}\sin\bar\omega\cos\bar\omega t\sin\big(\frac{\omega r}{i}\big) - \frac{Di\phi\omega e^{-t}\sin\bar\omega\cos\bar\omega t}{2\beta^3}\cos\big(\frac{\omega r}{i}\big)\big)\big\}u\frac{\partial}{\partial u}
\end{aligned} \tag{3.63}$$

The equation (3.63) simplifies to

$$[X_1, X_2] = \Big(\frac{4t^2e^{-2t}}{\beta^4\bar\omega}\sin^2\bar\omega\sin^2\big(\frac{\omega r}{i}\big)\Big)\frac{\partial}{\partial t} + 0\times\frac{\partial}{\partial r} + 0\times u\frac{\partial}{\partial u} = 0$$

as $\omega \to 0$. Thus we have that

$$[X_1, X_2] = 0 \tag{3.64}$$

Similarly we determine the other pairs of commutators that are defined and end with the Commutator Table given as in Figure (3.1).

The Commutator Table is determined for that instant $\omega \to 0$.

| [,] | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_8$ |
|---|---|---|---|---|---|---|---|
| $X_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Figure 3.1: Commutator Table

## 3.3 Invariant Solution for equation (3.6)

We consider the symmetry given by equation (3.56). The invariants are determined from solving the equation

$$\begin{aligned}
X_3 I = &\left(\frac{2te^{-t}}{\beta^2\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)\frac{\partial I}{\partial t} + \left(\frac{e^{-t}i}{\beta^2\bar{\omega}\omega}\sin\bar{\omega}\cos\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)\right)\frac{\partial I}{\partial r}\\
&+\left\{\frac{e^{-t}}{2\beta^2\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\cos\left(\frac{\omega r}{i}\right) - \frac{e^{-t}}{2\beta^3\bar{\omega}}\sin\bar{\omega}\cos\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)\right.\\
&\left.-\frac{e^{-t}\phi}{2\beta^3}\sin\bar{\omega}\sin\bar{\omega}t\cos\left(\frac{\omega r}{i}\right) - \frac{Di\omega e^{-t}\sin\bar{\omega}\cos\bar{\omega}t}{2\beta^3}\sin\left(\frac{\omega r}{i}\right)\right\}u\frac{\partial I}{\partial u} = 0
\end{aligned} \tag{3.65}$$

The characteristic equation of (3.65) is given by

$$\frac{dt}{\frac{2te^{-t}\sin\bar{\omega}\cos\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)}{\beta^2\bar{\omega}}} = \frac{dr}{\frac{e^{-t}i\sin\bar{\omega}\sin\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)}{\beta^2\bar{\omega}\omega}} = \frac{du}{ue^{-t}K} \tag{3.66}$$

where

$$\begin{aligned}
K = &\left\{\frac{\sin\bar{\omega}\cos\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)}{2\beta^2\bar{\omega}} - \frac{\sin\bar{\omega}\cos\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)}{2\beta^3\bar{\omega}}\right.\\
&\left.-\frac{\sin\bar{\omega}\sin\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)}{2\beta^3} - \frac{Di\omega\sin\bar{\omega}\cos\bar{\omega}t\sin\left(\frac{\omega r}{i}\right)}{2\beta^3}\right\}
\end{aligned} \tag{3.67}$$

From equation (3.66) we have that

$$\frac{dt}{\frac{2te^{-t}\sin\bar{\omega}\cos\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)}{\beta^2\bar{\omega}}} = \frac{dr}{\frac{e^{-t}i\sin\bar{\omega}\sin\bar{\omega}t\cos\left(\frac{\omega r}{i}\right)}{\beta^2\bar{\omega}\omega}} \tag{3.68}$$

simplifies to

$$\frac{dt}{t} = 2\frac{\omega}{i}dr \tag{3.69}$$

whose solution is

$$t = Ce^{\frac{2\omega r}{i}} \tag{3.70}$$

The first invariant is given by

$$\psi_1 = \frac{e^{\frac{2\omega r}{i}}}{t} \tag{3.71}$$

From equation (3.66) we also have that

$$\frac{dr}{\frac{e^{-t} i \sin\bar{\omega} \sin\bar{\omega} t \cos\left(\frac{\omega r}{i}\right)}{\beta^2 \bar{\omega}\omega}} = \frac{du}{Kue^{-t}} \tag{3.72}$$

where $K$ is given by (3.67). Equation (3.72) simplifies to

$$\left(\frac{\beta - 1 - \bar{\omega} - Di\omega\bar{\omega}\tan(\frac{\omega}{i})}{2\beta}\right)\left(\frac{\omega}{i}\right)dr = \frac{du}{u} \tag{3.73}$$

We integrate equation (3.73) and obtain

$$\frac{\beta\omega r}{2\beta i} - \frac{\omega r}{2\beta i} - \frac{\bar{\omega}\omega r}{2\beta i} - \frac{D\bar{\omega}\omega \ln\cos(\frac{\omega r}{i})|}{2\beta} + C = \ln u \tag{3.74}$$

We approximate

$$\frac{\beta\omega r}{2\beta i} - \frac{\omega r}{2\beta i} - \frac{\bar{\omega}\omega r}{2\beta i} - \frac{D\bar{\omega}\omega \ln\cos(\frac{\omega r}{i})|}{2\beta} \approx \frac{\omega r}{i} \tag{3.75}$$

Integrating equation (3.72) we obtain

$$\frac{u}{e^{\frac{\omega r}{i}}} = C \tag{3.76}$$

The equation (3.76) simplifies to

$$\frac{u}{e^{\frac{\omega r}{i}}} = \psi_2 \tag{3.77}$$

which is our second invariant. If we define

$$\psi_2 = \varphi(\psi_1) \tag{3.78}$$

where $\psi_1$ is given by equation (3.71), then an invariant solution is given by

$$u = e^{\frac{\omega r}{i}}\varphi(\psi_1) \tag{3.79}$$

We differentiate equation (3.79) with respect to $t$ and twice with respect to $r$ and get the following expressions for $u_t, u_r$ and $u_{rr}$.

$$u_t = -\frac{2\omega r}{i}\frac{e^{\frac{3\omega r}{i}}}{t^2}\varphi'(\psi_1) \tag{3.80}$$

$$u_r = \frac{\omega}{i}e^{\frac{\omega r}{i}}\varphi(\psi_1) + \frac{2\omega}{it}e^{\frac{3\omega r}{i}}\varphi'(\psi_1) \tag{3.81}$$

$$\begin{aligned} u_{rr} =& -\omega^2 e^{\frac{\omega r}{i}}\varphi(\psi_1) - \frac{2\omega^2}{t}e^{\frac{3\omega r}{i}}\varphi'(\psi_1) - \frac{6\omega^2}{t}e^{\frac{3\omega r}{i}}\varphi'(\psi_1) \\ & - -\frac{4\omega^2}{t^2}e^{\frac{5\omega r}{i}}\varphi''(\psi_1) \end{aligned} \tag{3.82}$$

We substitute equations (3.80), (3.81) and (3.82) into the original equation (3.6) and get the following equation

$$\begin{aligned} &-\frac{2\omega r}{i}\frac{e^{\frac{3\omega r}{i}}}{t^2}\varphi'(\psi_1) - \omega^2 e^{\frac{\omega r}{i}}\varphi(\psi_1) - \frac{A^2\omega^2}{t}e^{\frac{3\omega r}{i}}\varphi'(\psi_1) - \frac{3A^2\omega^2}{t}e^{\frac{3\omega r}{i}}\varphi'(\psi_1) \\ &-\frac{2A^2\omega^2}{t^2}e^{\frac{5\omega r}{i}}\varphi''(\psi_1) + \frac{D\omega}{i}e^{\frac{\omega r}{i}}\varphi(\psi_1) + \frac{2D\omega}{it}e^{\frac{3\omega r}{i}}\varphi'(\psi_1) - Ce^{\frac{\omega r}{i}}\varphi(\psi_1) = 0 \end{aligned} \tag{3.83}$$

Equation (3.83) is a second order equation in $\varphi(\psi_1)$. We rearrange it in the order of derivatives of $h(\psi_1)$, and apply equation (3.32)

$$\begin{aligned} &-\frac{4\beta}{t^2}e^{\frac{5\omega r}{i}}\varphi''(\psi_1) - \varphi'(\psi_1)\{\frac{2\omega r}{i}\frac{e^{\frac{3\omega r}{i}}}{t^2} + \frac{4\beta}{t}e^{\frac{3\omega r}{i}} + \frac{6\beta}{t}e^{\frac{3\omega r}{i}} \\ &-\frac{2D\omega}{i}e^{\frac{3)\omega r}{i}}\} + \varphi(\psi_1)\{\frac{Dk\omega}{it}e^{\frac{\omega r}{i}} - \beta e^{\frac{\omega r}{i}} - Ce^{\frac{\omega r}{i}}\} = 0 \end{aligned} \tag{3.84}$$

Letting $\omega \to 0$ equation (3.84) simplifies to

$$-\frac{4\beta}{t^2}\varphi''(\psi_1) - \frac{4\beta}{t}\varphi'(\psi_1) - \frac{6\beta}{t}\varphi'(\psi_1) - \beta\varphi(\psi_1) - C\varphi(\psi_1) = 0 \tag{3.85}$$

which can be simplified to

$$\frac{4\beta\varphi''(\psi_1)}{t^2} + \frac{10\beta\varphi'(\psi_1)}{t} + (\beta + C)\varphi(\psi_1) = 0. \tag{3.86}$$

We eliminate $t$ from equation (3.86) by applying the following change of variables. From equation (3.71) we let

$$\begin{aligned} d\lambda &= td\psi_1, \\ \varphi(\psi_1) &= h \end{aligned} \tag{3.87}$$

then

$$\varphi'(\psi_1) = t\varphi_\lambda \tag{3.88}$$

and

$$\varphi''(\psi_1) = \tfrac{d}{d\psi_1}\{t\varphi_\lambda\} \tag{3.89}$$
$$= \tfrac{dt}{d\psi_1}\varphi_\lambda + t\tfrac{d\varphi_\lambda}{d\psi_1}$$
$$= \tfrac{dt}{d\psi_1}\varphi_\lambda + t\tfrac{d\varphi_\lambda}{\frac{1}{t}d\lambda}$$
$$= t^2\{h_{\lambda\lambda} - \varphi_\lambda e^{-\frac{2\omega r}{i}}\}$$

For $\omega \to 0$ equation (3.89) simplifies to

$$\varphi''(\psi_1) = t^2\{\varphi_{\lambda\lambda} - \varphi_\lambda\} \tag{3.90}$$

The substitution of equations (3.88), (3.90) transforms equation (3.86) to

$$4\beta\varphi_{\lambda\lambda} + 6\beta\varphi_\lambda + (\beta + C)\varphi = 0. \tag{3.91}$$

The solution to equation (3.91) is given by

$$\varphi = C_1 e^{(\frac{-3\beta-\sqrt{9\beta^2-4\beta(\beta+C)}}{4\beta})(\lambda)} + C_2 e^{(\frac{-3\beta+\sqrt{9\beta^2-4\beta(\beta+C)}}{4\beta})(\lambda)} \tag{3.92}$$

Thus the invariant solution is

$$u = e^{\frac{\omega r}{i}}\{C_1 e^{(\frac{-3\beta-\sqrt{9\beta^2-4\beta(\beta+C)}}{4\beta})(\lambda)} + C_2 e^{(\frac{-3\beta+\sqrt{9\beta^2-4\beta(\beta+C)}}{4\beta})(\lambda)}\} \tag{3.93}$$

However the equation (3.93) can be expressed as

$$u = e^{\frac{\omega r}{i}}\{C_1 e^{(\frac{-3\beta-i\sqrt{-(9\beta^2-4\beta(\beta+C))}}{4\beta})(\lambda)} + C_2 e^{(\frac{-3\beta+i\sqrt{-(9\beta^2-4\beta(\beta+C))}}{4\beta})(\lambda)}\} \tag{3.94}$$

This simplifies to

$$u = e^{\frac{\omega r}{i}}\{C_1 e^{\frac{-3}{4}\lambda}e^{-i\Delta\lambda} + C_2 e^{\frac{-3}{4}\lambda}e^{i\Delta\lambda}\}$$
$$\text{where}\quad \Delta = \frac{\sqrt{-(9\beta^2 - 4\beta(\beta + C))}}{4\beta} \tag{3.95}$$

Since $\Delta < 0$, we express equation (3.95) as

$$u = e^{\frac{\omega r}{i}}\{C_1 e^{\frac{-3}{4}\lambda}\sin(\Delta\lambda) + C_2 e^{\frac{-3}{4}\lambda}\cos(\Delta\lambda)\} \tag{3.96}$$

We however advance the same reason that for equation (3.96) to return to the linear form as $\Delta \to 0$ it has to be transformed to be

$$u = e^{\frac{\omega r}{i}}\{C_1 e^{\frac{-3}{4}\lambda}\frac{\sin(\Delta\lambda)}{-i\Delta} + C_2 e^{\frac{-3}{4}\lambda}\phi\frac{\cos(\Delta\lambda)}{-i\Delta}\}$$
$$\text{where}\quad \phi = \sin(\frac{\Delta}{i}) \tag{3.97}$$

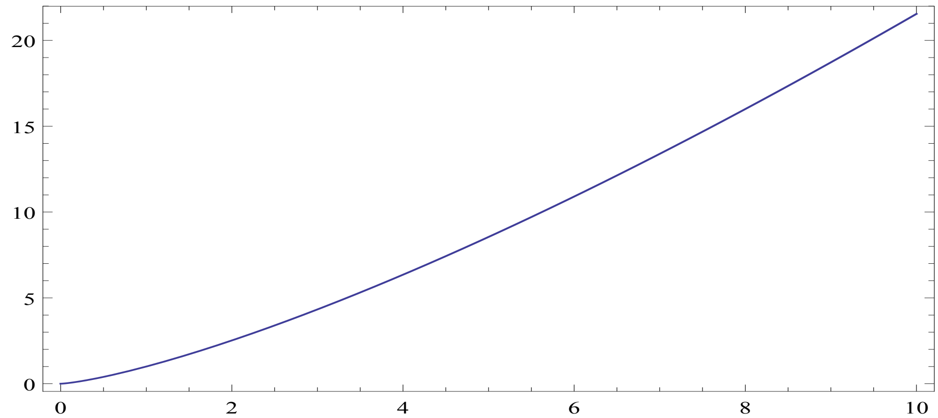


Figure 3.2: Plot for the solution (3.100) with dependent variable $u$ represented on the vertical axis and independent variable $t$ represented on the horizontal axis

### 3.3.1 Solutions for equation 3.97

This equation (3.97) has some few solutions as $\omega \to 0$. We recall that

$$\begin{aligned} \lambda &= t \int_{\psi_1}^{\psi_2} d\frac{e^{\frac{2\omega r}{i}}}{t} \\ &= e^{\frac{2\omega r}{i}} \int_{\psi_1}^{\psi_2} \frac{d\frac{1}{t}}{\frac{1}{t}} \\ &= C_0 - e^{\frac{2\omega r}{i}} \ln t \end{aligned} \tag{3.98}$$

#### Solution 1

$$\begin{aligned} u &= Ae^{\frac{\omega r}{i}} e^{-\frac{3}{4}\lambda} \frac{\sin(\Delta\lambda)}{-i\Delta} \\ &= Ae^{\frac{\omega r}{i}} e^{(C_0+\frac{3}{4}e^{\frac{2\omega r}{i}} \ln t)} \frac{\sin(\Delta(C_0-e^{\frac{2\omega r}{i}} \ln t))}{-i\Delta} \end{aligned} \tag{3.99}$$

as $\omega \to 0$, the solution becomes

$$u = \sqrt[3]{t^4} \tag{3.100}$$

**Solution 2**

$$u = Ae^{\frac{\omega r}{i}} e^{-\frac{3}{4}\lambda} \frac{\sin(\Delta\lambda)}{-i\Delta} \tag{3.101}$$
$$= A(\cos(\omega r) - \sin(\omega r) e^{(\frac{3}{4}\lambda)} \frac{\sin(\Delta\lambda)}{-i\Delta}$$

as $\omega \to 0$, the solution becomes

$$u = Ae^{(\frac{3}{4}\lambda)} \frac{\sin(\Delta\lambda)}{-i\Delta} \tag{3.102}$$

This result is comparable to (2.19).

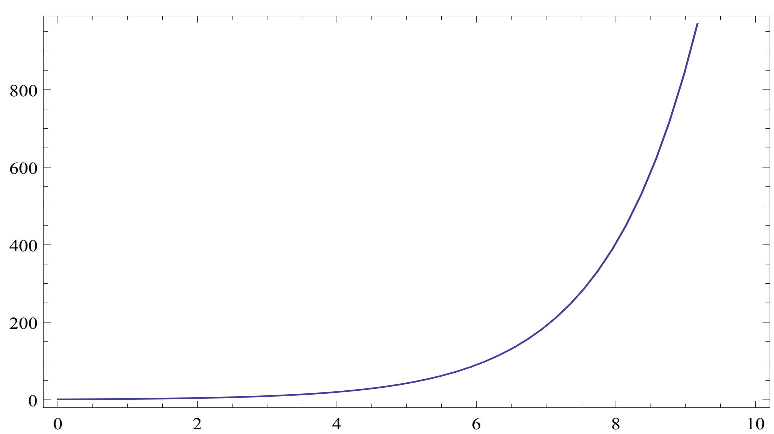


Figure 3.3: Plot for solution (3.102) with dependent variable $u$ represented on the vertical axis and independent variable $\lambda$ represented on the horizontal axis

**Solution 3**

$$
\begin{aligned}
u &= A\omega e^{\frac{\omega r}{i}} e^{-\frac{3}{4}\int_{\psi_1}^{\psi_1+\omega} d\psi_1} \frac{\sin(\Delta \int_{\psi_1}^{\psi_1+\omega} t d\psi_1)}{-i\Delta\omega} \\
&= A\omega e^{\frac{\omega r}{i}} e^{-\frac{3}{4}\int_{\psi_1}^{\psi_1+\omega} d\psi_1} \frac{d}{d\omega}[\frac{\sin(\Delta \int_{\psi_1}^{\psi_1+\omega} t d\psi_1)}{-i\Delta}] \\
&= A\omega e^{\frac{\omega r}{i}} e^{-\frac{3}{4}\int_{\psi_1}^{\psi_1+\omega} d\psi_1} \frac{d\psi_1}{d\omega}\frac{d}{d\omega}[\frac{\sin(\Delta \int_{\psi_1}^{\psi_1+\omega} t d\psi_1)}{-i\Delta}] \\
&= A\omega e^{\frac{\omega r}{i}} e^{-\frac{3}{4}\int_{\psi_1}^{\psi_1+\omega} d\psi_1} \frac{2re^{\frac{2\omega r}{i}}}{it}[\frac{\cos(\Delta \int_{\psi_1}^{\psi_1+\omega} d\psi_1)}{-i\Delta}]t \\
&= A\omega e^{\frac{\omega r}{i}} e^{-\frac{3}{4}\int_{\psi_1}^{\psi_1+\omega} d\psi_1} \frac{2re^{\frac{2\omega r}{i}}}{it} t \\
&= A\omega \frac{2re^{\frac{3\omega r}{i}}}{i}
\end{aligned} \tag{3.103}
$$

But

$$
\begin{aligned}
\omega e^{\frac{3\omega r}{i}} &= \omega\cos(3\omega r) - i\omega\sin(3\omega r) \\
\omega^2 e^{\frac{3\omega r}{i}} &= \omega^2\cos(3\omega r) - i\omega^2\sin(3\omega r) \\
&= \frac{1}{\omega}\{\omega^3\cos(3\omega r) - i\omega^3\sin(3\omega r)\}
\end{aligned} \tag{3.104}
$$

and

$$
\begin{aligned}
\omega^3 r^2 \cos(\frac{\omega r}{i}) &= \omega^2 e^{3\omega r} \\
\omega^2 e^{\frac{3\omega r}{i}} &= \frac{1}{\omega}\frac{1}{r^2}\omega^2 e^{3\omega r} \\
e^{\frac{3\omega r}{i}} &= \frac{1}{\omega}\frac{1}{r^2} e^{3\omega r}
\end{aligned} \tag{3.105}
$$

We substitute in equation (3.103) and obtain

$$
u = A\omega\frac{2r}{i}\frac{1}{\omega}\frac{1}{r^2}e^{3\omega r} \tag{3.106}
$$

as $\omega \to 0$, the solution becomes

$$
u = \frac{2A}{ir} \tag{3.107}
$$

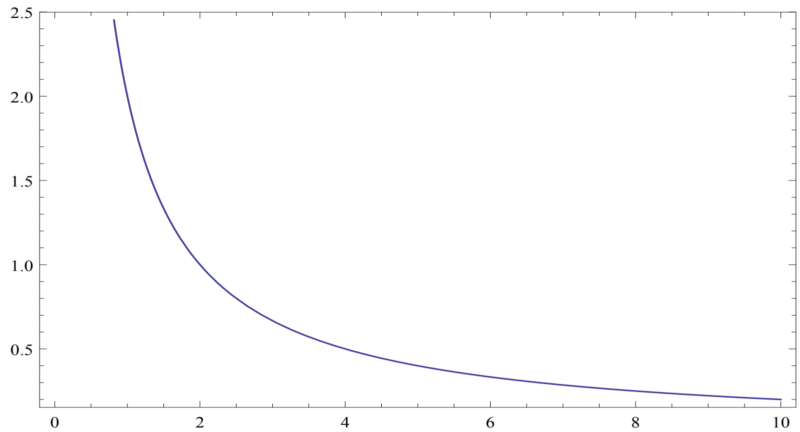


Figure 3.4: Plot for solution (3.107) with dependent variable $u$ represented on the vertical axis and independent variable $r$ represented on the horizontal axis

## 3.4 Two-Dimensional Black-Scholes Equation

### 3.4.1 Transformed two-dimensional Black-Scholes Equation

A two-dimensional Black-Scholes equation is given by

$$u_t + \frac{1}{2}A_{11}^2 x^2 u_{xx} + \frac{1}{2}A_{22}^2 y^2 u_{yy} + B_1 x u_x + B_2 y u_y - Cu = 0. \tag{3.108}$$

We apply change of variables similar to (3.4), and use equation (3.5) to reduce equation (3.108) to

$$u_t + \frac{1}{2}A_{11}^2 u_{rr} + \frac{1}{2}A_{22}^2 u_{ss} + D_1 u_r + D_2 u_s - Cu = 0. \tag{3.109}$$

where

$$\begin{aligned} D_1 &= B_1 - \frac{A_{11}^2}{2} \\ D_2 &= B_2 - \frac{A_{22}^2}{2} \end{aligned} \tag{3.110}$$

The infinitesimal generator for point symmetry admitted by equation (3.109) is of the form

$$X = \xi^1(t,r,s)\frac{\partial}{\partial t} + \xi^2(t,r,s)\frac{\partial}{\partial r} + \xi^3(t,r,s)\frac{\partial}{\partial s} + \eta(t,r,s)\frac{\partial}{\partial u} \tag{3.111}$$

Its first and second prolongations are given by

$$X^{(2)} = X + \eta_t^{(1)} \frac{\partial}{\partial u_t} + \eta_r^{(1)} \frac{\partial}{\partial u_r} + \eta_s^{(1)} \frac{\partial}{\partial u_s} + \eta_{rr}^{(2)} \frac{\partial}{\partial u_{rr}} + \eta_{ss}^{(2)} \frac{\partial}{\partial u_{ss}} \tag{3.112}$$

where $X$ is defined by equation (3.111).

### 3.4.2 Solution for determining equation for (3.109)

The determining equation for (3.109) is given by

$$\eta_t^{(1)} + \frac{1}{2} A_{11}^2 \eta_{rr}^{(2)} + \frac{1}{2} A_{22}^2 \eta_{ss}^{(2)} + D_1 \eta_r^{(1)} + D_2 \eta_s^{(1)} - C\eta = 0 \tag{3.113}$$

when

$$u_t = -\frac{1}{2} A_{11}^2 u_{rr} - \frac{1}{2} A_{22}^2 u_{ss} - D_1 u_r - D_2 u_s + Cu \tag{3.114}$$

where we define the following from ([5])

$$\begin{aligned}
\eta &= fu + g \\
\eta_t^{(1)} &= g_t + f_t u + [f - \xi_t^1] u_t - \xi_t^2 u_r - \xi_t^3 u_s \\
\eta_r^{(1)} &= g_r + f_r u + [f - \xi_r^2] u_r - \xi_r^1 u_t - \xi_r^3 u_s \\
\eta_s^{(1)} &= g_s + f_s u + [f - \xi_s^3] u_s - \xi_s^1 u_t - \xi_s^2 u_r \\
\eta_{rr}^{(2)} &= g_{rr} + f_{rr} u + [2f_r - \xi_{rr}^2] u_r - \xi_{rr}^1 u_t - \xi_{rr}^3 u_s + [f - 2\xi_r^2] u_{rr} \\
&\quad - 2\xi_r^1 u_{tr} - 2\xi_r^3 u_{rs} \\
\eta_{ss}^{(2)} &= g_{ss} + f_{ss} u + [2f_s - \xi_{ss}^3] u_s - \xi_{ss}^1 u_t - \xi_{ss}^2 u_r + [f - 2\xi_s^3] u_{ss} \\
&\quad - 2\xi_s^1 u_{ts} - 2\xi_s^2 u_{rs}
\end{aligned} \tag{3.115}$$

The substitutions of $\eta_t^{(1)}, \eta_r^{(1)}, \eta_r^{(1)}, \eta_{rr}^{(2)}$ and $\eta_{ss}^{(2)}$ in the determining equation (3.113) yield that

$$\begin{aligned}
&g_t + f_t u + [f - \xi_t^1](-\frac{1}{2}A_{11}^2 u_{rr} - \frac{1}{2}A_{22}^2 u_{ss} - D_1 u_r - D_2 u_s + Cu) \\
&- \xi_t^2 u_r - \xi_t^3 u_s + \frac{1}{2}A_{11}^2\{g_{rr} + f_{rr}u + [2f_r - \xi_{rr}^2]u_r \\
&- \xi_{rr}^1(-\frac{1}{2}A_{11}^2 u_{rr} - \frac{1}{2}A_{22}^2 u_{ss} - D_1 u_r - D_2 u_s + Cu) - \xi_{rr}^3 u_s \\
&+ [f - 2\xi_r^2]u_{rr} - 2\xi_r^1 u_{tr} - 2\xi_r^3 u_{rs}\} + \frac{1}{2}A_{22}^2\{g_{ss} + f_{ss}u + [2f_s - \xi_{ss}^3]u_s \\
&- \xi_{ss}^1(-\frac{1}{2}A_{11}^2 u_{rr} - \frac{1}{2}A_{22}^2 u_{ss} - D_1 u_r - D_2 u_s + Cu) - \xi_{ss}^2 u_r + [f - 2\xi_s^3]u_{ss} \\
&- 2\xi_s^1 u_{ts} - 2\xi_s^2 u_{rs}\} + D_1\{g_r + f_r u + [f - \xi_r^2]u_r \\
&- \xi_r^1(-\frac{1}{2}A_{11}^2 u_{rr} - \frac{1}{2}A_{22}^2 u_{ss} - D_1 u_r - D_2 u_s + Cu) - \xi_r^3 u_s\} \\
&+ D_2\{g_s + f_s u + [f - \xi_s^3]u_s \\
&- \xi_s^1(-\frac{1}{2}A_{11}^2 u_{rr} - \frac{1}{2}A_{22}^2 u_{ss} - D_1 u_r - D_2 u_s + Cu) - \xi_s^2 u_r\} \\
&- Cfu - Cg = 0
\end{aligned} \tag{3.116}$$

We set the coefficients of $u_{tr}, u_{ts}, u_{ts}, u_{rr}, u_{ss}, u_r, u_s, u$ and those free of these variables to zero. We thus have the following defining equations

$$u_{tr} \;:\; \xi_r^1 = 0, \tag{3.117}$$

$$u_{ts} \;:\; \xi_s^1 = 0, \tag{3.118}$$

$$u_{rs} \;:\; \xi_s^2 + \xi_r^3 = 0, \tag{3.119}$$

$$u_{rr} \;:\; \frac{1}{2}\xi_t^1 - \xi_r^2 = 0, \tag{3.120}$$

$$u_{ss} \;:\; \frac{1}{2}\xi_t^1 - \xi_s^3 = 0, \tag{3.121}$$

$$u_r \;:\; -\xi_t^2 + A_{11}^2 f_r - D_1\xi_r^2 - \frac{1}{2}A_{11}^2\xi_{rr}^2 + D_1\xi_t^1 = 0, \tag{3.122}$$

$$u_s \;:\; -\xi_t^3 + A_{22}^2 f_s - \frac{1}{2}A_{22}^2\xi_{ss}^3 - D_2\xi_s^3 + D_2\xi_t^1 = 0 \tag{3.123}$$

$$u \;:\; f_t + \frac{1}{2}A_{11}^2 f_{rr} + \frac{1}{2}A_{22}^2 f_{ss} + D_1 f_r + D_2 f_s - C\xi_t^1 = 0, \tag{3.124}$$

$$u^0 \;:\; g_t + \frac{1}{2}A_{11}^2 g_{rr} + \frac{1}{2}A_{22}^2 g_{ss} + D_1 g_r + D_2 g_s - Cg = 0 \tag{3.125}$$

From defining equation (3.120) we have that

$$\xi_{rr}^2 = 0 \tag{3.126}$$

Thus

$$\xi^2 = ar + b \tag{3.127}$$

which can be expressed using Manale's formula with infinitesimal $\omega$ as

$$\begin{aligned} \xi^2 &= \frac{a\ \sin(\frac{\omega r}{i}) + b\phi\ \cos(\frac{\omega r}{i})}{-i\omega} \\ \text{where} \quad & \phi = \sin(\frac{\omega}{i}), \end{aligned} \tag{3.128}$$

and also $a$ and $b$ are arbitrary functions of $t$. Similarly, from defining equation (3.121) we have that

$$\xi^3_{ss} = 0 \tag{3.129}$$

This implies that

$$\xi^3 = cs + h \tag{3.130}$$

which can be expressed using Manale's formula with infinitesimal $\omega$ as

$$\begin{aligned} \xi^3 &= \frac{c\ \sin(\frac{\omega s}{i}) + h\phi\ \cos(\frac{\omega s}{i})}{-i\omega} \\ \text{where} \quad & \phi = \sin(\frac{\omega}{i}), \end{aligned} \tag{3.131}$$

and also $c$ and $h$ are arbitrary functions of $t$. We differentiate equation (3.128) with respect to $r$ and $t$ and obtain expressions for $\xi^2_t, \xi^2_r$, and $\xi^2_{rr}$

$$\xi^2_t = \frac{\dot{a}\ \sin(\frac{\omega r}{i}) + \dot{b}\phi\ \cos(\frac{\omega r}{i})}{-i\omega} \tag{3.132}$$

$$\xi^2_r = a\ \cos(\frac{\omega r}{i}) - b\phi\ \sin(\frac{\omega r}{i}), \tag{3.133}$$

$$\xi^2_{rr} = \frac{-\omega}{i}\ a\ \sin(\frac{\omega r}{i}) - \frac{\omega}{i}\ b\ \phi\ \cos(\frac{\omega r}{i}), \tag{3.134}$$

Similarly we differentiate equation (3.131) with respect to $s$ and $t$ and obtain expressions for $\xi^3_t, \xi^3_s$, and $\xi^3_{ss}$

$$\xi^3_t = \frac{\dot{c}\ \sin(\frac{\omega s}{i}) + \dot{h}\phi\ \cos(\frac{\omega s}{i})}{-i\omega} \tag{3.135}$$

$$\xi^3_s = c\ \cos(\frac{\omega s}{i}) - h\phi\ \sin(\frac{\omega s}{i}), \tag{3.136}$$

$$\xi^3_{ss} = \frac{-\omega}{i}\ c\ \sin(\frac{\omega s}{i}) - \frac{\omega}{i}\ h\ \phi\ \cos(\frac{\omega s}{i}). \tag{3.137}$$

The defining equations (3.120) and (3.121) imply that $\xi_t^1 = 2\xi_r^2$ and $\xi_t^1 = 2\xi_s^3$ which translates to that

$$\xi_t^1 = 2\{a\ \cos(\frac{\omega r}{i}) + c\ \cos(\frac{\omega s}{i})\} - 2\phi\{b\sin(\frac{\omega r}{i}) + h\sin(\frac{\omega s}{i})\}. \tag{3.138}$$

The integration of equation (3.138) results in the infinitesimal

$$\xi^1 = 2t\{a\ \cos(\frac{\omega r}{i}) + c\ \cos(\frac{\omega s}{i})\} - 2t\phi\{b\sin(\frac{\omega r}{i}) + h\sin(\frac{\omega s}{i})\} + C. \tag{3.139}$$

We substitute equations (3.132), (3.133), (3.134) and (3.138) into the defining equation (3.122) to get the expression for $f_r$ given by

$$\begin{aligned} f_r = \cos(\frac{\omega r}{i})&\Big\{-\frac{b\omega\phi}{2i} - \frac{\dot{b}\phi}{A_{11}^2 i\omega} - \frac{D_1 a}{A_{11}^2}\Big\} \\ + \sin(\frac{\omega r}{i})&\Big\{-\frac{\omega a}{2i} - \frac{\dot{a}}{A_{11}^2 i\omega} + \frac{D_1\phi b}{A_{11}^2}\Big\} \\ - \frac{2D_1 c}{A_{11}^2}&\cos(\frac{\omega s}{i}) + \frac{2D_1\phi h}{A_{11}^2}\sin(\frac{\omega s}{i}) \end{aligned} \tag{3.140}$$

Similarly we substitute equations (3.135), (3.136), (3.137) and (3.138) into the defining equation (3.123) to get the expression for $f_s$ given by

$$\begin{aligned} f_s = \cos(\frac{\omega s}{i})&\Big\{-\frac{\omega h\phi}{2i} - \frac{\dot{h}\phi}{A_{22}^2 i\omega} - \frac{D_2 c}{A_{22}^2}\} \\ + \sin(\frac{\omega s}{i})&\Big\{-\frac{\omega c}{2i} - \frac{\dot{c}}{A_{22}^2 i\omega} + \frac{D_2 h\phi}{A_{22}^2}\Big\} \\ - \frac{2D_2 a}{A_{22}^2}&\cos(\frac{\omega r}{i}) + \frac{2D_2\phi b}{A_{22}^2}\sin(\frac{\omega r}{i}) \end{aligned} \tag{3.141}$$

Integrating equation (3.140) with respect to $r$ gives

$$\begin{aligned} \sin(\frac{\omega r}{i})&\Big\{-\frac{b\phi}{2} - \frac{\dot{b}\phi}{A_{11}^2\omega^2} - \frac{D_1 ia}{A_{11}^2\omega}\Big\} \\ + \cos(\frac{\omega r}{i})&\Big\{\frac{a}{2} + \frac{\dot{a}}{A_{11}^2\omega^2} - \frac{D_1 ib\phi}{A_{11}^2\omega}\Big\} \\ - \frac{2D_1 cr}{A_{11}^2}&\cos(\frac{\omega s}{i}) + \frac{2D_1 hr\phi}{A_{11}^2}\sin(\frac{\omega s}{i}) + k(t) \end{aligned} \tag{3.142}$$

Similarly integrating equation (3.140) with respect to $s$ yields

$$\begin{aligned} \sin(\frac{\omega s}{i})&\Big\{-\frac{h\phi}{2} - \frac{\dot{h}\phi}{A_{22}^2\omega^2} - \frac{D_2 ic}{A_{22}^2\omega}\Big\} \\ + \cos(\frac{\omega s}{i})&\Big\{\frac{c}{2} + \frac{\dot{c}}{A_{22}^2\omega^2} - \frac{D_2 ih\phi}{A_{22}^2\omega}\Big\} \\ - \frac{2D_2 as}{A_{22}^2}&\cos(\frac{\omega r}{i}) + \frac{2D_2 bs\phi}{A_{22}^2}\sin(\frac{\omega r}{i}) + k(t) \end{aligned} \tag{3.143}$$

Reconciling equations (3.142) and (3.143) we get the expression for $f$ given by

$$\begin{aligned} f = & \sin(\frac{\omega r}{i})\Big\{-\frac{b\phi}{2} - \frac{\dot{b}\phi}{A_{11}^2\omega^2} - \frac{D_1 ia}{A_{11}^2\omega} + \frac{2D_2bs\phi}{A_{22}^2}\Big\} \\ & + \cos(\frac{\omega r}{i})\Big\{\frac{a}{2} + \frac{\dot{a}}{A_{11}^2\omega^2} - \frac{D_1ib\phi}{A_{11}^2\omega} - \frac{2D_2as}{A_{22}^2}\Big\} \\ & + \sin(\frac{\omega s}{i})\Big\{-\frac{h\phi}{2} - \frac{\dot{h}\phi}{A_{22}^2\omega^2} - \frac{D_2ic}{A_{22}^2\omega} + \frac{2D_1hr\phi}{A_{11}^2}\Big\} \\ & + cos(\frac{\omega s}{i})\Big\{\frac{c}{2} + \frac{\dot{c}}{A_{22}^2\omega^2} - \frac{D_2ih\phi}{A_{22}^2\omega} - \frac{2D_1cr}{A_{11}^2}\Big\} \\ & + k(t) \end{aligned} \tag{3.144}$$

We differentiate equation (3.144) to obtain expressions for $f_t$, $f_{rr}$ and $f_{ss}$.

$$\begin{aligned} f_t = & \sin(\frac{\omega r}{i})\Big\{-\frac{\dot{b}\phi}{2} - \frac{\ddot{b}\phi}{A_{11}^2\omega^2} - \frac{D_1 i\dot{a}}{A_{11}^2\omega} + \frac{2D_2\dot{b}s\phi}{A_{22}^2}\Big\} \\ & + \cos(\frac{\omega r}{i})\Big\{\frac{\dot{a}}{2} + \frac{\ddot{a}}{A_{11}^2\omega^2} - \frac{D_1i\dot{b}\phi}{A_{11}^2\omega} - \frac{2D_2\dot{a}s}{A_{22}^2}\Big\} \\ & + \sin(\frac{\omega s}{i})\Big\{-\frac{\dot{h}\phi}{2} - \frac{\ddot{h}\phi}{A_{22}^2\omega^2} - \frac{D_2i\dot{c}}{A_{22}^2\omega} + \frac{2D_1\dot{h}r\phi}{A_{11}^2}\Big\} \\ & + cos(\frac{\omega s}{i})\Big\{\frac{\dot{c}}{2} + \frac{\ddot{c}}{A_{22}^2\omega^2} - \frac{D_2i\dot{h}\phi}{A_{22}^2\omega} - \frac{2D_1\dot{c}r}{A_{11}^2}\Big\} \\ & + k'(t) \end{aligned} \tag{3.145}$$

$$\begin{aligned} f_{rr} = & \sin(\frac{\omega r}{i})\Big\{-\frac{\omega^2 b\phi}{2} - \frac{\dot{b}\phi}{A_{11}^2} + \frac{D_1\omega a}{A_{11}^2 i}\Big\} \\ & + \cos(\frac{\omega r}{i})\Big\{\frac{\omega^2 a}{2} + \frac{\dot{a}}{A_{11}^2} + \frac{D_1 b\omega\phi}{iA_{11}^2}\Big\} \end{aligned} \tag{3.146}$$

$$\begin{aligned} f_{ss} = & \sin(\frac{\omega s}{i})\Big\{-\frac{\omega^2 h\phi}{2} - \frac{\dot{h}\phi}{A_{22}^2} - \frac{D_2\omega c}{A_{22}^2 i}\Big\} \\ & + \cos(\frac{\omega s}{i})\Big\{\frac{\omega^2 c}{2} + \frac{\dot{c}}{A_{22}^2} + \frac{D_2 e\omega\phi}{iA_{22}^2}\Big\} \end{aligned} \tag{3.147}$$

We substitute equations (3.133), (3.136) (3.138), (3.140), (3.142),(3.148), (3.149) and (3.150) into the defining equation (3.124) and solve the equation

$$\sin(\frac{\omega r}{i})\Big\{-\frac{\dot{b}\phi}{2}-\frac{\ddot{b}\phi}{A_{11}^2\omega^2}-\frac{D_1 i\dot{a}}{A_{11}^2\omega}+\frac{2D_2\dot{b}s\phi}{A_{22}^2}\Big\}+\cos(\frac{\omega r}{i})\Big\{\frac{\dot{a}}{2}+\frac{\ddot{a}}{A_{11}^2\omega^2}$$
$$-\frac{D_1 i\dot{b}\phi}{A_{11}^2\omega}-\frac{2D_2\dot{a}s}{A_{22}^2}\Big\}$$
$$+\sin(\frac{\omega s}{i})\Big\{-\frac{\dot{h}\phi}{2}-\frac{\ddot{h}\phi}{A_{22}^2\omega^2}-\frac{D_2 i\dot{c}}{A_{22}^2\omega}+\frac{2D_1\dot{h}r\phi}{A_{11}^2}\Big\}$$
$$+\cos(\frac{\omega s}{i})\Big\{\frac{\dot{c}}{2}+\frac{\ddot{c}}{A_{22}^2\omega^2}-\frac{D_2 i\dot{h}\phi}{A_{22}^2\omega}-\frac{2D_1\dot{c}r}{A_{11}^2}\Big\}$$
$$+\sin(\frac{\omega r}{i})\Big\{-\frac{A_{11}^2\omega^2 b\phi}{4}-\frac{\dot{b}\phi}{2}+\frac{D_1\omega a}{2i}\Big\}+\cos(\frac{\omega r}{i})\Big\{\frac{A_{11}^2\omega^2 a}{4}+\frac{\dot{a}}{2}+\frac{D_1 b\omega\phi}{2i}\Big\}$$
$$+sin(\frac{\omega s}{i})\Big\{-\frac{A_{22}^2\omega^2 e\phi}{4}-\frac{\dot{h}\phi}{2}+\frac{D_2\omega c}{2i}\Big\}+\cos(\frac{\omega s}{i})\Big\{\frac{A_{22}^2\omega^2 c}{4}+\frac{\dot{c}}{2}+\frac{D_2 h\omega\phi}{2i}\Big\}$$
$$+\cos(\frac{\omega r}{i})\Big\{-\frac{D_1 b\omega\phi}{2i}-\frac{D_1\dot{b}\phi}{A_{11}^2 i\omega}-\frac{D_1^2 a}{A_{11}^2}\Big\}+\sin(\frac{\omega r}{i})\Big\{-\frac{D_1\omega a}{2i}-\frac{D_1\dot{a}}{A_{11}^2 i\omega}+\frac{D_1^2\phi b}{A_{11}^2}\Big\}$$
$$-\frac{2D_1^2 c}{A_{11}^2}\cos(\frac{\omega s}{i})+\frac{2D_1^2\phi h}{A_{11}^2}\sin(\frac{\omega s}{i})+\cos(\frac{\omega s}{i})\Big\{-\frac{D_2\omega h\phi}{2i}-\frac{D_2\dot{h}\phi}{A_{22}^2 i\omega}-\frac{D_2^2 c}{A_{22}^2}\Big\}$$
$$+\sin(\frac{\omega s}{i})\Big\{-\frac{D_2\omega c}{2i}-\frac{D_2\dot{c}}{A_{22}^2 i\omega}+\frac{D_2^2 h\phi}{A_{22}^2}\Big\}-\frac{2D_2^2 a}{A_{22}^2}\cos(\frac{\omega r}{i})+\frac{2D_2^2\phi b}{A_{22}^2}\sin(\frac{\omega r}{i})$$
$$-2C\{a\ \cos(\frac{\omega r}{i})+c\ \cos(\frac{\omega s}{i})\}+2C\phi\{b\sin(\frac{\omega r}{i})+h\sin(\frac{\omega s}{i})\}$$
$$+\frac{2D_1^2\phi h}{A_{11}^2}+k'(t)=0 \tag{3.148}$$

We collect all the coefficients of sine function together and coefficients of cosine function together and equate each to zero. We thus have the following equations:

$$\begin{aligned}\sin(\frac{\omega r}{i}):&-\frac{\dot{b}\phi}{2}-\frac{\ddot{b}\phi}{A_{11}^2\omega^2}-\frac{D_1 i\dot{a}}{A_{11}^2\omega}+\frac{2D_2\dot{b}s\phi}{A_{22}^2}-\frac{A_{11}^2\omega^2 b\phi}{4}-\frac{\dot{b}\phi}{2}\\&+\frac{D_1\omega a}{2i}+\frac{2D_2^2\phi b}{A_{22}^2}-\frac{D_1\omega a}{2i}-\frac{D_1\dot{a}}{A_{11}^2 i\omega}+\frac{D_1^2\phi b}{A_{11}^2}+2C\phi b=0\end{aligned} \tag{3.149}$$

$$\begin{aligned}\sin(\frac{\omega s}{i}):&-\frac{\dot{h}\phi}{2}-\frac{\ddot{h}\phi}{A_{22}^2\omega^2}-\frac{D_2 i\dot{c}}{A_{22}^2\omega}+\frac{2D_1\dot{h}r\phi}{A_{11}^2}-\frac{A_{22}^2\omega^2 h\phi}{4}-\frac{\dot{h}\phi}{2}\\&+\frac{D_2\omega c}{2i}-\frac{D_2\omega c}{2i}-\frac{D_2\dot{c}}{A_{22}^2 i\omega}+\frac{D_2^2 h\phi}{A_{22}^2}+2C\phi h=0\end{aligned} \tag{3.150}$$

$$\cos(\frac{\omega r}{i}) : \frac{\dot{a}}{2} + \frac{\ddot{a}}{A_{11}^2\omega^2} - \frac{D_1 i\dot{b}\phi}{A_{11}^2\omega} - \frac{2D_2\dot{a}s}{A_{22}^2} + \frac{A_{11}^2\omega^2 a}{4} + \frac{\dot{a}}{2}$$
$$+ \frac{D_1 b\omega\phi}{2i} - \frac{D_1 b\omega\phi}{2i} - \frac{D_1\dot{b}\phi}{A_{11}^2 i\omega} - \frac{D_1^2 a}{A_{11}^2} + \frac{2D_2^2\phi b}{A_{22}^2} - 2Ca = 0 \tag{3.151}$$

$$\cos(\frac{\omega s}{i}) : \frac{\dot{c}}{2} + \frac{\ddot{c}}{A_{22}^2\omega^2} - \frac{D_2 i\dot{h}\phi}{A_{22}^2\omega} - \frac{2D_1\dot{c}r}{A_{11}^2} + \frac{A_{22}^2\omega^2 c}{4} + \frac{\dot{c}}{2}$$
$$+ \frac{D_2 h\omega\phi}{2i} - \frac{D_2\omega e\phi}{2i} - \frac{D_2\dot{h}\phi}{A_{22}^2 i\omega} - \frac{D_2^2 c}{A_{22}^2} - \frac{2D_1^2 c}{A_{11}^2} - 2Cc = 0 \tag{3.152}$$

Equation (3.149) splits into

$$\ddot{b} + \dot{b}A_{11}^2\omega^2 + \tfrac{A_{11}^4\omega^4 b}{4} - D_1^2\omega^2 b - 2A_{11}^2\omega^2 Cb = 0 \tag{3.153}$$

$$\tfrac{2D_2 s\dot{b}}{A_{22}^2} + \tfrac{2D_2^2 b}{A_{22}^2} = 0 \tag{3.154}$$

Solving equation (3.153) we proceed as in (3.1). We define

$$\beta_1 = \frac{A_{11}^2\omega^2}{2}, \quad \text{and}$$
$$j_1 = -\frac{D_1^2}{A_{11}^2} - 2A_{11}^2\omega^2 C \tag{3.155}$$

We also set

$$\alpha_1 = b\beta_1^2 - j_1, \quad \text{then}$$
$$\dot{\alpha_1} = \dot{b}\beta_1^2, \tag{3.156}$$
$$\ddot{\alpha_1} = \ddot{b}\beta_1^2$$

We express equation (3.153) as

$$\ddot{b} + 2\beta_1\dot{b} + \beta_1^2 b - D_1^2\omega^2 b - 4\beta_1 Cb = 0 \tag{3.157}$$

Equation (3.157) transforms to

$$\ddot{\alpha_1} + 2\dot{\alpha_1} + \alpha_1\beta_1^2 = 0. \tag{3.158}$$

To find the solution of equation (3.158) we proceed as in (3.1). We set

$$\alpha_1 = cz$$

where $c = c(t), z = z(t)$. This results in that

$$z = \frac{1}{\beta_1^2}\Big\{\Big(C_1\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\Big) + C_2\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\Big\} + \frac{D_{11}^2}{\beta_1^2 A_{11}^2} + \frac{4C}{\beta_1} \tag{3.159}$$

and

$$b = \frac{e^{-t}}{\beta_1^2}\left\{\left(C_1\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\right) + C_2\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\right\} + \frac{D_1^2}{\beta_1^2 A_{11}^2} + \frac{4C}{\beta_1} \tag{3.160}$$

where

$$\bar{\omega_1} = \sqrt{\beta_1^2 - 1}. \tag{3.161}$$

Solving equation (3.154) results in that

$$b = C_3 e^{-\frac{D_2}{s}t} \tag{3.162}$$

The general solution for $b$ is given by the linear combination of linearly independent solutions (3.161) and (3.162) as

$$b = \frac{e^{-t}}{\beta_1^2}\left\{\left(C_1\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\right) + C_2\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\right\} + C_3 e^{-\frac{D_2}{s}t} + \frac{D_1^2}{\beta_1^2 A_{11}^2} + \frac{4C}{\beta_1} \tag{3.163}$$

Similarly equations (3.150), (3.151) and (3.152) give rise to the splits

$$\ddot{h} + 2\beta_2\dot{h} + \beta_2^2 h - D_2^2\omega^2 h - 4\beta_2 C h = 0 \tag{3.164}$$

$$\tfrac{2D_1 r\dot{h}}{A_{11}^2} + \tfrac{2D_1^2 h}{A_{11}^2} = 0, \tag{3.165}$$

$$\ddot{a} + 2\beta_1\dot{a} + \beta_1^2 a - D_1^2\omega^2 a - 4\beta_1 C a = 0 \tag{3.166}$$

$$\tfrac{2D_2 s\dot{a}}{A_{22}^2} + \tfrac{2D_2^2 a}{A_{22}^2} = 0, \tag{3.167}$$

and

$$\ddot{c} + 2\beta_2\dot{c} + \beta_2^2 c - D_2^2\omega^2 c - 4\beta_2 C c = 0 \tag{3.168}$$

$$\tfrac{2D_1 r\dot{c}}{A_{11}^2} + \tfrac{2D_1^2 c}{A_{11}^2} = 0 \tag{3.169}$$

respectively, where

$$\beta_2 = \frac{A_{22}^2\omega^2}{2} \tag{3.170}$$

The general solutions for $h, c$ and $a$ in the equations (3.164), (3.165), (3.166), (3.167), (3.168) and (3.169) are given by

$$h = \frac{e^{-t}}{\beta_2^2}\left\{\left(C_4\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\right) + C_5\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\right\} + C_6 e^{-\frac{D_1}{r}t} + \frac{D_2^2}{\beta_2^2 A_{22}^2} + \frac{4C}{\beta_2}, \tag{3.171}$$

$$c = \frac{e^{-t}}{\beta_2^2}\left\{\left(C_7\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\right) + C_8\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\right\} + C_9 e^{-\frac{D_1}{r}t} + \frac{D_2^2}{\beta_2^2 A_{22}^2} + \frac{4C}{\beta_2}, \tag{3.172}$$

and

$$a=\frac{e^{-t}}{\beta_1^2}\Big\{\Big(C_{10}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\Big)+C_{11}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\Big\}+C_{12}e^{-\frac{D_2}{s}t}+\frac{D_1^2}{\beta_1^2A_{11}^2}+\frac{4C}{\beta_1} \tag{3.173}$$

respectively, where

$$\bar{\omega_2}=\sqrt{\beta_2^2-1}. \tag{3.174}$$

We differentiate equations (3.163), (3.171), (3.172), and (3.173), to get expressions for $\dot{b},\dot{h},\dot{c}$ and $\dot{a}$ respectively.

$$\begin{aligned}\dot{a}=&-\frac{e^{-t}}{\beta_1^2}\Big\{\Big(C_{10}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\Big)+C_{11}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\Big\}-C_{12}\frac{D_2}{s}e^{-\frac{D_2}{s}t}\\&-\frac{e^{-t}}{\beta_1^2}\Big(C_{10}\sin\bar{\omega_1}\sin\bar{\omega_1}t+C_{11}\cos\bar{\omega_1}t\Big)\end{aligned} \tag{3.175}$$

$$\begin{aligned}\dot{b}=&-\frac{e^{-t}}{\beta_1^2}\Big\{\Big(C_1\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\Big)+C_2\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\Big\}-C_3\frac{D_2}{s}e^{-\frac{D_2}{s}t}\\&-\frac{e^{-t}}{\beta_1^2}\Big(C_1\sin\bar{\omega_1}\sin\bar{\omega_1}t+C_2\cos\bar{\omega_1}t\Big)\end{aligned} \tag{3.176}$$

$$\begin{aligned}\dot{c}=&-\frac{e^{-t}}{\beta_2^2}\Big\{\Big(C_7\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\Big)+C_8\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\Big\}-C_9\frac{D_1}{r}e^{-\frac{D_1}{r}t}\\&-\frac{e^{-t}}{\beta_2^2}\Big(C_7\sin\bar{\omega_2}\sin\bar{\omega_2}t+C_8\cos\bar{\omega_2}t\Big)\end{aligned} \tag{3.177}$$

$$\begin{aligned}\dot{h}=&-\frac{e^{-t}}{\beta_2^2}\Big\{\Big(C_4\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\Big)+C_5\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\Big\}-C_6\frac{D_1}{r}e^{-\frac{D_1}{r}t}\\&-\frac{e^{-t}}{\beta_2^2}\Big(C_4\sin\bar{\omega_2}\sin\bar{\omega_2}t+C_5\cos\bar{\omega_2}t\Big)\end{aligned} \tag{3.178}$$

### 3.4.3 Infinitesimals for equation (3.109)

We substitute for equations (3.163), (3.171),(3.172),(3.173),(3.175),(3.176), (3.177), and (3.178) in equation (3.144) to obtain the expression for $f$ given as

$$\begin{aligned}f=&-\frac{e^{-t}\phi}{2\beta_1^2}C_1\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})-C_2\frac{e^{-t}\phi}{2\beta_1^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\\&-C_3\frac{1}{2}\phi e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i})-\frac{D_1^2}{2\beta_1^2A_{11}^2}\phi\sin(\frac{\omega r}{i})-\frac{2C}{\beta_1}\phi\sin(\frac{\omega r}{i})\\&+\frac{e^{-t}\phi}{2\beta_1^3}C_1\sin\bar{\omega_1}\sin\bar{\omega_1}t\sin(\frac{\omega r}{i})-C_2\frac{e^{-t}\phi}{2\beta_1^3}\cos\bar{\omega_1}t\sin(\frac{\omega r}{i})+\frac{e^{-t}\phi}{2\beta_1^3}C_1\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\\&+C_2\frac{e^{-t}\phi}{2\beta_1^3}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})+\frac{C_3}{2\beta_1}\phi\frac{D_2}{s}e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i})-\frac{D_1 i\omega}{2\beta_1^3}C_{10}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\end{aligned}$$

$$
\begin{aligned}
&- \frac{D_1 i\omega}{2\beta_1^3} C_{11} \frac{\sin\bar{\omega_1}t}{\bar{\omega_1}} \sin(\frac{\omega r}{i}) - \frac{D_1 i\omega}{2\beta_1} C_{12} e^{-\frac{D_2}{s}t} \sin(\frac{\omega r}{i}) \\
&- \frac{D_1^3 i\omega}{2\beta_1^3 A_{11}^2} \sin(\frac{\omega r}{i}) - \frac{2D_1 i\omega C}{\beta_1^2} \sin(\frac{\omega r}{i}) + \frac{2e^{-t}\phi D_2 s}{\beta_1^2 A_{22}^2} C_1 \frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}} \sin(\frac{\omega r}{i}) \\
&+ \frac{2e^{-t}\phi D_2 s}{\beta_1^2 A_{22}^2} C_2 \frac{\sin\bar{\omega_1}t}{\bar{\omega_1}} \sin(\frac{\omega r}{i}) + \frac{2D_1^2 D_2 s\phi}{\beta_1^2 A_{11}^2 A_{22}^2} \sin(\frac{\omega r}{i}) + \frac{8D_2 s\phi C}{\beta_1^2 A_{22}^2} \sin(\frac{\omega r}{i}) \\
&+ \frac{e^{-t}}{2\beta_1^2} C_{10} \frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}} \cos(\frac{\omega r}{i}) + C_{11} \frac{e^{-t}}{2\beta_1^2} \frac{\sin\bar{\omega_1}t}{\bar{\omega_1}} \cos(\frac{\omega r}{i}) + C_{12}\frac{1}{2} e^{-\frac{D_2}{s}t} \cos(\frac{\omega r}{i}) \\
&+ \frac{D_1^2}{2\beta_1^2 A_{11}^2} \cos(\frac{\omega r}{i}) + \frac{2C}{\beta_1} \cos(\frac{\omega r}{i}) - \frac{e^{-t}}{2\beta_1^3} C_{10} \frac{\sin\bar{\omega_1}\sin\bar{\omega_1}t}{\bar{\omega_1}} \cos(\frac{\omega r}{i}) \\
&+ \frac{e^{-t}}{2\beta_1^3} C_{11} \frac{\sin\bar{\omega_1}t}{\bar{\omega_1}} \cos(\frac{\omega r}{i}) - \frac{C_{12}}{2\beta_1} \frac{D_2}{s} e^{-\frac{D_2}{s}t} \cos(\frac{\omega r}{i}) - \frac{e^{-t}}{2\beta_1^3} C_{10} \sin\bar{\omega_1}\sin\bar{\omega_1}t \cos(\frac{\omega r}{i})
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{e^{-t}}{2\beta_1^3} C_{11} \cos\bar{\omega_1}t \cos(\frac{\omega r}{i}) - \frac{D_1 i\omega\phi e^{-t}}{2\beta_1^3} C_1 \frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}} \cos(\frac{\omega r}{i}) \\
&- \frac{D_1 i\omega\phi e^{-t}}{2\beta_1^3} C_2 \frac{\sin\bar{\omega_1}t}{\bar{\omega_1}} \cos(\frac{\omega r}{i}) - \frac{D_1 i\omega\phi}{2\beta_1} C_3 e^{-\frac{D_2}{s}t} \cos(\frac{\omega r}{i}) - \frac{D_1^3 i\omega\phi}{2\beta_1^3 A_{11}^2} \cos(\frac{\omega r}{i}) \\
&- \frac{2D_1 i\omega\phi C}{\beta_1^2} \cos(\frac{\omega r}{i}) - \frac{2D_2 s e^{-t}}{\beta_1^2 A_{22}^2} C_{10} \frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}} \cos(\frac{\omega r}{i}) - \frac{2D_2 s e^{-t}}{\beta_1^2 A_{22}^2} C_{11} \frac{\sin\bar{\omega_1}t}{\bar{\omega_1}} \cos(\frac{\omega r}{i}) \\
&- \frac{2D_2 s}{A_{22}^2} C_{12} e^{-\frac{D_2}{s}t} \cos(\frac{\omega r}{i}) - \frac{2D_2 D_1^2 s}{\beta_1^2 A_{11}^2 A_{22}^2} \cos(\frac{\omega r}{i}) - \frac{8D_2 sC}{\beta_1 A_{22}^2} \cos(\frac{\omega r}{i}) \\
&- \frac{e^{-t}\phi}{2\beta_2^2} C_4 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}} \sin(\frac{\omega s}{i}) - C_5 \frac{e^{-t}\phi}{2\beta_2^2} \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}} \sin(\frac{\omega s}{i}) - C_6 \frac{1}{2}\phi e^{-\frac{D_1}{r}t} \sin(\frac{\omega s}{i}) \\
&- \frac{D_2^2}{2\beta_2^2 A_{22}^2} \phi \sin(\frac{\omega s}{i}) - \frac{2C}{\beta_2} \phi \sin(\frac{\omega s}{i}) + \frac{e^{-t}\phi}{2\beta_2^3} C_4 \sin\bar{\omega_2}\sin\bar{\omega_2}t \sin(\frac{\omega s}{i}) + C_5 \frac{e^{-t}\phi}{2\beta_2^3} \cos\bar{\omega_2}t \sin(\frac{\omega s}{i}) \\
&+ \frac{e^{-t}\phi}{2\beta_2^3} C_4 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}} \sin(\frac{\omega s}{i}) + C_5 \frac{e^{-t}\phi}{2\beta_2^3} \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}} \sin(\frac{\omega s}{i}) + \frac{C_6}{2\beta_2} \phi \frac{D_1}{r} e^{-\frac{D_1}{r}t} \sin(\frac{\omega s}{i}) \\
&- \frac{D_2 i\omega}{2\beta_2^3} C_7 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}} \sin(\frac{\omega s}{i}) - \frac{D_2 i\omega}{2\beta_2^3} C_8 \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}} \sin(\frac{\omega s}{i}) - \frac{D_2 i\omega}{2\beta_2} C_9 e^{-\frac{D_1}{r}t} \sin(\frac{\omega s}{i}) \\
&- \frac{D_2^3 i\omega}{2\beta_2^3 A_{22}^2} \sin(\frac{\omega s}{i}) - \frac{D_2 i\omega C}{\beta_2^2} \sin(\frac{\omega s}{i}) + \frac{2e^{-t}\phi D_1 r}{\beta_2^2 A_{11}^2} C_4 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}} \sin(\frac{\omega s}{i}) \\
&+ \frac{2e^{-t}\phi D_1 r}{\beta_2^2 A_{11}^2} C_5 \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}} \sin(\frac{\omega s}{i}) + \frac{2D_2^2 D_1 r\phi}{\beta_2^2 A_{22}^2 A_{11}^2} \sin(\frac{\omega s}{i}) + \frac{8D_1 r\phi C}{\beta_2^2 A_{11}^2} \sin(\frac{\omega s}{i})
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{e^{-t}}{2\beta_2^2} C_7 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) + C_8 \frac{e^{-t}}{2\beta_2^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) + C_9 \frac{1}{2} e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) \\
&+ \frac{D_2^2}{2\beta_2^2 A_{22}^2}\cos(\frac{\omega s}{i}) + \frac{2C}{\beta_2}\cos(\frac{\omega s}{i}) - \frac{e^{-t}}{2\beta_2^3} C_7 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) \\
&- \frac{e^{-t}}{2\beta_2^3} C_8 \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) - \frac{C_9}{2\beta_2}\frac{D_1}{r} e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) - \frac{e^{-t}}{2\beta_2^3} C_7 \sin\bar{\omega_2}\sin\bar{\omega_2}t\cos(\frac{\omega s}{i}) \\
&- \frac{e^{-t}}{2\beta_2^3} C_8 \cos\bar{\omega_2}t\cos(\frac{\omega s}{i}) - \frac{D_2 i\omega\phi e^{-t}}{2\beta_2^3} C_4 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega r}{i}) \\
&- \frac{D_1 i\omega\phi e^{-t}}{2\beta_2^3} C_5 \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) - \frac{D_2 i\omega\phi}{2\beta_2} C_6 e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) - \frac{D_2^3 i\omega\phi}{2\beta_2^3 A_{22}^2}\cos(\frac{\omega s}{i}) \\
&- \frac{2D_2 i\omega\phi}{\beta_2^2}\cos(\frac{\omega s}{i}) - \frac{2D_1 r e^{-t}}{\beta_2^2 A_{11}^2} C_7 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) \\
&- \frac{2D_1 r e^{-t}}{\beta_2^2 A_{11}^2} C_8 \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) - C_9 \frac{2D_1 r}{A_{11}^2} e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) - \frac{2D_1 D_2^2 r}{\beta_2^2 A_{22}^2 A_{11}^2}\cos(\frac{\omega s}{i}) \\
&- \frac{8D_1 rC}{\beta_2 A_{11}^2}\cos(\frac{\omega s}{i}) + C_{14}
\end{aligned}
\tag{3.179}
$$

We substitute equations (3.163) and (3.173) into equation (3.128) to get the expression for $\xi^2$ given by

$$
\begin{aligned}
\xi^2 &= \frac{ie^{-t}}{\beta_1^2\omega} C_{10} \frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}) + \frac{ie^{-t}}{\beta_1^2\omega} C_{11} \frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}) \\
&+ \frac{iC_{12}}{\omega} e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i}) + \frac{D_1^2 i}{\omega A_{11}^2\beta_1^2}\sin(\frac{\omega r}{i}) + \frac{4iC}{\omega\beta_1}\sin(\frac{\omega r}{i}) \\
&+ \frac{ie^{-t}\phi}{\beta_1^2\omega} C_1 \frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i}) + \frac{ie^{-t}\phi}{\beta_1^2\omega} C_2 \frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i}) + \frac{iC_3\phi}{\omega} e^{-\frac{D_2}{s}t}\cos(\frac{\omega r}{i}) \\
&+ \frac{D_1^2 i\phi}{\omega A_{11}^2\beta_1^2}\cos(\frac{\omega r}{i}) + \frac{4iC\phi}{\omega\beta_1}\cos(\frac{\omega r}{i})
\end{aligned}
\tag{3.180}
$$

Similarly we substitute equations (3.171) and (3.172) into equation (3.131) to get the expression for $\xi^3$ given by

$$
\begin{aligned}
\xi^3 &= \frac{ie^{-t}}{\beta_2^2\omega} C_7 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) + \frac{ie^{-t}}{\beta_2^2\omega} C_8 \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) \\
&+ \frac{iC_9}{\omega} e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i}) + \frac{D_2^2 i}{\omega A_{22}^2\beta_2^2}\sin(\frac{\omega s}{i}) + \frac{4iC}{\omega\beta_2}\sin(\frac{\omega s}{i}) \\
&+ \frac{ie^{-t}\phi}{\beta_2^2\omega} C_4 \frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) + \frac{ie^{-t}\phi}{\beta_2^2\omega} C_5 \frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) + \frac{iC_6\phi}{\omega} e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) \\
&+ \frac{D_2^2 i\phi}{\omega A_{22}^2\beta_2^2}\cos(\frac{\omega s}{i}) + \frac{4iC\phi}{\omega\beta_2}\cos(\frac{\omega s}{i})
\end{aligned}
\tag{3.181}
$$

We substitute equations (3.163), (3.171), (3.172) and (3.173) into equation (3.138) to get the expression for $\xi^1$ given by

$$\begin{aligned}\xi^1 =& \frac{2te^{-t}}{\beta_1^2}C_{10}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i}) + C_{11}\frac{2te^{-t}}{\beta_1^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\\
&+ C_{12}e^{-\frac{D_2}{s}t}\cos(\frac{\omega r}{i}) + \frac{2tD_1^2}{\beta_1^2 A_{11}^2}\cos(\frac{\omega r}{i}) + \frac{8tC}{\beta_1}\cos(\frac{\omega r}{i})\\
&+ \frac{2te^{-t}}{\beta_2^2}C_7\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) + C_8\frac{2te^{-t}}{\beta_2^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) + C_9 e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})\\
&+ \frac{2tD_2^2}{\beta_2^2 A_{22}^2}\cos(\frac{\omega s}{i}) + \frac{8tC}{\beta_2}\cos(\frac{\omega s}{i}) - \frac{2t\phi e^{-t}}{\beta_1^2}C_1\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\\
&- C_2\frac{2t\phi e^{-t}}{\beta_1^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}) - C_3\phi e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i})\\
&- \frac{2t\phi D_1^2}{\beta_1^2 A_{11}^2}\sin(\frac{\omega r}{i}) - \frac{8t\phi C}{\beta_1}\sin(\frac{\omega r}{i}) - \frac{2t\phi e^{-t}}{\beta_2^2}C_4\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i})\\
&- C_5\frac{2t\phi e^{-t}}{\beta_2^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) - C_6\phi e^{-\frac{D_1\phi}{r}t}\sin(\frac{\omega s}{i}) - \frac{2t\phi D_2^2}{\beta_2^2 A_{22}^2}\sin(\frac{\omega s}{i})\\
&- \frac{8t\phi C}{\beta_2}\sin(\frac{\omega s}{i}) + C_{13}\end{aligned} \tag{3.182}$$

### 3.4.4 Symmetries for equation (3.109)

The Symmetries are

$$\begin{aligned}X_1 =& -\frac{2t\phi e^{-t}}{\beta_1^2}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\frac{\partial}{\partial t} + \frac{ie^{-t}\phi}{\beta_1^2\omega}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\frac{\partial}{\partial r}\\
&\{-\frac{e^{-t}\phi}{2\beta_1^2}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}) + \frac{e^{-t}\phi}{2\beta_1^3}\sin\bar{\omega_1}\sin\bar{\omega_1}t\sin(\frac{\omega r}{i})\\
&+ \frac{e^{-t}\phi}{2\beta_1^3}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}) + \frac{2e^{-t}\phi D_2 s}{\beta_1^2 A_{22}^2}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\\
&- \frac{D_1 i\omega\phi e^{-t}}{2\beta_1^3}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\}u\frac{\partial}{\partial u}\end{aligned} \tag{3.183}$$

$$\begin{aligned}X_2 =& -\frac{2t\phi e^{-t}}{\beta_1^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\frac{\partial}{\partial t} + \frac{ie^{-t}\phi}{\beta_1^2\omega}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\frac{\partial}{\partial r} +\\
&\{-\frac{e^{-t}\phi}{2\beta_1^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i} - \frac{e^{-t}\phi}{2\beta_1^3}\cos\bar{\omega_1}t\sin(\frac{\omega r}{i}) + \frac{e^{-t}\phi}{2\beta_1^3}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}))\\
&+ \frac{2e^{-t}\phi D_2 s}{\beta_1^2 A_{22}^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}) - \frac{D_1 i\omega\phi e^{-t}}{2\beta_1^3}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\}u\frac{\partial}{\partial u}\end{aligned} \tag{3.184}$$

$$X_3 = -\phi e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i})\frac{\partial}{\partial t} + \frac{i\phi}{\omega}e^{-\frac{D_2}{s}t}\cos(\frac{\omega r}{i})\frac{\partial}{\partial r} + \{ -\frac{1}{2}\phi e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i}) + \frac{\phi}{2\beta_1}\frac{D_2}{s}e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i}) - \frac{D_1 i\omega\phi}{2\beta_1}e^{-\frac{D_2}{s}t}\cos(\frac{\omega r}{i})\}u\frac{\partial}{\partial u} \tag{3.185}$$

$$X_4 = -\frac{2t\phi e^{-t}}{\beta_2^2}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i})\frac{\partial}{\partial t} + \frac{ie^{-t}\phi}{\beta_2^2\omega}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i})\frac{\partial}{\partial s} \{ -\frac{e^{-t}\phi}{2\beta_2^2}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) + \frac{e^{-t}\phi}{2\beta_2^3}C_4\sin\bar{\omega_2}\sin\bar{\omega_2}t\sin(\frac{\omega s}{i}) + \frac{e^{-t}\phi}{2\beta_2^3}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) + \frac{e^{-t}\phi D_2 i\omega}{2\beta_2^3}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) - \frac{2D_1 re^{-t}}{\beta_2^2 A_{11}^2}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i})\}u\frac{\partial}{\partial u} \tag{3.186}$$

$$X_5 = -\frac{2te^{-t}\phi}{\beta_2^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i})\frac{\partial}{\partial t} + \frac{ie^{-t}\phi}{\beta_2^2\omega}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i})\frac{\partial}{\partial s} + \{ -\frac{e^{-t}\phi}{2\beta_2^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) + \frac{e^{-t}\phi}{2\beta_2^3}\cos\bar{\omega_2}t\sin(\frac{\omega s}{i}) + \frac{e^{-t}\phi}{2\beta_2^3}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) + \frac{2e^{-t}\phi D_1 r}{\beta_2^2 A_{11}^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) - \frac{D_1 i\omega\phi e^{-t}}{2\beta_2^3}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i})\}u\frac{\partial}{\partial u} \tag{3.187}$$

$$X_6 = -\phi e^{-\frac{D_1\phi}{r}t}\sin(\frac{\omega s}{i})\frac{\partial}{\partial t} + \frac{i\phi}{\omega}e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})\frac{\partial}{\partial s} + \{ -\frac{1}{2}\phi e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i}) + \frac{\phi}{2\beta_2}\frac{D_1}{r}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i}) - \frac{D_2 i\omega\phi}{2\beta_2}e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})\}u\frac{\partial}{\partial u} \tag{3.188}$$

$$X_7 = \frac{2te^{-t}}{\beta_2^2}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i})\frac{\partial}{\partial t} + \frac{ie^{-t}}{\beta_2^2\omega}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i})\frac{\partial}{\partial s} + \{ -\frac{D_2 i\omega}{2\beta_2^3}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) + \frac{e^{-t}}{2\beta_2^2}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) - \frac{e^{-t}}{2\beta_2^3}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) - \frac{e^{-t}}{2\beta_2^3}\sin\bar{\omega_2}\sin\bar{\omega_2}t\cos(\frac{\omega s}{i}) - \frac{2D_1 re^{-t}}{\beta_2^2 A_{11}^2}\frac{\sin\bar{\omega_2}\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i})\}u\frac{\partial}{\partial u} \tag{3.189}$$

$$
\begin{aligned}
X_8 = &\frac{2te^{-t}}{\beta_2^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i})\frac{\partial}{\partial t} + \frac{ie^{-t}}{\beta_2^2\omega}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i})\frac{\partial}{\partial s} \\
&+ \big\{ - \frac{D_2 i\omega}{2\beta_2^3}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\sin(\frac{\omega s}{i}) + \frac{e^{-t}}{2\beta_2^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) \\
&- \frac{e^{-t}}{2\beta_2^3}\cos\bar{\omega_2}t\cos(\frac{\omega s}{i}) - \frac{e^{-t}}{2\beta_2^3}\frac{\cos\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i}) \\
&- \frac{2D_1 r e^{-t}}{\beta_2^2 A_{11}^2}\frac{\sin\bar{\omega_2}t}{\bar{\omega_2}}\cos(\frac{\omega s}{i})\big\}u\frac{\partial}{\partial u}
\end{aligned}
\tag{3.190}
$$

$$
\begin{aligned}
X_9 = &e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})\frac{\partial}{\partial t} + \frac{i}{\omega}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i})\frac{\partial}{\partial s} + \big\{\frac{1}{2}e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) \\
&- \frac{D_2 i\omega}{2\beta_2}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i}) - \frac{D_1}{2\beta_2 r}e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) - \frac{2D_1 r}{A_{11}^2}e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})\big\}u\frac{\partial}{\partial u}
\end{aligned}
\tag{3.191}
$$

$$
\begin{aligned}
X_{10} = &\frac{2te^{-t}}{\beta_1^2}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\frac{\partial}{\partial t} + \frac{ie^{-t}}{\beta_1^2\omega}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\frac{\partial}{\partial r} \\
&+ \big\{ - \frac{D_1 i\omega}{2\beta_1^3}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}) + \frac{e^{-t}}{2\beta_1^2}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i}) \\
&- \frac{e^{-t}}{2\beta_1^3}\sin\bar{\omega_1}\sin\bar{\omega_1}t\cos(\frac{\omega r}{i}) - \frac{e^{-t}}{2\beta_1^3}\frac{\sin\bar{\omega_1}\sin\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i}) \\
&- \frac{2D_2 s e^{-t}}{\beta_1^2 A_{22}^2}\frac{\sin\bar{\omega_1}\cos\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\big\}u\frac{\partial}{\partial u}
\end{aligned}
\tag{3.192}
$$

$$
\begin{aligned}
X_{11} = &\frac{2te^{-t}}{\beta_1^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\frac{\partial}{\partial t} + \frac{ie^{-t}}{\beta_1^2\omega}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i})\frac{\partial}{\partial r} \\
&+ \big\{ - \frac{D_1 i\omega}{2\beta_1^3}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\sin(\frac{\omega r}{i}) + \frac{e^{-t}}{2\beta_1^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i}) - \frac{e^{-t}}{2\beta_1^3}\cos\bar{\omega_1}t\cos(\frac{\omega r}{i}) \\
&- \frac{e^{-t}}{2\beta_1^3}\cos\bar{\omega_1}t\cos(\frac{\omega r}{i}) - \frac{2D_2 s e^{-t}}{\beta_1^2 A_{22}^2}\frac{\sin\bar{\omega_1}t}{\bar{\omega_1}}\cos(\frac{\omega r}{i})\big\}u\frac{\partial}{\partial u}
\end{aligned}
\tag{3.193}
$$

$$
\begin{aligned}
X_{12} = &e^{-\frac{D_2}{s}t}\cos(\frac{\omega r}{i})\frac{\partial}{\partial t} + \frac{i}{\omega}e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i})\frac{\partial}{\partial r} \\
&+ \big\{ - \frac{D_1 i\omega}{2\beta_1}e^{-\frac{D_2}{s}t}\sin(\frac{\omega r}{i}) + \frac{1}{2}e^{-\frac{D_2}{s}t}\cos(\frac{\omega r}{i}) - \frac{1}{2\beta_1}\frac{D_2}{s}e^{-\frac{D_2}{s}t}\cos(\frac{\omega r}{i}) \\
&- \frac{2D_2 s}{A_{22}^2}e^{-\frac{D_2}{s}t}\cos(\frac{\omega r}{i})\big\}u\frac{\partial}{\partial u}
\end{aligned}
\tag{3.194}
$$

$$
X_{13} = \frac{\partial}{\partial t} \tag{3.195}
$$

$$X_{14} = u\frac{\partial}{\partial u} \tag{3.196}$$

$$\begin{aligned} X_{15} = &\big(\frac{2tD_1^2}{\beta_1^2 A_{11}^2}\cos(\frac{\omega r}{i}) - \frac{2t\phi D_1^2}{\beta_1^2 A_{11}^2}\sin(\frac{\omega r}{i})\big)\frac{\partial}{\partial t} + \big(\frac{D_1^2 i}{\omega A_{11}^2 \beta_1^2}\sin(\frac{\omega r}{i}) \\ &+ \frac{D_1^2 \phi i}{\omega A_{11}^2 \beta_1^2}\cos(\frac{\omega r}{i})\big)\frac{\partial}{\partial r} big\{-\frac{D_1^2}{2\beta_1^2 A_{11}^2}\phi\sin(\frac{\omega r}{i}) - \frac{D_1^3 i\omega}{2\beta_2^3 A_{11}^2}\sin(\frac{\omega r}{i}) \\ &+ \frac{2D_1^2 D_2 s\phi}{\beta_1^2 A_{11}^2 A_{22}^2}\sin(\frac{\omega r}{i}) + \frac{D_1^2}{2\beta_1^2 A_{11}^2}\cos(\frac{\omega r}{i}) - \frac{2D_1^2 D_2 s}{\beta_1^2 A_{22}^2 A_{11}^2}\cos(\frac{\omega r}{i}) \\ &- \frac{D_1^3 i\phi\omega}{2\beta_2^3 A_{11}^2}\cos(\frac{\omega r}{i})\big\}u\frac{\partial}{\partial u} \end{aligned} \tag{3.197}$$

$$\begin{aligned} X_{16} = &\big(\frac{8tC}{\beta_1}\cos(\frac{\omega r}{i}) - \frac{8t\phi C}{\beta_1}\sin(\frac{\omega r}{i})\big)\frac{\partial}{\partial t} + \big(\frac{4iC\phi}{\omega\beta_1}\cos(\frac{\omega r}{i}) \\ &+ \frac{4iC}{\omega\beta_1}\sin(\frac{\omega r}{i})\big)\frac{\partial}{\partial r} + \big\{-\frac{2C}{\beta_1}\phi\sin(\frac{\omega r}{i}) + \frac{2C}{\beta_1}\cos(\frac{\omega r}{i}) \\ &- \frac{D_1 i\omega\phi C}{\beta_1^2}\cos(\frac{\omega r}{i}) - \frac{D_1 i\omega C}{\beta_1^2}\sin(\frac{\omega r}{i}) + \frac{8D_2 s\phi C}{\beta_1^2 A_{22}^2}\sin(\frac{\omega r}{i}) \\ &- \frac{8D_2 sC}{\beta_1^2 A_{22}^2}\cos(\frac{\omega r}{i})u\frac{\partial}{\partial u} \end{aligned} \tag{3.198}$$

$$\begin{aligned} X_{17} = &\big(\frac{2tD_2^2}{\beta_2^2 A_{22}^2}\cos(\frac{\omega s}{i}) - \frac{2t\phi D_2^2}{\beta_2^2 A_{22}^2}\sin(\frac{\omega s}{i})\big)\frac{\partial}{\partial t} \\ &+ \big(+\frac{D_2^2 i}{\omega A_{22}^2 \beta_2^2}\sin(\frac{\omega s}{i}) + \frac{D_2^2 i\phi}{\omega A_{22}^2 \beta_2^2}\cos(\frac{\omega s}{i})\big)\frac{\partial}{\partial s} \\ &+ \big\{-\frac{D_2^2}{2\beta_2^2 A_{22}^2}\phi\sin(\frac{\omega s}{i}) - \frac{D_2^3 i\omega}{2\beta_2^3 A_{22}^2}\sin(\frac{\omega s}{i}) + \frac{2D_2^2 D_1 r\phi}{\beta_2^2 A_{22}^2 A_{11}^2}\sin(\frac{\omega s}{i}) \\ &+ \frac{D_2^2}{2\beta_2^2 A_{22}^2}\cos(\frac{\omega s}{i}) - \frac{2D_2^2 D_1 r}{\beta_2^2 A_{22}^2 A_{11}^2}\cos(\frac{\omega s}{i}) - \frac{D_2^3 i\phi\omega}{2\beta_2^3 A_{22}^2}\cos(\frac{\omega s}{i})\big\}u\frac{\partial}{\partial u} \end{aligned} \tag{3.199}$$

$$\begin{aligned} X_{18} = &\big(\frac{8tC}{\beta_2}\cos(\frac{\omega s}{i}) - \frac{8t\phi C}{\beta_2}\sin(\frac{\omega s}{i})\big)\frac{\partial}{\partial t} + \big(\frac{4iC}{\omega\beta_2}\sin(\frac{\omega s}{i})\big) \\ &+ \frac{4iC\phi}{\omega\beta_2}\cos(\frac{\omega s}{i})\big)\frac{\partial}{\partial s}\big\{-\frac{2C}{\beta_2}\phi\sin(\frac{\omega s}{i}) + \frac{2C}{\beta_2}\cos(\frac{\omega s}{i}) - \frac{D_2 i\omega\phi C}{\beta_2^2}\cos(\frac{\omega s}{i}) \\ &- \frac{D_2 i\omega C}{\beta_2^2}\sin(\frac{\omega s}{i}) + \frac{8D_1 r\phi C}{\beta_2^2 A_{11}^2}\sin(\frac{\omega r}{i}) - \frac{8D_1 rC}{\beta_2^2 A_{11}^2}\cos(\frac{\omega r}{i})\big\}u\frac{\partial}{\partial u} \end{aligned} \tag{3.200}$$

$$X_{\infty} = g(t, r, s)u\frac{\partial}{\partial u} \tag{3.201}$$

### 3.4.5 Invariant Solution for equation (3.109)

We consider the symmetry given by equation (3.191). The invariants are determined from solving the equation

$$X_9 I = e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})\frac{\partial I}{\partial t} + \frac{i}{\omega}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i})\frac{\partial I}{\partial s} + \{-\frac{1}{2}e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) \\ + \frac{1}{2\beta_2}\frac{D_1}{r}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i}) - \frac{D_2 i\omega}{2\beta_2}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i}) - \frac{2D_1}{\beta_2 r}e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})\}u\frac{\partial I}{\partial u} = 0 \tag{3.202}$$

The characteristic equation of (3.202) is given by

$$\frac{dt}{e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})} = \frac{ds}{\frac{i}{\omega}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i})} = \frac{du}{uM} \tag{3.203}$$

where

$$M = \{\frac{1}{2}\phi e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i}) - \frac{1}{2\beta_2}\frac{D_1}{r}e^{-\frac{D_1}{r}t}\sin(\frac{\omega r}{i}) \\ - \frac{D_2 i\omega}{2\beta_2}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i}) - \frac{2D_1 r}{A_{11}^2}e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})\} \tag{3.204}$$

From equation (3.203) we have that

$$\frac{dt}{e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})} = \frac{ds}{\frac{i}{\omega}e^{-\frac{D_1}{r}t}\sin(\frac{\omega s}{i})} \tag{3.205}$$

simplifies to

$$dt = \frac{\omega}{i}\cot(\frac{\omega s}{i})ds \tag{3.206}$$

The solution to equation (3.206) is

$$B + t = \ln\sin|(\frac{\omega s}{i})| \tag{3.207}$$

which results in that the first invariant is given by

$$B_1 = e^{-t}\sin(\frac{\omega s}{i}) \tag{3.208}$$

Also from equation (3.203) we have that

$$\frac{dt}{e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})} = \frac{du}{uM} \tag{3.209}$$

where $M$ is given by (3.204). We simplify equation (3.209) by dividing the each term in the denominator by $e^{-\frac{D_1}{r}t}\cos(\frac{\omega s}{i})$ and letting $\omega \to 0$ to

$$\frac{1}{2}dt = \frac{du}{u} \tag{3.210}$$

The solution to equation (3.210) is

$$ue^{-\frac{1}{2}t} = B_2 \tag{3.211}$$

Since $B_1$ is independent of $u$, every invariant solution is of the form

$$ue^{-\frac{1}{2}t} = F(e^{-t}\sin(\frac{\omega s}{i})) \tag{3.212}$$

or equivalently

$$u = e^{\frac{1}{2}t}F(e^{-t}\sin(\frac{\omega s}{i})) \tag{3.213}$$

Differentiating equation (3.213) we obtain

$$u_t = \frac{1}{2}e^{\frac{1}{2}t}F - e^{-\frac{1}{2}t}\sin(\frac{\omega s}{i})F' \tag{3.214}$$

$$u_s = \frac{\omega}{i}F'e^{-\frac{1}{2}t}\cos(\frac{\omega s}{i}) \tag{3.215}$$

$$u_{ss} = \frac{\omega^2}{i^2}e^{-\frac{3}{2}t}F''\cos^2(\frac{\omega s}{i}) - \frac{\omega^2}{i^2}e^{-\frac{1}{2}t}F'\sin(\frac{\omega s}{i}) \tag{3.216}$$

$$u_r = 0 \tag{3.217}$$

$$u_{rr} = 0 \tag{3.218}$$

We substitute equations (3.214) to (3.218) into equation (3.109) and obtain

$$\begin{aligned}&\frac{1}{2}e^{\frac{1}{2}t}F - e^{-\frac{1}{2}t}\sin(\frac{\omega s}{i})F' + \frac{1}{2}A_{22}^2\frac{\omega^2}{i^2}e^{-\frac{3}{2}t}F''\cos^2(\frac{\omega s}{i}) - \\ &\frac{1}{2}A_{22}^2\frac{\omega^2}{i^2}e^{-\frac{1}{2}t}F'\sin(\frac{\omega s}{i}) + D_2\frac{\omega}{i}F'e^{-\frac{1}{2}t}\cos(\frac{\omega s}{i}) - Ce^{\frac{1}{2}t}F = 0\end{aligned} \tag{3.219}$$

If we apply equation (3.170) and let $\omega \to 0$ equation (3.219) simplifies to

$$-\beta_2 e^{-\frac{3}{2}t}F'' - (C - \frac{1}{2})e^{\frac{1}{2}t}F = 0$$

or

$$-\beta_2 F'' - (C - \frac{1}{2})e^{2t}F = 0 \tag{3.220}$$

Equation (3.220) simplifies to

$$F'' - \frac{(1-2C)e^{2t}}{2\beta_2}F = 0 \tag{3.221}$$

Solving equation (3.221) and substituting for equation (3.170) we arrive at two linearly independent solutions for $F$ given by

$$F = Ae^{\frac{\tilde{\alpha}e^{t}s}{\omega}} \tag{3.222}$$

$$F = Be^{-\frac{\tilde{\alpha}e^{t}s}{\omega}} \tag{3.223}$$

where

$$\tilde{\alpha} = \frac{\sqrt{2C-1}}{A_{22}}$$

The general solutions for $u$ is given by linearly independent solutions

$$u = e^{\frac{1}{2}t}(Ae^{\frac{\tilde{\alpha}e^{t}s}{\omega}}) \tag{3.224}$$

$$u = e^{\frac{1}{2}t}(Be^{-\frac{\tilde{\alpha}e^{t}s}{\omega}}) \tag{3.225}$$

with $A, B$ as constants. If $\omega \to 0$ equation (3.224) cannot be defined and equation (3.225) reduces to

$$u = He^{\frac{1}{2}t} \tag{3.226}$$

where $H$ is a constant. Hence the invariant solution is given by (3.226).The graphical solution is given in the accompanying Figure.

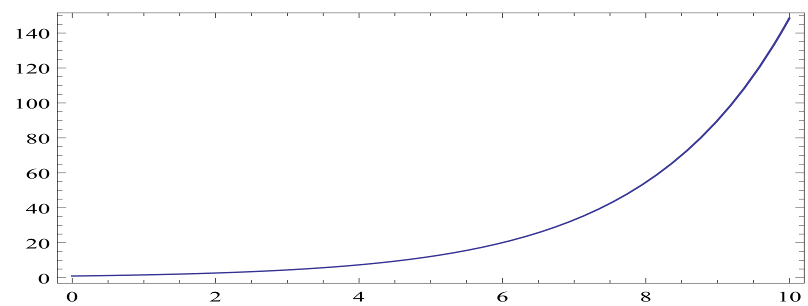


Figure 3.5: Plot for solution (3.226)with dependent variable $u$ represented on the vertical axis and independent variable $r$ represented on the horizontal axis

# Chapter 4

# Transformation of equation (3.6) to heat equation

In this chapter we show that the transformed Black-Scholes equation (3.6) is transformable to the classical heat equation (2.8) using an equivalence transformation for the independent and dependent variables given by

$$\begin{aligned} \tau =&\beta(t), \quad y = \alpha(t,r), \quad v = \gamma(t,r)u \\ &\alpha_r \neq 0, \quad \beta_t \neq 0. \end{aligned} \tag{4.1}$$

We recall the theorem from ([17]) without proof which goes as follows:

**Theorem 12** The parabolic equation

$$u_t - u_{xx} + a(t,x) + c(t,x) = 0$$

can be reduced to the heat equation

$$v_t - v_{xx} = 0$$

by an appropriate linear transformation of the dependent variable

$$u = ve^{-\varrho(t,x)} \tag{4.2}$$

without changing the independent variables $t$ and $x$ if and only if the semi-invariant

$$K = aa_x - a_{xx} + a_t + 2c_x = 0 \tag{4.3}$$

The function $\varrho$ in the transformation (4.2) is obtained by solving the equations

$$\frac{\partial \varrho}{\partial x} = -\frac{1}{2}a, \quad \frac{\partial \varrho}{\partial x} = -\frac{1}{4}a^2 - \frac{1}{2}a_x + C \tag{4.4}$$

## 4.1 Transformation

To determine the transformation of equation (3.6) to the classical heat equation (2.8) we consider some derivatives arising from (4.1).

$$\begin{aligned} v_t &= \frac{\partial v}{\partial \beta}\frac{d\beta}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} \\ &= v_\tau \beta_t + \alpha_t v_y \end{aligned}$$

but

$$v_y = \gamma u_y$$

thus

$$v_t = v_\tau \beta_t + \alpha_t \gamma u_y \tag{4.5}$$

We also have that

$$v_t = u\gamma_t + u_t\gamma \tag{4.6}$$

From

$$u_r = u_y \alpha_r$$

we have that

$$u_y = \frac{u_r}{\alpha_r} \tag{4.7}$$

Equations (4.5),(4.6) and (4.7) give the expression for $v_\tau$ as

$$v_\tau = \frac{\alpha_r u \gamma_t + \alpha_r u_t \gamma - \alpha_t u_r \gamma}{\alpha_r \beta_t} \tag{4.8}$$

We define

$$\begin{aligned} v_r &= \frac{\partial v}{\partial y}\frac{dy}{dr} \\ v_r &= v_y \alpha_r \end{aligned} \tag{4.9}$$

Thus

$$v_{rr} = \alpha_r^2 v_{yy} + v_y \alpha_{rr} \tag{4.10}$$

But from

$$v_r = \gamma_r u + u_r \gamma \tag{4.11}$$

we have that

$$v_{rr} = \gamma_{rr} u + 2u_r \gamma_r + u_{rr} \gamma, \tag{4.12}$$

also

$$v_r = v_y \alpha_r \tag{4.13}$$

results in that

$$v_y = \frac{\gamma_r u + u_r \gamma}{\alpha_r} \tag{4.14}$$

The equations (4.10),(4.12) and (4.14) imply that

$$v_{yy} = \frac{\gamma_{rr} + 2u_r \gamma_r + u_{xx} \gamma - \alpha_{rr} (\frac{\gamma_r u + u_r \gamma}{\alpha_x})}{\alpha_r^2} \tag{4.15}$$

The substitution of these derivatives, especially the equations (4.8) and (4.15) in the heat equation (2.8) result in the equation

$$\frac{\alpha_r^2 u \gamma_t + \alpha_r^2 u_t \gamma - \alpha_r \alpha_t u_r \gamma}{\beta_t} - \gamma_{rr} - 2u_r \gamma_x - u_{rr} \gamma + \frac{\alpha_{rr} \gamma_r u + \alpha_{rr} u_r \gamma}{\alpha_r} = 0 \tag{4.16}$$

We rearrange equation (4.16) in terms of $u$ and its partial derivatives, divide each term by $\gamma$ and this results in the equation

$$u_{rr} + \{\frac{2\gamma_r}{\gamma} + \frac{\alpha_r \alpha_t}{\beta_t} - \frac{\alpha_{rr}}{\alpha_r}\} u_r - \frac{\alpha_r^2}{\beta_t} u_t + \{\frac{\gamma_{rr}}{\gamma} - \frac{\alpha_r^2 \gamma_t}{\beta_t \gamma} - \frac{\alpha_{rr} \gamma_r}{\alpha_r \gamma}\} u = 0 \tag{4.17}$$

The equation (4.17) can be compared to equation (3.6) written in the form

$$u_{rr} + \frac{2}{A^2} u_t + \frac{2B}{A^2} u_r - \frac{2C}{A^2} u = 0 \tag{4.18}$$

We equate corresponding coefficients in the equations (4.17) and (4.18) and the results is the system of equations

$$\frac{\alpha_r^2}{\beta_t} = -\frac{2}{A^2} \tag{4.19}$$

$$\frac{2\gamma_r}{\gamma} + \frac{\alpha_r \alpha_t}{\beta_t} - \frac{\alpha_{rr}}{\alpha_r} = \frac{2B}{A^2} \tag{4.20}$$

$$\frac{\gamma_{rr}}{\gamma} - \frac{\alpha_r^2 \gamma_t}{\beta_t \gamma} - \frac{\alpha_{rr} \gamma_r}{\alpha_r \gamma} = -\frac{2C}{A^2} \tag{4.21}$$

The simplification of equation (4.19) implies that

$$\alpha_r = \frac{\sqrt{-2\beta_t}}{A},$$

however if we set $\sqrt{-2\beta_t} = \varphi(t)$, then we can define

$$\alpha_r = \frac{\varphi(t)}{A} \tag{4.22}$$

whence

$$\alpha = \frac{\varphi(t)r}{A} + \psi(t) \tag{4.23}$$

and

$$\beta_t = -\frac{1}{2}\varphi^2(t) \tag{4.24}$$

From the equations (4.22) and (4.37) we have that

$$\alpha_t = \frac{\varphi'(t)r}{A} + \psi'(t) \tag{4.25}$$

and

$$\alpha_{rr} = 0. \tag{4.26}$$

We substitute for (4.22), (4.24), and (4.26) in (4.20) and we have the simplification

$$2\frac{\gamma_r}{\gamma} = 2\frac{\varphi'(t)r}{A^2\varphi(t)} + 2\frac{B}{A^2}$$

which gives the expression for $\gamma$ as

$$\gamma(t,r) = f(t)e^{\frac{\varphi'(t)r^2}{2A^2\varphi(t)} + \left(\frac{B}{A^2} + \frac{\psi'(t)}{A^2\varphi(t)}\right)r - \frac{1}{2}}. \tag{4.27}$$

with $f(t)$ as some function of $t$. From the equation (4.27) we get the expressions for $\gamma_t, \gamma_r, \gamma_{rr}$ given respectively as

$$\gamma_t = f'(t)e^{\frac{\varphi'(t)r^2}{2A^2\varphi(t)} + \left(\frac{B}{A^2} + \frac{\psi'(t)}{A^2\varphi(t)}\right)r - \frac{1}{2}} + f(t)\Big(\frac{\psi''(t)}{\varphi(t)} - \frac{\psi'(t)\varphi'(t)}{(\varphi(t))^2}\Big)\frac{r}{A^2}e^{\frac{\varphi'(t)r^2}{2A^2\varphi(t)} + \left(\frac{B}{A^2} + \frac{\psi'(t)}{A^2\varphi(t)}\right)r - \frac{1}{2}}$$
$$+ f(t)\Big(\frac{\varphi''(t)}{\varphi(t)} - \Big(\frac{\varphi'(t)}{\varphi(t)}\Big)^2\Big)\frac{r^2}{2A^2}e^{\frac{\varphi'(t)r^2}{2A^2\varphi(t)} + \left(\frac{B}{A^2} + \frac{\psi'(t)}{A^2\varphi(t)}\right)r - \frac{1}{2}} \tag{4.28}$$

$$\gamma_r = f(t)\Big(\frac{\varphi'(t)r}{A^2\varphi(t)} + \frac{B}{A^2}\Big)e^{\frac{\varphi'(t)r^2}{2A^2\varphi(t)}+\left(\frac{B}{A^2}+\frac{\psi'(t)}{A^2\varphi(t)}\right)r-\frac{1}{2}} \tag{4.29}$$

$$\begin{aligned}\gamma_{rr} &= f(t)\Big(\frac{\varphi'(t)}{A^2\varphi(t)}\Big)e^{\frac{\varphi'(t)r^2}{2A^2\varphi(t)}+\left(\frac{B}{A^2}+\frac{\psi'(t)}{A^2\varphi(t)}\right)r-\frac{1}{2}} \\ &+ f(t)\Big\{\frac{\varphi''(t)}{\varphi(t)} - \Big(\frac{\varphi'(t)}{\varphi(t)}\Big)^2\Big\}e^{\frac{\varphi'(t)r^2}{2A^2\varphi(t)}+\left(\frac{B}{A^2}+\frac{\psi'(t)}{A^2\varphi(t)}\right)r-\frac{1}{2}}\end{aligned} \tag{4.30}$$

The substitutions in equation (4.21) yield the equation

$$\begin{aligned}&\frac{\varphi'(t)}{\varphi(t)} + \Big(\frac{\varphi'(t)}{\varphi(t)}\Big)^2\frac{r^2}{A^2} - \Big(\frac{\varphi'(t)}{\varphi(t)}\Big)^2\frac{r^2}{4A^2} + \frac{\psi''(t)}{\varphi(t)}\frac{r}{A^2} - \frac{\psi'(t)\varphi'(t)}{(\varphi(t))^2}\frac{r}{A^2} \\ &+ \Big(\frac{\varphi''(t)}{\varphi(t)}\Big)\frac{r^2}{4A^2} + \frac{2\varphi'(t)}{\varphi(t)}\frac{Br}{A^2} + \frac{B^2}{A^2} - \frac{f'(t)}{2f(t)} + 2C = 0\end{aligned} \tag{4.31}$$

The equation (4.31) splits into the following

$$\frac{\varphi''(t)}{\varphi(t)} - \Big(\frac{\varphi'(t)}{\varphi(t)}\Big)^2 = 0 \tag{4.32}$$

$$\frac{\psi''(t)}{\varphi(t)} - \frac{\psi'(t)\varphi'(t)}{(\varphi(t))^2} = 0 \tag{4.33}$$

$$\frac{\varphi'(t)}{\varphi(t)} + \frac{2\varphi'(t)}{\varphi(t)}\frac{Br}{A^2} + \Big(\frac{\varphi'(t)}{\varphi(t)}\Big)^2\frac{r^2}{A^2} + \frac{B^2}{A^2} + 2C = \frac{f'(t)}{2f(t)} \tag{4.34}$$

From equation (4.32) we have that

$$\frac{d}{dt}\frac{\varphi'(t)}{\varphi(t)} = 0,$$

thus

$$\frac{\varphi'(t)}{\varphi(t)} = K$$

for some function $K$. This results in that

$$\varphi = Le^{Kt} \tag{4.35}$$

for some function L. Also equation (4.33) implies that

$$\frac{d}{dt}\frac{\psi'(t)}{\varphi(t)} = 0,$$

and

$$\frac{\psi'(t)}{\varphi(t)} = M$$

whence

$$\psi = \frac{ML}{K}e^{Kt}, \quad K \neq 0. \tag{4.36}$$

Hence we have the expression for $\alpha$ given as

$$\alpha = \frac{L}{A}re^{Kt} + \frac{ML}{K}e^{Kt}. \tag{4.37}$$

The function $\frac{f'(t)}{2f(t)}$ from (4.34)is given as

$$\frac{f'(t)}{f(t)} = 2K + \frac{4KBr}{A^2} + \frac{2K^2r^2}{A^2} + \frac{2B}{A^2} + 4C \tag{4.38}$$

which implies that

$$\frac{df(t)}{f(t)} = 2Kt + \frac{4KBrt}{A^2} + \frac{2K^2r^2t}{A^2} + \frac{2Bt}{A^2} + 4Ct + K_1$$

Thus

$$f(t) = e^{2Kt+\frac{4KBrt}{A^2}+\frac{2K^2r^2t}{A^2}+\frac{2Bt}{A^2}+4Ct+K_1}$$

or

$$f(t) = K_1e^{2Kt+\frac{4KBrt}{A^2}+\frac{2K^2r^2t}{A^2}+\frac{2Bt}{A^2}+4Ct} \tag{4.39}$$

for some arbitrary constants $K$ and $K_1$. The transformation is given as

$$\begin{aligned} \tau = -\frac{1}{2}L^2e^{Kt}, \quad y = \frac{L}{A}re^{Kt} + \frac{ML}{K}e^{Kt}, \quad K \neq 0 \\ v = K_1e^{2Kt+\frac{4KBrt}{A^2}+\frac{2K^2r^2t}{A^2}+\frac{2Bt}{A^2}+4Ct}e^{\frac{K(t)r^2}{2A^2}+\left(\frac{B}{A^2}+\frac{M}{A^2}\right)r-\frac{1}{2}}u. \end{aligned} \tag{4.40}$$

# Chapter 5

# Further Applications

The chapter presents examples of areas where the method is applied successfully. We consider two examples of Gaussian type partial differential equation and a differential equation model of epidemiology of HIV and AIDS. We determine the symmetries using the method in Chapter 3 and calculate invariant solutions for operators providing them.

## 5.1 Gaussian type partial differential equation

In this section we look at the application of the method in Chapter 3 on the Gaussian type differential equation. The Gaussian function

$$\int_0^\infty e^{-ax^2} dx \tag{5.1}$$

is classified as an integral whose antiderivative cannot be expressed in closed form (i.e. cannot be expressed analytically in terms of a finite number of certain well known functions)[34].

The current undertaking seeks to determine the solution of its derived differential equation using Lie Symmetry method which is a mathematical theory that synthesizes symmetry of differential equation [13].

In order to apply Lie Symmetry method to the Gaussian type function, we need to first present it as a differential equation by substituting

$$a = t,$$

and letting

$$u = \int_0^\infty e^{-tx^2} dx \tag{5.2}$$

resulting in

$$u_x = -2txe^{-tx^2}. \tag{5.3}$$

If we differentiate equation (5.3) with respect to $t$ then the resulting partial differential equation becomes

$$u_{tx} = \frac{1}{t}u_x - x^2 u_x. \tag{5.4}$$

Equation (5.4) is a partial differential equation with independent variables $t$ and $x$, and differential variable $u$.

### 5.1.1 Solution of determining equation

The infinitesimal generator for point symmetry admitted by equation(5.4) is of the form

$$X = \xi^1(t,x)\frac{\partial}{\partial t} + \xi^2(t,x)\frac{\partial}{\partial x} + \eta(t,x)\frac{\partial}{\partial u} \tag{5.5}$$

Its first and second prolongations are given by

$$X^{(2)} = X + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_{tx}^{(2)}\frac{\partial}{\partial u_{tx}} \tag{5.6}$$

where $X$ is defined by equation (5.5). The invariance condition for (5.4) is given by

$$\begin{aligned} &X^{(2)}(u_{tx} - \frac{1}{t}u_x + x^2 u_x)|_{u_{tx}=\frac{1}{t}u_x - x^2 u_x} = \\ &(\eta_{tx}^{(2)} - \frac{1}{t}\eta_x^{(1)} + \frac{1}{t^2}\xi^1 u_x + 2x\xi^{(2)}u_x + x^2\eta_x^{(1)})|_{u_{tx}=\frac{1}{t}u_x - x^2 u_x} = 0 \end{aligned} \tag{5.7}$$

We define the following from ([6],[13])

$$\begin{aligned} \eta &= fu + g \\ \eta_t^{(1)} &= g_t + f_t u + [f - \xi_t^1]u_t - \xi_t^2 u_x \\ \eta_x^{(1)} &= g_x + f_x u + [f - \xi_x^2]u_x - \xi_x^1 u_t \\ \eta_{tx}^{(2)} &= g_{tx} + f_{tx}u + [f_t - \xi_{tx}^2]u_x + [f_x - \xi_{tx}^1]u_t \\ &\quad + u_{tx}[f - \xi_t^1 - \xi_x^2] - \xi_t^2 u_{xx} - \xi_x^1 u_{tt} \end{aligned} \tag{5.8}$$

The substitutions of $\eta_x^{(1)}$ and $\eta_{tx}^{(2)}$ in the invariance condition (5.7) yield the determining equation

$$
\begin{aligned}
&g_{tx} + f_{tx}u + [f_t - \xi^2_{tx}]u_x + [f_x - \xi^1_{tx}]u_t + (\frac{1}{t}u_x - x^2u_x)[f - \xi^1_t - \xi^2_x] \\
&- \xi^2_t u_{xx} - \frac{1}{t}g_x - \frac{1}{t}f_x u - \frac{1}{t}u_x[f - \xi^2_x] + \frac{1}{t}u_t\xi^1_x + \frac{1}{t^2}\xi^1 u_x - \xi^1_x u_{tt} \\
&+ 2x\xi^2 u_x + x^2 g_x + x^2 f_x u + x^2 u_x[f - \xi^2_x] - x^2\xi^1_x u_t = 0
\end{aligned}
\tag{5.9}
$$

We set the coefficients of $u_{xx}, u_{tt}, u_x, u_t, u$ and those free of these variables to zero. We thus have the following monomials which we call defining equations

$$u_{xx} \ : \ \xi^2_t = 0, \tag{5.10}$$

$$u_{tt} \ : \ \xi^1_x = 0, \tag{5.11}$$

$$u_t \ : \ f_x = 0, \tag{5.12}$$

$$u_x \ : \ f_t - \frac{1}{t}\xi^1_t + \frac{1}{t^2}\xi^1 + 2x\xi^2 + x^2\xi^1_t = 0, \tag{5.13}$$

$$u \ : \ f_{tx} = 0, \tag{5.14}$$

$$u^0 \ : \ g_{tx} = \frac{1}{t}g_x - x^2 g_x. \tag{5.15}$$

We differentiate defining equation (5.13) with respect to $t$ and apply equation (5.10) to obtain the equation

$$f_{tt} - (\frac{1}{t}\xi^1_t)_t + (\frac{1}{t}\xi^1)_t + x^2\xi^1_{tt} = 0 \tag{5.16}$$

The differentiation of equation (5.16) with respect to $x$ and the application of equations (5.11) and (5.12) result in that

$$2x\xi^1_{tt} = 0$$

whence

$$\xi^1_{tt} = 0 \tag{5.17}$$

Thus we have that

$$\xi^1 = at + b \tag{5.18}$$

which can be expressed using Manale's formula with infinitesimal $\omega$ as

$$
\begin{aligned}
&\xi^1 = \frac{a \ \sin(\frac{\omega t}{i}) + b\phi \ \cos(\frac{\omega t}{i})}{-i\omega} \\
&\text{where} \quad \phi = \sin(\frac{\omega}{i}) \quad \text{and} \quad a = a(x), b = b(x).
\end{aligned}
\tag{5.19}
$$

We differentiate equation (5.19) with respect to $t$ and obtain expressions for $\xi^1_t$, and $\xi^1_{tt}$

$$\xi^1_t = a\ \cos(\frac{\omega t}{i}) - b\phi\ \sin(\frac{\omega t}{i}), \tag{5.20}$$

$$\xi^1_{tt} = \frac{-\omega}{i}\ a\ \sin(\frac{\omega t}{i}) - \frac{\omega}{i}\ b\ \phi\ \cos(\frac{\omega t}{i}), \tag{5.21}$$

Similarly we differentiate defining equation (5.13) with respect to $x$ and obtain

$$(x\xi^2)_x = -x\xi^1_t \tag{5.22}$$

We integrate equation (5.22) with respect to $x$ and simplify to obtain the expression for $\xi^2$, given as

$$\xi^2 = -\frac{1}{2}x\xi^1_t + A \tag{5.23}$$

which translate to

$$\xi^2 = -\frac{1}{2}ax\ \cos(\frac{\omega t}{i}) + \frac{1}{2}bx\phi\ \sin(\frac{\omega t}{i}) + A. \tag{5.24}$$

The equation (5.16) imply that

$$f_{tt} = (\frac{1}{t}\xi^1_t)_t - (\frac{1}{t}\xi^1)_t - x^2\xi^1_{tt}$$

which translate to

$$\begin{aligned} f_{tt} = &-\frac{\omega a}{it}\ \sin(\frac{\omega t}{i}) - \frac{\omega b}{it}\phi\ \cos(\frac{\omega t}{i}) - \frac{a}{t^2}\cos(\frac{\omega t}{i}) + \frac{b\phi}{t^2}\sin(\frac{\omega t}{i}) \\ &- \frac{a}{t}\ \cos(\frac{\omega t}{i}) + \frac{b\phi}{t}\ \sin(\frac{\omega t}{i}) - \frac{a}{it^2\omega}\ \sin(\frac{\omega t}{i}) - \frac{b\phi}{it^2\omega}\ \cos(\frac{\omega t}{i}) \\ &+ \frac{ax^2\omega}{i}\ \sin(\frac{\omega t}{i}) + \frac{b\phi x^2\omega}{i}\ \cos(\frac{\omega t}{i}) \end{aligned} \tag{5.25}$$

The integration of equation (5.25) results in the expression for $f_t$ and $f$ given as

$$\begin{aligned} f_t = &\frac{a}{t}\ \cos(\frac{\omega t}{i}) - \frac{b}{t}\phi\ \sin(\frac{\omega t}{i}) - \frac{ai}{\omega t^2}\sin(\frac{\omega t}{i}) - \frac{bi\phi}{\omega t^2}\cos(\frac{\omega t}{i}) \\ &- \frac{ia}{\omega t}\ \sin(\frac{\omega t}{i}) - \frac{ib\phi}{t\omega}\ \cos(\frac{\omega t}{i}) + \frac{a}{\omega^2 t^2}\ \cos(\frac{\omega t}{i}) - \frac{b\phi}{\omega^2 t^2}\ \sin(\frac{\omega t}{i}) \\ &- ax^2\ \cos(\frac{\omega t}{i}) + b\phi x^2\ \sin(\frac{\omega t}{i}) \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} f = &\frac{ai}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{ib}{t\omega}\phi\ \cos(\frac{\omega t}{i}) - \frac{a}{\omega^2 t^2}\cos(\frac{\omega t}{i}) + \frac{b\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) \\ &- \frac{a}{\omega^2 t}\ \sin(\frac{\omega t}{i}) - \frac{ib\phi}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{ia}{\omega^3 t^2}\ \sin(\frac{\omega t}{i}) + \frac{ib\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i}) \\ &- \frac{iax^2}{\omega}\ \sin(\frac{\omega t}{i}) - \frac{bi\phi x^2}{\omega}\ \cos(\frac{\omega t}{i}) + B \end{aligned} \tag{5.27}$$

respectively. From the defining equation (5.11) we have that

$$\xi^1_x = \frac{\dot{a}\ \sin(\frac{\omega t}{i}) + \dot{b}\phi\ \cos(\frac{\omega t}{i})}{-i\omega} = 0 \tag{5.28}$$

This results in that

$$\begin{aligned} &\dot{a} = 0 \quad \text{and} \quad \dot{b} = 0 \\ &\text{Hence} \quad a = C_1 \quad \text{and} \quad b = C_2 \end{aligned} \tag{5.29}$$

The defining equation (5.12) $f_x = 0$ imply that the last terms of $f$ i.e.

$$-\frac{iax^2}{\omega}\ \sin(\frac{\omega t}{i}) - \frac{bi\phi x^2}{\omega}\ \cos(\frac{\omega t}{i}) = 0 \tag{5.30}$$

### 5.1.2 Infinitesimals

The linearly independent solutions of the defining equations (5.10) to (5.15) result in the infinitesimals

$$\xi^1 = -C_1\frac{\sin(\frac{\omega t}{i})}{i\omega} - C_2\frac{\cos(\frac{\omega t}{i})}{i\omega} \tag{5.31}$$

$$\xi^2 = -\frac{1}{2}C_1 x\cos(\frac{\omega t}{i}) + \frac{1}{2}\phi C_2 x\sin(\frac{\omega t}{i}) + A \tag{5.32}$$

$$\begin{aligned} f = &\frac{C_1 i}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{iC_2}{t\omega}\phi\ \cos(\frac{\omega t}{i}) - \frac{C_1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) \\ &+ \frac{C_2\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) - \frac{C_1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) - \frac{iC_2\phi}{t\omega}\ \sin(\frac{\omega t}{i}) \\ &+ \frac{iC_1}{\omega^3 t^2}\ \sin(\frac{\omega t}{i}) + \frac{iC_2\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i}) + B \end{aligned} \tag{5.33}$$

### 5.1.3 Symmetries

The symmetries according to infinitesimals (5.31) to (5.33) are:

$$\begin{aligned} X_1 = &-\frac{\sin(\frac{\omega t}{i})}{i\omega}\frac{\partial}{\partial t} - \frac{1}{2}x\cos(\frac{\omega t}{i})\frac{\partial}{\partial x} \\ &+ \{\frac{i}{t\omega}\ \sin(\frac{\omega t}{i}) - \frac{1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) \\ &- \frac{1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) + \frac{i}{\omega^3 t^2}\ \sin(\frac{\omega t}{i})\}u\frac{\partial}{\partial u} \end{aligned} \tag{5.34}$$

$$
\begin{aligned}
X_2 = & -\phi\frac{\cos(\frac{\omega t}{i})}{i\omega}\frac{\partial}{\partial t}+\frac{1}{2}x\phi\sin(\frac{\omega t}{i})\frac{\partial}{\partial x} \\
& +\{\frac{i}{t\omega}\phi\ \cos(\frac{\omega t}{i})+\frac{\phi}{\omega^2t^2}\sin(\frac{\omega t}{i}) \\
& -\frac{i\phi}{t\omega}\ \sin(\frac{\omega t}{i})+\frac{i\phi}{\omega^3t^2}\ \cos(\frac{\omega t}{i})\}u\frac{\partial}{\partial u}
\end{aligned} \tag{5.35}
$$

$$
X_3 = \frac{\partial}{\partial x} \tag{5.36}
$$

$$
X_4 = u\frac{\partial}{\partial u} \tag{5.37}
$$

The function $g(t,x)$ could not be determined and thus leads to an infinite symmetry generator

$$
X_\infty = g(t,x)u\frac{\partial}{\partial u} \tag{5.38}
$$

### 5.1.4 Invariant Solutions

**Invariant solution through the symmetry $X_2$**

We consider the symmetry given by equation (5.32). The invariants are determined from solving the equation

$$
\begin{aligned}
X_2I = & -\phi\frac{\cos(\frac{\omega t}{i})}{i\omega}\frac{\partial I}{\partial t}+\frac{1}{2}x\phi\sin(\frac{\omega t}{i})\frac{\partial I}{\partial x} \\
& +\{\frac{i}{t\omega}\phi\ \cos(\frac{\omega t}{i})+\frac{\phi}{\omega^2t^2}\sin(\frac{\omega t}{i}) \\
& -\frac{i\phi}{t\omega}\ \sin(\frac{\omega t}{i})+\frac{i\phi}{\omega^3t^2}\ \cos(\frac{\omega t}{i})\}u\frac{\partial I}{\partial u}=0
\end{aligned} \tag{5.39}
$$

The characteristic equation of (5.39) is given by

$$
\begin{aligned}
& -\frac{dt}{\phi\frac{\cos(\frac{\omega t}{i})}{i\omega}}=\frac{dx}{\frac{1}{2}x\phi\sin(\frac{\omega t}{i})} \\
& =\frac{du}{u\{\frac{i}{t\omega}\phi\ \cos(\frac{\omega t}{i})+\frac{\phi}{\omega^2t^2}\sin(\frac{\omega t}{i})-\frac{i\phi}{t\omega}\ \sin(\frac{\omega t}{i})+\frac{i\phi}{\omega^3t^2}\ \cos(\frac{\omega t}{i})\}}
\end{aligned} \tag{5.40}
$$

From equation (5.40) we have that

$$
-\frac{dt}{\phi\frac{\cos(\frac{\omega t}{i})}{i\omega}}=\frac{dx}{\frac{1}{2}x\phi\sin(\frac{\omega t}{i})} \tag{5.41}
$$

simplifies to

$$\frac{2}{x}dx = -\omega i \tan(\frac{\omega t}{i})dt \tag{5.42}$$

The solution to equation (5.42) is given by

$$C + 2\ln x = -\ln\cos|(\frac{\omega t}{i})| \tag{5.43}$$

for an arbitrary function C, which results in that the first invariant is given by

$$C_1 = x^2\cos(\frac{\omega t}{i}) \tag{5.44}$$

Also from equation (5.40) we have that

$$\begin{aligned} &-i\omega\frac{dt}{\phi\cos(\frac{\omega t}{i})}\{\frac{i}{t\omega}\phi\ \cos(\frac{\omega t}{i}) + \frac{\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) \\ &-\frac{i\phi}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{i\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i})\} = \frac{du}{u} \end{aligned} \tag{5.45}$$

We simplify left hand side of equation (5.45) by multiplying through by $\frac{-i\omega}{\cos(\frac{\omega t}{i})}$, and for smaller value of $\omega$ we have the approximation

$$dt\{\frac{1}{t} - \frac{1}{t^2} - 0 + \frac{1}{t^2}\} = \frac{du}{u}$$

Hence the equation becomes

$$\frac{dt}{t} = \frac{du}{u} \tag{5.46}$$

The solution to equation (5.46) is

$$\frac{u}{t} = C_2 \tag{5.47}$$

Since $C_1$ is independent of $u$, every invariant solution is of the form

$$\frac{u}{t} = F(x^2\cos(\frac{\omega t}{i})) \tag{5.48}$$

or equivalently

$$u = tF(x^2\cos(\frac{\omega t}{i})) \tag{5.49}$$

Differentiating equation (5.49) we obtain

$$u_x = 2xtF'\cos(\frac{\omega t}{i}) \tag{5.50}$$

$$u_{xt} = 2xF'\cos(\frac{\omega t}{i}) - 2xt\frac{\omega}{i}F'\sin(\frac{\omega t}{i}) - x^3t\frac{\omega}{i}F''\sin(\frac{2\omega t}{i}) \tag{5.51}$$

We substitute for equations (5.50) and (5.51) in equation (5.4) and obtain

$$\begin{aligned} &2xF'\cos(\frac{\omega t}{i}) - 2xt\frac{\omega}{i}F'\sin(\frac{\omega t}{i}) - x^3t\frac{\omega}{i}F''\sin(\frac{2\omega t}{i}) \\ &- 2xF'\cos(\frac{\omega t}{i}) + 2x^3tF'\cos(\frac{\omega t}{i}) = 0 \end{aligned} \tag{5.52}$$

If we let $\omega \to 0$ equation (5.52) simplifies to

$$2x^3tF' = 0$$

or

$$F' = 0 \tag{5.53}$$

Hence

$$F = A \tag{5.54}$$

The solution is given by

$$u = At \tag{5.55}$$

where $A$ is a constant.

### Invariant solution through the symmetry $X_1$

We consider the symmetry given by equation (5.31). The invariants are determined from solving the equation

$$\begin{aligned} X_1 I = &-\frac{\sin(\frac{\omega t}{i})}{i\omega}\frac{\partial I}{\partial t} - \frac{1}{2}x\cos(\frac{\omega t}{i})\frac{\partial I}{\partial x} \\ &+ \{\frac{i}{t\omega}\ \sin(\frac{\omega t}{i}) - \frac{1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) \\ &- \frac{1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) + \frac{i}{\omega^3 t^2}\ \sin(\frac{\omega t}{i})\}u\frac{\partial I}{\partial u} = 0 \end{aligned} \tag{5.56}$$

The characteristic equation of (5.56) is given by

$$\begin{aligned} &-\frac{dt}{\frac{\sin(\frac{\omega t}{i})}{i\omega}} = \frac{dx}{-\frac{1}{2}x\cos(\frac{\omega t}{i})} \\ &= \frac{du}{u\{\frac{i}{t\omega}\ \sin(\frac{\omega t}{i}) - \frac{1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) - \frac{1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) + \frac{i}{\omega^3 t^2}\ \sin(\frac{\omega t}{i})\}} \end{aligned} \tag{5.57}$$

From equation (5.57) we have that

$$-\frac{dt}{\frac{\sin(\frac{\omega t}{i})}{i\omega}} = \frac{dx}{-\frac{1}{2}x\cos(\frac{\omega t}{i})} \tag{5.58}$$

simplifies to

$$\frac{2}{x}dx = \omega i \cot(\frac{\omega t}{i})dt \tag{5.59}$$

The solution to equation (5.59) is given by

$$A + 2\ln x = -\ln\sin|(\frac{\omega t}{i})| \tag{5.60}$$

which result in that the first invariant is given by

$$A_1 = x^2 \sin(\frac{\omega t}{i}) \tag{5.61}$$

Also from equation (5.57) we have that

$$\begin{aligned} &- i\omega \frac{dt}{\sin(\frac{\omega t}{i})}\{\frac{i}{t\omega}\ \sin(\frac{\omega t}{i}) - \frac{1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) \\ &- \frac{1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) + \frac{i}{\omega^3 t^2}\ \sin(\frac{\omega t}{i})\} = \frac{du}{u} \end{aligned} \tag{5.62}$$

We simplify left hand side of equation (5.62) by multiplying through by $\frac{-i\omega}{\cos(\frac{\omega t}{i})}$, and for smaller value of $\omega$ we have the approximation

$$dt\{\frac{1}{t} - \frac{1}{t^2} - 0 + \frac{1}{t^2}\} = \frac{du}{u}$$

Hence the equation becomes

$$\frac{dt}{t} = \frac{du}{u} \tag{5.63}$$

The solution to equation (5.63) is

$$\frac{u}{t} = A_2 \tag{5.64}$$

Since $A_1$ is independent of $u$, every invariant solution is of the form

$$\frac{u}{t} = F(x^2\sin(\frac{\omega t}{i})) \tag{5.65}$$

or equivalently

$$u = tF(x^2\sin(\frac{\omega t}{i})) \tag{5.66}$$

Differentiating equation (5.66) we obtain

$$u_x = 2xtF'\sin(\frac{\omega t}{i}) \tag{5.67}$$

$$u_{xt} = 2xF'\sin(\frac{\omega t}{i}) + 2xt\frac{\omega}{i}F'\cos(\frac{\omega t}{i}) + x^3t\frac{\omega}{i}F''\sin(\frac{2\omega t}{i}) \tag{5.68}$$

We substitute for equations (5.67) and (5.68) in equation (5.4) and obtain

$$\begin{aligned} &2xF' \sin(\frac{\omega t}{i}) + 2xt\frac{\omega}{i}F' \cos(\frac{\omega t}{i}) + x^3 t\frac{\omega}{i}F'' \sin(\frac{2\omega t}{i}) \\ &- 2xF' \sin(\frac{\omega t}{i}) + 2x^3 tF' \sin(\frac{\omega t}{i}) = 0 \end{aligned} \tag{5.69}$$

If we let $\omega \to 0$ in equation (5.69) we get no solution.

### Invariant solution through the symmetry $X_3$

The invariant solution through symmetry $X_3 = \frac{\partial}{\partial x}$ yields that

$$u = H(t) \tag{5.70}$$

where $H(t)$ denotes some function of $t$, consistent with equation (5.71) The method produced symmetries which provided a linear invariant solutions. This is consistent with the result in [34] that

$$\int_0^\infty e^{-tx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}, \quad t > 0 \tag{5.71}$$

## 5.2 Symmetries in the epidemiology of HIV and AIDS

In the paper [35], Torrisi and Nucci apply Lie group analysis to a seminal model given by Anderson [2], which describes HIV transmission in male homosexual/bisexual cohorts. A technique introduced by Lie [**?**]. The equations have the form

$$\frac{dv_1}{dt} = \frac{-\beta_0 c v_1 v_2}{v_1 + v_2 + v_3} - \mu_0 v_1, \tag{5.72}$$

$$\frac{dv_2}{dt} = \frac{\beta_0 c v_1 v_2}{v_1 + v_2 + v_3} - (\nu + \mu_0) v_2, \tag{5.73}$$

$$\frac{dv_3}{dt} = \nu v_2 - \alpha v_3, \tag{5.74}$$

Here, the parameter $\mu_0$ is the per capita natural death rate of both susceptibles and infecteds, and $\alpha$ is the AIDS-related death rate. The term $\lambda$ is the per capita force of infection and is defined as:

$$\lambda = \frac{\beta_0 c v_2}{v_1 + v_2 + v_3},$$

where $\beta_0$ is the average probability that an infected individual will infect a susceptible partner over the duration of their relationship, and $c$ is the effective rate of partner change within the specified risk category. The dependent variables $v_1$, $v_2$, and $v_2$ divide the population into the population at time $t$ susceptibles (HIV negatives), infecteds (HIV positives), and AIDS patients, respectively.

Torrisi and Nucci's application of Lie group analysis to the system 5.72-5.74, led to the solutions

$$\begin{aligned}
v_1 &= \frac{e^{\nu t} c_2}{e^{\mu_0 s}\left[e^{\nu t}\left(\beta_0 c - \nu\right) c_1 + e^{\beta_0 c t}\beta_0 c\right]}\\
v_2 &= \frac{\left(\beta_0 c - \nu\right) I \beta_0 c c_2 + c_3}{e^{\mu_0 s + \nu t}}\\
v_3 &= \frac{\left[e^{\nu t}\left(\beta_0 c - \nu\right) c_1 + e^{\beta_0 c t}\beta_0 c\right]\left[e^{\nu t}\left(\beta_0 c - \nu\right) c_1 + e^{\beta_0 c t}\nu\right]}{e^{\beta_0 c t + \mu_0 t + \nu t}\left[e^{\nu t}\left(\beta_0 c - \nu\right) c_1 + e^{\beta_0 c t}\beta_0 c\right]\left(\beta_0 c - \nu\right)}\\
&+ \frac{\left(\beta_0 c - \nu\right) I \beta_0 c c_2 + c_3}{e^{\beta_0 c t + \mu_0 t + \nu t}\left[e^{\nu t}\left(\beta_0 c - \nu\right) c_1 + e^{\beta_0 c t}\beta_0 c\right]\left(\beta_0 c - \nu\right)}\\
&- \frac{e^{nut} c_2}{e^{\mu_0 t}\left[e^{\nu t}\left(\beta_0 c - \nu\right) c_1 + e^{\beta_0 c t}\beta_0 c\right]}
\end{aligned}$$

with

$$\int \frac{e^{\beta_0 c t + 2\nu t}}{\left(e^{\beta_0 c t}\beta_0 c + e^{\nu t}\beta_0 c c_1 - e^{\nu s} c_1 \nu\right)^2} dt. \tag{5.75}$$

Torrisi and Nucci noticed that if $\beta_0 c = 2\nu$ these solutions, constituting the general solution, assume a simpler form. But there is a problem. Other than the fact that this condition reduces the solutions to an integrable form, there is nothing else that does, meaning we have to patiently wait for experimental data to fit this condition, otherwise it cannot be supported. This then brings up to the purpose of our contribution. That being to integrate (5.75).

## 5.3 Integrating (5.75)

The integral (5.75) is a special case of

$$u = \int e^{sH} dt, \tag{5.76}$$

for $s = 1$ and

$$H = \frac{1}{s} \ln \left( \frac{e^{\beta_0 ct + 2\nu t}}{\left(e^{\beta_0 ct}\beta_0 c + e^{\nu t}\beta_0 cc_1 - e^{\nu t}c_1\nu\right)^2} \right), \tag{5.77}$$

However, it is well known that the Gaussian antiderivative cannot be established through quadrature. Our approach is to reduce this integral into a second order differential equation. Because such equations have more than one solution, the missing antiderivative can then be generated through transformations from the other solutions.

A simple differential equation possible from (5.76) is

$$u_{st} = Hu_t, \tag{5.78}$$

a partial differential equation with the dependent variable $u = u(s,t)$, depending on $s$, and $t$. The antiderivative then should result from this equation with $t$ assuming specific values. This we now solve through symmetries.

Traditional Lie symmetries do not assist much in this regard, What we do differently, is to introduce an infinitesimal complex parameter

$$\mu = \frac{\omega}{i}$$

into Lie's method. This parameter, discussed extensively in [23] and [28], facilitates evaluations through quadrature by invoking L'hopital's principle.

### 5.3.1 Applying the Lie symmetry generator to (5.78)

In order to generate point symmetries for equation (5.78), we first consider a change of variables from $s, t$, and $u$ to $s^*, t^*$, and $u^*$ involving an infinitesimal parameter $\epsilon$.

A Taylor's series expansion in $\epsilon$ near $\epsilon = 0$ yields

$$\left.\begin{array}{rcl} s^* & \approx & s + \epsilon S(s,t,u) \\ t^* & \approx & t + \epsilon \xi(s,t,u) \\ u^* & \approx & u + \epsilon \zeta(s,t,u) \end{array}\right\} \tag{5.79}$$

where

$$\left.\begin{array}{rcl} \frac{\partial s^*}{\partial \epsilon}|_{\epsilon=0} & = & S(s,t,u) \\ \frac{\partial t^*}{\partial \epsilon}|_{\epsilon=0} & = & \xi(s,t,u) \\ \frac{\partial u^*}{\partial \epsilon}|_{\epsilon=0} & = & \zeta(s,t,u) \end{array}\right\}. \tag{5.80}$$

The tangent vector field (5.80) is associated with an operator

$$X = S\frac{\partial}{\partial s} + \xi\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial u}, \tag{5.81}$$

called a symmetry generator. This in turn leads to the invariance condition

$$X^{[2]}\left(u_{st} - t^2 u_t\right)|_{\{u_{st}=t^2 u_t\}} = 0, \tag{5.82}$$

where $X^{[2]}$ is the second prolongation of $X$. It is obtained from the formulas:

$$\begin{array}{rcl} X^{[2]} & = & X + \zeta_s^{(1)}\frac{\partial}{\partial u_s} + \zeta_t^{(1)}\frac{\partial}{\partial u_t} + \zeta_{ss}^{(2)}\frac{\partial}{\partial u_{ss}} \\ & & + \zeta_{st}^{(2)}\frac{\partial}{\partial u_{st}} + \zeta_{tt}^{(2)}\frac{\partial}{\partial u_{tt}}, \end{array}$$

where

$$\zeta_s^{(1)} = \frac{\partial g}{\partial s} + u\frac{\partial f}{\partial s} + \left[f - \frac{\partial S}{\partial s}\right]u_s - \frac{\partial \xi}{\partial t}u_t, \tag{5.83}$$

$$\zeta_t^{(1)} = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial \xi}{\partial t}\right]u_t - \frac{\partial S}{\partial s}u_s, \tag{5.84}$$

$$\begin{array}{rcl} \zeta_{ss}^{(2)} & = & \frac{\partial^2 g}{\partial s^2} + u\frac{\partial^2 f}{\partial s^2} + \left[2\frac{\partial f}{\partial s} - \frac{\partial^2 S}{\partial s^2}\right]u_s - \frac{\partial^2 \xi}{\partial s^2}u_t \\ & & + \left[f - 2\frac{\partial S}{\partial s}\right]u_{ss} - 2\frac{\partial \xi}{\partial s}u_{st}, \end{array}$$

$$\begin{array}{rcl} \zeta_{tt}^{(2)} & = & \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t^2}\right]u_t - \frac{\partial^2 S}{\partial t^2}u_s \\ & & + \left[f - 2\frac{\partial s}{\partial t}\right]u_{tt} - 2\frac{\partial s}{\partial t}u_{tt}, \end{array}$$

and

$$\begin{array}{rcl} \zeta_{st}^{(2)} & = & \frac{\partial^2 g}{\partial s \partial t} + u\frac{\partial^2 f}{\partial s \partial t} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 S}{\partial s \partial t}\right]u_s \\ & & + \left[2\frac{\partial f}{\partial s} - \frac{\partial^2 \xi}{\partial s \partial t}\right]u_t - \left[f - \frac{\partial S}{\partial s} - \frac{\partial \xi}{\partial t}\right]u_{st} \\ & & - \frac{\partial S}{\partial t}u_{ss} - \frac{\partial \xi}{\partial s}u_{tt}. \end{array}$$

It is to be understood here that the simplification $\zeta(s,t,u) = uf(s,t) + g(s,t)$. The invariance condition (5.82) then leads to the equation

$$\begin{aligned}
&\frac{\partial^2 g}{\partial s\partial t} + u\frac{\partial^2 f}{\partial s\partial t} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 S}{\partial s\partial t}\right]u_s \\
&+\left[2\frac{\partial f}{\partial s} - \frac{\partial^2 \xi}{\partial s\partial t}\right]u_t - \left[f - \frac{\partial S}{\partial s} - \frac{\partial \xi}{\partial t}\right]u_{st} \\
&-\frac{\partial S}{\partial t}u_{ss} - \frac{\partial \xi}{\partial s}u_{tt} - 2t\xi u_t \\
&-t^2\left(\frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial \xi}{\partial t}\right]u_t - \frac{\partial S}{\partial s}u_s\right) = 0,
\end{aligned}$$

called determining equation, from which follows the monomials

$$\left.\begin{array}{ccc}
1 & : & g_{st} - Hg_t = 0, \\
u & : & f_{st} - Hf_t = 0, \\
u_s & : & 2f_t - S_{st} + HS_t = 0, \\
u_{ss} & : & S_t = 0, \\
u_{tt} & : & \xi_s = 0, \\
u_t & : & 2f_s - \xi_{st} \\
 & + & (-2f + S_s + \xi_t)H \\
 & + & H\xi_t - \xi H_t = 0.
\end{array}\right\} \tag{5.85}$$

called the defining equations.

The second equation in (5.85) leads to

$$\frac{1}{f_t}\frac{\partial f_t}{\partial s} = H, \tag{5.86}$$

so that

$$\ln f_t = Hs + \alpha, \tag{5.87}$$

or

$$\ln f_t = \frac{\alpha \sin\mu \cos(\mu s) + H\sin(\mu s)}{\mu}. \tag{5.88}$$

That is,

$$f = \int \exp\left(\frac{\alpha \sin\mu \cos(\mu s) + H\sin(\mu s)}{\mu}\right)dt + a. \tag{5.89}$$

Similarly,

$$g = \int \exp\left(\frac{\beta \sin\mu \cos(\mu s) + H\sin(\mu s)}{\mu}\right)dt + d. \tag{5.90}$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\mu = 0$, $a = a(s)$, and $d = d(s)$.

The third equation in (5.85) leads to

$$HS_s = -2f_t. \tag{5.91}$$

Substituting (5.88) into (5.91) gives

$$HS_s = -2\exp\left(\frac{\alpha\sin\mu\cos(\mu s) + H\sin(\mu s)}{\mu}\right). \tag{5.92}$$

That is,

$$S = \int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H} ds + A_0, \tag{5.93}$$

where $A_0$ is a constant.

The sixth equation in (5.85) leads to

$$\xi_t - \frac{H_t}{2H}\xi = \frac{2\left(f_t - f_s\right)}{H}. \tag{5.94}$$

Hence,

$$\xi = \sqrt{H}\int \frac{2\left(f_t - f_s\right)}{H\sqrt{H}} dt + B_0\sqrt{H}, \tag{5.95}$$

and

$$a = C_0 t + C_1 \tag{5.96}$$

where $B_0, C_0$, and $C_1$ are constants.

### 5.3.2 Infinitesimals

The linearly independent solutions of the defining equations (5.85) lead to the infinitesimals

$$S = \int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H}dt + A_0, \tag{5.97}$$

$$\xi = \sqrt{H}\int \frac{2\left(f_t - f_s\right)}{H\sqrt{H}}dt + B_0\sqrt{H}, \tag{5.98}$$

and

$$\begin{aligned} \zeta &= \int \exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)dt + a \\ &+ u\int \exp\left(\frac{\beta\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)dt \\ &+ ud. \end{aligned} \tag{5.99}$$

### 5.3.3 The symmetries

According to (5.81), the infinitesimals: (5.97), (5.98), and (5.99), lead to the generators

$$\begin{aligned} X_1 &= \left[\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H}ds\right]\frac{\partial}{\partial s} \\ &+ \left[\sqrt{H}\int \frac{2\left(f_t - f_s\right)}{H\sqrt{H}}dt\right]\frac{\partial}{\partial t} \\ &+ \left[\int \exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)dt\right]\frac{\partial}{\partial u}, \\ X_2 &= \frac{\partial}{\partial s}, \\ X_3 &= \sqrt{H}\frac{\partial}{\partial t}, \end{aligned} \tag{5.100}$$

and

$$
\begin{aligned}
X_4 &= \frac{\partial}{\partial u},\\
X_5 &= s\frac{\partial}{\partial u}.
\end{aligned}
$$

The functions $\beta = \beta(s)$, and $g = g(s,t)$ cannot be determined, as such, it leads to an infinite symmetry generator. That is,

$$
\begin{aligned}
X_\infty &= u\left[\int \exp\left(\frac{\beta \sin\mu\cos(\mu s) + H\sin(\mu s)}{\mu}\right) dt\right]\frac{\partial}{\partial u}\\
&+ d(s)u\frac{\partial}{\partial u}.
\end{aligned}
$$

### 5.3.4 Construction of invariant solutions

#### Invariant solutions through the symmetry $X_1$

The characteristic equations that arise from the symmetry $X_1$ :

$$
\begin{aligned}
&\frac{ds}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H} ds}\\
=\ &\frac{dt}{\sqrt{H}\int \frac{2(f_t - f_s)}{H\sqrt{H}} dt}\\
&= \frac{du}{\int \exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right) dt}. \qquad (5.101)
\end{aligned}
$$

The equation

$$
\begin{aligned}
&\frac{ds}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H} ds}\\
=\ &\frac{dt}{\sqrt{H}\int \frac{2(f_t - f_s)}{H\sqrt{H}} dt}, \qquad (5.102)
\end{aligned}
$$

arising from it, leads to the invariant $\eta$:

$$
\eta = t - \int\left(\frac{\sqrt{H}\int \frac{2(f_t - f_s)}{H\sqrt{H}} dt}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H} ds}\right) ds. \qquad (5.103)
$$

The other component:

$$\frac{ds}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H}ds} = \frac{du}{\int \exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)dt}, \tag{5.104}$$

leads to the invariant $\phi$, contained in the expression

$$u = \phi + \int \left( \frac{\int \exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)dt}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H}ds} \right) dt. \tag{5.105}$$

The invariant $\eta$, leads to

$$\eta_t = 1 - \frac{\partial}{\partial t}\int \left( \frac{\sqrt{H}\int \frac{2(f_t-f_s)}{H\sqrt{H}}dt}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H}ds} \right) ds. \tag{5.106}$$

$$\eta_s = -\frac{\sqrt{H}\int \frac{2(f_t-f_s)}{H\sqrt{H}}dt}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H}ds}, \tag{5.107}$$

and

$$\eta_{st} = -\frac{\partial}{\partial t}\left( \frac{\sqrt{H}\int \frac{2(f_t-f_s)}{H\sqrt{H}}dt}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H}ds} \right). \tag{5.108}$$

while $\phi$ gives

$$u_t = \dot{\phi}\eta_t + \psi_t, \tag{5.109}$$

$$u_{st} = \ddot{\phi}\eta_t\eta_s + \dot{\phi}\eta_{ts} + \psi_{ts}, \tag{5.110}$$

and

$$u_s = \dot{\phi}\eta_s + \psi_s, \tag{5.111}$$

with

$$\psi = \int \left( \frac{\int \exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)dt}{\int \frac{-2\exp\left(\frac{\alpha\sin\mu\cos(\mu s)+H\sin(\mu s)}{\mu}\right)}{H}ds} \right) ds. \tag{5.112}$$

Now substituting (5.109), (5.110) into (5.78) gives

$$\ddot{\phi}\eta_t\eta_t + \dot{\phi}\left(\eta_{ts} - H\eta_t\right) = H\psi_t - \psi_{ts}. \tag{5.113}$$

The function $\alpha$ is determined here by requiring $\eta_s = 0$, leading to the solution

$$u = \psi - \int \left( \frac{\psi_{ts} - H\psi_t}{\eta_{ts} - H\eta_t} \right) d\eta + \phi_0. \tag{5.114}$$

The function $\alpha$ is

$$\begin{aligned} &\alpha = \\ &e^{-\frac{t\mu\sin(\mu s)}{\cos(\mu s)}} \int \left( \left[ \frac{H\mu}{\sin\mu} - \frac{\dot{H}\sin(\mu s)}{\sin\mu\cos(\mu s)} \right] e^{-\frac{t\mu\sin(\mu s)}{\cos(\mu s)}} \right) d\eta \\ + \quad &\frac{\cos(\mu s)}{\sin(\mu s)} + D_0, \end{aligned} \tag{5.115}$$

where $\phi_0$, and $D_0$ are constants.

**Invariant solutions through the symmetry $X_2$**

The invariance condition from symmetry $X_1$ given as

$$\frac{\partial I}{\partial s} = 0 \tag{5.116}$$

leads to characteristic equation

$$\frac{ds}{1} = \frac{dt}{0} = \frac{du}{0}. \tag{5.117}$$

The characteristic equation (5.117)gives an invariant

$$u = \vartheta(t). \tag{5.118}$$

The substitution of (5.118) in (5.78) leads to that

$$t^2\vartheta'(t) = 0 \tag{5.119}$$

Whence

$$\vartheta(t) = K \tag{5.120}$$

where $K$ is a constant.

### Invariant solutions through the symmetry $X_3$

Similarly the invariance condition from symmetry $X_3$

$$\sqrt{H}\frac{\partial I}{\partial t} = 0 \tag{5.121}$$

leads to characteristic equation

$$\frac{dt}{\sqrt{H}} = \frac{dt}{0} = \frac{du}{0}. \tag{5.122}$$

The invariant $u = \chi(s)$ from (5.122) presents no invariant solution.

The symmetries $X_4$ and $X_5$ do not provide invariant solutions.

# Conclusion

In this project a different method to determine symmetries of one-dimensional and two-dimensional Black-Scholes equations was introduced. The method analyzes the behaviour of functions as the infinitesimal $\omega$ approaches zero. This method provided additional symmetries to the equations compared to the one in [9].

In carrying out the implementation of the method, Black-Scholes equation was transformed and expressed in terms of new variables. A method similar to the one in ([9]) was utilized to transform to the heat equation using Lie equivalence transformation. The symmetries generated from the transformed equation led to two additional operators as compared to the ones in ([9]). We determined an invariant solution using one of the operators and in the case of transformed one-dimensional Black-Scholes equation, further solutions were produced.

The method was further applied successfully on a Gaussian type differential equation, and symmetries for HIV/AIDS model type partial differential equations were determined. Invariant solutions for those operators that provided were worked out.

Future projects on the method would include to calculate all invariant solutions for all operators that provide them and work out an optimal system of subalgebras of the one-dimensional and two-dimensional Black-Scholes equations.

# Bibliography


[1] Adams,C.M., Masebe,T.P., Manale,J.M. Gaussian type differential equation: Proceedings of the 2014 International Conference on Mathematical methods and Simulation in Science and Engineering, Interlaken, Switzerland, Feb. 22 - 24, 2014. 28 - 31.

[2] Anderson,R.M. 1988. The role of Mathematical models in the study of HIV transmission and the epidemiology of AIDS. JAIDS Journal of Acquired Immune Deficiency Syndromes, 1(3):241256.

[3] Baumann, G. 1992. Lie Symmetries of Differential Equations: A mathematical program to determine Lie-Backland Symmetries. MathSource 0204-680. Illinois. Wolfram Research Inc., Champaign.

[4] Øksendal, B. 2003. Stochastic Differential Equations: An introduction with Applications.(6th ed). New York. Springer-Verlag.

[5] Bluman, G. W., Kumei, S. 1989. Symmetries and Differential Equations. New York, Springer-Verlag.

[6] Bluman,G.W., Anco,S.C. 2002. Symmetries and Integration methods for Differential Equations. New York. Springer-Verlag.

[7] Bluman, G. W., Cheviakov,A.F., Anco,S.C. 2010. Application of Symmetries methods to Partial Differential Equations. New York, Springer.

[8] Björk , T. 2004. Arbitrage theory in continuous time. Oxford. Oxford University Press.

[9] Gazizov, R. K., Ibragimov, N.H.1998. Lie Symmetry Analysis of Differential Equations in Finance. Nonlinear Dynamics 17:387 - 407. June.

[10] Gerald, C. F., Wheatley, P. O. 1992. Applied Numerical analysis.(4th ed). New York. Addison-Wesley Publishing Company.

[11] Gerald, C. F., Wheatley, P. O. 2003. Applied Numerical analysis.(7th ed). New York. Addison-Wesley Publishing Company.

[12] Hull, J. 2003. Options, Futures and other Derivatives.(5th ed). New Jersey. Prentice Hall.

[13] Hydon, P.E. 2000. Symmetry methods for Differential Equations. New York. Cambridge University Press .

[14] Ibragimov, N. H.1999. Elementary Lie Group Analysis and Ordinary Differential Equations. London. J. Wiley & Sons Ltd.

[15] Ibragimov, N. H., Kobalev, V.F. 2009. Approximate and Renormgroup Symmetries. Beijing. Springer.

[16] Ibragimov, N. H. 2010. A practical course in Differential Equations and Mathematical modelling. Beijing. Higher Education Press.

[17] Ibragimov, N.H. 2009. Selected Works: Extension of Euler method to parabolic equations,volume IV. Sweden. ALGA Publications.

[18] Ibragimov N. H. *et al* .(Editors). 1995. CRC Handbook of Lie Group Analysis of Differential Equations Vol 2: Applications in Engineering and Physical Sciences. Boca Ratan. CRC Press.

[19] Ibragimov, N.H. (1994). Sophus Lie and the harmony in Mathematical Physics on the 150th anniversary of his birth. Mathematical Intelligencer 16(1), 20-28.

[20] Ibragimov, N.H., Unal, G., Jogreus, C. (2004). Approximate Symmetries and Conservation Laws for Itô and Stratonovich dynamic Systems. Journal of Mathematical Analysis and Application. 297: 152 - 168.

[21] Joshi, M. S. 2003. The concepts and practice of mathematical finance. [Web:]http://books.google.co.za/books.[Date of access: 15 November 2011].

[22] Kishimoto, M. (2008). On the Black-Scholes Equation: Various Derivatives.http://www.stanford.edu/. japrims/Publications/OnBlackScholesEq.pdf

[23] Manale, J. M. New Symmetries of the heat equation and application to thin plate heat conduction:In Metin Demiralp, Constantin Udriste, and Gen Qi Xu, editors, Proceedings of the 2013 International Conference on Mathematics and Computation, Venice, Italy, Sep. 28 - 30, 2013. 90 - 103.

[24] Manale, J. M. 2000. Group classification of the two-dimensional Navier-Stokes-type equations. International journal of non-linear mechanics 35.4 : 627-644.

[25] Manale, J. M. 2014. New solutions of the heat equation. International Journal of Applied Mathematics and Informatics, 8:15-25.

[26] Manale, J. M. 2014. Lie symmetry solutions for the heat and Burgers equations. La Pensee 76.7 : 113-126.

[27] Manale, J. M. 2014. On a formula for the universal gravitational constant G: A classical mechanics approach. La Pensee 76. 10 : 267-275.

[28] Masebe, T .P., Manale,J. M. New Symmetries of Black-Scholes Equation:In Metin Demiralp, Constantin Udriste, and Gen Qi Xu, editors, Proceedings of the 2013 International Conference on Mathematics and Computation, Venice, Italy, Sep. 28 - 30, 2013. 221-231.

[29] Olver, P.J. 1991. Applications of Lie Groups to Differential equations. New York. Springer.

[30] Ömür,U. 2008. An introduction to Computational Finance.London. Imperial College Press.

[31] Ovsiannikov, L. V. 1982. *Group Analysis of Differential Equations*, English translation by Chapovsky Y. and Ames W. F. New York. Academic press.

[32] Silberberg, G. 2001. Derivative Pricing with Symmetry Analysis. http://www.econ.ceu.hu/download/thesis/Thesis-Silberberg.pdf

[33] Stephani, H. 1989. *Differential Equations*: their solutions using symmetries, Cambridge. Cambridge Press.

[34] Wikipedia Foundation. 2013. List of integrals[Online].en.wikipedia.org Available from http://www.wikipedia.org/wiki/list of integrals [Accessed:07/12/2013]

[35] Torrisi,V., Nucci, M. C. 2001. Application of lie group analysis to a mathematical model which describes HIV transmission. Contemporary Mathematics. 285:1120.

# Appendices

## Appendix A: Manale's formulas and the infinitesimal $\omega$

It is well-known that Lie's group theoretical methods seek to reduce procedures for solving differential equations of any challenging form to simple ones that may also have the form

$$a_0\ddot{y} + b_0\dot{y} + c_0 y \;=\; 0, \tag{5.123}$$

for $y = y(x)$, with parameters $a_0$, $b_0$ and $c_0$. It is also that accepted Euler's formulas are suitable for solving such equations. They are:

$$y = \begin{cases} e^{-\frac{b_0}{2a_0}x}\left(Ae^{-\tilde{\omega}x} + Be^{\tilde{\omega}x}\right), & b_0^2 > 4a_0c_0, \\ A + Bx, & b_0^2 = 4a_0c_0, \\ e^{-\frac{b_0}{2a_0}x}[A\cos(\tilde{\omega}x)] & \\ +Be^{-\frac{b_0}{2a_0}x}[\sin(\tilde{\omega}x)], & b_0^2 < 4a_0c_0 \end{cases} \tag{5.124}$$

where $\tilde{\omega} = \sqrt{b_0^2 - 4a_0c_0}/(2a_0)$.

But there is a problem with this system: It does not reduce to $y = A + Bx$ when $b_0 = c_0 = 0$. This is because Euler did not solve the equation to get the formulas. There has never been a need to do so, primarily because the formulas have been very successful in applications, and they still are.

The need for an exact solution here, is driven by the desire to understand solutions for equation (5.123) through symmetry methods. It is impossible through Euler's formulas. To get such exact formula, first let

$$y = \beta z,$$

with $\beta = \beta(x)$ and $z = z(x)$, so that

$$\dot{y} = \dot{\beta} z + \beta \dot{z},$$

and

$$\ddot{y} = \ddot{\beta} z + 2\dot{\beta}\dot{z} + \beta \ddot{z}.$$

These transform (5.123) into

$$a_0 \left( \ddot{\beta} z + 2\dot{\beta}\dot{z} + \beta\ddot{z} \right) + b_0 \left( \dot{\beta} z + \beta \dot{z} \right) + c_0 \beta z \ = \ 0.$$

That is,

$$a_0 \beta \ddot{z} + \left( 2a_0 \dot{\beta} + b_0 \beta \right) \dot{z} + \left( a_0 \ddot{\beta} + b_0 \dot{\beta} + c_0 \beta \right) z \ = \ 0. \tag{5.125}$$

Choosing $\beta$ to satisfy $2a_0\dot{\beta} + b_0\beta = 0$ simplifies equation (5.125). That is,

$$\beta = C_{00} e^{\frac{-b_0}{2a_0} x},$$

for some constant $C_{00}$. Equation (5.125) assumes the form

$$\ddot{z} \ = \ -\frac{a_0\ddot{\beta} + b_0\dot{\beta} + c_0\beta}{a_0\beta} z.$$

That is,

$$\ddot{z} \ = \ \left( \frac{b_0^2 - 4a_0c_0}{4a_0^2} \right) z.$$

But $\ddot{z}$ can be written as $\dot{z}dz/dx$. Therefore,

$$\dot{z}\frac{d\dot{z}}{dz} \ = \ \left( \frac{b_0^2 - 4a_0c_0}{4a_0^2} \right) z,$$

or

$$\dot{z}d\dot{z} \ = \ \left( \frac{b_0^2 - 4a_0c_0}{4a_0^2} \right) zdz.$$

That is,

$$\frac{\dot{z}^2}{2} \ = \ \left( \frac{b_0^2 - 4a_0c_0}{4a_0^2} \right) \frac{z^2}{2} + C_{01},$$

for some constant $C_{01}$. That is,

$$\dot{z} \;=\; \sqrt{\left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)\frac{z^2}{2}+C_{01}},$$

or

$$\frac{dz}{\sqrt{\left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)z^2+2C_{01}}}=dx.$$

That is,

$$\frac{dz}{\sqrt{A_{00}^2-z^2}}=\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}\;dx,$$

with $A_{00}^2=2C_{01}/\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}$. Hence,

$$z=\frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}}\;\sin\left(\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}\;x+C_{02}\right),$$

for some constant $C_{02}$. That is,

$$y=C_{00}e^{\frac{-b_0}{2a_0}x}\;\frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}}\;\sin\left(\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}\;x+C_{02}\right).$$

Letting

$$\bar{\omega}=\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}$$

we have

$$y=C_{00}e^{\frac{-b_0}{2a_0}x}\;\frac{2C_{01}}{\bar{\omega}}\;\sin\left(\bar{\omega}\;x+C_{02}\right),$$

or

$$y=C_{00}e^{\frac{-b_0}{2a_0}x}2C_{01}[\frac{\sin\left(C_{02}\right)}{\bar{\omega}}\cos\left(\bar{\omega}x\right)+\cos\left(C_{02}\right)\frac{\sin\left(\bar{\omega}\;x\right)}{\bar{\omega}}]$$

A reduction to the trivial case $\ddot{y}=0$ requires that $\sin(C_{02})=C_{03}\sin(\bar{\omega})$ and $\cos(C_{02})=C_{04}\cos(\bar{\omega})$. That is,

$$C_{03}^2+C_{04}^2=1.$$

Hence,

$$\begin{aligned}y&=C_{00}e^{\frac{-b_0}{2a_0}x}2C_{01}[\frac{C_{03}\sin\left(\bar{\omega}\right)}{\bar{\omega}}\cos\left(\bar{\omega}\;x\right)\\&+C_{04}\cos\left(\bar{\omega}\right)\frac{\sin\left(\bar{\omega}\;x\right)}{\bar{\omega}}]\end{aligned}\tag{5.126}$$

or simply

$$y = C_{00} e^{\frac{-b_0}{2a_0} x} 2C_{01} \frac{C_{03} \sin(\bar{\omega}) \cos(\bar{\omega} x)}{\bar{\omega}} + C_{00} e^{\frac{-b_0}{2a_0} x} 2C_{01} \frac{C_{04} \sin(\bar{\omega}\ x)}{\bar{\omega}}. \tag{5.127}$$

It is very vital to indicate that if the parameters $\bar{\omega}$ in the denominator and $\sin(\bar{\omega})$ are absorbed into the coefficients $C_{01}$ and $C_{03}$, then formula (5.127) would reduce to one of Euler's formulas. But the consequences would be fatal, as formula (5.127) would not reduce to $y = A + Bx$ when $b_0 = c_0 = 0$, that is, when $\bar{\omega} = 0$. Unfortunately, this result cannot be found in any university textbook.

# Appendix B: Useful limit results

It is true that

$$\lim_{\mu \to 0} \left\{ \frac{\sin\left(\frac{\mu t}{i}\right)}{\mu} \right\} = \frac{t}{i}. \tag{5.128}$$

This can be written in the form

$$\lim_{\mu \to 0} \left\{ \frac{\sin\left(\frac{\mu x}{i}\right)}{\mu} - \frac{t}{i} \right\} = 0,$$

or

$$\lim_{\mu \to 0} \left\{ \frac{\sin\left(\frac{\mu t}{i}\right)}{\mu} - \frac{t}{i} \cos\left(\frac{\mu t}{i}\right) \right\} = 0. \tag{5.129}$$

Removing the 'lim' for greater clarity:

$$\frac{\sin\left(\frac{\mu t}{i}\right)}{\mu} = \frac{t}{i} \cos\left(\frac{\mu t}{i}\right).$$

That is,

$$\sin\left(\frac{\mu t}{i}\right) = \frac{t}{i} \mu \cos\left(\frac{\mu t}{i}\right), \tag{5.130}$$

or

$$\cos\left(\frac{\mu t}{i}\right) = \frac{i}{t} \frac{\sin\left(\frac{\mu t}{i}\right)}{\mu}.$$

We then have

$$\frac{\cos\left(\frac{\mu t}{i}\right)}{\mu^q} = \mu \frac{\cos\left(\frac{\mu t}{i}\right)}{\mu^{q+1}}. \tag{5.131}$$

Carrying out the derivative on the right hand side:

$$\frac{\cos\left(\frac{\mu t}{i}\right)}{\mu^q} = \frac{-\mu\left(\frac{t}{i}\right)\sin\left(\frac{\mu t}{i}\right) + \cos\left(\frac{\mu t}{i}\right)}{\mu^{q+1}}. \tag{5.132}$$

Substituting (5.130)

$$\frac{\cos\left(\frac{\mu t}{i}\right)}{\mu^q} = \frac{-\mu^2\left(\frac{t}{i}\right)^2\cos\left(\frac{\mu t}{i}\right) + \cos\left(\frac{\mu t}{i}\right)}{\mu^{q+1}} \tag{5.133}$$

That is,

$$\mu \cos\left(\frac{\mu t}{i}\right) = \mu^2 t^2 \cos\left(\frac{\mu t}{i}\right) + \cos\left(\frac{\mu t}{i}\right), \tag{5.134}$$

which can be expressed in the form

$$\mu^2 \cos\left(\frac{\mu t}{i}\right) - \mu^3 t^2 \cos\left(\frac{\mu t}{i}\right) = \frac{i}{t} \sin\left(\frac{\mu t}{i}\right). \tag{5.135}$$

Since $\sin\left(\frac{\mu t}{i}\right) = 0$ for $\mu$ small, it follows then that

$$\mu^2 \cos\left(\frac{\mu t}{i}\right) = \mu^3 t^2 \cos\left(\frac{\mu t}{i}\right). \tag{5.136}$$

Since $e^{\mu t}$ can be expressed in the form $cos(\mu t/i) + i\sin(\mu t/i)$, then

$$\mu^2 e^{\mu t} = \mu^3 t^2 \cos\left(\frac{\mu t}{i}\right) \tag{5.137}$$

so that

$$\sqrt{\mu} e^{\mu t/4} = \left[\mu^3 \cos\left(\frac{\mu t}{i}\right)\right]^{\frac{1}{4}} \sqrt{t}, \tag{5.138}$$

or

$$\sqrt{\mu} e^{-\mu t/4} = \left[\mu^3 \cos\left(\frac{\mu t}{i}\right)\right]^{\frac{1}{4}} \sqrt{t}, \tag{5.139}$$

Therefore (5.138) and (5.139) can then be written in the form

$$u = \frac{\sqrt{\mu}}{\left[\mu^3 \cos\left(\frac{\mu t}{i}\right)\right]^{\frac{1}{4}} \sqrt{t}} \phi(\eta), \tag{5.140}$$

with $\mu = \omega^4(\omega^2 - 1)$ in the case of (5.137) and $\mu = \omega^4(\omega^2 + 1)$ for (5.138). That is,

$$u = \frac{1}{\sqrt{(\omega^2 - 1)t}\ \omega^2} \phi(\eta) \tag{5.141}$$

for (5.137), and

$$u = \frac{1}{\sqrt{(\omega^2 + 1)t}\ \omega^2} \phi(\eta) \tag{5.142}$$

for (5.139).

# Appendix C: Solution for determining equation (2.5)

The one-dimensional Black-Scholes equation is given by (2.1). Its admitted infinitesimal generator for point symmetry is of the form

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{5.143}$$

Its first and second prolongations are given by

$$X^{(2)} = X + \zeta^0\frac{\partial}{\partial u_t} + \zeta^1\frac{\partial}{\partial u_x} + \zeta^{11}\frac{\partial}{\partial u_{xx}} \tag{5.144}$$

where $X$ is defined by equation (5.143). The functions $\zeta^0, \zeta^1$and$\zeta^{11}$ are given by

$$\begin{aligned}\zeta^0 &= D_t(\eta) - u_t D_t(\xi^0) - u_x^i D_t(\xi^1)\\ &= \eta_t + u_t\eta_u - u_t\xi^0_t - u_t^2\xi^0_u - u_x\xi^1_t - u_t u_x\xi^1_u\end{aligned}$$

$$\begin{aligned}\zeta^i &= D_i(\eta) - u_t D_i(\xi^0) - u_x^i D_i(\xi^1)\\ \zeta^1 &= \eta_x + u_x\eta_u - u_t\xi^0_x - u_t u_x\xi^0_u - u_x\xi^1_x - u_x^2\xi^1_u\end{aligned}$$

$$\begin{aligned}\zeta^{ij} &= D_j(\zeta^i) - u_{tx^i}D_j(\xi^0) - u_{x^i x^k}D_j(\xi^k)\\ \zeta^{11} &= D_x(\zeta^1) - u_{tx}D_x(\xi^0) - u_{xx}D_x(\xi^1)\\ &= \eta_{xx} + 2u_x\ eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu}\\ &- 2u_{tx}\xi^0_x - u_t\xi^0_{xx} - 2u_t u_x\xi^0_{xu} - (u_t u_{xx} + 2u_x u_{tx})\xi^0_u - u_t u_x^2\xi^0_{uu}\\ &- 2u_{xx}\xi^1_x - u_x\xi^1_{xx} - 2u_x^2\xi^1_{xu} - 3u_x u_{xx}\xi^1_u - u_x^3\xi^1_{uu}\end{aligned}$$

where $D_x$ and $D_t$ are total derivatives with respect to the variables $x$ and $t$ respectively and are defined by

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots .$$

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots .$$

[9]

The determining equation is given by

$$\{\zeta^0 + \frac{1}{2} + A^2 x u_{xx}\xi^1 + \frac{1}{2}A^2x^2\zeta^{11} + Bu_x\xi^1 + Bx\zeta^1 - C\eta\}|_{u_t=-\frac{1}{2}A^2x^2u_{xx}-Bxu_x+Cu} = 0 \tag{5.145}$$

The substitutions of $\zeta_t^0, \zeta^1$ and $\zeta^{11}$ in the determining equation yields that

$$\begin{aligned}
&\eta_t + (-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)(\eta_u - \xi_t^0) - (-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)^2\xi_u^0 - \\
&\quad u_x\xi_t^1 - (-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)u_x\xi_u^1 + A^2xu_{xx}\xi^1 + \frac{1}{2}A^2x^2\eta_{xx} + A^2x^2u_x\eta_{xu} \\
&\quad + \frac{1}{2}A^2x^2u_{xx}(\eta_u - 2\xi_x^1 - 3u_x\xi_u^1) + \frac{1}{2}A^2x^2u_x^2(\eta_{uu} - 2\xi_{ux}^1) - A^2x^2u_{tx}\xi_x^0 \\
&\quad - \frac{1}{2}A^2x^2(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_{xx}^0 - A^2x^2(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_{xu}^0 \\
&\quad - \frac{1}{2}A^2x^2u_{xx}(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_u^0 - \frac{1}{2}A^2x^2(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_{uu}^0 \\
&\quad - \frac{1}{2}A^2x^2u_x^3\xi_{uu}^1 + Bv_x\xi^1 - A^2x^2u_{tx}u_x\xi_u^0 + Bxu_x(\eta_u - \xi_x^1) \\
&\quad - Bx(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_x^0 + Bx\eta_x \\
&\quad - Bxu_x(-\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu)\xi_u^0 - Bxu_x^2\xi_u^1 - C\eta = 0.
\end{aligned} \tag{5.146}$$

We set the coefficients of $u_x, u_{tx}, u_{xx}, u_x^2$ and those free of these variables to zero and solve the following monomials termed defining equations.

$$u_{tx} \;:\; -A^2x^2\xi_x^0 - A^2x^2u_x\xi_u^0 = 0, \tag{5.147}$$

$$u_{xx} \;:\; \xi_t^0 + \frac{2\xi^1}{x} - 2\xi^1 = 0 \tag{5.148}$$

$$u_x^2 \;:\; \xi_u^1 = \eta_{uu} = 0 \tag{5.149}$$

$$u_x \;:\; Bx\xi_t^0 - \xi_t^1 + A^2x^2\eta_{xu} - \frac{1}{2}A^2x^2\xi_{xx}^1 + B\xi^1 - Bx\xi_x^1 = 0 \tag{5.150}$$

$$u_x^0 \;:\; \eta_t + Cu(\eta_u - \xi_t^0) + Bx\eta_x + \frac{1}{2}A^2x^2\eta_{xx} - C\eta = 0 = 0, \tag{5.151}$$

We solve the defining equations.

- Defining equation (5.147) yields that

$$\xi^0 = a(t) \tag{5.152}$$

- The defining equation (5.148) is a linear first order ODE whose integrating factor is

$$e^{-\int \frac{1}{x}dx} = \frac{1}{x}.$$

Multiplying by the integrating factor and solving for $\xi^1$ results in

$$\frac{d}{dx}\frac{\xi^1}{x} = \frac{\xi^1_t}{2x}$$
$$\frac{\xi^1}{x} = \int \frac{\xi^1_t}{2x}dx = \{\frac{\xi^0_t}{2}\}\ln x + f(t)$$

Whence

$$\xi^1 = \{\frac{\xi^0_t}{2}\}x\ln x + xf(t) \tag{5.153}$$

We differentiate equation (5.153) with respect to $t$ and $x$ to obtain the following equations

$$\xi^1_x = \{\frac{\xi^0_t}{2}\}\ln x + \{\frac{\xi^0_t}{2}\} + f(t) \tag{5.154}$$
$$\xi^1_t = \{\frac{\xi^0_{tt}}{2}\}x\ln x + xf'(t) \tag{5.155}$$
$$\xi^1_{xx} = \frac{\xi^0_t}{2x} \tag{5.156}$$

- From defining equation (5.149) we have that

$$\eta = \alpha(t,x)u + \beta(t,x) \tag{5.157}$$

- From defining equation (5.150) we have that

$$\frac{1}{2}Bx\xi^0_t - \frac{\xi^0_{tt}}{2}x\ln x - xf'(t) - A^2x\frac{\xi^0_t}{4} + A^2x^2\alpha_x = 0 \tag{5.158}$$

We apply $D = B - \frac{1}{2}A^2$ thus have that

$$\frac{1}{2}Dx\xi^0_t - \frac{\xi^0_{tt}}{2}x\ln x - xf'(t) + A^2x^2\alpha_x = 0, \tag{5.159}$$

whence

$$\alpha_x = \frac{1}{A^2}\{\frac{\xi^0_{tt}}{2x}\ln x + \frac{2f'(t) - D\xi^0_t}{2x}\} \tag{5.160}$$

resulting in that

$$\alpha = \frac{1}{A^2}\{\frac{\xi^0_{tt}}{4}(\ln x)^2 + (\frac{2f'(t) - D\xi^0_t}{2})\ln x\} + q(t) \tag{5.161}$$

We differentiate equation (5.161) with respect to $t$ and equation (5.158) with respect to $x$ to get expressions for $\alpha_t$ and $\alpha_{xx}$ respectively given as

$$\alpha_t = \frac{1}{A^2}\{\frac{\xi^0_{ttt}}{4}(\ln x)^2 + (\frac{2f''(t) - D\xi^0_{tt}}{2})\ln x\} + q'(t) \tag{5.162}$$

and

$$\alpha_{xx} = \frac{1}{A^2}\{\frac{\xi^0_{tt}}{2x^2} - \frac{\xi^0_{tt}}{2x^2}\ln x + \frac{D\xi^0_t - 2f'(t)}{2x^2}\} \tag{5.163}$$

- The differentiation of defining equation (5.151) with respect to $u$ yields the following expression after simplification

$$\eta_{tu} - C\xi^0_t + Bx\eta_{xu} + \frac{1}{2}A^2x^2\eta_{xxu} = 0 \tag{5.164}$$

The substitution of equation (5.157) in (5.164) result in that

$$\alpha_t - C\xi^0_t + Bx\alpha_x + \frac{1}{2}A^2x^2\alpha_{xx} = 0 \tag{5.165}$$

The substitutions of equations (5.160),(5.163) and (5.164) into equation (5.165) yields

$$\begin{aligned}&\frac{\xi^0_{ttt}}{4A^2}(\ln x)^2 + \frac{f''(t)}{A^2}\ln x - \frac{D\xi^0_{tt}}{2A^2}\ln x + q'(t) - C\xi^0_t + \frac{B\xi^0_{tt}}{2A^2}\ln x \\ &+\frac{Bf'(t)}{A^2} - \frac{BD\xi^0_t}{2A^2} + \frac{\xi^0_{tt}}{4} - \frac{A^2\xi^0_{tt}}{4A^2}\ln x + \frac{A^2\xi^0_t}{2A^2} - \frac{A^2f'(t)}{2A^2} = 0\end{aligned} \tag{5.166}$$

Equation (5.166) simplifies to

$$\frac{\xi^0_{ttt}}{4A^2}(\ln x)^2 + \frac{f''(t)}{A^2}\ln x + q'(t) = -\frac{\xi^0_{tt}}{4} + (\frac{D^2}{A^2} + C)\xi^0_t - \frac{D}{A^2}f'(t) \tag{5.167}$$

We equate corresponding coefficients in equation (5.166) and obtain the following equations

$$\xi^0_{ttt} = 0 \tag{5.168}$$

$$f''(t) = \frac{D\xi^1_t}{4} \tag{5.169}$$

$$q'(t) = -\frac{\xi^0_{tt}}{4} + (\frac{D^2}{A^2} + C)\xi^0_t - \frac{D}{A^2}f'(t) \tag{5.170}$$

From the equations (5.168), (5.169) and (5.170)

$$\xi^0 = \frac{1}{2}C_1t^2 + C_2t + C_3 \tag{5.171}$$

$$\begin{aligned} f(t) &= C_4t + C_5 \\ q(t) &= -\frac{1}{2}C_1t + \frac{D^2}{A^2}C_1t^2 + CC_1t^2 + \frac{D^2}{A^2}C_2t \\ &+ CC_2t - \frac{D}{A^2}C_4t + C_6. \end{aligned} \tag{5.172}$$

Substituting for (5.172) in equation (5.161), gives

$$\begin{aligned} \alpha = \frac{C_1}{2A^2}(\ln x)^2 + (\frac{2C_4 - 2DC_1t + C_2}{2A^2})\ln x - \frac{1}{2}C_1t + \\ -\frac{1}{2}c_1t + \frac{D^2}{A^2}C_1t^2 + CC_1t^2 + \frac{D^2}{A^2}C_2t + CC_2t - \frac{D}{A^2}C_4t + C_6. \end{aligned} \tag{5.173}$$

Hence

$$\begin{aligned} \eta = \frac{C_1}{2A^2}(\ln x)^2u + (\frac{2C_4 - 2DC_1t + C_2}{2A^2})u\ln x - \frac{1}{2}C_1tu + \\ \frac{D^2}{2A^2}C_1t^2u + CC_1t^2u + \frac{D^2}{A^2}C_2tu + CC_2tu - \frac{D}{A^2}C_4tu + C_6u + \beta(t,x) \end{aligned} \tag{5.174}$$

## Infinitesimals

The infinitesimals are

$$\xi^0 = \frac{1}{2}C_1t^2 + C_2t + C_3 \tag{5.175}$$

$$\xi^2 = C_1tx\ln x + \frac{C_2}{2}x\ln x + C_4tx + C_5x \tag{5.176}$$

$$\begin{aligned} \eta = \frac{C_1}{2A^2}(\ln x)^2u + (\frac{2C_4 - 2DC_1t + C_2}{2A^2})u\ln x - \frac{1}{2}C_1tu + \\ \frac{D^2}{2A^2}C_1t^2u + CC_1t^2u + \frac{D^2}{A^2}C_2tu + CC_2tu - \frac{D}{A^2}C_4tu + C_6u + \beta(t,x) \end{aligned} \tag{5.177}$$