TEMPORAL LOGICS

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Abstract

We consider a number of temporal logics, some interval-based and some instant-based, and the choices that have to be made if we need to construct a computational framework for such a logic. We consider the axiomatisation of the accessibility relations of the underlying temporal structures when we are using a modal language as well as the formulation of axioms for distinguishing concepts like actions, events, processes and so on for systems using first-order languages. Finally, we briefly discuss the fields of application of temporal logics and list a number of fields that looks promising for further research.

Key terms:
Temporal logic; Temporal reasoning; Time structures; Interval logic; Axiomatisation; Causality

Dedicated to the memory of my late parents.

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Chapter 1

A range of knowledge representation languages

1.1 Introduction

The study of the nature of time has kept many scholars occupied over the last two thousand years. Greek philosophers like Aristotle and Diodorus Chronus, medieval Arabic logicians and Latin scholars in the Middle Ages were mainly interested in the formal representation of different tenses in natural language. In the late nineteen-fifties and early nineteen-sixties a number of publications on tense logic by Arthur Prior (e.g. [Pri57]) instigated a renewed interest in the subject especially when the possibility of the use of temporal logic for the representation of time in computer applications was realised.

Temporal logic has a wide variety of fields of application. It is used for example in natural language understanding, planning, databases, program verification, concurrent programming, expert systems and robotics. A great deal of work has been done in the development of temporal logics over the last thirty to forty years. To get an idea of the variety of these, we are going to consider a few of the well-known temporal logics that formed the bases for the development of other logics.

To make it easier for the reader, the same notation will be used throughout the text for the presentation of different logics even though this may differ from the notation used in the original articles. Since we are going to look at temporal logics using non-modal first-order languages as the object language as well as temporal logics that use propositional modal languages as the object language, we recapitulate the basic ideas of propositional logic as well as of first-order logic and modal logic in the subsequent sections. The main sources used for writing this chapter were [Lab98], [Ham88] and [Gol92].

1.2 Propositional languages

The sentences of *propositional languages* are built up of *atoms*, also called *atomic sentences,* that express basic facts like 'The sun is shining' and 'The table is red'. An alphabet for a propositional language may conveniently be taken to consist of the following symbols:

- a set Φ containing one or more atoms $p, q, \ldots$
- the punctuation symbols '(' and ')'
- the connectives $\neg$ (negation) and $\rightarrow$ (the conditional).

A propositional language $\mathcal{P}$ is a set of *sentences.* What constitutes a sentence? Well, φ is a sentence if and only if one of the following holds:

- φ is an atom
- φ is of the form $\neg\psi$ or $(\psi \rightarrow \chi)$, where ψ and χ are sentences.

It is often convenient to form an extended language $\mathcal{P}^*$ with the introduction of other connectives by the following abbreviations:

$(\psi \vee \chi)$	(disjunction)	abbreviates	$(\neg\psi \rightarrow \chi)$
$(\psi \wedge \chi)$	(conjunction)	abbreviates	$\neg(\psi \rightarrow \neg\chi)$
$(\psi \leftrightarrow \chi)$	(biconditional)	abbreviates	$(\psi \rightarrow \chi) \wedge (\chi \rightarrow \psi)$.

Example 1 *Sentences and nonsentences in a propositional language.*

Let $\Phi = \{p, q, r, s\}$. The following are sentences in the extended language $\mathcal{P}^*$ (where parentheses are used with pedantic strictness, although in the remainder of the dissertation we shall often omit them when no confusion would result):

$p,\ \neg p,\ (p \rightarrow q),\ ((p \vee \neg q) \rightarrow (s \wedge r))$.

But $(p \leftrightarrow \wedge\neg p)$ is not a sentence in $\mathcal{P}^*$.

How does one attach meaning to the sentences in a propositional language? In other words, how should the sentences be interpreted?

A *valuation* of $\mathcal{P}^*$ is an allocation of a truth value from the set $\{T,F\}$ to each of the atoms in Φ. Such a valuation results in the association of a truth value with every sentence in $\mathcal{P}^*$ in accordance with the following rules:

- $\varphi \in \Phi$ is true iff the truth value T was allocated to it by the valuation
- $\neg\varphi$ is true iff φ is false
- $(\varphi \rightarrow \psi)$ is true iff φ is false or ψ is true, or both.

One can regard a valuation as a representation of one *possible state* of some system of interest or one *possible world* that the sentences of $\mathcal{P}^*$ are intended to talk about. Let S be the set of all possible valuations of $\mathcal{P}^*$. If φ is true in $s \in S$, we say that s *satisfies* φ, or that s is a *model* of φ, denoted by $s \Vdash \varphi$ [1]. $Mod(\varphi)$ denotes the collection of all models of φ.

Any subset of S is called a *frame.* If a sentence φ is true in all valuations that are elements of a frame $\mathcal{F}$, we say that φ is *globally true* or *globally satisfied* over the frame $\mathcal{F}$. This is denoted by $\mathcal{F} \Vdash \varphi$. This in turn also means that, for all $s \in \mathcal{F}$, s is a model of φ, so $\mathcal{F}$ is a subset of the collection of all models of φ, i.e. $\mathcal{F} \subseteq Mod(\varphi)$.

A sentence is *globally true* over S if it is true in every possible valuation. Such a sentence is called a *tautology,* whereas a sentence that is false in every possible valuation is called a *contradiction.*

Example 2 *Global truth.*

Let $\Phi = \{p, q\}$. Then $S = \{s_0, s_1, s_2, s_3\}$, i.e. there are four possible valuations. Each valuation s_i can be represented by a sequence vw, where v is the truth value associated with p and w the truth value associated with q. Then $S = \{FF, FT, TF, TT\}$.

The proposition p is globally true over the frame $\mathcal{F}_0 = \{TT, TF\}$ but not over the frame $\mathcal{F}_1 = \{TF, FT\}$. So $\mathcal{F}_0 \Vdash p$ but $\mathcal{F}_1 \nVdash p$.

The sentence $p \vee \neg p$ is a tautology since it is globally true over S and the sentence $p \wedge \neg p$ is a contradiction since it is false in all four possible valuations.

A frame $\mathcal{F}$ represents knowledge about the system under consideration. More specifically, since a frame is formed by excluding members of S, it represents knowledge of states that cannot occur. Such knowledge can sometimes be expressed by a set of sentences Σ. We say that a set of sentences Σ *axiomatises* $\mathcal{F}$, and speak of Σ as a set of *axioms* for $\mathcal{F}$, if and only if $Mod(\Sigma) = \mathcal{F}$.

[1] Most of the older books use $\models$ to denote a *model* as well as for *semantic consequence* which will be discussed shortly.

Remark: The axiomatisation of which we speak here and in the chapters to follow has a semantic connotation. The term 'axiomatisation' is often used with a proof-theoretic (syntactic) connotation in contexts that involve simulating semantic consequence relations by means of effective procedures constructed with the aid of inference rules. This proof-theoretic connotation will not be relevant to our aims.

Example 3 *Axiomatising a frame.*

Suppose $\Phi = \{p, q\}$. The set of possible states is $S = \{FF, FT, TF, TT\}$. Let p and q express the notions of a light being on and the temperature being 40 $^\circ C$ respectively. Suppose we want to consider only those states in which exactly one of the atoms is true, i.e. we are interested in the frame $\mathcal{F} = \{TF, FT\}$. Then this frame can be axiomatised by the sentence

$$(p \wedge \neg q) \vee (\neg p \wedge q).$$

The first disjunct asserts that p is true and q is false, so the state TF is included in the class of models of the axiom. The second disjunct asserts that p is false and q is true, and this includes the state FT in the class of models. The states FF and TT are both excluded (in the sense that they falsify the sentence and thus do not belong to the class of models): if p is true then q cannot be true, so TT is excluded, and if p is false then q cannot be false so FF is excluded. Thus the sentence is globally true over the frame $\mathcal{F} = \{TF, FT\}$, and is true in no other valuation.

There are frames which are not axiomatisable, i.e. for which no set Σ of sentences can be found such that the frame is precisely $Mod(\Sigma)$. We will use an example to illustrate this.

Example 4 *Non-axiomatisable frames.*

Let $\mathcal{L}_A$ be the propositional language with infinitely many atoms $p_0, p_1, \ldots$ Then the set S of all valuations has uncountably many members which may be indexed by the elements of the cardinal number $2^{\aleph_0}$: $S = \{s_\beta | \beta \in 2^{\aleph_0}\}$. We assume that this indexing is done in such a way that s_0 is the valuation which makes every atom true. Let $\mathcal{G} = S - \{s_0\}$ be the frame that results from excluding s_0. Now it is not difficult to show that every set Σ of sentences globally true over $\mathcal{G}$ will also be globally true over S, so that $Mod(\Sigma) \neq \mathcal{G}$. Brink & Heidema [BH89] prove this by contradiction, using the following argument:

Let Σ be a set of sentences over $\mathcal{L}_A$, and let $Mod(\Sigma) = \{s \in S|$ for every $\alpha \in \Sigma, s \in Mod(\alpha)\}$. We want to show that the frame $\mathcal{G}$ is not axiomatisable, even by an infinite set of axioms.

Assume there exists a set of sentences Σ that axiomatises $\mathcal{G}$. Then every valuation in $\mathcal{G}$ must make every sentence in Σ true, and every valuation outside $\mathcal{G}$ must make some sentence in Σ false. There is only one valuation outside $\mathcal{G}$, namely s_0, so there is some sentence α in Σ which is false in s_0 but true in every other valuation. The sentence α is built up out of finitely many atoms. Let n be a number big enough so that the atoms in α are among $p_0, p_1, \ldots, p_n$ and consider the valuation s_γ which makes these atoms true and all other atoms false. So s_γ associates the value T with p_i iff $i \leq n$, in other words, $s_\gamma(p_0) = T, \ldots, s_\gamma(p_n) = T, s_\gamma(p_{n+1}) = F, \ldots$

Now s_γ must make α false, because s_0 makes α false and the two valuations give the same values to the atoms in α. But s_γ lives in $\mathcal{G}$, and we assumed that all the valuations in $\mathcal{G}$ satisfied α. So we have an absurdity - α is both satisfied and not satisfied by s_γ.

So, by *reductio ad absurdum*, we can claim that our initial assumption was invalid, i.e. there does not exist a set of sentences axiomatising $\mathcal{G}$.

Labuschagne [Lab98] discusses the significance of this result: a frame $\mathcal{G}$ represents our unverbalised knowledge about a system. If we succeed in axiomatising $\mathcal{G}$, then the axioms can be regarded as a verbalisation in the relevant formal propositional language of our previously unverbalised knowledge. The fact that such axiomatisations do not always exist just means that sometimes we have knowledge about a system that, while expressible in the metalanguage, cannot be expressed completely in the propositional object language.

A *schema* is a representation of the set of all sentences sharing some syntactic form. As an example, consider the language described in Example 3. If $(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)$ is regarded as a schema in which the metalinguistic variables φ and ψ stand for arbitrary sentences in the language, then $(p \wedge \neg q) \vee (\neg p \wedge q)$ is an *instance* of this schema.

We say that a sentence φ is a *semantic consequence* of the sentence α, denoted by $\alpha \models \varphi$, if φ is globally true over *Mod(*α*)*. In a similar way the notation $\Sigma \models \varphi$ means that the sentence φ is globally true over *Mod*(Σ).

Propositional languages such as those described above are not suited for the representation of sentences of which the truth value varies over time. In the following chapters we are going to look at some systems that use first-order languages as well as others that use propositional modal languages for this purpose.

1.3 First-order languages

The languages of first-order logic are similar to those of propositional logic but each is based on an alphabet $\mathcal{A}$ which consists of the following symbols:

- a set X of variables $x_0, x_1, \ldots$
- a set C consisting of zero, one, or more constants $a, b, \ldots$
- the connectives $\neg$ and $\rightarrow$
- the punctuation symbols '(' and ')'
- a set *FUN* consisting of zero, one, or more elements of the form (f, n), where f is a function symbol having arity n associated with it
- a set *PRED* consisting of one or more elements of the form (A, n), where A is a predicate symbol having arity n associated with it
- the *universal quantifier* $\forall$.

A *term* t over an alphabet is defined recursively as follows:

- if $t \in X$, then t is a term
- if $t \in C$, then t is a term
- if $(f, n) \in FUN$, and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

If $(A, n) \in PRED$ and $t_1, \ldots, t_n$ are terms, then $A(t_1, \ldots, t_n)$ is an *atomic formula*. We define the set Φ as the set of all atomic formulas.

A *wff* (i.e. a *well-formed formula*) over such an alphabet is a string conforming to the following criteria:

- if $\varphi \in \Phi$, then φ is a *wff*
- if φ and ψ are *wffs*, then $\neg\varphi$ and $(\varphi \rightarrow \psi)$ are *wffs*
- if φ is a *wff* and $x \in X$, then $\forall x(\varphi)$ is a *wff*.

The set $\mathcal{L}$ of all *wffs* over the alphabet $\mathcal{A}$ is called the *first-order language* over $\mathcal{A}$.

We can form an extended language $\mathcal{L}^*$ by the introduction of the connectives $\wedge, \vee$ and $\leftrightarrow$ by the usual abbreviations, and by the addition of the *existential quantifier* $\exists$, which is the dual of $\forall$ in the following sense:

$\exists x(\varphi)$ abbreviates $\neg\forall x(\neg\varphi)$.

Strings of the form $\forall x \forall y \forall z$ are abbreviated to $\forall x, y, z$.

Example 5 *Atomic formulas are transparent.*

In all logic languages the primitive entities, i.e. atoms, are analogues of simple English sentences like 'Jenny is pretty'. In propositional languages, atoms are indivisible and no attention is given to any internal structure they may have. In a first-order language, however, atomic formulas are built up from simpler things. To illustrate this informally, we give a natural language example. The sentence 'Jenny is pretty' may be considered as consisting of two separate parts, namely a *predicate symbol* and a *constant.* 'Jenny' would be the constant of the sentence and 'Is pretty' would be the unary predicate symbol. A more complex sentence like 'Jenny is dancing with John' would consist of three distinct parts, namely the binary predicate symbol 'Is dancing with' and the two constants 'Jenny' and 'John'.

Formal analogues of these sentences in a first-order language exhibit the structure more clearly. If, in an alphabet $\mathcal{A}$, the constant 'a' is thought of as representing the girl Jenny, and the predicate symbol 'A_1' as representing the property of being pretty, then $A_1(a)$ is an atomic *wff* expressing the information that Jenny is pretty.

Similarly, the information that Jenny is dancing with John may be expressed by a *wff* of the form $A_2(a, b)$, where A_2 is a binary predicate symbol and a and b are constants.

How do we attach meaning to sentences in a first-order language? This is done by means of an *interpretation* $I = (D, i)$ of $\mathcal{L}^*$ that consists of a non-empty set D called the *universe of discourse* or the *domain*, and a *meaning function* i which maps certain symbols of $\mathcal{L}^*$ to entities constructed from the domain. The function i must be such that:

$i(a) \in D$, for all $a \in C$
$i(f, n) : D^n \rightarrow D$, for all $(f, n) \in FUN$
$i(A, n) \subseteq D^n$, for all $(A, n) \in PRED$.

Given an interpretation $I = (D, i)$, we define a *variable assignment* v for I as a function that maps each variable in $\mathcal{L}^*$ to an element in the domain. So $v(x) \in D$, for all $x \in X$.

Let $I = (D, i)$ be an interpretation of $\mathcal{L}^*$ and let v be a variable assignment for I. The *denotation* of an arbitrary term t in $\mathcal{L}^*$ is calculated recursively in the following way:

- if $t \in X$, then t denotes $v(t)$
- if $t \in C$, then t denotes $i(t)$
- if t has the form $f(t_1, \ldots, t_n)$ for $(f, n) \in FUN$, then t denotes $i(f)(d_1, \ldots, d_n)$ where d_j is the element in D denoted by t_j.

The *valuation s* corresponding to I in the context of v associates a truth value from the set $\{T,F\}$ with the atomic formulas in $\mathcal{L}^*$ in the following way:

- $A(t_1,\ldots,t_n) \in \Phi$ is assigned the value T iff $(d_1,\ldots,d_n) \in i(A)$, where d_j is the element in D denoted by t_j.

Truth values can now be associated with all *wffs* in the language: For any *wff* $\varphi \in \mathcal{L}^*$, and for any variable assignment v for I, we say that I *satisfies* φ in the context of v, denoted by $(I,v) \Vdash \varphi$, iff one of the following cases applies:

- $\varphi = A(t_1,\ldots,t_n) \in \Phi$ and $(d_1,\ldots,d_n) \in i(A)$, where d_j is the element in D denoted by t_j
- $\varphi = \neg\psi$ and $(I,v) \nVdash \psi$
- $\varphi = (\psi \to \chi)$ and $(I,v) \nVdash \psi$ or $(I,v) \Vdash \chi$ or both
- $\varphi = \forall x(\psi)$ and $(I,v') \Vdash \psi$ for every variable assignment v' that differs from v at most on the variable x.

In a first-order language, the *scope* of a quantifier is defined as the shortest *wff* immediately following the quantifier, e.g. if we consider the *wff* $\forall x(\varphi)$, φ will be the scope of $\forall x$. Occurrences of x in $\forall x$ or in the scope of $\forall x$ are *bound*. If all occurrences of variables in a *wff* φ are bound, then φ is a *sentence*. If an occurrence of a variable is not bound, it is said to be *free*. If a sentence φ is satisfied by an interpretation I in the context of a variable assignment v, then φ will be satisfied by I in the context of all possible variable assignments. We say I satisfies φ, or I is a *model* of φ, and denote this by $I \Vdash \varphi$.

Example 6 *A model of a sentence.*

Let $\mathcal{L}_0$ be a first-order language with $PRED = \{(A_1,1),(A_2,1)\}$, $C = \{a,b,c\}$ and $FUN = \emptyset$. Let $I_0 = (D,i)$ be an interpretation of $\mathcal{L}_0$, where $D = \{1,2,3\}$, $i(a) = 1$, $i(b) = 2$, $i(c) = 3$, $i(A_1) = \{1,3\}$ and $i(A_2) = \{2\}$.

Is I_0 a model of the sentence $\varphi = \exists x(A_1(x))$? Let v be any variable assignment in I_0. I_0 satisfies $\exists x(A_1(x))$ in the context of v iff there is at least one variable assignment v', differing from v at most on x, such that I_0 satisfies $A_1(x)$ in the context of v'. (Remember that $\exists$ abbreviates $\neg\forall\neg$). Let v' be the same as v except that $v'(x) = 3$. Then $v'(x) \in i(A_1) = \{1,3\}$. So I_0 satisfies $A_1(x)$ in the context of v' and therefore I_0 satisfies $\exists x(A_1(x))$ in the context of v. Since v was arbitrary, we can claim that I_0 is a model of $\exists x(A_1(x))$.

Let S be the class of all interpretations of $\mathcal{L}^*$. A *frame* is any subset of S. If a sentence φ is true in all elements of a frame $\mathcal{F} \subseteq S$, we say that φ is *globally true* over $\mathcal{F}$, denoted by $\mathcal{F} \Vdash \varphi$. This also means that, for all $s \in \mathcal{F}$, s is a model of φ, so $\mathcal{F}$ is a subset of the collection of all models of φ, denoted by $Mod(\varphi) \supseteq \mathcal{F}$.

We say that a set Σ of sentences *axiomatises* $\mathcal{F}$ if and only if $Mod(\Sigma) = \mathcal{F}$. A sentence φ is a *semantic consequence* of the sentence α, denoted by $\alpha \models \varphi$, if φ is globally true over $Mod(\alpha)$. In a similar way, $\Sigma \models \varphi$ means that the sentence φ is globally true over $Mod(\Sigma)$.

As is the case in propositional languages, there are frames which are not axiomatisable, i.e. for which no set Σ of sentences can be found such that the frame is precisely $Mod(\Sigma)$.

Example 7 *Axiomatising a frame.*

Suppose we want to axiomatise the class of all interpretations whose domains contain only one element. This can be achieved by the axiom

$$\forall x, y(x = y).$$

In the discussion given above, all the variables in our language were of the same sort so we can speak of an unsorted first-order language. Most of the languages we are going to consider, however, contain variables that are of different sorts. Let's look at how many-sorted first-order languages differ from unsorted first-order languages.

1.3.1 Many-sorted languages

The application field for these languages comprises those systems in which the objects are of different *sorts*, for example agents, temperatures and time instants. Since we are primarily interested in temporal logics, most of the predicate symbols in the languages we are going to consider have an argument representing time.

Example 8 *Objects of different sorts.*

Suppose the sentence $A_2(a, b)$ expresses the information that Jenny is dancing with John. If we consider this sentence over a period of time, it may have different truth values at different times. The concept of time may be introduced by adopting a ternary predicate symbol admitting a new argument, say, $A_2'(a, b, t)$. We now have two different sorts of variables - one sort representing persons or agents and the other representing time.

Let TP be any finite set, say $\{1, \ldots, k\}$. The alphabet $\mathcal{A}$ of a *k-sorted language,* based on the set TP of sorts, comprises the following symbols:

- a set $X = X_1 \cup \cdots \cup X_k$ of variables, where each X_i is a countably infinite set containing all the variables of sort i
- a set $C = C_1 \cup \cdots \cup C_k$ of constants, where each C_i (which may be empty) contains all the constants of sort i
- a set FUN consisting of zero, one or more elements of the form (f, n), where f is a function symbol having arity n associated with it. Each f also has an $(n+1)$-tuple of sorts $(T_1, \ldots, T_n, T_{n+1})$ associated with it, called the sort of f
- a set $PRED$ consisting of one or more elements of the form (A, n), where A is a predicate symbol having arity n associated with it. Each A also has an n-tuple of sorts $(T_1, \ldots, T_n)$ associated with it, called the sort of A
- the connectives $\neg$ and $\rightarrow$
- the punctuation symbols '(' and ')'
- the quantifier $\forall$.

The set TM^i of *terms of sort i* is defined as follows:

- if $x \in X_i$, then $x \in TM^i$
- if $c \in C_i$, then $c \in TM^i$
- if f is an n-ary function with the sort $(T_1, \ldots, T_n, i)$ associated with it, and $t_1, \ldots, t_n$ are terms of sort $T_1, \ldots, T_n$ respectively, then $f(t_1, \ldots, t_n) \in TM^i$.

The set Φ of atomic formulas over $\mathcal{A}$ is the set of all strings of the form $A(t_1, \ldots, t_n)$ where A is an n-ary predicate symbol with the sort $(T_1, \ldots, T_n)$ associated with it, and $t_1, \ldots, t_n$ are terms of sort $T_1, \ldots, T_n$ respectively.

We say that φ is a *wff* over the k-sorted alphabet $\mathcal{A}$ iff one of the following holds:

- $\varphi \in \Phi$
- $\varphi = \neg\psi$, where ψ is a *wff* over $\mathcal{A}$
- $\varphi = (\psi \rightarrow \chi)$, where ψ and χ are *wffs*
- $\varphi = \forall x(\psi)$, where x is a variable of any sort and ψ is a *wff* over $\mathcal{A}$.

The set of all *wffs* over the alphabet $\mathcal{A}$ constitutes the k-sorted language $\mathcal{L}_k$.

It may again be convenient to form the extended language $\mathcal{L}_k^*$ by introducing the connectives $\wedge, \vee$ and $\leftrightarrow$, as well as the quantifier $\exists$, by the usual abbreviations.

An *interpretation* $I = (D, i)$ of $\mathcal{L}_k^*$ consists of some domain $D = D_1 \cup \cdots \cup D_k$, and a *meaning function* i that maps certain symbols in $\mathcal{L}_k^*$ to entities constructed from the domain. The function i must be defined in such a way that:

$i(a) \in D_j$, for all $a \in C_j$

$i(f, n) : D_{T_1} \times \cdots \times D_{T_n} \rightarrow D_{T_{n+1}}$, for all $(f, n) \in FUN$, where $(T_1, \ldots, T_n, T_{n+1})$ is the sort associated with f

$i(A, n) \subseteq D_{T_1} \times \cdots \times D_{T_n}$, for all $(A, n) \in PRED$, where $(T_1, \ldots, T_n)$ is the sort associated with A.

Given an interpretation $I = (D, i)$, we define a *variable assignment* v for I as a function that associates each variable in $\mathcal{L}_k^*$ with an element in the subdomain of the corresponding sort. This means that for all $x \in X_j$, $v(x) \in D_j$.

Let $I = (D, i)$ be an interpretation of $\mathcal{L}_k^*$ and let v be a variable assignment for I. The *denotation* of an arbitrary term t in $\mathcal{L}_k^*$ is calculated in the same way as for first-order languages. The concepts of a valuation, a model, a frame and global truth of a sentence over an interpretation and over a frame are defined as before. This also applies to the concepts of axioms and semantic consequences.

Example 9 *Truth values of sentences in a many-sorted language.*

Let $TP = \{agent,\ time\}$ be the set of sorts in a 2-sorted language $\mathcal{L}_2$. Let $C = C_{agent} \cup C_{time}$, where $C_{agent} = \{a_0, a_1\}$ of sort *agent* and $C_{time} = \{t_0, t_1, t_2\}$ of sort *time*, and let $PRED = \{(A_1, 3)\}$ of sort (*agent, agent, time*). So $(A_1, 3)$ is a ternary predicate symbol of which the first two arguments are of sort *agent* and the third argument is of sort *time*.

Let $I_0 = (D, i)$ be an interpretation of $\mathcal{L}_2$ such that $D = D_{agent} \cup D_{time}$, where $D_{agent} = \{\text{Jenny, John}\}$ and $D_{time} = \{1130, 1200, 1230\}$. Let $i(A_1) = \{(\text{Jenny, John, } 1200)\}$, $i(a_0) =$ Jenny, $i(a_1) =$ John, $i(t_0) = 1130$, $i(t_1) = 1200$ and $i(t_2) = 1230$.

(i) Does I_0 satisfy $A_1(a_0, a_1, t_0)$? The denotation of the predicate symbol A_1 is given by $i(A_1)$ so A_1 denotes the singleton set $\{(\text{Jenny, John, } 1200)\}$. The denotations of the three constants a_0, a_1 and t_0 are given by $i(a_0), i(a_1)$ and $i(t_0)$ respectively, so a_0 denotes Jenny, a_1 denotes John and t_0 denotes 1130. But $(\text{Jenny, John, } 1130) \notin i(A_1)$, so $I_0 \nVdash A_1(a_0, a_1, t_0)$. This means that I_0 does not satisfy $A_1(a_0, a_1, t_0)$.

(ii) Does I_0 satisfy $A_1(a_0, a_1, t_1)$? Since $i(t_1) = 1200$ and (Jenny, John, 1200) $\in i(A_1)$, it follows that $I_0 \Vdash A_1(a_0, a_1, t_1)$. So I_0 satisfies $A_1(a_0, a_1, t_1)$.

Suppose A_1 was included in the alphabet in order to symbolise the relation of one person dancing with another at a specific point in time. Then we can say that, using the interpretation I_0, the sentence $A_1(a_0, a_1, t_1)$ represents the English sentence 'Jenny is dancing with John at 12h00'.

1.4 Propositional modal languages

The languages for propositional modal logic are similar to languages for propositional logic with the addition of *modal operators.* Typically, such a language with one modal operator is a set of sentences built up in the context of the following syntax:

The alphabet $\mathcal{A}$ consists of the following symbols:

- a set Φ containing one or more atoms $p, q, \ldots$
- the punctuation symbols '(' and ')'
- the connectives $\neg$ and $\rightarrow$
- the modal operator $\Box$ (called *Box).*

The propositional modal language $\mathcal{L}$ over $\mathcal{A}$ is the set of sentences over $\mathcal{A}$, where φ is a sentence over $\mathcal{A}$ iff one of the following holds:

- $\varphi \in \Phi$
- $\varphi = \neg\alpha$, where α is a sentence over $\mathcal{A}$
- $\varphi = (\alpha \rightarrow \psi)$, where α and ψ are sentences over $\mathcal{A}$
- $\varphi = \Box\alpha$, where α is a sentence over $\mathcal{A}$.

The connectives $\vee, \wedge$ and $\leftrightarrow$ can be introduced as before. The modal operator $\Diamond$ (called *Diamond*) is the dual of $\Box$ in the following sense:

$\Diamond\varphi$ abbreviates $\neg\Box\neg\varphi$.

Example 10 *Sentences in a modal language.*

Let $\mathcal{L}_0$ be a propositional modal language and let $\Phi = \{p, q\}$.

The following, if we ignore the omission of inessential parentheses, are examples of sentences in $\mathcal{L}_0$:

$p \wedge q,\ \neg q \rightarrow \Diamond p,\ \Box(p \vee \neg p),\ \Box\neg\Box p \rightarrow \Box q.$

But $p\Box$ and $\Diamond q \rightarrow \Box$ are not sentences in $\mathcal{L}_0$.

Modal languages are typically equipped with a possible world semantics. An interpretation for a modal language is defined in terms of a *structure,* which is a pair $\mathcal{F} = (S, R)$, where S is a set, the members of which are called *possible worlds,* and R is a binary relation on S, $R \subseteq S \times S$. R is known as the *accessibility relation.* If $(s,t) \in R$, then we say that t is *accessible* from s in R.

We are specifically going to look at temporal logics, so in this context, structures are referred to as *temporal structures.* A temporal structure is typically defined as a pair $\mathcal{T} = (T, <)$, where T is a set, the elements of which are called time points, and $<$ is a transitive relation. An alternative to a set T of time points is to use a set of intervals, or even both, in the time structure.

A *model* is defined by equipping a structure with a valuation. More precisely, a model $\mathcal{M}$ based on the structure $\mathcal{F} = (S, R)$, is a triple (S, R, V), where V is a valuation function such that $V : \Phi \rightarrow 2^S$ [2]. Informally, V is a function that associates, with each atomic sentence $\varphi \in \Phi$, the possible worlds $s \in S$ at which φ is true. With the help of V, it is possible to define what it means to assert that $\mathcal{M}$ satisfies a sentence φ at s, i.e. φ is true at s in the model $\mathcal{M}$, abbreviated to $\mathcal{M} \Vdash_s \varphi$. The definition proceeds recursively on the format of φ. For arbitrary sentences $\alpha, \psi \in \mathcal{L}^*$ we can say that $\mathcal{M} \Vdash_s \varphi$ iff one of the following holds:

- $\varphi \in \Phi$ and $s \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_s \psi$
- $\varphi = \alpha \rightarrow \psi$ and $\mathcal{M} \nVdash_s \alpha$ or $\mathcal{M} \Vdash_s \psi$ or both
- $\varphi = \Box\psi$ and $\mathcal{M} \Vdash_{s'} \varphi$ for all $s' \in S$ such that sRs'.

---

[2] This approach to semantics is frequently referred to as Kripke semantics although it was discovered independently several times, first by Tarski, then by Kanger and Prior, and only thereafter almost simultaneously by Kripke and Hintikka [BS84].

Example 11 *Truth in a model.*

Let $\Phi = \{p, q\}$.

Let $\mathcal{M}_1 = (S, R, V)$ where $S = \{s, t, u\}$, $R = \{(s,t), (t,u)\}$, $V(p) = \{s, u\}$, and $V(q) = \{t, u\}$.

By way of illustration, we can determine the truth values of the sentences $(p \rightarrow \neg q)$, $\Diamond p$ and $\Box\Box p$.

(i) Is $(p \rightarrow \neg q)$ true at s in $\mathcal{M}_1$? $\mathcal{M}_1 \Vdash_s (p \rightarrow \neg q)$ iff $\mathcal{M}_1 \nVdash_s p$ or $\mathcal{M}_1 \Vdash_s \neg q$, or both. Consider the first condition: $s \in V(p)$ so $\mathcal{M}_1 \Vdash_s p$. Thus the first condition is false.

Let's look at the second condition. $\mathcal{M}_1 \Vdash_s \neg q$ can be written as $\mathcal{M}_1 \nVdash_s q$ and $s \notin V(q)$ so the second condition is true. This means that $\mathcal{M}_1 \Vdash_s (p \rightarrow \neg q)$ is true, i.e. $(p \rightarrow \neg q)$ is true at s in $\mathcal{M}_1$.

(ii) Is $\Diamond p$ true at t in $\mathcal{M}_1$? $\mathcal{M}_1 \Vdash_t \Diamond p$ abbreviates $\mathcal{M}_1 \Vdash_t \neg\Box\neg p$, and $\mathcal{M}_1 \nVdash_t \Box\neg p$ iff it is not the case that $\mathcal{M}_1 \Vdash_{s'} \neg p$ for all $s' \in S$ such that tRs'. From the definition of R, we know that u is accessible from t, and $u \in V(p)$, so $\mathcal{M}_1 \Vdash_u p$. It follows that $\mathcal{M}_1 \Vdash_t \Diamond p$, i.e. $\Diamond p$ is true at t in $\mathcal{M}_1$.

(iii) Is $\Box\Box p$ true at s in $\mathcal{M}_1$? $\mathcal{M}_1 \Vdash_s \Box\Box p$ iff $\mathcal{M}_1 \Vdash_{s'} \Box p$ for all $s' \in S$ such that sRs'. The world t is the only one accessible from s, and $\mathcal{M}_1 \Vdash_t \Box p$ iff $\mathcal{M}_1 \Vdash_{\tilde{s}} p$ for all $\tilde{s} \in S$ such that $tS\tilde{s}$. Well, u is the only world accessible from t and $u \in V(p)$, so $\mathcal{M}_1 \Vdash_u p$. This means that $\mathcal{M}_1 \Vdash_t \Box p$ and consequently that $\mathcal{M}_1 \Vdash_s \Box\Box p$. So $\Box\Box p$ is true at s in $\mathcal{M}_1$.

A sentence φ is *globally true over a model* $\mathcal{M} = (S, R, V)$, denoted by $\mathcal{M} \Vdash \varphi$, iff φ is true at all worlds in $\mathcal{M}$, i.e. $\mathcal{M} \Vdash_s \varphi$ for all $s \in S$.

If $\mathcal{F} = (S, R)$ is a structure, let the *frame* of $\mathcal{F}$ be $\mathcal{C}_\mathcal{F} = \{(S, R, V) | V \text{ is any valuation}\}$. A sentence φ is *globally true over a frame* $\mathcal{C}_\mathcal{F}$, denoted by $\mathcal{C}_\mathcal{F} \Vdash \varphi$, iff it is globally true over each $\mathcal{M} \in \mathcal{C}_\mathcal{F}$.

If $\mathcal{K}$ is any class of structures, let the frame of $\mathcal{K}$ be $\mathcal{C}_\mathcal{K} = \cup\{\mathcal{C}_\mathcal{F} | \mathcal{F} \in \mathcal{K}\}$. So $\mathcal{M} = (S, R, V) \in \mathcal{C}_\mathcal{K}$ iff there is some structure $\mathcal{F} = (S, R) \in \mathcal{K}$ such that $\mathcal{M} \in \mathcal{C}_\mathcal{F}$. A sentence φ is *globally true over a frame* $\mathcal{C}_\mathcal{K}$, denoted by $\mathcal{C}_\mathcal{K} \Vdash \varphi$, iff it is globally true over each $\mathcal{M} \in \mathcal{C}_\mathcal{K}$.

Given a class $\mathcal{K}$ of structures (typically those in which the accessibility relation has some desired set of properties), we are usually interested in whether $\mathcal{K}$ can be axiomatised, i.e. whether it is possible to find a set Σ of sentences that are globally true over the frame of $\mathcal{K}$ and no other frame.

1.4.1 Multimodal languages

The modal languages discussed in the previous section have only one modal operator, namely $\Box$. The alphabet for a multimodal language is similar to the alphabet for a modal language with only one modal operator but instead of having only one, we have several modal operators. If there are n modal operators, these can be denoted by $[1], [2], \ldots, [n]$, where the operator $[i]$ is similar to the $\Box$ operator. The dual operator $\langle i \rangle$, which is similar to $\Diamond$, abbreviates $\neg[i]\neg$.

The set of sentences is defined in a way similar to a propositional modal language with only one operator.

Example 12 *Sentences and nonsentences in a multimodal language.*

Let $\mathcal{L}_M$ be a multimodal language with $\Phi = \{p, q\}$ and with the modal operators $[1], [2]$ and $[3]$. The following will be sentences in $\mathcal{L}_M$:

$[1](p \to q)$, $[3]p \vee \langle 2 \rangle q$, $\neg[2](p \wedge [1]p)$.

But $[4]p$ and $[1]r$ are not sentences in $\mathcal{L}_M$, because there is no modal operator $[4]$, and $r \notin \Phi$.

Given a language with modal operators $[1], \ldots, [n]$, a *structure* is defined as a tuple $\mathcal{F} = (S, R_1, \ldots, R_n)$ where S is a set of possible worlds and we have a collection of n binary relations $R_i \subseteq S \times S$, one for each $[i]$. A model $\mathcal{M} = (S, R_1, \ldots R_n, V)$ on $\mathcal{F}$ is given by a valuation function $V : \Phi \to 2^S$ as before.

The truth value in $\mathcal{M}$ of atoms and sentences of the form $\neg\varphi$ and $(\varphi \to \psi)$, at a world $s \in S$, can be defined as before. The truth value of a sentence of the form $[i]\varphi$ is defined in terms of the accessibility relation R_i. We say that $[i]\varphi$ is true at s in $\mathcal{M}$, denoted by $\mathcal{M} \Vdash_s [i]\varphi$, iff $\mathcal{M} \Vdash_t \varphi$ for all $t \in S$ such that sR_it.

Example 13 *The truth of sentences in a multimodal language.*

Let $\mathcal{L}_M$ be a multimodal language with $\Phi = \{p, q\}$ and with the modal operators $[1]$ and $[2]$. Let $\mathcal{M}_1 = (S, R_1, R_2, V)$ where

$S = \{t_1, t_2, t_3\}$
R_1 is the relation $< = \{(t_1, t_2), (t_1, t_3), (t_2, t_3)\}$
R_2 is the relation $> = \{(t_2, t_1), (t_3, t_1), (t_3, t_2)\}$

$V(p) = \{t_1\}$
$V(q) = \{t_1, t_2\}$.

The truth value of a sentence of the form $[i]\varphi$ in $\mathcal{M}_1$ at the world $s \in S$, denoted by $\mathcal{M}_1 \Vdash_s [i]\varphi$, is defined as follows:

- $\mathcal{M}_1 \Vdash_s [1]\varphi$ iff $\mathcal{M}_1 \Vdash_{s'} \varphi$ for all $s' \in S$, such that $s < s'$
- $\mathcal{M}_1 \Vdash_s [2]\varphi$ iff $\mathcal{M}_1 \Vdash_{s'} \varphi$ for all $s' \in S$, such that $s > s'$.

Now we can determine the truth value of the sentences $[1]p$ and $\langle 2 \rangle p$:

(i) Is $[1]p$ true at t_2 in $\mathcal{M}_1$? $\mathcal{M}_1 \Vdash_{t_2} [1]p$ iff $\mathcal{M}_1 \Vdash_{t'} p$ for all $t' \in S$ such that $t_2 < t'$. Well, $t_3 \in S$ is the only world such that $t_2 < t_3$, and $t_3 \notin V(p)$. So $\mathcal{M}_1 \nVdash_{t_3} p$. It follows that $\mathcal{M}_1 \nVdash_{t_2} [1]p$, so $[1]p$ is not true at t_2 in $\mathcal{M}_1$.

(ii) Is $\langle 2 \rangle p$ true at t_2 in $\mathcal{M}_1$? $\mathcal{M}_1 \Vdash_{t_2} \langle 2 \rangle p$ iff there exists at least one $t' \in S$ such that $t_2 > t'$ and $\mathcal{M}_1 \Vdash_{t'} p$. (Remember that $\langle 2 \rangle$ abbreviates $\neg[2]\neg$.) We know that $t_2 > t_1$ and that $t_1 \in V(p)$, so $\mathcal{M}_1 \Vdash_{t_1} p$. This means that $\mathcal{M}_1 \Vdash_{t_2} \langle 2 \rangle p$, so $\langle 2 \rangle p$ is true at t_2 in $\mathcal{M}_1$.

As before, a sentence φ is globally true over a model $\mathcal{M} = (S, R_1, \ldots, R_n, V)$ iff $\mathcal{M} \Vdash_s \varphi$ for all $s \in S$, and φ is globally true over a frame $\mathcal{C}_\mathcal{F}$ iff it is globally true over each $\mathcal{M} \in \mathcal{C}_\mathcal{F}$, where $\mathcal{F} = (S, R_1, \ldots, R_n)$ is a structure, and $\mathcal{C}_\mathcal{F}$ is the frame of $\mathcal{F}$. If $\mathcal{K}$ is any class of structures and $\mathcal{C}_\mathcal{K}$ is the frame of $\mathcal{K}$, then a sentence φ is globally true over $\mathcal{C}_\mathcal{K}$ iff it is globally true over each $\mathcal{M} \in \mathcal{C}_\mathcal{K}$.

Given interesting properties that the accessibility relations $R_1, \ldots, R_n$ may possess, one may form the class $\mathcal{K}$ of all structures of which the relations have these properties and enquire whether this class is axiomatisable in the sense that a set Σ of sentences can be found which are globally true over all and only the frame of $\mathcal{K}$. An example that will play an important role in later chapters is the following: consider a language with two modal operators [1] and [2]. Consider the property that links R_1 and R_2 as mutual converses, i.e. $(x, y) \in R_1$ iff $(y, x) \in R_2$. Let $\mathcal{K}$ be the class of all structures (S, R_1, R_2) such that R_1 and R_2 are mutually converse. Can $\mathcal{K}$ be axiomatised? We defer the answer to Chapter 3.

1.5 Further chapters

In this chapter, we have given a brief outline of object languages that may be used in temporal logic. In the next chapter, we briefly look at the choices that have to be made if we need to construct a computational framework for such a logic.

We look at a number of different temporal logics in Chapters 3 to 6. What we are mainly interested in is the axiomatisation of these logics and in this respect two main aspects are considered. Firstly we consider the axiomatisation of the underlying temporal structures. When we are using a first-order language, the axiomatisation of the temporal structure does not usually pose a problem because we have a number of standard axioms for the definition of properties of relationships between time points or intervals. Some of these are given in section 2.3.1. When we are using a modal language, however, this is not the case. The semantics of the modal operators has to be taken into account. In Chapters 3 and 4, our emphasis is on the axiomatisation of the accessibility relations within the temporal structures for the relevant modal languages: for point-based systems in Chapter 3 and for interval-based systems in Chapter 4.

The second aspect we consider is the formulation of axioms for distinguishing concepts like actions, events, processes and so on. We do this for systems using first-order languages. Chapter 5 deals with point structures and Chapter 6 with interval structures.

In Chapter 7 we briefly discuss the fields of application of temporal logics and list a number of fields that looks promising for further research.

Chapter 2

A brief survey of computational frameworks

In order to establish a computational framework for reasoning about time and change, we need to establish an adequate basis for representing the objects to be reasoned about. This means that defining a temporal logic that will suit our requirements requires a number of important decisions to be made. These decisions depend to a large extent on the intended application. In this chapter we list the most important issues that have to be resolved and the various options that are available.

2.1 Dynamic approach vs temporal approach

We distinguish between two ways of handling time: the dynamic or change-based approach, and the temporal or time-based approach. In the *change-based approach*[1], we concentrate on entities, called *change-indicators,* that signify a change having taken place [SG88]. The system is described in terms of different states or events. Time is determined by events, and properties of time must be defined by investigating the properties of events [Vil94]. An example of this approach is the Situation Calculus, discussed in Chapter 5.

Although, in Lin's opinion [Lin91], the change-based approach is more interesting because it is based on the human perception of time, it has a number of limitations of which the following are notable:

- Actions do not have any duration.

[1]Galton [Gal95] refers to this as a *time-dependent approach.* With this approach, we use a *relational theory of time* [Vil94]

- The result of an action is immediate. We cannot distinguish actions that have delayed effects and assertions that are true for a time period only and then become false because of natural death or termination.
- Simultaneous actions, such as concurrent or overlapping actions, cannot be expressed.
- Continuous processes are not expressible.

In the *time-based approach*[2], time is given an autonomous status. So we have only one fundamental kind of change and that is the passage of time. This is a constant change unaffected by anything else. With this approach, a temporal structure is defined as primitive in the language we are using, and assertions are either true or false at points (or intervals) in this structure [SG88]. A set of structures, representing the various kinds of change that can occur in time, is then superimposed on the temporal structure. This approach dates back to Prior's tense logic which will be discussed in Chapter 3.

2.2 The object language

One of the most important decisions that has to be made is the choice of object language. We consider the two options non-modal first-order languages and modal languages.

In an *unsorted first-order language,* time is simply introduced as one or more (or even none) additional arguments to a predicate symbol. In such systems time is not accorded a special status so we cannot say as much about the temporal aspects of an assertion as we may wish to [SG88]. Therefore, we will not discuss the use of these languages further. *Many-sorted first-order languages* prevail in AI applications. In such a system, we separate the temporal and nontemporal components of an assertion [SG88]. Examples of systems using many-sorted first-order languages are discussed in Chapters 5 and 6.

The second option is the use of *modal languages.* Examples of systems using modal languages are discussed in Chapters 3 and 4.

The highly-developed theorem-proving techniques of first-order languages are frequently mentioned in their favour. However, relatively simple sentences in English may have undesirably complex representations in a first-order language. The following example used by Gabbay [GHR94] illustrates this:

[2]Galton [Gal95] uses the term *time-independent,* while Lin [Lin91] talks about using an *absolute theory of time*

Example 14 *A complex first-order sentence.*

Consider the following simple English sentence:

Since M became Prime Minister, the pound has steadily been going down, and in fact it will continue to go down for as long as she remains Prime Minister.

Let $VP(t,x)$ express the idea that the value of the pound is x at time t, and $PM(t,M)$ the idea that M is the Prime Minister at time t. Let *now* represent a time point that is regarded as the present moment (or the moment the sentence is spoken). The following first-order language sentence is a representation of the English sentence:

$$
\begin{aligned}
\exists t(t < now \rightarrow [&PM(t,M) \\
&\wedge \exists s(s < t \wedge \forall u(s < u < t \rightarrow \neg PM(u,M))) \\
&\wedge \forall u(t < u < now \rightarrow PM(u,M)) \\
&\wedge \forall u,v,x,y((t < u < v \leq now \wedge VP(u,x) \wedge VP(v,y)) \rightarrow x > y)]) \\
\wedge \forall s(s > now \rightarrow [&\forall u(now < u < s \rightarrow PM(u,M)) \\
&\rightarrow \forall u,v,x,y((now \leq u < v < s \wedge VP(u,x) \wedge VP(v,y)) \rightarrow x > y)])
\end{aligned}
$$

Suppose we have a modal language containing the connectives S and U, such that $S(A,B)$ means that B has been continuously true since A was true, and $U(A,B)$ means that B will continuously be true until A is true. The English sentence can be written in such a language as

$$
\begin{aligned}
&S(\neg PM(m), PM(m) \wedge PoundGoingDown) \wedge PM(m) \\
&\quad \wedge U(\neg PM(m), PM(m) \wedge PoundGoingDown)
\end{aligned}
$$

where *PoundGoingDown* means that 'if the value of the pound is x now, and if its value was y the day before, then $y > x$'.

A modal language, on the other hand, sometimes may lack expressiveness. Van Benthem [VB91] identifies a number of properties of the order relation of a time structure, such as irreflexivity and antisymmetry, that cannot be expressed in the modal language defined by him. This is discussed in Chapter 4.

We briefly look at typical applications of each of these languages in Chapter 7.

2.3 The temporal structure

Important decisions also have to be made about temporal representation. The first of these concerns the primitive elements in the temporal structure. The two contenders are points and intervals. Secondly we need to define a relation (or more often more than one relation) between primitive elements. When points are chosen as the primitive elements, we typically use the relation '$<$' (*earlier-than*) and its counterpart '$>$' (*later-than*). When intervals are chosen as the primitive elements, a wider variety of relations may be used, ranging from analogs of '$<$' to relations that take into account the possibility of intervals to overlap. Systems in which points constitute the primitive elements are treated in Chapters 3 and 5; those taking intervals as primitive are discussed in Chapters 4 and 6.

Historically, the first temporal logics were point-based. If we choose *points* as primitive, we have to make provision for deriving a representation of intervals in the system. We distinguish between the following two ways of representing intervals:

- Consider an interval to be a *pair of time points* [Sho88]. Suppose $t_1 \leq t_2$, then the interval with starting point t_1 and endpoint t_2 may be represented by the ordered pair (t_1, t_2). Nothing is said about the points between t_1 and t_2.

- Consider an interval to be a *convex set of points*. An open interval is thus defined as $(t_1, t_2) = \{t \in T : t_1 < t < t_2\}$ for a temporal structure $F = (T, <)$. Should these intervals be closed, open or half-open? This depends on whether we want to consider the starting point and the end point as part of the interval. This may lead to the so-called *dividing instant problem* [VB91]:

  - Suppose we choose open intervals. Let the assertion p be true over the interval (t_1, t_2) and false over the interval (t_2, t_3). What is the truth value of p at t_2? It is neither obvious that it should be true nor obvious that it should be false.

  - Suppose we choose closed intervals. Let, again, p be true over $[t_1, t_2]$ and false over $[t_2, t_3]$. What is the truth value of p at t_2? It would seem to inherit truth from the first interval and falsity from the second.

  - This problem is frequently solved by using half-open intervals. So, if p is true over $(t_1, t_2]$ and false over $(t_2, t_3]$, p will be true at t_2. This offers a solution which is, perhaps, somewhat superficial.

More recent temporal logics have tended to take intervals as basic [Nil98]. In [All84], Allen reasons that intervals should be defined as primitive in the language since 'it appears that we can always decompose times into subparts. Thus the formal notion of a time point which would not be decomposable is not useful.'

Allen's time structure is frequently used for approaches that take intervals as primitive. In this structure, there is a single time line and any two intervals on the time line are related by one of 13 mutually exclusive relations. Each of these interval-interval relations may be represented by a symbol in the relevant language. We will discuss two of these systems in detail, namely the system proposed by Shoham in Chapter 4, and Allen's system in Chapter 6. Figure 2.1 shows twelve of the thirteen different relations (excluding *equals*). As is apparent, there are six relations having six inverses. The dotted line indicates the interval i and the solid line the interval j.

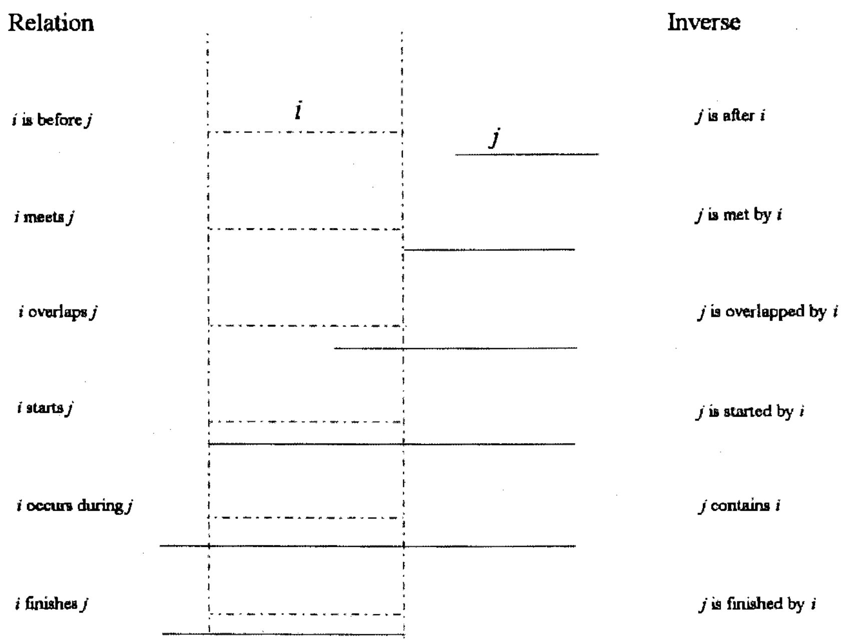

Figure 2.1: Allen's interval-interval relations

If *intervals* are chosen as primitive, we should provide a means of representing points, i.e. *durationless* intervals, in the system. To make provision for the representation of time points Allen, for example, allows for the existence of *point intervals* (or what Allen calls *zero-width intervals*) at both ends of an interval. This effectively provides each interval with a beginning point and an end point.

2.3.1 Properties of relations

Consider a point-based temporal structure $(T, <)$. Usually, a system uses either a *linear* or a *branching* time structure [Gal87]. Linear time is suitable for deterministic systems, whereas branching time is suitable for systems where more than one future, or more than one past, is possible, i.e. the future or past is indeterministic.

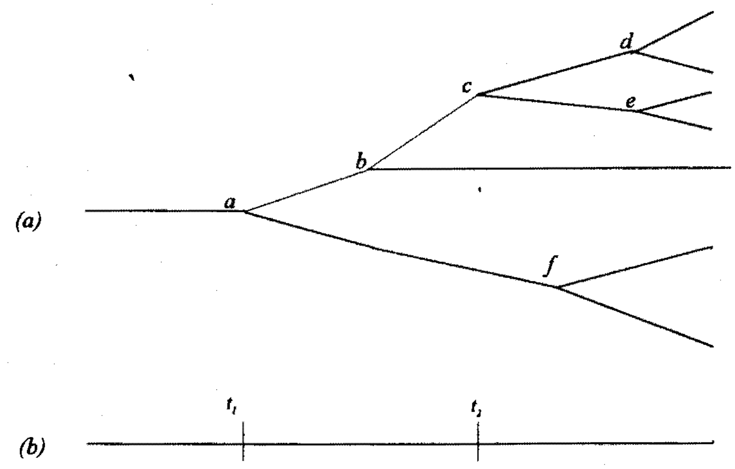

Figure 2.2: (a) Branching time, (b) Linear time

A linear time structure would be suitable when we want to consider a system where there is only one actual future and one actual past. Fig 2.2(b) is a representation of linear time. If we choose any two time points t_1 and t_2, the two points must be comparable. For the relation $<$, this means that exactly one of three possible relationships must hold between them:

$t_1 < t_2$
$t_2 < t_1$
$t_1 = t_2$.

So in Fig 2.2(b), $t_1 < t_2$.

A *forward branching* (or left-linear) time structure would typically be used in a planning system to determine the outcome of different possible futures, whereas a *backward branching* (right-linear) time structure may be used to represent different possible pasts (e.g. in medical diagnosis). Fig 2.2(a) represents the concept of branching time, or the property of left-linearity (forward branching time) in this particular case. We have one past and different possible futures. If one looks at the figure, it is clear that $a < b$ and $a < f$, but b and f are not comparable.

Two other possibilities are *parallel* time and *circular* time structures. A parallel time structure could be regarded as a model of the different subjective time-scales of different people. It might be difficult to imagine the idea of circular time, but the associated logic could be useful in reasoning about repetitive processes, such as the 'endless repetition of cycles' of a traffic light [Gal87]. However, one suspects that a temporal approach is less suitable for such an application than a dynamic (change-based) approach would be.

In a temporal structure $(T, <)$ the relation $<$ may, depending on its properties, allow different notions of nearness and hence different *topologies*. Whether topological aspects are of importance will depend on the application.

Which other properties might the order relation in the chosen time structure have? The following illustrate a few of the available options:

- Is time *discrete* or *dense*? If *dense*, is it *continuous* (like the real numbers), or not (like the rationals) [Gal95]? If time is discrete, we can talk about the next moment and the previous moment, two concepts that are widely used in computer science.
- Is time *bounded* or *unbounded* (*serial*, *infinite*, *successive*) in none, one or both directions [SG88]?
- Is the time structure *convex* (i.e. there are no gaps) or not?

A number of useful properties, together with their definitions, are listed below ([Gol92], [Ven95]). We frequently refer to these properties in subsequent chapters. Other less important properties are given as the need arises. The properties contained in the first group are formulations of relationships between time points, or between intervals that do not overlap. (In the context of intervals, the symbol $<$ is used to express the idea that one interval precedes another, so $x < y$ means that interval x is 'earlier' than interval y, and they do not overlap: interval x ends before y starts.)

Transitivity	$\forall x, y, z((xRy \wedge yRz) \rightarrow xRz)$
Reflexivity	$\forall x(xRx)$
Irreflexivity	$\forall x \neg(xRx)$
Symmetry	$\forall x, y(xRy \rightarrow yRx)$
Antisymmetry	$\forall x, y((xRy \wedge yRx) \rightarrow x = y)$
Asymmetry	$\forall x, y(xRy \rightarrow \neg(yRx))$
Euclideanness	$\forall x, y, z((xRy \wedge xRz) \rightarrow yRz)$
Partial functionality	$\forall x, y, z((xRy \wedge xRz) \rightarrow y = z)$
Confluency	$\forall y, z(\exists x(xRy \wedge xRz) \rightarrow \exists u(yRu \wedge zRu))$
Right-seriality (Successor)	$\forall x \exists y(xRy)$
Left-seriality (Predecessor)	$\forall x \exists y(yRx)$

Seriality (Infinity, unboundedness)	$\forall x(\exists y(xRy) \wedge \exists z(zRx))$
Linearity	$\forall x, y(x < y \vee x = y \vee y < x)$
Right-linearity (Backward branching)	$\forall x, y, z((x < y \wedge x < z) \rightarrow (y < z \vee y = z \vee z < y))$
Left-linearity (Forward branching)	$\forall x, y, z((y < x \wedge z < x) \rightarrow (y < z \vee y = z \vee z < y))$
Density	$\forall x, y(x < y \rightarrow \exists z(x < z \wedge z < y))$
Discreteness	$\forall x, y(x < y \rightarrow \exists z(x < z \wedge z \leq y \wedge \neg\exists u(x < u \wedge u < z)))$

A relation is said to be *functional* if it is both partially functional and right-serial. It is a *partial ordering* if it is reflexive, transitive and antisymmetric and it is an *equivalence relation* if it is reflexive, transitive and symmetric. A *strict partial ordering* is transitive and irreflexive. A linear partial ordering is a *total ordering.*

The second group of properties contains formulations of relationships between intervals that may overlap ([VB91], [Ven95]). We use the symbol $\subseteq$ to indicate interval inclusion, i.e. $x \subseteq y$ means that interval x is a subinterval of interval y. Properties of relations on overlapping intervals may include the following:

Monotonicity	$\forall x, y(x < y \rightarrow \forall z(z \subseteq x \rightarrow z < y))$ $\forall x, y(x < y \rightarrow \forall z(z \subseteq y \rightarrow x < z))$
Conjunctivity	$\forall x, y(\exists u(u \subseteq x \wedge u \subseteq y) \rightarrow \exists z(z \subseteq x \wedge z \subseteq y \wedge$ $(\forall v(v \subseteq x \wedge v \subseteq y) \rightarrow v \subseteq z)))$
Disjunctivity	$\forall x, y(\exists u(x \subseteq u \wedge y \subseteq u) \rightarrow \exists z(x \subseteq z \wedge y \subseteq z \wedge$ $(\forall v(x \subseteq v \wedge y \subseteq v) \rightarrow z \subseteq v)))$
Linearity	$\forall x, y(x < y \vee y < x \vee \exists u(u \subseteq x \wedge u \subseteq y))$
Freedom	$\forall x, y(\forall z(z \subseteq x \rightarrow \exists u(u \subseteq z \wedge u \subseteq y)) \rightarrow x \subseteq y)$
Convexity	$\forall x, y, z((x < y \wedge y < z) \rightarrow \forall u((x \subseteq u \wedge z \subseteq u) \rightarrow y \subseteq u))$
Left-linearity	$\forall x, y, z((x < z \wedge y < z) \rightarrow (x < y \vee y < x \vee$ $\exists u(u \subseteq x \wedge u \subseteq y)))$
Directedness	$\forall x, y \exists z(x \subseteq z \wedge y \subseteq z)$
Duration monotonicity	$\forall x, y, z((x < z \wedge y < z) \rightarrow \exists v(v < z \wedge x \subseteq v \wedge y \subseteq v \wedge$ $\forall u((x \subseteq u \wedge y \subseteq u) \rightarrow v \subseteq u)))$ $\forall x, y, z((z < x \wedge z < y) \rightarrow \exists v(z < v \wedge x \subseteq v \wedge y \subseteq v \wedge$ $\forall u((x \subseteq u \wedge y \subseteq u) \rightarrow v \subseteq u)))$

Conjunctivity expresses the idea that, if two intervals x and y overlap, then there exists a largest interval z representing the overlap. *Disjunctivity* expresses the idea that, if there exists an interval z of which two intervals x and y are subintervals, then there exists a smallest interval containing both x and y. *Duration monotonicity* makes sense only when disjunctivity and conjunctivity are satisfied and it implies both these properties [Haj96]

2.4 Facts, actions and events

The language we adopt should enable us to describe what holds or occurs in time. In particular, we need to talk about facts, actions and events. There are several different classifications of such notions some of which are discussed further in Chapters 5 and 6. An example of such a classification is given below [VB91]:

- The class of *states* (also called *facts* or *properties)*: These are static (permanent, invariant) in nature and represent a reality which is true or false at a particular instant or over a time interval. These primitives also have the property that they are *hereditary*, i.e. if a state is true over an interval, this truth value is inherited by all its subintervals. For example, if the assertion 'Julie lives in Durban' is true over an interval, it is true over all the subintervals. The essence of a state is that it cannot change its truth value by itself: a state change can only be caused by some 'event' or 'action'.
- The class of *activities:* These are also hereditary but dynamic in nature (some activity is involved). An example is the sentence 'Ben is swimming', which holds over all the subintervals. Activities are usually done for their own sake.
- The class of *events* (*achievements* or *actions* or *performances*)*:* These are also dynamic in nature but not hereditary. They occur and (most of the time) cause changes in states. An event takes place over an interval of time, i.e. it does not occur instantaneously. Consider the assertion 'the robot executed the *Navigate* procedure from start to finish' as an example. If it takes the robot half an hour to execute the *Navigate* procedure, the assertion does not hold for any proper subinterval of that half hour interval; it only holds over exactly the interval constituting one half hour. The duration of an event is generally short in comparison with the duration of a fact. Changes caused by events are sudden and discrete. They may represent actions consciously performed by a living agent. An achievement (or accomplishment) may be the result of an activity or process.

Now that we have an idea of some of the important issues that need to be resolved in formulating a suitable temporal logic for a particular application, we show how this is done in a number of temporal logics; firstly we consider systems using a modal language and then systems using a many-sorted first-order language.

Chapter 3

Modal approaches with instants

3.1 Priorean tense logic

Prior is regarded by many as the founding father of modern temporal logic [OH95]. He formulated a tense logic for natural language understanding based on the principle of having a current time instant, the present, from which one can consider the past and the future ([Pri57], [Pri67] and [Pri68]). Prior uses the term 'tense logic', reflecting the use in natural language of verb modifications to indicate temporal reference. Today the term 'temporal logic' is generally used when referring to the logic of time, reflecting the recognition that many other syntactic devices may be used to encode temporal information, such as temporal connectives and temporal adverbials. In this chapter, however, we talk about instant tense logic, a term used by Van Benthem who extended Prior's logic to an interval tense logic, discussed in the next chapter. We are primarily interested in the representation of time, so our emphasis is on the set of axioms used to characterise the properties of the time structures. We also look at a number of variations and extensions of this logic.

3.2 Past, present and future

The alphabet for instant tense logic consists of the following elements:

- a set Φ of one or more atoms $p, q, \ldots$
- the connectives $\neg$ and $\rightarrow$
- the punctuation symbols '(' and ')'
- the modal operators $[F]$ and $[P]$, where $[F]$ is read 'always in the future' and $[P]$ is read 'always in the past'.

The duals $\langle F\rangle$, read 'at least once in the future', and $\langle P\rangle$, read 'at least once in the past' of the modal operators $[F]$ and $[P]$, are defined in the usual way:

$$\langle F\rangle\varphi \quad \equiv_{def} \quad \neg[F]\neg\varphi$$
$$\langle P\rangle\varphi \quad \equiv_{def} \quad \neg[P]\neg\varphi$$

A graphical interpretation of the four modal operators is given in Figure 3.1.

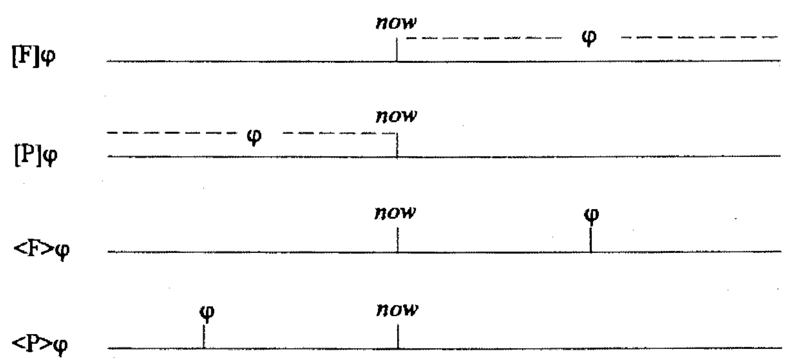

Figure 3.1: A graphical interpretation of Prior's modal operators.

The traditional literature on temporal logic uses G instead if $[F]$, H instead of $[P]$, F instead of $\langle F\rangle$ and P instead of $\langle P\rangle$. Our notation is based on that of Goldblatt [Gol92].

The set of sentences $\mathcal{G}$ of this language is, as one would expect, the smallest set such that

- $\Phi \subseteq \mathcal{G}$, and
- if $\varphi, \psi \in \mathcal{G}$, then $\neg\varphi, \varphi \rightarrow \psi, [F]\varphi, [P]\varphi \in \mathcal{G}$.

The connectives $\wedge$, $\vee$ and $\leftrightarrow$ are introduced by the usual abbreviations.

The language of instant tense logic, being multimodal, should be based on structures of the form (T, R_F, R_P). T is a set of time points or instants. The accessibility relations R_F and R_P on T are intended to capture the notions of *later* and *earlier:* if $(x, y) \in R_F$ we say 'y is in the future of, or is later than x' and if $(x, y) \in R_P$ we say 'y is in the past of, or is earlier than x'. Clearly the intention is that $[F]$ and $[P]$ should express properties of the same time-ordering, and thus that R_F and R_P should be converses, i.e. $(x, y) \in R_F$ iff $(y, x) \in R_P$. We thus restrict our attention to structures (T, R_F, R_P) in which R_F and R_P are interdefinable, which permits a simplification: we may take our structures to be pairs (T, R), where R is understood to be the accessibility relation R_F of $[F]$. Moreover, it is usual to require that R be some sort of ordering on T, or at least be transitive, and so we will henceforth use the symbol '$<$' for R (and '$>$' for its converse).

The reader will recall from Chapter 1 that a valuation $V : \Phi \rightarrow 2^T$ associates with each $p \in \Phi$ a set of instants $t \in T$, and these are taken to be the instants (or possible worlds) at which p is true. Let $\mathcal{M} = (T, <, V)$ be a model based on the time structure $\mathcal{F} = (T, <)$. A sentence φ is true relative to $\mathcal{M}$ at time point t, denoted by $\mathcal{M} \Vdash_t \varphi$, iff one of the following holds:

- $\varphi \in \Phi$ and $t \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_t \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_t \psi$ or $\mathcal{M} \Vdash_t \chi$
- $\varphi = [F]\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for all $t' \in T$ such that $t < t'$
- $\varphi = [P]\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for all $t' \in T$ such that $t > t'$.

Remark: $[F]$ represents the strict future; if $\mathcal{M} \Vdash_t [F]\psi$ then it need not be the case that $\mathcal{M} \Vdash_t \psi$. Similarly, $[P]$ represents the strict past.

A sentence φ is globally true over a model $\mathcal{M}$ iff $\mathcal{M} \Vdash_t \varphi$ for all $t \in T$. A sentence φ is globally true over the frame $\mathcal{C}_\mathcal{F}$ based on the structure $\mathcal{F} = (T, <)$ iff it is globally true over each $\mathcal{M} \in \mathcal{C}_\mathcal{F}$, where $\mathcal{C}_\mathcal{F} = \{(T, <, V) | V \text{ is arbitrary}\}$. A sentence φ is globally true over the frame $\mathcal{C}_\mathcal{K}$ of the class $\mathcal{K}$ of structures, iff it is globally true over each $\mathcal{M} \in \mathcal{C}_\mathcal{K}$, where $\mathcal{C}_\mathcal{K} = \cup \{\mathcal{C}_\mathcal{F} | \mathcal{F} \in \mathcal{K}\}$.

The following examples illustrate the range of temporal forms that a language of instant tense logic is capable of expressing [VB95]:

$\langle P \rangle [P]\varphi$	There was a time at which φ had always been the case
$\langle F \rangle \varphi \wedge \langle F \rangle \psi$	φ will be the case and so will ψ
$\langle F \rangle (\varphi \wedge \psi)$	φ and ψ will be the case simultaneously
$[F](\varphi \rightarrow \psi)$	φ will always 'guarantee' ψ
$[F](\varphi \rightarrow \langle F \rangle \psi)$	φ will always 'enable' ψ to become true afterwards.

3.2.1 Axiomatisation

Let $\mathcal{C}$ be a class of models. An important question is whether $\mathcal{C}$ is axiomatisable, i.e. whether there exists a set Σ of sentences such that the members of $\mathcal{C}$ are precisely the models globally satisfying the sentences of Σ. In Chapter 1 we saw that there are non-axiomatisable frames. In the case of modal (and thus temporal) logic, the rich semantic structure permits the notion of axiomatisation to be construed in more than one way. It is convenient to restrict attention to classes which are frames so we only consider the axiomatisation of frames in this dissertation. Fortunately, the classes in which one might be interested often may be seen as frames based on some class of structures. By way of illustration, recall that the models of interest in temporal logic are generally those whose accessibility relations are, at the very least, mutually converse. Take $\mathcal{C}$ to be class of all models such that $(x, y) \in R_F$ iff $(y, x) \in R_P$, where R_F is the accessibility relation corresponding to $[F]$ and R_P the accessibility relation corresponding to $[P]$. (We temporarily suspend the more intuitive notation that uses '<', as the discussion applies to multimodal logics in general.)

Proposition 1 *The class of models $\mathcal{C}$ is the frame $\mathcal{C}_{\mathcal{K}}$ of the class $\mathcal{K}$ of structures of the form $\mathcal{F} = (T, R_F, R_P)$, where R_F and R_P are converses.*

Pick any $\mathcal{M}_0 = (T, R_F, R_P, V) \in \mathcal{C}$. Then $(x, y) \in R_F$ iff $(y, x) \in R_P$, which means that R_F and R_P are converses. Let $\mathcal{F} = (T, R_F, R_P)$. Then $\mathcal{F} \in \mathcal{K}$ and therefore $\mathcal{M}_0 \in \mathcal{C}_{\mathcal{K}}$.

Conversely, pick an arbitrary $\mathcal{M}_0 \in \mathcal{C}_{\mathcal{K}}$. Then $\mathcal{M}_0 \in \mathcal{C}_{\mathcal{F}}$ for some structure $\mathcal{F} = (T, R_F, R_P)$ where R_F and R_P are converses, so $(x, y) \in R_F$ iff $(y, x) \in R_P$ and thus $\mathcal{M}_0 \in \mathcal{C}$.

$\mathcal{C}_{\mathcal{K}}$ is axiomatisable in a sense which takes into account the view of $\mathcal{C}_{\mathcal{K}}$ as the union of frames based on structures. More particularly, $\mathcal{C}_{\mathcal{K}}$ may be axiomatised by the following two schemas in the sense made explicit by Proposition 2.

TL.01 $\varphi \rightarrow [F] \langle P \rangle \varphi$.

(If φ is true now, then it is always going to be the case (for all points in the future) that φ was true at some point in the past.)

TL.02 $\varphi \rightarrow [P] \langle F \rangle \varphi$.

(If φ is true now, then it has always been the case (for all points in the past) that φ will be true at some point in the future.)

Proposition 2 *Let $\mathcal{K}$ be the class of structures $\mathcal{F} = (T, R_F, R_P)$.*

1. If, in each structure $\mathcal{F} \in \mathcal{K}$, R_F and R_P are mutually converse, then every model in $\mathcal{C}_{\mathcal{K}}$ globally satisfies every instance σ of each of axiom schemas TL.01 *and* TL.02.

2. If the frame $\mathcal{C}_{\mathcal{K}}$ of $\mathcal{K}$ globally satisfies axiom schemas TL.01 *and* TL.02, *then, for every $\mathcal{F} \in \mathcal{K}$, R_F and R_P are converses.*

1. Let $\mathcal{M}_0 = (T, R_F, R_P, V)$ be an arbitrary model in $\mathcal{C}_\mathcal{K}$. Assume that, in each structure $\mathcal{F} \in \mathcal{K}$, R_F and R_P are mutual converses. We have to show that $\mathcal{M}_0$ globally satisfies every instance σ of each of the two axiom schemas *TL.01* and *TL.02*.

Let $t_0 \in T$ be an arbitrary time point and suppose $\mathcal{M}_0 \Vdash_{t_0} \varphi$. We show that $\mathcal{M}_0 \Vdash_{t_0} [F]\langle P\rangle \varphi$.

Suppose $\mathcal{M}_0 \nVdash_{t_0} [F]\langle P\rangle \varphi$. This means that it is not the case that $\mathcal{M}_0 \Vdash_t \langle P\rangle \varphi$ for all $t \in T$ such that $t_0 R_F t$. Let t' be an arbitrary time point such that $t_0 R_F t'$ and $\mathcal{M}_0 \nVdash_{t'} \langle P\rangle \varphi$. Then there does not exist a $t'' \in T$ such that $t' R_P t''$ and $\mathcal{M}_0 \Vdash_{t''} \varphi$. But recall that $t_0 R_F t'$ iff $t' R_P t_0$, and we know that $\mathcal{M}_0 \Vdash_{t_0} \varphi$. So there does exist a $t'' \in T$, i.e. $t'' = t_0$, such that $t' R_P t''$ and $\mathcal{M}_0 \Vdash_{t''} \varphi$, thus $\mathcal{M}_0 \Vdash_{t'} \langle P\rangle \varphi$. This is a contradiction so $\mathcal{M}_0 \Vdash_t \langle P\rangle \varphi$ for all $t \in T$ such that $t_0 R_F t$. Thus $\mathcal{M}_0 \Vdash_{t_0} [F]\langle P\rangle \varphi$. It follows that $\mathcal{M}_0$ globally satisfies every instance σ of the schema *TL.01*, and, since $\mathcal{M}_0$ was chosen arbitrarily, that $\mathcal{C}_\mathcal{K}$ globally satisfies every instance of the schema *TL.01*.

It can be shown in a similar way that $\mathcal{C}_\mathcal{K}$ globally satisfies every instance σ of the schema *TL.02*.

2*a*. Assume that the frame $\mathcal{C}_\mathcal{K}$ of $\mathcal{K}$ globally satisfies the axiom schema *TL.01*. We have to show that, for every $\mathcal{F} \in \mathcal{K}$, it is the case that for all $t, t' \in T$, if $(t, t') \in R_F$ then $(t', t) \in R_P$.

Pick an arbitrary $\mathcal{F}_0 = (T, R_F, R_P) \in \mathcal{K}$ and suppose that there is some pair $(t_0, t_1) \in R_F$ such that $(t_1, t_0) \notin R_P$. Let $\mathcal{M}_0 \in \mathcal{C}_{\mathcal{F}_0}$ be the model such that $V(p) = \{t_0\}$ for every atom p. Thus $\mathcal{M}_0 \Vdash_{t_0} p$ for arbitrary p. From *TL.01*, this means that $\mathcal{M}_0 \Vdash_{t_0} [F]\langle P\rangle p$, i.e. $\mathcal{M}_0 \Vdash_t \langle P\rangle p$ for all $t \in T$ such that $t_0 R_F t$. Thus, for every $t \in T$ such that $t_0 R_F t$, there exists a $t' \in T$ such that $t R_P t'$ and $\mathcal{M}_0 \Vdash_{t'} p$, and thus that $t' \in V(p)$. Then it must be the case that $t' = t_0$ and consequently, that, for every t such that $t_0 R_F t$, we have $t R_P t_0$. In particular, since $t_0 R_F t_1$, we have $t_1 R_P t_0$, contradicting our earlier assumption. It follows that for all $(t, t') \in T$ if $(t, t') \in R_F$ then $(t', t) \in R_P$.

2*b*. It can be shown in a similar fashion that, if the frame $\mathcal{C}_\mathcal{K}$ of $\mathcal{K}$ globally satisfies the axiom schema *TL.02*, then for all $t, t' \in T$, if $(t, t') \in R_P$ then $(t', t) \in R_F$. From 2*a* and 2*b* it follows that R_F and R_P are mutually converse.

Remark: In this particular case direct proofs can also be given.

For the purpose of tense logic, all the frames of interest to us will be subclasses of the frame $\mathcal{C}_\mathcal{K}$ axiomatised by *TL.01* and *TL.02*. Consequently we shall revert to using temporal structures of the form $(T, <)$, where R_F is a synonym for $<$ and R_P for the converse $>$.

If we want to impose particular properties on the accessibility relation(s) of some frame of interest, the set Σ of sentences axiomatising the frame should include axiom schemas that are formulations of these properties. As we have seen in Proposition 2, the axiom schemas must be formulated in such a way that, if an accessibility relation R has a certain property, then all instances of the corresponding axiom schema will be globally true in the union of all frames $C_{\mathcal{F}}$, such that R is an accessibility relation of $\mathcal{F}$. We say that a time structure, and hence that a frame, has a property in the sense that the relevant accessibility relation has this property. For example, if the accessibility relation is linear, we can talk about linear structures and linear frames.

We can also say that a schema α *characterises* a class $\mathcal{K}$ of structures if, for every $\mathcal{F} \in \mathcal{K}$, and for no other $\mathcal{F}$, it holds that α is globally satisfied in $C_{\mathcal{F}}$ [1]. We say that a property is *definable* in a relevant language, if it is possible to formulate a schema in that language that characterises the class of structures having that property. Let us look at the way the underlying temporal structures of Priorean tense logic can be axiomatised.

Informally, axioms *TL.01* and *TL.02,* together with axioms *TL.03* to *TL.06* defined below, are used to express the idea that 'if one of the determinations past, present and future can ever be applied to [an event] N, then one of them has always been and always will be applicable, though of course not always the same one' [McT08]. The underlying principle is that, once a 'tense' is associated with a sentence at time t, then it is possible to associate some 'tense' with that particular sentence for all points t' in the time structure, for which a relationship exists between t and t' [VB91].

TL.03 $\langle P\rangle \varphi \rightarrow [P](\langle P\rangle \varphi \vee \varphi \vee \langle F\rangle \varphi)$ (Left-linearity of $>$ [2])

TL.04 $\langle P\rangle \varphi \rightarrow [F]\,\langle P\rangle \varphi$ (Transitivity of $>$)

TL.05 $\langle F\rangle \varphi \rightarrow [P]\,\langle F\rangle \varphi$ (Transitivity of $<$)

TL.06 $\langle F\rangle \varphi \rightarrow [F](\langle P\rangle \varphi \vee \varphi \vee \langle F\rangle \varphi)$ (Right-linearity of $<$).

Note that we have two distinct axioms for linearity, namely left-linearity and right-linearity. The reason for this is that linearity, as stated by the first-order sentence $\forall t, t'(t < t' \vee t = t' \vee t' < t)$, cannot be formulated in tense logic (see [VB75], [Gol75] and [VB95] for further discussion). By using the combination of left- and right-linearity, however, a form of linearity, sometimes referred to as *quasi-linearity* [Haj96], can be described. Quasi-linearity differs from linearity by admitting parallel time lines, but simulates linearity within each

[1] We regard the term characterisation as a synonym for axiomatisation as is sometimes done in the literature [Ven90].

[2] taking into account that $<$ and $>$ are converses

time line. So, if we choose any $t, t' \in T$ and $<$ is both left- and right-linear, we can claim that $t < t'$ or $t' < t$ or $t = t'$, as long as t and t' do not belong to two time lines running in parallel [Haj96].

Let us see how *TL.06* enforces right-linearity of the time structure:

Proposition 3 *1. Let $\mathcal{K}$ be the class of structures $\mathcal{F} = (T, <)$ such that $<$ is right-linear for every $\mathcal{F} \in \mathcal{K}$. Then the frame $\mathcal{C}_{\mathcal{K}}$ of $\mathcal{K}$ globally satisfies all instances of the axiom schema $\langle F \rangle \varphi \rightarrow [F](\langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi)$.*
2. If the frame $\mathcal{C}_{\mathcal{K}}$ of a class $\mathcal{K}$ of structures globally satisfies the axiom schema TL.06, *then, for every $\mathcal{F} = (T, <) \in \mathcal{K}$, $<$ is right-linear.*

1. Let $\mathcal{C}_{\mathcal{K}}$ be the class of models of the form $\mathcal{M} = (T, <, V)$ in which $<$ is right-linear, as expressed by the first-order sentence $\forall t, t', t''((t < t'' \wedge t < t') \rightarrow (t' < t'' \vee t' = t'' \vee t'' < t'))$. We want to show that all instances of the axiom schema $\langle F \rangle \varphi \rightarrow [F](\langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi)$ are globally true in $\mathcal{C}_{\mathcal{K}}$.

Let $\mathcal{M}_0 = (T, <, V) \in \mathcal{C}_{\mathcal{K}}$ be an arbitrary model. Pick any $t_0 \in T$ and suppose $\mathcal{M}_0 \Vdash_{t_0} \langle F \rangle \varphi$, so there exists a $t \in T$ such that $t_0 < t$ and $\mathcal{M}_0 \Vdash_t \varphi$. We have to show that $\mathcal{M}_0 \Vdash_{t_0} [F](\langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi)$.

Suppose $\mathcal{M}_0 \nVdash_{t_0} [F](\langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi)$, i.e. it is not the case that $\mathcal{M}_0 \Vdash_{t'} (\langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi)$ for all $t' \in T$ such that $t_0 < t'$. This means that it is not the case that ($\mathcal{M}_0 \Vdash_{t'} \langle P \rangle \varphi$ or $\mathcal{M}_0 \Vdash_{t'} \varphi$ or $\mathcal{M}_0 \Vdash_{t'} \langle F \rangle \varphi$) for all $t' \in T$ such that $t_0 < t'$. So $\mathcal{M}_0 \nVdash_{t'} \langle P \rangle \varphi$ and $\mathcal{M}_0 \nVdash_{t'} \varphi$ and $\mathcal{M}_0 \nVdash_{t'} \langle F \rangle \varphi$ for some $t' \in T$ such that $t_0 < t'$. But we know that $<$ is right-linear, and this means that, since $t_0 < t$ and $t_0 < t'$, we have ($t < t'$ or $t = t'$ or $t' < t$). We also know that $\mathcal{M}_0 \Vdash_t \varphi$. So we can claim that, if $t < t'$ (thus $t' > t$), it will be the case that $\mathcal{M}_0 \Vdash_{t'} \langle P \rangle \varphi$, if $t = t'$, we have $\mathcal{M}_0 \Vdash_{t'} \varphi$, and if $t' < t$ it follows that $\mathcal{M}_0 \Vdash_{t'} \langle F \rangle \varphi$. This contradicts our earlier assumption, so $\mathcal{M}_0 \Vdash_{t'} (\langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi)$ for all $t' \in T$ such that $t_0 < t'$, and consequently $\mathcal{M}_0 \Vdash_{t_0} [F](\langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi)$. But t_0 was chosen arbitrarily, and so was $\mathcal{M}_0$. Thus *TL.06* is globally true in $\mathcal{C}_{\mathcal{K}}$.

2. Let $\mathcal{C}_{\mathcal{K}}$ be the frame of $\mathcal{K}$, where $\mathcal{K}$ is any class of structures $\mathcal{F} = (T, <)$. Assume that all instances of the axiom schema $\langle F \rangle \varphi \rightarrow [F](\langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi)$ are globally true in $\mathcal{C}_{\mathcal{K}}$. We want to show that, for every $\mathcal{F} \in \mathcal{K}$, the accessibility relation $<$ is right-linear.

Pick an arbitrary $\mathcal{F}_0 = (T, <) \in \mathcal{K}$ and suppose there exists a $t_0, t_1, t_2 \in T$ such that $t_0 < t_1$ and $t_0 < t_2$ but ($t_1 \not< t_2$ and $t_1 \neq t_2$ and $t_2 \not< t_1$). Let $\mathcal{M}_0 \in \mathcal{C}_{\mathcal{F}_0}$ be the model such that $V(p) = \{t_1\}$ for every atom p. So $\mathcal{M}_0 \Vdash_{t_1} p$ for arbitrary p, and since $t_0 < t_1$, we have $\mathcal{M}_0 \Vdash_{t_0} \langle F \rangle p$.

From the schema it follows that $\mathcal{M}_0 \Vdash_{t_0} [F](\langle P\rangle p \vee p \vee \langle F\rangle p)$, i.e. $\mathcal{M}_0 \Vdash_t (\langle P\rangle p \vee p \vee \langle F\rangle p)$ for all $t \in T$ such that $t_0 < t$. This means that, for all $t \in T$ such that $t_0 < t$, $\mathcal{M}_0 \Vdash_t \langle P\rangle p$ or $\mathcal{M}_0 \Vdash_t p$ or $\mathcal{M}_0 \Vdash_t \langle F\rangle p$. So, for every $t \in T$ such that $t_0 < t$, (there exists a $t'' \in T$ such that $t > t''$ (thus $t'' < t$) and $\mathcal{M}_0 \Vdash_{t''} p$) or ($\mathcal{M}_0 \Vdash_t p$) or (there exists a $\tilde{t} \in T$ such that $t < \tilde{t}$ and $\mathcal{M}_0 \Vdash_{\tilde{t}} p$). But $V(p) = \{t_1\}$. From our choice of V it follows that, for every $t \in T$ such that $t_0 < t$, there exists a $t'' \in T$ such that $t'' < t$ and $t'' \in V(p)$ (so $t_1 < t$), or $t \in V(p)$ (so $t = t_1$), or there exists a $\tilde{t} \in T$ such that $t < \tilde{t}$ and $\tilde{t} \in V(p)$ (so $t < t_1$). In particular, since $t_0 < t_2$, we have $t_1 < t_2$ or $t_1 = t_2$ or $t_2 < t_1$. But this contradicts our earlier assumption, so it follows that the accessibility relation $<$ is right-linear in $\mathcal{K}$.

Axioms *TL.07* to *TL.09*, given below, formulate an idea expressed by Findlay (as reported by [HK80]): 'Every event taking place in time, will become past' [VB91].

TL.07 $\langle P\rangle \varphi \rightarrow \langle F\rangle \langle P\rangle \varphi$

TL.08 $\varphi \rightarrow \langle F\rangle \langle P\rangle \varphi$

TL.09 $\langle F\rangle \varphi \rightarrow \langle F\rangle \langle P\rangle \varphi$

Suppose $\mathcal{M}_0 \Vdash_t \varphi$ for some sentence φ in an arbitrary model $\mathcal{M}_0 = (T, <, V)$. These axioms express the idea that, whatever the 'tense' of the sentence φ at time t, there exists a point t' such that $t < t'$ (so t' is in the future) where $\mathcal{M}_0 \Vdash_{t'} \langle P\rangle \varphi$, i.e. where it holds that φ was true at some point in the past. Note that *TL.08* expresses the idea that such a future point exists, so right succession (unboundedness), as expressed by the first-order sentence $\forall t \exists t'(t < t')$, is implied. The following two first-order sentences express properties of $<$ axiomatised by *TL.07* and *TL.09* respectively:

$$\forall t, t'(t' < t \rightarrow \exists t''(t < t'' \wedge t' < t'')),$$
$$\forall t, t'(t < t' \rightarrow \exists t''(t < t'' \wedge t' < t'')).$$

The Priorean idea 'What will be the case, will be the case in between (its occurrence and the present)' can be expressed by *TL.10* and *TL.11*:

TL.10 $\langle F\rangle \varphi \rightarrow \langle F\rangle \langle F\rangle \varphi$ (Density of $<$)

If t is our present reference point and we know that φ will be true at some time in the future, at t' say, we may wish to be able to claim that there exists a point t'' between t and t' such that $\mathcal{M} \Vdash_{t''} \langle F\rangle \varphi$ for $\mathcal{M} = (T, <, V)$. This means that the accessibility relation must be dense, as expressed by the first-order sentence $\forall t, t'(t < t' \rightarrow \exists t''(t < t'' \wedge t'' < t'))$.

TL.11 $\langle F\rangle \langle F\rangle \varphi \rightarrow \langle F\rangle \varphi$ (Transitivity of $<$)

Note that *TL.05* and *TL.11* are different but equivalent axiom schemas for transitivity of $<$. *TL.05* involves two different modal operators wheras *TL.11* involves only one.

We take a closer look at *TL.11*:

Proposition 4 *1. Let $\mathcal{K}$ be the class of structures $\mathcal{F} = (T, <)$, such that $<$ is transitive in every $\mathcal{F} \in \mathcal{K}$, and let $C_{\mathcal{K}}$ be the frame of $\mathcal{K}$. Then every $\mathcal{M} \in C_{\mathcal{K}}$ globally satisfies all instances of the axiom schema $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$.*
2. If the frame $C_{\mathcal{K}}$ of some class $\mathcal{K}$ of structures globally satisfies the axiom schema TL.11, *then, for every $\mathcal{F} = (T, <) \in \mathcal{K}$, $<$ is transitive.*

1. Assume that $<$ is transitive in every $\mathcal{F} \in \mathcal{K}$. We have to show that $C_{\mathcal{K}}$ globally satisfies all instances of the axiom schema $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$.

Pick any $\mathcal{M}_0 = (T, <, V) \in C_{\mathcal{K}}$ and pick an arbitrary $t_0 \in T$. Suppose $\mathcal{M}_0 \Vdash_{t_0} \langle F \rangle \langle F \rangle \varphi$. Then there exists a $t' \in T$ such that $t_0 < t'$ and $\mathcal{M}_0 \Vdash_{t'} \langle F \rangle \varphi$, i.e. there exists a $t'' \in T$ such that $t' < t''$ and $\mathcal{M}_0 \Vdash_{t''} \varphi$. We have to show that $\mathcal{M}_0 \Vdash_{t_0} \langle F \rangle \varphi$. But this is immediate: by transitivity, $t_0 < t''$. It follows that $\mathcal{M}_0$ globally satisfies all instances of the schema and consequently, that $C_{\mathcal{K}}$ globally satisfies all instances of the schema.

2. Assume that the frame $C_{\mathcal{K}}$ of $\mathcal{K}$ globally satisfies the schema $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$. We have to show that $<$ is transitive for every $\mathcal{F} = (T, <) \in \mathcal{K}$.

Pick an arbitrary $\mathcal{F}_0 \in \mathcal{K}$ and suppose there exists a $t_0, t_1, t_2 \in T$ such that $t_0 < t_1$ and $t_1 < t_2$ but $t_0 \not< t_2$. Let $\mathcal{M}_0 \in C_{\mathcal{F}_0}$ be the model such that $V(p) = \{t_2\}$ for all atoms p. So $\mathcal{M}_0 \Vdash_{t_2} p$ for arbitrary p. But $t_1 < t_2$ so $\mathcal{M}_0 \Vdash_{t_1} \langle F \rangle p$. We also have $t_0 < t_1$ so $\mathcal{M}_0 \Vdash_{t_0} \langle F \rangle \langle F \rangle p$. Using the schema, we can claim that it must be the case that $\mathcal{M}_0 \Vdash_{t_0} \langle F \rangle p$. So there exists a $\tilde{t} \in T$, such that $t_0 < \tilde{t}$ and $\mathcal{M} \Vdash_{\tilde{t}} p$, which means that $\tilde{t} \in V(p)$. But $V(p) = \{t_2\}$. This means that $\tilde{t} = t_2$, so $t_0 < t_2$, which contradicts our earlier assumption. Thus, for all $t, t', t'' \in T$, if $t < t'$ and $t' < t''$, it follows that $t < t''$. Thus $<$ is transitive for every $\mathcal{F} = (T, <) \in \mathcal{K}$.

Remark: It is instructive to note that, if we take C to be the class of models globally satisfying all instances of the axiom schema $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$ rather than coupling the axiomatisation to a frame, then the accessibility relations are not necessarily all transitive. We can use the following example to illustrate:

Example 15 *The existence of rogue models.*

Let $\Phi = \{p\}$.

Let $\mathcal{M}_0 = (T, R_F, V)$, with $T = \{t_1, t_2\}$, $R_F = \{(t_1, t_2), (t_2, t_1)\}$ and $V_1(p) = \{t_1, t_2\}$. Although it is clear that R_F is not transitive, we are going to show that $\mathcal{M}_0$ globally satisfies all instances of the schema $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$. If we can show that $\mathcal{M}_0 \nVdash \langle F \rangle \langle F \rangle \varphi$ or $\mathcal{M}_0 \Vdash \langle F \rangle \varphi$, it follows that $\mathcal{M}_0 \Vdash \langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$.

From our choice of V_1 we know that $\mathcal{M}_0 \Vdash_{t_1} p$ and $\mathcal{M}_0 \Vdash_{t_2} p$. From our choice of R_F it follows that $\mathcal{M}_0 \Vdash_{t_2} \langle F \rangle p$ and $\mathcal{M}_0 \Vdash_{t_1} \langle F \rangle p$, so $\mathcal{M}_0 \Vdash \langle F \rangle p$ which means that $\mathcal{M}_0 \Vdash \langle F \rangle \langle F \rangle p \rightarrow \langle F \rangle p$. So $\mathcal{M}_0$ globally satisfies the axiom schema for arbitrary $\varphi \in \Phi$ (in this case $\Phi = \{p\}$). It remains to show that $\mathcal{M}_0$ globally satisfies the axiom schema for an arbitrary sentence φ, which easily follows by structural induction on the formation of φ. So $\mathcal{M}_0 \Vdash \langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$.

It follows that all instances of the schema are globally satisfied in $\mathcal{M}_0$ even though R_F is not transitive.

It is important to note that we cannot generalise this to the frame $\mathcal{C}_{\mathcal{F}}$ where $\mathcal{F} = (T, R_F)$. Let $\mathcal{M}_1 = (T, R_F, V_2) \in \mathcal{C}_{\mathcal{F}}$ with $V_2(p) = \{t_1\}$. We show that $\mathcal{M}_1 \nVdash \langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$ for arbitrary φ.

From our choice of V_2 we have $\mathcal{M}_1 \Vdash_{t_1} p$. From our choice of R_F it follows that $\mathcal{M}_1 \Vdash_{t_2} \langle F \rangle p$ and also that $\mathcal{M}_1 \Vdash_{t_1} \langle F \rangle \langle F \rangle p$. But it is not the case that $\mathcal{M}_1 \Vdash_{t_1} \langle F \rangle p$. Thus $\mathcal{M}_1 \nVdash_{t_1} \langle F \rangle \langle F \rangle p \rightarrow \langle F \rangle p$. So it is not the case that all instances of the axiom schema $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$ are globally satisfied by $\mathcal{M}_1$.

Now that we have an idea of how properties of the time structure for instant tense logic can be formulated, we look at a number of variations and extensions of this logic.

3.3 Variations and extensions of instant tense logic

There exists a wide variety of logics based on the principles of Priorean tense logic. The alphabets of these languages are similar to that of Priorean tense logic with the exception of the modal operators. In some cases the existing alphabet is extended by adding new modal operators to enhance the expressive power. In other systems, like the one defined by Kamp [Kam68] which we look at in the next section, entirely different modal operators are used. We discuss a few of the variations briefly.

3.3.1 Since and Until

It is not possible to represent the present perfect tense in the language of Priorean tense logic [VB91]. In an attempt to overcome this limitation, Kamp [Kam68] suggests the use of two new binary modal operators S and U, where S is read 'since' and U is read 'until' . So, if φ and ψ are sentences in the language, then $\varphi S \psi$ is read 'φ has been true since an instant in the past where ψ was true', and $\varphi U \psi$ is read 'φ will be true until some instant in the future where ψ is true'.

A graphical representation of S and U is given in Fig 3.2. If $\varphi U \psi$ holds at t then it means that φ holds until some instant t' in the future at which ψ holds, and if $\varphi S \psi$ holds at time t then it means that φ has been true since some instant t' in the past at which ψ was true.

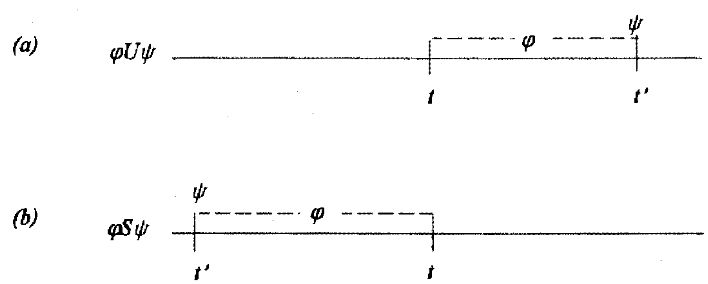

Figure 3.2: A graphical representation of S (since) and U (until).

The modal operators $\langle F\rangle, \langle P\rangle, [F]$ and $[P]$ of Priorean tense logic can all be defined in terms of S and U. The converse is not true: U cannot be defined in terms of $[F], [P], \langle F\rangle$ and $\langle P\rangle$ [GHR94].

As before, a model is a triple $\mathcal{M} = (T, <, V)$, with the accessibility relation $<$ corresponding to the modal operator U, and its converse $>$ corresponding to S. A sentence φ is true relative to $\mathcal{M}$ at time t, denoted by $\mathcal{M} \Vdash_t \varphi$, iff one of the following holds:

- $\varphi \in \Phi$ and $t \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_t \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_t \psi$ or $\mathcal{M} \Vdash_t \chi$
- $\varphi = \psi S \chi$ and there exists a $t' \in T$ such that $t > t'$ and $\mathcal{M} \Vdash_{t'} \chi$, and $\mathcal{M} \Vdash_{t''} \psi$ for every $t'' \in T$ such that $t > t'' > t'$

- $\varphi = \psi U \chi$ and there exists a $t' \in T$ such that $t < t'$ and $\mathcal{M} \Vdash_{t'} \chi$, and $\mathcal{M} \Vdash_{t''} \psi$ for every $t'' \in T$ such that $t < t'' < t'$.

The modal operators $\langle F \rangle, \langle P \rangle, [F]$ and $[P]$, can be now defined in terms of S and U in the following way [Bur82]:

$[P]\varphi$	$\equiv_{def}$	$\neg(\top S \neg\varphi)$	'φ has always been true'
$[F]\varphi$	$\equiv_{def}$	$\neg(\top U \neg\varphi)$	'φ is always going to be true'
$\langle P \rangle \varphi$	$\equiv_{def}$	$\top S \varphi$	'φ was true at some instant in the past'
$\langle F \rangle \varphi$	$\equiv_{def}$	$\top U \varphi$	'φ will be true at some instant in the future'

where	$\top$	$\equiv_{def}$	$p \vee \neg p$	(Tautology, i.e. true at every instant),
and	$\bot$	$\equiv_{def}$	$p \wedge \neg p$	(Contradiction, i.e. false at every instant).

How do we know that the modal operators that we have defined in terms of S and U correspond semantically to the modal operators of Priorean tense logic? We show that it does for $[P]$. It can be done in a similar way for the other modal operators.

Proposition 5 *The definition of $[P]$ in terms of S is semantically equivalent to the Priorean definition of $[P]$.*

Suppose $[P]$ is defined in the following way:

$[P]\varphi \equiv_{def} \neg(\top S \neg\varphi)$.

Let $\mathcal{F}_0$ be an arbitrary structure of the form $(T, <)$. Pick an arbitrary $\mathcal{M}_0 \in \mathcal{C}_{\mathcal{F}_0}$, and an arbitrary $t_0 \in T$. Suppose $\mathcal{M}_0 \Vdash_{t_0} [P]\varphi$. We have to show that $\mathcal{M}_0 \Vdash_{t'} \varphi$ for all $t' \in T$ such that $t_0 > t'$.

By definition, $\mathcal{M}_0 \Vdash_{t_0} [P]\varphi$ means that $\mathcal{M}_0 \Vdash_{t_0} \neg(\top S \neg\varphi)$, i.e. $\mathcal{M}_0 \nVdash_{t_0} \top S \neg\varphi$. So, it is not the case that (there exists a $t' \in T$ such that $t_0 > t'$ and $\mathcal{M}_0 \Vdash_{t'} \neg\varphi$, and $\mathcal{M}_0 \Vdash_{t''} \top$ for all $t'' \in T$ such that $t_0 > t'' > t'$). But we know that $\mathcal{M}_0 \Vdash_t \top$ for all $t \in T$. It follows that it is not the case that there exists a $t' \in T$ such that $t_0 > t'$ and $\mathcal{M}_0 \nVdash_{t'} \varphi$. So $\mathcal{M}_0 \Vdash_{t'} \varphi$ for all $t' \in T$ such that $t_0 > t'$. The converse is similar.

Burgess [Bur82] also defines the following variations of the new modal operators:

$[F]'\varphi \equiv_{def} \varphi U \top$ meaning 'φ is uninterruptedly going to be true for some time'

$[P]'\varphi \equiv_{def} \varphi S \top$ meaning 'φ has uninterruptedly been true for some time'

$\langle F \rangle' \varphi \equiv_{def} \neg(\neg\varphi U \top)$ meaning 'φ will be true, arbitrarily soon'

$\langle P \rangle' \varphi \equiv_{def} \neg(\neg\varphi S \top)$ meaning 'φ has been true, arbitrarily recently'

What are the 'meanings' of the modal operators $[P]', [F]', \langle P\rangle'$ and $\langle F\rangle'$? As an example, we will show how we arrived at the meaning of $\langle F\rangle' \varphi$. It can be shown for the other modal operators in a similar fashion.

Example 16 *The 'meaning' of the modal operator $\langle F\rangle'$.*

We defined $\langle F\rangle'$ as follows:

$$\langle F\rangle' \varphi \equiv_{def} \neg(\neg\varphi U \top).$$

We want to show that this means that φ will be true arbitrarily soon.

Let $\mathcal{F}_0$ be an arbitrary structure of the form $(T,<)$. Choose an arbitrary model $\mathcal{M}_0 \in \mathcal{C}_{\mathcal{F}_0}$ and an arbitrary $t_0 \in T$. Suppose $\mathcal{M}_0 \Vdash_{t_0} \langle F\rangle' \varphi$. By definition, this means that $\mathcal{M}_0 \nVdash_{t_0} (\neg\varphi U \top)$, i.e. it is not the case that (there exists a $t' \in T$ such that $t_0 < t'$ and $\mathcal{M}_0 \Vdash_{t'} \top$, and $\mathcal{M}_0 \nVdash_{t''} \varphi$ for all $t'' \in T$ such that $t_0 < t'' < t'$). But we know that $\mathcal{M}_0 \Vdash_t \top$ for all $t \in T$. This means that it is not the case that $\mathcal{M}_0 \nVdash_{t''} \varphi$ for all $t'' \in T$ such that $t_0 < t'' < t'$. So, for every possible choice of $t' > t_0$, there exists a $\tilde{t} \in T$ such that $t_0 < \tilde{t} < t'$ and $\mathcal{M}_0 \Vdash_{\tilde{t}} \varphi$. Even if we choose t' close to t_0, we can find a $\tilde{t}$ even closer to t_0 where φ holds. This means that φ will hold arbitrarily soon.

Burgess gives an axiomatisation for the class of linear structures, as well as for the class of dense and discrete structures [Bur82]. The axioms are given in pairs; each axiom in a pair forming the mirror image of the other member of the pair (the *mirror image* of a sentence φ being the result of replacing each U in the sentence by S and vice versa, and replacing each $[F]$ by $[P]$ and vice versa).

BU.01a $[F](\varphi \to \psi) \to (\chi U \varphi \to \chi U \psi)$

BU.01b $[P](\varphi \to \psi) \to (\chi S \varphi \to \chi S \psi)$

BU.02a $[F](\varphi \to \psi) \to (\varphi U \chi \to \psi U \chi)$

BU.02b $[P](\varphi \to \psi) \to (\varphi S \chi \to \psi S \chi)$

BU.03a $(\chi \wedge \varphi U \psi) \to \varphi U(\psi \wedge \varphi S \chi)$

BU.03b $(\chi \wedge \varphi S \psi) \to \varphi S(\psi \wedge \varphi U \chi)$

BU.04a $(\varphi U \psi \wedge \neg(\chi U \psi)) \to \varphi U(\varphi \wedge \neg\chi)$

BU.04b $(\varphi S \psi \wedge \neg(\chi S \psi)) \to \varphi S(\varphi \wedge \neg\chi)$

BU.05a $\varphi U \psi \to (\varphi \wedge \varphi U \psi) U \psi$

BU.05b $\varphi S \psi \to (\varphi \wedge \varphi S \psi) S \psi$

BU.06a $\varphi U(\varphi \wedge \varphi U \psi) \to \varphi U \psi$

BU.06b $\varphi S(\varphi \wedge \varphi S\psi) \rightarrow \varphi S\psi$

BU.07a $(\varphi U\psi \wedge \chi U\zeta) \rightarrow ((\varphi \wedge \chi)U(\psi \wedge \zeta) \vee (\varphi \wedge \chi)U(\psi \wedge \chi) \vee (\varphi \wedge \chi)U(\varphi \wedge \zeta))$

BU.07b $(\varphi S\psi \wedge \chi S\zeta) \rightarrow ((\varphi \wedge \chi)S(\psi \wedge \zeta) \vee (\varphi \wedge \chi)S(\psi \wedge \chi) \vee (\varphi \wedge \chi)S(\varphi \wedge \zeta))$

Another variation of the language based on S and U is discussed in [GHR94]. Two modal operators, S' and U', called *Stavi connectives,* are introduced with the following intuitions:

$\psi U'\varphi$ is true 'now' iff there is a gap in the future such that

- ψ is true until this gap;
- φ is true from the gap into the future for some uninterrupted stretch of time;
- in the future of the gap, ψ is false arbitrarily close to the gap.

S' is the mirror image of U'. So $\psi S'\varphi$ is true 'now' iff there is a gap in the past such that

- ψ was true since this gap;
- φ was true for some uninterrupted stretch of time from the past of the gap right up until the gap;
- in the past of the gap, ψ is false arbitrarily close to the gap.

3.3.2 Next and Previous

In a system based on a structure $\mathcal{F} = (T, <)$ in which the accessibility relation $<$ is discrete, we may want to address the point immediately preceding the current time point, as well as a point immediately succeeding the current time point. (Such systems are frequently used in temporal logics of programs [Haj96].) The notion of such a next and a previous point can be introduced by adding two new modal operators $\circ$, read 'at the next instant', and $*$, read 'at the previous instant', to the alphabet of Priorean tense logic (see [GHR94], [May83] and [AM85] for examples).

In such a system, a model is a triple $\mathcal{M} = (T, <, V)$, with the accessibility relation $<$ corresponding to $[F]$ and $\circ$, and its converse, $>$, corresponding to $[P]$ and $*$. A sentence φ is true relative to $\mathcal{M}$ at time point t, denoted by $\mathcal{M} \Vdash_t \varphi$, iff one of the following holds:

- $\varphi \in \Phi$ and $t \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_t \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_t \psi$ or $\mathcal{M} \Vdash_t \chi$

- $\varphi = [F]\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for all $t' \in T$ such that $t < t'$
- $\varphi = [P]\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for all $t' \in T$ such that $t > t'$
- $\varphi = \circ\psi$ and $\mathcal{M} \Vdash_{t'} \psi$, for some $t' \in T$ such that $t < t'$, and it is not the case that there exists a $t'' \in T$ such that $t < t'' < t'$
- $\varphi = *\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ where $t' \in T$ such that $t > t'$, and it is not the case that there exists a $t'' \in T$ such that $t > t'' > t'$.

The following two axioms describe the relationships between the primitive and the new operators: *NP.01* defines the relationship between $\circ$ and $[F]$, and *NP.02* defines the relationship between $*$ and $[P]$.

NP.01 $[F]\varphi \leftrightarrow \circ\varphi \wedge \circ[F]\varphi$. ($\varphi$ will always be true in the future iff φ is true at the next instant and at the next instant it is also true that φ will always hold in the future.)

NP.02 $[P]\varphi \leftrightarrow *\varphi \wedge *[P]\varphi$. ($\varphi$ has always been true in the past iff φ was true at the previous instant and at the previous instant it is also true that φ has always been true in the past.)

3.3.3 Progressive tense

The representation of the 'progressive' tense in English can be approximated by a modal operator $\prod$, which enables one to describe what happens between two distinct time points. This operator, whose semantics involves both the accessibility relation $<$ and its converse $>$, enables us to express an English sentence containing a verb such as 'being' in a language based on this alphabet.

We say that the sentence $\varphi = \prod \psi$ is true in a model $\mathcal{M} = (T, <, V)$ at time t, denoted by $\mathcal{M} \Vdash_t \prod \psi$, iff there exists $t', t'' \in T$ such that $t > t'$ and $t < t''$, and $\mathcal{M} \Vdash_{\tilde{t}} \varphi$ for all $\tilde{t} \in T$ such that $\tilde{t} > t'$ and $\tilde{t} < t''$.

A complete axiomatisation of this temporal logic for linear structures can be found in [She89].

3.3.4 Difference

Priorean tense logic can also be strengthened by the introduction of a modal operator that expresses the notion of 'truth in another world than the current one', i.e. truth at a different time (for examples see [Gor89], [Koy89] and [Sai88]).

The idea is to introduce a new modal operator to express the notion that a sentence φ is true in a model at a time point different from the current one.

To do this, we take $\prec$ as an accessibility relation associated with a new modal operator $\langle D \rangle$ read 'at a different time', and we say that $\mathcal{M} \Vdash_t \langle D \rangle \psi$ iff there exists some $t' \in T$ such that $t \prec t'$ and $\mathcal{M} \Vdash_{t'} \psi$. If we add the following set of axioms to the logic, then $\prec$ is 'as near to real inequality as possible' [VB95].

DF.01	$\langle D \rangle \neg \langle D \rangle \neg\varphi \rightarrow \varphi$	(Symmetry of $\prec$)
DF.02	$\langle D \rangle \langle D \rangle \varphi \rightarrow (\langle D \rangle \varphi \vee \varphi)$	(Pseudo-transitivity of $\prec$)
DF.03	$\langle F \rangle \varphi \rightarrow (\langle D \rangle \varphi \vee \varphi)$	$\forall t, t'((t < t') \rightarrow (t \prec t' \vee t = t'))$
DF.04	$\langle F \rangle \varphi \rightarrow \langle D \rangle \varphi$	(Irreflexivity of $\prec$)

3.3.5 Branching time structures

We might need a time structure where time is not linear but forward branching. In such a structure the accessibility relation is left-linear which means that a time point has only one past but different possible futures. A branching structure may be achieved by using *double modalities* as suggested by Mays [May83].

The alphabet used by Mays consists of the following elements:

- the set Φ of one or more atoms $p, q, \ldots$
- the connectives $\neg$ and $\rightarrow$
- the punctuation symbols '(' and ')'
- the modal operator $[P]$ read 'always in the past'
- the 'double' modal operators $[AG]$ and $\langle EG \rangle$, where $[AG]$ is read 'for every always in the future' and $\langle EG \rangle$ is read 'for some always in the future'.

Additional modal operators are defined in terms of $[AG]$, $\langle EG \rangle$ and $[P]$, in the following way:

$\langle AG \rangle \varphi$	$\equiv_{def}$	$\neg[AG]\neg\varphi$	For some eventually in the future
$[EG]\varphi$	$\equiv_{def}$	$\neg \langle EG \rangle \neg\varphi$	For every eventually in the future
$\langle P \rangle \varphi$	$\equiv_{def}$	$\neg[P]\neg\varphi$	Sometime past

Let $Elt(t, B)$ express the notion that $t \in B$. A *branch* can be defined in a first-order language as any subset $B \subseteq T$ such that,

$\forall t, t'((Elt(t, B) \wedge Elt(t', B)) \rightarrow (t < t' \vee t = t' \vee t' < t))$ and
$\forall t(\forall t'(Elt(t', B) \rightarrow ((t < t' \vee t = t' \vee t' < t) \rightarrow Elt(t, B))))$.

The first sentence formulates the idea that there exists a linear relationship between any two points on the same branch. The second is a formulation of the converse: for all $t \in T$, and for every point $t' \in T$ that is an element of some branch B, if there exists a linear relationship between t and t', then t is also an element of B.

In a model $\mathcal{M} = (T, <, V)$, we now have the accessibility relation $<$ corresponding to $[AG]$ and $\langle EG \rangle$, and its converse, $>$, corresponding to $[P]$. A sentence φ is true relative to $\mathcal{M}$ at time t, denoted by $\mathcal{M} \Vdash_t \varphi$, iff one of the following holds:

- $\varphi \in \Phi$ and $t \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_t \psi$
- $\varphi = (\psi \rightarrow \chi)$ and $\mathcal{M} \nVdash_t \psi$ or $\mathcal{M} \Vdash_t \chi$
- $\varphi = [AG]\psi$ and, for every branch B containing t, $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in B$ such that $t < t'$
- $\varphi = \langle EG \rangle \psi$ and there exists a branch B containing t where $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in B$ such that $t < t'$
- $\varphi = [P]\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in T$ such that $t > t'$.

The number of modal operators in the alphabet is relatively large and Mays does not provide an axiomatisation for the relevant frame. However, time can be assumed to be discrete since Mays also defines a modal operator denoting the set of next time points, namely $[AX]$ read 'at every next instant', and one for the previous time point, namely L read 'at the previous instant' (time is left-linear).

The modal operator $\langle AX \rangle$, read 'at some next instant', can again be defined in terms of its dual $[AX]$ in the usual way:

$$\langle AX \rangle \varphi \equiv_{def} \neg[AX]\neg\varphi$$

The semantics for the new operators can be defined as follows:

$\mathcal{M} \Vdash_t [AX]\varphi$ iff $\mathcal{M} \Vdash_{t'} \varphi$ for every $t' \in T$ such that $t < t'$, and, for every such t', it is not the case that there exists a $t'' \in T$ such that $t < t'' < t'$.

$\mathcal{M} \Vdash_t L\varphi$ iff $\mathcal{M} \Vdash_{t'} \varphi$ for some $t' \in T$ such that $t' < t$ and it is not the case that there exists a $t'' \in T$ such that $t' < t'' < t$.

Other examples of branching-time modal languages can be found in [Sti92] and [Zan91].

3.3.6 Metric tense logic

An interesting variation suggested by Prior is that of metric tense logic. In addition to the relation between time points, the distance between two distinct points is also taken into consideration. It is, of course, necessary to adjust the time structure to cater for this new aspect. We define a *distance structure* $\mathcal{C} = (C, 0, +)$ for representing the distance between points, where C is a set of distances, 0 represents the zero distance and $+$ is a binary operator on elements of C. The alphabets for these languages are similar to the alphabet for Priorean tense logic but, instead of $[F]$ and $[P]$, we have the two modal operators (also called *metric operators*) $[F]_n$ and $[P]_n$, where $[F]_n$ is read 'for all instants at a distance n into the future' and $[P]_n$ is read 'for all instants at a distance n in the past'.

We define the duals of these in the usual way:

$$\langle F\rangle_n \varphi \equiv_{def} \neg[F]_n\neg\varphi$$
$$\langle P\rangle_n \varphi \equiv_{def} \neg[P]_n\neg\varphi$$

A time structure is now a tuple $\mathcal{F} = (T, \mathcal{C}, <, \delta)$ where T is a set of time points, $<$ is an accessibility relation, $\mathcal{C}$ is a distance structure as defined above, and $\delta : T \times T \rightarrow C$ is a binary function expressing the distance between two time points. A model is an ordered tuple $\mathcal{M} = (T, \mathcal{C}, <, \delta, V)$ with the accessibility relation $<$ corresponding to the operator $[F]_n$, and the accessibility relation $>$ (the converse of $<$) corresponding to $[P]_n$. A sentence φ is true relative to $\mathcal{M}$ at time point t, denoted by $\mathcal{M} \Vdash_t \varphi$, iff one of the following holds:

- $\varphi \in \Phi$ and $t \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_t \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_t \psi$ or $\mathcal{M} \Vdash_t \chi$
- $\varphi = [F]_n\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in T$ such that $t < t'$ and $\delta(t, t') = n \in C$
- $\varphi = [P]_n\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in T$ such that $t > t'$ and $\delta(t, t') = n \in C$.

The duals $\langle F\rangle_n$ and $\langle P\rangle_n$ are defined as usual. Hajnicz [Haj96] suggests that the frame of interest is that characterised by the following set of axioms:

MTL.01 $\langle F\rangle_0 \varphi \rightarrow \varphi$

MTL.02 $\langle F\rangle_n (\varphi \rightarrow \psi) \rightarrow (\langle F\rangle_n \varphi \rightarrow \langle F\rangle_n \psi)$

MTL.03 $\langle F\rangle_n \langle F\rangle_m \varphi \rightarrow \langle F\rangle_{(n+m)} \varphi$

MTL.04 $\langle F\rangle_n \langle P\rangle_m \varphi \rightarrow \langle F\rangle_{(n-m)} \varphi$ for $m < n$

MTL.05 $\langle F\rangle_n \langle P\rangle_m \varphi \rightarrow \langle P\rangle_{(m-n)} \varphi$ for $n < m$

MTL.06 $\langle P \rangle_0 \varphi \rightarrow \varphi$

MTL.07 $\langle P \rangle_n (\varphi \rightarrow \psi) \rightarrow (\langle P \rangle_n \varphi \rightarrow \langle P \rangle_n \psi)$

MTL.08 $\langle P \rangle_n \langle P \rangle_m \varphi \rightarrow \langle P \rangle_{(n+m)} \varphi$

MTL.09 $\langle P \rangle_n \langle F \rangle_m \varphi \rightarrow \langle P \rangle_{(n-m)} \varphi$ for $m < n$

MTL.10 $\langle P \rangle_n \langle F \rangle_m \varphi \rightarrow \langle F \rangle_{(m-n)} \varphi$ for $n < m$

We note that, if the time structure is infinite and linear then $[F]_n = \langle F \rangle_n$ and $[P]_n = \langle P \rangle_n$ because, if we choose any $t \in T$, there is exactly one time point $t' \in T$ such that $t < t'$ and $\delta(t, t') = n$, and there is exactly one time point $t'' \in T$ such that $t > t''$ and $\delta(t, t'') = n$. This means that, in such a structure, we can restrict the modal operators to $\langle F \rangle_n$ and $\langle P \rangle_n$. We can also add the following two axiom schemas to express the idea that, if there exists an instant at a distance n into the future (*MTL.11*) where $\neg\varphi$, then there does not exist an instant at a distance n into the future where φ. *MTL.12* is relevant for the past.

MTL.11 $\langle F \rangle_n \neg\varphi \rightarrow \neg \langle F \rangle_n \varphi$

MTL.12 $\langle P \rangle_n \neg\varphi \rightarrow \neg \langle P \rangle_n \varphi$

The logic discussed in this section and the original Priorean tense logic can be connected by taking the operators $[F]$ and $[P]$ as primitive and then adding $[F]_n$ and $[P]_n$ by suitable definitions. The following two axioms can be added to the union of the two axiomatisations to express the relationship between $\langle F \rangle_n$ and $\langle F \rangle$, and between $\langle P \rangle_n$ and $\langle P \rangle$ respectively:

MTL.13 $\langle F \rangle_n \varphi \rightarrow \langle F \rangle \varphi$, for $n > 0$

MTL.14 $\langle P \rangle_n \varphi \rightarrow \langle P \rangle \varphi$, for $n > 0$.

3.3.7 Near past, near future, far past, far future

Rescher & Urquhart [RU71] define an alternative modal language based on a metric time structure.

The alphabet for this logic consists of the following elements:

- the set Φ of one or more atoms $p, q, \ldots$
- the connectives $\neg$ and $\rightarrow$
- the punctuation symbols '(' and ')'
- the modal (or metric) operators $[F]^\circ, [F]^*, [P]^\circ$ and $[P]^*$.

($^\circ$ and * should not be confused with the modal operators $\circ$ (next) and $*$ (previous) of section 3.3.2.) The duals of these operators can be defined in the usual way:

$\langle F\rangle^\circ \varphi \equiv_{def} \neg[F]^\circ\neg\varphi$
$\langle F\rangle^* \varphi \equiv_{def} \neg[F]^*\neg\varphi$
$\langle P\rangle^\circ \varphi \equiv_{def} \neg[P]^\circ\neg\varphi$
$\langle P\rangle^* \varphi \equiv_{def} \neg[P]^*\neg\varphi$

The semantics is based on a structure similar to the time structure used for the metric tense logic discussed in section 3.3.6, namely $\mathcal{F} = (T, \mathcal{C}, <, \delta)$, where the distance structure $\mathcal{C} = \langle C, 0, +\rangle$ is defined as before. Two arbitrary constants $c, c' \in C$ must be chosen which play an essential role that will become clear in a moment. A model is a tuple $\mathcal{M} = (T, \mathcal{C}, c, c', <, \delta, V)$ where the accessibility relation $<$ corresponds to the operators $[F]^\circ$ and $[F]^*$, and its converse, $>$, corresponds to the operators $[P]^\circ$ and $[P]^*$. A sentence φ is true relative to $\mathcal{M}$ at time t, denoted by $\mathcal{M} \Vdash_t \varphi$, iff one of the following holds:

- $\varphi \in \Phi$ and $t \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_t \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_t \psi$ or $\mathcal{M} \Vdash_t \chi$
- $\varphi = [F]^\circ\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in T$ such that $t < t'$ and $\delta(t, t') < c$
- $\varphi = [F]^*\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in T$ such that $t < t'$ and $\delta(t, t') > c'$
- $\varphi = [P]^\circ\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in T$ such that $t > t'$ and $\delta(t, t') < c$
- $\varphi = [P]^*\psi$ and $\mathcal{M} \Vdash_{t'} \psi$ for every $t' \in T$ such that $t > t'$ and $\delta(t, t') > c'$.

Note that, if $c < c'$, and there is some $t' \in T$ such that $t < t'$ and $\mathcal{M} \Vdash_{t'} \varphi$, but $c < \delta(t, t') < c'$, then neither $\langle F\rangle^\circ \varphi$ nor $\langle F\rangle^* \varphi$ will be satisfied. A similar problem could arise if $c' < c$. If $c' < \delta(t, t') < c$, then both $\langle F\rangle^\circ \varphi$ and $\langle F\rangle^* \varphi$ may be satisfied. To avoid any ambiguities this might cause, the best choice seems to be to choose $c = c'$.

3.4 Summary

In this chapter we considered various modal propositional languages with semantics described in terms of time structures based on time points (or instants). In the next chapter, we discuss modal propositional languages in which the possible worlds represent time intervals.

Chapter 4

Modal approaches with intervals

4.1 From instants to intervals

When reasoning about program specifications and verification, as well as in some areas of AI, it is sometimes more convenient to consider the truth value of sentences over intervals rather than time points. This is particularly relevant if we want to express the relationship between the truth value of a sentence over an interval and its truth value over some subinterval. In this chapter, we discuss the axiomatisation of time structures for modal logics based on intervals rather than points. Firstly, we consider *extended tense logic*, a modal logic suggested by Van Benthem that uses the principles of Priorean tense logic but with intervals as primitive [VB91]. Then we look at Halpern & Shoham's logic that has time points as primitive but where sentences are interpreted over intervals [HS91], and lastly we briefly discuss a spatial representation of relationships between intervals suggested by Venema [Ven88].

4.2 Extended tense logic

Extended tense logic, also called *interval tense logic*, uses a modal language and a time structure with intervals as primitive ([VB91] and [Haj96]). The alphabet consists of the following elements:

- a set Φ containing one or more atoms $p, q, \ldots$
- the punctuation symbols '(' and ')'
- the connectives $\neg$ and $\rightarrow$

- the modal operators $[F]$, read 'always in the future', $[P]$, read 'always in the past', and $\Box$, read 'for all subintervals'.

The duals $\langle F \rangle$, read 'sometime in the future', $\langle P \rangle$, read 'sometime in the past', and $\Diamond$, read 'for some subinterval', are defined in terms of $[F], [P]$ and $\Box$ as before: for any sentence φ,

$$\langle F \rangle \varphi \equiv_{def} \neg[F]\neg\varphi$$
$$\langle P \rangle \varphi \equiv_{def} \neg[P]\neg\varphi$$
$$\Diamond\varphi \equiv_{def} \neg\Box\neg\varphi.$$

The set of well-formed formulae and the connectives $\wedge$, $\vee$ and $\leftrightarrow$ are defined in the usual way. The semantics of the logic is based on a time structure $\mathcal{F} = (I, <, \subseteq)$ where I is a set, the elements of which are called intervals, with the accessibility relation $<$ corresponding to $[F]$, its converse, $>$, corresponding to $[P]$, and the accessibility relation $\subseteq$, also known as the *inclusion relation*, corresponding to $\Box$. Two of the relationships that may exist between distinct intervals i and j are relevant: $i < j$ expresses the notion that interval i precedes interval j (there is no overlap), and $i \subseteq j$ expresses the idea that interval i is a subinterval of interval j.

The valuation $V : \Phi \rightarrow 2^I$ associates, with each $p \in \Phi$, the set of intervals $i \in I$ over which p is true. A model is an ordered tuple $\mathcal{M} = (I, <, \subseteq, V)$ and we say that a sentence φ is true relative to $\mathcal{M}$ over an interval i, denoted by $\mathcal{M} \Vdash_i \varphi$, iff one of the following cases applies:

- $\varphi \in \Phi$ and $i \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_i \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_i \psi$ or $\mathcal{M} \Vdash_i \chi$
- $\varphi = [F]\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $i < j$
- $\varphi = [P]\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $i > j$
- $\varphi = \Box\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $j \subseteq i$.

The sentence φ is globally true over a model $\mathcal{M}$ iff $\mathcal{M} \Vdash_i \varphi$ for all $i \in I$. Global truth of a sentence over a frame is defined as before.

4.2.1 Axiomatisation

In [VB91] Van Benthem points out how properties of a temporal structure suitable for an intended application can be formulated. These axioms are based on a reformation of those suggested by Humberstone [Hum79]. We discuss the formulation of a number of these axioms.

The following two schemas axiomatise the class of structures in which $<$ and $>$ are converses:

ITL.01 $\varphi \rightarrow [F]\langle P\rangle\varphi$
ITL.02 $\varphi \rightarrow [P]\langle F\rangle\varphi$

Axioms *ITL.03* and *ITL.04* serve to define certain constraints on $\subseteq$:

ITL.03 $\Box\varphi \rightarrow \Box\Box\varphi$ (Transitivity of $\subseteq$)
ITL.04 $\Box\varphi \rightarrow \varphi$ (Reflexivity of $\subseteq$)

The two accessibility relations $<$ and $\subseteq$ are not completely independent. This means that for a number of properties we have to take the *interaction* between $<$ and $\subseteq$, or between $>$ and $\subseteq$, into account [VB91].

ITL.05 $\langle F\rangle\varphi \rightarrow \Box\langle F\rangle\varphi$ (Right-monotonicity of $<, \subseteq$, i.e. $\forall x, y(x < y \rightarrow \forall z(z \subseteq y \rightarrow x < z)))$
ITL.06 $\langle P\rangle\varphi \rightarrow \Box\langle P\rangle\varphi$ (Left-monotonicity of $>, \subseteq$, i.e. $\forall x, y(y > x \rightarrow \forall z(z \subseteq x \rightarrow y > z)))$

The following three axioms can be included for properties of $<$.

ITL.07 $[F]\varphi \rightarrow [F][F]\varphi$ (Transitivity of $<$)
ITL.08 $[F]\varphi \rightarrow \langle F\rangle\varphi$ (Right-unboundedness of $<$, i.e. $\forall x\exists y(x < y))$
ITL.09 $[P]\varphi \rightarrow \langle P\rangle\varphi$ (Left-unboundedness of $>$, i.e. $\forall x\exists y(x > y))$.

As was the case with instant tense logic, we again have the problem that there are some properties of $<$ and of the interaction between $<$ and $\subseteq$ for which it is not possible to formulate a schema in the language that characterises the class of structures with that property. These include convexity, freedom and disjunctivity (refer to Section 2.3.1). Van Benthem partially solves this problem by using axioms, or sometimes axiom pairs, to imply, in an indirect way, some of the required properties. This is done, for example, for *convexity* of the time structure, i.e. the notion that stretches of time are uninterrupted; there are no gaps [VB84]. We say that a structure is convex in the sense that $<$ and $\subseteq$ are such that

$\forall x, y, z((x < y \wedge y < z) \rightarrow \forall u((x \subseteq u \wedge z \subseteq u) \rightarrow y \subseteq u))$. Van Benthem formulates the following pair of axioms to imply a form of convexity:

ITL.10 $\diamond \langle F \rangle (\Box\varphi \wedge \psi) \rightarrow (\langle F \rangle \varphi \vee \diamond\psi)$, which can be expressed as $\forall i, j((j \subseteq i) \rightarrow \forall k((j < k) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (i < l)))))$, and

ITL.11 $\diamond \langle P \rangle (\Box\varphi \wedge \psi) \rightarrow (\langle P \rangle \varphi \vee \diamond\psi)$, which can be expressed as $\forall i, j((j \subseteq i) \rightarrow \forall k((k < j) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (l < i)))))$.

The combination of the two schemas imply convexity of the structure in an indirect way: by formulating a pair of related axioms, we can enforce a form of convexity on the relationships among intervals. This is shown in the next proposition.

Proposition 6 *1. Let $\mathcal{K}$ be the class of structures $\mathcal{F} = (I, <, \subseteq)$ where $<$ and $\subseteq$ are such that*
$\forall i, j[(j \subseteq i) \rightarrow \forall k((j < k) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (i < l))))]$ *and*
$\forall i, j[(j \subseteq i) \rightarrow \forall k((k < j) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (l < i))))]$.
Then the frame $C_{\mathcal{K}}$ of $\mathcal{K}$ globally satisfies all instances of both axiom schema ITL.10 *and* ITL.11.
2. If $\mathcal{K}$ is a class of structures $\mathcal{F} = (I, <, \subseteq)$ such that the frame $C_{\mathcal{K}}$ of $\mathcal{K}$ globally satisfies all instances of ITL.10 *and* ITL.11, *then for every $\mathcal{F} = (I, <, \subseteq) \in \mathcal{K}$, $<$ and $\subseteq$ satisfy*
$\forall i, j[(j \subseteq i) \rightarrow \forall k((j < k) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (i < l))))]$ *and*
$\forall i, j[(j \subseteq i) \rightarrow \forall k((k < j) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (l < i))))]$.

1*a*. Let $\mathcal{F}_0 = (I, <, \subseteq) \in \mathcal{K}$ be a structure in which $<$ and $\subseteq$ are such that

$$\forall i, j((j \subseteq i) \rightarrow \forall k((j < k) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (i < l))))).$$

We have to show that all instances of *ITL.10* are globally satisfied in $C_{\mathcal{K}}$.

Let $\mathcal{M}_0$ be an arbitrary model based on $\mathcal{F}_0$ and pick an arbitrary $i' \in I$.

Suppose $\mathcal{M}_0 \Vdash_{i'} \diamond \langle F \rangle (\Box\varphi \wedge \psi)$, i.e. there exists a j' such that $j' \subseteq i'$ and $\mathcal{M}_0 \Vdash_{j'} \langle F \rangle (\Box\varphi \wedge \psi)$. This means that there exists a k' such that $j' < k'$ and $\mathcal{M}_0 \Vdash_{k'} (\Box\varphi \wedge \psi)$, i.e. $\mathcal{M}_0 \Vdash_{k'} \psi$ and, for all l' such that $l' \subseteq k'$, we have $\mathcal{M}_0 \Vdash_{l'} \varphi$. (Figure 4.1 gives a graphical representation of the different possibilities.) We have to show that $\mathcal{M}_0 \Vdash_{i'} (\langle F \rangle \varphi \vee \diamond\psi)$.

Suppose $\mathcal{M}_0 \nVdash_{i'} (\langle F \rangle \varphi \vee \diamond\psi)$, i.e. $\mathcal{M}_0 \nVdash_{i'} \langle F \rangle \varphi$ and $\mathcal{M}_0 \nVdash_{i'} \diamond\psi$. So it is not the case that there exists an m such that $i' < m$ and $\mathcal{M}_0 \Vdash_m \varphi$, and it is also not the case that there exists an n such that $n \subseteq i'$ and $\mathcal{M}_0 \Vdash_n \psi$.

We know that $<$ and $\subseteq$ are such that

$$\forall i, j((j \subseteq i) \rightarrow \forall k((j < k) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (i < l))))).$$

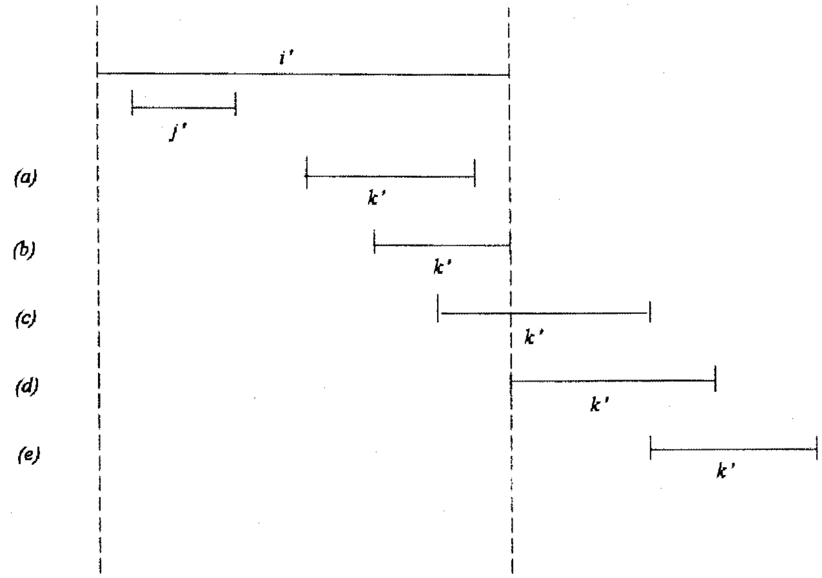

Figure 4.1: *A number of possible relationships exists between i' and k', where $j' \subseteq i'$ and $j' < k'$: (a) Interval k' ends before i' ends, (b) k' ends at the same time as i', (c) k' overlaps i' at the end, (d) k' starts where i' ends, and (e) k' starts after i' ends.*

Now $j' \subseteq i'$, so, for all k such that $j' < k$, we have $k \subseteq i'$, or there exists an l such that $l \subseteq k$ and $i' < l$. In particular, $k' \subseteq i'$, or there exists an l such that $l \subseteq k'$ and $i' < l$.

Suppose $k' \subseteq i'$. We know that $\mathcal{M}_0 \Vdash_{k'} \psi$, so there exists an n such that $n \subseteq i'$ and $\mathcal{M}_0 \Vdash_n \psi$, i.e. $\mathcal{M}_0 \Vdash_{i'} \Diamond\psi$ which contradicts our earlier assumption.

Suppose $k' \not\subseteq i'$. Then there exists an l such that $l \subseteq k'$ and $i' < l$. We know that $\mathcal{M}_0 \Vdash_{l'} \varphi$ for all $l' \subseteq k'$, so, in particular, $\mathcal{M}_0 \Vdash_l \varphi$. This means that there exists an m such that $i' < m$ and $\mathcal{M}_0 \Vdash_m \varphi$, i.e. $\mathcal{M}_0 \Vdash_{i'} \langle F \rangle \varphi$ which is also a contradiction. It follows that $\mathcal{M}_0 \Vdash_{i'} (\langle F \rangle \varphi \vee \Diamond\psi)$ and consequently, that all instances of *ITL.10* are globally satisfied in $\mathcal{C}_\mathcal{K}$.

1*b*. We can show in a similar way that, if $<$ and $\subseteq$ are such that

$$\forall i,j((j \subseteq i) \rightarrow \forall k((k < j) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (l < i))))),$$

then all instances of *ITL.11* are globally satisfied in $\mathcal{C}_\mathcal{K}$.

2*a*. Suppose the frame $\mathcal{C}_\mathcal{K}$ of $\mathcal{K}$ globally satisfies *ITL.10*. We have to show that, for every $\mathcal{F} = (I, <, \subseteq) \in \mathcal{K}$, $<$ and $\subseteq$ are such that

$$\forall i,j[(j \subseteq i) \rightarrow \forall k((j < k) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (i < l))))].$$

Pick an arbitrary $\mathcal{F}_0 \in \mathcal{K}$ and suppose there exists an $i_0, i_1, i_2 \in I$ such that $i_0 \subseteq i_1$ and $i_0 < i_2$, but $i_2 \not\subseteq i_1$ and there does not exist a k such that $k \subseteq i_2$ and $i_1 < k$. Let $\mathcal{M}_0$ be the model such that $V(p) = \{i | i \subseteq i_2\}$ for some $p \in \Phi$, and $V(q) = \{i_2\}$ for some $q \in \Phi$.

This means that $\mathcal{M}_0 \Vdash_{i_2} \Box p$ for some $p \in \Phi$, and $\mathcal{M}_0 \Vdash_{i_2} q$ for some $q \in \Phi$, i.e. $\mathcal{M}_0 \Vdash_{i_2} \Box p \wedge q$. Since $i_0 < i_2$, we have $\mathcal{M}_0 \Vdash_{i_0} \langle F \rangle (\Box p \wedge q)$. It also holds that $i_0 \subseteq i_1$, so $\mathcal{M}_0 \Vdash_{i_1} \Diamond \langle F \rangle (\Box p \wedge q)$.

From *ITL.10* it follows that $\mathcal{M}_0 \Vdash_{i_1} (\langle F \rangle p \vee \Diamond q)$, i.e. $\mathcal{M}_0 \Vdash_{i_1} \langle F \rangle p$ or $\mathcal{M}_0 \Vdash_{i_1} \Diamond q$. So there exists a $k \in I$ such that $i_1 < k$ and $\mathcal{M}_0 \Vdash_k p$, (so $k \subseteq i_2$ from our choice of $V(p)$), or there exists a $k \subseteq i_1$ such that $\mathcal{M}_0 \Vdash_k q$, (so $k = i_2$ from our choice of $V(q)$). It follows that either $i_2 \subseteq i_1$, or there exists a k such that $i_1 < k$ and $k \subseteq i_2$. This contradicts our earlier assumption. It follows that, for all $\mathcal{F} = (I, <, \subseteq) \in \mathcal{K}$, $<$ and $\subseteq$ are such that $\forall i, j[(j \subseteq i) \rightarrow \forall k((j < k) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (i < l))))]$.

2*b*. It can be shown in a similar way that, if all instances of *ITL.11* (which is the dual of *ITL.10* for the converse $>$ of $<$) are globally satisfied in the frame $\mathcal{C}_{\mathcal{K}}$ of $\mathcal{K}$, then for all $\mathcal{F} = (I, <, \subseteq) \in \mathcal{K}$, $<$ and $\subseteq$ are such that

$$\forall i, j[(j \subseteq i) \rightarrow \forall k((k < j) \rightarrow ((k \subseteq i) \vee \exists l((l \subseteq k) \wedge (l < i))))].$$

Van Benthem uses the axiom pair *ITL.12* and *ITL.13* for indirectly introducing the property of ***neighbourhood***, i.e. every interval that has an interval preceding it, also has a nearest neighbouring interval preceding it, and every interval that has an interval succeeding it, also a nearest neighbouring interval succeeding it. Thus $<$ is such that

$\forall x, y(x < y \rightarrow \exists z(x < z \wedge \forall u \neg(x < u \wedge u < z)))$ and
$\forall x, y(y < x \rightarrow \exists z(z < x \wedge \forall u \neg(z < u \wedge u < x)))$.

ITL.12 $(\varphi \wedge \Box\psi) \rightarrow \langle F \rangle [P](\langle F \rangle \varphi \vee \Diamond\psi)$
ITL.13 $(\varphi \wedge \Box\psi) \rightarrow \langle P \rangle [F](\langle P \rangle \varphi \vee \Diamond\psi)$

Directedness stipulates that, for every pair of intervals x and y, there exists an interval containing both x and y. This can be expressed in the metalanguage as $\forall x, y \exists z(x \subseteq z \wedge y \subseteq z)$. A form of directedness can be introduced by the following axiom pair:

ITL.14 $(\langle F \rangle \varphi \wedge \langle F \rangle \psi) \rightarrow \langle F \rangle (\Diamond\varphi \wedge \Diamond\psi)$
ITL.15 $(\langle P \rangle \varphi \wedge \langle P \rangle \psi) \rightarrow \langle P \rangle (\Diamond\varphi \wedge \Diamond\psi)$

It could be preferable for the structure to be dense, i.e. between any pair of intervals that does not overlap, we can fit in another one, i.e. $\forall x, y(x < y \rightarrow \exists z(x < z \wedge z < y))$. The following schema can be used to formulate a form of *density* indirectly:

ITL.16 $\Box\varphi \rightarrow \Diamond \langle F \rangle \varphi$

If the interval inclusion relation $\subseteq$ is transitive, then the *McKinsey schema*, i.e. $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$, is a formulation of the property of *atomicity*, i.e. the existence of atomic intervals. This means that $\subseteq$ is such that

$$\forall i \exists j (j \subseteq i \wedge \forall k (k \subseteq j \rightarrow k = j)).$$

Linearity of $<$ modulo $\subseteq$ can be introduced by the inclusion of a pair of axioms defining *left-linearity* and *right-linearity* respectively.

ITL.17 $(\langle F \rangle (\varphi \wedge \Box\psi) \wedge \langle F \rangle (\chi \wedge \Box\alpha)) \rightarrow (\langle F \rangle (\psi \wedge \alpha) \vee \langle F \rangle (\varphi \wedge \langle F \rangle \chi) \vee \langle F \rangle (\chi \wedge \langle F \rangle \varphi))$
i.e. $\forall i, j, k((i < j \wedge i < k) \rightarrow (k < j \vee j < k \vee \exists l(i < l \wedge (l \subseteq j \wedge l \subseteq k))))$

ITL.18 $(\langle P \rangle (\varphi \wedge \Box\psi) \wedge \langle P \rangle (\chi \wedge \Box\alpha)) \rightarrow (\langle P \rangle (\psi \wedge \alpha) \vee \langle P \rangle (\varphi \wedge \langle P \rangle \chi) \vee \langle P \rangle (\chi \wedge \langle P \rangle \varphi))$
i.e. $\forall i, j, k((j < i \wedge k < i) \rightarrow (k < j \vee j < k \vee \exists l(l < i \wedge (l \subseteq j \wedge l \subseteq k))))$.

4.2.2 Axiomatisation for superperiods

It may also be convenient to express the relationship between the truth of a sentence φ over an interval i and the truth of φ over an interval j of which i is a subinterval. The interval j is called a *superperiod* of i. In [VB95],Van Benthem discusses the use of a modal operator $\Box^{up}$, read 'over all superperiods'. Hajnicz [Haj96] proposes the extension of the alphabet for extended tense logic by adding such a modal operator, namely $\boxtimes$, corresponding to the accessibility relation $\supseteq$ which is the converse of $\subseteq$, so $(i, j) \in \supseteq$ iff $(j, i) \in \subseteq$. This gives a language based on this alphabet greater expressive power [Haj96]. The dual of $\boxtimes$, which we denote by $\otimes$, read 'for some superperiod', is defined as usual:

$$\otimes\varphi \equiv_{def} \neg \boxtimes \neg\varphi.$$

What about the truth value of sentences of the form $\boxtimes\psi$? If φ is a sentence of the form $\boxtimes\psi$, we say that φ is true relative to the model $\mathcal{M} = (I, <, \subseteq, V)$ over the interval i, denoted by $\mathcal{M} \Vdash_i \boxtimes\psi$, iff $\mathcal{M} \Vdash_j \psi$ for every $j \in I$ such that $i \subseteq j$.

The following axiom pair expresses the notion that $\subseteq$ and $\supseteq$ are mutually converse:

SPTL.01 $\varphi \rightarrow \Box \otimes \varphi$

SPTL.02 $\varphi \rightarrow \boxtimes\Diamond\varphi$.

With the addition of $\boxtimes$, some axioms are formulated in a way slightly different from the way they were formulated in the previous section. As an illustration, we give the axioms for left-linearity and right-linearity [Haj96]:

SPTL.03 $\langle P \rangle \varphi \rightarrow [P](\langle P \rangle \varphi \vee \Diamond \otimes \varphi \vee \langle F \rangle \varphi)$ (Left-linearity)

SPTL.04 $\langle F \rangle \varphi \rightarrow [F](\langle P \rangle \varphi \vee \Diamond \otimes \varphi \vee \langle F \rangle \varphi)$ (Right-linearity)

4.3 A modal language using Allen's interval-interval relations

There are areas in AI, such as qualitative physics and automatic planning, where we need to represent continuous change over a period of time. Halpern & Shoham [HS91] introduce a modal logic suitable for applications like these. A time structure is used with time points as primitive and, although the modal operators describe relationships between intervals, the semantics is formulated in terms of relationships between time points.

The alphabet of the language consists of the following elements:

- a set Φ containing one or more atoms $p, q, \ldots$
- the punctuation symbols '(' and ')'
- the connectives $\rightarrow$ and $\neg$
- the modal operators $\langle A \rangle$, read 'for some abutting interval', $\langle B \rangle$, read 'for some initial interval', $\langle E \rangle$, read 'for some terminal interval', $\langle \bar{A} \rangle$, read 'for some interval of which the current one is an abutting interval', $\langle \bar{B} \rangle$, read 'for some interval of which the current one is an initial interval', and $\langle \bar{E} \rangle$, read 'for some interval of which the current one is a terminal interval'.

The modal operators correspond to some of the interval-interval relations defined by Allen as discussed in Chapter 2 (see Figure 4.2). When considering the truth of a sentence over an interval $\langle t, t' \rangle$, we say that $\langle t, t' \rangle$ is the *current interval* [1]. An *abutting interval* is an interval that starts at the same time point at which the current one ends. By *initial interval* we mean an interval beginning at the same time as the current one but ending before it, and by *terminal interval* we mean an interval beginning after the current one but ending at the same time.

The set of sentences is defined in the usual way and the connectives $\vee$, $\wedge$ and $\leftrightarrow$ are defined in terms of $\rightarrow$ and $\neg$ as before.

Both $\langle B \rangle$ and $\langle E \rangle$ are defined for strict subintervals only, i.e. an interval cannot be a beginning or end subinterval of itself. The duals $[A]$, read 'for all abutting intervals', $[B]$, read 'for all initial intervals', $[E]$, read 'for all terminal intervals', $[\bar{A}]$, read 'for all intervals of which the current one is an abutting interval', $[\bar{B}]$, read 'for all intervals of which the

[1]This concept is sometimes referred to as *temporal indexicality*.

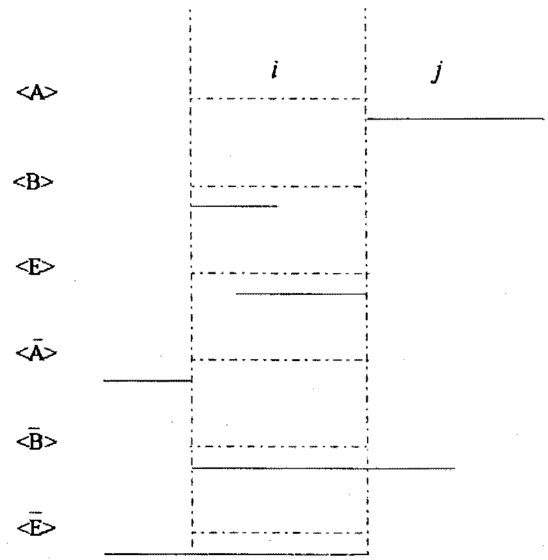

Figure 4.2: Halpern & Shoham's interval-interval relations

current one is an initial interval', and $[\bar{E}]$, read 'for all intervals of which the current one is a terminal interval', are defined in the usual way, e.g. $[A]\varphi \equiv_{def} \neg \langle A \rangle \neg\varphi$.

For the purpose of Halpern & Shoham's system, a temporal structure is taken to be a pair $\mathcal{F} = (T, \leq)$, where T is a set the elements of which are called time points and $\leq$ is an accessibility relation on T. An interval, denoted by $\langle t_1, t_2 \rangle$, is defined as a set of points $\{t : t_1 \leq t \leq t_2\}$. We say that $t_1 < t_2$ iff $t_1 \leq t_2$ and $t_2 \not\leq t_1$.

Sentences are interpreted over intervals. Let $\mathcal{M} = (T, \leq, V)$ be an arbitrary model based on $\mathcal{F}$, where V is a function that associates each $p \in \Phi$ with the set of intervals over which it is true. A sentence φ is said to be true relative to $\mathcal{M}$ over an interval $\langle t, t' \rangle$, denoted by $\mathcal{M} \Vdash_{\langle t,t' \rangle} \varphi$, iff one of the following cases applies:

- $\varphi \in \Phi$ and $\langle t, t' \rangle \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_{\langle t,t' \rangle} \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_{\langle t,t' \rangle} \psi$ or $\mathcal{M} \Vdash_{\langle t,t' \rangle} \chi$
- $\varphi = \langle A \rangle \psi$ and there exists a t'' such that $t' < t''$ and $\mathcal{M} \Vdash_{\langle t',t'' \rangle} \psi$
- $\varphi = \langle B \rangle \psi$ and there exists a t'' such that $t \leq t''$ and $t'' < t'$ and $\mathcal{M} \Vdash_{\langle t,t'' \rangle} \psi$
- $\varphi = \langle E \rangle \psi$ and there exists a t'' such that $t < t''$ and $t'' \leq t'$ and $\mathcal{M} \Vdash_{\langle t'',t' \rangle} \psi$

- $\varphi = \langle \bar{A} \rangle \psi$ and there exists a t'' such that $t'' < t$ and $\mathcal{M} \Vdash_{\langle t'',t \rangle} \psi$
- $\varphi = \langle \bar{B} \rangle \psi$ and there exists a t'' such that $t' < t''$ and $\mathcal{M} \Vdash_{\langle t,t'' \rangle} \psi$
- $\varphi = \langle \bar{E} \rangle \psi$ and there exists a t'' such that $t'' < t$ and $\mathcal{M} \Vdash_{\langle t'',t' \rangle} \psi$

A sentence φ is said to be globally true (or globally satisfied) over a model $\mathcal{M}$ iff $\mathcal{M} \Vdash_{\langle t,t' \rangle} \varphi$ for all intervals $\langle t, t' \rangle$ such that $t, t' \in T$. Global truth of a sentence over a frame is defined as before.

The system allows *point intervals* $\langle t, t \rangle$. A point interval cannot have any subintervals, i.e. for any point interval $\langle s, s \rangle$, there does not exist a time point t between s and s to form a subinterval $\langle s, t \rangle$ or $\langle t, s \rangle$. For point intervals of the form $\langle s, s \rangle$, $[B]\varphi$ will therefore be vacuously true. This means that $[B] \perp$ holds for point intervals only. This principle is used to define the following pair of modal operators, one for the *beginning point* and another for the *end point* of an interval:

$$[[BP]]\varphi \equiv_{def} (\varphi \wedge [B] \perp) \vee \langle B \rangle (\varphi \wedge [B] \perp)$$
$$[[EP]]\varphi \equiv_{def} (\varphi \wedge [B] \perp) \vee \langle E \rangle (\varphi \wedge [B] \perp)$$

where $[[BP]]\,\varphi$ expresses the notion that φ holds at the beginning point of the current interval, and $[[EP]]\,\varphi$ expresses the notion that φ holds at the end point of the current interval. The definition of the beginning point operator stipulates that φ is true at the beginning point of the current interval iff it either is the case that (φ is true over the current interval which has no beginning subintervals, i.e. the current interval itself is a beginning point) or (there exists a subinterval, which is a beginning point of the current interval, at which φ is true). The modal operators $[[BP]]$ and $[[EP]]$ are duals of themselves, i.e. $[[BP]]\varphi \equiv_{def} \neg[[BP]]\neg\varphi$ and $[[EP]]\varphi \equiv_{def} \neg[[EP]]\neg\varphi$: for an arbitrary interval there exists only one beginning point and one end point.

A number of more complex modal operators can be defined in terms of those primitive in the language. With the addition of these operators, we have a one-to-one correspondence between the set of modal operators for this language and Allen's set of interval-interval relations (excluding equality) as discussed in Chapter 2. Figure 4.3 gives a graphical representation of the new operators. The dotted line shows the current interval in each case.

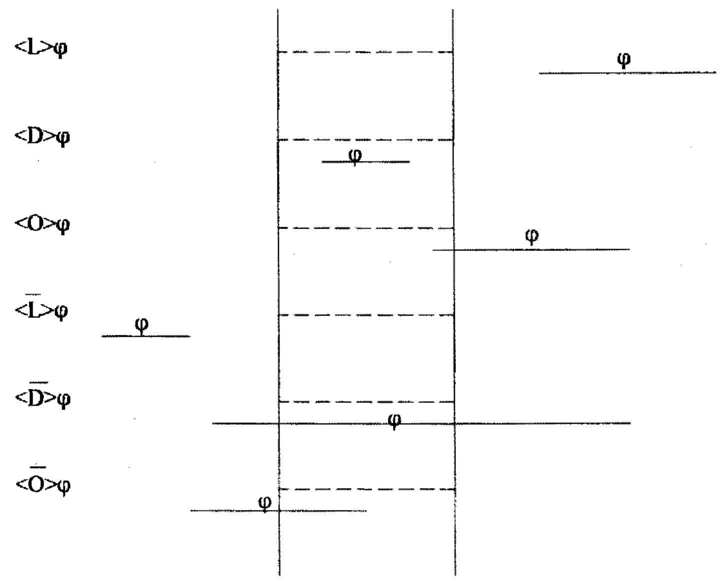

Figure 4.3: A graphical representation of the more complex modal operators defined by Halpern and Shoham.

These operators are defined as follows: For a sentence φ,

$\langle L\rangle\varphi$	$\equiv_{def}$	$\langle A\rangle\langle A\rangle\varphi$	There exists some later interval, i.e. an interval beginning after the current one ends (not abutting it) over which φ holds.
$\langle D\rangle\varphi$	$\equiv_{def}$	$\langle B\rangle\langle E\rangle\varphi$	There exists an interval during the current one (starting later than but ending before the current one ends) over which φ holds.
$\langle O\rangle\varphi$	$\equiv_{def}$	$\langle E\rangle\langle\bar{B}\rangle\varphi$	There exists an interval that overlaps with the current one at the end (beginning after and ending after the current one) over which φ holds.
$\langle\bar{L}\rangle\varphi$	$\equiv_{def}$	$\langle\bar{A}\rangle\langle\bar{A}\rangle\varphi$	There exists an interval earlier than the current one (not abutting it) over which φ holds.
$\langle\bar{D}\rangle\varphi$	$\equiv_{def}$	$\langle\bar{B}\rangle\langle\bar{E}\rangle\varphi$	There exists an interval that overlaps with the current one at both ends (beginning before and ending after the current one) over which φ holds.
$\langle\bar{O}\rangle\varphi$	$\equiv_{def}$	$\langle B\rangle\langle\bar{E}\rangle\varphi$	There exists an interval that overlaps with the current one at the beginning (beginning earlier and ending earlier than the current one) over which φ holds.

Do these modal operators we have defined in terms of the primitive ones really achieve their intended meaning by these definitions? Let us have a closer look at $\langle D \rangle$, using the given semantics, as an example.

Example 17 *The 'meaning' of* $\langle D \rangle$.

Let $\mathcal{M}_0 = (T, \leq, V)$ be an arbitrary model and let $\langle t, t' \rangle$ be an arbitrary interval, with $t, t' \in T$. Suppose $\mathcal{M}_0 \Vdash_{\langle t,t' \rangle} \langle D \rangle \varphi$. We want to show that this means that there exists an interval beginning later than the current interval, and ending before it, over which φ holds.

If $\mathcal{M}_0 \Vdash_{\langle t,t' \rangle} \langle D \rangle \varphi$ then $\mathcal{M}_0 \Vdash_{\langle t,t' \rangle} \langle B \rangle \langle E \rangle \varphi$, by definition. This means that there exists a $t'' \in T$ such that $t \leq t''$ and $t'' < t'$, and $\mathcal{M}_0 \Vdash_{\langle t,t'' \rangle} \langle E \rangle \varphi$, i.e. there exists a $t''' \in T$ such that $t < t'''$ and $t''' \leq t''$, and $\mathcal{M}_0 \Vdash_{\langle t''',t'' \rangle} \varphi$. So there exists a $t'', t''' \in T$ such that $t < t'''$ and $t'' < t'$, and $\mathcal{M}_0 \Vdash_{\langle t''',t'' \rangle} \varphi$. In other words, there exists an interval beginning later than the current interval and ending before it, over which φ holds.

Halpern & Shoham [HS91] define the following formulae which can be used in order to express properties like discreteness, denseness and linearity of the time structure in the object language:

length0	$\equiv_{def}$	$[B] \bot$
		(Point intervals have length = 0.)
length1	$\equiv_{def}$	$\langle B \rangle \top \wedge [B][B] \bot$
		(If an interval has length = 1, then that interval contains a beginning interval, and all beginning intervals are point intervals. So it only contains a beginning point.)

The accessibility relation $\leq$ in a time structure $\mathcal{F} = (T, \leq)$ is *discrete* if it is such that

$\forall x, y(x < y \rightarrow \exists z(x < z \wedge z \leq y \wedge \neg \exists u(x < u \wedge u < z)))$ and
$\forall x, y(y < x \rightarrow \exists z(y \leq z \wedge z < x \wedge \neg \exists u(z < u \wedge u < x)))$.

This property is introduced by Halpern & Shoham by means of the following definition:

discrete $\equiv_{def}$ length0 $\vee$ length1 $\vee$ ($\langle B \rangle$length1 $\wedge$ $\langle E \rangle$length1).

How does this correspond to discreteness? Suppose $\mathcal{M} \Vdash$ discrete. Then $\mathcal{M} \Vdash_{\langle s,t \rangle}$ discrete for every interval $\langle s, t \rangle$. Now exactly one of the following three possible conditions holds:

1. $\mathcal{M}_0 \Vdash_{\langle s,t \rangle}$length0: This means that $\langle s, t \rangle$ is a point interval, so $s \not< t$ $(s = t)$ and thus there does not exist any time point $r \in T$ such that $s < r < t$.

2. $\mathcal{M}_0 \Vdash_{\langle s,t\rangle}$length1: The only subinterval that $\langle s,t\rangle$ can contain is a beginning point interval $\langle s,s\rangle$. This means that the only subintervals that $\langle s,t\rangle$ can contain are $\langle s,s\rangle$ and $\langle t,t\rangle$. So there does not exist an $r \in T$ such that $s < r$ and $r < t$. (We will see later (axiom *VEN.07*) that, if an interval contains a beginning point, then it also contains an end point.)

3. $\mathcal{M}_0 \Vdash_{\langle s,t\rangle} \langle B\rangle$length1 $\wedge$ $\langle E\rangle$length1. Within $\langle s,t\rangle$, there exists a beginning interval with length1, i.e. an interval $\langle s,r\rangle$ such that $s < r < t$, but for which there does not exist an r' such that $s < r' < r$. It is also the case that there exists an end interval with length1, i.e. an interval $\langle u,t\rangle$ such that $s < u < t$, but for which there does not exist a u' such that $u < u' < t$.

It follows that discreteness of $\leq$ is ensured.

In contrast to this, *density* of $\leq$, i.e. $\forall x, y(x < y \rightarrow \exists z(x < z \wedge z < y))$, can be defined as follows:

dense $\equiv_{def} \neg$ length1.

From the definition of length1, it follows that, if an interval $\langle s,t\rangle$ is of length1, then $s < t$ and there does not exist any point r between s and t. If density is defined as above, it means that there does not exist an interval $\langle s,t\rangle$ of which we can say that there does not exist a point r between s and t. So, for every interval $\langle s,t\rangle$, it follows that there exists an r, such that $s < r < t$.

The accessibility relation $\leq$ is *unbounded* (infinite) if $\forall x(\exists y(x < y) \wedge \exists z(z < x))$. Halpern & Shoham define this property in terms of abutting intervals:

unbounded $\equiv_{def} \langle A\rangle \top \wedge \langle \bar{A}\rangle \top$

This means that, for an arbitrary interval, it is always the case that there exists an interval of which the current interval is an abutting interval, and there exists an abutting interval (succeeding the current one). The time structure is unbounded in both directions. *Right-unboundedness* and *left-unboundedness* can alternatively be defined as follows:

Right-unbounded $\equiv_{def} \langle A\rangle \top$
Left-unbounded $\equiv_{def} \langle \bar{A}\rangle \top$

Finally, the property of *linearity* of $\leq$ is introduced as follows:

linear $\equiv_{def} ((\langle A\rangle \varphi \rightarrow [A](\varphi \vee \langle B\rangle \varphi \vee \langle \bar{B}\rangle \varphi)) \wedge (\langle \bar{A}\rangle \varphi \rightarrow [\bar{A}](\varphi \vee \langle E\rangle \varphi \vee \langle \bar{E}\rangle \varphi))$

We want to ascertain whether this definition characterises the class $\mathcal{K}$ of quasi-linear structures, i.e. structures in which the accessibility relation $\leq$ is both right-linear and left-linear:

$\forall x, y, z(x < y \wedge x < z \rightarrow (y < z \vee y = z \vee z < y))$ and
$\forall x, y, z(y < x \wedge z < x \rightarrow (y < z \vee y = z \vee z < y))$.

Proposition 7 *1. Let $\mathcal{K}$ be the class of time structures of the form $\mathcal{F} = (T, \leq)$ such that $\leq$ is both right-linear and left-linear in every $\mathcal{F} \in \mathcal{K}$ and let $\mathcal{C}_{\mathcal{K}}$ be the frame of $\mathcal{K}$. Then every $\mathcal{M} \in \mathcal{C}_{\mathcal{K}}$ globally satisfies all instances of the axiom schema*
$(\langle A \rangle \varphi \rightarrow [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)) \wedge (\langle \bar{A} \rangle \varphi \rightarrow [\bar{A}] (\varphi \vee \langle E \rangle \varphi \vee \langle \bar{E} \rangle \varphi))$.
2. If the frame $\mathcal{C}_{\mathcal{K}}$ of the class $\mathcal{K}$ of structures globally satisfies the axiom schema
$(\langle A \rangle \varphi \rightarrow [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)) \wedge (\langle \bar{A} \rangle \varphi \rightarrow [\bar{A}] (\varphi \vee \langle E \rangle \varphi \vee \langle \bar{E} \rangle \varphi))$
then, for every $\mathcal{F} = (T, \leq) \in \mathcal{K}$, $\leq$ is both right-linear and left-linear.

1*a*. Let $\mathcal{K}$ be the class of structures of the form $\mathcal{F} = (T, \leq)$ such that $\leq$ is right-linear. Pick an arbitrary $\mathcal{F}_0 = (T, \leq) \in \mathcal{K}$ and an arbitrary $\mathcal{M}_0 \in \mathcal{C}_{\mathcal{F}_0}$. We have to show that $\mathcal{M}_0 \Vdash_{\langle t,t' \rangle} \langle A \rangle \varphi \rightarrow [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)$ for all $\langle t, t' \rangle$ such that $t, t' \in T$.

Pick an arbitrary $\langle t_0, t_1 \rangle$ with $t_0, t_1 \in T$. Suppose $\mathcal{M}_0 \nVdash_{\langle t_0,t_1 \rangle} \langle A \rangle \varphi \rightarrow [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)$, i.e. if there exists a $t_2 \in T$ such that $t_1 < t_2$ and $\mathcal{M}_0 \Vdash_{\langle t_1,t_2 \rangle} \varphi$, then $\mathcal{M}_0 \nVdash_{\langle t_0,t_1 \rangle} [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)$. So there exists a $t_3 \in T$ such that $t_1 < t_3$ and ($\mathcal{M}_0 \nVdash_{\langle t_1,t_3 \rangle} \varphi$ and $\mathcal{M}_0 \nVdash_{\langle t_1,t_3 \rangle} \langle B \rangle \varphi$ and $\mathcal{M}_0 \nVdash_{\langle t_1,t_3 \rangle} \langle \bar{B} \rangle \varphi$). But $\leq$ is right-linear and we have $t_1 < t_2$ and $t_1 < t_3$, so $t_2 < t_3$ or $t_2 = t_3$ or $t_3 < t_2$. If $t_2 = t_3$ then $\mathcal{M}_0 \Vdash_{\langle t_1,t_3 \rangle} \varphi$: a contradiction. We know that $\mathcal{M}_0 \Vdash_{\langle t_1,t_2 \rangle} \varphi$, so if $t_2 < t_3$, then $\mathcal{M}_0 \Vdash_{\langle t_1,t_3 \rangle} \langle B \rangle \varphi$, also a contradiction. Furthermore, if $t_3 < t_2$, then $\mathcal{M}_0 \Vdash_{\langle t_1,t_3 \rangle} \langle \bar{B} \rangle \varphi$. This also contradicts our earlier assumption, so $\mathcal{M}_0 \Vdash_{\langle t_0,t_1 \rangle} [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)$. Thus $\mathcal{M}_0 \Vdash_{\langle t_0,t_1 \rangle} (\langle A \rangle \varphi \rightarrow [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi))$.

1*b*. We can show in a similar fashion that, if $\leq$ is left-linear then $\mathcal{M}_0 \Vdash_{\langle t_0,t_1 \rangle} \langle \bar{A} \rangle \varphi \rightarrow [\bar{A}] (\varphi \vee \langle E \rangle \varphi \vee \langle \bar{E} \rangle \varphi)$.

It follows that, if $\leq$ is both right-linear and left-linear, then $\mathcal{M}_0 \Vdash_{\langle t,t' \rangle} (\langle A \rangle \varphi \rightarrow [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)) \wedge (\langle \bar{A} \rangle \varphi \rightarrow [\bar{A}] (\varphi \vee \langle E \rangle \varphi \vee \langle \bar{E} \rangle \varphi))$ for all intervals $\langle t, t' \rangle$, where $t, t' \in T$.

2*a*. Suppose all instances of the schema $(\langle A \rangle \varphi \rightarrow [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)) \wedge (\langle \bar{A} \rangle \varphi \rightarrow [\bar{A}] (\varphi \vee \langle E \rangle \varphi \vee \langle \bar{E} \rangle \varphi))$ are globally satisfied in the frame $\mathcal{C}_{\mathcal{K}}$ of the class $\mathcal{K}$ of structures of the form $\mathcal{F} = (T, \leq)$. We have to show that, for every $\mathcal{F} \in \mathcal{K}$, $\leq$ is both right-linear and left-linear.

Firstly, we show that, if all instances of $\langle A \rangle \varphi \rightarrow [A] (\varphi \vee \langle B \rangle \varphi \vee \langle \bar{B} \rangle \varphi)$ are globally satisfied in the frame $\mathcal{C}_{\mathcal{K}}$, then, for every $\mathcal{F} \in \mathcal{K}$, $\leq$ is right-linear.

Pick an arbitrary $\mathcal{F}_0 = (T, \leq) \in \mathcal{K}$ and suppose we have $t_0, t_1, t_2, t_3 \in T$ such that $t_0 < t_1$, $t_1 < t_2$ and $t_1 < t_3$, but $t_2 \neq t_3$ and $t_2 \not< t_3$ and $t_3 \not< t_2$, i.e. $\leq$ is not right-linear. Let $\mathcal{M}_0 = (T, \leq, V)$ be the model such that $V(p) = \{\langle t_1, t_2 \rangle\}$ for all $p \in \Phi$. We know that $\mathcal{M}_0 \Vdash_{\langle t_1,t_2 \rangle} p$ for arbitrary p, so $\mathcal{M}_0 \Vdash_{\langle t_0,t_1 \rangle} \langle A \rangle p$.

From the schema it follows that $\mathcal{M}_0 \Vdash_{\langle t_0,t_1\rangle} [A]\,(p \vee \langle B\rangle\, p \vee \langle \bar{B}\rangle\, p)$. So for an arbitrary $t' \in T$ such that $t_1 < t'$, it follows that $\mathcal{M}_0 \Vdash_{\langle t_1,t'\rangle} p$ or $\mathcal{M}_0 \Vdash_{\langle t_1,t'\rangle} \langle B\rangle\, p$ or $\mathcal{M}_0 \Vdash_{\langle t_1,t'\rangle} \langle \bar{B}\rangle\, p$. In particular, for $\langle t_1, t_3\rangle$, we have $\mathcal{M}_0 \Vdash_{\langle t_1,t_3\rangle} p$ or $\mathcal{M}_0 \Vdash_{\langle t_1,t_3\rangle} \langle B\rangle\, p$ or $\mathcal{M}_0 \Vdash_{\langle t_1,t_3\rangle} \langle \bar{B}\rangle\, p$. So $\mathcal{M}_0 \Vdash_{\langle t_1,t_3\rangle} p$ which means that $t_3 = t_2$ from our choice of V, or there exists a t'' such that $t_1 \le t'' < t_3$ and $\mathcal{M}_0 \Vdash_{\langle t_1,t''\rangle} p$, so $t'' = t_2$, or there exists a t''' such that $t_1 < t_3 < t'''$ and $\mathcal{M}_0 \Vdash_{\langle t_1,t'''\rangle} p$, so $t''' = t_2$. It follows that $t_2 = t_3$ or $t_2 < t_3$ or $t_3 < t_2$, which contradicts our earlier assumption. So, for every $\mathcal{F} \in \mathcal{K}$, $\le$ is right-linear.

2*b*. We can show in a similar way that, if $\langle \bar{A}\rangle\, \varphi \rightarrow [\bar{A}]\,(\varphi \vee \langle E\rangle\, \varphi \vee \langle \bar{E}\rangle\, \varphi)$ is globally satisfied in the frame $\mathcal{C}_{\mathcal{K}}$, then, for every $\mathcal{F} = (T, \le) \in \mathcal{K}$, $\le$ is left-linear.

It follows that, if all instances of the schema $((A)\, \varphi \rightarrow [A]\,(\varphi \vee \langle B\rangle\, \varphi \vee \langle \bar{B}\rangle\, \varphi)) \wedge (\langle \bar{A}\rangle\, \varphi \rightarrow [\bar{A}]\,(\varphi \vee \langle E\rangle\, \varphi \vee \langle \bar{E}\rangle\, \varphi))$ are globally satisfied in the frame $\mathcal{C}_{\mathcal{K}}$, then, for every $\mathcal{F} = (T, \le) \in \mathcal{K}$, $\le$ is both right-linear and left-linear.

Venema [Ven88] shows that the two modal operators $\langle A\rangle$ and $\langle \bar{A}\rangle$ can be defined in terms of other modal operators as follows:

$$\langle A\rangle\, \varphi \equiv_{def} [[EP]]\, \langle \bar{B}\rangle\, \varphi$$
$$\langle \bar{A}\rangle\, \varphi \equiv_{def} [[BP]]\, \langle \bar{E}\rangle\, \varphi$$

For an interval i to meet an interval j, they share one atomic interval which forms the end point of i and the beginning point of j. Since a point interval cannot meet and cannot be met by any other point interval, Hajnicz [Haj96] prefers the following alternative definitions:

$$[A]\varphi \equiv_{def} [E]\,([\bar{B}]\, \varphi \vee \langle E\rangle\, \top)$$
$$[\bar{A}]\, \varphi \equiv_{def} [B]\,([\bar{E}]\, \varphi \vee \langle B\rangle\, \top)$$

Proposition 8 *The Hajnicz definitions of* $[A]$ *and* $[\bar{A}]$ *given above correspond to those of Halpern & Shoham.*

(a) Hajnicz [Haj96] defines $[A]$ as follows:

$$[A]\varphi \equiv_{def} [E]\,([\bar{B}]\, \varphi \vee \langle E\rangle\, \top)$$

We have to show that this corresponds to Halpern & Shoham's definition of $[A]$.

Consider an arbitrary model $\mathcal{M}_0 = (T, \le, V)$ and an arbitrary interval $\langle t_0, t_1\rangle$ where $t_0, t_1 \in T$. Suppose $\mathcal{M}_0 \Vdash_{\langle t_0,t_1\rangle} [A]\varphi$, i.e. $\mathcal{M}_0 \Vdash_{\langle t_0,t_1\rangle} [E]\,([\bar{B}]\, \varphi \vee \langle E\rangle\, \top)$. This means that, for all terminal intervals $\langle t', t_1\rangle$ of $\langle t_0, t_1\rangle$, it must be the case that either $\mathcal{M}_0 \Vdash_{\langle t',t_1\rangle} [\bar{B}]\, \varphi$ or

$\mathcal{M}_0 \Vdash_{\langle t',t_1\rangle} \langle E\rangle\top$. Choose an arbitrary $\langle t_3,t_1\rangle$, such that $\langle t_3,t_1\rangle$ is a terminal interval of $\langle t_0,t_1\rangle$.

Suppose $\mathcal{M}_0 \nVdash_{\langle t_3,t_1\rangle} \langle E\rangle\top$. This means that $\langle t_3,t_1\rangle$ is an end point, i.e. $\langle t_3,t_1\rangle = \langle t_1,t_1\rangle$. It is also the case that $\mathcal{M}_0 \Vdash_{\langle t_3,t_1\rangle} [\bar{B}]\,\varphi$, so $\mathcal{M}_0 \Vdash_{\langle t_1,t_1\rangle} [\bar{B}]\,\varphi$, i.e. $\mathcal{M}_0 \Vdash_{\langle t_1,t''\rangle} \varphi$ for all intervals $\langle t_1,t''\rangle$ where $t_1 < t''$. So $\mathcal{M}_0 \Vdash_{\langle t_0,t_1\rangle} [A]\,\varphi$ as per Halpern & Shoham's definition.

Now suppose $\mathcal{M}_0 \nVdash_{\langle t_3,t_1\rangle} [\bar{B}]\,\varphi$. Then $\mathcal{M}_0 \Vdash_{\langle t_3,t_1\rangle} \langle E\rangle\top$, which means that $\langle t_3,t_1\rangle$ contains another end interval and is thus not relevant.

(b) Hajnicz defines $[\bar{A}]$ as follows:

$$[\bar{A}]\,\varphi \equiv_{def} [B]\,([\bar{E}]\,\varphi \vee \langle B\rangle\top)$$

It can be shown in a similar way that $\mathcal{M}_0 \Vdash_{\langle t,t'\rangle} [B]\,([\bar{E}]\,\varphi \vee \langle B\rangle\top)$ means that, for all intervals $\langle t'',t\rangle$, we have $\mathcal{M}_0 \Vdash_{\langle t'',t\rangle} \varphi$, so $\mathcal{M}_0 \Vdash_{\langle t,t'\rangle} [\bar{A}]\,\varphi$ as per Halpern & Shoham's definition.

Halpern & Shoham do not give an axiomatisation for their system so we present one suggested by Venema [Ven88].

4.3.1 Axiomatisation

Venema ([Ven88] and [Ven90]) maintains that Halpern & Shoham's system can be interpreted using one of two possible models. Firstly, we can use a model based on the time structure discussed in the previous section, i.e. points are primitive but sentences are interpreted over intervals. Secondly we can use a model based on a time structure with intervals as primitive.

Points as primitive

The following axioms express relationships between the modal operators $\langle B\rangle$ and $\langle\bar{B}\rangle$ and between $\langle E\rangle$ and $\langle\bar{E}\rangle$:

VEN.01a $\varphi \rightarrow [B]\,\langle\bar{B}\rangle\,\varphi$

VEN.01b $\varphi \rightarrow [E]\,\langle\bar{E}\rangle\,\varphi$

VEN.02a $\varphi \rightarrow [\bar{B}]\,\langle B\rangle\,\varphi$

VEN.02b $\varphi \rightarrow [\bar{E}]\,\langle E\rangle\,\varphi$

VEN.01 and *VEN.02* are formulated to express the relationship between $\langle B \rangle$ and $\langle \bar{B} \rangle$, and between $\langle E \rangle$ and $\langle \bar{E} \rangle$ respectively. As an example, we show the relationship between $[B]$ and $\langle \bar{B} \rangle$ from *VEN.01a*. Suppose $\mathcal{M} \Vdash_{\langle t,t' \rangle} \varphi$. From *VEN.01a* this means that $\mathcal{M} \Vdash_{\langle t,t' \rangle} [B] \langle \bar{B} \rangle \varphi$. So $\mathcal{M} \Vdash_{\langle t,t'' \rangle} \langle \bar{B} \rangle \varphi$ for all t'' such that $t'' < t'$, from which it follows that there exists a $\tilde{t}$ (i.e. t') such that $t'' < \tilde{t}$ and $\mathcal{M} \Vdash_{\langle t,\tilde{t} \rangle} \varphi$.

We want to show that a similar relationship exists between $\langle A \rangle$ and $\langle \bar{A} \rangle$ (if we use the Hajnicz definitions of $[A]$ and $[\bar{A}]$ given previously).

Proposition 9 *Let $[\bar{A}]$ and $[A]$ be defined as by Hajnicz:*

$$[\bar{A}]\,\varphi \equiv_{def} [B]\,([\bar{E}]\,\varphi \vee \langle B \rangle \top)$$
$$[A]\varphi \equiv_{def} [E]\,([\bar{B}]\,\varphi \vee \langle E \rangle \top)$$

Let $\mathcal{K}$ be the class of time structures of the form $\mathcal{F} = (T, \leq)$ such that every $C_{\mathcal{F}} \in C_{\mathcal{K}}$ globally satisfies all instances of the schemas VEN.01 *and* VEN.02 *where $C_{\mathcal{K}}$ is the frame of $\mathcal{K}$. Then every $\mathcal{M} \in C_{\mathcal{K}}$ globally satisfies all instances of the schemas $\varphi \rightarrow [\bar{A}]\,\langle A \rangle\,\varphi$ and $\varphi \rightarrow [A]\,\langle \bar{A} \rangle\,\varphi$.*

1. Let $[\bar{A}]\,\varphi$ and $[A]\varphi$ be defined as above. We have to show that all instances of the schema $\varphi \rightarrow [\bar{A}]\,\langle A \rangle\,\varphi$ are globally satisfied in the frame $C_{\mathcal{K}}$.

Well, $[\bar{A}]\,\varphi \equiv_{def} [B]\,([\bar{E}]\,\varphi \vee \langle B \rangle \top)$, so $[\bar{A}]\,\langle A \rangle\,\varphi = [B]\,([\bar{E}]\,\langle A \rangle\,\varphi \vee \langle B \rangle \top)$.

Using previous definitions and axioms, we firstly derive the following equality which is used in the proof:

$$\begin{aligned}
[B]\,([\bar{E}]\,\langle A \rangle\,\varphi \vee \langle B \rangle \top) &\leftrightarrow [B]\,([\bar{E}]\,(\neg [A]\,\neg\varphi) \vee \langle B \rangle \top) \\
&\leftrightarrow [B]\,([\bar{E}]\,(\neg [E]\,([\bar{B}]\,\neg\varphi \vee \langle E \rangle \top)) \vee \langle B \rangle \top) \\
&\leftrightarrow [B]\,([\bar{E}]\,(\neg [E]\,\neg(\neg\,[\bar{B}]\,\neg\varphi \wedge \neg\,\langle E \rangle \top)) \vee \langle B \rangle \top) \\
&\leftrightarrow [B]\,([\bar{E}]\,(\langle E \rangle\,(\langle \bar{B} \rangle\,\varphi \wedge \neg\,\langle E \rangle \neg\,\bot)) \vee \langle B \rangle \top) \\
&\leftrightarrow [B]\,([\bar{E}]\,(\langle E \rangle\,(\langle \bar{B} \rangle\,\varphi \wedge [E]\,\bot)) \vee \langle B \rangle \top) \\
&\leftrightarrow [B]\,([\bar{E}]\,(\langle E \rangle\,(\langle \bar{B} \rangle\,\varphi \wedge [B]\,\bot)) \vee \langle B \rangle \top)
\end{aligned}$$

(from *VEN.07* which follows)

Choose an arbitrary $\mathcal{M}_0 \in C_{\mathcal{K}}$ and an arbitrary $\langle t_1, t_2 \rangle \in T$. Suppose $\mathcal{M}_0 \Vdash_{\langle t_1,t_2 \rangle} \varphi$. We have to show that $\mathcal{M}_0 \Vdash_{\langle t_1,t_2 \rangle} [\bar{A}]\,\langle A \rangle\,\varphi$.

By *VEN.01a* we have $\mathcal{M}_0 \Vdash_{\langle t_1,t_2 \rangle} [B]\,\langle \bar{B} \rangle\,\varphi$. This means that $\mathcal{M}_0 \Vdash_{\langle t_1,t' \rangle} \langle \bar{B} \rangle\,\varphi$ for every $t' \in T$ such that $t_1 \leq t' < t_2$. For any such t', we now have two possibilities: $t_1 < t'$ or $t_1 = t'$. Let's look at each of these in turn.

Suppose $t_1 < t'$. We know that $\mathcal{M}_0 \Vdash_{\langle t_1, t_1 \rangle} \top$, so we can claim that $\mathcal{M}_0 \Vdash_{\langle t_1, t' \rangle} \langle B \rangle \top$.

Suppose $t_1 = t'$. Since $\langle t_1, t_1 \rangle$ is a point interval, $\mathcal{M}_0 \Vdash_{\langle t_1, t_1 \rangle} [B] \perp$, so $\mathcal{M}_0 \Vdash_{\langle t_1, t_1 \rangle} [\bar{E}] \langle E \rangle [B] \perp$ (by *VEN.02b*). It is also the case that $\mathcal{M}_0 \Vdash_{\langle t_1, t_1 \rangle} \langle \bar{B} \rangle \varphi$, so $\mathcal{M}_0 \Vdash_{\langle t_1, t_1 \rangle} [\bar{E}] \langle E \rangle \langle \bar{B} \rangle \varphi$ (again by *VEN.02b*). This means that $\mathcal{M}_0 \Vdash_{\langle t_1, t_1 \rangle} [\bar{E}] \langle E \rangle \langle \bar{B} \rangle \varphi \wedge [\bar{E}] \langle E \rangle [B] \perp$, so $\mathcal{M}_0 \Vdash_{\langle t_1, t_1 \rangle} [\bar{E}] (\langle E \rangle (\langle \bar{B} \rangle \varphi \wedge [B] \perp))$.

It follows that, for every $t' \in T$ such that $t_1 \leq t' < t_2$ we have $\mathcal{M}_0 \Vdash_{\langle t_1, t' \rangle} \langle B \rangle \top$ or $\mathcal{M}_0 \Vdash_{\langle t_1, t' \rangle} [\bar{E}] (\langle E \rangle (\langle \bar{B} \rangle \varphi \wedge [B] \perp))$, and consequently, that $\mathcal{M}_0 \Vdash_{\langle t_1, t' \rangle} [\bar{E}] (\langle E \rangle (\langle \bar{B} \rangle \varphi \wedge [B] \perp)) \vee \langle B \rangle \top$. From this we can claim that $\mathcal{M}_0 \Vdash_{\langle t_1, t_2 \rangle} [B]([\bar{E}](\langle E \rangle (\langle B \rangle \varphi \wedge [B] \perp)) \vee \langle B \rangle \top)$. From the equality shown earlier, it follows that $\mathcal{M}_0 \Vdash_{\langle t_1, t_2 \rangle} [\bar{A}] \langle A \rangle \varphi$.

2. We can show, in a similar fashion, that all instances of the schema $\varphi \rightarrow [A] \langle \bar{A} \rangle \varphi$ are globally satisfied in $\mathcal{C}_K$.

Axioms *VEN.03* and *VEN.04* formulate transitive relationships between initial intervals, and between intervals that share an initial interval, respectively.

VEN.03 $\langle B \rangle \langle B \rangle \varphi \rightarrow \langle B \rangle \varphi$

VEN.04 $\langle \bar{B} \rangle \langle \bar{B} \rangle \varphi \rightarrow \langle \bar{B} \rangle \varphi$

VEN.05 and *VEN.06* express the notion that each interval is either a beginning/end point itself or contains one.

VEN.05 $\langle B \rangle [B] \perp \vee [B] \perp$

VEN.06 $\langle E \rangle [E] \perp \vee [E] \perp$

Axiom *VEN.07* expresses the idea that an interval contains a beginning point iff it contains an end point.

VEN.07 $[B] \perp \leftrightarrow [E] \perp$

VEN.08, *VEN.09* and *VEN.10* express the relationships between $\langle B \rangle$ and $\langle E \rangle$, between $\langle B \rangle$ and $\langle \bar{E} \rangle$, and between $\langle \bar{B} \rangle$ and $\langle \bar{E} \rangle$ respectively.

VEN.08 $\langle B \rangle \langle E \rangle \varphi \leftrightarrow \langle E \rangle \langle B \rangle \varphi$

VEN.09 $\langle B \rangle \langle \bar{E} \rangle \varphi \rightarrow \langle \bar{E} \rangle \langle B \rangle \varphi$

VEN.10 $\langle \bar{B} \rangle \langle \bar{E} \rangle \varphi \rightarrow \langle \bar{E} \rangle \langle \bar{B} \rangle \varphi$

Axioms *VEN.11* and *VEN.12* express the idea that there exists a linear relationship between any two initial intervals of an interval (a form of left-linearity), as well as between any two terminal intervals of an interval (a form of right-linearity).

VEN.11 $(\langle B \rangle \varphi \wedge \langle B \rangle \psi) \rightarrow (\langle B \rangle (\varphi \wedge \langle B \rangle \psi) \vee \langle B \rangle (\varphi \wedge \psi) \vee \langle B \rangle (\langle B \rangle \varphi \wedge \psi))$

VEN.12 $(\langle E \rangle \varphi \wedge \langle E \rangle \psi) \rightarrow (\langle E \rangle (\varphi \wedge \langle E \rangle \psi) \vee \langle E \rangle (\varphi \wedge \psi) \vee \langle E \rangle (\langle E \rangle \varphi \wedge \psi))$

Intervals as primitive

In the second possibility pointed out by Venema [Ven88] we use a temporal structure with intervals rather than points as primitive. This way we get a direct correspondence between the modal operators and Allen's interval-interval relations. A time structure $\mathcal{F} = (J, ||, \subset_B, \subset_E)$ is used, where J is a set, the elements of which are called intervals, $||$ is a relation expressing the notion of 'meet of intervals' in J, and $\subset_B$ and $\subset_E$ are relations expressing the idea of initial and a terminal inclusion respectively, of intervals in J. So $j \subset_B i$ means that j is an initial interval of i and $j \subset_E i$ means that j is a terminal interval of i.

A model is an ordered tuple $\mathcal{M} = (J, ||, \subset_B, \subset_E, V)$, where V is a valuation: $\Phi \longrightarrow 2^J$ that associates each $\varphi \in \Phi$ with the set of intervals over which it is true. A sentence φ is true relative to $\mathcal{M}$ over an interval $i \in J$, denoted by $\mathcal{M} \Vdash_i \varphi$, iff one of the following cases applies:

- $\varphi \in \Phi$ and $i \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_i \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_i \psi$ or $\mathcal{M} \Vdash_i \chi$
- $\varphi = [A]\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $i||j$
- $\varphi = [B]\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $j \subset_B i$
- $\varphi = [E]\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $j \subset_E i$
- $\varphi = [\bar{A}]\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $j||i$
- $\varphi = [\bar{B}]\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $i \subset_B j$
- $\varphi = [\bar{E}]\psi$ and $\mathcal{M} \Vdash_j \psi$ for all j such that $i \subset_E j$,

where the modal operators are read as before.

The duals of the modal operators are also defined as in the previous section.

This system can be simplified by using the $||$ accessibility relation only. We use a time structure $\mathcal{F} = (J, ||)$ similar to the one used by Allen & Hayes in [AH89]. J is a set, the elements of which are called intervals, and $||$ again expresses the notion of two intervals meeting. The relations $\subset_B$ and $\subset_E$ can be defined in terms of $||$ as follows:

$$j \subset_B i \equiv_{def} \exists k, l, m((k||i \wedge i||m) \wedge (k||j \wedge j||l \wedge l||m))$$
$$j \subset_E i \equiv_{def} \exists k, l, m((k||i \wedge i||m) \wedge (k||l \wedge l||j \wedge j||m))$$

The language uses the same alphabet as the language discussed in the previous section with the exclusion of $[A]$ and $[\bar{A}]$. The semantics is defined as follows: A sentence φ is true relative to a model $\mathcal{M} = (J, ||, V)$ over the interval i, denoted by $\mathcal{M} \Vdash_i \varphi$, iff one of the following cases applies:

- $\varphi \in \Phi$ and $i \in V(\varphi)$
- $\varphi = \neg\psi$ and $\mathcal{M} \nVdash_i \psi$
- $\varphi = \psi \rightarrow \chi$ and $\mathcal{M} \nVdash_i \psi$ or $\mathcal{M} \Vdash_i \chi$
- $\varphi = [B]\psi$ and $\mathcal{M} \Vdash_j \psi$ for every $j \in J$ such that there exists a $k, l, m \in J$ where $(k||i \wedge i||m) \wedge (k||j \wedge j||l \wedge l||m)$
- $\varphi = [E]\psi$ and $\mathcal{M} \Vdash_j \psi$ for every $j \in J$ such that there exists a $k, l, m \in J$ where $(k||i \wedge i||m) \wedge (k||l \wedge l||j \wedge j||m)$
- $\varphi = [\bar{B}]\psi$ and $\mathcal{M} \Vdash_j \psi$ for every $j \in J$ such that there exists a $k, l, m \in J$ where $(k||i \wedge i||l \wedge l||m) \wedge (k||j \wedge j||m)$
- $\varphi = [\bar{E}]\psi$ and $\mathcal{M} \Vdash_j \psi$ for every $j \in J$ such that there exists a $k, l, m \in J$ where $(k||l \wedge l||i \wedge i||m) \wedge (k||j \wedge j||m)$.

4.3.2 A spatial representation of the interval structure

Venema [Ven90] also presents a different angle to viewing relationships between intervals where we get a spatial representation of the interval structure within the 'northwestern halfplane' of $\mathcal{F} \times \mathcal{F}$ in which an interval $\langle s, t \rangle$ is represented by the point (s, t) as depicted in Figure 4.4.

Venema shows that it is possible to construct an isomorphism between the interval structure $\mathcal{F} = (T, \leq)$ and $\mathcal{F}^{2NW} = \{(x, y) \in \mathcal{F}^2 | x \leq y\}$. A set of modal operators, which are the spatial versions of the Halpern-Shoham modal operators, are used to indicate the position of a point, or a set of points, relative to the position of the current point:

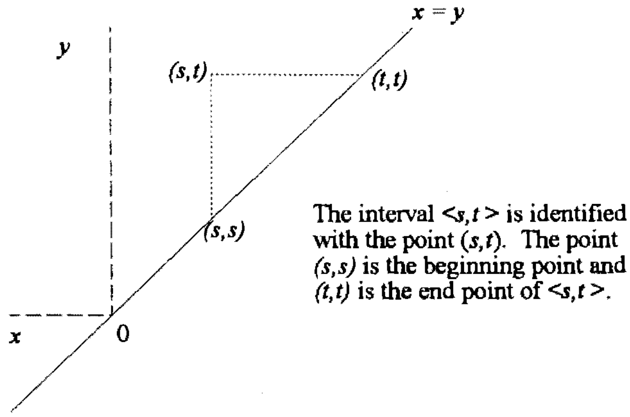

Figure 4.4: Mapping intervals to points on a plane.

$\Diamond\varphi \equiv_{def} \langle B \rangle \varphi$	φ holds at a point right below the current one. (The first coordinate is the same but the second coordinate of the current point is greater.)
$\Diamond\varphi \equiv_{def} \langle \bar{B} \rangle \varphi$	φ holds at a point right above the current one. (The first coordinate is the same but the second coordinate of the current point is smaller.)
$\Diamond\varphi \equiv_{def} \langle E \rangle \varphi$	Somewhere at a point to the right of the current one, φ holds. (The second coordinate is the same but the first coordinate of the current point is smaller.)
$\Diamond\varphi \equiv_{def} \langle \bar{E} \rangle \varphi$	Somewhere at a point to the left of the current one, φ holds. (The second coordinate is the same but the first coordinate of the current point is greater.)
$\Diamond\varphi \equiv_{def} \Diamond\varphi \vee \Diamond\varphi$	φ holds at a point with the same longitude and a different latitude as the current one. (The first coordinate is the same but the second coordinate differs.)
$\Diamond\varphi \equiv_{def} \Diamond\varphi \vee \Diamond\varphi$	φ holds at a point with the same latitude and a different longitude as the current one. (The second coordinate is the same but the first coordinate differs.)

We use the same definitions for $\langle D \rangle$, $\langle \bar{D} \rangle$, $\langle O \rangle$, $\langle \bar{O} \rangle$, $\langle L \rangle$ and $\langle \bar{L} \rangle$ as in section 4.3. We can now equate the 12 possible positions of one interval with respect to another with the equivalence classes as shown in Figure 4.5. For example, $\langle O \rangle \varphi$ means that φ holds at some point (s', t') somewhere in the block North-West of the current point. So $s < s'$ and $t < t'$, but $s' < t$, thus this block represents the intervals $\langle s', t' \rangle$ that overlaps the current interval $\langle s, t \rangle$ at the end. In contrast to this, $\langle L \rangle \varphi$ means that φ holds at some point (s', t') in the area where we have a greater latitude and a greater longitude than the current point. So $t < s'$ which means that this area represents the set of intervals later than $\langle s, t \rangle$, i.e. that starts after $\langle s, t \rangle$ ends.

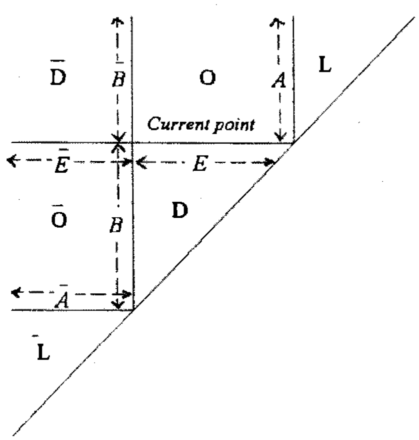

Figure 4.5: A spatial representation of the interval-interval relations.

4.4 Summary

In this chapter and the previous one, we considered the way the underlying time structure of a modal logic can be axiomatised. In the next two chapters we look at systems using first-order languages . Defining properties for the time structures of these systems are usually straight-forward because standard schemas can be used, so we rather concentrate on the way schemas are used to axiomatise the characteristics of actions and other occurrences.

Chapter 5

First-order approaches with instants

In the previous two chapters, the emphasis was on the representation of tenses in modal languages. In each case, there was a class of interpretations that were relevant, and we sought schemas that axiomatised this class. Conventional modal logics do not provide for the representation of actions and their effects [PR95]. In the ensuing two chapters, we want to look in more detail at the representation of concepts such as actions, events, facts and processes that occur in a system of interest. As James Allen put it: 'What do we know when we know an occurrence has occurred?' [All84].

Two of the early formalisms capable of expressing the effects of actions were Situation Calculus defined by McCarthy & Hayes [MH69] and McDermott's temporal logic [McD82]. We first give a short discussion of Situation Calculus and identify its limitations, then we consider Extended Situation Calculus as an example of a formalism that attempts to eliminate some of these limitations, and finally we discuss McDermott's approach.

Situation Calculi, like the other many-sorted languages we discuss in this chapter, are *reified* languages. What do we understand by the verb 'reify'? Let us illustrate this with the following example:

Example 18 *The reification of a property.*

Suppose that we want to talk about the sleeping habits of agents using a first-order language. An obvious first attempt would involve equipping the language with a unary predicate symbol *Sleeps* and constants such as, say, *Tony* so that the fact of an agent's somnolence may be conveyed by a sentence like *Sleeps(Tony)*. In an interpretation of the language, there will be an object in the universe of discourse serving as denotation for *Tony*, but no object that corresponds to the fact expressed by *Sleeps(Tony)*.

Reification is the trick of using a slightly more sophisticated language equipped with terms that represent facts such as the agent being asleep, so that in an interpretation, the universe of discourse would indeed contain an object corresponding to the fact. What previously was abstract - the sentence *Sleeps(Tony)* - has been replaced by something sufficiently concrete to correspond to an element of the universe of discourse.

5.1 Situation Calculus

Situation Calculus was defined by McCarthy and Hayes in 1969 [MH69]. Situation Calculi are many-sorted languages in which situation terms represent states of some system of interest. It is a change-based rather than temporal approach, i.e. the system is assumed to persist in one state until an action occurs that changes it to a new state [Sho87].

For our purposes, a *Situation Calculus* is a many-sorted language such that:

- the set of sorts on which the alphabet is based includes at least the sorts *situation*, *action* and *fact* [1]
- the set of function symbols includes at least the binary function symbol *Result* with sort (*situation, action, situation*)
- the set of predicate symbols includes at least the binary predicate symbol *Holds* with sort (*fact, situation*).

The following example illustrates the way a Situation Calculus is used to represent knowledge about a system and to show how we can represent the effect that a specific action or event[2] has on the system.

Example 19 *Defining a Situation Calculus.*

Consider the simple system of an air-conditioning unit in a car. There is a switch for the air-conditioner and a starter for the car engine. The air-conditioner can either be on or

[1] Some authors use the term *fluent*. This is a relic of McCarthy's original ungainly terminology.

[2] In Situation Calculus no distinction is made between actions and events. In the discussion of Situation Calculus as well as Extended Situation Calculus, which will be described in the next section, we will use the term action exclusively to avoid confusion.

off, and the engine can either be off or running. If the engine is running and we switch it off, the engine will be off, and conversely. The same applies to the air-conditioner. This means that we have four actions that can have an effect on the state of the system as we have described it and these are the actions of switching on or off either the engine or the air-conditioner. For simplicity, we rule out the possibilities of the engine not starting due to some mechanical problem or the air-conditioner malfunctioning, and so on.

To represent our knowledge of the engine/air-conditioning system, we choose a Situation Calculus the alphabet of which is based on the set of sorts $TP = \{situation, action, fact, unit\}$, and which contains the following symbols:

- a set $S = \{S_0, \ldots, S_3\}$ of constants of sort *situation*
- a set $U = \{U_0, U_1\}$ of constants of sort *unit*
- a set SV of variables $s_0, s_1, \ldots$ of sort *situation*
- a set AV of variables $a_0, a_1, \ldots$ of sort *action*
- a set PV of variables $p_0, p_1, \ldots$ of sort *fact*
- a set UV of variables $u_0, u_1, \ldots$ of sort *unit*
- the set $FUN = \{(Result, 2), (SwitchOn, 1), (SwitchOff, 1), (On, 1), (Off, 1)\}$ with the following associated sort tuples:
  - *Result* is of sort (*situation, action, situation*)
  - *SwitchOn* is of sort (*unit, action*)
  - *SwitchOff* is of sort (*unit, action*)
  - *On* is of sort (*unit, fact*)
  - *Off* is of sort (*unit, fact*)
- the set *PRED* contains only one element, namely (*Holds*, 2), with the sort (*fact, situation*) associated with it.

The usual first-order semantics applies to Situation Calculi. By way of illustration, we define the following interpretation $I_0 = (D, i)$. The domain $D = D_{sit} \cup D_{act} \cup D_{fact} \cup D_{unit}$ and these subdomains of D are defined as the following sets of strings:

- $D_{sit} = \{S\_0, \ldots, S\_3\}$
- $D_{act} = \{$*SwitchOnEng*, *SwitchOffEng*, *SwitchOnAC*, *SwitchOffAC*$\}$
- $D_{fact} = \{$*EngRunning*, *EngOff*, *ACOn*, *ACOff*$\}$
- $D_{unit} = \{$*Engine*, *AC*$\}$.

The meaning function i is defined as follows:

- $i(S_0) = S\_0$, the *initial situation*
- $i(S_j) = S\_j$, for $j = 1, 2, 3$
- $i(U_0) = Engine$, $i(U_1) = AC$
- $i(SwitchOn, 1)$ is a function of sort *(unit, action)* and consists of the following ordered pairs:
  - *(Engine, SwitchOnEng)*
  - *(AC, SwitchOnAC)*
- $i(SwitchOff, 1)$ is also a function of sort *(unit, action)* and consists of the following ordered pairs:
  - *(Engine, SwitchOffEng)*
  - *(AC, SwitchOffAC)*
- $i(On, 1)$ is a function of sort *(unit, fact)* and consists of the following ordered pairs:
  - *(Engine, EngRunning)*
  - *(AC, ACOn)*
- $i(Off, 1)$ is a function of sort *(unit, fact)* and consists of the following ordered pairs:
  - *(Engine, EngOff)*
  - *(AC, ACOff)*
- $i(Result, 2)$ is a function of sort *(situation, action, situation)* and consists of the following ordered pairs:
  - $((S\_0, SwitchOnEng), S\_2)$

- $((S\_0, \mathit{SwitchOffEng}), S\_0)$
- $((S\_0, \mathit{SwitchOnAC}), S\_1)$
- $((S\_0, \mathit{SwitchOffAC}), S\_0)$
- $((S\_1, \mathit{SwitchOnEng}), S\_3)$
- $((S\_1, \mathit{SwitchOffEng}), S\_1)$
- $((S\_1, \mathit{SwitchOnAC}), S\_1)$
- $((S\_1, \mathit{SwitchOffAC}), S\_0)$
- $((S\_2, \mathit{SwitchOnEng}), S\_2)$
- $((S\_2, \mathit{SwitchOffEng}), S\_0)$
- $((S\_2, \mathit{SwitchOnAC}), S\_3)$
- $((S\_2, \mathit{SwitchOffAC}), S\_2)$
- $((S\_3, \mathit{SwitchOnEng}), S\_3)$
- $((S\_3, \mathit{SwitchOffEng}), S\_1)$
- $((S\_3, \mathit{SwitchOnAC}), S\_3)$
- $((S\_3, \mathit{SwitchOffAC}), S\_2)$

- $i(\mathit{Holds}, 2) = \{(\mathit{EngOff}, S\_0), (\mathit{ACOff}, S\_0), (\mathit{EngOff}, S\_1), (\mathit{ACOn}, S\_1), (\mathit{EngRunning}, S\_2), (\mathit{ACOff}, S\_2), (\mathit{EngRunning}, S\_3), (\mathit{ACOn}, S\_3)\}$

Starting with the initial situation, we can construct a tree, called a *situation tree,* the branches of which consist of all possible sequences of situations. Each situation in a branch is the result of some action occurring in the previous situation. Each such sequence is called a *path.* Figure 5.1 shows the situation tree for the example discussed above. From the figure, it is clear that there are at least two possible paths from the initial situation S_0. One having S_2 as the next situation in the path and the other having S_1 as the next situation. Again, from S_2 there are two possible paths, one leading to S_0 and the other leading to S_3, and so on.

Although Situation Calculus provides a means for representing actions and outcomes of actions, several criticisms have been levelled at it:

- Consider actions such as 'The car engine was running for 20 minutes' and 'I spent the whole day driving around town'. These actions would have to be modelled, if at all, by an action such as *skip* which does not change the situation. However, there is then no way to express the differing durations of these *non-active actions* [All84].

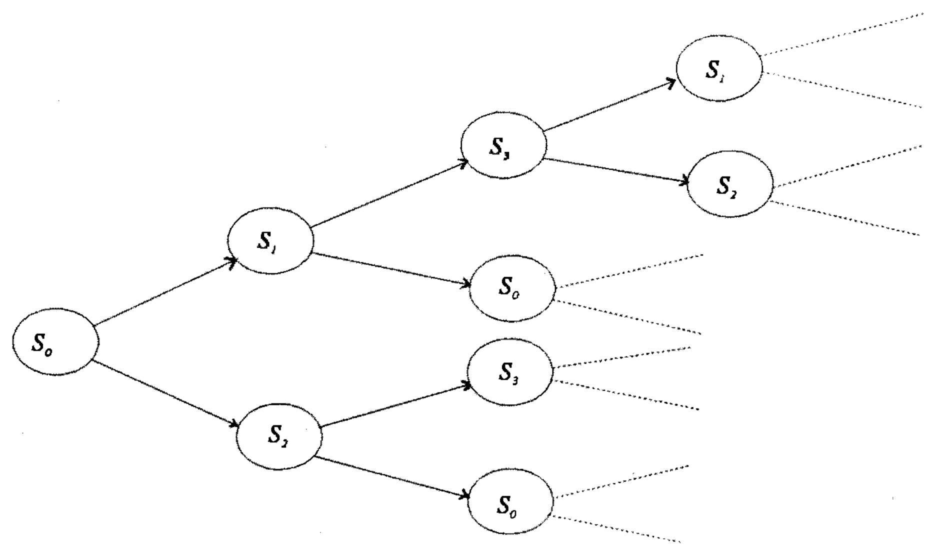

Figure 5.1: The situation tree for Example 19.

- It is not possible to express actions that occur simultaneously and may interact with each other, e.g. if action A moves block C one unit to the left and at the same time action B moves block C one unit to the right, perhaps block C should not move at all since the simultaneous occurrence of actions A and B results in each action cancelling the effect of the other ([All84] and [SG88]).
- When the same state of a system can recur at different times, the situations representing that state at the different times [3] would be equivalent insofar as the same facts would hold. The language offers no convenient way to make explicit the difference in times, and thus no convenient way to represent actions within a time structure.
- We have no way to describe how the world actually evolves [PR95].
- In most formulations of Situation Calculi, situations need to be named explicitly in terms of a sequence of actions following the initial state. As a result, it is not possible to represent such a sequence if we do not have complete information [KS94].
- Time is discrete, so it is not possible to describe continuous processes such as water flowing into a container [SM87].
- The *frame problem* is the best-known of a class of problems that may arise in applying

[3] McCarthy & Hayes [MH69] define a situation as a snapshot of the world at an instant in time.

change-based Situation Calculus [Mil93] and has been pointed out as a limitation of this approach, and indeed of the use of logic as a knowledge representation language [HM86]. These objections have been refuted by demonstrating that the frame problem, and related problems, may be avoided in a variety of ways, most notably by the use of nonmonotonic semantics [Lif87].

5.1.1 Extended Situation Calculus

McCarthy & Hayes suggest in [MH69] that the expressive power of modal logic can be gained using a first-order language by identifying each possible world of the relevant modal language with a situation in a suitable Situation Calculus. Pinto & Reiter built their Extended Situation Calculus on this principle by identifying the time structure (a set of possible worlds together with an accessibility relation) with a fixed infinite branch of states in the associated situation tree. In this way, the ontology of Situation Calculus is extended to provide for the representation of time, and thus for representing the occurrence of actions with no duration as well as of actions with some duration.

In principle, the notion of time is formalised within Extended Situation Calculus by identifying the time line, which may be thought of as the set of non-negative rational numbers[4] with the usual ordering, with a connected path in the situation tree. We introduce a sort *time,* to be interpreted as a time line. Situations will be associated with points on the time line by functions to be represented by the symbols *Start* and *End.*

The alphabet for an *Extended Situation Calculus,* or *ESC,* based on a set of sorts $TP = \{situation,\ action,\ fact,\ unit,\ time\}$, is similar to the alphabet for a Situation Calculus with the addition of a number of new predicate symbols and a number of new function symbols. In the spirit of [PR95], we use, instead of the function symbol *Result,* the function symbol *Do* of sort (*action, situation, situation*). Additionally, we have a binary predicate symbol '=' of sort (t, t) for each $t \in TP$, a binary predicate symbol '$\prec$' of sort (*time, time)* and another symbol '$<$' of sort (*situation, situation*). Additional function and predicate symbols will be introduced later.

Pinto & Reiter [PR95] also make use of a predicate variable to formulate an *induction axiom* :

$\forall\alpha[(\alpha(S_0) \wedge \forall s, a(\alpha(s) \rightarrow \alpha(Do(a, s)))) \rightarrow \forall s(\alpha(s))]$.

[4]Pinto & Reiter suggest using the real numbers. However, situations are built up from a starting point using functions like *Do* or *Result,* so there can only be countably many. This means that the rationals (countably infinite) are enough.

This axiom states that, for all properties α, the following must hold: if the initial situation has the property, and it is the case that, for all situations s and all actions a, if s has the property, then the situation resulting from performing action a in s also has the property, then it follows that all situations will have the property.

The predicate symbol '=' of sort (t, t) is interpreted as the identity relation on the relevant subdomain. The predicate symbol '<' of sort (*situation, situation*) is used to indicate the relative positions of situations in a branch of the situation tree. If $s_1 < s_2$, it means that s_1 precedes s_2; s_1 is a *predecessor* of s_2, and s_2 is a *successor* of s_1. We let $s_1 \leq s_2$ abbreviate $(s_1 < s_2) \vee (s_1 = s_2)$.

Pinto & Reiter propose the following basic axioms:

ESC.01 $\forall s_1, s_2, a(s_1 < Do(a, s_2) \leftrightarrow s_1 \leq s_2)$.

This axiom is intended to express the notion that there is no situation s that can be interpolated between a situation s_2 and the situation $Do(a, s_2)$ that results from performing action a in s_2. If $Do(a, s_2) = s_3$, we say that s_3 is an *immediate successor* of s_2 and that s_2 is an *immediate predecessor* of s_3.

ESC.02 $\forall a_1, a_2, s_1, s_2((Do(a_1, s_1) = Do(a_2, s_2)) \rightarrow (a_1 = a_2))$

The occurrence of two different actions in two situations that may or may not differ, cannot have the same immediate successor as a result.

ESC.03 $\forall s_1, s_2((s_1 < s_2) \rightarrow \neg(s_2 < s_1))$

If situation s_2 is a successor of situation s_1, then s_2 cannot be a predecessor of s_1. The idea is to produce a tree of situations by ruling out cycles.

The following is a list of consequences of the axioms listed above (Proofs of these are given in [PR95].):

- $\forall s(S_0 \leq s)$ (The initial situation is a predecessor of all situations other than itself.)
- $\forall s \neg(s < S_0)$. (The initial situation has no predecessor.)
- $\forall s \neg(s < s)$. (A situation cannot be a predecessor or a successor of itself.)
- $\forall s[(s \neq S_0) \rightarrow \exists a, s'(s = Do(a, s'))]$. (For all situations s other than S_0, s is the result of some action a occurring in some situation s'.)
- $\forall s_1, s_2, s_3((s_1 < s_2 \wedge s_2 < s_3) \rightarrow s_1 < s_3)$. (The relation $<$ is transitive.)
- $\forall a, s(s < Do(a, s))$. (A situation is a predecessor of its immediate successors.)

- $\forall a_1, a_2, s_1, s_2((Do(a_1, s_1) < Do(a_2, s_2)) \rightarrow s_1 < s_2)$. (If an immediate successor of some situation s_1 is a predecessor of an immediate successor of some situation s_2, then s_1 is a predecessor of s_2.)
- $\forall a, s_1, s_2((Do(a, s_1) \leq s_2) \rightarrow s_1 < s_2)$. (If an immediate successor of a situation s_1 is either a predecessor of some situation s_2 or s_2 itself, then s_1 is a predecessor of s_2.)
- $\forall a, s_1, s_2((Do(a, s_1) = Do(a, s_2)) \rightarrow s_1 = s_2)$. (If the same action is performed in two different situations, it is not possible for the two immediate successor situations to be the same.)

To represent the effect of actions on the states of the system of interest, i.e. to represent change in such a system, *effect* axioms may be introduced. These have the following general form:

ESC.04 $(Poss(A, s) \wedge \phi_{A,F}(s)) \rightarrow Holds(F, Do(A, s))$

where A and F are constants of sort *action* and *fact* respectively, *Poss* is a binary predicate symbol of sort *(action, situation)* which expresses the idea that all preconditions for performing action A in situation s have been met, and $\phi_{A,F}(s)$ may express any additional information about relevant relationships between action A and fact F.

One of the limitations of Situation Calculus was that it has no mechanism for distinguishing hypothetical courses of history from how the world actually evolves. This problem is addressed by Extended Situation Calculus with the introduction of a predicate symbol *Actual* of sort *(situation)*, which can be used to describe the actual sequence of situations. This sequence can be identified with a unique branch in the situation tree. The following set of axioms describes the behaviour of *Actual*:

ESC.05 $Actual(S_0)$.

In all interpretations, the initial situation is always part of the actual sequence.

ESC.06 $\forall a, s(Actual(Do(a, s)) \rightarrow (Actual(s) \wedge Poss(a, s)))$.

If the result of performing some action a in some situation s is in the actual sequence, then situation s is also in the actual sequence, and it is possible to perform action a in situation s.

ESC.07 $\forall a_1, a_2, s[(Actual(Do(a_1, s)) \wedge Actual(Do(a_2, s))) \rightarrow (a_1 = a_2)]$.

An actual situation has at most one actual immediate successor.

Does this also mean that each actual situation has at most one actual immediate predecessor? Assume $Actual(s)$. Suppose $s = Do(a_1, s_1)$ and $s = Do(a_2, s_2)$. We wish to infer that $a_1 = a_2$ and $s_1 = s_2$. If $Do(a_1, s_1) = Do(a_2, s_2)$, then, by *ESC.02*, $a_1 = a_2$. Hence from the consequences of *ESC.01* to *ESC.03*, it follows that $s_1 = s_2$. So s has at most one predecessor. Need this predecessor be actual? Indeed, by virtue of the following consequences of the axioms:

- $\forall s_1, s_2((s_2 \leq s_1) \rightarrow (Actual(s_1) \rightarrow Actual(s_2)))$. (If s_2 is a predecessor of s_1 and s_1 is actual, then s_2 will also be actual: all actual situations lie on the same path.)
- $\forall s_1, s_2((Actual(s_1) \land Actual(s_2)) \rightarrow ((s_1 < s_2) \lor (s_1 = s_2) \lor (s_2 < s_1)))$. (Actual situations are totally ordered.)

From the first consequence, it follows directly that each actual situation has at most one actual immediate predecessor.

The sequence of actual situations is called the *actual path.* This path completely describes the world in a way that parallels the time line. At this stage, we might ask how an association between a time structure and the actual path can be introduced. For the purpose of this association, we add to the alphabet two function symbols: *Start* of sort (*situation, time*) and *End* of sort (*situation, action, time).* The intention is that $Start(s)$ and $End(s, a)$ indicate the time span of situation s. Since the truth of atoms of the form $Holds(f, s)$ is unaffected by the introduction of the new function symbols, no fact can change its truth value during the time span of a situation. The association between situations and times allows us to stipulate an association between actions and times also: actions occur at the end time of situations.

The following axioms determine the behaviour of the *Start* and *End* functions:

ESC.08 $\forall s, a(End(s, a) = Start(Do(a, s)))$.

The end time of a situation is equal to the start time of its immediate successor.

ESC.09 $\forall a, s(Start(s) \prec End(s, a))$.

The start time of a situation is earlier than its end time, i.e. all situations have duration.

ESC.10 $Start(S_0) = 0$.

The start time of the initial situation is equal to the initial time.

From the above, it follows that

$\forall s_1, s_2((s_1 < s_2) \rightarrow (Start(s_1) \prec Start(s_2)))$.

The start time of a situation is earlier than the start time of all its successors and later than the start time of all its predecessors.

It is convenient to make explicit the idea of *occurrences.* The occurrence of an action can be expressed by introducing a binary predicate symbol *Occurs* of sort (*action, situation*), e.g. *Occurs(SwitchOn, S)* may represent the notion that the action of switching on something, occurs in situation S. Occurrences are defined in terms of the actual path as follows:

$Occurs(a, s) \leftrightarrow Actual(Do(a, s))$.

We can say that action a occurs in situation s iff the situation resulting from $Do(a, s)$ is in the actual path.

To link occurrences to times, we introduce a binary predicate symbol $Occurs_T$ of type (*action, time)* which may be defined as follows:

$Occurs_T(a, t) \leftrightarrow \exists s(Occurs(a, s) \wedge (Start(Do(a, s)) = t))$.

This means that action a occurs at time t iff the following holds: action a occurs in some actual situation s of which the immediate successor starts at t. (Remember that an action only occurs at the end time of a situation and t is also the end time of s.)

A relation between facts and time points can also be expressed by introducing a binary predicate symbol $Holds_T$ of sort (*fact, time)* such that

$Holds_T(p, t) \leftrightarrow \exists s(During(t, s) \wedge Holds(p, s))$

where

$During(t, s) \leftrightarrow (Actual(s) \wedge (Start(s) \prec t) \wedge \forall a(Occurs(a, s) \rightarrow (t \preceq End(s, a))))$.

$During(t, s)$, which is of sort (*time, situation*), holds iff the time point t falls within the time span of situation s: if s is actual and starts earlier than t, and all actions that occur in s occur later than t, then t falls within the time span of s. This means that $Holds_T(p, t)$ is true, i.e. the fact p is true at time t, iff there is some actual situation s such that p is true in s, and t falls within the time span of s.

It follows directly that, if a fact holds in a situation, it holds for the complete time span of that situation:

$\forall p, s, t_1, t_2((During(t_1, s) \wedge During(t_2, s)) \rightarrow (Holds_T(p, t_1) \leftrightarrow Holds_T(p, t_2)))$.

The occurrence of actions between two actual situations can be defined in the following way:

$OccursBet(a, s_1, s_2) \leftrightarrow \exists s((s_1 < s < s_2) \wedge Occurs(a, s))$.

Action a occurs between situations s_1 and s_2 iff action a occurs in some situation s, which has s_1 as a predecessor and s_2 as a successor.

Up to now, all actions were regarded as being instantaneous, i.e. of having no duration. How can an action that has duration, i.e. that occurs over a period of time, be modelled? This can be accomplished in a manner similar to the way temporal intervals are modelled in point-based time structures: by the introduction of two instantaneous action constants called *Start-action* and *End-action* respectively. We can use an example to illustrate this.

Example 20 *Actions that have duration.*

Think of the action of dancing that occurs over a period of time. We can introduce two instantaneous action constants, *StartDancing* and *EndDancing,* together with the fact *IsDancing,* where *IsDancing* expresses the notion of the action of dancing being in progress. The relationship between this fact and the corresponding instantaneous *start-end* actions, can be expressed by the following *effect* axioms:

ESC.11 $Poss(StartDancing, s) \rightarrow Holds(IsDancing, Do(StartDancing, s))$.

If it is possible for the action *StartDancing* to occur in situation s, then the fact *IsDancing* holds in the immediate successor of s resulting from *StartDancing.* This means that the action of dancing starts at the end of situation s and continues in its immediate successor.

ESC.12 $Poss(EndDancing, s) \rightarrow \neg Holds(IsDancing, Do(EndDancing, s))$.

If it is possible for the action *EndDancing* to occur in situation s, then the action of dancing cannot occur in its immediate successor resulting from *StartDancing.* So the action of dancing comes to an end at the end of situation s.

The introduction of *start-end* axioms makes it possible to model a restricted form of concurrency. Actions with duration can occur concurrently as long as their starting and ending points do not coincide: an instantaneous action occurs at the end of a situation and there can only be one actual successor. This restriction can be eliminated by allowing for concurrent actions in the languages of Situation Calculi. A number of such models have been developed, a list of which is given in [PR95].

Pinto & Reiter [PR95] show how the following applications may be implemented using the axioms defined above:

- explanation of observations that might be used for diagnostic reasoning. 'If the car was parked in the street last night and it is not there this morning, it might have been stolen or towed away'
- the expression of behavioral constraints that may be required to formulate constraints for agent-oriented programming. 'Never cross the street against the red light'
- the expression of the occurrence of external events over which an agent does not have any control. 'The agent boards the bus if he arrives at the bus stop before the last bus leaves'
- if the branching structure of Situation Calculus is preserved, a variety of hypothetical queries may be expressed and answered, although counterfactuals cannot be expressed: the use of only one actual path excludes the existence of hypothetical actual paths
- relations between events in a chain of events may be expressed using causal relationships. 'Shooting a loaded gun causes a noise to occur'.

Pinto & Reiter also show that a variation of instant tense logic discussed in Chapter 3 (that of Goldblatt [Gol92]) can be embedded in the *ESC* with the result that the *ESC* is at least as expressive as modal temporal logics. Moreover, by embedding the essential features of modal temporal logic within the *ESC* without losing the action-based ontology of Situation Calculus, the *ESC* is more expressive: conventional modal temporal logics do not provide for the representation of actions and their effects. However, Extended Situation Calculus has limitations of its own:

- *ESC.01* given by Pinto & Reiter is formulated in both directions but this does not allow for actions that have no effect. It is often useful, e.g. in modelling a busy-wait situation in Computer Science, to be able to include actions that have no effect, i.e. so-called *skip* actions. If $s_1 = s_2$, then it ought not to follow that $s_1 < Do(skip, s_2)$, contradicting one of the consequences of axioms *ESC.01* to *ESC.03* which states that a situation cannot be a successor or a predecessor of itself.
- consider a simple system like the engine/air-conditioning system of example 19. The same state of the system may recur at different times, and a given state may be reached via different routes. For example, suppose both the air-conditioner and the

engine are off in the initial situation. If we first switch on the air-conditioner and then switch on the engine, the system will be in the state where the air-conditioner is on and the engine is running. The system will be in this same state if we first switch on the engine and then the air-conditioner. This state is represented in the language by different situation terms, and there is no direct way to express the idea that these different situation terms are in fact equivalent in that they represent the same state of the system.

Although the *ESC* was not introduced primarily to allow for the representation of *events* instead of *actions*, Pinto & Reiter claim that 'the expanded ontology for Situation Calculus provides the essential features of the *Event Calculus* of Kowalski & Sergot [KS86] which in turn appears to be a variant of Allen's temporal logic.'[PR95]. We will not discuss the Event Calculus here, but rather consider the temporal logic for actions and events proposed by Allen, in the next chapter.

5.2 McDermott's system

The first-order temporal logic proposed by McDermott [McD82] is, like Situation Calculus, an attempt to capture the openness of the future since more than one thing can happen starting at a given instant, so there could be many possible futures. He also identifies 'an adequate treatment of continuous change as one of the key requirements for a system to be able to reason about time realistically.' [Gal90]. His system provides ways to name and prove things about facts, events, plans, and world histories.

McDermott pays particular attention to what he views as important problems of temporal representation and refers to as causality, flow (continuous change), and plans. In sections 5.2.3, 5.2.4 and 5.2.5 we briefly describe his treatment of these concepts. A many-sorted first-order language is used. Note that we have little to say on semantic issues, as McDermott does not explicitly discuss interpretations of the language. Shoham & Goyal [SG88] suggest an interpretation which we look at in the next chapter.

A system of interest is described in terms of a set of states. A *state* is, like a situation, an instantaneous snapshot of the universe. States are partially ordered by the relation $\leq$. If $s_1 \leq s_2$, it means that s_1 either precedes s_2, or is identical to s_2. The properties of $\leq$ may be expressed by axioms such as the following:

MD.01 $\forall s_1, s_2((s_1 \leq s_2 \wedge s_2 \leq s_1) \leftrightarrow (s_1 = s_2))$ (Anti-symmetry)

MD.02 $\forall s_1, s_2((s_1 < s_2) \rightarrow \exists s(s_1 < s < s_2))$ (Density)

MD.03 $\forall s_1, s_2, s_3((s_1 \leq s_2 \wedge s_2 \leq s_3) \rightarrow s_1 \leq s_3)$ (Transitivity).

The notion of an interval is introduced as a totally ordered, convex set of states. An interval is denoted by an ordered pair $[s_1, s_2]$, where s_1 is the state in which the interval starts and s_2 the state in which it ends. All intervals are assumed to be closed.

The set of states, ordered by $\leq$, is intended to represent all the possible ways in which the system might evolve over time. In order to permit distinctions in respect of the rate of evolution, a separate time structure is needed, together with a coupling between the set of states and the time structure.

The time structure is a pair $(T, \leq)$ where T is the set of real numbers and $\leq$ is the usual ordering. Thus, time is linear and continuous. A *scale* can be attached to the time line by assuming that there are objects, called scales, which occupy some constant amount of time. If *hour* is such a scale, $(5 * hour)$ is a length of time equal to 5 times the size of *hour*.

We can use a function d to associate an instant $t \in T$, known as a *date,* with each state, reflecting the time of occurrence of that state. So the *date function* d is an order-preserving function from states to time points. Since the set of states is partially ordered, many different sets of states may map onto the time line, each constituting a possible history of the system, and so the future is called open, or indeterminate.

MD.04 $(s_1 < s_2) \rightarrow (d(s_1) < d(s_2))$

The order that exists between two states also holds between the dates associated with them.

The notion of a *chronicle* is used to represent a complete possible history of the universe; a way events might happen. A chronicle c is a totally ordered set of states for which the date-function establishes a bijective correspondence with the time line. This can be formalised by the following two axioms ($Elt(s, c)$ means that state s is an element of the set c):

MD.05a $\forall c(\forall s_1, s_2((Elt(s_1, c) \wedge Elt(s_2, c)) \rightarrow (s_1 < s_2 \vee s_2 < s_1 \vee s_1 = s_2)))$

The states in a chronicle are totally ordered.

MD.05b $\forall c(\forall t \exists s(Elt(s, c) \wedge d(s) = t))$

For every chronicle it holds that, for every instant in the time structure, there exists a state that is part of that chronicle.

MD.05b expresses the surjectivity of the date-function restricted to a chronicle c; injectivity follows from the requirement that the date-function be order-preserving.

An immediate consequence of *MD.05* is the following:

$(Elt(s_1, c) \wedge Elt(s_2, c)) \rightarrow \forall s(s_1 < s < s_2 \rightarrow Elt(s, c))$ (a chronicle is convex).

Every state is in some chronicle. Futhermore, if the ordering on states compares two states, then there exists some chronicle c that contains both these states:

MD.06 $(s_1 \leq s_2) \rightarrow \exists c(Elt(s_1, c) \wedge Elt(s_2, c))$

It follows, by convexity, that every state between s_1 and s_2 is in the same chronicle.

Chronicles branch only into the future, so all the states in the system are arranged to form a chronicle tree.

MD.07 $(s_1 \leq s \wedge s_2 \leq s) \rightarrow (s_1 \leq s_2 \vee s_2 \leq s_1)$ (Left-linearity)

Thus we have only one past but more than one possible future from a given state. This is depicted in Figure 5.2. The dotted line indicates one chronicle. It is clear that s_2 and s_3 are in the same chronicle but that there is no relationship between s_4 and s_3, for example, since they are in different chronicles. The state s_1 maps to t_1 on the date line, s_4 maps to t_4, and so on.

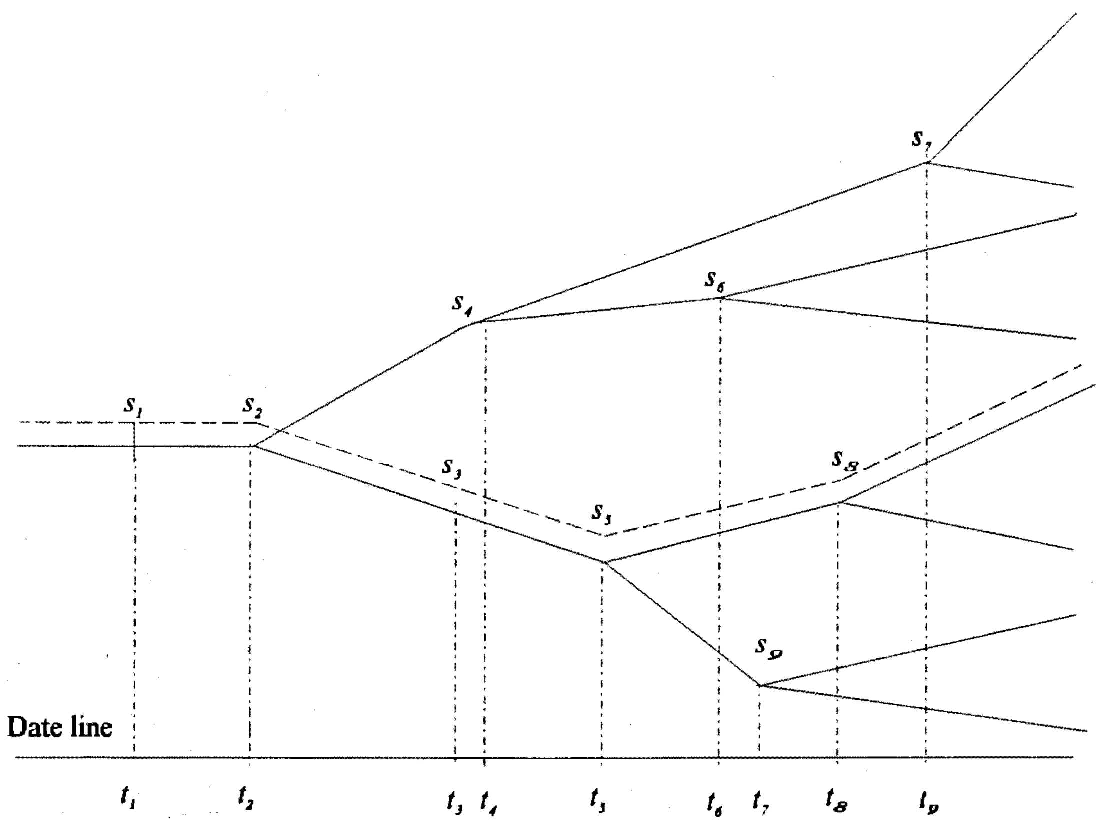

Figure 5.2 A chronicle tree

5.2.1 Representing facts

A *fact* can be represented as a set of states, intuitively those in which it is true. The fact *Always* is the set of all possible states and the fact *Never* is the empty set. A fact holds in an interval if it holds in all the states comprising the interval.

For the sake of clarity [5], we will write $True(s,p)$ instead of $Elt(s,p)$, meaning p holds in state s. The connectives '$\cap$', representing set intersection, '$\cup$' representing set union, and '$-$', representing the complement of a set with respect to the set *Always*, can be introduced in the usual way:

$True(s,(p \cap q))$ abbreviates $True(s,p) \wedge True(s,q)$

$True(s,(p \cup q))$ abbreviates $True(s,p) \vee True(s,q)$

$True(s,-p)$ abbreviates $\neg True(s,p)$.

Example 21 *Defining a fact.*

To express the notion that object a is on object b, we can use a binary function *On*, such that $On(a,b)$ represents the set of states in which a is on b. So *On* is a function from pairs of objects to sets of states of sort *fact*. This way of looking at facts is analogous to the logicians' trick of letting propositions denote sets of possible worlds [McD82].

The notion of a fact being true over a period of time is written as

$TT(s_1, s_2, p)$,

which means that fact p holds over the interval $[s_1, s_2]$, in other words, the interval $[s_1, s_2]$ is a subset of fact p.

5.2.2 Representing events and processes

McDermott attempts to define events in a way that includes both events that represent fact changes, like a block being moved from a to b, as well as events that are not fact changes, i.e. the large class of actions that are done for their own sake, like going for a walk [6]. We introduce the notion of an *event* as a set of intervals (thus a set of sets), intuitively those over which the event happens once, with no time 'left over' on either side. The occurrence of an event can be represented as

[5] Also called *syntactic sugaring*

[6] Galton [Gal87] calls these *atelic events* .

$Occurs(s_1, s_2, e)$,

which means that event e occurs over the interval $[s_1, s_2]$ (therefore $Elt([s_1, s_2], e)$ holds).

There is no event *Always* since events cannot intuitively have infinite duration. The empty set *Never* represents an event that can never happen. Consequently, the set *Never* can be used to represent both a fact and an event.

Certain facts and events are closely related. We say that the set consisting of all elements of intervals comprising an event e is the set over which e is *in progress* and write this as $InProgress(s, e)$ meaning e is in progress in s.

We want to restrict the set of facts to those that are well-behaved and to exclude the possibility of *intermingling* [7] [Gal90], i.e. a state of affairs where a proposition changes its truth value infinitely often in a finite period of time[8] [Gal96]. McDermott formulates the following axiom for this purpose:

MD.08 $\forall s, p(\exists s_0(TTopen(s_0, s, p) \vee TTopen(s_0, s, \neg p))$
$\wedge \forall c(Elt(s, c) \rightarrow \exists s_1(Elt(s_1, c) \wedge (TTopen(s, s_1, p) \vee TTopen(s, s_1, \neg p)))))$

where

$TTopen(s_1, s_2, p) \leftrightarrow (s_1 < s_2 \wedge \forall s(s_1 < s < s_2 \rightarrow True(s, p)))$ (p holds over the *open* interval (s_1, s_2)).

MD.08 states that, for an arbitrary state and fact, there is an interval preceding the state during which the fact is always true or always false. There is also such an interval succeeding the state. This holds for every chronicle of which the state is an element (we have more than one possible future).

The notion of a *sequence* or *chain* of abutting events can be written as $Chain(ee, n, s_1, s_2)$, i.e. there is a chain of length n of events from the set ee of events that reaches from s_1 to s_2. The following two axioms are used to formulate the properties of a chain:

MD.09a $Chain(ee, 0, s_1, s_2) \leftrightarrow (s_1 = s_2)$

There are no events in the chain, so the start state is equal to the end state.

MD.09b $Chain(ee, (n+1), s_1, s_2) \leftrightarrow \exists s, e(Elt(e, ee) \wedge Occurs(s, s_2, e) \wedge Chain(ee, n, s_1, s))$

i.e. if ee is a chain of events of length $(n+1)$ that starts in s_1 and ends in s_2, then there is some state s between s_1 and s_2 and some event e such that e occurs over $[s, s_2]$, and there exists a chain of events of length n that starts in s_1 and ends in s.

[7] This phenomenon is also known as a *fuzz* (a term used by Prior) or *clustered variation.*

[8] McDermott [McD82] uses the proposition '*the temperature in Cleveland is a rational number*' to illustrate the idea.

The idea that we can reach a certain state, s say, in a chronicle c if we follow the chain of events starting at a state s_1, can be expressed in the following way:

$$MD.10 \quad Reachable(ee, c, s_1, s) \leftrightarrow \exists n, s_2(n \geqslant 0 \wedge Elt(s_2, c) \wedge Chain(ee, n, s_1, s_2) \wedge (s_1 \leq s \leq s_2))$$

5.2.3 Causality

By causality McDermott means that 'one event always follows another event', and as illustration he gives the following example: If x is a loaded gun, pulling its trigger is followed by its firing. In his attempt to reason about causes and effects, he argues that events can cause two kinds of things: other events, and facts. He treats these two cases separately.

Firstly we consider the case where there exists a causal relationship between two different events. This is represented by

$$Ecause(p, e_1, e_2, rf, i)$$

which means that the occurrence of event e_1 is always followed by the occurrence of e_2, after a delay relative to the interval i, unless p becomes false before the delay is up. The following axiom is a formulation of this idea:

$$MD.11 \quad Ecause(p, e_1, e_2, rf, i) \rightarrow (Occurs(s_1, s_2, e_1) \rightarrow \forall c[Elt(s_2, c) \rightarrow \exists s_3(Elt(s_3, c) \wedge WithinDelay(s_3, rf, i, s_1, s_2) \wedge (\neg TT(s_2, s_3, p) \vee \exists s_4(Elt(s_4, c) \wedge Occurs(s_3, s_4, e_2))))])$$

where *WithinDelay* expresses the idea that state s_3 occurs after s_1, and possibly s_2, with a delay of i. In the above, rf is a real number in $[0, 1]$ that stipulates from which point through e_1 the delay is to be measured. If $rf = 0$, the delay must be measured starting at s_1; if $rf = 1$, the delay is to be measured starting at s_2. The i is a real interval from the time structure. For example, if $i_0 = (10\ sec, 15\ sec)$ then this indicates that the delay is at least 10 seconds and at most 15 seconds. So $Ecause\,(p, e_1, e_2, 0, i_0)$ means that the occurrence of e_1 will cause the occurrence of e_2 after a delay of at least 10 seconds and at most 15 seconds after e_1 starts ($rf = 0$) unless p becomes false before 15 seconds have elapsed after the start of e_1. The following holds for the notion of a delay:

$$WithinDelay(s, rf, i, s_1, s_2) \leftrightarrow Elt((d(s) - [d(s_1) + rf * (d(s_2) - d(s_1))]), i)$$

If an event is ever caused, then each of its occurrences is preceded (within a limited time interval) by an instance of one of its causes [9]. The following axiom formulates this idea:

$$MD.12 \quad Ecause(p, e_1, e_2, rf, i) \rightarrow \forall s_3, s_4[Occurs(s_3, s_4, e_2) \rightarrow \exists p', e', s_1, s_2, rf', i'(Ecause(p', e', e_2, rf', i') \wedge Occurs(s_1, s_2, e') \wedge$$

[9] McDermott calls this the *Principle of Paranoia.*

$WithinDelay(s_3, rf', i', s_1, s_2))]$

This principle enables us to infer that, if we know that an event e is the only cause for the occurrence of another event e', and e' occurs, then we know that e has occurred.

Secondly, an event can also cause a fact. This form of causation is represented by

$Pcause(p, e, q, rf, i, r)$,

which expresses the notion that event e is always followed by fact q, after a delay relative to the interval i, unless p becomes false before the delay is up. When q becomes true, it persists for a lifetime r. The concept of a *lifetime* is introduced because McDermott argues that certain facts will change their truth values autonomously (something dies) or we lose knowledge about its truth value after a period of time. He calls such a period the lifetime of a fact. The rules governing the causation of a fact by an event is contained in the following axiom:

MD.13 $Pcause(p, e, q, rf, i, r) \rightarrow [Occurs(s_1, s_2, e) \rightarrow \forall c(Elt(s_2, c) \rightarrow$
$\exists s_3(Elt(s_3, c) \wedge WithinDelay(s_3, rf, i, s_1, s_2) \wedge (\neg True(s_3, p) \vee Persist(s_3, q, r))))]$,

where $Persist(s, q, r)$ expresses the idea that the fact q remains true from state s for a lifetime r, i.e. until r has gone by or until q ceases to be true for some other reason.

5.2.4 Continuous change

In order to reason about continuous change, McDermott uses the term *fluent* (borrowed from McCarthy) for an object the *value* of which changes over time. All fluents have a range of numbers over which their values may vary. So $V(s, v)$ represents the value of a fluent v in a given state s. '*The temperature in Cleveland*' is an example of a fluent that takes on different values in a temperature space [McD82].

Vtrans is defined as the fundamental event involving fluents: $Vtrans(v, r_1, r_2)$ denotes the event consisting of all occasions on which the value of the fluent v changes from r_1 to r_2. An increase in inflation, for example, is a *Vtrans* of the fluent *Inflation* from one value to another [McD82]. If the quantity involved is continuous, the following axiom is formulated to capture this property:

MD.14 $Continuous(v) \rightarrow (Occurs(s_1, s_4, Vtrans(v, r_1, r_4)) \rightarrow$
$[\forall r_2, r_3 \exists s_2, s_3((s_1 \leq s_2 \leq s_3 \leq s_4) \wedge$
$((r_1 \leq r_2 \leq r_3 \leq r_4) \rightarrow$
$(Occurs(s_2, s_3, Vtrans(v, r_2, r_3)) \wedge \forall s(s_2 \leq s \leq s_3 \rightarrow r_2 \leq V(s, v) \leq r_3))) \wedge$
$((r_1 \geq r_2 \geq r_3 \geq r_4) \rightarrow$
$(Occurs(s_2, s_3, Vtrans(v, r_2, r_3)) \wedge \forall s(s_2 \leq s \leq s_3 \rightarrow r_2 \geq V(s, v) \geq r_3)))])$.

i.e. if v changes continuously from r_1 to r_4, and both r_2 and r_3 lie between them, then there exists a time interval in which v changes from r_2 to r_3 without going outside the bounds r_2 and r_3. This means that it spends a certain period of time in every subinterval between r_1 and r_4. (Note that the first of the innermost conjuncts applies when v is increasing in value and the second when v is decreasing.)

McDermott also adopts an 'abstract model' for representing other forms of continuous change over an interval and for representing rates of change of noncontinuous quantities which will not be discussed here.

5.2.5 Plans

Central to McDermott's reasoning about plans is the concept of an *action* which is defined as an event the occurrence of which involves an agent. McDermott does not create an environment in which multiple agents can operate, so the action function only has one argument and that is the action itself; the agent is not specified. A function can be used to define the performance of an action: $Do(a)$ represents the event of action a being performed. The system makes provision for actions that are straight-forward ('Put block A on block B') as well as for actions that are more complex, such as actions that express notions like prevention, allowing, proving, promising, etc. (e.g. 'Prevent e', where e is some event). Actions may be defined together with their preconditions, effects, and durations.

A composition of actions performed in parallel can be represented by $Par(e_1, \ldots, e_n)$ with the following axiom defining the resulting event:

MD.15 $Par(e_1, \ldots, e_n) = \{[s_1, s_2] | Occurs(s_1, s_2, e_1) \wedge \cdots \wedge Occurs(s_1, s_2, e_n)\}$

In McDermott's system, a *plan* is defined as a set of actions, often intended to carry out another action. Let *Plan (aa)* represent the action corresponding to the plan consisting of performing all the actions in the set *aa*. The notion of a plan carried out in the minimal time span in which all of its elements are carried out, can be represented by the following axiom:

MD.16 $Occurs(s_1, s_2, Do(Plan(aa))) \leftrightarrow [\forall a(Elt(a, aa) \rightarrow OccBetween(s_1, s_2, a))$
$\wedge \exists a, s(Elt(a, aa) \wedge Occurs(s_1, s, a)) \wedge \exists a, s(Elt(a, aa) \wedge Occurs(s, s_2, a))]$

where $OccBetween(s_1, s_2, a)$ expresses the notion that action a is performed in some subinterval of the interval $[s_1, s_2]$. Axiom *MD.16* stipulates that, if a plan is carried out, each of the actions comprising the plan is carried out; one of the actions starting in s_1 and another one ending in s_2.

The system proposed by McDermott also includes more complicated notions such as conflicts among actions done in the wrong order, and changing circumstances resulting in a possible forced change of plan.

McDermott's formulation, which can also be regarded as a reified temporal logic [Rei87], was the first to provide a mechanism for making a comparison within the logic between two possible futures; one in which a specific action is performed and another in which it is not [Sho87]. In particular, he gave meaning to the notion of prevention.

5.2.6 Some general comments on this system

Despite the expressiveness of McDermott's system, there are a number of limitations:

- One of the limitations, mentioned by McDermott, is that probabilities are not taken into account. This is particularly relevant in a planning system [McD82].
- Galton [Gal87] questions the propriety of identifying an event with a set of intervals. He argues that 'an event is what it is because of its internal characteristics', so one should not reject the internal characterisations of two different kinds of events in favour of one external characterisation.
- Shoham [Sho87] maintains that the dichotomy of facts/events is unnecessary at some times and insufficient at others, and that McDermott gives the semantics of what may be regarded as the propositional theory but does not always give his sentences a clear meaning.
- Ma & Knight [MK96] point out that McDermott's approach, which characterises intervals as structures derived out of time points, may face the *dividing instant problem.*

The approach used by Allen, one of the subjects of the next chapter, is compatible with McDermott's approach; the differences arise because of the intended application, and because of a different set of underlying assumptions in the two temporal logics.

Chapter 6

First-order approaches with intervals

One of the most important contributions to the representation of actions and events, together with their effects and the conditions that have to hold before an action or event can occur, is that of James Allen. We discuss this in section 6.1. However, Allen does not present an adequate model theory for his system. In section 6.2 we discuss Shoham & Goyal's [SG88] proposal in this regard.

6.1 Allen's theory of actions and events with time intervals

In [All84], the emphasis falls on the way in which the meaning of English sentences concerning actions and events can be expressed formally. If an event has taken place, we want to be able to represent the knowledge we have after the fact. For this purpose, it is necessary to represent both the effects of actions and events which are dynamic in nature ('James is painting the roof of the house') as well as the properties of an object ('The roof of the house is red') which are static in nature.

As object language, Allen uses a many-sorted first-order language based on a set *TP* of sorts that contains at least the subset {*interval, object, fact, event, process, action, agent, plan, goal*}.

The time structure is based on a set I of linear convex time intervals [1] and the thirteen irreducible relations between intervals introduced in Chapter 2. Each of these relations is represented by a binary predicate symbol in the alphabet. Suppose i and j are arbitrary time intervals in I. The relationship between i and j can be expressed by exactly one of the following atomic sentences [All84]:

1. *Before*(i, j) is intended to express the idea that i is completely before j and the intervals do not overlap in any way.
2. *After*(i, j) is intended to express that i is completely after j and they do not overlap in any way.
3. *Meets*(i, j) that i is before j and there is no interval between them, i.e. j starts where i ends.
4. *MetBy*(i, j) that i is after j and there is no interval between them, i.e. i starts where j ends.
5. *Overlaps*(i, j) that i starts before j and they overlap.
6. *OverlappedBy*(i, j) that j starts before i and they overlap.
7. *Starts*(i, j) that i starts at the same time as j but ends before j.
8. *StartedBy*(i, j) that i starts at the same time as j but ends after j.
9. *During*(i, j) that j starts before i and ends after i.
10. *Contains*(i, j) that i starts before j and ends after j.
11. *Finishes*(i, j) that j starts before i but they end at the same time.
12. *FinishedBy*(i, j) that i starts before j but they end at the same time.
13. *Equals*(i, j) that i and j start and end at the same time.

These relationships are depicted in Figure 6.1.

[1] 'Convex intervals are those that do not have any gaps within them, have duration, and are not necessarily indivisible.' [Bou95]

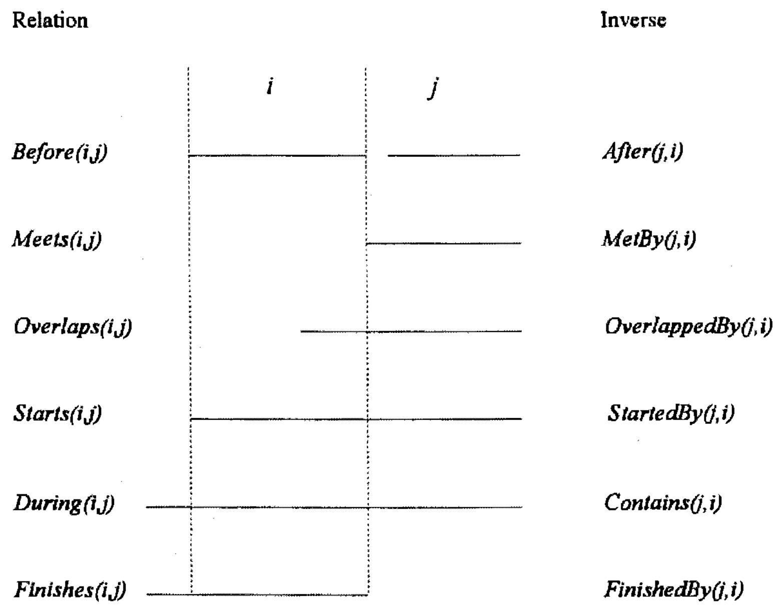

Figure 6.1: Interval-interval relationships

We choose the time structure for a specific application by the formulation of a set of axioms to describe the content of these relations. Transitivity plays a prominent role. Let r_m and r_n be interpretations of two of the predicate symbols used above. Given three intervals i, j and k, if ir_mj and jr_nk, then we might want to know what the relationship between i and k could be. Allen [All83] summarises the set of all possible relationships that may exist between i and k. This is given in the table on page 95 (omitting '='). If r_m holds between i and j, and r_n holds between j and k, then the entry associated with (r_m, r_n) in the table lists all the possible relationships that may hold between i and k.

To limit the size of the table to some extent, we use abbreviations to represent the relations. (Note the errors in the table on page 836 of [All83]: 'dur' should be 'd', and 'con' should be 'di'.)

We introduce these properties into the system by the formulation of a set of transitivity axioms. Consider the entry associated with ('o', 'oi') as an example. With the symbols used in the table, we can express the meaning of this entry as follows:

$$(i \text{ o } j) \wedge (j \text{ oi } k) \rightarrow ((i \text{ o } k) \vee (i \text{ oi } k) \vee (i \text{ d } k) \vee (i \text{ di } k) \vee (i = k)).$$

$j r_n k$ / $i r_m j$	<	>	d	di	o	oi	m	mi	s	si	f	fi
'before' <	<	no info	< o m d s	<	<	< o m d s	<	< o m d s	<	<	< o m d s	<
'after' >	no info	>	> oi mi d f	>	> oi mi d f	>	> oi mi d f	>	> oi mi d f	>	>	>
'during' d	<	>	d	no info	< o m d s	> oi mi d f	<	>	d	> oi mi d f	d	< o m d s
'contains' di	< o m di fi	> oi di mi si	o oi d di =	di	o di fi	oi di si	o di fi	oi di si	di fi o	di	di si oi	di
'overlaps' o	<	> oi di mi si	o d s	< o m di fi	< o m	o oi d di =	<	oi di si	o	di fi o	d s o	< o m
'over-lapped-by' oi	< o m di fi	>	oi d f	> oi mi di si	o oi d di =	> oi mi	o di fi	>	oi d f	oi > mi	oi	oi di si
'meets' m	<	> oi mi di si	o d s	<	<	o d s	<	f fi =	m	m	d s o	<
'met-by' mi	< o m di fi	>	oi d f	>	oi d f	>	s si =	>	d f oi	>	mi	mi
'starts' s	<	>	d	< o m di fi	< o m	oi d f	<	mi	s	s si =	d	< m o
'started by' si	< o m di fi	>	oi d f	di	o di fi	oi	o di fi	mi	s si =	si	oi	di
'finishes' f	<	>	d	> oi mi di si	o d s	> oi mi	m	>	d	> oi mi	f	f fi =
'finished-by' fi	<	> oi mi di si	o d s	di	o	oi di si	m	si oi di	o	di	f fi =	fi

In the language used by Allen, this property can be expressed by the following axiom:

TR.01 $Overlaps(i,j) \wedge OverlappedBy(j,k) \rightarrow (Overlaps(i,k) \vee OverlappedBy(i,k) \vee During(i,k) \vee Contains(i,k) \vee Equals(i,k)).$

We can express the transitivity of '<' in a similar way:

TR.02 $Before(i,j) \wedge Before(j,k) \rightarrow Before(i,k).$

Apart from *During*, Allen defines an alternative predicate symbol *In* so as to collapse the three 'during' relations denoted by *During*, *Starts* and *Finishes*, into one relation.

$$In(i,j) \leftrightarrow (During(i,j) \vee Starts(i,j) \vee Finishes(i,j))$$

means that i is a proper subinterval of j. In a similar way, we can define a new predicate symbol *Con* in order to collapse the three 'containment' relations, denoted by *Contains*, *StartedBy* and *FinishedBy*, into one.

$$Con(i,j) \leftrightarrow (Contains(i,j) \vee StartedBy(i,j) \vee FinishedBy(i,j)).$$

Now that we know what the time structure of the system looks like, we can consider the way Allen expresses the static and dynamic aspects of systems.

6.1.1 Static aspects

As in Situation Calculus, a predicate symbol *Holds* of sort *(fact, time)* is used to represent static aspects of the world. $Holds(p, t)$ is true if fact (or property) p holds over interval t. A set of axioms can be used to describe the properties of the *Holds* predicate symbol. The first axiom formulates the most important characteristic of static aspects of the system: if a fact holds over an interval, it holds over all subintervals of that interval (i.e. the *Holds* predicate symbol is *hereditary* [VB91]).

H.01 $Holds(p,t) \leftrightarrow \forall t'(In(t',t) \rightarrow Holds(p,t'))$.

A slightly stronger axiom can also be used from which the above-mentioned axiom is derivable.

H.02 $Holds(p,t) \leftrightarrow \forall t'(In(t',t) \rightarrow \exists s(In(s,t') \wedge Holds(p,s)))$.

Allen presumably introduces this axiom to exclude the possibility of *intermingling* [2] [Gal90].

The function symbols *and, or, not, all* and *exists* are used to abbreviate more complex logical expressions and are defined as follows:

H.03 $Holds(and(p,q),t) \leftrightarrow (Holds(p,t) \wedge Holds(q,t))$. (Conjunction of two atomic sentences of the form $Holds(p,t)$.)

H.04 $Holds(not(p),t) \leftrightarrow \forall t'(In(t',t) \rightarrow \neg Holds(p,t'))$. (Negation of the propositional term in an atomic sentence of the form $Holds(p,t)$, also referred to as *property negation* [3] [Gal90].)

Note that $Holds(not(p,t))$ is not the same as $\neg Holds(p,t)$. Using axiom *H.04*, if

$$Holds(not(p),t) \text{ then } \forall t'(In(t',t) \rightarrow \neg Holds(p,t')),$$

so, for every subinterval t' of t, p fails to hold over t'. Using the hereditary property of *Holds*, however, $\neg Holds(p,t)$ means that it is not the case that for all subintervals t' of t, we have $Holds(p,t')$. So, if there exists just one subinterval of t, say t'', where p fails to hold, it follows that $\neg Holds(p,t)$.

The function symbol *or* is defined in the following way:

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[2] This phenomenon is also known as a *fuzz* (a term used by Prior) or *clustered variation*.

[3] Also called *strong negation* [Sho88].

$H.05$ $Holds(or(p,q),t) \leftrightarrow Holds(not(and(not(p),not(q))),t)$. (Disjunction.)

From the definition of *or* given above, it can be shown that we can derive the following consequence (in fact, it can be shown that this consequence is equivalent to *H.05*):

$$Holds(or(p,q),t) \leftrightarrow \forall t'(In(t',t) \rightarrow \exists s(In(s,t') \wedge (Holds(p,s) \vee Holds(q,s))))$$

The *Holds* predicate is sufficient for the expression of static aspects of the world. Aspects that are dynamic in nature can vary considerably. We look at the way Allen caters for this variety in the next section.

6.1.2 Dynamic aspects

In contrast to Situation Calculus and Extended Situation Calculus, Allen distinguishes between events, processes and actions. We discuss the way Allen's system handles the notions of an event and a process in this section and in the next section, which deals with causality, the notion of an action is considered.

Events

The predicate symbol *Occurs* is used to express the occurrence of an *event*.

$$Occurs(e,t)$$

is intended to express the notion that event e occurs over interval t. The characteristic property of an event is that, if it occurs over an interval t, it does not occur over any subinterval of t :

$O.01$ $(Occurs(e,t) \wedge In(t',t)) \rightarrow \neg Occurs(e,t')$

Each occurrence of an event is called an *instance* of the event and such an instance can only be defined by specifying the unique interval over which it occurs. Different *event classes* can be defined by associating a set of necessary, or necessary and sufficient conditions, with each class. Such definitions are of the form

$$Occurs(e,t) \rightarrow P_t$$

for necessary conditions and

$$Occurs(e,t) \leftrightarrow P_t \wedge \forall t'(In(t',t) \rightarrow \neg P_{t'})$$

for necessary and sufficient conditions. For the sake of readability, we will only state the necessary conditions. Let us look at an example.

Example 22 *Defining an event class.*

Terms representing the events that involve a person travelling from one town to another may be constructed with the help of the function symbol *Travels* of sort *(agent, location, location, event)*. The sentence

$$Occurs(Travels(a,x,y),t)$$

expresses the notion that the event 'Agent a travels from town x to town y', takes place over interval t.

If agent a travels from x to y over interval t, it means that over the interval immediately preceding t, a is at location x and in the interval immediately succeeding t, a is at location y. These are necessary (and in fact sufficient) conditions for the class of events involving a *change of location* of an agent and this can be expressed in the following way:

$$Occurs(Travels(a,x,y),t) \leftrightarrow$$
$$\exists t',t''(Holds(At(a,x),t') \wedge Holds(At(a,y),t'') \wedge Meets(t',t) \wedge Meets(t,t''))$$

where $At(a,x)$ expresses the notion that agent a is at location x. The converse direction stipulates that, for person a to change location, an instance of the travelling event class has to occur.

Suppose we also have a class of event terms representing a person changing location by taking a flight from one town to another. The event term $Flies(a,x,y)$, where *Flies* is a function symbol of sort *(agent, location, location, event)*, expresses the idea of person a taking a flight from x to y. This class does not have the same necessary and sufficient conditions as the class of events involving a change of location: the conditions for change of location are also necessary for the class of events involving taking a flight, but not sufficient because there might now be different ways in which a person can travel from x to y.

We can also represent events that are composites of other events. In order to express the notion of an event e repeated at least twice in an interval t, the following necessary and sufficient conditions can be defined:

$$Occurs(Twice(e),t) \leftrightarrow$$
$$\exists t_1,t_2(In(t_1,t) \wedge In(t_2,t) \wedge (t_1 \neq t_2) \wedge Occurs(e,t_1) \wedge Occurs(e,t_2))$$

where *Twice* is a function symbol of sort (*event, event*).

The notion of a class of events such that each event e in the class is a sequence of at least two events e_1 and e_2, where e_2 occurs strictly after e_1, can be defined as follows:

$$Occurs(TwoSeq(e_1, e_2), t) \leftrightarrow \exists t_1, t_2(Starts(t_1, t) \wedge Finishes(t_2, t) \wedge After(t_2, t_1) \wedge Occurs(e_1, t_1) \wedge Occurs(e_2, t_2)).$$

where *TwoSeq* is a function symbol of sort (*event, event, event*).

The notion of a class of events that consists of events, each of which is the composite of two events e_1 and e_2 occurring simultaneously, can be defined as follows:

$$Occurs(Composite(e_1, e_2), t) \leftrightarrow (Occurs(e_1, t) \wedge Occurs(e_2, t))$$

where *Composite* is a function symbol of sort (*event, event, event*).

Processes

A *process*, like an event, is dynamic in nature. The predicate symbol *Occurring* of sort (*process, time*) can be used to express the notion of process r occurring over interval t, written as $Occurring(r, t)$.

How do processes differ from events? A suggestion of Allen's is that the number of times an event takes place can be counted whereas this cannot be done for a process. Furthermore, an event usually involves some product or outcome [All84] which is not the case for a process. Processes can be regarded as something between events and properties: while a process may be said to occur over a subinterval of the interval over which it occurs (unlike an event), it is not the case that it necessarily occurs over all the subintervals (unlike a property). As with events, a class of related processes can be defined by specifying the necessary, or necessary and sufficient conditions, for elements of that class.

Example 23 *A process.*

Let's look at the travel example again. Suppose person a travels by car from x to y over the interval t, so the event $Occurs(Travels(a, x, y), t)$ occurs over t, and suppose $Drives(a)$ expresses the idea that agent a is travelling by car (not necessarily driving). The process $Drives(a)$ will occur in at least one but probably most of the subintervals of t. If the interval grain size is small, or the travelling distance between x and y is great, some subintervals of t will be used for filling up the car or for getting something to eat and so on, so $Drives(a)$ will not be true in those. If the grain size is large or the distance between x and y is small, it might be the case that $Drives(a)$ occurs over all subintervals of t.

To define this formally: if a process is occurring over an interval t, it must be occurring over at least one subinterval, but not necessarily all subintervals, of t. This property of processes can be expressed by the axiom

$$O.02 \quad Occurring(r,t) \rightarrow \exists t'(In(t',t) \wedge Occurring(r,t'))$$

It might be necessary for some classes of processes to stipulate that the process does not only occur over at least one subinterval but over all subintervals. Let $Falling(a)$ express the notion that object a is in the process of falling. The idea that the process of falling occurs over all subintervals of the interval over which the process takes place, can be expressed by the following axiom:

$$O.03 \quad Occurring(Falling(a),t) \leftrightarrow \forall t'(In(t',t) \rightarrow Occurring(Falling(a),t'))$$

This is a good example for illustrating that an event can be a composite of an event and a process. Suppose we want to express the idea that object a falls from x to y. Let the term $ChangePos(a,x,y)$ express the notion that the position of object a changes from x to y. The event of falling can be defined as a composite of the event of changing position and the process of falling in the following way:

$$Falls(a,x,y) \leftrightarrow Composite(ChangePos(a,x,y), Falling(a)),$$

where the definition of *Composite* is extended to include processes. It is clear that it is frequently the case the there exists a close relationship between events and processes. An event has to take place for one or more of the properties of an object to change and during the interval that the event occurs, the occurrence of some process may result in the event taking place.

It is possible to define necessary and sufficient conditions for some processes, e.g. physical processes. However, there are many processes, especially those involving human activity, for which such definitions are not possible [All84].

In a language with the expressive power of Allen's, it is possible to represent metalinguistic notions that in less expressive languages are either not representable at all or else are conflated into one ambiguous sentence. Let us use an example to illustrate.

Example 24 *Expressiveness.*

Suppose we want to represent the following two English sentences that might erroneously be thought of as equivalent.

1. There are a number of people who have been complaining constantly, for the past n years, about the postal service during the Christmas season.

2. Over the past n years, a number of people have complained about the postal service during the Christmas season.

In the first instance, we actually want to say that the same people have complained every year for the past n years. In the second instance, we want to express the notion that a number of people have been complaining every year for the past n years, but not necessarily the same people from year to year. These two concepts can be expressed unambiguously by the following two sentences: Let $Occurring(Complain(a), i)$ express the notion that person a has complained (at least once) during interval i (where i represents an interval of one year.)

1. $\exists i_1, \ldots, i_n(Meets(i_1, i_2) \wedge \ldots \wedge Meets(i_{n-1}, i_n) \wedge$
   $\exists a_1, \ldots, a_m(Occurring(Complain(a_1), i_1) \wedge \ldots \wedge Occurring(Complain(a_m), i_1) \wedge \ldots \wedge$
   $Occurring(Complain(a_1), i_n) \wedge \ldots \wedge Occurring(Complain(a_m), i_n)))$.

2. $\exists i_1, \ldots, i_n(Meets(i_1, i_2) \wedge \ldots \wedge Meets(i_{n-1}, i_n) \wedge$
   $\exists a_1, \ldots, a_{m_1}(Occurring(Complain(a_1), i_1) \wedge \ldots \wedge Occurring(Complain(a_{m_1}), i_1)) \wedge \ldots \wedge$
   $\exists a_1, \ldots, a_{m_n}(Occurring(Complain(a_1), i_n) \wedge \ldots \wedge Occurring(Complain(a_{m_n}), i_n)))$.

6.1.3 Causality

The notion of causality, i.e. one event causing another, can be expressed by introducing a predicate symbol *Ecause* of sort (*event, time, event, time*), such that

$$Ecause(e, t, e', t')$$

expresses the notion that the occurrence of event e over interval t causes the occurrence of event e' over interval t'.

The causal relationship between events e and e', i.e. the idea that if e occurs and is a cause of e', then e' also occurs, can be expressed by the following axiom:

O.04 $Occurs(e, t) \wedge Ecause(e, t, e', t') \rightarrow Occurs(e', t')$

The notion that an event cannot cause other events prior to its own occurrence can be introduced as follows:

O.05 $Ecause(e, t, e', t') \rightarrow (In(t', t) \vee Before(t, t') \vee Meets(t, t') \vee Overlaps(t, t') \vee Equals(t, t'))$.

The *Ecause* relation, considered as a binary relation on pairs (e, t), is transitive and anti-symmetric. Allen also claims that this relation is anti-reflexive. Intuitively, one feels that this should be the case since events do not typically cause themselves, but this does not follow from the axioms. Axiom *O.05* allows t and t' to be equal, so, if we do not stipulate that $e \neq e'$, the relation might be reflexive.

Actions

Another important subclass of occurrences identified by Allen are those that involve an agent performing a specific *action*. Such an action is defined as an occurrence caused in a direct way by the agent. Such an action can either be a process ('James is dancing') or an event ('James switched on the light.').

Associated with this class, we introduce a new form of causality called *agentive causality*, which expresses the idea of an action by an agent being the direct cause of the occurrence of an event or a process. This can be expressed using the function symbol *Acause* of sort (*agent, occurrence, action*) such that

$Acause(a, o)$

expresses the notion of agent a causing occurrence o, where *occurrence* o can either be of sort *event* or of sort *process*.

As in the case of events and processes, we can identify a related class of actions by specifying the necessary, or necessary and sufficient conditions, for each class. Actions which cause an event to occur are called *performances,* whereas actions which cause the occurrence of a process are called *activities*.

Example 25 *Defining an action class.*

Suppose we want to introduce an agent as the driver of a car involved in the driving process discussed in Example 24. We can accomplish this by using the function symbol *DriveAction* of sort (*agent, agent, location, location, action*).

$DriveAction(a, b, x, y)$ abbreviates $Acause(a, Drives(b, x, y))$

expressing the notion that agent b travels by car, which is driven by a, from x to y.

'It is hypothesised that every action can be characterised as an agent *Acause*-ing an occurrence.' [All84] The following three axioms can be introduced for the occurrence of actions:

A.01 $Occurring(Acause(a,o),t) \rightarrow Occurring(o,t)$

If agent a causes the occurrence of o over interval t, then o occurs over t.

Every action c is the association of a unique agent a with a unique occurrence o caused by a. The following axiom is an expression of this idea:

A.02 $\forall c \exists! a \exists! o (c = Acause(a,o))$

For the class of performances, we have a stronger version of *A.01* using the *Occurs* predicate symbol:

A.03 $Occurs(Acause(a,e),t) \rightarrow Occurs(e,t)$,

i.e. if agent a causes the occurrence of event e over t, then e occurs over t. Constructors for composite actions can be used to describe actions consisting of a sequence of actions, and to describe actions consisting of actions being performed simultaneously.

Allen also proposes a way for representing actions that can be closely related, such as having two distinct actions where the one is performed in order to perform the other. This is relevant for planning.

6.1.4 Beliefs and plans

Allen's system makes provision for representing such notions as the beliefs and the intentions of an agent. These would be relevant in a planning model since they might play a role in the decisions the agent has to make. A sequential model of belief is used. In this model, a belief of an agent might be represented by a binary predicate symbol *Believes* of sort (*agent, proposition*). But where does time fit into this picture? Well, there are two relevant time indices to each belief: the time over which the belief is held, and the time over which the fact is believed to be true. ('I might believe today that it rained last weekend.') Thus the beliefs of an agent are atoms of the form $Believes(a,p,t_p,t_b)$, where the predicate symbol *Believes* is of sort *(agent, fact, time, time)*, and the atom expresses the notion that agent a believes over t_b, that p holds over time t_p.

The notion of a sequence of actions is particularly relevant in a planning model. One can think of a *plan* as a complex action, i.e. a set of decisions about performing or not performing an action. A plan occurs if all its decisions are carried out. The notion of an agent *committing* to a plan over a time interval is taken to mean that the agent believes it will act according to the plan. In order to represent this, we can introduce a predicate symbol *ToDo* of sort (*action, time, plan*), such that

$ToDo(c,t,l)$

expresses the notion that plan l includes performing action c at time t. In contrast to this, the predicate symbol *NotToDo* of sort *(action, time, plan)* can be introduced, where

$$NotToDo(c, t, l)$$

expresses the notion that plan l includes not performing action c at time t. This is much stronger than $\neg ToDo(c, t, l)$, which merely states that plan l does not include action c.

A desired goal is usually the reason for committing to a plan. The goal is not part of the plan but rather part of some desired world. This can be expressed by atoms of the form $IsGoalOf(a, g, t_g, t)$, where the predicate symbol *IsGoalOf* of sort *(agent, goal, time, time)* is used to record that the desired world (at time t) of agent a contains goal g holding over time t_g. The notion of an agent committing to a plan can now be introduced using the predicate symbol *Committed* of sort (*agent, plan, time*), so that

$$Committed(a, l, t_l)$$

expresses the notion that agent a is committed over the interval t_l to act in accordance with plan l.

We can talk of plan l occurring if all its actions are carried out. In order to do this, the *Occurs* predicate symbol is extended to also accept terms of sort *plan* as argument:

$$Occurs(l, t) \leftrightarrow \forall c, t_c[(ToDo(c, t_c, l) \rightarrow Occurs(c, t_c)) \wedge \\ \forall c, t_c(NotToDo(c, t_c, l) \rightarrow (\forall t'(In(t', t_c) \rightarrow \neg Occurs(c, t'))))]$$

Allen also describes ways in which a distinction can be made between actions that are performed intentionally and those that are performed accidentally.

The *event* of an agent committing to a plan can be represented by the term

$$Commit(a, l)$$

where *Commit* is a function symbol of sort (*agent, plan, event*). The class of such events can be defined as follows:

$$Occurs(Commit(a, l), t) \leftrightarrow \\ \exists t_1, t_2(Meets(t_1, t) \wedge Meets(t, t_2) \wedge \neg Committed(a, l, t_1) \wedge Committed(a, l, t_2))$$

We can also make provision for an agent changing his mind by defining the event class *ChangeMind* in the following way:

$$Occurs(ChangeMind(a, p, t_p), t) \leftrightarrow \\ \exists t_1, t_2(Meets(t_1, t) \wedge Meets(t, t_2) \wedge Believes(a, p, t_p, t_1) \wedge Believes(a, not(p), t_p, t_2))$$

The framework as proposed by Allen can be used to study general problem-solving behaviour and for natural language understanding. Allen & Koomen have investigated the application of this logic to plan-formation [Sad87]. The formalism has also been used for plan recognition and plan generation, as well as in a dialogue-understanding system that was implemented in LISP ([Sad87] and [All84]).

6.1.5 Refinements

The system proposed by Allen is not without limitations.

- Shoham [Sho88] points out that Allen's axiom *H.02* is not natural and actually allows some weird models such as models in which time has the structure of the reals, but in which a property holds over an interval only if it holds over those subintervals whose end points are rational.

- Galton suggests that Allen's property-negation does not always tally with our ordinary idea of the negation of a property [Gal90]. The interval-based time structure, as it stands, is not adequate for reasoning correctly about continuous change. Galton identifies the source of this inadequacy as Allen's determination to base his theory on intervals rather than on instants. There are properties such as 'X is not at P' which cannot be substituted for p in *H.02*. If a moving body X passes through P, it will be at P, but only for an instant, not for an interval, so we must conclude that 'X is not at P' cannot be expressed in the form *not*(q), nor can it legitimately be substituted for p in *H.02* [Gal90].

- Allen's system cannot distinguish between a body's being in a certain position for an instant, i.e. being true instantaneously, and its being at rest there, i.e. being true over an interval [Gal90]. Allen assumes that all facts (properties) can be treated similarly. Galton argues that one should at least distinguish between the following two kinds of facts:

  - *states of position* (These are states that can hold at isolated instants. If a state of position holds throughout an interval, then it must hold at the limits of that interval, e.g. a body being in a particular position, or moving at a particular speed or in a particular direction.)
  - *states of motion* (These are states that cannot hold at isolated instants. If a state of motion holds at an instant, then it must hold throughout some interval within which that instant falls, e.g. a body's being at rest or in motion.).

- Causality is handled in a rather simplistic way. To reflect the causal relationship between two events, one would normally expect to have a set of preconditions, similar to the set of preconditions we had for *ESC.04* in Extended Situation Calculus.

- No special formal semantics is provided for the temporal arguments. Haugh [Hau87] claims that, consequently, it does not ensure that absurd temporal consequences cannot be derived, or that all of the genuine temporal consequences will be derived.

Various attempts have been made to expunge some of the limitations and to refine Allen's system:

- Allen & Hayes [AH85] propose a system based on only one relation, namely *Meets*, that expresses the notion of two intervals meeting. This relation is formulated in such a way that the 12 relations that may hold between two distinct intervals are all definable in terms of the *Meets* relation.
- In [AH85], Allen & Hayes show how instants can be constructed as certain special sets of intervals (each instant may be identified either with the set of intervals that contain it or with the set of intervals that are bounded by it).
- Tsang [Tsa87] presents an alternative axiomatisation. His motivation is that Allen & Hayes' axiomatisation is not primitive enough to allow for extensions.
- Sadri [Sad87] proposes modifications to Allen's logic to incorporate some features for default reasoning such as default equality of intervals, and default completion of partially described events (as in Event Calculus).
- Galton's system attempts to treat instants and intervals on an equal footing. An instant is defined as the point where two intervals meet. Within such a temporal structure, it is possible to express the idea of a property's being true instantaneously as well as the idea of continuous change. He finds it unnecessary to introduce a category of processes separate from properties and events. The range of predicates assigning temporal locations to properties and occurrences is also diversified [Gal90].
- Allen & Ferguson [AF94] present a logic in which complex temporal relationships can be expressed and which is able to handle external events and simultaneous actions in a simple way. Their goal is the development of practical planning and natural language understanding systems.
- Ma & Knight [MKP94] define a revised theory using both the ideas of Allen and those of Galton. Time intervals (positive duration) and time points (zero duration) are regarded as primitive, and the temporal relations between intervals are extended to address points as well. They also do not distinguish a category of processes separate from properties and events but subsume processes under properties.

- In [SC96], Song & Cohen present a strengthened system for temporal reasoning about plans. Among other applications, this system can be applied to the process of plan recognition through the analysis of natural language input.

Allen does not define an adequate model theory for this system with the result that we do not have a clear interpretation of the system. In the next section, we look at a system proposed by Shoham & Goyal [SG88] that tries to rectify this. This system is particularly interesting since reification is done slightly differently to the way it is handled in Situation Calculus.

6.2 Shoham's reified language

In [SG88], Shoham & Goyal present a temporal logic for the propositional case as well as one for the first-order case. We discuss the first-order case, which uses a language similar to Allen's.

Shoham & Goyal's language is best understood as being many-sorted, with an alphabet $\mathcal{A}$ based on the set $TP = \{time,\ object,\ relation,\ proposition\}$ of sorts, and consisting of the following symbols:

- a set TC of zero, one, or more constants of sort *time*
- a set C of zero, one, or more constants of sort *object*
- a set REL of pairs (r, n) where r is a constant of sort *relation* and n is the arity associated with r. Each r also has a sort tuple $(object, \ldots, object)$ associated with it
- a set TV of variables $t_0, t_1, \ldots$ of sort *time*
- a set V of variables $x_0, x_1, \ldots$ of sort *object*
- a set $TFUN$ consisting of one or more elements of the form (tf, n) where tf is a temporal function symbol having arity n associated with it. Each tf also has a sort $(T_1, \ldots, T_n, T_{n+1})$ associated with it, where $T_j = time$ for $j = 1, \ldots, n+1$ (typically, the members of $TFUN$ would be arithmetic operators)
- a set FUN consisting of one or more elements of the form (f, n) where f is a nontemporal function symbol having arity n associated with it. Each f also has a sort $(T_1, \ldots, T_n, T_{n+1})$ associated with it, where $T_j \neq time$ is the sort of argument j of f for $j = 1, \ldots, n$ and $T_{n+1} \neq time$ is the resulting sort. In particular,

we have a function symbol ξ^{n+1} of sort $(relation, object, \ldots, object, proposition)$ which gives terms of the form $\xi^{n+1}(r, trm_1, \ldots, trm_n)$ of sort *proposition*. We write $\xi^{n+1}(r, trm_1, \ldots, trm_n)$ as $r(trm_1, \ldots, trm_n)$ for short. Every n has a unique ξ^{n+1} associated with it

- the set *PRED* of elements of the form (A, n) where A is a predicate symbol having arity n associated with it. Each A also has a sort $(T_1, \ldots, T_n)$ associated with it where T_j is the sort of argument j of A. *PRED* contains exactly three elements, namely $(=, 2)$ and $(\leq, 2)$, both of sort *(time, time)*, and the element (*True*, 3) of sort (*time, time, proposition*)

- the connectives $\neg$ and $\rightarrow$

- the quantifier $\forall$.

The set of *temporal terms* is defined as follows:

- if $t \in TC$, then t is a temporal term

- if $t \in TV$, then t is a temporal term

- if $t_1, \ldots, t_n$ are temporal terms and *tf* is an n-ary temporal function symbol, then $tf(t_1, \ldots, t_n)$ is a temporal term.

The set of *nontemporal terms* is defined as follows:

- if $c \in C$, then c is a nontemporal term of sort *object*

- if $x \in V$, then x is a nontemporal term of sort *object*

- if f is a nontemporal function symbol having arity n with associated sort tuple $(T_1, \ldots, T_n, T_{n+1})$, and $trm_1, \ldots, trm_n$ are nontemporal terms of the appropriate sorts, then $f(trm_1, \ldots, trm_n)$ is a nontemporal term of sort T_{n+1}. In particular, if r is a relation constant of arity n and if $trm_1, \ldots, trm_n$ are terms of sort *object*, then $\xi^{n+1}(r, trm_1, \ldots, trm_n)$, abbreviated as $r(trm_1, \ldots, trm_n)$, is a nontemporal term of sort *proposition*.

If t_a and t_b are temporal terms, and $r(trm_1, \ldots, trm_n)$ is a nontemporal term of sort *proposition*, then $(t_a = t_b)$, $(t_a \leq t_b)$ and $True(t_a, t_b, r(trm_1, \ldots, trm_n))$ are atomic *wffs* over $\mathcal{A}$. Let Φ be the set of all atomic *wffs* over $\mathcal{A}$.

The set of well-formed formulae or *wffs* is the smallest set such that:

- if $\varphi \in \Phi$, then φ is a *wff*
- if φ and ψ are *wffs*, then $(\varphi \rightarrow \psi)$ and $\neg\varphi$ are *wffs*
- if φ is a *wff*, and $x \in TV \cup V$ is a variable, then $\forall x(\varphi)$ is a *wff*.

The connectives $\wedge, \leftrightarrow$ and $\vee$, as well as the existential quantifier $\exists$, can be introduced by the usual abbreviations. The set of all *wffs* over the alphabet $\mathcal{A}$ constitutes the many-sorted language $\mathcal{L}_S$. A sentence is a *wff* that contains no free variables.

An interpretation is a pair $I = (D, i)$ such that, for every sort T, D has a subdomain of that sort. Let the subdomain of sort *time* be $D_{time} = TW$, the subdomain of sort *object* be $D_{object} = W$, the subdomain of sort *relation* be $D_{relation} = RL$ and the subdomain of sort *proposition* be $D_{proposition} = PL$. The meaning function i is defined as follows:

- $i(t) \in D_{time} = TW$ for all $t \in TC$
- $i(c) \in D_{object} = W$ for all $c \in C$
- $i(r) \in D_{relation} = RL$ for all $(r, n) \in REL$
- $i(tf) : TW^n \rightarrow TW$ for all $(tf, n) \in TFUN$
- $i(f) : TW \times TW \times D_{T_1} \times \cdots \times D_{T_n} \rightarrow D_{T_{n+1}}$ for all $(f, n) \in FUN$. In particular, $i(\xi^{n+1}) : TW \times TW \times RL \times W^n \rightarrow PL$
- $i(=)$ is the identity relation on TW
- $i(\leq) \subseteq TW \times TW$
- $i(True) \subseteq TW \times TW \times PL$.

Thus the meaning of a nontemporal function may vary with time, e.g. a function symbol defining the president of the United States of America would be permitted to have different meanings at different times.

A *variable assignment* v is defined as follows:

- $v(t) \in D_{time} = TW$ for all $t \in TV$
- $v(x) \in D_{object} = W$ for all $x \in V$.

The combination of the meaning function i and the variable assignment v induce a denotation function m on arbitrary terms. Since nontemporal terms may have meanings that vary over time, the denotation function m should be time-dependent. This may be made explicit either by defining a variety of denotation functions $m_{(\tau_1,\tau_2)}$ for all $\tau_1, \tau_2 \in TW$, or by introducing the parameters $\tau_1, \tau_2 \in TW$ as arguments of m. We follow the latter course. However, in many cases the denotation of a term is not time-dependent and in these cases we will simplify the notation by writing $m(t)$ instead of $m(\tau_1, \tau_2, t)$. Thus the denotation function m is defined as follows:

- if $t \in TV$, then $m(t) = v(t)$
- if $t \in TC$, then $m(t) = i(t)$
- if $t = tf(t_1, \ldots, t_n)$, for some $(tf, n) \in TFUN$, then $m(t) = i(tf)(m(t_1), \ldots, m(t_n))$
- if $t \in V$ and $\tau_1, \tau_2 \in TW$, then $m(\tau_1, \tau_2, t) = v(t)$
- if $t \in C$ and $\tau_1, \tau_2 \in TW$, then $m(\tau_1, \tau_2, t) = i(t)$
- if $(t, n) \in REL$ and $\tau_1, \tau_2 \in TW$, then $m(\tau_1, \tau_2, t) = i(t)$
- if $t = f(trm_1, \ldots, trm_n)$ for some $(f, n) \in FUN$, and if $\tau_1, \tau_2 \in TW$, then $m(\tau_1, \tau_2, t) = i(f)(\tau_1, \tau_2, m(\tau_1, \tau_2, trm_1), \ldots, m(\tau_1, \tau_2, trm_n))$. In particular, if $t = \xi^{n+1}(r, trm_1, \ldots, trm_n)$ then $m(\tau_1, \tau_2, t) = i(\xi^{n+1})(\tau_1, \tau_2, m(\tau_1, \tau_2, r),$ $m(\tau_1, \tau_2, trm_1), \ldots, m(\tau_1, \tau_2, trm_n))$.

Let t_a and t_b be temporal terms, $trm_1, \ldots, trm_n$ terms of sort *object*, and r any relation constant. We say that a *wff* φ is satisfied in an interpretation $I = (D, i)$, in the context of the variable assignment v, denoted by $(I, v) \Vdash \varphi$, iff one of the following cases applies:

- $\varphi = (t_a = t_b)$ and $m(t_a) = m(t_b)$
- $\varphi = (t_a \leq t_b)$ and $(m(t_a), m(t_b)) \in i(\leq)$
- $\varphi = True(t_a, t_b, r(trm_1, \ldots, trm_n))$ and $[m(t_a), m(t_b), m(m(t_a), m(t_b), \xi^{n+1}(r, trm_1, \ldots, trm_n))] \in i(True)$
- $\varphi = (\alpha \to \psi)$ and $(I, v) \nVdash \alpha$ or $(I, v) \Vdash \psi$, or both
- $\varphi = (\neg\alpha)$ and $(I, v) \nVdash \alpha$
- $\varphi = \forall x(\alpha)$ and $(I, v') \Vdash \alpha$ for every v' that differs from v at most on the variable x.

The predicate symbol *True* is known as a 'truth'-predicate symbol and is similar to the *Holds* predicate used in the systems discussed in the preceding section in this chapter. *True* associates a propositional term with a time interval.

Shoham & Goyal [SG88] distinguish different kinds of propositional terms such as terms that are fact-like, terms that are event-like and so on, not by introducing them as separate objects, but rather by specifying how the truth of the term over one interval is related to its truth over other intervals. They give the following categorisation of propositional terms for the representation of 'facts, properties, events, and other animals' [SG88]:

- A proposition is *downward-hereditary* if, whenever it holds over an interval, it also holds over all of its subintervals (possibly excluding its two end points) ('the robot travelled less than 2 miles'). This corresponds to Allen's characterisation of facts or properties, and it can be formulated by the axiom

  $\forall t_1, t_2, t_3, t_4(((t_1 \leq t_3 \leq t_4 \leq t_2) \wedge (t_1 \neq t_4 \wedge t_3 \neq t_2) \wedge \mathit{True}(t_1, t_2, \varphi)) \rightarrow \mathit{True}(t_3, t_4, \varphi))$.

- A proposition is *upward-hereditary* if, whenever it holds over all proper subintervals of a nonpoint interval (except possibly its end points), it also holds over the interval itself ('the robot travelled at a speed of two miles per hour'). This idea can be expressed by the axiom

  $\forall t_1, t_2((t_1 < t_2) \wedge \forall t_3, t_4[(t_1 \leq t_3 \leq t_4 \leq t_2) \wedge (t_1 \neq t_3 \vee t_4 \neq t_2) \wedge$
  $(t_1 \neq t_4 \wedge t_3 \neq t_2) \rightarrow \mathit{True}(t_3, t_4, \varphi)] \rightarrow \mathit{True}(t_1, t_2, \varphi))$.

- A proposition is *point-downward-hereditary* if, whenever it holds over an interval, it holds at all of its internal points.

- A proposition is *point-upward-hereditary* if, whenever it holds at all internal points of some nonpoint interval, it also holds over the nonpoint interval itself.

- A proposition is *interval-downward-hereditary* if, whenever it holds over an interval, it also holds over all of its nonpoint subintervals ('the robot travelled at a constant speed of 5mph' [Sho87]).

- A proposition is *interval-upward-hereditary* if, whenever it holds over all nonpoint subintervals of some nonpoint interval, it also holds over the nonpoint interval itself ('the robot was motionless' [Sho87]).

- A proposition is *liquid* if it is both upward- and downward-hereditary ('the robot's arm was in the *Grasping* state'). These are propositional terms that are homogeneous.

- A proposition is *concatenable* [4] if, whenever it holds over two consecutive intervals, it also holds over their union ('the robot travelled an even number of miles').

[4] Shoham & Goyal also use the term *clay-like*

- A proposition is *gestalt* if it never holds over two intervals, one of which properly contains the other ('exactly six minutes passed').

- A proposition is *solid* if it never holds over two properly overlapping intervals. This means that if the proposition is true over an interval, it is not true over any of the subintervals ('the robot executed the *Navigate* procedure from start to finish'). Allen's events correspond either to gestalt propositions, or to solid propositions, or to both [Sho88].

We could also give similar definitions for *point-point-liquid* propositional terms (coinciding with Allen's facts [Sho87]), *interval-interval-liquid, point-interval-liquid* and *interval-point-liquid* propositional terms.

Shoham & Goyal claim that this approach is richer and more flexible than either the property/event/process trichotomy of Allen or the fact/event dichotomy of McDermott. The categorisation of propositional terms has the advantage that a distinction need only be made when necessary, e.g. if we want to introduce the notion of causation, 'x causes y' say, then x and y may be facts or events or in some other category. Furthermore, if we want to categorise temporal propositions, we can do this in as fine a grain as we need to.

There has been some criticism of Shoham's system:

- Reichgelt [Rei87] feels that the categorisation, although rather elegant, leads to less expressive power and less logical clarity which may lead to a possible confusion between things that need to be distinguished.

- Bacchus et al. [BTK91] claim that the fact that the truth of a propositional term is expressed in terms of an interval (in this case, a pair of time points), is restrictive. They propose a system in which the terms of the language are partitioned into only two sorts, namely temporal and nontemporal, such that each atomic predicate symbol has a set of nontemporal arguments as well as *any* number of temporal arguments. They argue that this results in time being given a special syntactic and semantic status without having to resort to reification. They also show that this logic completely subsumes Shoham's logic.

- Vila [Vil94] states that the *True* predicate symbol necessarily includes everything in the nontemporal part of an atomic sentence in its scope, to be interpreted under the same time points. It is thus not clear how a sentence like 'Jordi is dancing today with the girl that was wearing the red skirt yesterday' can be expressed.

Chapter 7

Applications and related directions

In this dissertation we described a number of temporal logics, some using first-order languages and others using propositional modal languages as object language. We also discussed the various options available for deciding on the characteristics of the underlying temporal structure of such a logic. It is clear from the literature that these issues are to a great extent determined by the intended application.

7.1 Applications

There are a host of different applications where temporal logic may be used. These include applications from for example theoretical computer science, linguistics, artificial intelligence and physics:

- *Natural language understanding/processing:* The uses of the tense and aspect of a natural language need to be analysed to prepare it for the computer [GHR94]. Allen's system has been used for the representation of events, actions and interactions that arise in natural language dialogue and for speech recognition systems involving conversational agents [All98].

- *Planning:* 'We can treat planning as a form of reasoning about actions and plans in which time and modality play essential roles' [GHR94]. For example, we need the ability to look at different possible futures, i.e. for *forward* causal reasoning, 'what happens if I do this?', and to look at different possible pasts, i.e. for *backward* causal reasoning, 'what must I do to get that to happen?'.

- *Databases:* In many applications like project management and operational control, for example, the history of data in a changing world is relevant. Temporal databases provide ways of dealing with historical data (e.g. [BT98], [GHR94]).

- *Distributed systems:* Temporal logic can be used for describing communication protocols between stations and for describing the temporal conditions for turning stations on and off (see [GHR94] and [Gol92]).

- *Program specification and verification:* Burstall first introduced the idea of using modal logic to reason about computer programs [RN95]. Temporal logics are widely used in the field of program specification and verification. Manna and Pnueli, for example, made major contributions with their research in this field (see, for example [MP81b] and [MP82] and [MP81a]).Temporal logic can be used for

  - describing the properties of programs and the way in which they change states with execution
  - proving the properties of programs
  - controlling the execution of programs
  - proving the correctness of parallel programs, including both safety and liveness properties (see [Hai82], [GHR94] and [MP89]).

- *Expert systems:* In an expert system, the reasoning capability is equal in importance to the database. Let us consider a medical expert system for example. Many of the relevant facts are time-dependent. For diagnosis, the medical history of the patient, both long-term and short-term, may be important. We may also use temporal terminology to accurately describe prognoses and courses of treatment [GHR94].

A number of executable temporal logics, for example *Templog, Tempura, MetateM,* and *Tokio* [FO95], have been developed for the implementation of applications.

7.2 Related directions

Apart from the applications listed above, there are numerous related research fields based on the concepts discussed in this dissertation. We list a few of these that we hope to take a closer look at:

- *Distributed Temporal Logic.* Temporal logics are well-suited for specifying temporal properties of concurrent systems. Distributed temporal logic is a high-level specification technique that deals with distributed data and concurrent programs as well as with concurrent workflow and communication among different sites [ECSD98].

- *Graphical Interval Logic:* Dillon et.al. [DKM+94] claim that the specifications of even moderate-sized concurrent systems are complex and that most people find standard temporal logics difficult to understand and use for this purpose. Graphical interval logic is a tool set developed for use by software engineers for formal specification and verification of concurrent software systems. It is an interval-based temporal logic that has an intuitive graphical representation. (See [MS88], [Ram93] and [RMSM+93].)

- *Agent-oriented programming.* Temporal logic can be used as the basis for the formal description of dynamic agent behaviour in reactive systems. It has many advantages as a descriptive technique for agents, particularly in a multi-agent system [Fis94].

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