THE FORMATION OF THE CEREBROSPINAL FLUID

(A case study of the cerebrospinal fluid system)

by

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Dedication

This work is dedicated to God Almighty, for His mercy endureth forever. Also to my dear wife, **Faleye Adeseye Abimbola** and children (**Oluwatosin Paul, Esther Olajumoke**, and **Ayomide**), who throughout the period of this programme did not set their eyes on me. I, sincerely, thank them for their understanding, patience and love.

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Abstract

Key Words: *Navier Stokes, Blood Flow, Cerebrospinal Fluid, Permeability, Weak Solution.*

It was generally accepted that the rate of formation of cerebrospinal fluid (CSF) is independent of intraventricular pressure [**26**], until A. Sahar and a host of other scientists challenged this belief. A. Sahar substantiated his belief that the rate of (CSF) formation actually depends on intraventricular pressure, see A. Sahar, 1971 [**26**].

In this work we show that CSF formation depends on some other factors, including the intraventricular pressure. For the purpose of this study, we used the capillary blood flow model proposed by K.Boryczko et. al., [**5**] in which blood flow in the microvessels was modeled as a two-phase flow; the solid and the liquid volume phase.

CSF is formed from the blood plasma [**23**] which we assume to be in the liquid volume phase. CSF is a Newtonian fluid [**2**, **23**].

The principles and methods of "effective area" developed by N. Sauer and R. Maritz [**21**] for studying the penetration of fluid into permeable walls was used to investigate the filtrate momentum flux from the intracranial capillary wall through the pia mater and epithelial layer of the choroid plexus into the subarachnoid space. We coupled the dynamic boundary equation with the Navier-Stoke's constitutive equation for incompressible fluid, representing the fluid flow in the liquid volume phase in the capillary to arrive at our model.

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Chapter 1

Introduction

1.1 Background

For a long time, physiologists and medical scientists have been concerned with the study of the intracranial vault (ICV) dynamic systems. The study of the intracranial vault (ICV) came much later after people like Aristotle (384 - 322 BC), who first identified the role of blood vessels in transferring ‘animal heat’ from the heart to the periphery of the body. Galen (c.130 - 200 AD) was the first to observe the presence of blood in the arteries. Many others followed, who initiated the study of blood flow. The result of their various works was a forerunner to the study of intracranial (IC) dynamics. Ever since then, there have been much research work on pressure-volume [**12**, **15**, **18**], and pressure-cerebrospinal formation relationships of the cranial contents [**16**, **19**, **20**, **26**, **29**, **30**]. It was observed that the intracranial pressure (ICP) changes may have a serious impact in the course of various neurosurgical disorders such as caused by brain injury or brain tumors. The general focus was to understand the intracranial vault (ICV) better, so that the intracranial pressure (ICP) can be better managed.

In view of the above, we present a mathematical model in which all the parameters involved in CSF formation can be controlled. This may be a valuable tool for a better understanding of the cerebral vascular system. The previous mathematical modelling of the cerebral circulation focussed on the cranial arterial structure, which was considered either as an equating system for pressure and flow or as an anastomotic system which only function in pathological conditions. This belief was first proposed by Willis in 1664. The two hypotheses have given rise to several models by Rogers, 1947; Avmann and Bering, 1961; Murray 1964; Himwich et al, 1965; Clark et al, 1967, 1968; Himwich and Clark, 1971; Chao and Hwang, 1971.

Current researchers believe that the circle of Willis plays only a limited part of the cerebral dynamics. The pia mater vascular network, the intra cerebral arteries, the micro circulation (like cerebrospinal fluid) and the venous network, all play an important role in the regulatory process of the cerebral dynamics. Many papers have been published in recent times in line with this new belief, some of which are Anthony Marmarou et al, 1978; Eugeny I. Paltsev et al, 1982; H.A. Guess et all, 1984; Mokhtar Zagzoule and Jean-Pierre, 1986; Mauro Ursino, 1987, 1988; Z.M. Kadas et al, 1997; Andreas Jung, 2002; Linninger, A.A. et al, 2004; to mention but a few.

1.1.1 Whole blood

As presented by K. Rajagopal et al. [**25**, **31**], whole blood consists of gel-like 'cell' matter in an aqueous plasma solution. The cell matter (which makes up around 46% of the total blood volume) consist of red blood cells (RBCs) or erythrocytes, white blood cells (WBCs) or leukocytes, and platelets. The volume concentration of RBCs in whole blood is termed hematocrit. Plasma consists primarily in water (92% – 93%) in which various proteins are dissolved along with various ions (sodium (Na^+), potassium (K^+), calcium (Ca^{+2}), magnesium (Mg^{2+}), etc.). Plasma is a Newtonian liquid with a viscosity of approximately $1 \cdot 2$ cP. Erythrocytes are biconcave deformable discs with no nuclei. The RBC membrane comprises 3% by weight of the entire RBC and consist of proteins (spectrin) and lipids. The RBC cytoplasm is a solution of hemoglobin in water (32 g/100 ml). Evans and Hochmuth performed micropipette aspiration experiments which showed that RBCs display viscoelastic behaviour. They also claimed that the viscoelastic nature of the RBC is only due to the viscoelastic properties of the RBCs membrane. The leukocytes are further divided into granulocytes, monocytes and lymphocytes, and together form less than 1% of the blood volume of blood.

1.1.2 Blood as a Viscoelastic Fluid

Blood is a complex fluid with flow properties significantly affected by the arrangement, orientation and deformability of red blood cells (RBCs).

Viscoelasticity is a rheological parameter that describes the flow properties of complex fluids like blood. There are two components to the viscoelasticity; the viscosity and the elasticity. The viscosity is related to the energy dissipated during flow primarily due to slicking and deformation of red blood cells and red blood aggregates. The elasticity is related to the energy stored during flow due to orientation and deformation of red blood cells (RBCs).

Blood flow in the circulation is pulsatile. With each beat, the heart pumps energy into the blood. This energy is dissipated the stored. How the blood will dissipate and stored energy is related to both the viscosity and elasticity of the blood. Red blood cells (erythrocytes) plays a dominant role in blood viscoelasticity and its response to pulsatile flow.

Steady flow (in which we are interested) do not replicate the pulsatility in the circulation and is blind to the significant parameter of elasticity.

Blood is classified as a viscoelastic fluid with its rheological properties, viscosity and elasticity, dependend on the rate of flow or shear rate. The changes in viscosity and elasticity are as a result of changes in the arrangement, orientation and stretching of the red blood cells (RBCs).

1.1.3 Brief Description of Blood flow up to the Brain

The cardiovascular system is the portion of the circulatory system that includes the heart and blood vessels. The heart pumps blood to the body cells and organs of the integumentary, reproductive, musculoskeletal, digestive, respiratory and urinary systems that communicate with external environment. In performing this function, the heart acts as a pump that forces blood through the blood vessels.

The blood vessels are arranged as a closed system of ducts which transport the blood and allow exchange of gases, nutrients, and waste products between the blood and the body cells; see figure 1 below [**3**].

1.1.4 Paths of Blood Circulation

The blood vessels of the cardiovascular system can be divided into two major pathways. These are the pulmonary circuit and systemic circuit. The pulmonary circuit consists of those vessels that carry blood from the heart to the lungs and back to the heart. The systemic circuit is responsible for carrying the blood from the heart to all other parts of the body and back again to the heart [**3**].

Fig 1 The cardiovascular system

1.1.5 The Pulmonary Circuit

The blood enters the pulmonary circuit as it leaves the right ventricle through the pulmonary trunk. The blood in the arteries and arterioles of the pulmonary circuit has a relatively low concentration of oxygen and a relatively high concentration of carbon dioxide, gaseous exchange occurs in the lungs between the blood and inhaled air as the blood moves through the pulmonary capillaries. Because the right ventricle contracts with less force than the left ventricle, the arterial pressure in the pulmonary circuit is less than that in the systemic circuit. The pulmonary vascular system carries the same volume of blood as the systemic circulation. Pulmonary vascular resistance is lower. Thus the pulmonary pressure is lower. Consequently, the pulmonary capillary pressure is relatively low. Oxygenated blood is returned back into the left atrium of the heart, thereby completing the pulmonary circulation [**3**].

1.1.6 Systemic Circuit

The freshly oxygenated blood received by the left atrium is forced into the systemic circuit by the contraction of the left ventricle. The circuit includes the aorta and its branches that lead to all parts of the body tissues, as well as the companion system of veins that returns the blood to the right atrium, via the superior and inferior venae cavae [**3**].

1.1.7 Blood Flow in The Brain

The blood supply to the neck, head and the brain takes place via the left and right carotid arteries. (See figure 2.) Blood is supplied to these parts through the branches of the subclavian and common carotid arteries. The main divisions of the subclavian artery to these regions are the vertebral, thyrocervical and costocervical arteries. Within the cranial cavity, the left and right vertebral arteries unite to form a single basilar artery. This vessel passes along the ventral brain stem and gives rise to branches leading to the pons, midbrain and cerebellum.

Blood from these regions is drained by external and internal jugular veins which return deoxygenated blood to the heart, see figure 3. The fluid exchange in the intracranial vault (ICV) takes place in the cerebral capillaries, some of which produce the cerebrospinal fluid, see figure 4.

Thirty percent (30%) of the blood in our body flows through the brain to guarantee a sufficient supply with oxygen. The whole of the brain compartment is called the cranial vault. This vault can be divided into 4 compartments namely; brain parenchyma, arterial, cerebrospinal fluid ventricular and venous compartments. Figure 5 is a compartmental model of the cranial vault [**14**, **3**].

Fig 2 shows the subclavian and common carotid arteries that supply blood to upper parts of the body

Fig 3 shows the sinuses and the jugular veins that carry blood back to the heart

Fig 4 shows some of the cerebral arteries where fluid exchange takes place and some of these arteries produce the cerebrospinal fluid. The blood vessels in red are small cerebral arteries while the blue ones are the sinuses.

Fig 5 is the compartmental model of the cranial vaunt

Fig 6 is human red blood cells flowing in glass tubes of approximate diameters $4.5\,\mu$m (top) and $7\,\mu$m (bottom). The diameters of the tubes are comparable to that of the capillaries which is about $4\,\mu$m to $10\,\mu$m. This shows a single-file motion of red blood cells in the capillaries.

1.1.8 Blood Flow in the Capillaries

The capillaries are the terminal branches of the arteries and venous vascular trees, and the primary site of oxygen and other nutrients exchange with the tissues. They are blood vessels with very small diameter, of the order of micrometer. They can be as small as $4\,\mu$m. The RBCs whose unstressed diameter are about $8\,\mu$m must undergo large deformation in order to enter the smallest capillaries. In fact, the deformed RBCs almost entirely fill a capillary and typically move in a single file (see Figure 6) as given by T.W. Secomb, 2003 and S. Chinen et al., 1984. [**27**, **28**]

T.W. Secomb, 2003, noted that the analysis of blood flow in the microvessels present intricate problems combining fluid and solid mechanics.

Furthermore, K. Borycko . et. al., 2003, [**5**] also observed that blood dynamics in the microscale must be studied as a two-phase, nonhomogeneous fluid, consisting of a liquid plasma phase and the deformable RBCs phase. The RBCs flow stand for the phase volume with elastic properties while the rest of the blood (plasma) represent the colloidal suspension. The RBCs flow is referred to as the solid volume phase, while the plasma suspension is the liquid volume phase. Also see [**2**].

1.1.9 The Specialised Capillary System

Please note that in some capillaries, the exchange of nutrients and gasses does not take place. The fenestrated walls of the capillaries encourages the filtration of selected components of plasma across the capillary walls into the relevant tissues or systems, while the permeation of RBCs, nutrients and the remaining contents of the blood are hindered. We call this type of capillary system a specialised capillary.

The filtrate predominantly consists of water from the blood plasma. We discovered that this type of capillary system can only be found in

(i) sweat production system, in the dermis under the skin,

(ii) urine production system, in the kidney, and

(iii) the cerebrospinal fluid production system in the brain.

This work shall only be concerned with the production of cerebrospinal fluid.

1.1.10 Cerebrospinal Fluid Production and Circulation

Cerebrospinal fluid is a colorless fluid, low in cells and proteins, but generally similar to plasma in its ionic composition. It contains mainly Na^+ and k^+ ions. It is called the brain water. The fluid is produced by 'specialized' cerebral capillaries of the choroid plexus that selectively transfer certain substances from the plasma into the cerebrospinal fluid compartment by facilitated diffusion. The passage of water soluble substances, red blood cells and protein are hindered [**14**, **23**].

Figure 7 is the enlarged microscopic structure of the choroid plexus and the pathway of cerebrospinal fluid production. Most of the cerebrospinal fluid arises from the two lateral ventricles, from where it circulates slowly into the third and fourth ventricles and then fills the subarachnoid space.

Figure 8 shows the intracranial vault and its contents. The fourth ventricle where some CSF is formed is indicated by a plus sign.

About 0.35 milliliters of CSF is produced per minute and about 500 milliliters of CSF is produced daily. However, only about 140 milliliters is present around the nervous system at any time, because cerebrospinal fluid is continuously being reabsorbed through tiny, finger-like structures called arachnoid granulations that project from the subarachnoid space into the blood filled dural sinuses of the brain. The circulation of CSF is shown in Figure 10, [**14**].

Excessive CSF in the cranial vault results in a condition termed HYDROCEPHALUS. This may be caused by excessive production of CSF, blockage of the sinuses (the venous structures through which CSF is reabsorbed back into the blood system) or infection. This condition is common in infants. Also blockage of foramina (Magendi, Luschk, Monroe, aqenduct of sylcius).

Fig 7 is the microscopic structure of the region where CSF is produced. It is assumed that it is axisymmetrical.

Fig 8 The cranial vault and its content. The plus (+) sign indicates the region in the cranium where CSF is produced

Fig 9 shows the circulation and reabsorbtion of CSF back into the blood vessels. The white arrows indicate the circulation of fresh CSF while the black arrows indicate the CSF going into the sinuses.

Black arrows = production of CSF

White arrows = resorption of CSF

1.2 Intracranial Pressure

The space inside the cranium is occupied by the brain parenchyma and its coverings, blood vessels (arteries and veins), and Cerebrosphinal fluid. The sum of these four volumes is normally equal to zero, so that an increase in any one occurs at the expense of the others. Since the skull is rigid, the space available is finite, increase in intracranial mass causes an increase in intracranial pressure (ICP). As intracranial pressure rises, cerebral blood flow (CBF) falls, there is progressive depression of consciousness, increase in systemic arterial blood pressure leading to irregular breathing, with eventual deep coma [**18**].
Hence, good management of ICP is of great importance.

1.3 Objective of study

The cerebrospinal fluid completely surrounds the brain and spinal cord. In effect, the central nervous system float in the fluid that supports and protects them by absorbing mechanical forces that might otherwise jar and damage their delicate tissues.

Anthony Marmarou et al., [**18**], stated that the biological sequence of events leading to a relentless increase of intracranial pressure (ICP) and eventual neurological death are poorly understood. They further explained that if fluid is added to or withdrawn from the cerebrospinal fluid space, pressure will change transiently from its initial value, followed by a gradual return to the pre-disturbance level. In addition, they recognized that the magnitude of pressure change and overall stability of the system was somehow related to the IC elasticity and the persistent seepage of fluid into or out of the cerebrospinal fluid space. Therefore, rate of cerebrospinal fluid production and absorption is one of the major factors that cause changes in intracranial pressure (ICP).

It is in the light of the above that this research work was undertaken to study the production and filtration rate of the cerebrospinal fluid, so that a mathematical model, based on the fluid mechanics concept, can be formulated.

1.4 Aim of Study

Our aim is to formulate a mathematical model for the rates of formation of cerebrospinal fluid (CSF) in which all the parameters that are involved in the production are present. We shall also compare our result with previous clinical experimental results. The result of this research work may be of use to clinical practitioners.

1.5 Definition of Medical Terms Used

1. Aorta – the main artery of the body, from which all other systemic arteries derive.

2. Arachnoid membrane – the middle of the three membranes covering the brain and spinal cord, which has a fine, almost cobweb-like texture. Between it and the pia mater within lies the subarachnoid space,

containing cerebrospinal fluid (CSF) and some large vessels.

3. Arteriole – small branch of an artery leading into many smaller vessels called capillaries.

4. Artery – blood vessels carrying blood away from the heart.

5. Atrium – either of the two upper chambers of the heart.

6. Basilar artery – an artery in the base of the brain, formed by the union of the two vertebral arteries.

7. Capillary – an extremely narrow blood vessel, of approximately $5 - 20\,\mu$m in diameter.

8. Cardiovascular system – the heart together with the two networks of blood vessels (the systemic circulation and pulmonary circulation).

9. Carotid artery – either of the two main arteries in the neck whose branches supply the head and neck.

10. Cerebellum – the largest part of the hindbrain bulging back behind the pons and medulla oblongata and overhung by the occipital lobes of the cerebrum.

11. Cerebral tumor – an abnormal multiplication of brain cells.

12. Cerebrospinal fluid (CSF) – the clear watery fluid that surrounds the brain and spinal cord.

13. Choroid plexus – a rich network of blood vessels, derived from the pia mater, in each of the brain ventricles.

14. Circulatory system – network of blood vessels.

15. Cranium – the hard part of the skull enclosing the brain.

16. Cranial vault – within the skull.

17. Digestion – the process in which ingested food is broken down in the alimentary canal.

18. Duct – a tube-like structure or channel.

19. Epithelium – the tissue that covers the external surface of the body or organs.

20. Endothelium lines – the inside of all hollow organs, including blood vessels.

21. Filtration – the process of filtering liquid.

22. Granulations – small rounded outgrowths, made up of small blood vessels and connective tissue on the healing surface of a wound.

23. Hydrocephalus – an abnormal increase in the amount of cerebrospinal fluid (CSF) within the ventricles of the brain.

24. Hypertension – high blood pressure.

25. Integument – of a membrane or layer of tissue covering any organ of the body within the skull.

26. Intracranial pressure – pressure within the skull.

27. Jugular – veins draining blood from the head (and brain).

28. Jugular Vein – any of three veins that carry blood from the head and neck towards the heart.

29. Midbrain – the small portion of the brainstem excluding the pons and the medulla, that joins the hindbrain to the forebrain.

30. Morphology – study of form and structure.

31. Neurology – the study of the structure, function and diseases of the nervous system (including the brain, spinal cord, and all the peripheral nerves).

32. Parenchyma – the functional part of an organ, as opposed to the supporting tissue.

33. Physiology – the science of the functioning of living organisms and of their component parts.

34. Pia matter – the innermost of the three membranes surrounding the brain and spinal cord.

35. Plasma – the straw-colored fluid in which blood cells are suspended.

36. Pons – the part of the brainstem that links the medulla oblongata and the thalamus, bulging forwards in front of the cerebellum from which it separated by the fourth ventricle.

37. Pulmonary – relating, associated with or affecting the lungs.

38. Pulmonary Circuit – a system of blood vessels for transport of blood between the heart and lungs.

39. Respiratory system – the combination of organs and tissues associated with breathing.

40. Spinal cord – the portion of the central nervous system enclosed in the vertebral column, consisting of nerve cells and bundles of nerves connecting all parts of the body within the brain.

41. Subarachnoid space – the space between the arachnoid and pia meninges of the brain and spinal cord, connecting circulating cerebrospinal fluid (CSF) and large blood vessels.

42. Subclavian-artery – either of two arteries supplying blood to the neck and arms.

43. Systemic Circuit – the system of blood vessels that supplies all parts of the body except the lungs.

44. Tissue – a collection of cells specialized to perform a particular function.

45. Traumatic – a serious injury.

46. Trunk – the main part of a blood vessel, lymph vessel, or nerve, from which branches arise.

47. Tumor – any abnormal growth in or on a part of the body. A swelling may contain water and is not always a tumor.

48. Vein – a blood vessel conveying blood towards the heart.

49. Ventricle – either the two lower chambers of the heart, which have thick muscular walls. Ventricle is a cavity: eg in the heart or in the brain.

*Source: Concise Medical Dictionary (Third Edition), Oxford University Press, New York (1990) [**8**].*

Chapter 2

Cardiovascular and Intracranial Equations

The physiological description of the cardiovascular and intracranial vault that was given in Chapter One is hereby complemented by their corresponding mathematical equations for better morphological understanding.

2.1 Cardiovascular Equations

a. A geometrical argument

The relationship between linear mean velocity $\overline{v}$ and the blood flow in one unit of time, (the flux Q), is determined by the cross sectional area A:

$$Q = \overline{v}A$$

b. Bernoulli's equation

The Bernoulli's equation states that the total driving energy E_{total} , applied to a continuously flowing, small, ideal fluid volume dV, which is flowing frictionless and laminar, equals the sum of 3 types of energy the kinetic energy, and the laterally directed energy (ie, the lateral pressure, P, directed towards the walls).

$$E_{total} = dV \left(\frac{1}{2} \varphi v^2 + h\varphi G + P \right).$$

c. Poiseuilles law

The volume rate V^o is equal to the driving pressure. In the left ventricle, the blood flow is described by the cardiac output Q, so that the equation reads:

$$Q = \frac{\Delta P}{TPVR}.$$

Where the driving pressure ΔP is the mean arterial pressure. MAP minus the atrial pressure, and TPVR is the total peripheral vascular resistance. TPVR is directly related to the blood viscosity μ and to the length L of the vascular system, and inversely related to its radius in the 4th power:

$$TPVR = \frac{8\mu L}{\pi r^4}.$$

Doubling the length of the system only doubles the resistance, whereas doubling the radius increases $TPVR$ sixteen-fold.

d. Vascular Resistance in parallel organs.

In the systemic or peripheral circulation the resistance in single organs is mainly placed in parallel, and the resistance of all organs R_1 to R_n are related to the TPVR by the following relation:

$$\frac{1}{TPVR} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}.$$

e. Vascular Resistance in Portal Circulations

There are only a few serially connected elements (portal circulation): Spleen/ liver, gut/liver, pancreas/liver and hypothalamus/pituitary. For serial arranged resistance, the formula is:

$$R_{total} = R_1 + R_2 + ... + R_n.$$

f. The Law of Laplace

Since blood vessels are distenisible, an increase in pressure causes an increase in diameter.

Laplace law:

The larger the vessel radius, the larger the wall tension (**T**) required to withstand a given internal fluid pressure.

From this we deduced that, if $(r_1,\ r_2)$ are the two principal radii of the curvature of the wall, the tension (**T**) is given in terms of the pressure P, and radius (r), as follows:

$$T = P\left[\frac{r_1 r_2}{r_1 + r_2}\right]$$

and also the pressure is given by

$$P = T\left[\frac{1}{r_1} + \frac{1}{r_2}\right]$$

in the case of a sphere of radius r, it can be shown that the equation becomes

$$P = \frac{2\mathbf{T}}{r}.$$

In a cylinder such as a blood vessel, one of the two radii can be considered as being infinite and the effect on P and **T** would thus be defined entirely by the finite radius at right angles to the infinite one so that

$$P = \frac{\mathbf{T}}{r}.$$

(Source: http://hyperphysics-astrogsu.edu/Hbase/ptens.html#lap).

g. The Starling Equation

In 1896, Starling determined the transvascular fluid flow J_f by the combined effect of the Starling forces, shown in the following equation:

$$J_f = Cap_f[(P_c - P_t) - \sigma(\pi_c - \pi_t)]$$

where Cap_f is the capillary filtration coefficient (ml of fluid per min per kPa in 100g of tissue) and the Starling forces are the pressure differences in brackets. P_c is the capillary hydrostatic pressure, P_t is the tissue hydrostatic pressure (assumed to be zero), Π_c is the capillary colloid osmotic pressure (3.6kPa or 27mmHg), Π_t is the tissue colloid osmotic (0.5 kPa), and σ is the capillary protein reflection coefficient.

2.2 A Shear-Thinning Viscoelastic Constitutive Equation for describing the Blood Flow

A shear-thinning viscoelastic model with a deformation dependent relaxation time has been developed for describing blood flow [**4**, **25**]. This model (according to [**25**], page 25-27) is defined through the following set of equations:

Let $\kappa_R(B)$ and $\kappa_t(B)$ denote the reference and the current configuration of the body B at time t, respectively. We let $k_{p(t)}(B)$ denote the stress-free configuration that is reached by instantaniously unloading the body which is at the configuration $k_t(B)$. As the body continues to deform these natural configurations, $k_{p(t)}(B)$ can change, (the suffix $p(t)$ is used in order to highlight that it is the preferred stress-free state corresponding to the deformed configuration at time t).

By the motion of a body we mean a one-to-one mapping that assigns to each point $\mathbf{X} \in k_R(B)$, a point $x \in k_t(B)$, for each t, i.e.

$$x = \chi_{k_R}(\mathbf{X}_{k_R}, t),$$

where χ_{k_R} is the motion of the fluid at time t assigns to each position in a reference configuration.

We assume that the motion is sufficiently smooth and invertible. We suppress B in the notation $k_R(B)$, etc., for the sake of convenience.

The deformation gradients $\mathbf{F}_{k_R}$, and the left and right Cauchy-Green stretch tensors, $\mathbf{B}_{k_R}$ and $\mathbf{C}_{k_R}$, are defined through:

$$\mathbf{F}_{k_R} = \frac{\partial \chi_{k_R}}{\partial \mathbf{X}_{k_R}}, \quad \mathbf{B}_{k_R} = \mathbf{F}_{k_R}\mathbf{F}_{k_R}^T \text{ and } \mathbf{C}_{k_R} = \mathbf{F}_{k_R}^T\mathbf{F}_{k_R}.$$

The left Cauchy-Green stretch tensor associated with the instantaneous elastic response from the natural configuration $k_{p(t)}$ is defined in like fashion:

$$\mathbf{B}_{k_{p(t)}} = \mathbf{F}_{k_{p(t)}}\mathbf{F}_{k_{p(t)}}^T.$$

The shear-thinning viscoelastic stress tensor is given by

$$T = -p\mathbf{I} + \mathbf{S}$$

where

$$\mathbf{S} = \mu^b \mathbf{B}_{k_{p(t)}} + \eta_1^b \mathbf{D}.$$

The upper convected Oldroyd derivative of $\mathbf{B}_{k_{p(t)}}$, $\overset{\triangledown}{\mathbf{B}}_{k_{p(t)}}$, is given by

$$\begin{aligned}
\overset{\triangledown}{\mathbf{B}}_{k_{p(t)}} &= -2\left(\frac{\mu^b}{\alpha^b}\right)^{1+2n^b}\left(tr(\mathbf{B}_{k_{p(t)}}) - 3\lambda\right)^{n^b}\left[\mathbf{B}_{k_{p(t)}} - \lambda I\right] \\
\lambda &= \frac{3}{tr(B^{-1}_{k_{p(t)}})} \\
n^b &= \frac{\gamma^b - 1}{1 - 2\gamma^b} \; : \; n^b > 0.
\end{aligned}$$

The above model is a generalization of the classical Oldroyd-B model. Upon rearranging the terms in the above expression, the model can be expressed through:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}$$

$$\mathbf{S} + \frac{1}{\chi(\mathbf{S},1)}\overset{\triangledown}{\mathbf{S}} = \eta_1\left[\mathbf{D} + \frac{1}{\chi(S,\mathbf{D})}\overset{\triangledown}{\mathbf{D}}\right] + \frac{3}{tr(\mathbf{S} - \eta_1\mathbf{D})^{-1}} I. \tag{2.1}$$

Where

$$\mathbf{S} = \mu^b \mathbf{B}_{k_{p(t)}} + \eta_1^b \mathbf{D}. \tag{2.2}$$

This implies that

$$\begin{aligned}
\mathbf{B}_{k_{p(t)}} &= \frac{1}{\mu^b}\mathbf{S} - \frac{\eta_1^b}{\mu^b}\mathbf{D} \\
\chi(\mathbf{S},\mathbf{D}) &= \chi^*(\mathbf{B}_{k_{p(t)}}) = 2\left(\frac{\mu^b}{\alpha^b}\right)^{1+2n^b}\left(tr(\mathbf{B}_{k_{p(t)}}) - 3\lambda\right)^{n^b} \\
\lambda &= \frac{3}{tr(\mathbf{B}^{-1}_{k_{p(t)}})},
\end{aligned}$$

where $\mathbf{S}$ is the extral tensor, μ^b is the elastic modulus, η_1^b is viscosity and $\mathbf{D}$ is the deformation tensor.

The above model describes the flow of blood in the macroscale vessels. In the microscale, the flow is reduced to the capillary blood flow described in Chapter 1, Section 1.1.8, where two-phase flow is proposed for blood flow in the capillary. The solid phase flow is characterised by purely elastic behaviour induced when a RBC squeezes itself through the capillary tube and the liquid volume phase is dominated by incompressible Newtonian fluid.

2.2.1 The Solid Volume Phase

In the microvessels blood circulation, RBCs undergo large deformation and typically entirely fill the capillary. This situation forces the RBCs in the microvessels to flow in files and fill the entire capillary. This volume phase, therefore, exhibit purely elastic behaviour. The limiting case of the constitutive equation given in Section 2.2 as given by Rajagopal et al., 2000, [**24**] was applied to obtain a constitutive equation for blood flow with purely elastic behaviour as follows:

It is assumed that η tends to infinity while μ is finite, with all the other Kinematical quantities remaining finite. Thus, the deviatoric part of $\mathbf{T}$ is finite and $\mathbf{D}_{k_{p(t)}} \rightarrow 0$, and $\mathbf{B}_{k_{p(t)}} \rightarrow \mathbf{B}_{k_R}$. The stress tensor was given as

$$\mathbf{T} = \mu \mathbf{B}_{k_R} - p\mathbf{I}$$

It is assumed that the dissipation (ζ) vanishes and $\Delta_{k_{p(t)}} \rightarrow 0$, while $\mathbf{T}$

remains finite. The stress power was given as

$$\mathbf{T} \cdot \mathbf{D} = \dot{\mathbf{W}} \text{ where } \mathbf{W} = \widehat{\mathbf{W}}(\mathbf{B}_{k_R}).$$

The above equation also describes a neo-Hookean solid flow.

2.2.2 The Liquid Volume Phase

It is generally assumed that plasma has Newtonian rheology. We shall consider a 3D, Newtonian, incompressible flow satisfying the Navier-Stokes equation for this phase volume flow. Thus, the constitutive equation shall be

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}(\mathbf{v}).$$

The forces among the particles are also modeled, which we believe also play a major role in this phase volume.

K. Boryczko et al., 2003, [**5**] used the fluid-particle model (FPM) to model the flow in terms of the forces among the particles. The interaction between the particles are represented by the collision operator

$$\psi_i = \sum_{\text{Type}} \mathbf{F}_{\text{Type}}\left(r_{ij}, v_{ij}, w_{ij}\right) \cdot \left[\omega(r_{ij}) - \omega(r_{ij} - \mathbf{R}_{\text{out}})\right]$$

for two particles i and j interacting with each other by a collision operator ψ_{ij}, these particles have several attributes such as mass m_i, moment of inertial I_j, position r_{ij}, both translational v_i and angular w_j velocities and different types of forces. The collision operator is defined as the sum of vector forces $\mathbf{f}^C, \mathbf{f}^T, \mathbf{f}^R$, where $\mathbf{f}^C$ is the conservative force, $\mathbf{f}^T$ is the dissipative, noncentral, translational component force, $\mathbf{f}^R$ is the dissipative, noncentral, rotational component force and **f** is the Brownian force. Therefore, the collision operator can be expressed as

$$\begin{aligned}\psi_{ij} &= \{-p_i \cdot \mathbf{v}'(r_{ij}) \cdot e_{ij} - \lambda \cdot m(A(r_{ij})I + \mathbf{B}(r_{ij})e_{ij}) \\ &\circ(v_i + (r_{ij} \times (\tilde{w}_i + \tilde{w}_j))e_{ij}) + (A'(r_{ij})d\tilde{W}_{ij}^S + B'(r_{ij})\frac{1}{D}tr[dW]I \\ &+C'(r_{ij})d\tilde{W}_{ij}^A) \circ e_{ij}\} \cdot [\omega(r_{ij}) - \omega(r_{ij} - R_{\text{cut}})]\end{aligned}$$

where

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$$

is a vector pointing from particle i to particle j and

$$\mathbf{e}_{ij} = \frac{\mathbf{r}_i}{\mathbf{r}_j}$$

D is the model dimension, dt is the time step, $\mathbf{r}$ is a scaling factor for dissipation forces dW^A, dW^B, $tr[dW]I$ are respectively the symmetric, anti-symmetric and trace diagonal of the random matrices of the independent Wiener integral increment and $A(r), B(r), C(r), A'(r), B'(r), V'(r)$ are normalised weight functions dependant on the separation distance r_{ij}.

2.3 Intracranial Volume Equation

The intracranial vault (ICV) volume is governed by Moronkelli's hypothesis which states that:
if the intracranial vault (ICV) is divided into four compartments namely; artery, venous, tissue and cerebrospinal fluid (CSF) compartments. The constancy of the resulting total intracranial volume can be expressed by a differential equation given below:

V_a is the volume of the arterial compartment,

V_v is the volume of the venous compartment,

V_{tiss} is the tissue volume of the intracranial vault, (ICV) tissue

and

V_{csf} is the volume of the CSF compartment.

This means that:

$$\frac{dV_a}{dt} + \frac{dV_v}{dt} + \frac{dV_{tiss}}{dt} + \frac{dV_{CSF}}{dt} = 0 \tag{2.3}$$

If each compartment is considered separately, we then find the value of each compartment volume in terms of their compliance.

The arterial compartmental volume is given by:

$$\frac{dV_a}{dt} = C_{ai}\frac{d(P_a - P_{ic})}{dt}$$

$$C_{ai} = \frac{1}{K_a(P_a - P_{ic})}.$$

Therefore

$$\frac{dV_a}{dt} = \frac{1}{K_a}\frac{d(P_a - P_{ic})}{(P_a - P_{ic)dt}}$$

$$dV_a = \frac{1}{K_a}\frac{d(P_a - P_{ic})}{(P_a - P_{ic)}}$$

and

$$V_a = \frac{1}{K_a}\ln(P_a - P_{ic}) + V_{ai}.$$

For the venous compartment:

$$\frac{dV_v}{dt} = C_{vi}\frac{d(P_a - P_{ic})}{dt}$$

$$C_{vi} = \frac{1}{K_v(P_v - P_{ic} - P_{v1})}$$

$$\frac{dV_v}{dt} = \frac{1}{k_v(P_v - P_{ic} - P_{v1})}\frac{d(P_v - P_{ic})}{dt}$$

$$dV_v = \frac{d(P_v - P_{ic})}{K_v(P_v - P_{ic} - P_{v1})}$$

and

$$V_v = \frac{1}{K_v}\ln(P_v - P_{ic} - P_{v1}) + V_{vi}.$$

For the tissue compartment:

$$\frac{dV_{tiss}}{dt} = -C_{tiss}\frac{dP_{ic}}{dt}$$

$$C_{tiss} = \frac{1}{K_E[P_{ic} + (\frac{P_{ic}}{P_{o1}})^2]dt}.$$

Therefore,

$$\frac{dV_{tiss}}{dt} = -\frac{d(P_{ic})}{K_E[P_{ic} + (\frac{P_{ic}}{P_{o1}})^2]dt}$$

$$dV_{tiss} = -\frac{(P_{o1})^2}{K_E}[\frac{d(P_{ic})}{P_{ic}(P_{o1})^2 + (P_{ic})^2}]$$

and

$$V_{tiss} = -\frac{(P_{o1})^2}{K_E}\frac{\ln((P_{ic})^2 + (P_{o1})^2P_{ic})}{2P_{ic} + (P_{o1})^2} + V_{tiss}.$$

For the cerebrospinal compartment: Therefore, from equation (2.3)

$$V_{CSF} = -(V_a - V_{ai}) - (V_v - V_{vi}) - (V_{tiss} - V_{tiss}) + V_{CSFi}. \tag{2.4}$$

(See M. Ursino, [**29**].)

Chapter 3

Mathematical Formulation

3.1 Introduction

Blood flow in the capillaries was discussed in Section 1.1.8 as consisting of a two-flow-phase, as the deformed RBCs move in a single file along these microvessels. That is, the solid phase and the liquid phase. The flow of elastic deformable RBCs constitute the solid volume phase, while the flow of blood plasma and the rest of the blood composition form the liquid volume phase. In addition, we shall also recall from Section 1.1.10 where we stated that the CSF is formed from the blood plasma in such a way that the plasma filters through the specialised capillary walls and the RBCs remain in the capillaries. In view of the above, we shall assume that the formation of CSF takes place at the liquid volume phase of blood flow in the specialised capillary systems.

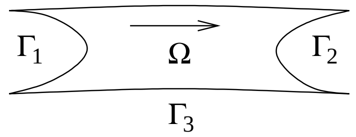
3.2 Method of Study

As stated above, we assumed that CSF is formed from the liquid volume phase of blood in the specialised capillaries and that the filtration of nutrients and the release of oxygen from the RBCs are inhibited in this type of capillaries [**14**, **23**]. Therefore, the theoretical effect of RBCs on the flow in this region is ignored in this study. The flow of CSF in the subarachnoid space is assumed to be very slow, to such an extent that we assume it is at rest in this study. Since filtration of CSF takes place mainly at the liquid volume phase, the portion of the capillary involved in our study shall be the space between two consecutive deformed RBCs in the capillary (diagram 3 refers). We shall apply the "freezed" method [**5**] to the segment to be studied. However, the fluid particles are frozen only along the flow direction and can move along the sagittal directions. We can now model the quasielastic nature of the capillary wall.The axial elasticity is less important and can be neglected. This method is considered to be physically correct because in the real blood vessels, the capillary wall consist of a layer of endothelial cells. These cells cannot move, but can deform. This method was also used by Boryczko K. et. al., [**5**] to study blood flow in the capillary.

3.3 Presentation of the Problem

The space, with its permeable wall, between two consecutive RBCs, where CSF is assumed to be formed, will be studied intensively. The interior of the specialised intracranial capillary, the space between two consecutive deformed

RBCs shall be denoted by Ω and the subarachnoid space (SAS), where the CSF is assumed to be at rest after formation, is denoted by Ω_o. The arterial end of Ω shall be denoted by Γ_1 while its venous end shall be denoted by Γ_2.



Γ_1 and Γ_2 are assumed to be artificial boundaries. Let $\mathbf{v}(x)_{\text{in}}$ be the fluid velocity in through Γ_1 at $x \in \Gamma_1$ and $\mathbf{v}(y)_{\text{out}}$ be the fluid velocity out through Γ_2 at $y \in \Gamma_2$. The permeable wall of the capillary together with the passage (the pia mater and epithellial layer) is referred to as the permeable interface, and it is denoted by Γ_3. We assume that this boundary condition takes into account a "type" of periodic flow in the capillary in such a way that we can assume "$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{out}}$". We shall take an arbitrary patch from Γ_1 which is congruent to a patch in Γ_2 to validate this assumption. If Γ_1^* and Γ_2^* be these two patches respectively, then

$$\int_{\Gamma_3} \rho\gamma_o \mathbf{v}_{\text{in}} \cdot \mathbf{n}\, ds = \int_{\Gamma_2} \rho\gamma_o \mathbf{v}_{\text{out}} \cdot \mathbf{n}\, ds = 0.$$

The net effect of various pressures responsible for filtration in the capillary is denoted by p and in the subarachnoid space by p_o.

The diagrams below give a visual representation of the geometry of the whole process.

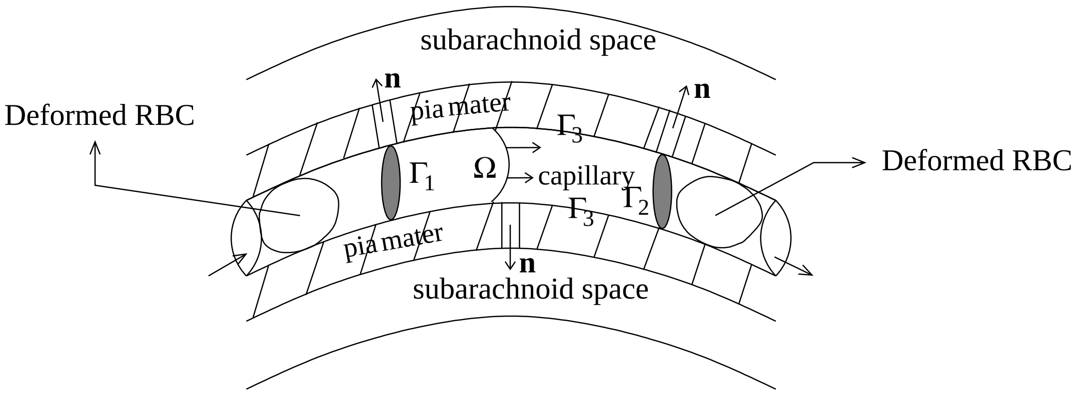


Diagram 1 A schematic diagram of CSF production

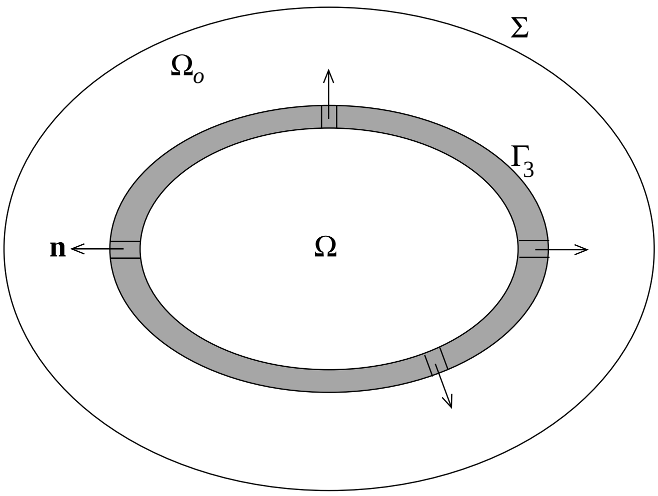


Diagram 2 Cross-sectional diagram of diagram 1

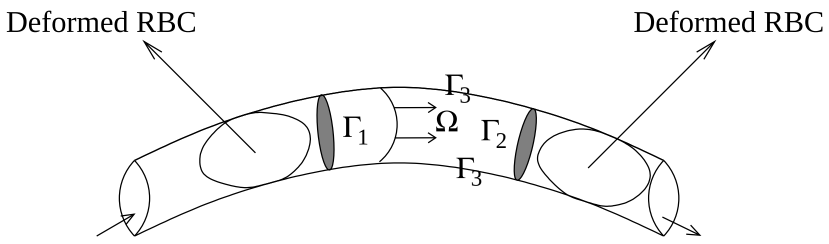


Diagram 3 Extract of the cerebral capillary from diagram 1.

3.4 Preliminaries

In the light of our discussion in Section 3.1, CSF is assumed to be formed only at the liquid volume phase of blood flow in the specialised capillaries. Specifically, it is formed from the blood plasma which is generally accepted to be a Newtonian fluid [**2**, **25**, **28**]. Hence, our constitutive equation for the flow of blood plasma inside Ω shall be the Navier-Stokes' constitutive equation for incompressible fluid.

Blood flow in the capillaries largely depend on the diameter of the capillaries. RBCs which are about 8 μm in diameter [**9**] undergo deformation to enter into capillaries of which the diameter can be as small as 4 μm [**2**]. It is important to note that RBCs will flow with less restriction in the capillaries with diameter of 8 μm and above. We shall limit the scope of this study to capillaries of diameter ranging from 4 μm to 8 μm. In which case, deformed RBCs will typically fill the capillaries and will be squeezed through the capillary tubes. The flow will be assumed to be greatly restricted and very slow. Our mathematical model for this flow shall be the linear Stokes' equation with the appropriate boundary conditions.

3.4.1 Assumptions

Movement of body fluid across membranes is a basic physiological requirement. Mathematical analysis of these movements would be highly complex without the use of simplifying models and assumptions about the micro-filtration. Therefore, the following assumptions are considered:

1. The cerebral capillary is modelled as a curved fixed tube.
2. The rate of flow across the wall of the cerebral capillary, per unit area of the wall surface, is small.
3. The flow of Cerebrospinal Fluid (CSF) in the Cerebrospinal Fluid (CSF) compartment is very slow, in such a way that we assume it to be constant, and the CSF compartment is always filled with fluid.
4. The (CSF) is a homogeneous Newtonian fluid that acts much like water.
5. The pathway of fluid across the capillary wall through the pia mater and epithelium into the subarachnoid space is distributed continuously through the walls of these specialized capillaries.
6. The motion of fluid is steady.
7. The intracranial pressure (the pressure of the cranium vault) is the same as the pressure of the CSF in the SAS.
8. At the wall of the tube, there is a no-slip condition of the fluid.
9. The flow through the permeable boundary is always in the direction of the outward normal, $\mathbf{n}$, to the boundary Γ_3.

10. The body forces which act on the fluid inside Ω and on the boundary Γ_3 will be modelled as the collision operator which is defined as the sum of vector forces $\mathbf{f}^C, \mathbf{f}^T, \mathbf{f}^R$, where $\mathbf{f}^C$ is the conservative force, $\mathbf{f}^T$ is the dissipative, noncentral, translational component force, $\mathbf{f}^R$ is the dissipative, noncentral, rotational component force. We assume that these forces are known entities and that each force will affect the force which acts on the boundary. We denote the total force inside Ω as a constant force at each $x \in \Omega$ as $\mathbf{f}_\Omega$ and the total force, associated with $\mathbf{f}_\Omega$ on the fluid inside the boundary as $\gamma_o(\mathbf{f}_\Omega) = f_{\Gamma_3}\mathbf{n}$. (This is discussed in section 2.2.2).

3.4.2 Notations, Definitions and Spaces

In this section we wish to explain the symbols and notations used in this work. In the second part of this section we focus on the structure or the environment to which the vector-valued function $\mathbf{v}$, its trace and derivatives belong.

The units for some quantities are in brackets.

Ω, Ω_o	:	bounded domain in $\mathbb{R}^3$
$x = (x_1, x_2, x_3)$	:	position in 3-dimensional space
$\mathbf{v}(x)$	:	the velocity field in the fluid (ms^{-1}) at $x \in \Omega$
ρ	:	blood volume density (kgm^3) in Ω
ρ_o	:	filtrate volume density in Ω_o

μ : coefficient of blood viscosity ($\mathrm{Nm^2 s}$) in Ω

μ_o : coefficient of filtrate viscosity ($\mathrm{Nm_s^2}$) in Ω_o

ds : Lebesgue measure of surface area on $\Gamma_3(\mathrm{m}^2)$

$da(x)$: the effective area measure on Γ_3

$\zeta(x)$: density function in terms of the area measure da,

i.e. $da = \zeta ds;\ 0 < \zeta(x) < 1$

$\delta(y)$: surface thickness at any point $y \in \Gamma_3(m)$

$\sigma(y)$: surface density of the fluid at any $y \in \Gamma_3(\mathrm{kg}^{-2})$,

i.e. $\sigma(y) = \delta(y)\zeta(y)\rho$

p : pressure (Nm^{-2}) of blood in Ω

p_o : pressure (Nm^{-2}) of CSF in Ω_o

$\eta_{\mathbf{v}}(y)\mathbf{n}$: the normal velocity component at $y \in \Gamma_3$

$[\nabla \mathbf{v}]_{i,j} = \partial_j \mathbf{v}_i$: the velocity gradient (s^{-1})

$\omega = \nabla \wedge \mathbf{v}$: the vorticity (s^{-1}) (wedge denotes vector product)

$\mathbf{f}_\Omega$: the body force acts on the fluid inside Ω

f_{Γ_3} : the body force acts on the fluid trough the boundary Γ_3.

Ω is a bounded domain in $\mathbb{R}^n,\ n = 2, 3$ with a smooth (at least C^2) boundary. Let $\mathbf{n} = \mathbf{n}(y)$ denote the unit exterior normal to Γ_3 at y. We shall be concerned with smooth vector fields $\mathbf{v} = \mathbf{v}(x)$ defined on Ω such that on Γ_3, it has the form

$$\gamma_o\, \mathbf{v}(y) = -\eta(y)\mathbf{n}(y),\ y \in \Gamma_3$$

where γ_o is the trace operator denoting boundary values and η is a smooth scalar field defined on Γ_3.

L^p and Sobolev spaces: $\mathbf{L}^p(\Omega)$ will denote the space of functions f, whose p^{th} power, $|\mathbf{f}|^p$, is integrable in Ω. The norm in $\mathbf{L}^p(\Omega)$ will be denoted by $||\cdot||_{\mathbf{L}^p(\Omega)}$. The scalar product and norm in $\mathbf{L}^2(\Omega)$ will be denoted with $(\cdot,\ \cdot)_{\mathbf{L}^2(\Omega)}$ and $||\cdot||_{\mathbf{L}^2(\Omega)}$ respectively (both for scalar and vector functions). For $s \in \mathbb{N}$, $\mathbf{H}^s(\Omega)$ will denote the Sobolev space of functions $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that all their (distributional) derivatives of order up to s are functions of $\mathbf{L}^2(\Omega)$.The norm and the inner product in $\mathbf{L}^2(\Omega)$ are defined as

$$(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega)} = \int_\Omega \mathbf{u}\mathbf{v}dx \text{ for } \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega)$$

and

$$\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} = \int_\Omega \mathbf{u}^2 dx \text{ for } \mathbf{u} \in \mathbf{L}^2(\Omega)$$

The norm in $\mathbf{H}^s(\Omega)$ will be denoted by $||\cdot||_s$.
$\mathbf{H}^m(\Omega)$ is the closure of $\mathbf{C}^m(\Omega)$ in $\mathbf{W}^m(\Omega)$ with respect to the norm of $\mathbf{W}^m(\Omega)$. $\mathbf{H}^m(\Omega)$ is a Hilbert space, $\mathbf{H}^m(\Omega) = \mathbf{W}^m(\Omega)$.

Trace spaces: If $\Gamma \subset \partial\Omega$ is open and non empty, then the trace space of $\mathbf{H}^s(\Omega)$, ($s \geq 1$), i.e. the space of functions defined on Γ which are traces of functions belonging to $\mathbf{H}^s(\Omega)$, is indicated by $H^{s-\frac{1}{2}}(\Gamma)$. The trace operator γ_o is such that

$$\gamma_o : \mathbf{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$$

is surjective and continuous and there exist an injective, linear, and continuous map $L : H^{s-\frac{1}{2}} \to \mathbf{H}^s(\Omega)$ called lifting, such that $\lambda = \gamma_o L\lambda$ for all $\lambda \in H^{s-\frac{1}{2}}(\Gamma)$. In particular, there exist a constant β such that the following trace inequality holds:

$$||\gamma_o\Phi||_{H^{\frac{1}{2}}(\Gamma_3)} \leq \beta||\Phi||_1 \ \ \forall\, \Phi \in \mathbf{H}^1(\Omega).$$

We assume $\mathbf{v} \in \mathbf{H}^1(\Omega)$, where $\mathbf{H}^1(\Omega)$ is a Sobolev space of vector valued functions.

We denote the scalar product and norm in $L^2(\Gamma)$ by $(\cdot,\cdot)_\Gamma$ and $||\cdot||_\Gamma$ respectively.

The norm and the inner product of the Trace spaces in $\mathbf{H}^m(\Gamma)$ will be indicated by $||\cdot||_{m,\Gamma}$ and $(\ ,\ )_{m,\Gamma}$, respectively.

Let $\mathbf{W}(\Omega)$ denote the subspace of test functions in $\mathbf{H}^1(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_1$. The weak solution will also be in $\mathbf{W}(\Omega)$. See [**1**].

Definition 3.1 *Let* $\mathbf{W}(\Omega)$ *be a space such that for* $\mathbf{v}(x) \in \mathbf{W}(\Omega)$, *then:*

(*i*) $\mathbf{v} \in \mathbf{H}^1(\Omega)$

(*ii*) $\nabla \cdot \mathbf{v} = 0$

(*iii*) $\gamma_o \mathbf{v} = -\eta_{\mathbf{v}} \mathbf{n}$ *on* Γ_3

(*iv*) $\gamma_o \mathbf{v} = 0$ *on* $\Gamma_1 \cup \Gamma_2$

(*v*) $\eta_{\mathbf{v}} \in H^1_o(\Gamma_3)$

The norm of $\gamma_o \mathbf{v} \in L^2(\Gamma_3)$ on the boundary Γ_3 is chosen as

$$\|\gamma_o \mathbf{v}\|^2_{\Gamma_3} = \|\eta\|^2_{\Gamma_3} = \int_{\Gamma_3} |\gamma_o \mathbf{v}|^2 ds.$$

The associated scalar product is

$$(\gamma_o \mathbf{u}, \gamma_o \mathbf{v})_{\Gamma_3} = \int_{\Gamma_3} \gamma_o \mathbf{u} \cdot \gamma_o \mathbf{v} ds.$$

We shall assume throughout that the mass density of the fluid in the interface $\sigma(x) = \zeta\rho$ is bounded and bounded away from zero: i.e. there exist constants s and S such that

$$0 < s \leq \sigma(x) \leq S \quad \text{for} \quad \text{all} \quad x \in \Gamma$$

and

$$\rho(\frac{1}{2} - \zeta) < S.$$

It is assumed that the function $\sigma \in C^\infty(\Gamma)$.

The constant C which appears in inequalities denotes a generic positive constant. This means that C may take different values even in the same calculation. Sometimes it is necessary to indicate the quantities on which a constant depends in brackets or by a subscript.

3.5 Equation of motion inside Ω

Let $\mathbf{v}$ and p denote the velocity and pressure fields at $x \in \Omega$. The constitutive equation of motion is giving by:

$$\rho D_t \mathbf{v} = \nabla \cdot \mathbf{T}(p, \mathbf{v}) + \mathbf{f}_\Omega \tag{3.1}$$

with the stress tensor $\mathbf{T}$

$$\mathbf{T}(p, \mathbf{v}) = -p\mathbf{I} + \mu\mathbf{A}(\mathbf{v}), \tag{3.2}$$

and the material derivative D_t is defined as

$$D_t := \partial_t + \mathbf{v} \cdot \nabla.$$

$\rho > 0$ denotes the density of the fluid, and $\mu > 0$ the viscosity, which are both assumed to be constant. $\mathbf{f}_\Omega$ is the the sum of the vector forces $\mathbf{f}^C$, $\mathbf{f}^T$, and $\mathbf{f}^R$ defined in Section 2.2.2.

$$\mathbf{D}(\mathbf{v}) = \frac{\mathbf{A}(\mathbf{v})}{2}$$

is the rate of the deformation tensor in the liquid phase and $\mathbf{A}(\mathbf{v})$ is defined as

$$\mathbf{A}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^T.$$

In the constitutive equation (3.1) we need to obtain the divergence of the stress tensor $\mathbf{T}$:

$$\begin{aligned}
\nabla \cdot \mathbf{T}(p, \mathbf{v}) &= \nabla \cdot (-p\mathbf{I} + \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T) \\
&= -\nabla \cdot (p\mathbf{I}) + \mu\nabla \cdot (\nabla\mathbf{v} + (\nabla\mathbf{v})^T) \\
&= -\nabla p + \mu(\Delta\mathbf{v} + \nabla(\nabla \cdot \mathbf{v})) \\
&= -\nabla p + \mu\Delta\mathbf{v} \qquad (3.3)
\end{aligned}$$

since $\nabla \cdot \mathbf{v} = 0$.

The assumption is a steady flow inside Ω, through the pia mater boundary layer, Γ_3. Our constitutive equation for describing blood flow in the capillary shall be given as follows:

Since we are considering capillaries with diameter less than 8 μm, we expect a slow, laminar and steady flow, i.e. creeping flow, hence the term

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = 0$$

and in view of (3.2) and (3.3), (3.1) becomes

$$-\mu\Delta\mathbf{v} = -\nabla p + \mathbf{f}_\Omega \text{ in } \Omega \qquad (3.4)$$

with the equation of continuity

$$\nabla \cdot \mathbf{v} = 0 \text{ in } \Omega.$$

3.6 At the Boundary

3.6.1 The Deformation Tensor at the Boundary

In this section we derive expressions at the boundary Γ_3 for the gradient of vector fields which satisfy $\gamma_o \mathbf{v} = -\eta_{\mathbf{v}} \mathbf{n}$ as well as other associated tensors. The unit outward normal to Γ_3 is given by $\mathbf{n}$ and γ_o is the trace operator used to restrict $\mathbf{v}$, the velocity field, to the boundary interface, Γ_3. Towards this goal, we assume there exist a local orthogonal system, in Γ_3, consisting of normal curves $y_1(s_1)$ and $y_2(s_2)$ where $y \in \Gamma_3$ and s_1, s_2 denote the arc length. Let $\tau_\kappa = y'_\kappa$ $(\kappa = 1, 2)$ denote the unit tangent vectors to the curve y_κ.

τ_1 and τ_2 are chosen such that $\tau_1 \wedge \tau_2 = \mathbf{n}$.

In such a coordinate system the surface gradient of a scalar field f on Γ_3 is defined as

$$\nabla_s f = \frac{\partial f}{\partial s_1}\tau_1 + \frac{\partial f}{\partial s_2}\tau_2. \tag{3.5}$$

For a function defined on Ω, the relationship between gradient and surface gradient is given by:

$$\gamma_o \nabla f = \nabla_s \gamma_o f + [\gamma_1 f]\mathbf{n}, \tag{3.6}$$

where $\gamma_1 f := \dfrac{\partial f}{\partial \mathbf{n}}$ denotes the normal derivative.

The surface gradient of vector field $\mathbf{f}$ on Γ_3 is defined as

$$\nabla_s \mathbf{f} = \frac{\partial \mathbf{f}}{\partial s_1} \otimes \tau_1 + \frac{\partial \mathbf{f}}{\partial s_2} \otimes \tau_2. \tag{3.7}$$

The relationship between gradient and surface gradient for the vector field $\mathbf{f}$ is given by:

$$\gamma_o \nabla \mathbf{f} = \nabla_s \gamma_o \mathbf{f} + [\gamma_1 \mathbf{f}] \otimes \mathbf{n}. \tag{3.8}$$

Similarly, for the vector field $\mathbf{f}$, we have the following definitions and relationships

$$\nabla_s \wedge \mathbf{f} \;:=\; \tau_1 \wedge \frac{\partial \mathbf{f}}{\partial s_1} + \tau_2 \wedge \frac{\partial \mathbf{f}}{\partial s_2} \tag{3.9}$$

$$\gamma_o \nabla \wedge \mathbf{f} \;=\; \nabla_s \wedge \gamma_o \mathbf{f} + \mathbf{n} \wedge \gamma_1 \mathbf{f} \tag{3.10}$$

$$\nabla_s \cdot \mathbf{f} \;=\; \tau_1 \cdot \frac{\partial \mathbf{f}}{\partial s_1} + \tau_2 \cdot \frac{\partial \mathbf{f}}{\partial s_2} \tag{3.11}$$

$$\gamma_o \nabla \cdot \mathbf{f} \;=\; \nabla_s \cdot \gamma_o \mathbf{f} + \mathbf{n} \cdot \gamma_1 \mathbf{f} \tag{3.12}$$

See [**21**].

By (3.6), (3.8), (3.10) and (3.12) taking into account, the Serret-Frenet formulas for curves without torsion as well as the chosen orientation of tangent vectors, we derive the following expressions:

$$\begin{aligned}
\nabla_s[\eta_{\mathbf{v}}\mathbf{n}] &= \frac{\partial(\eta_{\mathbf{v}}\mathbf{n})}{\partial s_1}\otimes\tau_1 + \frac{\partial(\eta_{\mathbf{v}}\mathbf{n})}{\partial s_2}\otimes\tau_2 \;\text{ from (3.10)} \\
&= \left(\frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\mathbf{n} + \eta_{\mathbf{v}}\frac{\partial\mathbf{n}}{\partial s_1}\right)\otimes\tau_1 + \left(\frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\mathbf{n} + \eta_{\mathbf{v}}\frac{\partial\mathbf{n}}{\partial s_2}\right)\otimes\tau_2 \\
&= \frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\mathbf{n}\otimes\tau_1 + \eta_{\mathbf{v}}\frac{\partial\mathbf{n}}{\partial s_1}\otimes\tau_1 + \frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\mathbf{n}\otimes\tau_2 + \eta_{\mathbf{v}}\frac{\partial\mathbf{n}}{\partial s_2}\otimes\tau_2 \\
&= \frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\mathbf{n}\otimes\tau_1 + \frac{\partial\eta_{\mathbf{v}}}{\partial s_2}\mathbf{n}\otimes\tau_2 + \eta_{\mathbf{v}}(-\kappa_1\tau_1)\otimes\tau_1 + \eta_{\mathbf{v}}(-\kappa_2\tau_2)\otimes\tau_2 \qquad (3.13) \\
&= \mathbf{n}\otimes\frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\tau_1 + \mathbf{n}\otimes\frac{\partial\eta_{\mathbf{v}}}{\partial s_2}\tau_2 - \eta_{\mathbf{v}}[\kappa_1\tau_1\otimes\tau_1 + \kappa_2\tau_2\otimes\tau_2] \\
&= \mathbf{n}\otimes\left[\frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\tau_1 + \frac{\partial\eta_{\mathbf{v}}}{\partial s_2}\tau_2\right] - \eta_{\mathbf{v}}\left[\kappa_1\tau_1\otimes\tau_1 + \kappa_2\tau_2\otimes\tau_2\right] \\
&= \mathbf{n}\otimes\nabla_s\eta_{\mathbf{v}} - \eta_{\mathbf{v}}[\kappa_1\tau_1\otimes\tau_1 + \kappa_2\tau_2\otimes\tau_2]
\end{aligned}$$

where κ_1, κ_2 are the curvatures of the orthogonal normal curves in Γ_3 and κ, the mean curvature, in the sense that $\kappa_1 + \kappa_2 = \kappa$. We also have:

$$\begin{aligned}
\nabla_s\wedge[\eta_{\mathbf{v}}\mathbf{n}] &= \tau_1\wedge\frac{\partial(\eta_{\mathbf{v}}\mathbf{n})}{\partial s_1} + \tau_2\wedge\frac{\partial(\eta_{\mathbf{v}}\mathbf{n})}{\partial s_2} \;\text{ from (3.12)} \\
&= \tau_1\wedge\left(\frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\mathbf{n} + \eta_{\mathbf{v}}\frac{\partial\mathbf{n}}{\partial s_1}\right) + \tau_2\wedge\left(\frac{\partial\eta_{\mathbf{v}}}{\partial s_2}\mathbf{n} + \eta_{\mathbf{v}}\frac{\partial\mathbf{n}}{\partial s_2}\right) \\
&= \tau_1\wedge\left(\frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\mathbf{n} - \eta_{\mathbf{v}}\kappa_1\tau_1\right) + \tau_2\wedge\left(\frac{\partial\eta_{\mathbf{v}}}{\partial s_2}\mathbf{n} - \eta_{\mathbf{v}}\kappa_2\tau_2\right) \\
&= \frac{\partial\eta_{\mathbf{v}}}{\partial s_1}(\tau_1\wedge\mathbf{n}) - \eta_{\mathbf{v}}\kappa_1(\tau_1\wedge\tau_1) + \frac{\partial\eta_{\mathbf{v}}}{\partial s_2}(\tau_2\wedge\mathbf{n}) - \eta_{\mathbf{v}}\kappa_2(\tau_2\wedge\tau_2) \\
&= \left(\frac{\partial\eta_{\mathbf{v}}}{\partial s_1}\tau_1 + \frac{\partial\eta_{\mathbf{v}}}{\partial s_2}\tau_2\right)\wedge\mathbf{n} \\
&= \nabla_s\eta_{\mathbf{v}}\wedge\mathbf{n}
\end{aligned} \qquad (3.14)$$

$$\begin{aligned}
\nabla_s \cdot [\eta_{\mathbf{v}}\mathbf{n}] &= \tau_1 \cdot \frac{\partial(\eta_{\mathbf{v}}\mathbf{n})}{\partial s_1} + \tau_2 \cdot \frac{\partial(\eta_{\mathbf{v}}\mathbf{n})}{\partial s_2} \text{ from (3.14)} \\
&= \tau_1 \cdot \left(\frac{\partial \eta_{\mathbf{v}}}{\partial s_1}\mathbf{n} - \eta_{\mathbf{v}}\kappa_1\tau_1\right) + \tau_2 \cdot \left(\frac{\partial \eta_{\mathbf{v}}}{\partial s_2}\mathbf{n} - \eta_{\mathbf{v}}\kappa_2\tau_2\right) \\
&= -\eta_{\mathbf{v}}\kappa_1 + (-\eta_{\mathbf{v}}\kappa_2) \\
&= -\eta_{\mathbf{v}}\kappa. \qquad (3.15)
\end{aligned}$$

From $\nabla \cdot \mathbf{v} = 0$, we have $\gamma_o \nabla \cdot \mathbf{v} = 0$.

Therefore,

$$\gamma_o \nabla \cdot \mathbf{v} = \nabla_s \cdot \gamma_o \mathbf{v} + \mathbf{n} \cdot \gamma_1 \mathbf{v} = 0. \text{ from (3.12)}$$

Hence,

$$\begin{aligned}
\mathbf{n} \cdot \gamma_1 \mathbf{v} &= -\nabla_s \cdot \gamma_o \mathbf{v} \\
&= -\nabla_s \cdot (-\eta_{\mathbf{v}}\mathbf{n}) \\
&= \nabla_s \cdot (\eta_{\mathbf{v}}\mathbf{n}) \\
&= -\eta_{\mathbf{v}}\kappa \qquad (3.16)
\end{aligned}$$

For the vorticity $\mathbf{w}$, we have from (3.10)

$$\begin{aligned}
\gamma_o \mathbf{w} &= \gamma_o \nabla \wedge \mathbf{v} \\
&= \nabla_s \wedge \gamma_o \mathbf{v} + \mathbf{n} \wedge \gamma_1 \mathbf{v} \\
&= \nabla_s \wedge (-\eta_{\mathbf{v}}\mathbf{n}) + \mathbf{n} \wedge \gamma_1 \mathbf{v} \\
&= -\nabla_s \wedge (\eta_{\mathbf{v}}\mathbf{n}) + \mathbf{n} \wedge \gamma_1 \mathbf{v} \\
&= -\nabla_s \eta_{\mathbf{v}} \wedge \mathbf{n} + \mathbf{n} \wedge \gamma_1 \mathbf{v} \\
&= -\mathbf{n} \wedge \nabla_s \eta_{\mathbf{v}} + \mathbf{n} \wedge \gamma_1 \mathbf{v} \\
&= \mathbf{n} \wedge [-\nabla_s \eta_{\mathbf{v}} + \gamma_1 \mathbf{v}]. \qquad (3.17)
\end{aligned}$$

Now

$$\begin{aligned}\gamma_o \mathbf{w} \wedge \mathbf{n} &= \mathbf{n} \wedge [\gamma_1 \mathbf{v} - \nabla_s \eta_{\mathbf{v}}] \\ &= \mathbf{n} \cdot \mathbf{n}(\gamma_1 \mathbf{v} - \nabla_s \eta_{\mathbf{v}}) - \mathbf{n} \cdot (\gamma_1 \mathbf{v} + \nabla_s \eta_{\mathbf{v}})\mathbf{n} \\ &= \gamma_1 \mathbf{v} - \nabla_s \eta_{\mathbf{v}} - (\mathbf{n} \cdot \gamma_1 \mathbf{v} + \mathbf{n} \cdot \nabla_s \eta_{\mathbf{v}})\mathbf{n} \\ &= \gamma_1 \mathbf{v} - \nabla_s \eta_{\mathbf{v}} - (\mathbf{n} \cdot \gamma_1 \mathbf{v})\mathbf{n}, \end{aligned} \tag{3.18}$$

From this, we have that

$$\begin{aligned}\gamma_1 \mathbf{v} &= \gamma_o \mathbf{w} \wedge \mathbf{n} - \nabla_s \eta_{\mathbf{v}} + (\mathbf{n} \cdot \gamma_1 \mathbf{v})\mathbf{n} \\ &= \gamma_o \mathbf{w} \wedge \mathbf{n} - \nabla_s \eta_{\mathbf{v}} - \eta_{\mathbf{v}} \kappa \mathbf{n}\end{aligned} \tag{3.19}$$

Therefore,

$$\begin{aligned}\nabla_s \gamma_o \mathbf{v} &= \nabla_s(-\eta_{\mathbf{v}} \mathbf{n}) \\ &= \eta_{\mathbf{v}}[\kappa_1 \tau_1 \otimes \tau_1 + \kappa_2 \tau_2 \otimes \tau_2] - \mathbf{n} \otimes \nabla_s \eta_{\mathbf{v}}\end{aligned} \tag{3.20}$$

But $\gamma_o \nabla \mathbf{v} = \nabla_s \gamma_o \mathbf{v} + [\gamma_1 \mathbf{v}] \otimes \mathbf{n}$.

Thus,

$$\begin{aligned}\gamma_o \nabla \mathbf{v} &= \eta_{\mathbf{v}}[\kappa_1 \tau_1 \otimes \tau_1 + \kappa_2 \tau_2 \otimes \tau_2] - \mathbf{n} \otimes \nabla_s \eta_{\mathbf{v}} \\ &\quad + [\gamma_o \mathbf{w} \wedge \mathbf{n}] \otimes \mathbf{n} - [\nabla_s \eta_{\mathbf{v}} + \eta_{\mathbf{v}} \kappa \mathbf{n}] \otimes \mathbf{n}\end{aligned} \tag{3.21}$$

From (3.21), we derive an explicit expression for $\gamma_o \mathbf{D(v)}$ as follows: We first observe that

$$\gamma_o \nabla^T \mathbf{v} = \eta_{\mathbf{v}}[\kappa_1 \tau_1 \otimes \tau_1 + \kappa_2 \tau_2 \otimes \tau_2] - \nabla_s \eta_{\mathbf{v}} \otimes \mathbf{n} + \mathbf{n} \otimes (\gamma_o \mathbf{w} \wedge \mathbf{n}) - \mathbf{n} \otimes [\nabla_s \eta_{\mathbf{v}} + \eta_{\mathbf{v}} \kappa \mathbf{n}].$$

Therefore

$$\begin{aligned}
\gamma_o \mathbf{D(v)} &= \frac{1}{2}[\gamma_o(\nabla \mathbf{v} + \nabla^T \mathbf{v})] \\
&= \eta_{\mathbf{v}}[\kappa_1 \tau_1 \otimes \tau_1 + \kappa_2 \tau_2 \otimes \tau_2 - \eta_{\mathbf{v}} \kappa \mathbf{n} \otimes \mathbf{n}] \\
&+ \frac{1}{2}[\gamma_o \mathbf{w} \wedge \mathbf{n} - 2\nabla_s \eta_{\mathbf{v}}] \otimes \mathbf{n} \\
&+ \frac{1}{2}\mathbf{n} \otimes [\gamma_o \mathbf{w} \wedge \mathbf{n} - 2\nabla_s \eta_{\mathbf{v}}]. \qquad (3.22)
\end{aligned}$$

Hence

$$\begin{aligned}
\gamma_o \mathbf{D}(\mathbf{v}) &= \eta_{\mathbf{v}}[\kappa_1 \tau_1 \otimes \tau_1 + \kappa_2 \tau_2 \otimes \tau_2 - \kappa \mathbf{n} \otimes \mathbf{n}] + \frac{1}{2}\psi \otimes \mathbf{n} + \frac{1}{2}\mathbf{n} \otimes \psi \\
&= -\eta_{\mathbf{v}}[\kappa \mathbf{n} \otimes \mathbf{n} - \kappa_1 \tau_1 \otimes \tau_1 - \kappa_2 \tau_2 \otimes \tau_2] + \frac{1}{2}(\psi \otimes \mathbf{n} + \mathbf{n} \otimes \psi) \qquad (3.23) \\
&= -\eta_{\mathbf{v}} \mathbf{M} + \frac{1}{2}\mathbf{N},
\end{aligned}$$

where the tensors $\mathbf{M}$ and $\mathbf{N}$ are defined by

$$\mathbf{M} := \kappa \mathbf{n} \otimes \mathbf{n} - \kappa_1 \boldsymbol{\tau}_1 \otimes \tau_1 - \kappa_2 \tau_2 \otimes \tau_2 \qquad (3.24)$$

$$\mathbf{N} := \mathbf{n} \otimes \Psi + \Psi \otimes \mathbf{n} \qquad (3.25)$$

and

$$\Psi = w \wedge \mathbf{n} - 2 \bigtriangledown_s \eta. \qquad (3.26)$$

It will be noted that Ψ is a sum of tangential vectors, and therefore tangential. In addition, $\gamma_o \mathbf{D}$ is symmetrical.

It is also necessary to obtain an expression for the normal component of deformation at the boundary Γ_3, we therefore deduce from (3.23) the following relations:

$$\gamma_o \mathbf{D}(\mathbf{v})\tau_1 = \eta_{\mathbf{v}}\kappa_1\tau_1 + \frac{1}{2}(\psi \cdot \tau_1)\mathbf{n} \tag{3.27}$$

$$\gamma_o \mathbf{D}(\mathbf{v})\tau_2 = \eta_{\mathbf{v}}\kappa_2\tau_2 + \frac{1}{2}(\psi \cdot \tau_2)\mathbf{n} \tag{3.28}$$

and

$$\begin{aligned}
\gamma_o[\mathbf{D}(\mathbf{v})]\mathbf{n} &= -\eta(\kappa\mathbf{n} \otimes \mathbf{n} - \kappa_1\tau_1 \otimes \tau_1 - \kappa_2\tau_2 \otimes \tau_2)\mathbf{n} \\
&+ \frac{1}{2}(\mathbf{n} \otimes \Psi + \Psi \otimes \mathbf{n})\mathbf{n} \\
&= -\eta(\kappa\mathbf{n} \wedge \mathbf{n} - \kappa_1\tau_1 \wedge \tau_1 - \kappa_2\tau_2 \wedge \tau_2)\mathbf{n} \\
&+ \frac{1}{2}(\mathbf{n} \wedge \Psi + \Psi \wedge \mathbf{n})\mathbf{n} \\
&= \eta\kappa\mathbf{n} + \frac{1}{2}\Psi + \frac{1}{2}\Psi \\
&= -\eta\kappa\mathbf{n} + \Psi.
\end{aligned} \tag{3.29}$$

From above and the tangentiality of Ψ it follows that

$$\begin{aligned}
\mathbf{n} \cdot \gamma_o[\mathbf{D}(\mathbf{v})]\mathbf{n} &= -\eta_{\mathbf{v}}\kappa + \mathbf{n} \cdot \Psi \\
&= -\eta_{\mathbf{v}}\kappa
\end{aligned} \tag{3.30}$$

We shall use the expression (3.27), (3.28) and (3.29) to obtain the matrix representation of the tensor $\gamma_o\mathbf{D}$, (deformation of the epithelial membrane and matter layer), as well as the symmetry of $\gamma_o\mathbf{D}$ as follows:

$$\gamma_o[\mathbf{Dv}] = \frac{1}{2}\begin{pmatrix} \tau_1\cdot\mathbf{D}\tau_1 & \tau_2\cdot\mathbf{D}\tau_1 & \mathbf{n}\cdot\mathbf{D}\tau_1 \\ \tau_1\cdot\mathbf{D}\tau_2 & \tau_2\cdot\mathbf{D}\tau_2 & \mathbf{n}\cdot\mathbf{D}\tau_1 \\ \tau_1\cdot\mathbf{Dn} & \tau_2\cdot\mathbf{Dn} & \mathbf{n}\cdot\mathbf{Dn} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2\eta_{\mathbf{v}}\kappa_1 & 0 & (\mathbf{n}\wedge\mathbf{w}-2\nabla\eta_{\mathbf{v}})\cdot\tau_1 \\ 0 & 2\eta_{\mathbf{v}}\kappa_2 & (\mathbf{n}\wedge\mathbf{w}-2\nabla\eta_{\mathbf{v}})\cdot\tau_2 \\ (\mathbf{n}\wedge\mathbf{w}-2\nabla\eta_{\mathbf{v}})\cdot\tau_1 & (\mathbf{n}\wedge\mathbf{w}-2\nabla\eta_{\mathbf{v}})\cdot\tau_2 & \nabla\cdot\mathbf{v}-2\eta_{\mathbf{v}}\kappa \end{pmatrix}$$

$$= -\eta_{\mathbf{v}}\begin{pmatrix} -\kappa_1 & 0 & 0 \\ 0 & -\kappa_2 & 0 \\ 0 & 0 & \kappa \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & \psi\cdot\tau_1 \\ 0 & 0 & \psi\cdot\tau_1 \\ \psi\cdot\tau_1 & \psi\cdot\tau_2 & 0 \end{pmatrix}$$

We make use of (3.2) and (3.30) to obtain

$$\begin{aligned} \gamma_o(\mathbf{n}\cdot\mathbf{Tn}) &= \mathbf{n}\cdot\gamma_o(-p\mathbf{I}+2\mu\mathbf{D(v)})\mathbf{n} \\ &= -\mathbf{n}\cdot(-\gamma_o p\mathbf{In}+2\mu\mathbf{D}(\mathbf{v})\mathbf{n}) \\ &= -\gamma_o p(\mathbf{n}\cdot\mathbf{I})+2\mu\mathbf{n}\cdot\gamma_o(\mathbf{D}(\mathbf{v}))\mathbf{n} \\ &= -\gamma_o p+2\mu(-\kappa\eta) \\ &= -(\gamma_o p+2\mu\kappa\eta). \end{aligned} \tag{3.31}$$

It is important to note that since the fluid in Ω_o is at rest for steady flow, the pressure p_o is a constant and

$$\mathbf{n}\cdot\gamma_o\mathbf{T}_o\mathbf{n} = p_o \tag{3.32}$$

3.6.2 At the Interface

We use the first Rivlin-Erickson tensor given in Section 3.6.1 to describe the flow through the interface. Let the unit outward normal to Γ_3 be denoted by $\mathbf{n}$, and the trace operator γ_o will be used to denote restriction of $\mathbf{v}$ to Γ_3. It is assumed that the flow through the permeable boundary is always in the direction of the outer normal, i.e.

$$\gamma_o \mathbf{v} = -\eta \mathbf{n} \text{ on } \Gamma_3. \tag{3.33}$$

The scalar-valued function η defined on Γ_3 is unknown. The interface is thought of as a grid with an irregular grating. This is modelled by introducing an effective surface measure da on Γ_3, related to the Lebesgue-induced measure ds by the relation

$$da = \zeta(y) ds(y)$$

with $\zeta(y)$, a measurable function defined on Γ_3 such that $0 \leq \zeta(y) \leq \frac{1}{2}$ for $y \in \Gamma_3$.

The effect of this modelling is that the mass density of the fluid in the interface is replaced by $\zeta(y)\rho$. In addition, we introduce the virtual thickness of the interface as a positive function $\delta > 0$ defined on Γ_3.

Having the above quantities, we define the virtual surface density of the fluid in Γ_3 as

$$\sigma(y) := \zeta(y)\delta(y)\rho.$$

We also impose the condition of "perfect contact" at the interface which states that the velocity field $\gamma_o \mathbf{v}$ at Γ_3 equals the velocity field in the interface layer. Thus, if the velocity field in the interface is of the form $\mathbf{v}^* = -\eta\mathbf{n}$ then

$$\eta = \eta_{\mathbf{v}}$$

for the material in the boundary.

It is assumed that the filtration is through holes.

For every measurable boundary patch $\Gamma_3' \subset \Gamma_3$, we propose the following balance of linear momentum, for a steady flow.

$$\int\limits_{\Gamma_3'} \zeta\rho\eta\eta_{\mathbf{v}}\mathbf{n} - \int\limits_{\Gamma_3'} \nabla_{\mathbf{s}} \cdot \mathbf{T}_{\Gamma_3} = \int\limits_{\Gamma_3'} \gamma_o\mathbf{T}\mathbf{n} + \int\limits_{\Gamma_3'} \gamma_o\mathbf{T}_o\mathbf{n} + \int\limits_{\Gamma_3'} f_{\Gamma_3}\mathbf{n} \tag{3.34}$$

where

$\mathbf{T}_{\Gamma_3}$ = stress tensor for material in the surface

$\mathbf{T}_o$ = stress tensor for the fluid in Ω_o

$\mathbf{T}_{\Gamma_3}$ is to be chosen: We define a "Rivlin-Erickson tensor" for the surface material by

$$\mathbf{A}_{\Gamma_3}(\eta) = 2[\nabla_{\mathbf{s}}\eta \otimes \mathbf{n} + \mathbf{n} \otimes \nabla_{\mathbf{s}}\eta]$$

and make the choice

$$\mathbf{T}_{\Gamma_3}(\eta) = \mu_{\Gamma_3}\mathbf{A}_{\Gamma_3}.$$

In addition, we take the scalar product of (3.34) with $\mathbf{n}$, substitute for $\mathbf{T}_{\Gamma_3}$ and $\mathbf{A}_{\Gamma_3}$ and use (3.31) to obtain the following for any arbitrary patch $\Gamma_3' \subset \Gamma_3$:

$$\zeta\rho\eta_{\mathbf{v}}^2 - 2\mu_{\Gamma_3}\Delta_{\mathbf{s}}\eta_{\mathbf{v}} + p_o - \mathbf{n} \cdot \gamma_o\mathbf{T}\mathbf{n} = f_{\Gamma_3} \tag{3.35}$$

with $\Delta_{\mathbf{s}} = \nabla_{\mathbf{s}} \cdot \nabla_{\mathbf{s}}$ the Laplace-Beltrami operator.

The positive constant μ_{Γ_3} differ in dimensionality from the corresponding 'volume' constant μ in the fluid, since μ is 3-dimensional and μ_{Γ_3} is 2-dimensional. Let

$$\theta_\mu = \frac{\mu_{\Gamma_3}}{\mu}.$$

We make use of (3.31) and substitute for μ_{Γ_3} so that (3.35) becomes

$$\zeta \rho \eta_{\mathbf{v}}^2 + 2\mu\kappa\eta_{\mathbf{v}} - 2\mu\theta_\mu \Delta_{\mathbf{s}} \eta_{\mathbf{v}} + \gamma_o p + p_o = \mathbf{f}_{\Gamma_3}.$$

Therefore, passage of fluid across the boundary interface, Γ_3, shall be represented by:

$$\zeta \rho \eta_{\mathbf{v}}^2 + 2\mu\kappa\eta_{\mathbf{v}} - 2\mu\theta_\mu \Delta_s \eta_{\mathbf{v}} + \gamma_o p = -p_o + \mathbf{f}_{\Gamma_3}. \tag{3.36}$$

The linearised form of (3.36) is derived by assuming that all the velocities are "small" so that products of terms with $\mathbf{v}$ in them may be considered negligible. The linearised form of (3.36) is therefore given as:

$$2\mu\kappa\eta_{\mathbf{v}} - 2\theta_\mu \mu \Delta_{\mathbf{s}} \eta_{\mathbf{v}} + \gamma_o p = -p_o + \mathbf{f}_{\Gamma_3}. \tag{3.37}$$

3.7 Existence Theory

In view of (3.4) and (3.37), we have

$$\left.\begin{array}{l} -\mu\Delta\mathbf{v} = -\nabla p + \mathbf{f}_\Omega \quad \text{in } \Omega \\ 2\mu\kappa\eta_{\mathbf{v}} - 2\mu\theta_\mu \Delta_s \eta_{\mathbf{v}} + \gamma_o p + p_o = \mathbf{f}_{\Gamma_3} \quad \text{on } \Gamma_3 \\ \nabla \cdot \mathbf{v}(x) = 0 \quad x \in \Omega \end{array}\right\} \tag{3.38}$$

subject to the constraints

$$\left.\begin{array}{l} \eta_{\mathbf{v}} = 0 \text{ on } \Gamma_1 \cup \Gamma_2 : \\ \eta_{\mathbf{v}} = 0 \text{ on } \partial\Gamma_3 = (\Gamma_1 \cup \Gamma_2) \cap \Gamma_3 \end{array}\right\} \tag{3.39}$$

Definition 3.2 *We denote the body force on the fluid in the boundary* Γ_3 *in a similar way as the velocity through the boundary, and that is as follows:*

$$\gamma_o(\mathbf{f}_\Omega) = f_{\Gamma_3}\mathbf{n}.$$

3.7.1 The Weak Formulation

$\mathbf{W}(\Omega) = \{\mathbf{v} \in H^1(\Omega) :\ \mathbf{v} \in C^\infty(\Omega),\ \nabla \cdot \mathbf{v} = 0$ in Ω, $\gamma_o\mathbf{v} = -\eta_\mathbf{v}\mathbf{n} \in L^2(\Gamma_3)$, $\gamma_o\mathbf{v} = 0$ at the curve c, where Γ_1 and Γ_2, Γ_2 and Γ_3 meet respectively, $\gamma_o\mathbf{v} = 0$ on Γ_1 and $\Gamma_2\}$.

Lemma 3.7.1a *For* $\mathbf{v} \in \mathbf{W}(\Omega)$,

$$||\mathbf{D}(\mathbf{v})||^2_{\mathbf{L}^2(\Omega)} = \frac{1}{2}\left[||\nabla\mathbf{v}||^2_{\mathbf{L}^2(\Omega)} + \int_{\Gamma_3} \kappa(y)\eta^2_\mathbf{v}(y)ds(y)\right].$$

Proof.

$$\mathbf{D}(\mathbf{v}) = \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right],$$

taking the inner product of $\mathbf{D(v)}$ with itself in $\mathbf{L}^2(\Omega)$ gives

$$\begin{aligned}
(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{v}))_{\mathbf{L}^2(\Omega)} &= \frac{1}{4}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T, \nabla\mathbf{v} + (\nabla\mathbf{v})^T\right)_{\mathbf{L}^2(\Omega)} \\
&= \frac{1}{4}\left[||\nabla\mathbf{v}||^2_{\mathbf{L}^2(\Omega)} + 2\left(\nabla\mathbf{v}, (\nabla\mathbf{v})^T\right)_{\mathbf{L}^2(\Omega)} + ||(\nabla\mathbf{v})^T||^2_{\mathbf{L}^2(\Omega)}\right] \\
&= \frac{1}{4}\left[2||\nabla\mathbf{v}||^2_{\mathbf{L}^2(\Omega)} + 2\left(\nabla\mathbf{v}, (\nabla\mathbf{v})^T\right)_{\mathbf{L}^2(\Omega)}\right].
\end{aligned}$$

Next, we derive an expression for $\left(\nabla\mathbf{v}, (\nabla\mathbf{v})^T\right)_{\mathbf{L}^2(\Omega)}$, using the Gauss divergence theorem:

$$\begin{aligned}
\left(\nabla\mathbf{v}, (\nabla\mathbf{v})^T\right)_{\mathbf{L}^2(\Omega)} &= \sum_{i,j}^{3}\int_\Omega \partial_i\mathbf{v}_j\partial_j\mathbf{v}_i dx \\
&= \sum_{i,j}^{3} \partial_i[\mathbf{v}_j\partial_j\mathbf{v}_i]dx
\end{aligned}$$

since $\nabla \cdot \mathbf{v} = 0$. However,

$$\int_{\Omega} \partial i[\mathbf{v}_j \mathbf{v}_j] dx = \int_{\Gamma_3} \mathbf{n}_i \mathbf{v}_j \partial_j \mathbf{v}_i ds$$

by the divergence theorem. Therefore,

$$\begin{aligned}
\left(\nabla \mathbf{v}, (\nabla \mathbf{v})^T\right)_{\mathbf{L}^2(\Omega)} &= \sum_{i,j}^{3} \int_{\Gamma_3} \mathbf{n}_i \mathbf{v}_j \partial_j \mathbf{v}_i ds \\
&= -\sum_{i,j}^{3} \int_{\Gamma_3} \eta_{\mathbf{v}} \mathbf{n}_i \mathbf{n}_j (\partial_j \mathbf{v}_i) ds \\
&= \int_{\Gamma_3} \eta_{\mathbf{v}} \mathbf{n} \cdot (\nabla \mathbf{v}) \mathbf{n} ds \\
&= \int_{\Gamma_3} \eta_{\mathbf{v}} \mathbf{n} \cdot \left[\frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right] \mathbf{n} ds \\
&= \int_{\Gamma_3} \eta_{\mathbf{v}} \mathbf{n} \cdot \mathbf{D}(\mathbf{v}) \mathbf{n} ds \\
&= \int_{\Gamma_3} \eta_{\mathbf{v}} (-\kappa \eta_{\mathbf{v}}) ds,
\end{aligned}$$

we then have

$$\begin{aligned}
(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{v}))_{\mathbf{L}^2(\Omega)} &= \frac{1}{4}\left[2||\nabla \mathbf{v}||^2_{\mathbf{L}^2(\Omega)} - 2\int_{\Gamma_3} \eta_{\mathbf{v}}(-\kappa \eta_{\mathbf{v}}) ds\right] \\
&= \frac{1}{2}\left[||\nabla \mathbf{v}||^2_{\mathbf{L}^2(\Omega)} + \int_{\Gamma_3} \kappa \eta_{\mathbf{v}}^2 ds\right] \\
(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{v}))_{\mathbf{L}^2(\Omega)} &= ||\mathbf{D}(\mathbf{v})||^2_{\mathbf{L}^2(\Omega)} \\
&= \frac{1}{2}\left[||\nabla \mathbf{v}||^2_{\mathbf{L}^2(\Omega)} + \int_{\Gamma_3} \kappa \eta_{\mathbf{v}}^2 ds\right]
\end{aligned}$$

■

Corollary 3.7.1 *For any* $\mathbf{v} \in \mathbf{W}(\Omega)$,

$$||\mathbf{A}(\mathbf{v})||^2_{\mathbf{L}^2(\Omega)} = 2\left(||\nabla \mathbf{v}||^2_{\mathbf{L}^2(\Omega)} + \int_{\Gamma_3} \kappa \eta_{\mathbf{v}}^2 ds\right).$$

Proof.

Since

$$\mathbf{D}(\mathbf{v}) = \frac{\mathbf{A}(\mathbf{v})}{2},$$

the proof is part of the proof for Lemma 3.7.1.

Lemma 3.7.1b *(A sharp Poincaré Inequality)*

There exist constants $C_p > 0$ *and* $c_p > 0$ *such that*

$$\begin{aligned} ||\nabla \mathbf{v}||^2_\Omega &\geq C_p||\mathbf{v}||^2_\Omega \\ ||\nabla_{\mathbf{s}}\eta_{\mathbf{v}}||^2_{\Gamma_3} &\geq c_p||\eta_{\mathbf{v}}||^2_{\Gamma_3} \textit{ for all } \mathbf{v} \in \mathbf{W}(\Omega). \end{aligned}$$

This is possible because $\gamma_o\mathbf{v} = 0$ on $\Gamma_1 \cup \Gamma_2$ and $\eta_{\mathbf{v}} \in H^1_o(\Gamma_3)$.

Definition 3.3 *We define the traditional norm in* $\mathbf{H}^1(\Omega)$ *as*

$$||\mathbf{v}||^2_1 = ||\nabla \mathbf{v}||^2_\Omega + C_p||\mathbf{v}||^2_\Omega.$$

In Ω:

We recollect from $(3.38)_1$ that the equation of the fluid flow inside Ω is given by

$$-\mu\Delta\mathbf{v} = -\nabla p + \mathbf{f}_\Omega \;\text{ in }\; \Omega.$$

Let $\varphi \in \mathbf{W}(\Omega)$ be a test function. It has the following properties:

(i) $\varphi \in \mathbf{C}^\infty(\Omega)$ sufficiently differentiable.

(ii) $\nabla \cdot \varphi = 0$ in Ω

(iii) $\gamma_o\varphi = 0$ on $\Gamma_1 \cup \Gamma_2$

(iv) $\gamma_o\varphi = 0$ on Γ_1 and Γ_2

(v) $\eta_\varphi \in H^1_0(\Gamma_3)$.

We take the scalar product of $(3.38)_1$, with $\varphi \in \mathbf{W}(\Omega)$ to obtain

$$-(\mu\Delta\mathbf{v}, \varphi)_{\mathbf{L}^2(\Omega)} = -(\nabla p, \varphi)_{\mathbf{L}^2(\Omega)} + (\mathbf{f}_\Omega, \varphi)_{\mathbf{L}^2(\Omega)}. \tag{3.40}$$

We now calculate each term of (3.40): The flow is divergence free, therefore

$$\begin{aligned} \nabla \cdot \mathbf{A}(\mathbf{v}) &= [\partial_i(\partial_j v_j) + \partial_i(\partial_i v_j)] \\ &= [\partial_j(\nabla \cdot \mathbf{v}) + \Delta v_j] = \Delta \mathbf{v}. \end{aligned}$$

Therefore,

$$\begin{aligned} (\Delta\mathbf{v}, \varphi) &= \int_\Omega \Delta\mathbf{v} \cdot \varphi dx = \int_\Omega \varphi \cdot \nabla \cdot \mathbf{A}(\mathbf{v}) dx \\ &= \int_\Gamma \gamma_o \varphi \cdot \gamma_o \mathbf{A}(\mathbf{v}) \mathbf{n} ds - \int_\Omega \nabla\varphi^{\mathbf{T}} : \mathbf{A}(\mathbf{v}) dx \\ &= -\int_{\Gamma_3} \eta_\varphi \mathbf{n} \cdot \gamma_o \mathbf{A}(\mathbf{v}) \mathbf{n} ds - \frac{1}{2} \int_\Omega \mathbf{A}(\mathbf{v}) : \mathbf{A}(\varphi) dx \\ &= 2 \int_{\Gamma_3} \kappa \eta_{\mathbf{v}} \eta_\varphi ds - \frac{1}{2} \int_\Omega \mathbf{A}(\mathbf{v}) : \mathbf{A}(\varphi) dx \end{aligned}$$

$$\begin{aligned} \therefore -(\mu\Delta\mathbf{v}, \varphi) &= -\mu \int_\Omega \Delta\mathbf{v} \cdot \varphi dx \\ &= -2\mu \int_{\Gamma_3} \kappa \eta_{\mathbf{v}} \eta_\varphi ds + \frac{\mu}{2} \int_\Omega \mathbf{A}(\mathbf{v}) : \mathbf{A}(\varphi) dx \end{aligned} \tag{3.41}$$

$$\begin{aligned} (\nabla p, \varphi) &= \int_\Omega \nabla p \cdot \varphi dx \\ &= \int_\Gamma \gamma_o p (\gamma_o, \varphi \cdot \mathbf{n}) ds - \int_\Omega p \nabla \cdot \varphi dx \\ &= -\int_{\Gamma_3} \gamma_o p \eta_\varphi ds \end{aligned}$$

$$\begin{aligned} \therefore -(\nabla p, \varphi)_\Omega &= -\int_\Omega \nabla p \cdot \varphi dx \\ &= \int_{\Gamma_3} \gamma_o p \eta_\varphi ds \end{aligned} \tag{3.42}$$

In view of (3.41) and (3.42), we rewrite (3.40) in scalar-product notation as

$$\frac{\mu}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\varphi))_\Omega - 2\mu(\kappa\eta_\mathbf{v}, \eta_\varphi)_{\Gamma_3} = (\gamma_o p, \eta_\varphi)_{\Gamma_3} + (\mathbf{f}_\Omega, \varphi)_\Omega \tag{3.43}$$

At the Boundary interface Γ_3:

The equation of the fluid flow across the boundary Γ_3 was given as $(3.38)_2$ which we recall as follows:

$$2\mu\kappa\eta_\mathbf{v} - 2\mu\theta_\mu\Delta_s\eta_\mathbf{v} + \gamma_o p + p_o = \mathbf{f}_{\Gamma_3} \text{ on } \Gamma_3.$$

The scalar product of the above with η_φ gives:

$$\begin{aligned} &2\mu(\kappa\eta_\mathbf{v}, \eta_\varphi)_{\Gamma_3} - 2\mu\theta_\mu(\Delta_s\eta_\mathbf{v}, \eta_\varphi)_{\Gamma_3} + (\gamma_o p, \eta_\varphi)_{\Gamma_3} + (p_o, \eta_\varphi)_{\Gamma_3} \\ &= (\mathbf{f}_{\Gamma_3}, \eta_\varphi)_{\Gamma_3} \end{aligned} \tag{3.44}$$

We combine (3.43) with (3.44) and rearrange to obtain

$$\frac{\mu}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\varphi))_\Omega - 2\mu\theta_\mu(\Delta_s\eta_\mathbf{v}, \eta_\varphi)_{\Gamma_3} = (\mathbf{f}_\Omega, \varphi)_\Omega + (\mathbf{f}_{\Gamma_3}, \eta_\varphi)_{\Gamma_3} \tag{3.45}$$

But

$$\nabla \cdot \varphi = 0$$

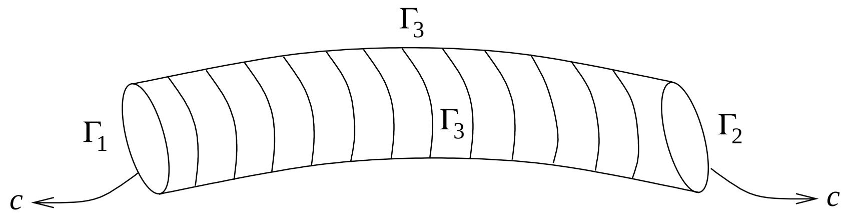
implies

$$\begin{aligned} 0 = \int_\Omega \nabla \cdot \varphi dx &= \int_\Gamma \gamma_o \varphi \cdot \mathbf{n} ds \\ &= -\int_{\Gamma_3} \eta_\varphi ds. \end{aligned}$$

Now

$$(\nabla_s\eta_\mathbf{v}, \eta_\varphi)_{\Gamma_3} = \int_C (\tau \cdot \nabla_s\eta_\mathbf{v}\eta_\varphi) d\ell - (\nabla_s\eta_\mathbf{v}, \nabla_s\eta_\varphi)_{\Gamma_3}$$

where $c =$ curves where Γ_1, Γ_2 and Γ_3 meet. We shall assume that on these curves $\eta_\varphi = 0$.



Then the line integrals vanish.

Thus, from (3.45) the weak formulation of (3.38) is given as

$$2\mu\theta_\mu(\nabla_s\eta_{\mathbf{v}}, \nabla_s\varphi)_{\Gamma_3} + \frac{\mu}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\varphi))_\Omega = (\mathbf{f}_\Omega, \varphi)_\Omega + (\mathbf{f}_{\Gamma_3}, \eta_\varphi)_{\Gamma_3} \tag{3.46}$$

with $\varphi \in \mathbf{W}(\Omega)$.

The Energy Identity

(3.46) holds for all $\mathbf{v}(x) \in \mathbf{W}(\Omega)$. We deduce from Corollary 3.7.1 that

$$(\mathbf{A}(\mathbf{v}), \mathbf{A}(\varphi))_\Omega = 2(\nabla\mathbf{v}, \nabla\varphi)_\Omega + 2(\kappa\eta_{\mathbf{v}}, \eta_\varphi)_{\Gamma_3}. \tag{3.47}$$

We substitute (3.47) into (3.46) to obtain

$$\mu(\nabla\mathbf{v}, \nabla\varphi)_\Omega + 2\mu\theta_\mu(\nabla_{\mathbf{s}}\eta_{\mathbf{v}}, \nabla_{\mathbf{s}}\eta_\varphi)_{\Gamma_3} + \mu(\kappa\eta_{\mathbf{v}}, \eta_\varphi)_{\Gamma_3} = (f_\Omega, \varphi)_\Omega + (\mathbf{f}_{\Gamma_3}, \eta_\mu)_{\Gamma_3} \tag{3.48}$$

(3.48) is our energy identity.

3.7.2 Existence of a Weak Solution

We shall base our definition of of weak solution on the energy identity (3.48) above. It should be noted, however, that (3.48) does not contain second order derivatives, and we observe that (3.48) makes sense in space $\mathbf{W}(\Omega)$. An additional restriction on $\mathbf{v}$ is that we look only at $\mathbf{v} \in \mathbf{H}^1(\Omega)$ for which $\eta_{\mathbf{v}} \in H_0^1(\Gamma_3)$.

We also observe that $\mathbf{v} \in \mathbf{W}(\Omega)$ satisfy the boundary condition (3.39). In fact, we endow $\mathbf{W}(\Omega)$ with the inner product

$$(\mathbf{v}, \varphi)_W = (\mathbf{v}, \varphi)_{H^1(\Omega)} + (\eta_{\mathbf{v}}, \eta_\varphi)_{H^1(\Gamma_3)}.$$

and denote the norm associated with $(\cdot)_W$ by $||\cdot||_W$ $(||\mathbf{v}||_W^2 = (\mathbf{v},\mathbf{v})_W)$.
Let B be a bilinear form defined by

$$B(\mathbf{v}, \varphi) := \mu(\nabla \mathbf{v}, \nabla \varphi)_\Omega + 2\mu\theta_\mu(\nabla_{\mathbf{s}}\eta_{\mathbf{v}}, \nabla_{\mathbf{s}}\eta_\varphi)_{\Gamma_3} + \mu(\kappa\eta_{\mathbf{v}}, \eta_\varphi)_{\Gamma_3}. \tag{3.49}$$

Lemma 3.7.2a *There exist a constant $C_B > 0$ such that*

$$|B(\boldsymbol{v}, \varphi)| \leq C_B||\boldsymbol{v}||_W||\varphi||_W \text{ for all } \boldsymbol{v}, \varphi \in \boldsymbol{W}(\Omega) \tag{3.50}$$

for B defined by (3.49).

Proof.
From (3.49),

$$\begin{aligned}
B(\mathbf{v}, \varphi) &= \mu(\nabla \mathbf{v}, \nabla \varphi)_\Omega + 2\mu\theta_\mu(\nabla_{\mathbf{s}}\eta_{\mathbf{v}}, \nabla_{\mathbf{s}}\eta_\varphi)_{\Gamma_3} + \mu(\kappa\eta_{\mathbf{v}}, \eta_\varphi)_{\Gamma_3} \\
|B(\mathbf{v}, \ell)| &\leq \mu||\nabla \mathbf{v}||_\Omega||\nabla \varphi||_\Omega + 2\mu\theta_\mu||\nabla_{\mathbf{s}}\eta_{\mathbf{v}}||_{\Gamma_3}||\nabla_{\mathbf{s}}\eta_\varphi||_{\Gamma_3} \\
&\quad +\mu\kappa||\eta_{\mathbf{v}}||_{\Gamma_3}||\eta_\varphi||_{\Gamma_3} \text{ by Cauchy Schwarz} \\
&\leq \mu||\mathbf{v}||_1||\varphi||_1 + 2\mu\theta_\mu||\eta_{\mathbf{v}}||_1||\eta_\varphi||_1 + \mu\kappa||\eta_{\mathbf{v}}||_1||\eta_\varphi||_1 \\
&\leq \mu||\mathbf{v}||_W\,||\varphi||_W + 2\mu\theta_\mu||\mathbf{v}||_W\,||\varphi||_W + \mu\kappa||\mathbf{v}||_W\,||\varphi||_W \\
&= (\mu + 2\mu\theta_\mu + \mu\kappa)||\mathbf{v}||_W||\varphi||_W.
\end{aligned}$$

Let there exist a positive constant $C_B > 0$ such that

$$C_B = \mu + 2\mu\theta_\mu + \mu\kappa$$

then

$$|B(\mathbf{v}, \varphi)| \leq C_B||\mathbf{v}||_W||\varphi||_W.$$

Lemma 3.7.2b *The space* $\mathbf{W}(\Omega)$ *equipped with the norm* $||\cdot||_{\mathbf{W}}$ *is complete and therefore a Hilbert space.*

Proof.

By definition

$$\begin{aligned} \mathbf{W}(\Omega) &= \{\mathbf{v} \in H^1(\Omega) : \nabla \cdot \mathbf{v} = 0, \gamma_o \mathbf{v} = -\eta_{\mathbf{v}} \mathbf{n} \text{ on } \Gamma_3, \\ &\quad \gamma_o \mathbf{v} = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \eta_{\mathbf{v}} \in H^1_o(\Gamma_3)\}. \end{aligned}$$

$\mathbf{W}(\Omega)$ is endowed with the inner product

$$(\mathbf{v}, \varphi)_{\mathbf{W}} := (\mathbf{v}, \varphi)_{H^1(\Omega)} + (\eta_{\mathbf{v}}, \eta_\varphi)_{H^1(\Gamma_3)}$$

and with the norm

$$\begin{aligned} ||\mathbf{v}||^2_{\mathbf{W}} &= ||\mathbf{v}||^2_1 + ||\eta_{\mathbf{v}}||^2_1 \\ &= ||\mathbf{v}||^2_\Omega + ||\nabla \mathbf{v}||^2_\Omega + ||\eta_{\mathbf{v}}||^2_{\Gamma_3} + ||\nabla_{\mathbf{s}} \eta_{\mathbf{v}}||^2_{\Gamma_3} \end{aligned}$$

For $\mathbf{W}(\Omega)$ to be complete, we need to show that every Cauchy sequence in $\mathbf{W}(\Omega)$ converges. Consider any Cauchy sequence in $\mathbf{W}(\Omega)$, i.e. there exist N such that for $m, n \geq N$,

$$||\mathbf{v}_m - \mathbf{v}_n||_W < \varepsilon \text{ for all } m, n > N.$$

This implies that

$$||\mathbf{v}_m - \mathbf{v}_n||_W^2 = ||\mathbf{v}_m - \mathbf{v}_n||_1^2 + ||\eta_{\mathbf{v}_m} - \eta_{\mathbf{v}_n}||_1^2 < \varepsilon,$$

for all $m, n > N$. Hence

$$||\mathbf{v}_m - \mathbf{v}_n||_1^2 < \varepsilon \text{ and } ||\eta_{\mathbf{v}_m} - \eta_{\mathbf{v}_n}||_1^2 < \varepsilon.$$

Thus $\{\mathbf{v}_m\}$ is a Cauchy sequence in $H^1(\Omega)$ and $\{\eta_{\mathbf{v}_m}\}$ is a Cauchy sequence in $H^1(\Gamma_3)$ which is complete.

Therefore, there exist $\mathbf{v} \in H^1(\Omega)$ and $\eta_{\mathbf{v}} \in H^1(\Gamma_3)$ such that

$$||\mathbf{v}_m - \mathbf{v}||_1^2 < \varepsilon \text{ and } ||\eta_{\mathbf{v}_m} - \eta_{\mathbf{v}}||_1^2 < \varepsilon \text{ for all } m > N$$

and

$$||\mathbf{v}_m - \mathbf{v}||_W^2 = ||\mathbf{v}_m - \mathbf{v}||_1^2 + ||\eta_{\mathbf{v}_m} - \eta_{\mathbf{v}}||_1^2 < 2\varepsilon \text{ for all } m > N.$$

Lemma 3.7.2c *If* $\mathbf{f}_\Omega \in L^2(\Omega)$ *and* $\mathbf{f}_{\Gamma_3} \in L^2(\Gamma_3)$ *and there exist a maximum value* $C_{\Omega F}$ *for* $\mathbf{f}_\Omega$ *and* $C_{\Gamma_3 F}$ *for* $\mathbf{f}_{\Gamma_3}$ *such that*

$$||\mathbf{f}_\Omega||_\Omega < C_{\Omega F}$$

and

$$||\mathbf{f}_{\Gamma_3}||_{\Gamma_3} < C_{\Gamma_3 F}.$$

Then linear functional $\mathbf{F} : \varphi \in \mathbf{W}(\Omega) \mapsto$

$$\langle \mathbf{F}, \varphi \rangle := (\mathbf{f}_\Omega, \varphi)_\Omega + (\mathbf{f}_{\Gamma_3}, \eta_\varphi)_{\Gamma_3}$$

is bounded with respect to $|| \cdot ||_W$.

Proof.

$$\begin{aligned}
\mathbf{F}(\varphi) &= \langle \mathbf{F}, \varphi \rangle = (\mathbf{f}_\Omega, \varphi)_\Omega + (\mathbf{f}_{\Gamma_3}, \eta_\varphi)_{\Gamma_3} \\
|\mathbf{F}(\varphi)| &\leq \|\mathbf{f}_\Omega\|_\Omega ||\varphi||_\Omega + \|\mathbf{f}_{\Gamma_3}\|_{\Gamma_3} ||\eta_\varphi||_{\Gamma_3} \\
&\leq C_{\Omega F} ||\varphi||_\Omega + C_{\Gamma_3 F} \|\varphi\|_\Omega \\
&\leq (C_{\Omega F} + C_{\Gamma_3 F}) ||\varphi||_W.
\end{aligned}$$

We shall define the weak solution as follows:

Definition 3.4 $\mathbf{v}(x) \in \mathbf{W}(\Omega)$ *is a weak solution of the boundary valued problem (3.38) and (3.39) if the functional equation*

$$B(\mathbf{v}, \varphi) = \langle \mathbf{F}, \varphi \rangle \ \textit{for all} \ \varphi \in \mathbf{W}(\Omega)$$

is satisfied.

However, to arrive at the existence of weak solution, we need find a suitable lower bound for the quadratic form which shall be denoted by $\widehat{B}$ and defined by

$$\widehat{B}(\mathbf{v}) = B(\mathbf{v},\mathbf{v}). \tag{3.51}$$

In view of (3.49), (3.51) is given by

$$\widehat{B}(\mathbf{v}) = B(\mathbf{v},\mathbf{v}) = \mu(\nabla \mathbf{v}, \nabla \mathbf{v})_\Omega + 2\mu\theta_\mu(\nabla_\mathbf{s} \eta_\mathbf{v}, \nabla_\mathbf{s} \eta_\mathbf{v})_{\Gamma_3} + \mu(\kappa \eta_\mathbf{v}, \eta_\mathbf{v})_{\Gamma_3}.$$

Lemma 3.7.2d *If the condition*

$$\sup_{y \in \Gamma_3} \left[\frac{-\kappa(y)}{c_p \theta_\mu} \right] \leq \delta < 1 \tag{3.52}$$

is satisfied, then there exist a constant C_μ such that

$$\widehat{B}(\boldsymbol{v}) \geq C_\mu \|\mathbf{v}\|_W^2 \ \ \textit{for all} \ \ \mathbf{v} \in W(\Omega). \tag{3.53}$$

Proof.

$$\begin{aligned}\widehat{B}(\mathbf{v}) &= B(\mathbf{v},\mathbf{v}) = \mu||\nabla\mathbf{v}||^2_\Omega + 2\mu\theta_\mu||\nabla_\mathbf{s}\eta_\mathbf{v}||^2_{\Gamma_3} + \mu\kappa||\eta_\mathbf{v}||^2_{\Gamma_3}.\\ &= \frac{\mu}{2}||\nabla\mathbf{v}||^2_\Omega + \frac{\mu}{2}||\nabla\mathbf{v}||^2_\Omega + \mu\theta_\mu||\nabla_\mathbf{s}\eta_\mathbf{v}||^2_{\Gamma_3}\\ &\quad +\mu\theta_\mu||\nabla_\mathbf{s}\eta_\mathbf{v}||^2_{\Gamma_3} + \mu\kappa||\eta_\mathbf{v}||^2_{\Gamma_3}.\end{aligned}$$

We now apply Lemma 3.7.1b, the Poincaré inequalities to obtain the following:

$$\begin{aligned}\widehat{B}(\mathbf{v}) &\geq \frac{\mu}{2}||\nabla\mathbf{v}||^2_\Omega + \frac{\mu}{2}C_p||\mathbf{v}||^2_\Omega + \mu\theta_\mu||\nabla_\mathbf{s}\eta_\mathbf{v}||^2_{\Gamma_3}\\ &\quad +\mu\theta_\mu c_p||\eta_\mathbf{v}||^2_{\Gamma_3} + \mu\kappa||\eta_\mathbf{v}||^2_{\Gamma_3}\\ &= \frac{\mu}{2}||\nabla\mathbf{v}||^2_\Omega + \frac{\mu}{2}C_p||\mathbf{v}||^2_\Omega + \mu\theta_\mu||\nabla_\mathbf{s}\eta_\mu||^2_{\Gamma_3}\\ &\quad +([\mu\theta_\mu c_p + \mu\kappa]\eta_\mathbf{v}, \eta_\mathbf{v})_{\Gamma_3}.\end{aligned}$$

This implies

$$\begin{aligned}\widehat{B}(\mathbf{v}) &\geq \frac{\mu}{2}||\nabla\mathbf{v}||^2_\Omega + \frac{\mu}{2}C_p||\mathbf{v}||^2_\Omega + \mu\theta_\mu||\nabla_\mathbf{s}\eta_\mathbf{v}||^2_{\Gamma_3}\\ &\quad +([\mu\theta_\mu c_p + \mu\kappa]\eta_\mathbf{v}, \eta_\mathbf{v})_{\Gamma_3}. \qquad (3.54)\end{aligned}$$

The last term in (3.54) will be positive if the condition

$$\sup_{y\in\Gamma_3}\left[\frac{-\kappa(y)}{c_p\theta_\mu}\right] \leq \delta < 1$$

is satisfied.

If (3.52) is satisfied, then

$$\begin{aligned}\widehat{B}(\mathbf{v}) &\geq \frac{\mu}{2}||\nabla\mathbf{v}||^2_\Omega + \frac{\mu}{2}C_p||\mathbf{v}||^2_\Omega + \mu\theta_\mu||\nabla_\mathbf{s}\eta_\mathbf{v}||^2_{\Gamma_3}\\ &\geq \frac{\mu}{2}||\nabla\mathbf{v}||^2_\Omega + \frac{\mu}{2}C_p||\mathbf{v}||^2_\Omega + \frac{\mu}{2}\theta_\mu\|\nabla_s\eta_\mathbf{v}\|^2_{\Gamma_3} + \frac{\mu}{2}\theta_\mu c_p\|\eta_\mathbf{v}\|^2_{\Gamma_3}\\ &= \frac{\mu}{2}(||\nabla\mathbf{v}||^2_\Omega + C_p||\mathbf{v}||^2_\Omega) + \frac{\mu}{2}\theta_\mu(||\nabla_s\eta_\mathbf{v}||^2_{\Gamma_3} + c_p\|\eta_\mathbf{v}\|^2_{\Gamma_3})\\ &= \frac{\mu}{2}\|\mathbf{v}\|^2_1 + \frac{\mu}{2}\theta_\mu\|\eta_\mathbf{v}\|^2_1.\end{aligned}$$

Let $C_\mu = \min\left(\frac{\mu}{2}, \frac{\mu}{2}\theta_\mu\right)$, then

$$\widehat{B}(\mathbf{v}) \geq C_\mu ||\mathbf{v}||^2_{\mathbf{W}} \quad \text{for all} \quad \mathbf{v} \in W(\Omega).$$

■

Now if the condition (3.52) is satisfied, we define the inner product

$$[\mathbf{v}, \varphi] := B(\mathbf{v}, \varphi) \text{ for } \varphi \in \mathbf{W}(\Omega).$$

Let $[|\cdot|]$ be the norm defined by the quadratic form $\widehat{B}$. That is

$$[|\mathbf{v}|]^2 := \widehat{B}(\mathbf{v}) = B(\mathbf{v},\mathbf{v}).$$

Theorem 3.7.2 *The functional equation*

$$B(\boldsymbol{v}, \varphi) = \langle \boldsymbol{F}, \varphi \rangle \quad \textit{for all} \quad \varphi \in \mathbf{W}(\Omega)$$

has a unique weak solution in $\mathbf{W}(\Omega)$*, provided that (3.52) is satisfied.*

Proof.

From (3.50) and (3.53), it is seen that the norms $||\cdot||_{\mathbf{W}}$ and $[|\cdot|]$ are equivalent. This means that the linear functional $\mathbf{F}$ considered above is also bounded in $[|\cdot|]$.

Also the functional equation

$$B(\mathbf{v}, \varphi) = \langle \mathbf{F}, \varphi \rangle$$

reduces to

$$[\mathbf{v}, \varphi] = \langle \mathbf{F}, \varphi \rangle \quad \text{for all} \quad \varphi \in \mathbf{W}(\Omega). \tag{3.55}$$

By Riesz Representation Theorem ([**22**], p. 346) there exist a unique $\mathbf{v}_F$ such that

$$[\mathbf{v}_F, \varphi] = \langle \mathbf{F}, \varphi \rangle.$$

Thus, the function $\mathbf{v}_F$ is the unique weak solution of the boundary valued problem (3.38) and (3.39). ■

Chapter 4

Result and Conclusion

4.1 Result

Firstly, we noted that the value of blood plasma viscosity (μ) is very important in the process of cerebrospinal fluid formation. The higher the value of μ in the liquid volume phase in the cerebral capillary, the parametric properties of the liquid volume phase tend to that of the solid volume phase, since the ratio of the shear stress and shear rate will increase.

Secondly, in addition to the viscosity of the blood plasma, we observe from equation (3.52) that the cerebrospinal fluid viscosity (μ_{Γ_3}), i.e. the filtrate, and the mean curvature (κ) of the permeable interface play a very important role in the cerebrospinal fluid formation.

4.2 Conclusion

Our models involve all the parameters that are naturally present in the cerebrospinal fluid formations. This is an extension of the work of A. Shara [**26**].

Understanding the basic parametric formation of CSF is paramount to understanding and treatment of various life-threatening diseases associated with cerebrospinal fluid and intracranial pressure p_o. In this work, we have presented a mathematical model of cerebrospinal fluid formation in which all the parameters can be controlled. Our concept can be compared with various previous publication listed in our bibliography. Since all the filtration parameters are present in the model, we believe that within prescribed morphological and physiological properties of the microvascular environment, our model is adaptable to real life situations. We therefore hope that the clinical researchers will find this piece useful and as an open way for further research work.

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