IMPLEMENTING INQUIRY- BASED LEARNING TO ENHANCE GRADE 11 STUDENTS' PROBLEM-SOLVING SKILLS IN EUCLIDEAN GEOMETRY

by

MOTSHIDISI MARLEEN MASILO

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Declaration

I declare that the thesis titled "*Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean geometry"* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged with complete reference

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M.M. Masilo

Dedication

I dedicate this project to God the Almighty, my creator, creator of heaven and earth. He travelled with me throughout the journey; we have safely reached the destination.

I also dedicate this work to my son, Pako, who has been affected in various ways by this venture. I thank you Pako for having inspired me to give my all to finish this project.

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ABSTRACT

Researchers conceptually recommend inquiry-based learning as a necessary means to alleviate the problems of learning but this study has embarked on practical implementation of inquiry-based facilitation and learning in Euclidean Geometry. Inquiry-based learning is student-centred. Therefore, the teaching or monitoring of inquiry-based learning in this study is referred to as inquiry-based facilitation. The null hypothesis discarded in this study explains that there is no difference between inquiry-based facilitation and traditional axiomatic approach in teaching Euclidean Geometry, that is, H_0: $\mu_{\text{inquiry-based facilitation}} = \mu_{\text{traditional axiomatic approach}}$. This study emphasises a pragmatist view that constructivism is fundamental to realism, that is, inductive inquiry supplements deductive inquiry in teaching and learning. Participants in this study comprise schools in Tshwane North district that served as experimental group and Tshwane West district schools classified as comparison group. The two districts are in the Gauteng Province of South Africa. The total number of students who participated is 166, that is, 97 students in the experimental group and 69 students in the comparison group. Convenient sampling applied and three experimental and three comparison group schools were sampled. Embedded mixed-method methodology was employed. Quantitative and qualitative methodologies are integrated in collecting data; analysis and interpretation of data. Inquiry-based-facilitation occurred in experimental group when the facilitator probed asking students to research, weigh evidence, explore, share discoveries, allow students to display authentic knowledge and skills and guiding students to apply knowledge and skills to solve problems for the classroom and for the world out of the classroom. In response to inquiry-based facilitation, students engaged in cooperative learning, exploration, self-centred and self-regulated learning in order to acquire knowledge and skills. In the comparison group, teaching progressed as usual. Quantitative data revealed that on average, participant that received intervention through inquiry-based facilitation acquired inquiry-based learning skills and improved (M= -7.773, SE= 0.7146) than those who did not receive intervention (M= -0.221, SE = 0.4429). This difference (-7.547), 95% CI (-8.08, 5.69), was significant at t (10.88), p = 0.0001, $p<0.05$ and represented a large effect size of 0.55. The large effect size emphasises that inquiry-based facilitation contributed significantly towards improvement in inquiry-based learning and

that the framework contributed by this study can be considered as a framework of inquiry-based facilitation in Euclidean Geometry. This study has shown that the traditional axiomatic approach promotes rote learning; passive, deductive and algorithmic learning that obstructs application of knowledge in problem-solving. Therefore, this study asserts that the application of Inquiry-based facilitation to implement inquiry-based learning promotes deeper, authentic, non-algorithmic, self-regulated learning that enhances problem-solving skills in Euclidean Geometry.

Key terms

Analysis; Cognitive processing; concepts; conceptualisation; conjecturing; cooperative learning; Differentiated teaching; Errors; Euclidean Geometry; formal deduction; geometric modelling; informal deduction; inquiry-based facilitation; inquiry-based learning; mathematical reasoning; mathematics; perception; pre-visualisation; problem-solving; rigour; Self-regulated learning; spatial abilities; spatial skills; traditional axiomatic approach; visualisation or imagination;

Acronyms

CAPS – Curriculum Assessment Policy Statement

CCSSI – Common Core State Standard Initiative

CREATE – Consortium for Research on Educational Access, Transition and Equity

DBE – Department of Basic Education

FET – Further Education and Training

GDE – Gauteng Department of Education

H_1 - Directional or Alternative Hypothesis

H_o - Null Hypothesis

IBE – Inquiry-based Education

IBF – Inquiry-Based Facilitation

IBL – Inquiry-Based learning

IBME – Inquiry-Based Mathematics Education

ICT - Integrated Computer Technology

NCTM – National Council of Teachers of Mathematics

LTSM – Learning and Teaching Support Material

RME – Realistic Mathematics Education

TIMSS – Trends in International Mathematics and Science Study

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CHAPTER 1
STUDY OVERVIEW

1.1 INTRODUCTION

The traditional way of teaching geometry in classrooms is apparent when teachers present geometry content and concepts through creating doubts (that is, formulating conjectures) in the minds of students in order to motivate a need for deductive proof (De Villiers, 2004). It is a view of this study that teachers should not formulate conjectures for students, nor motivate deductive proof, but teachers should guide and assist students to apply self-regulated learning and inquiry for exploration and discovery, in order to seek out patterns, formulate and validate conjectures and ultimately establish the truth. Self-regulated learning is also addressed as self-directed learning in this study. Inductive teaching that yields self-regulated learning is perceived in this study as a basis for deductive teaching and learning; therefore, learning by induction in a student-centred environment is perceived as essential to motivate deductive proof. It is recommended that in order to develop essential mathematics skills in Euclidean Geometry, the student should apply spatial skills to identify, pose, use properties of shapes, apply critical thinking and creativity to advance problem solving (Department of Basic Education [DBE], 2011). Therefore, teacher facilitation skills should enable students to apply relevant processes to control and own their learning in order to acquire mathematical knowledge through meaningful learning, that is, learning that will impact in and out of classroom context. This study focuses on the teaching of the content area of Euclidean geometry, with emphasis on the topic circle geometry including cyclic quadrilaterals. The concepts circle geometry and cyclic quadrilaterals will be used interchangeably in this study.

1.2 BACKGROUND

According to Gutierrez, Leder and Boero (2016), spatial reasoning skills are connected to human thought, action, geometric thinking and development of geometric knowledge. In addition, DBE (2011) highlights that, through acquiring geometric problem-solving skills, students are prepared to handle abstract concepts and improve their reasoning in problem solving as a result of experience they have acquired. In other words, experience, action and geometric thinking are basic to abstract thinking; learning from what one knows (experience) and learning by doing (action) induce

inductive geometric thinking and thought, resulting in deductive abstract thinking. Regardless of the fact that identifying, posing and solving problems creatively and critically are essential skills in Euclidean Geometry, research has shown that learning these skills is a challenge to many South African students who are at the Further Education and Training (FET) level (Alex & Mammen, 2014; Cassim, 2007). The FET band or level comprises Grades 10 – 12 high school classes. This study has shown that teaching these skills applying the traditional axiomatic approach only, contributes to the lack of learning these skills properly. However, inquiry-based facilitation (IBF) initiates inquiry-based learning (IBL) that equips students with skills such as creativity, critical thinking, student-centred learning and confidence in order enhance geometric thinking to identify, pose and solve problems.

When considering the three grades in the FET band, students in Grade 10 are new in the FET phase having been through the senior phase; therefore, they are still trying to find their place within the FET system. Students in Grade 10 are expected to have some experience with geometric shapes and objects, and to understand a few properties of these shapes and objects because they were oriented to geometric systems in the lower grades. However, in Grade 11 students should have settled into the FET phase and be preparing for the last level of Grade 12. Students are expected to be experienced enough, and to have accumulated knowledge, understanding and spatial skills to identify and create problems and ultimately apply creativity and critical thinking in problem-solving. It is at this stage that experience and level of understanding should enable the students to progress from being passive participants, as well as learning by rote and memorisation. Further, grade 11 students should be able to apply knowledge acquired through explorations and discoveries in lower grades to build corroborations based on how they understand (Cassim, 2007). In addition, Cassim highlighted that the students at FET levels do not need to learn ready-made substantiations by rote in order to give the proofs as they are in assessment. Cassim (2007) comments that very few students reach this stage of advanced reasoning, understanding and knowledge. It is also apparent from his study that the majority of Grade 11 students struggle in almost all levels of knowledge and understanding.

Clark (2012) suggests that for teachers to assist students to attain the requirements of the advanced reasoning stage, teachers should bear in mind that knowledge from

lower grades might not be in place to develop the advanced reasoning of geometric knowledge and understanding. In order to put knowledge and understanding of lower levels in place, Clark (2012) states that teachers must introduce definitions through a class discussion that advances gradually to the advanced levels of reasoning. The current study takes cognisance of the fact that students are still lacking in requisite lower-level knowledge and posits that teaching through IBF encourages IBL that would contribute to a reasonable number of students being able to master the concepts and skills they have missed in lower levels.

1.3 APPROACHES TO TEACHING EUCLIDEAN GEOMETRY

1.3.1 Open systems for acquiring knowledge and understanding in Euclidean Geometry

Teaching in Euclidean Geometry happens in different countries as either a systematised Euclidean Geometry based on few axioms or occurs as an open system to acquire skills for proof to advance solutions (Bankov, 2013; Baptist, 2012; Friesen & Scott, 2013; Kunimune & Nagasaki, 1996; Olkun, 2009; Siyepu & Mtonjeni, 2014).

Different countries use diverse names for these open systems of learning, for example, Inquiry-Based Mathematics Education (IBME); inspiring education; dynamic geometry application and Realistic Mathematics Education (RME) (Baptist, 2012; Friesen & Scott, 2013; Kunimune & Nagasaki, 1996; Olkun, 2009; Siyepu & Mtonjeni, 2014). For example, teaching Euclidean Geometry through inquiry is practised in Japan, and the recommended learning process is that learning begins with concrete realistic activities, secondly, application of analogy and inductive thinking for discovering properties that are proved deductively (Kunimune & Nagasaki, 1996). Similarly, in Netherlands the education system recognises RME as the essential teaching approach. RME is an inquiry course whereby a real problem is interpreted and solved mathematically (Siyepu & Mtonjeni, 2014; Wubbels, Korthagen & Broekman, 1997). Furthermore, RME encompasses consecutive steps such as constructing the problem, analysis of the constructed problem; finding solutions; application of solutions in real contexts; and reflecting on the implications of the solution (Wubbels, et al., 1997). In addition, Wubbels, et al. (1997) state that, in applying RME, teachers should utilise the students' constructions of knowledge in guiding them to build new knowledge and to solve problems. Inspiring education system is envisaged in Alberta, Canada to ensure that

students apply the process of discovering through inquiry to develop skills (Friesen & Scott, 2013). The education system relies on traits such as exploring, experimenting, probing, hypothesising and reflecting. Further, the system is envisioned at enabling students to apply co-operative learning in order to learn how to think critically, innovatively and make discoveries.

Dynamic geometry as an instructional approach means that instructional activities in geometry are designed to emphasise students' learning through exploration (Olkun, 2009). In addition, Olkun (2009) states that other activities behind the dynamic geometry approach comprise teacher asking questions to probe, students participating actively and making decisions by themselves. IBF activates metacognition and self-regulated critical learning that enable students to learn through exploration, experimentation and discovery.

Teaching and learning Euclidean Geometry through inquiry has been commented as productive because Netherlands has been considered to be one of the highest achieving countries in terms of international mathematics tests, which has been attributed to the implementation of RME during teaching and learning (Dickinson & Hough, 2012). In their research on RME, Dickinson and Hough (2012) found that teachers using RME testified that students who learn through RME acquire a positive attitude about mathematics than students who are exposed to traditional teaching and learning contexts. Further, Dickinson and Hough (2012) aver that students who are exposed to RME understand more, can explain the strategies they employ in problem solving and can solve problems in the approved manner. In light of this, this study employed inquiry-based teaching and learning to improve learning Euclidean Geometry in the South African context.

1.3.1.1 Guided inquiry

Guided inquiry is one teaching approach. Clark (2012) explains guided inquiry as a method that revolves around students other than focusing only on textbooks and the teachers' word. In addition, Clark (2012) highlights that guided inquiry engages students in class discussion where every student is involved. This study focuses on inquiry-based facilitation (IBF) as a foundation for inquiry-based learning (IBL) that empowers Grade 11 students to develop the experience, knowledge and understanding that they did not acquire in the lower grades and at entry level (Grade

10) to FET. The current study focuses on circle geometry. Further, this study motivates that IBF prompts IBL. In addition, IBL enables students to identify, pose and solve problems in circle geometry apply knowledge and understanding of geometric shapes, objects and vocabulary acquired in lower grades; and get empowered to solve problems at advanced abstract levels of problem-solving. Becoming equipped in geometric problem-solving skills not only makes students competent problem solvers, but prepares the students for the Grade 12 school exit examinations, and beyond.

Furthermore, Feza and Webb (2005) argue that the appropriate teaching strategy that can assist teachers to implement considerations stated is the inquiry approach. They highlight that, inquiry approach could assist teachers to generate teacher-student conversations where teachers would be able to establish students' social or educational background, pre-knowledge and level of knowledge and understanding.

1.3.1.2 Traditional axiomatic approach in Euclidean Geometry

The traditional axiomatic approach in Euclidean Geometry is defined as a systematised Euclidean Geometry based on limited number of priori axioms (Bankov, 2013). According to Houdement and Kuzniak (2003), the axiomatic teaching approach influences students to memorise axioms and theorems in order to produce proofs as solutions. Further, they assert that the axiomatic teaching approach can be without any relation to reality, where truth is needed to maintain facts. Research has indicated that the traditional axiomatic approach in the teaching and learning of Euclidean Geometry has been adhered to for long in South Africa and in more other countries.

For example, (Bankov, 2013) motivated that in Bulgaria teaching Euclidean Geometry revolves around presenting a set of agreed facts as a set of axioms and geometry is deductively build on these facts. This is similar to the traditional axiomatic approach. According to Bankov (2013) strict axiomatic approach contribute towards reasons that are more didactical and pedagogical in presenting solutions. Therefore, Bankov (2013) suggested that it is not appropriate to use traditional axiomatic approach for students at school level.

According to research in South African Euclidean Geometry, the teaching and learning style that is currently practised to deliver the content for problem-solving revolves much around traditional axiomatic approach. Much research has been conducted in

South Africa related to teaching and learning Euclidean Geometry using the traditional axiomatic approach (Alex & Mammen, 2014; du Plessis, 2011; Malati, 2014; Siyepu & Mtonjeni, 2014; Van Putten, Howie & Stols, 2010). Mostly, findings are that teaching methods currently applied in South Africa yield passive learning, that is, learning by memorisation and rote; and problem-solving skills are not developed by students for competence in the classroom or in the real world.

In the traditional axiomatic approach, students are supposed to listen to the teacher passively and memorise theorems for test and examination purposes (Siyepu & Mtonjeni, 2014). In the South African context, learning Euclidean Geometry is characterised by memorisation of rules, signs and procedures (Siyepu & Mtonjeni, 2014), leading to traditional axiomatic learning approach.

1.3.1.3 Differentiated teaching

In principle, the Department of Education in South Africa (DoE, 2010) recommends differentiated teaching as method of teaching or lesson presentation in various subjects including mathematics. DoE (2010) motivates that teaching methods should be inclusive and consider individual students' levels of participation, levels of thinking, and learning methods. The levels of thinking are expected to enable students to work on concepts that reflect their previous achievements (DoE, 2011). Further, DoE (2010) highlights that "differentiation or inclusivity assumes that students vary in their cognitive abilities" (p. 10); therefore, in order to cater for various cognitive abilities, teachers should consider teaching from most basic levels to complex levels; and present the content in multimedia format. However, in practice, the teaching strategy has not been successful as there is an outcry of learner's underperformance in geometry and mathematics in general; further, researchers continually recommend that South Africa needs a different angle when coming to the teaching of mathematics or diverse learning areas within the mathematics subject. This study embarked on a different angle of teaching geometry, that is, practical application of IBF.

1.3.1.4 Van Hiele's approach

This study emphasises that teaching that is based on students' experience or past achievements and students' levels thinking, or level of knowledge and understanding is in line with the actual learning method of Euclidean Geometry as reflected in the

Van Hiele (1959) theory. The Van Hiele levels of understanding geometry designate the students reasoning around geometric ideas, objects, figures and shapes; and further address the issue that students cannot memorise geometric systems, but must experience illustrations as models and patterns in order to be well conversant with the systems in geometry. (Van Hiele, 1986, 1959). The teaching approach is essential in guiding the students through their development in learning Euclidean Geometry.

Van Hiele's theory describes learning Euclidean Geometry in terms of learning by following the principles of visualisation, analysis, informal deduction, deduction and rigour. According to Clement and Battista (1992), pre-visualisation or pre-visualisation is more basic than the Van Hiele basic level of visualisation. Therefore, the current study operates on six levels of knowledge and understanding, that is, pre-visualisation; visualisation; analysis; informal deduction; formal deduction and rigour. Much research has been done on the Van Hiele levels of knowledge and understanding; however, the pre-visualisation stage has not been researched in any depth in combination with Van Hiele's levels of knowledge and understanding.

The six levels are sequential. The first level, that is, the pre- visualisation, is the stage where students consider the world as they perceive it with their normal senses, aided by instruments such as pictures, models, digital resources and other resources (Harre, 2002). Cassim (2007) contextualises the other levels according to Van Hiele's theory. Succeeding the pre-visualisation level, the visualisation level follows. Students visualise by recognising space in their context, considering geometric objects in entirety rather than basing judgements on properties, identify and reproduce specific shapes, and learn the appropriate geometric vocabulary. Thirdly, analysis follows where students unpack the properties of the visualised shapes through experimentation and observation and use the unpacked properties to conceptualise classes of shapes. In the fourth stage, students do informal deduction by using analysed properties to establish interrelationships that exist between and among figures. In the fifth stage, formal deduction occurs where students identify the problem, pose the exploration questions and construct proofs based on their understanding from the informal deduction. Lastly, students advance to the rigour level where they apply axioms from the formal deduction stage in problem-solving and apply knowledge and understanding of geometric systems in non-Euclidean systems such as spherical geometry.

Research related to students' understanding of Euclidean Geometry based on the Van Hiele levels of knowledge and understanding, has shown that students at FET (Grade 10-12) level have difficulties in solving Euclidean Geometry problems at deduction level, while their foundation knowledge (pre-visualisation, visualisation, analysis and informal deduction) is not well-established (Alex & Mammen, 2014; Van Putten, Howie & Stols, 2010). In order to address and improve the foundational knowledge of students, Feza and Webb (2005) suggest that teachers need to take into consideration the Van Hiele levels in conjunction with students' background and application of vocabulary from the home language. This would benefit students mostly in the pre-visualisation and visualisation levels where students will be assisted to be aware of the natural space around them, build concepts and vocabulary from what they know.

The importance of understanding geometric language is emphasised as important because distinction is accentuated in each level as each level of learning has its own language symbols and connections (Van Hiele, 1986). Olkun (2009) highlights that instruction should support students to learn across levels from exploratory phase, to concept construction and ability to link new and existing knowledge.

It has been argued that FET teachers and students are guided by the curriculum which directs them through questions and proof construction that requires the student to be on level three of the Van Hiele level of understanding (Van Putten, Howie & Stols, 2010). The authors further highlight that students who do not meet the level three requirements, can do proofs only by memorisation. However, it has been argued that learning by memorisation imposes limitations in terms of students' development of conceptual and problem-solving skills (Siyepu & Mtonjeni, 2014). While research continues to recommend open systems of teaching in principle, in practice the application of open systems teaching is still lacking. Malati (2014) states that the teaching of geometry should comprise steps such as: (1) the description and classification of plane figures (for example parallelograms, and cyclic quadrilaterals); (2) the study of the properties of the plane figures; (3) the direct comparison of these figures and their properties; (4) deduction using congruence of figures as a basic tool (some of the properties are deduced from others).

1.3.3 IBL teaching approach, learning environment and assessment

Research has found the IBL approach essential in learning Euclidean Geometry (Artigue, 2012; Bankov 2013; Clark, 2012; Feza & Webb, 2005; Kunimune & Nagasaki, 1996; Siyepu & Mtonjeni, 2014).

Research on IBF and IBL in Euclidean Geometry in the South African context, has been conducted by several researchers (Malati, 2014; Sanni, 2007; Siyepu & Mtonjeni, 2014) but applied research on IBF and IBL in mathematics, particularly in Euclidean Geometry, is still lacking. In emphasis, Botha (2017) highlight that there are extensive discussions in research literature about the application of inquiry based learning in South Africa. In a paper based on empirical study in South African schools, Botha, (2016) argue that IBL erupts as an essential strategy to improve learning. Similarly, Paulsen (2006) argue that the current approach to education in South Africa advocates constructivist teaching and learning; that is, classrooms must become "student-centred" and teachers must act as facilitators, instead of transmitters of knowledge (p.170). In light of the researchers' views, it is clear that the traditional axiomatic teaching approach is the method that is currently applied in teaching and learning geometry in most of the South African classrooms. While in principle researchers recommend IBF and IBL as strategies for alleviating the problems of learning, this study focuses on the practical application of IBF and IBL in Euclidean Geometry in order to contribute in assisting teachers with their teaching of circle geometry and support students to reach higher levels in geometry thinking, that is, formal deduction and rigour. Emphasising the idea that IBL is essential in teaching Euclidean geometry, Mason (1998) aver that the application of lecture and memorisation as the teaching method does not lead to effective and meaningful learning. In addition, Mason (1998) argue that meaningful and effective learning takes place when students engage in IBL activities such as engaging in discussions and reflection in order to acquire experience in the content they are studying. Therefore, the implementation of IBL through IBF in this study is grounded by the pragmatist view of inquiry-based learning aligned to John Dewey's philosophy of pragmatism that relates to IBL.

1.3.4 Use of technology in teaching Euclidean Geometry

The traditional axiomatic teaching approach differ with IBF in terms of teaching and learning methods; assessment and learning environment. How the teachers plan and

organise the learning environment has a major influence on learning. The teaching and learning environment may foster either traditional axiomatic approach or IBF. The use of technology is seen to be essential in the teaching of Euclidean Geometry. For example, it has been stated that mathematical laboratories like GEOMLAND[1] enable pupils to construct and experiment with Euclidean Geometry objects to investigate their properties, to formulate and verify conjectures (Checlarova & Sendova, 2012). Further, Checlarova and Sendova (2012) highlight that the usage of GEOMLAND has proved to be appropriate for materialising the abstract mathematical concepts and to enable students to adopt the style of discovery learning.

Friesen and Scott (2013) highlight that Alberta Education prioritises teaching that can utilise technology to assist students to successfully implement inspiring education. However, Friesen and Scott (2013) indicate that technology alone only provides a single flow of information; therefore, integrating technology in the classroom implies using multiple resources. According to Masilo (2015), multiple resources or multimedia that cater for all students' needs can be selected from different categories of resources such as visual, audio, audio-visual, text and digital resources. In addition, Masilo (2015) states that multimedia should be used in teaching in order to assist students to link concrete and abstract learning; to retain mathematical concepts and procedures; to approach problem-solving from various perspectives; and to play an active role in self-directed learning. Further, it has been stated that teachers should provide students with learning supports and rich, multiple sources of information to assist students in finding solutions (Indiana University, 2015). This study asserts that a combination of diverse resources fosters inquiry, self-regulated learning, metacognition and active students' participation in the learning process.

Chehlarova and Sendova (2012) maintain that the main purpose of integrating technology is to encourage students to "behave like working mathematicians, i.e. play with mathematical ideas and communicate their findings" (p. 114). Therefore, this study takes cognisance of the fact that the teaching and learning environment must

[1] GEOMLAND refers to a computer environment that supports exploration. In GEOMLAND students can explore geometric objects, investigate the properties of diverse geometric objets, create and verify conjectures.

be organised properly by making multiple resources available in order to allow students to explore and learn by discovery.

1.3.5 Concluding remarks on teaching approaches

The gap realised in this study is that traditional axiomatic teaching approach applied in South Africa has been commented as unproductive, while inquiry-based learning is recommended only in principle as an alternative method of teaching. To close this gap, , his study focuses on the application of inquiry- based facilitation in practice as an alternative teaching approach. The IBF practice applied has shown to have credits in assisting students to be able to apply inquiry to retrieve knowledge lost in lower grades. Further, teachers do not consider assisting students to recover knowledge lost from lower grades while teaching at grade 11 level. Lack of consideration of knowledge from lower grades leave students with one option, but memorising theorems. However, it is difficult for students to recall the memorised information during examination. Further, teaching and learning for memorisation is examination driven. This study had shown that inquiry based facilitation and learning strategies bridges the knowledge gap among levels of knowledge and understanding and students are able to acquire all the information from pre-visualisation to rigour. Essentially, the gap of learning for examination is closed in this study, because inquiry based learning had contributed towards conceptual development where the product produced by teaching and learning is knowledge that can be applied in both examination and other situations that require inquiry for problem solving.

The other essential factor contributed by this study is that testing through formal writing should not focus on questioning that require lower level demands of thinking such as completely procedural tasks and memorisation, but the focus of testing should be on high level cognitive demands of geometric thinking such as sense-making during mathematising or doing mathematics, comprehension of geometric concepts and exploration of concepts through linking inductive and deductive knowledge. The task compiled by the researcher (cf Annexure B) is a practical example of a formal task that links both higher level and lower level of cognitive demands. The tasks are developed according to Stein, Smith, Henningsen and Silver's (2000) idea of considering diverse levels of cognitive demands when structuring an assessment. The stipulated task used as a pre and post-test in this study represents multiple ways of knowledge, for

example, diagrams, symbols, figures representing real-life objects and conceptual ideas that underlie procedures of mathematising. The pre and post-test have contributed towards the development of students' understanding of all levels of knowledge and understanding through inquiry. The application of IBF is structured according to the Van Hiele levels of learning Euclidean Geometry, where the teaching and learning model (learning by inquiry, discovery, exploration and experience) according to Dewey's learning and teaching theories are integrated with the levels of knowledge in learning geometry.

While differences between the traditional axiomatic approach and inquiry-based approach have been enunciated, it is worth noting that assessment in inquiry-based approaches is executed differently compared to traditional axiomatic approaches. Assessment in traditional teaching and learning approach focuses on recalling what was memorised, while in IBL it focuses on the students' ability to find and use information, including discovering facts and relationships among concepts. Further, the learning environment that is resourced is vital in applying IBF successfully. Against this background, the study argues that IBF methods are necessary and the study is focused on the practical implementation of IBF to activate IBL in Euclidean Geometry. The IBF framework (cf. figure 5.4) contributed by this study stipulates that in order to facilitate problem solving through IBL, IBF strategies encompasses probing and following sequential steps specified in this study as coordinating orientation, orchestrating conceptualisation and guiding the conclusion stage. In addition, this study emphasises that through the stipulated IBF strategy, IBL in Euclidean geometry becomes a mode of inquiry, students are able to apply experience or prior knowledge and are able to establish connections among concrete and abstract geometric levels of knowledge.

1.4 PROBLEM STATEMENT

Teaching and learning Euclidean Geometry revolves around exploration through pre-visualisation, visualisation, analysis, informal deduction, formal deduction, and rigour (Clark, 2012; Clements & Battista, 1992; Van Hiele, 1986). In the Curriculum Assessment Policy Statement (CAPS) (DBE, 2011), Euclidean Geometry requires formal proof and also implies that students must perform at Van Hiele level 4, that is, at formal deduction with proof (Alex & Mammen, 2014, Malati, 2013). The key feature

of Van Hiele's theory is that students' comprehension of geometric concepts is classified according to an order of levels from 0 to 4 with level 0 being the lowest. The specific levels are outlined as follows (Alex & Mammen, 2014; Cassim, 2007): level 0 – visualisation where a student recognises a figure by its appearance, shape or form; level 1 – analysis, that is, students view a collective of shapes within a typology rather than as distinct shapes; level 2 – informal deduction where students realise and frame generalisations based on their pre-knowledge about properties of shapes; level 3 – Formal deduction where students prove theorems deductively and understand the structure of the geometric system; and level 4 – rigour, that is, where students can establish theorems in different systems, compare and analyse systems. Alex and Mammen (2014) conducted research based on the Van Hiele model of thinking with a sample of Grade 10 South African students studying Euclidean Geometry. The results of the study were that the majority of the students were at level 0 although the CAPS expectation is that students should perform at level 3 to be able to handle level 4. Students in Grades 10 and 11 are expected to have achieved level 0-3 of the Van Hiele levels of comprehension before they enter the FET phase. Grade 11 students should be in a position to handle problems at deduction level despite the fact that they have not yet achieved the informal deduction level 2.

In a report by Malati (2014) where primary school teachers were observed, it was argued that many of the teachers at primary level teach little or no geometry to their classes; and if the teaching is done, it is restricted to naming simple figures such as a square, triangle, circle and rectangle. Therefore, how teachers handle Euclidean geometry in their classrooms confirm that teachers encounter difficulties when teaching geometry. The report also shows that regardless of all the teachers' efforts in conveying Euclidean geometry content in an abstract way, students continue to encounter difficulties with formal deduction and proof which are more abstract. It was further reported that in trying to cope with difficulties students and teachers turn formal deduction and proof into something algorithmic, that is, many students simply memorise proofs or rules (Malati, 2014). In light of this, it appears that students at Grade 11 lack knowledge and understanding of lower levels in Euclidean Geometry; therefore, they learn by rote to operate with mathematical relations at the formal deduction level.

The main problem and question is: “Does IBF influence IBL to develop the students’ knowledge and understanding of formal deduction level through pre-visualisation, visualisation, analysis and formal deduction in Euclidean Geometry?”

1.5 RATIONALE

It has been stated that geometry is important because: (1) Shape and space form the world and geometry is part of mathematics that constitute space and shape (2) natural part of geometry is essential in assisting students to acquire experience and expertise in comprehending and applying concrete materials and activities; and (3) geometry can be applied in more fields of study (Teachers’ Lab, 2017). According to Teachers’ Lab (2017), learning Euclidean Geometry can benefit students in diverse ways. That is, experience with concrete materials can assist students with abstraction. Secondly, spatial understanding is necessary to assist student in understanding their natural geometric world. Thirdly, spatial sense enhances students’ perceptions about two- and three-dimensional shapes as well as their properties. Lastly, rigorous spatial reasoning enable students to learn by integrating topics within the mathematics subject. The benefits of learning as outlined by the Teachers’ Lab report align with the vision about learning Euclidean Geometry in South African. DoE (2011) states that students should apply critical and creative thinking skills to perceive, construct, critique and solve problems utilising spatial skills and applying properties of shapes. This will enable them to improve creative and reasoning skills for advancing more deductive mathematics. In addition, South African students should learn school geometry to acquire experience in mathematics, for them to have opportunities to study related courses at higher education and further to follow related career paths and occupations (Alex & Mammen, 2014). Related courses, career paths and occupations related to geometry include geology, architectural, engineering and various aspects of construction work. Geometric skill is important in all the mentioned fields and the fields are commented as important for the country’s development (Alex & Mammen, 2014). It is, therefore, essential that students should know the space around them, use the shape and space knowledge to develop spatial skills that will enable them to build comprehension of the schemes of connections between geometric concepts and linking to other subjects and to life in general. Notwithstanding the fact that geometry is an important branch of mathematics, research has proved that challenges are encountered in teaching and learning geometry in South Africa (Alex & Mammen,

2014). Using algorithmic styles in learning geometry, that is, memorising proofs or rules (Malati, 2013), does not enable students to create new knowledge and ultimately to solve problems in Euclidean Geometry.

1.6 HYPOTHESIS

Teachers teach Euclidean Geometry at formal deduction level when the students could not achieve the pre-visualisation, visualisation, analysis and informal deduction levels of understanding; therefore, students experience challenges with Euclidean Geometry (Alex & Mammen, 2014; Clark, 2012; Malati, 2014). The IBF method and strategies other than the traditional axiomatic teaching approach are needed to develop students' knowledge and understanding of formal deduction level through pre-visualisation, visualisation, analysis and informal deduction in Euclidean Geometry (Clark, 2012).

Therefore, this study addresses the following null hypothesis, there is no difference between IBF and traditional axiomatic approach in teaching Euclidean Geometry, that is, H_0: $\mu_{IBF} = \mu_{traditional\ axiomatic\ approach.}$ The directional hypothesis, also known as the alternative hypothesis states that IBF in Euclidean Geometry influences IBL, that is, H_1: $\mu_{IBF} \neq \mu_{traditional\ axiomatic\ approach.}$

1.7 AIMS AND OBJECTIVES

This study is aimed at investigating the influence of IBF on the students' knowledge and understanding of formal deduction level in geometry through pre-visualisation, visualisation, analysis and informal deduction.

The objectives of the study are to:

- investigate the impact of inquiry-based facilitation strategies on students' development of higher levels of geometric thinking, that is, formal deduction and rigour.
- determine the influence of interventions in activating IBL to assist students to acquire knowledge and understanding of the formal deduction level through Van Hiele's levels of visualisation, analysis and informal deduction.
- Explore whether IBF assists students to learn Euclidean Geometry successfully.

1.8 THE RESEARCH APPROACH ADOPTED IN THIS STUDY

This mixed-methods study with the aforementioned focus employs both quantitative and qualitative methodologies. Through the mixed-method design, namely, the quantitative methodology (quasi-experimental design) complemented by the qualitative methodology (phenomenographic approach through observations and interviews), clarification of the research questions is advanced. Through quasi-experimental design approach, this study is an empirical interventional study that estimates the causal impact of intervention on targeted grade 11 group without randomisation. To complement the quantitative quasi-experimental design, the qualitative phenomenographic or phenomenological approach is applied in this study to discover various ways in which the participants understand, conceptualise and realise the phenomenal aspects of teaching and learning around their world. Ultimately, the quasi-experimental approach and the phenomenological approach are triangulated to constitute the concurrent embedded mixed-method approach.

1.8.1 The research questions

The questions clarified in this study are: (1) To what extent does IBF assist students reach higher levels in geometry thinking? (2) Do inquiry-based interventions influence students' development in geometric thinking through Van Hiele's levels? (3) Does IBF help students to learn Euclidean Geometry?

Inferential statistical analysis is employed to clarify the question: "To what extent does IBF assist students reach higher levels in geometry thinking?" Descriptive statistical analysis is employed to elucidate the research question "Does inquiry-based intervention influence students' development in geometric thinking through Van Hiele's levels?" Differences in percentages as shown through the descriptive statistics demonstrate differences in achievement between the experimental and the comparison group. Therefore, evidence is accumulated regarding the influence of intervention on students' development in the experimental group or lack of development in the comparison group. Qualitative data are combined with the quantitative data to clarify both the questions. However, to respond to the question "Does IBF help students to learn Euclidean Geometry?" content analysis of observations and interviews applies.

Triangulation of qualitative data with quantitative data provides credibility to increase the integrity of the results. Qualitative data was collected through participatory observation during interventions in an experimental group and non-participatory observation of teachers teaching in a comparison group. Observation in the comparison group was necessary to ensure that the topic was taught before the students could take the post-test. Pre- and post-tests were administered to both experimental and comparison groups. Six conveniently-sampled schools from two districts (Tshwane North and West) participated in this study. Three schools (97 Grade 11 students) in Tshwane North served as the experimental group while three schools (69) students in Tshwane West were classified as the comparison group. The two districts are approximately 30 km apart. The researcher used IBF strategies to teach Euclidean Geometry to the experimental groups, while teachers of the comparison groups continued to use the regular traditional axiomatic approach to teach. The researcher played a participant observer role in the experimental groups, while the role was that of a non-participant observer in the comparison groups.

1.9 ELUCIDATION OF CONCEPTS

- Concept – mental representation of a category of some kind (things, actions, situations etc.). Concepts allow people to sort stimuli with similar characteristics into categories. Conceptualisation refers to the action or process of thinking in order to formulate a concept or idea of something. (Hamilton & Ghatala, 1994).
- Conjecturing – forming an opinion or supposition about (something) on the basis of incomplete information. A conjecture refers to a claim that an idea is true when there is no evidence (Haylock, 2001). In addition, Haylock (2001) motivate that a conjecture is followed by a mathematical process of confirming through checking. Confirming conjectures contribute as essential basis to mathematical reasoning.
- Differentiated teaching and learning means personalised instruction or inclusive teaching and learning. In differentiated teaching and learning, teacher applies different methods of facilitation to cater for all students' learning needs. Differentiated instruction is driven by the notion that students may learn in diverse ways, however, the content learned and skills developed remain the same, that is, students apply diverse methods of interest and expertise to reach a common goal (Tomlinson, 1999).

- Errors refer to misapplication of the rules in problem-solving (Luneta & Morapeli, 2017).
- Euclidean Geometry is the study of the relationships among points, lines, angles, surfaces and solids (Salim & Tiawa, 2015). In addition, Baroody and Coslick (1998) outline Euclidean Geometry as the study of stationery figures that have a rigid or fixed shape; Euclidean Geometry includes solid three dimensional and plane or two dimensional geometries.
- A figure is a conceptualised geometric object including its graphical representation. Figures are tools for solving problems and for constructing a geometrical theory (Hershkowitz, et al., 1996).
- Geometric modelling means to apply geometric concepts in modelling situations. In additions geometric modelling denotes the use of geometric shape, their measures and their properties to describe objects (CCSSI, 2017)
- Inquiry-based facilitation is the instruction conducted to teach students how to think, how to be attentive and reflective in thinking processes. The instruction follows an inquiry process, that is, explain inquiry process, present the inquiry problem, help students to formulate the hypothesis (conjecture), encourage students to collect data to test the hypothesis (prove the conjecture) and lastly help students to formulate explanations from proven conjectures (Arends, 2012).
- Inquiry-based learning (IBL) designates a learning approach where students have ownership of their learning. Students ask questions, collect and examine information, produce solutions, make decisions, rationalise conclusions and take actions (Friesen, 2014). In addition, IBL empowers students as self-directed and active knowledge creators where inquiry is driven by questions or problems (Spronken-Smith, 2012; Spronken-Smith & Walker, 2010).
- Inquiry describes an activity in which students explore situations and try to solve problems (Rusbult, 2007).
- Inquiry-Based Mathematics Education (IBME) means promoting mathematical learning with understanding through inquiry, to help students experience authentic mathematical activity (Artigue, 2012).
- Mathematics refers to a process of using symbolic notations for describing connections between numerals, geometrical concepts and graphical representations (DoE, 2011). Mathematicians seek out patterns, formulate new

conjectures and establish truth through studying quantity, structure, space and change.

- Perception – the ability to see, hear, or become aware of something through the senses; the way in which something is understood or interpreted (Harre, 2002).
- Problem-solving – the process of finding solutions to a problem. A problem bears a characteristic that it lacks an obvious way to find a solution or a solution is not immediately obvious. Problem solving process according to Polya (1887-1985) comprise four steps that are understanding the problem, devising a plan, carrying out the plan and looking back (Baroody & Coslick, 1998; Sonnabend, 2010).
- Self- regulated learning points to a classroom context wherein students take initiatives with the help of the teacher and other students to diagnose own learning needs, formulate goals, evaluate necessary resources and evaluate learning outcomes (Bergamin, Werlen, Siegenthaler & Ziska, 2012). Further, Bergamin et al. (2012) assert that self-regulated learning strategies involve cognition, metacognition, and resource based learning.
- Space and shape means real objects that occupy the real world. Shapes are visual representations of mathematical phenomena, processes and concepts (Hershkowitz, Parzysz and Van Dormolen,1996).
- Spatial reasoning and understanding concerns interpreting and understanding the inherently geometric world (Baroody & Coslick, 1998). Spatial reasoning is essential in real-world situations and relates to thinking about objects in three dimensions and to draw conclusions about those objects from limited information.
- Spatial skills are spatial abilities. Spatial abilities are defined as how students view space and shape, how they conceptualise and interpret space and shape in terms of two or three dimensional views. Spatial abilities also include how students react to visual stimulus material and the capability to manipulate visual patterns (Carroll, 1993).
- Traditional axiomatic approach of teaching involves teaching that revolves around presenting a set of agreed facts as a set of axioms and geometry is deductively build on these facts (Bankov, 2013).
- Visualisation or imagination implies knowledge representations formulated around physical objects and events (Hamilton & Ghatala, 1994); and visualisation is

grounded on the construction of symbolic representations of concepts and establishment of relations.

1.10 SCOPE AND ASSUMPTIONS

1.10.1 Scope

The scope of the study includes high schools in Tshwane North and West Districts of Gauteng Province in South Africa. The high schools were conveniently sampled according to their comparability based on the background, biographical and geographical factors. Six schools participated; i.e. three schools from each location. Grade 11 mathematics teachers and students were the main participants. Participation was limited to inquiry teaching and learning in Euclidean Geometry. The researcher interacted with teachers and students through the quasi-experimental design method, observations and interviews. Tshwane North teachers and their classes were located in the experimental group whereas the teachers in the Tshwane West were positioned in the comparison group; and both groups participated in pre- and post-tests. The researcher presented the lessons at the three experimental group schools, and teachers at comparison group schools presented lessons as normal while the researcher observed without participating.

Teaching and learning activities were evaluated according to Van Hiele's levels of understanding and John Dewey's theory of inquiry in this study. Of utmost importance is the use of IBF to achieve IBL in Euclidean Geometry. The teaching environment and resources that encourage IBL played a pivotal role in the study. Teaching Euclidean Geometry through inquiry fosters learning that moves from the known to critical self-discovery of the unknown in order to achieve higher levels of geometric thinking.

1.10.2 Assumptions

Assumptions are made in the study that all students who participated are full-time Grade 11 mathematics students. The students studied mathematics from Grade 1 through to Grade11 level. Further, it is assumed in the study that the methods of teaching and learning Euclidean Geometry need improvement in order to assist students to achieve higher levels of understanding in Euclidean Geometry.

1.11 OUTLINE OF CHAPTERS

- Chapter 1 – Study overview
- Chapter 2 – Review of literature and theoretical framework
- Chapter 3 – Research design (that is, methodology, method or approach, population and sample, data collecting tools and process, data analysis and interpretation, validity and reliability, and ethics issues).
- Chapter 4 – Presentation of results
- Chapter 5 – Interpretation of data, discussion of results, conclusion and recommendations

1.12 CHAPTER SUMMARY

This chapter provided the introduction and background to the thesis. Approaches to teaching Euclidean Geometry were reviewed with a focus on the difference between guided inquiry methods such as IBF and the traditional axiomatic approach. The problem statement, objectives, hypothesis and the research questions were set out and the adoption of a mixed-methods strategy using both qualitative and quantitative approaches was explained. The sample of participants was explained and the formation of a comparison group and experimental group was outlined. The key concepts used in the study were defined to provide the necessary content for the study. The next chapter provides a detailed literature review and theoretical framework for the study.

CHAPTER 2
LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.1 OUTLINE

The literature review in this study focuses on cognitive processing in Euclidean Geometry. Theoretical framework is centred on IBF and IBL in Euclidean Geometry. In this study, cognitive process or cognitive processing concern how student learn while cognition refers to understanding. Inferences are made from literature to support the hypothesis. Therefore, literature is reviewed from books, articles, journals, newspapers, the internet and theses based on (1) the methods of teaching; (2) how students learn Euclidean Geometry in South Africa; and (3) the contribution of both the traditional axiomatic approach and the IBL teaching approach to learning. Knowledge on how students learn is regarded as important in this study considering that when teachers know how learning takes place they will be able to practise relevant teaching methods. This study is not subjected to a comparison between the traditional axiomatic approach and IBL, but to support the alternative hypothesis that IBF teaching approach influences IBL and consequently increases students' knowledge and understanding of formal deduction level (that is, Van Hiele's level 4) through pre-visualisation, visualisation, analysis and informal deduction (that is, Van Hiele's levels 0-3).

2.2 COGNITIVE PROCESSING IN EUCLIDEAN GEOMETRY

Geometry comes from ancient Greek knowledge where it is described as "earth measure" where it was used to measure land in ancient times. (Hansen, 1993; Jones, 2002). Jones (2002) agrees that various ancient beliefs established a form of geometry appropriate to the associations among lengths, areas and volumes of physical objects. Further, Jones (2002) states that there are different types of geometry such as plane geometry, elliptic geometry, analytical geometry, coordinate geometry, Euclidean Geometry and others. Curriculum designers at schools choose a few types to be studied at school level, for example, transformation, analytical geometry and Euclidean Geometry.

The teaching and learning of Euclidean Geometry is crucial in this study. Hansen (1993) and Jones (2002) state that the understanding of geometry was accumulated

and ordered in a writing that was named Euclid's Elements. They posit that Euclid's Elements epitomise the axiomatic-deductive method and maintain that several theorems were proved by deductive logic. Jones (2002) states that in the nineteenth century, there were changes in the concepts of geometry. Euclid introduced abstract geometry, but according to British mathematician, Sir Zeeman (Jones, 2002), geometry is composed of mathematics that calls on visual intuition to perceive reality, to remember theorems, understand proofs, stimulate conjecture and give universal or inclusive insight. This study views Euclidean Geometry as an entity comprising both Euclid and Zeeman's views. That is, geometry begins with people perceiving and recognising the immediate world concretely, formulating conjectures about the environment and the world; and utilising the accumulated perceived and recognised knowledge as well as pre-conceived conjecture to seek validity and confirmation through existing axiomatic-deductive knowledge; that is, knowledge contained in axioms and theorems. It is the view of this study that inductive and deductive logic supplement each other in the teaching and learning of Euclidean Geometry.

The objectives of teaching geometry as summarised by Jones (2002) are: to encourage geometrical intuition, spatial awareness, and visualisation; provide geometrical experience with two and three-dimensional objects; develop knowledge of geometrical properties including theorems; develop the capability to apply the usage of properties of geometric figures and theorems in problem-solving; stimulate the progression and application of supposition, logical reasoning and proof; promote awareness of geometry in society and the contemporary applications of geometry; encourage expertise of applying geometric knowledge in modelling and problem-solving for real-world contexts; promote useful integrated technology (IT) skills in geometric contexts; and inculcate a positive attitude towards mathematics. Teaching influences learning, therefore, based on the objectives of teaching geometry, Jones (2002) outlines the objectives of learning geometry as: (1) developing the skills of perception, intuition, visualisation, critical thinking, conjecturing, logical argument, proof and deductive reasoning; (2) making sense of other sections of mathematics, for example, statistical representations of data through graphic illustrations; and (3) applying spatial sense or reasoning in curriculum for other areas of study, for example in technology, art, science, geography and other.

The teaching approach has a great impact on the development of the levels of learning in Euclidean Geometry. The impact is evident based on the fact that teachers teach at the deduction level, that is, level 4, while the students have not yet mastered the induction level. This study asserts that teaching must be based on leading students to learn by inquiry for them to apply inductive inquiry to develop knowledge and understanding at levels 0-3 in order for them to apply deductive inquiry in achieving level 4 of knowledge and understanding. Research has elaborated on mathematical reasoning as either deductive or inductive. For example, Polya in Gurung, Chick and Haynie (2009) state that practical mathematics seem as science that is inductive and experimental, but Euclidean mathematics appear as science that is deductive and systematic. Inductive and deductive reasoning are two forms of basic reasoning for mathematicians: through inductive reasoning students apply new observations to infer rules; thereafter they apply deductive reasoning to test their new observations by referring to existing rules (Prince & Felder, 2006). However, the two types of reasoning appear not to be appropriately integrated in mathematical teaching. This is consistent with the observation that students who take mathematics as a subject are not conscious nor oriented to the work of a mathematician which is to experiment, create and discover models, and use knowledge and inductive reasoning to solve new problems (Gurung, et al., 2009; Kinsey & Moore, 2015).

The common belief among mathematics teachers and students is that in Euclidean Geometry theorems can only be memorised and applied when tested for academic progression or promotions and cannot be applied in real-life problems. Gurung, et.al. (2009) outline the general conceptions of students towards mathematics as follows: (1) mathematics problems have only one solution, and there is only one correct method to find the correct answer; (2) memorisation of facts and processes will allow students to solve problems in a few minutes; (3) mathematics is an individual activity; and (4) the mathematics learned is not related to students' lives and the real world. However, they attribute all these statements to the currently-applied lecture method which is a teaching method that focuses on facts, concepts and procedures as knowledge to be memorised.

Research has shown that the teaching of Euclidean Geometry in the South African context is focused on the lecture method or the traditional axiomatic approach (Malati, 2014; Siyepu & Mtonjeni, 2014; Van Putten, Howie & Stols, 2010). Further, it has been

indicated through research that the traditional axiomatic approach yields learning by rote and memorisation. According to Luneta (2015), most teachers do not elucidate on concepts applying methods that empower students to acquire conceptual knowledge through thorough understanding of geometric concepts. Students commit more procedural errors in problem-solving, are not able to determine geometric relationships and, from the students answers in a geometry test, it was concluded that procedural errors committed by students outweighed conceptual errors. Further, Luneta (2015) attributes the errors and difficulties experienced by students to the conventional way of teaching. In light of his views, this study asserts that learning through memorisation does not afford the students a chance to acquire the skill of geometric problem-solving. With the traditional axiomatic approach, students are unable to acquire knowledge and understanding of the higher levels of geometric thinking as outlined in Van Hiele's theory of geometric understanding. Research has proved that countries that apply open systems of teaching and learning in Euclidean Geometry perform better in geometric problem-solving (Kunimune, et al., 1996; Wubbels, et al.,1997). In South Africa, the principle of open systems has been recommended, but the practice is still lacking.

Further, research has proved that South African students at Grade 11 have poor foundational knowledge in Euclidean Geometry (Alex & Mammen, 2014). This makes it difficult for students to acquire knowledge at higher levels of learning. Regardless of the fact that the students have not acquired foundational knowledge, teachers teach at higher levels without considering building knowledge and understanding that students may not have developed the requisite conceptual skills at lower levels. This study focuses on implementing IBF strategies that can improve learning by inquiry and, as a result, increase the students' knowledge and understanding of Euclidean Geometry at formal deduction level through visualisation, analysis and informal deduction levels of Van Hiele's theory. Therefore, this study hypothesises that IBF strategies are effective and can contribute positively towards IBL in developing inductive and deductive reasoning; namely, knowledge and understanding of formal deduction through pre-visualisation, visualisation, analysis and informal deduction. Consequently, students develop critical thinking skills to obviate learning Euclidean Geometry through memorisation.

To address the issue of how students learn Euclidean Geometry, Van Hiele's theory of geometric knowledge and understanding is applied. In addition, the following aspects of cognition are also examined: cognitive processing philosophy, cognitive science, cognitive load and the influence of using technology or multimedia in teaching Euclidean Geometry. The link between Van Hiele's theory of understanding in Euclidean Geometry and other aspects of cognition is outlined in this study. Van Hiele's theory also applies as an assessment theory in the alignment of the pre- and post-tests. Further, Van Hiele's theory also outlines the phases of learning in relation to instruction. These phases are also linked to methods of instruction that influence the development of geometric knowledge and understanding.

Literature in support of this study's hypothesis focuses on looking at IBL as a way of supporting students to achieve both inductive and deductive learning. As the directional hypothesis posits, much emphasis in this study is put on the process of IBL (the IBL framework), the essence of IBL, teacher and students' role in IBL and the challenges of IBL in Euclidean Geometry.

2.2.1 Methods of teaching to advance cognitive processing

The argument that Euclidean Geometry needs a deductive approach to mathematics instruction is supported by research. For example, Dell'Olio and Donk (2007) point out that traditional methods of teaching have been shown not to be effective in teaching mathematics. Knowledge is the product of inquiry which is a problem-solving process of moving from doubt to belief. Prawat (1999) highlights that, according to Dewey, truth is the relation between perception and the external object. Moving from doubt to belief is consistent with the idea that inductive reasoning (through perception) and deductive reasoning (through confirmation) supplement each other during teaching and learning. Inductive reasoning is basic to proofs. It is through inductive reasoning that hypotheses are formulated and tested. As it relies on inductive reasoning, deductive reasoning can be employed to arrive at correct conclusions. Looking at Van Hiele's theory of geometric understanding, the basic levels (that is, visualisation, analysis and informal deduction) are more concrete and inductive reasoning is needed. The knowledge accumulated in lower levels through inductive reasoning is helpful in higher levels (that is, formal deduction and rigour) which are more abstract and require deductive reasoning. The traditional method of teaching starts from the formal

deduction level, that is from an abstract level that relies more on deductive reasoning. Teachers and students start with belief; that is, they start problem-solving by attempting to make valid, reliable and correct conclusions without inquiring, that is, without moving from doubt. Moving from doubt to belief or learning by inquiry is done through manipulating reality or experimentation. The teaching of Euclidean Geometry needs first to encourage students' experimentation, exploration and discovery into space and shape, and then to encourage verification of results through experimentation, exploration and discovery by applying known facts.

The study emphasises that teaching should start with facilitating inquiry and end with confirming inquiry through facts. Based on the fact that the predominant method of teaching in South Africa is the traditional teaching method, comparisons are made between traditional axiomatic teaching and IBF. However, this study maintains a pragmatic point of view that traditional teaching where a teacher is present as the facilitator can encourage learning by inquiry, and teacher as a more knowledgeable agent is there to confirm knowledge constructed by students though inquiry. This means the teacher must use both inductive and deductive teaching techniques in order to ensure inductive and deductive learning in Euclidean geometry.

Despite the development of other approaches to teaching, the traditional axiomatic approach is predominant in most teaching. It is essential that it is linked to IBL in order to allow students also to be knowledge producers through the guidance of an expert who is a teacher. IBL as a means and process of capacitating students through creating doubt, aims to enable students to learn to retain concepts, exhibit deeper understanding of concepts, show superior abilities in higher-order thinking skills and a higher level of creativity through experiences in inquiry (Dell'Olio & Donk, 2007). However, to become capacitated and enabled to confirm their doubts through belief, Dell'Olio and Donk (2007) pointed out that direct instruction through teacher facilitation is needed as this will support students to focus their attention on the past, that is, on what has been already discovered by others to confirm their doubts. Further, Dell'Olio and Donk (2007) suggest that inquiry experiences will provide students with the tools to move into the future as producers of knowledge. Qualities produced by IBL are necessary to enable the Grade 11 students to learn Euclidean Geometry better; therefore, it is essential that teachers facilitate IBL in teaching Euclidean Geometry in Grade 11.

2.2.2 Geometric understanding of South African students

Euclidean Geometry at school level is a challenge, mostly at higher grades such as grade 11 and 12. The challenge results from the students' lack of experience as well as geometric knowledge and understanding from lower grades. Several studies within the South African context in grades lower than Grade 11 have proved that students are progressed from lower grades lacking experience and knowledge that contribute to the students' failure to cope with higher thinking in Grade 11. For example, Feza and Webb (2005) conducted a study investigating the Grade 7s competency in connection with the assessment criteria outlined by the curriculum in South Africa and the Van Hiele thinking levels. In their study, Feza and Webb (2005) deduced that most of the participants were not English native speakers, therefore, language proficiency remained a barrier that obstructed students from achieving the set requirements. Further, Alex and Mammen (2014) conducted a study with a sample of Grade 10 students in South Africa. The study conducted by Alex and Mammen (2014) focused on learning Euclidean Geometry based on the Van Hiele model of thinking. The results of the study revealed that regardless of CAPS expectation that Grade 10 students should perform at level 3, most students' level of knowledge was 0.

Challenges encountered in lower grades pose a problem in abstract, higher thinking levels; for example, Cassim (2006) also conducted a study focusing on the strategies that Grade 12 students employed to solve cyclic quadrilaterals and tangent problems. Cassim's study applied the Van Hiele levels of geometric knowledge and understanding as a method for classifying students' levels of understanding. Data gathered resulted in the following patterns: students' inappropriate usage of theorems in finding solutions to geometry problems; and students' responses that were based on visual form of the figures. Cassim's study concluded that the manner in which educators approach the solving of geometrical problems determined the strategies that the students applied to solve geometry problems.

Research confirms that it is in the lower grades where instruction leads the students to learn by memorisation, but they cannot recall the knowledge at higher levels. Despite lack of students' cognitive development in Euclidean Geometry, teachers teach at a higher cognitive level in Grade 11. The Grade 11 students cannot cope with the Euclidean Geometry at a very abstract formal deduction level as they cannot evoke

the basic knowledge from lower levels. To resolve this problem, learning by inquiry is necessary. Spronken-Smith and Walker (2010) highlighted that IBL is an example of experiential learning. Therefore, students should be instructed through IBF in order for them to apply IBL to draw from their experience and call to mind the information lost at lower grades. Through IBL, teachers allow students to learn by discovery and exploration, students can discover knowledge they could not acquire in lower grades and overcome the misconceptions about mathematics. If the students can overcome the misconceptions, then they will actively become engaged in searching and producing knowledge in order to develop their cognitive abilities. Spronken-Smith (2012) motivated that IBL enhances students' involvement in educational, accomplishment and higher level learning outcomes. Therefore, reconstructed cognitive abilities can contribute to higher-order critical thinking that will ease the problem-solving challenges at the abstract formal deduction level. IBL thus benefits the cognitive development of students.

This study avers that inductive inquiry coupled with deductive inquiry yield learning which produces critical thinkers in mathematics and in turn knowledgeable, skilled and competent mathematicians (Prince and Felder, 2006). Therefore, literature in this study provides information to answer the following question: 'Does IBF influence IBL to develop the students' knowledge and understanding of formal deduction level through pre-visualisation, visualisation, analysis and formal deduction in Euclidean Geometry?'. Further, answering the question posed, provide enough evidence to support the alternative hypothesis of this study

2.2.3 How do students learn?

In testing the hypothesis, the literature review focused on the influence of inquiry-based facilitation as a means to influence cognitive processes in IBL, that is, how instructional or teaching approaches influence the students' cognitive processes. According to Rubie-Davis (2011), there is a connection between human cognitive architecture and instructional design. Cognitive architecture refers to cognitive arrangements or structure in relation to the operation of the limited working memory. Essentially, teachers should know and understand how students learn (the process of learning, cognitive process or cognitive processing) in Euclidean Geometry. This idea directs the study to the integration of the cognitive processing philosophy (Table 2.1)

which emphasises the development of students' thinking abilities or how students learn (Dell'Ollio & Donk, 2007). According to Dell'Ollio and Donk (2007), cognitive processing philosophy entails factors that are outlined in Table 2.1.

Table 2.1: Cognitive processing philosophy

- Students' time and energies should be employed in developing intellectual skills such as analysis, inference, induction and evaluation, rather than focusing on developing academic content only.
- Learning should focus on students' development of cognitive skills and the application of the scientific method.
- Deductive teaching disrupts authentic students' learning
- Students should work as scientists, that is, through inductive teaching models.
- Inductive teaching models directs instruction and learning as follows: rather than students being given a scientific principle to observe through teacher direct demonstration and experimentation, students are presented with an engaging meaningful question; and through the inquiry process, students analyse the problem, generate the hypotheses, choose from among those hypotheses, and design an experiment to test the hypotheses.
- Students can induce principles of the subject through experience rather than absorb them passively in teacher directed lessons. These experiences strengthen students' intellectual abilities as they do the work of mathematicians
- Once the intellectual skills have been introduced in the early school grades, they should continue to be developed through learning experiences in all mathematical sections in higher grades.
- Critical thinking skills can also be viewed as instructional objectives.
- Traditional content organisation and teaching methods do not involve and promote cognitive abilities with deliberation.
- Teacher-centred methods do not promote students' thinking beyond basic levels of comprehension.
- Learning experiences can be designed to move students to higher levels of critical thinking
- Students possess an individual repertoire of cognitive skills when they enter the classroom
- Students' can produce knowledge at any age, they are not considered as empty to the extent that they just receive knowledge and replicate it.
- Traditionalist education focuses on acquisition and re-creation of knowledge, while cognitive processing philosophy outlines that through substantial experience as problem solvers, students will be able to create knowledge in future.

Adapted from: (Dell'Olio & Donk, 2007)

According to Dell'Olio and Donk (2007), cognitive processing philosophy gives both structure and directions to constructivist theory as it applies in the classroom. However, teaching and learning traits similar to those of the cognitive processing philosophy apply even in pragmatism as a philosophy of education. In relation to the teaching and learning traits of the cognitive processing philosophy, Ozmon (2012) emphasises that the pragmatist teaching and learning philosophy focuses on teaching that encourages students' active participation, rather than students' passive engagement of receiving knowledge imparted by the teacher.

This study addresses cognitive processing in Euclidean Geometry as the pragmatic process whereby self-directed students with minimal teacher facilitation, acquire knowledge and understanding through analysis, inferences, induction and evaluation; thereafter students construct and make sense of Euclidean Geometry knowledge through diverse cognitive levels (Van Hiele's levels of knowledge and understanding) in order to solve problems.

Merging teachers' consideration of students' cognitive architecture or cognitive processing abilities with instructional design is consistent with the idea in this study that instruction should motivate and equip students to utilise their intellectual abilities in self-directed learning. Self-directed learning assists students to learn by inquiry (that is, to develop authentic learning as well as both inductive and deductive inquiry) and the process of acquiring knowledge and understanding by inquiry is known as IBL.

2.2.4 IBL

IBL refers to the method of learning that involves students in activities that make sense to them, rather than students being provided with an easy way to the solution (Love, Hodge, Corritore & Ernst, 2015). IBL is a student-centred approach where the teacher only facilitates and directs the learning process. Therefore, IBL is a component of the learning paradigm rather than the teaching paradigm (Barret, 2005). In light of Barret's idea, the teaching approach that influences learning in this study is referred to as inquiry-based facilitation (IBF) or or IBL facilitation. Love et al. (2015) argue that in the process of IBF, the teacher directs learning employing well designed problems through adventurous activities, activities that supports discovery of concept and application of acquired knowledge in problem solving. Furthermore, Love et al. (2015) aver that in the IBF process teacher give students activities that engage them in problem solving

through communicating, experimenting, exploring, making suppositions and applying. Rubie-Davis (2011) emphasises that it is of essence that teachers know the students' cognitive processes and techniques as well as the students' cognitive architecture in order to select the correct methods for manipulating instructional processes and procedures as well as their interactions with students' cognitive structures and processes. In addition, teachers should know the contributors to cognitive load, that is, factors that hinder cognitive processing, in order to apply IBF to stimulate IBL that will ease the cognitive load.

Inquiry-based facilitation promotes IBL that empowers students to attain better cognitive processes in Euclidean geometry. Therefore, literature in this study is focused on cognitive processes based on cognitive science, pragmatism as a philosophy of education, Van Hiele's cognitive levels and cognitive processing and learning experiences according to Dale's theory. Further, literature in this study addresses cognitive load in Euclidean Geometry; traditional axiomatic approach as a supplement to IBL in Euclidean Geometry; challenges associated with the implementation of IBF to promote IBL in Euclidean Geometry; and IBL assessment and performance indicators in Euclidean Geometry. The main unit of analysis in this study is the teaching strategy; however, learning strategies are discussed at length in order to weigh the effect of the teaching strategies as applied in both the experimental and comparison group. Further, learning strategies are discussed as evidence of achievement in the experimental group, and non-achievement in the experimental group in order to single out IBF as an essential teaching strategy.

2.2.5 Cognitive processing according to cognitive science

Cognitive science is described as the "science of mind", where "mental operations" are regarded as "information-processing operations" (Bermúdez, 2010:3 & 457). Cognitive processing in geometry is mainly focused on "an account of how information is processed in individual cognitive systems" (Bermúdez, 2010, p. xix). Clements and Battista (1992) argue that cognitive science is another perspective applied to understand students' learning of geometry. Further, Clements and Battista (1992) discuss Anderson's and Greeno's models of cognitive science as follows:

2.2.5.1 Anderson's model

Anderson's model of cognitive science (called the Adaptive Control of Thought – Rational [ACT-R] model) depicts that learning geometry involves: acquisition of declarative knowledge; application of declarative knowledge to new situations by means of search and analogy; compilation of domain-specific productions and strengthening declarative and procedural knowledge.

Andersons' model of cognition declares that declarative knowledge means 'knowing that', for example, conjectures and theorems would be stored in schemas along with knowledge about the function of theorems; and procedural knowledge means 'knowing how' and is stored in the form of production systems or sets of condition-action pairs. According to Anderson (2000), declarative knowledge is knowledge about facts and things, that is, unequivocal understanding that individuals are knowingly aware of and can account on. Further, Anderson (2000) argued that procedural knowledge is inherent knowledge of how to do things, also knowledge about how to perform cognitive activities.

2.2.5.2 Greeno's model

Greeno's model of cognitive science emphasises procedural knowledge as production of systems. The model explains that for students to solve geometry problems, three domains of geometry are required: First, propositions are made in making inferences that establish the main steps in geometry problem-solving. For example, making familiar statements about geometric relations such as corresponding angles formed by parallel lines and a transversal are equal. Second, perceptual concepts are used to recognise patterns mentioned in the backgrounds of many propositions (for example, the corresponding angles). Thirdly, strategic principles are used in setting goals and planning; for example, when solutions require showing that two triangles are congruent, one approach is to use relations such as corresponding angles, proving that angles contained in the two triangles are equal, and that the triangles are therefore congruent.

In light of the two models it is clear that there is a link between declarative and procedural knowledge. The link is based on the fact that acquired declarative knowledge is essential in order to advance procedural knowledge.

2.2.5.3 Declarative knowledge and procedural knowledge

Clements and Battista (1992) argue that procedural knowledge occurs only in executing a skill, that is, learning by doing; and step-by-step interpretation is straightforward when declarative information is in the form of direct instruction. They further claim that in high school geometry, students may use declarative information to provide the data required by general problem-solving operations such as means-ends analysis, inferential reasoning and making analogies between worked examples and new problems. In light of these ideas, it appears that procedural knowledge depends on the compilation of previously-learned declarative knowledge. However, the authors indicate that it is a huge challenge when students have to acquire mathematical ideas only procedurally while they have not yet acquired declarative knowledge (this shows that deductive reasoning and inductive reasoning are supplementary). In this state, students cannot connect procedural knowledge to declarative knowledge, that is, students perform sequences of mathematical processes without being able to describe what they are doing or why.

Essentially the study addresses a gap in research on why students fail to apply declarative knowledge acquired in pre-visualisation, visualisation, analysis and informal deduction at advanced formal deduction through procedural knowledge. The main factor that places students in such a predicament is that teachers emphasise procedural knowledge at formal deduction level without leading students from the lost declarative knowledge. Teachers make students memorise, facts, names and rules to apply in proving theorems instead of guiding students to apply self-acquired schemata in reasonably proving and applying theorems. The study aims to close the gap by introducing IBF that will assist teachers to guide students to apply IBL id order to accumulate declarative knowledge in pre-visualisation, visualisation, analysis and informal deduction; and to apply the declarative knowledge in procedurally solving problems at the formal deduction and rigour levels.

2.2.6 Cognitive processing in Euclidean Geometry based on the Van Hiele cognitive levels

Van Hiele’s levels of geometric knowledge and understanding emphasise that students should understand and organise into schemata the system of associations that connects geometric concepts as well as processes; but not only memorise

formulae, rules, existing facts and names (Clements & Battista, 1992). A schema (plural schemata or schemas) is referred to as a cognitive structure that permits problem-solvers to distinguish a problem as fitting in a specific classification of problems that require defined methods (Sweller, 1988; 1994). In addition, Sweller (1994) proposes that the combination of discrete features into a single feature can efficiently increase the quantity of information that can be stored in the employed memory. Schemas organised by learning the network of relationships (that is, geometric concepts and processes) through Van Hiele levels contribute to the improvement of students' knowledge and understanding, and this is supported by De Villiers' (2004) idea that the exceptional characteristic of Van Hiele's theory is the dissimilarity of five detached levels of knowledge based on the students' growth of geometry understanding. As students build knowledge schemas in lower levels, the chances are high that students will competently achieve in higher levels. The five discrete levels of the Van Hiele model are outlined in Table 2.2.

2.2.7 Levels of knowledge and understanding (Van Hiele Model)

In the original Van Hiele model of geometric knowledge and understanding, the levels are numbered from 0 to 4. Mason (1998) states that the numbering in the model was altered by other researchers who numbered the levels 1 to 5. This was done in order to allow the additional level discovered, that is, the pre-visualisation stage which was numbered level 0 (Clements & Battista, 1992; Mason, 1998). First, the original Van Hiele model is shown in Table 2.2 and thereafter the additional opinions to the model by different authors are discussed.

Table 2.2: The Van Hiele levels of geometric knowledge and understanding

Level	Description
Visualisation(Recognition)	• Students distinguish figures by how they are formed • At this stage, students do not notice the properties figures. • Students base decision making on perception but not on cognition
Analysis	• Students identify and name characteristics of geometric figures • They designate an object according to properties they know, but cannot determine which properties

	are essential and which are adequate to define an object
Informal deduction(abstraction)	• Students observe connections among properties and among figures • Students can validate their thinking through constructing meaningful definitions • Group enclosures and rational effects are understood
Formal deduction	• Students can create verifications • They understand the meaning of adequate and essential characteristics and circumstances • Students comprehend the part of delineations axioms
Rigour	• Students understand how to compare mathematical networks as the formal feature of deduction • Students comprehend equally Euclidean and non-Euclidean systems

Adapted from: (Clements & Battista, 1992; De Villiers, 2004; Mason, 2009; Van Hiele, 1986)

Subsequently, Van Hiele developed a model where the original model was characterised into three levels (Clements & Battista, 1992; Teppo, 1991). Clements and Battista (1992) indicate that in Van Hiele's latest model the three levels of geometric knowledge and understanding are characterised as: (1) Visual (level 1 in the original model); (2) analytic (level 2 in the original model); and theoretical (level 3-5 in the original model). Teppo (1991) further summarises the visual, descriptive, and theoretical levels of thinking shown in Table 2.3. According to Teppo (1991), the Van Hiele model of geometric knowledge and understanding emphasises the importance of the teaching and learning act. He further argues that students progress sequentially across levels of knowledge as the result of decisive teaching planned according to five phases of sequenced activities that emphasise investigation, and amalgamation conversation.

Table 2.3: Three levels of Van Hiele's model

Level	Phases of learning	Description
Theoretical (Level 3)	None	Use deductive reasoning to prove geometric relationships

Descriptive ↑ **(Level 2)**	Integration Free orientation Explication Bound orientation Information	Awareness of objects by their geometric characteristics
Visual ↑ **(Level 1)**	Integration Free orientation Explication Bound orientation Information	Distinguish geometric objects universally

Source: (Teppo, 1991)

The three-level model developed by Van Hiele has been criticised by Clements and Battista (1992) as not being sufficiently refined to characterise thinking, which is consistent with their findings that students may progress to level 3 without having acquired axiomatic thinking. In addition, Clements and Battista (1992) argue that the combination of the levels 3-5 as theoretical may contribute to teaching and learning deficiencies at higher levels. However, in contrast to the view of Clements and Battista (1992), Teppo (1991) states that the three-level model is effective because: (1) in order to succeed in each level of knowledge, students advance diverse learning phases, for example, learning pertinent concepts in each level, essential language appropriate in each, progress from lower level to higher level of thinking; (2) According to Van Hiele's theory, each learning phase is apportioned to instruction to ensure improvement in student's understanding of geometry; (3) A teaching and learning programme is necessary in order to efficiently stimulate a shift from one level to the subsequent. According to Teppo (1991), the model characterises thinking as much as the model with five phases does.

This study postulates that Van Hiele's theory in general, that is, the theory of five levels and that of three levels, address the same issue of knowledge and understanding in Euclidean Geometry. The notion that there should be a pre-visualisation level is also viewed as significant in this study. Therefore, this study focuses on six levels of knowledge and understanding which are pre-visualisation, visualisation, analysis, informal deduction, formal deduction and rigour. This study emphasises that for students to advance the formal deduction level (which may be addressed as theoretical level), they should develop cognitive skills at the lower levels; and cognitive

development in this study is viewed as viable when each and every level is thoroughly addressed during teaching and learning. Lower levels such as pre-visualisation, visualisation, analysis and informal deduction need to be unfolded in Grade 11 in order to improve the students' perception skills (Harre, 2002) In support of developing the basic cognitive levels to enhance students' perceptual powers or perception skills, Harre (2002) outlines three distinct regions, shown in Table 2.4.

Table 2.4: Regions of perceptual power

Regions within the perceptual power of human beings	Contribution in the available world
What we can perceive	The world as we perceive it with our normal sense organs
What we can visualise	The world as we do experience it with sense extending instruments
What we can imagine	The world as we imagine it to be beyond the reach of all our powers of perception, aided or unaided by instruments

Adapted from: (Harre, 2002)

This study actualises the view by Harre (2002) and postulates that when the students interact with and experience the world or environment around them through perceiving, visualising, and imagining, levels such as pre-visualisation, Visualisation and analysis can be achieved by the students. Region one (that is, what we can perceive) supports the pre-visualisation stage identified by Clements and Battista (1992). This study further assumes that it is of essence for students to perceive the world around them in order to visualise this particular world. Imagination is supported by what has been perceived and visualised. These three regions of the students' material world are prioritised as items that can enhance what is perceived and what is visualised. This takes us to Dale's (1946) theory that learning begins with action learning, and action learning is possible when the student is exposed to diverse learning materials (Corpuz & Lucido, 2008). Ultimately, assumptions can be made in the study that pre-visualisation stage is very essential before visualisation can take place, and learning and teaching resources are essential in order for students to advance pre-visualisation and visualisation stages.

Based on their claim that that pre-visualisation is a level that is more basic than the Van Hiele's visualisation level, Clements and Battista (1992) numbered the levels in the original Van Hiele model as levels 1 to 5 in order to accommodate the pre-visualisation level. Therefore, according to Clements and Battista there are six

cognitive levels of geometric knowledge and understanding which are: level 0 – pre-visualisation (addressed as pre-visualisation in this study); level 1 – recognition; level 2 – descriptive or analytical; level 3 – abstract or relational; level 4 – formal deduction; and level 5 – rigour. The characteristics of the additional pre-visualisation level are: (1) students perceive geometric shapes, but may attend to only a subset of the shape's visual characteristics; (2) they may not be able to distinguish figures in the same class, even though they may be able to differentiate between figures that are curvilinear (shapely) and those that are rectilinear (uncurving or straight), for example, not being able to make a distinction between a three-sided figure and a four-sided figure (Clements & Battista, 1992). This study assumes that at Grade 11 level students are competent in the pre-visualisation level, that is, they can perceive geometric figures in their world and can, for example, distinguish between three- and a four-sided figure.

Pre-visualisation in this study explains that students can visualise what they have seen and perceived as a distinct object. According to Harre (2002), what an individual can perceive depends on the aspects of the material world that are available to their unaided senses; for example (1) the larger world available due to being adventurous in the exploration of the available environment; (2) the wider world because an individual has equipment; and (3) the richer world because an individual has more elaborate conceptual systems with which to recognise and classify things, properties and relations available to the unaided senses. Strong cognitive development in pre-visualisation can enable student to advance visualisation easily.

In concurrence with Harre's view that the world around the student is essential for visualisation, Hershkowitz et al., (1996) argue that geometry education starts with orientation in real space, that is, the space of which students themselves are part. In addition, the authors argue that experiencing space is about the student as the observer in space describing the relative position of objects in that space. Experiencing space by describing the position of objects in that space generates visualisation which is of essence in Euclidean Geometry. The main goal of visualisation in Euclidean Geometry has been outlined as inductive reasoning by Hershkowitz, et al. (1996); and they further highlight that knowing shapes and visual reasoning is considered as intuitive and underpinning the development of the higher levels of learning inVan Hiele's theory. Visualisation is propelled by the students' spatial skills grounded by pre-visualisation (The National Academy for Sciences,

2018). A report by the National Academy for Sciences (2018) further outlines that visualisation is evident in students' spatial abilities to apply diverse perspectives in constructing mental representations of objects. In order to activate students' pre-visualisation and visualisation skills, Hershkowitz, et al. (1996) aver that problem situations in geometry should be materialised with real objects. In light of these ideas on visualisation, it appears that instructional planning in the pre-visualisation and visualisation levels of knowledge should focus more on exposing students to sufficient and relevant learning resources. Instruction should also probe through questioning to enable students to relate with their world, environment or real space. Pre-visualisation and visualisation levels appear to be more concrete. These particular levels can enable students to build knowledge schemas that they can apply in higher levels which are more theoretical and abstract (that is, levels such as analysis, informal deduction, formal deduction and rigour).

2.2.8 Characteristics and the contribution of Van Hiele's model in teaching and learning

According to Fuys, Geddes and Tischler (1988), the Van Hiele model of geometric knowledge and understanding (that is, visualisation, analysis, informal deduction, formal deduction and rigour), supported by other researchers (De Villiers, 2004; Clements & Battista, 1992; Mason, 2009; Tempo, 1991), makes the following contribution to learning as presented in Table 2.5.

Table 2.5: Characteristics and contribution of Van Hiele's levels

• Levels are hierarchical in a fixed sequence.
• Students cannot skip a level, that is, they must be competent in lower levels in order to advance higher levels.
• Properties that are central in one level turn out to be fundamental in the succeeding level; for example, the properties of the analysis level exist as the visualisation level, however, the student becomes aware of the properties only when reaching the analysis level.
• Each level has its own language signs and system of relations.
• Progression to the succeeding level is dependent on educational background not on maturation or age.

Adapted from Clements & Battista,1992; De Villiers, 2004; Fuys, et al., 1988; Mason, 1998; Van Hiele, 1986).

The authors further observe the following: (1) difficulty in learning geometry (that is, cognitive load) is caused by teaching that occurs at the deduction level when the students have not yet attained knowledge and understanding at the informal deduction level; and (2) lack of understanding and difficulties may be experienced by students when the teacher reasons at a diverse Van Hiele level than that of the students during instruction. Further, Mason (1998), argues that in order to assess the Van Hiele levels, there are tests assigned to levels; for example, observation made by the teacher in order to find out how a student applies geometric language; and ultimately the teacher determines the students' thinking level through the analysis of the student's answers and reactions in the task allotted for a level.

Furthermore, emanating from the contribution of the Van Hiele levels in learning is a notion that learning should be conducted in a fixed sequence of the levels, and should not proceed if the preceding step has not been achieved (Van Hiele, 1986). In light of this, this study suggests that students through inquiry (reflecting on own experiences, predictions, experimentation, exploration, discovery, and drawing conclusions) should explore each level, and understand symbols and network of relationships in that level before progressing to the next level. Therefore, instruction or teaching methods have a great effect on the students' achievement on different Van Hiele levels. In concurrence, Clements and Battista (1992) highlight that the Van Hiele model emphasises that the process of development is subjective to the teaching or learning process. Mason (1998) states the implications of the Van Hiele model for teaching practices as follows: using the traditional teaching method of facilitating as the key methods, influences rote learning and obstructs meaningful learning; teachers should afford students chances to discuss appropriate capabilities; and teachers can assess the students' level of thought at each level and provide facilitation at each specific level to address gaps in their knowledge.

Mason (1998) outline that the agreement between the National Council of Teachers of Mathematics (NCTM) curriculum and evaluation standards and Van Hiele's theory is that students' geometric knowledge and understanding progresses through a hierarchy of levels. In addition, Mason's (1998) analysis of the NCTM standards in relation to Van Hiele's theory outline the following: learning starts by recognising whole shapes, analyse the properties of shapes and ultimately by specifying relationships among shapes and making deductions. Further, Mason's analysis motivates that the

the classroom context should contribute to students' active engagement in developing mathematical concepts through exploration, description, demonstration and discussion; and classroom context should enable students to communicate in order to discuss, share discoveries, formulate conjectures, confirm conjectures and acquire knowledge through reading, writing, speaking and listening. In summary, the pragmatist method of teaching and learning (for example, learning by exploration and discovery inquiry) promotes progression according to the hierarchy of the Van Hiele model.

Looking at the influence of instruction on learning, the Van Hiele model emphasises that teaching should be organised into five phases of learning in order for a student to progress through each level (Clements & Battista, 1992; Fuys et al., 1988; Teppo, 1991; Mason, 2009; Van Hiele, 1983). The phases are described in Table 2.6. The authors indicate that the student could go through the phases more than once in a particular topic.

Table 2.6: Van Hiele's phases of learning

Phase of learning	Description
Information	• Applying the discussion method, the teacher ascertains the students' pre-knowledge about the topic in order to engage students in the new topic. • Material connected to the presented level of study.
Guided orientation	• The teacher involves students and allow them to explore geometric concepts though active participation, for example in activities such as measuring, constructing etc.
Explication	• The teacher introduces essential geometric terms and allow students to use their own words to describe acquired knowledge.
Free Orientation	• Students explore open-ended activities by applying connections they have learnt to solve problems.
Integration	• Students review what they have learned and utilise it in impoving a new system of substances and connections.

Adapted from Clements and Battista (1992); Fuys, et al. (1988); Teppo (1991)

2.2.9 Cognitive processing and learning experiences

According to Bennet (2015), teachers can facilitate learning through inquiry by (1) encouraging students to connect to the subject; (2) identifying stimulating themes; (3)

applying metacognition by taking students into the real world; (4) using prompts to support students to advance analysis in order to extricate knowledge; and (5) encouraging students to inquire through asking thoughtful questions. Essentially, Bennet (2015), highlights that for students to advance inquiry through critical thinking, it depends on stimulating themes and prompts used by the teacher. Further, in light of Bennet's view of IBL, it shows that connection to the real world during learning demands metacognition, that is, students' knowledge about strategies and skills of identifying, selecting and monitoring cognitive strategies (Hamilton & Ghatala, 1994). Hamilton and Ghatala (1994) define cognitive strategies as mental processes for controlling learning and thinking. They state that students' ability to utilise cognitive strategies appropriately allows them to effectively manage their own learning, remembering and thinking. Carr (2010) agrees that the quality of conceptual knowledge is mostly advocated in mathematics; however, metacognition is necessary to determine how well and how quickly students will learn. In addition, Carr (2010) highlights that individuals' ability to evaluate the state of their knowledge supports the emergence of more conceptual knowledge and understanding. Tarricone (2011) states that metacognition is connected to reflection; and reflection and metacognition are dependent on self-knowledge. In addition, Tarricone (2011) also observes that reflection and self-knowledge as components of metacognition are essential for the identification, monitoring and control of problem-solving strategies. Reflection is an essential part of learning, where students develop skills such as awareness, monitoring and regulation of own learning. Metacognition through reflection remains essential for learning Euclidean Geometry.

It is postulated in this study that pre-visualisation and visualisation are important in students' cognitive growth, and the two levels rely on the adequate resources. It is at these two levels that students' metacognition or cognitive growth should be stimulated or prompted in relation to the students' real world. According to Tall (2004), construction of mathematical ideas in order to learn can develop in different ways such as: (1) from acting intuitively on the real-world; (2) increasing complexity of language to sustain representation of numbers and deductive concepts; and (3) increasing abstraction, explanation and meaning that end in formal axiomatic approaches. In addition, Tall (2004) categorises cognitive growth according to the following developments which he says are distinct but interdependent: (1) perception which

consists of thoughtfulness about objects sensed and observed in own intellectual world and in the real world; (2) manipulation of mathematical symbols to advance computations, for example, in algebra which is fundamental to geometry; (3) Formal expression of properties used in axiomatic approaches to postulate mathematical constructions. Perception or pre-visualisation, visualisation and analysis (that is evident through real-world enactive and iconic representations) get activated through actual and visual learning (Figure 1). Activating the pre-visualisation levels, visualisation and analysis can assure schema acquisition and ability to establish networks and relationships in order to link Euclidean Geometry concepts and processes. Formal definitions and deductions can be easily advanced based on acquired schema and cognition; and verbal learning can be applied to advance formal deduction and rigour which relies heavily on developed schemata and cognition.

The study aligns the necessity of supporting students' cognitive growth through diverse learning activities with a theory developed by Dale during the 1960s (Anderson, 2013; Corpuz & Lucido, 2008). The authors highlight that Dale's theory concerns a model that conceptualises three learning experiences, that is, action learning, visual learning and verbal learning. Learning experiences refer to ways of learning that can assist students to acquire knowledge, understanding and problem solving skills (Anderson, 2000; Corpuz & Lucido, 2008). In addition, Dale's theory of learning stipulates that students retain information and knowledge through what they do contrary to what they hear, read or observed (Anderson, 2013). In addition, Anderson (2013) motivate that Dale's theoretical model explains that the least effective method of learning (that is, reading and hearing) involves learning from information presented through verbal symbols, i.e., listening to spoken words (mostly practised in traditional approaches); and the most effective method (action learning) involves direct purposeful learning experiences, such as hands-on experience. In addition, Anderson (2013) highlights that learning according to learning experiences shows that action-learning techniques result in up to 90% retention of the content learned.

In addition to viewing action learning experiences as a means to better retention rate, Corpuz and Lucido (2008) view learning experience as being arranged according to the degree of abstraction of the learning experience. However, according to Molenda (2003), Dale's theory of learning emphasise that the classification of learning experiences according to levels should not be regarded as any type of hierarchy or

rank order. Further, Molenda (2003) agrees that Dale advocated the use of whatever method or medium was appropriate for the student and the task. In light of Dale's model, Molenda (2003) suggests that it is a misconception that there is a preference for concrete over abstract activities. Based on Molenda's argument, this study assumes that for IBL to take place effectively, teaching should move from concrete to abstract and vice versa in various stages of engaging students in conjecture and exploration. Therefore, in this study, integration of learning experience provides a means of assisting teachers in selecting relevant resources for both concrete and abstract activities (Anderson, 2000). Further, considering all learning experiences in teaching contributes to learning that integrates both inductive and deductive learning.

According to Corpuz and Lucido (2008), Dale's theory explains that: (1) maintaining the balance between concrete and abstract learning experiences could help students in their holistic development; and (2) the student should advance holistic improvement in learning by first acquiring basic experience in action learning, proceed to learning through visual learning, and lastly acquire knowledge by leaning through verbal learning; and (3) this means a student needs to be exposed to different media in order to have different learning experiences for holistic development. In addition, Corpuz and Lucido (2008) argue that in facilitating learning one medium is not enough, therefore, diverse medium of teaching and learning should be used in order for students to attain a maximum learning experience.

When resources or multimedia are combined with students' activities, they do not only serve as supporting mechanisms in the teaching of mathematics; they contribute in shaping the students' actions and therefore learning (Trigueross & Lozano, 2007). It is essential that teachers introduce more than one medium to assist students to acquire diverse learning experiences. Therefore, this study emphasises that Learning and Teaching Support Material (LTSM) must shift from verbal learning (text book, reading and hearing) to multiple resource usage in order for students to advance to other learning experiences. Assisting students with diverse learning experiences ensures a connection of sequenced activities from pre-visualisation to rigour. The LTSM that can be used are, for example, visual media, audio-visual material, textual material and digital material.

Linking cognitive processes and the learning experiences, the study suggests that it is of essence to combine learning experiences (action learning, visual learning and verbal learning) during IBL facilitation in order to enable students to (1) develop holistically; (2) acquire better retention rate; and (3) learn from the concrete to the abstract. Holistic development, better retention rate and learning from the concrete to the abstract impact learning positively and the positive impact thereof significantly improves students' ability to recognise and pose problems; and therefore apply creative thinking, critical thinking, spatial skills and acquired knowledge about properties of shapes to solve problems.

It is essential to combine media during the IBL facilitation in order to include all learning experiences. This will enable the teacher to facilitate learning in all levels of learning and understanding such as pre-visualisation, visualisation; analysis, informal deduction, formal deduction and rigour. Therefore, considering the challenge of cognitive growth from lower levels faced by Grade 11 students in Euclidean Geometry, it is of essence to link media combination and selection with the learning experiences, that is, action learning, verbal learning and visual learning focusing on the levels of learning Euclidean Geometry. For example, action learning by using visual aids, audio-visual aids, virtual manipulatives and digital manipulatives can support students in building schemas or mental structures in the pre-visualisation, visualisation, analysis and informal deduction levels. Verbal learning in combination with media like textual materials can support students in acquiring schema in formal deduction and rigour levels.

2.2.10 Cognitive processing and mathematical modelling

Cognitive processing involves observing reality and analysing the reality observed by using words, sketches, drawings and mathematical formulae. Reality can be modelled mathematically. Stockie (2014) states that a cognitive activity in which individuals think about and create models to describe how objects in the real-world function is called mathematical modelling. Furthermore, Stockie (2014) posits that, scientifically, mathematical modelling involves relating phenomena and behaviours observed in the external world to the conceptual world, that is, the world of the mind where understanding of the real-world is established. Three stages, that is, observation, modelling and prediction characterise conceptualisation, according to Stockie (2014).

The author further explains that in the conceptual world of mathematical modelling, observation measures the events in the real world, and further empirical evidence is collected directly by using the senses, or indirectly by taking readings or measurements of an event. The observations are analysed through developing models that describe the behaviour of observed results, explain why the behaviour and results occurred and predict future behaviours. In addition, Stockie (2014) recommends that successful modelling requires intense immersion in mathematics, that is to say, students need to be intensely engaged in mathematical activities as well as acquiring intense knowledge and understanding of mathematical concepts and processes. In order to afford students intense immersion in mathematical knowledge, understanding, concepts and processes, Lingefjard and Holmquist (2001) recommend that teachers must know both mathematics and computing tools in order to orientate students to mathematical modelling. Lingefjard and Holmquist (2001) further state that it is not enough for teachers in the 21st century to adhere to defining terms and implementing algorithms but teachers need an understanding of mathematics that allows them to (1) produce and interpret technology-generated results; (2) develop and evaluate different solution paths, and (3) recognise and understand the mathematical limitations of particular technological tools. However, Lingefjard and Holmquist (2001) emphasise that teachers must be informed about the place and role of technology in the pedagogical process in order for them to optimally apply technology in their practice. The current study suggests that technology or digital resources should be supplemented by other resources such as visual, audio, audio-visual and text. Mathematical modelling complemented by technology in facilitating and learning Euclidean Geometry adds to the strategies that can be used to equip students with geometric knowledge, understanding, concepts and processes in order to ease the cognitive load in Euclidean Geometry.

2.2.11 Cognitive load in Euclidean Geometry

According to Sweller (1994), learning mathematics can differ from being easy to difficult or impossible. Sweller (1994) posits that the features that make learning mathematics difficult concerns cognitive load in mathematical problem-solving. Students' cognitive load is evident mostly when students cannot perform as desired in examinations or in mathematical tasks in and out of the classroom. Committing mathematical errors during problem-solving is also evidence that students experience

difficulties. According to Luneta and Morapeli (2017) committing mathematical error refers to misapplication of mathematical rules. Further, Luneta and Morapeli (2017) argue that mathematical errors occur when (1) concepts are given to students in the form of concept names and definitions; (2) the students formulate their own conceptual images using cognitive structures linked to concepts that the teacher imparted to them; (3) new knowledge cannot be anchored to the existing knowledge or concept image; and (4) concept images are ill-constructed as a result of weak conceptual understanding. Furthermore, Sweller (1994) advances the following reasons for differences in how students acquiring mathematical knowledge: (1) changes in the amount of information, that is, difficulty in some concepts and procedures and (2) more information that students cannot acquire as expected. In addition, Rubie-Davis (2011) argues that the need to integrate more than one source in the mind, for example, integrating text and a diagram, inflicts a load in the working memory, therefore, learning is obstructed. These problems may be exacerbated by students' natural or artificial cognitive constructions. Thus, Dhlamini (2012) indicates that, in essence, teachers should consider the fact that students utilise limited working memory within the process of learning. Further, Dhlamini (2012) avers that instructional design should encourage students' efficient use of the limited working memory.

Efficient use of limited working memory is explained as the ability to increase the limited working memory through processing visual and auditory information simultaneously. Therefore, increased learning possibilities will be evident if teaching considers efficiency in students' utilisation of limited working memory (Dhlamini, 2012). In agreement, Rubie-Davis (2011) argues that increasing effective working memory capacity can ease the cognitive load. In addition, Rubie-Davis (2011) highlights that instructional procedures can alleviate cognitive load by formatting instructional material and cognitive resources in such a way that activities essential to learning are prioritised. This study collates the ideas on supporting cognitive construction by emphasising that instruction should be based on learning experiences such as orientating students to action learning, visual learning and verbal learning. Action learning is associated with active learning in this study. Dale's theory that refers in some sections of this study, suggests that exposure to more than one resource can enable individual students to cope by interacting based on individual style of learning,

e.g. one student can cope through action learning, another through visual or verbal learning, while others can cope by integrating more than one learning experience.

It is not only the students' cognitive architecture that can pose difficulties in learning, as artificial features like curriculum, environment and many other contextual factors can also contribute. According to the cognitive load theory, problem-solving can be enhanced by instructional design (Sweller, 1994). In Euclidean Geometry particularly, Grade 11 students cannot assimilate knowledge readily when they are taught at the formal deduction level while they could not achieve the pre-visualisation, visualisation, analysis and the informal deduction levels (Clements & Battista, 1992; De Villiers, 2004; Mason, 1998; Fuys, et al., 1988; Van Hiele, 1983). The authors further agree that when the teacher is reasoning at a different Van Hiele level than the students during instruction, students experience lack of understanding and difficulties in learning. However, to decrease the cognitive load by reducing the impact of contributing factors, Cathcart, Pothier, Vance and Bezuk (2014) advise that teachers should (1) develop a sequence of activities that will contribute to students' progress according to Van Hiele levels; (2) discern the students' geometric level of thinking by involving students in open-ended geometry surveys; and (3) design activities that will connect students with the real world or environment; for example, teachers could conduct tasks that will include students' reaction, collation of figures and processes on objects familiar to students. Pre-visualisation as well as visualisation and analysis play a major role in building fundamental knowledge and understanding of Euclidean Geometry. Fundamental knowledge in turn can support the students' natural or artificial cognitive construction and can also establish a good grounding for Euclidean Geometry problem-solving at more advanced levels such as formal deduction and rigour.

Natural and artificial cognitive construction arise from the ability to think creatively. Creative thinking initiates the generation of ideas; the ability to determine which ideas are best and when to use them; the ability to examine ideas from various perspectives; the ability to make inferences; and the ability to reflect on own learning processes, that is, metacognition (Cennamo, Ross & Ertmer, 2010). Further, Cennamo, et al. (2010) highlight that creative thinking involves diverse cognitive skills such as divergent thinking (from a common point to a variety of perspectives); convergent thinking (from diverse perspectives to a common understanding or conclusion); innovation

(producing something that is original and of value); critical thinking (also analytical thinking, thinking that determines the validity or value of something); inductive thinking (from parts to the whole); and deductive thinking (from generalisation to underlying concepts). In order to show that students possess creative thinking, the characteristics in Table 2.7 as outlined by Cennamo, et al., (2010) must be evident.

Table 2.7 Characteristics of Creative thinking

• Non-algorithmic.
• The process of actions is not specified in advance and multiple ways and solutions are available.
• Differences in judgement and interpretations.
• Conflicting information is evident; judgements and interpretation are necessary.
• Imposing meaning.
• Visible disorder needs to be structured and inconsistencies need to be addressed.
• Complex.
• Multiple vantage points are needed; Single way is not sufficient.
• Effortful.
• Substantial mental work is required and the students' elaborations and judgements involve higher-order thinking.
• Uncertainty.
• Irrelevant presented information, essential facts are not known
• Self-monitoring of problem-solving processes is essential and making choices based on learning goals.

Source: (Cennamo, et al., 2010)

2.2.12 Traditional axiomatic teaching approach versus IBL in Euclidean Geometry

Research has shown that IBL is more productive as compared to the traditional method of teaching. For example, Dell'Olio and Donk (2007) state that the traditional method of teaching has not been shown to be more effective than the inquiry method. The authors further suggest that, through the process of inquiry, students learn to retain concepts, exhibit deeper understanding of concepts, show superior abilities in higher-order thinking skills and demonstrate a higher level of creativity through experiences in inquiry. In addition, Dell'Olio and Donk (2007) emphasise that "direct instruction will continually focus students' attention on the past and on what has been already discovered by others, while IBL based on constructivist learning theory will provide students with the tools to move into the future as producers of knowledge" (p. 348-349). Table 2.8 shows the contrast between IBL and traditional axiomatic approach.

Table 2.8 Comparison of traditional axiomatic approach and IBL

Traditional axiomatic approach	IBL
• Traditionally organised classroom. • Teacher centred instruction.	• Positive classroom climate. • Student-centred instruction. • Open classroom organisation.
Teaching and learning are textbooks based.	Teaching is facilitation that uses spontaneous and practical learning methods
Use only of formal proof.	Posing problems, analysing examples, formulating conjectures, offering counterexamples, revising conjectures; answering significant questions and validating ideas that result in theorems.
Deductive reasoning based on axioms.	Inductive reasoning based on authentic, self-directed inquiry.
Students work individually, memorise proofs to formally produce during testing.	Students consider cooperative learning in order to formulate and validate conjectures; and further present evidence to argue about solutions they have obtained.
Communication is one way, that is, from teacher to students. Students remain passive.	Teacher-students and student-student communication and discussion are evident. Students are actively involved in classroom activities.
Student sit in a row to face the front, that is, the teacher and chalkboard. The chalkboard is the only resource.	Students sit in groups or in work stations to actively participate in scrutinising and discussing concepts using diverse resources.

Adapted from Dell'Olio & Donk, (2007); Battista & Clements (1995); Prince and Felder (2006).

Learning and teaching Euclidean Geometry that has been leaning much on traditional axiomatic approach has not contributed to cognitive growth of students if we consider fact that in Grade 11, students struggle with geometric concepts and processes that they have not acquired in lower grades. Therefore, students experience cognitive load that obstructs attaining knowledge and understanding during learning. Exposure to multiple learning experiences (that is, action learning, visual learning, and verbal learning) could possibly ease the cognitive load and improve the retention rate, degree of abstraction and hands-on experience (Anderson, 2013; Corpuz & Lucido, 2008; Dell'Olio & Donk, 2007). Using diverse learning experiences for inquiry is essential to enable students to learn Euclidean Geometry through explorations, discovery, guessing, and problem-solving.

2.2.13 The process and the essence of IBL in Euclidean Geometry

Inquiry or to inquire is to pursue an answer to a question that is not already known by the individual (Case, Gini-Newman, Gini-Newman, James & Tylor, 2015). In addition, Case et al. (2015) suggest that students' answers to an inquiry question require some examination or investigation on their part. Further, Case et al. (2015) state that the characteristics of IBL in mathematics include learning mathematical actions and language through co-operative involvement in mathematical discussions and reasoning while engaged in the process of finding solutions for unacquainted problems. According to March and Peters (2008), the inquiry process is as follows: questions should be asked about puzzlement, followed by formulating supposition about what might help explain the puzzlement; an actual experiment or activity might follow to test the supposition in real time; inquiry is limited to an informed discussion; inquiry requires students to apply thoughtful and strategic procedures; during the inquiry process, students behave like scientists investigating discrepant events and puzzlements, looking for reasonable solutions and explanations. In the inquiry process, a post-inquiry debrief helps students to reflect on what happened. This is the time for students to identify particularly successful strategies that should be repeated in subsequent inquiries, but students should also identify missteps that should be avoided the next time. Thus, inquiry can be useful when students need to apply creative and critical thinking to solve real-time or immediate problems and the inquiry process and its debrief are instructive, whether or not the students' suppositions were correct.

March and Peters (2008) state that inquiry lessons are classified according to designs which are hypothetical and real-time designs. Hypothetical designs are inquiries that address past events where students are expected to analyse underlying causes and effects and to analyse the relationships among relevant factors whereas real-time inquiry designs are inquiries that student can apply to solve actual problems. They further argue that the main objective of real-time inquiry designs (problem-solving inquiry and experimental inquiry) is for students to propose a solution or explanation and then test it as scientists would do. This study adhered to problem-solving inquiry-based on the notion that problem-solving is the core activity in mathematics. Problem-solving is a significant activity to the students and should be linked to the knowledge that must be assimilated (Grugnetti & Jaquet, 1996). In addition, Grugnetti & Jaquet

(1996) agree that the knowledge assimilated during the problem-solving activity must force the students to transform their methods of thinking and working. Real-time inquiry through exploration can teach students problem-solving skills that can transform them into critical thinkers.

The inquiry-learning process is essential in learning Euclidean Geometry at lower levels of learning such as pre-visualisation, visualisation and analysis with a view that learning should be natural and encompass “explorations, discovery, guessing, and problem-solving” (Cathcart et al., 2014, p. 295). In addition, Cathcart et al. (2014) emphasise the essence of IBL in Euclidean Geometry by highlighting that students must take part in more geometric practices prior to showing the ability to express exact accounts that can be described in terms of using symbols and concepts. In light of these ideas, a gap addressed in the current study is that there is lack of cognitive growth from lower grades (pre-visualisation, visualisation, analysis and informal deduction) and students cannot overcome the cognitive demands posed by Grade 11 problem-solving (that is, demand to solve problems at the formal deduction level).

2.2.14 Teacher and students’ role during IBL in Euclidean Geometry

IBL process involves a technique in which the teacher asks questions and the student gives answers so that one answer leads to the next question (Cotton, 1995). In addition, Cotton (1995) highlights that the IBL system works on the basis that the student, without realising it, already knows many of the answers, and the teacher uses questions to draw out clarity and understanding from the student. He further emphasises that the areas of ignorance which the teacher has to cover may be revealed during the process of inquiry. Students experience maximum participation during the inquiry process while the teacher becomes the facilitator. While the teacher as the facilitator must give some information as a base for the inquiry, Cotton (1995) cautions that, if too much information is given by the teacher, the session degenerates into the axiomatic style.

The process of inquiry as facilitated by the teacher affords the students opportunities to actively take part in learning and the process assists the students to own their learning. An outline of teacher and students’ roles during the IBL process by Cotton (1995) relates well with the stages of IBL as outlined by Dell’Olio & Donk (2007) in Table 2.9.

Table 2.9 Stages of the scientific method (IBL)

Stage	Teacher's role	Students' role
1.Developing a question	• Gives questions and facilitates working with the given questions as well as developing original learning. • Teacher uses a think aloud procedure	Work with given questions, eventually develop their own questions.
2.Generating a hypothesis	Ask students to verbalise or write the rationale for the hypothesis they have made.	Use their prior knowledge and understanding to answers questions asked at the beginning of the inquiry.
3.Developing an experimental design	• Facilitates experimental designs. • Uses think aloud procedure and provides multiple examples.	Work with given designs and ultimately develop their designs.
4.Collecting and recording data	Facilitates by using a think aloud procedure and by providing multiple examples.	Work with given systems for collecting and recording data and eventually develop their own systems.
5.Analysing data	Facilitates data analysis.	Analyse given data; analyse their own data.
6. Reaching conclusions, forming and extending generalisations	Facilitates by using a think aloud procedure providing multiple examples.	Reach conclusions, form and extend generalisations of given data as well as their own data.
7.Communicating results	Models multiple ways of communicating results.	Communicate results using given data. Communicate results of their own data.

Source: (Dell'Olio & Donk, 2007)

In addition to stages of the inquiry-based process, Pedaste, Mäeots, Siiman, de Jong, van Riesen, Kamp &Tsourlidaki (2015) developed the IBL framework (Figure 2.1) The IBL framework by Pedaste et al. (2015) shows three possible cycles of IBL which starts with orientation and arrows signifying different pathways during the IBL process: (1) orientation, that is, questioning through search and interpretation of data through to conclusion; (2) orientation, that is, producing the supposition, testing, interpreting data by possibility reviewing the created supposition, and concluding; and (3) orientation: questioning through creating the proposition, experimenting, data interpretation by possibility reviewing the questions posed or produced proposition. The IBL framework (Figure 2.1) also depicts the teacher's role as initiator and facilitator through questioning, and students' role as main actors and problem solvers through answering questions and formulating their own questions in order to conceptualise (by questioning and generating the hypothesis) and investigate (by exploring or experimenting and interpreting the data from exploration or experimentation).

In summary, the IBL framework by Pedaste, et al. (2015) applies as follows: the IBL process begins with orientation where the teacher introduces the students to the problem to be solved and the students get the idea about the topic to be investigated. Conceptualisation follows where students start from open questions to the exploration or experimentation of the phenomenon (investigation). Students will be expected to go back to conceptualisation to generate the hypothesis based on the data collected from the exploration. In this case, exploration or experimentation can be repeated (to test the hypothesis). Data can be interpreted to arrive at the conclusion. Pedaste et al. (2015) highlight that from the interpretation of data, it is probable to advance onwards to the conclusion stage or move backwards to the conceptualisation stage to review prevailing information or delineate new hypothesis or questions, and this makes IBL a cyclic process. Figure 2.1 also includes discussion, communication and reflection.

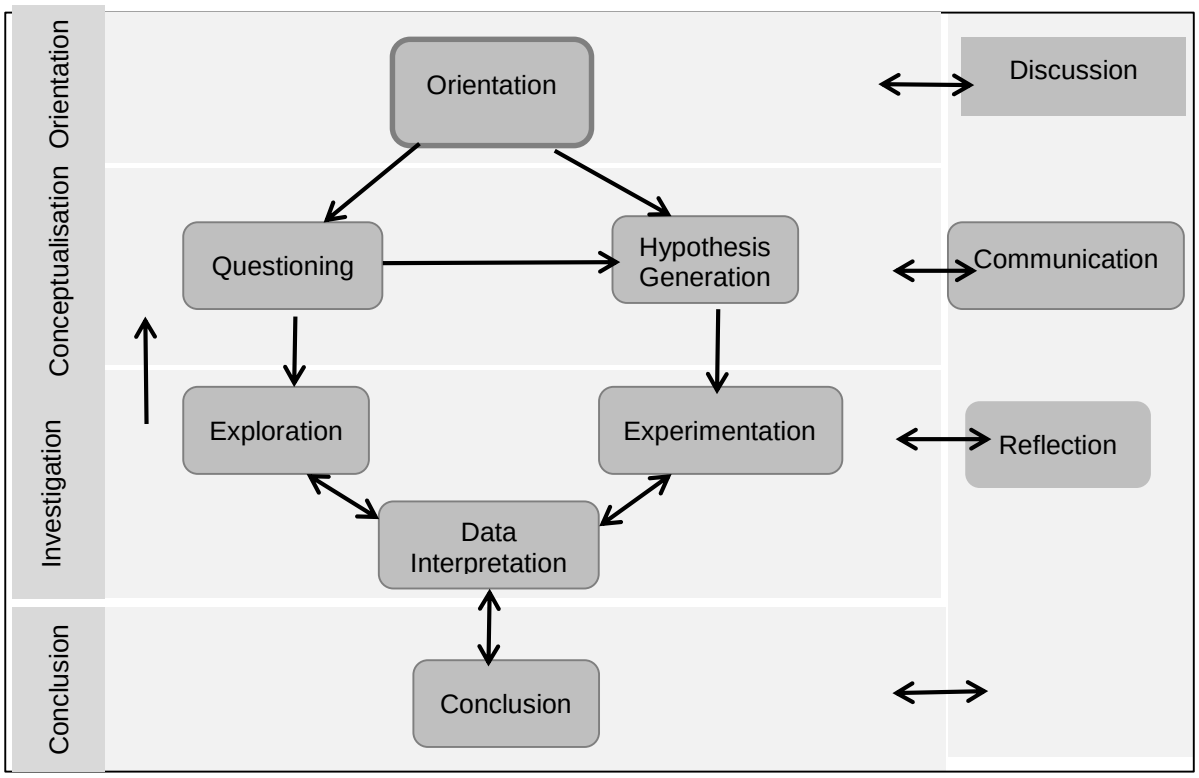


Figure 2.1: Inquiry-based learning framework Source: Pedaste et al. (2015)

According to Pedaste et al. (2015), in their IBL framework, discussion is an ongoing process. Further they argue that student-student and teacher-student communication can assist students to share outcomes and ideas in order to obtain feedback pertaining to their process of learning.

They further indicate that during the IBL process, communication can be “in action” or “on action” (p. 57), meaning that communication is regarded as a stage of inquiry, or can be completed separately as a task towards the conclusion of the inquiry stage. Reflection during the IBL process has been classified by Pedaste et al. (2015) as consideration in action and reflection on task, that is, students can assess their study progression and conduct tasks of a particular stage, further, students appraise their study development when they have completed the entire process of inquiry. In addition, Pedaste et al. (2015) point out that IBL consequences can be attained while there is no reflection or conversation when each student is considered as an individual. The idea of lack of communication and reflection during the IBL cycle will apply in the study mostly during assessment. Students are expected to go through cycles of orientation, conceptualisation, investigation and drawing conclusions individually for evaluation process, towards the concluding stage of the facilitation process. However, the current study has adopted the discussion process as a continuous process characterised by teacher facilitation that directs and monitors the students in each inquiry cycle during the teaching and learning process, that is, as the lesson unfolds in the classroom.

In essence, Cotton (1995), Dell’Olio & Donk (2007) and Pedaste et al. (2015) express the same notion that, during the IBL process, the student remains the main actor in providing the solution; whereas the teacher plays the role of the director and facilitator. Bennet (2015) explains the teacher role as the initiator. The teacher as the director and initiator facilitates the learning process by giving the students a chance to apply the IBL stages to advanced problem-solving such understanding the problem, conjecturing, judgements, verifications and validations (Grugnetti & Jaquet, 1996). In this sense, there is an association between IBL stages as presented in the IBL framework (Figure 2.1) and the mathematical problem-solving stages as presented by authors such as Grugnetti and Jaquet (1996), Polya (2004), Dewey and Wallas problem-solving phases presented by Natsusaka, (2007) and Baptist (2012). The comparison is shown in Table 2.10.

Table 2.10: Mathematical problem-solving stages

IBL framework (Pedaste et,al)	Problem-solving stages (Grugnetti & Jaquet, 1996)	Polya's four problem-solving phases	Dewey's five problem-solving phases	Walla's four problem-solving phases
Orientation	Understanding the problem	Understanding the problem	Experience a difficulty	Preparation
Conceptualisation (questioning and hypothesis generation)	Hypothesising	Devising a plan	Define the difficulty	Incubation
Investigation (Exploration or Experimentation)	Trials	Carrying out a plan	Produce a potential answer	Illumination
Conclusion	Verifications	Looking back	Assess the result by rationalising	Verification
----------	Justifications	----------	Confirm the answer	----------

In addition to Polya's four phases of mathematical problem-solving, Baptist (2012) introduce a fifth phase and suggests it as "looking forward" (p. 7). In this study, it is postulated that when problem-solving or inquiry ends by looking forward, it means new knowledge can be developed to advance conclusions in problem-solving and to make changes to the contemporary and future development of Euclidean Geometry.

In relating the IBL framework to mathematical problem-solving stages, Grugnetti and Jaquet (1996) recommend that teachers must organise the students' problem-solving activities in an approach that allow students to take responsibility for problem-solving. Natsusaka (2007) agrees that problem-solving oriented programs generally consist of of four of five stages as shown in Table 2.11.

Table 2.11: Structural model of a problem-solving oriented lesson

(1) Identifying the problem
- The teacher and students comprehend the problem by reading through the problem statement
- Students listen to the teacher's instruction to understand the problem clearly, and thereafter advance the student-student discussion.
- Students link prior knowledge and existing knowledge by examining the differences and similarities
- Collaboratively, students develop a standpoint about methods of solving a problem and finding answers.

(2) Development of a result or answer
- Students centre learning around themselves in thinking and finding the solutions.
- The teacher monitors each group providing instructions and clues to assist students who are clueless and motivate those who have clues about advancing solutions.

(3) Progress through conversation
- A group of students who got diverse solutions elucidate the group methods to the whole class.

- After group presentations, the class advance a general view of how to solve the problem through conversations about outstanding and common points of each approach presented by diverse groups

(4) Summarising
- Students summarise the essential facts agreed upon throughout the period.
- The teacher poses more and similar problems and student attempt to apply the knowledge a.

Source: (Natsusaka, 2007: 93)

Teacher-student as well as student-student communication is emphasised as important throughout the inquiry learning process. In concurrence, progression through discussion is highlighted in Table 2.11. Further, Natsusaka (2007) argues that the problem-based lesson should adopt an approach that focuses on fitting students' thinking as a group. Communicating and discussing as a group during inquiry is also referred to as collaboration or cooperative learning. Active student participation and cooperative learning through discussion is encouraged. The importance of teaching through discussion to ensure consistent communication, collaboration and co-operation is emphasised by Tanaka (2007) through the structural model of a discussion-oriented lesson (Table 2.12). In addition, Tanaka (2007) suggests that discussion as part of the problem-based lesson in essential because (1) as students explain their ideas to classmates and the teacher, students may come to realise their own errors, thereby facilitating a greater understanding of the issue; (2) one's own ideas can be confirmed through discussion; and (3) discussion is another way of cultivating students' mathematics skills and humanity.

Table 2.12: Structural model of a discussion-oriented lesson

1. Students develop their own approach. The teacher provides instruction about the problem. Students develop questions about the problem, discuss how to solve the problem they have been given. Further, students plan and find methods to solve the problem.
2. Students present their own approaches to the problem. Students give presentations on individual solutions, report what they did not understand and acquire communication skills.
3. Students listen to, understand and discuss each other's approaches. Students strive to understand and confirm each other's thoughts and processes. In addition, students debate among each other in order to examine or explore methods. Students acquire co-operative learning skill including ability to express themselves and listening skills.
4. Students compare various ideas and incorporate into their own thinking. Students' acquire improved thought processes, realise individual strength and ideas and learn new ways of looking at reality.

Source: (Tanaka, 2007:103)

The teacher plays a major role in planning and facilitating learning resources and providing a desirable environment to ensure students' maximum reference for inquiry, discussion and involvement during problem-solving. March and Peters (2007) emphasise that problem-solving inquiry refers to where students are presented with a problem to solve, including constraints about materials they may use and conditions under which they must work. In order to advance inquiry for problem-solving, teachers are challenged to plan the resources and the learning environment appropriately. Inquiry cannot be achieved if it is not stimulated. In addition, March and Peters (2008) argue that the teachers' use of resources in a productive learning environment and the questioning method applied can result in inquiry as a technique to advance problem-solving and discovery.

Through inquiry, students explore the material, make discoveries and ultimately solve problems. Dell'Olio and Donk (2007) argue that discovery learning, IBL and problem-based learning refers to the experience acquired by students as they apply a systematic procedure to solve a problem and communicate their findings with other students.Similarly, Prince and Felder (2006) state that teaching and learning through inductive methods is key in IBL and refers to constructivist teaching and learning methods such as problem-based learning, project-based learning, exploration, experimentation and discovery learning. In support of this notion, Capaldi (2015) agrees that IBL is "a pedagogical method that encourages students to conjecture, discover, solve, explore, collaborate and communicate" (p. 283). In light of the ideas

mentioned above, this study asserts that IBL through the influence of IBF is necessary to assist students to attain problem solving skills through inquiry.

Conjecturing is of essence in IBL. A conjecture during the inquiry process refers to a mathematical declaration which has been informally proven, however, seems to be true (Everything Maths and Science, 2012; 2015). The source further highlights that a conjecture can be thought of as the mathematician's way of saying "I believe that this is true, but I have no proof yet" (p. 5); that is, a conjecture is a virtuous speculation about a pattern. Conjecturing in this study is related to informal deduction level of knowledge and understanding, that is, Van Hiele's Level 3. This study emphasises that pre-visualisation and visualisation of objects and material can assist students in identifying problems and ultimately formulating conjectures that need proof. The proof will confirm the identified problem and conjecture to be true or false.

Discovery learning is grounded in the "discover it on your own" rule where students are not given texts, articles or any previously published material (Capaldi, 2015: 284). In light of this, it is evident that, according to Dale's theory, learning by discovery entails less verbal learning and more visual learning and action learning. Through visual learning, the teacher encourages students to accumulate direct purposeful experience through viewing pictures, watching demonstrations, audio-visual media, mathematical manipulatives and digital resources. Action learning refers to where the teacher inspires students to put the knowledge and experience accumulated through visual learning into practice, that is, through motor skills by participating in a problem-solving activity, simulating the activity through demonstrations. Discovery learning contributes to the students' "lasting effect on memory" (Cotton, 1995:13).

The 'discover it on your own rule' for discovery learning inspires student-centred learning rather than teacher-centred learning. According to Cotton (1995), teacher-centred learning reflects the following characteristics: teachers choose the learning experiences that suit the subject matter; they use a highly structured approach for essential principles; learning is limited by the teacher; teachers are stepping stones for students to achieve more quickly; and teachers work to their level best by helping students to progress.

2.2.15 Challenges associated with implementing IBL

Hwang, Chiu and Cheng (2014) found that researchers have indicated that IBL is confronted with various challenges: (1) the problem of large classrooms; (2) the time needed to develop the learning activities; and (3) the predicament of facilitating students' motivation in learning to organise information and to follow the learning context. In addition, Ramnarain (2016) found that, in South African township schools, significant constraints in implementing IBL comprise dynamics such as resource sufficiency, professional support, school's moral belief and time. Further, Hwang, Chiu and Cheng (2014) state that, as the IBL process unfolds, students with active learning styles (who learn by doing) out-perform those with reflective learning styles in terms of learning achievement. In agreement, Kinsey and Moore (2015) argue that stronger students often find using the discovery approach enjoyable, while other students are not motivated in using the approach. Regardless of the hurdles associated with the implementation of IBL, the model is still useful. For example, Kinsey and Moore (2015) found that in a study they conducted many students responded well to the IBL approach; and they concluded that students learned that they could discern and justify patterns in mathematics in nature, and even in their own lives.

2.2.16 Suggested solutions to challenges encountered

Hwang, Chiu and Cheng (2014) contribute the following as possible solutions to challenges faced: (1) the utilisation of computers and digital technologies and (2) gaming scenarios in order to instill active participation in activities and promote students' learning motivation. Further, they indicated that the tactic that is applied by the teacher in engaging students to explore in gaming activities, should challenge students and allow them to respond to the feedback from the system through reflective and active learning methods.

2.2.17 Assessment and performance indicators in IBL

Instruction is always aimed at encouraging individuals to learn, and the learning process should yield results. March and Peters (2008) highlight that instructional design commences with its endpoint, that is, "indicators of a successful performance or what students are expected to master" (p. 26). Performance indicators should be set in accordance with contemporary teaching and learning theories as outlined by

March and Peters (2008). In addition, they suggest that the performance indicators should be structured along the following pointers: (a) output (construction of meaning) focused, rather than input (characterising the curricular); (b) integration of the outcomes across diverse subjects (e.g. maths & earth science, Euclidean Geometry as a space science); (c) using high-order thinking and problem-solving skills to process information; (d) integrating skills from diverse topics rather than teaching one topic in isolation (for example, integrating number operations or patterns from algebra with Euclidean Geometry).

There should be indications that learning occurs during the process of inquiry. Hwang, Chiu and Cheng (2014) highlight that at the end of the IBL process, students should have developed higher-order thinking processes and self-directed learning skills. In addition, they argue that there should be evidence of authentic activities that afford students natural problem-solving contexts. According to Kreber (2006), assessment must be in part positivist (based on scientific facts) and partly constructivist (based on construction of knowledge). He further highlights that positivist assessment is appropriate when learning outcomes can be pre-specified, and constructivist assessment is proper when outcomes cannot be identified beforehand. This study postulates that content outcomes are predetermined as the focus in Euclidean Geometry; however, since the processes and procedures of problem-solving cannot be predetermined, students must establish their own processes and procedures of problem-solving through inquiry. Therefore, assessment partly requires both positivist and constructivist practices.

In combination of positivist and constructivist practices in assessment, Stein et al. (2000) highlights that tasks used for assessment should address different levels of cognitive needs. Stein et al (2000) define the levels of cognitive needs in tasks as lower-level demands and higher level demands as follows:

Lower-level demands Memorisation task - these are tasks that involve reproducing previously learnt definitions, rules, formulae and facts; tasks that disregards the connection of concepts and facts. Procedure without connection tasks – these tasks are algorithmic, no connection between meanings and procedures; are algorithmic and need limited cognitive effort.

Higher- level demands
Procedure with connection tasks – develops deeper levels of comprehension through procedures; proposes various ways towards problem solving; demands higher cognitive effort; tasks are presented in multiple ways, for example, as symbols, diagrams, concepts and more.
Doing Mathematics tasks – non-algorithmic and complex thinking is required; self-regulation is needed; fosters exploration comprehension of concepts, relationships and processes; analysis skills are required; unpredictable solution processes contribute anxiety and require higher cognitive effort.

Source: Stein et al (2000)

2.3 IBF AND LEARNING IN EUCLIDEAN GEOMETRY (DEWEY'S INQUIRY AND VAN HIELE'S THEORIES)

2.3.1 Overview

In analysing the current crisis in the learning of Euclidean Geometry in South Africa, this study bases its argument on the observation by Van Hiele (1986). Van Hiele (1986) stated that, in learning Euclidean Geometry: (1) students apply rote learning to operate with mathematical associations that they do not understand and have not seen the origin of such associations; (2) students experience the structure of mathematical connections as independent constructions having no relation to experiences acquired before; (3) the students know only what the teacher has taught them and what has been deduced from the teacher's ideas; (4) students do not learn to institute networks between the structures and the physical domain; and (5) students are clueless about utilising acquired information in original contexts (Van Hiele,1986).

In support of the statement by Van Hiele, research has shown that even in the 21st century the main contributor to rote learning is the mode of instruction that is applied by teachers. For example, Prince and Felder (2006) argue that traditional deductive teaching has been dominant for centuries; and this particular teaching approach has its foundation in positivism which explains that knowledge or unbiased certainty occurs autonomously of human insight. Hester (1994) agrees that the major educational process in most classrooms results in students avoiding the responsibility of independent thinking; and not trusting their own capacity of finding, creating and testing meaning for personal problem-solving. Prince and Felder (2006) believe that to alleviate the problem of objective teaching and learning; an alternative model of

constructivism should be adopted, explaining that regardless of the existence of objective reality, individuals make sense of their encounters in order to build own authenticity. In agreement, Hester (2004) believes that objective teaching needs to be de-emphasised, and suggests teaching and learning which involves critical thinking processes, methods such as diagnosis, speculation and hypothesis-testing. The constructivist model suggested by Prince and Felder (2006) highlights that, in the sense-making effort during learning, new evidence is clarified through intellectual constructions (schemas), and the schemas incorporate the students' pre-knowledge and conceptions. They further argue that new information can be integrated into schemas if it is consistent with these particular schemas; but, if the information is contradictory to the schemas, it may be memorised and not learned. In addition, Hester (2004) in support of applying inquiry in teaching and learning states that the constructivist technique of inquiry provides students chances to produce thoughts, challenge problems, assess the notions and use them in other situations.

Constructivism has been claimed to be an effective method of teaching and learning for educational reform. For example, in their research, Prince and Felder (2006) identify constructivism as a better learning method than the positivist approach. The constructivist philosophy is believed to be the best (in the current education era) at engaging students in learning and it is interpreted as a philosophy of teaching and learning that helps students to learn by exploration, investigation and discovery in order actively construct knowledge (Friesen & Scott, 2013; Prince & Felder, 2006; Tracey & Morrow, 2012). The authors specified the constructivist or inductive teaching methods as IBF, problem-based teaching, project-based teaching, case-based teaching and discovery-based teaching. However, Mougan (2013) argues that there are several shortfalls observed in constructivism; for example: (1) excessive subjectivism leads to arbitrariness and relativism; (2) constructivism states and investigates the origin of ideas, but does not point out the origin of sources for ensuring validity of the ideas. In addition, Priss (2014) emphasises that, in constructivism, learning is viewed as active creation of mental structures without making any philosophical claims. According to Priss (2014), Pierce as a pragmatist philosopher managed to find a middle way between constructivist (through idealism) and realist positions. He emphasises that pragmatism supports the advantages of a constructivist view of learning with a slightly more realist position. Further, Mougan (2013) describes

the similarities of constructivism and pragmatism as: (1) both theories do not ask for constructions for their own sake but look for solutions to problems of importance to people; and (2) both have established strong links between the ideas of democracy and construction of knowledge.

In this study, constructivism is viewed as fundamental to realism. A pluralistic position in teaching is promoted in this study. Furthermore, this study adopts the pragmatic stance that teaching should start by inquiry that leads to inductive reasoning; and inductive reasoning is fundamental and a prerequisite for deductive reasoning. Inductive IBF enables students to achieve lower levels of geometric knowledge and understanding, and they can then meet the requirements of learning through deductive inquiry at higher levels of geometric knowledge and understanding such as formal deduction level.

The framework of pragmatism has been outlined by Shields (2003) as follows: firstly, teaching is approached with a spirit of critical optimism; secondly, pragmatism takes a point of departure that involves inquiry which leads to successful learning in which the teachers' duty is to facilitate the transformation of inquiry by developing teaching tools; thirdly, a class of students should be treated as a community of inquiry; fourthly, teaching should focus on pragmatic consequences, in other words utilising inquiry to bring about changes; fifthly, teachers' and students' perspectives should be broadened by bridging dichotomies such as theory and practice; and finally, the teacher opens the classroom for public scrutiny.

Using the pragmatist framework, this study assumes that the teaching of Euclidean Geometry facilitates transformation through inquiry; turns classrooms into communities of inquiry and students are directed to think critically. The traditional axiomatic approach to teaching that has been used for most of students' academic life, has not contributed to their cognitive growth and critical thinking skills. Grade 11 students are confronted with challenges in geometric knowledge and understanding meaning that they could not master the lower Van Hiele levels and they cannot therefore cope at higher cognitive levels. A bridge between inductive learning and deductive learning (that is, pragmatist learning) is necessary. This is consistent with the idea that pragmatism bridges the gap between subjective and objective inquiry considering the fact that by applying pragmatist practices, excessive subjectivism is limited and justice is done to the objective aspect of human inquiry (Mougan, 2013).

Characteristics of pragmatism in learning as outlined by Mougan (2013) are that: (1) pragmatism rejects the impossibility of overcoming the isolation and lack of understanding; (2) what is different is recognised but ways are sought to understand and relate to what is different in a critical way.

Pragmatist teaching is a practice that has not been rolled out in most of our South African classrooms. Constructivism has been commented by many South African researchers in order to overcome the problems of positivist teaching, but positivist teaching still plays a major role in most classrooms. This study agrees with Ozmon (2013) about teaching and learning that considers pragmatism as a foundation. Ozmon (2013) agrees that pragmatism should be experimental; become an agent of change in the social context, promote civilisation in people; and promote an individual or collective efforts to finding answers to the contemporary social, economic and political problems. Table 2.13 presents more of Ozmon's ideas about pragmatism as a philosophy of education.

Table 2.13: Pragmatism as a philosophy of education

• Pragmatism believes that philosophy of education is the development of right psychological attitude and ethics that can be utilised in confronting modern problems, but not only the application of existing ideas to problems.
• Continuous experimentation with ideas and procedures should be adhered to; rejecting ideas that do not work in order to assist a child to learn a better way.
• Teaching consists of diverse methods, not only one way.
• Teaching methods should apply multiple teaching and learning resources.
• Students and teachers should realise the relation between disciplines and diverse approaches to teaching and learning even to problem solving.
• Pragmatism rejects separating knowledge from experience; facts are not supposed to be separated from experience.
• Attention is given to process rather than the outcome.
• Learning concerns studying and evaluating material and students draw conclusions and construct suitable generalisations concerning the problem
• Learning and growth are evaluated in order to set the next unit of study.
• Traditional disciplines are not ignored, but are used as knowledge background of the problem.
• Life in general provides lessons, however, due to the informal educational context of life students learn and acquire knowledge in an informal way. Furthermore, the disadvantage of formal education is its abstract nature and the education is more remote from the students' actual life experiences. In balancing the two, the teachers' role becomes critical.
• Subject matter should not be treated apart from the students' social life as the society is essential for the students' learning experience.
• Pragmatism believes that students have an inherent motivation to learn. Teachers must realise the enthusiasm that prevails and consider students' diversity in order to diversify the teaching methods to include all students.
• Teachers should serve as knowledgeable guides and resources for students, not as subject masters who drill the students in the subject matter.
• Drill as well as recitation have occasional uses but are not central to teaching

• The teacher should not appear as subject masters who centralise knowledge that can be provided for practise purpose, but teachers should play a role of well-informed facilitators who direct students towards meaningful learning.
• The educative process is fulfilled only when understanding and intelligent actions are promoted in order to help the students to advance reasoning.
• Learning tasks need to be utilised to address ideas and assist students to acquire skills as well as understanding; and the educational setting should allow students to act on and test what they have learned. The educational environment must be deliberately regulated by the teacher to ensure maximum educational effect.
• Teacher should display expertise of the knowledge area and enable students to link their experience and the subject matter.
• Teaching is a procedure to assist students recognise problems and collect ordered knowledge to comprehend how communal contexts came into being, where to maintain existing contexts and how to reform the existing contexts.

Adapted from Ozmon (2012:132-135)

The study contributes to educational practice in moving from the traditional axiomatic approach to IBL in teaching Euclidean Geometry based on Dewey's philosophy of education and Van Hiele's theory of teaching and learning Euclidean Geometry.

2.3.2 Dewey's philosophy of education

Ozmon (2012) reported on Dewey's view about the pragmatism philosophy. According to Dewey education occurs in both formal and informal settings and to achieve a balance between the two, the pragmatist philosophy must play a role (Ozmon, 2012). In addition, Ozmon (2012) elaborated on the pragmatist philosophy of education that education is a social endeavour and individuals must enter into social relations in order to be educated. In social contexts, communication becomes central to the education of individuals; and the centrality of communication stems from the fact that communication enlarges experience and makes it meaningful (Ozmon, 2012).

2.3.3 The origin of IBL focused on Socrates' and Dewey's inquiry

Self-directed inquiry is significantly addressed in pragmatism. Pragmatist theory underpins IBL, and IBL comprises methods like discovery learning, problem-based learning, experiential learning and exploration (Bruner, 1967; Kirschner, Sweller & Clark, 2006).

Inquiry as a student-centred approach is said to have a strong historical antecedent in Ancient Greek philosophy. It is characterised by the questioning method employed by Socrates when engaging in dialogue with his interlocutors by leading them through a series of questions (Friesen & Scott, 2013). The Socratic method of inquiry dates back from 400BC (Cotton, 1995; March & Peters, 2008). The Socratic philosophy is as

follows: “I shall only ask him, and not teach him, and he shall share the inquiry with me and do you watch and see if you can find me telling or explaining anything to him, instead of eliciting his opinion” (Socrates, 400 BC cited in Cotton, 1995: 46). Friesen and Scott (2013) indicate that the spirit of inquiry strongly emerged in the 13th century, and in the 20th century, the historical threads of inquiry found a home in the work of John Dewey. Dewey encouraged K-12 teachers to use inquiry as the primary teaching strategy in science classrooms. Inquiry according to the originator of IBL (namely, Socrates) is “the act of reflective thinking, prompted by probing, provocative questions” (March & Peters, 2008:235). In addition, March and Peters (2008) indicate that in Dewey’s view, Socratic teaching depends on two basic elements which are doubt and the act of probing which are essential in clarifying ideas and details.

IBL has been a method practised mostly in Science education, but “the term Inquiry-Based Mathematics Education (IBME) is of recent use in mathematics education” (Artigue, 2012:3). Artigue (2012) suggests that IBME stems from ambitions of innovation of research in mathematics education. He explains the goals of innovation in mathematics research as promoting mathematical learning with understanding and helping pupils to experience authentic mathematical activity. Dell’Olio and Donk (2007) argue that when students are involved in IBL, they are provided with opportunities to act in systematic ways in response to engaging questions. The students’ involvement in IBL encompass firstly, thinking that originates from a state of doubt, hesitation, perplexity and mental difficulty; and secondly learning that involves an act of searching, hunting, inquiring to find material that can resolve the doubt and dispose the perplexity (Dewey,1933 in Dell’Olio and Donk (2007). It is of essence to consider IBME in promoting and helping students to develop learning with understanding and experience in authentic Euclidean Geometry activities. Socratic and Dewey’s inquiry methods are categorised in this study as IBF. Similarities on Socratic and Dewey inquiry are outlined in Table 2.14.

Table 2.14: Socratic and Dewey’s Inquiry

Socratic Inquiry	Dewey Inquiry
- Inquiry is a way of living ethically in the world, that is, good life involves seeking knowledge as a means to live more ethically and consciously in the world.	- Active inquiry should be used not only to gain knowledge, but also to learn how to live

	- Through inquiry, education should help students realise their full potential, to strengthen democracy, and to promote the common good.
- Teacher is not transmitter of knowledge - Classroom experience is a shared dialogue between teacher and students in which both are responsible for pushing the dialogue forward through questioning	- Teacher should not simply stand in front of the class and transmit information to be passively absorbed by students. - Teacher's role is that of facilitator and guide - Students must be actively involved in the learning process and given a degree of control over what they are learning.
-Both teacher and students ask probing questions, meant to clarify basic assumptions underpinning a truth claim or the logical consequences of a particular thought.	- Process involves sensing perplexing situations, clarifying the problem, formulating a tentative hypothesis, testing the hypothesis, revising with rigorous tests, and acting on the solution.

Source: Friesen and Scott (2013: 5-7)

IBF informed by Socrates and Dewey inquiry approach is the teaching method used in this study; however, teaching in Euclidean Geometry is aligned to the Van Hiele's instructional levels. Students' development in Euclidean Geometry moves from the pre-visualisation stage to the rigour level; from the concrete to the abstract; and from inductive inquiry to deductive inquiry. Therefore, teaching by inquiry at all levels of learning Euclidean Geometry is crucial to ensuring critical thinking and optimism in spatial development.

2.3.4 Van Hiele's theory of learning geometry

The original Van Hiele's theory is a model that defines a chronological order of five levels for geometric thinking, which are visualisation, analysis, informal deduction, formal deduction and rigour. Further, Van Hiele's theory specifies that students have to pass through each level from global perception of geometric figures to, finally, an understanding of geometric proof (Teppo, 1991). In addition, Teppo (1991) argues that according to Van Hiele's theory, students develop from one level to the next as a result of purposeful instruction organised into five phases of sequenced activities which are information, guided (directed or bound) orientation, explication, free orientation and integration.

Table 2.15: Integration of Van Hiele's instructional phases and cognitive levels

Phase	Teacher activity	Student activity
Information or inquiry (level 0 - Visualisation)	• Teacher asks questions to let students recall relevant geometric information related to their real-world. • The teacher presents a new idea and allows the students to work with the new concept	Students get acquainted with the material and begin to discover its structure
Guided or directed orientation (level 1 – Analysis)	Teachers propose activities of a fairly guided nature that allow students to become familiar with the properties of the new concept which the teacher desires them to learn	Students do tasks that enable them to explore implicit relationships
Explication (level 2 – Informal deduction)	Teacher emphasises the vocabulary after students have had an opportunity to become familiar with the concept. The discoveries are made as explicit as possible	The students' experiences are linked to shared linguistic symbols
Free orientation (level 3- Formal deduction)	A teacher might present other problems for which students have not learned a fixed procedure	• Students do more complex tasks enabling them to master the network of relationships in the material. • They know the properties being studied, but need to develop fluency in navigating the network of relationships in various situations
Integration (Level 4 – Rigour)	The teacher may give the students an overview of everything they have learned. It is important that the teacher not present any new material during this phase, but only a summary of what has already been learned	• Students summarise what they have learned and commit it to memory • Supporters of the Van Hiele model point out that traditional instruction often involves only this last phase, which explains why students do not master the material.

Adapted from Fuys, D., Geddes, D., & Tischler, R. (1988: 224 - 251*)*

In this study, Van Hiele's theory is perceived as underpinned by Dewey's philosophy of inquiry because the teacher acts as a guide and facilitator more than the information disseminator in all levels and phases. Furthermore, inductive inquiry is emphasised more in fundamental levels, that is, from visualisation to informal deduction where students are guided to discover, explore and acquire experience. Clements and Battista (1992) discovered a cognitive level that is more basic than the Van Hiele's visualisation labelled the pre-visualisation level. This study includes the more basic level and supports the teaching of Euclidean Geometry according to six cognitive levels. Therefore, this study focuses on intervention where the teacher facilitates the learning of Euclidean Geometry by using IBF in order for students to learn circle geometry through IBL; at the end of the process, students must be able to perform at the formal deduction level. Further, this study focuses on integrating the five instructional phases of the Van Hiele model and the six cognitive levels (that is, pre-visualisation in addition to visualisation through to rigour). In this study, the following are incorporated: six cognitive levels (pre-visualisation to rigour), the five Van Hiele instructional phases and the IBL framework. The integration between the five instructional phases, cognitive levels and IBL framework is implemented as in Table 2.16. The implementation is focused on encouraging IBF that will foster IBL through: (1) discussions (students' peer discussions and teacher-student discussions); (2) students' orientation to the problem; (2) conceptualisation (questioning and hypothesis generation); (3) exploration and drawing conclusions to solve problems. In order to ensure that IBL has taken place, this study is guided among others by IBL indicators determined by Hester (1994), that is, teachers treat students as investigators; students are encouraged by the teacher to investigate the problem, generate solutions and answers that are relevant to the problem; inquiry method is used by students; and the information is new to students and requires the students' interpretation.

Table 2.16: Integration of instructional phases, cognitive levels and IBL

Instructional phase and the cognitive level	Teacher activity (IBF)	Student activity (IBL approach)
Information or inquiry (level 0 pre- visualisation)	For pre-visualisation teacher asks questions to let students recall relevant geometric	Students recall geometric figures in their real-world. For example, a hut is round, the earth is round. Other shapes

Instructional phase and the cognitive level	Teacher activity (IBF)	Student activity (IBL approach)
	information related to their real-world	they can recall such as road traffic signs, traffic circle and many more.
Information or inquiry (level 1– Visualisation)	• Teacher makes objects or materials available for students to view, identify, and classify according to what they see. • Probing questions are asked by the teacher to confirm students' orientation and conceptualisation	Through discussions students explore (through action learning) the presented material related to their pre-visualisation knowledge; and asks clarity seeking questions. (Orientation according to IBL framework). Students visualise the shapes according to what they see, associating with what they have recalled in the pre-knowledge
Guided or directed orientation (level 2 – Analysis)	Teachers guide the students and propose activities that allow students to become familiar with the properties of materials they explored during visualisation, and teacher can lead students to provide sketches for materials according to classes	Students do tasks that enable them to explore essential connections. They produce sketches denoting relations among shapes or figures, and outline the properties according to groups of materials. For example, four-sided figures, three-sided figures, round figures, etc. (Conceptualisation according to IBL framework)
Explication (level 3 – Informal deduction)	Teacher emphasises the vocabulary after students have had an opportunity to become familiar with the concept. The discoveries are made as explicit as possible	The students' experiences are linked to shared linguistic and symbols (for example, quadrilaterals as rhombus or kite, triangles as scalene, equilateral and individual figures are explored in relation to properties, for example properties of a square, rectangle, rhombus, isosceles triangle etc.

Instructional phase and the cognitive level	Teacher activity (IBF)	Student activity (IBL approach)
		(Conceptualisation stage according to IBL framework
Free orientation **(level 4- Formal deduction)**	A teacher might present other problems for which students have not learned a fixed procedure	Students do intricate tasks in order to master the system of connections. They know the properties of figures, but need to develop efficiency in applying the network of relationships in various situations. For example, understanding a cyclic quadrilateral, relating properties of a circle and a selected quadrilateral to formulate conjectures, and proof conjectures (Investigation stage according to IBL framework)
Integration **(Level 5 – Rigour)**	The teacher gives a summary of what was taught and what the students have learnt. New knowledge or material is not necessary at this stage.	Students summarise what they have learned and commit it to memory. Conclusions are made; students look forward and back to verify the solutions they got. (Conclusion stage according to the IBL framework)

2.4 CHAPTER SUMMARY

Education should aim at helping students to acquire skills, knowledge and the right attitude such as autonomy, authenticity, creativity and critical thinking. In addition, education needs to assist students to develop in all spheres of life. Test and examination should not only be the objective of teaching and learning. The focus of teaching and learning should be to empower students with skills of problem-solving in and out of the classroom context. Teaching and learning should further bridge the gap between theory and practice. Educational instruction that is aimed at developing a student in totality, progresses through probing, that is, facilitator asks open questions that are thought provoking to stimulate the thoughts of students. Tests, examinations

and tasks should be set in a sense that they assist students to develop higher-order thinking skills.

CHAPTER 3
RESEARCH DESIGN

3.1 INTRODUCTION

Philosophical assumptions about the nature of knowledge in research relies on ontology, that is, the nature and existence of social reality, and epistemology which explains what constitutes knowledge and ways of knowing (Ngulube, 2015). In addition, Ngulube (2015) outlines the basic research paradigms as realism (positivism); pluralism (pragmatism) and constructivism (interpretivism). According to Ngulube (2015) the realist ontology claims that knowledge is relative and there is only one objective reality. In relation to realism, the positivist epistemology yields a research framework that moves from the general to the specific asserting that an established theory can generate data; however, in contrast, the constructivist ontology relates to knowledge as a subjective reality that can be interpreted (Ngulube, 2015). The following summary emerges from Ngulube's (2015) work, firstly, relative to constructivism, the interpretivist epistemology outlines that knowledge exists, can be found, can be constructed and it can be used. Secondly, interpretivist research framework declares that theory emerges from data, that is, research moves from the specific to the general. Thirdly, the pluralist paradigm, that is, the pragmatist epistemology, combines realism and constructivism, that is, knowledge exists as objective reality and can be interpreted subjectively. Lastly, positivism is characterised by the quantitative methodology, interpretivism by the qualitative approach and pragmatism is considered as a mixed-methods approach.

The essential differences among the three methodologies were captured by Plano Clark and Creswell (2008), shown in Table 3.1 (adapted from Morgan, 2008).

Table 3.1: Comparison of positivism, interpretivism and pragmatism

	Qualitative methodology	**Quantitative methodology**	**Mixed-method methodology**
Connection to theory and data	Induction	Deduction	Abduction
Relationship to research process	Subjectivity	Objectivity	Inter-subjectivity
Inference from data	Context	Generality	Transferability

Source: Morgan (2008:58)

This study provides a discourse about pluralism, that is, a pragmatism view that focuses on the mixed-method research methodology (Figure 3.1).

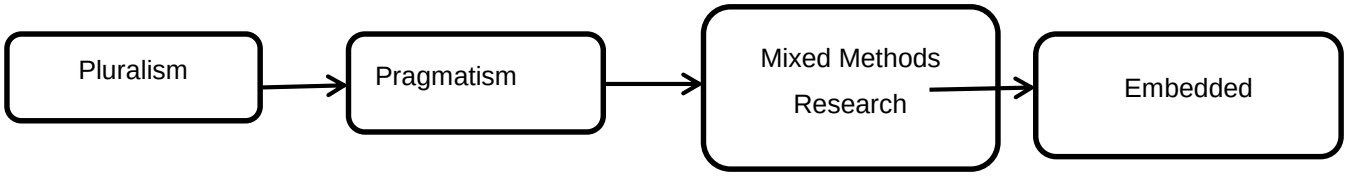


Figure 3.1: Research design

Abductive reasoning is advanced where the researcher works backwards and forwards between induction and deduction. Morgan (2008) defines abductive reasoning as research where observations are converted into theories and actions are used to assess theories. Inter-subjectivity is explained as research that works back and forth through various frames of reference, for example, working through both subjectivity and objectivity. He defines transferability as research where investigation is carried out on factors that have impact on whether the knowledge gained can be transferred to other settings rather than basing arguments on context-based results or generalisable results. Morgan (2008) argues that transferability as a mode of inference from data arises from a pragmatic focus on what can be done with knowledge produced, and further focuses on asking how much of the existing knowledge is usable in a new set of circumstances. In this study, a pragmatic point of reference is advanced in contributing interventions to produce knowledge through IBF in Euclidean Geometry. Facilitator's knowledge of IBF in Euclidean Geometry can be used in observable contexts where the students are assisted to learn through inquiry in order to acquire knowledge and understanding from pre-visualisation through to rigour.

3.2 HYPOTHESIS

In learning Euclidean Geometry, the teaching approach has an essential role. Teaching in Euclidean Geometry happens either as a traditional axiomatic approach or as an open system where students become participants with the teacher in acquiring knowledge and skills. This study is a mixed-methods study and posits the directional hypothesis: IBF influences IBL and consequently increases students' knowledge and understanding of formal deduction levels through visualisation, analysis and informal deduction.

The null hypothesis is that there is no difference between IBF and the traditional axiomatic approach in teaching Euclidean Geometry. It is expressed as follows:

H_0: $\mu_{\text{inquiry-based facilitation}} = \mu_{\text{traditional axiomatic approach}}$

3.3 METHODOLOGY

This study uses a mixed-methods research design. Mixed-method research combines both qualitative and quantitative methodologies of data collection and analysis (Newby, 2010). In addition, Ponce and Pagan-Maldonado (2014) agree that in a mixed-methods study, quantitative and qualitative approaches are combined and integrated at different points of the research process. In this study, quantitative and qualitative methodologies are combined, integrated and occur at the following points of the research process: (1) in integrating quantitative hypothesis and the qualitative research questions; (2) in generating quantitative and qualitative data by combining quantitative measurements with qualitative research interpretation; and (3) in integrating quantitative and qualitative data to present the results of the study. There are five purposes of mixed-methods research design, namely, triangulation, complementarity, development, initiation and expansion (Greene, Caracelli & Graham, 2008; Tashakkori & Teddlie, 1998). This study focuses on how IBF influences IBL to assist student to attain higher levels of knowledge and understanding in Euclidean Geometry. Therefore, the reason for mixing methods is that qualitative findings complement the quantitative findings in order to increase the credibility of the quantitative findings (McMillan, 2012); and to seek elaboration, illustration and clarification of results from quantitative methodology with results from qualitative methodology (Greene et al., 2008). McMillan (2012) states that credibility refers to accuracy and trustworthiness of the data, data analysis and conclusions. Accurate and trustworthy findings are essential in this study; therefore, quantitative analysis is complemented with qualitative analysis.

Ponce and Pagan-Maldonado (2014) emphasise that the other objective of a mixed-methods research study is to show the philosophy of the researcher. They state that the action of research in applying the mixed-methods approach is pragmatic, meaning the product is more important than the process. Therefore, this study is a pragmatic mixed-methods research study based on the fact that the whole study moves across

inductive and deductive reasoning as well as from concrete to abstract reasoning or vice versa.

3.3.1 Model of mixed-method research approach

According to Ponce and Pagan-Maldonado (2014), the mixed-methods approach consists of two models: model 1 uses the qualitative and quantitative approaches in an unconnected way, integrating data at the end of the study to answer the research questions; while in model 2, qualitative and quantitative data are connected and the integration of the approaches occur in the philosophical positioning of the study, methodology and data analysis. The basic structure of a mixed-method methodology comprises sequential phases design and parallel or convergent phases design. In sequential phases two research methods are used separately, that is, the researcher begins with one method and then integrate the other at a later stage (Ponce & Pagan-Maldonado, 2014).

Types of sequential mixed-method design are classified as explanatory, exploratory, whereas concurrent convergent design, multilevel design, emergent designs, triangulation design and complementary or embedded mixed-method design are classified as parallel phase designs (Creswell & Plano Clark, 2007; Ponce & Pagan-Maldonado, 2014). This study is constructed as a parallel phase design, structured out of the complementary or embedded mixed-methods approach where the study is framed by the quantitative data. The qualitative data play a supportive, secondary role. Quantitative data collection is advanced through pre- and post-tests, whereas the qualitative data is collected through observations and semi-structured interviews during the intervention in experimental groups and during usual lessons in comparison groups. Capturing, analysis and interpretation in the study incorporate quantitative data through a quasi-experimental design approach and qualitative data through phenomenographic approach (observation and interviews) being embedded into the quantitative data. The embedded experimental model (Figure 3.2) as adapted from Creswell and Plano Clark (2007) is the research approach used in this study.

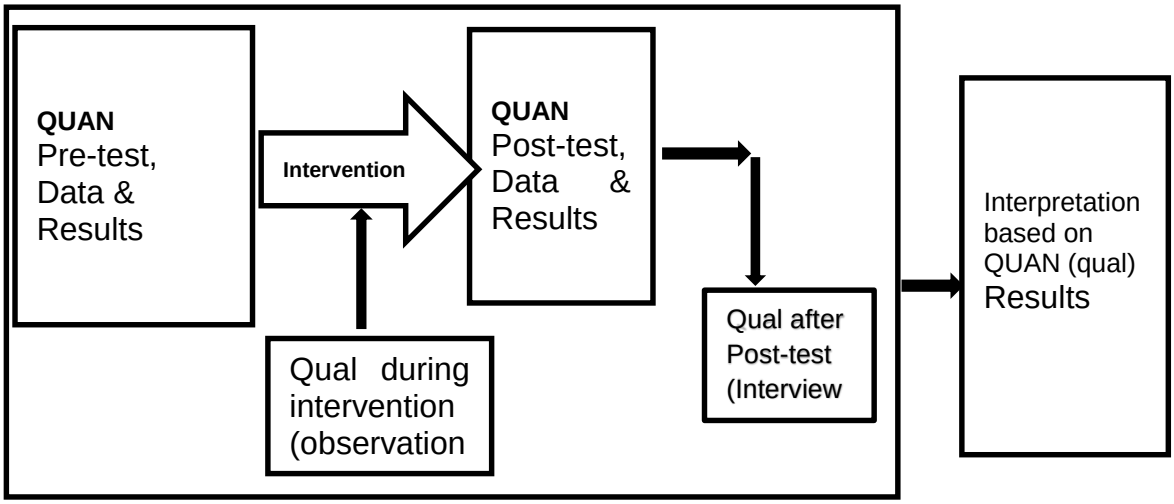


Figure 3.2: Research approach

Adapted from Creswell and Plano Clark (2007)

3.3.2 Population and sample

The population comprised the Grade 11 teachers and students of high schools in Tshwane North and West districts. Tshwane North district was more convenient as it is the most accessible area for the researcher, therefore, the area was marked as the experimental group. In any quasi-experimental design, a comparison group is necessary. Therefore, a district that is comparable to the experimental group was sought, and was found to be Tshwane West. Schools in the two districts, that is, Tshwane North and West are comparable in terms of biography, geographical standing and background factors. Three schools located in the Northern district served as the experimental group and three schools in the Western district as the comparison group. The two districts are about 30km apart. In any quantitative quasi-experimental approach, absolute control is not possible, therefore, the control group in this study is labelled as the comparison group.

According to Ponce and Pagan-Maldonado (2014) the two, dominant mixed-methods types of sampling are primary sampling and alternate sampling. Primary sampling was considered in this study based on the fact that the selected sample has been maintained up to the end of the study, and no additional sample were needed. Participants who were conveniently sampled consisted of six groups of Grade 11 students, that is, one classroom as one group from each of the six schools. Teachers teaching in the sampled classrooms were teacher participants in this research.

However, in order to maintain consistency in application of IBF on the side of experimental group, I maintained a teacher position in the three grade 11 classroom of the experimental group. Three teachers of the comparison group were participating by teaching applying the traditional axiomatic approach. The comparison group served for the purpose of contrast with the experimental group, an essential factor of quasi-experimental design. Therefore, there was no need for the comparison group to receive intervention and I could not play the dual role of teaching in both approaches. In addition, the traditional approach was seen to be ineffective, therefore, the effectiveness of IBF implemented as intervention this study is evaluated. Based on the fact that the success of intervention is measured through an achievement test, the differences in achievement in the pre and post-test results of the comparison and experimental group is crucial. The study does not compare teaching methods, but achievement test results to check the effect of the newly introduce IBF and IBL. Observation in the comparison group applied as a means to ensure that the topic circle geometry is taught through the traditional method before the students could take the post-test. Comparison of post-test results would not be valid if the comparison group did not receive any teaching, however, teaching approach in the comparison group could not be IBL, and the facilitator who facilitates IBL could not be the teacher who teaches the comparison group.

Cohen, Manion and Morrison (2005) refers to convenience sampling as accidental or opportunity sampling where the easily accessed group participates in the study. Convenience sampling applied in this study based on the reason that data collection occurred in a population that was conveniently available to participate in this research. Further, the application of convenience sampling contributed in attaining data and trends in this study without considering a randomised sample. A total of 18 students from the 166 were followed up as the study progressed. They were carefully observed, and follow-up interviews were conducted with them. Three teachers in the comparison group were observed and interviewed by the researcher. In experimental group, 107 students wrote the pre-test. However, only 97 participated in the post-test. Similarly, 97 students wrote a pre-test in the comparison group, while only 69 participated in the post-test. The aim of conducting pre- and post-tests was to evaluate the intervention. The evaluation of intervention is feasible if the improvement in the post-test is tracked against the pre-test. Therefore, a pre-test only would not be useful. Ten students in

the experimental group and 28 in the comparison group who did not write the post-test were excluded in reporting results. A high attrition rate was evident in C2 of the comparison group where 23 students in a class of 30 did not take the post-test. Students were absent from the class as research participants as a result of various factors such as absenteeism and involvement in other activities in the school. The statistics for students who participated in both pre- and post-tests are shown in Table 3.2.

Table 3.2: Number of students who participated in all research activities

Experimental group		
School	Number of students	Total
E1	33	97
E2	40	
E3	24	
Comparison group		
C1	31	69
C2	7	
C3	31	
	TOTAL	166

Students participated in pre- and post-test, observations and interviews. The student participants were grouped as the experimental and the comparison groups in order to determine the true effects of the intervention.

Schools in experimental and comparison groups were coded for anonymity. The schools in the experimental groups were coded as E1, E2 and E3 respectively. The letter E describes the category of the school which is experimental and the numbers 1 to 3 indicate the sequence of school visit. Schools in the comparison group were coded as C1, C2 and C3. The letter C denotes the category of the school in the comparison group and the numbers 1 to 3 indicate the sequence of visits in this category.

3.3.3 Measurement

Measurement is the assignment of numbers to indicate different values or variables. The purpose of measurement is to obtain information about the variables that are being studied. In this study, the dependent variable that is studied is learning. The success of intervention is measured through the differences in pre- and post-tests. Numbers are used to describe and differentiate achievement of students. The specific

technique or instrument used for measurement is an assessment task in a form of a test. A specific Euclidean Geometry content assessment task is used to provide a basis for measurement of problem-solving in Euclidean Geometry. The information collected through measurement forms the basis of the results, conclusions and significance of the study.

Measurement requires that variables be differentiated. The nature of differentiation can vary in four basic ways known as scales of measurement. The four scales of measurement are outlined by McMillan (2012) as: (1) nominal scale (numbers assigned to categories; for example, male=1; female=2); (2) ordinal scale (numbers rank-ordered, compared in terms of greater than or less than; for example, more creative, greater ability; (3) interval scale, that is, equal intervals between numbers; e.g. if J scores 90, U scores 80, and T scores 70 then the distance between them is twice the distance between U & J); and (4) ratio scale where numbers expressed as scores and ratios can be used in comparing and interpreting scores. The interval/ ratio scale applies in this study as the intervals between levels of knowledge and understanding have been measured in order to track the students' development from pre-visualisation to rigour levels of knowledge and understanding. Further, the ratio or interval scales in this study measure the difference in achievement before and after teaching has occurred in both experimental and comparison groups.

3.3.4 Research Methods

3.3.4.1 Quantitative component (Quasi-Experimental design)

The main purpose of doing experiments in research is to manipulate one variable (e.g., a teaching technique) to determine its effect on another variable (e.g. students on task behaviour in class) (Gall, Gall & Borg, 2010; Lodigo, Spaulding & Voegtle, 2010). This study puts more emphasis on manipulating the teaching technique (inquiry-based facilitation) as an independent variable to determine its effect on learning (problem-solving skills through IBL) as a dependent variable.

Gall et.al. (2010) define the three types of experimental methods as true experiment, quasi-experiment and single-case experiment. This study adopted a quasi-experimental design based on the fact that random assignments do not apply. Intact classrooms were used and participants did not have the option of being in a specific

group. Some classrooms were used for experimental purposes where some were used for comparison. Pre- and post-tests were administered to both experimental and comparison groups. I conducted intervention at the three grade 11classrooms in the experimental group by employing IBF strategies, and the researcher observed as a participant observer. Intervention was not employed at the three schools in the comparison group, that is, teachers were teaching as normal by applying the traditional axiomatic approach.

The facilitator who executes intervention in the experimental group cannot facilitate in the comparison group utilising a different method. The actual stage before intervention is that both groups were exposed to the traditional approach. Therefore, it is intervention that is new and need to be put on trial. Impartiality or biasness might be a result when one facilitator plays both roles of facilitating in IBL and in the traditional axiomatic approach. In order to see the effect of intervention at the experimental group, the comparison group has to proceed as normal. The only critical issue in the comparison group is that the topic circle geometry had to be taught before the students could participate in the post-test. Therefore, I was compelled to apply non-participant observations at the comparison group in order to confirm that the topic was taught before the students could participate in the post-test. Post –test evaluation could not proceed if the students were not taught the topic Circle geometry. I maintained stability and consistency in the three grade 11 classes of the experimental group. Similar data deductions were drawn from the three classes of the experimental group. Stability and consistency in terms of teaching and evaluation were also maintained in the three grade 11 classes of the comparison group. Similar data deductions were drawn from the three classes of the comparison group.

The experimental group students benefited from the research based on the fact that they participated and acquired IBL skills. However, to ensure that the comparison group benefit from the study as well, memoranda or feedback for tests and research results would be disseminated to teachers of both the experimental and the comparison groups at the end of the study.

Pre- and post-tests were conducted to measure the potential effects of an intervention by examining the differences between the pre-test and post-test results. Pre- and post-

tests were in a form of an individual assessment task based on the six levels of geometric knowledge and understanding, that is, pre-visualisation; visualisation; analysis; informal deduction, formal deduction and rigour. The main aim of the assessment task in this study was firstly to assess mathematical inquiry where students were expected to display the awareness of the connection between mathematical and extra-mathematical systems; to construct mathematical representations; and to search for structure, patterns, relationships and principal aim of generalisation; and secondly, the pre- and post-tests were aimed at assessing application of IBL to advance all levels of geometric knowledge and understanding.

An assessment task was administered by the researcher as both a pre- and post-test. According to Van Hiele's theory of geometric knowledge and understanding, students must achieve lower levels in order to successfully acquire knowledge and understanding in upper levels of knowledge and understanding. Therefore, test interpretations are based on how students compared with Van Hiele's levels of performance. The tests were based on the Grade 11 space and shape content area and the focus was Euclidean Geometry. The section concerned was circle geometry (cf Appendix B).

3.3.4.1.1 Inquiry-based assessment (pre- and post-test)

In implementing inquiry-based education (IBE), teaching strategies need to be coupled with the procedure and application of assessment. IBE in this study refers to application of IBF to promote IBL. Formative assessment was not practically executed in this study and the contributing barriers were mainly (1) large classes; (2) extensive curriculum; (3) the difficulty of meeting diverse students' needs; (4) formative assessment is very demanding when coming to resources; and (5) for the formative assessment to be practical, more time is needed (Bernholt, Ronnenbeck, Rophol, Koller & Parchmann, 2013). About summative assessment in mathematics inquiry, Bernholt et al. (2013) state that summative assessment emphasises constructed response or open-ended items as these items establish students' reasoning or problem-solving skills and their mathematical knowledge. Further, they highlight that, according to research, the links between formative and summative assessment could be strengthened in ways such as drawing on advances in the cognitive sciences; developing on-demand assessments; taking advantage of technology; developing

complementary diagnostic assessments for students at lower proficiency levels to identify specific learning difficulties; and ensuring that standards of validity, reliability, feasibility and equity are met. In this study, teaching (intervention) include informal assessment through questions that challenge students to think critically for problem-solving purposes. Probing through questioning that the facilitator adhered to during intervention influences inquiry and evaluates a certain level of knowledge and understanding of learning, and further determines progression to the next level of knowledge and understanding. The pre- and post-tests taken by students adhered to progression from lower levels to higher levels of knowledge and understanding.

Looney (2011) cited in Bernholt et al. (2013) argues that large-scale summative tests often do not reflect the development of higher-order skills such as problem-solving and reasoning which are key competences in IBE. The assessment task employed in this study is of a small scale so that it will reflect the development of higher-order skills such as problem-solving and reasoning. Further, Bernholt et al. (2013) agree that large-scale summative assessment data are often not detailed enough to diagnose individual students' needs and in some instances, they are not delivered in a time frame which enables students to receive feedback or demonstrate improvement at a later stage. The assessment task (pre- and post-test) in this study were aimed at assessing mathematical inquiry skills in learning and the impact of IBF in a form of intervention. Scoring or marking was done according to a memorandum and rating in pre- and post-tests was determined firstly based on the total mark according to the following categories of achievement: low achievement, LA (0-20 marks); average, AV (21-30 marks); and advanced, AD (31+ marks) (cf table 4.8). Secondly, rating is determined according to level of knowledge and understanding based on the following categories question not answered, low comprehension skills, average, advanced and proficient (cf table 3.3). The letter E appearing in each level or question, is used to explain that a student did not answer the question completely. Various levels possess diverse mark rating from low comprehension to proficient achievement levels.

Table 3.3 Rating the pre- and post-test

Level	Total	Did not answer the question	Low comprehension skills	Average	Advanced	Proficient
Pre-Visualisation	16	E	0	1-5	6-10	11-16
Visualisation	9	E	0	1-5	6-9	-------
Analysis	26	E	0	1-8	9-16	17 - 26
Informal deduction	13	E	0	1-4	5-8	9-13
Formal deduction	9	E	0	1-4	5-8	9-13
Rigour	3	E	0	1-3	--------	---------

Raw quantitative data was captured on SPSS and paired t-test was used to compare the variances and the means of the two groups. Quantitative data analysis in this study comprised both inferential statistical analysis and descriptive statistical analysis. The test for normality was performed by comparing the pre-test variances of the experimental and the comparison groups. The aim was to confirm the level of both groups before intervention and teaching in the comparison group. Further, inferential statistical analysis was used to compare the mean values of the two groups in order to track the most improved group. Descriptive statistical analysis employed charts and tables to illustrate the analysis of levels of knowledge and understanding, that is, pre-visualisation, visualisation, analysis, informal deduction, formal deduction and rigour. Charts were used to compare the pre- and post-test results of each group and comparison between the two groups was illustrated using charts to track the most improved group across the levels of knowledge and understanding.

3.3.4.1.2 Validity and reliability in the quantitative component

Students as participants completed cognitive tasks by responding to a standard set of questions; and the answers to the questions were summarised to obtain a numerical value that represented a cognitive characteristic; that is, scores were interpreted by comparison to a standard or criterion (Macmillan, 2012). Standardised achievement tests were used as pre and post-test to test knowledge and inquiry-based problem-solving skill in Euclidean Geometry (Macmillan, 2012). I have developed the achievement test based on the curriculum and focused on grade 11 content area

space and shape (cf Appendix B). Further the instrument was developed in line with Van Hiele's theory in order for it to cover all levels of understanding. In addition, the instrument was designed to address inquiry-based assessment that aligns with inquiry based learning. The validity and reliability of the instrument were tested in various ways, that is, validated through feedback from experts in research and mathematics education; conducting a pilot test on the instrument and through ensuring reliable error free scores.

Validity

Validity pertains to ensuring that the research instrument measures what it is designed to measure (Lodigo et.al, 2010). In addition, McMillan (2012) states that validity is a judgement of the appropriateness of a measure for the specific inferences or decisions that result from the scores generated by the measure. Further, McMillan (2012) maintains that validity is established by presenting evidence that the inferences are appropriate; and it is the inference that is valid or invalid not the measure. Lodigo et al. (2010) identify three types of validity, that is, (1) content validity to examine the degree to which an instrument measures the intended content area covered by the measure; (2) criterion validity which involves the correlation between two measures; and (3) construct validity involving a search for evidence that an instrument is accurately measuring an abstract trait or ability.

In the quantitative component, this study adhered to content validity. According to McMillan (2012), evidence based on test content or content validity demonstrates the extent to which the sample of questions in the instrument (pre- and post-tests in the study) are appropriately representative of the domain of content. Appendix B of this study shows the test according to six levels of knowledge and understanding.

In addition, McMillan (2012) states that, to accumulate evidence based on test content, experts are used to judge the instrument criticality and to evaluate the importance of various parts of the instrument. Furthermore, experts can judge whether the content is research-based, accurate, representative and adequate (McMillan, 2012). Content validity in the study was achieved by having experts in mathematics education and in research judge the instrument criticality and to evaluate the importance of various parts of the instrument, such as the Van Hiele levels of geometric knowledge and understanding. One teacher (the head of the mathematics department) was requested

to reviewed the instrument in order to ensure that the content was accurate and aligned with the Curriculum Assessment Policy Statement (CAPS) as the research was conducted in South African classroom where CAPS is practised. Two lecturers, one research professor and one mathematics education lecturer reviewed the instrument to check whether the instrument was research-based, adequate and representative for measuring the geometric knowledge and understanding of students based on the Van Hiele levels.

Further, validity was ensured in this study through internal and external validity. Internal validity was applied by ensuring that correspondence existed between the data collected and the research problem; and external validity applies as the data were applicable beyond the context of the study or in other samples that were not studied (Ponce & Pagan-Maldonado, 2015).

Pilot testing

A test was administered to a smaller group that did not take part in the main study. The school had only 16 grade 11 mathematics students, therefore, the group paticipated in only the pre-test as a pilot for the data collection instruments in a form of IBL test. The aim of piloting the test was to determine errors; whether the students can answer the questions correctly, whether the questions are oderly placed and whether the test will assist students to apply inquiry through deductive and inductive reasoning. The first instrument developed contained 5 tests, that is, a test on each level of knowledge. Therefore, the instruments did not address the issues mentioned, therefore, I developed another instrument that was piloted and taken the instrument back to experts for scrutiny. The second instrument (Appendix B) yielded desired results, that is, students completed the test at set time; could follow the order of questions and answer correctly and the test addressed both inductive and deductive levels of knowledge in a correct order based on six levels of knoweldge. All question assisted the student to utilised diverse levels of knowledge in resolving a central problem.

Reliability

Reliability is the consistency of scores or the extent to which participants and/or rate scores are free from error (McMillan, 2012). Further, McMillan (2012) outlined five

types of reliability estimates as: (1) stability (measured by giving the same instrument twice); (2) equivalence (correlation of two forms of the same test); (3) equivalence and stability estimate (two forms given at different times); (4) internal consistency (correlation of items measuring the same trait); and (5) agreement (existence of some type of co-efficient agreement, expressed as either a correlation or as percentage of agreement). This study applied stability, that is, test-retest reliability estimate. A set of questions as a pre-test was administered to the groups, and after a specified period of time, the same set of questions were administered as post-test. The connection of the pre- and post-tests was evaluated; and the consistency of the subjects' performance over time (before and after pre-test) was measured.

Gray (2009) states that another way of improving the reliability of a study is through the process of triangulation. He further maintains that, through triangulation, data can be interpreted from three or more different viewpoints in order to describe a particular area with accuracy. Triangulation to improve reliability was applied through the qualitative component of this study, that is, semi-structured observations and interviews. Trustworthiness was complied with in the qualitative component of this study.

3.3.4.2 The qualitative component (Phenomenological approach)

The objective of conducting the qualitative research in this study was to divulge concealed realities about both the IBF and the traditional teaching approach. Further, qualitative data play a significant role of complementing the quantitative data in this study. Phenomenographic approach was employed as a strategy of inquiry in the qualitative component of this study. Cohen, Manion and Morrison (2005) highlight that phenomenology is concerned with "the study of direct experience taken at face value; and one which sees behaviour as determined by the phenomena of experience rather than by external, objective and physically described reality" (p 23). Marton (1994) avers that phenomenography addresses what a phenomenon looks like and how is it seen. In addition, Marton (1994) motivated that in phenomenography experience is seen as an internal relation between the person and a phenomenon. In this study, social phenomenon, that is, teaching approach was explored and records were made from overt participatory and non-participatory observations as well as from semi-structured face-to-face interviews.

Observations and semi-structured interviews were employed to investigate the effect of IBL on learning in the experimental groups. Direct observation is explained as the type of observation that involves an observer collecting data while an individual is engaged in some form of behaviour or while an event is unfolding (Gall, et al., 2010). Participant observation applied in the study where the researcher applied the intervention of IBF while directly observing the students' interactions in all levels from pre-visualisation to rigour through IBL. I used an observation schedule (cf Appendix E) that defined each level from pre-visualisation to rigour of the levels of knowledge and understanding in Euclidean Geometry. In the comparison group, students' interactions were observed as the teachers applied the traditional axiomatic approach. The researcher was a non-participant observer in the comparison group and an observation scheduled that defined teacher and students' interactions was used. During the IBF intervention, actions were recorded on video camera and noted on the observation form. The participants' views were regarded as essential in this study, therefore, interviews were conducted, noted and recorded and transcribed. Selected students participated in semi-structured interviews. Verhoeven (2011) state that in semi-structured interviews, participants have freedom to contribute what they feel is relevant; however, I acted as the interviewer and utilised a list of questions that guides the interview session (cf Appendix F). Observations and the participants' views were compared to explain the performance of students in pre- and post-tests.

Trustworthiness in the qualitative component

Instead of referring to validity and reliability to establish the rigour of the qualitative data, the qualitative component in this study adhered to trustworthiness which consists of four components, that is, credibility, transferability; dependability and confirmability (Shenton, 2004). Firstly, to address the issue of credibility, the study presents a true picture of IBF and learning in Euclidean Geometry. Secondly, transferability has been addressed in this study by describing the research context in detail in order to enable readers of this study and other researchers to discern whether the environment of the study is similar to what is familiar to them and further to be able to justifiably apply the findings of this study in other settings. Thirdly, this study addresses confirmability by ensuring that the findings or results presented emerge from the data and not from the researcher's own predisposition. Finally, recommendations are made to enable continuation, further study or replicability. Shenton (2004) aligns validity and

trustworthiness as follows: credibility in preference to internal validity; transferability in preference to external validity or generalisability; dependability in preference to reliability; and confirmability in preference to objectivity. The alignment and combination of validity, reliability and trustworthiness was adhered to in order to ensure inference validity in this mixed-methods study.

3.3.4 Mixed-methods data analysis and interpretation

According to Ponce and Pagan-Maldonado (2014), analysing data means to extract meaning, implicit or explicit, from the information collected in the study. Further, they outline three methods of data analysis in mixed-method studies: (1) analysis of quantitative data, (2) qualitative data analysis and (3) analysis of mixed data. In addition, Ponce and Pagan-Maldonado (2014) state that the analysis of mixed data consists of organising and combining quantitative and qualitative data to achieve the following objectives in relation to the research topic: (1) triangulation of data; (2) complementing data; and (3) deepening the analysis. This study presents firstly the quantitative data analysis, secondly the analysis of qualitative data and thirdly a combination, that is, mixed-methods analysis. In the mixed-methods data analysis the qualitative data supports and complements the quantitative data. That is, the quantitative pre- and post-tests revealed the performance of the experimental and comparison group, whereas observations and interviews revealed causes of such performance. Quantitative and qualitative data were used to complement each other in presenting findings. The quantitative data determined the scope of the analysis and qualitative data deepened it through complementarity. Data analysis was conducted as in the embedded mixed-method design where the data was framed by the quantitative data, whereas the qualitative data provided a supportive secondary role (Creswell & Plano Clark, 2007). Accuracy and trustworthiness of study results were determined by embedding the qualitative data into the quantitative data.

3.3.4.1 Validity in this mixed-method research study

In order to ensure the investigative rigour of this study, validity and reliability in quantitative research was considered. Further, trustworthiness in qualitative research was also reflected upon. However, in this mixed-methods study, inference validity was applied in approaching and capturing the complexity of the research problem using quantitative and qualitative approaches (Ponce & Pagan-Maldonado, 2015).

According to Ponce and Pagan- Maldonado (2015), inference validity in a mixed-methods study implies that quantitative and qualitative data describe, explain or accurately capture the research problem and its complications.

3.4 DATA COLLECTION PROCESS

3.4.1 Pre-and post-tests

The principal instrument for data collection was a standardised (Euclidean Geometry) achievement test. Before the intervention, the researcher conducted a pre-test to both experimental and comparison groups in order to determine the participants' initial level of Euclidean Geometry knowledge and understanding in pre-visualisation, visualisation, analysis, informal deduction, formal deduction and rigour. A post-test was applied at the end of the intervention to observe change in the functionality of IBF strategies.

3.4.1.1 Experimental group

Three schools participated as an experimental group in District 1, and are labelled E1, E2, and E3 in this study. The pre-test was conducted and invigilated by the researcher in three schools classified as the experimental group. In all experimental group schools, the test was written during mathematics periods that were either in the morning or in the afternoon. Students could not be kept by the schools for after-school sessions; neither could the researcher. The total number of students who wrote the pre-test was 107 and the numbers per class at each school were as follows: E1 – 36; E2 – 40; E3 – 31.

Uniformity in schools was adhered to during the data collection process. The researcher facilitated IBL at the three schools comprising the experimental group. The IBL intervention was uniform in all schools, and mathematics periods were used for teaching circle geometry. This topic is part of the content area space and shape in Grade 11 in South Africa.

A post-test was conducted on a different day. The number of students who participated in post-test was less that the number in pre-test. An attrition rate of 9.3% was evident in the experimental group, that is, an unpredictable but controllable reduction of participants where 10 students did not participate in the post-test was evident. The

post-test in the experimental group was written by 97 students as 10 students did not participate in the post-test. Interviews were conducted after the post-test.

Experimental group intervention process

The total duration of intervention at each class of the experimental group encompassed two weeks where five days were classroom based, that is, 5 lessons of 1 hour each for facilitator and learner interaction. Other days were used for pre and post-test as well as interviews. The experimental group consisted of the three grade 11 groups from three schools, that is, E1, E2 and E3. I could not combine the three groups, therefore, I visited schools separately. Two full weeks, 10 days were reserved for experimental group visits. The schools in the experimental group are in close proximity, less than 5km away from each other, therefore, I managed to conduct a session of one hour at each school in a day. Cohesion was maintained by ensuring that same pace is maintained while dealing with the three groups separately. For example, introductions were done in one day separately at the three schools. Pre-tests were conducted in one day, one school after the other. All lessons were run concurrently, for example, lesson 3 could not proceed in one group while the other groups are still behind in lesson 2. After all groups participated in lesson 1, lesson 2 was carried out. Lessons followed each other until the fifth and last lesson was carried out in all the three experimental groups. I have followed a lesson plan from lesson 1 to 5. The lesson plan I have planned to teach in the experimental groups appear as Appendix A of this study. Intervention was conducted by me according to the schedule below in each of the three classes of the experimental group.

Day	Activities
1	• Introduction and request for approval from the principal. • Request for consent from teachers • Planning with the teacher involved. Checking time table to mark mathematics periods and do arrangements with other teachers where necessary. In the experimental group I have been facilitating, the teacher's role in each group was to connect me with the students in particular classes in every session.
2	• Class visit • Introducing myself to the students, explaining the rationale behind my being in their classroom. • Distribute consent forms for parents

3	• Receive consent forms signed by parents, request student to assent to participate by signing the assent form. • Students write a pre-test
4	Lesson 1 • The attempt in this lesson was to orientate students to the pre-visualisation stage • I facilitated through probing to engage students in recalling all information in their real world for learning in pre-visualisation stage • Further, I allowed student to manipulate a tangram in order explore the different shapes of a tangram.
5	Lesson 2 • My facilitation role in this lesson was probing to guide students in the visualisation level. Teacher-student and student-student interaction enabled students to apply free hand drawing and use symbols to represent objects outlined and explored in the pre-visualisation stage.
6	Lesson 3 • In this lesson I facilitated through probing and engaged students in the analysis level of knowledge and understanding. Students in this stage interacted to explore properties of shapes such as different types of quadrilaerals, circle, types of triangles. Other polygons such as pentagon, hexagon and octagon were also explored. The properties of separate figures were explored. Constructon of diverse figures was advanced and understanding of concepts developed around figures developed. For example, chord, diameter, diagonals bisect and more other concepts. • I allowed students to use different text books for reference and their smartphones (taking advantage of the wifi present at their schools) to search for concepts and properties in order to explore the properties of each figure.
7	Lesson 4 • In this lesson, I probed further to guide students to explore relationships of figures and relations of properties of various figures. In exploring relations few theorems were realised. I allowed students to use their cell phones and different textbooks at their exposure to explore theorems based on the concepts and properties of figures they have explored. Further, the concept of inscription of figures arised. Inscription of figures based on concepts such as circumcircle of a polygon; a square circumscribed about a circle, circumscribed circle in a triangle formed by tangents to a circle. Further, the idea of inscribing

	a quadrilateral in a cycle arised. That is where students realised the origin of a cyclic quadrilateral.
8	Lesson 5 • In this lesson, I probed to let students dwell deeper into the properties of a cyclic quadrilateral in order to discover theorems that describe a cyclic quadrilateral, a theme in circle geometry according to grade 11 syllabi. • For application of properties, concepts and theorems, I probed in order to guide students to apply in abstract problems as prescribed in their grade. • In addition, I engaged students in discussions pertaining to application of inscriptions in real world and real life situations. For example, in construction.
9	Post-test I conducted a post-test in all the three classes of the experimental group.
10	Interviews (only students in the experimental group) Interviews took place after post-test.

Teachers in the experimental group did not participate in teaching or doing any other activity during my presence in their classrooms, therefore, they could not participate as interviewees. While providing consent to participate, teachers in the experimental group indicated that they were not ready to teach utilising any new method contrary to their then practice, therefore, training the teachers to implement IBF did not materialise. In light of this. I continued with facilitation of IBF to engage students in IBL in all the three experimental groups.

Fidelity of intervention

Discussion of fidelity, that is, the extent to which the intervention is delivered as desired, is based on five components of intervention fidelity by Gutman and Murphy (2012). The five components are intervention design; training of providers; intervention delivery; receipt of intervention and inactmant of skills acquired from intervention. Intervention design is relevant based on the following facts: comparison group was used; number of intervention sessions in each school are outlined and were adhered to. Further, the intervention was conducted as desired based on the six levels of knowledge and understanding that constitute the theoretical framework of this study.

The skill of facilitation through probing as the major constituent of IBF for intervention was adhered to in all levels of facilitation. In line with the training of providers, I conducted the intervention to circumvent the inconsistencies that could be brought about by facilitation that is carried out by three different people and further to avoid lack of proper acquisition of IBF skills among the three teachers in the experimental group.

Intervention fidelity in line with intervention delivery pertains to methods used to ensure standardisation of the intervention. A common schedule was adhered to in three classes of the experimental group for the same duration. In addition, same pace and consistency of facilitation in lessons was adhered to in all three groups. Content delivered was based on the current grade 11 syllabi and the same content was observed at the comparison group classes for controlling differences during comparison. The receipt of intervention is accounted for in this study based on the fact that participants in the intervention group acquired knowledge in various steps and were able to apply in the next step. For example, students were able to contexualise the shapes and figures through out from the pre-visualisation level to rigour. Assessment recorded improvement, and student in the experimental group expressed their positive views about intervention as a relevant method of learning Euclidean geometry and as a method that can impact positively in every day learning. In essence, the intervention has shown that IBF promote IBL in order to enhance problem solvng skills in Euclidean geometry. I focused on the comparison group for a complete research process.

3.4.1.2 Comparison group

The pre-test was conducted at three schools (C1, C2 and C3) during normal mathematics periods. A total of 97 students wrote the pre-test. The numbers of students who participated per class in the pre-test in each school were as follows: C1 – 34; C2 – 32, C3- 31. All students in the comparison group participated in the pre-test, However, an attrition rate of 28.9% was evident, that is, reduction of participants was experienced where 28 students did not participate in the post-test. Therefore, 69 students of the comparison group participated in the post-test. Contributory factors are explained in section 3.3.2 of this study. The higher attrition rate contributed to reduction of participants from 204 to 166 when both groups are combined; and from

97 to 69 in the comparison group. Three teachers in the comparison group were observed while teaching circle geometry and cyclic quadrilaterals as usual. The researcher conducted non-participatory observations. Lesson observations were conducted at all comparison group schools after the pre-test had been written. Five lessons of 1 hour each were observed on five separate days. The lessons covered a number of theorems including cyclic quadrilaterals. Nine students and three teachers were interviewed in the comparison group.

The group is about 30km from the experimental group. What matters the most in the comparison group is the results of the pre and post-test in order to compare against the control group and the confirmation that circle geometry upto only cyclic quadrilaterals were taught in a traditional approach of teaching. Further, I visited schools to ensure that teaching in circle geometry happened before students participated in the post-test, as students could not take a post-test when they did not receive any teaching on the topic. The schools were in close proximity, therefore, I could attend a session in each school in one day, that is, three sessions in different groups of the comparison group. The total duration of my interaction in the comparison group was two full weeks, that is, 10 days. I have observed the teacher student activities as a non-participant onserver. Theorems about circle geometry were taught as expected in all the groups. Five days of observation were classroom based. The theorems and circle geometry topics concerned were not treated concurrently, however, all topics in cirle geometry were treated during the period I have observed the comparison groups. I followed the schedule below in engaging the comparison group.

Day	Activities
1	• Introduction and request for approval from the principal. • Request for consent from teachers • Planning with three teachers involved. Checking time tables to mark mathematics periods and do arrangements with other teachers where necessary. The teachers'role in the comparison group was to teach and control the class while I only observe lesson as a non-participant observer in all lessons.
2	• Class visit • Introducing myself to the students, explaining the rationale behind my being in their classroom.

	Distribute consent forms for parents
3	• Receive consent forms signed by parents, request student to assent to participate by signing the assent form. • Students write a pre-test
4	Lesson 1 Theorem 1 Centre of a cicle perpendicular to chord
5	Lesson 2 Theorem 2 & 3 Angles subtended by an arc at the circumference
6	Lesson 3 Theorem 4 Angles subtended by chord and diameter at the circumference
7	Lesson 4 Theorem 5 cyclic quadrilaterals – supplementary angles are equal
8	Lesson 5 Theorem 6 cyclic quadrilaterals- exterior angle = interiror opposite angle
9	Post-test I conducted a post-test in all the three classes of the comparison group after observing the lassons.
10	Interviews (students and teachers in the comparison group) Teachers in the coparison group were directly involved in teaching, therefore, they were interviewed Interviews took place after post-test.

3.4.2 Interviews

Interview in this study is grounded in a phenomenological research approach concerned with the meaning that people assign to phenomenon (Gray, 2009). Further, Gray (2009) motivates that interview "can be used as a means of gathering information about a persons' knowledge, values, preferences and attitudes (p 370). In light of Gray's (2009) ideas on the use of interviews in data collection, in this study, interviews were applied to gather information about the respondents' experiences, attitudes and preferences towards their daily teaching and learning practices. The researcher applied semi-structured face-to-face interviews to probe for clarity and to get more detailed responses from respondents who were comparison group teachers and students in both the experimental and the comparison groups. Further, semi-structured interviews were conducted after intervention and post-test in order to elaborate further, enhance, illustrate and clarify the results of pre- and post-tests. The qualitative data was embedded into the quantitative data for credibility purposes or to increase the integrity of the quantitative findings. The interview schedule of 10 standardised open-ended questions was drafted based on the research questions, hypothesis and the theoretical framework. Three students in each of the six schools were interviewed in a group by the researcher, while three teachers of the comparison group were also interviewed individually. The students' interview lasted for 30 minutes in each group in all six schools and the teachers individual interviews lasted 20 minutes. Respondents were interviewed at their respective schools after the post-test. Maximum participation was attained and almost all questions were answered by respondents. The initial plan was to select the students based on their performance in pre-tests. Classrooms had large numbers and the marking of pre-tests took a long time. Therefore, the researcher requested the teachers to identify the following students: one student whose performance was good, one whose performance was average and one whose performance was low. The students' participation was monitored during participatory observation in experimental group and during non-participatory observation in comparison group. A total of 18 students participated in interviews, that is, three students per school in both the experimental and the comparison group.

3.4.3 Observations

Participatory and non-participatory observations comprised an essential part of this study. Overt observations were conducted purported at “gathering ‘live’ data from ‘live’ situations” (Cohen, Manion and Morrison, 2005 p 305). Furthermore, the purpose of observational data in this study was for the researcher to move beyond opinion-based data obtained in interviews and to discover what respondents could not talk about in interviews (Cohen, Manion & Morrison, 2005). In addition, as suggested by Cohen, Manion and Morrison (2005), observation enabled the researcher in this study to gather data on human setting, physical setting, interactional setting and programme setting.

Comparison group teachers and students were observed by the researcher (who was not participating in teaching and learning activities) to ensure the teachers taught Euclidean Geometry, specifically circle geometry including the topic cyclic quadrilaterals. Furthermore, non-participatory observation in the comparison group was employed to confirm that the traditional axiomatic approach was the strategy applied during teaching. However, participatory observations were conducted in the experimental group. I participated in all three schools of the experimental group as the IBF facilitator while observing the students’ behaviour, attitude and level of participation in learning through IBL.

3.5 DATA ANALYSIS

3.5.1 Quantitative data analysis

According to McMillan (2008), a test is an instrument that requires participants to complete a cognitive task by responding to a standard set of questions. The answers to the questions are marked to obtain a numerical value that represents the cognitive level of the participants. All tests measure performance at the time a test is given. Furthermore, McMillan (2008) explains that tests differ, and the differentiating characteristics are classified into norm-referenced; criterion-based (standard based) and self-referenced. Self-referencing means that the reference point is the student’s own work; that is, comparison of the student’s achievements with what the student has done before. Criterion-referenced (standards-based) means comparing an individual’s

performance against an established level of performance or skill. The result is reported as a percentage of items answered correctly or as falling within categories such as pass, proficient, and advanced. Norm-referencing refers in this study to a situation where individual scores are compared to the scores of well-defined (reference group) of others who have taken the same test.

In this study, pre- and post-test were given in order to track the students' performance before and after the intervention. Criterion referencing is applied in this study because analysis of students' scores show how they compare to the following skills: mathematical inquiry and application of IBE to advance all levels of geometric knowledge and understanding. Further, criterion referencing is applied through interpreting scores by classifying the marks against a standard. The standards used for are low achievement, average and advanced.

Students belong to either experimental or comparison group. The group achievement of the experimental group was compared with the group achievement of the comparison group. Intervention was measured based on comparison of the test scores for experimental and comparison groups. Test or assessment task administered was designed in a way that questions are in an order from pre-visualisation to rigour. All questions were based on a central figure that constituted a problem. Students were guided through questions to apply all levels of knowledge and understanding in solving the problem posed (cf Appendix B). The achievement test administered was meant to evaluate students' inquiry skills in solving a problem through application of six levels of knowledge and understanding. The test was not focused on testing abstract knowledge, but in assisting students to develop concrete inductive knowledge in order to apply in abstract levels; or in order to assist students to advance abstract knowledge with ease and deeper understanding of concepts that are basic and fundamental to abstract thinking. Therefore, norm referencing applies in this study. Pre-visualisation was addressed by question 1.1. Question 1.2 represented visualisation whereas question 1.3 and 1.4.1 addressed the analysis stage of knowledge and understanding. Informal deduction was addressed by questions 1.4.2 to 1.6. Formal deduction was represented by questions 1.7 to 1.9. The last question which was question 1.10 addressed rigour where students needed to create a network of mathematical and extra-mathematical systems. The same test was administered as both pre- and post-tests.

Paired t-test statistical analysis was used to fairly compare the results of the experimental and comparison groups. T-tests were used to test the difference between two related variables, that is, pre- and post-test in the experimental and comparison group.

3.5.2 Qualitative data analysis

The analysis of qualitative data in this study adhered to the principles of content analysis. Trochim and Donelly (2008) describe three types of content analysis, classified according to purpose: conventional content analysis, directed content analysis and summative content analysis. This study employed directed content analysis. Directed content analysis is driven by a specific theory or model. In this study, qualitative data analysis complements the quantitative data analysis, based on Van Hiele's theory of geometric knowledge and understanding, coupled with Dewey's theory of inquiry. A systematic procedure of qualitative content analysis in this study followed Schilling's (2006) qualitative content analysis spiral as a model of analysis. Schilling's (2006) model of data analysis consists of four levels as shown in figure 3.3.

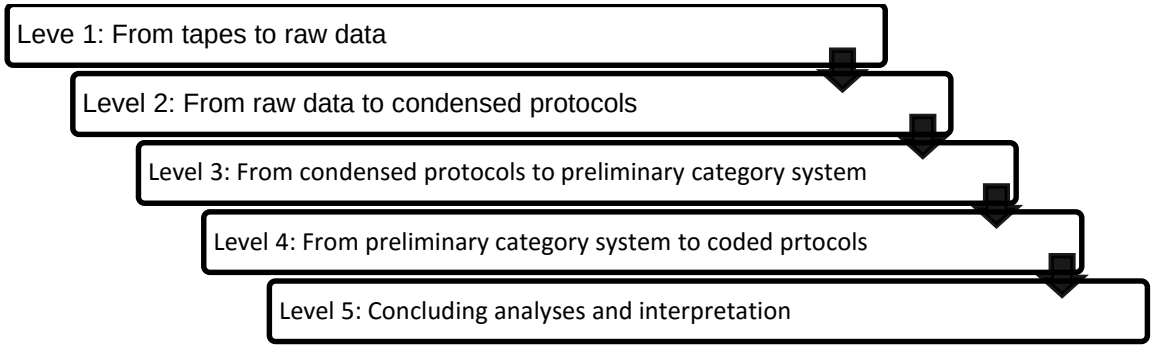


Figure 3.3: Schilling's model of content analysis, adapted from Trochim and Donelly (2008)

3.6 ETHICAL ISSUES

In order to adhere to research ethics, researchers require permission to collect data from individuals and sites. Creswell and Plano Clark (2007) indicate that individuals are those people who are in charge of research sites, people providing the data (teachers, students and their representatives such as parents) and campus-based institutional review boards. In light of this, ethical clearance was requested from the College of Science, Engineering and Technology's research and ethics committee of

the University of South Africa (UNISA). Further, DBE in South Africa Gauteng province was contacted to ask for permission to collect data at schools in Tshwane North and West districts and permission was granted. Approval was sought from principals of the participating schools to do research at the respective schools. Prior to data collection or engaging with participants, a clear explanation of the research purpose and procedures was given to the teachers and students as participants. English as the language of instruction in mathematics was used as the main language. However, code-switching was adhered to where necessary. Translation into the mother tongue, that is, Setswana language of the teachers and students was done by the researcher where there was a need in order to afford every participant a chance to understand expectations and their role in the study. Consent forms were provided to teachers in both experimental and comparison groups. Parents of students who participated also received consent forms, and assent forms were given to students. Each participant was requested to sign a consent form (assent form for students), and the following factors were clarified with participants: participants had a right to withdraw or refuse to participate if they so wished; anonymity and confidentiality would be considered when publishing the findings; and the findings might be published for academic purposes where applicable.

3.7 CHAPTER SUMMARY

This study is grounded in the pluralism paradigm which combines realism and constructivism. The main emphasis of this study is the pragmatist epistemology that maintains that facts exist and can be interpreted, and the ontology is based on the fact that pragmatist knowledge exists as an objective reality which can be interpreted subjectively. In this study, a pragmatic approach was adopted by using IBF as an intervention strategy to assist students to apply IBL in acquiring Euclidean Geometry knowledge and understanding.

The mixed-methods study is described, and a null hypothesis is explained. Sub-questions were used to answer the main question. Pre- and post-test measurements are explained and the qualitative data collection process through observations and interviews, is clearly elaborated on. The mixed-methods approach, that is, the integration of quantitative methodology (quasi-experimental design) and qualitative methodology (phenomenographic approach), is clarified.

CHAPTER 4
PRESENTATION OF RESULTS

4.1 INTRODUCTION

Results from data are presented in this chapter. Presentation of results focuses on firstly determining the impact of the model that employs IBF on students' development of higher levels of geometric thinking. In order to determine the impact of IBF on students' development, descriptive statistics are used to analyse the pre- and post-test results in order to track improvement of the experimental group and that of the comparison group. Description of the developmental patterns is done in terms of the geometric levels of knowledge and understanding and ultimately the most improved group is determined. Charts (bar and line) are used to compare the developmental patterns of the two groups. The extent of improvement and lack of improvement is measured by checking and comparing the average performance of both the experimental and the comparison groups.

Secondly, the results are presented to show the influence of IBL on students' acquisition of knowledge at formal deduction level through pre-visualisation, visualisation, analysis and informal deduction. Inferential statistics, that is, paired t-test statistical test, are used to analyse the pre- and post-test results to determine if a difference exist between the experimental and comparison group. Paired t-test statistical measure is used to determine the improvement rate between the pre- and post-test in each group. The variances and the mean values between the two groups are compared.

The third objective, that is, determining that IBF can divulge how students learn Euclidean Geometry better, is reported through results from participatory and non-participatory observations as well as from interviews. Two visual models of both the experimental and the comparison group are presented in order to present the results from data about how students learn Euclidean Geometry better.

Integrated findings of the mixed- methods design are presented, quantitative findings are presented as statistical and descriptive data while an analytical report on interviews and observations is also presented as qualitative findings.

4.2 QUANTITATIVE FINDINGS

4.2.1 Paired t-test statistical test

Paired t-test statistical test was performed on pre and post-test results of 97 students of the experimental group and 69 students of the comparison group. However, due to the SPSS system or user missing values, the number that participated as recorded on SPSS generated tables appear as 68 for the comparison group and in some instances 96 for the experimental group. Table 4.1 presents the results of the variables for pre-test between the experimental and the comparison groups.

Table 4.1: Paired t-test Variance test: Variable – Pre-test

Group	N	Mean	Std Dev	Std Err	Minimum	Maximum
Control	68	7.4265	6.4004	0.7762	0	27.0000
Experimental	97	8.4845	5.7302	0.5818	1.0000	27.0000
Diff (1-2)		-1.0581	6.0147	0.9513		

Group	Method	Mean	95% CL Mean		Std Dev	95% CL Std Dev	
Control		7.4265	5.8773	8.9757	6.4004	5.4763	7.7025
Experimental		8.4845	7.3296	9.6394	5.7302	5.0217	6.6734
Diff (1-2)	Pooled	-1.0581	-2.9365	0.8204	6.0147	5.4267	6.7469
Diff (1-2)	Satterthwaite	-1.0581	-2.9766	0.8605			

Method	Variances	DF	t Value	Pr > |t|
Pooled	Equal	163	-1.11	0.2677
Satterthwaite	Unequal	133.94	-1.09	0.2773

Equality of Variances				
Method	Num DF	Den DF	F Value	Pr > F
Folded F	67	96	1.25	0.3181

Variances between the pre-tests in the two groups were tested, with the null hypothesis that the variances of the experimental and comparison group pre-test are not equal. The pooled method has been used in testing the hypothesis. Table 4.1 reveals that the p-value for the pre-tests is not significant ($p > 0.05$, that is, 0.318). Therefore, the null hypothesis that the variances between the two groups are not equal fails to be rejected.

Further, paired t-test statistical test is used in this study to compare the post-test variances of the two groups (Table 4.2).

Table 4.2: Paired t-test Variance test: Variable – Post-test

Group	Method	Mean	95% CL Mean		Std Dev	95% CL Std Dev	
Control		7.7681	6.1351	9.4011	6.7978	5.8225	8.1686
Experimental		16.2577	14.4261	18.0893	9.0879	7.9642	10.5836
Diff (1-2)	Pooled	-8.4896	-11.0445	-5.9347	8.2162	7.4150	9.2129
Diff (1-2)	Satterthwaite	-8.4896	-10.9250	-6.0543			

Method	Variances	DF	t Value	Pr > |t|
Pooled	Equal	164	-6.56	<.0001
Satterthwaite	Unequal	163.56	-6.88	<.0001

Equality of Variances				
Method	Num DF	Den DF	F Value	Pr > F
Folded F	96	68	1.79	0.0120

The variances between the post-test of the experimental and the control groups were tested, with the null hypothesis that the variances of the post-tests for the two groups are equal. Satterthwaite test has been used to test the null hypothesis. Table 4.2 show that p-value is significant ($p<0.05$), the null hypothesis that the means are equal for post-test scores for the control and experimental group is rejected.

The paired t-test revealed a difference between the pre- and post-test in both experimental and the comparison group (Figure 4.1). However, in the experimental group most individuals scored higher on the post-test as compared to the comparison group. Paired t-test results for groups are shown in Table 4.3.

Table 4.3 Paired t-test results for the experimental group

Variable	Label	Mean	Std Dev	Minimum	Maximum
PRE_TEST	PRE-TEST	8.48	5.73	1.00	27.00
POST_TEST	POST-TEST	16.26	9.09	1.00	41.00

Table 4.3 provides the statistical analysis of the post- and pre-test mean scores and the standard deviation for the experimental group. The mean of the pre-test is 8.48 and the mean scores range between 1 and 27, while the mean scores for post-test scores range between 1 and 41 with the mean of 16.26. In comparing the pre- and post-test results of the experimental group, the null hypothesis that there is no difference between the pre- and post-test (H_0: $\mu_{Pre\_test} - \mu_{Post\_test = 0}$) was tested.

Table 4.4 displays the mean difference between the pre- and post-test in the experimental group as -7.7732 with standard deviation =7.0379.

Table 4.4: Mean difference between pre- and post-test of the experimental group

N	Mean	Std Dev	Std Err	Minimum	Maximum
97	-7.7732	7.0379	0.7146	-28.0000	10.0000

Mean	95% CL Mean		Std Dev	95% CL Std Dev	
-7.7732	-9.1916	-6.3548	7.0379	6.1677	8.1962

DF	t Value	Pr > |t|
96	-10.88	<.0001

The p-value =0 .0001< 0.05 (significance level), we reject the null hypothesis and conclude that the pre- and the post-test mean scores are statistically different. Agreement between the pre- and post-test results of the experimental group is shown on the scatter plot in Figure 4.1.

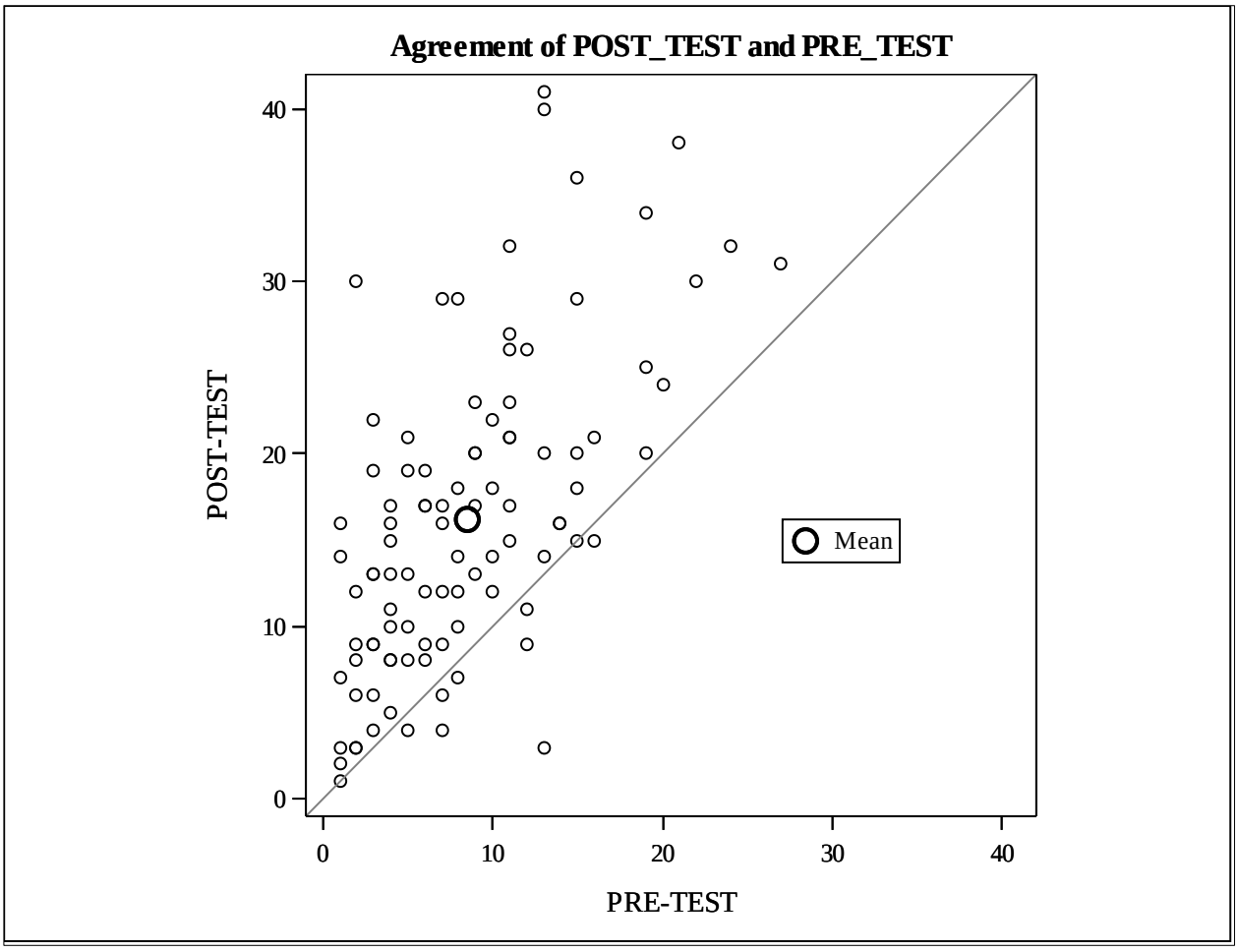


Figure 4.1 Agreement between pre- and post-test in the experimental group

It is evident from the chart that most of the student in the experimental group scored higher on the post-test compared to the pre-test at the following effect size:

$$r = \sqrt{\frac{t^2}{t^2+df}} = \sqrt{\frac{(10.88)^2}{(10.88)^2+96}} = 0.55$$

Small effect is signified by r= 0.1, medium effect is shown by r= 0.3 and a large effect is signified by r= 0.5. Therefore, the significance value is recorded as p=0.0001, p<0.05 at a larger effect of 0.55.

In the comparison group, the performance in the pre- and post-test was almost the same. Paired t-test results for the comparison group are shown in Table 4.5 below.

Table 4.5: Paired t-test results of the comparison group

Variable	Label	Mean	Std Dev	Minimum	Maximum
PRE_TEST	PRE-TEST	7.43	6.40	0.00	27.00
POST_TEST	POST-TEST	7.77	6.80	0.00	25.00

Table 4.5 provides the descriptive statistics of the post and pre-test mean scores and the standard deviation for the comparison group. The mean of the pre-test is 7.43 and the mean scores range between 0 and 27 while the mean scores for the post-test range between 0 and 25 with the mean 7.77. The null hypothesis tested about the pre- and post-test of the comparison group, was that there is no difference between the pre- and the post-test scores (H_0: $\mu_{Pre\_test} - \mu_{Post\_test = 0}$).

The tables below record the mean difference between the pre- and post-test results in the comparison group as - 0.02206 with the standard deviation =3.6522.

Table 4.6: Mean difference between pre- and post-test of the comparison group

N	Mean	Std Dev	Std Err	Minimum	Maximum
68	-0.2206	3.6522	0.4429	-10.0000	10.0000

Mean	95% CL Mean		Std Dev	95% CL Std Dev	
-0.2206	-1.1046	0.6634	3.6522	3.1249	4.3953

DF	t Value	Pr > |t|
67	-0.50	0.6201

The p-value =0.6201 > 0.05 (significance level) suggesting not-significant results. therefore, the null hypothesis fails to be rejected and the conclusion is that in the comparison groups the means for the pre- and post-test are not statistically different.

Agreement between pre- and post-test results in the comparison group is shown on the scatter plot (Figure 4.2) below.

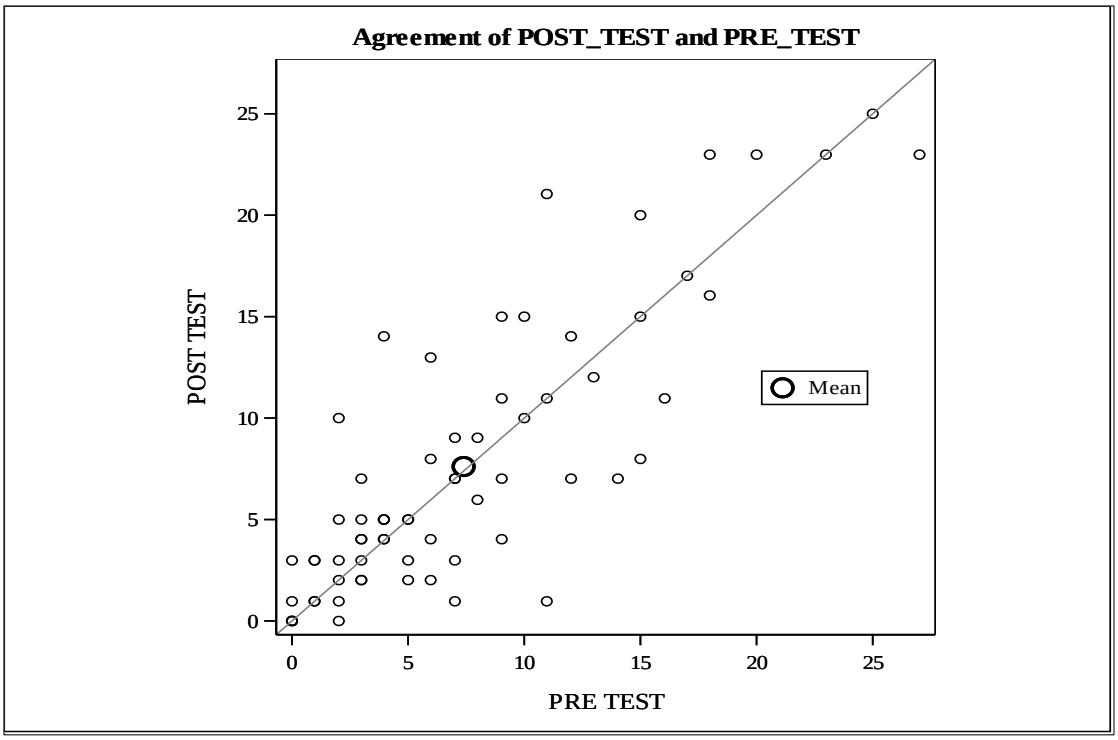


Figure 4.2: Agreement between pre- and post-test results of the comparison group

The chart shows an even distribution between the pre- and post-test results. Further, it is indicated in the scatter plot that there is not much difference in performance between the pre- and post-test results of the comparison group.

4.2.2 Summary of quantitative paired t-test findings

The normality test carried out by comparing the variances through the pooled standard error method, assumed that the variances of the pre-test between the experimental and the control were not significantly different; that is, they were equivalent ($p=0.2677$, that is, $p> 0.05$). However, verification was done through the Satterthwaite approximation of standard error which also revealed that the variances of the pre-test between the two groups were not significantly different ($p=0.2773$, that is, $p> 0.05$). Generally, the equality of variances was not significant at $p=0.3181$, that is, $p> 0.05$. Further, the pooled standard error method and the Satterthwaite approximation of standard error revealed that the variances of the post-test between the two groups was significantly different at $p= 0.0001$, $p< 0.05$. The paired t-test statistical tests indicate that the mean values for the pre- and post-test, when comparing the

experimental and the comaparison group, were statistically different. Higher scores were apparent in the post-test for experimental group. In comparison, the t-test statistical test proved that in the comparison group the mean values of the pre- and post-test were not statistically different. An improvement was not recorded between the comparison group pre- and post-test scores. The conclusion is that there is a significant difference between the experimental and the comparison group. Improvement was evident in the experimental group while the comparison group reflected no improvement. Quantitative data revealed that on average, participants that received intervention through IBF acquired IBL skills and improved (M= -7.773, SE= 0.7146) as opposed to those who did not receive the intervention (M= -0.221, SE = 0.4429). This difference (-7.547), 95% CI (-8.08, 5.69), was significant at t (10.88), p = 0.0001, $p<0.05$ and represented a large effect size of 0.55. The large effect size emphasises that IBF contributed significantly towards improvement of IBL and that the framework contributed by this study can be considered as a framework of IBF in Euclidean Geometry.

4.3 QUANTITATIVE DESCRIPTIVE STATISTICAL ANALYSIS

Descriptive statistical analysis employs charts and pecentages based on raw marks to report on pre and post-test results of the experimental and the comparison groups. Table

Table 4.7: Raw marks obtained by students

		Frequency (f) Experimental group		Frequency (f) Comparison group	
Marks Interval	Midpoint of interval	Pre-test	Post – test	Pre-test	Post - test
0-10	5	64	30	51	48
11-20	15.5	29	41	13	15
21 - 30	25.5	4	17	5	6
31 - 40	35.5	---	8	---	---
41 - 50	45.5	---	1	---	---
51 - 60	55.5	---	---	---	---
61 – 70	65.5	----	---	----	---
71 - 80	75.5	---	---	---	---
		97	97	69	69

4.3.1 Comparison of the experimental and the comparison group according to achievement levels

The students' achievement in the pre- and post-test were classified as Low Achievement, LA (0-20 marks); Average, AV (21-30 marks); and Advanced, AD (31+ marks).

The students' scores for the pre- and post-tests are shown in Table 4.8.

Table 4.8: Marks according to achievement levels

Groups (n=166)	Achievement category	Pre-test	Post-test
Experimental group (n=97)	LA	93 (95.9%)	71 (73.2%)
	AV	(4.1%)	17 (17.5%)
	AD	0 (0%)	9 (9.3%)
Comparison group (n=69)	LA	64 (92.8%)	63 (91.3%)
	AV	5 (7.2%)	6 (8.7%)
	AD	0 (0%)	0 (0%)

Table 4.8 reveals that in both groups there was a decrease in low performance in the post-test. In the experimental group, the number of students who performed in LA level students is 93, that is, 95.9% of students and the number decreased to 71 (73.2%) in the post-test. In the comparison group, 64 (92.8%) students are at LA level, and the number decreased to 63 (91.3%) in the post-test. A decrease rate of 22.6% in the experimental group was recorded against 1.5% in the comparison group. An improvement was recorded in both the experimental and the comparison groups for students who achieved an average score. In the experimental group, there were 4 (4.1%) students in the pre-test, while the post-test recorded 17 (17.5%) students who achieved at AV level. On the average achievement, the experimental group showed an increase rate of 13.4% against the 1.5% in the comparison group.

Considering students in the advanced category, the experimental group showed an increase. The experimental group recorded 0% of students who achieved at the advanced level in the pre-test, and the achievement increased to 9.3% after the intervention. By contrast, in the comparison group, achievement at the advanced level remained constant at 0% achievement in both tests. The differences between the pre- and post-test across levels of achievement in both groups were calculated in order to determine the decrease percentage in the LA level and the increase percentage in

both the AV and AD levels of achievement. Table 4.9 shows the decrease and increase percentages of the levels of achievement in the pre- and post-test for the experimental and the comparison group.

Table 4.9: Decrease and increase percentage in achievement levels

Group	Level	Decrease/increase percentage
Experimental group	LA - Decrease	22.6%
	AV – Increase	13.4%
	AD – Increase	9.3%
Comparison group	LA – Decrease	1.5%
	AV – Increase	1.5%
	AD – Increase	00%

Looking at table 4.9, a decrease in lower level of achievement and an increase in higher levels imply that the numbers in the lower levels were distributed to higher levels. It implies that 22.6% of students who were in the lower achievement category improved and achieved better that the lower level. The decrease affected the other levels and the number of those who performed better in categories such as average and advanced levels inceased. Graphical representation for the comparison of the pre- and post-test results as in Table 4.9 is shown on Figure 4.3.

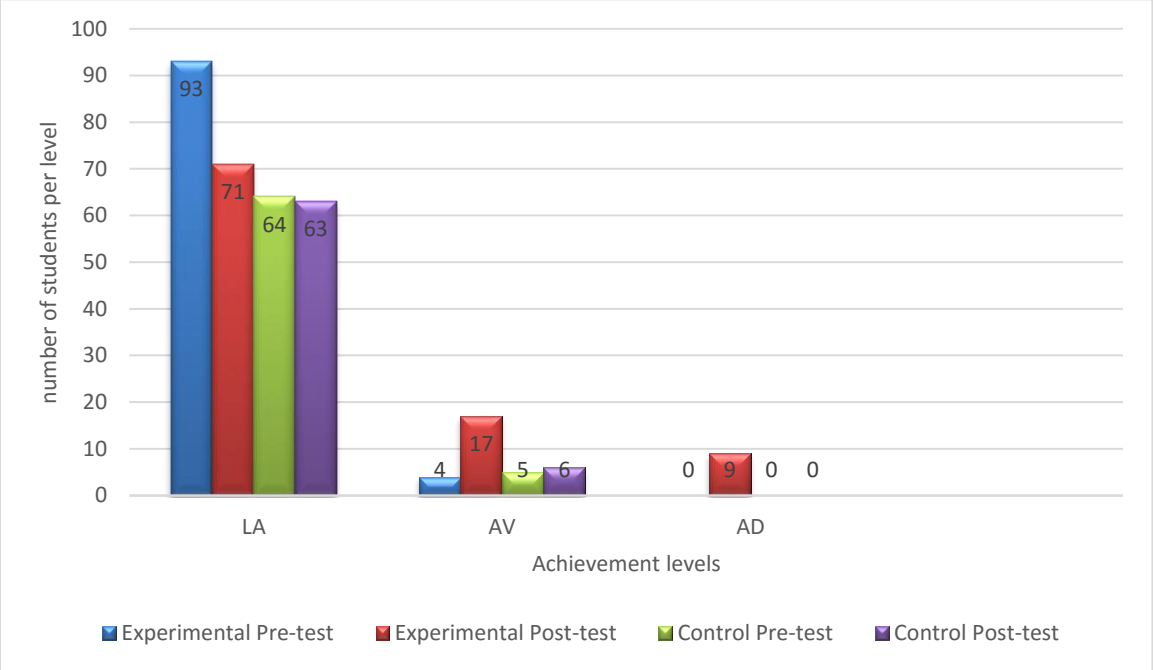


Figure 4.3: Achievement in pre- and post-test (experimental and comparison groups)

A conclusion is reached based on the general results for pre- and post-test in both groups, that IBF contributed to better results than the traditional axiomatic approach. IBF thus positively influences IBL in Euclidean Geometry.

4.3.2 Descriptive analysis per level of knowledge and understanding

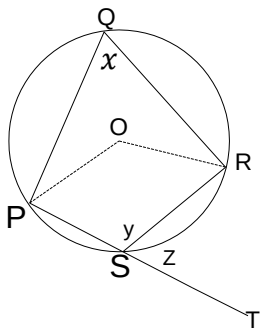
The main pre- and post-test problems are outlined and thereafter results are presented from pre-visualisation, visualisation, analysis, informal deduction, formal deduction and rigour.

4.3.2.1 The test problem

The problem was centralised to one main question in the pre- and post-test question paper (cf Appendix B). The sub-questions addressed all levels of knowledge and understanding in order to reach a solution of the main problem. Example 1 as shown below was presented as a problem statement as follows:

Figure 1 is given. Line PS is produced to a point T. Answer the questions that follow to prove whether x = y.

Figure 1



This problem was posed as a deductive and abstract problem. This is normally how problems are posed in classrooms for assessment purposes with expectations that students should recall the theorems they have memorised. In order to test memorisation of theorems in classrooms, students would have been asked a direct statement like “prove whether angles PSR and angle PQR are supplementary”. Alternatively, students could have been asked to “prove that angles of a cyclic quadrilateral are supplementary”. To answer the question by recalling theorems through memorisation, no critical thinking and analysis skills are necessary and a

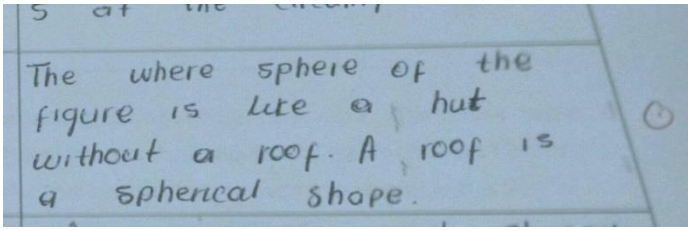
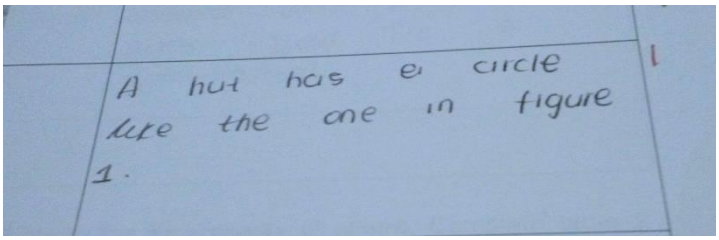
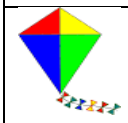
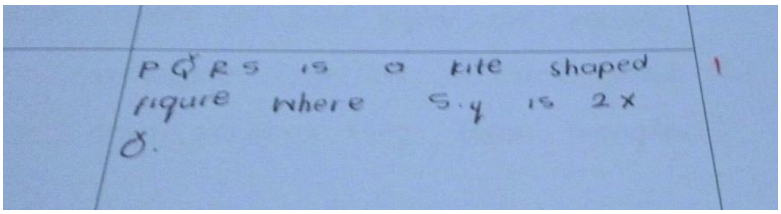
direct statement leads to a direct answer. In this study, the majority of students had in memory that opposite angles of a cyclic quadrilateral are supplementary which means the sum of x and y is 180°, but could not perceive that $x \neq y$ because in any cyclic quadrilateral, $x + y = 180°$. However, in this study students were led through inductive questioning in order to critically build facts to conclude deductively that $x \neq y$. Building facts revolves around concluding that the two angles are supplementary, therefore they are not equal. Questioning was structured in a way that it would enable students to firstly apply mathematical inquiry, that is, to display the awareness of the connection between mathematical and extra-mathematical systems and to construct mathematical representations. In addition, students were expected to search for structure, patterns, relationships and the principal aim of generalisation. Secondly, questioning was aimed at assisting students to apply IBE to improve all levels of geometric knowledge and understanding, that is, students were led to engage in sense-making activities, rather than applying a clear, smooth path to the solution. Concept discovery through making sense of concepts and exploring the relations of shapes and patterns was the focus. Critical thinking and self-discovery of concepts and relationships was encouraged through the questions posed. Students are supposed to know the origin of relations through exploration and self-discovery. However, IBF is the key to the success of students' discoveries and exploration.

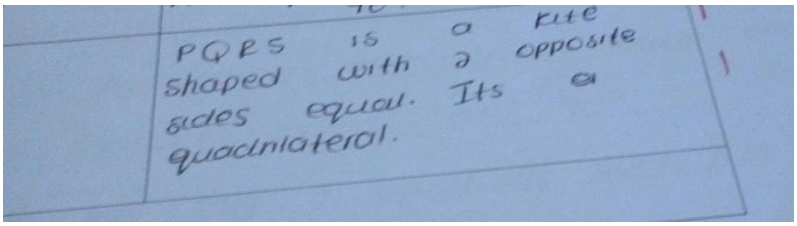
4.3.3 Descriptive analysis per level of knowledge and understanding

4.3.3.1 The pre-visualisation stage (question 1.1)

The pre-visualisation stage was evaluated through question 1.1. Students were expected to look at Figure 1, relate the figure with table 1 in the test and draw conclusions about whether there was an association based on what they saw or perceived. Students' experience is essential at this level. The problem in figure 1, is posed as a more abstract figure, and the solution can be attained at a formal deduction level. In order to arrive at the actual abstract solution of the problem, students are led to build facts around Figure 1 using their experience in pre-visualisation as the basis. Associating Figure 1 with other elements would lead to the ability to link the structure in question with other structures perceived through experience. Student struggled with this question. Using perceptual skills to create the basis or lay a setting of the problem-solving was a challenge. Students' responses are shown below:

How can you associate each of the following objects with Figure 1?

Object	Reasons for association with Figure 1
 Hut	Student EX4 Pre-test The problem is incorrect usage of the word sphere. Vocabulary is there, however, applied incorrectly.  Post-test The student is able to locate the shape of the two figures correctly 
 Kite	Pre-test The student knows a kite and has a clue that the main figure of problem statement associates with a kite. However, the abstract labelling of PQRS is incorrect and the reason of association is not clearly explained. 

	Post-test  In the post test, the students managed to label PQRS correctly. The reason of association is clear and extended to classification of the associated figures and a property.

In this study, emphasis is made on the fact that pre-visualisation precedes visualisation. Pre-visualisation as the basic level enhances the students' powers to perceive the world around them. In this sense, learning happens through perceiving real-life objects that have a certain effect on daily activities. If learning had occurred through pre-visualisation, students would (according to Clements and Battista, 1992): (1) perceive geometric shapes in their surroundings but not attend fully to visual characteristics of the particular shapes. In this study, students were expected to look at concrete shapes in Table 1 of the question paper and explain the association with Figure 1. This indicates the advanced perceptual abilities that students possess. Table 4.10 shows the results for responses for question 1.1, mainly based on assessing students' pre-visualisation level of knowledge and understanding.

Table 4.10: Responses to question 1.1 (pre-visualisation level)

		No. wrote	**E**	**0**	**1-5**	**6-10**	**11-16**
Experimental group	Pre – test	97	14	52	28	3	-----
	Post –test	97	8	54	25	8	2
Comparison group	Pre – test	69	4	44	19	2	-----
	Post –test	69	4	46	17	------	------

E = did not answer the question; 0 – low perception skills; 1-5- average; 6-10 = advanced; 11-16 = proficient

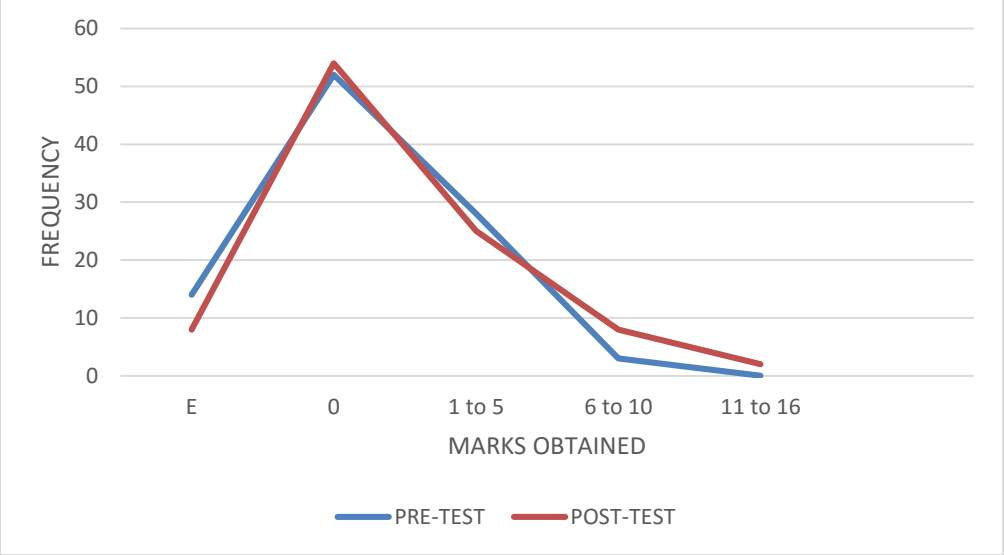


Figure 4.4: Pre-visualisation experimental group

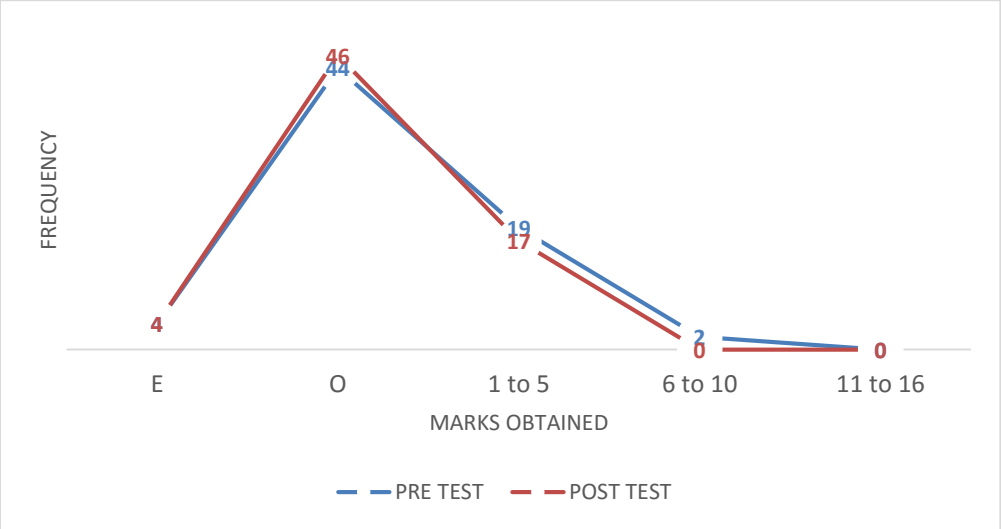


Figure 4.5: Pre-visualisation comparison group

Comparing the experimental and comparison group in the pre-visualisation stage, the number of students who did not answer the question in the comparison group was the same in both the control and experimental group, while there was a decrease in the experimental group. Students in the experimental group gained confidence in attempting the question. Low perception skills show a higher rate in post-test for both groups; average skills show a decline in both groups in the post-test, while in the experimental group there is an increase in the number of students who achieved both

the advanced and proficient perception skills in the post-test as compared to pre-tests. The advanced and proficient levels recorded an increase of 70 % in the experimental group. However, in the comparison group no student achieved at the advanced and proficient levels.

4.3.3 Visualisation stage

According to Way (2012) students' specific characteristics apply in the visualisation stage. The characteristics include reviewing the pre-visualised shapes through their appearance; application of verbal thinking and distinguishing the parts. Further, students should be able to formulate or recognise relationships between known objects and specific shapes. Kyllonen and Christal (1990) motivate that spatial imagery plays an essential role in yielding creative thinking, which is the critical skill in visualisation.

Question 1.2 addressed the visualisation stage. It does not measure abstract thinking, but the students' ability to apply spatial awareness through everyday demands of working memory to maintain and transform images (Kyllonen and Christal, 1990). In this question, students were expected to group the items in 1.1 according to similarity of shape, that is, match shapes in Figure 1 with shapes in Figure 2 (in column 1 of Table 4.11). For example, students were expected to notice that a pyramid has a base with four sides; but floor tiles, a garage door and a kite also have four sides. Therefore, they need to be able to visualise symbolic representations of the objects.

Table 4.11: Student responses

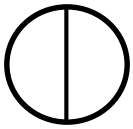
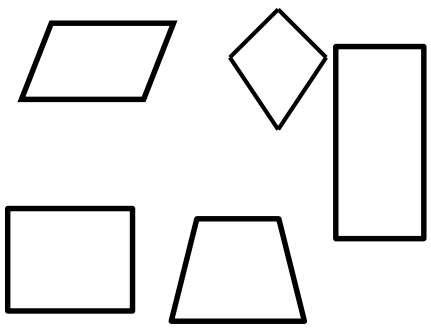
Common name of figures	**Shape**	**Properties**
Example (a) Circle – Examples: hut/ rondavel		The section of properties was evaluated as analysis level
(b) Quadrilaterals Examples: Kite, garage door and all other quadrilaterals appearing on table for question 1.1.		

Table 4.12: Results for experimental and comparison groups in question 1.2 (Visualisation level)

		No. Wrote	E	0	1-5	6-9
Experimental Group	Pre-test	97	24	39	34	----
	Post-test	97	35	20	42	1
Comparison group	Pre-test	69	20	25	52	-----
	Post-test	69	25	11	33	-----

E= did not write the question; 0= low visualisation skills; 1-5= average; 6-9 = advanced

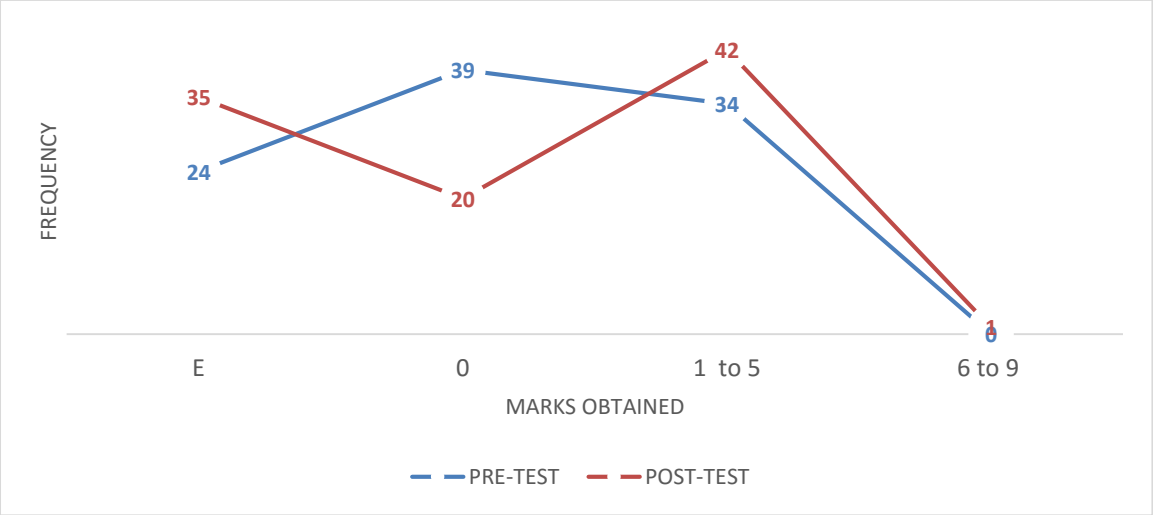


Figure 4.6: Visualisation experimental group

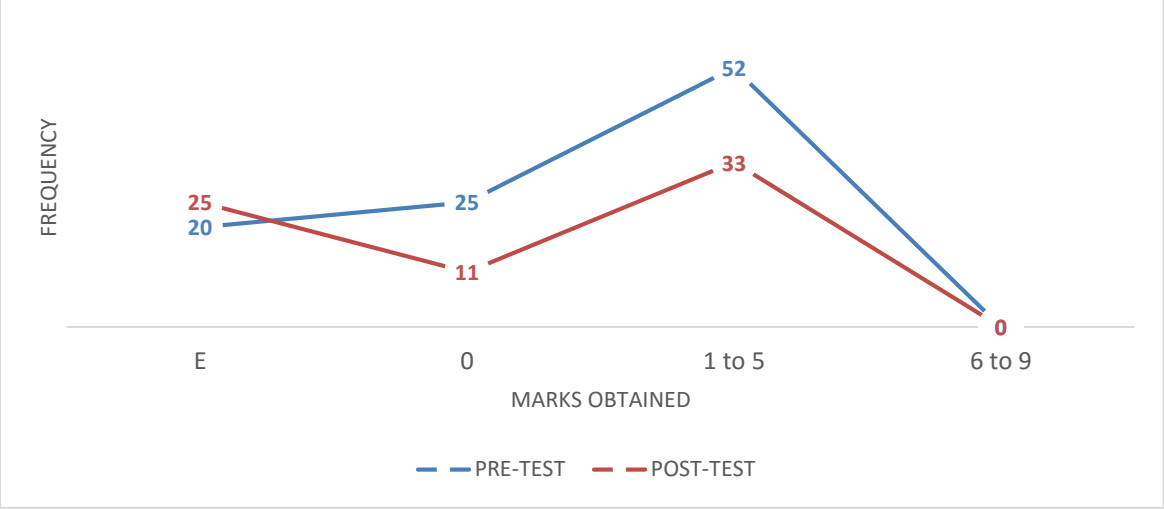


Figure 4.7: Visualisation comparison group

Group comparison indicates that there was an increase in a number of those who did not answer the question and a decrease in a number of those who showed low visualisation skills. Average and advanced stages increased in the experimental group, whereas there was a decline in the comparison group. In the experimental group, only one student achieved up to the advanced level; however, no student achieved at the advanced level of visualisation. A large number of those who did not answer the question confirm that students experience challenges in relation to application of spatial awareness and they lack skills to utilise the working memory in creatively transforming concrete pre-visualised images into mental concepts.

4.3.3.3 Analysis level

At this level, students start viewing figures in terms of their properties (Mason, 1998). For example, properties can be used to differentiate between a circle and a triangle, a triangle and a quadrilateral. This study has shown that students experience challenges in the analysis level, that is, they cannot specify the connections among properties of a figure and among various figures. Further, Mason (1998) outline that students can specify the properties but cannot state how and where they apply in problem solving.

Question 1.3 and 1.4.1 addressed analysis of the Van Hiele levels of knowledge and understanding. In this question, students were expected to give the general name of each group of shapes, and thereafter outline the general properties of each group. Further, in the analysis level of knowledge and understanding, students were expected to show ability to dissect isolated figures into parts and discover the properties. They were expected to apply own judgement to select and utilise relevant material or instruments to perform accurate construction, dissect the constructed figure into parts, name the parts and describe each part as in the post-test example displayed in Table 4.15. Results for analysis level pre and post-test for both the experimental and comparison group are shown in table 4.13.

Table 4.13: Results for experimental and comparison groups in question 1.3 and 1.4 (Analysis level)

		No. Wrote	E	0	1-8	9-16	17-26
Experimental Group	Pre-test	97	1	1	79	16	------
	Post-test	97	1	1	47	41	7
Comparison group	Pre-test	69	---	13	46	8	------
	Post-test	69	2	11	40	14	------

E= did not write the question; 0= low analysis skills; 1-8= average; 9-16 = advanced, 17-26= proficient

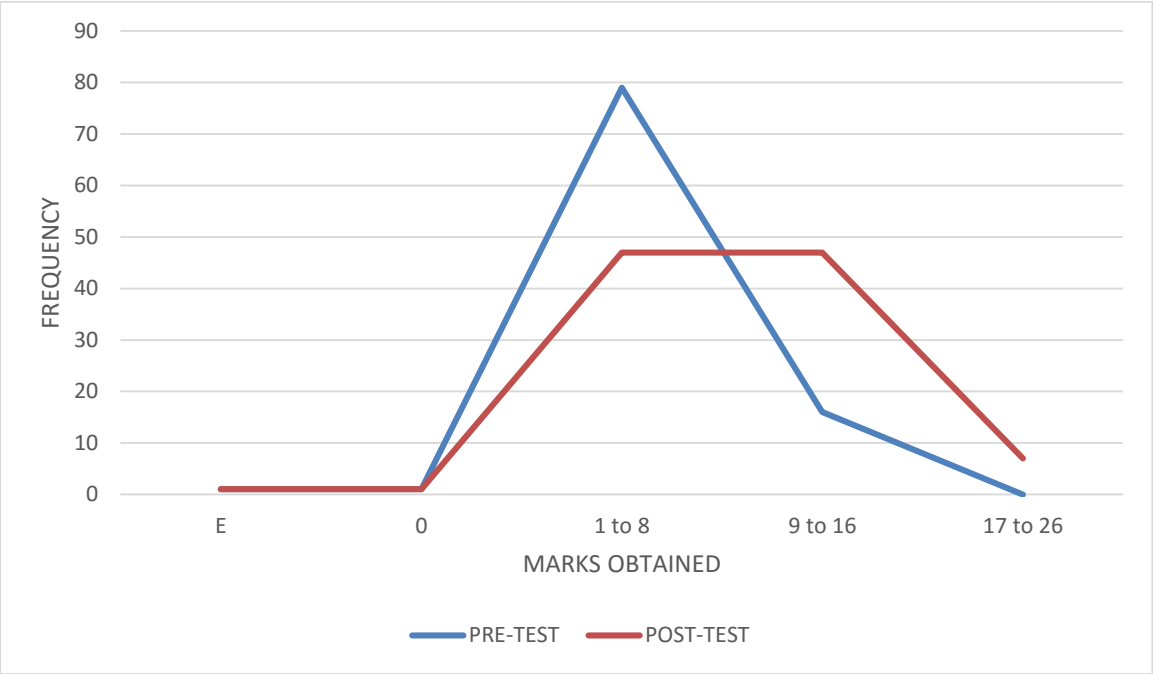


Figure 4.8: Analysis: Experimental group

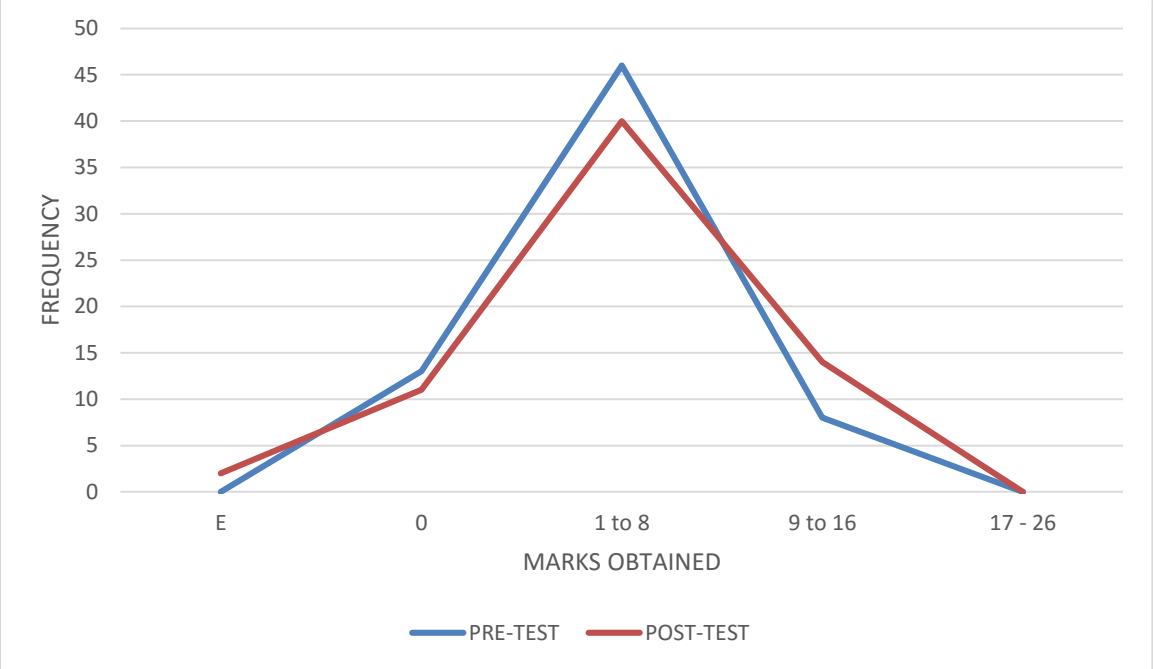
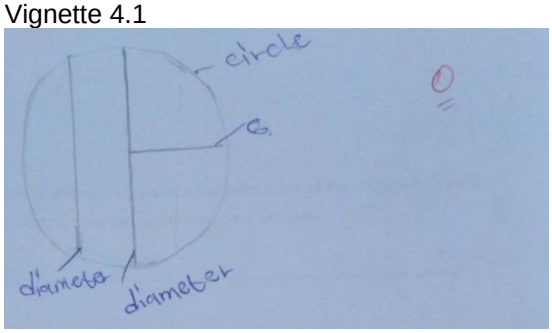


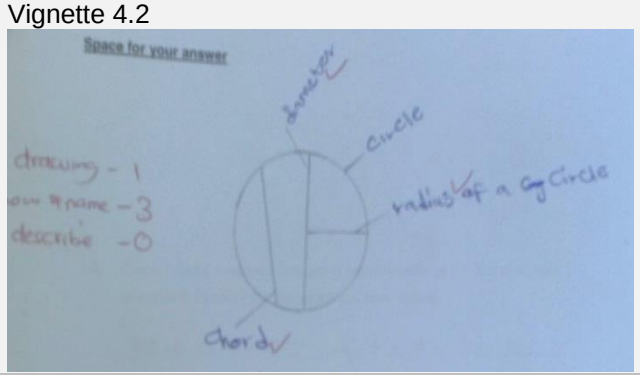
Figure 4.9: Analysis: Comparison group

Most of the students improved from the low analysis skills to average and through to advanced and proficient skills in the experimental group. The comparison group indicate improvement in the advanced skills; however, no student achieved proficient skill. The experimental group showed a huge improvement compared to the comparison group.

In question 1.3, Pre-test revealed that majority of students were unable to achieve in the analysis stage by showing the ability to perform correct constructions based on accurate dimensions in order to dissect isolated figures into parts and discover correct properties as shown in Table 4.14. In addition, pre-test revealed that constructions were not accurately performed due to lack of essential construction instruments.

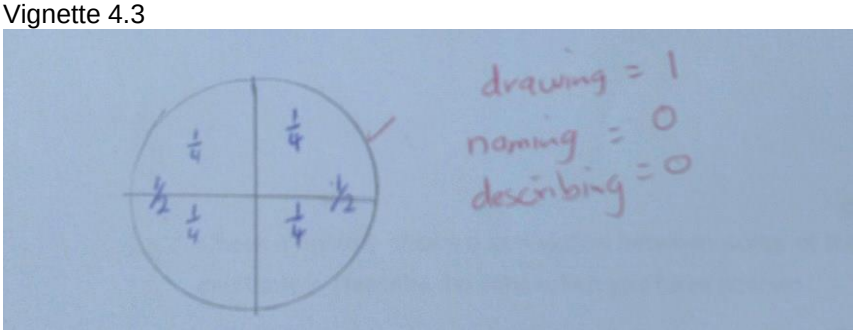
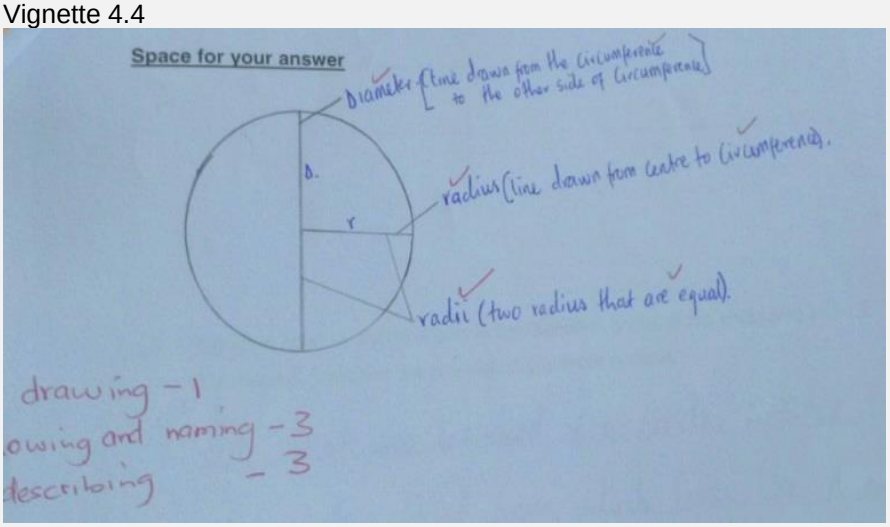
Table 4.14: Student EX2's construction of a circle

Pre-test (Before intervention)	Vignette 4.1 

Post-test (After intervention)	Vignette 4.2 

A free hand drawing of a figure that is similar to a circle or figure associated by students with a circle contributed to wrong analysis of the figure, that is, the circle. Conclusion cannot be reached that a figure in vignette 1 is a circle. The argument here is that a circle need to be constructed according to dimensions for it to maintain the correct centre, its position as a circle and its correct properties. For example, students could not understand that a diameter is not any line that divides any spherical, oval or elliptical figure, but should divide a circle resulting from accurate dimensions of a circle into two equal halves (vignette 4.1). The idea of construction of a circle that is accurate in order to present accurate information about a circle was not comprehended by students during the pre-test. However, improvement was shown in the post-test where accurate constructions were performed and correct properties were presented (vignette 4.2). Further, the line associated with a radius in vignette 4.1 is labelled by student VV as E. In vignette 4.2., student VV presented the radius correctly. Further, those who understood that a correct construction should be performed failed to dissect, label and describe the parts of a circle correctly during the pre-test (table 4.15, vignette 4.3). However, student WW could build on knowledge that accurate construction of a circle is essential. The student labelled and described the parts correctly (vignette 4.4). The same applied to most of the students in the experimental group.

Table 4.15: Student's EX19's answer: pre- and post-test

Pre-test (Before intervention)	Vignette 4.3 
Post-test (After intervention)	Vignette 4.4 

4.3.3.4 Informal deduction level

According to Way (2011) informal deduction level explains the students' ability to apply the acquired knowledge of properties in formulating connections among properties and among diverse figures. For example, students should be able to explain with reasons why some quadrilaterals can be explained in terms of a square (Way, 2011). Further, the author, motivated that students cannot apply acquired knowledge in as theorems and converses, however, informal deduction is basic to formal deduction.

Questions 1.4.2 to 1.6, addressed the informal deduction level. At this level, students are expected to move from generalising to being specific. Firstly, they had to work on the concept of a circle, that is, draw a circle, name and describe the parts of a circle. The teaching at this level guided the students to distinguish shapes in terms of their parts and properties. By describing the parts, students would be able to understand clearly the properties or characteristics of each specific shape. For example, students understood the group named quadrilaterals; in order to deduct informally, they had to

understand that, in a group, there specific shapes have underlying different features; for example, specific figures like rhombus, square or trapezium. At this level, students would be able to distinguish between, for example, angles in a triangle and angles in a quadrilateral in terms of size. For example, the sum of angles in a square is 360°, and angles of a square are equal, therefore, each angle is equal to 90°. Further, informal-deduction level is where relationships between shapes would be speculated.

Table 4.16: Results for experimental and comparison groups in question 1.4.2 to 1.6 (Informal deduction level)

		No. Wrote	E	0	1-4	5-8	9-13
Experimental Group	Pre-test	97	12	40	40	5	------
	Post-test	97	2	22	57	15	1
Comparison group	Pre-test	69	1	38	25	2	3
	Post-test	69	------	53	16	-----	-----

E=did not write the question; 0= low informal deduction skills; 1 to 4 = average; 5 to 8 = advanced; 9 to 13 = proficient

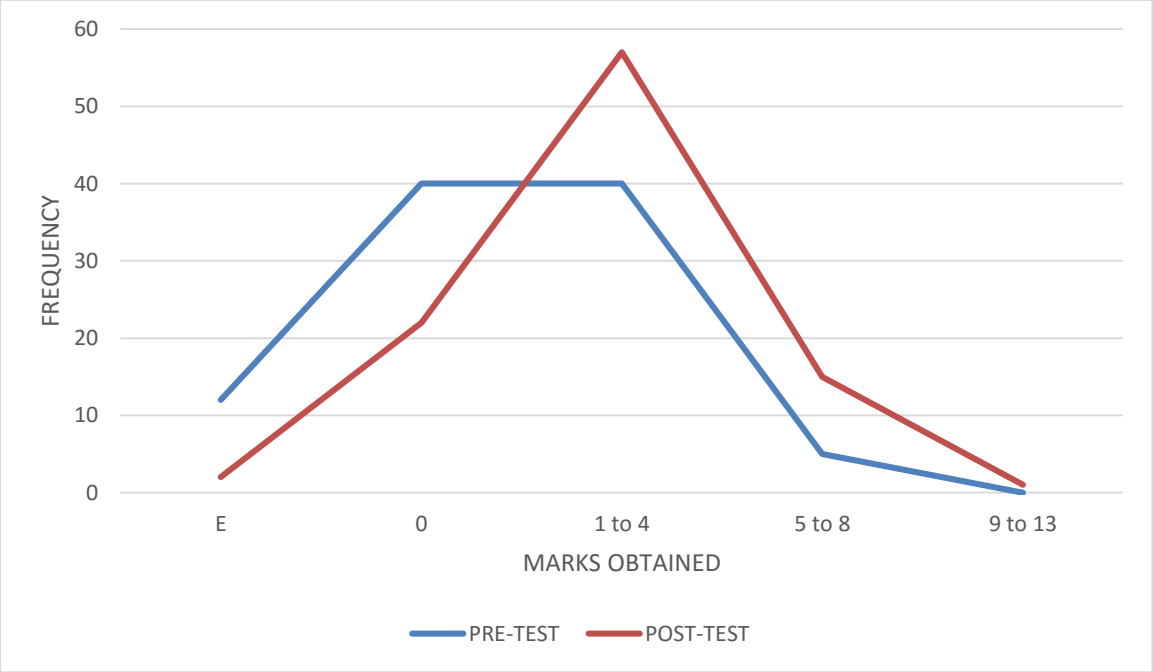


Figure 4.10: Informal deduction: Experimental group

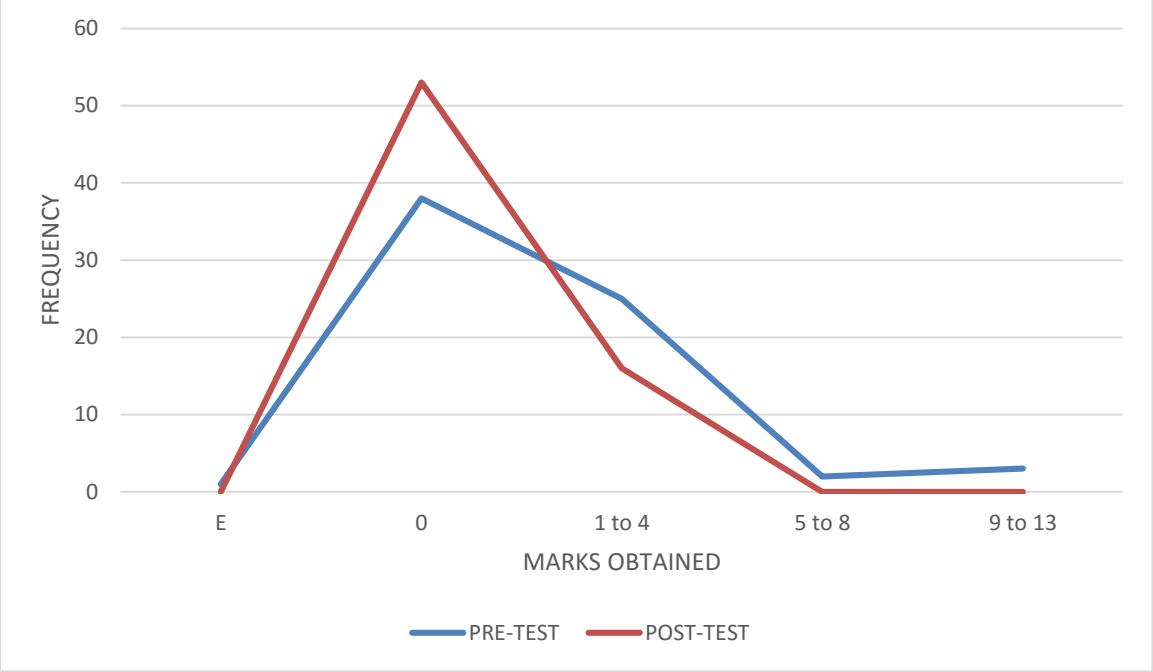


Figure 4.11: Informal deduction: Comparison group

In the comparison group, a large number of students who had low informal deduction skills was found. Few students obtained an average rating and no student achieved up to the advanced and proficient ratings in the post-test. An improvement from low informal deduction skills to advanced and proficient was recorded in the experimental group post-test. A large number of students who possessed low informal deduction skills improved to average, advanced and proficient.

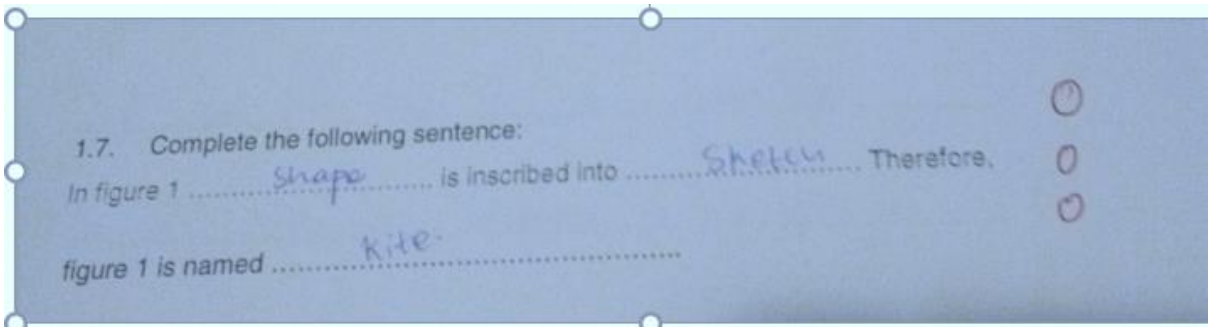
4.3.3.5. Formal deduction level

Questions 1.7 to 1.9 address the formal deduction stage, that is, level 4 of Van Hiele's levels of knowledge and understanding. This is the more abstract level of knowledge and understanding in Euclidean Geometry. The questioning was structured so that students would build on what they had experienced at level 0 to level 3. Students were expected to display a higher level of understanding of relationships of figures. Question 1.7 was posed as follows:

Complete the following sentence: In figure 1 …….. is inscribed into …… Therefore, figure 1 is named ……..

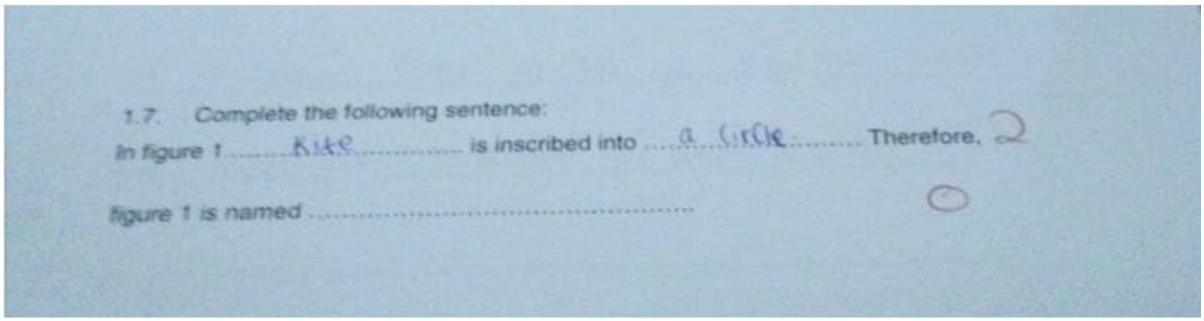
Student EX1 answered as follows in the pre-test:

Vignette 4.5



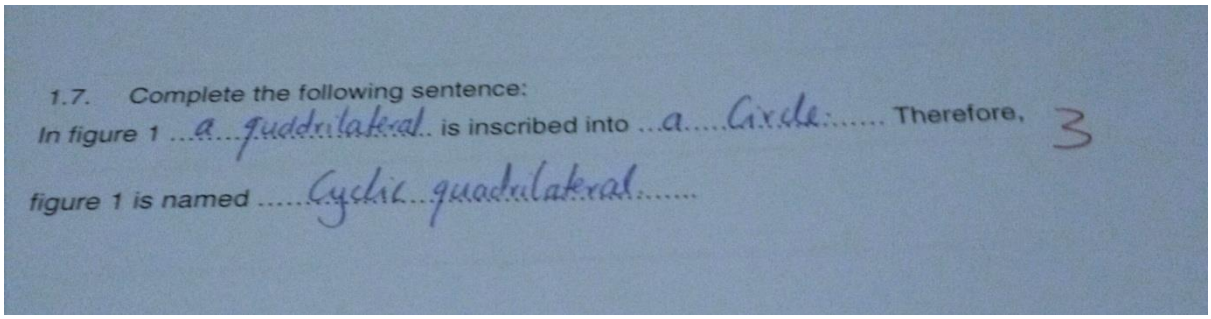
The same student answered as follows in the post-test:

Vignette 4.6



Student EX1 could not locate the name of the figure in the post-test; however, he could describe the combination better compared to the answer in the pre-test. The correct answer was presented by student EX15 in the post-test, as follows:

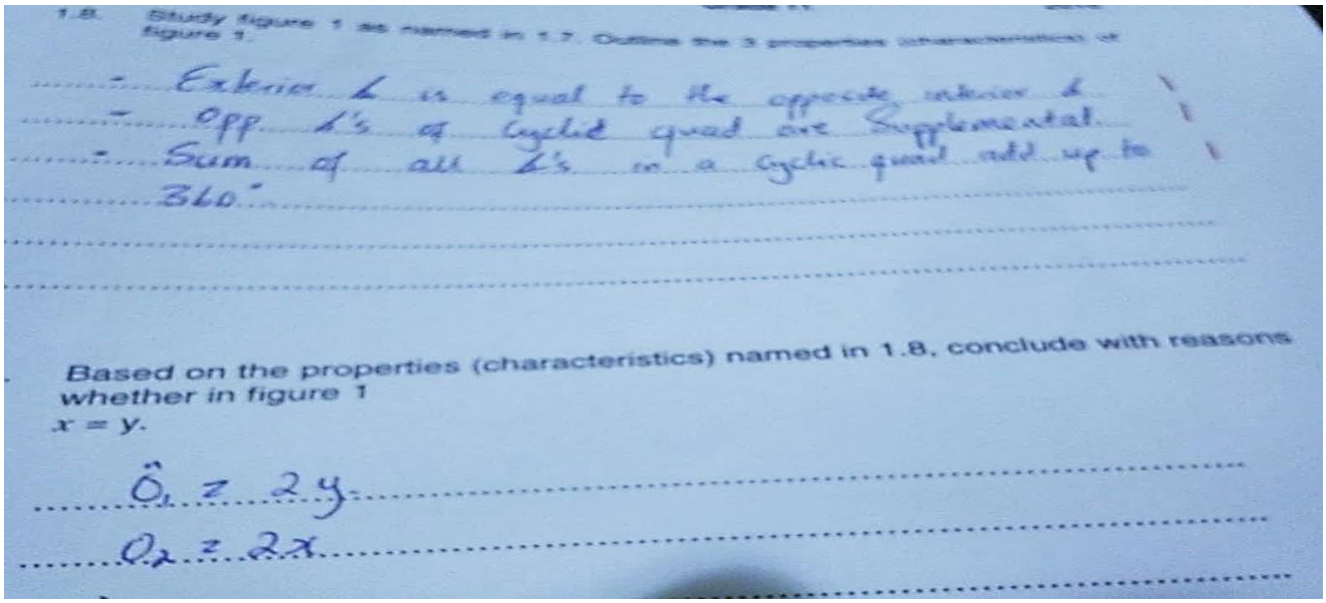
Vignette 4.7



A conclusion could not be reached about what type of a quadrilateral had been inscribed in a circle. It could be a kite or a rhombus, however, Therefore, Vignette 4.7 provides the most relevant answer.

Once the students had recognised Figure 1 and located its name, they were then asked to state the properties in question 1.8. Stating the properties would enable them to answer question 1.9, that is, they would be able to understand that, if opposite angles of a cyclic quadrilateral are supplementary, it means their sum must be $180°$, meaning that the two angles are not equal. They would be equal if they were right angles, but because they are not right angles, it means they are not equal. After the IBF intervention, some of the students were able to state that opposite angles of a cyclic quadrilateral are supplementary, however, they still struggled to use the statement or reason to argue that $x \neq y$ since x and y are supplementary. Student EX18 presented the properties of a cyclic quadrilateral correctly, but could not use the properties to draw a conclusion (Vignette 4.8).

Vignette 4.8



Student EX16 could reason and conclude that $x \neq y$ based on one of the properties discovered that if x and y are opposite angles of a cyclic quadrilateral, and the two angles are supplementary; then they cannot be equal (Vignette 9).

Vignette 4.9

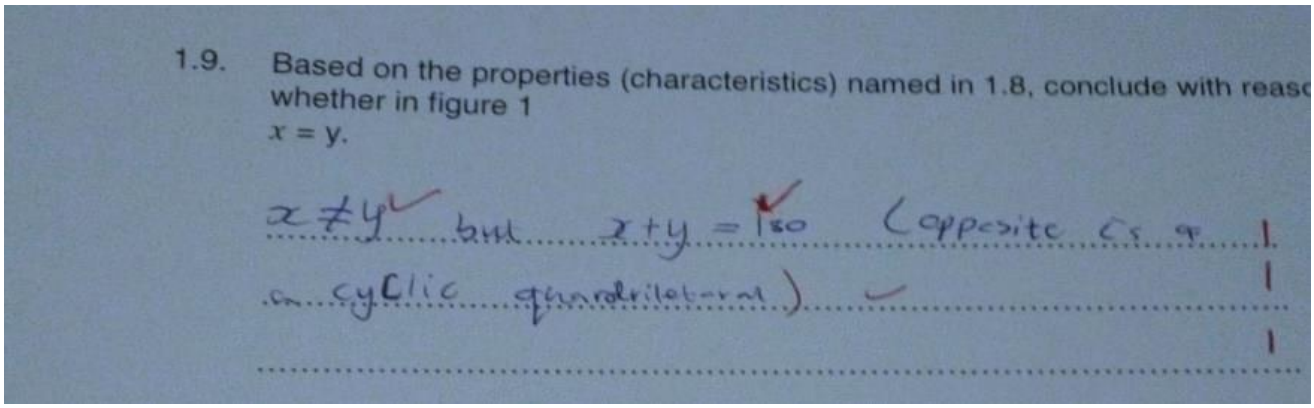


Table 4.17: Results for experimental and comparison groups in question 1.7 to 1.9 (Formal deduction level)

		No. Wrote	E	0	1-4	5-8	9-13
Experimental Group	Pre-test	97	29	46	21	1	----------
	Post-test	97	8	23	45	17	1
Comparison group	Pre-test	69	10	32	24	3	
	Post- test	69	9	34	25	1	---------

E=did not write the question; 0 = low informal deduction skills; 1 to 4 = average; 5 to 8 advanced; 9 to 13 = proficient

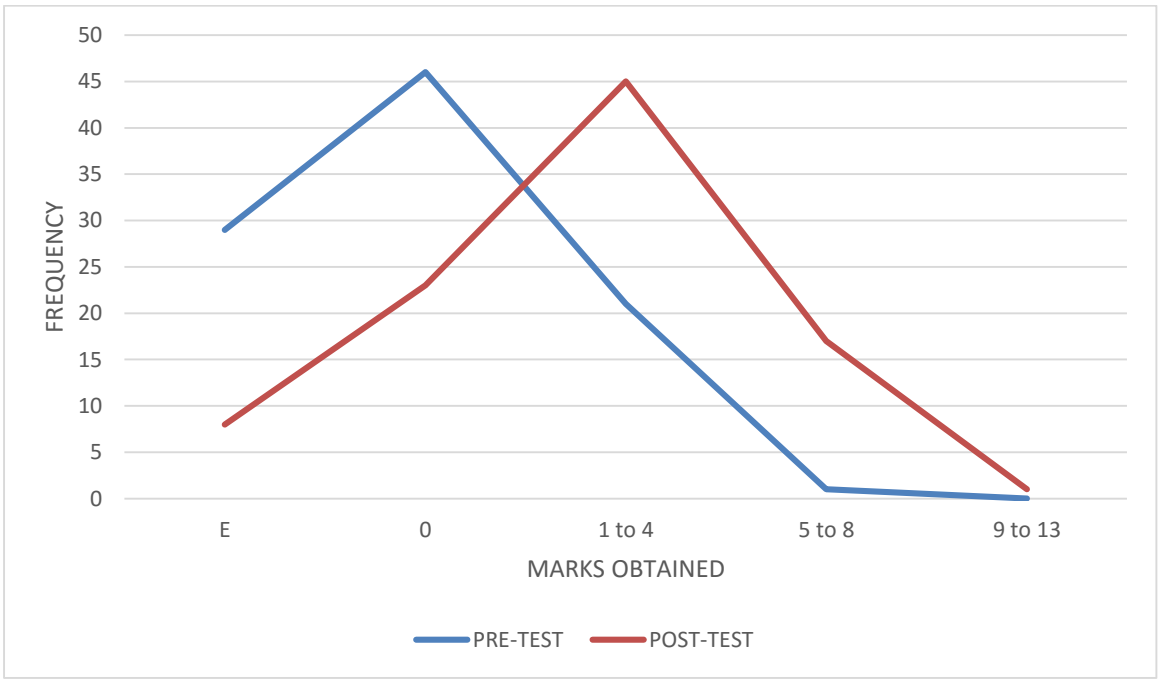


Figure 4.12: Formal deduction: Experimental group

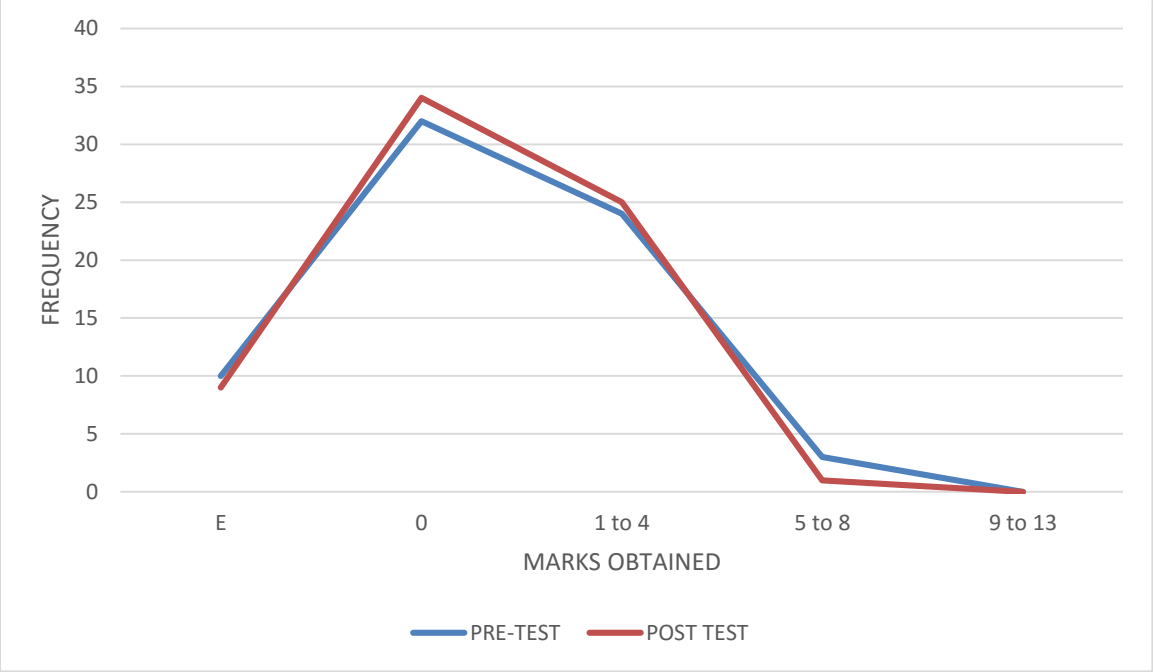


Figure 4.13: Formal deduction: Comparison group

In the comparison group, no improvement was recorded in the average, advanced and proficient ratings. A number of students who show low formal deduction skills increased. However, a great improvement from low formal deduction skills to average was shown in the experimental group. A number of students in the experimental groups achieved to advanced and proficient ratings.

4.3.3.6. Rigour

In question 1.10 students were expected to link mathematical and extra-mathematical systems. The question they were asked to respond to was: 'Figure 1 occurs every-time and everywhere in real life. Briefly discuss the importance of Figure 1 in construction of habitats and in architecture'. Table 4.18 shows the results of students' responses in the rigour level.

Table 4.18: Results for experimental and comparison groups in question 1.10 (Rigour level)

		No. Wrote	E	0	1-3	
Experimental Group	Pre-test	97	48	38	11	
	Post-test	97	23	50	24	
Comparison group	Pre-test	69	30	34	5	
	Post-test	69	13	53	3	

E=did not write the question; 0= low rigour skills; 1- 3 = advanced

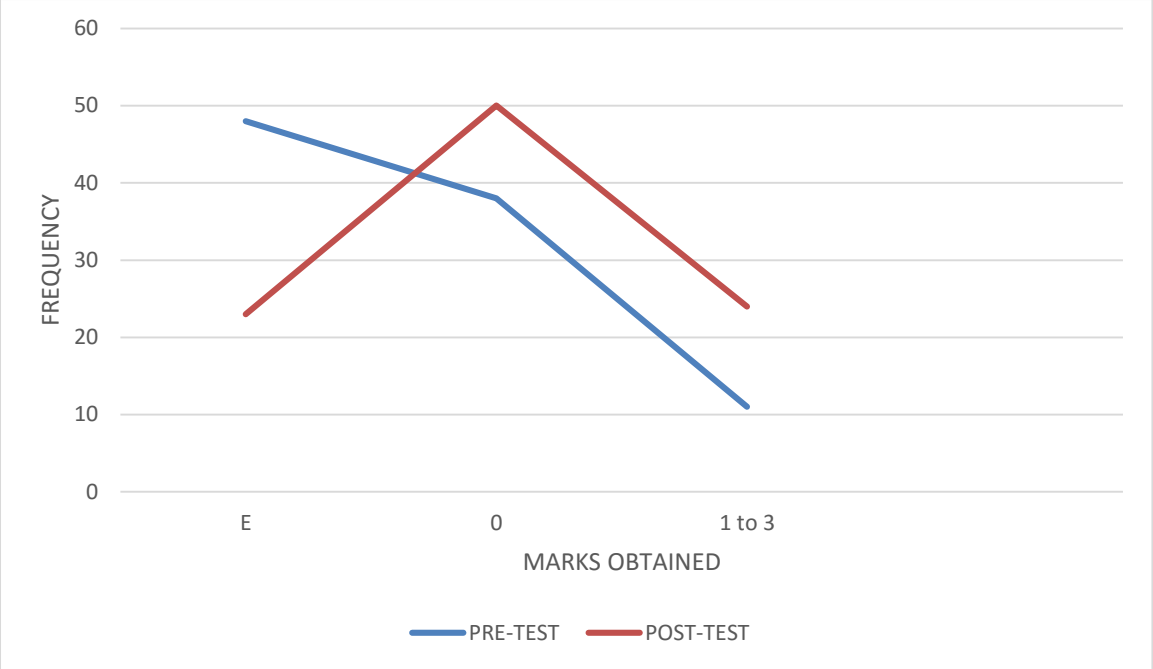


Figure 4.14: Rigour: Experimental group

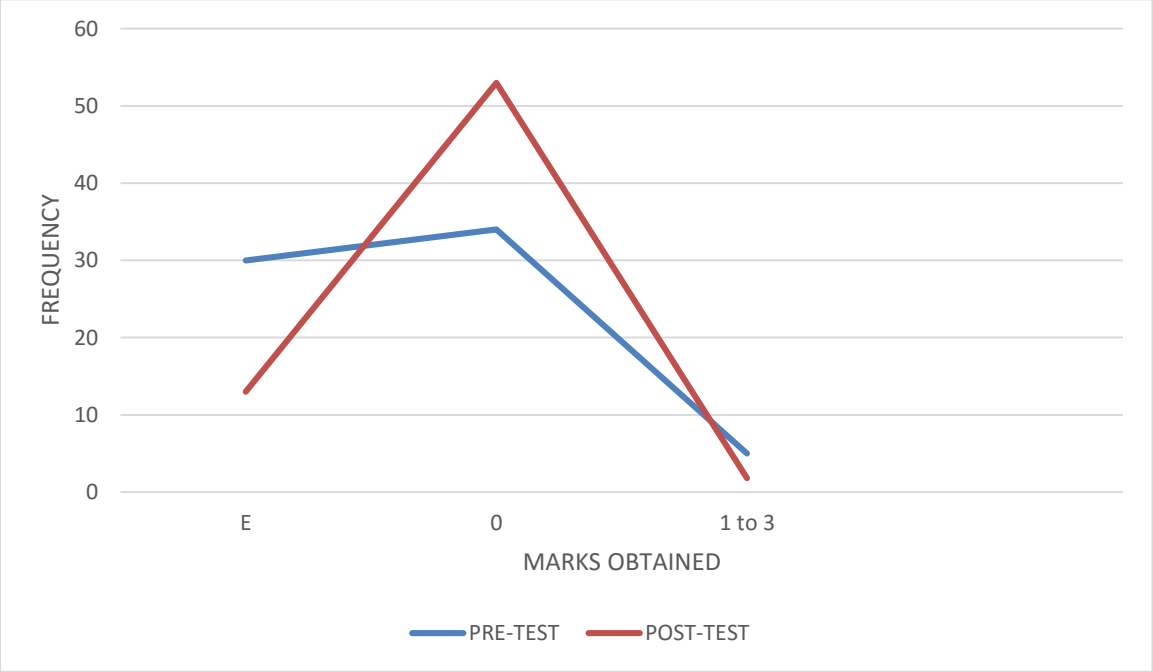


Figure 4.15: Rigour: Comparison group

At the rigour level of knowledge and understanding, the number of students who achieved at the advanced stage in the experimental group doubled. This shows improvement when comparing the pre- and post-test results. A decline is shown as fewer students achieved in the post-test compared to the pre-test in the comparison group.

A comparative summary of students' achievements in advanced and proficient levels for all levels of knowledge and understanding is outlined in Table 4.19.

Table 4.19: Summary of students' achievement in advanced and proficient levels

	Experimental group		**Comparison group**	
	Pre-test	Post -test	Pre-test	Post-test
Pre-visualisation	3 %	10.3%	2.9%	0%
Visualisation	0%	1%	0%	0%
Analysis	16.5%	49.9%	11.6%	20.3%
Informal deduction	5.1%	16.5%	7.2%	0%
Formal deduction	1%	18.9%	4.3 %	1.4%
Rigour	11.3%	24.7%	7.3%	4.3%

Table 4.19 reveals that there was a constant improvement across levels of knowledge and understanding in the experimental group between the pre- and post-test. However, the comparison group shows improvement only in the analysis level. A decline is recorded in the comparison group in most of the levels. Visualisation stage seem to be a challenging stage as both groups recorded 0% at pre-test. However, a change to 1% was realised in the experimental group, while the comparison group remained at 0%. Change was also realised in the formal deduction stage of the experimental group, that is, 1% in the pre-test and 18.9% in the post-test. Generally, in the experimental group, student development was perceived from the lower levels of knowledge and understanding through to higher levels of knowledge and understanding.

4.3.4 Common errors committed by students in the pre-test

Table 4.20 displays examples of errors committed by students in both experimental and control groups during the pre-test.

Table 4.20: Summary of errors in the pre-test

Error type	Example
Vocabulary, spelling and irrelevant words Students have the geometric vocabulary but do not know how to apply in problem-solving, for example	*Circle spelled as "*cycle"* *Quad (for quadrilateral) written as "quaard" *Response to question 1.7, "*a theorem is inscribed in a rider, therefore, the figure is named geometrical*"; instead of, *A quadrilateral is inscribed in circle, therefore, the figure is named a cyclic quadrilateral.*
Geometric language	Referring to vertices as corners
Labelling of figures Labelling the figures incorrectly, that is, incorrect positioning of alphabets used for naming figures	In Figure 1 of the problem, students labelled as follows: PQSR, RPSR instead of PQRS.

Error type	Example
Forcing solutions Applying memorised theorems where not applicable, therefore, forcing incorrect concepts to correspond in order to get solutions	Stating a midpoint theorem when not necessary. Stating more other memorised theorems wrongly, especially in attempting to answer questions 1.8 and 1.9.
Incorrect relation of figures with properties	"*opposite sides are equal in a hexagon*".
Instruments not used to construct figures	Drawing a free hand circle, resulting in a wrong radius, diameter. A pair of compass not used to draw the circle.
Wrong reasons	Visual appearance of diagrams used without analysis to solve problems.
Incorrect application of memorised theorems	Students write any theorem anywhere without doing much thinking to get meaning and apply correctly.

4.4. QUALITATIVE FINDINGS

Below is the analytical report for the qualitative findings. Four codes were used to record actions and skills observed, including views narrated by participants. The four codes are teaching strategies, learning strategies, skills acquired and content learned.

4.4.1 Codes for observations

The four codes seem to be very essential and interconnected as observed in this study. The essence emanates from the fact that the type of teaching carried out informs learning. Further, particular method of learning yields skills that support students in acquiring content in a form of knowledge. For example, teaching strategies observed as actions in the experimental group, promotes outlined IBL actions, and through IBL strategies, student acquire skills that enable to acquire content. Similarly, teaching strategies observed in the comparison group yield learning strategies outlined, promoting specific skills that support students to acquire content. However, the teaching strategy is observed to be the main determiner of the type of learning strategies and skills applied to acquire content. Table 4.21 below presents an analytical report of teaching and learning actions including skills observed in both experimental and comparison groups in this study.

Table 4.21: Analytical report (participatory and non-participatory observations)

Code	Actions observed in the experimental group	Actions observed in the comparison group
Teaching strategies	Discussion, probing by asking questions, explanation, facilitation and demonstration by showing how to manipulate a tangram, showing how gadgets to search for relevant mathematical concepts and information; writing on the chalkboard to emphasise some of the concepts.	Explanation; writing on the chalkboard and smartboard to show proofs of theorems and terminology; retrieving activities and exercises from the textbook, referring students to notes on worksheets provided)
Learning strategies	Discussion, group work and presentations, cooperative learning, experiential, explanation, Investigating, discovery, imagination, manipulating a tangram and think critically; peruse diverse textbooks to search for information; manipulate their electronic gadgets to search for information; write to note findings and group agreements, complete activities presented by the facilitator to try to commit what they have learned to memory.	Listening to the teacher's presentation; writing notes or examples as written by the teacher on the chalkboard; Complete activities to try to commit what they have learned to memory.
Skills acquired	Application of inquiry; formulating relationships between and among a network of concepts, discovery, applying knowledge gained from experience; creativity, critical thinking, developing patterns, exploration, identifying and naming, classifying, presentation; solving problems by completing activities deductively	Memorisation (memorising theorems, terminology, teachers' examples as on the chalk board); reviewing examples in the prescribed textbook and on worksheets, group work, solving problems by completing activities deductively; proving theorems.
Content learned	*Geometry information related to real-world *Properties of figures and application in formulating relationships of figures. *Relationships of figures or shapes *Confirming stated theorem about cyclic quadrilateral (inscription and properties of inscribed figures). *Mathematical vocabulary	*Theorem ((cyclic quadrilaterals; supplementary angles) and exterior angle = opposite interior angle. *Application of theorems in activities, that is, exercises in the textbook.

4.4.2 Interview codes

This section presents the students' and teachers' coded views on the teaching and learning of Euclidean Geometry. Students from experimental and comparison groups were interviewed, however, only teachers from the comparison groups were interviewed. Ideas from interviewees were based on their experiences in the four codes. Students in the experimental group commented more on teaching and learning strategies that support student-centred learning, while students in comparison group gave comments that show more interest in student centred learning, however, in some

instances their ideas about teaching and learning strategies are inclined to traditional approaches. The coded results of interviews are presented in Table 2.2 below.

Table 4.22: Analytical report (interviews)

Code	Experimental group	Comparison group
Teaching strategies	Students view the teaching of Euclidean Geometry as teaching that: • orientates to properties of shapes • allow exploration and self-centred learning • promotes cooperative learning and group work • asks challenging questions (probing) • guide and advise students	Students' expectations • Teacher must create more time and move at the students' pace. • Teacher must link students' experience. • Teacher must relate class work with examination (teach for examination) Teachers' views • Teachers confirmed that they teach for formal examination and for results by assisting students to memorise theorems. • Due to a rigid work-schedule, time constraints and lack of resources, teachers teach through textbook and question papers to prepare students for examinations
Learning strategies	Students view learning of Euclidean geometry as: • Exploration • Self-centred learning • Application of teacxhers' clues and taking the teacher's advice • Using manipulatives or resources	• Listening to the teachers' explanation and taking the teachers 'examples as notes • See questions before assessment (memorise the questions) • Memorise theorems *Teachers confirmed that students learn Euclidean Geometry by memorising theorems.
Skills acquired/envisaged	Students hinted on having acquired: • Visualisation and analysis skills • Creativity • Understanding of Euclidean Geometry • Confidence in Euclidean Geometry problem-solving • Exploration skills • Cooperative learning skills • Inductive and deductive inquiry skills	• Listening skills • Memorisation skills • Student outlined the following as a challenge, • Deductive reasoning • Analysis skills • Euclidean Geometry problem-solving skills
Content acquired/envisaged	Students indicated that they have acquired content such as: • Geometry that exist in the environment • previsualisation and visualisation • Shapes and how to classify them according to properties	• Theorems • Student outlined the following as lacking: • Knowledge of properties of figures • Basic knowledge (forgotten the knowledge accumulated at lower grades)

Code	Experimental group	Comparison group
	and characteristics (relationships of figures) • Properties of shapes • Theorems	Teachers confirmed that students at Grade 11 lack basics. Teachers cannot reteach the basics due to limited teaching and learning time, inflexible work schedule

Observation and interviews contribute to pragmatic inferences based on inductive and inductive categories of data. All codes encompass both deductive and inductive inferences. That is, teaching strategies and learning strategies include both inductive and deductive approaches. Skills and content acquired are either inductive or deductive. The coded data led to inferencing that yielded the analytical report. The analytical report presented in this study comprises the categorised, coded protocols, experimental and comparison groups' visual models, and the mixed-method analytical report. The analytical report is summarised in Table 4.23 and from the table emanate the two models for the experimental and comparison groups.

4.4.1 Coded protocols: summary of qualitative data

Table 4.23: Summary of qualitative analytical report

	Teaching strategies	Learning strategies	Skills acquired	Content learned
Inductive ***Information or inquiry (pre-visualisation and visualisation)** ***Guided or directed orientation (Analysis)** ***Explication (informal deduction)**	Discussion, probing by asking questions, explanation, facilitation and demonstration by showing how to manipulate a tangram, showing how gadgets to search for relevant mathematical concepts and information;	Discussion, group work and presentations, cooperative learning, experiential, explanation, Investigating, discovery, imagination, manipulating a tangram and think critically; peruse textbook to search for information; manipulate their electronic gadgets to search for information; write to note findings and group agreements,	Application of inquiry; formulating relationships between and among a network of concepts, discovery, applying knowledge gained from experience; creativity, critical thinking, developing patterns, exploration, identifying and naming, classifying, presentation;	Geometry information related to real-world Properties of figures and application in formulating relationships of figures. *Relationships of figures or shapes *Mathematical vocabulary *Linking geometric and non-geometric systems
Deductive ***Free Orientation (Formal deduction)** ***Integration (Rigour)**	*Explanation *writing on the chalkboard and/or smartboard to emphasise some of the concepts and to show proofs of theorems	Listening to the teacher's presentation; writing notes or examples as written by the teacher on the chalkboard; Complete activities to try to commit what had been learned to memory	Memorisation (memorising theorems, terminology, teachers' examples as on the chalk board); reviewing examples in the prescribed textbook and on worksheets, group work, solving problems by completing activities	Existing theorem about cyclic quadrilateral (inscription and properties of inscribed figures). *Theorem (cyclic quadrilaterals; supplementary angles) and exterior angle = opposite interior angle *Application of theorems in problem-solving, that is, exercises in the textbook

Teaching strategies in both experimental and comparison groups show the influence of specific learning strategies. Learning strategies contribute to students acquiring particular skills and content. The inductive category refers to the basic levels of knowledge and understanding, that is, pre-visualisation, visualisation, analysis and informal deduction. The deductive category covers the higher levels of knowledge and understanding that are formal deduction level and the rigour level. Based on the four analytical codes, visual models for both the control and experimental groups were developed (Figures 4.16.and 4.17).

4.4.2 Comparison group – qualitative results

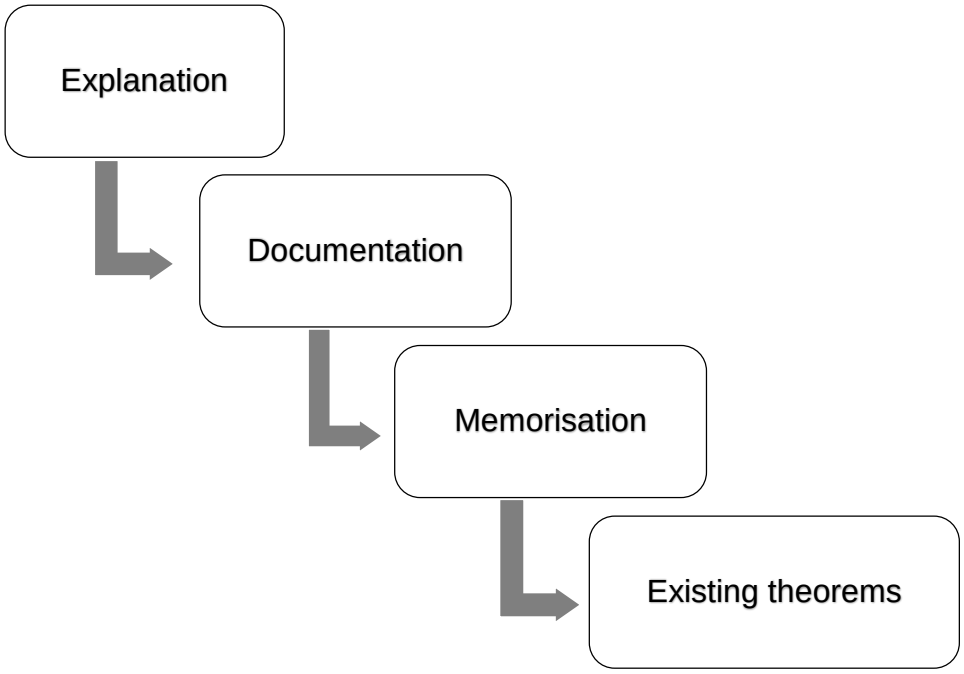


Figure 4.16: Visual model for the comparison group

Visual model for the comparison group indicate that teaching strategy applied is explanation. Teachers revealed during the interviews that the teaching strategies they prefer are question and answer, demonstration and explanation. However, in terms of demonstration as a teaching strategy, the teachers' views contradicted the observations where the textbook was observed as the only resource used in teaching. There is a lack of resources that hinders demonstration. In emphasis teacher B commented as follows, "*The problem is that we have only one textbook, as a teacher I cannot rely on one textbook, my lesson preparation is confined to only one textbook, if I want to demonstrate a sphere or a circle, we do not have those resources*. *As a*

teacher I could improvise, but it is only theoretical not practical; computers are available, but are not working". If there are no resources to advance demonstration, then demonstration cannot be a method used. Further, the teachers' preference for the question and answer method is based on the fact that the teachers asked questions to: (1) keep students alert in order to ensure that they are concentrating; and (2) make them focus throughout the lesson. This is confirmed by teacher C saying "*I prefer applying question and answer method. I apply this method to ensure concentration among students, and the advantage thereof is that students become alert and focus all the time. The disadvantages observed in this method is that as you ask students questions, they seem to have no knowledge of basics".* Teachers cannot use questions to probe because students lack basic knowledge. This confirms the observation where the teacher asked a question, students became alert but looked at the teacher clueless or not knowing how to answer the questions. In order to save time, the teacher answered her own question and explained further. Lack of resources for demonstration and teachers' challenges with using questions to probe confirms the observation that indicated that explanation is the commonly-used method in the comparison group. Explanation as a teaching method does not assist students to learn at their own pace. Individualisation or inclusivity is not considered. Teacher cannot cater for the needs of all students. This was confirmed when Laddie commented as follows: "*The teacher must be slow when teaching to help us to understand properly; the teacher is fast when teaching, teaches us everything, if two students understands, then she takes it that we all understand*". Explanation is necessary; however, based on observation and the teachers' and students' comments, the disadvantages of explanation outweigh its advantages for learning.

Explanation as a teaching strategy contributes to documentation as a strategy of learning. Students listen to the teacher's explanation of concepts or examples and write down in notebooks what the teacher writes on the chalkboard. Further, students commit to memory what they have documented and produce later for problem-solving. Most students do not realise the need for resources in learning; however, some showed awareness of the importance of learning resources, for example, Rod commented as follows: *"the teacher must liken constructions in class with constructions like of a house; teacher must associate with objects we know so that we can have practical in class; resources that we need for learning are things like stop*

sign, model of houses". Students and teachers view about the need for resources is challenged by the fact that teachers believe that there are no resources, therefore, learning cannot be hands-on. Text material such as textbook and question papers are a means to learn concepts and geometric problem-solving processes. For example, *Rod* echoed, appealed and said: "*Textbooks problems are not the same as problems in examination or test papers*". What is documented in the textbook and chalkboard is reproduced as a way of learning, according to the comparison group.

Documentation as a learning strategy leads to memorisation of information. Students stated during the interviews that the teachers must relate what they do in the class with examination and test during teaching. Students' mind-set is that teaching is aimed at preparing them for examination, that is, teaching must teach memorisation for examination. Students memorise questions and answers, that is, they see the test or examination questions beforehand and reproduce answers later when writing test. For example, Lebo commented: "*the pre-test was difficult, but the post-test was better because we have seen the questions before*". Lebo did not say the post-test was better because they have learned and acquired knowledge and understanding, but because they have seen the questions before.

Memorisation skill contribute to retaining existing concepts and theorems as content learned. Teachers confirmed that memorisation leads to learning existing theorems as content knowledge. Teacher A agreed that the other matter that teachers need to work on is to ensure that theorems are not memorised but applied in problem-solving. Teachers said that they would focus more on applying theorems in problem-solving as a teaching method rather than letting students memorise theorems. However, it was clear that geometric concepts that build up theorems were memorised during the teachers' explanations and proving of theorems. Memorisation of concepts and information contributes to difficulties and lack of confidence in handling Euclidean Geometry problems. All students interviewed in the comparison group attested to the fact that memorisation of theorems makes Euclidean Geometry difficult. Comparison group's students' comments about the difficulty of Euclidean geometry are summarised as follows:

"*My performance is good, but I do not always get more marks, but just a passing mark. I take Euclidean Geometry as difficult; Euclidean Geometry is complicated because if*

you claim about something you must be able to prove it; properties of figures are many, from ignorance we forget, and then we cannot solve the problems; when proving there are many reasons, when you do not know which reason to apply it becomes a problem. *Shapes and diagrams can confuse us, especially if the diagram is different from the teachers' example*".

4.4.3 Experimental group – qualitative results

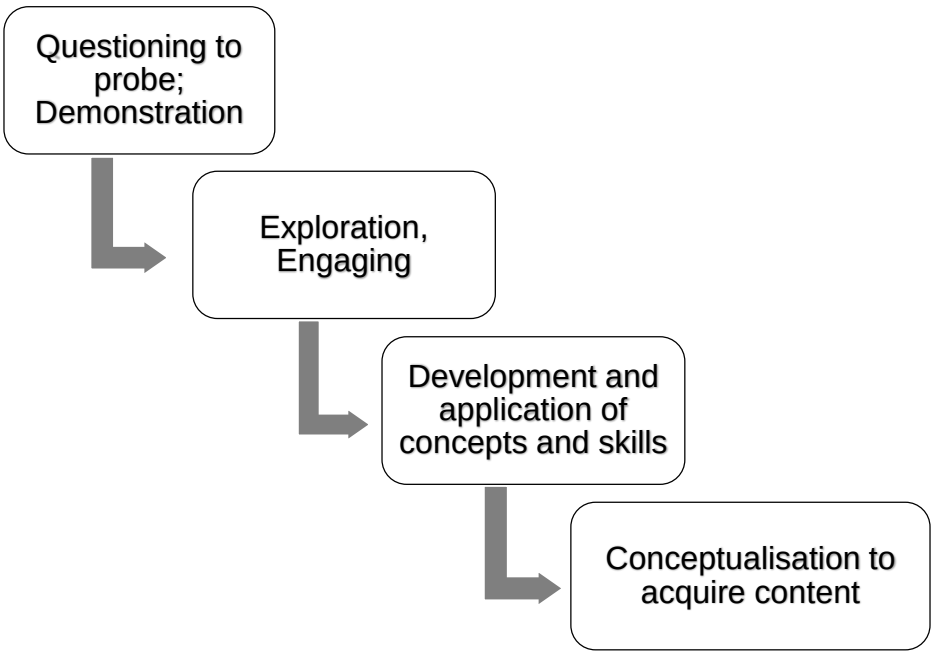


Figure 4.17: Visual mode for the experimental group

A visual model for the experimental group (Figure 4.17) displays that the facilitation strategy uses probing through questioning. The teacher facilitates by asking questions, explaining procedures, demonstrating how to manipulate a tangram, or demonstrating and guiding students on how to use their smartphones to search for relevant mathematical concepts and information. The teacher writes on the chalkboard to emphasise concepts when necessary. In facilitation, probing starts by moving from lower levels of knowledge and understanding through to higher levels of knowledge and understanding. Katli confirmed on an account about intervention and said: "*I enjoyed this lesson; mam was going through it bit by bit and step by step. I think every chapter should have a lesson like this. This lesson improved my way of understanding Euclidean Geometry*". In addition, most of the students responded as the facilitator was probing and they managed to recall the lost knowledge. Lemo agreed and said:

"*I learned the properties of shapes; I remembered something that I forgot*". Teaching through probing encouraged students to regulate their learning and recall lower level knowledge before they could advance deductive reasoning in the formal stage of knowledge and understanding.

The visual model for the experimental group further displays that questioning through probing yields learning by exploration and engaging. Students discuss in groups and engage in cooperative learning. In addition, the learning strategies of engagement and exploration include experiential learning, explaining ideas, investigating, discovery, imagination, using a smartphone and manipulating a tangram, writing to communicate ideas and presenting to share ideas. To give an account on how they should be taught, students hinted that they would understand concepts better if resources were provided for them, such as mathematical instruments, graph paper, calculators and shapes, that is, visual geometric models or manipulatives. Students have realised that learning should be hands-on and resources are needed for hands-on, self-regulated learning.

The third code of the visual model for the experimental group, that is, development and application of skills, is shown to depend on learning strategies. Specifically, the skills acquired include pre-visualisation, visualisation and analysis skills, creativity, application of experiential learning, exploration, inductive and deductive reasoning, establishing relationships between figures, problem-solving and presentation skills. *Didi* attested to having improved in learning and said: "*I learned how to be creative, how to solve things easily, especially in mathematics; I have now a knowledge of something based on Euclidean Geometry*".

Skills acquired influence the fourth code of the model, that is, conceptualisation. Tee commented as follows, "*I have learned a lot about shapes; how to classify them according to their properties and characteristics. Euclidean Geometry is not a difficult subject; it needs attention; that is all*". Conceptualisation orientates students to the application of concepts, vocabulary and relevant geometric language in problem-solving processes. In the experimental group, conceptualisation was evident and students acquired information related to real-world, geometric concepts and vocabulary, properties of figures, relationships between figures; inscription of figures, properties of inscribed figures and existing theorems about circles and cyclic quadrilateral.

Generally, the model shows that all codes adhere to both inductive and deductive categories. Inductively, facilitation probed and oriented students to inductive levels of

knowledge and understanding, that is, pre-visualisation, visualisation, analysis and informal deduction. Deductively, probing oriented students to deductive reasoning and understanding, and formal deduction where students understand the origin of facts that comprise theorems, for example, inscribing a quadrilateral in a circle to form a cyclic quadrilateral and thereafter exploring the properties of inscribed figures based on the knowledge of properties of the quadrilateral and circle separately. Students in the experimental group showed increased confidence in handling Euclidean Geometry problems through inquiry. This is evident in the following reflections: "*I have now a knowledge of something based on Euclidean Geometry; Euclidean Geometry is not a difficult subject; it needs attention; that is all; This lesson improved my way of understanding Euclidean Geometry; I remembered something that I forgot; I have learned and managed to focus based on shape and space; I would like the teacher to come back again and repeat on finding the volume of shapes*". The visual model for IBF and learning in the experimental group contributed to students' improvement in learning strategies, skills and conceptualisation.

4.4.4 Concluding qualitative findings

Table 4.24: **Contingency framework for the qualitative data**

Code		**Experimental**	**Comparison group**	
Teaching strategies	Inductive	✓		Only experimental
	Deductive	✓	✓	Both
Learning strategies	Inductive	✓		Only experimental
	Deductive	✓	✓	Both
Skills acquired	Inductive	✓		Only experimental
	Deductive	✓	✓	Both
Content acquired	Inductive	✓		Only experimental
	Deductive	✓	✓	Both
		Inductive and deductive in all four codes	Deductive in all four codes	

Based on Table 4.24, what emanated from the qualitative data is that in the experimental group, teaching strategies adhered to both inductive and deductive approaches; therefore, learning also progressed from inductive to deductive approach. A combination of both deductive and inductive learning yielded both deductive and inductive skills and content acquired. By contrast, Table 4.24 reveals that in comparison groups teaching strategy was more deductive which led to deductive

learning. Deductive learning in turn produced deductive skills and content. Table 4.21 outline the skills observed in each code while table 4.25 shows the classification of actions and skills as either inductive or deductive in each code.

Table 4.25: Comparison of inductive and deductive teaching and learning (cf table 4.21)

	Inductive	Deductive
Teaching strategies	Discussion, probing by asking questions, explanation, facilitation and demonstration by showing how to manipulate a tangram, showing how gadgets to search for relevant mathematical concepts and information;	*Explanation *writing on the chalkboard to emphasise some of the concepts and to show proofs of theorems.
Learning strategies	Discussion, group work and presentations, cooperative learning, experiential, explanation, Investigating, discovery, imagination, manipulating a tangram and think critically; peruse textbook to search for information; manipulate their electronic gadgets to search for information; write to note findings and group agreements,	Listening to the teacher's presentation; writing notes or examples as written by the teacher on the chalkboard; Complete activities to try to commit what had been learned to memory.
Skills acquired	Application of inquiry; formulating relationships between and among a network of concepts, discovery, applying knowledge gained from experience; creativity, critical thinking, developing patterns, exploration, identifying and naming, classifying, presentation;	Memorisation (memorising theorems, terminology, teachers' examples as on the chalk board); reviewing examples in the prescribed textbook and on worksheets, solving problems by completing activities
Content learned	Geometry information related to real-world Properties of figures and application in formulating relationships of figures. *Relationships of figures or shapes *Mathematical vocabulary *Linking geometric and non-geometric systems	*Confirming stated theorem about cyclic quadrilateral (inscription and properties of inscribed figures). *Theorem (cyclic quadrilaterals; supplementary angles) and exterior angle = opposite interior angle *Application of theorems in problem-solving, that is, exercises in the textbook.

4.5 TRIANGULATION OF RESULTS

Data revealed that the four codes, that is teaching strategies, learning strategies, skills acquired and the content acquired appear in both inductive and deductive stances. Table 4.25 indicates that in the experimental group, both inductive and deductive traits were developed in all codes while in the comparison group deductive traits dominated. The descriptive quantitative analysis in Table 4.19 shows that there is an improvement in pre-visualisation, visualisation and informal deduction whereas there is no constant improvement in the comparison group. In the experimental group, performance in pre- and post-tests was recorded as statistically significant at $p< 0.05$, $p= 0.0001$ with a large effect size of 0.55. This implies that the intervention worked towards improving students' performance, and the visual model for IBF and learning shows a better understanding of how students learn Euclidean Geometry.

In contrast, in the comparison group, the difference in performance on the pre- and post-tests was recorded as not significant, that is, $p>0.05$, $p = 6201$. The implication is that the traditional axiomatic approach did not contribute to any improvement in learning in the comparison group. Therefore, the visual model of teaching and learning in the comparison group does not provide a better method of learning Euclidean Geometry. That is, the way of learning in the comparison group revealed that Euclidean Geometry cannot be learned when facilitation is only done by means of explanation. Explanation leads to memorisation of documented facts which hones the memorisation skills, but the proving of theorems is learned only deductively. The way of learning promoted by explanation in the comparison group did not contribute to learning development as shown by the statistically not signif icant value of $p>0.05$, $p = 0. 6201$.

This study therefore found that facilitation in Euclidean Geometry should consider both inductive and deductive teaching through questioning to probe. Inductive and deductive IBF should focus on influencing inductive learning as a basis, and thereafter, influence deductive learning at a higher level of learning and understanding. Inductive and deductive learning should happen through exploration and students' engagement. Students' engagement through exploration influences development and application of both inductive and deductive skills in order to develop both inductive and deductive conceptualisation and content knowledge. The statistical results obtained in the

experimental group intergarted with the qualitative revealed that the significance value and large effect size were achieved through a combination of inductive and deductive teaching strategies. Further, the qualitative data revealed that, in the experimental group, learning progressed from lower levels (inductive levels) to higher levels (deductive levels) of knowledge and understanding.

In addition, the variances between the pre-tests in the two dependent groups were recorded by a pooled statistical method as not significant ($p > 0.05$, $p = 0.3181$).

Similarly, differences were observed through participatory and non-participatory observation. In the comparison group, learning commenced at the deductive level of knowledge and understanding. The learning that starts at deductive level has been confirmed by research (Alex & Mammen, 2014) to be a barrier to cognitive achievement in learning Euclidean Geometry. This study indicates that both inductive and deductive inquiry supplement each other in learning. The experimental group progressed from inductive to deductive learning which is shown to have contributed to improvement in cognition in higher levels of knowledge and understanding. The improved results of the experimental group show that IBF promoted self-centred learning, exploration, discovery, self-regulated learning, reflective thinking and metacognition. In addition, students acquired vocabulary and mathematical language in order to apply concepts and knowledge in problem-solving. The constant improvement in levels of knowledge and understanding reflected in the descriptive analysis is confirmed in Table 4.19, in that the students progressed from the concrete to the abstract in chronological order of the levels of knowledge and understanding. Similarly, the lack of improvement in the comparison group specifically in pre-visualisation and visualisation levels of knowledge and understanding (Table 4.19) is confirmed by the descriptive analysis, in that pre-visualisation and visualisation knowledge and understanding were not supported by the traditional teaching method and therefore, students attempted the highest deductive knowledge without proper knowledge and understanding of basic lower levels. The comparison group results that were proved to be not statistically significant ($p = 0.6201$, $p > 0,05$) resulted from the traditional axiomatic teaching approach that influenced learning through taking notes, listening to the teacher, recalling information presented by the teacher in order to answer questions, trying to follow the steps presented by the teacher to solve problems individually and in groups. In essence, students in the comparison group

relied on memorising what the teacher did in order to retrieve the information later. A difference was observed between the teaching methods in the two groups, and the diverse influence that the two teaching methods have on the two groups was noted.

However, the combination of inductive and deductive teaching yielded both inductive and deductive skills and content. IBF, IBL, skills and content acquired in the experimental group contributed to improvement as revealed by pre- and post-test results. The mean value for the pre-test in the experimental group was recorded as 8.48, and improved to 16.26 in the post-test. The percentage increase was recorded as 47.8%. The summary of students' achievement in advanced and proficient levels for all levels of knowledge and understanding (Table 14.19) showed an improvement in both inductive and deductive levels of knowledge and understanding. Further, the paired t-test analysis showed a significant improvement in post-test results to support the fact that IBL contributed to better performance.

Further, Table 4.24 revealed that in the comparison group, deductive teaching and learning were the prominent approaches. Deductive teaching influenced deductive learning and contributed to students acquiring deductive skills and content. The deductive skills and content affected the students' performance in the post-test, hence the statistically not significant value of $p= 0. 6201$, $p > 0.05$.

Both the teaching approaches used group dynamics to engage students in problem-solving. However, in the experimental group, cooperative learning including collaborative investigations, research, exploration, discussion and sharing of self-found ideas were evident. Furthermore, group work was compulsory and continuous in the experimental group. By contrast, group work was not compulsory and continuous in the comparison group. The shared ideas were based mostly on recalling what the teacher taught and referring to the notes they had written during the lesson. Some students worked as individuals although the classroom setting confined them to a group. The other common feature was that the facilitators used questioning. In the experimental groups, I applied probing as a form of questioning, while in the comparison group teachers applied the questioning technique in asking students to recall what they had learned previously. In experimental group, the students applied skills such as searching for information and exploration in order to provide correct answers, but in the comparison group, students struggled to recall the previously-

learned information without any clues in order to provide answers. Although, there was an improvement in both groups, with a 47,8% improvement in the experimental group, the improvement in the comparison group was only 2.9%. Lebo's comment that the pre-test was difficult, but the post-test was better, because they have seen the questions before, give clarity about why the 2.9% increase is obtained in the comparison group, and this is an indication that learning took place based on the notion that the question tackled before could be recalled, no matter how little. Lebo's statement hints that the improvement of 2.9% in the comparison group appears to have been contributed by few students who are good at recalling previous encounters, that is, students who are good at learning by memorising. Learning took place, based on traditional method of learning.

Table 4.19 shows development in all levels of knowledge and understanding in the experimental group. However, there was no consistency in the results of the comparison group. For example, 0% improvement was recorded at pre-visualisation, visualisation and informal deduction levels and no student achieved at advanced and proficient levels. These levels, mostly the pre-visualisation and visualisation, require more inductive teaching and learning. Inductive teaching and learning at the lower levels of knowledge and understanding were not evident in the comparison group (Table 4.24). This is confirmed by a comment from Rod that: "*when teaching us, the teacher must associate with objects we know so that we can have practical in class".* Further, teachers in comparison groups attested to the challenge experienced in teaching the inductive part of Euclidean Geometry. Challenges were restricted to lack of resources. Teacher B commented: "*the problem is that we have only one textbook, my preparation is confined to only one textbook*. *If I want to demonstrate a sphere or a circle, we do not have those resources"*. However, during the intervention in the experimental group, students used the internet available at school and on their cellular phones to search for the information they needed such as pictures, concepts and sketches.

Teacher X further confirmed that teaching begins at the formal deduction level and lower levels of knowledge and understanding are not developed. The teacher highlighted that a triangle is what the students learn about at primary school and as a Grade 11 teacher, the teacher assumed that students knew all about triangles. Students confirmed that shapes and diagrams confuse them, they do not know

properties of shapes. Petty, a student from the comparison group confirmed, "*at another school they taught us, but we have forgotten the properties of figures"*. In addition, Loli also from the comparison group commented, "*from ignorance we forget, and then we cannot solve the problems*". Therefore, lack of inductive skills affected the levels of knowledge and understanding negatively. Improvement was shown in analysis and rigour in the comparison group but the percentage increase was still lower than that of the experimental group.

In comparing the pre-test results in the advanced and proficient categories (Table 4.19) in the comparison and the experimental groups, the comparison group performed at a higher rate (4.3%) as compared to 1% of the experimental group. However, in the post-test, the comparison group declined to 1.4% and the experimental group achievement percentage increased to 18%. This research comments that the large number of attrition in the comparison group could be the reason of decline in a number of few students who performed at higher levels. Further, as teaching in the control group did not infuse pre-visualisation, visualisation and informal deduction, chances were high that students could not recall concrete spatial reasoning as they were stronger on abstract reasoning that they have learned before writing the post-test, therefore, a declined performance in some levels of knowledge and understanding. Performance in the comparison group was unsatisfactory, even in higher deductive levels of knowledge, irrespective of the fact that teaching in the comparison group started at deductive level as observations and interviews have revealed. This supports the literature that teachers teach at higher levels while the students have not understood the lower levels. Therefore, a conclusion can be drawn that with IBF facilitation, students learn by inquiry that assists them to recover the knowledge at inductive lower levels in order to perform better at higher deductive levels. In addition, a significance value of 0.002 ($p<0.05$) was recorded which confirms that IBL contributes to learning better than the traditional axiomatic approach.

4.5 CHAPTER SUMMARY

Inferential statistical analysis using the paired t-test statistical test showed that intervention through IBF is effective. The experimental group that received IBF intervention improved (M= -7.773, SE= 0.7146) while those who did not receive intervention (M= -0.221, SE = 0.4429) did not improve. This difference (-7.547), 95%

CI (-8.08, 5.69), was significant at t (10.88), p = 0.0001, p<0.05 and represented a large effect size of 0.55. Similarly, the descriptive statistical analysis across levels of knowledge and understanding as summarised in Table 4.19 reveal that there was a constant improvement across levels of knowledge and understanding in the experimental group between the pre- and post-test. However, the comparison group showed improvement only in the analysis level. On other levels, the comparison groups showed no improvement. IBF assisted students in the experimental group to improve in all levels of knowledge and understanding. Generally, in the experimental group, student development was perceived in lower levels of knowledge and understanding through to higher levels of knowledge and understanding. The quantitative results showed that, to a large extent, IBF assisted students to improve in the post-test and in all levels of knowledge and understanding.

Qualitative data analysis showed that the improvement in the experimental group (visual model for the experimental group) was caused by using probing questions as a teaching strategy. Probing teaching strategy influences learning by exploration and engagement that influenced development and application of skills such as conceptualisation in pre-visualisation, visualisation, analysis, creativity, application of experiential learning, exploration, inductive and deductive reasoning. On contrary, the qualitative data analysis showed that in the comparison group (visual model for the comparison group) lack of improvement was initiated by explanation as a teaching strategy that prompted documentation of facts as a learning strategy, and memorisation as a skill developed. Content memorised was existing concepts and theorems. Explanation, documentation and memorisation of existing concepts and theorems contributed only to deductive reasoning; inductive learning was not included in the comparison group. Students in the comparison group found deductive reasoning difficult for them, while students in the experimental group perceived that inductive reasoning is essential in deductive reasoning.

The triangulation of the quantitative and the qualitative data shows that the visual model for the experimental group is effective in facilitating the learning of Euclidean Geometry; the intervention contributed to improvement in the experimental group with a large effect size. The model supports the directional hypothesis as there was a significant difference between the experimental and the comparison group, that IBF

influences IBL and therefore enhances students' problem-solving skills in attaining higher deductive levels of geometric reasoning.

CHAPTER 5
INTERPRETATION OF DATA AND DISCUSSION

5.1 INTRODUCTION

This study argues that implementing IBL through IBF enhances students' problem-solving skills in Euclidean Geometry. Further, this study asserts that the Grade 11 students' knowledge and understanding of pre-visualisation, visualisation, analysis and informal deduction influence learning at formal deduction level. The interpretation and discussion of results involves elaborating on the hypothesis, responding to the research questions and evaluating the research objectives. Reference is made to the theories and literature adopted for this study. In addition, the interpretation and discussion of results focus on clarifying the argument and assertion presented in this study by contributing a model as a framework of IBF.

5.2 HYPOTHESIS

This study posits a null hypothesis that there is no difference between IBF and the traditional axiomatic approach. The related directional hypothesis states that IBF in Euclidean Geometry influences IBL and consequently increases the students' knowledge and understanding of the formal deduction level through lower levels of knowledge and understanding, that is, pre-visualisation, visualisation, analysis and informal deduction.

The paired t-test statistical test compared the variances between the experimental and the comparison group. Utilising paired t-test statistical test, the test for normality applied comparing the pre-test variances of the two groups. The comparison between the two groups revealed that pre-test variances were not significantly different ($p=0.3181$; $p> 0.05$), that is, they are statistically equivalent. Based on the equivalence of the pre-test in the experimental and the comparison groups, the pooled standard error method was relevant because the variances of the two groups are equivalent. The pooled standard error method confirmed that the variances were not statistically different and were equivalent at $p=0.2677$, that is, $p> 0.05$. The Satterthwaite approximation of standard errors verified the findings from the pooled standard error. The Satterthwaite approximation of the standard errors revealed the same result as the pooled standard error method confirming that the variances in the pre-test results

between the control and the experimental groups were not statistically different. These result confirmed that the groups were at the same level before the intervention. The test of normality before any form of intervention or teaching had a p-value > alpha ($p=0.3181$, $p> 0.05$). From these results, the assumption of normality is accepted, meaning that there is a normal distribution in the pre-tests between the two groups, that is, both experimental and control groups were at the same level at the beginning, at pre-test level before any form of teaching could commence. It is worth noting that at the pre-test level all students form both experimental and the comparison groups were departing from a traditional axiomatic approach situation. The normal distribution between that two groups initiated that the paired t-tests statistical test could proceed. On the same note of testing variances, the pooled method of standard error revealed that the variances in the post-test results were statistically significant ($p= 0.0001$, $p<0.05$). This shows that after intervention there was a non-normal distribution between the two groups in the post-tests, that is, the variances of the two groups were not equal. This also indicated that the paired t-test statistical analysis could proceed (Field, 2013; Rosner, 2006). Therefore, the paired t-test statistical test was an appropriate method of comparing the mean values of the experimental and the comparison group in order to track the most improved group.

Applying the paired t-test statistical test to compare the mean values between the experimental and the comparison group, focusing on pre- and post-tests, the conclusion was reached that, in the experimental group, the variances between the pre- and post-test mean scores (mean difference of 7.7732) were statistically significant ($p=0.001<0.05$). Most individuals in the experimental group scored higher in the post-test (range between 1 and 41) than in the pre-test (range between 0 and 27). In the comparison group, the variances in the mean scores (mean difference of 0.2206) in the pre- and post-test were not statistically significant ($p= 0.6201> 0.05$). The t-test revealed that improvement was not observed between the pre-test (range between 0 and 27) and the post-test (range between 0 and 25). The pre- and post-test scores were statistically the same for the comparison group. In addition, looking at the normality plot (figure 4.1) for the experimental pre and post-test, data deviates more to the side of the post-test, it is skewed to the post-test side. This indicates a non-normal distribution that the pre and post test scores are not equal. Much better performance is evident in the side of the post-test scores.

In the comparison group, the normality plot (figure 4.2) shows that data falls on the line and equally distributed and the sides of both pre and post-test. Therefore, data is skewed to neither side, shows a normal distribution that the comparison group pre and post-test mean scores are equal.

Therefore, the experimental group improved in performance but no improvement was evident in the comparison group. The means in the experimental group are significantly different, therefore, a conclusion is reached that the independent variable has an effect on the dependant variable (Field, 2016). The variable of the traditional axiomatic teaching approach was substituted with IBF. The significant difference between the means of the experimental and the comparison group shows that the IBF intervention led to improvement in performance of the experimental group.

The conclusion is reached that there is a statistically significant difference between the experimental and the comparison group when comparing the post-test results. Quantitative data revealed that on average, participants that received intervention through IBF acquired IBL skills (M= -7.773, SE= 0.7146) while the results for those who did not receive the intervention (M= -0.221, SE = 0.4429). This difference (-7.547), 95% CI (-8.08, 5.69), was significant at t (10.88), p = 0.0001, $p<0.05$ and represented a large effect size of 0.55. The large effect size emphasises that IBF contributed significantly towards improvement in IBL and that IBF can be considered as an appropriate method of teaching in Euclidean Geometry.

The paired t-test statistical test results in this study showed a statistically significant difference of $p< 0.05$, that is, 0.0001, in the experimental group as compared to the comparison group ($p>0.05$, that is, p= 0.6201). The statistical significant differences in this study, show that the IBF (intervention) and the traditional axiomatic approach have different impacts on student performance. The intervention was successful and contributed towards a significant improvement in the experimental group, whereas the traditional axiomatic approach contributed towards a not significant effect on the learning of the comparison group. Therefore, this study rejects the null hypothesis and the rejection of the null hypothesis provides enough evidence to support the alternative hypothesis, that is, IBF strategies in teaching Euclidean Geometry influence IBL; consequently, increasing students' understanding of the formal deduction level through pre-visualisation, visualisation, analysis and informal deduction.

The IBF intervention is shown to have contributed towards enhancing IBL as a means to problem-solving. The contribution of IBF includes guiding students through engagement in concept discovery and application (Love, et al., 2015). Through active involvement in their learning, students used learning by inquiry; that is, after discovering concepts, they applied the concepts in exploring, conjecturing, creating, and communicating in order to solve problems. Therefore, performance in the experimental group improved compared to the comparison group that received the traditional approach of teaching. Therefore, the null hypothesis that there is no difference between IBF and the traditional axiomatic is rejected and directional hypothesis is supported

Further, based on the fact that IBF increases IBL for student development in geometric problem-solving, this study provides answers to the following sub-questions: (1) To what extent does IBF assist students to reach higher levels in geometric thinking? (2) Does inquiry-based intervention influence students' development in geometric thinking through the Van Hiele levels? (3) How does IBF lead to a better understanding of how students learn Euclidean Geometry successfully?

5.3 RESPONDING TO THE RESEARCH QUESTIONS

5.3.1 To what extent does IBF assist students to reach higher levels in geometric thinking

A statistically significant difference of $p < 0.05$, that is, $p = 0.001$, and $p > 0.05$, that is, $p = 0.6201$ between the experimental and the comparison group confirms that IBF influences inquiry skills in geometric problem-solving. However, the question asked is how does this compare to the traditional axiomatic approach and to what extent does IBF equip students with problem-solving skills more than the traditional approach can offer. The test of normality after the pre-test showed that both the experimental and comparison groups were at the same level, that is, influenced equally by the traditional axiomatic approach. The pre-test results indicated that students in both groups committed the same errors such as inappropriate application of geometric vocabulary, language and wrong spelling of geometric words. For example, writing the word circle as "*cycle*", quad as "*quaard*" (cf Table 4.20). Wrong spelling and application of vocabulary show lack of conceptual understanding and meaning of geometric concepts, therefore, wrong application of concepts resulted in lack of meaning in

solutions. However, the post-test showed improvement in the experimental group, that is, errors were minimised and students in the experimental group improved in spelling, vocabulary and application of concepts (cf Vignettes). When students make mistakes or fail to apply rules correctly, it shows that learning occurred procedurally, but conceptual understanding of the rule is evident when a student is in a position to see why the rule does not apply (Luneta, 2017).

The visual model for the comparison group (Figure 4.16) shows that the teaching of Euclidean Geometry in the traditional axiomatic approach occurs through explanation which leads to learning by documentation and memorisation of existing theorems. This study asserts that the traditional axiomatic approach results in erroneous procedural problem-solving, and hence contributes to cognitive load. Cognitive load is evident in the pre-test that indicated that learning could not take place as procedural knowledge did not help students to apply concepts correctly in problem-solving. However, conceptual knowledge (attained through exploration and active engagement) as contributed by IBF eases the cognitive load.

The post-test in the experimental group showed that learning took place because students were able to apply concepts correctly in their solutions and errors committed in the pre-test were not committed in the post-test. Errors in the experimental group post-test problem-solving were minimised and the implication is that IBF assisted student to employ self-centred learning in order to formulate own concept images, utilising cognitive structures linked to self-explored concepts rather than utilising only cognitive structures linked to what was imparted by the teacher. Further, minimal errors imply that students could explore and discover concepts meaningfully and in totality rather than committing definitions, formulae and concepts presented by the teacher to memory. In addition, students attained the skill of connecting prior knowledge or existing concept image to new knowledge. Another milestone that the students in the experimental group accomplished was the ability to construct concepts well, and therefore, to demonstrate good conceptual understanding. Therefore, this study suggests that facilitation of Euclidean Geometry in Grade 11 should focus on probing through questioning to allow students to engage in exploration and develop conceptual understanding and skills for application in conceptualisation and ultimately to acquire Euclidean Geometry content knowledge.

This study affirms that teaching should start at lower levels of knowledge and understanding; that is, it should follow a chronological fixed order of levels to encourage students to progress from the lowest to the highest level of knowledge and understanding. This is in agreement with Van Hiele's idea that in learning Euclidean Geometry, students follow hierarchical levels that are in a fixed order. Visualisation appear to be the most difficult learning stage for most of the student. It is the level where students' scores are low in both pre and post-test. In visualisation specifically, the study has shown that students experience difficulties in making decisions based on perception, that is, inductive reasoning. Further, the lower performance in visualisation is accounted in this study that students struggle with spatial imagery therefore cannot apply critical thinking to critically represent spatial objects from diverse perspectives. The other stages especially those based on deductive reasoning appear to be better in performance. This study accounts on this that students are used to applying deductive thinking in problem solving. The students are acquainted of applying ready-made substantiations in geometric proofs, memorise the proofs in order to apply in problem solving. As this study attempted to assist students to use inductive reasoning to advance deductive reasoning, it is more evident that the part that was more easy to students seemed to be the part they are used to, that is, deductive reasoning. However, in the pre-test all students experienced challenges in tackling both the deductive and inductive part. Nonetheless, the analysis stage, where abstract reasoning coincides with concrete reasoning showed improvement. To show that learning by inquiry supported them in all levels, students in the experimental group recorded improvement in all levels, no matter how little in the visualisation stage.

The visualisation stage remains lower in achievement; however, improvement was recorded between the pre and post - test. A report by the National Academy of Sciences (2018) states that spatial abilities are essential in diverse fields, for example in creating visual images for processes occurring in the real world, in graphic technologies, in computing and more; nonetheless, spatial skills work in relation to other abilities such as verbal skills, memory retrieval and logical reasoning. Therefore, this study supports Van Hiele's statement that geometric levels of knowledge and understanding are hierarchical. Assertion is maintained in this study that a foundation from lower levels is necessary in order to achieve at higher levels of understanding.

In addition, the report by the National Academy of Science motivates that in order to relate diverse capabilities effectively, students should possess the capability to switch from one ability to another when necessary. Further, the report alerts that the deficit in one ability may be compensated by excellence in other abilities. It is evident in this study that spatial abilities are compensated by other abilities like memorisation. This is the reason that visualisation stage remained lower as students in the experimental group were also struggling to learn how to apply critical thinking based on space and shape; construct mental representations of real objects and present spatial objects from diverse perspectives. Further, compensation by memorisation is evident in the analysis stage that operates with concrete and lower level of abstraction. Teaching that allow students to explore the facets of spatial abilities, that is, to apply spatial orientation, spatial visualisation and explore spatial relations, impacts on the improvement of pre-visualisation and visualisation. This is evident in this study based on the finding that through inquiry, students' performance in the visualisation on stage in the experimental group display a shift from 0% to 1%, while no change (0%) was observed in the comparison group (cf Table 4.19). In light of reason furnished, this study asserts that IBF assisted student to apply inquiry skills and were able to develop in all levels and the more the knowledge students acquire in lower levels the more the chances are that they will be able to retrieve and apply at the higher levels of knowledge and understanding.

This study segregates the levels of knowledge and understanding in geometry by using the degree of concreteness and abstractness, and contributes a middle level between the lowest and highest levels of knowledge and understanding. That is, pre-visualisation and visualisation are more concrete and basic levels of geometric knowledge and understanding; analysis and informal deduction are a combination of both concrete and abstract factors at the middle levels; and informal deduction and rigour are abstract and occur at the highest levels (Figure 5.1). In addition, it is apparent in this study that at the basic levels of geometric knowledge and understanding, facilitation through utilising diverse sources or teaching and learning aids promotes learning by applying experience, perception and visualisation. This study observed that the middle levels of knowledge and understanding, that is, firstly analysis and secondly informal deduction lies between the concrete and the abstract. Hershkowitz et al. (1996) confirm that diverse perspectives of shape and space as well

as the interplay between concrete and abstract contribute to analytical thinking in the cognitive process. This study supports this principle because during analysis, IBF leaded students to employ analytical skills in order to explore what they have perceived and visualised through reference to diverse learning aids. Informal deduction also refers students to concrete material while they apply analytical thinking to formulate conjectures relating to properties and relationships of figures through abstract representations. The relation between concrete and abstract reasoning is emphasised by Hoekstra (2016) motivating that mental development relies on concrete reasoning which provides the appropriate base for abstract reasoning. Analysis and informal deduction allow students to utilise concrete connotations to establish deductive connotations.

This study views the middle stage as convenient to students as it is a stage that requires lower level of abstract thinking. This is evident as displayed on a summary of students' achievement in advanced levels (table 4.19). Both the experimental and the comparison group achieved better at analysis as compared to other levels. This also supports the finding that the comparison group showed improvement only at the analysis level (section 4.5). This middle level contributed by this study shows that analytical thinking is essential in geometric reasoning. Forster, Friedman and Lieberman (2004) highlight that analytical tasks need to be processed concretely whereas abstract mental representations require abstract reasoning. Basic, concrete levels (that is, pre-visualisation and visualisation) mean acquiring essential concepts, images and vocabulary for application at the middle and highest levels. In addition, both concrete and deductive connotations established at the middle levels are essential for conceptualisation and application at the highest abstract levels. Therefore, this study contributes a hierarchy of levels of knowledge and understanding that enables IBF and ultimately IBL that proceeds from the lowest inductive level through the middle levels to the highest deductive levels of knowledge and understanding (Figure 5.1).

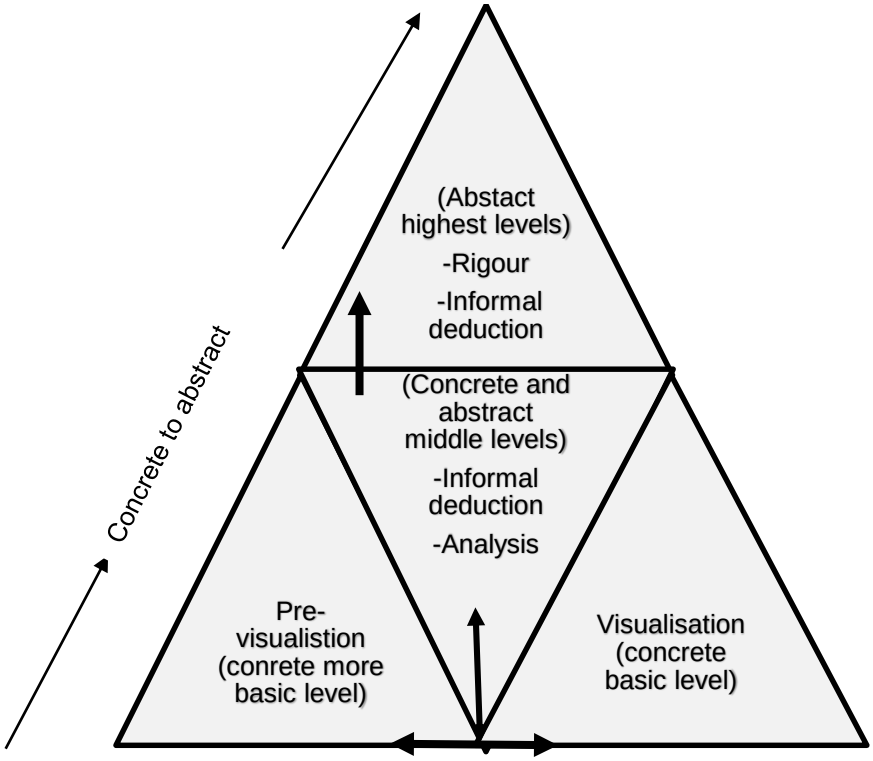


Figure 5.1: Hierarchical concrete to abstract levels of knowledge and understanding

Van Hiele's theory confirms that students cannot skip a level and they must be competent at the lower levels in order to advance to higher levels. This study also deduces that in both teaching and learning, knowledge acquired in the basic stage is important for the succeeding stage. Figure 5.1 displays interconnected relationships among the geometric levels of knowledge and understanding. Failure to connect the levels in a chronological order creates a gap in learning; for example, the inability to establish declarative or conceptual knowledge before the establishment of procedural knowledge. This is evident in the comparison group (cf. Table 4.24) where teaching did not consider basic and middle levels. This study confirms that the traditional axiomatic teaching approach operates on only abstract levels of knowledge and understanding and students strive to memorise in order to acquire procedural knowledge; as a result, learning does not take place particularly for the majority of students who are unable to memorise. Students' achievement in advanced and proficient levels (Table 4.19) indicates a difference in both groups in the pre- and post-test. The experimental group recorded a consistent improvement in all levels of knowledge and understanding, whereas in the comparison group an inconsistent and minimal improvement is evident. Further, it is deduced in this study that based on the improvement by the experimental group at all levels, teaching that commences by

assisting students to achieve lower inductive levels, leads to improvement at higher deductive levels.

Non-participatory observation, that is, observation in comparison group, revealed that abstraction is needed at all stages of teaching; for example, in order to assist students to recall prior knowledge, teachers ask questions that demand abstract thinking for students to recall from memory what they might have acquired in the lower grades or the previous lesson. Recalling the memorised information without any prompts or clues poses difficulties to students. Furthermore, observation (non-participatory) revealed that most students were unable to respond to questions related to recalling pre-knowledge, but the teachers proceeded with presenting new knowledge. Therefore, acquiring new knowledge becomes difficult and students are compelled to memorise the new information without linking it with existing knowledge. Thus, higher deductive levels such as the formal deduction level of knowledge and understanding remain a challenge when students are supposed to apply the memorised theorems. Van Hiele's theory of knowledge and understanding in Euclidean Geometry outlines that during learning in geometry, students should learn a network of relationships that link concepts and processes, which are eventually organised into schema (Clements & Battista, 1992). Further, Van Hiele's theory accentuates that students do not learn only facts, rules and names. Hamilton and Ghatala (1994) state that memory should contain the meaning of information, not its exact form. In concurrence, this study avers that learning facts, rule and names calls for students to memorise, that is, to commit to the schemata the exact form of information, whereas inductive inquiry in conjunction with deductive inquiry enables students to commit into schemata knowledge and understanding of concepts, network of relationships and processes, namely, the meaning of information acquired.

As compared to the traditional axiomatic approach, IBF equips students with problem-solving skills. IBF promotes perception and recognition, and leads students to conceptualise their perceptions and visualisation for analysis and conjecturing. Furthermore, IBF enables students to validate their conjectures and apply acquired knowledge and understanding in problem-solving.

The study revealed that the IBF teaching process encompasses six stages. The first stage involves orientation by probing and providing resources to promote perception.

Secondly, orientation by probing encourages recognition. The third stage involves probing to ensure conceptualisation through analysis. The fourth stage involves probing to encourage conjecturing; the fifth stage requires probing to encourage inferencing and confirmation of conjectures through existing concepts, axioms and theorems; and the sixth stage requires probing to allow students to apply concepts, vocabulary, mathematical language, axioms and theorems in problem-solving. The main teacher action is probing and providing resources, as IBL is student-oriented or student-centred. In the student-centred learning process, diverse resources are essential for facilitation of learning. Further, in IBF, probing is done through questioning. This study agrees with Chao, Murray and Star (2016), that facilitation through probing enables the facilitator to (1) understand the students' mathematical thinking; (2) evaluate the students' mathematical knowledge; (3) guide the students in their own thinking; and (4) encourage students to articulate their own ideas.

5.3.1.1 Orientation by probing and providing resources to promote perception

In the pre-test, 14.4% of students did not answer question 1.1 that focused on assessing the students' perception skills. However, in the post-test only 8.2% did not answer, and 3.1 % were at an advanced stage in the pre-test while the post-test recorded 10.3% at the advanced level. This shows that after the intervention, students gained confidence in perception and attempted the question in the post-test, and that some students improved to advanced and proficient levels. The comparison group recorded the same percentage, that is, 5.8% in both the pre- and post-test. The perception stage or pre-visualisation stage was not accomplished at all through the traditional teaching approach in the comparison group. In addition, no student achieved at the advanced and proficient levels in the post-test.

Constructions based on prior knowledge in memory rather than reproduction of stimuli is the focal point of perception (Hamilton & Ghatala, 1994). IBF contributed largely to the enhancement of students' perception skills in the experimental group; in other words, to remember environmental and experience matters, students were able to reconstruct their perceptions on diverse events (Hamilton & Ghatala, 1994). Participatory observation showed that, during the intervention, the facilitator commenced by orientating the students to engage their experience and prior knowledge in refining their perception skills. Dewey's ideology about thinking refers to

inquiry as a process that requires reflection on prior knowledge (Tarricone, 2011). However, the teacher remains the main challenger of student thinking through probing and questioning to enable students to draw on prior knowledge. In the perception stage, students acquire knowledge from what they see (environment) and can touch (manipulating available resources). Action, visual and verbal learning play a major role. Refining their perception skills involves reflecting on the environment, that is, learning by seeing, touching and sensing. Moreover, learning is deepened when students are able to link the environmental issues to the classroom context. Findings from the intervention revealed that students engage in reflective thought when the teacher probes and provides resources. Furthermore, recalling experience or prior knowledge and manipulating materials become essential activities in learning. Students share their experiences with one another and they can establish sense from shared experiences. The inquiry-based facilitator and students deliberated upon the meaning of space and shape in the perception or pre-visualisation state. Further, the students and facilitator considered and reflected on shapes in space; deliberated on the relation between space and shape and ultimately linked the environment to classroom learning. Students accumulated vocabulary in this stage. Knowledge that resulted from experience and perception is essential to recognition or visualisation where the teacher probes through questioning to challenge students to learn by visualisation and imagination. Students carry over the concepts and vocabulary that they establish during experiential learning at the perception stage to explore the visualisation stage.

5.3.1.2 Orientation by probing to encourage visualisation and imagination

The visualisation stage seemed to be very challenging to both groups. Table 4.19 revealed that, in the experimental group, only 1% achieved at the advanced level in the post-test from 0% in the pre-test. The comparison group recorded 0% in both pre- and post-test at the advanced level; in other words, no student achieved the advanced level in visualisation. The outcome from the comparison group resulted from the traditional teaching approach that did not apply any method to orientate students to the visualisation level; in other words, inductive or concrete teaching that is essential at the visualisation stage. In the experimental group, the facilitator probed to encouraged students to visualise and use their imagination to make sense of the mathematical setting. The facilitator guided the students to notice the similarity

between real objects and to relate the objects to geometric symbols; for example, to recognise that a garage door is four-sided and associate it with a four-sided figure. Students were also encouraged to name the figure; for example, a quadrilateral. The facilitator guided students to represent visualised objects symbolically. The facilitator's guidance in the visualisation stage initiated students' actions such as critical thinking (for imagination), using technology to verify their knowledge in the imagination level, sharing views, and the ability to use symbols to represent what they perceived and imagined. Student actions contributed to skills such as symbolic representations and visualisation (imagination or recognition). Knowledge acquired while students were involved in learning pertains to relational understanding, symbols for diverse real shapes and types of figures by symbolisation. Hamilton and Ghatala (1994) emphasise that competency in visualisation or imagery pertains to students' ability to represent knowledge formed around physical objects and events. Representation of knowledge in this study refers to using sketches and symbols to represent perceived objects. In addition, Hamilton and Ghatala (1994) motivated that problem-solving involves the use of images that enhance memory and aid the acquisition of information. The visualised information, that is, symbolic representations of perceived and experienced knowledge is essential at the analysis level for further problem-solving.

5.3.1.3 Probing to ensure conceptualisation through analysis

The analysis stage recorded better results in both the groups. A lower number was recorded for students who did not answer the question. Only 1 student for both pre- and post-tests in the experimental group, and 0 in the pre-test and 2 in post-test for comparison group (cf table 4.13). However, for advanced and proficient achievement levels in the analysis level, the experimental group recorded 16,5% in pre-test and improved to 49,9% in the post-test; whereas the comparison group recorded 11,6% in the pre-test and 20.3% in the post-test respectively (cf table 4.19). This stage is between concrete and abstract, where students still refer to concrete knowledge, but need to name abstractly the properties of geometric figures they know and search further for knowledge about the properties they do not know. The abstract knowledge of students in the comparison group enabled students to acquire some knowledge. However, in the experimental group, the facilitator applied the questioning and probing strategy to promote conceptualisation through analysis. Conceptualisation deals with

the development and clarification of concepts. In the analysis stage, the facilitator led students to explore and distinguish the component parts and properties of shapes. In addition to what they knew, the teacher probed to promote further search to engage students in in-depth understanding of the characteristics of diverse figures. The students' action triggered by IBF pertains to reflective thinking, exploring and sharing ideas, and using instruments to perform constructions. The skills and knowledge acquired at this stage encompass relational understanding, ability to distinguish among shapes, group characteristics; properties of separate shapes or figures and producing figures through accurate constructions. In the visualisation stage, only free hand symbolic representations apply without attaching any characteristics to figures. However, in the analysis stage, accurate constructions are encouraged by using relevant instruments and materials such as a pair of compasses, protractor, ruler and other instruments.

Students were not aware that any free hand constructed round shape is not a circle, but a figure similar to a circle. They were not knowledgeable that if a circle is not drawn according to the right dimension, then the centre is not correctly positioned, therefore, the radius and the diameter are not correct. Therefore, a large number of students could not achieve in this question. In the interviews, the students attributed their act of wrong or inability to perform correct constructions to having "*no instruments like a pair of compasses to use for construction*". They claimed that they had mathematical instruments but they left the instruments at home. They were asked by the researcher "*when do you bring your mathematical instruments to school*?" Student E22 answered, "*When Sir says that we are going to measure*". The researcher prompted further and asked, "*Does Sir always tell you when the section he is going to teach the next day will need instruments*?" Student E34 responded "*No*", and student E36 said, "*the problem is that when we bring the instruments to school, other students steal them, and we face a problem when there is a need to use them*". Students in the experimental group became aware during the intervention that relevant instruments are needed to produce accurate figures like a circle. Therefore, most of the students as shown by the post-test results improved their constructions, ability to dissect a figure in terms of properties and describing the properties of a figure. Further, students' improved in terms of awareness that accurate constructions yield accurate dissection of figures and hence correct analysis.

Van Hiele's theory outlines that during analysis, students can describe objects they know. In addition, this study asserts that students are supposed to describe each object and each figure in totality during analysis based on what they know and search further to find what they do not know. For this reason, students resort to inquiry to search for additional information on each object or figure they know. Holistic knowledge and understanding of each figure simplifies the conjecturing or informal deduction stage where students need to formulate propositions about individual figures, relate diverse figures and formulate conjectures about the network of relationships of objects.

5.3.1.4 Probing to encourage conjecturing

At this stage, students have to make informal deductions about the concepts or knowledge they have accumulated and analysed. According to Hamilton and Ghatala (1994), a proposition or a conjecture means that students compose statements of two connected concepts that asserts something about the world. In addition, Hamilton and Ghatala (1994) emphasise that a conjecture can be judged as true or false. The performance in the pre- and post-tests indicated that most students possess low to average informal deduction skills. The experimental group recorded 12.4% of students who did not answer the question but the percentage of non-responses was reduced to 2% in the post-test. Students in the experimental group gained confidence in attempting the question. In the comparison group, 1.4% did not answer the question in the pre-test and the number fell to 0% in the post-test. When comparing the advanced and proficient levels for the experimental group, 5.1% in the pre-test improved to 16.5% in post-test; whereas, in the comparison group, 7.2% in the pre-test declined to 0% in the post-test. The teaching approach in the comparison group oriented students to theorems that exist, but no conjecturing was encouraged and students were not offered a chance to connect concepts and find the origin of axioms and theorems, and the network of relations of concepts that result in theorems. Students' participation was minimal in comparison groups and students were presented with worksheets to memorise the theorems that appeared on the worksheets. The teaching approach required memorisation that led to difficulties in

answering 1.4.2. to 1.6 on the pre- and post-test. In the experimental group, the probing strategy applied by the facilitator encouraged students to formulate conjectures in terms of relationships of various figures based on relationships between properties. Through the facilitator's guidance, inscriptions were explored. Common properties of inscribed and related figures were explored. The action taken by students in order to deduce informally was to explore the information they had accumulated at lower levels and establish conjectures as facts. Skills and knowledge acquired by students at this stage include critical thinking, discovery and relationships of figures including properties that emanate when figures are inscribed. In support, Van Hiele's theory outlines that competence in analysis means students understand logical implications and class inclusions and can create meaningful definitions and justify their reasoning. Creative critical thinking in logical implications, class inclusions as well as justified reasoning that emanated during conjecturing is essential in confirming the conjures that leads to formal deductions through the facilitators' guidance.

5.3.1.5 Probing to encourage inferencing and confirmation of conjectures

Only deductive inquiry applies at this stage. Van Hiele's theory outlines that, at this stage, a formal system of relationships of objects is developed. This includes fundamental logical systems, suppositions, theorems and their proofs. Further, Van Hiele's theory affirms that at the formal deduction level, students can construct proofs and understand the role of axioms and definitions. Knowledge and understanding at the formal deduction level can assist students to confirm their conjectures. Questions 1.7 to 1.9 assessed students' formal deduction skills. The experimental group showed a great improvement of 29.9% from those who did not answer the question in the pre-test. The post-test recorded 8.2% who did not answer the question showing that most students gained confidence to attempt the question in the post-test. The comparison group recorded 14.5% that did not answer the question in the pre-test and 13% that did not answer in the post-test. When comparing achievement at advanced level in the pre- and post-tests of the two groups, the experimental group improved from 1% to 18,9%. A decline was recorded in the comparison group: 4.3% was recorded in the pre-test for those who had achieved the formal deduction level while this declined to 1.4% in the post-test. Teaching at the formal deduction stage in the comparison group progressed from where the teacher presented a theorem from the textbook and explained it to the students while writing the steps to prove it on the chalkboard.

Research indicates that the traditional axiomatic approach to teaching starts at this level. Table 4.24 confirms that teaching started at a more deductive level in the comparison group (Alex & Mammen, 2014). Student actions involved writing and documenting what the teacher wrote on the chalkboard.

Teaching in the experimental group included probing to encourage inferences and ability to search for existing information on theorems, axioms and geometric figures to confirm conjectures. Teacher allowed students to use diverse text materials, their cellular phones and the internet to search for more information about existing concepts, relevant axioms and theorems. The IBF intervention prompted students' actions of searching and exploring existing axioms and theorems to test their conjectures, the main conjecture being on the characteristics of a quadrilateral inscribed in a circle to form a cyclic quadrilateral. Students acquired deductive reasoning skills as well as geometric problem-solving skills at this stage. Knowledge accumulated comprise geometric theory, concepts and problem-solving technique. Hamilton and Ghatala (1994) consider propositions as units of declarative or conceptual knowledge, while productions (condition-action rules) are a way of representing procedural knowledge. Students confirm propositions, validate the information they have discovered in order to understand reality and know how to apply the principles in real-world problem-solving and for academic purposes. In essence, students' formal deduction knowledge and understanding are required at the rigour level of knowledge and understanding.

5.3.1.6 Probe to allow students to apply concepts, vocabulary and knowledge in problem-solving.

According to Van Hiele's theory, rigour level of knowledge and understanding refers to students being able to apply the principles of formal deduction where they are able to compare Euclidean and non-Euclidean systems. By way of example, Sherman (2012) outlines the importance of mathematics in real-life problem-solving. He states that the buildings that we stay in are not designed simply to look attractive; some are designed to withstand earthquakes and major disasters; some buildings bring, design, form and function to new levels; limitations of buildings can be determined by mathematical designs and the calculations of forces that apply to roofs, balconies and structures. Designs are available where building structures are inscribed into others

for various functions. Mathematical designs and computations (such as calculations of forces that apply on structures, balconies and roofs) could be used, for example, to determine limitations of buildings. In this stage, most of the students (49.9% and 43%) did not answer the question in the pre-test, but in both the experimental and comparison group, 27,7% and 18.8% of the students respectively gained confidence to answer the question in the post-test. However, the comparison group showed a decline in performance, that is, 7.3% in pre-test and 4.3% in the post-test. The experimental group recorded an improvement with 11.3% in the pre-test and 24.7% in the post-test.

The performance dropped in the comparison group because the teaching remained abstract focusing only on deductive problems from the textbook solely for classroom purposes. Out of the classroom problem-solving was not considered; therefore, students were challenged in showing how the knowledge on cyclic quadrilaterals apply in real-life problem-solving. Rodin (2012) agrees that contemporary mathematics is abstract and disconnected from physical experience; therefore, students cannot include physical experience in learning. Further, students confirmed their lack of knowledge and understanding in lower and concrete levels of knowledge and understanding. Common problems voiced by students are: (1) they do not know the properties of isolated figures; (2) they could not perceive the relations between shapes; (3) they had forgotten what they learned at primary level.

In the experimental group, abstract problem-solving for the classroom was considered, but the facilitator at some stage probed and led discussions that made students aware that knowledge and concepts in circle geometry and cyclic quadrilateral applies in problem-solving out of classroom context as well. The facilitation strategies in the rigour stage in the experimental group comprised of probing to allow students to apply concepts, vocabulary, axioms and theorems in problem-solving in exercises provided on worksheets and in discussions about application in a real-life context. Marzano (2014) confirms that teaching for rigour involves asking students to research, weigh evidence, allowing students to display authentic knowledge and skills, and guiding students to solve problems that are related to the real-world. The action taken by students in response to facilitation during the IBF intervention in this study involved discussions and solving problems in writing. Students presented the written solutions to other groups, and groups shared ideas. The skills and knowledge gained in the

rigour level pertains to knowledge of application of geometric concepts, vocabulary in geometric and non-geometric systems, and problem-solving skills. The relevance of axioms and theorems was realised by students.

To some extent, IBF enabled students to use basic inductive inquiry in order to handle advanced deductive inquiry. Furthermore, there was an indication that teaching that embraces inductive inquiry can to some extent promote learning ability in deductive formal inquiry. In addition, this study asserts that teaching that starts at the formal deduction learning does not promote knowledge of concepts but memorisation of concepts. In essence, teaching of Euclidean Geometry at Grade 11 should progress from inductive inquiry to deductive inquiry. The inductive to deductive order enables students to (1) bring the knowledge they have missed in lower levels into context; (2) to establish relations between concrete lower levels and abstract higher levels of knowledge and understanding; (3) link their experiences with the classroom context; and (4) apply prior knowledge into the new knowledge. It is the view of this study that the action of IBL the development of inductive and deductive inquiry skills needs integrated teaching and learning, metacognition, student-centred learning, cooperative learning and the use of technology and other relevant teaching and learning material. Autonomy and authentic learning are evident products of IBL.

The pre-test confirmed that students are lacking in basic knowledge of levels like pre-visualisation, visualisation and analysis and students confirmed this by indicating during interviews that they had forgotten what they had learned at primary level. Teachers also attested to the fact that students lack basic knowledge, and in Grade 11, teachers cannot go back and teach the basics that the students have missed. The reason for the teachers' inability to teach the basics and being forced to stick to the traditional axiomatic approach of teaching is due to rigid timeframes teachers and students operate under, and the prescribed Grade 11 syllabus that guides them to teach at more advanced formal deduction level. Furthermore, limited or lack of resources restricts them to the traditional teaching approach.

This study shows that to some extent, IBF influences both inductive and deductive inquiry. A constant improvement across the cognitive levels was evident in the experimental group while the comparison group showed inconsistency in the development across cognitive levels (Table 4.19). Therefore, development at pre-

visualisation, visualisation, analysis and informal deduction levels support development at formal deduction and rigour levels.

Teaching from the known to unknown, concrete to abstract, and inductive to deductive benefits students in a number of ways such as attaining geometric vocabulary and minimising errors when engaging in geometric problem solving. Feza and Webb (2005) deduced that language competency in general became a barrier to the attainment of higher levels of understanding among Grade 7s who were second language students. In addition, Makhubele, Nkhoma and Luneta (2015) found that in solving Euclidean Geometry problems students "misapplied rules, demonstrated signs of weak conceptual knowledge and had weak problem-solving skills" (p. 26). According to them, all the challenges served as barriers in the students' ability to solve problems or provide proofs that occur at a most formal deduction level. Furthermore, Cassim (2007) outlines a number of errors that also arose in the current study (cf Table 4.20). Teaching that does not close the gap between the known and the unknown, the concrete and the abstract; inductive and deductive learning cannot assist students to improve their geometric language or to understand geometric systems in order to avoid errors. Student highlighted that they had forgotten the basic and hinted that they would like to be assisted with basic knowledge like properties of figure; therefore, IBF that leads to IBL is a way of enabling students to learn the properties of shapes, relationships of figures, mathematical language and the vocabulary.

According to the pragmatist view, effective teaching that yields meaningful learning means bridging the gap between subjective and objective inquiry, connecting the contradictions between theory and practice and not separating facts from experience (Mougan, 2013; Ozmon, 2012; Shields, 2003). In light of this, this study confirms that it is possible to bridge the gap between subjective and objective learning during IBL through the application of IBF. Descriptively, the comparison of the pre- and post-test results shows a link between lower inductive levels and higher deductive levels. Improvement at lower levels positively affected improvement at higher levels. Further, concepts and vocabulary learned in lower inductive levels through inductive inquiry are of essence in applications in higher deductive levels. This is an indication that in any balanced lesson, there is a link between inductive and deductive learning. Learning by inquiry is necessary to bridge the subjective and objective gap. Table 4.24 revealed that IBF in the experimental group adhered to both inductive and deductive

facilitation and learning. Therefore, a constant improvement across inductive and deductive levels of knowledge and understanding was evident as shown in Table 4.19. By contrast, Table 4.24 shows that in the comparison group, teaching adhered to deductive teaching and learning. Therefore, no constant improvement was recorded in achievement of the comparison group as shown in Figure 4.19. Through the IBF intervention, a slight achievement was shown in pre-visualisation, visualisation and analysis; hence there was an improvement in informal deduction, formal deduction and rigour in the experimental group. Therefore, a gap between subjective and objective learning is evident. Through the teachers' guidance to merge objective and subjective inquiry, students could overcome lack of understanding at lower levels, recognise what is different in isolated parts, and through the teacher's guidance to critical inquiry, students could seek and understand the relationships between isolated parts.

The process of IBF, according to Socratic inquiry, implies that both teacher and students ask probing questions meant to clarify basic assumptions underpinning a claimed truth or the logical consequences of a particular thought. Teacher questioning and students asking questions to seek knowledge and clarity as applied during intervention in the experimental group, enabled both teacher and students to unpack the meaning of space and shape and to find isolated parts in space and shape, and ultimately question and determine the relationships between isolated parts. In addition to Socratic inquiry as a basis for IBF, Dewey's inquiry method highlights that the process of learning involves sensing perplexing situations, clarifying the problem, formulating a tentative hypothesis, testing the hypothesis, revising with rigorous tests, and acting on the solution. In their critical subjective inquiry, students could establish the relation between space and shape; understand the isolated parts of the shapes and establish the characteristics of each part. Further, students could critically establish relationships between isolated shapes and draw conclusions about the common characteristics of a combination of shapes. The understanding of space and shape that students have inductively acquired could lead them to understand and acquire deductive knowledge as well. Much as the process of learning equipped students with critical inquiry skills, in the pre- and post-tests, students dealt with perplexing situations of working on isolated parts of space and shape, and then attempted to determine the relationships among these isolated parts. Further, a

tentative hypothesis had to be created that some isolated parts can create one complete part (that is, inscribing figures like a quadrilateral within a circle); and the created bigger part has different characteristics from the parts that formed it; nonetheless, the characteristics of the isolated parts still influence the characteristics of the major part.

IBF contributed to the enhancement of students' problem-solving skills. Through IBF, the Grade 11 students learned to bridge the gap between theory and practice or subjective and objective reality. In this balanced form of learning, improvement was recorded in the results of the experimental group.

5.3.2 Does inquiry-based intervention influence students' development in geometric thinking?

Van Hiele's theory confirmed that in classrooms that are inclined to traditional teaching approach, students apply rote learning when they operate with mathematical relations that they do not understand; and students know only what the teacher has taught them and what has been deduced from it. In the non-participatory observation, teachers encouraged students to recall previous information through thinking hard to retrieve what was stored in their memory. Further, teachers led students to recall and repeat previously taught and learned theorems with the aim that recalled theorems would assist students in contextualising the new theorem introduced. In the activity provided to recall pre-knowledge, the teacher gave the answers without checking thoroughly whether each student had acquired the pre-knowledge. In this case, it would be hard for students to apply the new theorem introduced, as they were struggling to recall the terminology and concepts they had learned through repetition in the pre-knowledge section. The teachers in the comparison group were compelled to keep on giving correct answers to each question they asked as they progressed through the lesson. If the teacher did not provide the answer, students would not easily recall these from memory. Learning abstractly by repetition and memorising concepts does not enable students to retain concepts in long-term memory. This is consistent with the findings of Reagh and Yassa (2014) that repetition of facts and concepts improve recognition, but may reduce the reliability of representations in memory. They confirmed their hypothesis that repetition elicits a similar, but non-identical memory trace, and memorisation disadvantages students because some of the items in memory actually

are not recalled. Memorisation does not guarantee long term retention of information in the memory. In emphasis, Carr (2010) states that memorisation only teaches students procedures that they might not know where, how and when to use. In addition, non-participatory observations recorded spelling errors committed by a teacher during the lesson; for example, writing isosceles as “isoscelles”. Errors such as this contribute to students’ misconceptions and errors leading to the lower performance of the comparison group.

The IBF approach promoted conceptual understanding where declarative and procedural knowledge were regarded as supplementary. However, memorisation of concepts is not promoted by IBF. In support, Chao, Murray and Star (2016) state that contemporary mathematics is based on inquiry and the teaching thereof is characterised as inquiry and reform-oriented mathematics that emphasises conceptual understanding and procedural fluency rather than memorisation aided by prompts. As suggested by Shields (2003), the teachers’ duty as the inquiry-based facilitator encompasses developing teaching tools, treating students as a community of inquiry and opening the classroom for public scrutiny.

Intervention through IBF in the experimental group encouraged IBL. In other words, IBF encourages students to explore the problem on their own in order to understand what the facilitator was asking and to formulate a strategy (Chao et al., 2016); and through the facilitator’s guidance, the strategies that are formulated by students individually or in a group are linked to general strategies in the classroom. The intervention showed that, to ensure IBL in Euclidean Geometry, IBF must follow a number of steps. Firstly, the teacher leads students to start by engaging in concrete activities to develop conceptual understanding through inductive learning. Secondly, the IBL facilitator guides students to discover properties through analogical thinking. Thirdly, the teacher directs students to establish relationships among discovered properties. Lastly, students are encouraged to apply the established relationships and properties in deductive inquiry and problem-solving. IBF, in this study, revealed two essential components of learning Euclidean Geometry: a link between concrete or inductive inquiry and deductive inquiry and a link that is achieved through discovery, analogical thinking and ability to construct relations among properties of diverse mathematical structures. With IBF, students’ critical learning is evident, students’ creativity also manifests, students’ independence and self-reliance is established; and

finally, knowledge and understanding are acquired by students. IBF discourages drill and recitation in teaching and learning; therefore, IBL should replace learning by memorisation. IBF as effective teaching enables meaningful learning that is learning for a lifetime.

Implementation of IBF in this study has shown that to certain extent, some of the skills listed in table 5.1 as focal points in teaching and learning geometry are achievable. Through application of IBL, to some degree students acquire skills as stipulated as focal points. Acquiring skills according to the listed skills, and improvement was evident when comparing the pre- and post-test of the experimental group at all levels of knowledge and understanding.

Table 5.1: Focal points of teaching and learning geometry

• developing spatial awareness, geometrical intuition and the ability to visualise providing geometrical experiences in two and three-dimensional objects; • developing knowledge, understanding and application of geometrical properties and theorems • promoting the ability to formulate and use conjecture, deductive reasoning and proof • develop skills of applying geometry through modelling and problem-solving in real-world contexts • developing beneficial ICT skills in specific geometrical contexts; • stimulating a positive attitude to Euclidean Geometry • developing awareness of the geometry in society, and of the contemporary applications of geometry.

Adapted from Jones (2000)

5.3.2.1 Spatial awareness, geometrical intuition, experiences in two- and three-dimensional objects and the ability to visualise

IBF's focus on developing geometric intuition, spatial awareness, experience with objects and visualisation skills was successful as students in the experimental group performed better in post-test in questions 1.1 and 1.2. The facilitator's action of probing, providing resources and allowing students to manipulate resources to re-establish concepts functioned in building the geometric intuition and perception of three-dimensional objects. The intuition and perception enabled students to visualise the three-dimensional objects as two-dimensional through symbolic representation of real objects perceived from experience. A conclusion is reached in this study that

students in the experimental group were aided by IBF to develop spatial reasoning, that is, ability to draw conclusions about three-dimensional objects in terms of the objects' two-dimensional representations from own explored information. Furthermore, students developed spatial abilities where they could visualise how real objects would look when represented two-dimensionally.

5.3.2.2 Develop skills of applying geometry through modelling and problem-solving in real-world contexts

IBF assisted students to acquire necessary skills and concepts through experience and interaction with real-world contexts at the lower levels of knowledge and understanding. The concrete levels enabled students to interact with the real world they live in, in other words, to react to their intuition and learn from perception. The geometry perceived and observed by students daily in their real world is shown in this study to apply in the conceptual classroom world. Students' real-world observations were analysed and modelled into concepts and constructions or drawings; for example, properties and shapes that could assist students to formulate conjectures and predict the relationships between geometric concepts and figures. Through modelling, real-world knowledge, that is, knowledge and experience at the concrete levels, could be transformed into conceptual knowledge and abstraction could be established. Expressing the practical world in terms of the conceptual world by applying models to describe real objects contributes to bridging the gap between lower and higher levels of knowledge and understanding. Usage of resources and smartphone technology-based facilitation enhanced concrete interaction and therefore, modelling skills. The inquiry-based facilitator established an atmosphere that involved IBL that enabled students to recognise and understand the mathematical limitations of smartphone-based technology, produce and interpret technology-generated results and to use smartphone technology to develop and evaluate diverse approaches to solutions (Lingefjard & Holmquist, 2001). Making conceptual models in real and practical contexts enhanced problem-solving skills in the group that received the intervention. This study showed that visual representations (shapes), conceptualised geometric objects and symbolic or graphical representations of real objects are a means for constructing Euclidean Geometry theory, and therefore application in problem-solving. Hershkowitz et al. (1996) outlined the cognitive process in Euclidean Geometry as in Figure 5.2.

Figure 5.2: Cognitive process in Euclidean Geometry

Source: Hershkowitz et al. (1996)

In light of Figure 5.2, this study contributes the Euclidean Geometry modelling framework as a process shown in Figure 5.3.

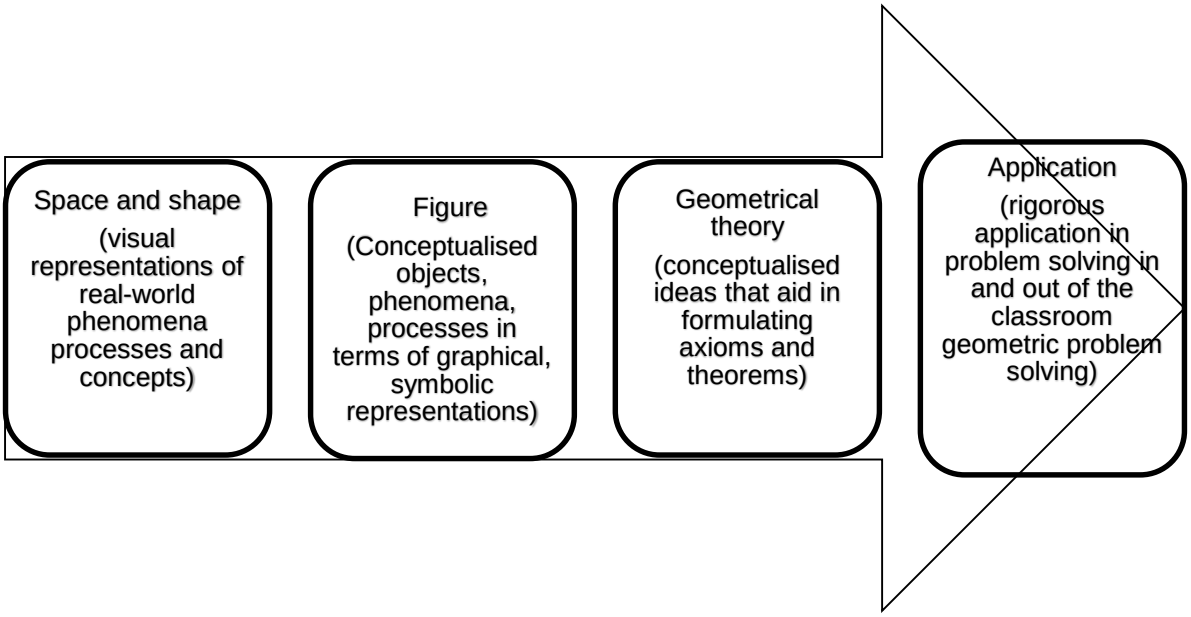


Figure 5.3: Geometric modelling framework

Emanating from the geometric modelling framework illustrated in figure 5.3, this study emphasises that cognitive processing in Euclidean Geometry is a modelling process of four stages. In the first stage, modelling to advance problem-solving considers interacting the real-world objects or phenomena and mathematics. The interaction leads to utilising drawings, figures, symbols and numeric representations to complete analysis, formulating conjectures, making predictions and translating the geometric modelling skill back to the real world, that is, applying geometric modelling in solving geometric problems out of the classroom context or in the real-world. Furthermore, this study avers that geometric modelling and technology are intertwined in the sense that students are bound to apply IBL skills to complete geometric modelling through using the internet to inquire more about geometric concepts. Evidence that geometric and mathematical modelling in general and technology are intertwined is provided in

this study as students in the experimental group utilised their smartphones and internet available at the school to search for more information to enhance their geometric knowledge and understanding.

Hershkowitz et al. (1996) agree that diagrams made on a sheet of paper are not the starting point of the cognitive process in Euclidean Geometry. It is evident in this study that, in the comparison group (traditional axiomatic teaching approach), the cognitive process commences with Euclidean Geometry theory, followed by application. The type of cognitive process that occurred in the comparison group contributed to cognitive load; and students did not achieve in the task assigned to them ($p > 0.05$, $p = 0.221$). Therefore, this study asserts that the cognitive process in Euclidean Geometry commences with visualising space and shape, followed by graphic and symbolic representations; and ultimately modelling the conceptualised and visualised symbolic representations into geometric theory. This cognitive process contributes to knowledge and understanding that allows for easy application in Euclidean Geometry problem-solving. In essence, this study emphasises that IBF is essential for teaching Euclidean Geometry and influences learning by modelling.

5.3.2.3 Developing beneficial ICT skills in specific geometrical contexts

Developing ICT skills is essential to stimulate the thinking of a technologically-oriented 21st century student. In this research, participating schools were equipped with computers and computer laboratories (computer labs) and Wi-Fi was available on the schools' premises. The availability of internet at schools and students having advanced cellular phones (smartphones) became an advantage to IBF that enhanced the development of IT skills in geometrical contexts. The use of smartphones during IBL was beneficial in this study. It was an advantage (as supported by Chao et al., 2016) that, firstly, smartphones are used every day by students; therefore, the phones were accessible and easy to use with the available internet. Secondly, smartphone technology is simple because access to video, images, and text is not linked to a particular operating system and access to information is immediate. Lastly, smartphones in this study were a digital technology used to inculcate inquiry, metacognition, self-regulated and student-centred learning. The inculcated metacognition supported students to use existing strategies; for example, students applied the declarative and procedural metacognitive knowledge to discern how, why

and when to use strategies and resources at all levels of knowledge and understanding. Lingefjard and Holmquist (2001) agree that it is not sufficient for a 21st century mathematics teacher to know only how to define terms and execute algorithms. They emphasised that teachers must be well informed about the role of technology in pedagogy in order to exploit current technology. In agreement, this study suggests that teachers need to be equipped to take advantage of the available resources without using the socio-economic status of areas where schools are located as a justification of non-usage of diverse resources in teaching. The facilitator in the experimental group encouraged students to utilise their smartphones and available internet in learning, while the teachers in the comparison group complained of lack of technological teaching and learning resources.

5.3.2.4 Stimulating a positive attitude to Euclidean Geometry

Table 4.8 reveal achievement according to levels, that is, low achievement, average and advanced in the pre- and post-test. For example, the experimental group pre-test recorded 95.9% of the learners who obtained 0-20 marks which is the lower level. The number of learners who achieved at a lower level decreased at a rate of 23.7% in the post test and 73.3% of learners achieved at a lower level. This is an indication that there was improvement from the lower to either the average or advanced achievement level. When comparing the experimental and the comparison group, the comparison group recorded 92.8% in pre-test and 91.3% in the post-test of learners who were at the lower achievement level, the difference being 1.2%. th comparison groupcomparison groupA large number of students in the experimental group gained confidence and expertise in answering questions through IBF compared to the comparison group. The advanced category as well show that 0% achieved in the pre-test, but 9.3% achieved in the post-test; while in the comparison group achievement in the advanced achievement category was 0% in both pre- and post-tests. Students' achievement in the experimental group recorded a shift from low to average and advanced levels. A decrease at lower achievement levels and an increase at average and advanced achievement levels in the experimental group shows that numerable students performed better at post-test and advanced to higher achievement levels (cf table 4.8). Therefore, it is evident that IBF influenced IBL in learning, and hence improvement in problem solving through IBL.

Inquiry-based intervention in this study showed that IBF promotes critical inquiry. This study showed that the traditional axiomatic teaching approach does not allow for remedial or re-teaching of the information that students have not acquired; therefore, the gap between geometric knowledge lost in lower levels and the deductive knowledge learned at Grade 11 is not bridged. This study confirmed that critical inquiry assisted students to recover geometric knowledge they had forgotten from lower levels. Table 4.19 shows improvement across all levels in the post-test, signifying that the inquiry-based intervention influenced students' development in geometric thinking. Students' comments in the interviews also indicate that the students' geometric thinking had developed. For example, students highlighted that they did not know the properties and relationships of figures and students were not aware that the knowledge they had missed in lower levels was necessary in Grade 11 to enable them to understand how to prove theorems. However, students in the experimental group realised that learning does not commence with memorising a stated proof of a theorem in order to solve Euclidean Geometry problems.

In a study conducted by Makhubele, Nkhoma and Luneta (2015), it was found that students' inability to solve problems or provide proofs is caused by misapplication of rules, weak geometric conceptual knowledge and weak problem-solving skills. In agreement, it is deduced in this study that knowledge missed in lower levels created a knowledge gap. Therefore, students struggled with problem-solving and made errors relating to vocabulary, geometric language, labelling or naming figures, forcing solutions, relating known properties to incorrect figures, free hand constructions where instruments were needed, giving known reasons where they did not apply and incorrect application of memorised theorems. Intervention through IBF assisted the students in minimising the errors through critical inquiry. For example, wrong spellings like "cycle" were improved to the correct spelling "circle". Naming or labelling of figures improved, for example, in Figure 1 of the problem, some students, who labelled the figure RPQS and PQSR, improved and labelled it correctly as PQRS. Lack of prior knowledge and the inability of teaching to bridge the gap contributes to lack of learning at higher levels. Therefore, students end up being forced to memorise concepts they are presented with in higher grades like Grade 11; and errors are evident as a result of learning by memorisation and trying to recall concepts during problem-solving.

Intervention through IBF influences students' development in geometric thinking. The influence emanates from the facilitator's role of ensuring that students are presented with material to use for inquiry purposes. Further, the influence is enabled by opening the classroom for participation and treating students as a community of inquiry where facilitator presence influences social presence in order influence cognitive presence. In addition, the facilitator's influence involves probing in order to help students to understand what the facilitator asked, requiring students to explore on their own and formulate a problem-solving strategy. The facilitator uses the students' mathematical thinking trends that emerge. These trends are utilised as clues to probe further to ensure holistic understanding of concepts and acquiring of knowledge. IBF revealed a better view of how students learn Euclidean Geometry successfully. This study asserts that inquiry-based intervention influences students' IBL skills. Therefore, the acquired IBL skills, contribute to development in geometric thinking and spatial awareness; geometrical intuition and the ability to visualise; providing geometrical experiences in two and three-dimensional objects; developing knowledge, understanding and application of geometrical properties and theorems; promoting the ability to formulate and use conjecture, deductive reasoning and proofs; developing skills of applying geometry through modelling and problem-solving in real-world contexts; developing beneficial smartphone technology skills in specific geometrical contexts; stimulating a positive attitude to Euclidean Geometry; and developing awareness of the geometry in society, and of the contemporary applications of geometry.

5.3.3 How does IBF divulge a better understanding of how students learn Euclidean Geometry successfully?

It is observed in this study that the learning of Euclidean Geometry occurs in diverse ways. In particular, IBF promotes IBL through mathematical modelling, reflective thought, relational understanding and metacognition. At lower levels of knowledge and understanding, students in the experimental group learned by observing reality and analysing their observations using words, sketches, symbols and drawings. IBL enabled students to get intensely immersed in the construction of concepts, knowledge and understanding. Facilitator flexibility in utlising digital and visual resources enabled students to use digital resources as well as other resources like visual and text in order to (1) become intensely immersed in acquiring concepts,

knowledge and understanding in Euclidean Geometry; and (2) become actively engaged in mathematical modelling, that is, engagement in activities involving thinking about and creating models in diverse forms (concepts, sketches, symbols) to describe real objects and how the objects function (including properties of diverse objects); for example, describing real figures such as buildings in terms of sketches and symbols. Further, sketches and symbols were modelled into axioms and theorems through conjecture and deductions. Reflective thought and relational understanding were prominent in the orientation and conceptualisation stages of the intervention as applied in this study. That is, students were able to grasp basic conceptual relationships during the investigations.

Tarricone (2011) agrees that reflective thinking is the basis for learning. For example, search and inquiry are processes that evolve, and these processes become part of problem-solving. Furthermore, Tarricone (2013) describes metacognition during learning in terms of Dewey's phases of reflective thinking. That is, firstly, the teacher challenges students to reflect and think about a problem in a state of doubt and puzzlement. Secondly, thinking and reflecting on the problem leads the student to search, explore and inquire, and attempt to find resources that will assist in finding the solution. Intervention through IBF called for the teacher to provide accessible resources, that is, text material and allowing students to use their smartphones, using the internet to search, explore and inquire. However, students judged for themselves when and at what stage to use text, digital or both resources to search for information. Azevedo and Aleven (2013) agree that leaving students to take control of the available support material assumes that students are "judges of their own learning process and possess the metacognitive skills to determine when and how to use the support material" (p. 188). Supporting student's creative thinking by using diverse material including digital material, contributes to authentic instruction. Authentic instruction refers to facilitation that applies real-world issues and authentic problems to inspire students' learning and creativity (Cennamo, Ross & Ertmer, 2010). The authors further state that authentic instruction builds and provides autonomy for students.

Authenticity, autonomy, cooperative learning and self-centred learning were traits evident at all stages of learning during the intervention. These traits initiated metacognition at all stages of learning. Students were aware of how they learned, that is, they had knowledge of their own cognition and process of learning; in addition,

students could regulate their learning process (Clarebout, Elen, Juarez Collazo, Lust & Jiang, 2013; Tarricone, 2011). For example, students realised that they were challenged in learning some of the concepts. This awareness fostered a need in them to search for further information as they were afforded a chance and guidance to do so. The use of technology and text materials to search for information was evident at all stages. This is evidence that students owned their learning. Azevedo and Aleven (2013) state that to foster metacognition students must have control over the use of tools and the facilitator's support. The students' ability to search for concepts and information is evidence that students became involved in exploration and discovery through self-centred learning. The teacher's act of probing encouraged students to learn by getting completely involved, that is, learning by applying intellect, senses, feelings and exercising personal preferences. Further, IBF yielded open-ended learning, that is, informal learning that is not based on specific formulae. According to Kilpatrick, Swafford and Findell (2001), teaching should support students' informal knowledge and move beyond it. For example, giving an instructional sequence to help students to identify specific classes of quadrilaterals and understand the relationships among the classes. In addition, Kilpatrick et al. (2001) state that instruction that assists students to move from informal to formal learning promotes students' ability to learn how to arrange the figures from the most to the least general members of the class (e.g. from quadrilaterals to squares), to embed hierarchies in the names they gave to shapes (e.g. "squares to rectangles"), and to examine the characteristics of figures. Through IBF, this study has shown that students learn by exploring the informal knowledge that enables them to acquire formal knowledge. In order to assist Grade 11 students to recall the knowledge they have lost at lower grades, Euclidean Geometry lessons should commence by teachers orientating students to explore their informal knowledge at levels such as pre-visualisation and visualisation. Without building on informal knowledge, formal knowledge becomes difficult to acquire as seen in the case of the comparison group where teaching did not link formal knowledge with students' informal knowledge.

IBF as applied in this study yielded IBL characterised by traits such as self-centred learning, authentic learning and autonomy. The traits acquired contributed to metacognition and ultimately to critical creative thinking. According to Cennamo et al. (2010), involving students in critical thinking tasks, the teacher facilitates and inspires

high levels of engagement in the content learned, and therefore, deeper learning is promoted. In addition, Cennamo et al. (2010) define creative thinking as higher-order thinking, considering multiple possibilities in problem-solving, Further, Cennamo et al. (2010) agree that types of critical thinking include inductive thinking (that is, moving from parts or examples to whole or generalisations) and deductive thinking (that is, moving from the whole or generalisations to the parts or underlying concepts and examples). Table 4.24 show that intervention, that is, facilitating through IBF fostered both types of critical thinking in IBL, that is, inductive and deductive thinking. However, Table 4.24 shows that, in the comparison group, only deductive reasoning applied in learning, and it was perpetuated by the traditional approach of teaching.

When critical thinking happens whether deductively or inductively, the following characteristics should be evident, according to Cennamo et al. (2010): non-algorithmic process, complexity, multiple solutions, nuanced judgements and interpretation, multiple criteria, uncertainty, self-direction, imposing meaning and conscious problem-solving. Intervention through IBF contributed to the characteristics of critical thinking through both inductive and deductive teaching (Tables 4.24 and 4.25). However, teaching through the traditional axiomatic approach that occurred only deductively (Tables 4.24 and 4.25) contributed to algorithmic approach to problem-solving, rote learning; passive problem-solving; lack of critical thinking with the teacher being the only disseminator of knowledge.

IBL enables students to learn how to organise information for them to comprehend how communal knowledge erupted from the ancient contexts, what knowledge should be retained and what essentials should be transformed (Ozmon, 2012). Linking knowledge and experience is essential in achieving inductive learning. The intervention conducted in this study, as well as pre-tests, oriented students to draw on their prior experience through pre-visualisation linking with visualisation. Knowledge and understanding of the world as perceived by an individual is very important. Perceptual powers (how an individual perceives the world) are essential in achieving the pre-visualisation level which lays the foundation for visualisation. Table 4.19 indicates that students struggled with pre-visualisation and visualisation was thus also affected. The two levels of knowledge and understanding that need real-life experience are lacking. However, the experimental group recorded an improvement in the post-test. Further, awareness of the importance of perceived real-life information

experience was evident after the pre-test as students in both groups stated that they were aware that there were shapes everywhere they went.

It is believed that the difficulties encountered by students in visualising and clarifying their thoughts is impacted by inexperience and immature rational status (Gutierrez, Leder & Boero, 2016: 111). Further, visualisation means the capability to characterize, text, produce, alter and build a conversation on facts that are visual. In visualising, visual insight is prioritised. However, Gutierrez et al. (2016) state that geometry exploits visual intuition which is the most dominant of our senses. The advantage experienced when visual intuition is exploited is that students do not perceive reality and provide global insight to inspire conjecture that will lead to understanding proof. Despite the essence of visualisation in the learning of geometry, students still struggled with visualisation skills as recorded in Table 4.19. The improvement in the experimental group showed a lower rate, and in the comparison group no improvement is recorded. Although minimal performane rate, IBF posed a good impact on the students' visualisation abilities.

Group work and cooperative learning appeared to be essential in the students' cognitive growth. Continued collaboration in groups and discussions, that is, teacher-student discussions and student-student discussions assisted students to share ideas and individual students to benchmark their own ideas against other students' ideas. Students could interact with each other to explore, discover and analyse in order to advance IBL. To maintain cooperative learning during the intervention, the facilitator maintained control of the learning environment, designed learning activities, structured teams work, and ensured that facilitation actions did not influence the students' views and participation (Li & Lam, 2013).

In general, this study revealed that IBF through questioning provides direction to students on learning Euclidean Geometry. The direction relates to enabling students to start by applying perception and intuition to build experience. Secondly, students put what is perceived into visual connotations. In the third stage, the visual connotations are analysed in order to identify concepts, vocabulary and mathematical language that will assist students to build conjectures. Fourthly, suppositions were tested and validated in accordance with existing and confirmed proofs, axioms and

theorems. Developed knowledge about geometric concepts, axioms and theorems could be applied later in problem-solving for classroom and real-world contexts.

While researchers conceptually comment IBF and learning as a necessary means to alleviate the problems of learning in the South African context, this study has embarked on practical application of IBF in promoting IBL in Euclidean Geometry. The confidence in implementing IBF is benchmarked against the good achievement recorded in countries like Netherlands, where the success is attributed to IBL. IBL is student-centred. Therefore, the teaching or monitoring of IBL in this study is mentioned as IBF. The focus in this study was to assist teachers with facilitation strategies to employ in their teaching of circle geometry in order to support students to reach higher levels in geometry thinking. IBF was applied to enhance IBL based on the pragmatist view of inquiry learning from John Dewey's theory of pragmatism as related to IBL.

This study has shown that teaching Euclidean Geometry applying the traditional approach only, contributes to drill and practice that leads to memorisation. Memorisation, in turn, contributes to lack of learning problem-solving skills properly. However, this study does not promote the complete exclusion of traditional disciplines in facilitation, although IBF initiates IBL that equips students with skills such as creativity, critical thinking, student-centred learning and confidence in order to identify, pose and solve problems.

5.4 IBF FRAMEWORK

IBL has been proven in this study to enhance the students' mathematical problem-solving skills in Euclidean Geometry (at statistical significance of $p=0.001$, $p< 0.05$ and a larger effect size of 0.55). Rather than ignoring the students' knowledge gap among levels on knowledge and understanding, IBL needs to be used to bridge the gap between lower inductive levels that students have missed and the higher deductive levels that students in Grade 11 are faced with.

This study found that in the first step of facilitating learning in Euclidean Geometry, the teacher should provide resources and engage students to intuitively interpret their experiences in their physical world. Secondly, the teacher should probe and question students to involve them in utilising what was perceived in the physical world and creating visual images in their mental world. Visualisation should lead to a collection

of ideas contributing to associating the physical world with the visual mental world. Thirdly, the visualised information should be applied at the analysis level to explore meaning and relationships between concepts. At this stage, more searching for information is necessary to acquire deeper knowledge and understanding of meanings and concepts. More vocabulary and geometric language is accumulated at this stage. In the fourth stage, analysed parts and the information or concepts that emerged are used to formulate conjectures based on relations of concepts. Fifthly, through search for information and supporting facts inferences are made, conjectures are tested, confirmed and validated. Conclusions are drawn from conjectures at this stage. This is the stage where conclusions as theorems are comprehended. The students do not only comprehend the theorems as memorised concepts, but understand the origin of concepts that combine to build a theorem. The last and the sixth stage is where the conclusions reached have to be applied in problem-solving and in external mathematical systems. Generally, the process of IBF influences IBL, that is, the facilitator: (1) directs orientation to perception and visualisation; (2) coordinates conceptualisation for analysis and conjecturing and (3) guides students to conclusions for validation and application. The steps of facilitation are: (1) orientation to perception; (2) orientation to visualisation; (3) coordinating conceptualisation through analysis; (4) directing conceptualisation for conjecturing; (5) guiding conclusions to validate conjectures; and (6) guiding conclusions for application of knowledge and understanding. The findings of this research study have led to the creation of the following model as the framework of IBF for geometry problem-solving (Figure 5.4).

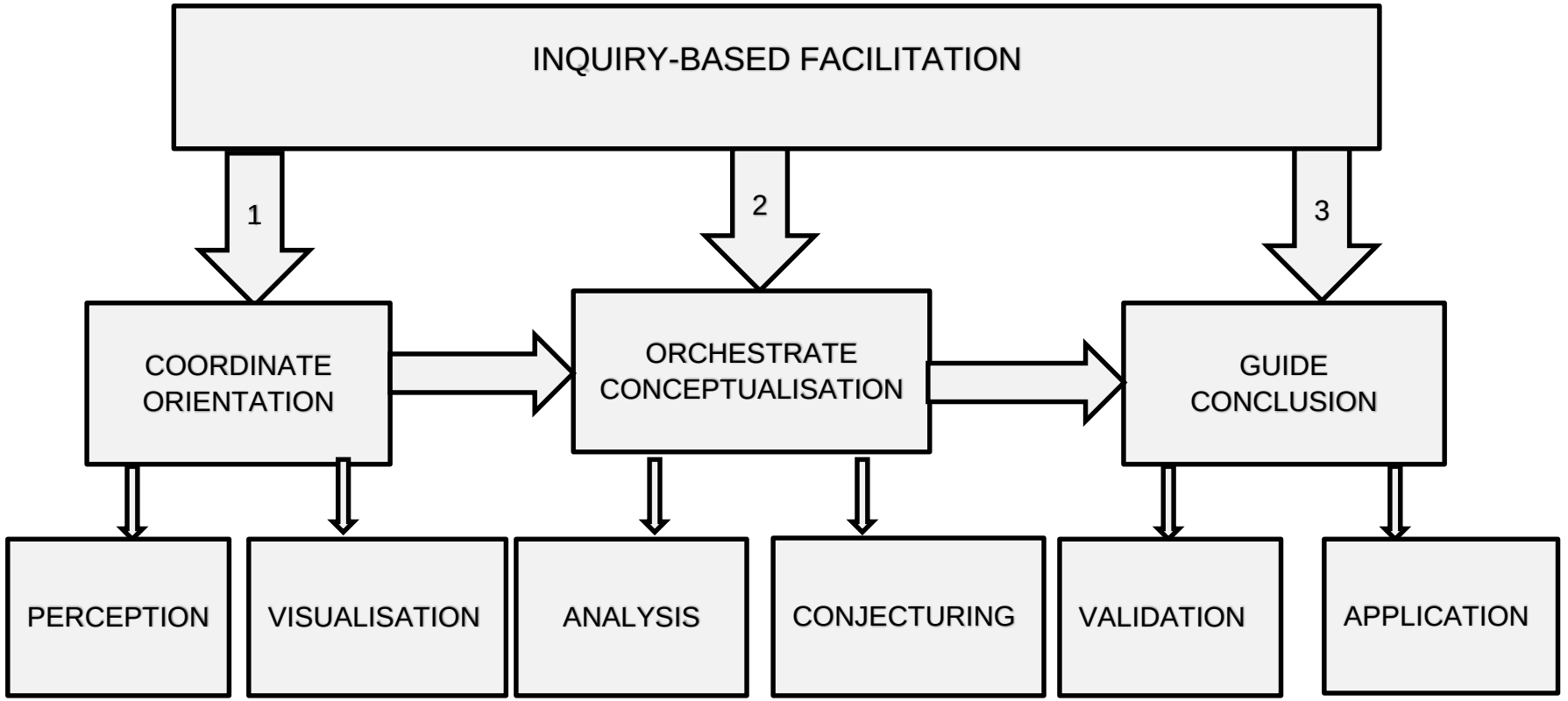


Figure 5.4: IBF framework for geometry problem-solving

In Figure 5.4, the facilitator follows the steps in chronological order, which is, step 1 should be completed, followed by step 2 and follows the conclusion step. In the whole process of IBL, concrete materials such as a tangram and exploration through digital media assist students with abstraction, building of spatial sense, collecting relevant and existing geometric concepts, exploring geometric concepts and acquiring the vocabulary. Teacher role through IBF is of essence to guide students through the process of inquiry. In the IBF framework for geometry problem- solving (Figure 5.4), the teacher role is displayed as coordinator, orchestrator and a guide. The teacher plays the role of facilitation mostly through probing, questioning as well as validating and concluding the ideas and geometric thinking contributed by students. Further, the teacher is the leader in terms of channelling the broad ideas that students accumulate in the process of inquiry. In order to achieve the IBF successfully, the teacher displays principal skills in geometric system. In addition, the teacher contributes in each step the students take. The students do the thinking, searching, analysing, conjecturing, testing, inferencing, conclusions and applications. The teacher guides them to think logically and confirms the students' conclusions.

Further, in the process of a strategic IBL, skills are acquired and there is a connection between and among the skills acquired. In addition, students acquire knowledge of content, concepts and vocabulary. A connection should be realised among the content concepts and vocabulary. Spatial understanding enable students to interpret, understand, and appreciate their innately geometric world. In addition, students acquired more insight and intuitions about two- and three-dimensional shapes and their characteristics, the interrelationships of shapes, and the effects of changes to shapes. Through spatial understanding and knowledge, students develop a strong sense of spatial relationships and acquire efficiency in concepts and language of geometry. Students are better prepared to learn advanced mathematical concepts such as axioms and theorems deductively because of having a rigorous spatial sense and being concretely efficient in geometric concepts and language.

The IBF framework in Figure 5.4 is envisaged at assisting teachers to conduct IBF for the successful implementation of IBL. The model benefits learning as students are active participants in discovering and creating knowledge in order to apply in formal and informal education systems.

5.5 EVALUATING THE STUDY OBJECTIVES

5.5.1 Objective 1: The impact of IBF strategies on students' development of higher levels of geometric thinking.

The improvement of lower levels of knowledge and understanding had influenced the development in higher levels of geometric thinking positively (Table 4.19). The lower the pre-visualisation and visualisation skills, the lower are the skills in the formal deduction level. For example, in the comparison group, the perception and visualisation skills of 0% achievement yielded 1.4% achievement at deduction level in the comparison group. These basic stages also affected the rigour level. Improved pre-visualisation and visualisation skills at post-test in the experimental group yielded a better improvement rate in the formal deduction and rigour level. Among other reasons, research has attributed the difficulties experienced by grade 11 students in learning geometry to lack of background from lower classes Alex & Mammen, 2014; Malati, 2014). Further, research affirmed that only higher levels are recognised at grade 11, regardless of the fact that students did not achieve the lower levels. The implication is that, in lower classes, lower levels of knowledge and understanding were either not addressed through teaching or not achieved by students. The improvement of formal deduction and rigour levels in this study is attributed to improvement in problem-solving at lower levels. Lower and more concrete levels, that is, pre-visualisation, visualisation; including middle levels of knowledge (analysis and informal deduction) as depicted by this study, are levels where students were guided through IBL to acquire vocabulary and conceptual understanding. This study has shown that the errors committed by experimental group students in the pre-test were lessoned in the post-test. Through IBL, students learnt the importance of accurate constructions in their interaction with the IBL facilitator during intervention (cf Vignettes). Formal deduction and rigour levels require a solid foundation in conceptual understanding, geometric vocabulary, ability to perform accurate constructions of figures, and ability to describe and contextualise the figures they construct or they are exposed to. In this study, IBF orientated students to most of the skills mentioned above required in problem solving at formal deduction and rigour levels. Therefore, this study has shown and asserts that good performance in problem solving at formal deduction and rigour levels is attainable on condition that lower levels were addressed through IBF and students achieved the levels not through memorisation, but through cognitive

processes such as IBL that enables students to retain knowledge and understanding in their long term memory.

5.5.2 Objective 2: The influence of intervention in activating IBL

IBL influenced by IBF has contributed to students' development of basic levels. In these basic levels, students acquired inductive learning skills that assisted them to acquire vocabulary and mathematical language that needs to be applied at the formal deduction level. For example, geometric vocabulary and concept understanding was accumulated which assisted in deductive inquiry to complete the problem in place.

5.5.3 Objective 3: If IBF can divulge how students learn Euclidean Geometry

The study had shown that students learn Euclidean Geometry through reasoning by linking the inductive and deductive parts of space and shape geometry. In essence, it had shown that exploration, discovery, modelling, creativity, metacognition and self-centred learning are key to both inductive and deductive inquiry that lead to better understanding of concepts. Further, students learn by applying critical thinking to utilise perception and experience in order to construct conjectures. After constructing the conjectures, they need to inquire deductively by utilising existing facts and knowledge (such as axioms and theorems) to confirm the conjectures they have constructed. After confirming the conjectures, that is, adopting what is right and rejecting what is wrong, students, can apply knowledge and understanding in problem-solving for geometric and non-geometric systems.

5.6 IMPLICATIONS OF THE STUDY TO THE PRACTICE

5.6.1 Theoretical implications

Study results supports the supplementary role of inductive teaching to deductive teaching and learning. Pragmatism is effective in terms of bridging inductive and deductive teaching. It is of essence that teachers consider re-teaching and fostering prior inductive knowledge. However, teaching that promotes long-term knowledge and understanding merges learning through inductive and deductive inquiry. Students are likely to retain information for long when they own their learning, that is, when learning is student regulated and centred, authentic and autonomous. The teacher is not regarded as the source of all information, however, due to lower levels of students'

expertise in Euclidean Geometry, the teacher remains the guide in all processes of learning. Therefore, traditional disciplines are not completely ignored, but are applied to guide dtudents through. Metacognition also plays a major role to assist students to be aware of how they progress and where they fall short in the process of acquiring knowledge and understanding.

5.6.2 Methodological implications

The pragmatist philosophy in this study expresses that a subjective discourse is essential to interpret objective reality. Both objectivity and subjectivity stances provided a frame of reference for this study. Pragmatically, this study addresses research that operates between induction and deduction. Further, pragmatic arguments in this study are context-based, generalizable and can be transferred to other settings. Mixed methods research followed an embedded approach. Nonetheless, only the observation part of the qualitative research was embedded during intervention at experimental groups and during teaching at comparison groups. Interviews were embedded, however, conducted at the end of the post-test for both the experimental and the comparison groups. This was convenient for all participants in the study. Quantitative data analysis through paired t-test addresses objective stance of the study, while qualitative data through content analysis addresses the objective stance. The two standpoints were triangulated in order to provide a pragmatic stance of this study. The dominance of the qualitative data led to the rejection of the null hypothesis, therefore, acquiring a stand to support the alternative hypothesis. Qualitative data provided evidence through observations and interviews to support the alternative hypothesis. In addition, the qualitative findings played a role of complementing quantitative findings, therefore, the credibility of the quantitative finding was increased. Further, results from the qualitative stance provided clarity of the results in the quantitative stance. Pragmatically, knowledge contributed as IBF processes and framework in improving IBL is an essential product towards teaching, facilitating and learning transformation in Euclidean geometry. In essence, the pragmatic stance of this study asserts that research needs to move back, forth and across objectivity and subjectivity in order contribute towards change.

5.6.3 Pedagogical implications

This study revealed that IBF is effective than the traditional axiomatic approach. IBF is the teaching method that begins by concretely tapping on students' experience to allow students to understand inductive and hence deductive stages better, and this method of teaching is proven to enhance problem-solving skills in Euclidean Geometry in this study. IBF allows students to improve through self-centred learning, therefore, learning by memorisation is limited. Further, students are led to explore and discover to acquire concepts and vocabulary. In the whole process of IBF, teacher operates according to the belief that students are motivated to learn. Therefore, the teacher, treats students as individuals who are at diverse levels of learning, and the implication thereof is that diverse facilitation methods should be used to accommodate all student.

The intervention revealed challenges encountered in implementing IBL. Firstly, time constraints remain a challenge. More time is needed to let the students learn at their own pace. Discussions should be allocated enough time to enable the IBF facilitator to perceive the students' thinking and guide them step by step leading them to metacognition and to allow self-centred learning. Secondly, learning through inquiry needs tools or resources and facilitator creativity. The socio-economic status is not supposed to serve as the facilitator's reason for not allowing students to make use of any available resources for inquiry purposes. For example, when interviewed, teachers complained about lack of resources. The schools that participated were equipped with Wi-Fi in the school premises. Students own smartphones, however, most schools' rules do not allow students to operate phones in classroom. The intervention in this study has revealed that if students are guided and given a clear direction while operating smartphones as the available technology tool, educational inquiry can be advanced through smartphones. In essence, guided inquiry while manipulating gadgets like smartphones contribute to authentic inquiry. Lester, Mott, Robison, Rowe and Shores (2013) outlined that authentic inquiry refers to a situation where students generate research questions and guide themselves through the problem-solving process, and Clark (2012) motivated that guided inquiry refers to situation where the teacher engages students in class discussions where every student will be involved. In this study's intervention, no irregularities were found when students were using their smartphones during the mathematics period under my supervision as the facilitator. Further, no inconsistencies were found during teacher-

student dialogue, student-student discussions, and students' self-regulated as well self-centred learning.

5.7 CONCLUSION

This section presents the conclusion from literature, qualitative and qualitative sections.

5.7.1 Conclusion from literature review

Literature show that declarative and procedural knowledge are both essential in learning. Therefore, facilitation need to recognise both. It is a challenge to students when facilitation orientates them to procedural knowledge only while they have missed declarative knowledge as a basic requirement. However, literature recommends learning through search and analogy as a means to close the gap between declarative and procedural knowledge. Search and analogy are components of inquiry. IBL is student-centred and instead of providing a straight and simple route to solutions, it engages students in sense making activities that causes them to apply critical thinking in problem-solving.

IBF technique directs teachers to utilise well-crafted problems and questioning techniques to guide students through an IBL process. The IBL process generated through students' response to IBF, encompass skills such as conjecturing, experiment, exploring, creating, applying and communicating. The mentioned skills perpetuate IBL process that contributes fully in assisting students to acquire declarative knowledge. For example, knowing what, that is, knowing and understanding geometric concepts and vocabulary. Declarative knowledge acquired can further assist students to advance procedural knowledge, that is, how to, or application of declarative knowledge in deductive inquiry.

Engaging in sense making activities inductively to acquire declarative knowledge for application in advancing procedural knowledge, enables students to learn a network of relationships that link geometric processes and concepts other than memorising facts, rules and names. The network of relationships in geometry, geometrical processes and concepts could be advanced in totality when students possess self-acquired knowledge and understanding in pre-visualisation, visualisation, analysis, informal deduction, formal deduction and rigour. IBF through questioning enables

students to adhere to a hierarchy of levels of knowledge and understanding without skipping any level. Facilitation that focuses on advanced levels while the students have not mastered the basic levels contribute to cognitive load. Further, the facilitator's reasoning that is different and higher than the levels of knowledge and understanding that students are exposed to, aggravates the cognitive load. Facilitation in each level of knowledge and understanding must address the needs of that level according to students' levels of learning.

Different levels of knowledge and understanding are influenced by diverse learning experiences such as action learning, visual learning and verbal learning. For example, pre-visualisation and visualisation concerns visual learning more than verbal learning, whereas analysis and informal deduction may need less visual learning and formal deduction and rigour concerns more of verbal and abstract learning. The learning experiences involves both inductive and deductive learning. Diverse facilitation resources, that is, visual, audio-visual, text and digital resources are necessary in order to assist students to advance diverse learning experiences in diverse levels of knowledge and understanding.

IBF orientates students to both formal and informal education. Further, communication and cooperative learning are of essence in IBL. Education acquired in both deductive and inductive ways assist students not only to gain classroom knowledge but to also learn how to live and how to apply critical thinking by solving problems that concerns out of the classroom context.

5.7.3 Conclusion from the Quantitative Component

A conclusion is presented based on what emanated from the intervention as well as pre- and post-test. Intervention has had an influence on the improvement of learning and has shown that inductive and deductive learning supplement each other in learning Euclidean Geometry. Descriptive analysis displays a constant improvement of all levels from pre- to post-test in experimental group as compared to the comparison group. Further, post-test results indicate that students gained confidence to attempt answering questions they could not answer in pre-test, regardless of a zero mark they have obtained. The number of students who obtained lower scores in different questions decreased in the post-test, while the number of students who obtained higher scores increased, specifically in the experimental group. In some

instances, the comparison group had shown an increase but lower than that in the experimental group. As the increase in experimental group is attributed to the intervention, a slight increase in some of the levels in the comparison group is inferenced to a comment given by one student that because they have seen the questions before, the post-test was not difficult as the pre-test. The t-test scale has recorded an improvement with a p-value of 0.0001, $p < 0.05$ in the experimental or intervention group. This shows that there is a chance that IBL enhances problem-solving skills in Euclidean Geometry.

5.7.4 Conclusion from the qualitative component

Based on observations, teachers and students' views, a conclusion for this section is presented. Optimum time is necessary for facilitation and to enable students to explore, discover new and old concepts, learn from experience in order to acquire successfully knowledge in all levels of learning in Euclidean Geometry. Students are able to evaluate their own thinking and learn from others through cooperative learning and teamwork. Basic concrete knowledge such as skills learned in pre-visualisation, visualisation and analysis, is necessary for students' success in abstract levels such as informal deduction, formal deduction and rigour. Facilitation that can bridge the gap among all levels of knowledge and understanding can enable students to display awareness of the connection between mathematical and extra-mathematical systems; to construct mathematical representations; and to search for structure, patterns, relationships and the main aim of generalisation.

However, availability of resources such as digital and visual resources enhances IBF and therefore IBL. Learning by inquiry encourages every student present in the classroom to participate actively. Participation becomes authentic because students have to search for knowledge by themselves through application of critical thinking. The more the facilitator prompts by asking questions, it is the more students will ask to seek clarity. The other important factor of IBF is that students are guided to explore and discover facts other than to memorise ready-made theorems. Most students cannot learn by memorisation but can readily discover facts and concepts comprising theorems in the Euclidean Geometry system. Facts and concepts learned through self-discovery can be retained for long in memory as compared to memorised facts and concepts. In addition, students are able to see the origin of concepts as they participate

in IBL and therefore, they are able to retain knowledge and understanding in their long-term memory. To some extent student interviews had shown that, (1) through IBL, students realised that they have full potential to solve problems in Euclidean Geometry; (2) they should be actively involved in the learning process; (3) it is important to concretise Euclidean Geometry in order to fully comprehend the abstract component.

5.8 LIMITATIONS

Only Grade 11 students in selected Tshwane West and South schools participated, hence generalised conclusions cannot be made in this study. For example, conditions of study in Tshwane West and South district might not be the same as at other schools in other parts of South Africa. Therefore, findings and conclusions refer mostly to Tshwane North and West areas. Only circle geometry part of geometry as chosen from topics in the Grade 11 syllabus was studied, therefore, conclusions cannot be made that IBL can be practised in all parts of mathematics. However, the IBF framework that developed from this study can be modified to suit application in other sections of geometry. The study was limited to intact classes that posed a threat to internal validity. Absolute control of research environment could not be maintained, therefore, high internal validity cannot be claimed. In addition, high internal reliability cannot be claimed as a result of test retest measure. Situational factors such as absenteeism or lack of students' interest especially in the comparison group, contributed to high attrition mainly during post-test period. Therefore, pre and post test results are limited to students who wrote both the test.

5.9 RECOMMENDATIONS

This mixed-method study focused on implementing IBL to enhance students' mathematical problem-solving skills in Euclidean Geometry. The implementation of IBL through IBF is envisaged to assist mathematics teachers and mathematics practitioners with strategies to teach or facilitate the learning of Euclidean Geometry or other sections of geometry in order to support students to acquire knowledge and understanding in both inductive and deductive levels of knowledge and understanding. This study has shown that IBF influences IBL that assist students to improve lower levels of learning in order to advance higher levels of knowledge and understanding. In addition, this study had shown that IBF equips students in concrete levels to enable

them to advance the abstract levels of geometric thinking. Therefore, to assist students to apply knowledge from lower concrete levels in higher abstract levels of geometric knowledge and understanding, IBL implemented through IBF based on integration of Van Hiele's theory of geometric knowledge and Dewey's theory of pragmatism is recommended by this study.

5.10 RECOMMENDATIONS FOR FURTHER RESEARCH

Teachers at experimental groups displayed lack of confidence in terms of facilitating IBL. Based on the fact that IBL enhances students' problem-solving skills, teachers need to be empowered to be able to facilitate IBL. Further, students are still struggling in basic levels of knowledge and understanding in Euclidean Geometry. For example, the visualisation level of knowledge and understanding reflected low performance. The IBF framework contributed by this study need to be applied in order to assist teachers to support students to learn meaningfully across all levels of knowledge and understanding in geometry. Therefore, further research on the model that the study contribute as the framework of IBF is recommended.

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APPENDICES

APPENDIX A: LESSON PLAN (EXPERIMENTAL GROUP)

Content area:

Space and shape (Euclidean Geometry)

Topic:

Circle geometry: focus on cyclic quadrilaterals

Specific aims applicable:

The lesson is aimed at implementing IBL to empower students to acquire problem solving skills and cognitive skills that will equip them to apply acquired knowledge in diverse contexts out of the classroom situations.

Specific skills appropriate

- Advance the appropriate utilisation of mathematical language.
- Assess and review results for quantitative data collected, organised, and analyse.
- Apply critical thinking skills to solve problem utilising mathematical processes.
- Usage of properties of figures and spatial skills as well as creative thinking skills in problem solving.

Context that is applicable:

Classroom, group work, students' self-centred and self-regulated learning skills, teacher Inquiry – based facilitation

Teacher's activity (applying IBF) Student activity (implementing IBL) Methods of teaching: Facilitation through questioning, probing, demonstration and explanation		
Instructional phase and the cognitive level	Teacher activity (facilitating IBF)	Student activity (IBL approach)
Information or inquiry (level 0 pre-visualisation)	For pre-visualisation teacher asks questions to let students recall relevant geometric information related to their real world.	Students recall geometric figures in their real world. Name prominent shapes and figures. For example, recognising figures at the students' exposure such as shapes of

	Name other prominent shapes in habitat construction, architecture, economy, industry Engage students in discussions through probing to rationalise space and shapes observed at the moment.	road signs, soccer pitch, buildings and more. Manipulate a tangram to explore different shapes contained in a tangram Discussions referring to prominent shapes to answer the following questions: What is geometry, what is space, what is shape? How is geometry relevant in everyday life? What is the relationship between space and shape?
Information or inquiry (level 1– Recognition/ Visualisation)	Teacher makes objects or materials available for students to view, identify, and classify according to what they see. Probing questions are asked by the teacher to confirm students' orientation and conceptualisation.	Through discussions students explore (through action learning) the presented material related with their pre-visualisation knowledge; and asks clarity seeking questions. (Orientation according to IBL framework). Student do free hand constructions shapes outlined in the pre-visualisation.
Guided or directed orientation (level 2 – Analysis)	Teacher facilitates through guiding students, affording them a chance to learn more about the properties of explored during visualisation, and teacher can lead students to provide sketches for materials according to classes	Students participate in activities empowering them to discover implied connections. Produce sketches denoting relations among shapes or figures, and outline the properties according to groups of materials. For examples, four-sided figures, three sided figures, round figure, etc. (Conceptualisation according to IBL framework)
Explication (level 3 – Informal deduction)	Teacher emphasises the terminology once the students have obtained a good conceptual background. The discoveries are made as explicit as possible Concept mapping and construction	The students' experiences are linked to shared linguistic and symbols (for example, quadrilaterals as rhombus kite, triangles as scalene, equilateral and individual figures are explored in relation to properties, for example properties of a square, rectangle, rhombus, isosceles triangle etc. *concept mapping and construction of figures must be considered. (Conceptualisation stage according to IBL framework
Free orientation (level 4- Formal deduction)	A teacher might present more information to afford the students a chance to learn what they could not acquire clearly. For example the relation of a circle and other figures like pentagon, triangles and quadrilaterals	Students advance difficult activities in order to get empowered with knowledge on systems of relations. For example understanding a cyclic quadrilateral, relating properties of a circle and a selected quadrilateral to formulate conjectures, and proof conjectures (Investigation stage according to IBL framework)
Integration (Level 5 – Rigour)	The teacher does not introduce new information, sum up what was taught, and review the acquired information together with students.	Students sum up the acquired information for retention. Conclusions are made; students look forward and back to verify the solutions they got. (Conclusion stage according to the IBL framework)

Content range that is applicable-

- Connecting mathematical structures
- Build mathematical illustrations
- Exploring patterns, constructions, relationships as well as principal aims of generalisation in circle geometry.
- Application of IBL to advance pre-visualisation, visualisation, analysis, informal deduction, formal deduction and rigour in circle geometry.

Teaching and learning support material necessary

- Tangram
- Mathematical instrument
- Projector
- Textbooks
- Cell phones and internet

Assessment (types of assessment, assessment approaches and assessment evidence collection techniques; also specify what will be assessed and how it will be assessed).

Formative assessment- short questions by the teacher to encourage inquiry and to lead student through different levels of learning.

Summative assessment

Students will write pre- and post-tests

What will be assessed?

Mathematical inquiry

- awareness of the connection amid mathematical structures
- ability to build mathematical illustrations
- Exploring patterns, constructions, relationships as well as principal aim of generalisation.

Application of Inquiry-Based Education (IBE) to advance pre-visualisation, visualisation, analysis, Informal deduction, formal deduction and rigour.

APPENDIX B: ASSESSMENT TASK

Euclidean Geometry

Student code: .....

Age: .....

Grade: .....

School code: .....

Date: .....

Duration: 60 minutes

Marks: 80

OBJECTIVES OF THE ASSESSMENT TASK

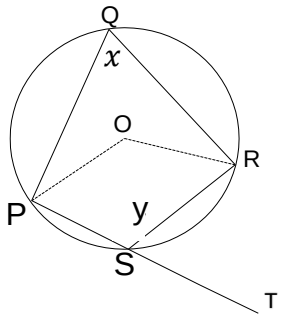
This assessment task is aimed at assessing:

- Mathematical inquiry
  awareness of the connection between mathematical and extra-mathematical systems
  ability construct mathematical representations
  ability to search for structure, patterns, relationships and principal aim of generalisation.
- Application of Inquiry-Based Education (IBE) to advance all levels of geometric knowledge and understanding

Question	Total Marks	Marks Obtained	Level of knowledge and understanding
1.1.	16		Pre-visualisation
1.2.	9		Visualisation
1.3.	26		Analysis
1.4.1.			
1.4.2.	13		Informal deduction
1.4.3.			
1.5.			
1.6			
1.7.	9		Formal deduction
1.8.			
1.9.			
1.10	3		Rigour
Total	76		

Problem

Figure 1 is given. Line PS is produced to a point T. Answer the questions that follow to prove whether x = y.



1.1. How can you associate each of the following objects with Figure 1?

Table 1

Object	Reasons for association with Figure 1
 Pyramid	
 Hut	
 Diamond	
 Floor tiles	
 Stop sign	

 Garage door	
 Traffic sign	
 Kite	

1.2. Group the items in 1.1 according to similarity of shape. Complete Table 2 (attached) and choose a general shape that represent each group of items. .

1.3. Name the shapes and discuss the properties or characteristics of each shape in Table 2.

1.4. Consider Table 3 and answer the following questions:

1.4.2. A diameter is a part of Figure F in Table 3. The diameter divides Figure F into two equal halves. Draw Figure F, show the other four parts, name them and describe the part.

Space for your answer

1.4.2. Figures A & C in Table 3 are the same as Figure G. Give reasons and discuss how true is the statement.

……..

……..

1.4.3. Are there chances that Figure G can be described in terms of Figure B? Name the figures and give reasons to support your doubts.

……

1.5. Check if Figure 1 shows a connection between some of the shapes in Table 2 and Table 3. Describe the connection you have noticed.

…….

1.6. Write down the relationship between properties of the connected shapes?

…….

Complete the following sentence:

1.7. In Figure 1 …….. is inscribed into …… Therefore, Figure 1 is named ……..

1.8. Study Figure 1 as named in 1.7. Outline the three properties (characteristics) of Figure 1.

……

1.9. Based on the properties (characteristics) named in 1.8, conclude with reasons whether in Figure 1

x = y.

……..

1.1.0. Figure 1 occurs every-time and everywhere in real life. Briefly discuss the importance of Figure 1 in construction of habitats and in architecture.

…….

Table 2

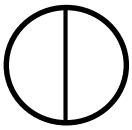
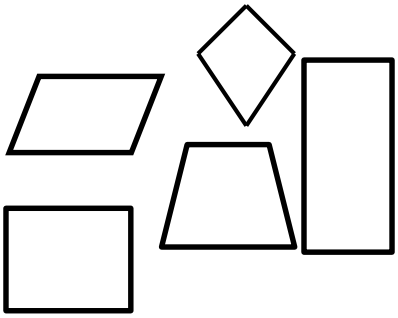
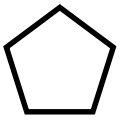
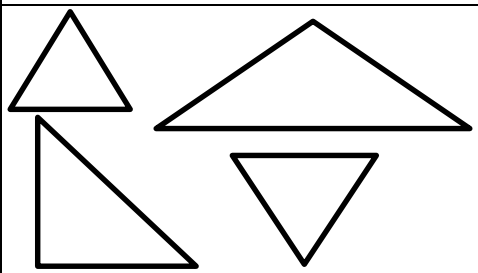
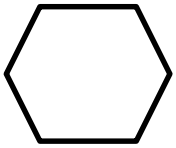
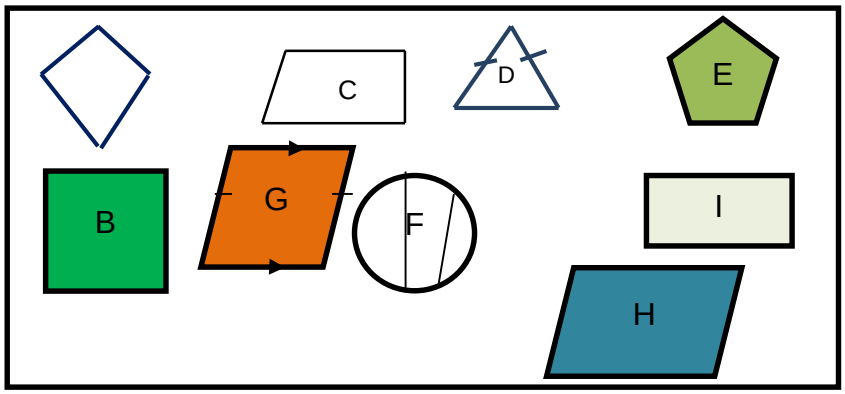
Common name of figures	Shape	Properties
Example Circle – Examples: hut/ rondavel		
(b)		
(c)		
(d)		
(e)		

Table 3



APPENDIX C: ETHICAL CLEARENCE CERTIFICATE

UNISA | college of science, engineering and technology

Dear Motshidisi Masilo (33773246)

Date: 2016-01-26

Application number:
2015_CGS/ISTE_017

REQUEST FOR ETHICAL CLEARANCE: (Implementing inquiry based learning to enhance grade 11 students` mathematical problem solving skills in Euclidean geometry)

The College of Science, Engineering and Technology's (CSET) Research and Ethics Committee has considered the relevant parts of the studies relating to the abovementioned research project and research methodology and is pleased to inform you that ethical clearance is granted for your research study as set out in your proposal and application for ethical clearance.

Therefore, involved parties may also consider ethics approval as granted. However, the permission granted must not be misconstrued as constituting an instruction from the CSET Executive or the CSET CRIC that sampled interviewees (if applicable) are compelled to take part in the research project. All interviewees retain their individual right to decide whether to participate or not.

We trust that the research will be undertaken in a manner that is respectful of the rights and integrity of those who volunteer to participate, as stipulated in the UNISA Research Ethics policy. The policy can be found at the following URL:
http://cm.unisa.ac.za/contents/departments/res_policies/docs/ResearchEthicsPolicy_apprvCounc_21Sept07.pdf

Please note that the ethical clearance is granted for the duration of this project and if you subsequently do a follow-up study that requires the use of a different research instrument, you will have to submit an addendum to this application, explaining the purpose of the follow-up study and attach the new instrument along with a comprehensive information document and consent form.

Yours sincerely

Prof Ernest Mnkandla
Chair: College of Science, Engineering and Technology Ethics Sub-Committee

Prof IOG Moche
Executive Dean: College of Science, Engineering and Technology

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2016 -01- 26
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College of Science, Engineering and Technology

University of South Africa
College of Science, Engineering and Technology
The Science Campus
C/o Christiaan de Wet Road and Pioneer Avenue,
Florida Park, Roodepoort
Private Bag X6, Florida, 1710
www.unisa.ac.za/cset

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APPENDIX D: GDE RESEARCH APPROVAL

For administrative use only:
Reference no: D2016/319
enquiries: Diane Buntting 011 843 6503

GAUTENG PROVINCE
REPUBLIC OF SOUTH AFRICA

GDE RESEARCH APPROVAL LETTER

Date:	**23 October 2015**
Validity of Research Approval:	**8 February 2016 to 30 September 2016**
Name of Researcher:	**Masilo M.M.**
Address of Researcher:	**P.O. Box 18812; Pretoria North; 0116**
Telephone / Fax Number/s:	**012 429 6154; 073 205 7671; 012 429 4922**
Email address:	**motshidisi.dm@gmail.com**
Research Topic:	**Implementing inquiry based learning to enhance Grade 11 student mathematical problem solving skills in Euclidean Geometry**
Number and type of schools:	**TEN Secondary Schools**
District/s/HO	**Tshwane North and Tshwane West**

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved. A separate copy of this letter must be presented to the Principal, SGB and the relevant District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted. However participation is VOLUNTARY.

The following conditions apply to GDE research. The researcher has agreed to and may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

CONDITIONS FOR CONDUCTING RESEARCH IN GDE

1. *The District/Head Office Senior Manager/s concerned, the Principal/s and the chairperson/s of the School Governing Body (SGB.) must be presented with a copy of this letter.*
2. *The Researcher will make every effort to obtain the goodwill and co-operation of the GDE District officials, principals, SGBs, teachers, parents and learners involved. Participation is voluntary and additional remuneration will not be paid;*

Office of the Director: Knowledge Management and Research
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506

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3. *Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal and/or Director must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.*
4. *Research may only commence from the second week of February and must be concluded by the end of the THIRD quarter of the academic year. If incomplete, an amended Research Approval letter may be requested to conduct research in the following year.*
5. *Items 6 and 7 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.*
6. *It is the researcher's responsibility to obtain written consent from the SGB/s; principal/s, educator/s, parents and learners, as applicable, before commencing with research.*
7. *The researcher is responsible for supplying and utilizing his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institution/s, staff and/or the office/s visited for supplying such resources.*
8. *The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research title, report or summary.*
9. *On completion of the study the researcher must supply the Director: Education Research and Knowledge Management, with electronic copies of the Research Report, Thesis, Dissertation as well as a Research Summary (on the GDE Summary template).*
10. *The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned;*
11. *Should the researcher have been involved with research at a school and/or a district/head office level, the Director/s and school/s concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.*

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

.....

Dr David Makhado

Director: Education Research and Knowledge Management

DATE: .....

APPENDIX E: OBSERVATION SCHEDULE

Research Title

IMPLEMENTING INQUIRY-BASED LEARNING TO ENHANCE GRADE 11 STUDENT PROBLEM-SOLVING SKILLS IN EUCLIDEAN GEOMETRY

Data collection instrument

Qualitative data (Observation)

[A] Experimental group

E0. Observation Focus Variable: Visualisation (level 0)

Variable themes		Observation Results (Data)
Teacher Activity	Students' activity	
Resources in the form of geometric figures (or other forms related to Euclidean Geometry) are availed to students Seating arrangement is well organised Teacher presents a question and allow students to brainstorm and find the possible answers from existing knowledge Teacher presents a different notion and permits students to engage in the idea posed.	Student identifies and operates figures (for example, squares, circles,) and more geometry configurations (for example, angles, lines,) according to the shapes' appearance.	

E1. Observation Focus Variable: Analysis (level 1)

Variable themes		Observation Results (Data)
Teacher Activity	Students' activity	
The teacher gives clarification based on proof assembled throughout the probing procedure Teacher provides a method and poses questions that will lead to students unpacking properties of diverse geometric shapes	Students review the shapes based on properties and connections among properties, (brainstorming applying existing knowledge and working with new concepts) Students perform activities leading them to exploration of hidden relationships.	

E2. Observation Focus Variable: Informal deduction (level 2)

Variable themes		Observation Results (Data)
Teacher Activity	Students' activity	
2.1. The teacher poses questions to confirm knowledge and understanding of vocabulary and properties. The main goal is to design the method of investigating by formulating conjectures.	2.1 Student use properties to formulate arguments and conjecture. 2.2 The students' skills are connected to general language networks.	

E3. Observation Focus Variable: Formal deduction (level 3)

Variable themes		Observation Results (Data)
Teacher Activity	Students' activity	
3.1 Teacher poses a question that will provide students with methods of testing the given question and prove conjectures they have made. 3.2. Teacher encourages students to apply inquiry in order to search for information and construct their own enquiries.	3.1 Student establishes, within a postulation system, theorems and interrelationships between networks of theorems. 3.2 Students understand the characteristics of shapes they have learnt, however, they attempt improve on how to link the systems of connecting the characteristics to address diverse contexts. 3.3 Students perform intricate activites empowering with skills to apply systems of connections they gave acquired.	

E4. Observation Focus Variable: Rigour (level 4)

Variable themes		Observation Results (Data)
Teacher Activity	Students' activity	
4.1. Teacher must allow open inquiry and students must present their results at the end of the process. 4.2. Teacher lead presentations and allow open discussions 4.3. Teacher gives explanations and conclusions based one investigative findings and proofs	4.1 Students rigorously do a comparison and analysis on supposition networks in order to build statements as theorems. 4.2 Students sum up the acquired information for retention. 4.3 Students present their work and share ideas with others. 4.4 Students present their ideas for correction by the teacher.	

[B] Comparison group

C1. Observation Variable: Recalling pre-knowledge and actualisation of pre-knowledge

Teacher	Students	Observation results (Data)
1.1 How did the teacher assist student to recall relevant pre-knowledge? 1.2 How was pre-knowledge made relevant to the topic in place? 1.3 Were there any resources used? 1.4 How were the resources used? 1.5 How was the classroom seating planned?	1.1 How was the students' reaction or response to the teacher's style of recalling pre-knowledge 1.2 Was the students' pre-knowledge relevant? 1.3 Could the students' pre-knowledge be actualised or linked to the new topic?	

C2. Observation Variable: New Knowledge

Teacher	Students	Observation results (Data)
2.1 How did the teacher introduce the new topic and content? 2.2 What material/resources were used to deliver the new content? 2.3 What teaching method did the teacher employ?	2.1 Did the students participate in finding new knowledge? 2.2 Were they dependent on the teacher? 2.3 How was the students' reaction to the teaching method and resources?	

C3. Observation Variable: Functionalisation (New knowledge)

Teacher	Students	Observation results (Data)
What type of examples and questions/problems did the teacher use to apply new knowledge in problem-solving?	3.1 Did the students understand and grab the new knowledge? 3.2 Did they successfully apply new knowledge in problem-solving? 3.3 Did the students work individually or as a group?	

C4. Observation Variable: Conclusion

Teacher	Students	Observation results (Data)
4.1. How did the teacher link the whole content with real-life problems? 4.2. How was assessment planned to evaluate student knowledge and understanding?	4.1. How were the students engaged in finding solutions to real-life problems posed to them? 4.2. How as the students' response or performance in the assessment?	

APPENDIX F: INTERVIEW QUESTIONS

Research Title

IMPLEMENTING INQUIRY-BASED LEARNING TO ENHANCE GRADE 11 STUDENT PROBLEM-SOLVING SKILLS IN EUCLIDEAN GEOMETRY

Data collection instrument

Qualitative data

Interview questions

Questions for students

2 What do you understand by the word Euclidean Geometry?
3 Can you outline the challenges you encounter in learning Euclidean Geometry?
4 What strategies do you apply when solving problems in Euclidean Geometry?
5 How would you prefer to be taught in order to improve knowledge and understanding in Euclidean Geometry?
6 What material or resources do you need to help you understand geometry concepts better?
7 Do you think Euclidean Geometry can be applied in problem-solving out of the classroom?
8 If yes, how?
9 If no, explain why?
10 How can you rate your performance in Euclidean Geometry (poor, average, good)?
11 What is the main cause of such performance?
12 How would you like to your teacher to help you to improve in learning Euclidean Geometry?
13 Are you looking forward to any future through learning Euclidean Geometry?
14 Questions for teachers
15 Can you outline the importance of teaching Euclidean Geometry at school?
16 What strategy/teaching method do you apply in teaching cyclic quadrilaterals (or Euclidean Geometry in general)?
17 What challenges do you encounter in teaching Euclidean Geometry?
18 Do you prefer a specific classroom setting/seating during you Euclidean Geometry lessons?

19 If yes, what setting/seating?

20 If no, how specifically do you organise your classroom?

21 What material/ or resources do you find necessary to teach Euclidean Geometry topics?

22 Do you use any of the mentioned materials? How and what was the outcome in your last lesson?

23 How can you rate your teaching and your students' performance in Euclidean Geometry last year and this year (poor, average, good)?

24 What is the main cause of such performance?

25 What improvements would you like to make for better facilitation and learning in Euclidean Geometry?

APPENDIX G: PRINCIPAL'S PERMISSION LETTER

College of Graduate Studies

Institute of Science and Technology Education

UNISA

Date.....

The Principal

.....

(Name of School)

Dear Sir/Madam

Research title: Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean Geometry.

My name is Motshidisi Masilo. I am a Doctor of Philosophy (Mathematics education) student at the University of South Africa. I am conducting a research study that will involve mathematics teachers and students in Grade 11 at your school. I request your permission to conduct research at your school. The research study will focus on teaching strategies applied in Euclidean Geometry in order to improve the students' problem-solving skills.

Your school is one of the selected schools representing your district in Gauteng, and it is classified as the experimental group. Your role in the research study will involve allowing the researcher to execute pre- and post-test in one Grade 11 Mathematics classroom and to observe teachers and students as they will be engaging in classroom activities. Your permission to train and interview the subject teacher on inquiry-based teaching in the selected classroom is requested. Further, I request permission to interview only three students whose progress will be tracked during the lesson. I am looking forward to spending one or two weeks at your school in a very productive interaction with one teacher and his/her mathematics class in Grade 11.

This research study will not only benefit the institution involved, but will contribute to the improvement of the teacher's teaching strategies as well as students' problem-solving skills in Euclidean Geometry. Please note that if you allow one teacher and his/her Grade 11 Mathematics class to participate in the research study, the following ethical values will apply: the teacher and students' participation is voluntary; all information will be treated with confidentiality and anonymity in order to ensure that no harm or bad effect will be caused to participants by the research study; all observation videos and interview recordings will be destroyed at the end of the study; participants will be granted the right to withdraw when they so wish, they may also refrain from answering questions when they see it necessary.

I will avail to you the summary of the study results at the time of completion if you would wish to have the summary.

Thank you in advance for your support

Yours sincerely

Motshidisi Masilo

(Please complete the consent form below and return to me)

Permission to conduct research

I..... the principal of ..... understand the context of the research study and I grant permission that the research study (title: Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean Geometry) may be conducted at the school. I am aware that the teacher and students' participation is voluntary; all information will be treated with confidentiality and anonymity in order to ensure that no harm or bad effect will be caused to participants by the research study; all observation videos and interview recordings will be destroyed at the end of the study; participants will be granted the right to withdraw when they so wish, they may also refrain from answering questions when they see it necessary.

Principal's signature: .....

Date: .....

APPENDIX H: TEACHER CONSENT LETTER

College of Graduate Studies

Institute of science and Technology Education

UNISA

Date.....

Dear Teacher

Re: Request for your participation in a research study

My name is Motshidisi Masilo. I am a Doctor of Philosophy (Mathematics education) student at the University of South Africa. I request you to participate in a research study titled: Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean Geometry. Your school is classified as the experimental group. Your interaction with the researcher will involve allowing the researcher to execute pre- and post-test in your classroom, to work with you in administering inquiry-based learning in your classroom. I will further request to interview only three students whose development will be tracked during the lesson. I am looking forward to spending two weeks in your classroom engaging in a very productive interaction with your students.

Your participation is voluntary. You may discontinue participation at any time if you so wish. You may also refrain from answering some interview questions when there is a need to do so. I am looking forward to your participation in the research study. I request you to sign the consent form provided if you accept my request to participate.

Thank you in advance for your support

Yours sincerely

Motshidisi Masilo

(Please complete the consent form below and return to me)

Teacher's participation consent

I ..... understand the context of the research study titled:
(Name & Surname)

Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean Geometry. I am aware that optionally I can allow video and audio recording during participation. I am aware that anonymity and confidentiality will be adhered to in this study. I am informed that I may withdraw my consent to participate at any time without penalty by advising the researcher. I agree on my free will to participate in the research study.

Participant's signature: .....

Date: .....

APPENDIX I: PARENT/GUARDIAN CONSENT LETTER

College of Graduate Studies

Institute of Science and Technology Education

UNISA

Date.....

Dear Parent/Guardian

Re: Request for your child to participate in a research study

My name is Motshidisi Masilo. I am a Doctor of Philosophy (Mathematics education) student at the University of South Africa. I request you to allow your child to participate in a research study titled: Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean Geometry. Your child's role in the research study will be to participate in problem-solving activities. His/her progress will be observed during participation in problem-solving. I will also conduct pre- and post-test in order to track the progress of your child. Further, interviews will be conducted to find out how your child experienced the problem-solving activities during lessons. At the end of the research study your child is expected to show improved problem-solving skills in Euclidean Geometry.

If you allow your child to participate in the research study, take note that the following ethical values will apply: your child's participation is voluntary; he/she may discontinue participation at any time if a need arise; he/she may also refrain from answering some interview questions when there is a need to do so. I am looking forward to your child's participation in the research study. I request you to sign the consent form provided if you give permission that your child may participate.

Thank you in advance for your support

Yours sincerely

Motshidisi Masilo

(Please complete the consent form below and return to school)

Participation consent

I:....., parent/guardian of ..... understand

the context of the research study titled: Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean Geometry. I am aware that optionally I can allow video and audio recording of my child's participation during problem-solving sessions. I am aware that anonymity and confidentiality will be adhered to in this study. I am informed that I may withdraw my consent for my child to participate at any time without penalty by advising the researcher. I agree on my free will that my child will participate in the research study.

Parent's/Guardian's signature: .....

Date: .....

APPENDIX J: STUDENTS' ASSENT LETTER

College of graduate studies

Institute of science and Technology Education

UNISA

Date.....

Dear Student

Re: Request for your assent to participate in a research study

My name is Motshidisi Masilo. I am a Doctor of Philosophy (Mathematics education) student at the University of South Africa. I request you to participate in a research study titled: Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean Geometry. Your role in the research study will be to participate in classroom problem-solving activities. Your progress will be observed during participation in problem-solving. I will also conduct pre- and post-test in order to track your progress of. Further, interviews will be conducted to find out how you have experienced the problem-solving activities during the lesson. At the end of the research study you are expected to show improved problem-solving skills in Euclidean geometry.

If you agree to participate in the research study, take note that the following ethical values will apply: your participation is voluntary; you may discontinue participation at any time if a need arise; you may also refrain from answering some interview questions when there is a need to do so. I am looking forward to your participation in the research study. I request you to sign the assent form provided if you agree to participate in the study.

Thank you in advance for your support

Yours sincerely

Motshidisi Masilo

(Please complete the assent form below and return to school)

Participation assent

I·....., understand the context of the research study titled: Implementing inquiry-based learning to enhance Grade 11 students' problem-solving skills in Euclidean Geometry. I am aware that optionally I can allow video and audio recording of my participation during problem-solving sessions. I am aware that anonymity and confidentiality will be adhered to in this study. I am informed that I may withdraw my assent to participate at any time without penalty by advising the researcher. I agree on my free will to participate in the research study.

Student's signature: .....

Date: .....