EQUILIBRIUM PROBLEM IN THE TRANSITION FROM A CENTRALIZED ECONOMY TO A COMPETITIVE MARKET

by

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Preface

This work is intended to discuss an Equilibrium problem in the transition from one type of economic mechanism to another. The general equilibrium theory is applied to model the transition from a centralized (budget-controlled) economy to a competitive market based on the publication of V.I. Arkin and A.D. Slastnikov (Moscow). Also there is an introduction into the following topics of theoretical economics:

2. Economical Models, short descriptions.
3. Economic Performance During Transition. (The paper of Hans Pitlik in INTERECONOMICS, January/February 2000, has been studied.)

Mathematical Foundations of theory of Sets, Topological Spaces, Probability and Equilibrium Analysis have been exposed.

The general equilibrium theory is also applied to prove the existence of an equilibrium of transition process that, in a certain sense, makes it possible for agents to "adapt" the choice of technology to the change in economic mechanism.

At the end of the paper there is an example of a transition process in attempt to describe effects that arise in alternative methods of transition from one economic mechanism to another.

Key terms: Centralized Economy; Competitive Market; General Equilibrium; Utility Function; Maximum Theorem; Kuhn-Tacker Theorem; Walrasian Equilibrium Model; Demand function; Supply function; Budget-controlled economy; Transition Process.
Introduction

In all cases of change it must be remembered that changes are disturbing and usually cause losses during the period of adjustment. And for this reason it seems desirable to place some obstacles in the way of changes so that they will not be embarked upon in a capricious spirit. When it is clear that they are wanted or necessary, they should be made with decision.

"The economic system in a Socialist State." [16]


Transition from centrally planned to a market-based system in Central and Eastern Europe (CEE) and in the former Soviet Union started about a decade ago. Following a breakdown of the communist block in 1989 virtually all post-communist countries undertook market-oriented reforms.

1. Centralized Economy

Central planning has been a core future of socialism, in any socialist society it would be difficult to avoid assigning the central role in coordinating production to the state, since socialism aspires to escape the political, social, and economic anarchy of markets by vesting ownership and control over production in the hands of the community. In the absence of markets, the state is the only institution capable of organizing such an endeavor.
The main advantage of a centralized strategy is its capacity to carry out a limited number of high-priority tasks quickly. But the complexity of managerial structures and operations, the excessive volume and poor reliability of information, the difficulty of controlling more than a limited number of priority tasks, and the lack of incentive for middle management to engage in risk exert strong pressures for a delegation of authority. Efforts to promote rational organization are usually conducted simultaneously with a centralist strategy, and a portion of the intelligentsia is engaged in enhancing central control over administrators. But a centralist strategy usually drives a wedge between middle managers and rationalizers due to the conflicting administrative pressures it generates.¹

The basic objective of the authorities in the Soviet-type economy is the production of goods that maximize the rate of growth of wealth in the economy. The goods that make up wealth are evaluated by both the government and the consumers, with the preferences of the former overriding those of the latter. The government has certain output preferences for goods it wants produced in a given year: for example, 120 millions tons of steel, 530 millions tons of coal, 1.2 millions automobiles, etc. within the constraints of these output targets, expressed in aggregates, the government wants the output of disaggregated assortments and goods not included in the plan to reflect consumer preferences.

To minimize waste and hence increase growth, the authorities have two subobjectives: (1) production of a consistent set of outputs, and (2) production which is efficient or economical.

The plan is initiated by the top authority, whose preferences are final. The group of technicians draw up the plan according to the preferences of the top authority.²

Talking about the legacy of communism, it is straightforward to start by drawing on those indicators that used to constitute the characteristics of centrally planned economies and might have left a mark on economic structures because they could not be changed quickly: this is first of all the absence of a financial system that would serve to allocate savings to investments; second, the missing legal and institutional framework which commonly underpins the market economy; third, the marked preference of central planners for heavy industry and the neglect of services; and last, the excessive rates of investment, both in physical and human capital.³

2. The Competitive Market

The solution of the economic problem which is based on the institutions of free contract and the private ownership of property, is called the competitive solution.

The essentials of free competition are that everyone shall be allowed to use or to exchange whatever he possesses of economic significance (whether goods or resources capable of producing them) in any way he pleases; and that no-one takes account in his transactions of the immediate effect that these have on the rates at which he can carry them out. The first condition means that there is no imposition of a co-ordinated scheme upon the individual: the order which is obtained


³"Is Transition Over?" Marc Suhrcke, INTERECONOMICS, May/June 2000.[21]
is an incidental result of the behaviour chosen by each member of the society. The second condition means that no-one acts as a monopolist: Whether he controls an appreciable part of the supply of anything or not, he acts as if his own supply were too small a part of the whole amount to have an appreciable effect on the price.\footnote{The Economic System in a Socialist State\textsuperscript{,} R.L. Hall, London.\cite{Footnote16}}

The conditions of competition imply freedom of independent action on the part of business firms to increase profits by offering customers incentives to do business with them. These same customers are free to accept alternative incentives offered by rival firms. Such competition assumes freedom of business action and counteraction; freedom for the independent firm to offer whatever goods and services and special buying inducements it judges the customer will want and respond to; freedom of the customer to accept or reject what is offered; freedom for everyone to make their own mistakes and learn from them rather than having to do what other people think is best for them or happens to suit those in control.

A competitive economy does not necessarily mean a free-of-all economy. Certain rules must be observed in order to preserve the best elements of competition and to protect society from harmful or noxious activities, cut-throat practices, fraud and misrepresentation. It is the responsibility of the State to create the conditions most favourable to business growth and greater competitive efficiency, to create the economic climate within which business units are left to prepare their own plans in the light of their own best information, planning ability and competitive advantage.\footnote{Marketing in a Competitive Economy.\textsuperscript{,} L.W. Rodger, 1974, London.\cite{Footnote31}}

\footnote{The Economic System in a Socialist State\textsuperscript{,} R.L. Hall, London.\cite{Footnote16}}

\footnote{Marketing in a Competitive Economy.\textsuperscript{,} L.W. Rodger, 1974, London.\cite{Footnote31}}
1 Economic Performance During Transition

While after the collapse of the communist bloc virtually all of its former member countries embarked on market-oriented reforms, the individual countries followed different routes and experienced different outcomes. In all cases, however, output declined steeply during the early years of transition. What were the main causes behind the severe contraction of output? Why have some countries managed to overcome the transformation crisis far better than others?

Perhaps no other geographical area and no other period in history experienced a comparably radical change of economic policies. On the one hand, a number of reforming countries progressed impressively towards establishing a market order. On the other hand, some countries are still have a very long way to go on the rocky road of reforms. In some of these countries market-friendly reforms are still in their infancy.

During the transformation from plan to market the nations followed various routes to reform and experienced different outcomes. A common pattern, though, is that reforming countries in their entirety witnessed a serious output decline during the early years of transition. Of course, no serious economist expected the post-socialist economies to enjoy an immediate recovery or even that they would rapidly catch up with the Western nations.

In the 1960s the model of socialist planned economies appeared to be very successful. Output growth typically reached more than 5 per cent, average investment equalled about 30 per cent or more of GDP and open unemployment was virtually unknown. Central planning was often expected to lead

6"Explaining Economic Performance During Transition: What do we know?" Hans Pitlik, INTERECONOMICS, January/February 2000.[20]
to a fast catching up with Western market economies' standards. During the next two decades, the socialist economic system lost much of its seeming attractiveness, however. As predicted by Hayek and Mises forty years earlier, socialist planning proved to be increasingly incapable of solving the coordination problems of a modern economy. An oversized state industrial sector was unable to overcome serious shortages and black market activities became progressively more important. The quality of both consumer and investment goods was very low and so was productivity in all sectors. In short, the static and dynamic inefficiencies of socialist planning caused the visible poor performance of the official economic system. Moreover, economic deterioration in the 1980s contributed a great deal to the breakdown of the establishment one-party political systems in the communist states.

The events of 1989/90 set the stage for a comprehensive revision of the previous economic systems. Reform-minded politicians faced enormous challenges in transforming the centrally planned economies. The most important task of micro-level reforms is getting the institutions right. Enhancing the quality of the legal framework and making property rights safer in a well-designed legal environment is consequently a central element of systematic transformation.

The challenge of transformation proved to be eminently tough, as the patterns of GDP growth, Table 1 illustrate. After implementing the first reform steps all countries witnessed sharp output losses.

The fall in output reached an enormous amount. With respect to the labour market, the intensively reforming economies of Central Europe showed a distinct rise in official unemployment rates. This obviously points more to the flaws of official data than to a successful labour market during the reforms. At the beginning of the transformation, inflation in some countries
exploded to rates of more than 1,000 per cent, although Table 1 documents that inflation experience varied considerably.

The facts show that all transition economies exhibited a worsening of macroeconomic conditions during the first years. Policy reforms were accompanied by a serious deterioration of growth and a rise in official registered unemployment. Even for successful reformers it took years to recover.
Table 1.  

Economic Performance of Economies in Transition.  
*(Sources: European Bank for Reconstruction and Development; International Monetary Fund.)*

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Central planning led to a serious misallocation of resources, but removing distorting policies opened up new opportunities for individuals. Private actors were now free to shift their privately controlled resources to more advantageous purposes. Structural reforms implied a massive relocation to previously discriminated employment or to newly discovered uses. Factors of production formerly employed in an over-industrialised state sector shifted toward the emerging private sector. But structural reforms appeared to have a harmful effects on output and employment in the short run. Reallocation and restructuring affected the shrinking state sector and the expanding private sector in opposite ways. The main argument is that reallocation and restructuring are essentially investment decisions. If a policy change lacks credibility from the viewpoint of private actors, new investments will be withheld. The lack of credibility then has a harmful impact on investment response. The key problem is that a disappointing reallocation of resources in turn enhances the probability of policy reversals. Rodrik\textsuperscript{7} demonstrates that these effects can lead to multiple equilibria, which are either conducive or detrimental to the success of a market-friendly policy change. Reforms can collapse, even if a government which is reform-minded but lacking in credibility follows a correct economic recipe. The sharp initial decline in production in a number of transition countries and even the failure of reforms may thus be the result of a self-fulfilling prophecy.

**Gradualism and Shock Therapy**

To many observers, the choice of the speed and the comprehensiveness of reforms are the key features of the economic transition. The discussion on an appropriate strategy for reform, however, comes to inconclusive results. On

the one hand, proponents of a gradual approach claim that a severe output collapse can be avoided if reform policy measures are not too tight. Consequently, gradualism is said to lead to less sharp output deterioration and moderate initial unemployment. On the contrary, a higher pace of policy changes is supposed to cause an immediate breakdown of state firms while private businesses are not yet capable of absorbing capital outflows and layoffs. As seen from this view, reform policies were too fast and often too slow.

On the other hand, the case for gradualism is less convincing if it is considered that incentives to reallocate resources and to restructure state firms are seriously weakened. According to the proponents of a big bang, gradual reforms always carry with them the danger of preserving inefficiencies far too long. Slowing down the speed of change is then supposed to be detrimental to growth and employment.

Shock therapy is also often said to be unsuitable for the reform of institutional arrangements. The old institutions are invalidated but new arrangements need time to develop. Murrell points out that fast changes destroy informational networks and capacities of economics. As seen from this viewpoint, the displacement of institutional arrangements is an erroneous strategy because it causes a loss of valuable knowledge, never available to the designers of new institutions. At the beginning of the transition, market-supportive institutions hence work at best imperfectly - radical institutional reforms often create the risk of an institutional vacuum. Advocates of gradualism therefore propose a smooth substitution of previous arrangements by new ones in order to manage the severe problems of discontinuity. The main

\[\text{P. Murrell: Evolutionary and Radical Approaches to Economic Reform, in; Economics of Planning, Vol. 25, pp. 79-95.}[26]
dilemma of this proposition is that central planning institutions in 1989 lost simply all of their trustworthiness, as sharply increasing shadow market activities indicate. It was even better for people to rely on a highly insecure environment than to trust in a corrupt and incompetent state bureaucracy. In that case a fast and comprehensive change of rules seemed to be superior to moderate reforms.

Concluding remarks

Almost 10 years after the start of the transition economic theory is beginning to understand the mechanisms that caused the serious output collapse in the post-communist nations. It seems that both supply-side and demand-side responses to the policy changes as well as bad initial conditions contributed to the dramatic output losses during the early years of reform. Countries that initiated comprehensive reforms were able to return to positive growth rates. Delays in policy changes and the failure to continue with reforms after some minor initial steps appear to be the central causes of the ongoing bad performance in a number of transition economies.
2 Economic Models and General Equilibrium

The ultimate goal of economics is to discover truth about economic relationships in the real world, but there are many ways to seek the truth. Some look for empirical verification as the evidence that must support a theory before it can be accepted. Others assert that economic truths can be discovered only by logical deduction. The methods used by economists vary tremendously, but the common element that runs throughout economic analysis is the reliance on models. Economic models act as an analogy to the real world. The world is complex and difficult to understand, and the economist's model is a simplified representation of it. The economist understands how the model works, and if it works analogously to a certain aspect of the real world, then the economist understands how that aspect of the real world works. The world is too complex for anyone to ever hope to understand all of its interrelationships simultaneously, but small aspects of the world can be represented by comprehensible models. Economists use models because by understanding various models of the economy, the economy that is too complex to be understood in its entirety can be understood by knowing how its component parts work and how they are interrelated.

The common characteristic that pervades modern economics is the reliance on economic models as a foundation for understanding economic phenomena. A model is a simplified representation of its subject that provides a framework for analysis. This description naturally points to the subject of analysis, and economics analyzes economic phenomena from the real world.

Economic models are designed for the purpose of depicting the essential aspects of some economic phenomena. Many details of the real world will necessarily be left out of any economic model, because it is simply not possible to take account of everything. To do so would produce a model as complex as
the real world, and presumably the reason the model was constructed in the first place was that the real world was too complex to understand by itself, outside the framework of a model. Thus, the fact that a model abstracts from some aspects of reality, while necessary, is also a virtue, because it is an aid to understanding the process being modeled. By abstracting from the complications that are only of secondary importance, the economic model focuses its attention on the most important economic factors.

A direct consequence of this is that no single economic model will ever be appropriate for analyzing all economic phenomena. The very nature of a model precludes this. A model abstracts from some futures of reality.

To summarize, economic models are simplified analytical frameworks for depicting particular economic phenomena. The simplification is both necessary and desirable. By using a model, aspects that are unimportant to the phenomenon being studied can be abstracted, allowing attention to be focused on the important relationships of the problem under study. Because of the nature of a model, no one economic model can hope to describe all economic activity. The appropriate model depends upon the problem to which it will be applied. The best model for one purpose will be inappropriate for some others.

Within a positive framework, there are three distinct uses for economic models. First, they can be used for prediction. This application is straightforward; a model used for this purpose implies that if event A happens, than event B will happen as a result of the occurrence of A. The second use for an economic model is to describe and organize observed phenomena. The model acts as an analogy to some real-world phenomena, but since the model is simpler than the real-world, understanding the model helps the observer to understand the essence of the real-world phenomena. The third use of a
model is to logically deduce the existence of phenomena that have not been observed.

The three roles of a model - as a predictor, an explainer, and a revealer - are closely related. The elements in the model are not analogous to the elements in the real world; rather, the relationships among the elements of the model are analogous to the relationships among the elements in the real world. It is because of the analogous relationships between the model and the real world that the model is useful as a device for understanding how the real world works.

A methodological individualism was recommended as a foundation to all economic models. Methodological individualism refers to the use of the individual as the fundamental unit of analysis in a social science. The individuals have utility functions that exhibit certain properties. They are made up of indifference curves that cover every point in the commodity space and they order all points in the space. The ordering is transitive. Indifference curves exhibit diminishing marginal rates of substitution. Beginning with the utility function, neoclassical micro theory then derives the individual demand curve. The utility function or the demand curve derived from the utility function is used as the fundamental building block of neoclassical model.

The methods of representing economic phenomena in models is to examine general equilibrium and partial equilibrium. The concept of equilibrium in economics which at first seems straightforward is in fact viewed in different ways in different types of models. Milton Friedman says, "An equilibrium position is one that if attained will be maintained." This vision of equilibrium, which depicts a situation that is unchanging through the time, can be
contrasted with the rational expectations view of equilibrium which considers an individual to be in equilibrium if the individual considers himself to be in the optimum position at a point in time, given what he knows at that time. The rational expectations view allows for the possibility that an individual can remain in equilibrium in a changing economy, always adjusting to the changing circumstances as they arise.10

**General Equilibrium**11

*General equilibrium* theory is in contrast with partial equilibrium theory where some specified part of an economy is analyzed while the influences impinging on this sector from the rest of the economy are held constant. In general equilibrium the influences which are treated as constant are those which are considered to be noneconomic and thus beyond the range of economic analysis.

The institutions whose phenomena are the primary subject of economic analysis is the market, made up of a group of economic agents who buy and sell goods and services to one another. In general equilibrium theory all the agents involved in exchanges with each other should ideally be included and all their sales and purchases should be allowed for. However, it may happen that the activities of many agents are only treated in the aggregate and the list of goods and services may be reduced by aggregation. The aggregation of the agents and the commodities into a few categories is especially important when general equilibrium theory is applied to special areas of public policy such as the government budget, money and banking, or foreign trade. Much of the theory developed for these subjects is general equilibrium theory in


aggregated form.

The general equilibrium implies that all subsets of agents are in equilibrium and in particular that all individual agents are in equilibrium. The conscious development of a formal general equilibrium theory stated in mathematical terms seems to have been inspired by a formal theory of the equilibrium of the individual consumer faced with a given set of trading opportunities or prices. This theory was developed by the marginal utility, or neo-classical, school of economists in the third quarter of the 19th century, independently, by Gossen (1854), Jevons (1871), and Walras (1874-7), who used mathematical notations, and by Menger (1871) who did not. The step was taken in the most effective way by Walras.

Walras assumed that the utility derived from the consumption of a good was given as a function of the amount of that good along that was consumed and independent of the amounts consumed of other goods. He also assumed that the first derivative of the utility function was positive and decreasing up to a point of satiation when one exists. He then gave a rigorous derivation of the demand for a good by a consumer from the maximization of utility subject to a budget constraint. The demand functions give the equilibrium quantities traded by the consumer as a function of a general equilibrium theory for an economy. It has remained in a generalized form the cornerstone of general equilibrium theory since Walras.
3 Mathematical Foundations

3.1 Families of Sets

A set $A$ is a collection of objects of any kind (for example points in a plane, real numbers, functions) which are called the elements (or points) of $A$; in general sets are denoted by capital Latin letter and elements are denoted by small Latin letters.

In certain cases a set can be determined by means of a list or, more generally, by means of a property of its elements; for example, the set of positive rational numbers, which we denote by $\mathbb{R}^+_*$, is the collection of positive numbers $x$ which have the following property: $x$ is the quotient of an integer $p$ by an integer $q$, where $q$ is not zero.

A one-one correspondence between two sets $A$ and $B$ is a rule in which there is associated with each element $a$ of $A$ an element $b$ of $B$, this being denoted by $a \rightarrow b$, such that for each $b \in B$ there exists one and only one $a \in A$ for which $a \rightarrow b$.

The product $\mathbb{R} \times \mathbb{R} = \{(x_1, x_2) | x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$, which can be represented by the points of a plane, is called the Euclidean plane, and is denoted by $\mathbb{R}^2$.

A set $I$ and a correspondence $i \rightarrow a_i$ in which there corresponds to each $i$ in $I$ an element $a_i$ of set $A$, is called a family of elements in $A$ and is denoted by $(a_i / i \in I)$; $I$ is called the index set. In the case in which $I = \{1, 2, ..., n\}$ we have a family of elements called an $n$-tuple and if $n = 2$, this $n$-tuple is called a pair.
3.2 Topological Concepts in $\mathbb{R}^l$

The topological structure of $\mathbb{R}^l$ is built on the topological properties of the real numbers $\mathbb{R}$. Let us recall the fundamental properties of $\mathbb{R}$.

**Proposition (A1).** Every non-empty subset $S$ of $\mathbb{R}$ which is bounded from below (above)$^{12}$ has an infimum (supremum). The infimum of $S$, written $\inf(S)$, is defined by the properties:

1. (a) $\inf(S) \leq x$ for every $x \in S$,
   
   (b) for every $\epsilon > 0$ there exists $x \in S$ with $x < \inf(S) + \epsilon$.

The supremum of $S$, written $\sup(S)$, is defined by the properties:

1. (a) $x \leq \sup(S)$ for every $x \in S$,
   
   (b) for every $\epsilon > 0$ there exists $x \in S$ with $\sup(S) - \epsilon < x$.

If the numbers $\inf(S)$ ($\sup(S)$) belongs to a set $S$ then we express this by writing $\min(S)$ instead of $\inf(S)$ ($\max(S)$ instead of $\sup(S)$).

**Definition 1.** A sequence $(\lambda_n)_{n=1,\ldots}$ in $\mathbb{R}$ is said to be convergent to $\lambda \in \mathbb{R}$ if for every $\epsilon > 0$ there exists an integer $\bar{n}$ such that

$$\lambda - \epsilon < \lambda_n < \lambda + \epsilon \quad \text{for all } n > \bar{n}.$$ 

This can be written as

$$|\lambda_n - \lambda| < \epsilon \quad \text{for all } n > \bar{n}.$$ 

$^{12}$A subset $S$ of $\mathbb{R}$ is called bounded below if there is a point $\underline{x}$ of $\mathbb{R}$ such that

$$x \geq \underline{x} \quad \text{for every } x \in S,$$

and bounded above if there is a point $\overline{x}$ of $\mathbb{R}$ such that

$$x \leq \overline{x} \quad \text{for every } x \in S.$$
Definition 2. A sequence \((x_n)_{n=1,2,...}\) of points in \(\mathbb{R}^l\) is said to converge to the point \(x \in \mathbb{R}^l\) (written \(x_n \to x\)) if the sequence \((x_n)\) converges coordinate-wise, that is to say, if for every \(h = 1,...,l\) the sequence \((x_n^h)_{n=1,...}\) of real numbers converges to the real number \(x^h\).

The point \(x\) is called the limit of the sequence \((x_n)\); this is also written \(x = \lim_{n \to \infty} x_n\).

3.2.1 Closed and Open Sets

Definition 3. A subset \(F\) of \(\mathbb{R}^l\) is said to be closed if the limit of every convergent sequence in \(F\) belongs to \(F\), i.e., if \(x_n \to x\) and \(x_n \in F\) for \(n = 1, 2,...\) then \(x \in F\).

Definition 4. A subset \(G\) of \(\mathbb{R}^l\) is said to be open if its complement \(\mathbb{R}^l\setminus G\) is closed.

Proposition (2A). A subset \(G\) of \(\mathbb{R}^l\) is open if and only if for every \(z \in G\) there exists a ball with centre \(z\) and radius \(r > 0\) which is contained in \(G\).

Definition 5. A neighbourhood \(U\) of a subset \(S\) of \(\mathbb{R}^l\) is a subset containing an open set \(G\) which contains \(S\), i.e.,

\[S \subset G \subset U,\] where \(G\) is open.

Definition 6. A point \(x\) is called a boundary point of the set \(S \subset \mathbb{R}^l\) if every neighbourhood \(U\) of \(x\) has a non-empty intersection with \(S\) and \(\mathbb{R}^l\setminus S\). The set of all boundary points of \(S\) is called the boundary of \(S\) and is denoted by \(\partial S\).
3.2.2 Convex Sets\textsuperscript{13}

A segment connecting two end points \( x \) and \( y \) in \( \mathbb{R}^n \) is a point set represented by the expression

\[
\alpha x + \beta y; \quad \alpha, \beta \geq 0, \quad \alpha + \beta = 1.
\]

This set is denoted notationally by \([x, y]\), just like a closed interval of real numbers. Individual points of the segment \([x, y]\) are called convex linear combinations of \( x \) and \( y \). A subset \( X \) of \( \mathbb{R}^n \) is called a convex set if a segment \([x, y]\) with any two end points \( x \) and \( y \) in \( X \) is always included in \( X \). In fig.1, the set in the right-hand figure is convex, but the one on the left is not.

(Fig.1)

We count an empty set as a convex set. Hence, in order to prove that a set \( X \) is convex, it suffices to show that \( x, y \in X \implies \alpha x + \beta y \in X \) (for any \( \alpha \geq 0, \beta \geq 0, \) and \( \alpha + \beta = 1 \)).

\textsuperscript{13}Hukukane Nikaido, Introduction to Sets and Mappings in Modern Economics.[27]
3.3 The Commodity Space\footnote{"Equilibrium Analysis", Variations on themes by Edgeworth and Walras, W. Hildenbrand and A. P. Kirman, 1988, Advanced Textbooks in Economics, North-Holland.[18]}

A commodity is anything which may be used or consumed. It may be a physical good such as bread or a service such as the use of some object. Services may also be viewed as elementary commodities as long as they can be quantified by units of measurements. Let us assume that there is only a finite number \( l \) of commodities. This does not impose any real restriction, since all that assumed is that the agents in an economy are only capable of distinguishing between a finite number of commodities. Quantities of commodities are given by non-negative real numbers.

A commodity consists of a vector \( x \in R^l \) which describes the quantity \( x_h \) of each elementary commodity \( h = 1, \ldots, l \).

We consider then a "bundle" of commodities as \( l \) non-negative real numbers. Indexing the commodities from 1 to \( l \) we may then describe such a bundle as a vector \( x = (x_1, x_2, \ldots, x_l) \) in \( l \)-dimensional Euclidean space. Thus, in summary:

\[
\begin{array}{|c|c|}
\hline
\text{the commodity space} & \text{is given by } R^l \\
\hline
\text{a commodity bundle} & \text{is given by } x = (x_1, x_2, \ldots, x_l), \text{ an element of } R^l. \\
\hline
\end{array}
\]

3.4 Consumption Set

The features of an economic agent are simply that he possesses a commodity bundle, his initial resources, and is supposed to make choices. His choices are based on his preferences which extend over a subset of the commodity space called the consumption set.
Let us consider that he will be interested in \( X(a) \subseteq \mathbb{R}^l_+ \). A possible consumption plan for the agent \( a \) is then any commodity bundle in his consumption set, i.e., \( x \in \mathbb{R}^l_+ \). Then we have

<table>
<thead>
<tr>
<th>the consumption set ( X(a) )</th>
<th>is the positive orthant of ( \mathbb{R}^l_+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a consumption plan ( x )</td>
<td>is an element of ( \mathbb{R}^l_+ )</td>
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</tbody>
</table>

### 3.5 Simplexes

For each number \( n = 0, 1, 2, \ldots \) one has the simplex \( \Delta^n \) which is the convex hull (the smallest convex set containing the points) of the unit vectors \( e_1, e_2, \ldots, e_n \) in the Euclidean \( (n + 1) \)-space \( \mathbb{R}^{n+1} \). Hence \( \Delta^0 \) is a point, \( \Delta^1 \) is an interval, \( \Delta^2 \) a triangle, \( \Delta^3 \) a tetrahedron, and so on.

The dimension of \( \Delta^n \) is \( n \). A point \( x \in \Delta^n \) is given by barycentric coordinates,

\[
x = \sum_{i=0}^{n} t_i e_i \quad \text{with} \quad \sum_{i=0}^{n} t_i = 1 \quad \text{and} \quad t_i \geq 0.
\]

The name simplex describes an object which is supposed to be very simple; indeed, natural numbers and simplexes both have the same kind of innocence. \(^{15}\)

### 3.6 Continuity. Monotonicity

#### 3.6.1 Continuity

**Definition 1.**

\(^{15}\)Handbook of Algebraic Topology, edited by I.M. James, 1996.[17]
A function \( f : R \rightarrow R \) is said to be continuous at a point \( p \in R \) if whenever \((a_n)\) is a real sequence converging to \( p \), the sequence \((f(a_n))\) converges to \( f(p) \).

**Definition 2.**

A function \( f \) defined on a subset \( D \) of \( R \) is said to be continuous if it is continuous at every point \( p \in D \).

We also have the following.

**Definition 3.**

A real valued function \( f \) defined on a subset \( S \) of \( R \) is said to be continuous if it is continuous at all points of \( S \).

**Definition 4.**

If \( f \) and \( g \) are functions from \( R \) to \( R \), we define the function \( f + g \) by \((f + g)(x) = f(x) + g(x)\) for all \( x \) in \( R \).

Similarly we may define the difference, product and quotient of functions.

**Theorem 1.**

If \( f \) and \( g \) are continuous at a point \( p \) of \( R \), then so are \( f + g \), \( f - g \), \( f \cdot g \) and (provided \( g(p) \neq 0 \)) \( f/g \).

**Proof**

This follows directly from the corresponding arithmetic properties of sequences.

For example: to prove that \( f + g \) is continuous at \( p \in R \)

Suppose \((x_n) \rightarrow p\). We are told that \((f(x_n)) \rightarrow f(p)\) and \((g(x_n)) \rightarrow g(p)\)

and we must prove that \((f + g)(x_n)) \rightarrow (f + g)(p)\).

But the LHS of this expression is \( f(x_n) + g(x_n) \) and the RHS is \( f(p) + g(p) \)

and so the result follows from the arithmetic properties of sequences. **Q.E.D.**

**Theorem 2.**

The composite of continuous functions is continuous.
Proof

Suppose \( f : R \to R \) and \( g : R \to R \). Then the composition \( gf \) is defined by \( gf(x) = g(f(x)) \).

We assume that \( f \) is continuous at \( p \) and that \( g \) is continuous at \( f(p) \). So suppose that \( (x_i) \to p \). Then \( (f(x_i)) \to f(p) \) and then \( (g(f(x_i))) \to g(f(p)) \) which is what we need. Q.E.D.

3.6.2 Monotonicity

Definition 1.

A sequence \( \{a_j\}_{j=1}^{\infty} \) is called monotone increasing if \( a_{j+1} \geq a_j \) for all \( j \). A sequence \( \{a_j\}_{j=1}^{\infty} \) is called monotone decreasing if \( a_j \geq a_{j+1} \) for all \( j \).

A sequence is called a monotonic if it is either increasing or decreasing.

Definition 2.

A function \( u^l(c_s) \) is called increasing if \( u^l(c_s) \leq u^l(c_{s+1}) \) for all \( 1 \leq s \leq l \), that is \( u^l(c_1) \leq u^l(c_2) \leq u^l(c_3) \leq \ldots \). It is called decreasing if \( u^l(c_s) \geq u^l(c_{s+1}) \) for all \( 1 \leq s \leq l \). It is called a monotonic if it is either increasing or decreasing.

3.7 Infimum, Supremum

Recall that any finite set of real numbers has a greatest element (maximum) and a least element (minimum). Example, \( \{-2.5, 3.1, -4.4, 4.5, 5\} \).

However, this property does not necessarily hold for infinite sets. Example, \( \{1, 2, 3, 4, \ldots \} \).

Definition 1. A real number \( M ( \neq \pm \infty ) \) is called the least upper bound or supremum of a set \( E \) if

(i) \( M \) is an upper bound of \( E \), i.e., \( x \leq M \) for every \( x \in E \), and
(ii) if \( M' < M \), then \( M' \) is not an upper bound of \( E \) (i.e., there is an \( x \in E \) such that \( M' < x \)).

We write \( M = \sup E \).

**Remark.**

(i) \( \sup E \) is unique whenever it exists.

(ii) The main difference between \( \sup E \) and \( \max E \) is that \( \sup E \) may not be an element of \( E \), whereas \( \max E \) must be an element of \( E \) if it does exist.

(iii) If \( E \) has a maximum, then \( \sup E = \max E \).

**Example.** 1. Let \( E = \{ r \in \mathbb{Q} \mid 0 \leq r \leq \sqrt{2} \} \). Then \( \sup E = \sqrt{2} \) but \( \max E \) does not exist because \( \sqrt{2} \) is not a rational number, that is, \( \sup E \notin E \).

2. Let \( E = \{1/2, 2/3, 3/4, 4/5, 5/6, \ldots \} \). Then \( \sup E = 1 \) and \( \max E \) does not exist.

3. Let \( E = \{1, 1/2, 1/3, 1/4, 1/5, \ldots \} \). Then \( \max E = 1 = \sup E \).

**Definition 2.** A real number \( m(\neq \pm \infty) \) is called the greatest lower bound or \( \text{infimum} \) of a set \( E \) if

(i) \( m \) is a lower bound of \( E \), i.e., \( m \leq x \) for every \( x \in E \), and

(ii) if \( m' > m \), then \( m' \) is not a lower bound of \( E \) (i.e., there exists an \( x \in E \) such that \( x < m' \)).

We write \( m = \inf E \).

**Remark.**

(i) \( \inf E \) is unique whenever it exists.

(ii) The main difference between \( \inf E \) and \( \min E \) is that \( \inf E \) may not be an element of \( E \), whereas \( \min E \) must be an element of \( E \) if it does exist.

(iii) If \( E \) has a minimum, then \( \inf E = \min E \).

**Example.** 1. Let \( E = \{1, 1/2, 1/3, 1/4, \ldots \} \). Then \( \inf E = 0 \) but
min \ E \) does not exist.

2. Let \( E = \{ r \in Q \mid 0 \leq r \leq \sqrt{2} \} \). Then \( \min E = \inf E = 0 \).

An important property of the set of real numbers is the following:

**Theorem 1.** (Completeness Axiom of \( R \)). The following statement hold for subsets of real numbers:

(i) If \( E \) is bounded above, then \( \sup E \) exists.

(ii) If \( E \) is bounded below, then \( \inf E \) exists.

Recall that a set \( E \) is bounded if and only if it is bounded above and bounded below. Thus the Completeness Axiom leads to

**Corollary 1.** If \( E \) is bounded, then both \( \sup E \) and \( \inf E \) exist.

### 3.7.1 Monotone Sequences

**Definition 3.** \( \{a_n\} \) is called *monotone increasing* (decreasing) if

\[
a_n \leq (\geq) a_{n+1}
\]

for every \( n \), that is,

\[
a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots
\]

\((a_1 \geq a_2 \geq a_3 \geq \cdots)\).

**Example.** 1. The sequence \( \{1/n\} \) is monotone decreasing.

2. The sequence \( \{1/2, 2/3, 3/4, 4/5, 5/6, \ldots\} \) is monotone increasing.

**Proposition 1.** A monotone increasing (decreasing) sequence is bounded below (above).

**Proof.** Let \( \{a_n\} \) be a monotone increasing sequence, that is,

\[
a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots
\]

Then \( a_1 \) is a lower bound for \( \{a_n\} \) and hence the result.
Theorem 2. (Monotone Convergence Theorem). Let \( \{a_n\} \) be a sequence.

(i) If \( \{a_n\} \) is monotone increasing and bounded above, then \( \{a_n\} \) is convergent and

\[
\lim_{n \to \infty} a_n = \sup_n a_n.
\]

(ii) If \( \{a_n\} \) is monotone decreasing and bounded below, then \( \{a_n\} \) is convergent and

\[
\lim_{n \to \infty} a_n = \inf_n a_n.
\]

Proof. (i). Suppose \( \{a_n\} \) is monotone increasing and bounded above. Then by the Completeness Axiom of \( \mathbb{R} \), \( \sup_n a_n \) exists (finite). Now, given \( \epsilon > 0 \), since

\[
\sup_n a_n - \epsilon < \sup_n a_n,
\]

it follows that \( \sup_n a_n - \epsilon \) is not an upper bound of \( \{a_n\} \). In other words, there exists an integer \( N \) such that \( a_N > \sup_n a_n - \epsilon \). Then for all \( n > N \), we have

\[
\sup_n a_n - \epsilon < a_N \leq a_n \leq \sup_n a_n < \sup_n a_n + \epsilon \quad \text{(since } n > N).\]

Equivalently, \( |a_n - \sup_n a_n| < \epsilon \) for all \( n > N \) and so \( \lim_{n \to \infty} a_n = \sup_n a_n \) (exists). The proof of (ii) is similar. Q.E.D.

Example. Let \( a_n = \frac{n}{n+1} \), that is, \( \{a_n\} = \{1/2, 2/3, 3/4, \ldots\} \). Then \( a_n \) is monotone increasing and bounded above. Thus \( \sup a_n = \lim_{n \to \infty} a_n = 1 \).

Corollary 2. If \( \{a_n\} \) is monotone increasing (decreasing), then either

(i) \( \{a_n\} \) is convergent or

(ii) \( \lim_{n \to \infty} a_n = +\infty (-\infty) \).

Proof. Suppose \( \{a_n\} \) is monotone increasing, then either \( \{a_n\} \) is bounded above or not bounded above.
Case (a): If \( \{a_n\} \) is bounded above, then by the Monotone Convergence Theorem, \( \{a_n\} \) converges.

Case (b): If \( \{a_n\} \) is not bounded above, then \( \{a_n\} \) has no upper bounds. Thus for any given \( k > 0 \), \( k \) is not an upper bound of \( \{a_n\} \). In other words, there exists \( N \) such that \( a_N > k \). Since \( \{a_n\} \) is monotone increasing, it follows that for all \( n > N \), \( a_n \geq a_N > k \). Therefore, \( \lim_{n \to \infty} a_n = +\infty \).

The proof for the case when \( \{a_n\} \) is monotone decreasing is similar. Q.E.D.

3.7.2 The \( \lim \sup \) and \( \lim \inf \) of a sequence


Given a sequence \( \{a_n\} \), we can form another sequence \( \{b_n\} \) given by

\[
b_n = \sup_{k \geq n} a_k = \sup \{a_n, a_{n+1}, a_{n+2}, \ldots\}.
\]

Example. Let \( \{a_n\} = \{1, -1, 1, -1, \ldots\} \). Then

\[
b_n = \sup_{k \geq n} a_k = \sup \{\pm 1, \mp 1, \pm 1, \mp 1, \ldots\} = 1.
\]

Proposition 2. For any sequence \( \{a_n\} \), the associated sequence \( \{b_n\} = \{\sup_{k \geq n} a_k\} \) is always monotone decreasing.

Proof. For each \( n \),

\[
b_n = \sup \{a_n, a_{n+1}, a_{n+2}, \ldots\} \geq \sup \{a_{n+1}, a_{n+2}, \ldots\} = b_{n+1}.
\]

Q.E.D.

Definition 4. The limit superior of \( \{a_n\} \), denoted by \( \limsup_{n \to \infty} a_n \) or \( \lim \sup_{n \to \infty} a_n \) is defined to be \( \lim b_n \), i.e.

\[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} \sup_{k \geq n} a_k.
\]
Example. 1. Let \( \{a_n\} = \{1, -1, 1, -1, 1, -1, \ldots\} \).

\[
\overline{\lim_{n \to \infty}} a_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} 1 = 1.
\]

2. Let \( \{a_n\} = \{1, 2, 3, \ldots\} \). Then

\[
b_n = \sup_{k \geq n} a_k = \sup\{n, n+1, \ldots\} = +\infty
\]

and so \( \overline{\lim_{n \to \infty}} a_n = \lim_{n \to \infty} b_n = +\infty \).

3. Let \( \{a_n\} = \{-1, -2, -3, \ldots\} \). Then

\[
b_n = \sup_{k \geq n} a_k = \sup\{-n, -n-1, \ldots\} = -n
\]

and so \( \overline{\lim_{n \to \infty}} a_n = \lim_{n \to \infty} b_n = -\infty \).

**Proposition 3.** Given any \( \{a_n\} \), \( \lim \sup a_n \) always exists (either finite, \( +\infty \) or \( -\infty \)).

**Proof.** If \( \{a_n\} \) is not bounded above, then each \( b_n \) is \( +\infty \), and thus

\[
\overline{\lim_{n \to \infty}} a_n = \lim_{n \to \infty} b_n = +\infty.
\]

If \( \{a_n\} \) is bounded above, then each \( b_n \) is finite. Since \( \{b_n\} \) is monotone decreasing, \( \{a_n\} \) converges (to a finite limit), or \( \lim_{n \to \infty} b_n = -\infty \). Q.E.D.

Similarly, given any sequence \( \{a_n\} \), we can form another sequence \( \{c_n\} \) given by

\[
c_n = \inf_{k \geq n} a_k = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}.
\]

**Definition 5.** The *limit inferior* of \( \{a_n\} \), denoted by \( \liminf a_n \), is defined to be \( \lim_{n \to \infty} c_n \), i.e.

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = \lim_{n \to \infty} \inf_{k \geq n} a_k.
\]
Example. 1. Let \( \{a_n\} = \{1, -1, 1, -1, 1, \cdots\} \).

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = \lim_{n \to \infty} (\inf\{\pm 1, \pm 1, \pm 1, \cdots\}) = \lim_{n \to \infty} -1 = -1.
\]

2. Let \( \{a_n\} = \{1, 2, 3, \cdots\} \). Then

\[
c_n = \inf_{k \geq n} a_k = \inf\{n, n+1, \cdots\} = n
\]

and so \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = +\infty. \)

3. Let \( \{a_n\} = \{-1, -2, -3, \cdots\} \). Then

\[
c_n = \inf_{k \geq n} a_k = \inf\{-n, -n-1, \cdots\} = -\infty
\]

and so \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = -\infty. \)

Proposition 4. (i). As in Proposition 2, for any given sequence \( \{a_n\} \),
the associated sequence \( \{c_n\} = \{\inf_{k \geq n} a_k\} \) is always monotone increasing.

(ii). As in Proposition 3, for any given \( \{a_n\} \), \( \lim \inf a_n \) always exists
(either finite, \( +\infty \) or \( -\infty \)).

Remark. We always have \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \) because \( c_n \leq b_n. \)

Proposition 5. (i). If \( \lim_{n \to \infty} a_n = B \) with \( B \neq -\infty \), then given \( \epsilon > 0 \),
there exists \( N \) such that \( a_n < B + \epsilon \) for all \( n > N \).

(ii). \( \lim_{n \to \infty} a_n = C \) with \( C \neq +\infty \), then given \( \epsilon > 0 \), there exists \( N \) such
that \( a_n > C - \epsilon \) for all \( n > N \).

Proof. (i). If \( B = +\infty \), the assertion is obvious and so we assume that
\( B \) is finite. Since \( \lim_{n \to \infty} a_n = B \), given any \( \epsilon > 0 \), there exists \( N \) such that for
all \( n > N \),

\[
|b_n - B| < \epsilon \implies b_n < B + \epsilon \implies \sup\{a_n, a_{n+1}, \cdots\} < B + \epsilon,
\]

i.e. \( a_n, a_{n+1}, \cdots < B + \epsilon \) for all \( n > N \). Proof of (ii) is similar.

Remark. Roughly speaking, Proposition 5 says that for any sequence
\( \{a_n\} \), the \( a_n \)'s are eventually not much smaller than \( \lim_{n \to \infty} a_n \) and not much
bigger than \( \lim_{n \to \infty} a_n \).
Proposition 6. If \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n = A \) (finite), then \( \{a_n\} \) converges and \( \lim_{n \to \infty} a_n = A \).

**Proof.** Let \( b_n = \sup\{a_n, a_{n+1}, \ldots\} \) and let \( c_n = \inf\{a_n, a_{n+1}, \ldots\} \). Then
\[
c_n = \inf\{a_n, a_{n+1}, \ldots\} \leq a_n \leq b_n = \sup\{a_n, a_{n+1}, \ldots\}.
\]

By the assumption, we have
\[
\lim_{n \to \infty} c_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n.
\]

The sequence \( \{a_n\} \) converges and
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n.
\]

**Proposition 7.** If \( \{a_n\} \) converges, then \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n. \)

**Proof.** Let \( A = \lim_{n \to \infty} a_n. \) For any \( \epsilon > 0 \), there exists \( N \) such that \( |a_n - A| < \frac{\epsilon}{2} \) or \( A - \frac{\epsilon}{2} < a_n < A + \frac{\epsilon}{2} \) for \( n > N \). For each (fixed) \( n > N \), let \( k = n, n+1, n+2, \ldots \), we have
\[
A - \frac{\epsilon}{2} \leq \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} = c_n \leq b_n = \sup\{a_n, a_{n+1}, \ldots\} \leq A + \frac{\epsilon}{2}.
\]

It follows that
\[
|b_n - A| \leq \epsilon < \frac{\epsilon}{2} < \epsilon \quad |c_n - A| \leq \frac{\epsilon}{2} < \epsilon
\]
for all \( n > N \). By the \( \epsilon - N \) definition,
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = A = \lim_{n \to \infty} c_n = \lim_{n \to \infty} a_n
\]
and hence the result. Q.E.D.

### 3.8 Convex and Concave functions

Economic analysis sometimes requires a maximization and sometimes a minimization problem. Due to their importance in applications, both convex and
concave functions, along with their generalizations to quasiconvex and quasiconcave functions, are discussed frequently in economics. There are many papers devoted to optimization problems in microeconomic theory, most of which are concerned with convex minimization or, equivalently, concave maximization problems.

Convexity plays a crucial role in the study of minimization problems.

Let $X$ be a vector space.

**Definition 1.** [6] A function $f$ from $X$ to $\mathbb{R} \cup \{+\infty\}$ is **convex** if for any convex combination $x = \sum_{i=1}^{n} \lambda_i x_i$ of elements $x_i \in X$ we have the inequality

$$f \left( \sum_{i=1}^{n} \lambda_i x_i \right) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

A function $f$ is **concave** if $-f$ is convex, and that $f$ is **affine** if $f$ is both convex and concave.

**Proposition 1.** [6] Let $f$ be a function from $X$ to $\mathbb{R} \cup \{+\infty\}$. The following conditions are equivalent

a) $f$ is convex;

b) for any $x, y \in X$, and any $\alpha \in ]0, 1[$

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$$

**Definition 2.** [13] A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **quasiconvex** if

$$f(\alpha x + (1 - \alpha)y) \leq \max \{f(x), f(y)\}$$

holds for all $x, y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$.

**Definition 3.** [13] A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **strictly quasiconvex** if the inequality

$$f(\alpha x + (1 - \alpha)y) \leq \max \{f(x), f(y)\}$$

holds for all $x, y \in \mathbb{R}^n$ and $\alpha \in (0, 1)$.
holds for all \( x, y \in \mathbb{R}^n, x \neq y \) and \( \alpha \in ]0, 1[ \).

It is clear that any convex function is a quasiconvex, but the converse is not necessarily true. If \( f \) is quasiconvex then \( -f \) is called quasiconcave.

**Lemma 1.** [13] A function \( f : \mathbb{R}^n \to \mathbb{R} \) is quasiconvex if and only if the set

\[
L_c(f) = \{ x \in \mathbb{R}^n \mid f(x) < c \}
\]

is convex for all \( c \in \mathbb{R} \).

**Lemma 2.** [13] Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a quasiconvex and differentiable function. Then the inequality \( f(x) \leq f(y) \) for \( x, y \in \mathbb{R}^n \) implies that

\[
\langle f'(y), x - y \rangle \leq 0,
\]

where \( f' \) denotes the gradient and \( \langle ., . \rangle \) denotes the scalar product of two vectors.

### 3.9 Utility Functions

A function which assigns numbers to bundles in such a way that a bundle receives a higher number than another if, and only if, the former is preferred to the latter. For some purposes it is convenient to work with "utilities" instead of the actual bundles associated with these utilities. A utility function is a continuous function \( u : X \to \mathbb{R} \) which represents these preferences in the following sense. Such a function may be associated with a partial order \( \geq \) (called the partial order of preferences) as follows

\[
(x_1, y_1) \in \mathbb{R} \times \mathbb{R} \text{ is preferred to } (x_2, y_2) \in \mathbb{R} \times \mathbb{R}
\]

if and only if \( u(x_1, y_1) \geq u(x_2, y_2) \).
Definition 1. A utility function for a preferred relation is a continuous function \( u : X \to \mathbb{R} \) with the property that \( u(x) \geq u(y) \) if and only if \( x \) is preferred to \( y \).

**Theorem 1.** Any preference relation which is reflexive, complete, transitive and continuous can be represented by a continuous utility function.

Note that there will be many utility functions representing a preference relation.

### 3.10 Particular Types of Utility functions

For the particular properties of \( u \) we have

additive utility, i.e.,

\[
u(x_1, x_2, ..., x_l) = u_1(x_1) + ... + u_l(x_l);
\]

differentiable utility;

The property of 'admitting derivatives' in the space \( \mathbb{R}^n \) has been advantageously replaced, by Stolz and Frechet, by the now classical notion of 'differentiability'.[6]

Let \( u \) be a numerical function, with values \( u(x) = u(x_1, x_2, ..., x_l) \), defined in an open set \( G \) of \( \mathbb{R} \). It is said that \( u \) is differentiable in \( G \) if for all \( x = (x_1, x_2, ..., x_l) \) in \( G \) and all \( \Delta x = (\Delta x_1, \Delta x_2, ..., \Delta x_l) \) such that \( x + \Delta x \in G \), we have

\[
u(x + \Delta x) - u(x) = \alpha_1(x)\Delta x_1 + \alpha_2(x)\Delta x_2 + ... + \alpha_l(x)\Delta x_l + \beta(x, \Delta x) \|\Delta x\|,
\]

where the numerical functions \( \alpha_1, \alpha_2, ..., \alpha_l \) are finite and the numerical function \( \beta(x, \Delta x) \) tends to 0 whenever \( \Delta x \) tends to 0 (the point \( x \) remaining fixed).
It is said that $u$ admits a partial derivative with respect to $x_1$ if

$$
\frac{u(x + h\delta_1) - u(x)}{h} = \frac{u(x_1 + h, x_2, ..., x_l) - u(x_1, x_2, ..., x_l)}{h}
$$

tends to a limit when $h$ tends to 0; this limit, which we denote by $u'_1 = \frac{\partial u}{\partial x_1}$, is called the **partial derivative** of $u$ with respect to $x_1$.

concave utility, i.e.,

$$
u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y), \quad 0 \leq \lambda \leq 1.
$$

Any utility function for a convex preference is quasiconcave, i.e., \{x \in \mathbb{R}^l \mid u(x) \geq k\} is convex for all $k$.

### 3.11 Semi-continuous Functions

Let $X$ be a metric space.

Recall that a function $f$ from $X$ to $\mathbb{R} \cup \{+\infty\}$ is **continuous** at a point $x_0$ (which necessarily belongs to the domain of $f$) if, for all $\varepsilon > 0$, there exists $\eta > 0$ such that for any $x \in B(x_0, \eta)$ we have both $\lambda := f(x_0) - \varepsilon \leq f(x)$ and $f(x) \leq f(x_0) + \varepsilon$. Demanding only one of these properties leads to a notion of **semi-continuity** introduced by Rene Baire.

**Definition.** We shall say that a function $f$ from $X$ to $\mathbb{R} \cup \{+\infty\}$ is lower semi-continuous at $x_0$ if for all $\lambda < f(x_0)$, there exists $\eta > 0$ such that

$$
\forall x \in B(x_0, \eta), \lambda \leq f(x).
$$

We shall say that $f$ is lower semi-continuous if it is lower semi-continuous at every point of $X$. A function is upper semi-continuous if $-f$ is lower semi-continuous.
3.12 Mappings. Semi-continuous mappings

Let $X$ and $Y$ be two sets. If with each element $x$ of $X$ there is an associated subset $\Gamma(x)$ of $Y$, it is said that the correspondence $x \rightarrow \Gamma(x)$ is a **mapping** of $X$ into $Y$; the set $\Gamma(x)$ is called the **image** of $x$ under the mapping $\Gamma$. Where no confusion is possible this set is denoted indifferently by $\Gamma x$ or $\Gamma(x)$.

If the mapping $\Gamma$ of $X$ into $Y$ is such that the set $\Gamma x$ always consists of a single element, it is said that $\Gamma$ is a single-valued mapping of $X$ into $Y$.

A topological space $X$ is said to be **compact** if it is separated and if the following axiom is satisfied:

(1) (Borel-Lebesgue axiom). Every family of open sets $(G_i / i \in I)$ forming a covering of $X$, contains a finite covering:

\[
(G_{i_1}, G_{i_2}, \ldots, G_{i_k}).
\]

A compact set $K$ is a subset of a topological space $X$ with $\mathcal{G} = (G_i / i \in I)$ as a topology such that $(K, \mathcal{G}_K)$ is compact; in other words, every family of open sets whose union contains $K$ has a finite sub-family whose union contains $K$.

Let $\Gamma$ be a mapping of a topological space $X$ into a topological space $Y$ and let $x_0$ be a point of $X$. It is said that $\Gamma$ is **lower semi-continuous** at $x_0$ if for each open set $G$ meeting $\Gamma x_0$ there exists a neighbourhood $U(x_0)$ such that

\[
x \in U(x_0) \implies \Gamma x \cap G = \emptyset.
\]

It is said that $\Gamma$ is **upper semi-continuous** at $x_0$ if for each open set $G$ containing $\Gamma x_0$ there exists a neighbourhood $U(x_0)$ such that

\[
x \in U(x_0) \implies \Gamma x \subseteq G.
\]
It is said that the mapping $\Gamma$ is **continuous** at $x_0$ if it is both lower and upper semi-continuous at $x_0$.

If $\Gamma$ is a single-valued mapping, the definition given above for lower semi-continuity coincides with the ordinary definition of continuity; the same is true for upper semi-continuity.

It is said that $\Gamma$ is lower semi-continuous in $X$ if it is lower semi-continuous at each point of $X$. It is said that $\Gamma$ is upper semi-continuous in $X$ if it is upper semi-continuous at each point of $X$ and if, also, $\Gamma x$ is a compact set for each $x$.

If $\Gamma$ is both lower semi-continuous and upper semi-continuous in $X$, then it will be called continuous in $X$.

**Theorem 1.** A necessary and sufficient condition for $\Gamma$ to be lower semi-continuous is that for each open set $G$ in $Y$, the set $\Gamma^- G = \{x \mid x \in X, \Gamma x \cap G \neq \emptyset\}$ is open.

**Proof.** [8]

**Theorem 2.** A necessary and sufficient condition for $\Gamma$ to be upper semi-continuous is that the set $\Gamma x$ is compact for each $x$ and that for each open set $G$ in $Y$ the set $\Gamma^+ G = \{x \mid x \in X : \Gamma x \subset G\}$ is open.

**Proof.** [8]

Mapping of $X$ into $Y$ which are upper semi-continuous have the following fundamental property:

**Theorem 3.** If $\Gamma$ is upper semi-continuous, the image $\Gamma K$ of a compact subset $K$ of $X$ is also compact.

**Proof.** [8]

In addition to the two types of semi-continuity, it is sometimes convenient to consider a third topological property. It is said that $\Gamma$ is a **closed mapping** of $X$ into $Y$ if whenever $x_0 \in X$, $y_0 \in Y$, $y_0 \notin \Gamma x_0$ there exist two
neighbourhoods $U(x_0)$ and $V(y_0)$ such that $x \in U(x_0) \implies \Gamma x \cap V(y_0) = \emptyset$.

It is observed that an immediate consequence of the definition is that if $\Gamma$ is a closed mapping then the set $\Gamma x$ is closed in $Y$.

**Example.** If $f$ is a continuous numerical function in $X \times Y$, the mapping defined by $\Gamma x = \{ y \mid y \in Y, f(x, y) \leq 0 \}$ is a closed mapping of $X$ into $Y$, for, in $X \times Y$, the graphical representation

$$\sum_{x \in X} \Gamma x = \{ (x, y) \mid f(x, y) \leq 0 \}$$

is closed.

### 3.13 Maximum Theorem

**Theorem 1.** If $\phi$ is a lower semi-continuous numerical function in $X \times Y$ and $\Gamma$ is an lower-semi-continuous mapping of $X$ into $Y$ such that, for each $x$, $\Gamma x \neq \emptyset$, the numerical function $M$ defined by

$$M(x) = \sup \{ \phi(x, y) \mid y \in \Gamma x \}$$

is lower semi-continuous.

**Proof.** Suppose that $x_0 \in X$ and let $y_0$ be such that

$$y_0 \in \Gamma x_0; \phi(x_0, y_0) \geq M(x_0) - \varepsilon.$$

There exist neighbourhoods $U(x_0)$ and $V(y_0)$ such that

$$(x, y) \in U(x_0) \times V(y_0) \implies \phi(x, y) \geq \phi(x_0, y_0) - \varepsilon \geq M(x_0) - 2\varepsilon;$$

there exists a neighbourhood $U'(x_0)$ such that

$$x \in U'(x_0) \implies \Gamma x \cap V(y_0) \neq \emptyset.$$

---

16C. Berge, "Topological Spaces", 1963, Oliver and Boyd Ltd. [8]
Therefore
\[ x \in U(x_0) \cap U'(x_0) \implies M(x) \geq M(x_0) - 2\varepsilon. \]

Q.E.D.

**Theorem 2.** If \( \phi \) is an upper semi-continuous numerical function in \( X \times Y \) and \( \Gamma \) is an upper semi-continuous mapping of \( X \) into \( Y \) such that, for each \( x, \Gamma x \neq \emptyset \), the numerical function \( M \) defined by
\[ M(x) = \max \{ \phi(x, y) \mid y \in \Gamma x \} \]
is upper semi-continuous.

*Proof.* Suppose that \( x_0 \in X \); to each \( y \) in \( \Gamma x_0 \) there correspond neighbourhoods \( U_y(x_0) \) and \( V(y) \) such that
\[ (x, z) \in U_y(x_0) \times V(y) \implies \phi(x, z) \leq \phi(x_0, y) + \varepsilon. \]

Since \( \Gamma x_0 \) is compact, it can be covered by a finite number of neighbourhoods of the form \( V(y) \), say \( V(y_1), V(y_2), \ldots, V(y_n) \). Putting \( U'(x_0) = \bigcap_{i=1}^{n} U_{y_i}(x_0) \) and \( V(\Gamma x_0) = \bigcup_{i=1}^{n} V(y_i) \), we have
\[ x \in U'(x_0), \; y \in V(\Gamma x_0) \implies \phi(x, y) \leq \max_i \phi(x_0, y_i) + \varepsilon \leq M(x_0) + \varepsilon. \]
Moreover there exists a neighbourhood \( U(x_0) \) such that
\[ x \in U(x_0) \implies \Gamma x \subset V(\Gamma x_0) \]
and so
\[ x \in U(x_0) \cap U'(x_0) \implies M(x) = \max_{y \in \Gamma x} \phi(x, y) \leq M(x_0) + \varepsilon. \]

Q.E.D.
**Maximum theorem.** If $\phi$ is a continuous numerical function in $Y$ and $\Gamma$ is a continuous mapping of $X$ into $Y$ such that, for each $x$, $\Gamma x \neq \emptyset$, then the numerical function $M$ defined by $M(x) = \max \{\phi(y) / y \in \Gamma x\}$ is continuous in $X$ and the mapping $\Phi$ defined by $\Phi x = \{y / y \in \Gamma x. \phi(y) = M(x)\}$ is a upper-semi-continuous mapping of $X$ into $Y$.

**Proof.** The function $\phi$ is continuous in $X \times Y$ and so $M$ is a continuous numerical function; moreover the mapping $\Delta$ given by

$$\Delta x = \{y / M(x) - \phi(x) \leq 0\}$$

is closed (by the Example following Theorem 3 of the previous section) and hence $\Phi = \Gamma \cap \Delta$ is upper-semi-continuous. Q.E.D.

### 3.14 Kuhn-Tucker Theorem

Consider the general nonlinear programming problem:

$$\max(\min) f_0(x),$$

$$f_i(x) \geq 0, \ i = 1, m, \ x \in C,$$

$f_0(x)$— an objective function.

If we are solving a problem of maximization, then $f_0(x)$ is a concave, if minimization then it is convex. $f_i(x)$— the function of the restrictions of convex function, the set $C$ - convex set, non-functional restriction.

Study of these functions includes a number of steps:

1. Examine a question about an existence of the solution. (For example, by the Weierstrass Theorem, it reaches a min or a max on the compact set.)

2. Identifying the solutions, i.e. getting the necessary or/and (if possible) sufficient conditions for the solution of the problem. Example of
such conditions: if $f_0$ is continuously-differentiated, then the necessary condition is that the gradient equals to zero.

3. Choosing the numerical method for the solution of the problem.

A problem which often occurs in questions of economics is the following:

**MAXIMUM PROBLEM.** Given concave functions $f, g_1, g_2, ..., g_n$ defined in $\mathbb{R}^m$, find a point $x \in \mathbb{R}^m$ such that

1. $g_j(x) \geq 0$ ($j = 1, 2, ..., n$),
2. $f(x)$ is maximal with respect to these constraints.

Let us show that this problem can be reduced to another which is much easier to solve; the method used is a generalization of the well-known method of Lagrange multipliers and is due to Kuhn and Tucker.

Let $y = (y_0, y_1, y_2, ..., y_n)$ be a variable point of $\mathbb{R}^{n+1}$, and let us associate with the above maximum problem a function, called the **Lagrange function**, given by

$$F(x, y) = y_0f(x) + \sum_{j=1}^{n} y_jg_j(x).$$

Now writing $\bar{y} = (y_1, y_2, ..., y_n)$ and $\bar{g}(x) = (g_1(x), g_2(x), ..., g_n(x))$, the Lagrangian function can also be written in the form

$$F(x, y) = y_0f(x) + \langle \bar{y}, \bar{g}(x) \rangle.$$

**LAGRANGE PROBLEM.** Find an $x \in \mathbb{R}^m$ and a $y \in \mathbb{R}^{n+1}$ such that

1. $F(\zeta, y)$ is maximal in $\mathbb{R}^m$ for $\zeta = x$,
2. $\bar{g}(x) \geq 0$, $y \geq 0$, $\sum_{j=0}^{n} y_j = 1$,
3. $\langle \bar{y}, \bar{g}(x) \rangle = 0$.

Whenever the functions under consideration are differentiable, the Lagrange problem reduces to a system of inequalities which can be solved,
since (1) can be replaced by

\[ (1') \quad y_0 \frac{\partial f}{\partial x_i} + \left\langle \frac{\partial g}{\partial x_i}, \bar{y} \right\rangle = 0 \quad (i = 1, 2, \ldots, m). \]

The fundamental result is given in the following theorem:

**Theorem of Kuhn and Tucker.** Let \( f, g_1, g_2, \ldots, g_n \) be concave functions in \( \mathbb{R}^m \) (differentiable or otherwise); for each solution \( x \in \mathbb{R}^m \) of the maximum problem, there exists a \( y \in \mathbb{R}^{n+1} \) such that \((x, y)\) is a solution of the Lagrange problem; for each solution \((x, y)\) of the Lagrange problem with \( y_0 \neq 0 \), the point \( x \) is a solution of the maximum problem.

**Proof.** Let \( x \) be a solution of the maximum problem. By hypothesis, the system

\[
\begin{align*}
g_j(\zeta) & \geq 0 \quad (j = 1, 2, \ldots, n) \\
f(\zeta) & > f(x)
\end{align*}
\]

does not admit a solution \( \zeta \in \mathbb{R}^m \). Therefore, by the first fundamental theorem\(^{17}\) there exist coefficients \( y_0, y_1, y_2, \ldots, y_n \geq 0 \) with sum 1, such that

\[
y_0 [f(\zeta) - f(x)] + \sum_{i=1}^{n} y_i g_i(\zeta) \leq 0 \quad (\zeta \in \mathbb{R}^m),
\]

\(^{17}\) **First Fundamental Theorem** [8]. Let \( C \) be a convex set \( \subset \mathbb{R}^m \) and let \( f_1, f_2, \ldots, f_m \) be convex functions. If the system

\[
\begin{align*}
f_i(x) & \leq 0 \quad (i = 1, 2, \ldots, k) \\
f_i(x) & < 0 \quad (i = k + 1, k + 2, \ldots, m)
\end{align*}
\]

admits no solution \( x \in C \), there exists a function \( f \), given by

\[
f(x) = \sum_{i=1}^{m} p_i f_i(x), \quad \text{where} \quad (p_1, p_2, \ldots, p_m) \in \mathbb{P}_m,
\]

such that

\[
f(x) \geq 0, \quad (x \in C).\]
or

\[ y_0 f(\zeta) + \langle \overline{y}, \overline{g}(\zeta) \rangle \leq y_0 f(x) \quad (\zeta \in \mathbb{R}^m). \]

Putting \( \zeta = x \), we get \( \langle \overline{y}, \overline{g}(x) \rangle \leq 0 \). Hence, since the opposite inequality is also satisfied,

\[ \langle \overline{y}, \overline{g}(x) \rangle = 0. \]

Therefore

\[ y_0 f(\zeta) + \langle \overline{y}, \overline{g}(\zeta) \rangle \leq y_0 f(x) + \langle \overline{y}, \overline{g}(x) \rangle \quad (\zeta \in \mathbb{R}^m). \]

Thus it shows that the Lagrange function \( y_0 f(\zeta) + \langle \overline{y}, \overline{g}(\zeta) \rangle \) is maximal in \( \mathbb{R}^m \) for \( \zeta = x \) and the first part of the theorem is established.

Let us suppose that \((x, y)\) is a solution of the maximum problem. To do this, the argument given above to prove the first part of the theorem will be reversed. By hypothesis,

\[ y_0 f(\zeta) + \langle \overline{y}, \overline{g}(\zeta) \rangle \leq y_0 f(x) \quad (\zeta \in \mathbb{R}^m) \]

or

\[ y_0 [f(\zeta) - f(x)] + \sum_{i=1}^{n} y_i g_i(\zeta) \leq 0 \quad (\zeta \in \mathbb{R}^m). \]

Since \( y_0 > 0 \), the system

\[
\begin{cases}
g_i(\zeta) \geq 0, \\
f(\zeta) > f(x)
\end{cases}
\]

does not admit a solution \( \zeta \in \mathbb{R}^m \) and \( f(x) \) is a solution of the maximum problem.

Example.

Q.E.D.
For the following problem, verify that the Kuhn-Tucker necessary conditions are satisfied at the optimal point:

\[
\text{minimize } z = x_1^2 - 2ax_1 + x_2 \quad (a > 0),
\]

subject to

\[
\begin{aligned}
x_1 + 4x_2 & \leq 2a, \\
x_1 + x_2 & \geq a, \\
x_1, x_2 & \geq 0.
\end{aligned}
\]

**Solution.**

Consider the equivalent problem of maximizing

\[
-z = -x_1^2 + 2ax_1 - x_2 \quad (a > 0).
\]

Obviously, \( x^* = [a, 0] \). The Lagrangian function is

\[
F(x, \lambda) \equiv -x_1^2 + 2ax_1 - x_2 + \lambda_1 (2a - x_1 - 4x_2) + \lambda_2 (a - x_1 - x_2).
\]

Hence

\[
\begin{aligned}
\frac{\partial F}{\partial x_1} & = -2x_1 + 2a - \lambda_1 - \lambda_2, \\
\frac{\partial F}{\partial x_2} & = -1 - 4\lambda_1 - \lambda_2, \\
\frac{\partial F}{\partial \lambda_1} & = 2a - x_1 - 4x_2, \\
\frac{\partial F}{\partial \lambda_2} & = a - x_1 - x_2.
\end{aligned}
\]

\((*)\)

Since the first constraint is passive, we have \( \lambda_1^* = 0 \). Also, from the Kuhn-Tucker theorem,

\[
\frac{\partial F}{\partial x_1(x^*, \lambda^*)} = 0
\]

\((**)\)

which gives \( \lambda_2^* = 0 \). The optimal point \((x^*, \lambda^*)\) is therefore

\[
[x_1^*, x_2^*, \lambda_1^*, \lambda_2^*] = [a, 0, 0, 0],
\]

\((***)\)
and from (*) we now find

\[
\frac{\partial F}{\partial x_2}(x^*, \lambda^*) = -1, \quad \frac{\partial F}{\partial \lambda_1}(x^*, \lambda^*) = a, \quad \frac{\partial F}{\partial \lambda_2}(x^*, \lambda^*) = 0. \tag{****}
\]

It is easily verified that the values given by the equations (*) - (****) satisfy Kuhn-Tucker.

Kuhn-Tucker theory is an extension of the theory of classical optimization to the case where inequality constraints are present.

3.14.1 Definition Of Lagrange Multiplier[9]

Let \( x^* \) be a local minimum. Then \( \lambda^* = (\lambda^*_1, ..., \lambda^*_m) \) and \( \mu^* = (\mu^*_1, ..., \mu^*_r) \) are Lagrange multipliers if

\[
\mu^*_j \geq 0, \text{ for any } j = 1, ..., r,
\]

\[
\mu^*_j = 0, \text{ for any } j \text{ with } g_j(x^*) < 0,
\]

\[
\nabla_x L(x^*, \lambda^*, \mu^*)'y \geq 0, \text{ for any } y \in T_X(x^*),
\]

where \( L \) is the Lagrangian function

\[
L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^r \mu_j g_j(x)
\]

When \( X = \mathbb{R}^n \), then \( T_X(x^*) = \mathbb{R}^n \) and the Lagrangian stationarity condition becomes

\[
\nabla f(x^*) + \sum_{i=1}^m \lambda^*_i \nabla h_i(x^*) + \sum_{j=1}^r \mu^*_j \nabla g_j(x^*) = 0.
\]

CLASSICAL ANALYSIS

- Necessary condition at a local minimum \( x^* \):

\[
-\nabla f(x^*) \in T(x^*)^*.
\]

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Assume linear equality constraints only

\[ h_i(x) = a_i^T x - b_i, \ i = 1, \ldots, m. \]

- Tangent cone

\[ T(x^*) = \{ y \mid a_i^T y = 0, \ i = 1, \ldots, m \}. \]

\( T(x^*)^\ast \) is the range space of the matrix having as columns the \( a_i \), so for some scalars \( \lambda_i^\ast \)

\[ \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^\ast a_i = 0. \]

### 3.15 Convex Analysis and Optimization[9]

#### 3.15.1 Optimization Problems

Optimization problems:

\[
\text{minimize } f(x), \\
\text{subject to } x \in C.
\]

Cost function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), constraint set \( C \), e.g.,

\[ C = X \cap \{ x \mid h_1(x) = 0, \ldots, h_m(x) = 0 \} \]
\[ \cap \{ x \mid g_1(x) = 0, \ldots, g_r(x) \leq 0 \} \]

- Examples of problem classifications:
- Continuous vs discrete;
- Linear vs nonlinear;
- Deterministic vs stochastic;
- Static vs dynamic.
Convex programming problems are those for which $f$ is convex and $C$ is convex (they are continuous problems).

However, convexity permeates all of optimization, including discrete problems.

### 3.15.2 Why Is Convexity So Special In Optimization?

- A convex function has no local minima that are not global.
- A convex set has a nonempty relative interior.
- A convex set is connected and has feasible directions at any point.
- A nonconvex function can be convexified while maintaining the optimality of its global minima.
- The existence of a global minimum of a convex function over a convex set is conveniently characterized in terms of directions of recession.
- A polyhedral convex set is characterized in terms of a finite set of extreme points and extreme directions.
- A convex function is continuous and has nice differentiability properties.
- Closed convex cones are self-dual with respect to polarity.
- Convex, lower semicontinuous functions are self-dual with respect to conjugacy.
3.15.3 Convexity and Duality

- A multiplier vector for the problem minimize \( f(x) \) subject to \( g_1(x) \leq 0, \ldots, g_r(x) \leq 0 \) is a \( \mu^* = (\mu_1^*, \ldots, \mu_r^*) \geq 0 \) such that

\[
\inf_{g_j(x) \leq 0, j = 1, \ldots, r} f(x) = \inf_{x \in \mathbb{R}^n} L(x, \mu^*),
\]

where \( L \) is the Lagrangian function

\[
L(x, \mu) = f(x) + \sum_{j=1}^{r} \mu_j g_j(x), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^r.
\]

- Dual function (always concave)

\[
q(\mu) = \inf_{x \in \mathbb{R}^n} L(x, \mu).
\]

- Dual problem: Maximize \( q(\mu) \) over \( \mu \geq 0 \).

**KEY DUALITY RELATIONS**

- Optimal primal value

\[
f^* = \inf_{g_j(x) \leq 0, j = 1, \ldots, r} f(x) = \inf_{x \in \mathbb{R}^n} \sup_{\mu \geq 0} L(x, \mu).
\]

- Optimal dual value

\[
q^* = \sup_{\mu \geq 0} q(\mu) = \sup_{\mu \geq 0} \inf_{x \in \mathbb{R}^n} L(x, \mu).
\]

- We always have \( q^* \leq f^* \) (weak duality - important in discrete optimization problems).

- Under favorable circumstances (convexity in the primal problem):
- We have $q^* = f^*$.

- Optimal solutions of the dual problems are multipliers for the primal problem.

This opens a wealth of analytical and computational possibilities, and insightful interpretations.

Note that the equality of "sup inf" and "inf sup" is a key issue in minimax theory and game theory.

### 3.15.4 Min-Max Problems[9]

Given $\phi : X \times Z \rightarrow \mathbb{R}$, where $X \subseteq \mathbb{R}^n$, $Z \subseteq \mathbb{R}^m$, under what conditions do we have

$$\sup_{x \in X} \inf_{z \in Z} \phi(x, z) = \inf_{z \in Z} \sup_{x \in X} \phi(x, z)?$$

This equality is called a minimax equality.

Given also a perturbation function $p : \mathbb{R}^m \rightarrow [-\infty, \infty]$

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \{\phi(x, z) - u'z\}, \text{ } u \in \mathbb{R}^m.$$
MIN COMMON/MAX CROSSING FRAMEWORK

Generally, for $q(\mu) = \inf_{u \in \mathbb{R}^m} \{p(u) + \mu' u\}$:

$$\sup_{x \in Z} \inf_{z \in X} \phi(x, z) \leq \sup_{\mu \in \mathbb{R}^m} q(\mu) = q^*$$

$$\leq w^* = p(0) = \inf_{z \in X} \sup_{x \in Z} \phi(x, z).$$
If $X$ is convex and $\phi(\cdot, z)$ is convex for each $z \in Z$, then $M$ is a convex set.

If $Z$ is convex and $\phi(x, \cdot)$ is closed and concave for each $x \in X$, then

$$q(\mu) = \inf_{x \in X} \phi(x, \mu) \text{ for } \mu \in Z.$$ Also

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = q^*.$$ 

**MINIMAX THEOREM I** Assume that:

1. $X$ and $Z$ are convex.
2. $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$ is finite.
3. For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.
4. For each $x \in X$, the function $-\phi(x, \cdot) : Z \to \mathbb{R}$ is closed and convex.

Then, the minimax equality holds if and only if the function $p$ is lower semicontinuous at $u = 0$.

**Proof:** The convexity/concavity assumptions guarantee that the minimax equality is equivalent to $q^* = w^*$ in the min common/max crossing framework.

Furthermore, $w^*$ is finite by assumption, and the set $\overline{M}$ [equal to $M$ and $\text{epi}(p)$] is convex.

By the 1st Min Common/Max Crossing Theorem [D.Bertsekas], we have $w^* = q^*$ iff for every sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \liminf_{k \to \infty} w_k$. This is equivalent to the lower semicontinuity assumption on $p$:

$$p(0) \leq \liminf_{k \to \infty} p(u_k)$$

for all sequences $\{u_k\}$ with $u_k \to 0$. **Q.E.D.**
MINIMAX THEOREM II Assume that:

1. $X$ and $Z$ are convex.
2. $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty$.
3. For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.
4. For each $x \in X$, the function $-\phi(x, \cdot) : Z \to \mathbb{R}$ is closed and convex.
5. 0 lies in the relative interior of $\text{dom}(p)$.

Then, the minimax equality holds and the supremum in $\sup_{z \in Z} \inf_{x \in X} \phi(x, z)$ is attained by some $z \in Z$. [Also the set of $z$ where the sup is attained is compact if 0 is in the interior of $\text{dom}(f)$.]

Proof: [see D. Bertsekas[9]]

EXAMPLE I

Let $X = \{(x_1, x_2) | x \geq 0\}$ and $Z = \{z \in \mathbb{R} | z \geq 0\}$, and let

$$\phi(x, z) = e^{\sqrt{x_1^2 + z}} + x_1,$$

which satisfy the convexity and closedness assumptions. For all $z = 0$,

$$\inf_{x \geq 0} \{e^{-\sqrt{x_1^2 + z}} + zx_1\} = 0,$$

so $\sup_{z \geq 0} \inf_{x \geq 0} \phi(x, z) = 0$. Also, for all $x \geq 0$,

$$\sup_{x \geq 0} \{e^{-\sqrt{x_1^2 + z}} + zx_1\} = \begin{cases} 1 & \text{if } x_1 = 0, \\ \infty & \text{if } x_1 > 0, \end{cases}$$

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so \( \inf_{x \geq 0} \sup_{z \geq 0} \phi(x, z) = 1 \).

\[
p(u) = \inf_{x \geq 0} \sup_{z \geq 0} \left\{ e^{-\sqrt{xz^2}} + z(x_1 - u) \right\}
= \begin{cases} 
\infty & \text{if } u < 0, \\
1 & \text{if } u = 0, \\
0 & \text{if } u > 0.
\end{cases}
\]

**EXAMPLE II**

Let \( X = \mathbb{R} \), \( Z = \{ z \in \mathbb{R} \mid z \geq 0 \} \), and let \( \phi(x, z) = x + z x^2 \), which satisfy the convexity and closedness assumptions. For all \( z = 0 \),

\[
\inf_{x \in \mathbb{R}} \{ x + z x^2 \} = \begin{cases} 
-1/(4z) & \text{if } z > 0, \\
-\infty & \text{if } z = 0,
\end{cases}
\]

so \( \sup_{z \geq 0} \inf_{x \in \mathbb{R}} \phi(x, z) = 0 \). Also, for all \( x \in \mathbb{R} \),

\[
\sup_{z = 0} \{ x + z x^2 \} = \begin{cases} 
0 & \text{if } x = 0, \\
\infty & \text{otherwise},
\end{cases}
\]
so $\inf_{x \in \mathbb{R}} \sup_{z \geq 0} \phi(x, z) = 0$. However, the sup is not attained.

\[ p(u) = \inf_{x \in \mathbb{R}} \sup_{z \geq 0} \{ x + uz^2 - uz \} \]
\[
\begin{cases} 
-\sqrt{u} & \text{if } u \geq 0, \\
\infty & \text{if } u < 0.
\end{cases}
\]
4 A Walrasian equilibrium model

As the first step in studying the existence problem of an equilibrium solution, an exact formulation of a mathematical model that represents of the economy as a whole should be given.

Let us begin with the very simple problem.

The pure exchange problem. Though dressed in a very simple garment, this problem is equipped with all essential features of the existence problem of an equilibrium solution. The general model to be analyzed later is only sophisticated elaboration of this simple model.

Goods are numbered by $j$ ($j = 1, 2, ..., n$). Assume that there are $l$ ($\geq 2$) consumers, numbered by $i$ ($i = 1, 2, ..., l$). Let the amount of good $j$ held by consumer $i$ to be

$$a^i_j \geq 0 \ (j = 1, 2, ..., n). \quad (a1)$$

We further assume that each $i$ holds at least one good in positive amount. Hence, the vector $a^i$ of consumer $i$'s holdings with components (a1) is

$$a^i \geq 0 \ (i = 1, 2, ..., l). \quad (a2)$$

(a2) represents consumer $i$'s holdings of goods.

Let us consider the problem of exchanging these holdings $a^i$ in the market. In this case, we assume that

$$a = \sum_{i=1}^l a^i > 0 \quad (a3)$$

without loss of generality.

Exchange takes place because people wish to increase the satisfaction of their wants by exchanging their current holdings for baskets of goods that

\[18\] H. Nikaido, Introduction to Sets and Mappings in modern economics. [27]
stand higher in their preference orderings. If one's satisfaction of wants is reduced by exchange, he would not participate in the market transactions. For he can be better off simply by consuming his current holdings.

Let i's utility indicator be \( u_i(x) \) that is defined on the positive orthant \( R^n_+ \). It is assumed to be continuous, quasi-concave and strictly increasing on \( R^n_+ \).

Let \( \tilde{x}^i \) be i's holdings after the market exchanges. Obviously, \( u_i(\tilde{x}^i) \geq u_i(a_i) \). The larger \( u_i(\tilde{x}^i) \), the more i would gain from the exchanges.

The ratios of exchange among goods are determined as the inverses of the price ratios, once the prices are given. However, exchanges are not necessarily barter exchanges of one good for another and between one consumer and another. Just as in our daily experiences, a consumer can first sell all his holdings in the market and then buy \( \tilde{x}^i \) in the market out of his proceeds.

Now let \( p = (p_1, p_2, \ldots, p_n)' > 0 \) be a given price vector. Then the market value of consumer i's holdings is

\[
I_i(p) = (p, a^i) > 0, \quad (a4)
\]

which is his income if all his holdings are sold. He attempts to purchase \( \tilde{x}^i \) that will give him the highest utility out of his income. Thus, his most desired purchase \( \tilde{x}^i \) is such as, under the constraint

\[
(p, x) \leq I_i(p), \quad x \geq 0, \quad (a5)
\]

maximizes his utility, i.e.

\[
\max u_i(x) = u_i(\tilde{x}^i). \quad (a6)
\]

Aggregate demand is the sum total of individual demands, i.e. \( \sum_{i=1}^{l} \tilde{x}^i \), while aggregate supply is the sum total of individual holdings, i.e. \( \sum_{i=1}^{l} a^i \).
Exchanges (the act of each consumer selling $a^i$ and purchasing $\hat{x}^i$) are exactly carried out if and only if there is an equilibrium

$$\sum_{i=1}^{l} \hat{x}^i = \sum_{i=1}^{l} a^i \quad (a7)$$

between aggregate demand and supply. The equilibrium equation (a7) does not necessarily hold for an arbitrary price system $p$. For instance, if the price of good $i$ is too low, consumers' demand for it would be so large that aggregate demand $\sum_{i=1}^{l} \hat{x}^i$ exceeds aggregate supply $\sum_{i=1}^{l} a^i$. It is only with an appropriate price vector $\hat{p}$ and appropriate demand vectors $\hat{x}^i$ associated with it that (a5), (a6), and (a7) hold simultaneously. Every consumer's utility can then be maximized under the constraint of his initial holdings. Such a price system $p$ is called an equilibrium price system. Whether there exists such a price system or not cannot be answered by appealing merely to our naive intuition. An unambiguous answer is possible only with detailed mathematical analysis.

A general equilibrium model involving production. A general equilibrium model is constructed by incorporating production possibilities into this model of pure exchanges.

Arrow and Debreu\textsuperscript{19} constructed a model that is typical modern version of the Walrasian general equilibrium system. They synthesized findings of mathematical formulations of the equilibrium system, ranging from Walras and Pareto to Hicks, eliminated classical assumptions inessential to the existence problem (differentiability of utility indicators and production functions) and modernized the system. In order to make it easy to understand, the model that describes an economy consisting of households and firms will be taken.

\textsuperscript{19}K.J. Arrow and G. Debreu, Existence of an equilibrium for a competitive economy, Econometrica 22 (1954),[4]

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In the model of pure exchanges, consumers' holdings of goods are redistributed among consumers without further processing. The model that is to be formulated below allows for the existence of various factors of production (in particular, labor), raw materials and semi-finished goods in addition to consumer goods in the initial holdings of goods.

Household purchase consumer goods out of their income acquired by selling their initial holdings of goods (and that part of income which is profits distributed by firms). Supplies for such consumers' purchases come partly from their own holdings and partly from firms' outputs.
Firms employ workers (i.e. purchase labor from households that hold labor) and purchase raw materials and semi-processed goods from households and firms. They carry out production with these goods as inputs, sell their outputs, pay wages to workers and other expenses to other firms out of their total revenue, and disburse profits as dividends to households. Before giving an exact formulation of this model, let us schematize the exchange flow of goods in fig.2. A thick clockwise arrow represents a flow of goods and a fine counterclockwise arrow denotes a flow of money.

Let us construct our model. In this model, households and firms are called consumer units and producer units respectively.

Suppose that the economy is composed of $l$ consumer units, suffixed by $i$ ($i = 1, 2, ..., l$), and $m$ producer units, suffixed by $k$ ($k = 1, 2, ..., m$). There are $n$ goods, suffixed by $j$ ($j = 1, 2, ..., n$). We enumerate below the assumptions on production and consumption.

1. Production. Each production unit $k$ has a set of processes $Y_k$, which satisfies the following conditions:

   (a) $Y_k$ is a closed convex set containing 0.
   (b) $Y \cap R^m_+ = \{0\}$ for $Y = \sum_{k=1}^{m} Y_k$.
   (c) $Y \cap (-Y) = \{0\}$.

Positive and negative components of a process $y \in Y_k$ represent outputs and inputs respectively. This formulation of $Y_k$ includes a special case where it is represented by an input-output matrix in the form $y = Ax$, $x \geq 0$, where $A$ is the input-output matrix. $1a$ restricts $Y_k$ to be merely a convex set, not necessarily a convex cone, in order to permit us to deal with the case of diminishing returns and the case.
where bottleneck factors constrain the production possibility set. $Y_k$ is thus formulated fairly loosely so that it can represent the production possibility set\(^{20}\) as well as its subsets determined by the constraints of bottleneck factors.

$Y$, of course, denotes the set of processes feasible in the economy as a whole. 1b indicates that this economy is not the land of Cockaigne\(^{21}\) (rejects the unrealistic situation), while 1c shows that no process but for 0 is reversible\(^{22}\) in this economy.

2. **Preference fields.** The preference field $X_i$ of each consumer unit $i$ and the utility indicator $u_i(x)$ satisfy the following conditions:

(a) $X_i = b^i + R^+_i$, i.e. $X_i$ is a parallel translation of the positive orthant by $b^i$.

(b) $u_i(x)$ is continuous.

(c) $u_i(x)$ is quasi-concave.

(d) $u_i(x)$ is strictly increasing.

3. **Distributive ratios of profits.** For each $i$ and $k$, there is a given constant $\alpha_{ik} \geq 0$ such that

$$\sum_{i=1}^{l} \alpha_{ik} = 1 \quad (k = 1, 2, ..., m).$$

\(^{20}\)All points $y$ in $R^n$ that can be expressed in the form $y = Ax$, $x \geq 0$, compose the collection of production possibilities technically available to the economic unit. This is designed as the set $Y$ and called the production possibility set.

\(^{21}\)Cockaigne is an imaginary Utopia in the medieval Western Europe where wine flowed in rivers, houses were made of candles, every good was free and people lived in utmost luxury.

\(^{22}\)For a production process $y \in Y$, $-y$ is a process that completely reverses inputs and outputs of $y$ into outputs and inputs.
When the producer unit \( k \) earns profits \( \pi_k \), the consumer unit \( i \) receives a dividend \( \alpha_{ik} \pi_k \). (a8) shows that \( \pi_k \) is distributed completely among all consumer units.

4. *Initial holdings.* Each consumer unit \( i \) holds \( a_i^j \) units of each good \( j \) \((j = 1, 2, ..., n)\) as its initial holdings. It is assumed that \( a^i \geq b^i \) where \( a^i \) is a vector whose components are \( a^i_j \). In this case it is said, that initial holdings are non-negative. If \( a^i > b^i \), initial holdings are said to be positive.

5. There exists \( \hat{y}^k \in Y_k \) \((k = 1, 2, ..., m)\) such that

\[
\sum_{i=1}^{l} b^i < \sum_{i=1}^{l} a^i + \sum_{k=1}^{m} \hat{y}^k.
\]

This means that the economy as a whole is able to produce positive excess supplies for all goods if initial holdings and outputs are added together.

Let us now define the equilibrium conditions for this general equilibrium model. \([\hat{p}, \bar{x}^1, \bar{x}^2, ..., \bar{x}^l, \hat{y}^1, \hat{y}^2, ..., \hat{y}^m]\), composed of a price vector \( \hat{p} > 0 \), the demand of consumer units \( \bar{x}^i \in X_i \), and the output of producer units \( \hat{y}^k \in Y_k \), is called the equilibrium solution of the general equilibrium model 1-4 when it satisfies the following conditions:

(i) *Profit maximization of producer units.* For each producer unit \( k \),

\[
\pi_k(\hat{p}) = \max(\hat{p}, y) = (\hat{p}, \hat{y}^k) \text{ (the maximum for all } y \in Y_k). \quad (a9)
\]

(ii) *Utility maximization of consumer units.* For each consumer unit \( i \),

\[
\max u_i(x) = u_i(\bar{x}^i) \quad (a10)
\]
subject to the budget constraint
\[
x \in X_i, \ (\tilde{p}, x) \leq (\tilde{p}, a^i) + \sum_{k=1}^{m} \alpha_{ik} \pi_k(\tilde{p}). \tag{a11}
\]

(iii) Equilibrium of demand and supply.
\[
\sum_{i=1}^{l} \tilde{x}^i = \sum_{i=1}^{l} a^i + \sum_{k=1}^{m} \tilde{y}^k. \tag{a12}
\]
4.1 Demand and Supply functions

Since the time of Walras, the problem of equilibrium has traditionally been discussed as that of solving a system of equations with prices as unknowns. These equations are obtained by equating demand and supply functions for individual goods.

Demand and supply equations for the economy as a whole are derived as the aggregates of individual demand and supply functions of economic units (consumer and producer units). These individual demand and supply functions are obtained as solutions of individual optimization problems. We must raise a fundamental question as to whether the problem of maximizing utility or profit can be solved for any given price system. In order to overcome certain difficulties, it is necessary to add modifications to the utility and profit maximization problems conceived of on directly economic grounds. Supply and demand functions are to be derived from them. This procedure will prove very effective in that of presenting the existence proof of an equilibrium solution.

It is very useful to recall the fact that "a continuous function always attains a maximum on a compact set"\(^{23}\). Thus, the next step that we must take is to restrict the domains of the maximum problems within bounded regions by some appropriate methods. For this purpose let us first verify that, if there exist equilibrium solutions, all equilibrium demands \(x^i\) and equilibrium outputs \(y^k\) are located within a certain bounded domain.

Let \(X = \sum_{i=1}^{l} X_i\) and \(Y = \sum_{k=1}^{m} Y_k\). If there is an equilibrium solution \([\bar{p}, \bar{x}^1, \bar{x}^2, ..., \bar{x}^l, \bar{y}^1, ..., \bar{y}^m]\), it satisfies condition (iii) of (a12) in the preceding

\(^{23}\)H. Nikaido, *Introduction to Sets and Mappings in modern economics.* [27]
section, now we take a relation

\[ a + \sum_{k=1}^{m} y^k \geq \sum_{i=1}^{l} x^i \quad \text{where} \quad a = \sum_{i=1}^{l} a^i, \] (b1)

and consider all \((l + m)\)-tuples that satisfy (b1), i.e.

\[ [x^1, x^2, ..., x^l, y^1, y^2, ..., y^m], \quad x^i \in X_i, \ y^k \in Y_k. \] (b2)

Let \( \tilde{X}_i \) denote the set of those points in \( X_i \) that appear as \( x^i \) in any of the \((l + m)\)-tuples (b2). \( \tilde{Y}_k \) similarly represents the set of those points in \( Y_k \) that appear as \( y^k \) in any of the \((l + m)\)-tuples (b2). Thus, we have

\[ \tilde{X}_i = X_i \cap (a + Y - \sum_{s \neq i} X_s - R^n_+) \quad (i = 1, 2, ..., l) \] (b3)

and

\[ \tilde{Y}_k = Y_k \cap (X - a - \sum_{l \neq k} Y_l + R^n_+) \quad (k = 1, 2, ..., m). \] (b4)

If there exists an equilibrium solution, then obviously we have

\[ \tilde{x}^i \in \tilde{X}_i \text{ and } \tilde{y}^k \in \tilde{Y}_k. \]

Let us now examine the properties\(^{24}\) of these sets.

(i) \( \tilde{X}_i \) and \( \tilde{Y}_k \) are non-empty sets.

(ii) \( \tilde{X}_i \) and \( \tilde{Y}_k \) are convex sets.

(iii) \( \tilde{X}_i \) and \( \tilde{Y}_k \) are bounded sets.

\( \tilde{X}_i \subset X_i \) and \( \tilde{Y}_k \subset Y_k. \)

**Construction of demand and supply functions.**

We introduce a set in \( R^n \) that is called a hypercube. This set is given by

\[ E = \{ x | x \in R^n, \ c_j \leq x_j \leq d_j \quad (j = 1, 2, ..., n) \} \] (c1)

\(^{24}\)See proof in H. Nikaido, *Introduction to sets and mappings in modern economics.*[27]
where $c_j$ and $d_j$ are constants subject to $c_j < d_j$. $E$ is a convex set. It is also compact. Any bounded set can be included within the interior of a hypercube $E$ if $E$ is sufficiently large.

$\tilde{Y}_k (k = 1, 2, ..., m)$ are bounded. Let us select a hypercube $E$ large enough to include all of them in its interior. In other words,

$$E^0 \supset \tilde{Y}_k (k = 1, 2, ..., m). \quad (c2)$$

Form an intersection $Y_k \cap E$ for each $k$. It is not an empty set because of (c2). It is compact, convex set$^{25}$.

We now define supply function by restricting the domain $Y_k$ of the profit maximization problem to be $Y_k \cap E$. This restriction enables us to relax the condition on the price vector from $p > 0$ to $p \geq 0$. Unless otherwise noted, the price vector is assumed to be $p \geq 0$.

The supply function $\psi^k(p)$ and the profit function $\pi_k(p)$ of the producer unit $k$ are defined as

$$\psi^k(p) = \{y^k | \max(p, y) = (p, y^k) \text{ for } y \in Y_k \cap E\} \quad (c3)$$

and

$$\pi_k(p) = \max(p, y) \quad (\text{subject to } y \in Y_k \cap E). \quad (c4)$$

As $Y_k \cap E$ is compact and as the function $(p, y)$ is continuous in $y$, we know that there always exists a maximum and that (c3) and (c4) are consistently defined for any $p$. $\pi_k(p)$ is an ordinary, single-valued function, but there are in general many maximizers $y^k$ of $(p, y)$ so that $\psi^k(p)$ is a set-valued function.

Let us form another hypercube in order to define demand functions. Let (c1) be the hypercube $E$ that was already selected. Let $d$ be a vector whose

$^{25}$As $Y_k$ is closed and $E$ is compact, their intersection is compact. As $Y_k$ and $E$ are convex, their intersection is convex.
components are $d_j$. Select a vector $h^i$ that satisfies

$$h^i > a^i + m\delta$$  \hspace{1cm} (c5)

and

$$h^i > x \text{ (for any } x \in \tilde{X}_i) .$$  \hspace{1cm} (c6)

As $\tilde{X}_i$ is bounded, this procedure is permissible.

We have $b^i \in \tilde{X}_i$, then $b^i < h^i$ by (c6). Let us take this $h^i$ and form a hypercube

$$E_i = \{ x \mid x \in R^n, b^i \leq x \leq h^i \} \quad (i = 1, 2, ..., l).$$  \hspace{1cm} (c7)

Thus, it immediately follows that

$$X_i \supset E_i \supset \tilde{X}_i \quad (i = 1, 2, ..., l).$$  \hspace{1cm} (c8)

The demand function of the consumer unit $i$ is defined by

$$\varphi^i(p) = \left\{ x^i \mid \max u_i(x) = u_i(x^i) \quad \text{for} \quad x \in E_i, \quad (p, x) \leq (p, a^i) + \sum_{k=1}^{m} \alpha_{ik} \pi_k(p) \right\} .$$  \hspace{1cm} (c9)

As $Y_k \cap E \ni 0$, we have $\pi_k(p) \geq 0$. Further, $\alpha_{ik} \geq 0$, so that

$$\sum_{k=1}^{m} \alpha_{ik} \pi_k(p) \geq 0 .$$

Thus, by setting $x = a_i$, we see that the budget constraint

$$(p, x) \leq (p, a^i) + \sum_{k=1}^{m} \alpha_{ik} \pi_k(p)$$  \hspace{1cm} (c10)

$$\frac{\sum_{m} m d = d + \ldots + d}{m} .$$

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is fulfilled. On the other hand, as \( a^i \in E_i \), the domain of this maximization problem is not an empty set. As this domain is an intersection of a hyper-cube and a half-space, it is compact. \( u_i(x) \) is continuous and has always a maximum. (c9) is defined consistently for any \( p \geq 0 \). (c9) is, in general, a set-valued function.

Individual demand and supply functions have now been defined. Thus, the aggregate demand function is given by

\[
\varphi(p) = a + \sum_{i=1}^{l} \varphi^i(p),
\]

while the aggregate supply function is determined by

\[
\psi(p) = a + \sum_{k=1}^{m} \psi^k(p).
\]

Both of them are in general set-valued functions.

*Equilibrium of demand and supply functions.* The state of equilibrium in the system is expressed by equation "demand=supply":

\[
\varphi(p) = \psi(p).
\]

But in the general case, the equilibrium is expressed by

\[
\varphi(p) \cap \psi(p) \neq \emptyset,
\]
which means that \( \varphi(p) \) and \( \psi(p) \) contain a common element for some price vector \( p \). (c14) is equivalent to

\[
\psi(p) - \varphi(p) \ni 0,
\]
which states that the vectorial difference of \( \psi(p) \) and \( \varphi(p) \), i.e. the excess supply function \( \psi(p) - \varphi(p) \), contains 0. In the case of single-valued functions, (c14) and (c15) reduce to (c13).
The Walras Law.

Individual demand and supply functions represent the schedules of economic units at given prices $p$. These schedules are based on individuals’ decisions made independently of those of other economic units. But there are certain indirect mutual constraints on them. We now state this point in a clear-cut form.

The consumer unit $i$ decides on his demand programme $x^i$ so as to maximize $u_i(x)$ subject to a budget constraint (c10) within the hypercube $E_i$. $\varphi^i(p)$ is the set of all demand programmes. Though (c10) is an inequality constraint, the equality sign must hold for points $x^i$ of $\varphi^i(p)$. To demonstrate this, assume that

\[ (p, x) < (p, a^i) + \sum_{k=1}^{m} \alpha_{ik} \pi_k(p). \]  \hspace{1cm} (c16)

As

\[ \pi_k(p) = (p, y^k) \quad \text{(for any } y^k \in \psi^k(p)), \]  \hspace{1cm} (c17)

we have

\[ \sum_{k=1}^{m} \alpha_{ik} \pi_k(p) = \sum_{k=1}^{m} \alpha_{ik} (p, y^k) = \left( p, \sum_{k=1}^{m} \alpha_{ik} y^k \right). \]

As $y^k, 0 \in Y_k \cap E$ where $Y_k \cap E$ is a convex set and as $0 \leq \alpha_{ik} \leq 1$,

\[ \alpha_{ik} y^k = (1 - \alpha_{ik}) 0 + \alpha_{ik} y^k \in Y_k \cap E. \]

Hence, $\alpha_{ik} y^k \leq d$ so that we get

\[ a^i + \sum_{k=1}^{m} \alpha_{ik} y^k < h^i \]

because of (c5). Taking its inner product with $p \geq 0$, we have

\[ \left( p, a^i + \sum_{k=1}^{m} \alpha_{ik} y^k \right) < \left( p, h^i \right), \]

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from which it follows that

\[(p, a^i) + \sum_{k=1}^{m} \alpha_{ik} \pi_k(p) < (p, h^i).\] (c18)

Because \(x^i \in E_i\), we have \(h^i \geq x^i\). As \(h^i \neq x^i\) from (c16) and (c18), we see that \(h^i \geq x^i\). Hence,

\[u_i(x) > u_i(x^i)\] (c19)

holds for such \(x\) as those we have referred to above. On the other hand, if such \(x\) is sufficiently close to \(x^i\), it satisfies the budget constraint (c10) because of its continuity in view of (c16) which is a strict inequality. This contradicts the fact that \(x^i\) maximizes utility. Hence, the equality sign must hold in (c16) so that

\[(p, x^i) = (p, a^i) + \sum_{k=1}^{m} \alpha_{ik} \pi_k(p) \ (i = 1, 2, ..., l).\] (c20)

Now, aggregate (c20) and take (c17) into consideration. Then, because of

\[\sum_{i=1}^{l} \alpha_{ik} = 1.\]

we get

\[
\left( p, \sum_{i=1}^{l} x_i \right) = \left( p, \sum_{i=1}^{l} a_i \right) + \sum_{k=1}^{m} \pi_k(p) = \\
= \left( p, \sum_{i=1}^{l} a_i \right) + \left( p, \sum_{k=1}^{m} y_k \right) = \left( p, \sum_{i=1}^{l} a_i + \sum_{k=1}^{m} y_k \right).
\]

From this, we observe that

\[(p, x) = (p, y) \ (\text{for any } p \geq 0, \ x \in \varphi(p), \ y \in \psi(p)).\] (c21)

(c21) reveals the identity of revenue and expenditure and is called the Walras law. This shows that revenue arising from the supply of goods (production and sales of initial holdings) is entirely spent on purchases of goods.
It is an important relation representing complete circular flows of income. It should particularly be noted that (c21) is an identity holding for any $p$. When the demand and supply functions are single-valued, it reduces to an identity

$$(p, \varphi(p)) = (p, \psi(p))$$

for $p \geq 0$.

The Walras law is one of the important keys for solving the existence problem of an equilibrium solution.

**NB. Single-valued function.** If the mapping $F$ of $X$ into $Y$ is such that the set $F(x)$ always consists of a single element, we say that $F$ is a *single-valued function* or a *single-valued mapping* of $X$ into $Y$.

*Multi-valued function* or mapping, or a point-to-set mapping associates with each point a set.
5 Existence of Equilibrium Transient Processes in Economical Mechanism Change\textsuperscript{27}

In the following work the general equilibrium theory is applied to model the transition from a centralized (budget-controlled) economy to a competitive market and to prove the existence of an equilibrium transition process that, in a certain sense, makes it possible for agents to "adapt" the choice of technology to the change in economic mechanism.

5.1 Introduction

The objective of this study is to model an economy with a finite number of agents who produce or consume certain types of goods (products) and services. Each consumer has his own system of preferences (utility functions), and a producer is characterized by a feasible technological set (the set of input-output pairs). All products are assumed to be comeasurable, i.e., to have prices, and every agent at each point in time has a certain amount of money at his disposal (a budget).

The analysis is conducted within the framework of dynamic equilibrium\textsuperscript{28}

\textsuperscript{27} V.I. Arkin and A. D. Slastnikov (Moscow). An Equilibrium Model of Transition from a Centralized Economy to a Competitive Market, Spring 1995, MATEKON, pp. 27-47.

\textsuperscript{28} A dynamic equilibrium may be defined as a state which each agent (weakly) prefers to that which would result from any unilateral move. The dynamic equilibrium (state to which there is a movement) is an extension of the concept of competitive equilibrium (this concept is static, in that it does not involve any laws of movement) in that it allows
models. However, unlike most studies in this field, the budget constraints on consumers as well as producers are imposed. This enables to link various principles of agent budget formation to various principles of agent budget formation to various functioning mechanisms of the economy, and to consider transition from one economic mechanism to another.

Two economic mechanisms were examined. The first assumes the existence of a central planning authority ("the center") and is identified with state control of the economy. The functioning of the system in this case is described as follows.

Suppose that the producers provide a certain output at instant t. The center fixes the price of this output and determines the total budget for the economy as the cost for all goods manufactured at instant t. This budget is then allocated to consumers and producers in accordance with some central priorities. A consumer uses his budget to purchase at given prices a certain bundle of goods that maximize his economic utility. A producer selects a production process (an input-output pair, where, as usual, inputs (costs) are incurred at time t and outputs are produced at time t + 1) by purchasing, subject to budget constraints, a vector of inputs at the prices prevailing at instant t so as to maximize the value of the output at the prices of the next instant (t + 1). The same cycle is then repeated.

In the second mechanism, the system functions without a central authority and the allocating functions of the center are assumed by a competitive market. The income of the producer at instant t includes only money received

_all commodities to be monopolistic, or more importantly, some to be monopolistic and some competitive. It is also an extension in the sense that it is more likely to exist: every 'economy' which has a competitive equilibrium has a dynamic equilibrium, but not conversely, though of course this only has meaning for economies which can be compared in both forms._

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from the sale of his own products manufactured (and sold) at that instant at prevailing market prices. This income is subdivided into two parts. One is distributed to consumer-shareholders in the form of dividends on shares in the given enterprise. The other part is reinvested in resources for the next production cycle. The production process is chosen so as to maximize income by the end of the next period. The consumer budget (income) is equal to the sum of dividends received from enterprises in which the consumer owns shares, and consumer demand is driven by maximization of the utility function subject to given income.

Let us now assume that at time $t < 0$ the system functions in accordance with the centralized mechanism and at time $t = 0$ a decision is made to introduce radical changes associated with transition to a competitive market. Our objective is to construct a transition process with a number of "good" properties.

Let us consider some alternative transitions. The first is an unexpected instantaneous transition ("shock") in which all agents who operate at time $t$ within the framework of a centralized economy find themselves at time $t + 1$ within the framework of a competitive market. Since at time $t$ the agents do not anticipate the impending transition and orient their actions to budget financing, some producers might have zero (or very small) income at time $t + 1$, which automatically forces them to halt production (leading to bankruptcy). In this case, the consumer can avoid a zero budget only if he owns shares of producer enterprises with nonzero income or with some assets. Thus, establishment of equilibrium in a shock transition will cause agent bankruptcies.

The second alternative is also an instantaneous transition, but one declared in advance. In other words, the agents know with certainty that at
time $t+1$ the system will take on the framework of a competitive economy. In this case, the system may fail to reach an equilibrium, for instance, because of wild speculative demand and demand for products not manufactured under the centralized economy. Note that both instantaneous transition alternatives as a rule involve an abrupt increase in prices.

Let us now consider the case of a gradual transition, which is the object of the study.

Qualitatively, this alternative can be described as follows. The state (the center) at instant $t=0$ declares the possibility of a future change in the economic mechanism without fixing a specific time. From the agent’s point of view, the change point is a random variable. Thus, at each instant $t$ there is a certain probability that the economic mechanism will change at $t+1$. Allowing for this probability, producers naturally choose a technology in interval $(t, t+1)$ by considering both "state" and market prices at time $t+1$. The main assumption of producer behavior is that producers choose a technology that maximizes their income at weighted-average prices, the "weights" being equal to the conditional probabilities of a change in the economic mechanism at the next instant. The assumed transition process is free from the above-mentioned shortcomings of instantaneous transition and, under appropriate conditions, ensures equilibrium paths with positive budgets for all agents. Note that the proposed alternative also provides gradual adjustment (adaptation) of producer plans to market prices.
5.2 Description of the system

Let us consider a closed economic system that functions during discrete time $t = 0, 1, ..., T$, whose states are characterized by bundle $l$ of various goods (products). The finite number of agents are divided into the set of producers ($I$) and the set of consumers ($J$). The same agent can simultaneously act as a producer and consumer of a certain bundle of products.

The technological opportunities of the producers, which will be denoted by index $i, i \in I$, are described in period $t, t + 1$ (denoted $t$ in what follows) by set $Q_t^l \subseteq \mathbb{R}_+^l \times \mathbb{R}_+^l$ of input-output pairs $(x, y)$. Inputs $x$ are incurred at time $t$, and output $y$ is produced at time $t + 1$.

The consumers (indexed by $j, j \in J$) are characterized at time $t$ by their utility functions, $u^l_t(c)$, defined on the set of consumer goods and services, $C_t^l \subseteq \mathbb{R}_+^l$.

As mentioned above the set of assumptions is necessary in a model to make it an understandable depiction of the reality. With domain assumptions, the model is asserted to be an accurate depiction of reality only within the assumed domain.\textsuperscript{29}

A formal model of the competitive economy, presented in the form of series of axioms and assumptions, was developed in the 1950s. It was intended that the axioms should be interpretable to apply to real economic systems. However, as a formal mathematical model the implications of the axioms could be developed independently of the applications. The selection of the axioms and assumptions was influenced by the possibility of making useful interpretations, but also with the facility with which results

\textsuperscript{29}Economic Models and Methodology, Randall G. Holcombe, Contributions in Economics and Economic History, Number 99, Greenwood Press.[19]
can be derived.

Two closely related sets of assumptions were developed. The assumptions are made for the consumption sector and for the production sector. Also assumptions made to relate the consumption and the production sectors.\textsuperscript{30}

Let us make the following assumptions:

1. sets $Q_i^t$ are unbounded in all coordinates, convex, closed, contain point $(0,0)$, and are locally bounded (i.e., for any bounded $A$ set $\{y : (x, y) \in Q_i^t, x \in A\}$ is bounded);

Convexity of $Q_i^t$ is equivalent to the production set being generated by linear activities. It means that if $y$ and $y'$ are producible, that is, elements of $Q_i^t$, then $\alpha y + \beta y'$ is also producible, that is, an element of $Q_i^t$, for any non-negative numbers $\alpha$ and $\beta$. Thus producible goods are divisible. Closedness is needed for the compactness of the feasible set. Assumption that $Q$ contains point $(0,0)$ is not restrictive. It is a recognition that goods which are never scarce are irrelevant to problems of economizing.\textsuperscript{31}

2. sets $C_i^j$ are convex, closed, and $0 \in C_i^j$;

Convexity of $C_i^j$ implies that a good is divisible if someone can consume it in more than one quantity. $C_i^j$ bounded below means that the consumer is not able to supply an indefinite quantity of any good. Closedness and boundness are needed to provide compact feasible sets.\textsuperscript{32}


\textsuperscript{31}see: The New Palgrave, General Equilibrium.[36]

\textsuperscript{32}see: The New Palgrave, General Equilibrium.[36]
3. nonnegative functions \( u_i^j \) are continuous and quasiconcave (i.e.,

\[
  u_i^j(\alpha c^1 + (1 - \alpha)c^2) \geq \min \{ u_i^j(c^1), u_i^j(c^2) \},
\]

for any \( c^1, c^2 \in C_i^j \), and \( 0 \leq \alpha \leq 1 \).

By convention, for vectors \( x^k = (x_1^k, x_2^k, ..., x_n^k) \), \( k = 1, 2 \), inequality \( x^1 \geq x^2 \) implies \( x_i^1 \geq x_i^2 \) for any \( i \); \( x^1 > x^2 \) implies \( x_i^1 \geq x_i^2 \) and \( x^1 \neq x^2 \); and \( x^1 \gg x^2 \) implies \( x_i^1 > x_i^2 \) for any \( i \).
5.3 A model of a Centralized budget-controlled economy

Let us now describe the behaviour of agents in an economic system with centralized budget control. At the beginning of each time period, the center sets agent budget on the total income of the entire system during the preceding period. The agents then operate independently during one time interval, but subject to their budget constraints. This is a model of a centralized economy (CE).

Suppose that $p_t$ is the nonnegative vector of product prices at time $t$, when producers and consumers are allocated budgets $\rho^i_t$, $i \in I$, and $\pi^j_t$, $j \in J$. The producer problem is to maximize income (the value of the output) at the end of the current time interval; the consumer problem is to maximize current utility of consumption. In both cases, maximization is carried out subject to corresponding budget constraints, i.e.,

$$ p_{t+1} y \rightarrow \max, $$

$$(x, y) \in Q^i_t, \quad p_t x \leq \rho^i_t, \quad i \in I, $$

$$ u^i_t(c) \rightarrow \max, $$

$c \in C^j_t, \quad p_t c \leq \pi^j_t, \quad j \in J, $$

where prices $p_{T+1}$ are given.

The budget allocation mechanism in this model is allocation $\rho^i_t(K) = \alpha^i_t K_t$, $\pi^j_t(K) = \beta^j_t K_t$, where the coefficients $\alpha^i_t$ and $\beta^j_t$ characterize the "priorities" of the agents (from the point of view of the Center), and $K_t$ is the total income in the system at time $t$.

The system functions as follows. At time $t = 0$, given initial state $(y^i_0, i \in I)$, the center determines total budget $K_0 = \sum_i p_0 y^i_0$ and thus also $\rho^i_0$ and
\( \pi^t_0 \). Given these respective budgets, the agents solve problems (1) and (2) at \( t = 0 \). Solutions \((x^0_t, y^0_t)\) are used to construct the total income at the next time instant, \( K_1 = \sum_i p_i y^1_i \), to set budgets \( \rho^1_t \) and \( \pi^1_t \), and so on.

An equilibrium in the CE model on finite time interval \( T \) with initial state \((y^0_i, i \in I)\), is tuple \(^{33}\) \(((\tilde{x}^t_i, y^t_{i+1}), \tilde{c}^t_i, \tilde{p}_t, 0 \leq t \leq T, i \in I, j \in J)\), where \((\tilde{x}^t_i, y^t_{i+1}) \in Q^t_i, \tilde{c}^t_i \in C^t_i, \) and \( \tilde{p}_t > 0 \) for \( 0 \leq t \leq T \) satisfy the following conditions:

\[ \tilde{p}_{t+1} y_{t+1} \geq \tilde{p}_{t+1} y \text{ for any } (x, y) \in Q^t_i : \tilde{p}_t x \leq \tilde{p}_t, \quad \tilde{p}_{T+1} = p_{T+1}. \]  

\[ u^t_i(\tilde{c}^t_i) \geq u^t_i(c) \text{ for any } c \in C^t_i : \tilde{p}_t c \leq \tilde{p}_t, \]  

\[ \tilde{p}_t = \alpha^t_i \tilde{K}_t, \tilde{\pi}_t = \beta^t_i \tilde{K}_t, \tilde{K}_t = \sum_i \tilde{p}_t \tilde{y}^t_i, \quad \tilde{y}_0 = y^t_0, \]  

\[ \sum_j \tilde{x}^t_i + \sum_i \tilde{x}^t_i = \sum_i \tilde{y}^t_i. \]

Relationships (3) and (4) imply that \((\tilde{x}^t_i, \tilde{y}^t_{i+1})\) and \(\tilde{c}^t_i\) are solutions of the producer and the consumer problem, respectively, given system of prices \(\tilde{p}_t\) and budgets \(\tilde{p}_t\) and \(\tilde{\pi}_t\); relationships (5) indicate that agent budgets are allocated from the total income at the end of the preceding period; relationship (6) is the condition of material balance.

To prove the existence of an equilibrium in this model, we need some additional (not overly restrictive) assumptions that must hold for all \( 0 \leq t \leq T \):

(P1) There exist technologies \(({0,x}^t_0, {0,y}^t_{i+1}) \in Q^t_i\) such that \(\sum_i {0,y}^t_{i+1} \gg 0\) (at every time instant each produced by one of the producers).

\(^{33}\)Tuple - a finite sequence, that the order in which the elements appear is essential.
(P2) For any $i \in I$ and $(x, y) \in Q_i^t$ there exists technology $(x', y') \in Q_i^t$ such that $y' > y$ (nonsatiation of producers).

(C1) For any $s$, $1 \leq s \leq l$, there exists $j \in J$ such that function $u_s^j(c_1, ..., c_l)$ is strictly monotonic in $c_s$ (nonsatiation of consumption of each product).

(C2) For any $j \in J$ and $c \in C^j_i$ there exists $c' \in C^j_i$ such that $u_i^j(c') > u_i^j(c)$ (nonsatiation of consumers).

(D) $\alpha^j_i, \beta^j_i > 0$, $\sum_i \alpha^j_i + \sum_j \beta^j_i = 1$.

**Satiation.** When an equilibrium price prevails, some consumer, because the prices of his resources are very high, may be able to purchase a consumption bundle such that he is satiated. (The nonsatiation assumption is needed to establish the lower semicontinuity of the demand function). Every consumer is nonsatiated in his consumption set. This is a strong assumption.

**Definition (nonsatiation):** The $i$th consumer is said to be nonsatiated at the point $\hat{x}_i$ if there exists an $x_i \in X_i$ such that $x_i$ is preferred to $\hat{x}_i$.

**Theorem (1).** Assume that the conditions (P1)-(D) are satisfied. Then for any initial states $(y^0_i, i \in I)$, such that $\sum_i y^0_i > 0$, the CE model has an equilibrium with positive prices ($\hat{p}_i > 0$).

To prove this and similar theorems, we use the following generalization of Gale's lemma\(^{34}\).

According to this lemma, mapping $F : X \to 2^Y$, where $X$ and $Y$ are finite-dimensional sets, is called standard if it is convex-valued, closed (i.e., $x^n \to x \in X$, $y^n \to y \in Y$, $y^n \in F(x^n)$ implies $y \in F(x)$), and maps compacta from $X$ into compacta.

\(^{34}\)See Appendix 1 for the original formulation of the Gale's lemma.
Suppose that $\sigma = \{x = (x_1, ..., x_I) \in \mathbb{R}_+^I : |x| := x_1 + ... + x_I = 1\}$ is the unit simplex in $\mathbb{R}_+^I$, $\Delta = \text{int} \sigma = \{x \in \sigma : x \gg 0\}$, and $P_m = \Delta \times \cdots \times \Delta$ is the Cartesian product of $m$ open simplexes $\Delta$.

**Theorem (2)**. Let $F : P_m \to 2^{\mathbb{R}^m}$ be a standard mapping, and assume that the following conditions hold:

1. for any $p_t^n \in \Delta$, $p_t^n \to p_t$, $1 \leq t \leq m$, where $p_r \in \sigma \setminus \Delta$ for some $r$, and $x^n = (x_1^n, ..., x_m^n) \in F(p_1^n, ..., p_m^n)$, there exist $t$ and $s$, $1 \leq t \leq m$, $1 \leq s \leq I$, such that $p_{t,s} = 0$ and $\limsup_n x^n_{t,s} > 0$;

2. $p_t x_t = 0$ for any $p = (p_1, ..., p_m) \in P_m$, $(x_1, ..., x_m) \in F(p)$, $1 \leq t \leq m$.

Then $0 \in F(P_m)$, i.e., $0 \in F(\widehat{p})$ for some $\widehat{p} \in P_m$.

**Proof (of the Theorem 1).**

We first note that, without loss of generality, we can assume that sets $Q_t^i$ and $C_t^i$ are bounded. Indeed, let $(\widehat{p}_t, (x_t^i, \widehat{y}_{t+1}^i), (\widehat{c}_t^i), i \in I, j \in J, 0 \leq t \leq T)$ be an equilibrium with initial point $(y_0^i, i \in I)$. Then, by (6), we have

\[
\begin{align*}
\widehat{x}_0^i & \leq \sum_i y_0^i, \quad \widehat{c}_0^i \leq \sum_i y_0^i, \quad i \in I, \quad j \in J, \\
\widehat{y}_t^i & \leq b_t, \quad \widehat{x}_t^i \leq d_t, \quad \widehat{c}_t^i \leq d_t,
\end{align*}
\]

for some $b_t, d_t \gg 0$ that exist by local boundedness of technologies $Q_t^i$. Thus, there exist vectors $\widehat{x}$ and $\widehat{y}$ such that $x_t^i \leq \widehat{x}$ and $\widehat{y}_{t+1}^i \leq \widehat{y}$ for all $t$ and $i$. It

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35Simplexes, Open simplexes - Ref.: Mathematical Foundation, Simplexes.
36Proof of the Theorem 2 for $m = 1$ can be found in Appendix 2.
therefore suffices to consider only technologies \( \bar{Q}_i = \{(x, y) \in Q_i : \hat{x}, y \leq \hat{y}\} \) and sets of consumer goods \( \bar{C}_i = \{c' \in C_i : c \leq \bar{c}\} \), taking vector \( \bar{c} \) such that \( \bar{c} \gg \hat{y}N \),

\[ (7) \]

where \( N \) is the total number of producers.

We now construct the excess demand function (EDF)\(^{37}\). From the structure of producer and consumer problems (1) and (2) and the agent budgets, we can see that it suffices to consider prices \( p_t \) from simplex \( \sigma \). By \( P = \Delta \times \ldots \times \Delta \) we denote the product of \( T + 1 \) open simplexes. For \( p = (p_0, \ldots, p_T) \in P \) we denote \( p^t = (p_k, 0 \leq k \leq t) \). Take arbitrary \( p = (p_0, \ldots, p_T) \in P \) and construct the EDF successively, step by step.

Let \( K_0 = \sum_i p_i \psi_0(p^0) = \alpha_0^i K_0 \) and \( \pi_0^i(p^0) = \beta_0^i K_0 \), and define \( \varphi_0^i(p^1) \) as set of solutions \( (\bar{x}_0^i, \bar{y}_0^i) \) of producer problem (1) when \( t = 0 \) and prices \( p^1 \). Similarly, \( \varphi_0^i(p^0) \) is the set of optimal solutions of consumer problem (2) when \( t = 0 \) and prices \( p^0 \). Now let \( K_1 = \sum_i p_i \bar{y}_0^i \), where

\[ \begin{align*}
(\bar{x}_0^i, \bar{y}_0^i) & \in \psi_0^i(p^1), \\
\rho_0^i(p^1) & = \alpha_0^i K_1,
\end{align*} \]

and

\[ \pi_0^i(p^1) = \beta_0^i K_1. \]

Find \( \psi_1^i(p^2) \) and \( \varphi_1^i(p^1) \) as in the first step. Continuing this process to successively compute total income \( K_t = \sum_i p_i \bar{y}_t^i \), where \( (\bar{x}_{t-1}^i, \bar{y}_t^i) \in \psi_{t-1}^i(p^t) \), budgets \( p_t^i(p^t) \) and \( \pi_t^i(p^t) \), \( t \leq T \), and so on.

The EDF is now defined as \( \chi(p) = (\chi_0, \ldots, \chi_T) \), where

\[ \chi_t = \sum_j \bar{c}_j^i + \sum_i \bar{x}_t^i - \sum_i \bar{y}_t^i, \bar{c}_j^i \in \varphi_t^i(p^t), \]

\[ \text{An excess demand function shows the relation for the difference between quantities demanded and quantities supplied.} \]

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and
\[(x_t, y_{t+1}) \in \psi_i(p^{t+1}), \ 0 \leq t \leq T.\]

Let us show that \(\chi(p)\) is a standard mapping of \(P\) to \(2^{R(T+1)}\). The convex valuedness of \(\chi\) follows from the convexity of sets \(Q^i_t\) and \(C^j_t\). To prove closure of the mapping, we use the following lemma.

**Lemma.** Let \(X\) and \(Y\) be finite-dimensional spaces where \(Y\) is convex and closed, and let \(F\) and \(g\) be continuous functions on \(X \times Y\), where \(g(x, y)\) is convex in \(y\). If for any \(x \in X\) the set
\[D(x) = \{y \in Y : g(x, y) \leq 0\}\]
is bounded and \(g(x, y') < 0\) for some \(y' \in Y\), then function
\[\Phi(x) = \max_{y \in D(x)} F(x, y)\]
is continuous and mapping
\[G(x) = \{y^* \in D(x) : F(x, y^*) = \Phi(x)\}\]
is closed.

**Proof.** By the maximum theorem\(^{38}\), it suffices to prove that mapping \(D(x)\) is continuous. Its closure follows from the continuity of function \(g\). Now let \(x_n \rightarrow x, y \in D(x)\), i.e., \(g(x, y) \leq 0\). Take
\[y_n = \vartheta_n y + (1 + \vartheta_n) y',\]
where
\[g(x, y') = -\varepsilon < 0, \ 0 < \vartheta_n \leq 1.\]

---

\(^{38}\)Ref.: Mathematical Foundations, Maximum Theorem.
Then \( y_n \in Y \) and
\[
g(x_n, y_n) \leq \vartheta_n g(x_n, y) + (1 - \vartheta_n)g(x_n, y')
\leq \vartheta_n \delta_n^1 + (1 - \vartheta_n)(\varepsilon + \delta_n^2), \quad \delta_n^1, \delta_n^2 \to 0.
\]

For
\[
\vartheta_n = (\varepsilon - \delta_n^1)/(\varepsilon + \delta_n^1 - \delta_n^2),
\]
we have
\[
g(x_n, y_n) \leq 0,
\]
i.e., \( y_n \in D(x_n) \) and \( y_n \to y \). This proves lower semicontinuity, and with it the entire lemma.

Let us return to the proof of Theorem 1. Since budgets \( p_0^i(p^0) \) and \( \pi_0^i(p^0) \) are positive and sets \( Q_i^2 \) and \( C_i^2 \) contain 0, Lemma 1 implies that mappings \( \psi_0^i(p^1) \) and \( \varphi_0^i(p^0) \) are closed, and functions
\[
\nu_i^1(p^1) = p_1 \tilde{y}_i^1,
\]
where
\[
(\tilde{x}_0^i, \tilde{y}_1^i) \in \psi_0^i(p^1)
\]
are continuous. The mappings \( \psi_t^j(p^{t+1}) \) and \( \varphi_t^j(p^t) \), \( 0 \leq t \leq T \), are closed (by Lemma). Hence, \( \chi(p) \) is a closed mapping. The image of a compact set is compact because the mapping is closed and because it suffices to consider bounded sets \( \tilde{Q}_i^1 \) and \( \tilde{C}_i^2 \). Thus, \( \chi \) is a standard mapping.

We now show that the conditions 1 and 2 of the Theorem 2 are satisfied. Let
\[
(p_0(n), ..., p_T(n)) \to (p_0, ..., p_T).
\]

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where \( p_t \in \sigma \setminus \Delta \) for some \( t \). Take
\[
    r = \min \{ t : p_t \in \sigma \setminus \Delta \}.
\]

Let us consider two cases:

1) Let \( r = 0 \), i.e., \( p_{0,s} = 0 \) for some \( s \), and
\[
    \eta_0(n) = \sum_j c_0^j + \sum_i x_0^i - \sum_i y_0^i,
\]
where
\[
    c_0^j \in \varphi_0^j(p^0(n)), \ (x_0^i, y_0^i) \in \psi_0^i(p^1(n)).
\]

By the condition (C1), there exists \( j \in J \) such that \( u_0^j(c_1, \ldots, c_t) \) is strictly monotonic in \( c_s \). Then
\[
    u_0^j(c_0^j) = \max \{ u_0^j(c) : c = (c_1, \ldots, c_t) \in \tilde{C}_0^j, \}
\]
\[
    p_0(n)c = \sum_{k \neq s} p_{0,k}(n)c_k + p_{0,s}(n)c_s \leq \pi_0^j(n),
\]
where
\[
    \pi_0^j(n) = \beta_0^j \sum_i p_0(n)y_0^i.
\]

Since \( \sum_i y_0^i \to 0 \) and \( p_0(n) \to p_0 > 0 \), we have \( \lim \pi_0^j(n) > 0 \) and by the strict monotonicity of \( u_0^j \), we have \( C_{0,s}^j \to \tilde{c}_s \), where \( \tilde{c}_s \) is defined in (7). Therefore,
\[
    \limsup_n \eta_{0,s}(n) \geq \tilde{c}_s - \sum_i y_0^i > 0.
\]

2) Let \( r \geq 1 \) and
\[
    p_{r,s} = 0, \ \eta_t(n) = \sum_j c_t^j + \sum_i x_t^i - \sum_i y_t^i,
\]

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where
\[ \exists^i_t \in \varphi^i_t(p^i_t(n)), \]
\[ (\bar{x}^i_t, \bar{y}^i_{t+1}) \in \psi^i_t(p^{i+1}_t(n)). \]
Let us take \( j \in J \) such that \( w^j \) is strictly monotonic in \( c_s \), and technologies
\[ (\bar{x}^i_t, \bar{y}^i_{t+1}), t = 0, \ldots, r - 1, \]
from condition (P1). Since \( p_t \gg 0 \) for \( t \leq r - 1 \), for some \( 0 < \vartheta_t < 1 \) and for all sufficiently large \( n \) we have
\[ \vartheta_t p_t(n)^0 x^i_t \leq \alpha^i_t \sum_i p_t(n)^0 y^i_t, \]
\[ i \in I, \ 0 \leq t \leq r - 1, \]
and \( ^0 y^i_0 = y^i_0 \).

Then
\[ \vartheta_0 p_0(n)^0 x^i_0 \leq \rho^i_0(n), \]
and thus
\[ p_1(n) \bar{y}^i_1 \geq \vartheta_0 p_1(n)^0 y^i_0. \]
Hence,
\[ \vartheta_0 \vartheta_1 p_1(n)^0 x^i_1 \leq \rho^i_1(n). \]
Finally, continuing this chain, we have
\[ \lambda p_{r-1}(n)^0 x^i_{r-1} \leq \rho^i_{r-1}(n), \]
where \( \lambda = \vartheta_0, \ldots, \vartheta_{r-1} \). Hence,
\[ p_r(n) \bar{y}^i_r \geq \lambda p_r(n)^0 y^i_r \]
and
\[ x^i_r(n) \geq \lambda \beta^i_r \sum_i p_t(n) \bar{y}^i_t. \]
Thus,

\[ \liminf_n \pi^i_t(n) > 0, \]

and by the analogical case 1,

\[ \limsup_n \eta_{r,s}(n) \geq \hat{c}_s - \hat{y}_s > 0. \]

Condition 1 of the Theorem 2 is thus true.

To check condition 2, note that, if

\[ \tilde{c}^i_t \in \varphi^i_t(p'), \]
\[ (\tilde{x}^i_t, \tilde{y}^i_{t+1}) \in \psi^i_t(p^{t+1}), \]

then

\[ p_t \tilde{x}^i_t = \rho^i_t(p') \]
and
\[ p_t \tilde{c}^i_t = \pi^i_t(p'). \]

Indeed, let \( p_t \tilde{x}^i_t < \rho^i_t(p') \). Then by (P2) there exists \( (x, y) \in Q^i_t \), such that \( y > \tilde{y}^i_{t+1} \). For \( 0 < \vartheta < 1 \), technology

\[ (x_\vartheta, y_\vartheta) = (\vartheta x + (1 - \vartheta)\tilde{x}^i_t, \]
\[ \vartheta y + (1 - \vartheta)\tilde{y}^i_{t+1}) \in Q^i_t, \]

and for sufficiently small \( \vartheta \) we have

\[ p_t x_\vartheta \leq \rho^i_t(p'). \]

But

\[ p_{t+1} y_\vartheta > p_{t+1} \tilde{y}^i_{t+1}, \]
which contradicts the optimality of $\tilde{y}_{t+1}$. Using (C2), there is a similar characteristic for the consumer budget.

Finally, if

$$\eta = (\eta_0, ..., \eta_T) \in \chi(p),$$

then

$$p_t\eta_t = \sum_j p_t\tilde{x}_t^j + \sum_i p_t\tilde{\gamma}_t^i - \sum_i p_t\tilde{\rho}_t^i$$

$$= \sum_i \pi_t^i(p^f) + \sum_i \rho_t^i(p^f) - \sum_i p_t\tilde{\rho}_t^i = 0$$

by (D). We have verified all the conditions of Theorem 2, which directly implies the assertion of Theorem 1. Q.E.D.
5.4 A model of a competitive market economy

We now describe a different model of an economic system with the same agents that is similar to the previous system but has a totally different budget formation mechanism. In this system, the agents themselves determine their budgets based on income from goods produced (and sold) during the preceding period (for producers) or based on participation in producer income (for consumers). This model is called a competitive market economy (ME).

The behaviour of producers and consumers is described, respectively, by problems (1) and (2) with budgets \( \rho_i^t \) and \( \pi_i^t \), which are defined as follows:

\[
\rho_i^t = (1 - \gamma_i^t)p_t y_i^t, \quad 0 \leq t \leq T,
\]
\[
\pi_0^t = p_0 \omega_0^t + \sum_i \alpha_0^t \gamma_i^t p_0 y_0^t,
\]
\[
\pi_i^t = p_i \omega_i^t + \sum_i \alpha_i^t \gamma_i^t p_i y_i^t, \quad 1 \leq t \leq T,
\]

where \( \omega_0^j \in \mathbb{R}_+ \) is the initial endowment of consumer \( j \); \( \omega_i^j \in \mathbb{R}_+ \) is the property of consumer \( j \) for the moment \( t \); the number \( \gamma_i^t \), and \( 0 < \gamma_i^t < 1 \) describe the share of income of producer \( i \) at time \( t \) allocated to consumption; and nonnegative \( \alpha_i^t \) (\( \sum_j \alpha_i^t = 1 \)) represent the share of consumer \( j \) in producer income (dividend distribution).

An equilibrium in the ME model on a finite time interval can be defined as above, similar to a previous model: this is tuple \( ((\tilde{x}_i^t, \tilde{y}_i^{t+1}), \tilde{z}_i^t, \hat{p}_t, 0 \leq t \leq T, i \in I, j \in J) \) that satisfies relationships (3) and (4) with budgets \( \hat{\rho}_i^t \) and \( \hat{\pi}_i^t \) calculated using \( p_t = \hat{p}_t \) and \( y_i^t = \tilde{y}_i^t \), as well as material balances (6) for \( t \geq 1 \) and

\[
\sum_j \tilde{z}_0^j + \sum_i \tilde{x}_0^i = \sum_i y_0^i + \sum_j \omega_0^j.
\]
Denote by \( I_t(j) = \{ i \in I : \alpha^i_t > 0 \} \) the set of producers who pay positive dividends to consumer \( j \). We assume that for all \( j \in J \) and \( 0 \leq t \leq T \) the sets \( I_t(j) \) are nonempty. Moreover,

\[ (P3) \text{ There exist } (0, x_{t-1}, 0, y_t^i) \in Q_{t-1}^i \text{ such that } \sum_{i \in I} I_t(j) \cdot y_t^i > 0 \text{ ("widely" distributed shares, i.e., consumers receive dividends from many producers).} \]

We also assume that the following condition holds:

\[ (E) \omega_0^j \gg 0 \text{ for all } j \in J. \]

This requirement can be relaxed if we use this positivity property for initial states \( y_0^i \). Then, assuming that \( \sum y_0^i \gg 0 \), we may take \( \omega_0^j = 0 \) for any \( j \in J \). However, we specifically do not impose any additional assumptions on initial state \( y_0^i \), so that this model can be used in what follows as the second stage in the model of transition from CE to ME.

**Theorem 3.** Assume that assumptions \((P2),(P3),(C1),(C2)\), and \((E)\) are satisfied. Then for any nonzero initial states \( y_0^i, i \in I \) the ME model has an equilibrium with positive prices \((\hat{p}_t \gg 0)\).

**Proof.** Theorem 3 is basically proved like Theorem 1. Vector \( \hat{c} \) in (7) should be chosen so that \( \hat{c} \gg N \hat{y} + \hat{\omega} \), where \( \sum y_0^i \leq \hat{\omega} \).

The construction of the excess demand function (EDF), like the proof of its standard property, is carried out by the same scheme as above. Condition 2 of Theorem 2 is checked similarly. To check condition 1, we keep the notation used in the proof of the Theorem 1 and take \((p_0(n),...,p_T(n)) \to (p_0,...,p_T)\), where \( p_t \in \sigma \backslash \Delta \) for some \( t \), setting \( r = \min \{ t : p_t \in \sigma \backslash \Delta \} \).

If \( r = 0 \), then there exist \( 1 \leq s \leq l \) and \( j \in J \) such that \( p_{0,s} = 0 \), and \( u_0^j(c) \) is strictly increasing in \( c_0 \). By condition \((E)\), \( \lim \pi_0^j(n) > 0 \), and as previously,

\[ \limsup_n \eta_{0,s}(n) \geq \hat{c}_s - \sum_i y_0^i - \sum_j \omega_0^j \gg 0. \]

Let \( r \geq 1 \), \( p_{t,s} = 0 \), for some \( 1 \leq s \leq l \), and \( u_0^j(c) \) is strictly monotonic in \( c_0 \).
By (P3), \( p_r^0 y_r^i > 0 \) for some \( i \in I_r(j) \). Let
\[
\eta_0(n) = \sum_i \tilde{c}^i_0 + \sum_i \tilde{x}^i_0 - \sum_i y^i_0 - \sum_j \omega^j_0
\]
and
\[
\eta_t(n) = \sum_j \tilde{c}^j_t + \sum_i \tilde{x}^i_t - \sum_i y^i_t, \quad t \geq 1.
\]

Since \( p_{r-1} \gg 0 \), there exist \( 0 < \phi < 1 \) such that
\[
\phi p_{r-1}^0 x_{r-1}^i \leq p_{r-1}(n) y_{r-1}^i
\]
and therefore,
\[
p_r(n) y_r^i \geq \phi p_r^0 y_r^i.
\]

Hence,
\[
\liminf \pi_r^i(n) \geq \liminf_n \alpha_r^j \gamma_r^i p_r(y_r^i) \geq \phi \alpha_r^i \gamma_r^i p_r^0 y_r^i,
\]
and as previously,
\[
\limsup \eta_{r,n}(u) \geq \tilde{c}_s - N \tilde{y}_s > 0.
\]

Thus, condition 1 of the Theorem 2 holds, the use of which finalizes a proof.
5.5 Probability Foundations

Prior to initiating a discussion on a model of transition from a centralized economy to a competitive market economy, it is necessary to establish a set of the specific interpretations of probability and game theory.

Under the classical interpretations of probability, probability is definable as a single, unique value, being the ratio of actual (favorable) to possible types of occurrences in a sufficiently well-defined sequence of events, where each case is assumed a priori equally possible.

5.5.1 Review of Probability

Let us review the basic rules of probability and then derive the notion of expected value. Let us also develop the notion of expected utility as an alternative to expected payoffs.

Probabilistic analysis arises when we face uncertainty.

In situations where events are uncertain, a probability measures the likelihood that a particular event (or set of events) occurs, e. g.:

- The probability that a roll of a die comes up 6.
- The probability that two randomly chosen cards add up to 21 (Blackjack).

Sample Space or Universe

Let $S$ denote a set (collection or listing) of all possible states of the environment known as the sample space or universe; a typical state is denoted as $s$. For example:

- $S = \{s_1, s_2\}$; success/ failure, or low/ high price.
- $S = \{s_1, s_2, ..., s_{n-1}, s_n\}$; number of $n$ units sold or $n$ offers received.
- $S = [0, \infty)$; stock price or salary offer (continuous positive set space).

Events
An event is a collection of those states \( s \) that result in the occurrence of the event. An event can be that state \( s \) occurs or that multiple states occur, or that one of several states occurs (there are other possibilities).

Event \( A \) is a subset of \( S \), denoted as \( A \subset S \).

Event \( A \) occurs if the true state \( s \) is an element of the set \( A \), written as \( s \in A \).

\( S \) is the sample space and \( A_1 \) and \( A_2 \) are events within \( S \).

Event \( A_1 \) does not occur. Denoted \( A_1' \) (Complement of \( A_1 \))

Event \( A_1 \) or \( A_2 \) occurs. Denoted \( A_1 \cup A_2 \) (For probability use Addition Rules)

Event \( A_1 \) and \( A_2 \) both occur, denoted \( A_1 \cap A_2 \) (For probability use Multiplication Rules).

To each uncertain event \( A \), or set of events, e.g. \( A_1 \) or \( A_2 \), we would like to assign weights which measure the likelihood or importance of the events in a proportionate manner. Let \( P(A_i) \) be the probability of \( A_i \). We further assume that:

\[
\bigcup_{\text{all } i} A_i = S,
\]
\[
P\left(\bigcup_{\text{all } i} A_i\right) = 1,
\]
\[
P(A_i) \geq 0.
\]

**Additional Rules**

The probability of event \( A \) or event \( B : P(A \cup B) \)

If the events do not overlap, i.e. the events are disjoint subsets of \( S \), so that \( A \cap B = \emptyset \) then the probability of \( A \) or \( B \) is simply the sum of the two probabilities. \( P(A \cup B) = P(A) + P(B) \).
If the events overlap, (are not disjoint) \( A \cap B \neq \emptyset \) use the modified addition rule: \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \).

**Multiplication Rules**

The probability of event \( A \) and event \( B \) : \( P(A \cap B) \)

Multiplication rule applies if \( A \) and \( B \) are independent events.

\( A \) and \( B \) are **independent events** if \( P(A) \) does not depend on whether \( B \) occurs or not, and \( P(B) \) does not depend on whether \( A \) occurs or not.

\[ P(A \cap B) = P(A) \times P(B) = P(AB) \]

Conditional probability for non independent events. The probability of \( A \) given that \( B \) has occurred is \( P(A|B) = P(AB)/P(B) \).

### 5.5.2 Conditional probabilities. Independence

Let \((\Omega, \mathcal{A}, P)\) — (bounded) probability space and \( A \) - any event (e.g., \( A \in \mathcal{A} \)).

**Definition 1.** A **conditional probability of the event** \( B \) **under condition of the event** \( A \) with \( P(A) > 0 \) (denoted as \( P(B|A) \)) is a value of

\[
\frac{P(AB)}{P(A)}.
\]

(p1)

In a case of classical method of defining probabilities \( P(A) = \frac{N(A)}{N(\Omega)} \), \( P(AB) = \frac{N(AB)}{N(\Omega)} \) and, therefore

\[
P(B|A) = \frac{N(AB)}{N(A)}.
\]

(p2)

**Definition 2.** Two events \( A \) and \( B \) are called **independent or statically independent** (with respect to the probability \( P \)), if

\[ P(AB) = P(A)P(B). \]

In the theory of probability we often discuss independency not only of the events (sets), but of the systems of events (sets).
Definition 3. Two algebras of events (sets) $A_1$ and $A_2$ are called independent or statically independent (with respect to the probability $P$), if any two sets $A_1$ and $A_2$ belonging to $A_1$ and $A_2$ relatively, are independent.

The definition of independency of two sets and two algebras is applicable to the case of any finite number of sets and algebras of sets.

Precisely, we say, that the sets $A_1, \ldots, A_n$ are independent or statically independent combined (with respect to the probability $P$), if for any $k = 1, \ldots, n$ and $1 \leq i_1 < i_2 < \ldots < i_k \leq n$

$$P(A_{i_1} \ldots A_{i_k}) = P(A_{i_1}) \ldots P(A_{i_k}).$$

The algebras of sets $A_1, \ldots, A_n$ are called independent or statically independent combined (with respect to the probability $P$), if any sets $A_1, \ldots, A_n$ belonging to $A_1, \ldots, A_n$ relatively, are independent.

5.5.3 Bayes Rule

Used for making inferences: given a particular outcome, event $A$, can we infer the unobserved cause of that outcome, some event $B_1, B_2, \ldots, B_n$. Suppose we know the prior probabilities, $P(B_i)$ and the conditional probabilities $P(A|B_i)$. Suppose that $B_1, B_2, \ldots, B_n$ be a complete partition of the sample space $S$, so that $\bigcup_i B_i = S$ and $B_iB_j = \emptyset$ for any $i \neq j$. In this case we have that:

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i) \quad (1)$$

Bayes rule is a formula for computing the posterior probabilities, e.g. the probability that event $B_k$ was the cause of outcome $A$, denoted $P(B_k|A)$:

$$P(B_k|A) = P(B_k \cap A)/P(A).$$
Using the conditional probability rule

\[ P(A|B_k)P(B_k)/P(A). \]

Using expression (1) above

\[ \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}. \]

This is Bayes Rule.

5.5.4 Expected Value

One use of probabilities to calculate expected values (or payoffs) for uncertain outcomes.

Suppose that an outcome, e.g. a money payoff is uncertain. There are \( n \) possible values, \( X_1, X_2, ..., X_N \). Moreover, we know the probability of obtaining each value. The expected value of the outcome is then given by:

\[ P(X_1)X_1 + P(X_2)X_2 + ... + P(X_N)X_N. \]

**Expected Utility**

The assumption that individuals treat expected payoffs the same as certain payoffs (i.e. that they are risk neutral) may not hold in practice.

Example:

A risk neutral person is indifferent between $25 for certain or a 25% chance of earning $100 and a 75% chance of earning 0. Many people are risk averse and prefer $25 with certainty to the uncertain gamble, and so treating expected payoffs the same as certain payoffs may yield misleading results. What can we do to fix this problem? A simple solution is to transform payoffs using a utility function, and then consider the expected utility of an action choice.
Utility Function Transformation

Let \( x \) be a payoff amount in dollars. Let \( U(x) \) be a continuous, increasing function of \( x \).

The function \( U(x) \) gives an individuals level of satisfaction in fictional units from receiving payoff amount \( x \), and is known as a utility function.

If the certain payoff of $25 is preferred to the gamble, (due to risk aversion) then we want a utility function that satisfies:

\[
U(\$25) > .25U(\$100) + .75U(\$0).
\]

The left hand side is the utility of the certain payoff and the right hand side is the expected utility from the gamble.

Any concave function \( U(x) \) will work. For example, \( U(x) = \sqrt{x} \)

\[
\sqrt{25} > .25\sqrt{100} + .75\sqrt{0}, \iff 5 > 2.5.
\]

5.5.5 Random variables and their characteristics

**Definition 4.** Any numerical function \( \xi = \xi(\omega) \), defined on the (finite) space of the elementary events \( \Omega \), will be called (simple) random variable.

One of the example of the random variable \( \xi \) is the characteristic function of some set \( A \in \mathcal{A} \):

\[
\xi = I_A(\omega),
\]

where

\[
I_A(\omega) = \begin{cases} 
1, & \omega \in A, \\
0, & \omega \notin A.
\end{cases}
\]

Let us denote \( \mathcal{X} \) - the union of all subsets of the set \( X = \{x_1, ..., x_m\} \), and let \( B \in \mathcal{X} \).
Let probability \( P_\xi(.) \) on \((X, \mathcal{X})\), induced by the random variable \( \xi \), be described by the formula

\[
P_\xi(B) = P \{ \omega : \xi(\omega) \in B \}, \quad B \in \mathcal{X}.
\]

It is clear that the values of these probabilities are fully defined by the probabilities

\[
P_\xi(x_i) = P \{ \omega : \xi(\omega) \in x_i \}, \quad x_i \in X.
\]

The set of numbers \( \{P_\xi(x_1), \ldots, P_\xi(x_m)\} \) is called the distribution of random variable \( \xi \), where \( x_1, \ldots, x_m \) describe all the values of the random variable \( \xi \).

**Definition 5.** Let \( x \in R^1 \). The function

\[
F_\xi(x) = P \{ \omega : \xi(\omega) \leq x \}
\]

is called a function of distribution of the random variable \( \xi \).

## 5.6 Game Theory Foundations

### 5.6.1 What is a Game?

There are many types of games, board games, card games, video games, field games (e. g. football), etc.

**Definition 1.** Game theory is a formal way to analyze interaction among a group of rational agents who behave strategically.

This definition contains a number of important concepts which are discussed in order:

**Group:** In any game there is more than one decision maker who is referred to as player. If there is a single player the game becomes a decision problem.
**Interaction:** What one individual player does directly affects at least one other player in the group. Otherwise the game is simple a series of independent decision problems.

**Strategic:** Individual players account for this interdependence.

**Rational:** While accounting for this interdependence each player chooses her best action. This condition can be weakened and we can assume that agents are boundedly rational. Behavioural economics analyzes decision problems in which agents behave boundedly rational. **Evolutionary game theory** is game theory with boundedly rational agents.

We focus on games where:

- There are 2 or more players.
- There is some choice of action where strategy matters.
- The game has one or more outcomes, e. g. someone wins, someone loses.
- The outcome depends on the strategies chosen by all players; there is strategic interaction.

What does this rule out?

- Games of pure chance, e. g. lotteries, slot machines. (Strategies don't matter).
- Games without strategic interaction between players, e. g. Solitaire.

### 5.6.2 Why Do Economists Study Games?[12]

Game theory has found numerous applications in all fields of economics:
1. Trade: Levels of imports, exports, prices depend not only on your own tariffs but also on tariffs of other countries.

2. Labor: Internal labor market promotions like tournaments: your chances depend not only on effort but also on efforts of others.

3. IO: Price depends not only on your output but also on the output of your competitor (market structure ...).

4. PF: My benefits from contributing to a public good depend on what everyone else contributes.

5. Political Economy: who/what I vote for depends on what everyone else is voting for.

Games are a convenient way in which to model the strategic interactions among economic agents. Many economic issues involve strategic interaction:

- Behaviour in imperfectly competitive markets, e. g. Coca-Cola versus Pepsi.

- Behaviour in auctions, e. g. Investment banks bidding on U. S. Treasury bills.

- Behaviour in economic negotiations, e. g. trade.

Game theory is not limited to Economics. 

*Three Elements of a Game:*

1. The players:
   - how many players are there?
   - does nature/chance play a role?

2. A complete description of the *strategies* of each player

3. A description of the *consequences (payoffs)* for each player for every possible profile of strategy choices of all players.
5.6.3 The Prisoners’ Dilemma Game

Two players: prisoners 1, 2.

Each has two strategies:

Prisoner 1: Don’t Confess, Confess
Prisoner 2: Don’t Confess, Confess

Payoff consequences quantified in prison years.

fewer years = greater satisfaction.

Prisoner 1 payoff first, followed by prisoner 2 payoff.

*Prisoners Dilemma in Normal or Strategic Form*

<table>
<thead>
<tr>
<th></th>
<th>Prisoner 1</th>
<th>Prisoner 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Don’t Confess</td>
<td>1.1</td>
<td>15.0</td>
</tr>
<tr>
<td>Confess</td>
<td>0.15</td>
<td>5.5</td>
</tr>
</tbody>
</table>

*Prisoners’ Dilemma is an example of a Non-Zero Sum Game*

- A zero-sum game is one in which the players’ interests are in direct conflict, e.g. in football, one team wins and the other loses.

- A game is non-zero sum, if players interests are not always in direct conflict, so that there are opportunities for both to gain.

- For example, when both players choose Don’t Confess in Prisoners’ Dilemma.

*The Prisoners’ Dilemma is applicable to many other situations:*

- Nuclear arms races.

- Dispute Resolution and the decision to hire a lawyer.

- Corruption/ political contributions between contractors and politicians.
5.6.4 Simultaneous versus Sequential Move Games

- Games where players choose actions simultaneously are **simultaneous move games**. Examples: Prisoners’ Dilemma, Sealed- Bid Auctions.

  Must anticipate what your opponent will do right now, recognizing that your opponent is doing the same.

- Games where players choose actions in a particular sequence are **sequential move games**. Examples: Chess, Bargaining/ Negotiations.

  Must look ahead in order to know what action to choose now.

- Many strategic situations involve both sequential and simultaneous moves.

  The Investment Game is a Sequential Move Game.

5.6.5 One- Shot versus Repeated Games.

- One- shot: play of the game occurs once.

  - Players likely to not know much about one another.
  
  Example - tipping on your vacation

- Repeated: play of the game is repeated with the same players.

  - Indefinitely versus finitely repeated games
  
  - Reputational concerns matter; opportunities for cooperative behavior may arise.

- Advise: If you plan to pursue an aggressive strategy, ask yourself whether you are in a one- shot or in a repeated game. If a repeated game, think again.
5.6.6 Strategies

- A strategy must be a comprehensive plan of action, a decision rule or set of instructions about which actions a player should take.

- It is the equivalent of a memo, left behind when you go on vacation, that specifies the actions you want taken in every situation which could conceivably arise during your absence.

- Strategies will depend on whether the game is one-shot or repeated.

Examples of one-shot strategies:
- Prisoners' Dilemma: Don't Confess, Confess
- Investment Game: Sender: Don't Send, Send. Receiver: Keep, Return

- How do strategies change when the game is repeated?

5.6.7 Repeated Game Strategies

In repeated games, the sequential nature of the relationship allows for the adoption of strategies that are contingent on the actions chosen in previous plays of the game. Most contingent strategies are of the type known as "trigger" strategies.

Example trigger strategies

In prisoners' dilemma: Initially play Don't confess. If your opponent plays Confess, then play Confess in the next round. If your opponent plays Don't confess, then play Don't confess in the next round. This is known as the "tit for tat" strategy.

In the investment game, if you are the sender: Initially play Send. Play Send as long as the receiver plays Return. If the receiver plays Keep, never play Send again. This is known as the "grim trigger" strategy.

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Information

Players have perfect information if they know exactly what has happened every time a decision needs to be made, e.g., in Chess. Otherwise, the game is one of imperfect information.

Example: In the repeated investment game, the sender and receiver might be differentially informed about the investment outcome. For example, the receiver may know that the amount invested is always tripled, but the sender may not be aware of this fact.

Assumptions Game Theorists Make

- Payoffs are known and fixed. People treat expected payoffs the same as certain payoffs (they are risk neutral).

Example: a risk neutral person is indifferent between $25 for certain or a 25% chance of earning $100 and a 75% chance of earning 0. We can relax this assumption to capture risk averse behavior.

- All players behave rationally.

They understand and seek to maximize their own payoffs. They are flawless in calculating which actions will maximize their payoffs.

- The rules of the game are common knowledge:

Each player knows the set of players, strategies and payoffs from all possible combinations of strategies: call this information X.

Common knowledge means that each player knows that all players know X, that all players know that all players know X, and so on, ..., *ad infinitum*.
5.6.8 **Equilibrium**

- The interaction of all (rational) players' strategies results in an outcome that we call "equilibrium."

- In equilibrium, each player is playing the strategy that is a "best response" to the strategies of the other players. No one has an incentive to change his strategy given the strategy choices of the others.

- Equilibrium is not:
  - The best possible outcome.
  - A situation where players always choose the same action.

Equilibrium in the one-shot prisoners' dilemma is for both players to confess.

Sometimes equilibrium will involve changing action choices (known as a mixed strategy equilibrium).

5.6.9 **Strategic Behavior in Markets**

A market is any arrangement in which goods or services are exchanged.

For example, auctions, supermarkets, job markets, meet markets. Those supplying goods are suppliers or sellers. Those demanding goods are demanders or buyers. Do individual buyers or sellers act strategically? If just one (or a few) buyers and just one (or a few) sellers, then strategic behavior is very likely as in a bargaining game. If there are many buyers and sellers, individuals still act strategically, in the sense that they seek prices that maximize their individual surplus (they play individual best responses).

**Buyers and Sellers Objectives in the Market for a Certain Good/Service**
Each buyer has a value $v$ that they attach to each unit of some good. The buyers surplus is the value $v$ of a unit of the good less the price paid to the seller for that unit: $p$:

\[
\text{Buyer's surplus} = v - p
\]

Each seller has a cost, $c$ of bringing a unit of the good to market. The sellers surplus is the price received for a unit of the good, $p$ less the per unit cost, $c$:

\[
\text{Seller's surplus} = p - c.
\]

**Example: 1 Buyer and 1 Seller**

Suppose a buyer wants to buy one unit of a good, say a used car. The buyer values it at $v$, and the sellers cost (or reserve value) $c$, is such that $c < v$.

Suppose that $v$ and $c$ are common knowledge (a big assumption, we will relax later).

Then any price, $p$, such that $v > p > c$ will provide a positive surplus to both the buyer and the seller:

$p$ could be determined via bargaining, but the important point is that there are gains from trade: the buyer gets $v - p > 0$ and the seller gets $p - c > 0$. 

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5.7 A model of transition from a centralized economy to a competitive market economy[3]

In this model a decision is made at time $t = 0$ requiring transition from a centralized economy (CE) to a market economy (ME). The problem is how to adapt the system to a new economic mechanism while remaining within the CE framework.

Let us introduce a transition period during which all agents (primarily producers), while continuing to act by CE rules, change their behaviour based on the information about prices in the future ME and the uncertain change point.

The length of the transition period ($\theta$) is treated by the agents as a random variable with given probability distribution $P\{\theta = t\}, t = 1, ..., T$. The behaviour of agents in this transition model is somewhat more complicated than in the previous models.

Denote $\tau_0 = \max \{t : P\{\theta \geq t\} = 1\}$ and $\tau_1 = \max \{t : P\{\theta \leq t\} = 1\}$. In other words, interval $[\tau_0, \tau_1]$ is the carrier of the distribution of random variable $\theta$, i.e., the minimum interval that contains $\theta$ with probability 1.

Let price system

$$\{p_t, t \leq \tau, p_\tau(\tau), \tau \leq t \leq T\}$$

be given for some $\tau$. If $\tau$ is interpreted as the change point (from one economic mechanism to another), then $p_t$ are product prices at time $t$ given that at this instant the system is still functioning according to the CE model, and $p_\tau(\tau)$ are the prices at time $t$ given that the economic mechanism changed at instant $\tau$. The prices $p_t$ are correctly defined for $0 \leq t \leq \tau_1 - 1$, and prices $p_\tau(\tau)$ are correctly defined for $\tau_0 \leq \tau \leq \tau_1$, $\tau \leq t \leq T$. Agent budgets $\rho^i_t, \pi^i_t, \rho^i_\tau(\tau)$, and $\pi^i_\tau(\tau)$ are interpreted similarly. We now consider two types
of producer and consumer problems at time $t$ depending on the particular economic mechanism that prevails at that time:

$$ p_{t+1}^* y \rightarrow \max, \quad (8) $$

$$(x, y) \in Q_i^t, \quad p_t x \leq \rho_i^t, \quad i \in I, \quad t \leq \tau - 1,$$

where

$$ p_{t+1}^* = p_{t+1} q_{t+1} + p_{t+1}(t + 1)(1 - q_{t+1}), $$

$$ q_{t+1} = P\{\vartheta > t + 1 \mid \vartheta \geq t + 1\}, $$

$$ u_i^t(c) \rightarrow \max, \quad (9) $$

$$ c \in C_i^t, \quad p_t c \leq \pi_i^t, \quad j \in J, \quad t \leq \tau - 1; $$

$$ p_{t+1}(\tau) y \rightarrow \max, \quad (10) $$

$$(x, y) \in Q_i^\tau, \quad p_t(\tau) x \leq \rho_i^t(\tau), \quad i \in I, \quad \tau \leq t \leq T; $$

$$ u_i^t(c) \rightarrow \max, \quad (11) $$

$$ c \in C_i^\tau, \quad p_t(\tau) c \leq \pi_i^t(\tau), \quad j \in J, \quad \tau \leq t \leq T. $$

All prices are given at the moment $T + 1$.

The main producer problem during the transition period (8) differs from producer problem (1) in the CE model in that the producer now must calculate the income at the end of the current period $t$ from expected (weighted-average) prices $p_{t+1}^*$ and not from $p_{t+1}$.
When the producer chooses a technology at the beginning of period \( t \) under CE conditions, it is still uncertain that the economic mechanism will change by the end of that period. The probability of a change at time \( t + 1 \) given that no change occurred previously is \( 1 - q_{t+1} \). Then the prices become \( p_{t+1}(t + 1) \). If no change occurs at time \( t + 1 \), the prices at the instant are \( p_{t+1} \), and the probability of this event is \( q_{t+1} \). Thus, \( p_{t+1} \) are the weighted-average prices, with the weights equal to the conditional probability that the economic mechanism does or does not change at time \( t + 1 \).

Relationships (9)-(11) are the standard agent problems in the corresponding models (CE for (9) and ME for (10) and (11)).

An equilibrium in the transition model with initial producer states \((y'_{0i}, i \in I)\), and consumer endowments \((\omega'_{ij}, 1 \leq \tau \leq T, \tau \leq t \leq T; j \in J)\), at the change point is defined as the tuple

\[
\{ ((x'_{t}, y'_{t+1}), \hat{c}'_{t}, \hat{p}_{t}, 0 \leq t \leq \tau - 1) , \\
((x'_{t}(\tau), y'_{t+1}(\tau)), \hat{c}'_{t}(\tau), \hat{p}_{t}(\tau), \tau \leq t \leq T), \\
1 \leq \tau \leq T; i \in I, j \in J \}
\]

that satisfies the following conditions:

(a) \((x'_{t}, y'_{t+1})\) is the solution of problem (8) given prices \( \hat{p}_{t} \), \( \hat{p}_{t+1} \), and \( \hat{p}_{t+1}(t + 1) \), and budget \( \hat{n}_{t} \);

(b) \( \hat{c}'_{t} \) is the solution of problem (9) given prices \( \hat{p}_{t} \) and budget \( \hat{n}'_{t} \);

(c) \( \hat{p}_{t} = \omega_{t}(\hat{K}_{t}), \hat{n}_{t} = \omega_{t}(\hat{K}_{t}) \), \( \hat{K}_{t} = \sum \hat{p}_{t} \hat{g}_{t}, \hat{y}_{0} = y'_{0} \);

(d) \((x'_{t}(\tau), y'_{t+1}(\tau))\) is the solution of problem (10) given prices \( \hat{p}_{t}(\tau) \) and budget \( \hat{n}'_{t}(\tau) \);

(e) \( \hat{c}_{t}(\tau) \) is the solution of problem (11) given prices \( \hat{p}_{t}(\tau) \) and budget \( \hat{n}_{t}(\tau) \);

(f) \( \hat{p}_{t}(\tau) = (1 - q_{t})(\hat{p}_{t}(\tau)\hat{g}_{t}(\tau)). \)
\[ \hat{\pi}_i(t) = \hat{p}_t(\tau) \omega_i^t(\tau) + \sum_i \alpha_i^j(\tau) \gamma_i^t(\tau) \hat{p}_t(\tau) \tilde{y}_i^t(\tau), \quad \tilde{y}_i^t(\tau) = \tilde{y}_i^t; \]

\[ \sum_j \alpha_j^t + \sum_i \tilde{x}_i^t = \sum_i \tilde{y}_i^t; \]
\[ \sum_j \alpha_j^t(\tau) + \sum_i \tilde{x}_i^t(\tau) = \sum_i \tilde{y}_i^t(\tau) + \sum_j \omega_j^t(\tau); \quad \tau + 1 \leq t. \]

Let assumptions (P1)-(P3), (C1), (C2), and (D) hold. Then for any producer initial states \((y_0^i, i \in I)\), such that \(\sum_i y_0^i > 0\) and positive consumer endowments at the change point \((\omega_j^t, j \in J)\), the transition model has an equilibrium (with positive prices \(\hat{p}_t\) and \(\hat{p}_t(\tau)\)).

Let
\[ P = \Delta^{(\tau_1 - 1)} \prod_{\tau = \tau_0}^{\tau_1} \Delta^{(\tau - \tau_1 + 1)}. \]

Then for \(p = ((p_t, 0 \leq t \leq \tau_1 - 1), (p_t(\tau), \tau_0 \leq \tau \leq \tau_1, \tau \leq t \leq T)) \in P\), and we can successively define, as before, sets \(\psi_i^t, \varphi_i^t, \psi_i^t(\tau), \text{and} \varphi_i^t(\tau)\) and define the excess demand function (EDF) as

\[ \chi = ((\chi_t, 0 \leq t \leq \tau_1 - 1), (\chi_t(\tau), \tau_0 \leq \tau \leq \tau_1, \tau \leq t \leq T)), \]

where \(\chi\) is a standard mapping, and
\[ \chi_t = \sum_j \alpha_j^t + \sum_i \tilde{x}_i^t - \sum_i \tilde{y}_i^t, \quad (\tilde{x}_i^t, \tilde{y}_i^{t+1}) \in \psi_i^t, \quad \tilde{z}_i^t \in \varphi_i^t; \]
\[ \chi_t(t) = \sum_j \alpha_j^t(t) + \sum_i \tilde{x}_i^t(t) - \sum_i \tilde{y}_i^t - \sum_j \omega_j^t; \]
\[ \chi_t(\tau) = \sum_j \alpha_j^t(\tau) + \sum_i \tilde{x}_i^t(\tau) - \sum_i \tilde{y}_i^t(\tau), \]
\[ \tilde{z}_i^t \in \varphi_i^t(\tau), \quad (\tilde{x}_i^t(\tau), \tilde{y}_i^{t+1}(\tau)) \in \psi_i^t(\tau). \]
5.8 Optimal properties of equilibria

The notion of equilibrium in Walrasian models is traditionally associated with Pareto-optimality. Pareto-optimality is the expression of a basic principle according to which none of the agents (consumers) can improve his state (utility) without causing damage to another agent. In budget-constrained models, which include the CE and ME models described above, the equilibrium is generally not Pareto-optimal. However, as will be shown, the equilibrium has some "pleasant" properties and can be represented as a solution of a dynamic optimization model of the following form:

$$\sum_{t=0}^{T} [U_t(c_t) + F_t(z_t)] \rightarrow \max$$

over all paths $c_t \in C_t = \sum_j C^j_t$ and $z_t = (x_t, y_{t+1}) \in Q_t = \sum_i Q^i_t$: $c_t + x_t \leq y_t$ given initial state $y_0 = \sum_i y^i_0$. The prices $(p_t, 0 \leq t \leq T)$ are called stimulating prices for path $(\hat{c}_t, \hat{z}_t)$ if for all $0 \leq t \leq T$ we have

(i) $U_t(\hat{c}_t) - p_t \hat{c}_t = \max_{c \in C_t} [U_t(c) - p_t c]$,  
(ii) $F_t(\hat{z}_t) + p_{t+1} \hat{y}_{t+1} - p_t \hat{x}_t = \max_{z \in Q_t} [F_t(z) + p_{t+1} y - p_t x], p_{T+1} = 0$,  
(iii) $p_t (\hat{y}_t - \hat{x}_t - \hat{c}_t) = 0$.

The path $(\hat{c}_t, \hat{z}_t)$ stimulated by some prices is optimal in problem (12).

Below we consider general budget-constraint (BC) models in which the objectives of agents are described by problems (1) and (2) and where budgets $\rho^j_t$ and $\pi^j_t$ depend on the entire "history" of prices and producer problem solutions, i.e., $\rho^j_t = \rho^j_t(p^t, \omega^t)$, $\pi^j_t = \pi^j_t(p^t, \omega^t)$, where

$$p^t = (p_k, 0 \leq k \leq t),$$
$$\omega^t = ((x^i_k, y^i_{k+1}), 0 \leq k \leq t - 1, i \in I), t \geq 1,$$
$$\omega^0 = (y^i_0, i \in I);$$

and $\rho^j_t(\cdot)$ and $\pi^j_t(\cdot)$ are given functions. An equilibrium in this model is
defined similarly to an equilibrium in the CE model, with one difference: relationship (5) is replaced by
\[ \hat{\rho}_k = \rho_k(\hat{\sigma}, \hat{\omega}), \hat{\pi}_i = \pi_i(\hat{\sigma}, \hat{\omega}), \]
\[ \hat{\sigma}^t = (\hat{x}_k^t, \hat{y}_{k+1}^t), 0 \leq k \leq t - 1, i \in I, t \geq 1, \]
\[ \hat{\omega}^0 = (y_0^i, i \in I). \]

An allocation is a tuple \((x_k^t, y_{k+1}^t, c_t^i, 0 \leq t \leq T, i \in I, j \in J)\) such that \((x_k^t, y_{k+1}^t) \in Q_i^t, c_t^i \in C_i^t, \) and \(\sum_j c_t^j + \sum_i x_t^i \leq \sum_i y_t^i, 0 \leq t \leq T.\)

**Theorem (4).** Let \((\hat{\rho}_k, (\hat{x}_k^i, \hat{y}_{k+1}^i), c_t^i, i \in I, j \in J, 0 \leq t \leq T)\) be an equilibrium in a BC model with initial state \((y_0^i, i \in I)\), and positive budgets \(\hat{\rho}_k^i\) and \(\hat{\pi}_i^t.\) If nonsatiation conditions (P2) and (C2) are satisfied, then there are numbers \(\mu_i^t > 0, 0 \leq t \leq T, \psi_i^t > -1, 0 \leq t \leq T - 1, \psi_t^i > 0\) such that \(((\hat{x}_k^t, \hat{y}_{k+1}^t), c_t^i)\) is an optimal solution of the problem
\[ \sum_{t=0}^T \left[ \sum_j \mu_i^t u_i^t(c_t^i) + \sum_i \psi_i^t p_{t+1} y_{t+1}^i \right] \rightarrow \max \] among all allocations \(((x_k^t, y_{k+1}^t), c_t^i)\) with initial state \(y_0^i, i \in I,\) and \(\hat{\rho}_k\) are stimulating prices for path
\[ \left(\sum_i (\hat{x}_k^{t}, \hat{y}_{k+1}^t), \sum_j c_t^j, 0 \leq t \leq T\right) \]
in problem (12) with
\[ U_t(c_t) = \max \left\{ \sum_j \mu_i^t u_i^t(c_t^i) : c_t^i \in C_i^t, \sum_j c_t^j = c_t \right\}, \]
\[ F(z_t) = \max \left\{ \sum_i \psi_i^t p_{t+1} y_{t+1}^i : (x_k^t, y_{k+1}^t) \in Q_i^t, \sum_i (x_k^t, y_{k+1}^t) = z_t \right\}. \]

**Proof.** Since \(\pi_i^t > 0,\) by the Kuhn-Tucker theorem\(^{39}\) there exist \(\lambda_i^t \geq 0\)

\(^{39}\text{Ref. Mathematical Foundations in this paper.}\)
such that, for any $c \in C^t_i$,

$$u^t_i(c) - \lambda^t_i \hat{p}_t c \leq u^t_i(\hat{c}_t) - \lambda^t_i \hat{\pi}^t_i \leq u^t_i(\hat{c}_t) - \lambda^t_i \hat{p}_t \hat{c}_t.$$  

By (C2), $\lambda^t_i > 0$. Thus, for all $c^t_i \in C^t_i$,

$$U_t(c^t_i; j \in J) - \sum_j \hat{p}_t c^t_j \leq U_t(\hat{c}^t_i; j \in J) - \sum_j \hat{p}_t c^t_j,$$  

where $U_t(c^t_i; j \in J) = \sum_j \mu^t_i u^t_i(c^t_j)$ and $\mu^t_i = 1/\lambda^t_i$.

Similarly there exist $\eta^t_i > 0$ such that, for all $z^t_i = (x^t_i, y^t_{i+1}) \in Q^t_i$,

$$\hat{p}_{t+1} y^t_{i+1} - \eta^t_i \hat{p}_t x^t_i \leq \hat{p}_{t+1} \hat{y}^t_{i+1} - \eta^t_i \hat{p}_t \hat{x}^t_i, \quad 0 \leq t \leq T - 1,$$

$$p_{T+1} y^t_{T+1} = \eta^t_T p_T x^t_T \leq \hat{p}_{T+1} \hat{y}^t_{T+1} - \eta^t_T \hat{p}_T \hat{x}^t_T.$$

Hence,

$$F_t(z^t_i, i \in I) + \sum_i (\hat{p}_{t+1} y^t_{i+1} - \hat{p}_t x^t_i)$$

$$\leq F_t(\hat{z}^t_i, i \in I) + \sum_i (\hat{p}_{t+1} \hat{y}^t_{i+1} - \hat{p}_t \hat{x}^t_i),$$

$$F_t(z^t_i, i \in I) = \sum_i (1 - \eta^t_i) \hat{p}_{t+1} y^t_{i+1}, \quad 0 \leq t \leq T - 1,$$

$$F_T(z^t_T, i \in I) = \sum_i (1/\eta^t_T) \hat{p}_{T+1} y^t_{T+1}.$$

Moreover, by (6),

$$\hat{p}_t \left( \sum_i \hat{y}^t_i - \sum_i \hat{x}^t_i - \sum_j \hat{c}^t_j \right) = 0, \quad 0 \leq t \leq T.$$  

Relationships (14)-(16) prove the assertion of the theorem. Q.E.D.

Using the view of equilibrium BC models as optimization problems, we can explain the origin of the structure of prices $p^*_t$ in the CE-to-ME transition model. Assume that the CE model is represented by problem (12) with some functions $U^t_i(c)$ and $F^t_i(z)$, and the ME model by problem (12) with functions
$U_t^2(c)$ and $F_t^2(z)$. If $\theta$ is the change point from CE to ME, the transition model described above is naturally associated with the maximization problem for functional

$$F_\theta \left[ \sum_{t=0}^{\theta-1} F_t^1(c_t^1, z_t^1) + \sum_{t=\theta}^T F_t^2(c_t^2, z_t^2) \right],$$

where $F_t^k(c, z) = U_t^k(c) + P_t^k(z)$, $k = 1, 2$, and expectation $F_\theta$ is taken over the distribution of random variable $\theta$. On the other hand, we know that stimulating prices in this problem have the same weighted structure as prices $p_t^*$ in (8). This property suggests that the price structure in the proposed transition model is "optimal" in a certain sense and that the producer behaves "optimally".
5.9 Example of a transition process[3]

In this section we consider an example of an economy in which transition from one economic mechanism to another induces essential changes in production activity. Although the theorems (1)-(4) are formally not applicable to this example (for instance, we allow zero prices), it is highly useful in describing effects that arise in alternative models of transition from one economic mechanism to another.

Consider a system with three products \((x_1, x_2, x_3)\) and two producers with technologies

\[
Q^1 = \{((x_1, x_2, x_3), (y_1, 0, y_3)), y_1 \leq f_1(x_1, x_2'), y_3 \leq f_3(x_2', x_3), x_2' + x_2'' = x_2\},
\]

as well as

\[
Q^2 = \{((x_1, x_2, x_3), (0, y_2, 0)), \text{ where } y_2 \leq f_2(x_2)\}.
\]

The producer problem is to maximize income subject to corresponding budget constraints:

\[
p_{t+1,1}f_1(x_1, x_2') + p_{t+1,3}f_3(x_2', x_3') \rightarrow \max, \tag{17}
\]

\[
p_{t,1}x_1 + p_{t,2}(x_2' + x_2'') + p_{t,3}x_3 \leq \rho_t^1,
\]

\[
f_2(x_2) \rightarrow \max,
\]

\[
p_{t,3}x_2 \leq \rho_t^2,
\]

where \(p_{t,s}\) is the price of product \(s\), \(s = 1, 2, 3\). With this system we associate the CE and ME models depending on the method of budget formation.

Assume that functions \(f_i, i = 1, 2, 3\), are concave, \(f_1(x_1, x_2)\) and \(f_3(x_2, x_3)\) are strictly increasing in \(x_2\), and moreover, \(f_1(x_1, 0) = f_3(0, x_3) = 0\).
The CE model has an equilibrium with prices $\tilde{p}_t = (\tilde{p}_{t,1}, \tilde{p}_{t,2}, 0)$. Problem (17) thus reduces to the following static problem:

$$ f_1(x_1, x_2) \rightarrow \max, \\
\tilde{p}_{t,1}x_1 + \tilde{p}_{t,2}x_2' \leq \rho_t^1,$$

and $\tilde{p}_{t,1}$ and $\tilde{p}_{t,2}$ are equilibrium prices in the exchange model with two products $(x_1, x_2)$ and two agents with utility functions $f_1(x_1, x_2)$ and $f_2(x_2)$ and fixed budgets $\alpha_t^1$.

The competitive market economy (ME) model, on the other hand, has an equilibrium with prices $\tilde{p}_t = (0, \tilde{p}_{t,2}, \tilde{p}_{t,3})$. The first-agent problem thus takes this form:

$$ f_3(x_2, x_3) \rightarrow \max, \\
\tilde{p}_{t,2}x_2 + \tilde{p}_{t,3}x_3 \leq \rho_t^1.$$

Since $\rho_t^1 = p_{t,3}y_t^1$ and $\rho_t^2 = p_{t,2}y_t^1$, prices $\tilde{p}_{t,2}$ and $\tilde{p}_{t,3}$ are equilibrium prices in the standard exchange model with two products $(x_2, x_3)$ and two agents with utility functions $f_3$ and $f_2$ and initial stock $(y_t^1, y_t^1)$.

The first agent's behaviour depends on the particular model used: in the CE model the first agent produces only the first product, while in the ME model he produces only the third product.

Let us now consider alternative transitions from CE to ME. In case of an unexpected shock, the first agent is left with zero budget because the price of the product that he produces drops to zero. If the instantaneous transition is declared, say, at time $t + 1$, then the first agent allowing for future prices at time $t$ solves the following problem:

$$ f_3(x_2, x_3) \rightarrow \max, \\
\tilde{p}_{t,2}x_2 \leq \rho_t^1. \quad (18)$$

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If function $f_3$ is unbounded in $x_3$, then this problem is unsolvable. If $f_3$ is bounded and solution $\tilde{x}_3$ is non-zero, no equilibrium exists at time $t$, because beforehand the supply of the third product was zero. Thus, any kind of instantaneous transition in this example does not yield satisfactory results.

These effects are eliminated in the proposed transition model. Indeed, main producer problem (8) for the first agent takes the following form:

$$p_{t+1,1} f_1(x_1, x_2') + (p_{t+1,3} + \hat{p}_{t+1,3}(1 - q_{t+1})/q_{t+1}) f_3(x_2'', x_3) \rightarrow \max,$$

$$p_{t,1} x_1 + p_{t,2} (x_2' + x_2'') + p_{t,3} x_3 \leq \rho^1_t.$$  

To establish that $f_3(\tilde{x}_2'', \tilde{x}_3) > 0$, we rewrite the last problem in more general form:

$$a f(x, y) + bg(z, w) \rightarrow \max,$$

$$cx + d(y + z) + ew \leq \rho,$$

where $a = p_{t+1,1}$, $f(x, y) = f_1(x_1, x_2')$, $b = (p_{t+1,3} + \hat{p}_{t+1,3}(1 - q_{t+1})/q_{t+1})$, $g(z, w) = f_3(x_2'', x_3)$, and $b > 0$ because $\hat{p}_{t+1,3} > 0$. Using the Lagrange multiplier method, it can be shown that, if $\partial g(0, w)/\partial z = +\infty$, then the optimal $z^*$ is positive. Thus, if we assume that $\partial f_3(0, x_3)/\partial x_2 = +\infty$ for any $x_3$, then $f_3(\tilde{x}_2'', \tilde{x}_3)$ is positive on the optimal solution.

Thus, in our example, the agent during the transition period should produce a positive quantity of a product that is unprofitable in the Centralized Economy model, but will be profitable in the future Competitive Market model.
Appendix 1

**Theorem [28]** (*Gale* [15], *Nikaido* [27], *Debreu* [4]).

Let $P_n = \left\{ p \mid p \geq 0, \sum_{j=1}^{n} p_j = 1 \right\}$ — standard simplex in $\mathbb{R}^n$, $\Gamma$ — some compact convex set in $\mathbb{R}^n$, and let fix some multi-valued mapping $\chi : P_n \to 2^{\Gamma}$, which for simplicity we will call an excess demand function. Let us assume that this mapping satisfies the following two conditions:

(a) a mapping $\chi : P_n \to 2^\Gamma$ is closed and maps each point of the simplex $P_n$ into nonempty convex subset of the set $\Gamma$.

(b) Walrasian law in a broad sense holds, i.e.

$$\langle p, u \rangle \geq 0$$

with $u \in \chi(p)$. Then there exists a vector $\tilde{p} \in P_n$ such that

$$\chi(\tilde{p}) \cap R_+^n \neq \emptyset.$$
Appendix 2

1. The Main Theorem of Existence

1.1. Notations.

Further by $\mathbb{R}^n$ we have $n$-dimensional Euclidean space with a scalar product $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, norm $\|x\| = \sqrt{\langle x, x \rangle}$ and metric $\rho(x, y) = \|x - y\|$ ($x = (x_i)$, $y = (y_i) \in \mathbb{R}^n$). Let us adopt the following notations: $int X$ - interior in $\mathbb{R}^n$; $\partial X$ - boundary in $\mathbb{R}^n$; $\overline{X}$ - closure; $X'$ - set of limit points; $co X$ - convex hull; $\Pi(X)$ - set of all non-empty subsets of some set $X \subseteq \mathbb{R}^n$.

**Definition 1.2.** Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ - some sets. The point-set mapping $F: X \to \Pi(Y)$ is called standard if following conditions hold:

1) mapping $F$ is closed, i.e. the conditions $x^* \to x$, $y^* \to y$, $y^* \in F(x^*)$ imply that $y \in F(x)$;

2) for any $x \in X$ the set $F(x)$ is convex;

3) for any compacta $X_1 \subseteq X$ the set $F(X_1) = \bigcup_{x \in X_1} F(x)$ is also compact.

In particular, any continuous single-valued function is standard.

**Definition 1.3.** The set $X \subseteq \mathbb{R}^n$ is called $V$-set, if an interior of $X$ is not empty and convex.

**Lemma 1.4.** Let $X \subseteq \mathbb{R}^n$ - convex body and $0 \in int X$. Let us take the standard mapping $F: X \to \Pi(Y)$ for which $0 \notin F_\lambda(X)$. Then there exist vector $x \in \partial X$ and index $\lambda > 0$ such that $\lambda x \in F(x)$.

**Lemma 1.5.** Let us consider $V$-set $Y \subseteq \mathbb{R}^n$ and bounded set $Y_0 \subseteq Y$ for which $\partial Y \cap Y_0 = \emptyset$. Then there exists such convex body $Y_1 \subseteq Y$ that $\partial Y_1 \cap Y_0 = \emptyset$, moreover $Z = Y_0 \cap \partial Y \subseteq int Y_1$.

1.6. For an arbitrary point-set mapping $F: X \to \Pi(R^n)$, where $X \subseteq R^n$ and an arbitrary point $x^0 \in \mathbb{R}^n$ we define the following sets:

$$R_{F, x^0} = \{x \in X \mid \lambda(x - x^0) \in F(x), \exists \lambda > 0\}$$
and
\[ Q_{F,x^0} = \{ x \in X \mid -\lambda x^0 \in F(x), \exists \lambda > 0 \}. \]

**Theorem 1.7 (Main Theorem).** Let \( F : X \to \prod(R^n) \) \-- standard mapping defined on \( V \)-- set \( X \subseteq R^n \). Let us assume that with some \( x^0 \in \text{int} \ X \) there we have

1) \( \partial X \cap R'_{F,x^0} = \emptyset \);

2) set \( R_{F,x^0} \) is bounded.

Then \( 0 \in F(x) \).

**Proof.** Let us consider set \( Y = X - x^0 \) and the standard mapping \( \tilde{F} : Y \to \prod(R^n) \) such that
\[ \tilde{F}(y) = F(y + x^0) \ (y \in Y). \]
Then the following statements are valid: \( Y \) is a \( V \)--set; \( 0 \in \text{int} \ Y \); set
\[ R_{\tilde{F},0} = R_{F,x^0} - x^0 \] is bounded and \( \partial Y \cap R'_{F,x^0} = \emptyset \).

By the Lemma 1.5 for the set \( Y_0 = \tilde{F} \circ \{0\} \subseteq Y \) we have that there exists a convex body \( Y_1 \subseteq Y \) such that
\[ \partial Y_1 \cap R_{\tilde{F},0} = \emptyset, \] (a)
and moreover \( 0 \in \text{int} \ Y_1 \). Let us assume that \( 0 \notin \tilde{F}(Y_1) \). Then applying Lemma 1.4 for the mapping \( \tilde{F} : Y_1 \to \prod(R^n) \), we have a contradiction in (a).

Therefore, \( 0 \in \tilde{F}(Y_1) \subseteq F(X) \). **Q.E.D.**

2. **Regularity on the boundary of mapping.**[35]

**Definition 2.1.** Mapping \( F : X \to \prod(R^n) \), defined on the \( V \)--set \( X \subseteq R^n \), is called regular on some set \( X_1 \subseteq \partial X \), if:

1) for any \( x \in X_1 \cap X \) and any \( y \in F(x) \) there is a hyperplane of support \( (p,\beta)^{40} \) to a set \( X \) in a point \( x \), such that \( \langle p, y \rangle \geq 0 \);

\[ ^{40} \text{By the hyperplane of support to a set } X \text{ in a point } x \in \partial X, \text{ we understand a pair } (p, \beta), \text{ where } p \in R^n, \beta \in R^1, \text{ such that } \langle p, x \rangle = \beta \text{ and } \langle p, x' \rangle \geq \beta \text{ for any } x' \in X. \]

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2) for any sequences \( x^s \rightarrow x \in X_1 \setminus X \) \((x^s \in X)\) and \( y^s \in F(x^s) \) there is a hyperplane of support \((p, \beta)\) to a set \(X\) in a point \(x\), such that \(\limsup_{s \to \infty} (p, y^s) > 0\).

**Definition 2.2.** If \( X = R^n_+ \setminus T \), where \( T \subseteq \partial R^n_+ \) and \( X_1 = \partial R^n_+ \), then conditions 1) and 2) of the definition 2.1 are taking the following equivalent form:

1) for any \( x \in \partial R^n_+ \setminus T \) and any \( y \in F(x) \) there is an index \( i \in K(x) = \{i \in \{1, 2, \ldots, n\} \mid x_i = 0\} \) such that \( y_i \geq 0\);

2) for any sequences \( x^s \rightarrow x \in T \) and \( y^s \in F(x^s) \) there is an index \( i \in K(x) \) such that \(\limsup_{s \to \infty} y^s_i > 0\).

**Definition 2.3.** \( V \)-set \( K \subseteq R^n \), which is a cone with a vertex in 0, we call a \( V \)-cone. Any hyperplane of support \((p, \beta)\) to a \( V \)-cone is going through the origin (i.e. \( \beta = 0 \)). There are two conclusions from here: (\( \alpha \)) (invariance) if a mapping \( F : K \to \prod(R^n) \), defined on the \( V \)-cone \( K \subseteq R^n \), is regular on some set \( X_1 \subseteq \partial K \), then for any real bounded function \( \lambda(x) \) defined on \( K \), the mapping \( \tilde{F} : K \to \prod(R^n), \tilde{F}(x) = F(x) - \lambda(x)x \) \((x \in K)\), is also regular on \( X_1 \); (\( \beta \)) let \( K \subseteq R^n \) - closed \( V \)-cone, then any mapping \( F : K \to \prod(R^n) \) such that \( F(X_1) \subseteq K \) \((X_1 \subseteq \partial K)\) is regular on \( X_1 \) (in particular, a mapping of the cone \( K \) "into itself" \( F : K \to \prod(K) \) is regular on the all bound \( \partial K \)).

**Theorem 2.4 (special case of the Theorem 1.7).** Let \( F : X \to \prod(R^n) \) - standard mapping defined on the \( V \)-set \( X \subseteq R^n \) and \( X_1 \subseteq \partial X \). Assume that

1) mapping \( F \) is regular on \( X_1 \);

2) \((\partial X \setminus X_1) \cap R'_{F,x^0} = \emptyset\), and

3) set \( R_{F,x^0} \) is bounded with some \( x^0 \in \text{int} X \). Then \( 0 \in F(X) \).

**Proof.** Let us show that \( X_1 \cap R'_{F,x^0} = \emptyset \). Presume that it is not true.
Then there exist such sequences of vectors $x^s \in X$ and numbers $\lambda^s > 0$ such that

$$\lambda^s (x^s - x^0) \in F(x^s), \ s = 1, 2, \ldots, \quad (b)$$

moreover $x^s \to x \in X_1$.

Assume that $x \in X_1 \cap X$. Let us consider a compact set $X_0 = \{x, x^1, \ldots, x^s, \ldots\}$. By the condition of the theorem a mapping $F$ is standard, therefore $F(X_0)$ is compact, and from (b) we have that the sequence $\lambda^s$ is bounded. Without loss of generality we can say that $\lambda^s \to \lambda \geq 0$. Taking a limit in (b), we have $\lambda(x - x^0) \in F(x)$. If $\lambda = 0$, then theorem is proved. Let $\lambda > 0$. The regularity of the mapping $F$ on the set $X_1$ implies that there exists a hyperplane of support $(p, \beta)$ to a set $X$ in a point $x$, such that $\langle p, y \rangle \geq 0$, where $y = \lambda(x - x^0) \in F(x)$. But for the same $y$ the following holds

$$\langle p, y \rangle = \lambda \left( \langle p, x \rangle - \langle p, x^0 \rangle \right) < 0,$$

since $\langle p, x^0 \rangle > \beta$. Using the condition 2) of the definition 2.1 we have a similar contradiction, but in a limit form, if we assume that $x \in X_1 \setminus X$.

Thus, $\partial X \cap R'_{F, x^0} = \emptyset$, and therefore, all the conditions of the theorem 1.7 hold. Q.E.D.

3. Generalized Walrasian Condition.\[35\]

**Definition 3.1.** A continuous real function $\Phi(x, y)$ defined on $K \times R^n$, where $K \subseteq R^n$—closed $V$—cone, is called a Walrasian function (in the restricted sense) if the following conditions apply:

1) a function $\Phi(x, y)$ is homogeneous of the nonnegative power on any of the variables $x$ and $y$ individually;

2) for any $x \in K \setminus \{0\}$ $\Phi(x, x) > 0$;

3) for any $x \in K \setminus \{0\}$ and any $y \in \text{int} (-K)$ $\Phi(x, y) < 0$. 

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Definition 3.2. A mapping \( F : K \to \prod (\mathbb{R}^n) \) defined on the \( V^- \) cone \( K \subseteq \mathbb{R}^n \) is subject to a generalized Walrasian condition in a restricted sense, if there exists a Walrasian function \( \Phi(x, y) \) defined on \( K \times \mathbb{R}^n \) such that the following fulfilled
\[
\Phi(x, y) = 0, \text{ for any } x \in K, \ y \in F(x).
\]

(*)

3.3. Examples. The link between introduced concepts and the name Walras is explained by the fact that in a case, when \( \Phi(x, y) = \langle x, y \rangle \), \( K = \mathbb{R}^n_+ \), a condition (*) is really a Walrasian condition, which plays a fundamental role in theory of economical equilibrium.

Functions
1) \( \Phi(x, y) = \sum_{i=1}^{n} x_i^\alpha y_i, \ \alpha \geq 0, \ K = \mathbb{R}^n_+ \);
2) \( \Phi(x, y) = \max_{i=1,2,\ldots,n} y_i, \ K = \mathbb{R}^n_+ \);
3) \( \Phi(x, y) = \langle x, y \rangle, \ K \subseteq K^* = \{ y \in \mathbb{R}^n \ | \ \langle x, y \rangle \geq 0 \ \text{for any} \ x \in K \} \);
4) \( \Phi(x, y) = \langle x, y \rangle, \ \bar{x} \in \text{int} \ K^* \)

are also Walrasian functions.

Theorem 3.1. Let \( F : K \to \prod (\mathbb{R}^n) \) standard mapping defined on the \( V^- \) cone \( K \subseteq \mathbb{R}^n \). Assume that 1) mapping \( F \) is regular on \( \partial K \setminus \{0\} \) and 2) satisfies a generalized Walrasian condition in a restricted sense. Then \( 0 \in F(K) \).

Proof. Let us fix an arbitrary vector \( x^0 \in \text{int} \ K \), then for some Walrasian function \( \Phi(x, y) \) defined on \( K \times \mathbb{R}^n \) and any \( x \in R_{F,x^0} \)
\[
\Phi(x, x - x^0) = 0.
\]

(***)

Let us assume that \( 0 \in R_{F,x^0} \), and therefore exists such sequence \( x^s \in R_{F,x^0} \) that \( x^s \to 0 \) and \( x^s \in K \setminus \{0\} \). Because \( 0 \in x^0 - \text{int} \ K \), then exists a number \( S \), for which \( x^s \in x^0 - \text{int} \ K \), or \( x^s - x^0 \in \text{int} (-K) \) and, therefore \( \Phi(x^s, x^s - x^0) < 0 \). But this contradicts to formula (***) with \( x = x^s \in R_{F,x^0} \).
Let us assume now, that the set $R_{F,x_0}$ is not bounded, that is there will be some sequence $x^s \in R_{F,x_0}$ such that $\|x^s\| \to \infty$. Using the homogeneity of the function $\Phi(x,y)$ we can rewrite an equality (**) in the following form

$$\Phi \left( \frac{x^s}{\|x^s\|}, \frac{x^s}{\|x^s\|} - \frac{x^0}{\|x^s\|} \right) = 0, \ s = 1, 2, \ldots$$

(***)

Without loss of generality we can say that $\frac{x^s}{\|x^s\|} \to x \in \overline{K} \setminus \{0\}$.

Taking now a limit in the equality (***) , we have $\Phi(x,x) = 0$. But this contradicts to the condition 2) of the definition 3.1 (Walrasian function).

Thus, all the conditions of the theorem 2.4 with $X = K$ and $X_1 = \partial K \setminus \{0\}$ are satisfied. Therefore, $0 \in F(K)$.

In particular, the theorem 3.1, definition 2.2 and the statement 3.3, imply the following assertion of the Gale’s lemma.

**Theorem**$^{41}$. Let for standard mapping $F : \text{int} \ R^n_+ \to \prod(R^n)$ the following conditions hold:

1) for any sequences $x^s \to x \in \partial R^n_+$ and $y^s \in F(x^s)$ there is an index $i \in K(x)$ such that $\limsup_{s \to \infty} y^s_i > 0$;

2) for any $x \in \text{int} \ R^n_+$ and $y \in F(x)$ the following equality holds

$$\langle x, y \rangle = 0.$$

Then $0 \in F \text{ int} \ R^n_+$.

This theorem makes it easier to prove the existence of the equilibrium in some models, because there will be no need in extension of the excess demand function to the closure of the price domain.

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$^{41}$General case of the Theorem 2 in a section 5.3 of this work.
CONCLUSION

In this work an Equilibrium problem in the transition from one type of economic mechanism to another was discussed. The general equilibrium theory is applied to model the transition from a centralized (budget-controlled) economy to a competitive market based on the publication of V. I. Arkin and A.D. Slastnikov (Moscow)[3]. A many-partner system with two possible modes of functioning was studied. The first one involves centralized budget distribution according to certain priorities of the "center". In the second one, the partners' budgets are formed as a result of their own activity (a competitive market). The problem consists of constructing a transition process that develops within the framework of a centralized economy and adapts the system to a market mechanism.

Also the following topics of theoretical economics were thoroughly studied:

2. Economical Models, short descriptions.
3. Economic Performance During Transition. (The paper of Hans Pitlik[20] in INTERECONOMICS, January/February 2000, has been studied.)

Mathematical Foundations of theory of Sets, Topological Spaces, Probability, Game theory, and Equilibrium Analysis have been exposed.

The general equilibrium theory was also applied to prove the existence of an equilibrium in the transition process that, in a certain sense, makes it possible for agents to "adapt" the choice of technology to the change in economic mechanism.

At the end of the paper there is an example of a transition process in attempt to describe effects that arise in alternative methods of transition from one economic mechanism to another.
In this dissertation I have made an attempt to look at the mathematics as *motivated mathematics*, i.e. a mathematics motivated by economics and game theory.
References


/London School of Economics. 1975, The MacMillan Press LTD.


[23] Jie Wu. Lecture Notes on Advanced Calculus II. Department of Mathematics, National University of Singapore.


[28] Nikaido Hukukane, "Vipuklie structuri i matematicheskaja economika." Moscow, "Mir", 1972


[34] Shirjayev A. H. Verojatnost. 1989, M. - NAUKA. (Russian Edition.)


