AN INVESTIGATION INTO THE SOLVING OF POLYNOMIAL EQUATIONS AND THE IMPLICATIONS FOR SECONDARY SCHOOL MATHEMATICS

by

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I declare that, AN INVESTIGATION INTO THE SOLVING OF POLYNOMIAL EQUATIONS AND THE IMPLICATIONS FOR SCHOOL MATHEMATICS, is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

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SUMMARY:

This study investigates the possibilities and implications for the teaching of the solving of polynomial equations. It is historically directed and also focusses on the working procedures in algebra which target the cognitive and affective domains. The teaching implications of the development of representational styles of equations and their solving procedures are noted. Since concepts in algebra can be conceived as processes or objects this leads to cognitive obstacles, for example: a limited view of the equal sign, which result in learning and reasoning problems. The roles of sense-making, visual imagery, mental schemata and networks in promoting meaningful understanding are scrutinised. Questions and problems to solve are formulated to promote the processes associated with the solving of polynomial equations, and the solving procedures used by a group of college students are analysed. A teaching model/method, which targets the cognitive and affective domains, is presented.

KEY TERMS:

Solving of polynomial equations; historical perspective, subject didactical perspective; understanding in mathematics, procedural and structural thinking; mental schemata; approaches/methods of teaching; procedures in algebra; secondary school mathematics; factorisation and simplification; mathematics education.

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CHAPTER 1
BACKGROUND AND OVERVIEW OF THE STUDY

1.1 INTRODUCTION

The purpose of this introduction and section 1.2 below is to provide a motivation for, and a scope of, the whole study.

There is a perception among many pupils and parents that mathematics is a mystifying subject which is difficult for many people to learn. Various mathematics educationists (eg. Silver 1985, Davis 1988 & 1991, Sfard 1991, Cangelosi 1996) have noted the factors that contributed to this perception. Some of these factors are:

- a failure to link school mathematics to its historical origins,
- a failure by teachers to appreciate the difficulties experienced by students when new concepts and algorithms have to be learned and applied, and
- the use of teaching approaches which are not process-directed.

Many of the roots of research studies in mathematics can be traced to the two disciplines mathematics and psychology (Kilpatrick 1992:5). However, the broad study terrain of mathematics education is described by the subject didactics of mathematics, which is the science of the teaching and learning of mathematics (Wessels 1993:52). The discipline mathematics indicates *what* mathematical content is to be taught and learned, while the discipline subject didactics of mathematics focusses on *how* this content should be taught and learned. It follows that the subject diadactics of mathematics amongst other things, also deals with learning problems in the teaching and learning of mathematics.

The content that has been chosen for this investigation is the solving of polynomial equations. A study of recent Matric Examiners Reports for the Senior Certificate Examinations in Mathematics (eg. House of Delegates, Department of Education and

Culture 1992, 1994 & 1996) indicates that many candidates experience difficulties when solving polynomial equations of degree two and three. These difficulties include:

- Recognising the type of equation that they are confronted with.
- Carrying out solving procedures that are required to solve a given equation.
- Interpreting of graphs drawn in order to solve quadratic and cubic equations.

For these reasons it was decided to focus on the solving of polynomial equations.

As a result of the points noted in the above three paragraphs, this investigation into the solving of polynomial equations will focus on the following:

- a historical review with interpretations of the solving of polynomial equations,
- working procedures in algebra, and
- possible approaches regarding the teaching of equations.

Justification for such an investigation, from a historical and psychological perspective, was recently provided by the NCTM (1997:11), and Garcia and Piaget (both cited in Sfard & Linchevski 1994:195).

An investigation into the solving of equations will be of use to teachers for the following reasons:

- The linking of the solving of equations to its historical origins will expose students to the fact that mathematics is not really a cold subject, but has been created and contributed to by different peoples and cultures.
- The working procedures in algebra will give insight into how understanding in algebra, and in particular the solving of polynomial equations, occurs. Teachers will become aware of the cognitive obstacles that are likely to occur when new knowledge and algorithms are taught to students.
- The possible teaching approaches for equations will present approaches/methods which will promote the learning processes and procedures in algebra in a meaningful manner.

Each of the above will impact on the meaningful teaching and learning of the solving of polynomial equations.

1.2 RELEVANCE OF THE STUDY

A case for the motivation of this study has already been presented in the introduction. Attention will now be focussed on the relevance to school mathematics content, the fact that the study addresses a real problem in mathematics teaching and how this study contributes to the achievement of both essential and specific outcomes as envisaged in Curriculum 2005.

With reference to the Interim Core Syllabuses for Mathematics (in South Africa), the solving of polynomial equations of degree one (linear equations), two (quadratic equations) and three (cubic equations) are explicitly addressed in grades 8, 9, 10 and 11. The solving of such equations is also required in order to work out problems relating to a number of sections in the syllabuses for grades 11 and 12. These include:

- Solving word problems that lead to quadratic equations.
- Curve sketching of parabolas (quadratic functions) and third degree polynomial functions [to determine the x-intercepts].
- The solving of certain trigonometric, exponential and logarithmic equations.
- Maximization and minimization problems in calculus [grade 12].

It was noted in 1.1 above that the recent Matric Examiners' Reports for the Senior Certificate Examinations in Mathematics, which were administered by the Department of Education in the House of Delegates, indicated that many candidates experienced difficulties when solving polynomial equations of degree two and three. These difficulties relate to working procedures in algebra.

Presently in South Africa, planning by both the National and Provincial Education Departments is in progress to phase in Curriculum 2005 from 1998. This curriculum, which is also known as Outcomes Based Education (OBE), focusses on both essential and specific outcomes. Outcomes are the results of the learning process and refer to what the learner knows and can do (Morris 1996:4). Essential (or critical) outcomes are general things which matter in all areas of learning, like communicating by using mathematical and language skills, problem-solving, and working with others as part of a group (Morris 1996:10). The possible teaching approaches which are discussed in chapter 4 presents models/methods and outlines learning activities that can be implemented to promote these essential outcomes. In the school context specific outcomes refer to special skills, knowledge, attitudes and understanding in a particular subject or learning area (Morris 1996:4). With regard to the solving of polynomial equations some of these outcomes are discussed in chapter 3. Section 4.4 present some learning activities which promote these specific outcomes. In particular this study gives some ideas as to how the following specific outcomes for the learning area Mathematical Literacy, Mathematics and Mathematical Sciences (Department of Education 1997:114, 125) can be achieved by activities relating to the solving of polynomial equations.

> *Specific Outcome 1:* Demonstrate understanding of the historical development of mathematics in various social and cultural contexts.
>
> *Specific Outcome 9:* Use mathematical language to communicate mathematical ideas, concepts, generalisations and thought processes.

It follows from the above that the topic of this dissertation, namely: *An investigation into the solving of polynomial equations and implications for secondary school mathematics*, is a relevant study.

1.3 STATEMENT OF THE RESEARCH QUESTION

From sections 1.1 and 1.2 it follows that an investigation into the solving of polynomial equations, which has implications for the teaching, should concentrate on the following aspects: the historical development of the solving of polynomial equations, the working procedures that are required to solve such equations, and teaching approaches that could promote the attainment of these procedures. With this in mind the following sub-questions can be formulated:

- Of what use could a review of the historical development to the solving of polynomial equations be to the teaching and learning of secondary school mathematics?
- What are the working procedures in the solving of polynomial equations?
- Which teaching approaches could promote these working procedures, and the

specific and some of the essential outcomes as envisaged in outcomes based education?

These sub-questions lead to the research question:

What possibilities and implications, for the teaching of the solving of polynomial equations, can be exposed by carrying out an investigation which is historically directed and also focusses on the working procedures in algebra which target the cognitive and affective domains?

1.4 AIMS AND OBJECTIVES OF THIS STUDY

The following are the three aims of this study:

1. To review and interpret the historical development of the solving of polynomial equations.
2. To discuss the working procedures in algebra.
3. To present possible teaching approaches for the solving of equations.

From the first aim the following objectives can be formulated:

- To note how and why representational forms for the statement and solution procedures of polynomial equations developed.
- To note how and why polynomial equations of degree 2 or more originated.
- To note how the development of the solving of polynomial equations influenced other developments in mathematical knowledge.
- To note how and why particular methods to solve polynomial equations developed.
- To note which polynomial equations are solvable by radicals.

The following objectives can be formulated from the second aim:

- To note how understanding in mathematics and in algebra in particular occurs.
- To note the findings of research studies on the working procedures and cognitive obstacles with respect to the use of symbols in algebra, algebraic expressions, the solving of equations and functions.

- To discuss the processes involved in the solving of quadratic and cubic equations.

From the third aim the following objectives can be formulated:

- To note the theories/models/methods/strategies that can be used in the solving of equations in algebra.
- To outline some models/methods which promote the working procedures in algebra.
- To present a model/method for the solving of polynomial equations which target both the cognitive and affective domains.
- To note some of the working procedures used by students, and the implications for good teaching and learning.

1.5 THE RESEARCH METHODS USED

The research methods that were used were guided by the research question, and the aims and objectives of the study.

Surveys were conducted of the relevant literature, on the existing knowledge base, relating to the aspects noted in the first aim and the related objectives. This led to a study of existing literature on the knowledge base relating to the historical development of solving procedures for equations of degree two and more. A historical review with interpretation of solving procedures for polynomial equations was then documented. Then literature dealing with the psychological aspects with regard to the learning and teaching of secondary school mathematics was studied. Particular attention was focussed on research studies in algebra that looked at the analogy between historical and psychological developments in algebra, and cognitive obstacles in the learning of algebra. This led to the writing of the working procedures in algebra that pupils should be exposed to when solving equations.

Then attention was focussed on some theories/models/methods for teaching mathematics, which could promote the working procedures in algebra. This led to the formulation of a teaching model with three levels for the solving of polynomial equations, which targets both the cognitive and affective domains. The Interim Core Syllabuses for Mathematics (KwaZulu-Natal Department of Education: 1995) were studied in order to formulate some teaching goals or outcomes relating to the solving of polynomial equations. Learning tasks were then designed to illustrate how specific and essential outcomes can be promoted in each of the three levels of the model.

During the course of the study certain ideas dealing with the solving of equations were tried out with student-teachers at Springfield College of Education. Some of the findings were included in the discussion of the processes involved in the solving of quadratic and cubic equations. This was done to motivate certain ideas. A limited empirical investigation was also done on the solving procedures used by a group of student-teachers at Springfield College of Education. This limited investigation and its implications for teaching were noted.

1.6 DEFINITION OF TERRAIN

The historical review focussed on polynomial equations of degree 2 and more in order to give an insight into the working procedures used by mathematicians to gradually develop solving procedures for the different equations and to illustrate the nature of the development of mathematical knowledge.

The teaching implications for secondary school mathematics confines itself to the teaching of algebra in grades 8 and 9, and in particular to the solving of polynomial equations in grades 10, 11 and 12.

Student-teachers provided an insight into the teaching conditions in the classrooms of government secondary schools in KwaZulu-Natal and this was noted. With this in mind a workable model for the solving of polynomial equations was formulated to promote meaningful teaching and learning.

1.7 DEFINITION OF TERMINOLOGY

1.7.1 Polynomial equation

A *polynomial equation in the unknown* x is an equation of the form

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$$

where

$a_n, a_{n-1}, \dots, a_1, a_0$ are coefficients which could be rational, real or complex numbers; a_n is non-zero, a_0 is the constant term, and n is a natural number.

The highest power of x with non-zero coefficient gives the *degree of the polynomial* equation.

1.7.2 Solution or root of an equation

A *solution* or *root of an equation* is a value of the unknown which satisfies the equation. Solutions could be complex numbers. The *solution set* of an equation refers to the set of all possible values of the unknown which satisfy the equation.

1.7.3 Solving of polynomial equations

This refers to the methods and procedures used to determine solutions or the solution sets of polynomial equations.

1.7.4 Algebraic solution or solution by radicals

The terms *algebraic solution* or a *solution by radicals* of a given polynomial equation refers to solutions that are obtained by performing a finite number of operations on the coefficients of the equation. These operations are addition, subtraction, multiplication, division and extraction of roots.

1.7.5 Secondary school mathematics

This refers to the mathematics that is covered in grades 8 to 12 at South African schools.

1.8 OVERVIEW OF THE STUDY

Chapter 2 gives a historical review with interpretation of the solving of polynomial equations of degree 2 and more. This includes the development of and the need for modern symbolic algebra. The solution of quadratic equations (degree 2) as handled by the Babylonians, the Greeks who used geometry, and the contributions by the Hindus and Arabs who used more advanced methods are also discussed. Tartaglia's and Cardano's solutions for cubic equations (degree 3) are discussed. Then attention is focussed on the developments in the theory of polynomial equations. The contributions of Gauss and Lagrange are discussed. Finally the solution of equations of degree 5 or more are discussed by focussing on the contributions of Ruffini, Abel and Galois.

In chapter 3 working procedures in algebra are discussed. The chapter begins by looking at how understanding in mathematics and concept formation in algebra occurs. The findings of various research studies in algebra which focussed on the use of symbols in algebra, algebraic expressions, equation-solving and functions are discussed. Finally the processes involved in the solving of quadratic and cubic equations, and their teaching implications are discussed.

Chapter 4 discusses possible methods (or approaches) regarding the teaching of equations at secondary schools. The theories/models/methods/strategies in solving equations are noted. Then some general models/methods for teaching are discussed. Included in this discussion is a model with three levels which targets both the affective and cognitive domains. This model is then used to present a teaching model for the solving of polynomial equations. Also included in this chapter is a limited empirical investigation on the solving procedures used by students.

In chapter 5 the conclusions and recommendations of this research, and themes for further research are noted.

CHAPTER 2

HISTORICAL REVIEW WITH INTERPRETATION OF THE SOLVING OF POLYNOMIAL EQUATIONS

2.1 INTRODUCTION

This chapter gives a historical review, with interpretation, of the solving of polynomial equations of degree 2 or more. The notation that was used to state these types of equations and the solution of these equations will also be discussed. With reference to the school mathematics syllabus, the solution of polynomial equations (up to the third degree) are found in the algebra syllabus of the senior secondary school phase. Subsection 2.1.1 focusses on the question: *What is algebra?* In chapter 2, an attempt is made to answer the following questions:

- *How and why did polynomial equations of degree 2 or more originate?*
- *How and why did particular methods to solve polynomial equations develop?*
- *How did the solving of polynomial equations contribute to the development of the number system?*
- *Which polynomial equations are solvable by radicals?*

In section 2.2 the rise of symbolic algebra is discussed. This includes the development of and the need for modern symbolic algebra. The contributions made by Cardano, Stevin, Harriot, Descartes and Wallis are noted. Then some developments that arose from the use of symbolic notation are noted.

Section 2.3 focusses on quadratic equations and the development of the solution to quadratic equations. The solution of quadratic equations as handled by the Babylonians, the Greeks who used geometry, and the contributions by the Hindus and Arabs who used more advanced methods are also discussed. Finally the development of the quadratic formula to solve any quadratic equation is discussed.

In section 2.4 the solutions of Tartaglia and Cardano for solving cubic equations are discussed. The questions that arose from Cardano's Formula are also noted. Sections 2.3 and 2.4 also include a discussion of how the number system developed to include irrational numbers, and finally unreal numbers. In section 2.5 the method used by Ferrari to solve quartic equations is discussed.

Section 2.6 focusses on the developments in the theory of polynomial equations. Included in the discussion are the following:

- The relationship between roots and coefficients.
- Investigations on the factorisation of real polynomials.
- The ambiguity of the nth roots of a number.
- The fundamental theorem of algebra.
- Lagrange's contribution to the solving of equations.
- Binomial or Cyclotomic Equations.

Finally in section 2.7 attention is focussed on the solving of equations of degree 5 or more. The contributions of Ruffini, Abel and Galois are noted.

2.1.1 What is algebra?

A good starting point is to clarify what is meant by the term *algebra*. The word algebra is a Latin variant of the Arabic word *al-jabr* which means completion (Baumgart 1989:233). In the early developments of algebra one of the two main themes was the search for a *general* solution of equations, the other was the "concept and the definition of the realm of numbers" (Novy 1973:25). Many authors agree on the early origins of algebra because they spot "algebraic thinking whenever an attempt is made to treat computional processes in a somewhat general way" (Sfard 1995:18). The definition used by Baumgart (1989:234) suggests that a definition of the term algebra requires a two-phased approach:

- early elementary algebra - which refers to the study of equations and the methods for solving them,
- modern abstract algebra - which refers to the study of mathematical structures

such as groups, rings and fields.

According to Kieran (1992:391)

> A study of the historical development of algebra suggests that currently algebra is conceived as the branch of mathematics that deals with symbolizing general numerical relationships and mathematical structures and with operating on those structures.

It follows from these descriptions that generality is one of the characteristics of algebra (compare with 3.3.2.3). There is also an important link between algebraic thinking and notation (see 3.3.2.2).

Since the division of phases suggested by Baumgart is both chronological and conceptual, it is convenient to trace the development of algebra in terms of these phases. As this chapter deals with the historical review of polynomial equations, the discussion will concentrate on the elementary phase which seems to have began around 2000 BC. The elementary phase was characterised by the gradual invention of symbolism and the solving of equations with numerical coefficients by various methods.

Modern algebraic notation, which we are used to today, is the finished product. Baumgart (1989:234) notes that the development of algebraic notation progressed through three stages:

- the rhetorical (or verbal), from about 2000 BC
- the syncopated in which abbreviated words were used, from about 600 AD, and
- the symbolic, which began to emerge around 1500 AD.

This development of stages, with regard to the formulation of algebraic notation, has important implications for the teaching of equation-solving (refer to 4.2 and 4.4 for details).

Examples of each of these stages will be given as the historical development of solutions to *quadratic*, *cubic* and *quartic* equations is reviewed. Also, the historical development of the representations and solving of equations will be presented as a sequence of steps towards greater generality, yet at the same time moving towards greater structurality. The rise of symbolic algebra will first be discussed since modern algebraic notation will be used in order to aid understanding of the rhetorical and syncopated styles and the algebra that is implied by each of these styles.

2.2 THE RISE OF SYMBOLIC ALGEBRA

2.2.1 The impact of printing

The invention of printing with movable type (1450) led to a need for standardisation of symbolism in algebra (Baumgart 1989:243). This was necessary for improved communications since printing made it possible for written communication to be distributed widely. Commenting on the impact of printing McQualter (1988:4) makes the following observation:

> Intellectual mathematics was at last available as interpretative mathematics to users in commerce, travelling, surveying and manufacturing.

2.2.2 Constraints posed by the rhetorical and syncopated styles

Rapid development in the theory of equations occurred around the middle of the sixteenth century. Cardano who solved the general cubic equation in the 16th century (refer to 2.4.2) used symbolism which was in the form of abbreviations such as 'p' for 'plus', 'm' for 'minus' and 'R' for 'radix' (root). Tignol (1988:36) makes the following remark:

> In the solution of cubic and quartic equations, Cardano was straining to the utmost the capabilities of the algebraic system that was available to him.

The result of this was that the rhetorical and syncopated styles, that were used to state and solve the equations, posed constraints on the full understanding and exploration of the new ideas that arose from the solutions. As a result of this, progress during the next two centuries (17th and 18th) was rather slow.

History therefore indicates that a lack of understanding of and the ability to use modern symbolic notation, could pose serious constraints on the understanding and use of mathematical ideas (compare with 3.2.2 and 3.4.1). Teaching should therefore take into account the role of algebraic symbolism in the teaching and learning of algebra (refer to 3.3.1 and 3.4.1 for further details).

2.2.3 The need for more efficient notations and its development

Various studies (eg. Groza 1968, Resnikoff & Wells 1973, Baumgart 1989, Cummins 1989, Heinke 1989, Read 1989) have focussed on the development of algebraic notation. To fully understand and explore the new ideas arising from the solution of cubic and quartic equations there was a need for more efficient notations to be developed. The algebraic renaissance in Europe began in Italy and modem symbolism began to emerge around 1500 AD (Baumgart 1989:243). The following examples illustrate the initial poverty and later diversity of symbols, gradual improvement and the standardisation of notation. Beneath each of the older forms the equivalent in modem notation is indicated.

Cardano(1545): cubus $\bar{p}$ 6 rebus aequalis 20.

$x^3 + 6x = 20.$

Bombelli(1572): $\overset{\overset{6}{\smile}}{1}$. p . $\overset{\overset{3}{\smile}}{8}$. Eguale à 20.

$x^6 + 8x^3 = 20.$

Stevin(1585): 3② + 4 egales à 2① + 4.

$3x^2 + 4 = 2x + 4.$

Viète(1591): I QC - 15 QQ + 85 C - 225 Q + 274 N aequatur 120.

$x^5 - 15x^4 + 85x^3 - 225x^2 + 274x = 120.$

Harriot(1631): aaa - 3bba = +2.ccc

$x^3 - 3b^2x = 2c^3.$

Descartes(1637): $x^3 - 6xx + 13x - 10 \propto 0.$

$x^3 - 6x^2 + 13x - 10 = 0.$

Wallis(1693): $x^4 + bx^3 + cxx + dx + e = 0.$

From the above it seems that Wallis in 1693 was the first to write polynomial equations in the way we are used to today. Wallis' contribution to the development of modem symbolic notation and the resulting advantages are noted by Kline (1972:281) who makes the following observation:

> John Wallis, influenced by Vieta [Viète], Descartes, Fermat, and Harriot, went far beyond these men in freeing arithmetic and algebra from geometric representation. .... He saw in algebra brevity and perspicuity.

Although Descartes and Wallis both used x^3, note that they used xx for x^2. A possible reason for this is that mathematicians avoided using x^2, since a second power was used to represent area (Kline 1972:281). This indicates that in the initial development of algebra there was a close link between algebraic and geometric representations - a link which had important implications for introducing algebra (refer to 3.3.2.3). According to Kline (1972:261) the notation x^2 for xx was adopted by Gauss in 1801.

History of mathematics should be discussed in class when appropriate. Such discussions will help to expose pupils to the fact that mathematics has a human dimension to it - mathematics is a human activity which has developed as a result of contributions from different cultures and peoples. This is an essential outcome in Outcomes Based Education (OBE) which is to be phased in at schools from 1998.

There was a need to develop modem mathematical notation. A discussion of this should get pupils to appreciate and see the need to be able to use and interpret modern mathematical notation. It took a long time to develop the modem notation (that we are used to today) and to explore the new ideas that emerged. From a teaching and learning point of view more time and effort should be spent on mathematical symbolism. An interpretation of the surface and deep structure meanings that emerge from mathematical symbolism as used in different mathematical contexts will promote mathematical maturity among pupils and teachers (refer to 3.4.5.1 and 4.4.3). In the long run this will be time well spent (see 1.2, Specific Outcome 9).

2.2.4 Developments arising from the use of symbolic notation

The new notations helped to create a new mathematical object: *polynomials*, which

Simon Stevin's referred to as 'multinomials' or 'integral algebraic numbers' (Tignol 1988:38). Viète's ideas initiated a completely formal treatment of algebraic expressions. His use of letters for denoting all quantities, known or unknown, in a problem made it

> ... possible to replace various numerical examples by a single 'generic example', from which all the others could be deduced by assigning values to letters (Tignol 1988:41).

It seems then that Viète's ideas led to algebra developing into a form of generalised arithmetic. His ideas made it possible to study an entire class of expressions or equations. Kline (1972:262) notes that improvements in Viète's use of symbols were due to Descartes. Descartes introduced the modern practice of using the first letters of the alphabet for known quantities and the last letters for the unknowns. Perhaps he realised that the different roles that letters play in algebra could lead to confusion (see 3.3.1). However, Descartes' use of letters was restricted to positive numbers only. In 1657, John Hudde used letters for positive and negative numbers (Kline 1972:262).

In the period 1545 to 1693 algebraic notations were dramatically improved (refer to the illustrations in 2.2.3 above). They reached the same level of generality and versatility as the present notations. The illustrations indicate that René Descartes in 1637 and John Wallis in 1693 were instrumental in shaping these notations. The advances in algebraic notation led to the following (Tignol 1988:42):

- a deeper understanding of the nature of equations
- advancement in the theory of equations, for example: the number of roots of an equation and the relationship between roots and coefficients of equations (refer to 2.6.1).

With this background on the development of mathematical symbolism, the development of the solving of quadratic, cubic, quartic and quintic equations will be discussed. Modern symbolism will be used to illustrate the statements and solutions to problems in order to demonstrate similarities that may exist between modern and traditional methods.

2.3 SOLVING QUADRATIC EQUATIONS

In modern notation, the standard form of the quadratic equation (as dealt with in the grade 11 mathematics syllabus) is as follows:

$ax^2 + bx + c = 0$, where a, b and c are rational numbers and $a \neq 0$.

The solution of this equation is given by the quadratic formula, namely: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Pupils in grade 11 are familiar with the quadratic formula which enables them to determine the solutions to quadratic equations with rational coefficients.

However, the solution of the general quadratic equation could not be written in the form given by the quadratic formula before the seventeenth century. Mathematicians were able to solve quadratic equations for about 40 centuries before this (Tignol 1988:6). The different phases leading to the development of the final form given by the quadratic formula will now be discussed.

2.3.1 Contributions from Babylonian Algebra (2000 BC)

Edwards (1984:3) notes that

> Archeological research in the twentieth century has revealed the surprising fact that the peoples of Mesopotamia in the period around 1700 B.C. had an advanced mathematical culture, ...

Studies of Babylonian tablets reveal that this ancient culture was familiar with stating and solving problems which were essentially quadratic equations or which could be put in a quadratic form for solution.

2.3.1.1 Practical problems that led to normal forms

Studies by McQualter (1988:3) and Berriman (1956:187-191) suggest that the problems and their method of solution were concerned with land tenure, building, engineering and military situations. These problems of social organisation were of practical significance in societies which used agriculture and trade as their source of wealth. McQualter (1988:3) notes that the major concerns of these societies were with the erection of buildings, fortifications, irrigation works, roads and harbours in order to house, protect and promote the generation of such wealth (compare with 3.4.5.3). Kline (1972:11) gives the following example to illustrate the close relationship between Babylonian mathematics and practical problems:

> A canal whose cross-section was a trapezoid and whose dimensions were known was to be dug. The amount of digging one man could do in a day was known, as was the sum of the number of men employed and the days they worked. The problem was to calculate the number of men and the number of days of work.

From such practical situations it seems that Babylonian pupils were confronted with the need to make preliminary calculations in order to produce the coefficients required for a solution to given problems.

Tignol (1988:7) claims that the " .... first known solution of a quadratic equation dates from about 2000 BC". He cites the following example of a procedure, found on a Babylonian tablet, which illustrates the *rhetorical (or verbal)* stage in the development of algebraic notation:

> I have subtracted from the area the side of my square: 14.30. Take 1, the coefficient. Divide 1 into two parts: 30. Multiply 30 and 30:15. You add to 14.30, and 14.30.15 has the root 29.30. You add to 29.30 the 30 which you have multiplied by itself: 30, and this is the side of the square.

The rhetorical style of this text describes the procedure to find the side (say x) of a square when the difference between the area and the side is given. Using modern algebraic notation, the text gives the solution of $x^2 - x = b$.

In order to make sense of the text, it should be noted that in Babylonian arithmetic the base for numeration is 60, although the reason for using base 60 is uncertain (Vogeli 1989:37). Using base 60, 14.30 means 14(60) + 30 ie. 870. *Divide 1 into two parts: 30* means 30.60^{-1} which is 0,5. In *multiply 30 and 30:15*, the 15 represents (0,5)(0,5) which is 0,25. It seems that the author of the text used the method of completing the square to solve the equation $x^2 - x = 870$, and got $x = 30$, which is one of the roots of the equation.

Practical problems do not always lead to solutions which are rational numbers. A natural question to ask then is "What did the Babylonians do if a root was irrational?" Based on the discussions of Neugebauer (1957:35,47) it seems that the Babylonians used geometrical ideas to compute approximations for irrational numbers. Neugebauer (1957:35) cites as an example a tablet which shows a square with side the number 30. On the diagonal an approximation appears. This example also shows that the Pythagorean theorem was known to the Babylonians more than a thousand years before Pythagoras (Neugebauer 1957:36).

Note that the solution routines were rhetorical. Commenting on the solution routines Neugebauer (1957:43) wrote that

> ... it becomes obvious that it was the general procedure, not the numerical result, which was considered important.

In the solution routines variables were expressed in practical terms such as 'length', 'width', and 'area'. Therefore length, width and area were considered as numbers, which could be added and multiplied without any restrictions, since no real geometric situation was envisaged (Aaboe 1964:25). From a teaching point of view the indiscriminant addition, subtraction and multiplication of numbers representing length, area and volume should be discouraged. The dimensions given by the appropriate unit of measure should be used to guide reasoning.

The use of practical terms for variables excluded the possibility of a negative solution arising. Perhaps it was the nature of these practical and real life problems that restricted the numerical system of the Babylonians to positive rational numbers.

The lack of negative numbers led to the Babylonians developing various normal forms of problems (Edwards 1984:3), which arose because of the frequent occurrence of the practical problems which could be reduced to the following types (Tignol 1988:8, 10):

- to find the length and the breadth of a rectangle when the perimeter and the area of the rectangle are given. In modern algebraic notation the normal form of this problem amounts to the solution of the system of equations $x + y = a$ and $xy = b$.
- to find the length (x) and the breadth (y) of a rectangle, when the excess of the length on the breadth and the area are given. This problem amounts to the

solution of the system of equations $x - y = a$ and $xy = b$.

The system of equations which model the first type of problem will be referred to as normal form 1. Normal form 2 will be used to refer to the system of equations which model the second type of problem.

2.3.1.2 Normal form 1 and rhetorical styles

Using modern algebraic notation NORMAL FORM 1 is as follows (Edwards 1984:3):

Given two numbers a and b, and that $x + y = a$, $xy = b$, find x and y.

According to Edwards (1984:3) the Babylonians outlined the procedure to solve this problem by stating the following steps:

1. Take half of *a*.
2. Square the result.
3. From this subtract *b*.
4. Take the square root of the result.
5. Add this to half of *a*; this is one of the numbers.
6. *a* minus the result is the other number.

Using the two equations of the normal form we arrive at the following quadratic equation $x^2 - ax + b = 0$ in modern algebraic notation. Steps 5 and 6 above give the results

$$\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} \quad \text{and} \quad \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

which give the solutions of the quadratic equation $x^2 - ax + b = 0$, where the coefficients *a* and *b* were restricted to positive rational numbers, which is only a special type of the standard form of a quadratic equation as stated in 2.3 above.

2.3.1.3 Normal form 2

Using modern algebraic notation NORMAL FORM 2 is as follows (Tignol 1988:8):

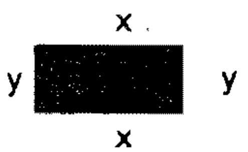
Given two numbers a and b such that $x - y = a$

and $xy = b$ ....[2.1]

where x and y represent the length and breadth respectively of a rectangle, find x and y.

The quadratic equations $x^2 - ax = b$ and $y^2 + ay = b$ are equivalent to the system of equations in [2.1] above, after setting $y = x - a$ and $x = y + a$ respectively (Tignol 1988:8). From this it follows that the normal form represented by system [2.1] enabled the Babylonians to solve quadratic equations of the types $x^2 - ax = b$ and $y^2 + ay = b$, where a and b are positive rational numbers.

For normal form 1 (refer to 2.3.1.2) the procedure to find x and y was outlined. The question arises as to how the Babylonians arrived at the various steps. The following approach dealing with normal form 2 is based on the suggestions by Tignol (1988:9).



The rectangle with sides x and y has perimeter 2x + 2y.

The square which has the same perimeter as the rectangle, has sides of length z where z is the arithmetic mean of x and y.

Geometrically the following representation is obtained from normal form 2 :

```
              x
|<---------------------->|
|       z                |
|<------------------>|   |
|                    |   |
|<------------><----->|
       y          a
```

The following equations are arrived at

$$z = y + \frac{a}{2} \qquad \text{......[2.2]}$$

$$z = x - \frac{a}{2} \qquad \text{......[2.3]}$$

Area of square = z^2

Area of rectangle = xy

Using equations [2.2] and [2.3] we have

$$xy = (z + \frac{a}{2})(z - \frac{a}{2}).$$

$$\therefore \quad b = z^2 - (\frac{a}{2})^2$$

$$\therefore \quad z = \sqrt{(\frac{a}{2})^2 + b}$$

Substituting for z in equations [2.2] and [2.3] we get

$$x = \frac{a}{2} + \sqrt{(\frac{a}{2})^2 + b} \qquad \text{.....[2.4]}$$

$$\text{and } y = -\frac{a}{2} + \sqrt{(\frac{a}{2})^2 + b} \qquad \text{.....[2.5]}$$

Such normal forms developed by the Babylonians to solve the types of practical problems mentioned in 2.3.1.1 therefore gave, in the rhetorical style (refer to 2.3.1.2), procedures which led to the solution of the following types of quadratic equations:

$x^2 + ax = b$; $x^2 - ax = b$ and $x^2 + b = ax$

where a and b were restricted to *positive rational numbers*, because of the initial context of the problems. Under sections 2.3.2 and 2.3.4 it will be seen how (irrational) real numbers evolved and when they were included.

In normal form 1, discussed in 2.3.1.2, note that the statement of the problem and the procedure for the solution were clearly stated. It is because of this that researchers were able to make sense of the contributions of the Babylonians to solutions to quadratic equations. From a teaching point of view it is important that any problem dealt with and its solution(s) are clearly indicated, since this serves to enhance communication. The rhetorical style to state the problem and to outline procedures could be used to promote mathematical thinking, in the sense that they aid in the understanding of the problem and the solution procedure (compare with 4.4.1). Teachers should therefore effectively model settings with regard to presentation of written work, and the processes that are involved in the understanding of problems and the solution procedures (refer to 3.4.4 for further details). This should result in pupils making use of such modelling in their own work and thinking (see 4.4.2).

2.3.2 Contributions from Greek Geometric Algebra (about 300 BC)

Greek algebra was mainly formulated by the Pythagoreans (540 BC) and Euclid (300 BC), and was geometric in nature (Baumgart 1989:237). Van der Waerden (1983:72) notes that unlike the Babylonians, the Greeks made a clear distinction between numbers and geometric quantities (line segments, areas and volumes). The result was that the Greeks did not add line segments to areas. Greek algebra followed the same method of solution as the Babylonians, but these methods were phrased in terms of line segments and areas, and the methods were illustrated by geometric figures (refer to 2.3.2.1 for an illustration).

Why did the Greeks transform these algebraic methods into the geometric form we find in Euclid's Elements? Van der Waerden (1983:88) notes that one reason for this "geometrization" seems to be the discovery of irrational lines (line segments of irrational lengths) by the Pythagoreans. If a square has side 1 unit length, then by the Theorem of Pythagoras the length of the diagonal would satisfy the equation $x^2 = 2$. This equation could not be solved by the use of rational numbers $\frac{m}{n}$.

Although the Greeks knew approximate solutions to $x^2 = 2$, they wanted to have exact solutions. Kline (1972:599) notes that their criterion for existence was constructibility. It is for this reason that the Greeks used the length of line segments to represent numbers, and irrational numbers in particular (compare with 3.2.2.1).

2.3.2.1 Geometric Algebra based on Euclid

The identity $(a + b)^2 = a^2 + 2ab + b^2$, which is in the South African grade 9 mathematics syllabus, was taught by the Greeks with the aid of the following diagram:

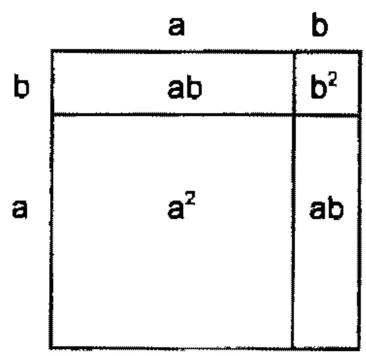


Fig.2-1: The identity $(a + b)^2 = a^2 + 2ab + b^2$.

Baumgart (1989:237) notes that Euclid in his Elements, Book 11, Proposition 4 states this identity as:

> If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the parts.

Note the use of geometry and the concept of area to illustrate the identity. Geometry and algebra are integrated. This idea can be used to teach simplification of certain types of algebraic expressions (refer to 3.3.2.3 for further details). The idea can also be used to teach the method of completing the square (refer to 2.3.4.2 for an illustration).

2.3.2.2 Contributions to the Solution of Quadratic Equations

Tignol (1988:14) notes that

> Although Euclid does not explicitly deal with quadratic equations, the solution of these equations can be detected under some geometrical garb in some propositions of the Elements.

The propositions provide solutions to systems of equations given by normal form 1 and normal form 2, as developed by the Babylonians (refer to 2.3.1.2 and 2.3.1.3).

Van der Waerden (1983:78) notes that in Euclid's Elements, Book 11, Proposition 6 reads:

> If a straight line be bisected and a straight line be added to it, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

The following modern transcription of the proposition is based on the arguments put forward by Van der Waerden (1983:78-80).

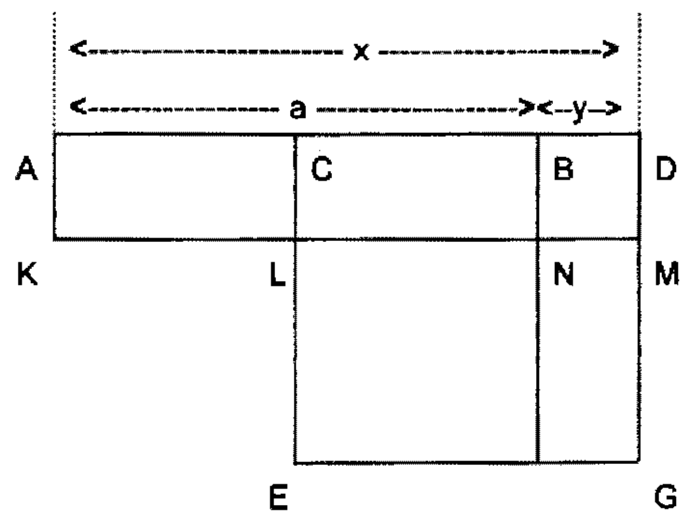


Fig.2-2: An illustration of Proposition 6.

In the figure AB = a is the straight line which is bisected at point C. Therefore AC = $\frac{1}{2}a$. BD = y is the straight line which is added to AB and AD = x. Then AKMD, with area xy, represents *the rectangle contained by the whole with the added straight line and the added straight line.*

Let CD = z. Then from the figure

$$x - y = a$$

$$x = z + \frac{1}{2}a \qquad \text{......[2.6]}$$

$$y = z - \frac{1}{2}a. \qquad \ldots\ldots[2.7]$$

Proposition 6 can now be stated as follows:

$$xy + (\frac{1}{2}a)^2 = (\frac{1}{2}a + y)^2$$

i.e. $$xy + (\frac{1}{2}a)^2 = z^2 \qquad \ldots\ldots[2.8]$$

According to Van der Waerden, the Greeks used Proposition 6 to construct two line segments x and y when the difference $x - y = a$ and the area $xy = b$ were given. [Note that this is the same as normal form 2 discussed under Babylonian algebra (refer to 2.3.1.3).]

They then argued as follows: By [2.8], $z^2 = b + (\frac{1}{2}a)^2$ since $xy = b$ (given). Hence z^2 is a known quantity, since both the terms on the right hand side are known. (In Fig.2-2, the square CEGD referred to in Proposition 6 has area z^2.) But from z^2, z can be obtained by a geometric construction explained below. This then enabled them to construct x and y, using [2.6] and [2.7] respectively.

To construct z from z^2 we proceed as in Fraleigh (1982:357) using constructions and concepts found in the grade 9 interim mathematics syllabus.

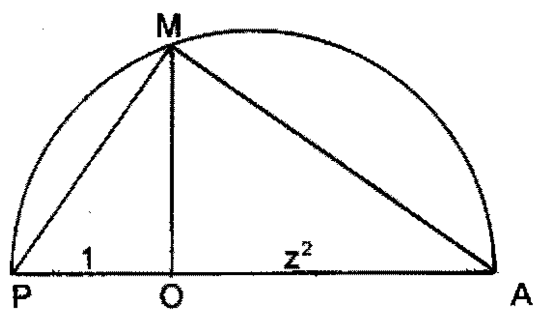


Fig.2-3: Constructing z from z^2.

Fig.2-3 and the following steps show how the number z is constructed from z^2.

- Let line segment OA have length z^2.
- Find a point P on AO produced so that OP has length 1.
- Find the midpoint of PA.
- Use this midpoint to draw a semicircle with PA as diameter.
- Erect a perpendicular to PA at O, intersecting the circle at M.

From Fig.2-3, the triangles MOP and AOM are similar.

Therefore $\frac{OM}{PO} = \frac{AO}{OM}$ so $OM^2 = AO.PO = z^2$ and $OM = z$.

Note that to solve the system of equations given by normal form 2, the Greeks and the Babylonians obtained a quadratic equation in z, which can be solved by extracting a square root. Therefore the Greek method which is based on geometric foundations is essentially the same as the Babylonian method. However Greek geometric algebra reached a higher level of generality in the solution of quadratic equations, since equations with (positive) real coefficients were considered.

From a teaching point of view both approaches can be used to illustrate how geometry can be effectively used to give meaning to, what at first glance seems to be an algebra problem (see 4.4.2). A discussion of solutions to systems of equations of the type $x - y = a$ and $xy = b$, which is done in grade 10, provides the teacher with an ideal opportunity to bring in the relevant history of mathematics. Referring to the discussion on the square roots of constructible numbers being constructible (refer to Fig.2-3), the procedure outlined can be used to construct line segments of given irrational lengths, for example $\sqrt{5}$ cm, in grade 9. The procedure integrates the work done in grade 9 on constructions with ruler and compass, and similar triangles.

2.3.3 Contributions from Chinese Algebra (206 BC to 221 AD)

Greece, Egypt, India and China had a completely developed decimal counting system, and rules for operations with rational numbers $\frac{m}{n}$ (Van der Waerden 1983:45).

The Chinese collection "Nine Chapters of the Mathematical Art" was written between 206 BC and 221 AD (Van der Waerden 1983:36). In this book Chapter 9 contains 16 problems on right-angled triangles. The relation $x^2 + y^2 = z^2$ between the sides of a right-angled triangle is presupposed (Van der Waerden 1983:49).

In all cases the problems are stated with definite numbers, but the solutions are presented in the form of general rules (Van der Waerden 1983:50). This is an important distinction between the Babylonian and Chinese contributions. The Chinese texts present the methods of solution as general rules whereas the Babylonians formulated general rules in only a few cases (refer to 2.3.1.2).

Problem 9:1 in chapter 9 in which the hypotenuse has to be found given the lengths of the other two sides reads:

> The shorter leg is 3 chi'ih, and the longer leg is 4 chi'ih. What is the hypotenuse?

The method of solution is presented in the form of the following general rule (Van der Waerden 1983:50):

> Multiply the shorter and the longer leg each by itself, add, extract the square root.
> This is the hypotenuse.

Note that the rule is equivalent to the correct formula that we are used to, namely $z = \sqrt{x^2 + y^2}$.

The implication for mathematics teaching is that when we solve problems in mathematics we should verbalise the general principles that are involved in the solution of the problem(s). This could lead to the formulation of general rules and an extension of the knowledge base of pupils (see 4.4.2). Furthermore, pupils should be encouraged to state methods of solutions in the verbal form (refer to 3.4.4). The aim here is to be able to work out certain types of problems. This is a higher level of learning. Stating the procedure to work out a particular type of problem is a much higher ability than the working of the problem itself. The development of a basic framework for a mathematical theory is an important aspect of problem solving (compare with 3.4.5.2).

2.3.4 Contributions from Hindu and Arabic Algebra (628 AD to 830 AD)

Hindu, Chinese and Arab mathematicians of the period 500 to 1200 AD took the ideas of Greek mathematicians and explored other approaches to algebra in general (McQualter 1988:3) . The numerous invasions of India facilitated the exchange of ideas. The Hindus and then the Arabs developed techniques of calculation with irrational numbers (Tignol 1988:16). For example, Bhaskara gave the following correct procedure to add two irrational numbers (Kline 1972:185):

Given the irrationals $\sqrt{3}$ and $\sqrt{12}$,

$$\sqrt{3} + \sqrt{12} = \sqrt{(3+12)+2\sqrt{3.12}} = \sqrt{27} = 3\sqrt{3}.$$

Using modern notation the general principle for the addition of irrational numbers, according to the above example is $\sqrt{a} + \sqrt{b} = \sqrt{(a+b)+2\sqrt{ab}}$. Since $(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b$, it follows that the general principle that Bhaskara gave for adding two irrational numbers is correct. Bhaskara also gave rules for multiplication, division and the square root of irrational numbers (Kline 1972:186).

The Hindu number system and operations formed the basis for the development of the decimal number system. McQualter (1988:3) notes that Aryabata (474 AD), Brahmagupta (598 AD) and Bhaskara (114 AD) worked with this system. It seems that the Hindu mathematicians, besides using mathematics to solve practical problems, *also studied mathematics for its own sake*. Perhaps this is what led to some rapid developments by them during this phase. Some of these developments are as follows:

- Lal (1985a:8) notes that "... negative numbers were first introduced by the Hindus and the first known use is by Brahmagupta about 628 A.D."
- McQualter (1988:3) notes that Brahmagupta contributed the most to the solution of the quadratic equation in that he provided general solutions and even accepted two roots even if one was negative.

- According to Baumgart (1989:241), Hindu mathematicians solved quadratic equations by completing the square, accepted negative and irrational roots, and realised that a quadratic equation (with real coefficients) has two solutions. However they did not always bother to find the second root (Wrestler, 1989:303).
- Solving techniques developed from a geometrical approach to a more algebraic approach.

2.3.4.1 Brahmagupta's Method

Brahmagupta gave an interesting rule for finding one of the two positive roots of the equation $x^2 - 10x = -9$ (Wrestler 1989:301). In the original records this equation is written using a syncopated style as follows:

$$\begin{array}{l} \textit{ya} \;\; \textit{v} \;\; 1 \;\; \textit{ya} \;\; \dot{10} \\ \quad \textit{ru} \;\; \dot{9} \end{array}$$

The *ya* refers to the unknown, *v* means 'squared', the dot above the number means the negative of the number. The first line gives the left-hand side of the equation and the second line the right-hand side, *ru* means the 'absolute' or 'plain' number (independent of the unknown). While Brahmagupta used a syncopated style to state the problem, an examination of the records kept indicated that he used a rhetorical style to describe his method to find one of the roots of the equation.

His method is essentially the same as what is today known as the method of completing the square (for details refer to Wrestler 1989:301-303). Al-Khowarizmi, whose work was largely based on Brahmagupta's work (Lal 1985a:7) used geometry to justify the procedure on which Brahmagutpa's method was based [refer to 2.3.4.2, Fig.2-4].

2.3.4.2 Al-Khowarizmi's Operations and Methods

The book *Al-jabr w' al muqabala* written by Mohammed ibn Musa al-Khowarizmi (830 AD) marks an important landmark in the theory of equations (Tignol 1988:16), because of the introduction of the concept of *equivalence* (see 3.3.3.3). Algebra is a Latin variant of the arabic word al-jabr (Baumgart 1989:233). The book dealt with calculation rules of completion (al-jabr) and balancing (muqabalah). The purpose of these calculations were to find a number (solution) to problems arising out of inheritance, trade, partitions, law-suits, measurement of land and digging of canals (McQualter 1988:4).

The operation *al-jabr* refers to the transposition of subtracted terms to the other side of an equation. Given the equation $x^2 + 12x + 7 = 12 - 3x$
al-jabr gives

$$x^2 + 15x + 7 = 12.$$

The operation *al-muqabalah* refers to the elimination of like terms on opposite sides of an equation. So in the equation

$$x^2 + 15x + 7 = 12$$

al-muqabalah gives

$$x^2 + 15x = 5.$$

It is important to note that these two operations led to the concept of *equivalent equations*. As a result of this al-Khowarizmi was able to reduce old methods of solving equations to a few standardised procedures (compare with 4.4.2). He solved the three 'types' of quadratic equations: $x^2 + ax = b$, $x^2 + b = ax$, $x^2 = ax + b$ by the method of completing the square. However his style was *rhetorical*, "and his work was not as good as that of the Babylonians or the Hindus" (Baumgart 1989:242).

Tignol (1988:17) cites the following equation as an example:

$$x^2 + 10x = 39.$$

The statement of this problem and the procedure to find x was explained as the Babylonians did. The statement of the problem is as follows:

a square and 10 roots are equal to 39 units

This al-Khowarizmi gave as an example of the problem type *Roots and Squares equal to numbers* (in modern algebraic notation $x^2 + ax = b$).

The procedure used by al-Khowarizmi to solve the above problem (refer to Van der Waerden 1985:7 for details) amounts to what is today known as the method of completing the square. Intuitive geometric evidence was used afterwards to justify the procedure as follows:

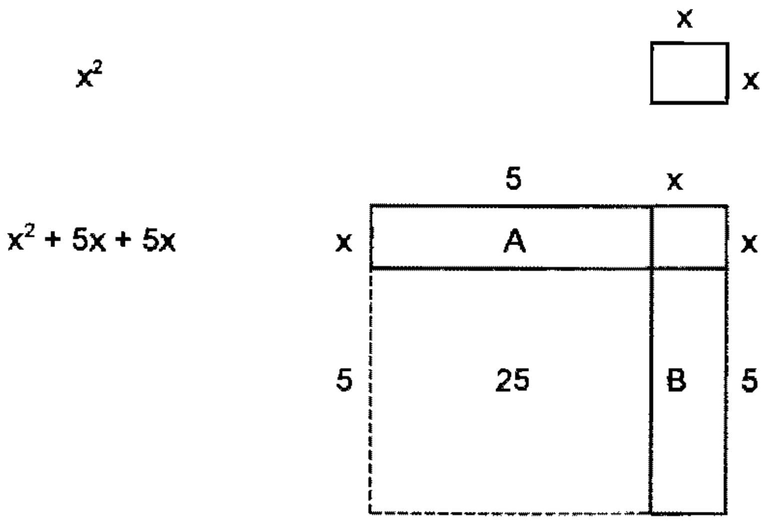


Fig.2-4: Geometric illustration of the method of completing the square.

Now from the sketch

$$(x + 5)^2 = x^2 + 10x + 25$$
$$= 39 + 25 \qquad \text{(since } x^2 + 10x = 39\text{)}$$
$$= 64$$

Therefore $x + 5 = \sqrt{64}$ which gives $x = 3$.

Note that since the geometrical justification is dimension based, $x + 5$ which represents the length of the side of the square cannot take the numerical value $-\sqrt{64}$. So although this geometric approach can be used to bridge the gap between algebra and geometry in grade 11, it is necessary to treat $x + 5$ as a number when the step $(x + 5)^2 = 64$ is arrived at. This will result in the implication that the number $x + 5 = \pm\sqrt{64}$, from which the two roots of the given equation can be found (see 4.4.2, Question 11).

2.3.5 The Quadratic Formula (17^{th} to 19^{th} centuries)

In subsection 2.3.1 it was noted that the Babylonians first solved only certain types of quadratic equations and considered only positive rational coefficients. The Greeks made a clear distinction between numbers and geometric quantities, they introduced irrational lines and their solving approach was geometrical in nature (refer to 2.3.2). Solving techniques then developed to a more algebraic approach as a result of contributions from Chinese, Hindu and Arabic Algebra (refer to 2.3.3 and 2.3.4). It was recognised that two solutions exist - one or both could be positive or negative real numbers. However, solving styles were rhetorical and syncopated.

Symbolism developed to its modern form, as we know it today, in the 17th century (refer to 2.2.3). This together with the discovery of complex numbers in the 16th century (refer to 2.4.2.1) made it possible to express the general quadratic equation in the form $ax^2 + bx + c = 0$ where the coefficients $a \neq 0$, b and c represent real or complex numbers. The first person to write the general quadratic equation in this form was John Napier, around 1589 (McQualter 1988:3).

Keeping in mind the development of symbolic notation, it is not surprising to note that the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which gives the solution of quadratic equations could not have been written in this form before the 17^{th} century (Tignol 1988:6). According to Smith (1953:450), of the modern methods used to obtain the quadratic formula, Euler and Bezout used determinants (in the context of matrices). Smith notes that Sylvester in 1840 and Hesse in 1844 improved on the methods used by Euler and Bezout. However the coefficient of x^2 was taken to be 1. For details of the method used refer to Smith (1953:450). So it seems that finally in the 19^{th} century the quadratic formula in the above form was deduced, and this formula could represent rational, real or complex roots (compare with 4.4.2, Question 17).

2.4 SOLVING CUBIC EQUATIONS (16th century)

According to Lal (1985a:8), $y^3 = ab^2$ is perhaps the first cubic equation which arose in the problem of the duplication of a cube, around 460 BC. This problem of the duplication of a cube amounts to finding an edge of a cube whose volume is twice the volume of a given cube. The story of its origin is that the Athenians appealed to the oracle at Delos to know how to stay the plague which visited their city in 430 BC (Smith 1953:298). According to the story the oracle replied that they must double in size the altar of Apollo, which was the shape of a cube. This led to the problem of the duplication of a cube, one of the problems that the Greeks could not solve by using only an unmarked ruler and compass.

Omar Khayyam, a Persian poet, philosopher, mathematician and astronomer who lived in the 11th century (AD) used graphical methods (Lal 1985a:8) to solve cubics of the types

$$x^3 + b^2x = b^2c,\ \ x^3 + ax^2 = c \ \text{ and } x^3 + ax^2 + b^2x = b^2c.$$

Omar solved cubics of the type $x^3 + b^2x = b^2c$ by finding the intersection of the graphs of $x^2 = by$ and $y^2 = x(c - x)$. An illustration of the method is given by Tignol (1988:19). Omar believed that the general cubic equation could only be solved graphically. [From a teaching point of view, the types of graphs dealt with in the grade 10, 11 and 12 syllabuses could be exploited to solve certain types of equations. An approach that could be used is given in 3.4.2. Also see 4.4.2, Questions 7 to 10.]

In the meantime Leonardo da Pisa (Fibonacci) an Italian merchant in the 12th century, as a result of his trade travels, composed several works (refer to Van der Waerden 1985:33) which were published in the period 1202 to 1225. He introduced medieval scholars to the Hindu-Arabic numerals, methods of calculations with integers and fractions (Van der Waerden 1985:35), propositions (without proofs) from Euclid's Elements and applications of these propositions, and a systematic treatment of linear and quadratic equations based on al-Khowarizimi's arithmetical and algebraic writings (refer to Van der Waerden 1985:33-42).

In the 14th and 15th centuries it seems that in Italy loan problems of the following type led to cubic equations and a need for their solutions (Van der Waerden 1985:42-49):

Someone lends to another one 100 Lira, and after 3 years receives 150 Lira with *annual capitalisation of the interest. One asks at what monthly rate of interest the loan was given.* (Van der Waerden 1985:49)

Using modern notation and the compound increase formula, which is done in the grade 12 (standard grade) interim mathematics syllabus, the following cubic equation is arrived at: $150 = 100[1 + r/100]^3$, where r is the annual interest rate.

The discovery of the algebraic solution of cubic equations is still surrounded by mystery since a lot of secrecy existed amongst mathematicians, who used their methods for personal advantage in mathematical duels and tournaments (Hood 1989:277). Nevertheless, the real break-through in the solution of the cubic is found in the work of Italian algebraists of the 16th century.

2.4.1 Tartaglia's Solution

The algebraic solution of $x^3 + mx = n$ was discovered in 1535 by Niccolo Fontana nicknamed "Tartaglia" which means 'Stutterer' (Van der Waerden 1985:55). Tartaglia discovered the solution when he was challenged to a public problem-solving contest. The problems that he was given, all led to solutions of equations of the type $x^3 + mx = n$. In 1539, Tartaglia handed to Girolama Cardano the solution, in verses, of the following forms of cubic equations:

$x^3 + mx = n$, $x^3 = mx + n$ and $x^3 + n = mx$

where the coefficients m and n are *positive* (Tignol 1988:21).

The following two verses give the equation of the form $x^3 + mx = n$, and the procedure to solve the equation (Tignol 1988:22).

- *the cube and the things* ($x^3 + mx$) *equal to a number* (n).
- *find two numbers which differ by the given number* (n) *and such that their product is equal to the cube of the third of the number of things. Then the difference between the cube roots of these numbers is the unknown* (x).

Using modern symbolic notation, to find the solution of $x^3 + mx = n$, the procedure is to determine x as follows:

(1) find two numbers t, u such that $t - u = n$

and $tu = (\frac{m}{3})^3$

(2) $x = \sqrt[3]{t} - \sqrt[3]{u}$.

To determine the values of t and u the solution of normal form 2 is used (refer to subsection 2.3.1.3):

$$t = \sqrt{(\frac{n}{2})^2 + (\frac{m}{3})^3} + (\frac{n}{2})$$

$$u = \sqrt{(\frac{n}{2})^2 + (\frac{m}{3})^3} - (\frac{n}{2}).$$

Substituting in (2) above we get a solution of the equation $x^3 + mx = n$. Note that known knowledge (normal form 2) was used to solve this type of equation. This demonstrates Tartaglia's use of the problem solving strategy of relating the new problem to a known simpler problem (compare with 4.3.2).

2.4.2 Cardano's Formula

Cardano found justification for Tartaglia's formulae. He also solved particular cubic equations of the types

$x^3 + mx = n$, $x^3 = mx + n$, $x^3 + mx + n = 0$ and $x^3 + n = mx$.

Kline (1972:265) cites two reasons as to why Cardano treated each of these equations separately. Up to this time the Europeans wrote equations so that only terms with positive coefficients appeared in them. Another reason as to why Cardano treated each of the above equations separately is that he had to give a separate geometrical justification for the rule in each case. Cardano also devised a method to solve the general cubic equation, $x^3 + ax^2 + bx + c = 0$, by eliminating the term in x^2. His method as outlined by Tignol (1988:24) is as follows:

Put $x = y - \frac{a}{3}$, then we get the following cubic which lacks the second degree term

$y^3 + py + q = 0$ ....[1]

where $p = b - \dfrac{a^2}{3}$ ....[2]

and $q = c - \dfrac{ab}{3} + 2(\dfrac{a}{3})^3$. ....[3]

Let $y = \sqrt[3]{t} + \sqrt[3]{u}$ ....[4]

then $y^3 = t + u + 3(\sqrt[3]{t} + \sqrt[3]{u})\sqrt[3]{tu}$.

Equation [1] now becomes

$$(t + u + q) + (3\sqrt[3]{tu} + p)(\sqrt[3]{t} + \sqrt[3]{u}) = 0.$$

This equation holds if

$$t + u + q = 0$$

and $$3\sqrt[3]{tu} + p = 0.$$

The following system of two equations follow:

$$t + u = -q \text{ and } tu = -(\frac{p}{3})^3$$

which is in the form of normal form 1 under the solution of quadratic equations (refer to 2.3.1.2). Using the appropriate formulae from this subsection we get

$$t,u = -(\frac{q}{2}) \pm \sqrt{(\frac{p}{3})^3 + (\frac{q}{2})^2}\ . \qquad ....[5]$$

Substituting in equation [4] above, the value of y is obtained. Using this value in

$x = y - \dfrac{a}{3}$ a solution of the general cubic equation is obtained.

When a = 0 we get

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{(\frac{p}{3})^3 + (\frac{q}{2})^2}} + \sqrt[3]{-\frac{q}{2} - \sqrt{(\frac{p}{3})^3 + (\frac{q}{2})^2}}$$

which is known as *Cardano's formula* for a solution of $x^3 + px + q = 0$. Note that in this equation and equation [4] no mention is made of which cube root to take, since the ambiguity of nth roots was not yet discovered (refer to 2.6.3).

2.4.2.1 Questions arising from Cardano's Formula

The use of *Cardano's formula* to get a solution for equations of the form $x^3 + px + q = 0$ led to important investigations. Some questions that arose from these investigations were not fully settled until 1732 (Hood 1989:278).

How many roots does a cubic equation have? This question was prompted by the following observation: 2 is a solution of $x^3 - 12x = -16$, but Cardano's formula yields $x = -4$. This led Cardano to observe that a cubic equation may have three solutions. Cardano noted that these solutions could include negative numbers. In Chapter 1 of the *Ars Magna*, Cardano presents a complete discussion of the number of positive or negative roots of cubic equations of the type $x^3 + px + q = 0$ (Van der Waerden 1985:56).

The development of a more efficient notation which was necessary to explore the new ideas arising from the solution of cubic and quartic equations, led to the concept of a polynomial (refer to 2.2.4). According to Tignol (1988:51) the following fact, which was based on observations, is credited to Descartes:

A number b *is a root of a polynomial* P(x) *if and only if* x - b *divides* P(x).

Around the middle of the 17th century the following fact viz.

the number of solutions of an equation is equal to the degree of the equation,

was accepted without proof and never questioned (Tignol 1988:52). These ideas based on observations were explained and accepted by a kind of incomplete induction.

By inspection, 4 is a solution of $x^3 = 15x + 4$. Use of Cardano's formula yields the following solution $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$. The discussion of the case $(\frac{p}{3})^3 + (\frac{q}{2})^2 < 0$ in Cardano's formula led to the use of complex numbers. According to Van der Waerden (1985:56), Cardano knew about the difficulty of extracting a square root from a negative number. In Chapter 37 of the *Ars Magna* Cardano poses the problem: *Divide 10 into two parts, the product of which is 40.* Although Cardano writes that "It is clear that this is impossible", he goes on to argue and verify that these parts must be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Van der Waerden (1985:56) states that to the best of his knowledge Cardano was the first to introduce complex numbers $a + \sqrt{-b}$ into algebra.

The next steps in the development of complex numbers was taken by Bombelli in 1572 (Van der Waerden 1985:59,177). Bombelli used the expressions which in modern notation we know as i (for $\sqrt{-1}$) and $-i$ (for $-\sqrt{-1}$). He also gave rules for calculating with complex numbers. The term imaginary number was introduced by Descartes (1637). Leading mathematicians, after Descartes, made free use of complex numbers. Some of these mathematicians were Abraham de Moivre (refer to 2.6.3) and Leonard Euler. So it seems that just as there was a reluctance by the Greeks to accept the existence of irrational numbers, there was initially a similar reluctance by mathematicians to accept the concept of the new 'object' complex numbers. However as these new numbers were used they came to be accepted and their existence was acknowledged. This is an illustration of reification (Sfard 1995:16) of the concept of complex number (refer to 3.2.2.1 for further details).

The questions arising from Cardano's formula and the later developments in mathematics indicate the role and the power of problem solving and problem posing in the development of mathematical knowledge. Where possible these strategies should be exploited in the classroom. The problem "*Divide* 10 *into two parts the product of which is* 40", could serve as an ideal problem to introduce pupils to complex numbers (see 4.4.3, Question 22). In the teaching of the solution of cubic equations the following type of question could be given:

<u>Question 1</u>

Consider the cubic equation $x^3 = -x + 2$

1.1 What are the different approaches that could be used to solve this equation?

1.2 How many roots does this equation have?

1.3 Determine all the roots.

History of mathematics provides an insight into how mathematical knowledge arises and develops. For example, what we today know as the factor theorem arose out of observations by Descartes. The fact that the number of solutions of a polynomial equation is equal to the degree of the equation, also arose out of observations. From a teaching point of view the power of observations in mathematics should not be underestimated. Problems that are normally given to pupils for application or in assignments could be formulated with appropriate sub-questions aimed at particular observations or 'discoveries' (compare with 4.4.2 and 4.4.3). The next step could be to provide suitable proofs (where possible) for the observations or general statements.

2.5 SOLVING QUARTIC EQUATIONS (16th century)

A solution to the general quartic equation was found soon after that for the cubic equation. In 1540 Ludovic Ferrari, a pupil of Cardano, solved a challenging problem that his teacher could not solve (Hood 1989:278).

2.5.1 Ferrari's Method

His method of solving the general quartic equation can be described as follows:

i. Use a change of variable to eliminate the term of the third degree.
ii. Rearrange the terms so that the terms of degree 4 and 2 are on the left-hand side of the equation, and add a suitable polynomial quantity to both sides so that the left-hand side is a perfect square.
iii. The polynomial quantity is then determined so that the right-hand side is also a square.
iv. This condition leads to a cubic, which can be solved.
v. The quartic can then be easily solved.

Using modern notation, the following discussion of Ferrari's solution to the general quartic equation $x^4 + ax^3 + bx^2 + cx + d = 0$ is based on the work by Tignol (1988:33) and follows the procedure outlined above.

(1) Put $x = y - \frac{a}{4}$, then $y^4 + py^2 + qy + r = 0$ where p, q and r are in terms of a, b, c or d.

(2) $y^4 + py^2 + qy + r = 0$

$\Leftrightarrow y^4 + py^2 = -qy - r$

$\Leftrightarrow (y^2 + \frac{p}{2})^2 = -qy - r + \frac{p^2}{4}$

$\Leftrightarrow [(y^2 + \frac{p}{2}) + u]^2 = -qy - r + \frac{p^2}{4} + 2uy^2 + pu + u^2$

In the second last equation $\frac{p^2}{4}$ is added so that the left-hand side is a perfect square. The addition of $2uy^2 + pu + u^2$ to both sides of this equation gives a perfect square on the left-hand side as is evident from the last equation under (2). Therefore the quantity that has been added is $\frac{p^2}{4} + 2uy^2 + pu + u^2$.

(3) $2uy^2 - qy - r + \frac{p^2}{4} + pu + u^2 = [\sqrt{2u}y - \frac{q}{2\sqrt{2u}}]^2$

where $(-r + \frac{p^2}{4} + pu + u^2) = \frac{q^2}{8u}$.

(4) Equating the independent terms ($-r + \frac{p^2}{4} + pu + u^2$) and $\frac{q^2}{8u}$ leads to the following cubic equation in u

$8u^3 + 8pu^2 + (2p^2 - 8r)u - q^2 = 0,$

which can be solved.

(5) From (2) and (3)

$[(y^2 + \frac{p}{2}) + u]^2 = [\sqrt{2u}y - \frac{q}{2\sqrt{2u}}]^2$

therefore

$$y^2 + \frac{p}{2} + u = \pm[\sqrt{2u}y - \frac{q}{2\sqrt{2u}}],$$

which gives two quadratic equations in y that can be solved.

Substitute in $x = y - \frac{a}{4}$ in order to get x.

Note that if q = 0 then the quartic equation is $y^4 + py^2 + r = 0$, which is a quadratic equation in y^2.

In the last two subsections it was noted that by 1540 the general cubic and quartic equations were solved by Cardano and Ferrari respectively. Some of the questions that arose from the use of Cardano's formula were not fully settled until 1732 (refer to 2.4.2.1). In order to fully understand the new ideas arising from the solution of the cubic and quartic equations there was a need to develop a more efficient notation. The symbolic notation that was developed led to further developments in mathematics. Some of these developments were discussed in 2.2.4. Investigations into the solving of polynomial equations led to developments in the theory of equations (compare with 4.4.2 and 4.4.3). Some of these developments are discussed below.

2.6 DEVELOPMENTS IN THE THEORY OF POLYNOMIAL EQUATIONS

The solution of polynomial equations was one of the investigations that continued in the 16^{th} and 17^{th} centuries. Interest focused on *better methods to solve equations* of any degree and in *proving* that every nth degree polynomial equation has n roots.

2.6.1 The relationship between roots and coefficients

According to Tignol (1988:47), Viète in 1615 stressed the importance of understanding the relationship between the roots of a polynomial equation and the coefficients of the equation. Albert Girard in 1629 is credited with discovering the relationship between the roots and coefficients of polynomial equations (Tignol 1988:48). Using modern notation his observation is as follows:

$x^n - s_1x^{n-1} + s_2x^{n-2} - s_3x^{n-3} + \ldots\ldots + (-1)^n s_n = 0$

has n roots $x_1, x_2, x_3, \ldots\ldots, x_n$ such that

$s_1 = x_1 + x_2 + x_3 + \ldots\ldots + x_n$

$s_2 = x_1x_2 + x_1x_3 + \ldots\ldots + x_1x_n$

$s_3 = x_1x_2x_3 + x_1x_4x_5 + \ldots\ldots + x_{n-2}x_{n-1}x_n$

. . . .

$s_n = x_1x_2x_3 \ldots\ldots x_n$

Note that these relationships easily follow by expanding the right-hand side and equating the coefficients of the monic polynomial in the following:

$x^n - s_1x^{n-1} + s_2x^{n-2} - s_3x^{n-3} + \ldots\ldots + (-1)^n s_n = (x - x_1)(x - x_2)(x - x_3)\ldots\ldots(x - x_n).$

According to Tignol (1988: 48-50), Girard in 1629 showed how the coefficients of any monic polynomial are related to the zeros of the polynomial (compare with 4.5.2, Problems 2 and 3). Note that for the above relationships between the roots and the coefficients of a polynomial equation to apply, the coefficient of the leading term must be 1. For the quadratic equation $x^2 + bx + c = 0$, if x_1 and x_2 are the roots then $x_1 + x_2 = -b$ and $x_1x_2 = c$.

Efforts to solve polynomial equations of degree higher than four concentrated on the general polynomial equation. This led to subsidiary work on *symmetric functions* which proved to be important (Kline 1972:600). The expression $x_1x_2 + x_2x_3 + x_3x_1$ is a symmetric function of x_1, x_2 and x_3 since the interchanging throughout of any x_i by a x_j and any x_j by a x_i leaves the expression unchanged.

Note that Girard observed that the coefficients of a polynomial equation are the various sums of the products of the roots. These various sums are symmetric functions of the roots of the polynomial equation. With regard to the roots of a polynomial equation, Kline (1972:600) notes that Newton *proved*

> ... that the various sums of the products of the roots of a polynomial equation can be expressed in terms of the coefficients...

and that Vandermonde in 1771 showed

> ... that any symmetric function of the roots can be expressed in terms of the

coefficients of the equation.

These results that are based on the roots of polynomial equations proved to be important (refer to 2.6.5 for further details).

2.6.2 Investigations on the factorisation of real polynomials

The technique of factorisation is now often used to solve polynomial equations (see 4.4.2, Question 6). This technique took a long time to develop. According to Smith (1953:448) the first important treatment of the solution of quadratic and other equations by factorisation was given by Harriot in 1631. Tignol (1988:32, 86) notes that in 1637 Descartes recommended the following way of tackling equations of any degree:

> First, try to put the given equation into the form of an equation of the same degree obtained by multiplying together two others, each of lower degree.

Note that the technique of factorisation was made possible after the development of modern symbolic notation, which created the concept of a polynomial around 1585 (refer to 2.2.4).

Leibniz wrote a paper in 1702 on the integration of rational fractions (Tignol 1988:98). In this paper Leibniz pointed out the usefulness of the decomposition of rational fractions into partial fractions. This decomposition requires that the denominator be factorised into a product of irreducible polynomials with real coefficients. So this technique of integration of rational functions thus contributed to the investigation of factorisation of real polynomials.

While working on an integration problem Leibniz could not find a real divisor of $x^4 + a^4$ (Tignol 1988:99). Note that this expression can be factorised with real coefficients by adding and subtracting $2a^2x^2$ and then working with the difference of two squares as follows:

$$\begin{aligned} x^4 + a^4 &= x^4 + 2a^2x^2 + a^4 - 2a^2x^2 \\ &= (x^2 + a^2)^2 - 2a^2x^2 \\ &= [x^2 + a^2 - \sqrt{2}ax][x^2 + a^2 + \sqrt{2}ax]. \end{aligned}$$

According to Tignol (1988:99) this technique was shown by Bernolli in 1719.

The following question arose from investigations on the factorisation of real polynomials. *Can a polynomial with real coefficients be factorised into a product of irreducible polynomials with real coefficients?* This question was eventually answered by Gauss in 1799 (refer to 2.6.4).

2.6.3 The ambiguity of the nth roots

The result $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ where n is a natural number is known as De Moivre's Formula. In 1739 De Moivre used trigonometric representations of complex numbers and his formula to extract the nth root of a complex number (Tignol 1988:105). His work resulted in the following important result on the roots of a complex number:

Let u be a non-zero complex number. Then $u = r(\cos\theta + i\sin\theta)$ has n distinct nth roots $u_0, u_1, \ldots\ldots, u_{n-1}$ given by

$$u_k = \sqrt[n]{r}[\cos(\frac{\theta + k360°}{n}) + i\sin(\frac{\theta + k360°}{n})] \text{ where } k = 0, 1, 2, \ldots, n-1.$$

An important implication of the above result is that the nth root of any non-zero number is ambiguous. This implies that every formula that involves the extraction of a root needs some clarification as to which root must be chosen. In Cardano's formula which is given in 2.4.2 no clarification was given as to which cube roots should be taken. If the various cube roots are considered then the three solutions of the cubic equation are obtained. So it seems that De Moivre's work led to a deeper insight on the ambiguity of the roots of (complex) numbers. This insight was particularly relevant to the problem of solving equations by radicals as will be seen in 2.6.5.

2.6.4 The Fundamental Theorem of Algebra

It was noted in 2.6.2 that the method of partial fractions for integration led to investigations on the factorisation of real polynomials. An important question that arose was the following: *Can any polynomial with real coefficients be decomposed into a product of linear factors or a product of linear and quadratic factors with real coefficients?* The crux to this question was to prove that every non-constant polynomial has at least one real or complex zero. This latter fact, which is now known as *the fundamental theorem of* *algebra*, became a major goal. The significance of this for the solving of polynomial equations is that *every* polynomial equation with rational coefficients is solvable over the complex numbers, but *not* by radicals, as Abel was able to prove (refer to 2.7.2).

The first substantial proof of the fundamental theorem was given in 1799 by Gauss, who later gave three further proofs of the theorem, with each proof improving in rigour as mathematics developed (Kline 1972:598) . His first proof used graphical arguments and concentrated on demonstrating the existence of a root instead of just calculating a root. The second proof dispensed with geometrical arguments but assumed that a polynomial could not change signs at two different values of x without taking on the value 0 in between. The proof of this fact was beyond the rigour of the times (Kline 1972:599). The full statement of the fundamental theorem includes the case of complex numbers as coefficients of the polynomial. It was Gauss's fourth proof which used complex numbers freely, as by then they were regarded as common knowledge.

Note that the fundamental theorem of algebra leads to the following result which is now known as the complete factorisation theorem (Fleming & Varberg 1989:328):

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1 + a_0$ is an nth degree polynomial with $n>0$, then there are n numbers c_1, c_2, ..., c_n, not necessarily distinct, such that $P(x) = a_n (x - c_1)(x - c_2) \ldots\ldots (x - c_n)$. The c's are the zeros of P(x); they may or may not be real numbers.

From the complete factorisation theorem it follows that a polynomial equation of degree n has exactly n roots, provided multiple roots are counted as many times as their multiplicity. Gauss proved this result in 1799 as part of his doctoral thesis (Lal 1985b:31).

In 2.3.2 it was noted that the Greeks regarded constructibility (in the geometric sense) as their criterion for existence. In the formal work of succeeding centuries existence in mathematics was established by actually obtaining or exhibiting the quantity in question (Kline 1972:599). For example, in the case of a quadratic equation the existence of solutions is established by obtaining quantities that satisfy the equation. The methods given by Tartaglia, Cardano and Ferrari are such that the solutions could be obtained by using the four basic operations and extraction of roots. Existence is therefore an important factor in mathematics. The existence of mathematical entities must be first established before theorems about them can be considered (compare with 4.3.4). Note that Gauss' approach to the fundamental theorem led to a new approach to the question of mathematical existence. Also his 'proofs' introduced greater mathematical rigour and abstraction. However, the starting point was his first 'proof' which relied on graphical representations and arguments (compare with 4.5.4, problem 3). These visual representations enabled him to introduce greater degrees of abstraction in his later proofs.

2.6.5 Lagrange's contribution to the solution of equations

Lagrange, an Italian mathematician, and Vandermonde are credited with doing outstanding work in the 18th century on the problem of the solution of equations by radicals (Kline 1972:600). Lagrange analysed the solutions of equations of degree 2, 3 and 4 from all angles (Edwards 1984:19). He wanted to know why the methods for solving the 3rd and 4th degree equations worked. He also wanted to determine what clues these methods could provide for solving equations of degree 5 or more.

Commenting on his reasons for examining the known methods to solve polynomial equations of degree 3 and 4, Lagrange as quoted by Tignol (1988:168) notes

> This examination will have a double advantage: on one hand, it will shed greater light on the known solutions of the third and the fourth degree; on the other hand, it will be useful to those who will want to deal with the solution of higher degrees, by providing them with the various views to this end and above all by sparing them a large number of useless steps and attempts.

Lagrange's examination of the previously known methods can be described as a unification and reassessment of these methods. In his examination of the various methods for solving equations he noted that they have common features. They all reduce the problem by some clever transformation to the solution of an auxiliary or 'reduced' (the term used by Lagrange) equation. The auxiliary equation has a smaller degree or can be decomposed into other equations of smaller degree than the proposed equation. The required solutions of the proposed equation are then obtained from the roots of the auxiliary equation. For example, Ferrari in his method reduced the general quartic equation to the equation in step 5 (refer to 2.5.1). This equation split into two quadratic equations which yield the four roots of the proposed quartic equation. Lagrange examined Cardano's method of solving the cubic and noted that his method also depends on an auxiliary equation (refer to Tignol 1988:169-170 for details, or use the substitution $y = t + u$ instead of the substitution given by equation [4] in 2.4.2).

After observing these common features of the various methods of solving equations, Lagrange analysed *why* each method works. Lagrange's idea involves reversing the steps and determining the roots of the auxiliary equations as functions of the roots of the proposed equations. He noted that

> ... the secret must lie in the relation that expresses the solutions of the reduced equation in terms of the solutions of the proposed equation (Kline 1972:601).

For each of the known methods Lagrange showed that the roots of the auxiliary equations can be expressed in terms of the roots of the proposed equation (refer to Tignol 1988:170-173 for details relating to the cubic equation; 177-179 for details relating to the quartic equation).

For example let $x^3 + nx + p = 0$ ...... [1]
be the proposed equation. The transformation $x = y - (n/3y)$
gives the auxiliary equation $y^6 + py^3 - n^3/27 = 0$ ...... [2]
which is a quadratic in y^3.
Let w be a cube root of unity other than 1. Then each of the roots of the auxiliary equation [2] can be obtained from the expression $y_1 = (1/3)(x_1 + wx_2 + w^2x_3)$ by permutations of x_1, x_2 and x_3 which are the roots of the proposed equation (refer to Tignol 1988:170-173 for details). Lagrange further observed that certain permutations of the roots of the proposed equation give the degree of the reduced equation. The reason for this is that some permutations of the roots of the proposed equation result in the expression taking on the same values (for details relating to the cubic refer to Tignol 1988:173-174). This is why the reduced equation has a smaller degree or can be decomposed into equations of smaller degree than the proposed equation.

The general results behind Lagrange's arguments are given by the following propositions which deal with the roots of the general equation:

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0 \text{ of degree } n.$$

Proposition 1
Let f and g be functions of the roots of the general equation of degree n. If f admits all permutations of the roots x_i that g admits (and even permutations that g does not admit), then f can be expressed rationally in terms of g and the coefficients of the general equation of degree n.

Note that when a function does not change under a permutation performed on its variables, one says that the function *admits* the permutation. Thus the function $x_1 + x_2$ admits the permutation of interchanging x_1 and x_2, but the function $x_1 - x_2$ does not.

Consider the quadratic equation $x^2 + bx + c = 0$. Let x_1 and x_2 be the roots of this equation. Then the function $x_1 - x_2$ admits only the identity permutation. Therefore the function x_1 (the f) admits all permutations that $x_1 - x_2$ (the g) admits. By proposition 1, x_1 can be expressed rationally in terms of $x_1 - x_2$ and the coefficients of the quadratic equation as is evident from the following:

$$x_1 = \frac{-b + (x_1 - x_2)}{2} \qquad \dots\dots [3]$$

Proposition 2
Let f and g be rational functions of the n roots of the general equation of degree n. If f takes on m different values by the permutations which g admits, then f is a root of an equation of degree m whose coefficients are rational functions in g and in the elementary symmetric polynomials $s_1, s_2, \dots, s_n$. This equation of degree m can be constructed.

For example, for the quadratic equation stated above, the function $x_1 - x_2$ (the f) does not admit all the permutations that $x_1 + x_2$ (the g) admits, but takes on the two different values $x_1 - x_2$ and $x_2 - x_1$. Therefore by proposition 2, $x_1 - x_2$ is a root of a second degree equation whose coefficients are rational functions of $x_1 + x_2$ and of b and c [which can be expressed in terms of elementary symmetric polynomials (refer to 2.6.1)]. Since $b^2 - 4c = (x_1 - x_2)^2$, $x_1 - x_2$ is a root of $t^2 - (b^2 - 4c) = 0$. By taking the value of this root, namely $\sqrt{b^2 - 4c}$, the value of x_1 can be found by using equation [3].

For the cubic equation [1], the expression (the f)

$$(x_1 + wx_2 + w^2x_3)^3,$$

where w is a cube root of unity other than one, takes on two values under the six possible permutations of the roots, while $x_1 + x_2 + x_3$ (the g) admits all permutations. Let the two values be A and B, then it can be shown that A and B are the roots of a second degree equation whose coefficients are rational in p and n (for further details refer to Kline 1972:603-604).

For the fourth degree equation, Lagrange used the function (the f)

$$x_1x_2 + x_3x_4. \qquad \ldots\ldots [4]$$

This takes on three different values under the 24 possible permutations of the roots, whereas $x_1 + x_2 + x_3 + x_4$ (the g) admits all 24. Therefore [4] is a root of a third degree equation whose coefficients are rational functions of the coefficients of the original equation.

Note that in all three examples above, g itself is an elementary symmetric polynomial and can therefore be expressed in terms of the coefficients of the original equation (refer to 2.6.1 for further details).

Lagrange was able to demonstrate that his method worked for the general second, third and fourth degree equations. For the cubic and quartic equations he was able to find resolving functions that satisfied equations of degree 2 and 3 respectively, so these could be solved. (Note that the term resolving function refers to the function f in proposition 2, therefore in each of the three examples above the function f denotes the resolving function.) For the quintic equation Lagrange had to solve a sixth degree equation and he could not find a resolving function that would satisfy an equation of degree less than 5 (Kline 1972:605). Lagrange then concluded that the solution by radicals of polynomial equations of degree 5 and higher was *likely to be impossible*.

Note that Lagrange's method applied only to the general equation since both his basic propositions assume that the roots are independent. Further, his work did not give any criterion for picking the function f (in proposition 2) that would satisfy an equation which is solvable by radicals. However, his idea that one must consider the number of values that a rational function takes when its variables are permutated led to the theory of permutation or substitution groups.

2.6.6 Binomial or Cyclotomic Equations

A *binomial equation* is an equation of the type $x^n - 1 = 0$, where n is a natural number. This type of equation is also called a *cyclotomic equation*. The problem of solving equations of degree 5 or more focused for a while on the special case of the binomial equation. In 1801, Gauss made his second contribution to the theory of equations by studying the solution of cyclotomic equations (Tignol 1988:219).

Gauss considered the equation $x^p - 1 = 0$ where p is prime. ...... [1]

Note that by De Moivre's work the roots of this equation are given by

$$x_k = \cos\frac{k360^\circ}{p} + i\sin\frac{k360^\circ}{p} \quad \text{where } k = 0, 1, 2, \ldots\ldots, p-1.$$

Now the complex numbers x_k when plotted in the complex plane form the vertices of a p-sided regular polygon and these vertices lie on the unit circle. As a result of these facts equation [1] is often called the cyclotomic equation or the equation for the division of a circle.

According to Kline (1972:753), Gauss showed that the roots of the cyclotomic equation [1] can be rationally expressed in terms of the roots of a sequence of equations

$Z_1 = 0,\ Z_2 = 0,\ \ldots\ldots$ ...... [2]

Gauss then made the following observations

- The coefficients in each of the equations in the sequence are rational in the roots of the preceding equations of the sequence.
- The degrees of the equations in [2] are the prime factors of p-1.
- For each prime factor of p-1 there is a Z_i.
- Each of the equations $Z_i = 0$ can be solved by radicals.

▸ Therefore equation [1] can be solved by radicals.

Consider the cyclotomic equation $x^5 - 1 = 0$. A primitive root satisfies the equation $(x^5 - 1)/(x - 1) = x^4 + x^3 + x^2 + x + 1 = 0$. Since zero is not a root of this equation, dividing by x^2 and noting that $(x + x^{-1})^2 = x^2 + 2 + x^{-2}$ gives the equation

$$(x + x^{-1})^2 - 2 + (x + x^{-1}) + 1 = 0.$$

This gives the equation $y^2 + y - 1 = 0$ ...... [3]

where $y = x + x^{-1}$. ...... [4]

Equation [4] is equivalent to the equation

$$x^2 - xy + 1 = 0. \qquad \text{...... [5]}$$

Now consider the sequence of equations given by [3], [5]. Using $p = 5$, the equations in this sequence satisfy the observations made by Gauss as listed above.

Gauss' result is of great significance for the problem of solving the general nth degree polynomial equation by radicals. The result shows that for each prime greater than or equal to 5 there is a cyclotomic equation (over the rationals), which although it cannot be solved completely over the rationals, can be solved by radicals. Therefore Gauss' work on cyclotomic equations showed that *there are some equations of degree 5 or more that are solvable by radicals.*

2.7 THE SOLUTION OF EQUATIONS OF DEGREE 5 OR MORE

It was noted in 2.6.5 that Lagrange concluded that the solution, by radicals, of polynomial equations of degree 5 or more was unlikely. Gauss showed that there are equations of degree 5 or more that are solvable by radicals (refer to 2.6.6). Significant further work on the solvability of equations of degree 5 or more was done by Ruffini, Abel and Galois. Various writers (eg. Kline 1972, Lal 1985b, Bell 1986, Stewart 1992, Rosen 1995) have documented the contributions made by these mathematicians.

There are some quintics which are obviously solvable by radicals. For example any quintic $p(x)$ of the form $p(x) = f(x)g(x)$, where the degrees of both f and g are less than 5, is solvable by radicals. The question is: *Is the general quintic solvable by radicals?* Abel showed that the answer is to this question is *no* (refer to 2.7.2).

2.7.1 Ruffini's contribution

Paolo Ruffini (1765-1822), an Italian mathematician, extended Lagrange's method to equations of degree 5 or more. He believed that he had proved that an algebraic solution (by radicals) of general polynomial equations of degree 5 and more was impossible. However his efforts were not conclusive and his arguments were not clear. In his proof on the impossibility of the solution of the general quintic, Ruffini used the following result without actually proving it (Lal 1985b:36):

> The roots of an equation solvable by radicals can be given such a form that each of the radicals occurring in the expressions for the roots is expressible in terms of rational functions of the roots of the equation and roots of unity.

Ruffini's use of this result without proving it led to his work, on the insolvability of polynomial equations of degree 5 or more by radicals, as being regarded as inconclusive.

2.7.2 Abel's contribution

Niels Henrik Abel, a poor Norwegian mathematician, became interested in the problem of solving, by radicals, equations of degree five or higher. He believed that he had solved the general fifth degree equation by radicals (Kline 1972:754). Abel sent his paper to the Danish Academy for publication. The Danish mathematician Degen could not discover any faults in Abel's arguments. However he requested that Abel illustrate his method by an example (Lal 1985b:34).

While constructing an example, Abel discovered that his method was wrong (compare with 4.5.4, problem1). Abel then took up the reverse view of the problem. Lal (1985b:34) notes that

> ... by the Christmas of 1823 he was able to demonstrate convincingly that it was not possible to solve the general equation of degree five or more by radicals.

According to Lal (1985b:34), Abel was unaware of the work done by Ruffini, some 20 years earlier, on the impossibility of the solution of the general quintic equation by radicals. Kline (1972:754) notes that Abel first succeeded in proving the result stated in

2.7.1 above. In 2.7.1 it was noted that Ruffini used this result without proving it. After proving this theorem Abel used it to conclusively establish that the general polynomial equation of degree 5 or higher was not solvable by radicals.

Therefore, Abel proved by 1823 that the general polynomial equation of degree 5 or higher is not solvable by radicals. However, excluding the case of the cyclotomic equations, there was no method to determine whether a given polynomial equation of degree 5 or higher was solvable by radicals or not. This led to the following problem: *Which polynomial equations are solvable by radicals?*

2.7.3 The Contributions of Galois

Evariste Galois (born 1811), a French mathematician, read the memoir of Lagrange: *The Resolution of Algebraic Equations* (Rothman 1982:115). It was here that Galois came across the problem of solving by radicals the general quintic equation. The mathematicians Gauss, Euler and Lagrange had worked on this problem, but none with complete success.

Galois worked on the problem and thought that he had found a 'method' to solve the general quintic. However, he found a mistake in his arguments and after several attempts believed that not every polynomial equation is solvable by radicals. Galois then tackled the following problem which is regarded as one of the greatest problems in algebra:

> Find a necessary and sufficient condition for the solvability by radicals of an algebraic equation of any degree.

Galois therefore tackled the problem of *characterizing all equations which are solvable by radicals*. He tackled this problem by introducing the concept of a *group*.

With every polynomial equation Galois associated a group in such a way that the properties of the group and the nature of the roots of the equation were closely related. This group, which reflects the properties of the roots of the polynomial equation, is now called the *Galois group of the polynomial equation*. Let $p(x) = 0$ be a polynomial equation of degree n. Note that Gauss proved that such an equation has n roots, provided multiple roots are counted as many times as they occur. The Galois group of $p(x) = 0$ is a subgroup of $S(n)$, the symmetric group of degree n. This subgroup consists of only those permutations in $S(n)$ which preserve the algebraic properties of the roots of the equation $p(x) = 0$.

Determining the roots of a polynomial equation can be a tedious and unrewarding task. In order to overcome this problem Galois devised a method to determine the Galois group of a polynomial equation without actually knowing the roots explicitly (compare with 4.5.2, Problems 2 and 3). This method makes use of the concept of a *solvable group*. Galois then formulated the following necessary and sufficient condition for the solvability, by radicals, of a polynomial equation of any degree:

> *A polynomial equation $p(x) = 0$ is solvable by radicals if and only if its Galois group is a solvable group.*

The following are some of the consequences of Galois' discoveries:

Let $p(x) = 0$ be a polynomial equation of degree n.

- It is known that for $n \leq 4$, $S(n)$ and all its subgroups are solvable. Therefore all polynomial equations of degree 4 or less are solvable by radicals.
- It is known that abelian groups are solvable. Therefore if the Galois group of a polynomial equation is abelian, in particular if it is cyclic, then the equation is solvable by radicals.
- The following results are also known:

  (1) For $n \geq 5$, $S(n)$ is not solvable.

  (2) For each n there is a polynomial equation of degree n whose Galois group is $S(n)$.

From (1) and (2) it follows that not every polynomial equation of degree 5 or more is solvable by radicals. So Galois' theory took care of Abel's result that the general quintic equation is not solvable by radicals. It also conclusively indicates that general polynomial equations of degree 6 and higher are not solvable by radicals. Further, polynomial equations of degree 5 or more whose Galois group is solvable are solvable by radicals.

Galois' introduction of the concept of a group led to the study of group theory. The study of the structure of groups marked the beginning of a new movement for the abstraction of algebra, the so-called *Abstract Algebra.*

2.8 SUMMATION

All polynomial equations of degree 4 or less are solvable by radicals. The general polynomial equation of degree 5 or more is not solvable by radicals. A polynomial equation of degree 5 or more is solvable by radicals if and only if its Galois group is a solvable group.

Solving procedures and representational forms for polynomial equations gradually developed from being verbal, graphical and finally symbolic (compare with 3.2.2). A parallel can also be found in the extension of the number system (positive rational numbers, irrational numbers, real numbers to finally complex numbers) and the solving of polynomial equations. The solution of the cubic equation and its related ideas led to development of modern symbolic notation. This in turn gave insight into the ideas related to the solving of quadratic, cubic and quartic equations which resulted in developments in the theory of equations (compare with 4.4.2 and 4.4.3). Therefore the historical review of the solving of polynomial equations reveals a sequence of steps towards greater generality, yet at the same time moving towards greater structurality, whereby new algebraic objects came into being.

The sequence in the solving of polynomial equations goes hand in hand with the development of the number system, different representational forms and finally modern algebraic symbolism. Each of these were contributed to by various peoples and cultures. All of these have important teaching and learning implications for secondary school mathematics, which are discussed in chapters 3 and 4.

CHAPTER 3

WORKING PROCEDURES IN ALGEBRA

3.1 INTRODUCTION

This chapter focuses on the working procedures and processes in school algebra with regard to the content, the learning of the content and their implications for teaching. An attempt is made to answer the following question which has important implications for teaching: *How can the learning and teaching of algebra be improved?* In order to answer this question attention is focussed on answering the following questions:

- How does understanding in algebra occur?
- What do research studies in algebra indicate?

In section 3.2 attention is focussed on how understanding in mathematics and concept formation in algebra occurs. A theory on how understanding in mathematics occurs is outlined and the implications for teaching are noted. Attention is then focussed on concept formation in algebra from which it is noted that concepts in algebra are first conceived as processes and then as objects.

In section 3.3 the theoretical framework for understanding and concept formation in algebra, developed in 3.2, is used to document the results of various research studies in algebra, and their implications for learning and teaching are noted. The research findings dealing with the following secondary school algebra sections are noted:

- Use of symbols in algebra
- Algebraic expressions
- Solving equations
- Functions

The observations made in sections 3.2 and 3.3 are then used to focus on the working procedures and processes involved in equation-solving and their implications for teaching. Section 3.4 focusses on the following important question:

How can the learning and the teaching of the solving of quadratic and cubic equations be improved at secondary school level?

In answering this question attention is focussed on the following processes:

- Recognising the type of equation
- Verbalising, visualising or reading the situation
- Transforming equations to standard forms
- Devising, understanding and applying algorithms
- Appreciating algorithms and the steps in an algorithm
- Representing, connecting and justifying

3.2 UNDERSTANDING

Several studies in mathematics education (eg. Davis 1986, Davis 1991, Davis 1992a, Davis 1993a, Hiebert & Carpenter 1992, Sfard 1994) have focussed on how understanding in mathematics occurs. Some of these studies (Hiebert & Carpenter 1992, Dreyfus 1991, Davis 1992a) have proposed a framework for reconsidering understanding, based on internal and external representations. These studies give an insight into how understanding in algebra occurs.

Davis (1992a:229) concludes that

> "Understanding" occurs when a new idea can be fitted into a larger framework of previously-assembled ideas. ... its inclusion helps to make sense of the entire picture.

It follows that previously-built-up understanding can either hinder or promote the understanding of new ideas. This has implications for both learning and teaching. Since what pupils learn is built upon the foundation of previous understanding it follows that future learning could be limited by the form of this previous understanding. Teachers should therefore make efforts to determine whether their pupils have the relevant previous knowledge or abilities which is/are required for the understanding of new ideas.

Since understanding is guided by prior learning, pupils must be provided with experiences that lead them to connect each new topic to previously learned topics (Cangelosi 1996:28). This implies that lesson preparation for topics in algebra should not be done on a piece-meal basis, but rather there should be section preparation which takes into account connections between related topics. This is possible since many concepts in algebra are related [examples: quadratic equations, roots of quadratic equations, the graphs of quadratic functions with respect to the x-axis, the nature of the roots of quadratic equations, and the fact that $x^2 + x + 1$ is always positive].

Further, since sense-making is an important part of understanding, pupils should at some time during the teaching of a topic be exposed to problems that they consider to be relevant to real-life.

3.2.1 Internal and external representations

Hiebert and Carpenter (1992) present a new theory for understanding based on the formation and interplay of internal and external representations. The key ideas of this theory are as follows:

a. In order to think about and communicate mathematical ideas, we need to represent them in some way.

b. Communication requires that the representations be external. *External representations* could take the form of spoken or written language, written symbols (algebraic symbolism), pictures (diagrams, graphs, tables) or physical objects.

c. In order to think about mathematical ideas there is a need to represent them internally (*internal representations*), in a way that allows the mind to operate on them.

d. As relationships are constructed between internal representations of ideas, they produce networks which could be structured like vertical hierarchies or webs.

It follows that this theory is based on the following assumptions:

- some relationship exists between external and internal representations [from (a) above].
- Internal representations can be related or connected to one another in useful ways

[from (c) and (d) above].

The following question has important implications for both teaching and learning: *How can internal representations be connected?* Hiebert and Carpenter (1992:66) note that

> ... connections between internal representations can be stimulated by building connections between corresponding external representations.

This can be done by looking at different external representational forms of the same mathematical idea or between related ideas within the same representational form. In the latter case attention can be focussed on similarities or differences when looking at examples or non-examples of a concept, say quadratic equations given in the form of algebraic symbolism. The research findings on the use of different external representation forms to introduce the concept of a function is given in 3.3.4.

With regard to learning mathematics with understanding Hiebert and Carpenter(1992:67) conclude that

> A mathematical idea or procedure or fact is understood if it is part of an internal network ... the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of connections.

The notion of connected representations forming structured, cohesive networks gives useful insight into how understanding in mathematics and in algebra occurs. The teaching implications of this, with regard to equation-solving, is discussed in 3.4.5. In the following sub-section attention will be focussed on how concepts in mathematics (especially algebra) are conceived and developed.

3.2.2 Conceiving concepts operationally and structurally

According to Sfard (1991:1) abstract mathematical notions (concepts) can be conceived in two fundamentally different ways: *operationally* (as processes) or *structurally* (as objects).

The same is true for algebraic concepts. As an example consider an algebraic expression, say 3x + 1 which is represented using algebraic symbolism. This expression can be interpreted as a *computational process* in which the expression is seen as a sequence of instructions: multiply the number at hand by 3 and then add 1 to the result. On the other hand $3x + 1$ can be seen as a certain number, in which case it is the *product of a computation process* rather than the process itself. These two interpretations (there are many others) illustrate the process-object duality of algebraic concepts. They also illustrate that the interpreting of information, given in the form of algebraic symbolism, is an important procedure in algebra (compare with 2.2.2). The interpretation of quadratic and cubic equations given in the form of algebraic symbolism is discussed in 3.4.1 and 3.4.2.

3.2.2.1 Stages in the formation of algebraic concepts

Using Sfard's terminology, note that certain algebraic concepts such as irrational number (refer to 2.3.2) and solving procedures for quadratic equations (refer to 2.3) evolved from being conceived operationally to structurally. The existence of such historical stages during which algebraic concepts evolved from being operational (procedural) to structural led Sfard (1991) to create a three-phase model of concept development, which is supported by several research findings (Kieran 1992:392).

Sfard suggests that the stages in concept formation represent a transition from computational operations to abstract objects. This transition which is a long and difficult process (Sfard 1991:1) is achieved in the following three phases:

- *interiorization* - where some operation or process is performed on already familiar mathematical objects.
- *condensation* - where the operation or process is squeezed into more manageable units. This phase lasts as long as a new entity is conceived operationally.
- *reification* - which refers to the sudden ability to see something familiar in a new light; a process solidifies into an object.

The idea of reification combines with the new theory of understanding (Sfard 1994:44, 46). Reification "is an act of turning computational operationals into permanent object-like entities" Sfard (1995:16). Therefore reification, which refers to a transition from an operational to a structural mode of thinking, is a basic phenomenon in the formation of an algebraic concept since it brings an algebraic concept "into existence and thereby deepens our understanding" (Sfard 1994:54). With regard to understanding in algebra Sfard and Linchevski (1994:202) note that

> ... algebra is a hierarchial structure in which what is perceived operationally at one level must be perceived structurally at a higher level.

The structural conception is described as being static, instantaneous and integrative while the operational conception is dynamic, sequential and detailed (Sfard 1991:4). It follows then that both operational (procedural) and structural thinking is important in algebra - both contribute to the hierarchial structure of algebra. With regard to internal representations of concepts, the operational conception "is supported by verbal representations" while the structural conception "is supported by visual imagery" (Sfard 1991:33). This implies that both verbalisation and visualisation is important when teaching concepts in algebra (refer to 3.4.2 and 3.4.5 for further details).

The theoretical framework on understanding and concept formation in algebra will now be used to examine some of the findings of research studies in algebra.

3.3 RESEARCH STUDIES IN ALGEBRA

Various studies (eg. Kieran 1992, Esty 1992, Sfard & Linchevski 1994, Bell 1995, Linchevski & Herscovics 1996) have focussed on the teaching and learning of school algebra. These studies indicate some important sources of students' difficulties in algebra. Kieran (1992) considers a student's inability to acquire an indepth sense of the structural aspects of algebra to be the main obstacle. Sfard and Linchevski (1994) have analysed the nature and growth of algebraic thinking from an epistemological perspective supported by historical observations. They have noted that the development of algebraic thinking is a sequence of ever more advanced transitions from operational to structural outlooks.

3.3.1 Use of symbols in algebra

Symbols are used in many different ways in algebra, for example: to represent technical concepts (eg. unknown, coefficient, variable), to represent operations (eg. +, - , $\sqrt{}$) and to represent expressions (eg. $3x + 1$) or equations (eg. $ax^2 + bx + c = 0$).

The many uses of literal symbols in algebra has been documented by Philipp (1992:560). Some of the different uses of literal symbols are given in Table 3-1.

labels	km, m in 1km = 1000m	*generalised number*	$3x + 2x = 5x$
constants	c in $3x + c$	*varying quantities*	x, y in $y = 3x + 1$
unknowns	x in $3x + 1 = 0$	*parameters*	m, b in $y = mx + b$

Table 3-1: The many uses of literal symbols in algebra.

It follows from this that literal symbols could represent various concepts and also take on varying roles when algebraic symbolism is used. The context is important in determining the role of the literal symbol (Philipp 1992:560). The correct reading of the context could pose a problem to learners of algebra (see Descartes convention for the use of letters in 2.2.4). This view is supported by Kieran (1992:396) who notes that

> Discriminating the various ways in which letters can be used in algebra can present difficulties to students.

Therefore the different notions of letters in the context of algebraic symbolism could imply different levels of difficulty for students. From a cognitive point of view, tasks such as grouping algebraic terms and using algebraic expressions demand "quite an advanced perception of literal symbols" (Linchevski & Herscovics 1996:43).

With regard to the order in which meanings are established Hiebert and Carpenter (1992:72) note that

> Once meanings are established for individual symbols, it is possible to think about creating meanings for rules and procedures that govern actions on these symbols.

The teaching implication is that before students are required to use and manipulate algebraic expressions, the meanings of the symbols must be established.

In algebra, different classes of symbols can be used to distinguish and to reveal the essentially different identities of an object which is being treated in two different ways (Harel & Kaput 1991:89). For example, the identity element for addition is denoted by the symbol 0 when working in the real number system, while the symbol 0 + 0i is used when working in the complex number system. Harel and Kaput (1991:91) also note that certain symbols

> ... include features that reflect the structure of mathematical objects, relations or operations that they stand for.

Examples of such elaborated symbols are (x;y) for an ordered pair of numbers, and $f(x) = 3x^2$ for a specific real-valued function.

Noting the different uses of symbols in algebra and the importance of correct interpretations, the obvious teaching implication is that attention must be focussed on establishing the meaning of symbols as they appear in the different contexts when and where algebraic symbolism is used (compare with 2.2.2).

3.3.2 Algebraic expressions

Prior studies (eg. Sfard & Linchevski 1994, Kieran 1992) provide a detailed analysis of cognitive obstacles due to the dual interpretation of algebraic expressions - operational and structural. Linchevski and Herscovics (1996:42) note that research studies show that

> ... simplification of algebraic expressions creates serious difficulties for many students.

3.3.2.1 Deletion error

This type of error is illustrated when students simplify an expression, say 9x - 4, to 5x. Carry, Lewis and Bernard (Kieran 1992:398) observed this type of error in a study of the equation-solving processes used by college students. These researchers attributed the deletion error to the over-generalisation (or false generalisation) of certain mathematically valid operations. A further example of a false generalisation pertains to the zero product (refer to 3.3.3.4). The source of the deletion error can be traced back to arithmetic, where simplification gives a single numerical value. So it seems that these students simplified 9x - 4 by first 'deleting' x and treating the expression as 9 - 4, and then tacking on x again. This is supported by Kieran (1992:398) who notes that some students therefore tend to

> ... simplify algebraic expressions by computing according to the rules of arithmetic and then tack on the letters.

3.3.2.2 Conventions of algebraic syntax

Students often when simplifying, say 3(x - 2), write 3(x -2) = 3x -2. This can be explained by the fact that beginning algebra students tend to read expressions from left to right and therefore do not see the need for brackets (Kieran 1992:398). Further, they do not read and apply the equal sign as an equivalence relation in the context of equations (Sfard & Linchevski 1994:208). Students must be aware of the conventions of algebraic syntax since this gives meaning to algebraic expressions and equations (refer to 3.4.2). They must also learn where to use brackets and where not, since bracketing structures the text.

Teaching must therefore focus on the conventions of algebraic syntax (see 2.1.1). This implies many advantages for learners of algebra. In this regard Cangelosi (1996:137) notes that, provided

> ... a person has been taught the meanings of the symbols and has become accustomed to using them, the compact form with the shorthand notation makes it easier to recognise critical relationships ...

The correct interpretations of these conventions reveal the power of algebraic symbolism. For example, when simplifying the product of two binomials, say (x + 2)(x - 3), particular attention should be focussed on the relationship between coefficients and signs in the factors and those that occur in the quadratic product (Roebuck 1997:206). This promotes the interpretation of the equal sign as an equivalence relation.

3.3.2.3 Bridging the gap between arithmetic and algebra

When introducing algebra the use of letters should be withheld until it is evident that pupils are ready for their use, and teaching must recognise and prepare pupils for the various uses of letters in algebra as the need arises (Harper 1987:85). These suggestions are also supported by Stols (1996).

The structurality of geometry and the visual overview that it provides facilitate thinking and effective investigation (Sfard 1995:23). In 2.2.3 it was noted that in the initial development of algebra there was a close link between algebraic and geometric representations. The formulae for determining the areas of squares and rectangles can be used to introduce algebraic expressions. Such an approach could help students to make links between arithmetic and algebra as noted below.

A teaching sequence which allowed students to develop a procedural (operational) meaning for algebraic expressions such as 4x + 4y was designed by Chahouh and Herscovics (Kieran 1992:397). One of the questions posed to the subjects was: *Can you write down the area of the following rectangle?*

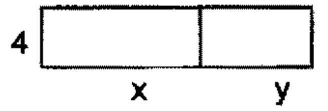


Fig.3-2: Developing a procedural meaning for 4(x + y).

The researchers noted that students tended to regard expressions such as 4x + 4y as incomplete unless they formed part of an equality such as 4x + 4y = Area. This suggests that a procedural conception for algebraic expressions requires in the mind of the student a final result as the end product of the procedure (as in arithmetic). Such reluctance by students to accept a lack of closure should therefore be appreciated by teachers.

Kieran (1991) describes the research findings of a teaching approach developed by Peck and Jencks "that helps students make explicit links between their arithmetic and the nonnumerical notation of algebra" (Kieran 1991:49). Students were exposed to an approach for simplifying 24x26 (arithmetic), based on areas of rectangles and squares using the following geometric illustration (compare with 2.3.2.1).

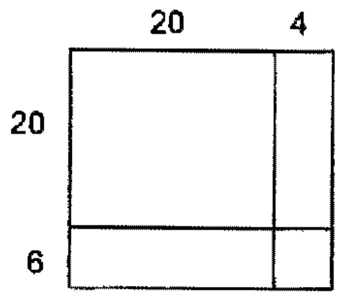


Fig.3-3: Linking 24x26 to a geometrical illustration.

This approach enabled students to record statements such as

$$(x + 4)(x + 6) = x^2 + 10x + 24.$$

Peck and Jencks also observed that this approach led students to handle expressions of the type

$$(x + 9)(x - 4) \qquad \text{or} \qquad (x + 5)(x - 5).$$

This approach also allows for generalisations such as

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad \text{and} \qquad (a + b)(a - b) = a^2 - b^2$$

to be viewed as simple variations of the same conceptual theme (Kieran 1991:50).

Various studies (eg. Ernest 1987, Burton 1988, Pegg & Redden 1990, Wessels 1990, Oldfield 1996) suggest that there should be a greater focus on language. Burton (1988:2) notes that

> ... a major component of student difficulty with algebra is the inability to make sense of the algebraic system as a language, and accordingly that remedies should be sought by considering algebra in a linguistic sense.

Continuing on this theme, Pegg and Redden (1990:19) suggest that an approach to algebra that is often neglected is the

> ... role of language as the link between experiences with number patterns and the emergence of algebraic notation.

They suggest an approach to algebra based on the following three-part process (refer to Pegg & Reddan 1990:19-21 for details):

- Experiencing activities with number patterns.
- Expressing the rules which govern particular number patterns in English sentences.
- Writing the rule(s) which govern number patterns in an abbreviated form.

Contributing further to this theme of language, Bell (1995:50-57) describes how the generality or non-generality of proposed patterns observed on a calendar page can be used to introduce algebra and appropriate symbolic notation.

A necessary condition for reification to occur is the introduction of a symbolic notation (Sfard & Linchevski 1994:197). Each of the above approaches is aimed at a transition from a procedural (operational) to a structural conception of the relevant concepts in algebra. Future success in algebra requires a structural interpretation and the ability to use modern algebraic notation (compare with 2.2.2). Both of these are also required when solving equations.

3.3.3 Solving equations

Various studies (eg. Kieran 1992, Herscovics & Linchevski 1994, Linchevski & Herscovics 1996) have focussed on the solving of linear equations. Linchevski and Herscovics (1996:44) note that

> ... for a large number of high-school students, there are many cognitive obstacles involved in perceiving an equation as a mathematical object on which they can perform operations.

Some of these obstacles are: a limited view of the equal sign, the idea of equivalent equations, interpreting the structure of equations and constructing meaning for formal solution procedures. The teaching implications of these with regard to quadratic and cubic equations are discussed in 3.4.

3.3.3.1 Limited view of the equal sign

Consider the equation $2x + 3 = 11$. Some students see the expression on the left-side as a process and the expression on the right-side as the result (Linchevski & Herscovics 1996:51). If algebraic expressions are seen as processes rather than objects, then the equality sign is interpreted as a "do something symbol" (Sfard & Linchevski 1994:208). Several researchers have noted that such a limited view of the equal sign exists among some students in secondary school (Herscovics & Linchevski 1994:65) and also at college level (Bell 1995:47).

3.3.3.2 Didactic cut

Research on the solution of linear equations shows that while pupils can solve equations of the type ax + b = c, they have difficulty solving equations in which the unknown appears on both sides of the equation - for example equations of the type ax + b = cx + d (Sfard & Linchevski 1994:210). Filloy and Rojano (Herscovics & Linchevski 1994:61) refer to this as a 'didactic cut' between arithmetic and algebra. The reason is that in the former type of equation the equality still functions as in arithmetic - operations on one side and the result on the other. In the latter type of equation the equal sign represents an equivalence relation.

3.3.3.3 Formal solving methods

Formal methods of equation solving, which requires that an equation be treated as a mathematical object, include transposing and the performing of the same operation on both sides of the equation. While performing the same operation on both sides of an equation makes use of and emphasises the symmetry of an equation, this is absent in the procedure of transposing (compare with 2.3.4.2).

Substitution is one of the methods used to solve equations. However, research seems to suggest that once pupils learn a formal method for solving an equation, they tend to drop the use of substitution for verifying the correctness of their solution (Kieran 1992:400).

Research evidence seems to suggest that students who view equations as entities with symmetric balance find it easier to operate on the structure of an equation by performing the same operation on both sides (Kieran 1992:400). This suggests that for formal equation-solving the following order of instruction could help students treat equations as algebraic objects: first establish that the equality sign is a symbol that denotes the equivalence between the left and the right sides of an equation, followed by instruction on performing the same operation on both sides, and then instruction on the use of substitution for verifying.

3.3.3.4 Structural features of equations

Kieran (1992:402-403) documents the results of various studies that provide evidence of the inability of students to distinguish structural features of linear equations. A brief description of these follow:

- Studies in 1982 and 1984 by Kieran showed that beginning algebra pupils do not regard $x + 4 = 7$ and $x = 7 - 4$ as equivalent equations.
- Wagner and his colleagues in 1984 noted that some high school students did not regard $7w + 22 = 109$ and $7n + 22 = 109$ as equivalent equations.
- With reference to a study by Wagner and his colleagues done in 1984 Kieran (1992:403) writes that

  > The findings of this study show that most algebra students have trouble dealing with multiterm expressions as a single unit and suggest that students do not perceive the basic surface structure of, for example, say, $4(2r + 1) + 7 = 35$ is the same as, say, $4x + 7 = 35$.

- Greeno in 1982 noted that beginning algebra students lack knowledge of the constraints that determine whether a transformation is permissable. For example, students were not able to use the equivalence constraint to show that an incorrect solution is wrong. Kieran (1992:403) found that a group of "competent high school solvers also lacked this knowledge".
- Citing research by Lewis and Bernard done in 1980, Kieran (1992:401) notes that

  > Students have generally been found to lack the ability to generate and maintain a global overview of the features of an equation that should be attended to in deciding upon the next algebraic transformation to be carried out.

It can be concluded from the above studies that students seem to have difficulty in recognising equivalent equations, interpreting the basic surface structure of equations, dealing with multiterm equations (including ones in which the unknown occurs on both sides - refer to 3.3.3.2), and decision making with regard to which transformations are permissable and should be made in the context of the given equation.

A study of recent Examiners Reports (eg. House of Delegates, Department of Education and Culture 1992, 1994 &1996) for the Senior Certificate Examination in mathematics, written in South Africa, suggest that the above conclusions are also true for quadratic and cubic equations. As a specific area of weakness the Matric Examiners Report for

Mathematics (1992:28) states

> Attempting to solve a quadratic equation without first writing it in standard form:
> eg. $(x - 3)(2x + 1) = 4$
> $\therefore\ x - 3 = 4$ or $2x + 1 = 4$.

Another illustration of the lack of comprehending the structure of an equation is cited in the Matric Examiners Report for Mathematics for 1995 (House of Delegates, Department of Education and Culture 1996:20) which states

> Candidates would prefer to solve $2x^2 - 5x - 3 = 0$ than $(x - 3)(2x + 1) = 0$.

The first illustration indicates that many students falsely generalise the rule for the zero product, which suggests that they do not appreciate or understand the need for the various steps in an algorithm for solving a quadratic equation. The second illustration shows that some students do not maintain an overview of the features of an equation and this results in inappropriate algebraic transformations being carried out. Note that both these difficulties are a result of students not interpreting the overall structure of the given equations and the relationship between their component parts.

The research evidence noted above clearly indicates that a structural conception of a given equation is " a prerequiste for the comprehension of the strategy that must be used" (Sfard & Linchevski 1994:211). The teaching implication is that instruction should focus on and emphasise the structural features of equations and their implications (see 4.4.3, Questions 18 and 19).

3.3.3.5 Solving word problems that lead to equations

Various studies (eg. Kintsh & Greeno 1985, Burton 1988, Kieran 1992, Bell 1995) have focussed on the solving of word problems in algebra. These studies imply that students encounter many difficulties when they are exposed to word problems. According to these studies the difficulties that students experience could relate to one or more of the following:

- Comprehending the word problem.
- Specifying and expressing relations among variables.
- Representing the information correctly by using a table of relations.

- Detecting and using the correct verb (eg. *is* or *exceeds*) for the problem statement.
- Noticing the structural similarities when problems have different cover stories.
- Correctly translating the word problem into an equation or equations containing numbers, variables and operations.
- Interpreting the result after solving the equation that was set up.

Solving word problems are important since such problems continuously expose pupils to the full activity of beginning with a problem, *formulating* the equation, *solving* this equation and then *interpreting* the result (Bell 1995:61). The teaching implication is that pupils should at some stage be exposed to word problems. Perhaps word problems should be used to introduce quadratic and possibly cubic equations to students. A focus on formulating the problem statement and transforming it into the relevant equation may give students a deeper insight into the structural features of equations, and the need for transforming them to equivalent equations.

3.3.4 Functions

Various studies (eg. Eisenberg 1991, Dubinsky 1991, Widmer & Sheffield 1994) have focussed on the function concept. Eisenberg (1991:140) notes that the function concept is "one of the most difficult concepts to master in the learning of school mathematics". A possible reason for this is that algebraic symbolism is usually used to represent functions. The concept of a function presented in the form of algebraic symbolism is an abstract concept. Any difficulty that a student has with the conceptualisation of the symbolic representation or the context in which symbols are used will therefore impact on his/her understanding of the function concept (refer to 3.3.1). Further, note that often movement is involved in a function concept and that this is an advanced idea. There is always a relationship between two or more variables. Therefore the variable concept should be well developed before functions are introduced, and this is very often a complex process (see 3.3.1). In the secondary school syllabus displacement, velocity, and rates and changes are popular appearances of functions.

However, research has shown that some success can be achieved by introducing the function concept in a variety of contexts (Eisenberg 1991:141,152). For example: by using visual representations in the form of arrow diagrams, tables, input-output boxes or graphs, or by using algebraic representations in the form of ordered pairs or algebraic descriptions. Dubinsky (1991:104) notes that research has shown that an important way to understand the concept of a function is to construct a process. In the case of specific examples, say $y = x^2$, the individual may respond by *constructing* in his or her mind a mental process which relates to the function's process. This is an example of interiorisation which is a prerequisite for total understanding. By making use of function machines and function games together with calculators and computers, Widmer and Sheffield (1994) have also shown that the learning difficulties associated with the function concept can be addressed to a large extent.

It follows from these research studies that different representations of a concept in a variety of contexts, and the processes that they imply, could aid the achievement of interiorisation and thus promote understanding.

3.3.4.1 Representations, switching representations and translating

In 3.2.1 and above it was noted that different representations (both internal and external) play an important part in understanding algebraic concepts. Dreyfus (1991:32) notes that although

> ... it is important to have many representations of a concept, their existence by itself is not sufficient to allow flexible use of the concept in problem solving.

However, if the various representations are correctly linked then it becomes possible to switch from one representation to another which is more efficient to use. For example, the quadratic function which is an abstract concept could have an algebraic representation, say $g(x) = x^2 + x + 1$, or a graphical representation. Research by Sneldon and his colleagues (Eisenberg 1991:146) has shown that students often approach problems analytically without utilizing the visual interpretation of the givens.

Consider the following question:

Show that $x^2 + x + 1$ is always positive.

Students often ignore the power of the graphical representation when faced with such a question. At school level both graphical and analytical arguments are acceptable. It should be noted that although the analytical argument is deductive and logical, it "may not be appropriate for the cognitive development of the learner" (Tall 1991:3).

Translating is a process which is closely connected to switching representations. Dreyfus (1991:33) notes that one meaning of *translating* is "going over from one formulation of a mathematical statement to another ". The inclusion of translating in the primary school syllabus (KwaZulu-Natal Department of Education and Culture 1996:67), indicates its importance in mathematics. Ignoring the visual formulation of aspects of algebra (where possible) could lead to learning problems as noted by Eisenberg (1991:152)

> ... the unwillingness to stress the visual aspects of mathematics in general, and of functions in particular, is a serious impediment of students' learning.

The teaching implications are that instruction should focus on using different representations in order to help students to understand concepts. Further, students must be exposed to activities that require them to switch representations and to focus on different formulations of algebraic statements. Each of these procedures could also be helpful when teaching the solving of quadratic and cubic equations.

3.4 PROCESSES IN THE SOLVING OF QUADRATIC AND CUBIC EQUATIONS

This section focuses on the processes involved in solving quadratic and cubic equations, and their implications for learning and teaching.

3.4.1 Recognising the type of equation

At the secondary school level students are normally required to solve linear, quadratic and cubic equations in one unknown. Therefore students should be able to classify a given equation as linear, quadratic or cubic. Each of these should bring into play a representation of a certain algorithm that has to be followed (refer to 3.4.4). Correctly classifying depends on correctly comprehending the structure of the equation (refer to 3.3.3.4). This in turn requires the correct interpretation of the symbols (refer to 3.3.1) that represent the following structural features of the equation:

- the unknown that has to be solved for,
- the coefficients of the equation,
- the degree of the equation,
- the various operations between the component parts of the equation,
- the equality sign as an equivalence relation, and
- permissable algebraic transformations that lead to equivalent equations (refer to 3.3.3.3 and 3.3.3.4).

The teaching implication is that instruction should focus on each of these aspects (compare with 2.2.2). There is a need for students to be exposed to questions of the following types:

- What is an equation?
- What is meant by a solution of an equation as opposed to the solution set of an equation?
- Questions which require students to classify mixed types of equations, even if the equations are not in standard forms. Equations in which the unknown appears on both sides of the equation must be included (refer to 3.3.3.2).
- Questions which require students to detect the basic underlying structure of equations. These questions could include disguised equations of the type:
  * $(x^2 - x)^2 = 8(x^2 - x) - 12$
  * $2^{2x} - 5.2^x + 4 = 0$
  * $2\sin^2\theta - \sin\theta = 1$

3.4.2 Verbalising, visualising or reading the situation

Suppose that students are required to solve equations which are represented in the form of algebraic symbolism. In order to be able to answer such questions the students must be able to read and extract meaning from the algebraic situation which confronts them. While reading such an algebraic situation it is important to have an "understanding of the different processes" and the connections between them. This helps to extract meaning from the algebraic situation (De Morgan as cited by Arcavi & Bruckheimer 1989:37). Students must correctly interpret the connections of algebraic syntax (refer to 3.3.2.2) since this gives meaning to an equation. In the context of equations, the conceptualisation of the information given by an equation could be procedural or structural (refer to 3.2.2).

"Students are often unable to read an algebraic sentence in the sense of extracting meaning from it" (Burton 1988:3). Verbalising the situation could help students to make sense of certain equations and also to extract meaning from them. Consider the following equations

$x^2 = x$ ...... (1)

$x^2 = 4$ ...... (2)

$x(x + 1) = 12$ ...... (3)

For each of these equations, verbalising the situation may help students to solve the equation correctly. Each of these equations can be solved if the situation that each depicts is *correctly verbalised*. For example, in the case of equation (1): *The square of a number is equal to the same number.* This gives 0 and 1 as being the only possibilities for x. Substituting shows that these values are correct.

During a lecture on the solving of equations, equation (3) was given to a group of third year student-teachers specialising in mathematics at Springfield College of Education. The students were asked to solve the given equation. All but one of the students used a formal algebraic approach. This student pointed out the numbers are 3 and 4 *since we want two consecutive numbers whose product is 12.* When asked if this was *the solution* of the given equation, he soon realised that 4 is not a solution. Noting that the equation depicted a quadratic he gave another pair -3 and -4, but pointed out that -3 is not a solution. This illustrates the role of verbalising or reading the situation *correctly* in representing (formulating) , connecting and justifying (refer to 3.4.5 for further details).

With regard to quadratic equations in standard form, say in the unknown x, verbalisation in the form: *the coefficient of x^2 is ..., the coefficient of x is ..., the constant term is ...,* could help students to detect the correct values of the coefficients that must be substituted in the quadratic formula. Of course in this case students must be able to state and apply the quadratic formula correctly. Even here verbalisation could play an important role in understanding the structure of the formula which is expressed in the form of algebraic symbolism. Once the formula is stated correctly, correct substitution of the coefficients of the equation into the formula is required, followed by simplification using the rules of algebra.

The role of verbalisation and substitution should not be under-estimated in the teaching of the solving of equations. Correctly reading an equation represented in the form of algebraic symbolism gives meaning to the situation represented by the equation. This could have implications for the next algebraic transformation that should be carried out.

Note that each of the equations (1), (2) and (3) above can also be solved graphically. For example, equation (1) can be solved by drawing the graphs of $f(x) = x^2$ and $g(x) = x$ on the same system of axes, and noting the values of x where they intersect. Such a representation also gives a powerful visual interpretation of the equal sign as it is used in the context of equations. For equation (3), a correct reading of the algebraic sentence that is represented indicates that x is not zero. By making use of this fact the equivalent equation $x + 1 = 12/x$ can be arrived at. Finally, equation (3) can be solved by drawing the graphs of $h(x) = x + 1$ and $m(x) = 12/x$ on the same system of axes, and noting the values of x where they intersect.

The teaching implication is that verbalising, visualising or reading the situation are important procedures when solving equations. Therefore instruction should focus on these procedures.

3.4.3 Transforming equations to standard forms

At secondary school level many quadratic and cubic equations that students have to solve require algebraic manipulations that transform the equations to their respective standard forms. The standard form of the quadratic equation is $ax^2 + bx + c = 0$, while the standard form of the cubic equation is $Ax^3 + Bx^2 + Cx + D = 0$ (where the coefficients a and A are not zero). Consider the equations (1), (2), (3) in 3.4.2 above and the following equation:

$$x(x + 10) = 9 \qquad \ldots\ldots (4)$$

These quadratic equations are not in standard form. All of these equations can be solved by writing them in the standard form of a quadratic equation. This requires the algebraic processes of simplifying, for example $x(x + 10)$, and adding the same quantities to both sides of the equation or transposing terms. Note that, from a conceptual perspective, transposing is an operation that results after the process of adding the same quantities to both sides is understood.

When solving equations, students have difficulty "in deciding upon the next algebraic transformation to be carried out" (Kieran 1992:401). Greeno (Kieran1992:403) noted that students also seem to lack knowledge of the constraints that determine whether a transformation is permissable. For example, when students are faced with equation (1) above some of them tend to divide by x. The teaching implication is that instruction should focus on legal algebraic operations or manipulation skills that could lead to the standard forms of equations. In subsection 3.3.3.4 it was noted that a structural conception of a given equation is a prerequisite for determining the strategy that must be used to solve the equation.

3.4.4 Devising, understanding, appreciating and applying algorithms

An algorithm is a multistep procedure for obtaining a result *based on relationships* (Cangelosi 1996:144). The part in italics is important since many of these relationships in the context of equations are represented in the form of algebraic symbolism. Therefore the student must be able to interpret and attach meaning to the symbols in the context of the structure of equations.

The use of algorithms in mathematics is opposed by some mathematics educators while others encourage their use (refer to Sfard 1991: 9 for details). The usefulness of algorithmic activities in the learning of mathematics is noted by Ervynck (1991:43) who writes

> ... algorithmic activity is an essential part of the learning of mathematics because such processes must be interiorized and become routinised before they can be reflected upon as manipulable mental objects in a higher order theory.

With regard to the solving of quadratic and cubic equations the following types of algorithmic activities are required:

- *Reforming algebraic expressions.*

  For example, writing the following expressions in factorised form:

  $3x^2 + 2x - 5$, $x^3 + 2x - 3$.

- *Translating statements of relationships.*

  For example, the translating of the quadratic equation $x(3x + 2) = 5$ to its standard form.

- *Formulating key steps in the solving procedure for quadratic or cubic equations.*

  For example, the following could be an algorithm to solve quadratic equations by using a factorisation technique.

  Step 1. Write the equation in standard form.

  Step 2. Factorise the quadratic trinomial by using the cross-method.

  Step 3. Use the principle that the product of two numbers is zero if and only if one of the numbers is zero.

Note that each of the steps in the previous algorithm relies on other algorithmic skills. This observation could serve to illustrate to students the need for and usefulness of developing the other algorithmic skills (for example factorising quadratic trinomials).

From the above it should be clear that the use of algorithms to work out certain types of problems in algebra certainly has advantages. However, it is important that students are actively engaged in the devising of an algorithm, are exposed to appreciating the importance of each step in the algorithm and are able to apply the algorithm correctly. Instruction should focus on each of these aspects.

Traditional teaching focussed exclusively on developing algorithmic skills (Cangelosi 1996:144). Aspects that were neglected include the following: constructing a concept, discovering a relationship, comprehending, deductive reasoning and creativity. If students are meaningfully exposed to *devising*, *understanding*, *appreciating* and *applying* algorithms then many of the neglected aspects (noted above) can be addressed. In some primary schools in South Africa these aspects (KwaZulu-Natal Department of Education and Culture 1996:3-4) are being addressed by the problem-centred approach to teaching and learning.

Students in secondary schools should also be encouraged to devise and analyse their own algorithms before such algorithms are used as a recipe to get answers (compare with 2.3.1.3 and 2.3.3). This requires activities which encourage students to write down the various steps in an algorithm and to then justify their inclusion or exclusion. The posing of the following types of questions followed by appropriate discussion could help pupils devise, reformulate and appreciate the steps in an algorithm:

Question 1

Outline your procedure to solve the equation $x(x + 1) = 1$. Justify the inclusion of each step in your procedure.

Question 2

Solve the following equations:

2.1 $(x + 3)^2 = 9$

2.2 $(x + 2)(x - 1) = 0$

2.3 $(x + 2)(x + 1) = 4$

2.4 $x^2(x - 2) - x + 2 = 0$

2.5 $x^3 + 2x = 3$

Question 3

How would you determine the nature of the roots of the equation $x(x - 3) = 5$, without solving the equation?

Note that the development of algorithmic skill requires that students remember how to execute the algorithm based on relationships. However applying an algorithm involves more cognitive processes since students are required to determine *when and how* to use a relationship. The questions relating to *how and when* to use relationships require *flexibility* which is considered to be a source of competence and a trait of algebraic thinking (Sfard & Linchevski 1994:203). In this regard Sfard and Linchevski (1994: 203-204) note that flexibility

> ... seems to be a function of two parameters: the *versatility* of the available interpretations, and the *adaptability* of the perspective.

Note that the equations given in question 2 above require flexibility in the sense that versatility and adaptability of the perspectives are required. For question 2.1 a correct verbalisation could lead to the step $x + 3 = 3$ or $x + 3 = -3$ from which the equation could be easily solved. This interpretation does not require that the given quadratic equation be written in standard form. Similarly for question 2.2, the first few steps in an algorithm for solving quadratic equations using a factorisation technique have to be by-passed. Question 2.4 does not require the use of the factor theorem while question 2.5 could require an application of the factor theorem. In question 2.5 a student could easily detect that 1 is a root of the given equation, therefore $(x - 1)$ is a factor of the polynomial $x^3 + 2x - 3$. Therefore some form of *decision making* is still required from the students.

Developing algorithmic skill, applying algorithms correctly and still being able to be flexible are important when learning how to solve quadratic and cubic equations. Therefore the design of lesson units must include appropriate activities for these important aspects.

3.4.5 Representing, connecting and justifying

In equation-solving, representation, connection and justification can occur at different levels, for example: the different steps in the formal solution of an equation, the procedures for solving different types of equations, and the utility value of equation solving. Each of these levels promotes understanding.

3.4.5.1 The different steps in equation-solving

In 3.3.3.4 it was noted that many students find it difficult to maintain an overview of the structural features of an equation and in deciding on the next algebraic transformations to be carried out. One way of improving the situation is to connect the different steps in the formal solution of an equation.

This can be achieved by emphasising the language concepts that are involved in equation-solving. The write up of formal solutions should be such that the "process of solving an equation creates a mathematical *paragraph*" (Esty (992:41). In a paragraph the different sentences lead on and are connected in a logical manner to form a coherent whole. The same should be true when formal solutions to equations are written down. This implies that where possible explanations should be given and the different steps in the solution procedure must be related by the use of logical connectives. Very often a number of unconnected steps is accepted and even given by some teachers as a solution of an equation. It could be argued that this promotes a lack of understanding and decision making in students. In 2.2.3 it was noted that an interpretation of the surface and deep surface meanings that emerge from mathematical symbolism, as used in different mathematical contexts, will promote mathematical maturity among pupils and teachers.

Consider the following examples:

Example 1

Solve for x: $(x + 5)(x - 2) = -6$

Consider the following 'solution':

$(x + 5)(x - 2) = -6$	
$x^2 + 3x - 10 = -6$	[1.1]
$x^2 + 3x - 4 = 0$	[1.2]
$(x + 4)(x - 1) = 0$	[1.3]
$x + 4 = 0$ or $x - 1 = 0$	[1.4]
$x = -4$ or $x = 1$	[1.5]

This solution consists of a number of unrelated steps. Do students really understand why $x = -4$ or $x = 1$ is *the solution of the given equation*? To improve understanding logical connectives must be used to connect the different steps. The use of the symbol $\Leftrightarrow$ (if and only if) to connect each of the steps in this case can be used to show that the given (condition) equation is equivalent to step [1.5]. By logically connecting them the unrelated steps are now changed into a mathematical paragraph.

Example 2

Solve for x in the equation $1 - x = \sqrt{x + 11}$.

Consider the following 'solution':

$(1 - x)^2 = x + 11$	[2.1]
$x^2 - 2x + 1 = x + 11$	[2.2]
$x^2 - 3x - 10 = 0$	[2.3]
$(x - 5)(x + 2) = 0$	[2.4]
$x = 5$ or $x = -2$	[2.5]

Some students stop here while others may check by substitution in the original equation and conclude that $x = -2$. *Why is there a need to check in example 2 but not in example 1?* These examples clearly indicate that there is a need for a proper emphasis on the use of logical connectives when solving equations. Without such an emphasis "students lose track of which steps are guaranteed to work and which are not" (Esty 1992:44).

It seems then that in order to help students maintain an overview of the features of an equation, and in deciding on permissable and logical transformations the following could help:

- Teaching students the importance of logical connectives in the equation-solving process.
- Providing reasons for the steps in the solution process.

  In example 1, step [1.1] denotes the use of the identity

  $(x + 5)(x - 2) = x^2 + 3x - 10$.

  Step [1.2] follows by transposing -6 and simplifying the L.H.S.

  Step [1.3] uses the identity $x^2 + 3x - 4 = (x + 4)(x - 1)$.

  Step [1.4] is obtained by using the zero product rule.

  Step [1.5] follows by solving the linear equations.

  Such an approach will also enable students to appreciate the relationship between the different sections in algebra. The utility value of simplifying and factorising in order to transform to different yet equivalent forms is contextualised.
- Changing the emphasis of some homework activities. Some tasks need to be given where students are asked to use logical connectives explicitly and to cite reasons which justify the steps.

All of these procedures could help students to represent, connect and justify the different steps in the equation-solving process.

3.4.5.2 Solution procedures for different types of equations

It has been argued above (3.4.4) that algorithms could guide thinking and understanding in algebra. In subsection 3.1 it was noted that representations of concepts and the formation of connections between representations are important for understanding to occur. The teaching of equation-solving at the secondary school level should build up to the type of framework (or schema), given in Fig.3-4 below, in order to help students structure their thinking (see 2.3.3).

Such a schema gives a strategy (or plan) for equation-solving at the secondary school level. Some researchers strongly advocate "teaching problem-solving strategies like any other part of mathematics" (Silver 1985:402). The devising of plans, a form of representation, which could be used to solve problems is an important part of problem-solving and should therefore be taught. In order to develop into successful equation-solvers students need to spend time analysing equations and the directions that could be taken (Fernandez, Hadaway and Wilson 1994:195). This should lead to the development of frameworks (schemata) aimed at pinpointing the different phases and the processes that are necessary when solving equations. In short this involves representing, connecting and justifying equation-solving procedures.

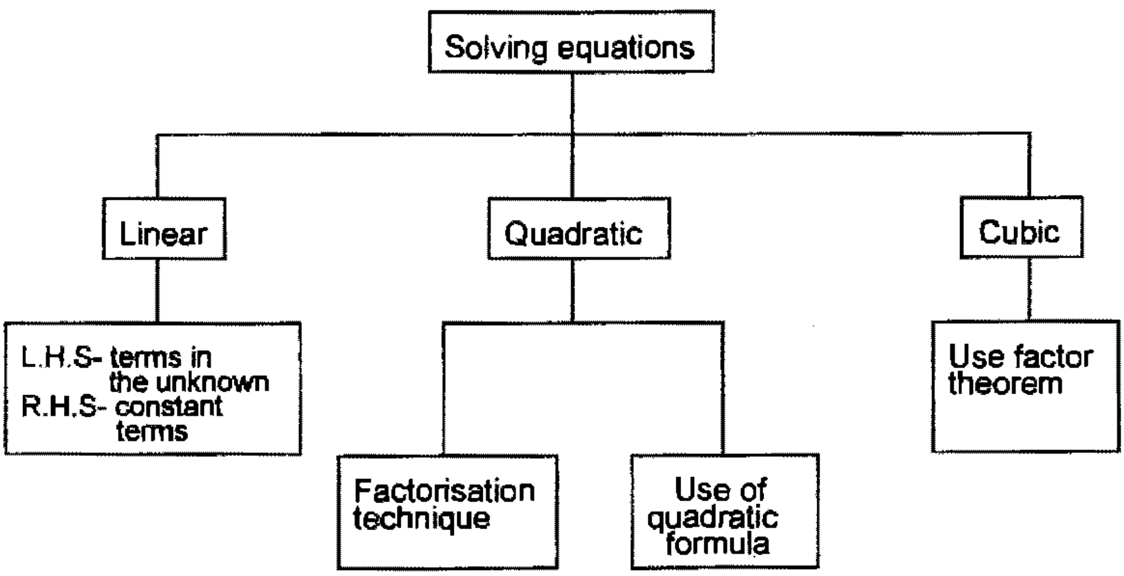


Fig.3-4: A schema for algebraically solving equations.

The schema for algebraically solving equations, represented in Fig.3-4 above, associates with each of the algebraic objects (linear, quadratic and cubic equations) particular solving procedures. Such a schema requires that students first identify the type of equation that they are confronted with. Then an algorithm [for example: the factorisation technique for quadratic equations (refer to 3.4.4)] should follow which outlines a particular procedure. Such a schema which includes a more or less coherent collection of objects and processes can aid students to "understand, deal with, organise, or make sense out of a perceived problem situation" (Dubinsky 1991:102), which in this case is the solving of equations in one unknown of degree one, two or three. By focussing on the structural nature of such schemata, explicit descriptions of possible relationships between schemata can be detected.

Instruction should therefore focus on the need for, the usefulness of and the development of such schemata to aid thinking and bringing into play encoded labels for algorithms.

3.4.5.3 The utility value of equation-solving

Sense-making is an important part of learning to understand and teaching for understanding. The following type of question can be given to students:

> *What is the purpose of equation-solving? Try to give a representation of your schema.*

During the course of 1997 this question was given to a group of third year student-teachers specialising in mathematics at Springfield College of Education. Their responses (verbal and written) made use of flow diagrams, tree networks, and listing and connecting sections which included the following:

- solving word problems
- drawing graphs of quadratic and cubic functions: x-intercepts
- linear programming: x- and y- intercepts
- verifying graphical solutions
- solving equations in algebra, trigonometry, physical science
- real-life applications: in the construction of parabolic arches and supports for bridges

Such an activity, besides getting students to represent and connect, also develops in students an appreciation of the utility value of equation-solving (compare with 2.3.1.1).

3.5 SUMMATION

Many working procedures and processes are involved in the understanding of school algebra. A teacher who functions at the structural level, and ignores the fact that concepts in algebra are first conceived operationally, is unlikely to meaningfully develop in his or her students an understanding of algebraic concepts. Further, he or she is unlikely to appreciate the cognitive obstacles experienced by students with regard to the formation of concepts and the achieving of understanding in algebra.

Instruction should take into account the links between arithmetic and algebra, and teaching implications of research studies in algebra. Students must be encouraged to seek meaning when dealing with algebraic expressions and equations. Although structural conceptions are difficult to achieve, proper planning and appropriate instruction (taking into account how understanding occurs) could overcome many of the problems encountered by students. Concerted efforts must be made in order to help students to develop "appropriate ideas and powerful methods of thinking" in algebra (Davis 1992b:359). The next chapter focusses on teaching approaches to achieve this.

CHAPTER 4

POSSIBLE METHODS/APPROACHES REGARDING THE TEACHING OF EQUATIONS

4.1 INTRODUCTION

This chapter focusses on the following:

- Theories/models/strategies in solving equations.
- Four methods/models for teaching and learning.
- A model/method for solving polynomial equations.
- Solving procedures used by students.

In section 4.2 an attempt is made to answer the question: *What are the theories/models/methods/strategies that could be used to teach the solving of equations?* The answer to this question is informed by the working procedures in algebra which were discussed in chapter 3.

Section 4.3 focusses on the following methods of teaching and learning: guided discovery, problem solving and inquiry. The question that is focussed on is: *What type of teaching model could include these methods and also promote the working processes in algebra?* In trying to answer this question a guided problem solving model is presented which comprises of three levels, namely: inductive reasoning, inductive and deductive reasoning, and deductive reasoning. The model targets both the cognitive and affective domains.

In section 4.4 the guided problem solving model is used to design learning activities for the solving of polynomial equations. An attempt is made to answer each of the following questions:

- How can the network in the brain for equation solving be meaningfully developed?
- How can this network be modified, reorganised and linked to other networks?
- What type of tasks, relevant to the solving of polynomial equations, need to be

formulated for each of the three levels in the guided problem solving model?

- How can the history of the development of the solving of polynomial equations be used?
- Do the tasks target both the cognitive and affective domains?

With this in mind examples of tasks are formulated and discussed for each of the three levels.

Section 4.5 focusses on a limited investigation on the solving procedures used by students for three problems in algebra. An attempt is made to answer the following questions:

- Do the solving procedures used by students demonstrate and confirm the working processes in algebra (discussed in chapter 3)?
- What are the shortcomings of the normal traditional teaching approach? How can these shortcomings be overcome?
- What are the specific implications for good teaching?

4.2 THEORIES/MODELS/METHODS/STRATEGIES IN SOLVING EQUATIONS (ALGEBRA)

There are basically two theories for the starting point in the teaching of equations. In one theory the starting point is equations of a particular type given in the form of algebraic symbolism, followed by formal methods to solve such equations. However a theory based on the importance of and on how conceptualisation occurs, requires word problems as a starting point. The rest of the discussion in this subsection is based on the latter theory.

Any theory/model/method/strategy for the solving of equations must take into consideration the findings of research studies in the solving of equations (refer to 3.3.3), and the processes involved in the solving of quadratic and cubic equations (refer to 3.4). These sections suggest that many of the problems encountered by students, when solving equations, can be overcome if students correctly interpret the structure of given equations. If the starting point is word problems that lead to the formulation of expressions and equations, then this will result in mastery of the language, terminology and symbolisation. The formulation of expressions and equations in terms of symbols must not be rushed. An approach which takes into consideration the stages in the development of modern algebraic notation (see 2.1.1) must be used. This implies that the formulation of an equation must translate through stages which could include words, sentences, some symbols and finally a formal symbolic representation. For example, consider the following verbal statement which could be given to pupils in grade 8: *Two times a number plus seven gives 14.* In the formulation of the equation the following intermediate steps can be used: *2 times number plus 7 gives 14, or 2xnumber + 7 = 14, or 2x□ + 7 = 14.* Pupils must be encouraged to translate from the verbal form to these intermediate forms of symbolic representations before formally writing $2x + 7 = 14$. Note that in fig. 4-1 below, the stage indicating the formulation of the equation includes these intermediate steps. Such an approach is more intuitively driven and promotes inductive reasoning.

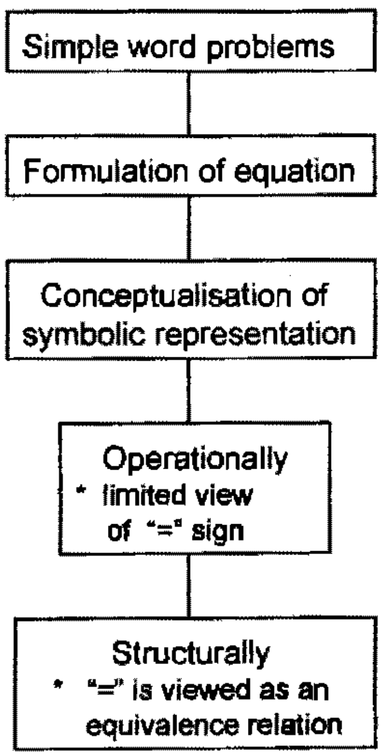


Fig. 4-1: A model for teaching the solving of equations.

Further, note that from a conceptual point of view the symbolic representation of an equation must first be conceived operationally (as representing processes) and then structurally (as representing an object) [refer to 3.2.2 for further details]. In view of the above, a simple model is presented in fig. 4-1 which can be used to introduce particular types of equations to students in grades 8 to 11.

Simple word problems must be used to introduce equations of a particular type to students. A focus on simple word problems leading to the formulation of the problem statements and transforming each of these into the relevant equation will give students a deeper insight into the structural features of equations (refer to 3.3.3.5 for justification). This will also develop in students an appreciation of the need to solve such equations. Students must also be given simple equations in the symbolic form and asked to write down, for each equation, a suitable word problem that leads to the equation. Such a teaching model will develop in students the ability to translate between the verbal and symbolic representations, and also provide them with an opportunity to be creative.

In fig.4-2, the different methods for solving equations in the different grades (from 8 to 12) are indicated. After students arrive at the symbolic representation of equations (linear, quadratic or cubic type), then attention must be focussed on the methods/strategies that can be used to solve such equations. The following methods/strategies can be used to solve equations:

- trial and error, which includes substitution
- verbalisation (refer to 3.4.2 for an example)
- visualisation of the situation which could include diagrams, tables or the use of graphs (refer to 3.4.2 for an example)
- formal methods which require that the equation be viewed as an object.

Note that verbalisation and visualisation are implicitly part of the higher order thinking processes. Once they are developed by a person they become part of that person's thinking. If the method is a graphical one then verbalisation and visualisation are even more important.

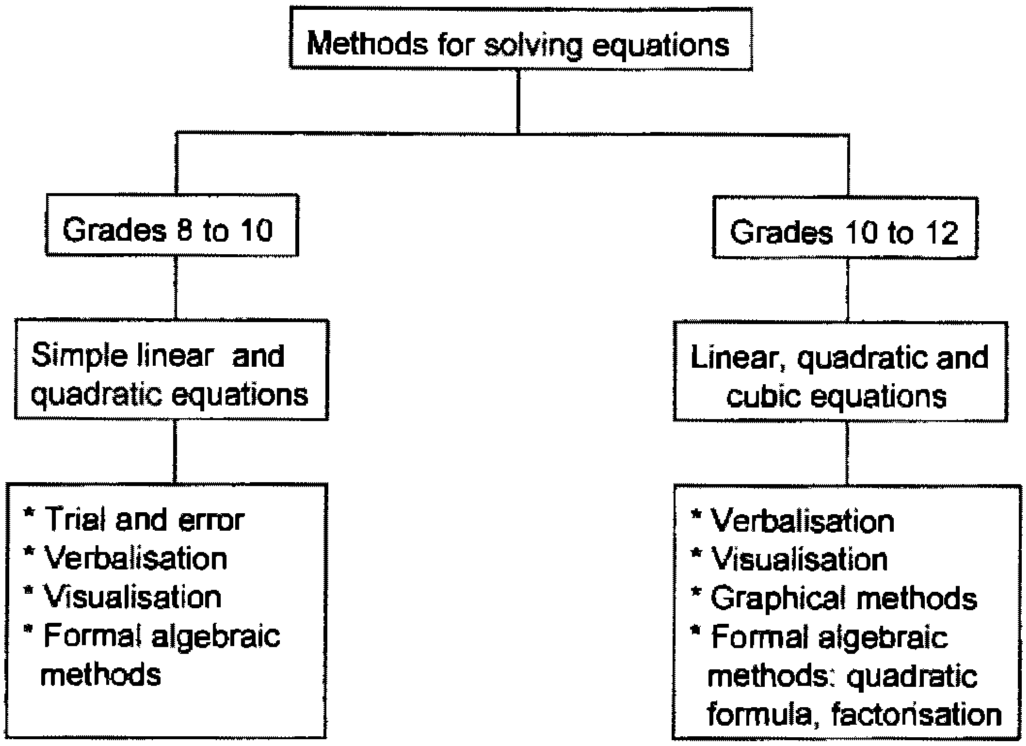


Fig.4-2: Grades and the methods for solving equations

When solving linear equations (grades 8 to 10), quadratic equations (grade 10) and cubic equations (grade 11) the focus should *first* be on informal methods/strategies. Students need to be exposed to small group activities which encourage the use of informal methods/strategies. The advantages and disadvantages of such methods need to be discussed.

Since visual imagery supports the structural conception (Sfard 1991:33) and the visual interpretation of the given information are often not utilised by students (Eisenberg 1991:146), more time needs to be spent on visual methods in grades 9 and 10. This will help to develop among students a more structural conception of equations. Such a conceptualisation can be promoted by using the graphs of the relevant functions to solve a given equation (refer to 3.4.2 for an example). Note that the graphical method requires a good foundation in order for it to be appreciated as a powerful method for solving equations.

Grade 10 is a crucial transition phase for equation solving. Gradually in the solving of linear and quadratic equations the transition must be made to graphical and formal algebraic methods (not necessarily in this order). The type of graphs dealt with in the grade 10 syllabus together with the work done on factorisation allows for such a transition. Solutions obtained graphically should be verified algebraically.

In order for formal algebraic methods/strategies to be used, students must view the given equation as an object. Such a conceptualisation requires that the equal sign be interpreted as an equivalence relation. In order to promote the conceptualisation of an equation as an object, teaching strategies should focus on the following:

- recognising the type of equation that is given,
- transforming equations with justification (refer to 3.4.5.1) to their respective standard forms,
- the devising of algorithms to solve particular types of equations.

Since success in grades 10 to 12 requires that equations be treated as objects, the focus of teaching in these grades should be on graphical and formal algebraic methods.

This subsection has indicated that the starting point for the teaching of equation solving must be word problems. The methods/strategies in solving equations and the grades in which the focus should be on the different methods/strategies have also been discussed. Attention will now be focussed on broader methods of teaching/learning within which the model/methods/strategies in solving equations could be developed.

4.3 FOUR METHODS FOR TEACHING MATHEMATICS

Various writers (eg. Bell 1978, Ernest 1991, Hyde & Hyde 1991, Schoenfeld 1992, Silver 1994, Cangelosi 1996) have written on methods that can be used to teach mathematics. Some of the methods that are suggested are guided discovery, problem solving and inquiry (investigatory). The role of the teacher and student in each of these methods is given in table 4-3, which has been adapted from Ernest (1991:286). Note that the shift from guided discovery, via problem solving, to inquiry approaches requires more processes. Also this shift implies a change in the role of the teacher, from leader to moderator.

Method	Teacher's role	Student's role
Guided discovery	• Poses problem or chooses situation. • Designs activity to guide student towards solution or goal.	• Follows guidance. • Is at a stage "forced" to close the gap or make the jump!
Problem solving	• Outlines stages in the model selected. • Poses problem. • Leaves method of solution open. • Moderator.	• Finds own way to solve the problem. • Justifies the stages followed. • Gradually learns and applies different strategies. • Develops algorithms for problem solving.
Inquiry	• Outlines phases in the model selected. • Chooses starting point or approves student's choice. • Moderator.	• Defines or formulates own problem(s) within situation. • Attempts to solve in his/her own way. • Develops algorithms for the process of inquiry.
Guided problem solving	• Outlines stages in the model. • Designs activities, poses problems or approves student's choice. • Facilitates inductive and deductive reasoning. • Moderator.	• Constructs concepts, discovers relationship. • Gradually formulates, refines and applies strategies. • Develops algorithms for problem solving and the process of inquiry.

Table 4-3: A comparison of methods for teaching mathematics.

For the problem solving and inquiry methods the stages/phases of the model that is selected must be discussed by the teacher. Some of these models are outlined in the subsections below. Note that from table 4-3, the algorithms developed by the students for the process of inquiry/problem solving need not be identical to that suggested by the teacher. It is also evident that problems play a crucial role in each of the methods for teaching mathematics. Therefore it is important that the teacher chooses carefully the type of problem, situation or activity on which the students will focus their attention when engaged in a particular learning activity.

4.3.1 Guided Discovery

As the name suggests, this method requires the planning of suitable activities that will lead students to concepts, relationships and procedures that result in the development of mathematical knowledge (Bell 1978:241). Therefore in this method the teacher has to construct a process that will lead students to the intended discovery.

Some strategies that can be used to promote guided discovery are:

- the sequential enumeration of examples and questions to facilitate inductive generalisations.
- the rephrasing of a question by the framing of suitable sub-questions that will lead students to answer the original question.

4.3.2 Problem solving (Polya, Cangelosi)

In his problem solving model Polya (1973:xvi) suggests the following four phases in the process of problem solving:

- understanding the problem
- devising a plan
- carrying out the plan
- looking back.

During each of these phases some form of mathematical thinking [reasoning, communication, metacognition (decision making) and problem solving] takes place (refer to Hyde & Hyde 1991:29-42 for further details). Note that each of Polya's phases imply certain processes that are discussed in chapter 3 (for example sense making, representing, following a schema, connecting). For further reading on the problem solving model refer to Bell (1978:308-318). The strategies for teaching problem solving that are suggested by Bell (1978:318) include the following:

- Give students a list of questions to answer as they attempt to solve the problem.
- Demonstration by the teacher on the use of different strategies: draw a table/graph/sketch to represent the situation, make a model of the situation, guess and check, introduce variables.
- Group problem solving sessions - teacher and students come up with suggestions on how to solve the problem.
- Help students to formulate questions which might be of use in finding a solution.

Such indirect approaches help students to learn how to generalise approaches to solve entire classes of problems (Bell 1978:318). For example, Tartaglia solved a certain type of cubic equation by using the problem solving strategy of relating the new problem to a known simpler problem (see 2.4.1).

A problem solving model which is suggested by Cangelosi (1996:50-51) has the following nine stages:

1. A person is confronted with a relevant question or questions about how to do or explain something.
2. Clarify the question or questions posed by the problem, in terms of more specific questions (about quantities).
3. Identify the principal variable or variables to be solved.
4. Visualise the situation so that relevant relationships involving the principal variable(s) are identified.
5. Finalise solution plan, including (a) selection of measurements (how data is to be collected), (b) identification of relationships to establish, and (c) selection of algorithms to execute.
6. Collect data (for example measurements if necessary).
7. Execute processes, formulae, or algorithms.
8. Interpret results to shed light on the original question or questions.
9. Make a value judgement (about the original question or questions).

Note that Cangelosi's model is an extended version of Polya's model. For example, stages 1 to 4 above reflect the processes involved in understanding the problem (the first phase of Polya's model). Further note that Cangelosi's model reflects many of the processes involved in algebra (refer to chapter 3), and addresses both the cognitive and affective domains.

4.3.3 Inquiry Model (A. W. Bell)

A model of the process of inquiry/investigation with the following four phases is proposed by A. W. Bell (Ernest 1991:285):

- problem formulating
- problem solving
- verifying
- integrating.

The term investigation is used by Bell to embrace the whole variety of means of acquiring knowledge. This implies that the characteristics of mathematical inquiry include problem posing, abstracting, representing, modelling, generalising, proving and symbolising. Therefore Bell's approach specifies a number of the mental processes (discussed in chapter 3) that are involved in mathematical investigation and problem solving.

Note that the four components of Polya's model are included in Bell's and Cangelosi's models. The latter two models also include problem formulating (or problem posing), which precedes problem solving (Ernest 1991:286).

4.3.4 The guided problem solving model

The learning of mathematics requires the mastery of concepts, relationships (which include conventions, for example $\sqrt{3}$), algorithms and then the application of these. Section 3.2 suggests that the learning of mathematics must be meaningful to students and result in a gradual expansion and modification of networks in the brain. How teaching can promote this with the aim of simultaneously addressing the cognitive and affective domains is discussed by Cangelosi (1996:49-174). From his discussion a guided problem solving model with the following three levels/phases can be formulated:

1. *Inductive reasoning* (conceptual level)
2. *Inductive and deductive reasoning* (simple knowledge and knowledge of a process level)

3. *Deductive reasoning* (application level)

However, note that there is always an interplay between inductive and deductive thinking. They are continuously present and constantly following each other in mathematical thinking and in the developing solving processes. For example, in an inductive process very often a preliminary 'generalising' step is reached. This is exactly when one gets to the deductive part. When one gets to a conclusion or the finalisation of an inductive part, one is at the beginning of the deductive part. Therefore generalising at each of the different levels implies that the deductive mode of thinking comes into play.

In the *inductive reasoning level*, inductive learning activities (which stimulate students to reason inductively) must be used to *construct a concept* or *discover a discoverable relationship*. The learning activities for constructing a concept could have the following four stages:

- sorting (examples and non-examples) and categorising,
- reflecting and explaining the rationale for categorising,
- generalising by describing the concept in terms of attributes (that is, what sets examples of the concept apart from non-examples), and
- verifying and refining (the description or definition is tested and refined if necessary).

Inductive learning activities for discovering a discoverable relationship (for example the square of a real number) could have the following four stages:

- experimenting or reasoning inductively,
- reflecting on and explaining outcomes,
- conjecturing, generalising and
- verifying and refining.

The design of inductive learning activities should evoke in students a willingness to try to construct the concept or to discover the relationship. Activities should also develop in students an appreciation of the concept constructed, or relationship discovered. Guided inquiry instruction can be used to get students to construct a concept or discover a discoverable relationship.

The *inductive and deductive reasoning level* is the intermediate level. While the basis for learning meaningful mathematics is to get students to construct concepts and to discover relationships, students also have to remember conventional names for the concepts that they have constructed and for the relationships that they have discovered. They therefore have to be exposed, by means of a direct instruction process, to certain mathematical information which they have to remember. Information that students have to remember include the following:

- Simple knowledge which requires the remembering of a specific response (but not multi-step process) to a specific stimulus.

  Example: *State the standard form of the quadratic equation.*

  Learning activities must target the end product, that is, the correct statement of the simple knowledge.

- Comprehension which requires the extracting and interpreting of meaning from symbolic representations.

  Example: *What do the following mean to you?*

  (a) $x(x + 1) = 12$ (b) $a.b = 0 \Rightarrow a = 0$ or $b = 0$.

- Knowledge of a process (algorithms). Here the emphasis is on the process involved. So the target of strategies or learning activities is to help students achieve algorithmic skills.

The direct instruction process for the teaching of simple knowledge, comprehension or knowledge of a process should also have a number of suitable stages. Guided instructional activities should enable a willingness to try and appreciation in students of the simple knowledge, comprehension skills and algorithmic skills that they have to master.

In the final level, deductive reasoning is required. Cangelosi (1996:157) notes that

> ...deductive reasoning is the cognitive process by which people determine whether what they know about a concept or abstract relationship is applicable to some unique situation.

Therefore in the *deductive reasoning level* the rules for logic have to be in place. Application level lessons in this phase will require students to put into practise previously developed or acquired concepts, relationships, information and algorithms. Students must determine whether or not a known concept, relationship, information or algorithm applies to a given problem. Therefore in this level, deductive learning activities (which stimulate students to reason deductively) must be planned.

The guided problem solving model is based on the theory of moving from the known to the unknown, taking into consideration how conceptual development occurs. A movement from the inductive reasoning level to the next two levels requires the gradual development of schemata and networks in the brain. This movement in levels is characterised by greater abstraction. During each of the levels well formulated problems (or questions) will form an integral part of the design of learning activities. The teacher has to design suitable learning activities that deal with the relevant aspects (constructing concepts, discovering relationships, formulating knowledge and algorithms, and the application of these) as indicated in each of the levels. These learning activities must guide students to attain the relevant objectives for each level.

4.4 A MODEL/METHOD FOR SOLVING POLYNOMIAL EQUATIONS

In this section the guided discovery model will be used to outline a model/method for the teaching of the solving of polynomial equations. The levels and the processes to be targeted are indicated in fig. 4-4 below.

Note that in each of the levels, the learning activities must target both the cognitive and affective behaviours. In order for this to happen the lessons for each level must include objectives that have two distinct components, namely the content and how the students are to relate to the content.

The discussions in section 4.2 and subsection 4.3.4 suggest the following order for teaching the solving of a particular type of polynomial equation:

- simple word problems
- mastery of language and terminology

- symbolic formulation leading to coded information
- interpretation of formal coded information, for example expressions and equations
- development of theory to solve equations
- application including more word problems (both contrived and real)

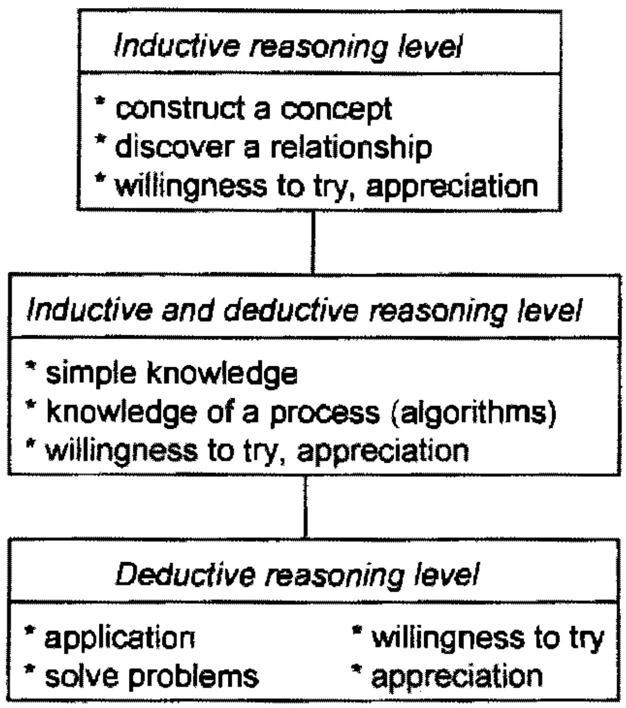


Fig. 4-4: A teaching model for the solving of polynomial equations.

4.4.1 Inductive reasoning (conceptual level)

In this level inductive learning activities must be designed to lead students to the concept to be constructed or relationship to be discovered. This discussion will focus on the concepts *polynomial in one variable* and *polynomial equation in one unknown*. The introduction of the following concepts will also be discussed: polynomial equations of degree one, two and three.

Some students 'solve' polynomial expressions. This suggests that they falsely generalise the concept of an equation and the related processes to that of an expression. The inductive learning activity below is designed to overcome this confusion.

The concept of a polynomial equation of degree one and in one unknown (linear equation) is done in grade 8. The relevant sub-concepts, namely variable, constant and replacement set for variables is also done in grade 8. In grade 7 the writing of number sentences for word problems is done. Now in grade 8, when the concept of a linear equation in one unknown is to be constructed the following type of learning activity can be designed:

- *Design a suitable word problem that leads to a polynomial in one variable and a polynomial equation in one variable (both of degree one), for example:*

  Question 1

  Consider the following two sentences

  (a) A number increased by one.

  (b) A number increased by one is 4.

  1.1 Use symbols to represent the information given in (a) and (b) above.

  1.2 How many replacement values are there for "a number" in (a)?

  1.3 How many replacement values are there for "a number" in (b)?

  1.4 How do the symbolic representations for (a) and (b) differ?

- *The next type of activity should deal with suitable examples and non-examples of linear equations in one unknown, for example:*

  Question 2

  Study the following examples and non-examples of linear equations in one unknown given in 2.1 and 2.2 below:

  2.1 Examples of linear equations in one unknown.

  (a) $3x + 1 = 4$ (b) $3x - x = 4 - 1$

  (c) $4x + 2 = x + 5$ (d) $2(x + 1) = x + 3$

  2.2 Non-examples of linear equations in one unknown.

  (a) $3x + 1$ (b) $3x - x + 4 - 1$

  (c) $x^2 = 4$ (d) $x^3 = 27$

  2.3 In your own words, what to you understand by a linear equation in one unknown?

- *The next activity should expose students to a sorting exercise, for example:*

Question 3

For each of the following, indicate whether it is a linear equation in one unknown or not. If it is a linear equation indicate the unknown.

3.1 $5y - 2 = 3$

3.2 $5x + 2 - 3$

3.3 $6m + 2 = 4m - 3$

3.4 $x^2 - 4 = 0$

3.5 $3^x = 9$

- *This could be followed by a question which focusses on the difference between an equation and an expression, for example:*

Question 4

How does a mathematical expression differ from an equation?

If students are exposed to an inductive learning activity of the type given by the above set of questions, they would gradually construct and refine the concept of a linear equation in one unknown. Such an activity would also get students to appreciate the difference between an expression and an equation. Note that each of the learning activities must be followed by a class discussion. The following general principles should apply to small group setting and class discussions:

- the work must be directed by specific assignments/questions/worksheets
- everybody's opinion must be valued equally
- only one person should speak at a time
- one should feel free to conjecture
- everyone has the right to differ as long as he/she can substantiate

The discussion will now focus on establishing the concept of a quadratic equation in one unknown, which students encounter in grade 10. Again the order that is suggested is suitable word problems followed by inductive learning activities for constructing the concept.

- *Design a suitable word problem leading to a polynomial in one variable and an equation in one unknown, both of degree 2, for example:*

Question 5

Consider the following two sentences

(a) The square of a number.

(b) The square of a number is 4.

5.1 Use symbols to represent the information given in (a) and (b) above.

5.2 How many values are there for "a number" in (a)?

5.3 How many values are there for 'a number" in (b)?

5.4 How do the symbolic representations for (a) and (b) differ?

This should be followed by an inductive learning activity for constructing the concept of a quadratic equation. Questions similar to that for constructing the concept of a linear equation in one unknown can be designed. The examples and non-examples chosen, and the sorting exercise must focus on the differences between linear and quadratic equations in one unknown. Included should be equations of the types in which the unknown (not necessarily x) appear on both sides of the equal to sign, and letters used to represent coefficients. The network in the brain, for equations in one unknown, has to be gradually expanded.

The solving of polynomial equations of degree three is done in grade 11. The concept of a polynomial of degree three and a polynomial equation of degree three should be clarified by a suitable word problem. For example,

Consider the following two sentences:

(a) The cube of a number.

(b) The cube of a number is equal to the original number.

This can be followed by suitable sub-questions to clarify the differences between the symbolic representations for the two sentences. Questions similar to that for constructing the concept of a linear equation in one unknown, can be designed. In this regard the pre-knowledge that the students have with regard to equations must be noted and used. The network in the brain for polynomial equations in one unknown has to be further expanded. Therefore the examples and non-examples that are chosen, and the sorting exercises must focus on the following aspects: differences between linear, quadratic and cubic equations; equations that lead to each of these forms; equations in which the unknown appears on both sides of the equal to sign; equations which have literal coefficients; the number of roots that each type of equation has.

It is advisable that the culmination of the inductive learning activities for the construction of the concepts linear, quadratic or cubic equation lead to the symbolic representation of the respective standard forms. This is the formal simple knowledge that is required in the intermediate level (refer to 4.4.2 below).

4.4.2 Inductive and deductive reasoning level

The main goals (outcomes) for this level are:

- knowledge of conventional names [for example: simple knowledge relating to equations and parts of an equation],
- broadening the perspective of an equation from a process to an object, and
- devising algorithms to solve linear, quadratic and cubic equations in the relevant grades.

Students have to remember the conventional names for the concepts that they have constructed and for the statements of relationships that they have discovered, in the previous level. These conventional names (simple knowledge) will form the labels to encode the relevant information within the networks in the brain. Examples of simple knowledge that have to be in place before progress can be made in this level are:

- a root (or a solution) of an equation
- linear equation (in grades 8, 9 and 10)
- factorisation of quadratic trinomials (grade 10)
- the zero product relationship (grade 10)
- quadratic equation (grades 10 and 11)
- the factor theorem (grade 11)
- polynomial equation of degree 3 (grade 11)

In the relevant grades students should also know the respective standard forms of linear, quadratic and cubic equations. The teaching of the conventional names for concepts or the statements of relationships may require a direct instruction process. If the inductive learning activities in the previous level led to the establishment of conventional names for concepts, statements of relationships or symbolic forms, then suitable questions can be designed to test that the relevant simple knowledge is in place. For example, in grade 10 the following type of questions can be posed when dealing with quadratic equations:

- State the standard form of a quadratic equation (*simple knowledge of a concept*)
- State the zero product relationship for any two real numbers a and b (*simple knowledge relating to the statement of a relationship*)

This discussion will now focus on the use of *inductive and deductive reasoning* to develop knowledge of a process for the solving of a particular type of polynomial equation. Using the terminology of section 3.4.4 the focus will be on devising and appreciating algorithms. As students are exposed to learning activities for particular types of problems, one of the outcomes should be to verbalise the generalise principles that are involved in the solution of the problem (compare with 2.3.3). A rhetorical style could also be used to state problems that lead to the formulation of equations (compare with 2.3.1.3 and 4.2).

In grade 10 the solving of quadratic equations, with integral coefficients, by means of the factorisation technique is done. For the purpose of this discussion, assume that the students have the following previous knowledge/abilities:

- An algorithm to factorise a quadratic trinomial (*knowledge of a process*)
- The standard form of a quadratic equation (*simple knowledge*)
- Statement of the zero product relationship (*simple knowledge*)

If these previous knowledge/abilities are in place, then students can be given the following type of tasks, A to G below, based on the word problems in question 6.

<u>Question 6</u>

Consider the following word problems:

6.1 The square of a number is 4. Find the number(s).

6.2 The square of a number is 2. Find the number(s).

6.3 The square of a number is equal to the original number increased by 2. Find the number(s).

6.4 Five times the square of a number is equal to three times the original number increased by two. Find the number(s).

Required:

[A] Use symbols to write the information given in each of the word problems above, 6.1 to 6.4.

[B] What type of equation(s) did you arrive at, for each of 6.1 to 6.4, in [A] above?

[C] Write each of these equations in the standard form, and factorise their left hand sides.

[D] Write down the zero product relationship for any two real numbers m and q.

[E] How can the zero product relationship be used to solve each of the equations in [C] above?

[F] Solve each of the equations in [C] above.

[G] Write down the steps you will follow when solving a quadratic equation by means of the so-called factorisation technique (refer to [C], [E] and [F] above).

[H] Are their other methods that can be used to solve the quadratic equations that you arrived at in [A]?

Students must be given adequate time to work through tasks [A] to [G] above. They could work in small groups. Note that the tasks are process driven in the sense that they require students to:

- translate from the verbal to the symbolic forms,
- recognise the type of equations arrived at,
- transform these equations to the relevant standard forms,
- deduce how the zero product relationship can be used to solve certain quadratic equations, and
- attempt to devise an algorithm for solving quadratic equations by the factorisation technique.

After students have attempted the task there must be a class discussion. When discussing task [C], al-Khowarizmi's operations and his use of the concept of equivalent equations to solve quadratic equations (refer to 2.3.4.2) can be discussed. The importance of equivalent equations and its value in equation solving must be emphasised. The solving of equations by use of factors and its development as a technique should also be discussed (refer to 2.6.2 for details). Students must appreciate that they are working with fairly modern techniques.

Note that task [E] is designed to get students to appreciate the role of the zero product relationship when solving equations by means of the factorisation technique. Task [G] requires that students devise an algorithm to solve quadratic equations by the factorisation technique. Students should be questioned on the number of steps that their algorithms have, and the need for each step. The steps in their algorithms may have to be refined. Students must be given suitable exercises to practise the algorithmic skills that are required in the application of the algorithms.

Task [H] provides an opportunity for the student to note and reflect on the other methods (for example: verbalisation, trial and error, graphical methods). The advantages and disadvantages of these methods should be discussed. Task [H] also provides an opportunity to expose students to the graphical method to solve equations.

The visual imagery provided by graphs supports structural conceptions (Sfard 1991:33). Research has shown that some high school students have a limited view of the equal to sign (refer to 3.3.3.1). Further students tend to approach problems analytically without making use of the visual interpretations of the givens (refer to 3.3.4.1). A thorough foundation for the graphical method to solve equations could overcome these shortcomings. The Interim Core Syllabus for Mathematics for grade 10 includes graphs defined by equations of the types $ax + by + c = 0$, $xy = k$ and $y = ax^2 + c$. Assume that students have the knowledge of a process to draw such graphs. Now, the following teaching goals can be formulated before students are exposed to the graphical method to solve equations:

- knowledge of the process to solve equations,
- to develop a broader interpretation of the equal to sign,
- the importance of equivalent equations in the context of solving equations,
- the use of graphs to solve equations has limitations,

- there is a need to develop other methods to solve equations,
- graphs can be used to solve inequalities, and
- there is a need to verbalise the mathematical situation represented by graphs.

In order to develop a knowledge of the process that is involved in the graphical approach to solving equations and to address these teaching goals, students can be exposed to the following types of questions:

Question 7

7.1 Sketch on the same system of axes the graphs defined by

$f = \{(x;y) : y = x^2\}$ and $g = \{(x;y) : y = 4\}$

7.2 Now, use your graphs to solve the equation $x^2 = 4$.

Question 8

Draw appropriate graphs to solve for x in the following equations:

8.1 $x^2 = x + 2$

8.2 $x^2 - 1 = x + 1$

8.3 $x^2 - x = 2$

Check algebraically the solution(s) that you have obtained.

Question 9

Draw appropriate graphs to solve for x in each of the following equations, and check whether the solutions you obtain satisfy the equation.

9.1 $5x^2 = 3x + 2$

9.2 $x^2 = 6 - 4x$

Question 10

10.1 Consider the equation $x^2 - x = 2$. Write down the steps in the procedure to solve this equation when using a graphical method.

10.2 What is the role of the concept of equivalent equations when solving equations?

10.3 What are the advantages and disadvantages of the graphical approach to solving equations?

10.4 How can graphs be used to solve for x in the inequality $x^2 - 1 \geq x + 1$?

Note that as students work through the tasks outlined by the questions, of the types given by 7 to 10 above, they would be exposed to inductive and deductive learning activities. These learning activities together with suitable class discussion would lead to the following:

- devising of an algorithm for the graphical method to solve equations
- appreciation of the concept of equivalent equations in the context of equation solving
- appreciation of the advantages and disadvantages of the graphical approach
- create the need to look for other methods to solve quadratic equations (refer to question 9.2 above).

The graphical solution of inequalities becomes a natural extension of the graphical approach that is used to solve equations. The use of appropriate graphs and the verbalisation of the inequalities in terms of the graphs drawn could improve the teaching of the solution of inequalities. As an example consider the inequality given in question 10.4 above.

Let $f(x) = x^2 - 1$ and $g(x) = x + 1$.

Draw the graphs of f and g on the same system of axes.

$x^2 - 1 \geq x + 1$ is the same as $f(x) \geq g(x)$.

Verbalisation: *We require all the values of* x *for which the graph of f is above or on the graph of g.*

[These values can now be easily determined from the graphs drawn.]

In section 2.4 it was noted that Omar Khayyam used graphs to solve cubic equations. Note that the graphical approach views an equation (or inequality) as a comparison of two functions of the same variable on the same domain, for example $f(x) = g(x)$. Such an approach can help students to develop a deeper understanding of algebra (Chazan 1992:372).

In grade 11 the roots of $ax^2 + bx + c = 0$ where a, b and c are rational ($a \neq 0$) is done. Instruction in grade 11 could begin with a recap of simple knowledge by posing the following type of questions, probably in an assignment.

<u>Question 11</u>

11.1 What is the standard form of a quadratic equation?

11.2 State the different methods that can be used to solve quadratic equations.

11.3 Solve for x in the following quadratic equations:

11.3.1 $x^2 + 10x = 39$

11.3.2 $x^2 = 6 - 4x$

11.4 Verify the solutions that you obtained for the equations in 11.3 above.

11.5 Are there other methods to solve the quadratic equation in 11.3.2 above?

After the students have attempted the task outlined by question 11 above, there must be a class discussion. The first equation in question 11.3 can be solved by the factorisation technique. However, the second equation cannot be solved by the factorisation technique. A graphical method to solve this equation yields only approximate solutions. Therefore there is a need to focus on other methods to solve quadratic equations.

This sets up an ideal opportunity to discuss al-Khowarizmi's method to solve the quadratic equation $x^2 + 10x = 39$ by the method of completing the square, with a geometrical justification (refer to 2.3.4.2 for details). After the second equation in 11.3 is written in the equivalent form $x^2 + 4x = 6$, the same procedure with geometrical justification can be used to solve this equation. From these two examples an algorithm can be developed for solving quadratic equations by the method of completing the square. Gradually quadratic equations with the coefficient of x^2 not equal to 1 should be introduced. The algorithm that was developed must be refined. This must be followed by suitable quadratic equations (with numerical coefficients) to be solved by the method of completing the square, in order to develop the algorithmic skills that are required when applying the algorithm.

A study of the secondary school algebra syllabus suggests that the following teaching goals should be formulated when the section on the solving of third degree polynomial equations is done in grade 11:

- Statement of the remainder and factor theorems *(simple knowledge)*
- Application of the factor theorem to factorise third degree polynomial *(knowledge of a process)*
- To broaden the perspective of the equal to sign.
- Use of the concept of an identity.
- Solve third degree polynomial equations *(knowledge of a process)*

The realisation of these goals can be set up by inductive and deductive learning activities as outlined below:

Question 12

Consider the word problem.

The cube of a number is equal to the original number. What are the possibilities for the original number?

12.1 Translate the verbal form of this word problem into its symbolic representation.

12.2 Now, determine the possibilities for the given number.

12.3 Are there any other methods to find a solution to the word problem?

Question 13

Consider the third degree polynomial equation $x^3 + 2x^2 - 5x - 6 = 0$.

13.1 What are the different methods that could be used to solve this equation?

13.2 Solve the equation.

Once students have attempted the task outlined by questions 12 and 13, there must be a class discussion. This discussion should focus on the methods to solve third degree polynomial equations, namely: trial and error, verbalisation, graphical methods and algebraic approaches (from sub-questions 12.1 and 12.2 above). Very briefly the type of approaches used by Omar Khayyam, Tartaglia and Cardano to solve third degree polynomial equations can be noted (refer to 2.4, 2.4.1 and 2.4.2). The importance of

Descartes contribution to factorising polynomials should also be noted (refer to 2.6.2).

The equation in question 13 should lead to a discussion on the remainder and factor theorems (*simple knowledge*), and then an algorithm to factorise fully third degree polynomials of the type $x^3 + 2x^2 - 5x - 6$. Note that the Interim Core Syllabus for Ordinary Grade Mathematics 160407707 (grade 9) had to be phased in by January 1998. Under the section "2.5 Operations on polynomials" the syllabus states that for division the divisors are restricted to monomials. Therefore as students from grade 9 (in 1998 onwards) progress to grade 11, long division will not be a readily available technique to get the corresponding quadratic factor after a linear factor is obtained for the third degree polynomial.

An alternate approach based on the concept of an identity can be set up by the following type of question, which promotes guided discovery.

<u>Question 14</u>

Let $f(x) = x^3 + 2x^2 - 5x - 6$.

14.1 Show that $(x - 2)$ is a factor of $f(x)$.

14.2 Now $f(x)$ can be written in the form $x^3 + 2x^2 - 5x - 6 = (x - 2)(ax^2 + bx + c)$.
Do you know why?

14.3 By inspection, what are the numerical values of a and c?

14.4 Can you now determine the numerical value of b? Explain.

After students have attempted the task outlined by question 14, a class discussion must follow. Note that from 14.2, by making use of the equality of the polynomial in the left hand side and the factored form on the right hand side, it is easy to determine by inspection the coefficients of x^2 and the constant term for the quadratic factor. The numerical value of b can be found by treating $x^3 + 2x^2 - 5x - 6 = (x - 2)(x^2 + bx + 3)$ as an identity. By substituting for x, any number besides 2 in this identity, the numerical value of b can be determined.

For example, substituting $x = 1$ gives $1 + 2 - 5 - 6 = (-1)(1 + b + 3)$ from which it follows that $b = 4$.

Guided by question 14 and the above discussion an algorithm can be devised to outline the procedure to factorise third degree polynomials. Students can then be assigned the task outlined by the following question:

<u>Question 15</u>

Consider the third degree polynomial equation $x^3 + 2x^2 = 5x + 6$.

15.1 Write this equation in the standard form.

15.2 Now solve the equation by making use of the factor theorem.

15.3 Outline the procedure that you used to solve the given third degree polynomial equation.

After students have attempted the task outlined by question 15, this must be followed by a class discussion. The focus should be on the steps in the algorithm for solving third degree polynomial equations by use of the factor theorem, and the formal setting of the write up of the solution (including the use of logical connectives). This must be followed by a suitable task which requires students to apply and refine the steps in the algorithm to solve third degree polynomial equations.

Note that the inductive and deductive learning activities that have been outlined above, for the solving of polynomial equations, target both the cognitive and affective domains. They also gradually require that the relevant network in the brain be further developed and modified. For example, the type of schema that should be developed for algebraically solving polynomial equations is given in fig.3-3 (refer to 3.4.5.2).

4.4.3 Deductive reasoning (application level)

During this level the type of lessons that are designed include the following:

- the solving of equations with literal coefficients where operations are performed on letters instead of numbers,
- focussing on the justification of each step and the use of logical connectives when solving equations,
- evaluating a given algebraic solution to an equation and detecting the point of breakdown (if any),

- formulating new theory based on the literal coefficients of equations,
- application including the use of contrived and real-life word problems which can be solved by the use of equations, and
- finding problems in real-life which can be solved by setting up an appropriate polynomial equation.

Suppose that in grade 10 the students have developed, in the intermediate level, an algorithm for algebraically solving equations with numerical coefficients which lead to linear equations. In order to promote deductive reasoning students can be given the following type of task in grade 10:

Question 16

Consider the equation $mx + px - b = a$.

16.1 Solve for x in the above equation.

16.2 What do m, p, a and b represent in the given equation?

16.3 Are there any restrictions on m, p, a and b?

16.4 Are there any other methods to solve the given equation?

To solve the equation with literal coefficients in question 16, students have to apply their knowledge of factorisation and generalise the algorithm for algebraically solving linear equations (with numerical coefficients). Sub-question 16.4 is designed to get students to appreciate the shortcomings of all other methods when solving equations with literal coefficients.

In grade 11 suppose that students have devised and refined an algorithm for the method of completing the square in order to solve quadratic equations with numerical coefficients (refer to the activity based on question 11 in 4.4.2). Deductive reasoning can be promoted by the task outlined in the following question:

Question 17

Consider the standard form of a quadratic equation, namely $ax^2 + bx + c = 0$ where $a \neq 0$. Solve for x in terms of a, b and c in the given quadratic equation.

In order for students to answer this question they will have to select and apply the algorithm for the method of completing the square that they used for solving quadratic equations with numerical coefficients (devised in the intermediate level). This should result in the derivation of the quadratic formula, which then can be used to solve any quadratic equation. (This development parallels the historical development, see 2.3.5.) The network in the brain for solving quadratic equations has to be modified and refined. The different algebraic methods that are available and the advantages/disadvantages of each method should be noted.

Suppose that students in grade 11 have developed algorithms for algebraically solving equations that lead to quadratic equations. Students can be given a detailed solution to an equation (refer to Example 1 and Example 2 in 3.4.5.1) and asked to give reasons for each of the steps in the solution. Other tasks, based on the ideas of connecting, justifying and sense making, which could promote deductive reasoning are given by the following questions:

<u>Question 18</u>

Consider the following equation: $x(x - 1) = 20$.

18.1 Solve the given equation algebraically.

18.2 In 18.1 above use the logical connectives, '$\Rightarrow$' or '$\Leftarrow$', to link the different steps in the write up of your solution.

18.3 Give a reason to justify each step in your solution in 18.1.

<u>Question 19</u>

Study the following attempt of a student to the equation $x^2 - 4x = 4$. Is the working correct or incorrect? Explain.

Solution: $x^2 - 4x = 4$

$x(x - 4) = 4$

$x = 0$ *or* $x - 4 = 4$

i.e. $x = 8$

Note that in question 18 students have to connect and justify the different steps in their solution procedure, while in question 19 they have to evaluate a given attempt. These type of tasks which require students to connect and justify (including the accepting or rejecting of) the different steps in a formal solution to a problem, promote *deductive reasoning* and give students a deeper insight into the mathematics that is involved (compare with 3.3.3.4 and 2.2.3). This in turn should lead to an *appreciation* of the mathematics and the rules of logic.

Suppose now that in grade 11, during the intermediate level, students have devised and used an algorithm based on the application of the quadratic formula to determine the roots of quadratic equations with numerical coefficients. Then the following type of question can be formulated to promote deductive reasoning:

Question 20

20.1 Write down the formula used to solve the quadratic equation

$ax^2 + bx + c = 0,\ a \neq 0.$

20.2 Which expression in terms of a, b and c determines the number of roots of the quadratic equation? Explain.

20.3 Discuss how this expression (in 20.2 above) affects the nature of the roots of the quadratic equation.

In grade 11 after the work on the sketching of the quadratic function is done, in the intermediate level, the following type of question (which promotes deductive reasoning) can be formulated to connect or appreciate the work on the theory of quadratic equations and the graphs of quadratic functions:

Question 21

Consider the quadratic function $f(x) = ax^2 + bx + c,\ a \neq 0$.

21.1 Explain why solving for $f(x) = 0$ gives the x-intercepts of f.

21.2 Suppose $a > 0$. Draw sketches to illustrate the effect of the discriminant on the position of the turning point of f with respect to the x-axis.

21.3 Explain why the discriminant and the sign of a (positive or negative) determines the number of x-intercepts of f.

21.4 Suppose you are the function f. Explain how you feel when the discriminant is zero and a takes on any non-zero value.

Deductive reasoning can also be promoted by activities which require students to decide whether certain word problems can be solved by setting up equations. The following type of word problems can be given to promote deductive reasoning, since students are required to reflect on, integrate and apply the relevant previous knowledge/abilities developed in the intermediate level.

Question 22

Determine two numbers whose sum is 10 and whose product is 40.

Question 23

The area of a rectangle is *p* square units. The sum of the length (x) and the breadth (y) is *s* units. Is it possible to determine the length and the breadth of the rectangle? Explain.

After students have attempted the above problems, the class discussion based on question 22 could include how this problem led to the development of complex numbers (refer to 2.4.2.1). Note that question 23 presents a problem in its general form. Such questions promote problem solving. Question 23 could set up an ideal opportunity to briefly outline the type of problems that the Babylonians had to solve, their use of normal forms and development of procedures to solve such problems (refer to 2.3.1 for details).

The following problem which is adapted from *Professional Standards for Teaching Mathematics* (NCTM 1991: 157) presents, in a general form, a problem which students can identify with:

Question 24

A textbook is opened at random. The product (p) of the numbers of the facing pages is known. Is it possible to determine to which pages the book is opened?

The above three questions require students to set up appropriate equations and to note the advantage of being able to solve such equations algebraically. Students must also be exposed to real-life problems which require the solving of given equations in order to answer questions. This can be set up by getting students to work in small groups in which they could explore strategies to answer the question(s). These strategies could include the following:

- Guess and check for specific cases [questions 22 to 24].
- Drawing a table of relationships.
- Drawing suitable graphs.
- Diagrams to visualise given and implied information.
- Using variables to model the problem (by setting up an equation or simultaneous equations to represent the situation).
- Manipulating equations in order to solve for an unknown quantity.

Examples of such problems which could get students to appreciate the need for being able to solve quadratic equations are given by the following questions:

Question 25 (*Business and Economics*)

If P rands is invested at r percent compounded annually, at the end of two years it will grow to $A = P[1 + \frac{r}{100}]^2$. At what interest rate (r) will an amount of R100 grow to R144 in two years.

Question 26 (*Safety Research*)

It is of considerable importance to know the least number of feet d in which a car can be stopped, including reaction time of the driver, at various speeds v (in miles per hour). Safety research has produced the formula $d = 0{,}044v^2 + 1{,}1v$. If it took a car 550 feet to stop, estimate the car's speed in kilometres per hour at the moment the stopping process started.

Note that in the above question the use of the given formula will provide the car's speed in miles per hour. However, an estimated speed is required in kilometres per hour. Besides involving more thinking processes, such an answer will be more meaningful to South African students. Question 26 which is relevant to the social sciences has been adapted from the book by Barnett and Ziegler (1989:72). The book also gives a problem from ecology which leads to a quadratic equation. In order to further emphasise the use of equation solving across the traditional school subjects, suitable problems based on equations, can be given to students. In physical science, problems on the equations relating to motion, for example $v^2 = u^2 + 2as$ and $s = ut + \frac{1}{2}gt^2$, can be given.

Such real-life problems would promote problem solving and get students to appreciate the role of equations and solving techniques in the different fields. To get students to deduce, connect and appreciate this role the following type of task can be designed:

Question 27

Reflect on the different sections in the school syllabus and situations in real-life, where equations and their solving techniques are important.

27.1 List the different sections or situations.

27.2 Give examples of real-life problems that require equation solving techniques in order to find solutions to them.

In grade 12 students are required to sketch the graphs of third degree polynomial functions. Assume that students have the following previous knowledge/abilities relating to third degree polynomial functions with numerical coefficients:

- procedure to determine the coordinates of the stationary points
- procedure to determine the x-intercepts and y-intercepts
- procedure to sketch the graph of a given polynomial function
- knowledge of the theory of the nature of the roots of quadratic equations.

Students can be given the following type of problem which requires them to generalise the relevant algorithms and to apply the relevant knowledge in order to deduce a theory relating to the coefficients of a third degree polynomial function/equation:

Question 28

How do the coefficients A, B and C of the third degree polynomial function $y = Ax^3 + Bx^2 + Cx + D$ affect the number of stationary points and hence the x-intercepts?

This investigation question serves the following purposes:

- to apply the procedure to determine the stationary points of third degree polynomial functions with numerical coefficients to a function with literal coefficients,

- to select and apply the theory on the nature of the roots of quadratic equations, done in grade 11, in order to determine the number of stationary points that a third degree polynomial function has,
- to use the number of stationary points (supported by suitable visual imagery relating to the position of the stationary point(s) with respect to the x-axis) to determine the number of possible roots that the corresponding third degree equation has, and
- to develop a theory on the nature of the roots of cubic equations.

Therefore such a question serves to deductively extend the mathematical knowledge base of students in grade 12, and also demonstrates the power of higher mathematics. If students work through the processes indicated above then this could result in the development of new knowledge (refer to 4.5.4, the discussion on problem 3).

For equation-solving the network in the brain has to be continually reorganised and expanded in order to accommodate new developments relating to methods/theory. If suitable small group work settings are used, then the learning activities targeted by the type of problems/questions that have been formulated above would help to develop both essential and specific outcomes as advocated in Outcomes Based Education (refer to Morris 1996 for further details). Note that in all three levels of the model, that has been suggested, both the cognitive and affective domains are targeted. This should always be the case.

4.5 SOLVING PROCEDURES USED BY STUDENTS

This section concentrates on an investigation of a limited scope on the solving procedures used by a group of students at Springfield College of Education. The following were the objectives for the limited investigation that was carried out:

- To determine if students were aware of the different methods that could be used to solve a given quadratic equation, and their reason for selecting a particular method.
- To determine if students use algorithms correctly and appreciate the steps in the

algorithms.

- To determine if students connect symbolic and graphical representations.
- To provide students with opportunities to write (Davis 1993b) and talk about their thinking.
- To determine the implications, if any, for the teaching and learning of mathematics.

4.5.1 Small empirical setting and background of students

In 1997, fourteen fourth year mathematics student-teachers at Springfield College of Education participated in this limited investigation of two $1\frac{1}{2}$ hour sessions. These students were specialising to become senior secondary mathematics teachers. The mathematics teaching course that they were studying in 1997 concentrated on the mathematics syllabuses for grades 11 and 12. These students had already passed a Mathematics I course which includes calculus.

4.5.2 Problems from algebra

The following three problems were selected:

<u>PROBLEM 1</u> (10 minutes)

Consider the following equation $x^2 - 4x = 4$.

1.1 Solve this equation.

1.2 Are there other methods to solve this equation? Outline these (if any).

1.3 Explain why you chose the method in 1.1 above, and not the other possible methods.

1.4 Discuss your responses to the above sub-questions with other members in your group. Make a note of important points for our class discussion.

<u>PROBLEM 2</u> (15 minutes)

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers but a is non-zero.

2.1 Which expression in terms of a, b and c determines the nature of the roots of this equation? Substantiate your answer.

2.2 How does the expression affect the nature of the roots of the quadratic equation?

2.3 For the corresponding quadratic polynomial function $y = ax^2 + bx + c$, how does the expression (in 2.1 above) affect the position of the graph with respect to the x-axis?

<u>PROBLEM 3</u> (40 minutes)

Consider the cubic polynomial function $y = Ax^3 + Bx^2 + Cx + D$ where A, B, C and D are rational numbers, and $A > 0$.

3.1 Determine an expression in terms of A, B, C or D which affects the number of stationary points .

3.2 How does this expression affect the number of x- intercepts of the graph?

3.3 For the corresponding cubic equation $Ax^3 + Bx^2 + Cx + D = 0$, how does the expression (in 3.1 above) affect the roots?

Note that with regard to the model for the solving of polynomial equations discussed in 4.4 above, problem 1 deals with a task at the inductive and deductive reasoning level. The tasks outlined by problems 2 and 3 focus on the deductive reasoning level, since both the problems deal with equations/polynomials with literal coefficients.

4.5.3 Procedures

Students worked with the tasks outlined by problems 1 and 2 in the first session and the task outlined by problem 3 in the next session. Copies of the 3 problems were given to the students in the first session. Students were asked to form their own groups, with not more than 5 members, in which they liked to work. Each group had to elect a scribe who recorded the responses and reported back to the class during the class discussion. While students worked individually and in their groups, their progress was monitored. A tape recording was made of the second session when students worked on problem 3. The written responses of students and the notes made by scribes were collected after each session. A scribe made a note of important points that were discussed during the class discussion times. An evaluation of each of these sources led to the write up in sections 4.5.4 and 4.5.5 below.

4.5.4 Students at work: problems, responses

Although 10 minutes was budgeted for *problem 1*, students requested for more time. This was because of the nature of the sub-questions 1.2, 1.3 and 1.4. Their request indicated a willingness to try.

With the exception of one student, all the other students answered sub-question 1.1 by using the quadratic formula. The other student tried factorisation and then the method of completing the square. All students were able to recognise the type of equation correctly. A study of their responses indicated that they had the necessary knowledge of the process to solve quadratic equations by using the quadratic formula. They were also able to apply this knowledge correctly. Reflecting on her reason for using the quadratic formula, one student (group 1: student 1) wrote:

> I could not factorise the trinomial using the cross method thus I use the quadratic formula which guarantees under the circumstances an answer, as basic substitution is involved.

This response indicates that the student had a schema for solving quadratic equations which first required her to try the factorisation technique. The next link in her schema required her to try the quadratic formula, if factorisation failed.

In group 2 there was a discussion on whether $x(x - 4) = 4$ implies $x = 4$ or $x - 4 = 4$. Four of the students in this group falsely generalised the zero product relationship. A student in this group (student 4) used the graphical representation of $xy = 4$ to question their conjecture. After reflecting on their reasoning students in this group realised that it was *crucial for the product to be zero.*

Students were able to state the other possible methods to solve the equation and to motivate their use of the quadratic formula. Group 3 suggested that a graphical method could be used, although they did not know which graphs to use. After some reflection students from group 1 indicated that the graphs defined by $f(x) = x^2 - 4x$ and $g(x) = 4$ could be drawn and used to determine the values that satisfy the given equation. However, students pointed out that the graphical method should not be used as this method would not yield precise x-values, since for the given quadratic equation the discriminant was not a perfect square.

Student responses to *problem 2* will now be discussed. Although all the students knew that for sub-question 2.1 the expression is $b^2 - 4ac$, there were students who did not know *why* this expression determined the nature of the roots of the quadratic equation. For example, one student (group 2: student 1) made the following remark:

> At school I was just told that this expression determined the nature of the roots.

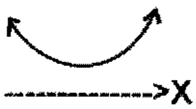
When other students in the group explained, placed and highlighted the expression in the context of the quadratic formula, the same student remarked

> Oh now I see, ... of course that now makes sense.

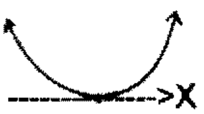
This display of appreciation by the student illustrates the importance of the processes of connecting, justifying and sense-making which are crucial for meaningful learning in mathematics to occur.

Sub-question 2.2 was well answered by the students in all three groups. Students had the knowledge relating to how the expression $b^2 - 4ac$ affects the nature of the roots of a quadratic equation. Some students had difficulty in interpreting sub-question 2.3. However, after discussing the sub-question in their groups they were able to draw appropriate sketches to *visualise and explain* the relationship. For example, assuming $a > 0$, the following sketches were drawn:

$b^2 - 4ac < 0$



$b^2 - 4ac = 0$



$b^2 - 4ac > 0$

Student responses to *problem 3* will now be discussed. Note that class discussions were held, after students worked on each sub-question, as deemed appropriate. For sub-question 3.1, students in group 1 reasoned as follows:

> A cubic polynomial function has this shape

> therefore there must always be two stationary points.

One student (group 1: student 3) highlighted the words *affects the number of stationary points* and reasoned that there may not be two stationary points. Other students in this group asked for an illustration. After some time one student (group 1: student 5) asked: *What about* $y = x^3$*?* A point by point method was used to sketch the graphical representation. Students then questioned whether this graph had a stationary point.

It was evident from the discussions in all the groups that these students regarded stationary points to be the same as turning points. This incomplete construction of the concept and mental schema for stationary points was a stumbling block to the task outlined by problem 3. A class discussion was held and the following was noted with the aid of suitable graphical illustrations: *Stationary points could be turning points or points of inflection.* Once this simple knowledge was in place students continued with sub-question 3.1. Using polynomial functions with numerical coefficients students outlined the procedure to determine the stationary points. Then for the given polynomial function with literal coefficients students generalised the procedure as follows:

> Find the first derivative.
> For stationary points the first derivative is zero.

Students in group 2 could not explain why the first derivative is zero and not any other number. After some discussion, with the aid of sketches, students realised that the first derivative gives the gradient function for any point on the curve of the original function. This illustrates that visualisation promotes structural conceptions. Students correctly verbalised the situation by linking the visual and symbolic representations for the first derivative. They then appreciated why for stationary points the first derivative is equated to 0 and not any other number.

Students soon were able to deduce that the number of stationary points is determined by the number of roots that the quadratic equation $3Ax^2 + 2Bx + C = 0$ has. They concluded that the number of stationary points is determined by the value of the following expressions: $4B^2 - 12AC$ or $B^2 - 3AC$. Students decided to use the latter expression since it was easier to remember and closely resembled the formula for the discriminant of a quadratic equation.

They then proceeded to work on sub-questions 3.2 and 3.3 in their groups. The three possibilities for the expression $B^2 - 3AC$ (namely positive, zero or negative) were considered by students. Using sketches they soon realised that the position of the stationary points with respect to the x-axis was crucial to answer these sub-questions, (the stationary points could be above, below or on the x-axis). The possibilities that arose were verbalised and visualised by students (compare with 2.6.4, Gauss' attempts at proving the fundamental theorem of algebra). For the case $B^2 - 3AC < 0$ students were unsure of what happens. Students were asked to further examine this case by constructing cubic functions that satisfied the given constraints. In group 1 students constructed the function $y = x^3 + x^2 + x$. After writing this function in the form $y = x(x^2 + x + 1)$ they proceeded to analyse the sign diagram of this function.

The class discussion for sub-question 3.2 concentrated on the different visual (graphical) illustrations that arose for the three possibilities for the expression $B^2 - 3AC$. Students then proceeded to answer sub-question 3.3 in their groups. Their conclusions which followed from the class discussion are summarised below:

For the cubic equation $Ax^3 + Bx^2 + Cx + D = 0$

(1) $B^2 - 3AC > 0$ implies 3, 2 or 1 real root(s).

(2) $B^2 - 3AC = 0$ implies 1 real root.

(3) $B^2 - 3AC < 0$ implies 1 real root.

4.5.5 Implications and reflections

These students had an incomplete construction of the concept of stationary points. Some of these students also tried to incorrectly apply the zero product relationship. The incomplete construction of concepts or discoverable relationships could become a major stumbling block to problem solving in the intermediate and deductive reasoning levels (refer to student responses to problem 3, above).

The student responses to problems 2 and 3 indicate that they do use algorithms when working (for example: algorithms to solve quadratic equations; an algorithm to determine the stationary points of polynomial functions). In some cases these algorithms seem to be linked to a larger schema for solving quadratic equations. Incorrect applications of some step(s) in an algorithm is possible (for example, the zero product relationship). If students are actively engaged in the devising, refining and appreciating of the different steps in an algorithm then incorrect applications could be eliminated. The students' algorithm to determine the stationary points of polynomial functions seemed to be restricted to polynomials with numerical coefficients. This implies that tasks need to be designed to enable algorithms to be generalised and incorporated into a larger network.

There were two clear illustrations of students writing down correct mathematics, but without proper understanding of the mathematics involved. These were $b^2 - 4ac$ determines the nature of the roots of a quadratic equation, and at a stationary point the first derivative is zero. The teaching and learning of mathematics should not be recipe-like. There is a need for students to meaningfully conceptualise and establish mathematical knowledge, before attempting to use such knowledge.

The linking of symbolic and visual (graphical) representations, in problems 2 and 3, helped students to conceptualise. For example, students felt that the structure of problem 2 enabled them to get a deeper understanding of the sections on the nature of the roots of a quadratic equation and the sketches of quadratic polynomial functions. The general feeling was that at the end of the task they had a good overview of what was going on in these sections. It seems that the two (separate) schemata were linked and integrated into a larger network.

The implication for teaching in grade 11, from the above two paragraphs, is that the section on the quadratic formula and its application to determine the roots of quadratic equations with numerical coefficients must be done. This must set up a discussion on the nature of the roots of a quadratic equation. Pupils must also be given a mental picture of what is happening with regard to the nature of the roots of quadratic equations and the graphs of quadratic polynomial functions. There must be integration of sections where possible.

This paragraph outlines the general teaching style that should be used. The posing of well formulated and structured tasks implemented in small group settings seem to be conducive to problem solving. During the group-work sessions, as students discuss and conjecture, they must be given enough time to think through their ideas, argue these ideas verbally by providing suitable illustrations and to note important counterexamples and conclusions. Where misconceptions are detected, these can be effectively dealt with in small group settings or during class discussion time (as deemed appropriate). At the end of a given task it is important that a class discussion is held to summarise conclusions, to identify and remedy misconceptions, to tie up loose ends, and to help students to modify and reorganise networks.

4.6 SUMMATION

The teaching approach to solving equations should begin with word problems which lead to equations. Then algorithms can be devised to solve particular types of equations.

For each of the three levels of the guided problem solving model suitable learning activities can be formulated based on guided discovery, problem solving and inquiry methods. Activities for each level must target both the cognitive and affective domains. Inductive learning activities should be planned to lead to the construction and discovery of relationships relevant to polynomial equations. Inductive and deductive learning activities must be designed to formalise simple knowledge, devise algorithms and to develop algorithmic skills relevant to polynomial equations. The interplay between inductive and deductive thinking will improve the reasoning abilities of students. Deductive learning activities need to be planned in order to promote deductive reasoning, which includes the selection and application of concepts/relationships/algorithms relevant to polynomial equations. The symbolic and graphical representations must be linked in order to help students to conceptualise. At each level the network for equation solving in the brain will be reorganised and modified to accommodate increasing degrees of abstraction.

Well formulated and structured tasks implemented in a group work setting promotes problem solving. Given sufficient time students can construct useful mathematical knowledge largely on their own. However, the scene must be carefully planned for and set for this to happen. Well planned group work sessions could lead to students debating ideas, forming and reformulating useful mathematical ideas, and even enjoying doing mathematics. Well formulated tasks and suitable group work sessions, which focus on each of the three levels of the guided problem solving model, can promote the attainment of both essential and specific outcomes as envisaged in Outcomes Based Education.

CHAPTER 5

FINDINGS, CONCLUSIONS, RECOMMENDATIONS AND FURTHER RESEARCH THEMES

This chapter notes the conclusions that have emanated from chapters 2, 3 and 4. Based on these conclusions, recommendations dealing with teaching implications for secondary school mathematics and teacher training programmes are presented. Then some themes for further research are noted.

5.1 FINDINGS AND CONCLUSIONS

In order to present the findings and conclusions in a structured manner, they are noted under the following headings:

- The historical perspective.
- The subject didactical perspective.
- Research findings in algebra.
- Word problems.
- Modern mathematical symbolism.
- Equal to sign.
- Formal equation-solving.
- Understanding in mathematics.
- Integration of sections and experiences.
- Working procedures and approaches/methods of teaching.

5.1.1 The historical perspective

The historical review of the solving of polynomial equations indicated that solving procedures developed from being verbal or rhetorical (refer to 2.3.1.2) to geometric or graphical (refer to 2.3.2.1, 2.4) and finally to the form of algebraic symbolic notation (refer to 2.3.5). As solutions were found for different equations there was a gradual extension of the number system from natural numbers to rational numbers, then to include irrational numbers and finally complex numbers (refer to 2.3.1.1, 2.3.2, 2.4.2.1). The recording of problems and the statement of the solving procedures were first represented by rhetorical styles, then syncopated styles and finally algebraic symbolism. Therefore the gradual development of the solving of equations resulted in the gradual development of the number system and styles for representing problems, their solution procedures and solutions.

History of the solving of polynomial equations indicates that the original impetus for each of these developments were practical motivations to solve practical (real-life) problems encountered by the different people from various cultures (refer to 2.3.1.1, 2.4). Later mathematicians also worked on mathematics for its own sake (refer to 2.3.4, 2.6, 2.7). This also resulted in some rapid developments in equation-solving procedures, the number system, concepts in algebra and styles for representing problems, the solution procedures and concepts in algebra. Contributions to each of these were made by the different peoples and cultures of the world. All of the above aspects have important implications for teaching.

5.1.2 The subject didactical perspective

From a subject didactical point of view, concepts in algebra (refer to 3.2.2) are first conceived operationally (as processes) and then structurally (as objects). So it seems that there is a parallel in the way concepts developed historically and the way in which understanding of concepts occur. Research has shown that verbalisation supports operational thinking while visual imagery supports structural thinking. While appropriate symbolic notation supports structural thinking, this is only possible if meaning is given to the symbolic notation. In other words the symbolic notation must make sense to the user. Both operational and structural thinking reinforce each other and promote the development of algebraic thinking.

Further internal and external representations are important for understanding in mathematics to occur (refer to 3.2.1). For a mathematical concept to be understood it must form part of a network of representations. The degree of understanding depends on the number and strengths of the connections in the network. This has important implications for teaching and learning mathematics.

5.1.3 Research findings in algebra

Research studies in algebra (refer to 3.3) has shown that information that is represented symbolically, if it makes sense to the user, could be interpreted as depicting processes or an object. This results in the so-called *name-process dilemma.* Although modern algebraic notation is a pre-requisite for structural thinking it could also result in *cognitive obstacles* to understanding. Many of these obstacles can be traced to the reading of the structure of expressions and equations represented by modern algebraic symbolism. Some of the obstacles that have been identified are

- a gap between arithmetic and algebra,
- a limited view of the equal to sign which can be traced to arithmetic,
- the deletion error which can be explained by the incorrect generalisation to operations in arithmetic,
- a lack of appreciation of the conventions of algebraic syntax (for example the use of brackets) which structures the text,
- a didactic cut when solving equations in which the unknown appears on both sides of the equation, and
- incorrect reading of the structure (including the components) of equations.

The last obstacle noted seems to result in learners displaying a lack of knowledge of the next appropriate algebraic procedure that should be carried out in order to solve a given equation.

The above indicates that the ability to correctly read the information structured and encoded by algebraic symbolism contributes towards success in equation-solving. Besides this pupils should be able to read and represent information in the following representational forms: verbal, symbolic and graphical. The ability to be able to switch or translate between these representational forms is also important. This requires flexibility.

5.1.4 Word problems

Word problems should be the starting point for teaching equation-solving. Such an approach will gradually and meaningfully develop the formulation of expressions and equations (see 4.2). Students will experience the different representational styles which lead to the development and conceptualisation of modern mathematical symbolism.

5.1.5 Modern mathematical symbolism

Modern mathematical symbolism is the finished product. This notation has many advantages and is a powerful aid to structural thinking. However, this depends on how the coded information is interpreted by the user. Literal symbols could represent various concepts and also take on varying roles when algebraic symbolism is used. The context in which symbols are used and interpreted is important. If the notation is not interpreted or used in a suitable manner then this will result in cognitive obstacles. Therefore the introduction and use of modern mathematical symbolism must not be rushed.

Learners must be exposed to mathematical symbolism through learning experiences which take into consideration the three stages in the historical development of modem mathematical symbolism (see 2.1.1). Word problems will play an important part (see 5.1.4). It is important for meanings to be first established for individual symbols as they are introduced. Then only can meanings be created for rules and procedures that govern action on these symbols (see 3.3.1).

5.1.6 Equal to sign

In the context of equations the equal to sign could be interpreted as *it gives* or *an equivalence relation*. The former is a limited view which can be traced to arithmetic. Such a limited interpretation of the equal sign poses a cognitive obstacle to success in equation-solving. Therefore an interpretation of the equal sign as an equivalence relation has to be developed in secondary school mathematics students.

5.1.7 Formal equation-solving

This requires that an equation be treated as an object. The following order of instruction could help students to conceptualise equations as algebraic objects: first establish that the equality sign is a symbol that denotes the equivalence between the left and the right sides of an equation, followed by performing the same operation on both sides of the equation, and then instruction on the use of substitution for verifying. Note that performing the same operation on both sides makes use of and emphasises the symmetry in an equation. This is absent in the procedure of transposing terms. Therefore the latter procedure should be introduced to students when they are ready for it.

5.1.8 Understanding in mathematics

When a new idea is meaningfully fitted into a larger framework of previously-assembled ideas, then understanding occurs. Understanding can be promoted by making use of the relationship that exists between external and internal representations (see 3.2.1). Understanding of an idea, procedure or fact in mathematics occurs if the corresponding mental representation is part of a network of representations. The degree of understanding depends on the number and strength of the connections. Therefore for understanding in mathematics to occur the networks in the brain have to be continually modified and linked. This has important implications for both teaching and learning.

5.1.9 Integration of sections and experiences

The integration of algebra and geometry will promote understanding in mathematics (see 2.3.2.1, 2.3.4.2, 3.3.2.3 and 4.4.2). Such an integration will also help to bridge the gap between algebra and geometry.

By focussing on the use and application of a particular mathematical concept in the different disciplines and in real-life students will get to appreciate the usefulness of the concept. Integration across sections, disciplines and real-life experiences will impact on the affective domain which is often neglected.

5.1.10 Working procedures and approaches/ methods for teaching

Research on how understanding in mathematics occurs and can be promoted, together with research studies in algebra, have exposed a number of thinking and working procedures in algebra in general, and equation-solving in particular (refer to chapter 3 for details). A study of the syllabus considerations for the solving of polynomial equations implies that a number of teaching goals or outcomes can be formulated, which could promote the achievement of the thinking and working procedures in algebra. It seems that the guided problem solving model can be used to advantage in the classroom context to promote the realisation of teaching outcomes, which target both the cognitive and affective domains (refer to 4.4.4).

5.2 RECOMMENDATIONS

The recommendations for teaching are noted under the following headings:

- Use the implications from the historical perspective.
- Bridge the gap between arithmetic and algebra.
- Problems must be used as growth points.
- Integrate sections and formulate teaching goals.
- Plan for processes involved in equation-solving.
- Promote operational and structural thinking.
- Help pupils to connect and justify.
- The guided problem solving model.
- Teacher education.

5.2.1 Use the implications from the historical perspective

The historical perspective to the solving of equations should not be ignored. Suitable word problems that lead to equations must be the starting point for the teaching of equation-solving. The formulation of the resulting equation must include intermediate steps which use words, sentences, some symbols and finally formal algebraic symbolism (see 4.2, fig. 4-1). Representational forms for expressions and equations must be gradually developed and refined.

The solving of equations must be used to extend the number system and to gradually develop symbolic notation. Solving procedures that are gradually developed must include trial and error, verbalisation and visualisation, and finally formal algebraic methods (see fig. 4-2 in section 4.2). Such an approach more or less parallels the historical development of solving procedures for equations, the development of mathematical symbolism and the extension of the number system. The approach will expose students to a number of working processes and procedures in mathematics (see 3.4 and 4.4).

Suitable anecdotes and illustrations from history can be used to convey to pupils that the solution of equations and solving procedures developed, were developed by and were of concern to different cultures and peoples (refer to chapter 1 and section 4.4 for examples). All of the above will impact on the affective domain which is often neglected.

5.2.2 Bridge the gap between arithmetic and algebra

There is a need in secondary schools to bridge the gap between arithmetic and algebra. Teachers must be aware of the obstacles that exist and plan to help overcome them by exposing students to process-directed experiences. This implies that there will be a need to change the teaching approach to topics like algebraic expressions and the simplification of polynomial expressions (refer to 3.3.2.3). The teaching of simplification of polynomial expressions must be linked to suitable geometrical illustrations (see fig. 3-2, and fig. 2-1 in 2.3.2.1).

The concept of a variable is more abstract and cognitively demanding than the concept of an unknown. Therefore teachers should reflect on the order in which they are introduced to students. In order to introduce the concept of a variable suitable geometric illustrations dealing with patterns, input and output machines, and images can be used. To bridge the gap between arithmetic and algebra the teaching and learning approach needs to be highly process orientated.

5.2.3 Problems must be used as growth points

Word problems should be the starting point for equation solving (refer to 4.2). These problems should lead to the formulation of symbolic representations for equations.

Where possible practical problems related to the solving of equations should be given to pupils. Suitable problems from the social sciences, economics and business, and the natural sciences must be used to focus on the need to develop solving procedures for polynomial equations. Problems must serve as growth points for the development of mathematical knowledge and solving procedures, and justification for the need to develop such knowledge and procedures.

5.2.4 Integrate sections and formulate teaching outcomes

There is a need to integrate sections in the syllabus, where possible, and to formulate explicit teaching outcomes. Activities that are planned for pupils must be pupil-centred. They must encourage pupils to explore, question, conjecture, generalise and justify (refer to 3.4.5). Each of these should be realised by planning and implementing suitable inductive and deductive learning activities to teach equation-solving (refer to 4.4 for examples).

5.2.5 Plan for processes involved in equation-solving

The teaching approaches used must expose pupils to algebra not as a product, but rather as a process leading to generalisations. With regard to the solving of equations appropriate activities must be planned for in order to promote the following processes: constructing a concept or discovering a relationship; recognising the type of equation; verbalising, visualising or reading the situation given in the form of algebraic symbolism; transforming equations to standard forms; devising, appreciating and applying algorithms; representing, connecting and justifying (refer to 3.4 and 4.4).

The formulation of teaching outcomes must focus efforts at promoting the above processes. These outcomes can be realised by planning and implementing suitable learning tasks, for equation-solving, which target each of the three levels in the model that has been suggested for the solving of polynomial equations. Suitable questions must be formulated, types that do not appear in textbooks, in order to promote the achievement of the cognitive processes involved in equation-solving and to also impact on the affective domain. In order for this to happen the objective(s) for each lesson must specify the mathematical content and also state how students are to relate to this content. The learning activities that are designed must then lead to the achievement of such objectives which target both the cognitive and affective domains.

5.2.6 Promote operational and structural thinking

Verbalisation supports operational thinking, while visual imagery supports structural thinking. Therefore both verbalisation and visualisation must play an important role in attempts to help pupils to make sense of and understand equations and solving procedures. There needs to be a greater emphasis on graphical approaches to solve equations, which makes use of the concept of equality of functions (refer to 4.4.2). Where possible the use of suitable computer software and graphic calculators must also be exploited to promote structural thinking. Several studies (eg. Kaput 1986, Mercer 1995) give ideas relating to the use of such technology in order to promote understanding in algebra.

In order to develop a broader interpretation of the equal to sign, the concept of an identity and its implications must be thoroughly discussed when the section on factorisation is taught in grades 9 and 10 (refer to 4.4.2, question 14 for ideas).

5.2.7 Help pupils to connect and justify

In the write up of formal solutions to equations, teaching must focus on the use of logical connectives in order to connect the different steps in the solution procedure. Further there is a need to focus on the justification of each of the steps in the solution procedure. Both of these could get pupils to make more sense of the solution procedure and the mathematics that is involved (refer to 3.4.5, and 4.4.3 - questions 18 and 19).

The devising of algorithms is an important process in the learning of mathematics. One of the end products of problem-solving must be the devising of algorithms. Pupils must appreciate the need for the different steps in an algorithm and they must also be able to apply algorithms correctly. Teaching must focus on and plan for the processes involved in devising, appreciating and applying algorithms (refer to 3.4.4, 4.4.2 and 4.4.3 for details).

Since schemata and networks play an important part in understanding and equation-solving, the teaching implication is that attention must be focussed on representing, formulating and modifying schemata, reorganising and linking networks. Students must be encouraged to develop schemata, for each of the following:

- solution procedures for different types of equations,
- for the utility value of equation-solving with regard to the different sections in the school mathematics syllabus, and
- for the uses of equations and equation-solving in real-life situations.

All of the above can promote sense-making and understanding with regard to the solving of equations (refer to 3.4.5.1 and 3.4.5.3).

5.2.8 The guided problem solving model

The three levels of this model should be used to *guide* the planning and implementation of learning activities to promote the working processes and procedures in algebra. For each of the three levels of the guided problem solving model suitable problems could serve as growth points to develop the mathematical knowledge related to equation-solving and also knowledge relating to how mathematical knowledge can be developed. In this regard problem posing (including problem formulating) has an important role to play in each of the levels both as a means and a goal of instruction.

Note that this teaching model for solving polynomial equations (refer to 4.4 for details) requires new and innovative forms of assessment. Many workable ideas can be found in *Assessment in the Mathematics Classroom* (Webb 1993).

Finally note that the guided problem solving model that has been presented is still teacher-centred, especially in the sense that the learning activities for each of the stages is designed by the teacher. Teachers should try to investigate the possibilities of other more student-centred variations/approaches. The guided problem solving approach is this writer's own preference, but it is not necessarily the only one that will work.

5.2.9 Teacher education

If the quality of teaching and learning in mathematics is to improve, then it is important that teachers are exposed to the findings and teaching implications of research studies on school mathematics. Sadly there seems to be a "communication gap at the level of reporting the results of research" (Pateman 1989:3). It is important that both researchers and teachers make efforts to change this situation.

Note that the formulation of the teaching outcomes (goals) for each of the three levels of the model for solving polynomial equations in chapter 4 requires a thorough knowledge of the school mathematics syllabus as a whole. The formulation of some of the teaching outcomes and ideas were informed by research studies in algebra (refer to 3.3). In addition to this it is important that a grade 9 mathematics teacher has a good knowledge of the role of mathematical concepts, knowledge and techniques dealt with (in grade 9), the utility value of these and the function that they will serve in the further development and application of the mathematical ideas/knowledge/concepts of students in later years. This together with the relevant findings of research studies should guide the teacher in formulating specific short term and long term goals (outcomes) for the teaching of particular topics in the syllabus. These outcomes must focus on both the cognitive and affective domains.

The same is true for teachers in grades 10, 11 and 12. If this is accepted then there are important implications for the training of mathematics teachers. Mathematics teachers must be exposed to the findings and teaching implications of research studies. Further, they need to know much more mathematics than they will be required to teach in a particular grade. These points should be taken into account when the relevant authorities, for example in colleges of education and universities, design and reflect on the mathematics curriculum for the training of pre-service and in-service mathematics teachers.

5.3 THEMES FOR FURTHER RESEARCH

Some possible themes for further research are noted below:

- Historical review or investigation on other topics from the school mathematics syllabus, and the implications for teaching mathematics.
- Research on what minimal notation and concepts could be used within the confines of the school syllabus, and how pupils can construct their own notation to represent ideas.
- The role of creativity in school mathematics. Creativity dealing with the processes involved in the formulation of conjectures or observations by students and their justification for accepting or rejecting the conjectures or observations.
- Investigations on the type of schemata and networks that pupils have for problem-solving and particular sections in the mathematics syllabus. Instruction units that could be designed to bridge the gap between schemata and networks that exist in the pupil's mind, and the type of schemata and networks that are required.
- How teachers' philosophies of what mathematics is and their knowledge of mathematics influences the type of instruction that is planned.
- Testing of ideas. The teaching ideas on the processes involved in the solving of polynomial equations still need to be implemented in the classroom situation. Whether or not these ideas together with the guided problem solving model for the teaching of polynomial equations work, and lead to an improved understanding of mathematical concepts, knowledge, and techniques still has to be tried out in mathematics classrooms in secondary schools.
- Investigate the literacy level of secondary school pupils relating to the interpreting of information given in the form of algebraic symbolism, and the relationship between this and their success in learning mathematics.
- Investigate the views of teachers with regard to historical and psychological perspectives to teaching mathematics. Determine whether the teaching practices of teachers take into account such perspectives.
- Investigate the level of awareness of teachers on the ground to research studies relating to school mathematics and their teaching implications.
- Investigate the impact of computer software and graphic calculators in developing

structural thinking among pupils.

- It seems that the majority of research in mathematics education has been done in elementary and middle-level grades. Research is required at the secondary and post secondary levels to determine whether the theories of learning and teaching would be viable in more complex mathematical situations, with older students and teachers who usually have a more thorough mathematics background.

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