CRITICAL CONCEPTS IN DOMINATION, INDEPENDENCE AND IRREDUNDANCE OF GRAPHS

by

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Contents

Summary

The lower and upper independent, domination and irredundant numbers of the graph $G = (V, E)$ are denoted by $i(G)$, $\beta(G)$, $\gamma(G)$, $\Gamma(G)$, $ir(G)$ and $IR(G)$ respectively. These six numbers are called the *domination parameters*. For each of these parameters π, we define six types of criticality. The graph G is *π-critical* (*π^+-critical*) if the removal of any vertex of G causes $\pi(G)$ to decrease (increase), G is *π-edge-critical* (*π^+-edge-critical*) if the addition of any missing edge causes $\pi(G)$ to decrease (increase), and G is *π-ER-critical* (*π^--ER-critical*) if the removal of any edge causes $\pi(G)$ to increase (decrease). For all the above-mentioned parameters π there exist graphs which are π-critical, π-edge-critical and π-ER-critical. However, there do not exist any π^+-critical graphs for $\pi \in \{ir, \gamma, i, \beta, IR\}$, no π^+-edge-critical graphs for $\pi \in \{ir, \gamma, i, \beta\}$ and no π^--ER-critical graphs for $\pi \in \{\gamma, \beta, \Gamma, IR\}$. Graphs which are γ-critical, i-critical, γ-edge-critical and i-edge-critical are well studied in the literature. In this thesis we explore the remaining types of criticality.

We commence with the determination of the domination parameters of some well-known classes of graphs. Each class of graphs we consider will turn out to contain a subclass consisting of graphs that are critical according to one or more of the definitions above. We present characterisations of γ-critical, i-critical, γ-edge-critical and i-edge-critical graphs, as well as of π-ER-critical graphs for $\pi \in \{\beta, \Gamma, IR\}$. These characterisations are useful in deciding which graphs in a specific class are critical. Our main results concern π-critical and π-edge-critical graphs for $\pi \in \{\beta, \Gamma, IR\}$. We show that the only β-critical graphs are the edgeless graphs and that a graph is IR-critical if and only if it is Γ-critical, and proceed to investigate the Γ-critical graphs which are not β-critical. We characterise β-edge-critical and Γ-edge-critical graphs and show that the classes of IR-edge-critical and Γ-edge-critical graphs coincide. We

also exhibit classes of Γ^+-critical, Γ^+-edge-critical and i^--ER-critical graphs.

Key terms: domination, independence, irredundance, vertex-critical graphs, edge-critical graphs, edge-removal-critical graphs.

Chapter 1
Introduction

This chapter contains the basic definitions and notations needed in this thesis, including some basic results on independence, domination and irredundance and their related parameters ir, γ, i, β, Γ and IR. For each of these parameters we define six notions of criticality and determine for which of them there exist graphs that are critical. Finally, we briefly discuss existing results on criticality and outline the scope and main results of this thesis. For all undefined graph-theoretical terms we refer the reader to [16] .

1.1 Independence, domination and irredundance

A *graph* G is an ordered pair (V_G, E_G) where V_G is a finite set of *vertices* and E_G is a set of two-element subsets of V_G, called the *edges* of G. We will often denote an edge $\{u, v\}$ by uv. If $uv \in E_G$, we say that u and v are *adjacent*, or that they are *neighbours* in G. The *open neighbourhood of a vertex* v, denoted by $N_G(v)$, is the set $\{u \in V_G | uv \in E_G\}$ and the *closed neighbourhood* $N_G[v]$ is the set $N_G(v) \cup \{v\}$. The *degree* of $v \in V_G$ is the cardinality $|N_G(v)|$ of $N_G(v)$ and is denoted by $\deg_G(v)$. The *minimum degree* $\delta(G)$ and the *maximum degree* $\Delta(G)$ is, respectively, the minimum and the maximum of the degrees of the vertices of G. A vertex of degree 0 is an *isolated vertex* and a vertex of degree $|V_G| - 1$ is a *universal vertex* of G, i.e. an isolated vertex is adjacent to no vertices of G and a universal vertex is adjacent to every vertex of G other than itself. If $\delta(G) = \Delta(G) = r$, then G is called an r*−regular graph*. An *edgeless graph* and a *complete graph* are r-regular graphs with $r = 0$ and $r = |V_G|-1$, respectively.

Whenever the graph G is clear from the context, the sets V_G, E_G, $N_G(v)$ and $N_G[v]$ will often be denoted by V, E, $N(v)$ and $N[v]$ and the numbers $\deg_G(v)$, $\delta(G)$ and $\Delta(G)$ by $\deg(v)$, δ and Δ respectively. This applies throughout this thesis, i.e. the subscript G may be dropped and a parameter $\pi(G)$ may become π if there is no ambiguity.

Two graphs G and H are *isomorphic* if there exists a bijection $\phi : V_G \rightarrow V_H$ such that u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. Clearly this is an equivalence relation on graphs and we will write $G \cong H$ if G and H are isomorphic. A *copy* of G is a graph isomorphic to G. A graph H is a *subgraph* of the graph G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. A graph that is isomorphic to a subgraph of G will also be called a subgraph of G. The *subgraph* $G\langle S\rangle$ of G *induced by a vertex-set* S of G has vertex-set S and edge-set $\{uv \in E_G \mid u, v \in S\}$.

Two graphs G and H are *disjoint* if V_G and V_H are disjoint. The *union* $G \cup H$ of G and H has $V_{G\cup H} = V_G \cup V_H$ and $E_{G\cup H} = E_G \cup E_H$. The *disjoint union* of G and H is the union of disjoint copies of G and H. The disjoint union of n copies of G will be denoted by nG.

A graph G is *connected* if for any partition $\{V_1, V_2\}$ of V_G, there exist $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1v_2 \in E_G$; otherwise G is *disconnected*. A *component* of G is a maximal connected subgraph of G.

Suppose G is disconnected and let $\{V_1, V_2\}$ be a partition of V_G such that no vertices of V_1 are adjacent to any vertices of V_2. Clearly G is the disjoint union of the induced subgraphs $G\langle V_1\rangle$ and $G\langle V_2\rangle$ of G. Each of these subgraphs, if disconnected, can in turn be written as the disjoint union of two induced subgraphs. This procedure terminates in the decomposition of G into its components.

An *automorphism* of a graph G is an isomorphism of G onto itself. G is *vertex-transitive* if, for any two vertices u and v, there is an automorphism Φ of G such that $\Phi(u) = v$. G is *edge-transitive* if, for any two edges u_1v_1 and u_2v_2, there is an automorphism Φ of G such that $\Phi(\{u_1, v_1\}) = \{u_2, v_2\}$.

The *complement* $\overline{G}$ of the graph G has $V_{\overline{G}} = V_G$ and $uv \in E_{\overline{G}}$ if and only if $uv \notin E_G$.

The *closed neighbourhood of a set* $S \subseteq V_G$, denoted by $N_G[S]$, is the set $\bigcup_{s\in S} N_G[s]$ and the *open neighbourhood* $N_G(S)$ is the set $N_G[S] - S$. Clearly $N_G[\{v\}] = N_G[v]$ and $N_G(\{v\}) = N_G(v)$. If $S,T \subseteq V_G$, then S *dominates* T in G if $T \subseteq N_G[S]$ and if $v \in N_G[S]$, then we say that S *dominates* v. If $S \subseteq V_G$ and $s \in S$, then s is an *isolated vertex* of S in G if $N_G(s) \cap S = \emptyset$, i.e. s is an isolated vertex of the graph $G\langle S\rangle$. $S \subseteq V_G$ is an *independent set* of G if $N_G(s) \bigcap S = \emptyset$ for every $s \in S$, i.e. no two vertices in S are adjacent in G. Note, however, that $N_G(S) = \bigcup_{s\in S} N_G(s)$ only if S is an independent set of G. Observe further that independence is a hereditary property, i.e. every subset of an independent set is independent. Consequently, an independent set S of G is maximal independent if and only if $S \cup \{v\}$ is not independent for every $v \in V_G - S$. The *independence number* $\beta(G)$ and the *lower independence number* $i(G)$ are the largest and the smallest number of vertices in a maximal independent set of G, respectively.

$S \subseteq V_G$ is a *dominating set* of G if S dominates V_G, i.e. every vertex of $V_G - S$ is adjacent to at least one vertex of S. Domination is clearly a super-hereditary property, i.e. every superset of a dominating set is dominating. It follows that a dominating set S of G is minimal dominating if and only if $S - \{s\}$ is not dominating for every $s \in S$. The *domination number* $\gamma(G)$ and the *upper domination number* $\Gamma(G)$ are the smallest and the largest number of vertices in a minimal dominating set of G, respectively.

If $S \subseteq V_G$ and $s \in S$, then the *private neighbourhood of* s relative to S, denoted by

$PN_G(s, S)$, is the set $N_G[s] - N_G[S - \{s\}]$. The vertices of $PN_G(s, S)$ are called the *private neighbours* of s *relative to* S. If $PN_G(s, S) = \emptyset$, then s is a *redundant vertex* of S, otherwise it is an *irredundant vertex* of S. Note that $s \in PN_G(s, S)$ if and only if s is an isolated vertex of S. Thus, if s is an isolated vertex of S, then it is an irredundant vertex of S and if s is a non-isolated vertex of S, then $PN_G(s, S) \subseteq V_G - S$. We often refer to the vertices of $PN_G(s, S) - S$ as the *external private neighbours of s relative to* S.

$S \subseteq V_G$ is an *irredundant set* of G if $PN_G(s, S) \neq \emptyset$ for every $s \in S$, i.e. every non-isolated vertex of S has an external private neighbour. Note that independent sets are irredundant. As in the case of independence, irredundance is a hereditary property and therefore an irredundant set S of G is maximal irredundant if and only if $S \cup \{v\}$ is not irredundant for every $v \in V_G - S$. The *irredundance number* $IR(G)$ and the *lower irredundance number* $ir(G)$ are the largest and the smallest number of vertices in a maximal irredundant set of G, respectively.

Suppose S is an independent set of G and $v \in V - S$. Then $S \cup \{v\}$ is not independent if and only if S dominates v. Therefore

Proposition 1.1 (Berge [4]) *S is a maximal independent set of G if and only if S is an independent dominating set of G.*

Because of the existence of Proposition 1.1, the lower independence number i is also called the *independent domination number.* Suppose S is a dominating set of G and $s \in S$. Then $S - \{s\}$ is not dominating if and only if $PN(s, S) \neq \emptyset$. Therefore

Proposition 1.2 (Cockayne and Hedetniemi [9]) *S is a minimal dominating set of G if and only if S is an irredundant dominating set of G.*

We present a characterisation of maximal irredundance after the following lemma.

Lemma 1.3 *Suppose $S \subseteq V$, $s \in S$ and $v \in V - S$. Then*

(i) $PN(v, S \cup \{v\}) = N[v] - N[S]$; *hence v is a redundant vertex of $S \cup \{v\}$ if and only if $N[v] \subseteq N[S]$.*

(ii) $PN(s, S \cup \{v\}) = PN(s, S) - N[v]$; *hence s is a redundant vertex of $S \cup \{v\}$ if and only if $PN(s, S) \subseteq N[v]$.*

If $PN(s, S) \subseteq N[v]$, we often say that v *annihilates* s relative to S.

Proposition 1.4 *Suppose S is an irredundant set of G and $R = V - N[S]$. Then S is maximal irredundant if and only if for every $v \in N[R]$ there exists an $s_v \in S$ such that $PN(s_v, S) \subseteq N[v]$.*

Proof. Note that $v \notin N[R]$ if and only if $N[v] \subseteq N[S]$. Therefore, by Lemma 1.3 (i), S is maximal irredundant if and only if for every $v \in N[R]$ there exists an $s_v \in S$ such that s_v is a redundant vertex of $S \cup \{v\}$. Lemma 1.3 (ii) now completes the proof. ■

Proposition 1.5 (Berge [5]) *If S is a maximal independent set of G, then S is a minimal dominating set of G.*

Proof. Note that any independent set of G is irredundant. The proof now follows from Propositions 1.1 and 1.2. ■

Proposition 1.6 (Cockayne, Hedetniemi and Miller [10]) *If S is a minimal dominating set of G, then S is a maximal irredundant set of G.*

Proof. By Proposition 1.2, S is an irredundant dominating set of G. The proof now follows from Proposition 1.4 since $R = V - N[S] = \emptyset$. ■

A consequence of Propositions 1.5 and 1.6 is the following well-known string of inequalities, first mentioned in [10] .

Proposition 1.7 *For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

These six parameters will be called the *domination parameters*; ir, γ and i will be called the *lower domination parameters,* while β, Γ and IR will be called the *upper domination parameters*. By a *π-set* of G we mean a vertex-set of G realising $\pi(G)$, *eg.* a *β-set* of G is a maximal independent set X of G with $|X| = \beta(G)$.

In the following observations, G and H are arbitrary disjoint graphs.

Each of the six domination parameters π mentioned above has the properties

P1 $\pi(G \cup H) = \pi(G) + \pi(H)$.

P2 $1 \leq \pi(G) \leq |V_G|$.

P3 $\pi(G) = |V_G|$ if and only if G is edgeless.

Each lower domination parameter π has the property

P4 $\pi(G) = 1$ if and only if G has a universal vertex.

Each upper domination parameter π has the property

P5 $\pi(G) = 1$ if and only if G is complete.

Suppose G has k components C_1, C_2, ..., C_k. **P1** implies that for any domination parameter π,

$$\pi(G) = \sum_{i=1}^{k} \pi(C_i).$$

Let K_n denote the complete graph on n vertices. Then $\overline{K_n}$ is the edgeless graph on n vertices. **P3** implies that $\pi\left(\overline{K_n}\right) = n$ and **P4**, **P5** imply that $\pi(K_n) = 1$ for all domination parameters.

1.2 Criticality

When studying a particular graph parameter it is worthwhile to investigate those graphs that are in some sense critical with respect to the parameter, the reason being that knowledge of the structure of such graphs often results in a deeper insight into the parameter. Also, considerations of criticality play an important role in induction arguments, and these abound in graph theory.

For each of the six domination parameters π, we define six types of criticality. The graph G is

C1 π*-critical* if $\pi(G-v) < \pi(G)$ for all $v \in V_G$.

C2 π^+*-critical* if $\pi(G-v) > \pi(G)$ for all $v \in V_G$.

C3 π*-edge-critical* if $\pi(G+uv) < \pi(G)$ for all $uv \in E_{\overline{G}}$.

C4 π^+*-edge-critical* if $\pi(G+uv) > \pi(G)$ for all $uv \in E_{\overline{G}}$.

C5 π*-ER-critical* if $\pi(G-uv) > \pi(G)$ for all $uv \in E_G$.

C6 π^-*-ER-critical* if $\pi(G-uv) < \pi(G)$ for all $uv \in E_G$.

For any domination parameter π, all edgeless graphs with more than one vertex are both π-critical and π-edge-critical. If π is an upper parameter, then all complete graphs with more than one vertex are π-ER-critical and if π is a lower parameter, then all stars $K_{1,n}$ $(n \geq 1)$ are π-ER-critical. This establishes the existence of π-critical, π-edge-critical and π-ER-critical graphs for all domination parameters π. The following three propositions show that there exist no π^+-critical graphs for $\pi \in \{ir, \gamma, i, \beta, IR\}$, no π^+-edge-critical graphs for $\pi \in \{ir, \gamma, i, \beta\}$ and no π^--ER-critical graphs for $\pi \in \{\gamma, \beta, \Gamma, IR\}$.

Proposition 1.8 (Topp, [24]) *For any graph* G,

(a) $\beta(G-v) \leq \beta(G)$ *for all* $v \in V_G$.

(b) $IR(G-v) \leq IR(G)$ *for all* $v \in V_G$.

(c) $\gamma(G+uv) \leq \gamma(G)$ *for all* $uv \in E_{\overline{G}}$.

(d) $\beta(G+uv) \leq \beta(G)$ *for all* $uv \in E_{\overline{G}}$.

(e) $\gamma(G-uv) \geq \gamma(G)$ *for all* $uv \in E_G$.

(f) $\beta(G-uv) \geq \beta(G)$ *for all* $uv \in E_G$.

Proof. (a) A β-set S of $G-v$ is independent in G; hence $\beta(G-v) = |S| \leq \beta(G)$.
(b) An IR-set S of $G-v$ is irredundant in G; hence $IR(G-v) = |S| \leq IR(G)$.
(c) A γ-set S of G dominates $G+uv$; hence $\gamma(G+uv) \leq |S| = \gamma(G)$.
(d) A β-set S of $G+uv$ is independent in G; hence $\beta(G+uv) = |S| \leq \beta(G)$.
(e) A γ-set S of $G-uv$ dominates G; hence $\gamma(G) \leq |S| = \gamma(G-uv)$.
(f) A β-set S of G is independent in $G-uv$; hence $\beta(G) = |S| \leq \beta(G-uv)$. ■

The following proposition implies that there are no π^+-critical graphs if π is a lower parameter. We first prove a lemma.

Lemma 1.9 *For any graph* G *and* $v \in V_G$, *if* S *is a maximal irredundant set of* G *and an irredundant set of* $G-v$, *then* S *is a maximal irredundant set of* $G-v$.

Proof. Consider any $x \in V_{G-v} - S$. Then $x \in V_G - S$ and since S is a maximal irredundant set of G, $S \cup \{x\}$ is not an irredundant set of G ; hence $S \cup \{x\}$ is not an irredundant set of $G-v$. The result follows since S is an irredundant set of $G-v$.
■

Proposition 1.10 *Let π be a lower domination parameter. For any graph G with more than one vertex, $\pi(G-v) \le \pi(G)$ for at least one $v \in V_G$.*

Proof. If G has an isolated vertex v, then clearly $\pi(G-v) = \pi(G) - 1$. So assume G has no isolated vertices; hence $\pi(G) < |V_G|$.

(i) Let S be a γ-set of G and $v \in V_G - S$. Since S is a dominating set of $G - v$, it follows that $\gamma(G-v) \le |S| = \gamma(G)$.

(ii) Let S be an i-set of G and $v \in V_G - S$. Since S is an independent dominating set of $G - v$, it follows that $i(G-v) \le |S| = i(G)$.

(iii) Let S be an ir-set of G. If S is a dominating set of G, then S is a γ-set of G; hence by (i), $ir(G-v) \le \gamma(G-v) \le \gamma(G) = ir(G)$ for some $v \in V_G$. If S is not a dominating set of G, let $v \in V_G - N_G[S]$. Then S is an irredundant set of $G - v$. It follows from Lemma 1.9 that S is a maximal irredundant set of $G - v$; hence $ir(G-v) \le |S| = ir(G)$. ■

The next proposition implies that there are no π^+-edge-critical graphs if π is a lower parameter. We first prove a lemma.

Lemma 1.11 $ir(G) = \gamma(G)$ *if and only if there exists an ir-set S of G and an $x \in V_G$ such that $S \cup \{x\}$ is a dominating set of G.*

Proof. If $ir(G) = \gamma(G)$, then any γ-set S of G is an ir-set of G (by Proposition 1.6) and for any $x \in V_G$, $S \cup \{x\}$ is clearly a dominating set of G.

Now suppose S is an ir-set of G and $S \cup \{x\}$ is a dominating set of G. If $x \in S$, then S is a dominating set of G; hence $\gamma(G) \le |S| = ir(G)$. If $x \in V_G - S$, then $S \cup \{x\}$ is dominating but not irredundant in G, that is, $S \cup \{x\}$ is a dominating but not minimal dominating set of G. It follows that

$$\gamma(G) < |S| + 1,$$

hence

$$\gamma(G) \leq |S| = ir(G).$$

In both cases it follows from Proposition 1.7 that $ir(G) = \gamma(G)$. ■

Proposition 1.12 *Let π be a lower domination parameter. For any graph G which is not complete, $\pi(G+uv) \leq \pi(G)$ for at least one $uv \in E_{\overline{G}}$.*

Proof. If G has a universal vertex, then clearly $\pi(G+uv) = \pi(G) = 1$ for all $uv \in E_{\overline{G}}$ and if G is edgeless, then $\pi(G+uv) = \pi(G) - 1$ for all $uv \in E_{\overline{G}}$. So assume that G has no universal vertices and at least one edge.

(i) Let S be an i-set of G. By the assumption above, there exists $uv \in E_{\overline{G}}$ with $u \in V_G - S$. Since S is an independent dominating set of $G + uv$, it follows from Proposition 1.1 that

$$i(G+uv) \leq |S| = i(G).$$

(ii) For $\pi = \gamma$ the statement follows from Proposition 1.8 (c).

(iii) Let S be an ir-set of G. If $ir(G) = \gamma(G)$, then it follows from Propositions 1.7 and 1.8 (c) that $ir(G+uv) \leq ir(G)$ for all $uv \in E_{\overline{G}}$. So assume that $ir(G) < \gamma(G)$. By Lemma 1.11 there exists $uv \in E_{\overline{G}}$ with $u, v \in V_G - N_G[S]$. It follows from Proposition 1.4 that S is a maximal irredundant set of $G + uv$; hence

$$ir(G+uv) \leq |S| = ir(G). \quad ■$$

The following proposition shows that there are no π^--ER-critical graphs if π is an upper parameter.

Proposition 1.13 *Let π be an upper domination parameter. For any graph G with at least one edge, $\pi(G-uv) \geq \pi(G)$ for at least one $uv \in E_G$.*

Proof. (i) For $\pi = \beta$ the statement follows from Proposition 1.8 (f).

(ii) Let S be a Γ-set of G. If S is independent, then

$$\Gamma(G) = |S| \leq \beta(G) \leq \beta(G - uv) \leq \Gamma(G - uv)$$

for all $uv \in E_G$ (by Proposition 1.8 (f)). If S is not independent, then there exists $uv \in E_G$ with $u, v \in S$. Since S is a dominating irredundant set of $G - uv$,

$$\Gamma(G) = |S| \leq \Gamma(G - uv).$$

(iii) Let S be an IR-set of G. If S is independent, then

$$IR(G) = |S| \leq \beta(G) \leq \beta(G - uv) \leq IR(G - uv)$$

for all $uv \in E_G$. If S is not independent, then there exists $uv \in E_G$ with $u, v \in S$. Since S is an irredundant set of $G - uv$,

$$IR(G) = |S| \leq IR(G - uv). \quad \blacksquare$$

The types of criticality for which the existence or not remains to be decided are Γ^+-criticality, Γ^+- and IR^+-edge-criticality, and ir^-- and i^--ER-criticality.

Observe that Property P1 (see Section 1.1) implies that a graph is π-critical (π-ER-critical, π^--ER-critical, respectively) if and only if each of its components is either π-critical (π-ER-critical, π^--ER-critical) or isomorphic to K_1. A graph is π^+-critical if and only if each of its components is π^+-critical.

1.3 Outline

Graphs which are γ-critical were first studied by Brigham, Chinn and Dutton [8] in 1988 and γ-edge-critical graphs by Sumner and Blitch [21] in 1983. Further work appears in about 20 papers. These include [12, 13, 14, 15, 19, 20, 22, 23, 27, 28]. In 1994 Ao [2] extended this to a theory of i-critical and i-edge-critical graphs and made a thorough comparison of these four notions of criticality. In 1979 Walikar and Acharya [25] gave a characterisation of γ-ER-critical graphs, which was extended by Ao [2] to i-ER-critical graphs:

Theorem 1.14 [2] *A graph is i-ER-critical if and only if each of its components is a star.*

These are the only forms of domination criticality that have been studied. The purpose of this thesis is to explore the remaining ones, especially those that involve the three upper parameters.

In Chapter 2 we present some well-known bounds for the lower domination parameters and proceed to determine the domination parameters of some classes of graphs. Each class of graphs we consider will turn out to contain a subclass that consists of graphs that are critical according to one or more of the definitions in Section 1.2.

Chapter 3 deals with vertex-critical graphs. In Section 3.1 we list the subclasses of the graphs in Chapter 2 that are well-known to be γ-critical or i-critical. We present a characterisation of γ- and i-critical graphs in terms of so-called "singular isolated vertices" and find a new class of i-critical graphs. In Section 3.2 we show that the only β-critical graphs are the edgeless graphs and that a graph is IR-critical if and only if it is Γ-critical. We proceed to investigate the Γ-critical graphs which are not β-critical. In Section 3.3 we exhibit a class of Γ^+-critical graphs.

The development of Chapter 4 is analogous to that of Chapter 3 and deals with edge-critical graphs. In Section 4.1 we find characterisations of γ-edge- and i-edge-critical graphs in terms of the existence of γ-sets and i-sets with certain properties and then determine which of the graphs of Chapter 2 are γ-edge- or i-edge-critical. In Section 4.2 we characterise β-edge- and Γ-edge-critical graphs and show that a graph is IR-edge-critical if and only if it is Γ-edge-critical. In Section 4.3 we present a class of graphs that are Γ^+-edge-critical. The existence or not of IR^+-edge-critical graphs remains unresolved.

Chapter 5 deals with ER-critical graphs. In Section 5.1 we present characterisations of π-ER-critical graphs for π an upper parameter and then determine which of the graphs of Section 2.2 are π-ER-critical. In Section 5.2 we find necessary conditions for a connected graph to be ir-ER-critical and characterise the connected 2-ir-ER-critical graphs. In Section 5.3 we exhibit three classes of i^--ER-critical graphs. The existence or not of ir^--ER-critical graphs also remains unresolved.

Chapter 6 contains a brief list of open problems.

Chapter 2
Preliminaries

In this chapter we present some well-known bounds for the lower domination parameters and determine the domination parameters of some classes of graphs. The bounds are useful, either in their direct applications or because the constructions in their proofs provide insight into the relevant types of sets. The classes of graphs we consider will prove to contain subclasses which are critical according to one or more of the definitions in Section 1.2.

2.1 Some useful bounds for domination parameters

The following inequality was obtained independently by Allan and Laskar in [1] and by Bollobás and Cockayne in [6] .

Proposition 2.1 $[1, 6]$ *For any graph,* $\gamma \leq 2ir - 1$.

Proof. Let S be an ir-set of G and construct T by choosing one private neighbour for each non-isolated vertex of S. Clearly, $|T| \leq |S|$ and, by Proposition 1.4, every vertex not dominated by S is dominated by T. Therefore $S \cup T$ is a dominating set of G. However, $S \cup T$ is not a minimal dominating set of G, since it properly contains the maximal irredundant set S. Hence

$$\gamma(G) < |S| + |T| \leq 2\,|S| = 2ir(G)$$

and thus

$$\gamma \leq 2ir - 1. \quad \blacksquare$$

The next three results give lower bounds for γ and ir in terms of the maximum degree Δ. We use the following notation in their proofs: If S is an irredundant set of the graph G, then Z and Y denote the isolated and the non-isolated vertices of S, respectively, and C, B and R denote the sets of vertices of $V - S$ which are adjacent to at least two vertices, exactly one vertex and no vertices of S, respectively, *i.e.*

$$\begin{aligned} R &= V - N[S], \\ B &= \left(\bigcup_{s \in S} PN(s, S) \right) - S \quad \text{and} \\ C &= N(S) - B. \end{aligned}$$

Clearly, Z, Y, C, B and R form a partition of V and $R = \emptyset$ if and only if S is a dominating set of G.

Proposition 2.2 (Walikar, Sampathkumar and Acharya [26]) *For any n-vertex graph G,*

$$\gamma \geq \frac{n}{\Delta + 1}.$$

Proof. Consider a γ-set S of G. Each vertex of S dominates at most $\Delta + 1$ vertices of G and the bound follows. $\blacksquare$

The next two results by Bollobás and Cockayne, and Cockayne and Mynhardt, respectively, are presented here with simplified proofs.

Theorem 2.3 (Bollobás and Cockayne [7]) *For any n-vertex graph G with $\Delta \geq 2$,*

$$ir \geq \frac{n}{2\Delta - 1}.$$

Proof. Consider an ir-set S of G. Each vertex of Z dominates at most Δ vertices of $V - S$ and each vertex of Y dominates at most $\Delta - 1$ vertices of $V - S$. Therefore

$$|N[S]| \leq (\Delta + 1)\,|Z| + \Delta\,|Y|\,;$$

hence

$$n - |R| \leq (\Delta + 1)\,|Z| + \Delta\,|Y|\,. \tag{2.1}$$

Each vertex of R annihilates at least one vertex of Y. Therefore, if r_y denotes the number of vertices of R that annihilate $y \in Y$, then

$$|R| \leq \sum_{y \in Y} r_y\,.$$

For each $y \in Y$ and $w \in PN(y, S)$, note that w is adjacent to y and to r_y vertices of R. Thus

$$\sum_{y \in Y} r_y \leq (\Delta - 1)\,|Y|$$

and it follows that

$$|R| \leq (\Delta - 1)\,|Y|\,. \tag{2.2}$$

From (2.1) and (2.2),

$$\begin{aligned} n &\leq (\Delta + 1)\,|Z| + (2\Delta - 1)\,|Y| \\ &\leq (2\Delta - 1)\,|S| \quad \text{since} \quad \Delta \geq 2. \end{aligned}$$

Hence

$$ir = |S| \geq \frac{n}{2\Delta - 1}. \quad \blacksquare$$

Theorem 2.4 (Cockayne and Mynhardt [11]) *For any n-vertex graph G with $\Delta \geq 2$,*

$$ir \geq \frac{2n}{3\Delta}.$$

Proof. Consider an ir-set S of G and let l be the number of edges between S and C. Since every vertex of C sends at least two edges to S, $l \geq 2\,|C|$. Since the vertices of S send at most $\Delta\,|Z| + (\Delta - 1)\,|Y| - |B|$ edges to C,

$$l \leq \Delta\,|Z| + (\Delta - 1)\,|Y| - |B|\,.$$

Hence

$$|B| + 2\,|C| \leq \Delta\,|Z| + (\Delta - 1)\,|Y|\,. \tag{2.3}$$

Each vertex of R annihilates at least one vertex of Y. Therefore, if r_y is the number of vertices of R that annihilate $y \in Y$, then

$$|R| \leq \sum_{y \in Y} r_y\,.$$

Let $Y_1 = \{y \in Y | r_y > 0\}$ and $Y_2 = \{y \in Y | r_y = 0\}$. Then $|Y| = |Y_1| + |Y_2|$. For each $y \in Y_1$ and $w \in PN(y, S)$, note that w is adjacent to y and to r_y vertices of R. This implies that

$$1 + r_y \leq \deg(w) \leq \Delta,$$

hence

$$\sum_{y \in Y_1} r_y \leq (\Delta - 1)|Y_1|.$$

It follows that

$$|R| \leq \sum_{y \in Y_1} r_y + \sum_{y \in Y_2} r_y \leq (\Delta - 1)|Y_1| + 0. \tag{2.4}$$

Let

$$\begin{aligned} E_1 &= \bigcup_{y\in Y_1} PN(y,S), \\ E_2 &= \bigcup_{y\in Y_2} PN(y,S) \quad \text{and} \\ F &= B-(E_1\cup E_2)\,. \end{aligned}$$

Each vertex of E_1 annihilates at least one vertex of Y. Therefore, if k_y is the number of vertices of E_1 that annihilate $y \in Y$, then

$$|E_1| \leq \sum_{y\in Y} k_y\,.$$

For each $y \in Y_1$ and $w \in PN(y,S)$, since w is adjacent to y, to r_y vertices of R and to $k_y - 1$ or k_y vertices of E_1, depending on whether w annihilates y or not, it follows that

$$1 + r_y + k_y - 1 \leq \text{ deg}(w) \leq \Delta$$

and hence

$$\sum_{y\in Y_1} k_y \leq \Delta|Y_1| - |R|.$$

For each $y \in Y_2$ and $w \in PN(y,S)$, since w is adjacent to y and to k_y vertices of E_1, it follows that

$$1 + k_y \leq \text{ deg}(w) \leq \Delta,$$

hence

$$\sum_{y\in Y_2} k_y \leq (\Delta-1)|Y_2|.$$

It now follows that

$$\begin{aligned} |E_1| &\leq \sum_{y\in Y_1} k_y + \sum_{y\in Y_2} k_y \\ &\leq \Delta|Y_1| - |R| + (\Delta - 1)|Y_2|. \end{aligned}$$

Furthermore, $|E_2| \leq (\Delta - 1)|Y_2|$ and $|F| \leq \Delta|Z|$. Therefore

$$|B| + |R| \leq \Delta|Z| + \Delta|Y_1| + (2\Delta - 2)|Y_2|. \tag{2.5}$$

From (2.3), (2.4) and (2.5) it follows that

$$\begin{aligned} 2n &\leq 2|Z| + 2|Y| + \Delta|Z| + (\Delta - 1)|Y| \\ &\quad +(\Delta - 1)|Y_1| + \Delta|Z| + \Delta|Y_1| + (2\Delta - 2)|Y_2| \\ &= (2\Delta + 2)|Z| + 3\Delta|Y_1| + (3\Delta - 1)|Y_2| \\ &\leq 3\Delta|S| \quad \text{since } \Delta \geq 2. \end{aligned}$$

Hence

$$ir = |S| \geq \frac{2n}{3\Delta}. \quad \blacksquare$$

2.2 Domination parameters of some well-known classes of graphs

The *complete multipartite graph* $K_{n_1,n_2,\ldots,n_m}$ is the complement of the disjoint union $K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m}$. That is, the vertex-set of $K_{n_1,n_2,\ldots,n_m}$ has partition $\{V_1, V_2, \ldots, V_m\}$ with $|V_i| = n_i$ for $1 \leq i \leq m$ and uv is an edge of $K_{n_1,n_2,\ldots,n_m}$ if and only if u and v do not belong to the same partite set. If $m = 2$, then this graph is known as the *complete bipartite graph* and $K_{1,n}$ is called the *star* on $n + 1$ vertices.

Proposition 2.5 *If* $G = K_{n_1,n_2,...,n_m}$ *with* $m \geq 2$, *then*

$$ir = \gamma = \begin{cases} 2 & \text{if } n_i > 1 \text{ for } 1 \leq i \leq m \\ 1 & \text{otherwise} \end{cases},$$

$$i = \min\{n_i \mid 1 \leq i \leq m\},$$

$$\beta = \Gamma = IR = \max\{n_i \mid 1 \leq i \leq m\}.$$

Proof. The only independent sets of G are the partite sets V_1, V_2, ..., V_m. This establishes $i = \min\{n_i \mid 1 \leq i \leq m\}$ and $\beta = \max\{n_i \mid 1 \leq i \leq m\}$. If $n_i = 1$, then G has a universal vertex; hence $ir = \gamma = 1$. If $n_i > 1$ for $1 \leq i \leq m$, then the only non-independent irredundant sets of G are obtained by choosing one vertex from each of any two partite sets. Therefore $ir = \gamma = 2$ and $\beta = \Gamma = IR$ $(= \max\{n_i\})$. ■

The *product* $G_1 \times G_2$ of two graphs G_1 and G_2 has vertex-set $V_{G_1} \times V_{G_2}$ and two vertices $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are adjacent in $G_1 \times G_2$ if and only if $u_1 = v_1$ and $u_2v_2 \in E_{G_2}$, or $u_2 = v_2$ and $u_1v_1 \in E_{G_1}$.

Let

$$V = \{v_{ij} | i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}$$

and

$$E = \{\{v_{ij}, v_{kl}\} | v_{ij}, v_{kl} \in V, i = k \text{ and } j \neq l, \text{ or } j = l \text{ and } i \neq k\}$$

be the vertex and edge sets of the graph $K_m \times K_n$, respectively. Furthermore, let

$$X_i = \{v_{ik} | k = 1, 2, ..., n\}$$

for each $i = 1, 2, ..., m$ and

$$Y_j = \{v_{kj} | k = 1, 2, ..., m\}$$

for each $j = 1, 2, ..., n$. Note that $\langle X_i \rangle \cong K_n$ for each $i = 1, 2, ..., m$ and $\langle Y_j \rangle \cong K_m$ for each $j = 1, 2, ..., n$. See Figure 2.1, p. 21, but note that not all edges of $K_m \times K_n$

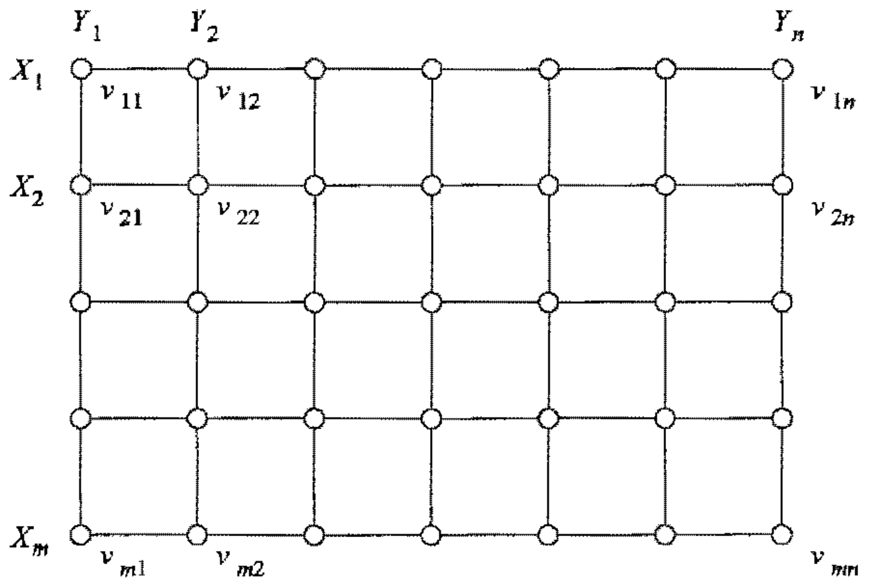


Figure 2.1

are shown.

Theorem 2.6 *Let* $G = K_m \times K_n$ *for* $n \geq m \geq 2$. *Then*

$$ir(G) = \gamma(G) = i(G) = \beta(G) = m,$$

$$\Gamma(G) = n$$

and

$$IR(G) = \begin{cases} n & \text{if } m \leq 4 \\ m + n - 4 & \text{if } m \geq 4. \end{cases}$$

Proof. Consider any maximal independent set S of G. Since S is independent, $|X_i \cap S| \leq 1$ for all $i = 1, 2, ..., m$. Therefore

$$|S| = \sum_{i=1}^{m} |X_i \cap S| \leq m.$$

Since S is dominating, $X_i \cap S \neq \emptyset$ for all $i = 1, 2, ..., m$ or $Y_j \cap S \neq \emptyset$ for all

$j = 1, 2, ..., n$. Therefore

$$|S| = \sum_{i=1}^{m} |X_i \cap S| \geq m.$$

or

$$|S| = \sum_{j=1}^{n} |Y_j \cap S| \geq n \geq m.$$

It follows that $|S| = m$ for every maximal independent set S of G ; hence

$$i(G) = \beta(G) = m.$$

Consider any minimal dominating set S of G. Again, since S is dominating, $X_i \cap S \neq \emptyset$ for all i or $Y_j \cap S \neq \emptyset$ for all j. If $X_i \cap S \neq \emptyset$ for all i, then choose $x_i \in X_i \cap S$ for each i. Since $\{x_1, x_2, ..., x_m\}$ is a dominating subset of the minimal dominating set S, it follows that $S = \{x_1, x_2, ..., x_m\}$; hence $|S| = m$. Similarly, if $Y_j \cap S \neq \emptyset$ for all j, then $|S| = n$. It follows that $|S| \in \{m, n\}$ for every minimal dominating set S of G. Furthermore, Y_1 and X_1 are minimal dominating sets with cardinalities m and n, respectively; hence

$$\gamma(G) = m \quad \text{and} \quad \Gamma(G) = n.$$

To complete the proof, we show that $n \leq |S| \leq m + n - 4$ for any maximal irredundant set S of G that is not dominating, and that there exists one with cardinality $m + n - 4$ if $m \geq 4$. To be more precise, we show that for each $c \in \{n, n+1, ..., m+n-4\}$ there exists a non-dominating maximal irredundant set with cardinality c.

Consider any maximal irredundant set S of G that is not dominating. Assume without loss of generality that $X_m \cap S = \emptyset$ and $Y_n \cap S = \emptyset$, *i.e.* the vertex v_{mn} is not dominated by S.

If $|X_i \cap S| \leq 1$ for all i, let $T = S \cup \{v_{m1}\}$. Since $Y_n \cap S = \emptyset$ and $v_{m1} \notin Y_n$, we see that $v_{in} \in PN(v_{ij}, T)$ for every $v_{ij} \in T$. Therefore T is an irredundant superset of the maximal irredundant set S, which is impossible. Hence assume, without loss of generality, that $|X_{m-1} \cap S| > 1$ and, similarly, that $|Y_{n-1} \cap S| > 1$.

Let r be the number of sets X_i for which $|X_i \cap S| = 1$ and s the number of sets Y_j for which $|Y_j \cap S| = 1$. It follows that $r \leq m-2$ and $s \leq n-2$. Assume without loss of generality that

$$\begin{aligned} |X_i \cap S| &= 1 \quad \text{for all} \quad i = 1, 2, ..., r \\ |X_i \cap S| &\neq 1 \quad \text{for all} \quad i = r+1, ..., m \\ |Y_j \cap S| &= 1 \quad \text{for all} \quad j = 1, 2, ..., s \\ |Y_j \cap S| &\neq 1 \quad \text{for all} \quad j = s+1, ..., n. \end{aligned}$$

Since S is irredundant, $|X_i \cap S| = 1$ or $|Y_j \cap S| = 1$ for every $v_{ij} \in S$. Therefore

$$\bigcup_{i=r+1}^{m} (X_i \cap S) \subseteq \bigcup_{j=1}^{s} (Y_j \cap S).$$

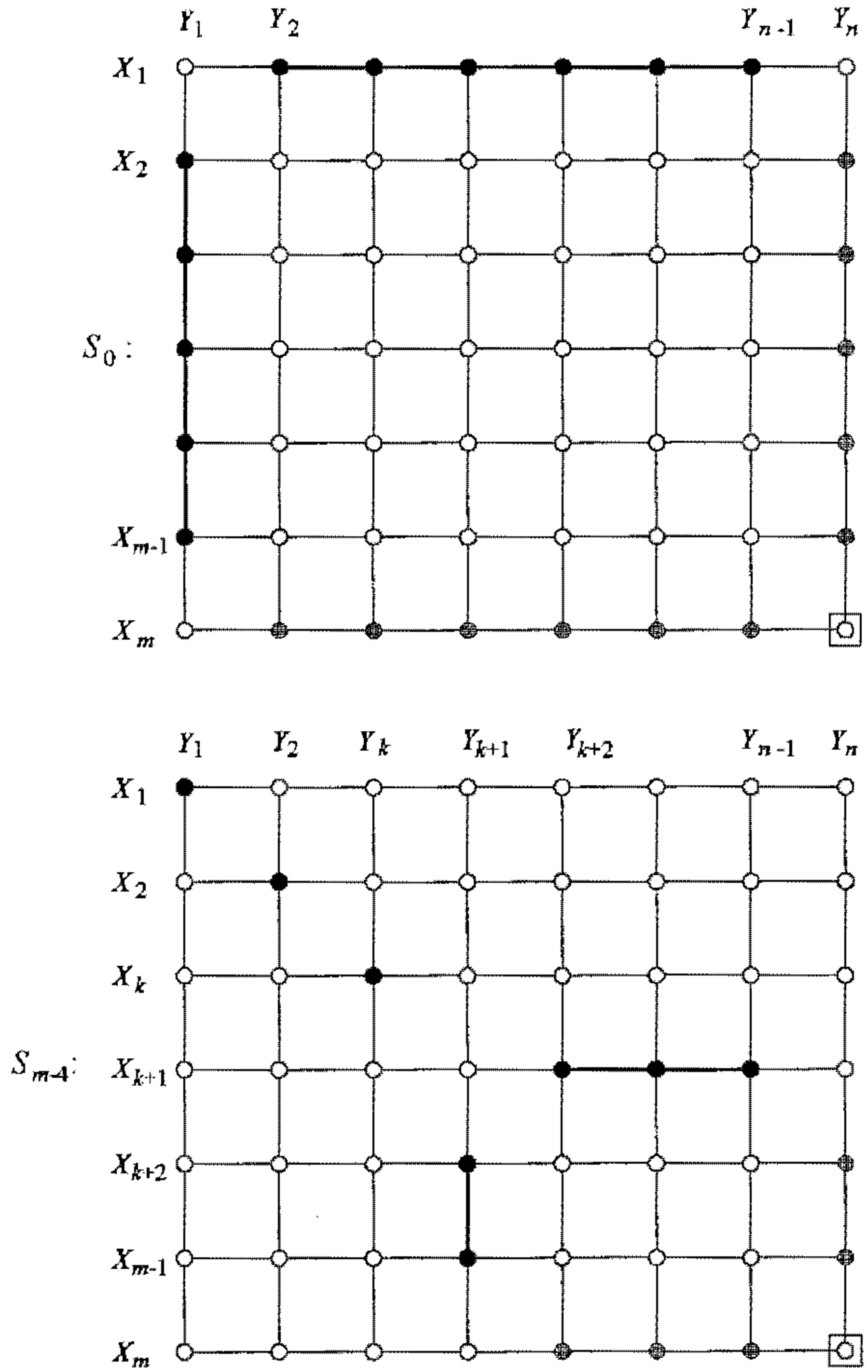
These unions are disjoint, so

$$|S| - r = \sum_{i=r+1}^{m} |X_i \cap S| \leq \sum_{j=1}^{s} |Y_j \cap S| = s.$$

Hence

$$|S| \leq r + s \leq (m-2) + (n-2) = m + n - 4.$$

Furthermore, suppose $Y_k \cap S = \emptyset$ for $k \neq n$. Then v_{mk} is not dominated by S and therefore $v_{(m-1)k}$ annihilates some $v_{ij} \in S$. If $|Y_j \cap S| = 1$, then $v_{mj} \in PN(v_{ij}, S)$; thus v_{mj} is adjacent to $v_{(m-1)k}$ and so $j = k$, which is impossible since $Y_k \cap S = \emptyset$. Consequently, $|X_i \cap S| = 1$ and it follows that $v_{in} \in PN(v_{ij}, S)$. This implies that v_{in} is adjacent to $v_{(m-1)k}$; hence $i = m-1$, which is impossible since $|X_{m-1} \cap S| > 1$.



Y_1
Y_2
Y_{n-1}
Y_n
X_1
X_2
S_0:
X_{m-1}
X_m
Y_1
Y_2
Y_k
Y_{k+1}
Y_{k+2}
Y_{n-1}
Y_n
X_1
X_2
X_k
S_{m-4}:
X_{k+1}
X_{k+2}
X_{m-1}
X_m

Figure 2.2

It now follows that $|Y_j \cap S| > 1$ for all $j = s+1, ..., n-1$ and therefore

$$\begin{aligned} |S| &= \sum_{j=1}^{n} |Y_j \cap S| \\ &= \sum_{j=1}^{s} |Y_j \cap S| + \sum_{j=s+1}^{n-1} |Y_j \cap S| + |Y_n \cap S| \\ &\geq s + 2(n-s-1) + 0 \\ &= 2n - s - 2 \\ &\geq n \quad \text{since} \quad s \leq n-2. \end{aligned}$$

For each $k \in \{0, 1, ..., m-4\}$, let

$$S_k = \{v_{11}, v_{22}, ..., v_{kk}\} \cup \{v_{(k+1)(k+2)}, ..., v_{(k+1)(n-1)}\} \cup \{v_{(k+2)(k+1)}, ..., v_{(m-1)(k+1)}\}.$$

See Figure 2.2, p. 24 for S_0 and S_{m-4} and note that k is the number of isolated vertices of S_k. (Black dots denote the vertices of the set, grey dots the external private neighbours and white squares the vertices not dominated by the set. This notation is used in all the figures in this section.) For each $k \in \{0, 1, ..., m-4\}$, S_k is a maximal irredundant set which is not dominating and $|S_k| = m + n - 4 - k$. Therefore the non-dominating maximal irredundant sets have cardinalities $n, n+1, ..., m+n-4$. ■

Theorem 2.7 *Let $G = \overline{K_m \times K_n}$ for $n \geq m \geq 2$. Then*

$$ir(G) = \gamma(G) = \min\{3, m\},$$
$$i(G) = m,$$
$$\beta(G) = \Gamma(G) = IR(G) = n.$$

Proof. The only maximal independent sets of G are X_i for $i = 1, 2, ..., m$ and Y_j for $j = 1, 2, ..., n$; hence $i(G) = m$ and $\beta(G) = n$.

We now consider the maximal irredundant sets of G that are not independent. Suppose S is such a set and assume without loss of generality that S contains the adjacent

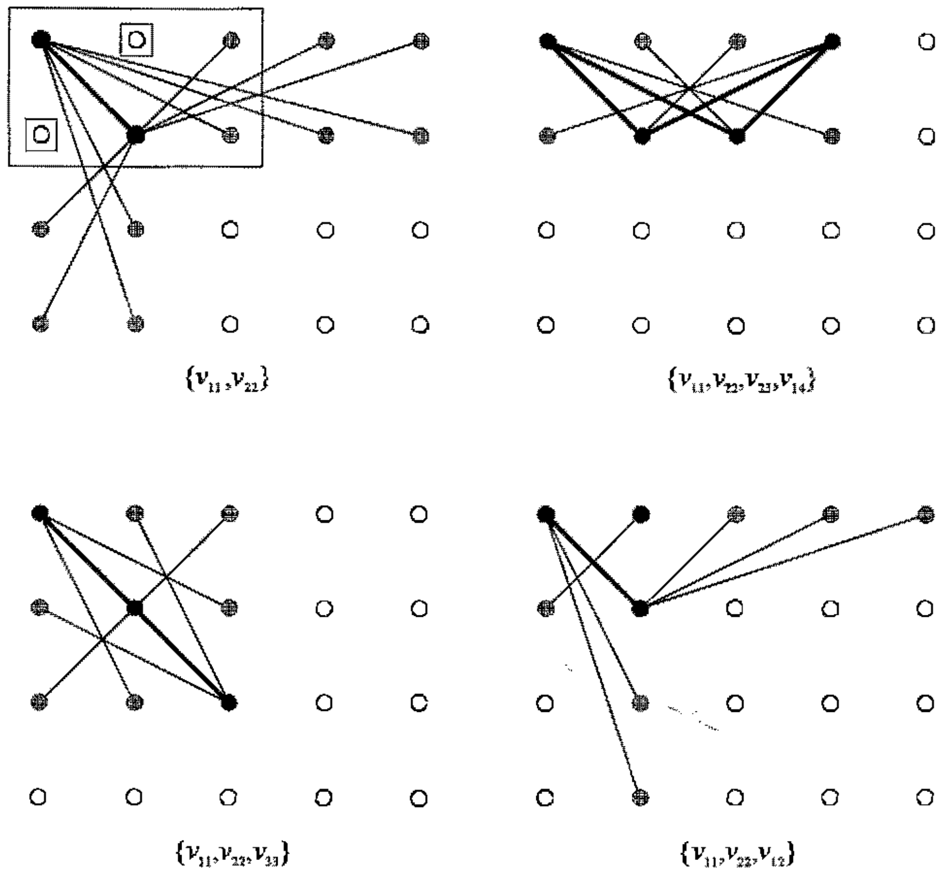


Figure 2.3

vertices v_{11} and v_{22}. See Figure 2.3, p. 26.

If $m = n = 2$, then neither v_{11} nor v_{22} has private neighbours, contradicting the irredundance of S. Therefore $n \geq 3$. If $m = 2$ and $n = 3$, then $\{v_{11}, v_{22}\}$ is a maximal irredundant set ; hence $S = \{v_{11}, v_{22}\}$ and thus $|S| = 2$.

Suppose now that $m \geq 3$ or $n \geq 4$. Then $\{v_{11}, v_{22}\}$ is not maximal irredundant and therefore is a proper subset of S. Consider $s \in S - \{v_{11}, v_{22}\}$. There are three possibilities:

Case 1: $s \in N(v_{11}) \cap N(v_{22})$. Assume without loss of generality that $s = v_{33}$. Then

$\{v_{11}, v_{22}, v_{33}\}$ is minimal dominating ; hence $S = \{v_{11}, v_{22}, v_{33}\}$ and $|S| = 3$.

Case 2: s is not dominated by $\{v_{11}, v_{22}\}$. Assume without loss of generality that $s = v_{12}$. Then $\{v_{11}, v_{22}, v_{12}\}$ is minimal dominating ; consequently $S = \{v_{11}, v_{22}, v_{12}\}$ and $|S| = 3$.

Case 3: s is adjacent to exactly one vertex of $\{v_{11}, v_{22}\}$. Assume without loss of generality that $s = v_{23}$. Then $\{v_{11}, v_{22}, v_{23}\}$ is irredundant and v_{21} is the only vertex not dominated by $\{v_{11}, v_{22}, v_{23}\}$. Now v_{21} annihilates v_{22} and v_{23}. The only vertices adjacent to v_{21} that annihilate no vertex of $\{v_{11}, v_{22}, v_{23}\}$ are $v_{14}, v_{15}, ..., v_{1n}$. Therefore, without loss of generality, S must contain v_{14} also. Now $\{v_{11}, v_{22}, v_{23}, v_{14}\}$ is minimal dominating ; hence $S = \{v_{11}, v_{22}, v_{23}, v_{14}\}$ and $|S| = 4$. ■

We define the *circulant* $C_n\langle a_1, a_2, ..., a_l\rangle$ with $0 < a_1 < a_2 < ..., a_l < n$ by specifying the vertex and edge sets, where the arithmetic is performed modulo n:

$$V = \{1, 2, ..., n\}$$

$$E = \{\{i, i+j\} \mid i = 1, 2, ..., n \text{ and } j = a_1, a_2, ..., a_l\}.$$

Consider the circulant $G = C_n\langle 1, 2, ..., r\rangle$ for $n \geq 3$ and $1 \leq r \leq \lfloor n/2 \rfloor$. Note that G is the cycle C_n if $r = 1$ and the complete graph K_n if $r = \lfloor n/2 \rfloor$. Also, if $r = n/2$, then $\Delta = 2r - 1$ and if $r < n/2$, then $\Delta = 2r$.

For each $i \in V$, $N[i] = \{i - r, ..., i - 1, i, i + 1, ..., i + r\}$. Let $LN(i) = \{i - r, ..., i - 1\}$ and $RN(i) = \{i + 1, ..., i + r\}$ and call these sets the *left* and the *right neighbourhoods* of i respectively. Clearly, $N(i) = LN(i) \cup RN(i)$ and the union is disjoint except when $r = n/2$, in which case $LN(i) \cap RN(i) = \{i - r\} = \{i + r\}$.

Theorem 2.8 *If* $G = C_n\langle 1, 2, ..., r\rangle$ *for some* $n \geq 3$ *and* $1 \leq r \leq \lfloor n/2 \rfloor$, *then*

$$ir(G) = \gamma(G) = i(G) = \lceil n/(2r+1) \rceil$$

and

$$IR(G) = \Gamma(G) = \beta(G) = \lfloor n/(r+1) \rfloor.$$

Proof. If $r = \lfloor n/2 \rfloor$, then $G = K_n$; hence all the domination parameters equal $1 = \lceil n/(2r+1) \rceil = \lfloor n/(r+1) \rfloor$. Assume henceforth that $r < \lfloor n/2 \rfloor$. Then $\Delta = 2r$ and for every $v \in V$, $LN(v) \cap RN(v) = \emptyset$.

We first prove that $ir(G) = \gamma(G) = i(G)$ and $IR(G) = \Gamma(G) = \beta(G)$ by showing that, for every maximal irredundant set S of G, there exists a minimal dominating set T with $|T| = |S|$, and for every minimal dominating set S, there exists a maximal independent set T with $|T| = |S|$.

Suppose S is an irredundant set of G and consider any two adjacent vertices x and y of S with $y \in RN(x)$. Clearly, $PN(x,S) \subseteq LN(x)$ and $PN(y,S) \subseteq RN(y)$; hence no other vertex of S is adjacent to x or y. It follows that $\Delta(\langle S \rangle) \leq 1$ and that

$$\begin{aligned} I &= \{i \in S | i \text{ is an isolated vertex of } S\} \\ X &= \{x \in S | PN(x,S) \subseteq LN(x)\} \quad \text{and} \\ Y &= \{y \in S | PN(y,S) \subseteq RN(y)\} \end{aligned}$$

are mutually disjoint sets with $|X| = |Y|$, $I \cup X$ and $I \cup Y$ independent and $S = I \cup X \cup Y$. Let

$$Z = \{x + r + 1 | x \in X\}$$

and

$$W = \{y - r - 1 | y \in Y\}.$$

For adjacent vertices $x \in X$ and $y \in Y$, $x + r + 1 \in PN(y,S)$ and $y - r - 1 \in PN(x,S)$. Therefore $Z \cap S = W \cap S = Z \cap W = \emptyset$, $I \cup Z$, $X \cup Z$, $I \cup W$ and $Y \cup W$ are independent sets and $|W| = |X| = |Y| = |Z|$. Furthermore, $N[x] \cup N[y]$ is a subset of each of the sets $N[x] \cup N[x+r+1]$, $N[y-r-1] \cup N[y]$ and $N[y-r-1] \cup N[x+r+1]$. Therefore $N[S]$ is a subset of each of the sets $N[I \cup X \cup Z]$, $N[I \cup W \cup Y]$ and $N[I \cup W \cup Z]$.

Suppose S is minimal dominating but not independent and let $T = I \cup X \cup Z$ or $I \cup W \cup Y$. It is now clear that T is an independent dominating set with $|T| = |S|$.

Suppose S is maximal irredundant but not dominating and let $T = I \cup W \cup Z$. It is clear that $|T| = |S|$ and since $x \in PN(y-r-1, T)$ and $y \in PN(x+r+1, T)$ for every adjacent pair $x \in X$ and $y \in Y$, T is an irredundant set. It remains to be proved that T is dominating.

If $v \in N[S]$, then $v \in N[T]$. If $v \notin N[S]$, then there exist adjacent vertices $x \in X$ and $y \in Y$ such that $PN(x, S) \subseteq N(v)$ or $PN(y, S) \subseteq N(v)$. Therefore $y - r - 1 \in N(v)$ or $x + r + 1 \in N(v)$; hence v is dominated by W or Z and it follows that T is a dominating set.

To complete the proof, we show that $\gamma(G) = \lceil n/(2r+1) \rceil$ and $\beta(G) = \lfloor n/(r+1) \rfloor$. By Proposition 2.2, $\gamma(G) \geq \lceil n/(2r+1) \rceil$. Let

$$S = \{(2r+1), 2(2r+1), ..., \lceil n/(2r+1) \rceil (2r+1)\}.$$

Then S is a dominating set and the only possible non-isolated vertices of S are

$$\lceil n/(2r+1) \rceil (2r+1) \quad \text{and} \quad (2r+1).$$

These vertices have r and $3r+1$ as private neighbours respectively. Therefore S is a minimal dominating set of G and it follows that $\gamma(G) \leq \lceil n/(2r+1) \rceil$.

Suppose S is a β- set of G. Let l be the number of edges between S and $V - S$. Since S is independent, each $s \in S$ sends $2r$ edges to $V - S$; hence $l = 2r|S|$. Consider any $v \in V - \dot{S}$. Since S is independent, $LN(v)$ and $RN(v)$ each contains at most one vertex of S; hence v is adjacent to at most two vertices of S. Therefore $l \leq 2|V - S|$. It follows that

$$\begin{aligned} 2r|S| &\leq 2|V - S| \\ \therefore \quad (r+1)|S| &\leq |V| \end{aligned}$$

$$\begin{aligned} \therefore \quad (r+1)\beta &\le n \\ \therefore \quad \beta &\le \lfloor n/(r+1) \rfloor . \end{aligned}$$

Finally,

$$S = \{(r+1), 2(r+1), ..., \lfloor n/(r+1) \rfloor (r+1)\}$$

is an independent set with $|S| = \lfloor n/(r+1) \rfloor$ and we have thus proved that $\beta = \lfloor n/(r+1) \rfloor$. ■

Consider the circulant $G = C_n \langle 1, 3, 5, ..., 2r-1 \rangle$ for $1 \le r \le (n-1)/2$. For each $v \in V$,

$$N[v] = \{v - 2r + 1, v - 2r + 3, ..., v - 1, v, v + 1, ..., v + 2r - 3, v + 2r - 1\}.$$

Let

$$LN(v) = \{v - 2r + 1, v - 2r + 3, ..., v - 1\}$$

and

$$RN(v) = \{v + 1, ..., v + 2r - 3, v + 2r - 1\}.$$

Since $r \le (n-1)/2$, both $LN(v)$ and $RN(v)$ has cardinality r and neither contains v. Therefore $N(v) = LN(v) \cup RN(v)$.

If n is odd, then $LN(v) \cap RN(v) = \emptyset$; hence $\Delta = 2r$. In this case, each $r \in \{1, 2, 3, ..., (n-1)/2\}$ gives a different graph G. If $r = 1$, then $G = C_n$ and if $r = (n-1)/2$, then $G = K_n$.

If n is even, then we assume $r \le (n+2)/4$, for

$$C_n \langle 1, 3, ..., 2r - 1 \rangle = C_n \langle 1, 3, ..., 2\lfloor (n+2)/4 \rfloor - 1 \rangle$$

whenever $r > (n+2)/4$. If $r < (n+2)/4$, then $LN(v) \cap RN(v) = \emptyset$; hence $\Delta = 2r$. If $r = (n+2)/4$, then $LN(v) \cap RN(v) = \{v - 2r + 1 = v + 2r - 1\}$;

hence $\Delta = 2r - 1$.

Suppose $n = \Sigma_{i=1}^{k} n_i$ is a partition of n such that each n_i is odd and $n_i \geq 2r + 1$. Let $V_G = \cup_{i=1}^{k} N_i$ be a partition of V_G such that each N_i consists of n_i consecutive vertices of V_G. This is illustrated in Figure 2.4 with $G = C_{19} \langle 1, 3 \rangle$ and the partitions $5 + 5 + 9$, $5 + 7 + 7$ and 19, of 19.

For each i, if

$$N_i = \{v + 1, v + 2, v + 3, ..., v + n_i\},$$

let

$$S_i = \{v + r + 1, v + r + 3, v + r + 5, ..., v + n_i - r\}.$$

Then $S_i \subseteq N_i$ and $|S_i| = (n_i - 2r + 1)/2$. It is not difficult to check that $S = \cup_{i=1}^{k} S_i$ is an independent dominating set of G. We call S an *independent dominating set of* G *induced by the partition* $n = \Sigma_{i=1}^{k} n_i$ and we say that n_i *contributes* $(n_i - 2r + 1)/2$ vertices to S. Now

$$|S| = \sum_{i=1}^{k} (n_i - 2r + 1)/2 = [n - (2r - 1)k]/2.$$

Theorem 2.9 *Let* $G = C_n \langle 1, 3, ..., 2r - 1 \rangle$, *where* $1 \leq r \leq (n-1)/2$, *and let* $n = (2r + 1)m + q$ *for some integer* m *and where* $0 \leq q \leq 2r$. *Then*

$$\beta(G) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 - r & \text{if } n \text{ is odd} \end{cases}$$

$$i(G) = \begin{cases} m + q/2 & \text{if } q \text{ is even} \\ m + r + (q-1)/2 & \text{if } q \text{ is odd} \end{cases}$$

Proof. Consider any independent set S of G. Clearly $|S| \leq n/2$ and if $|S| = n/2$, then the only independent dominating sets of G are $\{2, 4, ..., n\}$ and $\{1, 3, ..., n-1\}$.

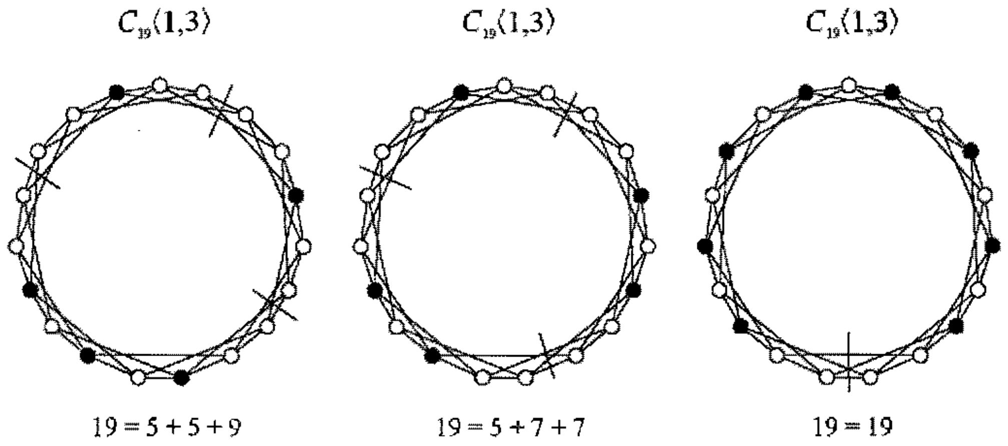


Figure 2.4

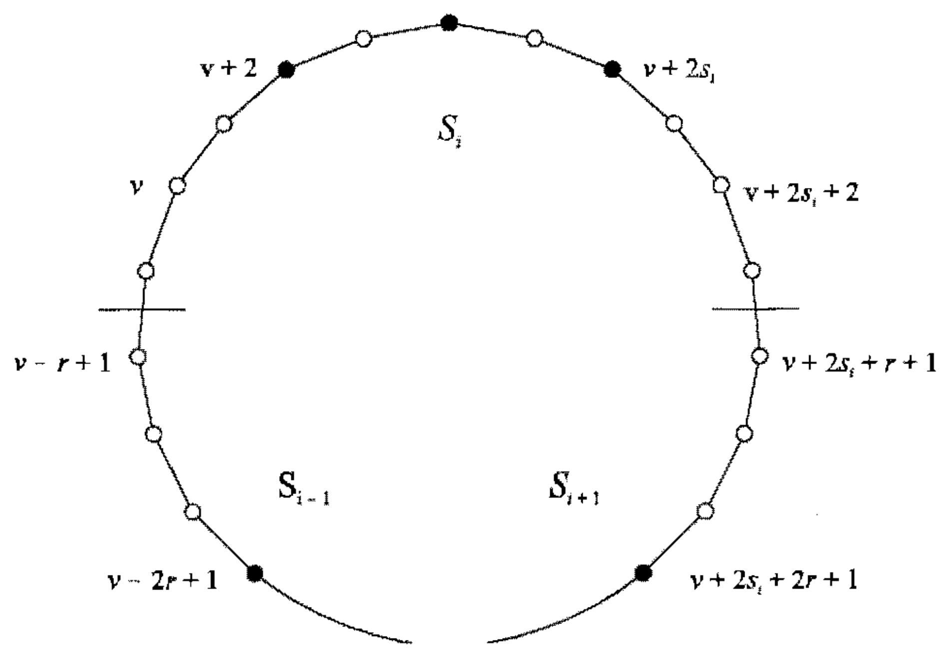


Figure 2.5

Assume now that $|S| < n/2$. Then S has a partition $S = \cup_{i=1}^{k} S_i$ such that each S_i $(1 \leq i \leq k)$ has the form

$$S_i = \{v + 2, v + 4, ..., v + 2s_i\}$$

with $s_i = |S_i|$, $v \in V_G$ and $\{v, v + 2s_i + 2\} \subseteq V_G - S$. See Figure 2.5, p. 32. Since v is not dominated by S_i, it must be dominated by the rightmost vertex of S_{i-1}, which is $v - 2r + 1$. Therefore

$$\{v - 2r + 2, ..., v - r + 1, v - r + 2, ..., v + 1\} \subseteq V_G - S.$$

Similarly, $v + 2s_i + 2$ is dominated by the leftmost vertex $v + 2s_i + 2r + 1$ of S_{i+1}; hence

$$\{v + 2s_i + 1, ..., v + 2s_i + r, v + 2s_i + r + 1, ..., v + 2s_i + 2r\} \subseteq V_G - S.$$

Let

$$N_i = \{v - r + 2, ..., v + 2s_i + r\}$$

and $n_i = |N_i|$. Then $V_G = \cup_{i=1}^{k} N_i$ is a partition of V_G and $n_i = 2r + 2s_i - 1$. It is now clear that $n = \Sigma_{i=1}^{k} n_i$ is a partition of n into k odd numbers $n_i \geq 2r + 1$ and that S is induced by this partition of n.

Thus, the only independent dominating sets of G are those induced by the partitions $n = \Sigma_{i=1}^{k} n_i$ of n into k odd numbers $n_i \geq 2r + 1$, and also $\{2, 4, ..., n\}$ and $\{1, 3, ..., n - 1\}$ if n is even. It follows that the cardinalities of the independent dominating sets of G are

$$\{[n-(2r-1)k]/2 \mid k \in \{0,2,...,m\}\} \quad \text{if } n \text{ and } q \text{ are both even,}$$

$$\{[n-(2r-1)k]/2 \mid k \in \{0,2,...,m-1\}\} \quad \text{if } n \text{ is even and } q \text{ is odd,}$$

$$\{[n-(2r-1)k]/2 \mid k \in \{1,3,...,m\}\} \quad \text{if } n \text{ is odd and } q \text{ is even,}$$

$$\{[n-(2r-1)k]/2 \mid k \in \{1,3,...,m-1\}\} \quad \text{if } n \text{ and } q \text{ are both odd.}$$

Note that $n-(2r-1)m = 2m+q$ and $n-(2r-1)(m-1) = 2m+2r-1+q$. The theorem now follows by evaluating the least and the greatest elements of these sets. ■

Theorem 2.10 *Let* $G = C_n(1,3,...,2r-1)$, *where* $1 \le r \le (n-1)/2$, *and let* $n = (2r+1)m+q$ *for some integer* m *and where* $0 \le q \le 2r$.

(a) *If* $q = 0$, *then* $\gamma(G) = m$.

(b) *If* $q = 2$, *then* $\gamma(G) = m+1$.

(c) *If* q *is odd, then* $\gamma(G) = m+1$.

Proof. Suppose $r = (n+2)/4$. Then $\Delta = 2r-1$ and $r \ge 2$. By Proposition 2.2,

$$\gamma \ge n/(\Delta+1) = (4r-2)/2r = 2-1/r.$$

Since G has no universal vertices, $\gamma(G) \ne 1$ and thus $\gamma(G) = 2$.

Assume henceforth that $r \ne (n+2)/4$. Then $\Delta = 2r$. By Proposition 2.2,

$$\gamma(G) \ge n/(\Delta+1) = m + q/(2r+1).$$

If $q = 0$, then $\gamma(G) \ge m$. By Theorem 2.9, $i(G) = m$. Therefore

$$m \le \gamma(G) \le i(G) = m$$

and (a) holds. Similarly, if $q = 2$, then

$$m+1 \le \gamma(G) \le i(G) = m+1$$

and thus (b) also holds.

Now suppose that q is odd. For $1 \leq j \leq m-1$, let

$$N_j = \{(j-1)(2r+1)+1, (j-1)(2r+1)+2, ..., j(2r+1)\}$$

and

$$N_m = \{(m-1)(2r+1)+1, (m-1)(2r+1)+2, ..., n\}.$$

Then

$$|N_j| = 2r+1 \text{ for } 1 \leq j \leq m-1,$$

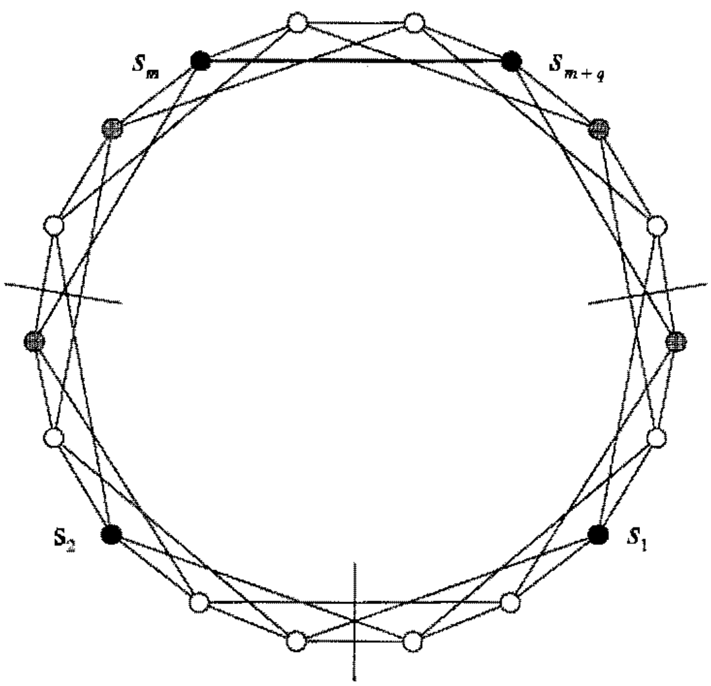
$$|N_m| = 2r+1+q$$

and $V = \cup_{j=1}^{m} N_j$ is a partition of V. For $1 \leq j \leq m$, let $s_j = j(2r+1) - r$ and let $S = \{s_1, s_2, ..., s_m, s_m + q\}$. This is illustrated in Figure 2.6, p. 35 with $G = C_{18}(1,3)$. Clearly S is a minimal dominating set of G with s_m and $s_m + q$ the only non-isolated vertices. It follows that

$$\gamma(G) \leq |S| = m+1.$$

By Proposition 2.2, $\gamma(G) \geq m+1$ and hence $\gamma(G) = m+1$. This proves (c). ■

$C_{18}\langle 1,3\rangle$



$18 = 5 + 5 + 8$

Figure 2.6

Chapter 3
Vertex-Critical Graphs

In this chapter we discuss π-critical and π^+-critical graphs. We have seen that the edgeless graphs $\overline{K_n}$, $n \geq 2$ are n-π-critical and that there exists no π^+-critical graphs for $\pi \in \{ir, \gamma, i, \beta, IR\}$. Also, recall that a graph is π-critical (π^+-critical, respectively) if and only if each of its components is either π-critical or K_1 (π^+-critical, respectively).

3.1 Lower domination parameter critical graphs: vertex removal

Graphs that are γ-critical were first studied by Brigham, Chinn and Dutton in [8] , where they showed that the only 2-γ-critical graphs are $\overline{nK_2}$, $n \geq 1$. They presented some properties of γ-critical graphs and a method of constructing them, and concluded their study with the following questions:

(1) If G is γ-critical, is $|V| \geq (\delta + 1)(\gamma - 1) + 1$?

(2) If G is γ-critical and $|V| = (\Delta + 1)(\gamma - 1) + 1$, is G regular?

(3) Is $i = \gamma$ for all γ-critical graphs?

(4) Let d be the diameter of the γ-critical graph G. Is it always true that $d \leq 2(\gamma - 1)$?

These questions were answered by Fulman, Hanson and MacGillivray in [15] . They proved that (2) and (4) are true, but presented $C_{17}\langle 1, 3, 5, 7\rangle$ as a counter-example to (1) and (3).

Graphs that are i-critical were studied by Ao in [2] , in which she obtained results analogous to those in [8] and [15] . For example, she showed that the graphs $\overline{nK_2}$, $n \geq 1$, are also the only 2-i-critical graphs. This result can be generalised to

Proposition 3.1 *For π a lower parameter, the only 2-π-critical graphs are $\overline{nK_2}$, $n \geq$ 1.*

Proof. Let $G = \overline{nK_2}$, $n \geq 1$. That $\pi(G) = 2$ and $\pi(G - v) = 1$ for every $v \in V_G$ is clear if $n = 1$ and follows from Proposition 2.5 if $n \geq 2$. Suppose now that $\pi(G) = 2$ and $\pi(G - v) = 1$ for every $v \in V_G$. Then, for every v$\in V_G$, there exists $f(v) \in V_G$ which is adjacent to all vertices of V_G except v. Clearly, $f(f(v)) = v$ for all $v \in V_G$ and therefore $G \cong \overline{nK_2}$ for some $n \geq 1$. ■

Various classes of i-critical or γ-critical graphs are exhibited in [2] . These include the graphs

$$\begin{aligned}
G_1 &= K_n \times K_n \text{ with } n \geq 2, \\
G_2 &= \overline{K_3 \times K_3}, \\
G_3 &= C_n\langle 1, 2, 3, ..., r\rangle \text{ with } n = m(2r+1)+1 \text{ and } m, r \geq 1, \\
G_4 &= C_n\langle 1, 3, 5, ..., 2r-1\rangle \text{ with } n = 4r+1 \text{ and } r \geq 2, \\
G_5 &= \overline{mK_n} \text{ with } m \geq 2 \text{ and } n \geq 3, \\
G_6 &= \overline{K_m \times K_m} \text{ with } m \geq 4
\end{aligned}$$

and

G_7 which is illustrated in Figure 3.1, p. 38.

The graphs G_1, G_2, G_3 and G_4 are i-critical and γ-critical; G_1, G_2 and G_3 satisfy $\gamma = i = n$, $\gamma = i = 3$ and $\gamma = i = m + 1$ respectively, while $\gamma(G_4) = 3$ and $i(G_4) = r + 1$. The graphs G_5 and G_6 are examples of graphs which are i-critical but

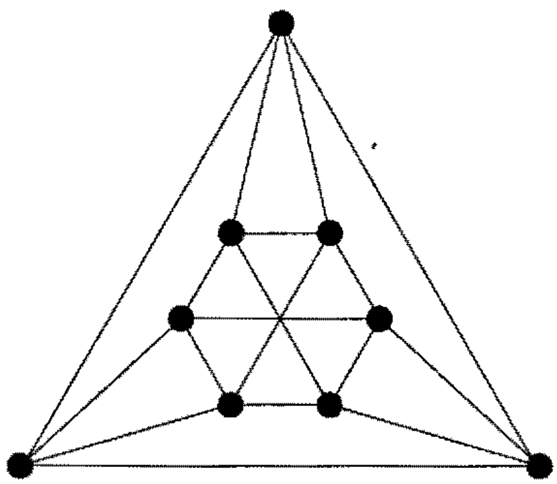


Figure 3.1

not γ-critical and G_7 is an example of a graph which is γ-critical but not i-critical. Note that $\gamma(G_5) = 2$ and $i(G_5) = n$, $\gamma(G_6) = 3$ and $i(G_6) = m$ and $\gamma(G_7) = i(G_7) = 3$. The proofs can be found in [2]. Observe that the statements for G_2 and G_6 are true also for $\overline{K_3 \times K_n}$ with $n \geq 3$ and for $\overline{K_m \times K_n}$ with $n \geq m \geq 4$ respectively.

In Theorem 3.4 we give a new class of i-critical graphs. We begin by giving a characterisation of γ- and i-critical graphs in terms of so-called "singular isolated vertices". Consider a graph G and let $v \in S \subseteq V_G$. We say that v is a *singular isolated vertex* of S if $PN_G(v, S) = \{v\}$, *i.e.* v is an isolated vertex of S which has no external private neighbours.

The following result implies that if $\pi \in \{\gamma, i\}$, then G is π-critical if and only if every vertex of G is a singular isolated vertex of some π-set of G. Proposition 3.2(a) is well-known and appears in [2, 8, 25].

Proposition 3.2 *Let* $\pi \in \{\gamma, i\}$. *For any* n*-vertex graph* G *with* $n \geq 2$,

(a) $\pi(G - v) \geq \pi(G) - 1$ *for all* $v \in V_G$;

(b) $\pi(G - v) = \pi(G) - 1$ *if and only if* v *is a singular isolated vertex of some* π*-set* *of* G.

Proof. For $\pi = i$, let $v \in V_G$ and consider an i-set S of $G-v$. Then S is an independent set of G and $S \cup \{v\}$ is a dominating set of G. If S dominates v, *i.e.* if v is not an isolated vertex of $\langle S \cup \{v\}\rangle$, then S is an independent dominating set of G; hence

$$i(G) \leq |S| = i(G-v).$$

If S does not dominate v, *i.e.* if v is a singular isolated vertex of $S \cup \{v\}$, then $S \cup \{v\}$ is an independent dominating set of G; hence

$$i(G) \leq |S| + 1 = i(G-v) + 1.$$

Furthermore, if $i(G) = i(G-v) + 1$, then $S \cup \{v\}$ is an i-set of G. This establishes (a) and necessity in (b). For sufficiency in (b), suppose v is a singular isolated vertex of the i-set T of G. Since $T - \{v\}$ is an independent dominating set of $G - v$,

$$i(G-v) \leq |T| - 1 = i(G) - 1 \leq i(G-v);$$

hence $i(G-v) = i(G) - 1$.

For $\pi = \gamma$, let $v \in V_G$ and consider a γ-set S of $G-v$. Since $S \cup \{v\}$ is a dominating set of G,

$$\gamma(G) \leq |S| + 1 = \gamma(G-v) + 1.$$

Furthermore, if $\gamma(G) = \gamma(G-v) + 1$, then $S \cup \{v\}$ is a γ-set of G and since S is a dominating set of $G - v$ which does not dominate v, v is a singular isolated vertex of $S \cup \{v\}$. This establishes (a) and necessity in (b). For sufficiency in (b), suppose v is a singular isolated vertex of the γ-set T of G. Since $T - \{v\}$ is a dominating set of $G - v$,

$$\gamma(G-v) \leq |T| - 1 = \gamma(G) - 1 \leq \gamma(G-v);$$

thus $\gamma(G-v)=\gamma(G)-1$. ■

Corollary 3.3 *Let* $\pi\in\{\gamma,i\}$. *Then*

(a) *G is π-critical if and only if $\pi(G-v)=\pi(G)-1$ for all $v\in V_G$.*

(b) *G is π-critical if and only if every vertex of G is a singular isolated vertex of some π-set of G.*

(c) *If a vertex-transitive graph G has a π-set which contains a singular isolated vertex, then G is π-critical.*

Observe that the statements concerning the graphs G_1 to G_6 above now follow easily from Corollary 3.3 and the results in Section 2.2.

Theorem 3.4 *Let $G=C_n\langle 1,3,...,2r-1\rangle$, where $1\leq r\leq(n-1)/2$ and let $n=(2r+1)m+q$, where $0\leq q\leq 2r$. Then G is i-critical if and only if $q\notin\{0,2\}$.*

Proof. Suppose first that q is odd. If $m=1$, then it follows from Theorem 2.9 that

$$i(G)=r+(q+1)/2=n/2.$$

Therefore $S=\{2,4,...,n\}$ is an i-set of G. Since every vertex of S is a singular isolated vertex, it follows from Corollary 3.3(c) that G is i-critical.

Assume henceforth that $m\geq 2$ and consider the partition

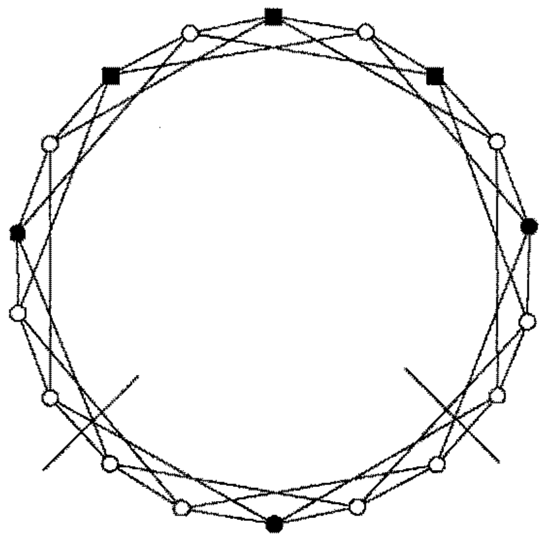
$$n=(2r+1)(m-2)+(4r+2+q)$$

of n into $m-1$ odd numbers. Let S be an independent dominating set induced by this partition of n. Since

$$|S|=[n-(2r-1)(m-1)]/2=m+(q+2r-1)/2,$$

S is an i-set of G. Furthermore, $(4r+2+q)$ contributes $(2r+3+q)/2$ vertices to S, and since $(2r+3+q)/2\geq(2r+4)/2\geq 3$, S has at least one singular isolated vertex. In fact, it is not difficult to see that of these $(2r+3+q)/2$ vertices, only the

$C_{18}\langle 1,3\rangle$



18 = 5 + 13

Figure 3.2

two extreme ones are not singular isolated vertices of S. This is illustrated in Figure 3.2, p. 41 with $C_{18}\langle 1,3\rangle$; the singular isolated vertices are denoted by squares. It now follows from Corollary 3.3(c) that G is i-critical.

Suppose next that q is even, $q \notin \{0,2\}$ and consider the partition

$$n = (2r+1)(m-1) + (2r+1+q)$$

of n into m odd numbers. Let S be an independent dominating set induced by this partition of n. Since

$$|S| = [n-(2r-1)m]/2 = m + q/2,$$

S is an i-set of G. Furthermore, $2r+1+q$ contributes $1+q/2$ vertices to S. Since $q \notin \{0,2\}$, $1+q/2 \geq 3$. With an argument similar to the one above it now follows that $2r + 1 + q$ contributes at least one singular isolated vertex ro S. Therefore G is i-critical by Corollary 3.3(c). See Figure 2.4, p. 31: $19 = 5 + 5 + 9$ induces an i-set of $C_{19}\langle 1, 3\rangle$ with singular isolated vertices; $19 = 5 + 7 + 7$ also induces an i-set, but it has no singular isolated vertices.

Finally, suppose $q = 0$ or $q = 2$. Then any i-set S of G is induced by respectively $n = (2r + 1)m$ or $n = (2r + 1)(m - 1) + (2r + 3)$. Since $(2r + 1)$ and $(2r + 3)$ contribute respectively 1 and 2 vertices to S, S has no singular isolated vertices, and by Corollary 3.3(b), G is not i-critical. ■

Since $ir \leq \gamma \leq i$, it follows that if G is i-critical and $\gamma = i$, then G is γ-critical, and if G is γ-critical and $ir = \gamma$, then G is ir-critical. The γ-critical graphs G_1 to G_4 above all have $ir = \gamma$ and are therefore also ir-critical. It remains an open problem to find ir-critical graphs which are not γ-critical and γ-critical graphs which are not ir-critical.

3.2 Upper domination parameter critical graphs: vertex removal

In this section we show that the only β-critical graphs are the edgeless graphs and that a graph is IR-critical if and only if it is Γ-critical. We then investigate the Γ-critical graphs which are not β-critical.

Proposition 3.5 *The graph G is β-critical if and only if G is edgeless with more than one vertex.*

Proof. We know that edgeless graphs with more than one vertex are β-critical. Suppose G has at least one edge and let S be a β-set of G. Then there exists a vertex $v \in V_G - S$. Now, S is an independent set of $G - v$, hence $\beta(G) = |S| \leq \beta(G - v)$. It follows that G is not β-critical. ■

Consider a graph G with n vertices. A partition $\{S, T\}$ of V_G is called a *one-to-one perfect matching*, abbreviated to $1-1$ *p.m.*, of G if each $s \in S$ is adjacent to exactly one $t \in T$, and each $t \in T$ is adjacent to exactly one $s \in S$. If G has a $1-1$ p.m. $\{S, T\}$, then clearly G has an even number of vertices. Furthermore, S is an irredundant dominating set of G (every vertex of S has a unique private neighbour in T); hence $\Gamma(G) \geq n/2$.

Proposition 3.6 *If a graph G with n vertices has a $1-1$ p.m., then $\Gamma(G) = n/2$ and the class of Γ-sets of G is the same as the class of IR-sets of G.*

Proof. Let $\{S, T\}$ be a $1-1$ p.m. of G. Consider any IR-set B of G and let $W = V_G - B$. Define the sets

$$\begin{aligned} L_2 &= \{st \in E_G | s \in S \cap B, t \in T \cap B\}, \\ L_1 &= \{st \in E_G | s \in S \cap B, t \in T \cap W; \text{ or } s \in S \cap W, t \in T \cap B\} \\ &\quad \text{and} \\ L_0 &= \{st \in E_G | s \in S \cap W, t \in T \cap W\}. \end{aligned}$$

Then $|L_2| + |L_1| + |L_0| = n/2$ and $|B| = 2|L_2| + |L_1|$. Furthermore, each vertex incident with an edge of L_2 is a non-isolated vertex of B and therefore has a private neighbour in W and, since the vertices incident with edges of L_1 are dominated by vertices of B, this private neighbour is incident with an edge of L_0. Therefore $|L_2| \leq |L_0|$ and hence

$$|B| = 2|L_2| + |L_1| \leq |L_2| + |L_1| + |L_0| = n/2.$$

It follows that

$$n/2 \leq \Gamma(G) \leq IR(G) = |B| \leq n/2$$

and therefore

$$\Gamma(G) = IR(G) = |B| = n/2.$$

Since $|B| = n/2$, it follows that $|L_2| = |L_0|$. This implies that each vertex incident with an edge of L_0 is the private neighbour of some vertex incident with an edge of L_2; hence B is a dominating set.

We have proved that every IR-set of G is a dominating set of G and that $\Gamma(G) = IR(G)$. It follows that B is a Γ-set of G if and only if B is an IR-set of G. ■

Theorem 3.7 *If G is a connected graph with n vertices, then the following statements are equivalent.*

(a) *G is Γ-critical.*

(b) *$n > 2$ and for every Γ-set S of G and $T = V_G - S$, $\{S, T\}$ is a $1-1$ p.m. of G.*

(c) *$\Gamma(G) = n/2$ and no Γ-set S of G has any isolated vertices.*

(d) *G is IR-critical.*

Proof. We first prove that (b) and (c) are equivalent and then use this equivalence to prove (a) ⇒ (b) ⇒ (d) ⇒ (a).

(b) ⇒ (c): Consider any Γ-set S of G and let $T = V_G - S$. By (b), $\{S, T\}$ is a $1-1$ p.m. of G and by Proposition 3.6, $\Gamma(G) = n/2$. Suppose s is an isolated vertex of S and let t be the vertex of T adjacent to s. Since G is connected and $n > 2$, it follows that t is adjacent to some $u \in T$. Let $R = (T - \{t\}) \cup \{s\}$. Then R is a dominating set of G, while s is an isolated vertex of R and each $r \in R - \{s\}$ has a private neighbour (relative to R) in S. Therefore R is a minimal dominating set of G and, since $|R| = |S| = \Gamma(G)$, it follows that R is a Γ-set of G. This contradicts (b), since the vertices s and u of R are both adjacent to t and $t \notin R$. We conclude that S has no isolated vertices.

(c) ⇒ (b): Consider any Γ-set S of G and let $T = V_G - S$. Since S has no isolated vertices, each $s \in S$ has at least one private neighbour in T and since $\Gamma(G) = n/2$, it follows that $|S| = |T|$ and therefore each $t \in T$ is the unique private neighbour of some $s \in S$. It follows that $\{S, T\}$ is a $1-1$ p.m. of G.

(a) $\Rightarrow$ (b): Consider any Γ-set S of G and let $T = V_G - S$. For every $t \in T$, the set S is a dominating set of $G - t$ and since $\Gamma(G-t) < \Gamma(G)$, S is not an irredundant set of $G - t$; hence t is the unique private neighbour of some non-isolated vertex $s \in S$. Let

$$S' = \{s \in S |\ s \text{ has a unique private neighbour } t \in T \text{ with respect to } S\}.$$

Then $S - S'$ consists of all the isolated vertices of $\langle S \rangle$. By the above, these vertices are isolated in G and since G is connected, it follows that $S = S'$. Therefore each $s \in S$ has a unique private neighbour in T. Consequently, $\{S, T\}$ is a $1-1$ p.m. of G.

(b) $\Rightarrow$ (d): Suppose to the contrary that $v \in V_G$ and that S is an irredundant set of $G - v$ such that $|S| \geq IR(G)$. Since S is an irredundant set of G, we have that $|S| \leq IR(G)$ and necessarily S is an IR-set of G. By (b) and Proposition 3.6, S is a Γ-set of G. Therefore, by (b) and (c), $\{S, T\}$ with $T = V_G - S$ is a $1-1$ p.m. of G and S has no isolated vertices. Note that $v \in T$ and let $s \in S$ be adjacent to v. Since v is the unique private neighbour in G of s and since s is not isolated in S, we see that s is a redundant vertex of S in $G - v$. This is impossible since S is an irredundant set of $G - v$.

(d) $\Rightarrow$ (a): We first prove that $\Gamma(G) = IR(G)$. Let S be an IR-set of G. For each $v \in V_G - S$, $IR(G - v) < IR(G)$; hence S is not irredundant in $G - v$ from which it follows that S dominates v. Therefore S is an irredundant dominating set of G so that $|S| \leq \Gamma(G)$ and thus $\Gamma(G) = IR(G)$. Now, for each $v \in V_G$,

$$\Gamma(G - v) \leq IR(G - v) < IR(G) = \Gamma(G);$$

consequently, G is Γ-critical. ■

In Theorem 3.7 (c) it is necessary to require that $\Gamma(G) = n/2$: The graph in Figure 3.3, p. 46 has only the two Γ-sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ and neither has isolated

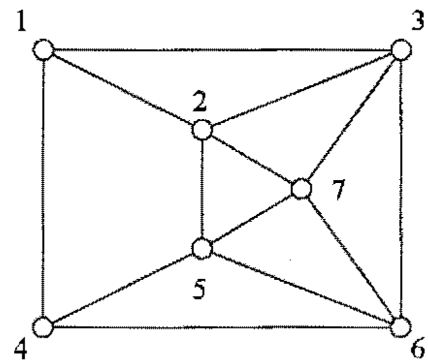


Figure 3.3

vertices, but the graph does not have a $1-1$ p.m.

Corollary 3.8 *If G is a Γ-critical graph with n vertices, k of which are isolated, then each component of G is either K_1 or it has a $1-1$ p.m. with more than two vertices; hence $\Gamma(G) = \frac{n-k}{2} + k$.*

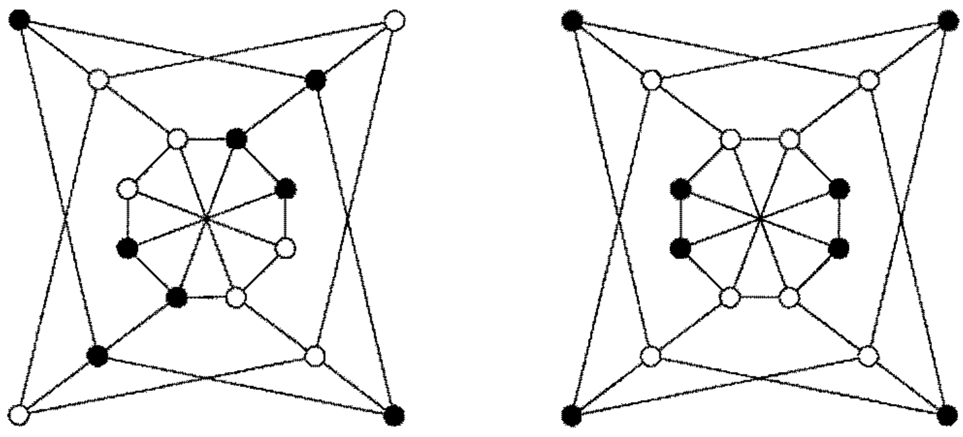
Corollary 3.9 *The graph G is Γ-critical if and only if it is IR-critical.*

Corollary 3.10 *If a connected graph G with n vertices is Γ-critical, then it has a $1-1$ p.m., while $\delta(G) \geq 2$ and $\beta(G) < n/2$.*

Proof. By Theorem 3.7 (b), G has a $1-1$ p.m. $\{S,T\}$. By Theorem 3.7 (c), since both S and T are Γ-sets of G, neither has any isolated vertices. It follows that $\delta(G) \geq 2$. The fact that $\beta(G) < n/2$ follows directly from Theorem 3.7 (c). ■

The converse of Corollary 3.10 does not hold: The connected graph in Figure 3.4, p. 47 has a $1-1$ p.m. (Figure 3.4 (a)) with 16 vertices, $\beta = 7$ and $\delta = 3$, but it is not Γ-critical since it has a Γ-set (the dark vertices in Figure 3.4 (b)) with isolated vertices.

The following two propositions give sufficient conditions, in terms of $\delta(G)$, for a connected graph G with a $1-1$ p.m. to be Γ-critical. We first prove a lemma.



(a) (b)

Figure 3.4

Lemma 3.11 *Suppose G is a graph with a $1-1$ p.m. If a Γ-set of G has isolated vertices, then it has at least $2(\delta(G)-1)$ isolated vertices.*

Proof. If $\delta(G)=1$, then the statement is trivial. Assume therefore that $\delta(G)\geq 2$. Consider a $1-1$ p.m. $\{S,T\}$ of G and a Γ-set B of G that has isolated vertices. Let $W=V_G-B$ and let k and l denote the numbers of isolated and non-isolated vertices of B respectively. By Proposition 3.6, $k+l=\Gamma(G)=n/2$.

Suppose all k isolated vertices of B are in S. Since $\{S,T\}$ is a $1-1$ p.m., each isolated vertex of B in $S\cap B$ sends one edge to $T\cap W$ and at least $\delta(G)-1$ edges to $S\cap W$, and the neighbours in $T\cup W$ of distinct vertices in $S\cap B$ are all distinct. Therefore $S\cap B$ dominates at least $k+\delta(G)-1$ vertices of W. Since each of the l non-isolated vertices of B needs a private neighbour in W, it follows that

$$|W|\geq l+k+\delta(G)-1;$$

hence

$$n/2 \geq n/2 + \delta(G) - 1,$$

from which it immediately follows that $\delta(G) \leq 1$. This contradicts our assumption. Therefore T and (similarly) S both contain isolated vertices of B.

Let $x \in S$ and $y \in T$ be isolated vertices of B. Since $\{S, T\}$ is a $1-1$ p.m., x sends one edge to $T \cap W$ and at least $\delta(G) - 1$ edges to $S \cup W$. Similarly, y sends at least $\delta(G) - 1$ edges to $T \cap W$. Therefore $\{x, y\}$ dominates at least $2(\delta(G) - 1)$ vertices of W. Since each of the l non-isolated vertices of B needs a private neighbour in W, it follows that

$$k + l = |W| \geq l + 2(\delta(G) - 1);$$

hence

$$k \geq 2(\delta(G) - 1). \;\blacksquare$$

Proposition 3.12 *Let G be a connected graph with n vertices, a $1-1$ p.m. and $\delta(G) \geq \lfloor n/4 \rfloor + 2$. Then G is Γ-critical.*

Proof. By Proposition 3.6, $\Gamma(G) = n/2$ If G is not Γ-critical, then it follows from Theorem 3.7 that G has a Γ-set B with isolated vertices. Hence, by Lemma 3.11, B has at least

$$2(\delta(G) - 1) \geq 2(\lfloor n/4 \rfloor + 1) > n/2$$

isolated vertices, which is impossible since $|B| = n/2$. ■

Corollary 3.13 *If G is a connected graph with a $1-1$ p.m. and $\delta(G) \geq \lfloor n/4 \rfloor + 2$, then $\beta(G) < \Gamma(G)$.*

The result in Proposition 3.12 can be improved slightly if we know beforehand that $\beta(G) < \Gamma(G)$:

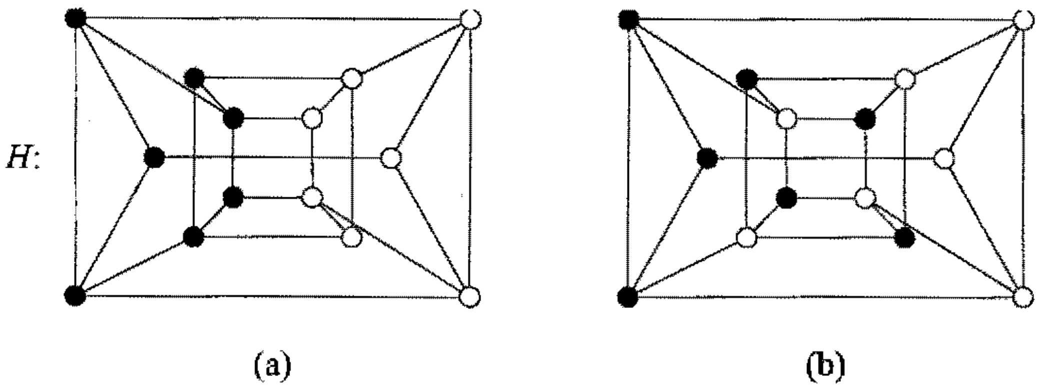


Figure 3.5

Proposition 3.14 *Let G be a connected graph with $n > 2$ vertices, a $1-1$ p.m. and $\delta(G) \geq \lfloor n/4 \rfloor + 1$. If $\beta(G) < n/2$, then G is Γ-critical.*

Proof. By Proposition 3.6, $\Gamma(G) = n/2$. Suppose G is not Γ-critical. By Theorem 3.7, G has a Γ-set B with isolated vertices. Hence, by Lemma 3.11, B has at least $2(\delta(G)-1) \geq 2(\lfloor n/4 \rfloor) \geq n/2 - 1$ isolated vertices. Therefore B is independent; hence $\beta(G) = n/2$. ■

This last result is the best possible, in the sense that if $\delta(G) < \lfloor n/4 \rfloor + 1$ and $\beta(G) < n/2$ then it does not necessarily follow that G is Γ-critical: The connected graph H in Figure 3.5, p. 49 has $\delta(H) = \lfloor n/4 \rfloor$ and $\beta(G) < n/2$ and the black vertices in Figure 3.5 (a) form one part of a $1-1$ p.m. of H. However, H is not Γ-critical since the black vertices in Figure 3.5 (b) form a Γ-set of H with isolated vertices.

Also, in Proposition 3.14 it is necessary to require that G has at least one $1-1$ p.m.: The connected graph K in Figure 3.6, p. 50 has $\delta(K) = \lfloor n/4 \rfloor + 1$ and $\beta(K) < n/2$, but it is not Γ-critical since the black vertices form a Γ-set of K with isolated vertices. (Note that K does not have a $1-1$ p.m.)

K:

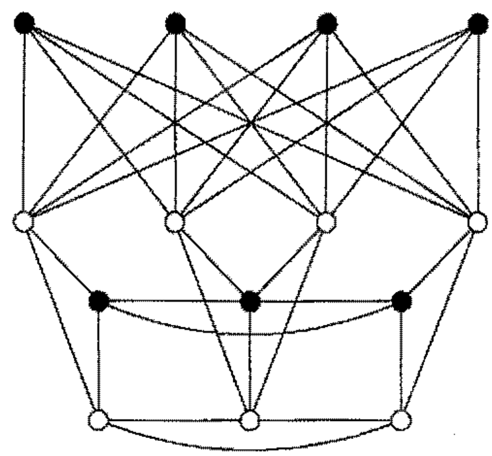


Figure 3.6

Proposition 3.15 *If G is a connected r-regular graph with $\beta(G) < \Gamma(G) = n/2$, then G is Γ-critical.*

Proof. Suppose to the contrary that G is not Γ-critical. By Theorem 3.7 (c), G has a Γ-set S such that $\langle S \rangle$ has isolated vertices. Since $\beta(G) < n/2$, $\langle S \rangle$ also has non-isolated vertices. Let $K \neq \emptyset$ and $L \neq \emptyset$ be the sets of isolated and non-isolated vertices of $\langle S \rangle$ respectively. Let $L' = \bigcup_{s \in L} PN(s, S)$ and $K' = (V_G - S) - L'$. Clearly $\{K, L\}$ and $\{K', L'\}$ are partitions of S and $V_G - S$ respectively.

Let q be the number of edges between K and K'. Since G is r-regular and since K sends all its edges to K', $q = r|K|$. Suppose K' does not send all its edges to K. Then $q < r|K'|$ and therefore $|K| < |K'|$. Since $\Gamma(G) = n/2$, $|K| + |L| = |K'| + |L'|$. Therefore $|L| > |L'|$, which is impossible since every vertex of L has a private neighbour in L'. It follows that K' sends all its edges to K. Therefore $K \cup K'$ sends no edges to $L \cup L'$. This contradicts the assumption that G is connected. ■

Proposition 3.16 *Let G be a graph on n vertices with a $1-1$ p.m. $\{S, T\}$ such that*

$\langle S\rangle$ and $\langle T\rangle$ are connected r-regular graphs with

$$\beta(\langle S\rangle)+\beta(\langle T\rangle)<n/2.$$

Then G is Γ-critical.

Proof. Clearly, G is a connected r-regular graph and $\Gamma(G)=n/2$ by Proposition 3.6.

Consider a β-set B of G. Then $B\cap S$ and $B\cap T$ are independent sets of $\langle S\rangle$ and $\langle T\rangle$ respectively; hence

$$\begin{aligned}\beta(G) &= |B|=|B\cap S|+|B\cap T|\\ &\leq \beta(\langle S\rangle)+\beta(\langle T\rangle)\\ &< n/2.\end{aligned}$$

It follows from Proposition 3.15 that G is Γ-critical. ■

We illustrate the usefulness of Proposition 3.16 with some examples:

(i) In the proposition, let $\langle S\rangle$ and $\langle T\rangle$ be complete graphs; then

$$\beta(\langle S\rangle)+\beta(\langle T\rangle)=1+1=2.$$

If $n\geq 6$, it follows that $\beta(\langle S\rangle)+\beta(\langle T\rangle)<n/2$; hence G is Γ-critical. If $n=2$ or 4, then G is not Γ-critical, *i.e.* P_2 and C_4 are not Γ-critical.

(ii) Again in the proposition, let $\langle S\rangle$ and $\langle T\rangle$ be cycles. By Theorem 2.8,

$$\beta(\langle S\rangle)+\beta(\langle T\rangle)=\lfloor n/4\rfloor+\lfloor n/4\rfloor.$$

If n is not divisible by 4, it follows that $\beta(\langle S\rangle)+\beta(\langle T\rangle)<n/2$; hence G is Γ-critical. If n is divisible by 4, then G is not Γ-critical. (It is easy to construct an independent set of G with $n/2$ vertices.)

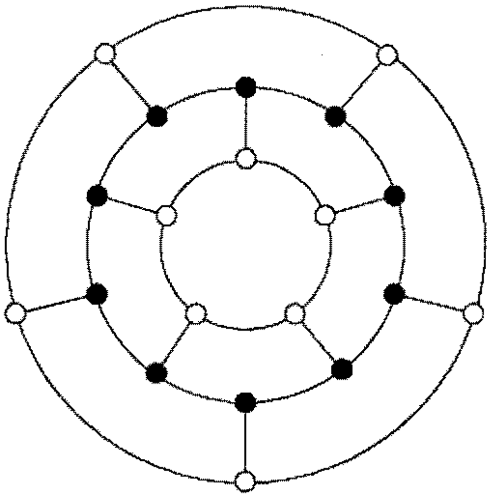


Figure 3.7

(iii) Figure 3.7 shows that the dodecahedron has a $1-1$ p.m. $\{S,T\}$ with $\langle S\rangle = C_{10}$ (indicated by the black vertices) and $\langle T\rangle = 2C_5$. By Theorem 2.8,

$$\beta(\langle S\rangle) + \beta(\langle T\rangle) = 5+2+2 = 9 < 10 = n/2.$$

It follows from Proposition 3.16 that the dodecahedron is Γ-critical.

We conclude this section with a method for constructing Γ-critical graphs that need not be regular and for which $\lfloor n/4\rfloor - \delta$ (see Propositions 3.12 and 3.14) may be made arbitrarily large.

Begin with four empty vertex-sets V_1, V_2, V_3, V_4 and an empty edge-set. Increase the number of vertices and edges by applying R1 and R2 any number of times :

R1 Add a cycle $C_{4r} (r \geq 1)$ with consecutive vertices in $V_1, V_2, V_3, V_4, V_1, \ldots$.

R2 Add P_2 with its two vertices in V_1 and V_3 or in V_2 and V_4.

This construction ensures that our graph has two $1-1$ perfect matchings, namely $\{V_1 \cup V_2, V_3 \cup V_4\}$ and $\{V_2 \cup V_3, V_4 \cup V_1\}$. For each $i \in \{1,2,3,4\}$, join all vertices in V_i so that $\langle V_i\rangle$ is complete and ensure that the graph is connected with at least 10 vertices and that $|V_i| \geq 2$ for each $i \in \{1, 2, 3, 4\}$ by applying R1 and R2 enough times and R1 at least once. Graphs so constructed are Γ-critical :

Proposition 3.17 *Suppose G is a connected graph with $n \geq 10$ vertices and two distinct $1-1$ perfect matchings $\{A, B\}$ and $\{C, D\}$ such that each of the subgraphs $\langle A \cap C \rangle$, $\langle A \cap D \rangle$, $\langle B \cap C \rangle$ and $\langle B \cap D \rangle$ of G is a complete graph with more than one vertex. Then G is Γ-critical.*

Proof. By Proposition 3.6,

$$|A| = |B| = |C| = |D| = \Gamma(G) = n/2.$$

Let $|A \cap C| = |B \cap D| = p$ and $|A \cap D| = |B \cap C| = q$. Then $p, q \geq 2$ and $p + q = n/2 \geq 5$.

Suppose to the contrary that G is not Γ-critical. By Theorem 3.7 (c), G has a Γ-set S that has isolated vertices.

For any $x \in A \cap C$, since $\{A, B\}$ is a $1-1$ p.m. and $\langle A \cap C \rangle$ is complete with more than one vertex, $\deg(x) \geq 2$. A similar argument holds for $A \cap D$, $B \cap C$ and $B \cap D$; hence $\delta(G) \geq 2$. It follows from Lemma 3.11 that S contains at least two isolated vertices x_1 and x_2.

Let $x_1 \in X_1$ and $x_2 \in X_2$ for $X_1, X_2 \in \{A \cap C, A \cap D, B \cap C, B \cap D\}$. Since $\langle X_1 \rangle$ and $\langle X_2 \rangle$ are complete graphs, $S \cap X_1 = \{x_1\}$, $S \cap X_2 = \{x_2\}$ and $X_1 \cup X_2$ does not contain private neighbours of non-isolated vertices of S.

Since $|S| = p + q \geq 5$, assume without loss of generality that $|S \cap A \cap D| \geq 2$. Since $\langle A \cap D \rangle$ is complete, $S \cap A \cap D$ consists of non-isolated vertices of S, and $A \cap D$ contains no private neighbours of vertices of S. Therefore all the private neighbours of the non-isolated vertices of S lie in

$$Y \in \{A \cap C, B \cap C, B \cap D\} - \{X_1, X_2\}.$$

Since $\langle Y \rangle$ is complete, $S \cap Y = \emptyset$. It follows that x_1 and x_2 are the only isolated vertices of S and that $A \cap D$ contains all $p+q-2$ non-isolated vertices of S ; hence $q \geq p+q-2$. Therefore, since $p \geq 2$ and $p+q \geq 5$, it is evident that $p = 2$ and $q \geq 3$. It follows that $A \cap D$ is the set of non-isolated vertices of S, $Y = B \cap C$ is the set of the private neighbours of the non-isolated vertices of S and $\{X_1, X_2\} = \{A \cap C, B \cap D\}$.

Now, since $\{A, B\}$ and $\{C, D\}$ are $1-1$ perfect matchings of G and each vertex in $A \cap D$ is adjacent to precisely one vertex in $B \cap C$ and *vice versa*, it follows that no vertex in $(A \cap D) \cup (B \cap C)$ is adjacent to any vertex in $(A \cap C) \cup (B \cap C)$. Therefore G is disconnected. ■

Note that if we apply R1 (with $r = 1$) in the construction only once and ensure $|V_1| = |V_3| = 2$, then $\delta = 2$ while $n/4$ may be made arbitrarily large.

3.3 Γ^+-critical graphs

If $\Gamma(G) = IR(G)$, then it follows from Propositions 1.7 and 1.8 that $\Gamma(G-v) \leq IR(G-v) \leq IR(G) = \Gamma(G)$ for all $v \in V_G$. Therefore, of all the graphs considered in Section 2.2, only $K_m \times K_n$ with $m, n \geq 5$ are candidates for Γ^+-criticality. The following proposition states that they are indeed Γ^+-critical. We first prove a lemma.

Lemma 3.18 *If S is an IR-set of G which dominates all vertices of G except $v \in V_G$, then $\Gamma(G-v) = IR(G)$.*

Proof. S is a dominating set of $G-v$ and, since $v \in V_G - N_G[S]$, it follows that S is an irredundant set of $G-v$. Therefore $IR(G) = |S| \leq \Gamma(G-v)$. Propositions 1.7 and 1.8 imply that $\Gamma(G-v) = IR(G-v) = IR(G)$. ■

Proposition 3.19 *For any $m, n \geq 5$, the graph $K_m \times K_n$ is Γ^+-critical.*

Proof. Let $G = K_m \times K_n$ for $n \geq m \geq 5$ and consider any $v \in V_G$. Since G is vertex-transitive, assume without loss of generality that $v = v_{mn}$. Now, S_0 in the proof of Theorem 2.6 (see Figure 2.2, p. 24) is an IR-set of G that dominates all vertices of G except v. Therefore, by Lemma 3.18 and Theorem 2.6,

$$\Gamma(G-v) = IR(G) = m+n-4 > n = \Gamma(G). \quad \blacksquare$$

If $G = K_m \times K_n$ with $m, n \geq 5$, then there exists, for every $v \in V_G$, an IR-set of G that dominates all vertices of G except v. Is this a general property of Γ^+-critical graphs? (If this is so, then it will follow from Lemma 3.18 that every Γ^+-critical graph G has the property that $\Gamma(G-v) = IR(G)$ for each $v \in V_G$.) Are the graphs $K_m \times K_n$ with $m, n \geq 5$ the only vertex-transitive Γ^+-critical graphs? Are there Γ^+-critical graphs that are not vertex-transitive? The results in the rest of this section may be useful in the investigation of these questions, which remain unanswered.

Proposition 3.20 *Suppose G is Γ^+-critical. For every $v \in V_G$ and every Γ-set S_v of $G-v$, S_v is a maximal irredundant set of G that dominates all vertices of G except v. Furthermore, each $w \in N_G[v]$ annihilates at least two vertices of S_v.*

Proof. Since S_v is an irredundant set of G, S_v is not a dominating set of G, for otherwise $\Gamma(G-v) = |S_v| \leq \Gamma(G)$. Hence S_v dominates all vertices of G but not v.

Consider any $w \in N_G[v]$. Since $S_v \cup \{w\}$ is a dominating set of G, it follows that $S_v \cup \{w\}$ is not an irredundant set of G, for otherwise

$$\Gamma(G-v) + 1 = |S_v| + 1 \leq \Gamma(G).$$

It follows from Proposition 1.4 that S_v is a maximal irredundant set of G. Moreover, if w annihilates only one vertex x of S_v, then the set $S = (S_v - \{x\}) \cup \{w\}$ is a dominating irredundant set of G; hence

$$\Gamma(G-v) = |S_v| = |S| \leq \Gamma(G),$$

which contradicts the fact that $\Gamma(G-v) > \Gamma(G)$. Therefore each $w \in N_G[v]$ annihilates at least two vertices of S_v. ■

Proposition 3.21 *If G is Γ^+-critical, then*

(a) *G has at least $|V_G|$ maximal irredundant sets.*

(b) $\delta(G) \geq 3$.

(c) *$N(v) \nsubseteq N(u)$ for any two adjacent vertices u and v.*

(d) *If $N(v) \subseteq N(u)$ for two non-adjacent vertices u and v, then* $\deg(u) \geq \deg(v)+2$.

Proof. (a) For any two distinct vertices u and v of G, let S_u and S_v be Γ-sets of $G-u$ and $G-v$, respectively. By Proposition 3.20, S_u and S_v are maximal irredundant sets of G and since S_u dominates v and S_v does not dominate v, the sets S_u and S_v are distinct.

(b) Let $v \in V_G$ and consider a Γ-set S_v of $G-v$. Since v annihilates some vertex of S_v, we see that $N(v) \neq \emptyset$. Let $w \in N(v)$ and consider a Γ-set S_w of $G-w$. Since v annihilates at least two vertices of S_w, and since v is adjacent to w, it is clear that $\deg(v) \geq 3$.

(c) Let u and v be two adjacent vertices. Consider a Γ-set S_u of $G-u$. Since S_u dominates v but not u, $N(v) \nsubseteq N(u)$.

(d) Suppose $N(v) \subseteq N(u)$ for two non-adjacent vertices u and v. Consider a Γ-set S_u of $G-u$. Since S_u dominates v but not u and since $N(v) \subseteq N(u)$, v is an isolated vertex of S_u. But u annihilates two non-isolated vertices of S_u. Therefore $\deg(u) \geq \deg(v)+2$. ■

Corollary 3.22 *If G is Γ^+-critical and $\Delta(G) - \delta(G) \leq 1$, then $N(v) \nsubseteq N(u)$ for any distinct vertices u and v of G.*

Chapter 4
Edge-Critical Graphs

Recall that the edgeless graphs $\overline{K_n}$, $n \geq 2$, are n-π-edge-critical for all the domination parameters π and that there do not exist π^+-edge-critical graphs if $\pi \in \{ir, \gamma, i, \beta\}$.

4.1 Lower domination parameter critical graphs: edge addition

The study of γ-edge-critical graphs was initiated by Sumner and Blitch in [21], where they showed that G is 2-γ-edge-critical if and only if $\overline{G}$ is the disjoint union of non-trivial stars. They also obtained several properties of 3-γ-edge-critical graphs. Hamiltonian properties of 3-γ-edge-critical graphs were studied in [27] and [28] and k-γ-edge-critical graphs with $k \geq 4$ were studied in [12] and [19]. For a recent survey on γ-edge-critical graphs we refer the reader to [22].

Graphs that are i-edge-critical were studied by Ao in [2], where she obtained results analogous to those in [21]. For example, G is 2-i-edge-critical if and only if $\overline{G}$ is the disjoint union of non-trivial stars. Since $\gamma(G) = 2$ implies $ir(G) = 2$, it is evident that the same characterisation holds for ir-edge-critical graphs.

Considerable interest and subsequent papers concerning γ-edge-critical graphs were generated by the following two conjectures.

Conjecture 1 [21] If G is k-γ-edge-critical, then $\gamma(G) = i(G)$.

Conjecture 2 [27] If G is 3-γ-edge-critical and $\delta(G) \geq 2$, then G is hamiltonian.

For $k \geq 4$ Conjecture 1 was shown to be false by Ao, Cockayne, MacGillivray and Mynhardt [3] and this conjecture can therefore be reformulated as

Conjecture 1′ If G is 3-γ-edge-critical, then $\gamma(G) = i(G)$.

This conjecture remains unsolved, although some progress has been made. It was shown in [21] that the conjecture holds for disconnected graphs, for graphs with $\delta \leq 2$ and for graphs with diameter 3. The only other known result on this conjecture is one by Favaron, Tian and Zhang [13]. They first proved that $\beta \leq \delta + 2$ for any 3-γ-edge-critical graph, and then showed that if equality holds, then $i = 3$.

Conjecture 2 has recently been proved. The eventual solution is the result of a combined effort by five researchers and the proof, which is long, difficult and technical, is contained in the three papers [13, 14, 23]. The first contributions were made by Flandrin, Tian, Wei and Zhang in [14]. This includes the result that a 2-connected 3-γ-edge-critical graph G of order n has $c(G) \geq n - 1$, where $c(G)$ denotes the circumference of G. Using the results in [14], Favaron, Tian and Zhang next proved in [13] that connected 3-γ-edge-critical graphs with $\delta \geq 2$ and $\beta \leq \delta + 1$ are hamiltonian. That Conjecture 2 also holds if $\beta = \delta + 2$ was finally proved by Tian, Wei and Zhang in [23].

For an exposition of all the work done on Conjectures 1 and 2 we refer the reader to the Master's dissertation by Moodley [17].

In the remainder of this section we present a characterisation of γ- and i-edge-critical graphs in terms of the existence of γ-sets and i-sets with certain properties, and determine which of the graphs in Section 2.2 are π-edge-critical for π a lower domination parameter.

Proposition 4.1 *Let* $\pi \in \{\gamma, i\}$. *For any non-complete graph* G,

(a) $\pi(G + uv) \geq \gamma(G) - 1$ *for all* $uv \in E_{\overline{G}}$,

(b) $\pi(G + uv) = \pi(G) - 1$ *if and only if there exists a* π*-set* T *of* G *such that* $\{u, v\} \subseteq T$ *and one of* u *and* v *is a singular isolated vertex of* T.

Proof. For $\pi = i$, let $uv \in E_{\overline{G}}$ and consider an i-set S of $G + uv$. Then S is an independent set of G and $S \cup \{u\}$ or $S \cup \{v\}$ dominates G. If S dominates G, then S is an independent dominating set of G; hence $i(G) \leq |S| = i(G + uv)$. Suppose now that S does not dominate G and assume without loss of generality that $S \cup \{u\}$ dominates G. Then $u \notin S$, $v \in S$ and u is a singular isolated vertex of $S \cup \{u\}$ in G. It follows that $S \cup \{u\}$ is an independent dominating set of G and thus

$$i(G) \leq |S| + 1 = i(G + uv) + 1.$$

Furthermore, if $i(G) = i(G + uv) + 1$, then $S \cup \{u\}$ is an i-set of G. This establishes (a) and necessity in (b). For sufficiency in (b), suppose T is an i-set of G such that $\{u, v\} \subseteq T$ and u is a singular isolated vertex of T. Since $T - \{u\}$ is an independent dominating set of $G + uv$,

$$i(G + uv) \leq |T| - 1 = i(G) - 1 \leq i(G + uv).$$

It follows that $i(G + uv) = i(G) - 1$.

For $\pi = \gamma$, let $uv \in E_{\overline{G}}$ and consider a γ-set S of $G + uv$. Then $S \cup \{u\}$ or $S \cup \{v\}$ dominates G; hence $\gamma(G) \leq |S| + 1 = \gamma(G + uv) + 1$. Furthermore, if $\gamma(G) = \gamma(G + uv) + 1$, then S does not dominate G. Assume without loss of generality that $S \cup \{u\}$ dominates G. Then $u \notin S$, $v \in S$ and u is a singular isolated vertex of $S \cup \{u\}$ in G. This establishes (a) and necessity in (b). For sufficiency in (b), suppose T is a γ-set of G such that $\{u, v\} \subseteq T$ and u is a singular isolated vertex of T. Since $T-\{u\}$ is a dominating set of $G+uv$, it follows that

$$\gamma(G+uv) \leq |T|-1=\gamma(G)-1 \leq \gamma(G+uv)$$

and therefore $\gamma(G+uv)=\gamma(G)-1$. ■

Corollary 4.2 *Let $\pi \in \{\gamma, i\}$. Then*

(a) *G is π-edge-critical if and only if $\pi(G+uv)=\pi(G)-1$ for all $uv \in E_{\overline{G}}$.*

(b) *G is π-edge-critical if and only if for every $uv \in E_{\overline{G}}$, there exists a π-set T of G such that $\{u,v\} \subseteq T$ and one of u or v is a singular isolated vertex of T.*

We remark that by Corollary 4.2, if $\pi \in \{\gamma, i\}$ and u and v are nonadjacent vertices in a k-π-edge-critical graph G, then there exists a set $S \subseteq V(G)-\{u,v\}$ of cardinality $k-2$ such that $S \cup \{u\}$ dominates $G-v$ but not v, or $S \cup \{v\}$ dominates $G-u$ but not u. This was also observed in [2, 21] for i and γ respectively.

Consider the graphs G_1 to G_6 of Section 3.1. G_1 and G_2 are i-edge-critical and γ-edge-critical. G_5 and G_6 are i-edge-critical but not γ-edge-critical. Detailed proofs of these statements appear in [2]. Alternative proofs can be found by considering the γ- and i-sets of the graphs described in the proofs of Proposition 2.5 and Theorems 2.6 and 2.7 and applying Corollary 4.2.

If $n>m$, then $\overline{K_m \times K_n}$ is neither i-edge-critical nor γ-edge-critical (apply Corollary 4.2(b) to $u=v_{11}$ and $v=v_{12}$). If $m=1$, then $G_3=\overline{(r+1)K_2}$; hence $\overline{G_3}$ is a disjoint union of stars. If $m>1$, then G_3 is neither i-edge-critical nor γ-edge-critical (apply Corollary 4.2(b) to vertices u and $v=u+2r+1$). G_4 is neither i-edge-critical nor γ-edge-critical (apply Corollary 4.2(b) to vertices u and $v=u+2r$).

The γ-edge-critical graphs G_1 and G_2 have $ir=\gamma$ and are therefore also ir-edge-critical. It remains an open problem to find ir-edge-critical graphs which are not γ-edge-critical, and γ-edge-critical graphs which are not ir-edge-critical (or to show that these classes of graphs coincide).

4.2 Upper domination parameter critical graphs: edge addition

In this section we find characterisations of β- and Γ-edge-critical graphs and show that a graph is IR-critical if and only if it is Γ-edge-critical.

For any graph G and integer $q \geq 0$, let $G + q$ denote the graph obtained by adding q universal vertices to $V(G)$, *i.e.*, $G + q \cong G + K_q$.

Lemma 4.3 *Suppose π is an upper parameter and $q \geq 0$. Then G is π-edge-critical if and only if $G + q$ is π-edge-critical.*

Proof. We make three observations.

O1. $E_{\overline{G+q}} = E_{\overline{G}}$.

O2. If $uv \in E_{\overline{G}}$, then $(G + uv) + q = (G + q) + uv$.

O3. $\pi(G + q) = \pi(G)$.

O1 and O2 are obvious. We prove O3. Obviously, G and $G + q$ are either both complete or both non-complete. In the first case $\pi(G) = \pi(G + q) = 1$. In the second case, S is an independent (dominating, irredundant) set of G if and only if S is an independent (dominating, irredundant) set of $G + q$. It follows that if S is a π-set of G, then $\pi(G + q) \leq |S| = \pi(G)$ and if S is a π-set of $G + q$, then $\pi(G) \leq |S| \leq \pi(G + q)$; hence $\pi(G + q) = \pi(G)$.

It now follows from O1 and O3 that $\pi(G + uv) < \pi(G)$ for all $uv \in E_{\overline{G}}$ if and only if $\pi((G + uv) + q) < \pi(G + q)$ for all $uv \in E_{\overline{G+q}}$. Applying O2 completes the proof of the lemma. ■

Proposition 4.4 *The graph G is β-edge-critical if and only if $G = H + q$, where H is an edgeless graph with more than one vertex.*

Proof. Suppose G is β-edge-critical and consider a β-set S of G. Clearly $H = \langle S \rangle$ is an edgeless graph with more than one vertex. Let $q = |V_G - S|$. For each $u \in V_G - S$

and $v \in V_G - \{u\}$, $uv \in E_G$, for otherwise S is an independent set of $G + uv$, in which case $\beta(G) = |S| \leq \beta(G + uv)$. It follows that $G = H + q$.

Conversely, if H is edgeless with more than one vertex, then H is clearly β-edge-critical. It now follows from Lemma 4.3 that $H + q$ is β-edge-critical for any $q \geq 0$.

■

Theorem 4.5 *The following statements are equivalent for any graph G:*

(a) *G is Γ-edge-critical.*

(b) *$G = H + q$, where* (i) *H is an edgeless graph with more than one vertex, or* (ii) *the non-isolated vertices of H induce a graph M with at least six vertices and a $1-1$ perfect matching $\{S, T\}$ such that $\langle S \rangle$ and $\langle T \rangle$ are complete graphs.*

(c) *G is IR-edge-critical.*

Proof. (a) ⇒ (b): Consider a Γ-set X of G and let Z and S be the sets of isolated and non-isolated vertices of X, respectively. Let T be the set of external private neighbours of vertices in X and let W be the set of vertices of $V_G - X$ that are adjacent to more than one vertex of X. Clearly, $X = Z \cup S$ and $\{Z, S, T, W\}$ is a partition of V_G.

If $uv \in E_{\overline{G}}$ and X is an irredundant set of $G + uv$, then X is a minimal dominating set of $G + uv$. Therefore $\Gamma(G) = |X| \leq \Gamma(G + uv)$, which contradicts the Γ-edge-criticality of G. Hence

$$\text{for all } uv \in E_{\overline{G}},\ X \text{ is not an irredundant set of } G + uv. \tag{4.1}$$

Furthermore, if $\beta(G) = \Gamma(G)$, then G is β-edge-critical and it follows from Proposition 4.4 that $G = H + q$, where H is a graph satisfying (i). Henceforth, assume that

$$\beta(G) < \Gamma(G). \tag{4.2}$$

Suppose $x \in X$ and $|PN_G(x, X)| > 1$. Let $u \in X - \{x\}$ and $v \in PN_G(x, X)$. Then X is an irredundant set of $G + uv$, which contradicts (4.1). Therefore $|PN_G(x, X)| =$

1 for all $x \in X$. It follows that Z is the set of isolated vertices of $H = \langle Z \cup S \cup T \rangle$ and that $\{S, T\}$ is a $1-1$ p.m. of $M = \langle S \cup T \rangle$.

Suppose $uv \in E_{\overline{G}}$ and $\{u, v\} \subseteq S$ or $\{u, v\} \subseteq T$. Then X is an irredundant set of $G + uv$, which contradicts (4.1). Therefore $\langle S \rangle$ and $\langle T \rangle$ are complete graphs.

Suppose $|S| = 2$ and let $s \in S,\ t \in T$ with s and t nonadjacent. Then $Z \cup \{s, t\}$ is an independent set of G. Therefore

$$\Gamma(G) = |X| = |Z \cup \{s, t\}| \leq \beta(G),$$

which contradicts (4.2). Consequently $|S| \geq 3$ and so M has at least six vertices.

Finally, suppose $uv \in E_{\overline{G}}$ with $u \in W$ and $v \in V_G - \{u\}$. Then X is an irredundant set of $G + uv$, which contradicts (4.1). Therefore u is a universal vertex of G. It thus follows that $G = H + q$ with H satisfying (ii).

(b) $\Rightarrow$ (c): In the case of (i) it is clear that H is IR-edge-critical. Hence by Lemma 4.3, $H + q$ is IR-edge-critical. Now consider condition (ii) and let Z be the set of isolated vertices of H. Since $S \cup Z$ and $T \cup Z$ are IR-sets of H,

$$IR(G) = |S| + |Z| = |T| + |Z|.$$

Consider any $uv \in E_{\overline{H}}$ and let B be an IR-set of $H + uv$.

Case 1: $B \cap S \neq \emptyset$ and $B \cap T \neq \emptyset$. Let $s \in B \cap S$ and $t \in B \cap T$. Since $\langle S \rangle$ and $\langle T \rangle$ are complete graphs, $B \cap S = \{s\}$ and $B \cap T = \{t\}$, for otherwise B is not irredundant in $H + uv$. Therefore $B \subseteq \{s, t\} \cup Z$ and so

$$IR(H + uv) = |B| \leq 2 + |Z| < |S| + |Z| = IR(H).$$

Case 2: $B \cap S = \emptyset$ or $B \cap T = \emptyset$. Assume without loss of generality that $B \cap S = \emptyset$. Then $B \subseteq T \cup Z$. Since $\langle T \rangle$ is complete, $T \cup Z$ is not irredundant in $H + uv$. Therefore

$$IR(H + uv) = |B| < |T| + |Z| = IR(H).$$

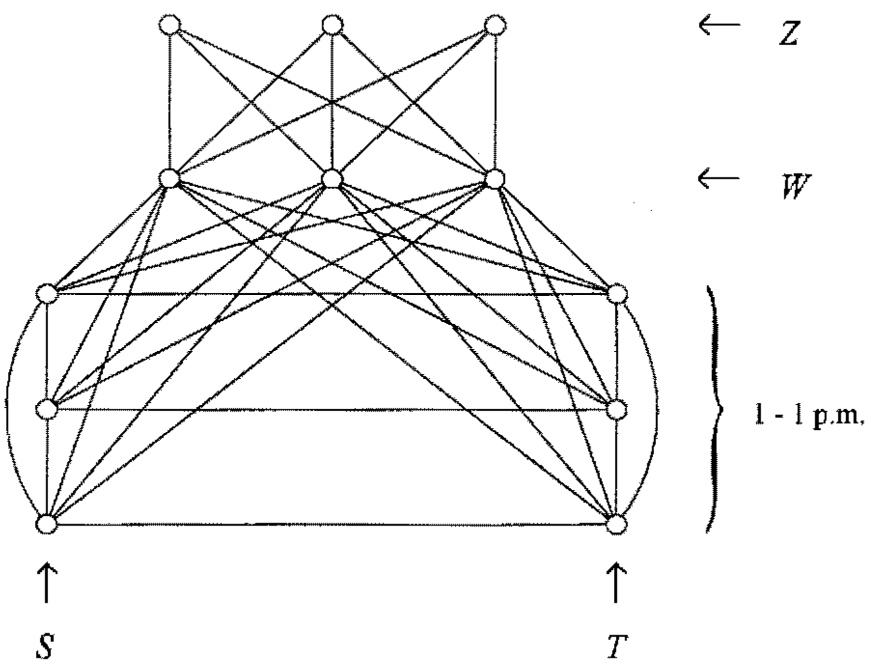


Figure 4.1

This proves that H is IR-critical and by Lemma 4.3, $H+q$ is IR-edge-critical.

(c) $\Rightarrow$ (a): Let S be an IR-set of G and let $s \in S$. If $r \in V_G - N_G[S]$, then $sr \in E_{\overline{G}}$ and S is irredundant in $G+sr$; hence $IR(G) = |S| \leq IR(G+sr)$, which contradicts the IR-edge-criticality of G. Therefore S is a dominating irredundant set of G and thus $\Gamma(G) = IR(G)$. It follows that for all $uv \in E_{\overline{G}}$,

$$\Gamma(G+uv) \leq IR(G+uv) < IR(G) = \Gamma(G).$$

■

Figure 4.1 shows the Γ-edge–critical graph $G = H + q$ with $|Z| = |S| = |T| = |W| = 3$.

4.3 Γ^+- and IR^+-edge-critical graphs

In this section we show that $K_m \times K_n$ is Γ^+-edge-critical if $m, n \geq 5$. Whether there exist IR^+-edge-critical graphs or not remains an open problem.

Proposition 4.6 *For any $m, n \geq 5$, the graph $K_m \times K_n$ is Γ^+-edge-critical.*

Proof. Let $G = K_m \times K_n$ where $n \geq m \geq 5$ and consider any $uv \in E_{\overline{G}}$. Since $\overline{G}$ is edge-transitive, assume without loss of generality that $u = v_{12}$ and $v = v_{mn}$. Now S_0 in the proof of Theorem 2.6 (see Figure 2.2, p. 24) is a minimal dominating set of $G + uv$. Therefore $\Gamma(G) = n < m + n - 4 = |S_0| \leq \Gamma(G + uv)$. ■

Although we have not found an IR^+-edge-critical graph, we show that for such a graph G, if it exists, the upper irredundance number increases by exactly one whenever an edge is added to G.

Proposition 4.7 *Suppose G is IR^+-edge-critical. For every $uv \in E_{\overline{G}}$ and every IR-set S of $G + uv$, $u \in S$ or $v \in S$. If $u \in S$, then $PN_{G+uv}(u, S) = \{v\}$ and $S - \{u\}$ is an IR-set of G.*

Proof. S is not an irredundant set of G, for otherwise $IR(G + uv) = |S| \leq IR(G)$. It follows that $u \in S$ and $PN_{G+uv}(u, S) = \{v\}$, or $v \in S$ and $PN_{G+uv}(v, S) = \{u\}$. Furthermore, if $u \in S$, then $S - \{u\}$ is an irredundant set of G. Therefore

$$IR(G + uv) - 1 = |S| - 1 \leq IR(G) < IR(G + uv)$$

and hence $S - \{u\}$ is an IR-set of G. ■

Corollary 4.8 *If G is IR^+-edge-critical, then $IR(G + uv) = IR(G) + 1$ for each $uv \in E_{\overline{G}}$.*

Chapter 5
ER-Critical Graphs

If π is an upper parameter, then K_n, $n \geq 2$, is π-ER-critical and if π is a lower parameter, then $K_{1,n}$, $n \geq 1$ is π-ER-critical. For $\pi \in \{\gamma, \beta, \Gamma, IR\}$ there do not exist π^--ER-critical graphs. Also recall that a graph is π-ER-critical (π^--ER-critical, respectively) if and only if each of its components is either π-ER-critical or isomorphic to K_1 (π^--ER-critical or isomorphic to K_1, respectively).

5.1 Upper domination parameter critical graphs: edge removal

Let π be an upper parameter. In this section we present a class of non-complete π-ER-critical graphs. We first find a useful characterisation of β-ER-critical graphs.

Proposition 5.1 *For any graph G with at least one edge,*

(a) $\beta(G - uv) \leq \beta(G) + 1$ *for all* $uv \in E_G$;

(b) $\beta(G - uv) = \beta(G) + 1$ *if and only if there exists a β-set T of G such that $u \in T$ and $v \in PN_G(u, T)$.*

Proof. Let $uv \in E_G$ and consider a β-set S of $G - uv$. Since $T = S - \{v\}$ is an independent set of G,

$$\beta(G - uv) - 1 = |S| - 1 \leq |T| \leq \beta(G).$$

Furthermore, if $\beta(G - uv) - 1 = \beta(G)$, then T is a β-set of G and since S is independent in $G - uv$, $v \in PN_G(u, T)$. This establishes (a) and necessity in (b). For sufficiency in (b), suppose T is a β-set of G such that $u \in T$ and $v \in PN_G(u,T)$. Since $T \cup \{v\}$ is an independent set of $G - uv$,

$$\beta(G) + 1 = |T| + 1 \leq \beta(G - uv).$$

It follows from (a) that $\beta(G - uv) = \beta(G) + 1$. ■

Corollary 5.2 (a) *G is β-ER-critical if and only if $\beta(G - uv) = \beta(G) + 1$ for all $uv \in E_G$.*

(b) *G is β-ER-critical if and only if for every $uv \in E_G$, there exists a β-set T of G such that $u \in T$ and $v \in PN_G(u.T)$.*

Observe that it follows from Proposition 1.7 that if G is β-ER-critical and $\beta = \Gamma$, then G is Γ-ER-critical, and if G is Γ-ER-critical and $\Gamma = IR$, then G is IR-ER-critical.

Proposition 5.3 *Let $G = C_n\langle 1, 2, ..., r\rangle$, where $1 \leq r \leq (n-2)/2$, and let $n = (r+1)m + q$ for some integer m and where $0 \leq q \leq r$. Then G is β-ER-critical if and only if $q = r$.*

Proof. Suppose $q = r$ and consider any $uv \in E_G$. Assume without loss of generality that $u = r + 1$ and $v \in LN(u) = \{1, 2, ..., r\}$. Let

$$T = \{(r+1), 2(r+1), ..., m(r+1)\}.$$

Then T is a β-set of G, and since $n = (r+1)m + r$, $v \in PN(u,T)$. It follows from Corollary 5.2 that G is β-ER-critical.

Conversely, suppose that G is β-ER-critical. Let $u = r + 1$ and $v = 1$. Then $uv \in E_G$. By Corollary 5.2 there exists a β-set T of G such that $u \in T$ and $v \in PN(u,T)$. Since $v \in PN(u,T)$, none of the vertices of

$$LN(v) = \{n - r + 1, ..., n - 1, n\}$$

are in T. Therefore, since T is an independent dominating set of G, $n-r \in T$. It follows that $n-r=(r+1)\,m$ and hence $q=r$. ■

In order to find a characterisation of Γ-ER-critical graphs, we need a definition and a lemma.

Suppose $uv \in E_G$. An irredundant set T of G is a *uv-irredundant set* if $u \in T$ or $v \in T$, and if $u \in T$, then $v \in PN_G(u,T)$ and either

(i) u is an isolated vertex of T and $PN_G(t,T) \not\subseteq N_G[v]$ for all $t \in T-\{u\}$, or

(ii) there exists an $s \in N_G[v]-T$ such that $PN_G(t,T) \not\subseteq N_G[v]$ for all $t \in T$.

Lemma 5.4 (a) *If $uv \in E_G$ and S is an irredundant set of $G-uv$ but not of G, then there exists a uv-irredundant set T of G with $|T|=|S|-1$. Furthermore, if S dominates $G-uv$, then T dominates G.*

(b) *If $uv \in E_G$ and T is a uv-irredundant set of G, then there exists an irredundant set S of $G-uv$ with $|S|=|T|+1$. If T dominates G, then S dominates $G-uv$.*

Proof. (a) Clearly $u \in S$ or $v \in S$. Suppose first that $\{u,v\} \subseteq S$. Then u or v is a singular isolated vertex of S in $G-uv$; assume without loss of generality that v is one. Let $T=S-\{v\}$. Then T is an irredundant set of G, $u \in T$ and $v \in PN_G(u,T)$. Furthermore, if u is also a singular isolated vertex of S in $G-uv$, then u is an isolated vertex of T and $PN_G(t,T) \not\subseteq N_G[v]$ for all $t \in T-\{u\}$. If u is not one, then $PN_G(t,T) \not\subseteq N_G[v]$ for all $t \in T$.

Suppose next that $\{u,v\} \not\subseteq S$ and assume without loss of generality that $u \in S$ and $v \notin S$. Then there exists an $s \in S-\{u\}$ such that $PN_{G-uv}(s,S)=\{v\}$. Let $T=S-\{s\}$. Then T is an irredundant set of G, $u \in T$ and $v \in PN_G(u,T)$. Also, $s \in N_G[v]-T$ and $PN_G(t,T) \not\subseteq N_G[v]$ for all $t \in T$.

In both cases it is clear that $|T|=|S|-1$ and that T dominates G if S dominates $G-uv$.

(b) Assume without loss of generality that $u \in T$. Suppose first that u is an isolated vertex of T and $PN_G(t,T) \not\subseteq N_G[v]$ for all $t \in T - \{u\}$. Let $S = T \cup \{v\}$. Then $PN_{G-uv}(t,S) \neq \emptyset$ for all $t \in T - \{u\}$ and u and v are isolated vertices of S in $G - uv$. Therefore S is irredundant in $G - uv$. Clearly $|S| = |T| + 1$ and S dominates $G - uv$ if T dominates G.

Suppose next that there exists an $s \in N_G[v] - T$ such that $PN_G(t,T) \not\subseteq N_G[v]$ for all $t \in T$ and let $S = T \cup \{s\}$. Then $PN_{G-uv}(t,S) \neq \emptyset$ for all $t \in T$ and $v \in PN_{G-uv}(s,S)$. This implies that S is irredundant in $G - uv$. Clearly $|S| = |T| + 1$. If T dominates G, then T dominates all vertices of $G - uv$ except v, and s dominates v; hence S dominates $G - uv$. ■

Proposition 5.5 *Let $\pi \in \{\Gamma, IR\}$. For any graph G with at least one edge,*

(a) $\pi(G - uv) \leq \pi(G) + 1$ *for all* $uv \in E_G$.

(b) $\pi(G - uv) = \pi(G) + 1$ *if and only if there exists a π-set T of G such that T is uv-irredundant.*

Proof. For $\pi = \Gamma$, let $uv \in E_G$ and consider a Γ-set S of $G - uv$. Then S is a dominating set of G. If S is an irredundant set of G, it follows that S is a minimal dominating set of G; hence $\Gamma(G - uv) = |S| \leq \Gamma(G)$. Suppose now that S is not an irredundant set of G. By Lemma 5.4(a) there exists a dominating uv-irredundant set T of G with $|T| = |S| - 1$. Therefore

$$\Gamma(G - uv) - 1 = |S| - 1 = |T| \leq \Gamma(G).$$

Furthermore, if $\Gamma(G - uv) - 1 = \Gamma(G)$, then T is a Γ-set of G. This establishes (a) and necessity in (b). For sufficiency in (b), suppose T is a uv-irredundant Γ-set of G. By Lemma 5.4(b) there exists an irredundant dominating set S of $G - uv$ with $|S| = |T| + 1$. Therefore

$$\Gamma(G) + 1 = |T| + 1 = |S| \leq \Gamma(G - uv).$$

It follows from (a) that $\Gamma(G-uv)=\Gamma(G)+1$.

For $\pi = IR$, let $uv \in E_G$ and consider an IR-set S of $G-uv$. The rest of the proof is similar to that for Γ; the only difference is that S and T need not be dominating sets of $G-uv$ and G respectively. ■

Corollary 5.6 *Let* $\pi \in \{\Gamma, IR\}$. *Then*

(a) G *is* π*-ER-critical if and only if* $\pi(G-uv)=\pi(G)+1$ *for all* $uv \in E_G$.

(b) G *is* π*-ER-critical if and only if for every* $uv \in E_G$, *there exists a* uv*-irredundant* π*-set of* G.

With the aid of Corollaries 5.2 and 5.6 we are now able to determine which of the graphs of Section 2.2 are π-ER-critical.

(i) $G = K_{n_1,n_2,\ldots,n_m}$ with $m \geq 2$.

Consider any π-set T of G. If T is independent, then all vertices of T are singular isolated vertices. If T is not independent, then T has two vertices and both are annihilated by their private neighbours. It follows from Corollaries 5.2 and 5.6 that G is not π-ER-critical.

(ii) $G = K_m \times K_n$ with $n \geq m \geq 2$.

There exists no β-set T of G with $v_{11} \in T$ and $v_{21} \in PN(v_{11}, T)$; hence G is not β-ER-critical (by Corollary 5.2). For $\pi \in \{\Gamma, IR\}$, consider any non-independent π-set T of G. For any $u \in T$ and $v \in PN(u,T)$, $PN(u,T) \subseteq N[v]$. (If T is dominating, this is clear; if not, see S_0 in Figure 2.2, p. 24.) Therefore G is not π-ER-critical (by Corollary 5.6).

(iii) $G = \overline{K_m \times K_n}$ for $n \geq m \geq 2$.

If $n = m = 2$, then G is the π-ER-critical graph $K_2 \cup K_2$. Suppose now that $n \geq m$ if $m = 2$. If T is an independent π-set of G, then all vertices of T are singular isolated vertices. If T is a non-independent π-set of G, then $m = 2$, $n = 4$ and we may choose $T = \{v_{11}, v_{22}, v_{23}, v_{14}\}$ (see Figure 2.3, p. 26), or $m = n = 3$ and we may assume $T = \{v_{11}, v_{22}, v_{33}\}$ or $T = \{v_{11}, v_{22}, v_{12}\}$. In all these cases, every external private neighbour of any vertex in T annihilates some vertex of T. It follows from Corollaries 5.2 and 5.6 that G is not π-ER-critical.

(iv) $G = C_n\langle 1, 3, ..., 2r-1\rangle$, where $1 \leq r \leq (n-2)/2$.

If n is odd and $r = 1$, then G is β-ER-critical by Proposition 5.3. Suppose that n is odd and $r > 1$. Recall that the only β-sets of G are the independent dominating sets of G induced by the partition $n = n$ of n. It is now easy to check that, for any β-set T of G and $u \in T$, $u + 1 \notin PN(u, T)$. Hence G is not β-ER-critical in this case. If n is even, then the only β-sets of G are $\{2, 4, 6, ..., n\}$ and $\{1, 3, 5, ..., n-1\}$. These sets have only singular isolated vertices; hence in this case G is not β-ER-critical either.

5.2 Lower domination parameter critical graphs: edge removal

Graphs that are γ-ER-critical and i-ER-critical have been characterised by Walikar and Acharya [25] and Ao [2] respectively.

Proposition 5.7 [2, 25] *Let $\pi \in \{\gamma, i\}$. The graph G is π-ER-critical if and only if G is a disjoint union of stars.*

Proof. If G is a disjoint union of stars, then G is clearly π-ER-critical. Suppose G is i-ER-critical and consider an i-set S of G and let $uv \in E_G$. Since S is independent in G, S is independent in $G - uv$. Therefore, if S dominates $G - uv$, then S is a maximal independent set of $G - uv$; hence $i(G - uv) \leq |S| = i(G)$. This contradicts the criticality of G. Therefore, for every $uv \in E_G$, S does not dominate $G - uv$. It follows that G is a disjoint union of stars. The proof for γ is the same except that we do not require the set S to be independent. ■

Clearly disjoint unions of stars are also ir-ER-critical. The next proposition gives necessary conditions for a connected graph that is ir-ER-critical but not γ-ER-critical. We will use the following notations in its proof: Suppose S is an irredundant set of the graph G. Let C, B and R denote the sets of vertices of $V_G - S$ which are adjacent to at least two vertices, exactly one vertex and no vertices of S, *i.e.*

$$\begin{aligned} R &= V_G - N_G[S], \\ B &= \left(\bigcup_{s \in S} PN(s,S)\right) - S \text{ and} \\ C &= N_G(S) - B. \end{aligned}$$

For each $s \in S$, let $B(s) = PN_G(s,S) - S$, i.e. $B(s)$ in the set of external private neighbours of s. Furthermore, let

$$\begin{aligned} Z &= \{z \in S |\, z \text{ is an isolated vertex of } S\}, \\ X &= \{x \in S |\, x \text{ is annihilated by some } r \in R\} \text{ and} \\ Y &= S - (Z \cup X). \end{aligned}$$

Let

$$\begin{aligned} X_1 &= \{x \in X |\, |B(x)| = 1\}, \\ X_2 &= \{x \in X |\, |B(x)| > 1\}, \\ Y_1 &= \{y \in Y |\, |B(y)| = 1\} \text{ and} \\ Y_2 &= \{y \in Y |\, |B(y)| > 1\}. \end{aligned}$$

Lastly, let

$$E_1 = \bigcup_{x \in X_1} B(x), \quad E_2 = \bigcup_{x \in X_2} B(x),$$

$$\begin{aligned} F_1 &= \bigcup_{y \in Y_1} B(y), \quad F_2 = \bigcup_{y \in Y_2} B(y), \\ E &= E_1 \cup E_2 \text{ and} \\ F &= F_1 \cup F_2. \end{aligned}$$

Proposition 5.8 *Suppose G is a connected ir-ER-critical graph other than a star. Then every ir-set S of G has the following properties.*

(a) *Every $r \in R$ annihilates exactly one vertex $\alpha(r)$ of S and $N_G(r) = B(\alpha(r))$.*

(b) *Every $v \in E_2$ annihilates exactly one vertex $\alpha(v)$ of S and $\alpha(v) \in Y_2$.*

(c) *If u and v are adjacent vertices of $V_G - (S \cup R)$, then $u \in E_2$ and $v \in B(\alpha(u))$, or $v \in E_2$ and $u \in B(\alpha(v))$.*

(d) *If $v \in F_2$, then v annihilates no vertices of S.*

(e) *Every vertex of C has exactly two neighbours and each neighbour is annihilated by a vertex of $R \cup E$.*

(f) *$\langle S \rangle$ is a disjoint union of stars. Furthermore, if $s \in S$ has more than one neighbour in S, then each of its neighbours is annihilated by a vertex of $R \cup E$. If s has one neighbour in S, then s or its neighbour is annihilated by a vertex of $R \cup E$.*

(g) *If $s \in X_1 \cup Y_1$, then s has only one neighbour in $\langle S \rangle$.*

Proof. Consider any ir-set S of G. If $uv \in E_G$ and S is a maximal irredundant set of $G - uv$, then $ir(G - uv) \le |S| = ir(G)$. This contradicts the criticality of G; hence

$$\text{for every } uv \in E_G,\ S \text{ is not a maximal irredundant set of } G - uv. \qquad (5.1)$$

Let $uv \in E_G$ with $\{u, v\} \subseteq V_G - S$. Since $PN_{G-uv}(s, S) = PN_G(s, S)$ for all $s \in S$, it follows from the irredundance of S in G that S is irredundant in $G - uv$. Note that $V_{G-uv} - N_{G-uv}[S] = R$ and $N_{G-uv}[R] \subseteq N_G[R]$. By (5.1) and Proposition 1.4, there exists $w \in N_G[R]$ such that w annihilates no vertices of S in $G - uv$.

Since S is a maximal irredundant set of G, w annihilates some $s_w \in S$ in G. It follows that $w = u$ and $v \in PN_G(s_w, S)$, or $w = v$ and $u \in PN_G(s_w, S)$. In both cases, s_w is the only vertex of S annihilated by w in G. We have proved:

For every $uv \in E_G$ with $\{u,v\} \subseteq V_G - S,\ u \in N_G[R]$ and there exists $s_u \in S$ such that $v \in PN_G(s_u, S) \subseteq N_G[u]$ and s_u is the only vertex of S annihilated by u, or $v \in N_G[R]$ and there exists $s_v \in S$ such that $u \in PN_G(s_v, S) \subseteq N_G[v]$ and s_v is the only vertex of S annihilated by v. (5.2)

Let $r \in R$ and $v \in N_G(r)$. Then $rv \in E_G$ with $\{r, v\} \subseteq V_G - S$. By (5.2), r annihilates exactly one vertex $\alpha(r)$ of S and $v \in B(\alpha(r))$. This is true for every $v \in N_G(r)$. Therefore $N_G(r) \subseteq B(\alpha(r))$ and hence $N_G(r) = B(\alpha(r))$. This completes the proof of (a). Note that (a) implies $N_G[R] = R \cup E$.

Let $x \in X_2$ and $v \in B(x)$. Since $PN_{G-xv}(x, S) = B(x) - \{v\} \neq \phi$ and $PN_{G-xv}(s, S) = PN_G(s, S) \neq \phi$ for all $s \in S - \{x\}$, S is an irredundant set of $G - xv$. Note that

$$V_{G-xv} - N_{G-xv}[S] = R \cup \{v\} \quad \text{and} \quad R \cup E \subseteq N_{G-xv}[R \cup \{v\}].$$

By (5.1), S is not maximal irredundant in $G - xv$. Therefore there exists a vertex $u \in N_{G-xv}[R \cup \{v\}]$ such that u annihilates no vertices of S in $G - xv$. But since S is maximal irredundant in G, every $w \in R \cup E$ annihilates some vertex of S in G and hence in $G - xv$. Therefore $u \notin R \cup E$ and hence $u \in F$. It follows from (5.2) that v annihilates exactly one vertex $\alpha(v)$ of S and $u \in B(\alpha(v))$. Since u does not annihilate $\alpha(v)$, $\alpha(v) \in Y_2$. This completes the proof of (b). Furthermore, (5.2) and (b) clearly imply (c).

Let $y \in Y_2$ and $v \in B(y)$. Since $PN_{G-yv}(y, S) = B(y) - \{v\} \neq \phi$ and $PN_{G-yv}(s, S) = PN_G(s, S) \neq \phi$ for all $s \in S - \{y\}$, S is an irredundant set of $G - yv$. Note that

$$V_{G-yv} - N_{G-yv}[S] = R \cup \{v\} \text{ and, by (5.2), } N_{G-yv}[R \cup \{v\}] = R \cup E \cup \{v\}.$$

Since S is maximal irredundant in G, every vertex of $R \cup E$ annihilates some vertex of S in G and hence in $G-yv$. Therefore, by (5.1), v annihilates no vertices of S. This completes the proof of (d).

Let $c \in C$ and let u and v be distinct neighbours of c. By (5.2), $\{u,v\} \subseteq S$. Since $PN_G(s,S) \subseteq PN_{G-uc}(s,S)$ for all $s \in S$, it follows from the irredundance of S in G that S is irredundant in $G-uc$. Note that

$$V_{G-uc} - N_{G-uc}[S] = R \quad \text{and} \quad N_{G-uc}[R] = R \cup E.$$

By (5.1), there exists $w \in R \cup E$ such that w annihilates no vertices of S in $G-uc$. But w annihilates the vertex $\alpha(w)$ of S in G. Therefore $v = \alpha(w)$ and $N_G(c) = \{u,v\}$. A similar argument with $G-vc$ shows that u is also annihilated by some $w \in R \cup E$. This completes the proof of (e).

By (c) and (e), if $z \in Z$, then $\langle N_G[z] \rangle$ is a component of G. Therefore G is either disconnected or a star. This contradicts our assumptions about G and therefore $Z = \emptyset$, i.e. S has no isolated vertices. Let u and v be adjacent vertices of S. Since $PN_G(s,S) \subseteq PN_{G-uv}(s,S)$ for all $s \in S$, it follows from the irredundance of S in G that S is irredundant in $G-uv$. Note that

$$V_{G-uv} - N_{G-uv}[S] = R \quad \text{and} \quad N_{G-uv}[R] = R \cup E.$$

By (5.1), there exists $w \in R \cup E$ such that w annihilates no vertices of S in $G-uv$. But w annihilates the vertex $\alpha(w)$ of S in G. Therefore, either u is an endvertex of $\langle S \rangle$ and $u = \alpha(w)$, or v is an endvertex of $\langle S \rangle$ and $v = \alpha(w)$. This completes the proof of (f).

Let $s \in X_1 \cup Y_1$ and $B(s) = \{v\}$. If s has more than one neighbour in $\langle S \rangle$, then $(S - \{s\}) \cup \{v\}$ is a maximal irredundant set of $G - sv$, and therefore $ir(G-sv) \le |S| = ir(G)$. This contradicts the criticality of G; hence s has only one neighbour in $\langle S \rangle$. This completes the proof of (g). ■

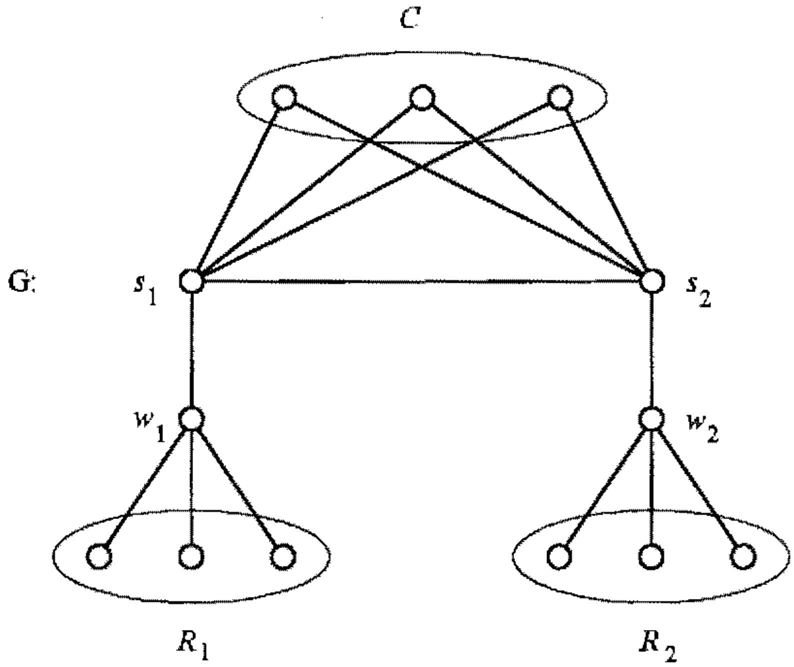


Figure 5.1

With the aid of Proposition 5.8 we are now able to characterise the connected 2-ir-ER-critical graphs. Define the graph G as follows. The vertex-set of G has partition

$$V_G = R_1 \cup R_2 \cup C \cup \{s_1, s_2, w_1, w_2\},$$

where R_1, R_2 and C are non-empty, and s_1, s_2, w_1 and w_2 are all distinct. The edge-set E_G of G is defined as follows.

(i) For each $c \in C$, $\{cs_1, cs_2\} \subseteq E_G$.

(ii) For each $i = 1, 2$ and each $r \in R_i$, $rw_i \in E_G$.

(iii) $\{s_1s_2, s_1w_1, s_2w_2\} \subseteq E_G$.

An example of G is illustrated in Figure 5.1. The class of all such graphs G will be denoted by $\mathcal{G}$.

Proposition 5.9 G is 2-ir-ER-critical if and only if $G \in \mathcal{G}$.

Proof. Suppose G is a connected 2-ir-ER-critical graph and let $S = \{s_1, s_2\}$ be an ir-set of G. By Proposition 5.8 (f), $s_1 s_2 \in E_G$ and $R \neq \emptyset$. Let R_1 and R_2 be the sets of vertices in R that annihilate s_1 and s_2 respectively and assume without loss of generality that $R_1 \neq \emptyset$, *i.e.* $s_1 \in X$. We prove that $s_1 \in X_1$. Suppose to the contrary that $s_1 \in X_2$ and let $u \in B(s_1)$. By Proposition 5.8 (b), u annihilates $s_2 = \alpha(u) \in Y_2$. This is true for every $u \in B(s_1)$; therefore $v \in B(s_2)$ annihilates s_1. But this contradicts Proposition 5.8 (d); hence $s_1 \in X_1$. Let $B(s_1) = \{w_1\}$.

If $R_2 = \emptyset$, then $\{w_1, s_1\}$ is an independent dominating ir-set of G. This contradicts Proposition 5.8 (f) and thus $R_2 \neq \emptyset$. With a proof similar to that in the case of s_1 we now have that $s_2 \in X_1$. Let $B(s_2) = \{w_2\}$.

If $C = \emptyset$, then $\{w_1, w_2\}$ is an independent dominating ir-set of G; again a contradiction of Proposition 5.8 (f). Therefore $C \neq \emptyset$. Note that Y, F, X_2 and E_2 are all empty sets and that $E_1 = \{w_1, w_2\}$.

It is now clear that the vertex-set of G corresponds to the description of the graphs in the class $\mathcal{G}$ above. That all the edges of G are given by (i), (ii) and (iii) follows from Proposition 5.8 (e), (a) and (c). Therefore $G \in \mathcal{G}$.

Conversely, it is not difficult to check that $G \in \mathcal{G}$ has $ir(G) = 2$ and $ir(G - uv) = 3$ for each $uv \in E_G$. ■

5.3 Graphs that are ir^-- or i^--ER-critical

Recall that π^--ER-critical graphs can only exist if $\pi \in \{ir, i\}$. In this section we exhibit three classes of i^--ER-critical graphs. Whether there exist ir^--ER-critical graphs remains an open problem.

Observe that if a graph G is i^--ER-critical, then it follows from Proposition 1.8(e) that $\gamma(G) < i(G)$.

Lemma 5.10 *Suppose $\gamma(G) < i(G)$ and, for any $uv \in E_G$, G has a γ-set S such that u and v are the only non-isolated vertices of S. Then G is i^--ER-critical.*

Proof. Since S is an independent dominating set of $G - uv$,

$$i(G - uv) \leq |S| = \gamma(G) < i(G). \quad \blacksquare$$

We begin by considering complete multipartite graphs.

Proposition 5.11 *If $m \geq 2$ and $n_i \geq 3$ for $1 \leq i \leq m$, then $K_{n_1,n_2,\ldots,n_m}$ is i^--ER-critical.*

Proof. This follows directly from Proposition 2.5 and Lemma 5.9. $\blacksquare$

We now turn to the complement of the cartesian product of two graphs.

Proposition 5.12 *If $m, n \geq 4$, then $\overline{K_m \times K_n}$ is i^--ER-critical.*

Proof. Let $G = \overline{K_m \times K_n}$ with $n \geq m \geq 4$. By Theorem 2.7, $i(G) = m > 3 = \gamma(G)$. Consider any $uv \in E_G$. Since G is edge-transitive, assume without loss of generality that $u = v_{11}$ and $v = v_{22}$. The set $\{v_{11}, v_{12}, v_{22}\}$ (see Figure 2.3, p. 26) is a γ-set of G with u and v the only non-isolated vertices. It follows from Lemma 5.10 that G is i^--ER-critical. $\blacksquare$

Finally, we show that some circulants are i^--ER-critical.

Proposition 5.13 *Let $G = C_n\langle 1, 3, \ldots, 2r-1\rangle$, where $1 \leq r \leq (n-1)/2$ and let $n = (2r+1)m + q$ for $0 \leq q \leq 2r$. If q is odd and $q > 3 - 2r$, then G is i^--ER-critical.*

Proof. By Theorems 2.9 and 2.10,

$$\gamma(G) = m + 1 < m + r + (q-1)/2 = i(G).$$

Let S be a γ-set of G induced by the partition

$$n = (2r + 1)\,(m - 1) + (2r + 1 + q)$$

of n. Each term $(2r + 1)$ contributes one isolated vertex to S and the term $(2r + 1 + q)$ contributes two adjacent vertices to S. Lemma 5.10 and the edge-transitivity of G now complete the proof. ■

Chapter 6
Conclusion

We conclude this thesis with a short list of open problems.

1. Are there ir-critical graphs which are not γ-critical, or γ-critical graphs which are not ir-critical?

2. Are there ir-edge-critical graphs which are not γ-edge-critical, or γ-edge-critical graphs which are not ir-edge-critical?

3. $K_m \times K_n$ with $m, n \geq 5$ are the only known Γ^+-critical graphs and also the only known Γ^+-edge-critical graphs. Are these the only Γ^+-critical or Γ^+-edge-critical graphs? (This would be very surprising.) Do these two types of criticality coincide? Do they imply vertex-transitivity? (This also seems unlikely.)

4. Are there IR^+-edge-critical or ir^--ER-critical graphs?

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Index