STRUCTURE OF HYPERNUCLEI STUDIED WITH THE INTEGRODIFFERENTIAL EQUATIONS APPROACH

BY

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Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in the subject

PHYSICS

at the

UNIVERSITY OF SOUTH AFRICA

SUPERVISOR: PROF. G.J. RAMPHO

JUNE 2012
DECLARATION

I declare that "Structure of Hypernuclei Studied with the Integrodifferential Equations Approach" is my own work and that all sources that I have used or quoted have been indicated and acknowledged by means of references.

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Mr J.S. Nkuna                      Date
Acknowledgements

First and foremost I thank the Almighty God for creating such a wonderful world that is filled with so many questions for the curious individual.

It is not possible to acknowledge everyone who directly or indirectly contributed to the realization of this dissertation. Should your name not be in this list, kindly accept my apologies but your help and support is greatly appreciated - continue helping others.

I would like to convey my sincere gratitude to the following people: Prof. G.J. Rampho, my supervisor for his guidance and encouragement, it is greatly appreciated. Prof. M.L. Lekala for the positive words of encouragement. If it were not for Prof. Sofianos, who asked me “how many feet does a lame chicken have” - this work would never have seen the light of day, thank you for encouraging my curiosity. I would also like to thank members of the physics department at the University of South Africa for their support and encouragement.

Finally a word of appreciation for the support from my mother and the rest of the Nkuna family. To Ntombifuthi Khumalo - the mother of my daughter - I appreciate your understanding and the time you gave me. “Sengatsi Umdali losetulu ezulwini anganibusisa kakhulu”.
A two-dimensional integrodifferential equation resulting from the use of potential harmonics expansion in the many-body Schrödinger equation is used to study ground-state properties of selected few-body nuclear systems. The equation takes into account two-body correlations in the system and is applicable to few- and many-body systems. The formulation of the equation involves the use of the Jacobi coordinates to define relevant global coordinates as well as the elimination of center-of-mass dependence. The form of the equation does not depend on the size of the system. Therefore, only the interaction potential is required as input. Different nucleon-nucleon potentials and hyperon-nucleon potentials are employed to construct the Hamiltonian of the systems. The results obtained are in good agreement with those obtained using other methods.

Keywords:
Integrodifferential Equation, Hypernuclei, Schrödinger equation, Adiabatic Approximation, Potential Harmonics, Hyperspherical Harmonics, Few-Body Systems
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Chapter 1

INTRODUCTION

The interaction between baryons, of which protons and neutrons are the lightest, is very strong. This is in contrast to the weak and electromagnetic forces that are responsible for keeping bosons and atoms together, respectively. In terms of their mass, baryons such as the lambda particle are heavier relative to the mass of the proton and neutron. In some modelling approaches the interactions that are observed between baryons, which are classified as such because they are spin−1/2 particles, are always effective interactions. The fundamental microscopic interactions that are governed by quantum chromodynamics (QCD) [1, 2] cannot be accessed directly, but microscopic models can be developed to describe baryon-baryon interactions in small nuclear systems. The interaction between baryons is not only very strong, but also of very short range in comparison with the long range electromagnetic force. This interaction is, however, responsible for the structure of all matter in our daily lives, from small to large scales in the known universe.

In its attempt to understand the fundamental building blocks of matter, the study of nuclear structure has revealed the existence of smaller constituents of matter such as quarks which have also been found to be composite systems. The developments on this subject have attracted attention from theoretical and experimental investigations. Early studies of matter have a concept of the atom with a dense central nucleus surrounded
by a cloud of electrons. Now we know that the nucleus itself can be decomposed into protons and neutrons collectively known as nucleons. Experiments have been performed which indicate that protons and neutrons belong to a family of particles called hadrons which interact via the strong force. Furthermore, in the late sixties the quark model [3] established that all known hadrons could be described as systems of quarks [4].

In nuclear physics, the understanding of how nuclear binding, structure, and stability arise from fundamental interactions between individual nucleons or other nuclear constituents is important in constructing realistic nuclear interaction models. Even though nuclear physics has a long tradition of theoretical and experimental investigations the exact nature of the nuclear force is not yet known. Thus a number of different phenomenological models are in use. Properties of nuclear systems are studied as $A$-body systems interacting via two-body potentials. The use of proven nucleon-nucleon interaction potentials that are supplemented by three-body potentials for the purpose of calculating nuclear properties using a quantum many-body framework that does not violate symmetries of the potentials is known as $ab$ $initio$ calculation. $ab$ $initio$ calculations have been realized to be numerically challenging [5]. In the absence of experimental data, results from $ab$ $initio$ calculations provide a basis against which other models can be measured. Another advantage of $ab$ $initio$ approaches is that they allow for the accurate calculation of the nuclear matrix elements that can be used for investigating processes that can not be accessed presently by using experimental techniques. However, such approaches also require the knowledge of the Hamiltonian of the system. Hence, a phenomenological Hamiltonian is used to describe the nuclear system [6].

The theoretical framework that is adopted in the present work views the nucleus as made up of nucleons and hyperons (which are strange particles that can coexist with nucleons in a nucleus). This requires the assumption that all other subnucleonic degrees of freedom be suppressed in favour of two-body operators acting on the hyperons’ and nucleons’ coordinates. This simplified description suggests that color-carrying quarks and gluons be assembled into colourless clusters (the nucleons and hyperons). The validity of this
approximation is based on results that it has produced in the quantitative prediction of many nuclear structure observables (in comparison with experimental measurements) [7]. Therefore it is necessary to also consider corrections to this premise by taking into account the degrees of freedom associated with colourless quark-gluon clusters (like hyperons) other than the nucleons as additional constituents of the nucleus.

The idea that nucleons and hyperons in the nucleus interact primarily through two-body interactions is sufficient at low \((E/A \leq 20 \text{ MeV})\) to intermediate energies \((20 \leq E/A \leq 200 \text{ MeV})\) and momentum transfer [8]. It is much more sensible to make such an assumption since nuclear properties are dominated by two-nucleon interactions and one- and two-nucleon couplings to electrons probes. This description is not recent, it is an idea that was present in the 1960s in the works of Hamada and Johnson [9] and Reid [10] about nucleon-nucleon interactions which include short-range repulsion, intermediate-range attraction, and long-range attraction due to one-pion exchange. Within the framework of light nuclear systems that have few degrees of freedom, accurate solutions can be obtained for a wide range of nuclear properties directly when using realistic models of NN interactions. From this simple description we can test our understanding of nuclear structure and other dynamics from low energies in few keV to medium energies in the MeV regime to even higher energies in the range of hundreds or even thousands of MeV in experiments. Advances in computational power and facilities which allow for complex data analysis and numerical calculations in the past few decades have benefited progress in experimental and theoretical studies of nuclei. In the low energy regime there has been important advances made in investigating structure of light nuclei, nuclear spectra, the response of light nuclei to external forces, hadronic scattering, and electroweak reactions that affect few-nucleon systems [7].

A large amount of NN scattering data has been collected over the past decades and has been used in the construction of potentials that yield accurate results of nuclear structure information [7]. It has been a long-standing question in nuclear physics to attempt to understand the nature of the nuclear force. It was Yukawa [11] who first described the
pair-wise NN interaction successfully as an exchange of a boson. Modern NN interaction models are based on Yukawa’s idea which provides an excellent approach for the description of high-quality databases of neutron-proton and proton-proton scattering and properties of $^2$H $[^{12}]$. However, even for the simplest tri-nucleon system $^3$H, three-body calculations that use NN forces do not provide good results as compared with experimental data, this suggests that it is not sufficient to merely use NN interactions to give an accurate description of a three-nucleon system. A technique used to overcome these discrepancies is the introduction of an additional three-nucleon force $[^{13}]$. The precision that is required for most of these potentials means they include degrees of freedom such as spin, isospin, spin-orbit, tensor, quadratic momentum-dependent, and charge-independence-breaking terms. However, even with all their complexity these modern NN potentials have not been able to precisely reproduce the binding energies of few-body nuclei such as $^3$H and $^4$He without the use of three-nucleon (3N) potentials $[^{6}]$. In principle, the 3N interaction could have a more complicated formulation that is dependent on spin, isospin, and other parameters such as momenta of nucleons but the lack of information means that there are constraints in the formulation of the nuclear models.

It is possible nowadays to calculate NN phase shifts which compare well with experimental data $[^{14}]$, however the foundation in which the phase shifts and the effective nucleon-nucleon interactions are built is not well grounded $[^{15}]$. As an example it can be noted that when one uses “realistic” NN interactions in nuclear structure calculations the results yield underbinding and leads to overbinding when 3N interactions are included. This is true even when excellent methods like the Green function Monte Carlo (GFMC) method is employed $[^{16}]$. Instances like this require the use of an effective potential that is not identical to the bare NN interaction that is fitted to the data of a particular nucleus $[^{8}]$. Irrespective of the choice of an effective or a realistic interaction, one still has to select a method to solve the Schrödinger equation for the few- or many-body system. This involves selecting a finite model space which is a subspace of a Hilbert space. A very prominent feature of nuclear structure computations is that the more realistic the potential and the less restrictive the subspace, the smaller the nuclei that can be treated. Calculations
are performed taking into account the fact that the number of combinations of spin and isospin $z$-projections available to the system increase exponentially with increasing nucleon number [8, 17].

A number of theoretical methods have been employed in the past to study properties of few-body and many-body nuclear systems. In general, theoretical methods that are applicable to many-body bound-state problems can be categorized into two families. One set of approaches considers the assumption that the potential can be written as a sum of pairwise forces that results in wave functions that are written as sums of amplitudes for the pairs that fulfill a Faddeev-type equation. Faddeev’s approach [18] to the few-body problem can be applied to systems with $A = 4$ but beyond this the resulting equations become very complex and numerically challenging. A consequence of this is that assumptions such as clustering of the system and effective interactions have to be considered to reduce the size of the system. Other approaches that can be used for such systems include variational methods, perturbative theory, G-matrix theory, Resonating Group model, and Monte Carlo based techniques, to mention a few. At times some of the approaches employ combinations of these methods [19].

Calculations that use techniques other than the variational or Monte Carlo based method also need to be explored since these approaches are not capable of determining all the parameters of interest in a system. Examples of variational methods include the Variational Monte Carlo (VMC) [16, 20] and Correlated Harmonic Oscillator Expansion [21] which both involve the Ritz principle to determine variational parameters. An example of a perturbative method is the Brueckner-Hartree-Fock (BHF) [22, 23]. Results from the BHF method can be improved by maximizing the overlap between the wave function and Harmonic Oscillator basis function which can be achieved by selecting an optimum value of the Harmonic Oscillator parameter [8]. The Factor-Aviles-Hartog-Tolhoek (FAHT) [24] coupled cluster method and the Resonating Group Method [25] are both examples of cluster methods. A formal development of the solution to the three-body problem was formulated by Faddeev [18] and it correctly treated the boundary conditions and provided
unique solutions for scattering states. This approach was extended by Alt, Grassberger, and Sandhas (AGS) [26, 27] and by Faddeev and Yakubovsky [28] for applications to four-body systems. Formulated in momentum-space as integral equations, the AGS technique is widely applicable in studies of three- and four-body scattering and break-up reactions [29, 30].

In this work we perform a theoretical study of the structure of hypernuclei by using the Integrodifferential Equations Approach (IDEA). This is a method that is based on an extension of the Faddeev equations for $A \geq 4$, that is, for four or many-bodies. In solving the many-body Schrödinger equation, the many-body spherical harmonics or hyperspherical harmonics are employed. This results in a system of large numbers of coupled hyperradial equations. For an example, the sum of two-body potentials when considered in a many-body problem, is generally not invariant by rotation in the $D$-dimensional space. This procedure leads to a degeneracy factor that causes problems in achieving converged solutions [31]. The Potential Harmonics [32, 33] expansion method (PHEM) gives good approximations to the solution of the convergence problem to some degree by selecting the hyperspherical harmonics that describe only two-body correlations. Faster converging solutions can also be improved by bringing in functions that limit the number of coupled equations [34]. Increasing the number of particles for the concerned system can, however, result in a very large system of coupled differential equations especially when good accuracy is required [31, 35]. The Integrodifferential Equation Approach (IDEA) as introduced by Fabre de la Ripelle et al. [33, 36, 37, 38] aims at dealing with this difficulty. Whatever the number of particles in the system, the form of the equations is the same. Therefore only the two-body potentials are required. The IDEA takes into account two-body correlations and transforms the Schrödinger equation into a two-variable integrodifferential equation.

A variety of methods can be employed to study properties of hypernuclei. In Refs. [39, 40, 41, 42] the Faddeev approach is used to calculate ground-state energies and dissociation energies of some single-lambda and double-lambda light hypernuclei. In these studies the
systems are considered in the cluster model. In Refs. [43, 44, 45] the Gaussian Expansion Method and Variational methods are used to determine ground-state properties of single-lambda and double-lambda hypernuclei. In these studies the systems are also treated in the cluster approach [46]. The Integrodifferential Equations Approach was also used to study light hypernuclei [47]. However, in these studies the systems were treated in the cluster approach as well as non-cluster (A-body systems). In this dissertation we employ the integrodifferential equations approach to investigate the dependence of the ground-state energies and root-mean-square radii of the $^5\Lambda$He and $^6\Lambda\Lambda$He hypernuclei on the input interaction potentials.

This dissertation is organized as follows: in Chapter 2 we introduce the concepts of hypernuclear physics. We discuss some properties of hyperons and give a short description of their production in hypernuclear experiments; Chapter 3 describes the formalism of the Integrodifferential Equations Approach. We explain how the integrodifferential equations are constructed from the Potential Harmonics expansion on the many-body wave functions. In Chapter 4 we present a technique used to solve the integrodifferential equations. We apply the technique to solve the equations for selected hypernuclei. Conclusions are given in Chapter 5.
Chapter 2

HYPERNUCLEAR PHYSICS

2.1 Introduction

Stable nuclear matter is composed of protons and neutrons with protons made up of $udu$ quarks and neutrons of $udd$ quarks. The existence of strange nuclear matter has become the focus of many investigations and poses many challenges in nuclear physics. Strange nuclear matter is composed not only of the usual ”up” ($u$) and ”down” ($d$) quarks that form the substance of ordinary atomic nuclei, but also of strange ($s$) quarks. Hyperons are made up of three quarks, one of which is the $s$ quark. Relative to the $u$ and $d$ quarks, the $s$ quark is much heavier and a consequence of this is that hyperons are unstable (mean lifetime of $\approx 10^{-10}$ seconds) [48]. The $\Lambda$ hyperon was first reported as a hyperfragment (unstable nucleon containing one or two $\Lambda$ hyperons) in a nuclear high-energy cosmic ray emulsion experiment [49].

Nuclei that consist (in addition to nucleons) of at least one type of the baryon hyperons ($\Xi$, $\Sigma$, $\Lambda$ or $\Omega$) are referred to as hypernuclei. Our understanding of nuclear matter as hadronic systems provides new concepts and clues about the physical properties of hadrons. As a result of the presence of hyperons in nuclei, interactions between all baryons can not
be ignored. Some knowledge about the interactions between hyperons and nucleons has been established. For example, in addition to the nuclear force, the potential depth in $\Lambda N$ interactions is approximately $-30\text{MeV}$ [50, 51]. This suggests that $\Lambda N$ bound states may exist. Another investigated feature of the $\Lambda$ hyperon is its apparent “gluing” role in the nucleus. In the presence of a $\Lambda$ hyperon, nuclei that are loosely bound, such as $^6\text{Li}$, are expected to decrease in size due to the interaction between a $\Lambda$ and a nucleon. Moreover, information such as the strength of the $\Lambda N$ interaction is also of great interest [52].

Recent hypernuclear experiments have revealed many features of hyperons. As an example, it has been observed that the $\Lambda$ hyperon retains its identity in a nucleus. This means that the hyperon still displays its properties such as the appearance of its non-mesonic decays and the extremely small spin-orbit strength [53]. Relativistic heavy ion collision experiments performed at ALICE$^1$ in CERN$^2$ and by the STAR$^3$ collaboration [54] have yielded interesting results in the field of hypernuclear physics such as the observation of hypernuclei and antihypernuclei (see Figure 2.2). Despite some of these achievements, hypernuclear physics still faces many challenges especially in areas such as $\gamma$-spectroscopy of $\Lambda\Lambda$ hypernuclei, observation of more antihyperons, and the precise measurement of ground state masses. Some ongoing, performed and planned hypernuclear experimental investigations are indicated in Figure 2.1 and a comprehensive review of current and planned hypernuclear experimental activities is provided in [55].

A hypernuclei nowadays provide a laboratory in which properties of $\Lambda N$ interaction in nuclear matter can be studied. In condensed matter studies, the $\Lambda$ particle may be considered as an impurity in a nucleus and be used for studies of other structural properties of the nucleus [56, 57]. Nuclei with strange degrees of freedom can be plotted in a three-dimensional nuclear chart instead of the more familiar two-dimensional chart of nuclei in which nucleon numbers are plotted according to increasing atomic number $Z$. The

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$^1$A Large Ion Collider Experiment (part of the Large Hadron Collider, LHC).

$^2$Centre Européen pour la Recherche Nucléaire, European Laboratory for Particle Physics.

$^3$Solenoidal Tracker At RHIC. RHIC is the Relativistic Heavy Ion Collider.
strangeness number \((−S)\) is plotted in the third-axis. The antinuclei that are indicated on the chart were discovered from CERN experiments. There exists only a few experimental data points with only 39 for \(Λ\) hypernuclei in the \(S = −1\) plane and two or three double-lambda \(ΛΛ\) points in the \(S = −2\) plane \([50]\).

In as much as our concentration is on \(Λ\) hypernuclei in this study, other properties of (\(Ξ\), \(Σ\), or \(Ω\)) hypernuclei have also been studied \([58, 59, 60]\). However, these will not be discussed here. As an example of some of the research that has been done with these
hyperons, it was found that the only bound state of a Σ hyperon exists with $^4_2$He and is bound by isospin forces [51]. Studies of Σ-nuclei reveal evidence of a repulsive potential in the nuclear core. On the contrary, Ξ-nucleon interactions have been found to be attractive and a number of hypernuclear states have been found to exist [61, 62].

![Figure 2.2: A three-dimensional chart of (hyper) nuclei.](image)

The identification of the double-Λ hypernucleus $^6_{\Lambda\Lambda}$He event in Japan by [63] has drawn a lot of attention in the field of hypernuclear physics. This event identified the $^6_{\Lambda\Lambda}$He to contain a smaller binding energy as compared to an earlier experimental identification [64] of a similar hypernucleus. Inconsistencies between the two data sets mentioned above has led the earlier event (ref. [64]) to be stated under suspicion of misinterpretation.
Measurements of the binding energy of two Λ hyperons, $B_{\Lambda\Lambda}$, and the $\Lambda - \Lambda$ interaction energy $\Delta B_{\Lambda\Lambda}$ are obtained by measurements of the masses of double-Λ nuclei, where the interaction energy is given by

$$\Delta B_{\Lambda\Lambda}(A_{\Lambda\Lambda}Z) = B_{\Lambda\Lambda}(A_{\Lambda\Lambda}Z) - 2B_{\Lambda}(A_{\Lambda} - 1 Z). \quad (2.1)$$

Our knowledge of nuclear physics in the traditional model of few-nucleon systems is based on the assumptions [57, 68] that

- nuclei consist only of nucleons (the isospin degree of freedom is suppressed),
- the motion of nucleons within nuclei is slow enough to employ effective non-relativistic dynamics and
- nucleon interactions are dominated by pairwise interactions resulting in a Hamiltonian of the form

$$H = \sum_i \frac{p_i^2}{2m_i} + \sum_{i<j} V_{ij} \quad (2.2)$$

where $p_i$ is the momentum of nucleon $i$, $m_i$ the mass of the nucleon and $V_{ij}$ the two-body interaction.

A consequence of this model is an extremely large simplification of the problem. However, this simplified model gives excellent results. As an example, calculations have been performed [69] using only the two-body AV18 potential [70] and the results correctly predicted important properties of nuclear structure such as the sudden saturation of the binding energy above $^4$He and the correct ordering of excited states. It is, however, important to note that this potential has its disadvantages. For instance, without the exception of $^2$H, it underbinds all nuclei and this behaviour grows rapidly with increasing $A$. To account for low energy electromagnetic interactions, in addition to the pairwise interaction in equation (2.2) one can add a small three-body force (3BF) [57].
2.2 Hyperon Interactions in Nuclear Matter

The development of new interpretations, ideas and understanding of baryon-baryon interaction in nuclear matter is one of the essential goals of hypernuclear physics. The baryon-baryon interaction forms a fundamental and crucial problem of nuclear physics. In general, data from nuclear scattering experiments are used to construct interaction models. However, the lack of sufficient hyperon-nucleon ($YN$) and hyperon-hyperon ($YY$) scattering data makes it difficult to model such interactions. The lack of data is a result of the fact that $YN$ scattering experiments are difficult to perform [71, 72]. This is a result of the unstable nature of hypernuclei (lifetime $\approx 10^{-10}$ seconds) meaning that the hypernuclear path length is very short. While there are many nucleon-nucleon (NN) scattering data studied extensively since the 1950s, this is not the case with hyperon-hyperon scattering. Direct experimental investigations of hypernuclei are still very sparse. The reason for this is, as indicated above, the short lifetimes, the unavailability of hyperon targets and also with low momentum hyperons being difficult to produce [73].

2.2.1 Production of Hypernuclei

There are two classes of reactions that are used to produce and identify hypernuclei. The first one depends on the detection of the decay products and the second reaction uses kinematic information of the production process to identify the hypernucleus [74]. It is possible to use violent hadronic interactions to produce secondary hypernuclei from delayed weak decay processes [50, 75]. The information from hypernuclear spectra is obtained by analysing decay products in the formation of $\Lambda\Lambda$ hypernuclei, (i) in cosmic ray interactions in an emulsion experiment or bubble chamber, (ii) and in single hypernuclei produced in proton or heavy ion induced reactions, to give but a few examples. Of great interest are measurements of lifetimes by $\gamma$ decay, spectroscopic level structures, ground-state binding energies and other observables [55]. When employing kinematic techniques, one analyses the missing-mass of the incident beam and observed meson. For example, the
double charge exchange reaction \((\pi^+, K^+)\), the conversion of a neutron into a \(\Lambda\) hyperon by the reactions \((\pi^+, K^+)\) and \((K^-, \pi^-)\), and the conversion of a proton into a hyperon by the reactions \((\gamma, K^+)\) and \((e, e' K^+)\) [55]. Only a small number of hypernuclei can be observed using these reactions because they require stable target nuclei. Some of the possible channels that can be used to produce hypernuclei are illustrated in Figure (2.3).

Figure 2.3: Hypernuclei accessible by missing mass experiments for the different production channels. The boundaries at the neutron and proton rich side mark the predicted drip lines by a nuclear mass formula extended to strange nuclei. This figure is adopted from reference [55].

Since it is not practical to perform direct scattering experiments between two hyperons, we rely on the precise spectroscopy of multi-strange hypernuclei to provide us with unique
chances to explore hyperon-hyperon interactions. Angular distributions and polarisations have been measured at some energies, particularly for the $\Lambda N$ system, but data are still too sparse [76]. The substantial progress that has already been made in nuclear structure [73, 77] is certainly expected to contribute detailed information on excitation spectra of double hypernuclei. The structure of such spectra will contribute unique information on the hyperon-hyperon interaction.

The $\Lambda$ hyperon that is bound in a nucleus will ultimately decay from its ground state through a process of weak interaction. The decay is either a $\pi$ mesonic or a non-mesonic decay process. In the $\pi$ mesonic decay process a $\Lambda$ hyperon in a hypernucleus decays to a nucleon ($N$) and a pion via $\Lambda \to N\pi$, this is how a free $\Lambda$ particle decays. The Pauli blocking strongly suppresses $\pi$ mesonic decay of the $\Lambda$ in heavy hypernuclei due to the process releasing small momentum [78]. Despite this process, there exist counter-balancing nuclear processes (like nuclear distortion effect), to the blocking which still have a share of about 10 percent of the production of single and double hypernuclei [79]. As a result this process can be observed experimentally and can be used to provide valuable information about $\pi$ meson processes deep inside the nucleus. Another $\Lambda$ decay process that occurs inside the nucleus is non-mesonic weak decay, $\Lambda + N \to N + N$. This process is dominant when the mass of the hypernucleus increases due to its large momentum bearing at approximately 400 MeV/c, branching ratios for non-mesonic decay processes reach almost 80 percent for the $^{12}_\Lambda$C hypernucleus and about 90 percent for the $^{28}_\Lambda$Si hypernucleus [80]. Confirmations of these processes have been done experimentally in the Korea-Japan collaboration experiments KEK-PS E307 and E369 [67]. Non-mesonic decay mechanisms are not entirely understood and experimental data of partial decay rates have large uncertainties and fail to provide clear guidelines for theoretical studies [55].
2.2.2 Hyperon-Nucleon Potential Models

Nuclear interaction models are often constructed by fitting nucleon-nucleon (NN) scattering data. The difference between the $\Lambda N$ and $NN$ interaction models is that in contrast to $YN$ data, $NN$ scattering data is abundant (including important spin observables) which means that spin-dependent partial wave amplitudes can be readily obtained. $YN$ scattering data are lacking with less than 600 events below 300 MeV/c and less than 300 events between 300 and 1500 MeV/c [74]. Again very few spin-dependent data for $YN$ exists. We only have differential and total cross section measurements [57].

In place of reliable data on $\Lambda$-nucleon interactions, the Nijmegen [81, 82, 83, 84, 85], Jülich [86], Ehime [87] and other groups have developed $\Lambda N$ interaction potential models based on the one-boson exchange (OBE). Within the framework of the quark cluster model (QCM) potentials have been developed by groups in Tokyo, Tübingen, Kyoto-Niigata and others [71]. The $NN$, $YN$, and $YY$ systems in the OBE model are connected by $SU(3)$ or $SU(6)$-symmetry relations [88, 89]. When performing calculations in the OBE model and the QCM the model parameters are determined by utilizing readily available $NN$ scattering data and the few $YN$ (especially $\Lambda N$) available data. The difference between OBE and QC models of the baryon-baryon interaction is apparent at short range as a result of model parameters, which are fixed by fitting to the precision $NN$ scattering data, leading to qualitative differences in the sector where the strangeness is not zero [76]. As an example we note that, the antisymmetric spin orbit ($ALS$) forces differ qualitatively between quark models and OBE models [90]. One interesting distinction in the prediction of the component of the $\Lambda N$ interaction is that the quark model [91] predicts that the antisymmetric spin-orbit component of the $\Lambda N$ is strong enough to cancel the effect of the spin-orbit coupling. The OBE models [83, 84, 85] on the other hand predict much smaller antisymmetric spin-orbit contributions. It is important to note that the strange quark $s$ is much heavier than the $u$ and $d$ quarks. A consequence of this is that $\Lambda$ is measurably heavier than the proton and neutron which are made up of the $uud$ and $udd$ quarks, respectively.
The various potential models that have been proposed by the Nijmegen group (excluding \( pp^- \), \( np^- \) [92] and optical potentials) are given in their various classes as

- **Hard-core potentials** - significant examples of these are the ND [83, 84] and NF [85] potentials
- **Soft-core potentials** - examples are the NSC, Nijm78 [93] and its updated form the Nijm93 [12] potentials
- **Extended soft-core potentials** - an example is the ESC [94, 95] model which is an improvement of the soft-core potentials.

Hard-core potential models are short-range interactions that are described by assuming an infinitely strong repulsion. Soft-core potential models, on the other hand, assume Gaussian form factors, and thus describe both the long or intermediate and short range forces when taking mesons into consideration. The Jülich and Kyoto-Niigata groups have also proposed other versions of \( YN \) interactions [71]. The \( \Lambda N \) and \( \Lambda\Lambda \)-potentials are commonly constructed in such a way that the calculations with these potentials reproduce experimentally known bound states of certain hypernuclei. In particular, the Nijmegen and Jülich are models constructed by extending \( NN \) interaction models on the basis of flavour SU(3) symmetry using the barely available \( \Lambda N \) and \( \Sigma N \) scattering data to adjust parameters. It is, however, a known fact [96] that even with the complete availability of scattering data, it is not possible to construct a potential in a unique way.

A unique feature of the \( \Lambda N \) interaction compared to \( NN \) interactions is that the \( \Lambda \) has isospin 0. This prevents one-pion exchange (OPE) contributions due to isospin conservation requirements for the strong interaction. This suggests that the OPE, which is the long range component in the \( NN \) interaction, is missing in the \( \Lambda N \) interaction [97]. As a consequence the study of the \( \Lambda N \) interaction is more appealing in exploring features of the short-range components in the strong attraction. Nonetheless, the full understanding of the \( \Lambda \) hypernuclear structure and \( \Lambda N \) interaction is a process that is difficult to analyse
and requires solving a many-body problem. In reference [98] it is reported that if one assumes that a Λ hypernuclear wavefunction can be decomposed into a core nucleus and Λ hyperon, then one can express the hypernuclear Hamiltonian in the form

$$H = H_{\text{core}} + T_{\Lambda} + \sum V_{\Lambda N}^{\text{eff}}$$  \hspace{1cm} (2.3)

where $H_{\text{core}}$ is the Hamiltonian for the core nucleus, $T_{\Lambda}$ is the kinetic energy of the Λ hyperon and $V_{\Lambda N}^{\text{eff}}$ is the effective ΛN interaction. Starting from a two-body interaction in free space, G-matrix theory calculations are utilised to construct the effective interaction. The Nijmegen and Jülich one-boson exchange models (for an example), are broadly used to account for elementary two-body interactions.

The effective potentials for the singlet-even (-odd) and triplet-even (-odd) channels are often written in analytic form as a Gaussian sum of three terms [71],

$$V_{\Lambda N}(r) = \sum_i (a_i + b_i k_f + c_i k_f^2) \exp(-r^2/\beta_i^2)$$ \hspace{1cm} (2.4)

where $a_i$, $b_i$, and $c_i$ are parameters and $k_f$ is the Fermi momentum [99]. The parameters that appear in (2.4) are adjusted so that $V_{\Lambda N}(r)$ is phase equivalent to the Nijmegen hard core interaction [83, 84]. From the effective potential one can extract a wide variety of hypernuclear properties such as level structure and reaction cross-sections and compare with experiments. Theoretical results produced using equation (2.4) are reliable to a good degree. This is expected because no anti-symmetrisation against the nucleon is required and the ΛN is much weaker than the NN interaction [97]. The properties of any few-body hypernuclear system with $A \leq 5$ can be calculated directly from the two-body interaction and comparisons of binding energies of ground and excited states with experimental data have shown good agreement [42, 43, 100, 101]. There also exist phenomenological approaches [102, 103] to studies of the ΛN effective interaction in p-shell Λ hypernuclei. Spin dependence of the effective interaction between a Λ in the 0s orbit and a nucleon in the 0p orbit have been examined. The resulting ΛN interaction...
has the form

\[ V_{AN}(r) = V_0(r) + V_\sigma(r)s_N \cdot s_\Lambda + V_\Lambda(r)l_{NA} \cdot s_N + V_N(r)l_{NA} \cdot s_N \]

\[ + V_T(r) [3(\sigma_N \cdot \hat{r})(\sigma_\Lambda \cdot \hat{r}) - \sigma_N \cdot \sigma_\Lambda]. \]  

where

- \( V_0(r) \) represents the contribution of the spin averaged central force
- \( V_\sigma(r)s_N \cdot s_\Lambda \) is the spin-spin force
- \( V_\Lambda(r)l_{NA} \cdot s_N \) is the contribution of the \( \Lambda \)-spin-dependent spin-orbit force
- \( V_N(r)l_{NA} \cdot s_N \) is the nucleon-spin-dependent force
- and \( V_T(r) [3(\sigma_N \cdot \hat{r})(\sigma_\Lambda \cdot \hat{r}) - \sigma_N \cdot \sigma_\Lambda] \) is the contribution to the tensor force.

One can determine low-lying energy levels of p-shell hypernuclei from radial integrals over the \( s_Np_N \) wavefunction for each of the five terms in equation (2.5). The integrals are calculated from available \( p \)-shell \( \Lambda \) hypernuclear data and then compared with theoretical predictions of \( \Lambda N \) interactions from G-matrix calculations [103]. The past decades of studying hypernuclei have revealed many features of the \( \Lambda N \) interaction. However, comparison of the theoretical and experimental spectra remains inconclusive and filled with flaws. One of the discrepancies is that various potentials lead to the same spectra of hypernuclei [50].

The demand for improved hyperon-hyperon and hyperon-nucleon potentials is not only restricted to theoretical and experimental nuclear physics, but is also studied in astrophysics. In reference [51, 104] it is indicated that studies of neutron stars show that they are very dense and are composed of compact hypernuclei. Since the core structure of a neutron star is not precisely known, several models [105] have been developed about the possible constituent of these compact objects. The strangeness in neutron stars undoubt-
edly plays an important role on the properties of neutron stars. The two most important properties of neutron stars are their maximum masses and typical radii which are not yet well known, but they can be obtained from appropriate equation of states that are able to describe dense matter. With regard to the astrophysical scale the presence of hyperons in the dense core of neutron stars has been a subject that received much attention since the early days of neutron star research [106]. Hyperons seem to appear in neutron stars at densities that are two to three times the normal nuclear density and that the type of hyperons that dominate depends on the hyperon-nucleon interaction. This happens irrespective of the hyperon-nucleon interactions, incompressibility, and symmetry parameter used. The presence of an extra degree of freedom, the strangeness, tends to soften the equation-of-state resulting in a reduced maximum mass of a neutron star compared to a purely nucleonic equation-of-state (EOS). The recent observation [107] of a neutron star with about twice the solar mass eliminates many EOSs including those that predict the presence of hyperons. That said, there are still several high-density EOSs that can be specified which allow for or give results about neutron stars with two times or even more than that of the solar mass. As expected, one has to account for the origin of the extra repulsion which is needed to stiffen the EOS at high densities. This has been cited to be possibly related to the extra repulsion that is similar to the three-body repulsion in ordinary nuclear systems or possibly to some vector meson couplings [55]. Further studies of hypernuclear physics and related fields can provide clear and meaningful answers to these and other unanswered questions.
Chapter 3

THE INTEGRODIFFERENTIAL EQUATIONS APPROACH

3.1 Introduction

A theoretical study of hypernuclei requires a realistic Hamiltonian and a good wave function (WF) that includes all dynamical correlations induced by the interactions in the Hamiltonian. However, the use of a realistic Hamiltonian results in computational complexities that increase with increasing particle number $A$, irrespective of the many-body techniques involved [108]. It is possible to reduce these complexities by using computational approaches that simplify the problem. The study of $A$–particle bound state problems can be approached, broadly speaking, via two families of techniques. In one of these techniques one assumes that the interaction between particles can be written as a sum of pairwise forces which motivates for the wave function of the system to be written as a sum of pairwise amplitudes that fulfil a Faddeev-type equation. The second technique involves variational methods. Many-body correlation effects can be included in these techniques [109].
• The first approach is the Faddeev [18] (for three-body) or Faddeev-Yakubovsky decomposition of the wave function (for four-body), which result in an infinite set of coupled integrodifferential equations with two or three variables. The Faddeev equations are restricted to three- and four-body systems. In order to investigate nuclear systems that consist of more than four nucleons one has to resort to cluster approaches and use effective intercluster interactions. Numerous calculations [110, 47] with the Faddeev equations have been performed for different hypernuclei like $^6\Lambda\Lambda$He, $^9\Lambda$Be, $^{13}\Lambda$C, $^{10}\Lambda$Be, where the systems were considered as three- and four-body cluster systems.

• The second method involves transforming the Schrödinger equation to a variational problem using various techniques such as the Hyperspherical Harmonic-Expansion Method (HHEM) [32], Variational Monte Carlo (VMC) or Green-Function Monte Carlo (GFMC) [19, 111]. The HHEM technique also results in the conversion of the Schrödinger equation into an infinite set of coupled second-order differential equations.

It is significant to stress that the expansion of hyperspherical harmonics functions results in an infinite set of equations. The infinite set is truncated to a finite number $N$ because you cannot solve for infinite variables. The choice of the number $N$ is based on the significance of the HH required to give a good description of the function expanded.

The introduction of the HHEM for solving three-body problems also has its challenges. Attaining convergence with HHEM is difficult due to a large number of independent harmonics of the grand orbital quantum number $L$. This problem is aggravated by the fact that each equation is associated with one hyperspherical harmonic. A solution to this problem was achieved by the introduction of the Potential Harmonics (PH) expansion approach [32, 112]. The advantage of the PH expansion is that one can select only the harmonics that are relevant to describe the two-body correlations. For a three-body system in an $S$-state only one harmonic is selected for each even grand orbital number $L$. A consequence of this approach is the extreme reduction of the number of coupled equations
that have to be solved [113]. One can obtain equivalent two-variable integrodifferential equations for \( A \)-particle systems from the corresponding Schrödinger equation by expanding the wave function in Faddeev amplitudes for the numerous particle pairs, as well as in potential harmonics as will be explained in the following sections.

3.1.1 Centre of Mass Effects

In studies of properties of many-body systems only internal motions in the system are relevant. The internal motions of the system are required to be translationally and rotationally invariant. The condition of translational invariance is obtained by choosing a basis in which we can separate out the centre-of-mass motion. This is achieved by introducing the Jacobi coordinates [36, 114]:

\[
\xi_N = \left[ \frac{2Am_1m_2}{M(m_1 + m_2)} \right]^{1/2} (x_2 - x_1)
\]

\[
\xi_{N-1} = \left[ \frac{2A(m_1 + m_2)m_3}{M(m_1 + m_2 + m_3)} \right]^{1/2} \left( x_3 - \frac{m_1x_1 + m_3x_2}{m_1 + m_2} \right)
\]

\[\vdots\]

\[
\xi_{N-i+1} = \left[ \frac{2A(\sum_{j=1}^{i-1} m_j)m_{i+1}}{M \sum_{j=1}^{i} m_j} \right]^{1/2} \left( x_{i+1} - \frac{\sum_{j=1}^{i} m_jx_j}{\sum_{j=1}^{i} m_j} \right)
\]

\[\vdots\]

\[
\xi_1 = \left[ \frac{2A(M - m_A)m_A}{M^2} \right]^{1/2} \left( x_A - \frac{\sum_{j=1}^{A-1} m_jx_j}{M - m_A} \right)
\]

\[
X = \frac{1}{M} \sum_{j=1}^{A} m_jx_j \tag{3.1}
\]

for a system of \( A = N + 1 \) particles. In equation (3.1), \( M = \sum_{j=1}^{A} m_j \) represents the total mass of the system, \( x_i \) the position vector, \( m_i \) the mass of particle \( i \), and \( X \) the centre of mass of the system. In equation (3.1) the coordinates \( \xi_i \) are carefully constructed so that the Laplace operator results in a total kinetic energy operator \( T \) of the form
Figure 3.1: A diagrammatic representation of one of the partitions of Jacobi coordinates for six particles.

\[ T = -\frac{\hbar^2}{2} \sum_{i=1}^{A} \frac{1}{m_i} \nabla^2 \]

\[ = -\sum_{j=1}^{A-1} \frac{\hbar^2}{M} \nabla^2_{\xi_j} - \frac{\hbar^2}{2M} \nabla^2_X \]

\[ = T_{\text{internal}} + T_{\text{cm}} \]  

where

\[ T_{\text{internal}} = -\frac{\hbar^2}{M} \sum_{j=1}^{A-1} \nabla^2_{\xi_j} \quad \text{and} \quad T_{\text{cm}} = -\frac{\hbar^2}{2M} \nabla^2_X. \]  

Our interest is in the structural properties of the systems and therefore we will employ the internal coordinates only. This is accomplished by assuming that the centre of mass is in the ground state and, as a result only hyperspherical harmonics of degree zero in \( X \) are dominant [8]. The solutions of equations where the Laplace operator is applied can
be separated into a product of the centre-of-mass and translationally invariant internal components.

The \((A - 1)\) Jacobi coordinates given by equation (3.1) describe the relative motion of \(A\) particles that have \(3(A - 1)\) degrees of freedom. We can also define a similar set of hyperspherical variables that have the hyperradius \(r\) with \(2(A - 1)\) spherical polar angles of \(\{\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_N\}\) together with \((N - 1)\) hyperangles \(\{\phi_2, \phi_3, \ldots, \phi_N\}\) that generate the magnitude of the Jacobi vectors \(\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_N\). Let us define the hyperradius which is independent of the centre-of-mass \(X\) [33]:

\[
r = \left[\sum_{i=1}^{A-1} \xi_i^2\right]^{1/2} = \left[\frac{2A}{M^2} \sum_{i<j \leq A} m_i m_j r_{ij}^2\right]^{1/2}
\]

(3.4)

where \(r_{ij} = x_i - x_j\). Then the Jacobi vectors and the hyperspherical coordinates are related by [47, 33]

\[
\begin{align*}
\xi_N &= r \cos \phi_N, \\
\vdots \\
\xi_{N-1} &= r \sin \phi_N \cos \phi_{N-1}, \\
\vdots \\
\xi_i &= r \sin \phi_N \ldots \sin \phi_{i+1} \cos \phi_i, \\
\xi_1 &= r \sin \phi_N \ldots \sin \phi_2 \quad (\phi_1 = 0).
\end{align*}
\]

(3.5)

The angular components of these coordinates represent a set of \(3N - 1\) coordinates describing the position of a point on the unit hypersphere \(r = 1\) [32, 115]:

\[
\Omega(\omega_1; \omega_2; \ldots; \omega_N, \phi_N)
\]

(3.6)

where \(\omega_i = (\theta_i, \varphi_i)\) is the set of two angular spherical coordinates of Jacobi vector with
the hyperangles defined by [32]

\[ \tan \phi_i = \frac{1}{\xi_i} \left( \sum_{j=1}^{i-1} \xi_j^2 \right). \]  

(3.7)

The Laplace operator in hyperspherical coordinates has the form [32, 111, 116]

\[ \nabla^2 = \sum_{i=1}^{N} \nabla^2 \xi_i = \frac{\partial^2}{\partial r^2} + \frac{3A - 4}{r} \frac{\partial}{\partial r} + \frac{L^2(\Omega_N)}{r^2} \]  

(3.8)

where

\[ L^2(\Omega) = 4(1 - z^2) \frac{\partial^2}{\partial z^2} + 5 [2 - (A - 1)(1 + z)] \frac{\partial}{\partial z} + \frac{L^2(\omega_{ij})}{1 + z} + 2 \frac{L^2(\Omega_{N-1})}{1 - z}, \]  

(3.9)

\( z = \cos 2\phi \), and as usual \( N = (A - 1) \). The term \( L^2(\Omega) \) is the grand orbital operator in the \( D = 3(A - 1) \) dimensional space, and \( L^2(\Omega_{N-1}) \) is the grand orbital operator in the \( D - 3 \) dimensional space. The volume element \( d\tau \) in terms of Jacobi coordinates is given by [109]

\[ d\tau = \prod_{j=1}^{N} \xi_j^2 \cos \phi_j d\phi_j d\omega_j = \prod_{j=1}^{N} \xi_j^2 d\xi_j d\omega_j \]  

(3.10)

with \( d\omega_j = \cos \phi_j d\phi_j \). The coordinates in equation (3.5) can now be used to write the volume element as

\[ d\tau = d\omega_1 \prod_{j=2}^{N} \cos^2 \phi_j (\sin \phi_j)^{3j-4} d\phi_j d\omega_j r^{3N-1} dr. \]  

(3.11)

The variable \( z_j = \cos 2\phi_j \) can be used in equation (3.11) to write the surface element \( d\Omega \) in the form

\[ d\Omega = \cos^2 \phi_j (\sin \phi_j)^{3j-4} d\phi_j d\omega_j r^{3N-1} dr \]  

(3.12)
where $W_j(z_j)$ are weight functions given by
\[ W_j(z_j) = (1 - z_j^{(3j-5)/2})(1 + z_j)^{1/2}. \] (3.13)

3.1.2 Hyperspherical Harmonics Expansion

The hyperspherical harmonics (HH) expansion method first found its application to describe three-body channels in nuclear reactions by [117, 118]. Earlier attempts to use the HH expansion to solve the tri-nucleon bound state problem were met with the difficulty of a large degeneracy of the HH basis which results in unsatisfactorily slow convergence when using realistic potentials without a selection of the HH involved in a particular problem [119]. This drawback was solved [120] by the introduction of a set of HH that consider the symmetry required with respect to the exchange of a pair of fermions to construct a complete HH basis. It was possible [112, 117, 121] to further decrease the number of HH needed to solve the trinucleon bound state problem by introducing the potential harmonics basis to expand the wave functions.

3.2 Potential Harmonics

The many-body Schrödinger equation in polar coordinates has the form
\[ \left[ -\frac{\hbar^2}{m} \left( \frac{\partial^2}{\partial r^2} + \frac{L^2(\Omega)}{r^2} \right) + V(r, \Omega) - E \right] u(r, \Omega) = 0, \] (3.14)

where $m$ represents the effective mass, defined in terms of individual particle masses. A first attempt to find an accurate solution of the problem was achieved by Simonov [120] by constructing $S$-state basis for three bodies defined only by the two quantum numbers $L$ (the degree) and an integer $\nu$ that is associated with the symmetry by permutation of the coordinates. Another improvement from this followed when it was noticed that
an important correction to an independent particle model ought to come from two-body correlations which can be described by a harmonic basis of the interaction potential [123]. This basis is called the Potential Basis and has elements known as the Potential Harmonics (PH). The theoretical framework of the PH expansion method was first adopted by [32, 111] as an extension of the hyperspherical harmonics expansion method with the intention of applying the method to fermionic systems. The degeneracy problems associated with Hyperspherical Harmonics expansion method are the reason for the introduction of the relatively more efficient PH expansion method [33, 111]. One particular advantage of the PH $P_{2K+l}^{\ell,m}(\Omega_{ij})$ is that they form a complete basis for continuous functions that depend only on the relative coordinates $r_{ij}$.

The potential basis $P_{2K+l}^{\ell,m}(\Omega_{ij})$ are defined as the set of hyperspherical harmonics such that $r^{2K+l}P_{2K+l}^{\ell,m}(\Omega_{ij})$ is a harmonic polynomial of degree $2K + \ell$ [32, 33]. The $P_{2K+l}^{\ell,m}(\Omega_{ij})$ are the eigenfunctions of $\hat{L}^2(\Omega)$ when the eigenvalues of $L^2(\Omega^{N-1})$ equal zero and fulfil the eigen-equation

$$
\left[ \hat{L}^2(\Omega) + L(L + D - 2) \right] P_{2K+l}^{\ell,m}(\Omega_{ij}) = 0, \text{ where } L = 2K + \ell.
$$

(3.15)

Observe that $L^2(\Omega)$ and $L^2(\Omega^{N-1})$ correspond to the $D = 3(A - 1) = 3N$- and $D = 3(A - 2) = 3(N - 1)$- dimensional spaces respectively. For each particle pair $(i, j)$, the potential harmonics are denoted by the expansion

$$
P_{[L]}^{\ell,m}(\Omega_{ij}) = \mathcal{N}^l_K Y^m_\ell(\omega_{ij})(1 + \cos 2\phi_{ij})^{l/2} P^K_{2\alpha,\beta,\gamma}(\cos 2\phi_{ij}),
$$

(3.16)

where the term $\mathcal{N}_K^l$ is the normalisation constant, $Y^m_\ell(\omega_{ij})$ the spherical harmonic, and $P^{\alpha,\beta,\gamma}(\cos 2\phi)$ is a Jacobi polynomial [19, 124]. In this form the pair $(ij)$ is in an $\ell$-state and other pairs are in an $S$-state. This can be written as [19, 111]

$$
P_{2K+l}^{\ell,m}(\Omega_{ij}) = \mathcal{N}^l_K Y^m_\ell(\omega_{ij}) \left( \frac{r_{ij}}{r} \right)^l P^K_{\alpha,\beta,\gamma} \left( \frac{2r_{ij}^2}{r^2} - 1 \right)
$$

(3.17)

where $[L] = 2K + l$, $\alpha = (D - 5)/2$, $\beta = 1/2$, and $D = 3(A - 1)$. The potential harmonics
are normalised in the form
\[
\int_{r=1}^{\ell,m} P_{2K+\ell}^\ell,m(\Omega_{ij}) P_{2K'+\ell'}^{\ell',m'}(\Omega_{ij}) d\Omega = \delta_{KK'}\delta_{\ell\ell'}\delta_{mm'}. \quad (3.18)
\]

In reference [125] it was demonstrated that the exclusive use of the potential harmonics expansion gives extremely accurate results for describing three-body bound state systems in $S$-states. Thus we assume that this method can be applied to any bound state system. The convergence problems associated with the use of hyperspherical harmonics are still present when potential harmonics are used to transform the Schrödinger equation into coupled differential equations (CDEs) [111]. We circumvent this problem by transforming the CDEs into integrodifferential equations [19].

### 3.3 The Integrodifferential Equations

Consider a system consisting of $A$ particles with mass $m_i$, where $i = 1, \ldots, A$ and $n_j$ of the particles being of type $j$. Let the system consist of $J$ distinct types such that
\[
A = \sum_{j=1}^{J} n_j. \quad (3.19)
\]

The total number of particle pairs in the system is [47]
\[
N_p = \frac{1}{2} A(A - 1) = \sum_{j=1}^{J} \frac{1}{2} n_j(n_j - 1) + \sum_{j<i} n_i n_j. \quad (3.20)
\]

Furthermore, divide the system into sets where $n_a$ particles have mass $m_a$ forming the set $a$, $n_b$ particles forming the set $b$ have mass $m_b$ and $n_c$ particles that form the set $c$ have mass $m_c$; then we explicitly have
\[ N_a = \frac{1}{2}n_a(n_a - 1) \] possible pairs in the set \( a \) that form the particle channel 1,
\[ N_b = \frac{1}{2}n_b(n_b - 1) \] possible pairs in the set \( b \) that form the particle channel 2, and
\[ N_c = n_an_b \] possible pairs formed from the set \( a \) and \( b \) forming channel 3.

Our interest is in systems consisting of \( \Lambda \) hyperon and nucleons \( N \) where the proton and neutron are assumed to have the same mass. Thus we have to deal pairs of the form \( \Lambda_i\Lambda_j, N_kN_l \) and \( \Lambda_mN_n \) only.

For a system of particles interacting via pair-wise potentials the Schrödinger wave function \( \Psi(x) \) can be decomposed in Faddeev two-body amplitudes as

\[ \Psi(x) = \sum_{i<j} \psi_{ij}(x) \] (3.22)

where \( \psi_{ij} \) are two-body amplitudes and \( x \) is the position vector in the \( 3(A-1) \)-dimensional space. The Faddeev decomposition of the Schrödinger equation takes the usual form

\[ (T - E) \psi_{ij}(x) = -V(r_{ij})\Psi(x) \] (3.23)

where the total kinetic energy operator is represented by the \( T \), the binding energy by \( E \), \( V(r_{ij}) \) is the potential operator, and \( r_{ij} \) is the relative position vector for the set of particles \( i \) and \( j \).

For identical particles equation (3.23) reduces to \( \frac{1}{2}A(A-1) \) identical equations. Summing up all these equations yields

\[ (T - E) \sum_{i<j} \psi_{ij}(x) = - \left[ \sum_{i<j} V(r_{ij}) \right] \Psi(x), \] (3.24)

which is the Schrödinger equation.

Let us denote a given pair of particles under consideration by \( \varrho \). Then for a system of non-identical particles we require a system of \( N_p \) coupled Faddeev-type equations of the
The Integrodifferential Equations Approach

form [47]

\[ (T - E)\psi_{ij}^{(\varrho)}(x) = -V^{(\varrho)}(r_{ij}) \sum_{i<j} \psi_{ij}(x) \]  

(3.25)

We now consider amplitudes \( \psi_{ij}^{(\varrho)}(x) \) for states that are invariant under rotation in the 
\((D - 3)\)-dimensional space spanned by the \( N - 1 \) vectors \( \vec{\xi}_2, \vec{\xi}_3, \ldots, \vec{\xi}_N \). The invariance is
reflected in the condition given by equation

\[ \hat{L}^2(\Omega_{N-1})\Psi_{ij}(x) = 0, \]  

(3.26)

this suggests that \( \psi_{ij}(r) \) depends only on the vector \( r_{ij} \) and the collective hyperradius \( r \).

Then \( \psi_{ij}^{(\varrho)}(x) \) can be written as a function of the collective variables \( r \) and \( r_{ij} \) as

\[ \psi_{ij}^{(\varrho)}(x) = F_{ij}^{(\varrho)}(r_{ij}, r) \quad \text{where} \quad ij \in \varrho. \]  

(3.27)

This results in the coupled equations [19]

\[ (T - E)F_{ij}^{(\varrho)}(r_{ij}, r) = -V_{ij}^{(\varrho)}(r_{ij}) \sum_{\varrho'} \left[ \sum_{kl} F_{kl}^{(\varrho')} (r_{kl}, r) \right] \]  

(3.28)

where \( kl \) in the summation includes all possible pairs in each particle channel \( \varrho' \). Expanding the amplitude \( F_{ij}^{(\varrho)}(r_{ij}, r) \equiv F_{ij}^{(\varrho)}(r_{ij}, r) \) in potential harmonics we obtain [19]

\[ F_{ij}^{(\varrho)}(r_{ij}, r) = \sum_{K=0}^{\infty} P_{2K+\ell}^{\ell,m}(\Omega_{ij}) U_{K}^{\varrho,\ell}(r) \]  

(3.29)

where

\[ U_{K}^{\varrho,\ell}(r) = \int P_{2K+\ell}^{\ell,m}(\Omega_{ij}) F_{ij}^{\varrho,\ell}(r_{ij}, r) d\Omega \]  

(3.30)

are radial functions and \( P_{2K+\ell}^{\ell,m}(\Omega_{ij}) \) the potential harmonics.

To isolate the equation for the amplitude \( F_{ij}(r_{ij}, r) \), the total wave function is projected
onto the $r_{ij}$ space. This requires the evaluation of the integral [32, 33]

$$
\langle r_{ij} \mid F(r_{kl}, r) \rangle = \sum_K \langle r_{ij}, K, \ell \mid r_{kl}, K, \ell \rangle \mathcal{P}_{2K+\ell}^{m}(\Omega_{ij}) \int \mathcal{P}_{2K+\ell}^{m}(\Omega_{kl}) F^\ell(r_{kl}, r) d\Omega \tag{3.31}
$$

where $\langle r_{ij}, K, \ell \mid r_{kl}, K, \ell \rangle$ are overlap integrals given by [47]

$$
\langle r_{ij}, K, \ell \mid r_{kl}, K, \ell \rangle = \left| \mathcal{P}_{2K+\ell}^{m}(\Omega_{ij}) \right| \mathcal{P}_{2K+\ell}^{m}(\Omega_{kl}) \tag{3.32}
$$

$$
= \left( \cos \varphi_{ij}^{kl} \right)^{\ell} \frac{P_{K}^{\alpha,\beta}(\cos 2\varphi_{ij}^{kl})}{P_{K}^{\alpha,\beta}(1)}. \tag{3.33}
$$

The angle $\varphi_{ij}^{kl}$ is determined as

$$
\varphi_{ij}^{kl} = \begin{cases} 
+1 & \text{if } (kl) = (ij), \\
\frac{m_i m_k - m_j (m_i + m_j + m_k)}{m_i m_k + m_j (m_i + m_j + m_k)} & \text{if } (kl) \text{ and } (ij) \text{ are joint}, \\
-1 & \text{if } (kl) \text{ and } (ij) \text{ are disjoint}.
\end{cases}
$$

Using the notation

$$
z = \frac{r_{ij}^2}{r^2} - 1 \tag{3.34}
$$

we define the projection function

$$
f(z, z', \cos 2\varphi_{ij}^{kl}) = W(z') \sum_{K=0}^{\infty} P_{K}^{\alpha,\beta}(z) P_{K}^{\alpha,\beta}(z') \frac{P_{K}^{\alpha,\beta}(\cos 2\varphi_{ij}^{kl})}{P_{K}^{\alpha,\beta}(1)} h_{K}^{\alpha,\beta} \tag{3.35}
$$

where $h_{K}^{\alpha,\beta}$ are the normalisation constants of the Jacobi polynomials and $W(z)$ the weight functions. The normalisation is given by

$$
h_{K}^{\alpha,\beta} = \int_{-1}^{+1} W(z) \left[ P_{K}^{\alpha,\beta}(z) \right]^2 dz, \tag{3.36}
$$

and the weight functions by

$$
W(z) = (1 - z)^{\alpha}(1 + z)^{\beta} \tag{3.37}
$$
where $\alpha = (D - 5)/2$, $\beta = 1/2$, and $D = 3(A - 1)$. The projection functions $f^{(q,q')}(z, z')$ arise from projecting the amplitudes $F^{(q)}(r_{ij}, r)$ onto the $r_{ij}$ space. When $q = q'$ then the two pairs are of the same type. The projection functions are given by [47]

$$f^{(q,q')}(z, z') = (n_j - 2) \left[ 2f(z, z', -\frac{1}{2}) + \frac{1}{2}(n_j - 3)f(z, z', -1) \right]$$

(3.38)

where $n_j$ is the total number of particles of the type comprising the pair. The projection functions $f(z, z')$ for a particular channel $q$ are given by

$$f^{q}(z, z') = \sum_{\varrho' = 1}^{N_p} \sum_{(kl) \in \varrho} f^{q}_{kl}(z, z', \cos 2\varphi_{ij}^{kl}) = \sum_{\varrho' = 1}^{N_p} Q^{q}_{\varrho'}(z, z').$$

(3.39)

We introduce the new functions $P(z, r)$ through

$$F^{(q)}(r_{kl}, r) = \frac{P^{(q)}(z, r)}{r^{(D-1)/2}}$$

(3.40)

in the coupled differential equation (3.28). The new functions satisfy the integrodifferential equations [47, 109]

$$\left\{ \frac{\hbar^2 A}{M} \left[ T_r - \frac{4}{r^2} T_z \right] - E \right\} P^{(q)}(z, r) = -V^q \left( \frac{r}{\mu^q} \sqrt{1 + z^2} \right) \Pi^{(q)}(z, r)$$

(3.41)

where $\mu$ is the reduced mass of the interacting pair,

$$T_r = -\frac{\partial^2}{\partial r^2} + \frac{L(L+1)}{r^2}, \quad T_z = \frac{1}{W(z)} \frac{\partial}{\partial z} (1 - z^2) W(z) \frac{\partial}{\partial z},$$

(3.42)

and

$$\Pi^{(q)}(z, r) = P^{(q)}(z, r) + \sum_{q' = 1}^{N_p} \int_{-1}^{+1} Q^{q'}_{\varrho'}(z, z') P^{(q')}(z', r) dz'.$$

(3.43)

The reduced mass, given by

$$\mu^q = \left[ \frac{2A m_i m_j}{M(m_i + m_j)} \right]^{1/2},$$

(3.44)
The Integrodifferential Equations Approach

is a result of the non-identical particle constituents of the systems [19]. The equations in (3.41) are referred to as the S-projected integrodifferential equations (SIDE).

For the system of unequal mass particles the total wave function in the Faddeev formalism is expanded as [19]

$$\Psi(x) = \sum_{ij} \psi_{ij}^1(x) + \sum_{kl} \psi_{kl}^2(x) + \sum_{mn} \psi_{mn}^3(x) + \ldots \equiv \sum_{\varrho} \sum_{ij} \psi_{ij}^{(\varrho)}(x)$$ (3.45)

where $\varrho$ depends on the number of types of particles in the system. For a system consisting of only two distinct types of particles there are only three channels: $ii$, $jj$, and $ij$. In matrix form the integrodifferential equation in (3.41) for the three channels has the form

$$\begin{bmatrix}
H_{zr} + V_{AA}^1 - E & 0 \\
0 & H_{zr} + V_{AN}^2 - E
\end{bmatrix}
\begin{bmatrix}
P^1 \\
P^2
\end{bmatrix} = -
\begin{bmatrix}
V_{AA}^1 & 0 \\
0 & V_{AN}^2
\end{bmatrix}
\begin{bmatrix}
\hat{Q}^{11} & \hat{Q}^{12} \\
\hat{Q}^{21} & \hat{Q}^{22}
\end{bmatrix}
\begin{bmatrix}
P^1 \\
P^2
\end{bmatrix},$$ (3.46)

$$\begin{bmatrix}
H_{zr} + V_{AA}^1 - E & 0 \\
0 & H_{zr} + V_{NN}^3 - E
\end{bmatrix}
\begin{bmatrix}
P^1 \\
P^3
\end{bmatrix} = -
\begin{bmatrix}
V_{AA}^1 & 0 \\
0 & V_{NN}^3
\end{bmatrix}
\begin{bmatrix}
\hat{Q}^{11} & \hat{Q}^{13} \\
\hat{Q}^{31} & \hat{Q}^{33}
\end{bmatrix}
\begin{bmatrix}
P^1 \\
P^3
\end{bmatrix},$$ (3.47)

$$\begin{bmatrix}
H_{zr} + V_{AN}^2 - E & 0 \\
0 & H_{zr} + V_{NN}^3 - E
\end{bmatrix}
\begin{bmatrix}
P^2 \\
P^3
\end{bmatrix} = -
\begin{bmatrix}
V_{AN}^2 & 0 \\
0 & V_{NN}^3
\end{bmatrix}
\begin{bmatrix}
\hat{Q}^{22} & \hat{Q}^{23} \\
\hat{Q}^{32} & \hat{Q}^{33}
\end{bmatrix}
\begin{bmatrix}
P^2 \\
P^3
\end{bmatrix},$$ (3.48)

where we have introduced the operators $\hat{Q}^{\varrho\varrho'}$ such that

$$\hat{Q}^{\varrho\varrho'} P^{\varrho'} = \int_{-1}^{+1} Q^{\varrho\varrho'}_{\varrho'}(z, z') P^{\varrho'}(z', r) dz'$$ (3.49)

and

$$H_{zr} = \frac{\hbar^2 A}{M} \left[ T_r - \frac{4}{r^2} T_z \right].$$ (3.50)

To include effects of higher partial waves, a hypercentral potential is introduced to the S-projected integrodifferential equations. This results in integrodifferential equations of
the form [8, 47, 109, 124]

\[
\left\{ H_{zr} + \sum_{\varrho} \nu_{\varrho} V_{0}^{(\varrho)}(r) - E \right\} P(\varrho)(z, r) = - \left[ V^{(\varrho)} \left( \frac{r}{\mu_{\varrho}} \sqrt{(1 + z)/2} \right) - V_{0}^{(\varrho)}(r) \right] \Pi^{(\varrho)}(z, r) \quad (3.51)
\]

where \( V_{0}(r) \) is the central potential term. The hypercentral potential for each channel \( \varrho \) is given by [19]

\[
V_{0}^{(\varrho)}(r) = \frac{1}{h_{0}} \int_{-1}^{+1} W(z) V^{(\varrho)} \left( \frac{r}{\mu_{\varrho}} \sqrt{(1 + z)/2} \right) dz \quad (3.52)
\]

where only the \( K = 0 \) multipole is considered.

Introducing the short-hand notation

\[
\mathcal{H}_{zr}^{(\varrho)} = H_{zr} + \sum_{\varrho'} \nu_{\varrho'} V_{0}^{(\varrho')} + \Delta V^{(\varrho)}
\]

where

\[
\Delta V^{(\varrho)}(z, r) = V^{(\varrho)} \left( \frac{r}{\mu_{\varrho}} \sqrt{(1 + z)/2} \right) - V_{0}^{(\varrho)}(r), \quad (3.54)
\]

the matrix equations for the integrodifferential equations of the three channels take the form

\[
\begin{bmatrix}
H_{zr}^{1} - E & 0 \\
0 & H_{zr}^{2} - E
\end{bmatrix}
\begin{bmatrix}
P^{1} \\
P^{2}
\end{bmatrix}
= - \begin{bmatrix}
\Delta V_{AA}^{1} & 0 \\
0 & \Delta V_{AN}^{2}
\end{bmatrix}
\begin{bmatrix}
\hat{Q}_{11} & \hat{Q}_{12} \\
\hat{Q}_{21} & \hat{Q}_{22}
\end{bmatrix}
\begin{bmatrix}
P^{1} \\
P^{2}
\end{bmatrix}, \quad (3.55)
\]

\[
\begin{bmatrix}
H_{zr}^{1} - E & 0 \\
0 & H_{zr}^{2} - E
\end{bmatrix}
\begin{bmatrix}
P^{1} \\
P^{3}
\end{bmatrix}
= - \begin{bmatrix}
\Delta V_{AA}^{1} & 0 \\
0 & \Delta V_{NN}^{2}
\end{bmatrix}
\begin{bmatrix}
\hat{Q}_{11} & \hat{Q}_{13} \\
\hat{Q}_{31} & \hat{Q}_{32}
\end{bmatrix}
\begin{bmatrix}
P^{1} \\
P^{3}
\end{bmatrix}, \quad (3.56)
\]

\[
\begin{bmatrix}
H_{zr}^{2} - E & 0 \\
0 & H_{zr}^{3} - E
\end{bmatrix}
\begin{bmatrix}
P^{2} \\
P^{3}
\end{bmatrix}
= - \begin{bmatrix}
\Delta V_{AN}^{1} & 0 \\
0 & \Delta V_{NN}^{3}
\end{bmatrix}
\begin{bmatrix}
\hat{Q}_{21} & \hat{Q}_{23} \\
\hat{Q}_{32} & \hat{Q}_{33}
\end{bmatrix}
\begin{bmatrix}
P^{2} \\
P^{3}
\end{bmatrix}. \quad (3.57)
\]

It is these equations that we solve for hypernuclear systems of interest.
3.4 Wavefunction Symmetries

In nucleon systems, the potential is spin-isospin dependent and is therefore different for spin singlet (1) or triplet (3) and even (+) or odd (−) states [113]. Consider a system of three particles \(a, b,\) and \(c\) of unequal masses which interact via spin- and isospin dependent forces. Let us denote the three possible pairs by \(\alpha = (ab), \beta = (ca),\) and \(\gamma = (bc).\) The general form of the central nuclear potential is

\[
V(r_{ij}, \sigma, \tau) = V_{1+}(r_{ij})P_{ij}^{1+} + V_{3+}(r_{ij})P_{ij}^{3+} + V_{1-}(r_{ij})P_{ij}^{1-} + V_{3-}(r_{ij})P_{ij}^{3-};
\]

so that the total potential for the system is given by [19, 127]

\[
V(r) = \left[ V_{1+}(r_\kappa)P_{ij}^{1+} + V_{3+}(r_\kappa)P_{ij}^{3+} + V_{1-}(r_\kappa)P_{ij}^{1-} + V_{3-}(r_\kappa)P_{ij}^{3-} \right] = \sum_\kappa V_\kappa
\]

where \((\alpha\beta\gamma) \in \kappa;\) and the projection operators \(P_{ij}^{3\pm}, P_{ij}^{1\pm}\) act on spin-triplet- and spin-singlet- even and odd states. The two-body Faddeev-like amplitudes are constructed in the form

\[
\mathcal{W}_\kappa = |A'_\kappa\rangle \psi_{\kappa}^{S'} + |A\rangle \psi_{\kappa}^{S} + |S'_\kappa\rangle \psi_{\kappa}^{A'} + |S\rangle \psi_{\kappa}^{A};
\]

where \(|S'\rangle, |S\rangle, |A'\rangle,\) and \(|A\rangle\) are respectively the mixed symmetric, fully symmetric, mixed antisymmetric, and fully antisymmetric spin-isospin \((\sigma\tau)\) states. The total wavefunction is given by

\[
\Psi(x, \sigma, \tau) = \mathcal{W}_\alpha + \mathcal{W}_\beta + \mathcal{W}_\gamma.
\]
we can write the Schrödinger equation in the coupled Faddeev equations of the form

\[(H_0 - E)W_\alpha = -V_\alpha (W_\alpha + W_\beta + W_\gamma),\]

\[(H_0 - E)W_\beta = -V_\beta (W_\alpha + W_\beta + W_\gamma),\]

\[(H_0 - E)W_\gamma = -V_\gamma (W_\alpha + W_\beta + W_\gamma).\]  

(3.62)

We consider only the even states \(\psi^S_{\kappa}\) and \(\psi^{S'}_{\kappa}\) and project each of the coupled equations (3.62) on to the space of the interacting pair. As an example, for the \(\alpha\) pair we project with \(\langle A'_{\alpha} | r_\alpha \rangle\) to obtain [19]

\[(H_0 - E)\psi^{S'}_{\alpha} = -\sum_{n=+1,+3} V_\alpha^n \left[ \langle A'_\alpha | P^n_{\alpha} | A'_{\alpha} \rangle \psi^{S'}_{\alpha} + \langle A'_\alpha | P^n_{\alpha} | A \rangle O^S_{\alpha\beta} + \langle A'_\alpha | P^n_{\alpha} | A'_\beta \rangle O^{S'}_{\alpha\beta} \right.\]

\[\left. + \langle A'_\alpha | P^n_{\alpha} | A \rangle O^S_{\alpha\gamma} + \langle A'_\alpha | P^n_{\alpha} | A'_\gamma \rangle O^{S'}_{\alpha\gamma} + \langle A'_\alpha | P^n_{\alpha} | A \rangle O^S_{\alpha\gamma} \right] (3.63)\]

for the \(\langle A'_\alpha | r_\alpha \rangle\) projection, and

\[(H_0 - E)\psi^{S}_{\alpha} = -\sum_{n=+1,+3} V_\alpha^n \left[ \langle A | P^n_{\alpha} | A'_\alpha \rangle \psi^{S'}_{\alpha} + \langle A | P^n_{\alpha} | A \rangle \psi^{S}_{\alpha} + \langle A | P^n_{\alpha} | A'_\beta \rangle O^{S'}_{\alpha\beta} \right.\]

\[\left. + \langle A | P^n_{\alpha} | A \rangle O^S_{\alpha\beta} + \langle A | P^n_{\alpha} | A'_\gamma \rangle O^{S'}_{\alpha\gamma} + \langle A | P^n_{\alpha} | A \rangle O^S_{\alpha\gamma} \right] (3.64)\]

for \(\langle A | r_\alpha \rangle\) projection, where

\[O^n_{\alpha\gamma} = \int_{-1}^{+1} f(z, z'; \cos 2\varphi_\alpha)\psi^n_{\gamma}(z')dz', \quad n = S' \text{ or } S. \quad (3.65)\]

The mixed symmetric spin-isospin states in channels \(\beta\) and \(\gamma\) can be generated using the relations given in equations (A.7) and (A.8) in appendix A. The coupling of symmetry relations (Appendix A) can be used in equations (3.55), (3.56), and (3.57) to generate the equations:

\[(-H_0 + E)\psi^{S'}_{\alpha} = G^+_\alpha \left( \psi^{S'}_{\alpha} + u^- O^{S'}_{\alpha\beta} + u^+ O^{S'}_{\alpha\gamma} \right) + G^-\alpha \left( \psi^{S}_{\alpha} + O^S_{\alpha\beta} + O^S_{\alpha\gamma} \right), \quad (3.66)\]

\[(-H_0 + E)\psi^{S}_{\alpha} = G^+_\alpha \left( \psi^{S'}_{\alpha} + u^- O^{S'}_{\alpha\beta} + u^+ O^{S'}_{\alpha\gamma} \right) + G^+_\alpha \left( \psi^{S}_{\alpha} + O^S_{\alpha\beta} + O^S_{\alpha\gamma} \right)\]
\(-H_0 + E\psi^S_\beta = G^+_\beta \left( \psi^S_\beta + u^-O^S_{\beta\alpha} + u^+O^S_{\beta\gamma} \right) + G^-_\beta \left( \psi^S_\beta + O^S_{\beta\alpha} + O^S_{\beta\gamma} \right), \tag{3.67}\)

\(-H_0 + E\psi^S_\gamma = G^+_\gamma \left( \psi^S_\gamma + u^-O^S_{\gamma\alpha} + u^+O^S_{\gamma\beta} \right) + G^-_\gamma \left( \psi^S_\gamma + O^S_{\gamma\alpha} + O^S_{\gamma\beta} \right), \tag{3.68}\)

\(-H_0 + E\psi^S_\gamma = G^+_\gamma \left( \psi^S_\gamma + u^-O^S_{\gamma\alpha} + u^+O^S_{\gamma\beta} \right) + G^-_\gamma \left( \psi^S_\gamma + O^S_{\gamma\alpha} + O^S_{\gamma\beta} \right)\)

for the respective \(\alpha, \beta\) and \(\gamma\) channels; where, for example, we have

\[G^\pm_\alpha(z, r) = \frac{1}{2} \left[ V^+_\alpha \left( r/\mu_\alpha \sqrt{(1+z)/2} \right) \pm V^3_\alpha \left( r/\mu_\alpha \sqrt{(1+z)/2} \right) \right] \tag{3.69}\]

for the \(\alpha\) channel, and

\[u^\pm = \frac{1}{2} \left( -1 \pm \sqrt{3} \right). \tag{3.70}\]

Using the transformation

\[\psi^S_\alpha(z, r) = r^{-(D-1)/2}P^N_\alpha(z, r), \tag{3.71}\]

the S-projected integrodifferential equations (3.66), (3.67), and (3.68) take the form

\[
\begin{align*}
\left[ \frac{\hbar^2 A}{M} \left( -T_r + \frac{4}{r^2} T_z \right) + E \right] P^S_\alpha(z, r) &= G^+_{\Pi^S_\alpha}(z, r) + G^-_{\Pi^S_\alpha}(z, r), \tag{3.72} \\
\left[ \frac{\hbar^2 A}{M} \left( -T_r + \frac{4}{r^2} T_z \right) + E \right] P^S_\beta(z, r) &= G^+_{\Pi^S_\beta}(z, r) + G^-_{\Pi^S_\beta}(z, r), \tag{3.73} \\
\left[ \frac{\hbar^2 A}{M} \left( -T_r + \frac{4}{r^2} T_z \right) + E \right] P^S_\gamma(z, r) &= G^+_{\Pi^S_\gamma}(z, r) + G^-_{\Pi^S_\gamma}(z, r), \tag{3.74} \\
\end{align*}
\]
where $T_r$, $T_z$ are given in (3.42) and

$$
\Pi_{\alpha}^S(z,r) = P_{\alpha}^S(z,r) + O_{\alpha}^S(z,r) + O_{\alpha}^S(z,r),
$$

(3.75)

for the $\alpha$ channels. Similarly for the $\beta$ channels we have

$$
\Pi_{\beta}^S(z,r) = P_{\beta}^S(z,r) + O_{\beta}^S(z,r) + O_{\beta}^S(z,r),
$$

(3.76)

and for $\gamma$ channels

$$
\Pi_{\gamma}^S(z,r) = P_{\gamma}^S(z,r) + O_{\gamma}^S(z,r) + O_{\gamma}^S(z,r).
$$

(3.77)

A formulation of the IDEA equations is now simple. For the $\alpha$ channel we have

$$
\left[ \frac{\hbar^2 A}{M} \left( -T_r + \sum_{\kappa} V_{\kappa}(r) + \frac{4}{r^2} T_z \right) + E \right] P_{\alpha}^S(z,r) = \left[ G^+(z,r) - V_{\alpha}(r) \right] \Pi_{\alpha}^S(z,r) + G^-(z,r) \Pi_{\alpha}^S(z,r),
$$

(3.78)

$$
\left[ \frac{\hbar^2 A}{M} \left( -T_r + \sum_{\kappa} V_{\kappa}(r) + \frac{4}{r^2} T_z \right) + E \right] P_{\alpha}^S(z,r) = \left[ G^+(z,r) - V_{\alpha}(r) \right] \Pi_{\alpha}^S(z,r) + G^-(z,r) \Pi_{\alpha}^S(z,r).
$$

(3.79)
For the $\beta$ we have

$$\frac{\hbar^2 A}{M} \left( -T_r + \sum_{\kappa} V_{0,\kappa}(r) + \frac{4}{r^2} T_z \right) + E \right] P^S_\beta(z,r) \tag{3.80}$$

$$= \left[ G^+(z,r) - V_{0,\beta}(r) \right] \Pi^S_\beta(z,r) + G^-(z,r) \Pi^S_\beta(z,r),$$

and the IDEA equations for $\gamma$ channels are

$$\frac{\hbar^2 A}{M} \left( -T_r + \sum_{\kappa} V_{0,\kappa}(r) + \frac{4}{r^2} T_z \right) + E \right] P^S_\gamma(z,r) \tag{3.81}$$

$$= \left[ G^+(z,r) - V_{0,\gamma}(r) \right] \Pi^S_\gamma(z,r) + G^-(z,r) \Pi^S_\gamma(z,r),$$

A System with aab Particles

We are now interested in the channels $\alpha = (12)$, $\beta = (31)$ and $\gamma = (23)$ where the latter two channels produce similar projections. From the systems of equations in (3.66), (3.67),
and (3.68), and again by considering only singlet and triplet even states, we have

\[
(-H_0 + E)\psi^S_{\alpha} = G^+_\alpha \left( \psi^S_{\alpha} - O^{S'}_{\alpha\beta} \right) + G^-_\alpha \left( \psi^S_{\alpha} + 2O^S_{\alpha\beta} \right),
\]

\[
(-H_0 + E)\psi^{S'}_{\alpha} = G^+_\alpha \left( \psi^{S'}_{\alpha} - O^{S'}_{\alpha\beta} \right) + G^-_\alpha \left( \psi^{S'}_{\alpha} + 2O^{S'}_{\alpha\beta} \right),
\]

\[
(-H_0 + E)\psi^S_{\beta} = G^+_\beta \left( \psi^S_{\beta} + u^- O^S_{\beta\gamma} + u^+ O^{S'}_{\beta\alpha} \right) + G^-_\beta \left( \psi^S_{\beta} + O^S_{\beta\gamma} + O^{S'}_{\beta\alpha} \right),
\]

\[
(-H_0 + E)\psi^{S'}_{\beta} = G^+_\alpha \left( \psi^{S'}_{\beta} + u^- O^{S'}_{\beta\gamma} + u^+ O^S_{\beta\alpha} \right) + G^-_\alpha \left( \psi^{S'}_{\beta} + O^{S'}_{\beta\gamma} + O^S_{\beta\alpha} \right),
\]

\[
(-H_0 + E)\psi^S_{\gamma} = G^+_\gamma \left( \psi^S_{\gamma} + u^- O^S_{\gamma\alpha} + u^+ O^{S'}_{\gamma\beta} \right) + G^-_\gamma \left( \psi^S_{\gamma} + O^S_{\gamma\alpha} + O^{S'}_{\gamma\beta} \right),
\]

\[
(-H_0 + E)\psi^{S'}_{\gamma} = G^+_\gamma \left( \psi^{S'}_{\gamma} + u^- O^{S'}_{\gamma\alpha} + u^+ O^S_{\gamma\beta} \right) + G^-_\gamma \left( \psi^{S'}_{\gamma} + O^{S'}_{\gamma\alpha} + O^S_{\gamma\beta} \right).
\]

Above, we assumed that for $\psi^S_{\beta}$, $n = S$ or $S'$. 
Chapter 4

PROPERTIES OF LAMBDA HYPERNUCLEI

4.1 Numerical Techniques

Among the very successful techniques that are used for the study of few-body nuclear systems are the Faddeev integral equation methods [128, 129] and Schrödinger variational methods [119, 130], these families of methods have yielded excellent results for bound state systems. The method of separability of the Schrödinger equation or adiabatic approximation was first introduced by Macek [131]. In mathematical terms it is apparent that the Schrödinger equation and its boundary conditions are non-separable equations in the \( r = \sqrt{r_1^2 + r_2^2} \) and \( \alpha = \tan(r_2/r_1) \) coordinates. Nonetheless, the separable condition is proposed in the sense that the total wavefunction \( \psi \) can be approximately written in terms of the five hyperspherical coordinates \( \Omega \) and \( r \) as \( \psi \sim u(r)\phi(r, \Omega) \); where the \( r \) in \( \phi \) is considered as a parameter. In previous calculations [134] the Extreme Adiabatic Approximation (EAA) has been used to find nuclear properties like ground-state and continuum state energies. The EAA, together with the Uncoupled Adiabatic Approximation (UAA) method have been used to numerically solve for problems of bound systems for
various values of central two-body interactions and their accuracy has been compared with the corresponding solution of coupled differential equations [132, 133].

4.2 The Adiabatic Approximation

It is possible to find a direct solution of the integrodifferential equation in (3.51) and extract properties of the system. However, such approaches are sensitive to the definition and properties of the grid points and often require more computer memory. The use of the extreme and uncoupled adiabatic approximations to solve the integrodifferential equations is less sensitive to the distribution and density of the grid points and it has been shown to generate accurate reliable solutions. In most nuclear systems a comparison of the radial and rotational energies shows that the radial energy of the ground state is of the order of half the monopolar excitation energy which is about 10 MeV, compared to 10 times or hundreds times more energy generated by rotation which can be measured from the Fermi gas model. The adiabatic approximation takes this behavior into account by assuming that the rotational and the radial (vibration) parts can be decoupled making the application of the adiabatic approximation method possible since this freezes the radial \( r \)-motion and at the same time solve the equation of rotational motion for each \( r \) to find an eigenpotential that is eventually in the radial equation [31, 135]. In molecular systems, the application of the adiabatic approximation assumes that the velocity of the electrons is large compared to the velocities of the nuclei in such a way that for each distance \( r \) between two nuclei the total energy of the electronic cloud determines the potential between two atoms. Thus the assumption is that the radial motion is very slow and it contains most of the energy [133, 135].

Finding a solution to the integrodifferential equation in (3.51) is a challenging task. We will approach the attempt of finding a solution by simplifying this equation by means of the adiabatic approximation method. This method reduces the equation into an integrodifferential equation in \( z \) and a differential equation in \( r \), where the rotational motion in
the $D$-dimensional space is weakly coupled to the hyperradial vibrational motion. Thus the adiabatic approximation consists of factoring out the wave function onto the two-dimensional space $r \oplus z$ into an $r$-dependent part and a $z$-dependent part where the $r$ appears as a parameter. This is done by writing \[8, 36, 33, 31, 37\] the amplitude in product form as

$$P^{(e)}(z, r) = P^{(e)}_{\lambda}(z, r)u_{\lambda}(r),$$  \hfill (4.1)

where we assume that $P^{(e)}_{\lambda}(z, r)$ varies slowly or weakly with the $r$ motion and $u_{\lambda}(r)$ is a radial function. The $P^{(e)}_{\lambda}(z, r)$ is an eigensolution of the one-variable integrodifferential equation obtained by substituting in (3.51) $P^{(e)}_{\lambda}(z, r)$ and $U^{(e)}_{\lambda}(r)$ in place of $P^{(e)}(z, r)$ and $E$, respectively; that is, it is a solution of the equation given by

$$\left[ \frac{4}{r^2} \hbar^2 m \left\{ \frac{1}{W_{[Lm]}(z)} \frac{\partial}{\partial z} (1 - z^2) W_{[Lm]}(z) \frac{\partial}{\partial z} \right\} + U^{(e)}_{\lambda}(r) \right] P^{(e)}_{\lambda}(z, r) = \left[ V(r\sqrt{(1 + z)/2}) - V_{[Lm]}(r) \right] \left[ P^{(e)}_{\lambda}(z, r) + \int_{-1}^{1} f_{[Lm]}(z, z') P^{(e)}_{\lambda}(z', r) \, dz' \right]$$  \hfill (4.2)

where the derivatives of $P^{(e)}_{\lambda}(z, r)$ with respect to $r$ given by

$$\frac{\hbar^2}{m} \left[ - \frac{d^2}{dr^2} + \frac{L(L + 1)}{r^2} \right] P^{(e)}_{\lambda}(z, r) + \frac{A(A - 1)}{2} V_0(r) P^{(e)}_{\lambda}(z, r)$$  \hfill (4.3)

have been omitted to obtain the radial equation given by

$$\left\{ \frac{\hbar^2}{m} \left[ - \frac{\partial^2}{\partial r^2} + \frac{L(L + 1)}{r^2} \right] + \frac{1}{2} A(A - 1) V_0^{(e)}(r) + U^{(e)}_{\lambda}(r) - E_{\lambda, n} \right\} u_{\lambda, n}(r) = 0.$$  \hfill (4.4)

We solve equation (4.2) by freezing the value of $r$ to obtain the eigenpotential $U^{(e)}_{\lambda}(r)$ which we then plug in the radial equation (4.4) to obtain the total energy $E$ of the system and the radial function $u_{\lambda}(r)$. Therefore the $P^{(e)}_{\lambda}(z, r)$ is an eigenamplitude and the $U^{(e)}_{\lambda}(r)$, $\forall \lambda = 0, 1, 2, \ldots$ are eigenvalues associated with it. Generally speaking, each eigenvalue $U^{(e)}_{\lambda}(r)$ is associated with the amplitude $P^{(e)}_{\lambda}(z, r)$ which has a definite number of nodes\footnote{We can denote by $n$ the number of nodes, then $n \neq 0$.} in the interval $-1 < z < 1$. It is now possible to determine the wave function
by using this EAA, we obtain

$$\Psi_{EAA}(x) = \frac{u_\lambda(r)}{r^{3A/2-2}} \sum_{i<j\leq A} P^{(q)}_\lambda(2r^2/\rho^2 - 1, r). \quad (4.5)$$

The Extreme Adiabatic Approximation (EAA), in summary, corresponds to the equations in (4.2) and (4.4) where the wave function is determined by (4.5). Previously it has been shown that by using the hyperspherical harmonics expansion a lower-bound for the binding energy $E_{EAA}$ is obtained by this method which implies that the condition $U_\lambda^{(q)}(r) < U_{\lambda+1}^{(q)}(r)$ is fulfilled. Furthermore, this suggests that the ground state potential corresponds to the lowest eigenpotential $U_0^{(q)}(r)$. To determine an upper-bound of the potential we let $\Psi_{EAA}(x)$ be a variational solution and introduce the eigenfunctions

$$B_\lambda(r, \Omega_N) = \sum_{i<j\leq A} P^{(q)}_\lambda(2r^2/\rho^2 - 1, r) \quad (4.6)$$

which are normalised by

$$\int |B_\lambda(r, \Omega_N)|^2 d\Omega_N = \langle B_\lambda(r, \Omega) | B_\lambda(r, \Omega) \rangle = 1. \quad (4.7)$$

The solution

$$\Psi(x) = B_\lambda(r, \Omega)u_{\lambda,n}(r)/r^{3A/2-2} \quad (4.8)$$

is the adiabatic separation for the full wave functions since equation (4.1) is for a particular Faddeev component. The radial wave function $u_{\lambda,n}(r)$ is the solution of the uncoupled radial equation given by

$$\left\{ \frac{\hbar^2}{m} \left[ -\frac{d^2}{dr^2} + \frac{L_0(L_0 + 1)}{r^2} \right] + \frac{A(A-1)}{2} V_0(r) + U_\lambda^{(q)}(r) - \frac{\hbar^2}{m} \langle B_\lambda | \frac{d^2 B_\lambda}{dr^2} \rangle - E_{U_{EAA}} \right\} u^{(U_{EAA})}_{\lambda,n}(r) = 0 \quad (4.9)$$

where we used the orthogonal condition

$$\int B_\lambda(r, \Omega) \frac{dB_\lambda(r, \Omega)}{dr} d\Omega = \langle B_\lambda | \frac{dB_\lambda}{dr} \rangle = 0. \quad (4.10)$$
From the formulation above we can conclude that

$$\left\langle B_\lambda \left| \frac{d^2 B_\lambda}{dr^2} \right. \right\rangle = - \left\langle \frac{dB_\lambda}{dr} \left| \frac{dB_\lambda}{dr} \right. \right\rangle < 0.$$  \hspace{1cm} (4.11)

The substitution of the equation in (4.8) into the integrodifferential equation, followed by adiabatic separation gives the equation in (4.2) for the z-motion and the equation in (4.9) for the hyperradial (r) motion. Observe that equations (4.10) and (4.11) are direct consequences of the normalisation condition in equation (4.7) and have been used to obtain equation (4.9).

Moreover, notice that from equation (4.11) the effective potential for the uncoupled adiabatic approximation $U_\lambda(r) = \frac{\hbar^2}{2m} \left\langle B_\lambda \left| \frac{d^2 B_\lambda}{dr^2} \right. \right\rangle$ that appears in the radial equation (4.4) is always larger than $U_\lambda(r)$ and that for ground states (with $\lambda = 0$) the Extreme Adiabatic binding energy $E^{EAA}$ is always less than the binding energy obtained by the Uncoupled Adiabatic Approximation which is an upper-bound of the exact binding energy. This proves that in general we have the inequality

$$E^{EAA} < E^{\text{exact}} < E^{UAA},$$ \hspace{1cm} (4.12)

where an accurate estimate of the exact binding energy is provided by the interpolation formula given by

$$E^{\text{exact}} \simeq 0.2 \left( E^{AA} - E^{UAA} \right) + E^{UAA}. \hspace{1cm} (4.13)$$

We can apply the property that when a function $F(r_{kl}, r)$ is projected on the space given by $|r_{ij}\rangle$, we find that any function of $r_{ij}$ and $r$ fulfils the orthogonal condition of these functions. This is applicable in the normalization of $B_\lambda(r, \Omega)$ which we write as

$$\langle B_\lambda(r, \Omega) | B_\lambda(r, \Omega) \rangle = \frac{A(A-1)}{2} \int_{-1}^{1} dz P_\lambda^*(z, r)w_\alpha(z) \left[ P_\lambda(z, r) + \int_{-1}^{1} f_{[0]}(z, z')P_\lambda(z', r)dz' \right]. \hspace{1cm} (4.14)$$
The differentiation of (4.14) with respect to \( r \) yields
\[
\left\langle \frac{dB_\lambda}{dr} \bigg| \frac{dB_\lambda}{dr} \right\rangle = A(A - 1) \int_{-1}^{1} dz \frac{dP_\lambda(z, r)}{dr} w_\alpha(z) \left[ \frac{dP_\lambda(z, r)}{dr} + \int_{-1}^{1} f_{[0]}(z, z') \frac{dP_\lambda(z', r)}{dr} dz' \right].
\] (4.15)

### 4.2.1 Coupled Channels in the Adiabatic Approximation

Recalling the definitions in equation (3.42) and (3.50) and by introducing
\[
V_{11} = V_{22} = \frac{V^{1+} + V^{3+}}{2}, \quad V_{12} = V_{21} = \frac{V^{1+} - V^{3+}}{2},
\] (4.16)
taking \( \hbar = m = 1 \) and after the introduction of the hypercentral potential, the integrodifferential equations for the coupled channels have the explicit form
\[
(H_r + U_\lambda(r)) P_{0\lambda}^S(z, r) = \left( \frac{V^{1+}_{\Lambda\Lambda} + V^{3+}_{\Lambda\Lambda}}{2} - V_0(r) \right) \Pi_{0\lambda}^S(z, r) + \left( \frac{V^{1+}_{\Lambda\bar{\Lambda}} - V^{3+}_{\bar{\Lambda}\Lambda}}{2} \right) \Pi_{0\lambda}^S(z, r),
\] (4.17)
\[
(H_r + U_\lambda(r)) P_{0\lambda}^{S'}(z, r) = \left( \frac{V^{1+}_{\Lambda\bar{\Lambda}} - V^{3+}_{\bar{\Lambda}\Lambda}}{2} \right) \Pi_{0\lambda}^S(z, r) + \left( \frac{V^{1+}_{\Lambda\Lambda} + V^{3+}_{\Lambda\Lambda}}{2} - V_0(r) \right) \Pi_{0\lambda}^{S'}(z, r),
\] (4.18)
\[
(H_r + U_\lambda(r)) P_{0\lambda}^S(z, r) = \left( \frac{V^{1+}_{\Lambda\Lambda} + V^{3+}_{\Lambda\Lambda}}{2} - V_0(r) \right) \Pi_{0\lambda}^S(z, r) + \left( \frac{V^{1+}_{\bar{\Lambda}\bar{\Lambda}} - V^{3+}_{\bar{\Lambda}\bar{\Lambda}}}{2} \right) \Pi_{0\lambda}^{S'}(z, r),
\] (4.19)
\[
(H_r + U_\lambda(r)) P_{0\lambda}^{S'}(z, r) = \left( \frac{V^{1+}_{\bar{\Lambda}\bar{\Lambda}} - V^{3+}_{\bar{\Lambda}\bar{\Lambda}}}{2} \right) \Pi_{0\lambda}^S(z, r) + \left( \frac{V^{1+}_{\Lambda\Lambda} + V^{3+}_{\Lambda\Lambda}}{2} - V_0(r) \right) \Pi_{0\lambda}^{S'}(z, r).
\]
The binding energy is calculated from the radial equation

\[
\left( \frac{\hbar}{m} \left( \frac{d}{dr^2} - \frac{\mathcal{L}_0(\mathcal{L}_0 + 1)}{r^2} \right) - A(A - 1) V_0(r) - U_\lambda(r) + E \right) u_\lambda(r) = 0. \tag{4.20}
\]

Solutions to the equations in (4.17), (4.18), (4.19) and (4.20) are calculated numerically to determine the eigenpotential \( U_\lambda(r) \) and the binding energy \( E \) by using the extreme adiabatic approximation technique described above.

### 4.3 Results and Discussion

#### 4.3.1 Application to Hypernuclear Systems

A numerical study of the \(^5\Lambda\text{He}\) and \(^6\Lambda\Lambda\text{He}\) hypernuclear systems to obtain exact calculations was performed by [47] using numerous potentials. In this study, the respective hypernuclei will be considered as five- and six-body systems. With the recent experiments at KEK for the identification of the \(^6\Lambda\Lambda\text{He}\), calculations are performed here to attempt to reproduce the experimental \( \Lambda - \Lambda \) binding energy of \( B_{\Lambda\Lambda} = 7.25 \pm 0.19^{+0.18}_{-0.11} \) MeV as calculated in the \textit{Nagara} event [63]. Various \( NN \) potentials are used, namely the hard core S3 potential

<table>
<thead>
<tr>
<th>Potential</th>
<th>S3</th>
<th>Volkov V7</th>
<th>B1</th>
<th>MTV</th>
<th>MT (I+III)/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>E (MeV)</td>
<td>29.09</td>
<td>28.76</td>
<td>38.51</td>
<td>30.68</td>
<td>29.39</td>
</tr>
</tbody>
</table>
used by Afnan and Tang [138], the soft core, Volkov-V7 potential [139], the Malfiet-Tjon
(MTV) and Malfiet-Tjon (I+III)/2 spin averaged potentials [137]. The ΛΛ interaction
used is that of Filikhin and Gal [141]. Most calculations [100, 47, 109] in literature have
used the Dalitz [110] ΛΛ interaction with the above \(NN\) potentials. The \(ΛN\) potential
of Bassichis-Gal [136] has purely attractive terms and is composed of Gaussian terms.
The S3 and Volkov-V7 potentials have repulsive short-range and attractive long-range
Gaussian terms. The Malfiet-Tjon (MTV) together with the spin-averaged MT(I+III)/2
potentials are semi-realistic potentials composed of short-range repulsive and long-range
attractive Yukawa-type terms [47].

Calculations for hypernuclear systems were performed using the computational techniques
as outlined in chapter 4 using the EAA and UAA. We have applied the generalized
integrodifferential equation approach to the following systems:

- a five-body (4\(N\))\(Λ\) system of the \(^5_Λ\)He hypernucleus that requires two-channel inte-
grodifferential equations corresponding to two distinct pairs (\(ΝΛ\)) and (\(NN\))

- and a six-body (4\(N\))\(ΛΛ\) system of \(^6_ΛΛ\)He that requires three-channel integrodifferen-
tial equations that correspond to three distinct pairs of (\(ΝΛ\)), (\(ΛΛ\)) and (\(NN\)).

The results given in Table 4.3 are for the five-body \(^5_Λ\)He calculations. Table 4.3 gives the
\(Λ\)-dissociation energies of the ground state of \(^5_Λ\)He obtained by calculating the binding
energies of the five-baryon system (\(ΛΝΝΝΝ\)) for non-identical particles and subtracting
from them the corresponding \(^4\)He ground-state binding energies given in Table 4.1. The
ground-state energies in Table 4.1 have been calculated using the IDEA for different
Wigner-type \(NN\) potentials.

There are a number of features that can be observed from the results in Tables 4.3 and 4.2.
One of these features is that the results span a wide range of values that do not correspond
to experimental results. This could be caused by our choice of the ΛΛ potential of Filikhin
and Gal [141], but also not leaving out the possibility of numerical errors in the code and
Table 4.2: Calculations for the $^6_{\Lambda\Lambda}$He hypernucleus considered as a six-body system. The $V_{\Lambda\Lambda}$ potential used is that of Bassichis and Gal [136] and the $\Lambda\Lambda$ potential is that of Filikhin and Gal [141]. Energy values given in units of MeV.

<table>
<thead>
<tr>
<th>$NN$ potential</th>
<th>$E_{EAA}$</th>
<th>$E_{\text{exact}}$ [47]</th>
<th>$\sqrt{\langle r^2 \rangle}$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S3</td>
<td>49.5817</td>
<td>41.99</td>
<td>1.5205</td>
</tr>
<tr>
<td>MTV</td>
<td>52.5063</td>
<td>44.36</td>
<td>1.4971</td>
</tr>
<tr>
<td>MT(I+III)/2</td>
<td>52.5236</td>
<td>42.94</td>
<td>1.4970</td>
</tr>
<tr>
<td>Volkov-V7</td>
<td>54.4080</td>
<td>42.93</td>
<td>1.5136</td>
</tr>
</tbody>
</table>

Table 4.3: Calculations for the $^5_{\Lambda}$He hypernucleus considered as a five-body system. The $V_{\Lambda\Lambda}$ potential used is that of Bassichis and Gal [136]. Energy values given in units of MeV.

<table>
<thead>
<tr>
<th>$NN$ potential</th>
<th>$E_{EAA}$</th>
<th>$E_\alpha$</th>
<th>$E_{\text{diff}} = (E_{EAA} - E_\alpha)$</th>
<th>$\sqrt{\langle r^2 \rangle}$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S3</td>
<td>39.1912</td>
<td>29.09</td>
<td>10.1012</td>
<td>1.5031</td>
</tr>
<tr>
<td>MTV</td>
<td>41.3323</td>
<td>30.68</td>
<td>10.6523</td>
<td>1.4840</td>
</tr>
<tr>
<td>MT(I+III)/2</td>
<td>41.3498</td>
<td>29.39</td>
<td>11.9598</td>
<td>1.4839</td>
</tr>
<tr>
<td>Volkov-V7</td>
<td>41.1367</td>
<td>28.76</td>
<td>12.3767</td>
<td>1.5539</td>
</tr>
</tbody>
</table>

approximations in the IDEA equation. A second feature of the results is their dependence on the type of $NN$ potential.
We now focus our attention on the $^6\Lambda\Lambda$He hypernucleus which is considered here as a six-body $\Lambda\Lambda$–NNNN system and was studied previously as a three-body system by Filikin and Gal [141] with the aid of the Faddeev method. For these particular calculations we employ a Gaussian potential

$$V_{\Lambda\Lambda}(r) = \sum_{i} v_i e^{-r^2/\beta_i^2}$$

(4.21)

where the parameters for this potential are given in Table 4.4. It is satisfying to observe that using the integrodifferential equations yields results that are comparable to literature [36] and experiments [63].

The effective adiabatic potentials are described by

$$V_{\text{eff}}(r) = U_{\text{eaa}}(r) + \frac{\mathcal{L}(\mathcal{L} + 1)}{r^2}$$

(4.22)

for the S3, Volkov-V7 and MT NN interactions. Figures 4.1 and 4.2 give results of the plotted effective interaction for the spin-averaged MT(I+III)/2 and Volkov-V7 phenomenological potentials. Finding a solution of the equations in (3.51) using the extreme adiabatic approximation gives rise to a number of eigenpotentials $U_\lambda(r)$ very close to each other. This is the reason that the results, though not deviating from each other by a large value, depend on the eigenpotential $U_\lambda(r)$ used. This feature is what is shown by

<table>
<thead>
<tr>
<th>$i$</th>
<th>$v_i$ (MeV)</th>
<th>$\beta$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-21.49</td>
<td>1.342</td>
</tr>
<tr>
<td>2</td>
<td>-379.1×γ</td>
<td>0.777</td>
</tr>
<tr>
<td>3</td>
<td>9324.0</td>
<td>0.350</td>
</tr>
</tbody>
</table>

Table 4.4: Parameters of the $\Lambda\Lambda$ interaction used in the calculation for the $^6\Lambda\Lambda$He hypernucleus. The $\gamma$ parameter depends on the NN potential that is used.
Figure 4.1: Two eigenpotentials (denoted by $\lambda_1$ and $\lambda_2$) for the $^6\Lambda\Lambda$ He hypernucleus corresponding to the spin-averaged MT(I+III)/2 potential. The potentials converge to binding energies of about $-8.0$ MeV, the experimental value is $-7.6$ MeV [63].

plots of the effective potential $V_{\text{eff}}(r)$ in Figures 4.1 and 4.2. It of importance to note the asymptotic behavior of the Volkov-V7 and MT(I+III)/2 potentials in the adiabatic approximation. We do expect the potentials to converge to a certain value of the binding energy. A consequence of this is that we can ascertain that the extracted eigenpotential $U_{\text{exa}}(r)$ is correct.
Figure 4.2: Two eigenpotentials (denoted by $\lambda_1$ and $\lambda_2$) for the $^6_{\Lambda\Lambda}$He hypernucleus corresponding to the Volkov-V7 potential. The eigenpotentials converge, as expected, closer to the experimentally predicted binding energy value of $-7.6$ MeV.
Chapter 5

CONCLUSIONS

We have shown how the integrodifferential equation approach is used in theoretical nuclear physics to study hypernuclear bound-state systems. The approach is a huge simplification because for any number of particles the equations are reduced to depend on only two variables $z$ and $r$. The Faddeev decomposition of the Schrödinger wave function is used as a starting point with the assumption of two-body interactions. An interesting feature of the integrodifferential equation approach is that little effort is required even when the $A$-particle systems dramatically increases in number.

The IDEA formalism presented is based on the expansion of Faddeev amplitudes for non-identical particles in terms of Potential Harmonics. Expanding in terms of Potential Harmonics and projecting on a similar basis yields differential equations which are complicated to solve numerically. This difficulty is overcome by first projecting the Faddeev-type equations for amplitudes $\Psi(r_{ij}, r)$ with $\langle r_{ij} \rangle$ and then expanding, resulting in coupled differential equations that solve quantum mechanical systems of $A = N - 1$ non-identical particles. These equations become single two-variable integrodifferential equations with a Faddeev-like structure and can be treated with the adiabatic approximation or as two-variable systems. The former method can provide us with the dynamics of the systems or scattering states as required.
The term $\mathcal{L}(\mathcal{L}+1)/2$ that is present in the nuclear interaction of the particles depends on the size of the systems $A$. When $A$ increases, the centrifugal part $\mathcal{L}(\mathcal{L}+1)/2$ becomes large and expands outward while the potential remains constant and is restricted to smaller $r$. This is what is suggested by the adiabatic approximation, that is, that the main contribution in the effective interaction originates from the centrifugal potential.

In this work, the IDEA was applied to the $^5_3$He and $^6_{3\Lambda}$He hypernuclear systems. The EAA results for spin-dependent interactions show pleasing agreement with experimental data and other literature results, and in particular, the overall results for single and double-lambda hypernuclei presented here, by comparison, show excellent agreement with literature and experimental values. This stresses the proposition that this technique can be used as an alternative to competing methods like variational and hyperspherical harmonics methods.

As final remarks we point out that, when desired, the IDEA can also be solved by means of three-body methods developed for two-variable integrodifferential equations to obtain higher accuracy. One can also include tensor and spin-orbit components of the interaction.
Appendix A

Three-Body Nucleon Systems

Spin-isospin states for three nucleons may be constructed from the direct product of the \([21]\)-spin and \([21]\)-isospin states. Considering spin, when the three-body system is either in the triplet (symmetric \(S\)) state or in the singlet (antisymmetric \(A\)) state, we know that the particle pair \((ij)\) is coupled to the particle \(k\) to give a total spin of \(\sigma = 1/2\). The states are

\[
|\sigma\rangle^n \equiv |\sigma, [21]^n\rangle, \quad \text{where } n = A \text{ or } S, \quad (A.1)
\]

\[
|\tau\rangle^n \equiv |\tau, [21]^n\rangle, \quad \text{where } n = A \text{ or } S. \quad (A.2)
\]
We can then construct the following recoupling symmetry relations

\[ |A⟩ \equiv |A; στ; [111]⟩ \]
\[ = \frac{1}{\sqrt{2}} \left( |σ⟩^A |τ⟩^S - |σ⟩^S |τ⟩^A \right) \text{ is completely antisymmetric ,} \]  
\[ |A'_{ij}⟩ \equiv |A'_{ij}; στ; [21]^A⟩ \]
\[ = \frac{1}{\sqrt{2}} \left( |σ⟩^S |τ⟩^S + |σ⟩^A |τ⟩^A \right) \text{ is completely antisymmetric with respect to } ij, \]
\[ |S'_{ij}⟩ \equiv |S'_{ij}; στ; [21]^S⟩ \]
\[ = \frac{1}{\sqrt{2}} \left( |σ⟩^S |τ⟩^S - |σ⟩^A |τ⟩^A \right) \text{ is completely antisymmetric with respect to } ij, \] and
\[ |S⟩ \equiv |S; στ; [3]⟩ \]
\[ = \frac{1}{\sqrt{2}} \left( |σ⟩^S |τ⟩^S + |σ⟩^A |τ⟩^A \right) \text{ is completely symmetric.} \]

One can also construct four-body spin-isospin states for nucleon systems by following the method described above.

Now we can use the recoupling relations to write the states \( |A'_{ij}⟩ \) and \( |A'_{ij}⟩ \) in terms of \( (ij) \)

\[ |A'_{ij}⟩ = -\frac{1}{2} |A'_{ij}⟩ + \frac{\sqrt{3}}{2} |S'_{ij}⟩, \text{ and} \]
\[ |A'_{ij}⟩ = -\frac{1}{2} |A'_{ij}⟩ - \frac{\sqrt{3}}{2} |S'_{ij}⟩ \]

The expressions for the singlet and triplet projection operator functions is given by

\[ P^{1+} = \frac{1}{2} (1 - P^σ), \quad P^{3+} = \frac{1}{2} (1 + P^σ) \]

where \( P^σ_{12} \) is the spin-exchange operator given by

\[ P^σ_{12} = \frac{1}{2} [1 + σ_1, σ_2], \quad P^σ |σ_{12}⟩^S = |σ_{12}⟩^S, \quad P^σ |σ_{12}⟩^A = -|σ_{12}⟩^A. \]
we find that

$$\mathcal{P}^{1+} |\sigma_{12}\rangle^A = 1 |\sigma_{12}\rangle^A, \quad \mathcal{P}^{1+} |\sigma_{12}\rangle^S = 0$$  \hspace{1cm} (A.11)

$$\mathcal{P}^{3+} |\sigma_{12}\rangle^A = 0, \quad \mathcal{P}^{3+} |\sigma_{12}\rangle^S = 1 |\sigma_{12}\rangle^S.$$  \hspace{1cm} (A.12)
Bibliography


therein


