

ERGODIC TYPE THEOREMS IN OPERATOR  
ALGEBRAS

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# Chapter 0

## Introduction

### 0.1 Overview of Ergodic Type Theorems and positioning of the work

Ergodic type theorems is a branch of Ergodic Theory. In a narrow sense ergodic theorems usually mean limit properties of certain averages of the iterates of some operators.

Traditionally, Ergodic Type Theorems include the following topics:

- a) Mean Ergodic Theorems and weak convergence
- b) Positive Contractions in  $L_1$
- c) Pointwise Ergodic Theorems for general groups
- d) Local Ergodic Theorems and differentiation
- e) Convergence of subsequences and generalized means

For more detailed information I refer the reader to the monograph of prof. U. Krengel [44].

This work contributes to the following three topics of the Ergodic Type theorems in the context of von Neumann algebras:

Criteria relating weak convergence of iterates to norm convergence of the regular averages, are established

Pointwise ergodic theorems for the action of a free group of a finite number of generators is provided.

Lastly, Convergence averages over Besicovitch subsequences of iterates is considered.

## 0.2 Historical remarks and detailed results of the work

The first Ergodic Theorems for actions of the general semigroups are probably due to Alaoglu and Birkhoff [4], who replaced the iterates  $T_i$  by a semigroup  $G$  of linear transformations and showed that the convergence of certain general means of transforms of an element  $x$  of  $E$  is equivalent to the existence (and uniqueness) of a fixed point  $y$  in the closed convex hull of the orbit of  $x$  under  $G$ .

There are a number of monographs devoted (in part) to the topic of Ergodic Theorems in Operator Algebras. Book [44] by Krengel contains chapter 9 devoted to an exposition of the subject of Ergodic Theorems in Operator Algebras as it was on 1984. Monograph [36] by R. Jajte is devoted to an exposition of the (quasi)-Super-additive Ergodic Theorem for von Neumann Algebras.

One of the most citable mathematical books [12] by Alain Connes contains a section V.6 devoted to Noncommutative Ergodic Theory. The main subject of the

section is a classification of the automorphisms of von Neumann algebras up to internal or external conjugacy that is later applied to the classification of von Neumann algebras.

Importance of Ergodic Type Theorems from a mathematical physics point of view is in the existence and uniqueness of the vacuum state of a quantum system, see for example books by Bratteli and Robinson [9], Benatti [7], or the paper by Narnhofer, Thirring, Wiclicki [45].

The first results in the field of non-commutative ergodic theory were obtained independently by Sinai and Anshelevich [63] and Lance [48]. Developments of the subject are reflected in the monographs of Jajte [36] and Krengel [44] (see also [20], [21], [30], [55]).

Chapter 1 of this work is devoted to a presentation of some results concerning ergodic type properties of weak convergence of iterates of operators acting in  $L_1$  space for general von Neumann algebras and  $JBW$ -algebras, as well as Segal - Dixmier  $L_p$ -spaces ( $1 \leq p < \infty$ ) of operators affiliated with semifinite von Neumann algebras and semifinite  $JBW$ -algebras. The results of the section 2, chapter 1 complete arguments of [40].

Ergodic Theorems for actions of arbitrary countable groups were obtained by Oseledets [53], who followed an idea of Kakutani [39]. For actions of free groups Guivarc'h [33] considered uniform averages over spheres of increasing radii in a group and proved a related mean ergodic theorem; Grigorchuk [31] announced a Pointwise Ergodic Theorem for Česaro averages of the spherical averages. Nevo [51], and Nevo and Stein [52] published a proof of the Pointwise Ergodic Theorem. In [32] Grigorchuk announced an Ergodic Theorem for Actions of Free Semigroups. In [11] Bufetov

generalized classical and recent Ergodic Theorems of Kakutani, Oseledets, Guivarc'h, Grigorchuk, Nevo, Nevo and Stein for measure-preserving actions of free semigroups and groups.

In chapter 2, Bufetov's results from [11] are generalized to the non-commutative case to obtain non-commutative Ergodic Theorems for the actions of finitely generated semigroups on von Neumann algebras with faithful normal finite tracial state.

The chapter 2 proceeds as follows. Operator ergodic type theorems for finite von Neumann algebras are formulated in Section 1 of the chapter. Section 2 of the chapter is devoted to the proof of convergence for the theorems of section 1. Section 3 is devoted to the application of Guivarch's approach to establishing the invariancy of the limit. Section 4 contains extensions of Kakutani's results to the setting of finite von Neumann algebras. Section 5 contains an example of application of the results of the chapter.

In chapter 3 the Stochastic Banach Principle is established . The Banach Principle is one of the most useful tools in "classical" point-wise ergodic theory. The Banach principle was used to give an alternative proof of the Birkhoff- Khinchin individual ergodic theorem. Typical applications of the Banach Principle are Sato's theorem for uniform subsequences [62] and the individual ergodic theorem for the Besicovitch Bounded sequences [59] .

Non-commutative analogs for (bilateral) almost everywhere convergence may be found in papers [23], [13]. In chapter 3 a Banach Principle for convergence in measure (Stochastic Banach Principle) is established. Based on this principle a simplified proof of Stochastic Ergodic Theorem (compare with [30]) is offered. Stochastic convergence for Sato's uniform subsequences (Theorem 3.3.6) and a Stochastic Ergodic Theorem

for the Besicovitch Bounded sequences (Theorem 3.3.5) are established.

Note that the Stochastic Banach Principle results are new even in the commutative case. Indeed, it is well known (see for example [44]) that there are Cesaro averages constructed for automorphisms that converge in measure and do not converge almost everywhere. This implies that although the condition of Stochastic Ergodic Theorem is satisfied in this case, the pointwise Banach Principle condition is not.

The results of the chapter 1 were published in [28] and the results of the chapter 2 were published in [25]. Results of the chapter 3 were accepted for publication in *Studia Mathematica* [24].

Text of the papers is based on the text of the thesis, chapters 1, 2 and 3 accordingly.

Results of the work were presented in referred conferences [26] and [29].

### 0.3 Preliminary

This section contains notions and notations that are used further in the thesis.

We will use facts and terminology from the general theory of von Neumann algebras ([9], [15], [54], [60], [65]), and the theory of non-commutative integration ([61],[70]).

Let  $M$  be a von Neumann algebra, acting on a separable Hilbert space  $H$ ,  $M_*$  is a pre-dual space of  $M$ , which always exists according to the Sakai theorem [60]. It is well known that  $M_*$  can be identified with  $L_1$ -space for  $M$ .

Spaces  $L_1$  and  $L_2$  of the operators affiliated with the semifinite von Neumann algebra  $M$  with semifinite faithful trace  $\tau$  were introduced by Segal (see [61]). This result was extended to  $L_p$  spaces of operators affiliated with von Neumann algebra  $M$ ,  $\tau$  by Dixmier (see [14]). For an alternative exposition of building  $L_p$  based on

Grothendieck's idea of using rearrangements of functions see also [70]. The theory of  $L_p$  spaces was extended further to von Neumann algebras with faithful normal semifinite weight  $\rho$ . However, these spaces lack some of the properties inherent in the commutative case, for example, in general, these spaces have a trivial intersection.

Recall some standard terminology ([20], [21], [30], [44]).

**Definition 0.3.1.** A linear mapping  $T$  from  $M_*$  into itself is called a **contraction** if its norm is not greater than one.

**Definition 0.3.2.** A contraction  $T$  is said to be **positive** if

$$TM_{*+} \subset M_{*+}. \quad (0.3.1)$$

Let the pair  $(M, \tau)$  be a **non-commutative probability space**, where  $M$  is a von Neumann algebra with a faithful, normal tracial state  $\tau$ .

Let

$$\alpha_1, \alpha_2, \dots, \alpha_m : M \longmapsto M$$

be **positive kernels** or linear maps satisfying following conditions:

$$(\alpha_i(M_+) \subset M_+; \quad \alpha_i \mathbf{1} \leq \mathbf{1}; \quad \tau \circ \alpha_i \leq \tau). \quad (0.3.2)$$

All the  $\{\alpha_i\}$ 's could be extended to operators

$$L_1(M, \tau) \longmapsto L_1(M, \tau),$$

which we will also call  $\{\alpha_i\}$ . The meaning of  $\{\alpha_i\}$  will be clear from the context.

Denote by  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ .

**Definition 0.3.3.** A densely defined closed operator  $x$  affiliated with von Neumann algebra  $M$  is called  **$(\tau)$  measurable** if for every  $\epsilon > 0$  there exists projection  $e \in P(M)$  with  $\tau(\mathbb{I} - e) < \epsilon$  such that  $e(\mathfrak{H}) \subset \mathfrak{D}(x)$ , where  $\mathfrak{D}(x)$  is a domain of  $x$ .

Recall the following definitions (combined from the papers by Segal [61], Nelson [50], Yeadon [70], Fack and Kosaki [18]):

**Definition 0.3.4.** A sequence  $\{x_n\} \subset L_1(M, \tau)$  is said to converge to  $x_0 \in L_1(M, \tau)$  bilateral almost everywhere if for every  $\epsilon \geq 0$  and  $\delta \geq 0$  there exists  $N \in \mathbb{N}$  and projection  $e \in M$  such that  $\tau(\mathbb{I} - e) < \delta$  and  $e(x_n - x_0)e \in M$  and for  $n \geq N$

$$\|e(x_n - x_0)e\|_\infty \leq \epsilon \quad (0.3.3)$$

*Remark 0.3.1.* We mention that another name for bilateral almost everywhere convergence is double side almost everywhere convergence.

Space of all  $(\tau)$  **measurable operators affiliated with  $M$**  is denoted by  $S(M)$ .

For convenience for a self-adjoint  $x \in S(M)$  we denote by  $\{x > t\}$  the **spectral projection of  $x$  corresponding to the interval  $(t, \infty]$** .

**Definition 0.3.5.** Sequence  $\{x_n\}_{n=1}^\infty$  **converges to 0 in measure (stochastically)** if for every  $\epsilon > 0$  and  $\delta > 0$  there exists an integer  $N_0$  and a set of projections  $\{e_n\}_{n \geq N_0} \subset P(M)$  such that  $\|x_n e_n\|_\infty < \epsilon$  and  $\tau(\mathbb{I} - e_n) < \delta$  for  $n \geq N_0$ .

*Remark 0.3.2.* We will use terms **converges in measure** and **converges stochastically** interchangeably.

# Chapter 1

## Weak Convergence of Iterates of Operators Acting in Preduals of von Neumann Algebras

### 1.1 Preliminaries

We will consider the two topologies on the space  $M_*$ : the *weak topology* , or the  $\sigma(M_*, M)$  topology, and the *pointwise norm topology* of the  $M_*$ -space norm convergence .

**Definition 1.1.1.** A matrix  $(a_{n,i})$  ,  $i, n = 1, 2, \dots$  of real numbers is called **uniformly regular** , if:

$$\sup_n \sum_{i=1}^{\infty} |a_{n,i}| \leq C < \infty; \quad (1.1.1)$$

$$\lim_{n \rightarrow \infty} \sup_i |a_{n,i}| = 0; \quad (1.1.2)$$

$$\lim_{n \rightarrow \infty} \sum_i a_{n,i} = 1. \quad (1.1.3)$$

## 1.2 The case of non-commutative $L_1$ -spaces

The following theorem is valid:

**Theorem 1.2.1.** *The following conditions for a positive contraction  $T$  in the pre-dual space  $M_*$  of a complex von Neumann algebra  $M$  are equivalent:*

- i). *The sequence  $\{T^i x\}_{i=1,2,\dots}$  converges weakly, for every  $x \in M_*$*
- ii). *For each strictly increasing sequence of natural numbers  $\{k_i\}_{i=1,2,\dots}$ ,*

$$n^{-1} \sum_{i \leq n} T^{k_i}, \quad (1.2.1)$$

*converges pointwise in norm topology,*

- iii). *For any uniformly regular matrix  $(a_{n,i})$ , the sequence  $\{A_n(T)\}_{n=1,2,\dots}$ ,*

$$A_n(T) = \sum_i a_{n,i} T^i, \quad (1.2.2)$$

*converges pointwise in norm topology.*

We first prove the following Lemma:

**Lemma 1.2.2.** *Let there exist a uniformly regular matrix  $(a_{n,i})$  such that for each strictly increasing sequence  $\{k_i\}_{i=1,2,\dots}$  of natural numbers,*

$$B_n = \sum_i a_{n,i} T^{k_i}, \quad (1.2.3)$$

*converges pointwise in norm topology. Then the sequence  $\{T^i x\}_{i=1,2,\dots}$  converges weakly for every  $x \in M_*$ .*

*Proof.* Let  $(a_{n,i})$  be a matrix with the aforementioned properties. Then the limit  $B_n$  is not dependent upon the choice of the sequence  $\{k_i\}_{i=1,2,\dots}$ . In fact, let  $\{k_i\}_{i=1,2,\dots}$  and  $\{l_i\}_{i=1,2,\dots}$  be increasing sequences for which limits  $B_n$  are different. This means that for some  $x \in M_*$ ,

$$\sum_i a_{n,i} T^{k_i} x \rightarrow x_1, \quad (1.2.4)$$

and

$$\sum_i a_{n,i} T^{l_i} x \rightarrow x_2, \quad (1.2.5)$$

for  $n \rightarrow \infty$ , where  $x_1 \neq x_2$ . For the matrix  $(a_{n,i})$  let us build increasing sequences  $\{i_j\}_{j=1,2,\dots}$  and  $\{n_j\}_{j=1,2,\dots}$ , such that

$$\lim_{j \rightarrow \infty} \left( \sum_{i < i_{j-1}} |a_{n_j, i}| + \sum_{i > i_j} |a_{n_j, i}| \right) = 0. \quad (1.2.6)$$

Let

$$m_i = k_i \text{ for } i \in [r_{4p}, r_{4p+1}) \text{ and } m_i = l_i \text{ for } i \in [r_{4p+2}, r_{4p+3}), j = 1, 2, \dots \quad (1.2.7)$$

Here the sequence  $\{r_{4p+q}, q = \overline{0, 3}, p = 1, 2, \dots\}$  is an increasing subsequence of the sequence  $\{i_j\}_{j=1,2,\dots}$ .

Intervals  $(r_{4p+q}, r_{4p+q+1}), q = \overline{0, 3}$  are chosen to ensure that the sequence  $\{m_i\}_{i=1,2,\dots}$  is increasing.

This condition may be proved as shown by the following construction. Choose  $r_4 = i_4$  and  $r_5 = i_5$  (and for  $p = 1, q = 0, 1$ ). For  $i \in [r_4, r_5)$  we set  $m_i = k_i$ . Skipping if necessary a few indexes  $i_j$  we can ensure that  $l_i > k_{r_5}$  for  $i \geq i_{j'}$  and  $j' \geq 6$ . Set  $r_{4p+2} = i_{j'}, r_{4p+3} = i_{j'+1}$  for  $p = 1$ .

Suppose that sequence  $\{r_{4p+q}, q = \overline{0, 3}\}$  is built for  $p = \overline{1, s-1}$ . In order to build  $r_{4p}$  and  $r_{4p+1}$  for  $p = s$ , we skip if necessary a few indexes of  $i_j$  to ensure that

$k_i > l_{r_{4p-1}}$  for  $i \geq i_{j'}$  and  $i_{j'} > r_{4p-1}$  (this is possible since sequence  $\{k_i\}$  is increasing to infinity). Set  $r_{4p} = i_{j'}$  and  $r_{4p+1} = i_{j'+1}$ . Repeating previous arguments, in order to build  $r_{4p+2}$  and  $r_{4p+3}$  for  $p = s$ , we skip if necessary a few indexes of  $i_j$  to ensure that  $l_i > k_{r_{4p+1}}$  for  $i \geq i_{j'}$  and  $i_{j'} > r_{4p+1}$  (this is possible since sequence  $\{l_i\}$  is increasing to infinity). Set  $r_{4p+2} = i_{j'}$  and  $r_{4p+3} = i_{j'+1}$ . Hence the required sequence  $m_i$  is built.

On replacing sequence  $i_j$  with subsequence  $r_p$ , 1.2.6 and 1.2.7 then imply

$$\lim_p \left\| \sum_r a_{n_{4p}, r} T^{m_r} x - x_1 \right\| = 0, \quad (1.2.8)$$

$$\lim_p \left\| \sum_r a_{n_{4p+2}, r} T^{m_r} x - x_2 \right\| = 0, \quad (1.2.9)$$

which contradicts (1.2.3), and therefore  $x_1 = x_2$ .

Suppose now that

$$\sum_i a_{n,i} T^{k_i} x \rightarrow x_1, \quad (1.2.10)$$

to the same  $x_1$  for every sequence  $\{k_i\}$ , however that the sequence  $T^n x$  does not converge weakly to  $x_1$ , or, in other words, there exists  $y \in M$  such that  $(T^n x - x_1, y)$  has an accumulation point  $\gamma$  other than 0. We choose a subsequence  $\{k_i\}$  such that

$$(T^{k_i} x - x_1, y) \rightarrow \gamma \neq 0, \quad (1.2.11)$$

where  $\gamma$  is a complex number. Then, from the uniform regularity of the matrix  $(a_{n,i})$

it follows that

$$\lim_n \left( \sum_i a_{n,i} T^{k_i} x - x_1, y \right) = \gamma, \quad (1.2.12)$$

which contradicts the choice of the matrix  $(a_{n,i})$  and (1.2.10).  $\square$

*Proof of the Theorem 1.* The implication  $iii) \implies ii)$  is trivial, because the matrix  $(a_{n,i})$ ,  $a_{n,i} = \frac{1}{n} \sum_{j \leq n} \delta_{j,k_i}$  is uniformly regular. Applying the above Lemma 1.2.2 to the matrix  $a_{n,i} = \frac{1}{n}$ ,  $i \leq n$  and  $a_{n,i} = 0$  for  $i > n$ , we get the implication  $ii) \implies i)$ .  $\square$

To prove the implication  $i) \implies iii)$ , we need the following Lemma:

**Lemma 1.2.3.** *Let  $Q$  be a contraction in the Hilbert space  $H$ . Then the weak convergence of  $Q^n x$  in  $H$ , where  $x \in H$ , implies the norm convergence of*

$$\sum_i a_{n,i} Q^i x \quad (1.2.13)$$

for any uniformly regular matrix  $(a_{n,i})$ .

*Proof.* If the weak limit  $Q^n x$  exists and is equal to  $x_1$ , then

$$Qx_1 = Q(\lim_{n \rightarrow \infty} Q^n x) = x_1, \quad (1.2.14)$$

where the limit is considered in the weak topology, i.e.  $x_1$  is  $Q$ -invariant. Replacing  $x$  by  $x - x_1$  (if necessary), we may suppose that  $Q^n x$  converges weakly to  $\mathbf{0}$ , and hence  $(Q^n x, x) \rightarrow 0$ . We are going to show that  $\sum_n a_{i,n} Q^n x \xrightarrow{\|\cdot\|} \mathbf{0}$ , where  $(a_{i,n})$  is a uniformly regular matrix. One can see that

$$\left\| \sum_i a_{n,i} Q^i x \right\|^2 \leq \sum_i \sum_j a_{n,i} a_{n,j} (Q^i x, Q^j x) \leq \sum_i \sum_j |a_{n,i} a_{n,j} (Q^i x, Q^j x)|. \quad (1.2.15)$$

Let us fix  $1 > \varepsilon > 0$ . Because  $Q$  is a contraction, the limit  $\lim_{n \rightarrow \infty} \|Q^n x\|$  does exist.

Now, we can find  $K > 0$ , such that for  $k > K$  and  $j \geq 0$ ,

$$\|Q^k x\|^2 - \|Q^{k+j} x\|^2 \leq \varepsilon^2 \quad (1.2.16)$$

and

$$|(Q^k x, x)| \leq \varepsilon. \quad (1.2.17)$$

Then,

$$\begin{aligned} & |(Q^k x, x) - (Q^{k+j} x, Q^j x)| = |(Q^k x, x) - (Q^{*j} Q^{k+j} x, x)| \leq \\ & \leq \|Q^k x - Q^{*j} Q^{k+j} x\| \cdot \|x\| = (\|Q^k x - Q^{*j} Q^{k+j} x\|^2)^{\frac{1}{2}} \cdot \|x\| = \\ & = (\|Q^k x\|^2 - 2\|Q^{k+j} x\|^2 + \|Q^{*j} Q^{k+j} x\|^2)^{\frac{1}{2}} \cdot \|x\| \leq \\ & \leq (\|Q^k x\|^2 - \|Q^{k+j} x\|^2)^{1/2} \cdot \|x\| \leq \varepsilon \cdot \|x\|, \end{aligned} \quad (1.2.18)$$

and therefore

$$|(Q^{k+j} x, Q^j x)| \leq \varepsilon \cdot (1 + \|x\|) \quad (1.2.19)$$

for all  $k > K$  and  $j \geq 0$ , ie. for  $|i - j| \geq k$  the inequality

$$|(Q^i x, Q^j x)| \leq \varepsilon \cdot (1 + \|x\|) \quad (1.2.20)$$

is valid. We will fix  $\eta > 0$ , and let  $N$  be such a natural number that

$$\sup_i |a_{n,i}| < \eta, \quad (1.2.21)$$

for  $n \geq N$ . Then the expression 1.2.15 for  $n \geq N$  could be estimated the following

way:

$$\begin{aligned} & \sum_i \sum_j |a_{n,i} a_{n,j} (Q^i x, Q^j x)| = \\ & = \sum_{|i-j| \leq k} |a_{n,i} a_{n,j} (Q^i x, Q^j x)| + \sum_{|i-j| > k} |a_{n,i} a_{n,j} (Q^i x, Q^j x)| \leq \\ & \leq \sum_i |a_{n,i}| \cdot \eta \cdot \|x\|^2 \cdot (2k + 1) + \sum_i \sum_j |a_{n,i} a_{n,j}| \cdot \varepsilon \cdot (1 + \|x\|) \leq \end{aligned}$$

$$\leq C \cdot \eta \cdot \|x\|^2 \cdot (2k + 1) + C^2 \cdot \varepsilon \cdot (1 + \|x\|). \quad (1.2.22)$$

From the arbitrariness of the values of  $\varepsilon$  and  $\eta$  it follows that strong convergence is present and the Lemma is proven.  $\square$

*Proof of the Theorem 1 (cont.)* Let us prove the implication  $i) \implies iii)$ . Let  $x \in M_{*+}$  and the sequence  $\{T^i x\}_{i=1,2,\dots}$  converges weakly. Without the loss of generality we can consider  $\|x\| \leq 1$ , and let

$$\bar{x} = \lim_{n \rightarrow \infty} T^n x, \quad (1.2.23)$$

where the limit is understood in the weak sense.

By  $s(z)$  we denote the support of the positive normal functional  $z$ .

For  $y \in M_{*+}$  the set

$$\mathfrak{L}_y = \{w \in M_{*+}, w \leq \lambda y, \text{ for some } \lambda > 0\}, \quad (1.2.24)$$

is dense in the set

$$\mathfrak{S}_y = \{w \in M_{*+}, s(w) \leq s(y)\}, \quad (1.2.25)$$

in the norm of the space  $M_*$  (see for example [60], Theorem 1.24.3, proof).

An alternative proof of the density statement could be obtained by applying theorem 1.6 of [22] to  $s(y)Ms(y)$ , equipped with the weight obtained by restricting the action of  $y$  to  $s(y)Ms(y)$ .

Let us introduce the following notion: For  $\mu \in M_*$ , we will denote by  $\mu.E$ , where  $E$  is a projection from the algebra  $M$ , the functional

$$(\mu.E)(A) = \mu(EAE), \quad (1.2.26)$$

where  $A \in M$ .

Let us fix  $1 > \varepsilon > 0$ . Since  $(T^n x)(\mathbf{1} - s(\bar{x})) \rightarrow 0$ , we can find a number  $N$ , such that

$$(T^n x)(\mathbf{1} - s(\bar{x})) < \varepsilon^2 \quad (1.2.27)$$

for  $n > N$ .

Then

$$\begin{aligned} & \|T^N x.s(\bar{x}) - T^N x\| = \\ & \sup_{\substack{A \in M \\ \|A\|_\infty \leq 1}} |(T^N x)((\mathbf{1} - s(\bar{x}))A(\mathbf{1} - s(\bar{x}))) + \\ & + (T^N x)((s(\bar{x}))A(\mathbf{1} - s(\bar{x}))) + (T^N x)((\mathbf{1} - s(\bar{x}))A(s(\bar{x})))| \leq \\ & \leq \varepsilon \cdot (\varepsilon + 2\|x\|^{\frac{1}{2}}), \end{aligned} \quad (1.2.28)$$

because

$$|\mu(A^*B)|^2 \leq \mu(A^*A) \cdot \mu(B^*B), \quad (1.2.29)$$

where  $\mu \in M_{*+}$  and  $A, B \in M$ .

Let us now approximate  $T^N x.s(\bar{x})$ . Note, that  $T^N x.s(\bar{x}) \in \mathfrak{G}_{\bar{x}}$ . Let  $w \in \mathfrak{L}_{\bar{x}}$  be such that

$$w \leq \lambda \bar{x} \quad (1.2.30)$$

for some  $\lambda > 0$  and

$$\|T^N x.s(\bar{x}) - w\| \leq \varepsilon' < \varepsilon \quad (1.2.31)$$

for some  $0 < \varepsilon' < \varepsilon$ .

Then, for  $n > N$ , the following is valid:

$$\begin{aligned} \|T^n x - T^{n-N} w\| &\leq \|T^{n-N}(T^N x - T^N x.s(\bar{x}))\| + \\ &+ \|T^{n-N}(T^N x.s(\bar{x}) - w)\| \leq 3 \cdot \varepsilon + \varepsilon' < 4 \cdot \varepsilon. \end{aligned} \quad (1.2.32)$$

By taking the weak limit in the inequality (1.2.40), and because the unital sphere of  $M_*$  is closed weakly, we will get

$$\|\bar{x} - \bar{w}\| < 4 \cdot \varepsilon, \quad (1.2.33)$$

where

$$\bar{w} = \lim_{n \rightarrow \infty} T^n w. \quad (1.2.34)$$

Let us now consider the algebra  $M_{s(\bar{x})}$ . The functional  $\bar{x}$  is faithful on the algebra  $M_{s(\bar{x})}$ . We will consider the representation  $\pi_{\bar{x}}$  of the algebra  $M_{s(\bar{x})}$  constructed using the functional  $\bar{x}$  [15]. Because the functional  $\bar{x}$  is faithful, we can conclude that the representation  $\pi_{\bar{x}}$  is faithful on the algebra  $M_{s(\bar{x})}$ , and therefore  $\pi_{\bar{x}}$  is an isomorphism of the algebra  $M_{s(\bar{x})}$  and some algebra  $\mathfrak{A}$ . The algebra  $\mathfrak{A}$  is a von Neumann algebra, and its pre-conjugate space  $\mathfrak{A}_*$  is isomorphic to the space  $M_* . s(\bar{x})$  ([60], the proof of Theorem 1.24.3). Let us note now that

$$TM_* . s(\bar{x}) \subset M_* . s(\bar{x}). \quad (1.2.35)$$

In fact, since an inequality  $w \leq \lambda \bar{x}$  implies  $Tw \leq \lambda \bar{x}$ , we get

$$T\mathfrak{L}_{\bar{x}} \subset \mathfrak{L}_{\bar{x}}. \quad (1.2.36)$$

Therefore, by taking the norm closure, we get

$$T\mathfrak{G}_{\bar{x}} \subset \mathfrak{G}_{\bar{x}}; \quad (1.2.37)$$

and by taking the linear span, we get based on 1.2.25 and the fact that every  $y \in M_*$  maybe decomposed as a linear combination of four  $y_i \in M_{*+}$ ,  $i = 1, 2, 3, 4$

$$TM_{*.s}(\bar{x}) \subset M_{*.s}(\bar{x}). \quad (1.2.38)$$

Let us denote by  $\bar{T}$  the isomorphic image of the operator  $T$ , acting on the space  $\mathfrak{A}_*$ .

Let

$$u \in \mathfrak{A}_{*+} \text{ and } u \leq \lambda \pi_{\bar{x}}(\bar{x}) \quad (1.2.39)$$

for some  $\lambda > 0$ . Then there exists an operator  $B \in \mathfrak{A}'_+$  with  $\|B\|_\infty < \lambda$ , where  $\mathfrak{A}'$  is a commutant of  $\mathfrak{A}$ , such that

$$(AB\Omega, \Omega) = u(A) \quad (1.2.40)$$

for all  $A \in \mathfrak{A}$  [60] (here  $\Omega$  is such that  $(A\Omega, \Omega) = \bar{x}(A)$ ), Theorem 1.24.3, proof (we refer to the statement about the existence of the derivative from commutant for the normal bounded state ).

For bounded positive contraction  $\bar{T}$  defined on  $\mathfrak{A}_*$  let

$$(\bar{T}u)(A) = u((\bar{T})^*A) = (((\bar{T})^*A)B\Omega, \Omega) = (A((\bar{T}^*)'B)\Omega, \Omega), \quad (1.2.41)$$

The operator  $(\bar{T}^*)'$  defined by equality (1.2.41) was introduced for the first time in [20]. It is established in [20] that  $(\bar{T}^*)'$  is a positive contraction in  $\mathfrak{A}'$ .

Also, from

$$\bar{T}\mathfrak{A}_{*+} \subset \mathfrak{A}_{*+}, \|\bar{T}u\| \leq \|u\| \text{ and } \bar{T}\bar{x} = \bar{x} \quad (1.2.42)$$

it follows that

$$(\bar{T})^*\mathfrak{A}_+ \subset \mathfrak{A}_+; (\bar{T}^*)\mathbf{1} \leq \mathbf{1} \text{ and } \|(\bar{T})^*A\|_\infty \leq \|A\|_\infty \quad (1.2.43)$$

for all  $A \in \mathfrak{A}$ . Based on the Lemma 1.3 of [20] stating that operator  $\overline{T}^{*'}$  satisfies 1.2.43 for  $\mathfrak{A}'$  under condition that operator  $\overline{T}^*$  satisfies 1.2.43 for  $\mathfrak{A}$ , we now conclude that

$$\left\| (\overline{T}^{*'} B) \right\|_{\infty} \leq \|B\|_{\infty}; \overline{T}^{*'} \mathfrak{A}'_+ \subset \mathfrak{A}'_+; \overline{T}^{*'} \mathbf{1} \leq \mathbf{1} \quad (1.2.44)$$

for all  $B \in \mathfrak{A}'$ .

The space  $\mathfrak{A}'_{sa}$  is a pre-Hilbert space of the self adjoint operators from  $\mathfrak{A}'$  with the scalar product

$$(B, C)_{\bar{x}} = (CB\Omega, \Omega), \quad (1.2.45)$$

and, using the Kadison inequality [9], Proposition 3.2.4 we have

$$((\overline{T}^{*'} B)(\overline{T}^{*'} B)\Omega, \Omega) \leq (\overline{T}^{*'}(B^2)\Omega, \Omega) \leq (B\Omega, B\Omega), \quad (1.2.46)$$

i.e. the operator  $\overline{T}^{*'}$  is a contraction in the pre-Hilbert space  $(\mathfrak{A}'_{sa}, (\cdot, \cdot)_{\bar{x}})$ . Denote closure of the  $(\mathfrak{A}'_{sa}, (\cdot, \cdot)_{\bar{x}})$  by  $\mathfrak{B}$ . Since operator  $\overline{T}^{*'}$  is a contraction on the dense subset, it may be extended by continuity to the contraction on whole  $\mathfrak{B}$ . We use the same notation for the extension as for the operator  $\overline{T}^{*'}$ .

We will identify  $M_{*,s}(\bar{x})$  and  $\mathfrak{A}_*$ . Because  $w \in \mathfrak{L}_{\bar{x}}$ , i.e.

$$w \leq \lambda \bar{x} \quad (1.2.47)$$

for some  $\lambda > 0$ , then since  $\bar{x}$  is  $T$  invariant and  $T$  is a contraction

$$\bar{w} \leq \lambda \bar{x} \quad (1.2.48)$$

as well. Let

$$w(A) = (BA\Omega, \Omega) \text{ and } \bar{w}(A) = (\overline{BA}\Omega, \Omega) \quad (1.2.49)$$

for all  $A \in \mathfrak{A}$ , where  $B, \bar{B} \in \mathfrak{A}'$ .

Let us show that weak convergence  $T^n w \rightarrow \bar{w}$  and the fact that  $T$  (and hence from the previous note  $\bar{T}^{*l}$ ) is a contraction implies that  $\bar{T}^{*lm} B$  converges weakly for  $B \in \mathfrak{B}$ .

Indeed, let  $\eta \in \mathfrak{B}$ . It is enough to show that  $(\eta, \bar{T}^{*lm} \eta)$  is Cauchy sequence.

Since

$$\begin{aligned}
& |(\eta, \bar{T}^{*ln} \eta) - (\eta, \bar{T}^{*lm} \eta)| \leq \tag{1.2.50} \\
& |(\eta, \bar{T}^{*ln} \eta) - (A_k \Omega, \bar{T}^{*ln} \eta)| + |(A_k \Omega, \bar{T}^{*ln} \eta) - (A_k \Omega, \bar{T}^{*ln} B_l \Omega)| + \\
& |(A_k \Omega, \bar{T}^{*ln} B_l \Omega) - (A_k \Omega, \bar{T}^{*lm} B_l \Omega)| + |(A_k \Omega, \bar{T}^{*lm} B_l \Omega) - (A_k \Omega, \bar{T}^{*lm} \eta)| + \\
& |(A_k \Omega, \bar{T}^{*lm} \eta) - (\eta, \bar{T}^{*lm} \eta)| \leq 2 \|\eta - A_k \Omega\| \sup_n \|\bar{T}^{*ln}\| \|\eta\| + \\
& 2 \|A_k \Omega\| \sup_n \|\bar{T}^{*ln}\| \|\eta - B_l \Omega\| + |(A_k \Omega, \bar{T}^{*ln} B_l \Omega) - (A_k \Omega, \bar{T}^{*lm} B_l \Omega)| \leq \epsilon
\end{aligned}$$

for  $A_k \in \mathfrak{A}$ ,  $B_l \in \mathfrak{A}'$  chosen in such a manner that  $\|\eta\| < 1$ ,  $\|A_k \Omega\| < 1$ ,  $\|A_k \Omega - \eta\| \leq \epsilon/5$  and  $\|B_l \Omega - \eta\| \leq \epsilon/5$ , and  $N \in \mathbb{N}$  chosen in such a manner that  $|(A_k \Omega, \bar{T}^{*ln} B_l \Omega) - (A_k \Omega, \bar{T}^{*lm} B_l \Omega)| \leq \epsilon/5$  for  $m, n \geq N$ . Possibility of latter choice follows from the fact that sequence  $(\bar{T}^{*lm} A_k \Omega, B_l \Omega)$  converges. Hence,  $(\eta, \bar{T}^{*lm} \eta)$  is Cauchy sequence.

Let now  $(a_{n,i})$  be a uniformly regular matrix. Since  $T^i w(A) = (T^{*i}(B)A\Omega, \Omega)$ , then using Lemma 1.2.3 we can find  $K \in \mathbb{N}$  so that

$$\begin{aligned}
& \left\| \sum_i a'_{k,i} T^i w - \bar{w} \right\| = \sup_{\substack{A \in \mathfrak{A} \\ \|A\|_\infty = 1}} \left| \left( \sum_{i=1}^{\infty} a'_{k,i} ((\bar{T}^{*l})^i (B - \bar{B}) A \Omega, \Omega) \right) \right| \leq \\
& \leq \left( \sum_{i=1}^{\infty} a'_{k,i} ((\bar{T}^{*l})^i (B - \bar{B}) \Omega, \Omega) \right)^{\frac{1}{2}} \cdot \sup_{\substack{A \in \mathfrak{A} \\ \|A\|_\infty \leq 1}} (A \Omega, A \Omega)^{\frac{1}{2}} \leq \\
& \leq (\bar{x}(\mathbf{1}))^{\frac{1}{2}} \cdot \left\| \sum_{i=1}^{\infty} a'_{k,i} ((\bar{T}^{*l})^i (B - \bar{B})) \right\|_{(\cdot, \cdot)_{\bar{x}}} < \epsilon \tag{1.2.51}
\end{aligned}$$

for  $k > K$ , where by  $(a'_{n,i})$  we will denote a matrix with the elements

$$a'_{n,j} = \left( \sum_{i>N} a_{n,i} \right)^{-1} a_{n,j+N}. \quad (1.2.52)$$

Choose  $N$  in such a manner that (1.2.40) takes place.

Given  $0 < \epsilon_0, \epsilon_1 < \epsilon/2$  if we choose  $K$  big enough so that  $|a_{k,i}| < \epsilon_0/N$  and  $|\sum a_{k,i} - 1| < \epsilon_1$  for all  $k \geq K$ , we will have  $|\sum_{i>N} a_{k,i} - 1| < \epsilon_1 + \epsilon_0$ . If also  $\epsilon_1 + \epsilon_0 < 1$ , then  $\sum_{i>N} a_{k,i} > 0$ . It is easy to see that the matrix  $(a'_{n,i})$  will be uniformly regular as well.

Then, for a big enough  $k > K$  we will have

$$\begin{aligned} & \left\| \sum_i a_{k,i} T^i x - \bar{x} \right\| \leq \underbrace{\sum_{i \leq N} |a_{k,i}| \|T^i x - \bar{x}\|}_1 + \underbrace{\sum_{i > N} |a_{k,i}| \|T^i x - T^{i-N} w\|}_2 \\ & + \underbrace{\left| 1 - \left( \sum_{i > N} a_{k,i} \right)^{-1} \right| \sum_{i > N} |a_{k,i}| \sup_{i > N} \|T^{i-N} w\|}_3 + \underbrace{\left\| \sum_{j=1}^{\infty} a_{k,j+N} \cdot \left( \sum_{i > N} a_{k,i} \right)^{-1} T^j w - \bar{w} \right\|}_4 \\ & + \underbrace{\left\| \left( \sum_{i \leq N} a_{k,i} \right) \cdot \bar{w} \right\|}_5 + \underbrace{\left| \sum_{i > N} a_{k,i} \right| \|\bar{w} - \bar{x}\|}_6 + \underbrace{\left( 1 - \sum_i a_{k,i} \right) \|\bar{w}\|}_7 + \underbrace{\left( 1 - \sum_i a_{k,i} \right) \|\bar{x}\|}_8 \leq \\ & \leq \sum_{i \leq N} 2 \cdot \frac{\epsilon}{N} + \sum_{i > N} |a_{k,i}| \cdot 4\epsilon + \sum_{i > N} |a_{k,i}| \max\{(1 - (1 + \epsilon)^{-1}), ((1 - \epsilon)^{-1} - 1)\} \cdot (1 + 4\epsilon) + \epsilon \\ & + \sum_{i \leq N} (1 + 4\epsilon) \cdot \frac{\epsilon}{N} + (1 + \epsilon) \cdot 4\epsilon + \epsilon(1 + 4\epsilon) + \epsilon \end{aligned} \quad (1.2.53)$$

$$\leq 2\varepsilon + C \cdot (1 + \varepsilon) \cdot 4\varepsilon + C \cdot 2\varepsilon \cdot 2 + \varepsilon + 2\varepsilon + (1 + \varepsilon) \cdot 4\varepsilon + \varepsilon(2 + 4\varepsilon) \leq (20 + 12 \cdot C)\varepsilon, \quad (1.2.54)$$

for  $1/4 > \varepsilon > 0$ . Here inequalities for terms 1 and 5 follows from the choice of  $K$  and uniform regularity of matrix  $(a_{k,i})$  and, hence, for fixed  $N$  and sufficiently large  $K$  for  $k > K, i \leq N$  holds  $|a_{k,i}| < \varepsilon/N$ ; inequality for term 2 follows from (1.2.40) (the constant  $C$  was defined in the definition of the uniformly regular matrix (1.1.1)); inequality for term 3 follows from regularity of matrix  $(a_{n,i})$ , hence, for fixed  $N$  and sufficiently large  $K$  and  $k > K$  holds  $\sum_{i>N} a_{k,i} > 0$  and  $|1 - \sum_{i>N} a_{k,i}| < \varepsilon$ ; inequality for term 4 follows from (1.2.59); inequality for term 6 follows from (1.2.41); and inequalities for terms 7 and 8 follow from regularity of the matrix  $(a_{n,i})$ .

The arbitrariness of  $\varepsilon$  proves the needed statement. This completes the proof of the theorem.  $\square$

### 1.3 The case of $L_1$ -spaces for $JBW$ -algebras

The  $L_1$ -spaces for semifinite  $JBW$ -algebras were considered in [8] (see also [1], [37]), where it has been proven that they do coincide with predual spaces. In the case of a semifinite  $JBW$ -algebra, algebra  $A$  does not have a direct summand of type  $I_2$ , hence  $A$  is represented as

$$A = A_{sp} \dot{+} A_{ex}, \quad (1.3.1)$$

where  $A_{sp}$  is isometrically isomorphic to an operator  $JW$ -algebra, that is, a self-adjoint part of a Real von Neumann algebra  $R(A_{sp})$ , whose complexification

$$R(A_{sp}) \dot{+} iR(A_{sp}) = M, \quad (1.3.2)$$

where  $M$  is the enveloping von Neumann algebra of  $A_{sp}$ , and  $A_{ex}$  is isometrically isomorphic to the space  $C(X, M_3^8)$  of all continuous mappings from a Hyperstonean

compact topological space  $X$  onto the exceptional Jordan algebra  $M_3^8$ . The predual space of  $A$ , is the space

$$A_* = (A_{sp})_* \dot{+} (A_{ex})_*, \quad (1.3.3)$$

where  $(A_{sp})_*$  is the predual space of  $A_{sp}$ , and  $(A_{ex})_*$  is the predual space of  $A_{ex}$  (see, for example [34] and [5]). The main result for the summand  $A_{ex}$  follows immediately from the result for  $C(X)$ , and the fact that the algebra  $M_3^8$  is finite-dimensional. So, without the loss of generality we may pass to the operator case only. But in the operator case, the space  $(A_{sp})_*$  is a self-adjoint part of  $R_* = (R(A_{sp}))_*$ , and

$$M_* = R_* \dot{+} iR_*, \quad (1.3.4)$$

(see [5] and [49] for details). But the main result for  $R_*$  thus follows from the complex case by restriction.

## 1.4 The case of non-commutative $L_p$ -spaces, ( $1 < p < \infty$ )

In this section  $M$  denotes a semifinite JBW algebra with faithful normal trace  $\tau$ . By  $L_p$  we denote the space of the operators affiliated with the JBW algebra  $M$  and integrated with  $p$ -th power ( $p > 1$ , see for example [14], [61], [2]). Space  $L_q$  (here  $q = \frac{p}{p-1}$ ) is dual to  $L_p$  as a Banach space (see for example [61], [1]). The following theorem takes place:

**Theorem 1.4.1.** *The following conditions for a positive contraction  $T$  in  $L_p$  are equivalent:*

- i) The sequence  $\{T^i x\}_{i=1,2,\dots}$  converges in  $\sigma(L_p, L_q)$  topology, for every  $x \in L_p$*

ii) For each strictly increasing sequence of natural numbers  $\{k_i\}_{i=1,2,\dots}$ ,

$$n^{-1} \sum_{i < n} T^{k_i}, \quad (1.4.1)$$

converges in pointwise norm topology with respect to norm of  $L_p$ ,

iii) For any uniformly regular matrix  $(a_{n,i})$ , the sequence  $\{A_n(T)\}_{n=1,2,\dots}$ ,

$$A_n(T) = \sum_i a_{n,i} T^i, \quad (1.4.2)$$

converges in pointwise norm topology with respect to norm of  $L_p$ .

For the sake of completeness we give the following definitions (see [71]), and a sketch of the proof. Let  $\phi$  be a gauge function ( $\phi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ , with  $\phi(0) = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ ). The Hahn-Banach theorem implies for strictly convex Banach spaces  $E$  with conjugate  $E'$  that there exists a duality map  $\Phi : E \mapsto E'$  associated with  $\phi$  such that  $\langle x, \Phi(x) \rangle = \|x\| \|\Phi(x)\|$ , and  $\|\Phi(x)\| = \phi(x)$ . Map  $\Phi$  is said to satisfy property (S) uniformly if for every  $\epsilon > 0$  there exists  $\delta(\epsilon) > 0$  such that for any  $x, y \in E$

$$|\langle x, \Phi(y) \rangle| < \delta(\epsilon) \text{ implies } |\langle y, \Phi(x) \rangle| < \epsilon \quad (1.4.3)$$

*Proof.* From [37] section 4, it follows that duality map defined as  $\Phi(a) = s|a|^{p-1}$  for  $a = s|a| \in M$ , (here  $a = s|a|$  is a polar decomposition of element  $a$ ) satisfies property (S) uniformly, hence the statement of the theorem follows from [71] Theorem 3.1.  $\square$

## Chapter 2

# Ergodic Type Theorems for Finitely Generated Groups Acting in von Neumann Algebras

### 2.1 Non-commutative Operator Ergodic Theorems

Let

$$\Omega_m = \{\omega = \omega_1\omega_2\dots\omega_n\dots : \omega_i = 1, \dots, m\} \quad (2.1.1)$$

be the space of all one-sided infinite sequences of  $m$  symbols  $1, \dots, m$ . We denote by  $\sigma_m$  the shift on  $\Omega_m$ , defined by the formula

$$(\sigma_m\omega)_i = \omega_{i+1}. \quad (2.1.2)$$

Consider the set

$$W_m = \{w = w_1w_2\dots w_n : w_i = 1, \dots, m; n = 1, \dots\} \quad (2.1.3)$$

of all finite words of  $m$  symbols  $1, \dots, m$ .

Denote by  $|w|$  the length of the word  $w$ . For each

$$w \in W_m,$$

let

$$C(w) \subset \Omega_m$$

be the set of all sequences starting with the word  $w$ . For an arbitrary Borel measure  $\mu$  on  $\Omega_m$ , set

$$\mu(w) = \mu(C(w)). \quad (2.1.4)$$

Measure  $\mu$  on  $\Omega_m$  invariant with respect to shift  $\sigma_m$  we call  $\sigma_m$ -invariant.

For each

$$w \in W_m,$$

introduce the operator

$$\alpha_w = \alpha_{w_n} \alpha_{w_{n-1}} \dots \alpha_{w_1}. \quad (2.1.5)$$

Let  $\mu$  be a Borel  $\sigma_m$ -invariant probability measure on  $\Omega_m$ . Consider the words  $w$  with

$$|w| = l \quad (2.1.6)$$

and the sum of the corresponding operators  $\alpha_w$  with the weights  $\mu(w)$ ,

$$s_l^\mu(\alpha) = \sum_{|w|=l} \mu(w) \alpha_w. \quad (2.1.7)$$

Here  $\alpha$  is just a notational device to remind the reader that this operator is built from the  $\alpha_w$ s. For the sake of convenience, we set  $\alpha_w$  with  $|w| = 0$  equal to the identity operator on  $M$ .

Average  $s_l^\mu(\alpha)$  over  $l = 0, \dots, n-1$ ,

$$c_n^\mu(\alpha) = \frac{1}{n} \sum_{l=0}^{n-1} s_l^\mu(\alpha). \quad (2.1.8)$$

Suppose  $\mu$  is a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ . We will show that the averages  $c_n^\mu(\alpha)\varphi$  converge both bilateral almost everywhere and in  $L_1(M, \tau)$  for any operator

$$\varphi \in L_1(A, \tau).$$

**Definition 2.1.1.** A matrix  $Q$  with non-negative entries is said to be irreducible if, for some  $n > 0$ , all entries of the matrix

$$Q + Q^2 + \dots + Q^n \tag{2.1.9}$$

are positive (if  $Q$  is stochastic, then this is equivalent to saying that in the corresponding Markov chain any state is attainable from any other state).

**Definition 2.1.2.** A matrix  $P$  with non-negative entries is said to be strictly irreducible if  $P$  and  $PP^T$  are irreducible (here  $P^T$  stands for the transpose of the matrix  $P$ ).

**Definition 2.1.3.** A Markov chain is said to be strictly irreducible if the corresponding transition matrix is strictly irreducible.

**Theorem 2.1.1.** *Let  $(M, \tau)$  be a non-commutative probability space,*

$$\alpha_1, \dots, \alpha_m : M \longmapsto M$$

*positive kernels, and*

$$\alpha_1, \dots, \alpha_m : L_1(M, \tau) \longmapsto L_1(M, \tau)$$

*their corresponding extensions. Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ .*

*Then, for any element*

$$\varphi \in L_1(M, \tau),$$

there exists an element

$$\bar{\varphi} \in L_1(M, \tau),$$

such that

$$c_n^\mu(\alpha)\varphi \rightarrow \bar{\varphi} \tag{2.1.10}$$

both bilateral almost everywhere and in  $L_1(M, \tau)$  norm as  $n \rightarrow \infty$ .

The following equality holds whenever  $\alpha_1, \dots, \alpha_m$  preserve the state  $\tau$ :

$$\tau(\varphi) = \tau(\bar{\varphi}). \tag{2.1.11}$$

If the measure  $\mu$  is strictly irreducible, then

$$\alpha_j \bar{\varphi} = \bar{\varphi}$$

for  $j = 1, \dots, m$ .

If

$$\varphi \in L_p(M, \tau),$$

$p > 1$ , then  $\bar{\varphi} \in L_p(M, \tau)$  and

$$c_n^\mu(\alpha)\varphi \rightarrow \bar{\varphi} \tag{2.1.12}$$

both in  $L_p(M, \tau)$  norm and bilateral almost everywhere as well.

Theorem 2.1.1 generalizes the Ergodic Theorems of Grigorchuk [32], Nevo [51], Nevo and Stein [52], and Bufetov [11] to the non-commutative case.

The following is a generalized version of the Mean Ergodic Theorem for operators on Hilbert spaces.

**Theorem 2.1.2.** *Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ , let*

$$H = L_2(M, \tau)$$

be the Hilbert space constructed using non-commutative probability space  $(M, \tau)$ , and let the linear operators

$$\alpha_1, \dots, \alpha_m : L_2(M, \tau) \longmapsto L_2(M, \tau)$$

be contractions. Then, for each operator

$$h \in L_2(M, \tau)$$

there exists an operator

$$\bar{h} \in L_2(M, \tau)$$

such that

$$c_n^\mu(\alpha)h \rightarrow \bar{h} \tag{2.1.13}$$

in  $L_2(M, \tau)$  as  $n \rightarrow \infty$ .

If the measure  $\mu$  is strictly irreducible, then

$$\alpha_i \bar{h} = \bar{h} \tag{2.1.14}$$

for all  $i = 1, \dots, m$ .

The following is a non-commutative version of the Ergodic Theorem for operators on  $L_p(M, \tau)$ .

**Theorem 2.1.3.** *Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ . Let  $(M, \tau)$  be a non-commutative probability space and let  $p > 1$ . Suppose that all operators*

$$\alpha_1, \dots, \alpha_m : L_p(M, \tau) \longmapsto L_p(M, \tau)$$

are positive contractions. Then for each

$$\varphi \in L_p(M, \tau)$$

there exists

$$\bar{\varphi} \in L_p(M, \tau)$$

such that

$$c_n^\mu(\alpha)\varphi \rightarrow \bar{\varphi} \tag{2.1.15}$$

as  $n \rightarrow \infty$  in  $L_p(A, \tau)$ .

If the measure  $\mu$  is strictly irreducible, then

$$\alpha_i \bar{\varphi} = \bar{\varphi} \tag{2.1.16}$$

for all  $i = 1, \dots, m$ .

## 2.2 Convergence of Multiparametric Česaro Averages

In this section we discuss the convergence of time averages in Theorems 2.1.1 and 2.1.3. The main idea here is to use the operator  $\alpha_\mu$  introduced later in this section.

Let  $L$  be a Real or Complex linear space, let

$$\alpha_1, \dots, \alpha_m : L \longmapsto L$$

be linear operators, and let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$  with initial distribution  $(p_1, \dots, p_m)$  and transition probability matrix

$$P = (p_{ij}).$$

We always assume in what follows that  $p_i > 0$  for any  $i = 1, \dots, m$ .

Consider the weighted sum of operators  $\alpha_w$  over all words of length  $l$  with given last symbol  $i$ ,

$$s_l^{\mu,i}(\alpha) = \sum_{\{w:|w|=l,w_l=i\}} \mu(w)\alpha_w. \quad (2.2.1)$$

For the sake of convenience, we set  $\alpha_w$  with  $|w| = 0$  equal to the identity operator on  $L$ .

Now we average  $s_l^{\mu,i}(\alpha)$  over  $l = 0, \dots, n-1$ ,

$$c_n^{\mu,i}(\alpha) = \frac{1}{n} \sum_{l=0}^{n-1} s_l^{\mu,i}(\alpha). \quad (2.2.2)$$

The following lemma describes the relation between  $s_l^{\mu,i}(\alpha)$  and  $s_{l+1}^{\mu,j}(\alpha)$ .

**Lemma 2.2.1.** *For any positive integer  $l$  and any  $j \in \{1, \dots, m\}$ , we have*

$$s_{l+1}^{\mu,j}(\alpha) = \sum_{i=1}^m p_{ij} \alpha_j s_l^{\mu,i}(\alpha). \quad (2.2.3)$$

*Proof.* The proof of the lemma follows directly from Definition (2.2.1) of  $s_l^{\mu,i}(\alpha)$  and  $\alpha_w$ . □

We can formulate the expression from the above lemma as follows:

$$\frac{s_{l+1}^{\mu,j}(\alpha)}{p_j} = \sum_{i=1}^m \frac{p_i p_{ij}}{p_j} \alpha_j \left( \frac{s_l^{\mu,i}(\alpha)}{p_i} \right). \quad (2.2.4)$$

Now we consider the space  $L^m$ , i.e., the  $m$ -th Cartesian power of  $L$ . We introduce operators

$$\alpha_\mu : L^m \longmapsto L^m$$

defined by the formula

$$\alpha_\mu(v_1, \dots, v_m) = (\tilde{v}_1, \dots, \tilde{v}_m), \quad (2.2.5)$$

where

$$\tilde{v}_j = \sum_{i=1}^m \frac{p_i p_{ij}}{p_j} \alpha_j v_i. \quad (2.2.6)$$

**Lemma 2.2.2.** *For any  $v \in L$  and  $n \in \mathbb{N}$  (or  $n \geq 1$ ), we have*

$$\alpha_\mu^n(v, \dots, v) = \left( \frac{s_n^{\mu,1}(\alpha)v}{p_1}, \dots, \frac{s_n^{\mu,m}(\alpha)v}{p_m} \right). \quad (2.2.7)$$

*Proof.* Follows by induction from the formulae above.  $\square$

**Corollary 2.2.3.** *For any  $v \in L$  and  $n \in \mathbb{N}$ ,*

$$\frac{1}{n} \sum_{l=0}^{n-1} \alpha_\mu^l(v, \dots, v) = \left( \frac{c_n^{\mu,1}(\alpha)v}{p_1}, \dots, \frac{c_n^{\mu,m}(\alpha)v}{p_m} \right). \quad (2.2.8)$$

*Proof.* Follows from the previous lemma.  $\square$

Applying the classical non-commutative Individual Ergodic Theorem of Goldstein [20], Theorem 1.1 to the operator  $\alpha_\mu$  and using Corollary 2.2.3, we obtain statements on the convergence of the averages  $c_n^m(\alpha)$ .

**Lemma 2.2.4.** *Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ , let*

$$H = L_2(M, \tau)$$

*be the Hilbert space constructed using non-commutative probability space  $(M, \tau)$ , and let the linear operators*

$$\alpha_1, \dots, \alpha_m : L_2(M, \tau) \longmapsto L_2(M, \tau)$$

*be contractions. Then, for any operator*

$$h \in L_2(M, \tau)$$

and  $i \in \{1, \dots, m\}$ , the sequence

$$\left(\frac{1}{p_i}\right)c_n^{\mu, i}(\alpha)h \rightarrow \bar{h}_i \quad (2.2.9)$$

in  $H$  as  $n \rightarrow \infty$ , where

$$\bar{h}_i \in L_2(M, \tau),$$

and

$$\alpha_\mu(\bar{h}_1, \dots, \bar{h}_m) = (\bar{h}_1, \dots, \bar{h}_m). \quad (2.2.10)$$

*Proof.* If  $\alpha_1, \dots, \alpha_m$  are contractions on  $L_2(M, \tau)$ , then  $\alpha_\mu$  is a contraction on

$$(L_2(M, \tau))^m.$$

Corollary 2.2.3 and the Mean Ergodic Theorem for  $\alpha_\mu$  complete the proof.  $\square$

The relation

$$c_n^\mu(\alpha) = c_n^{\mu, 1}(\alpha) + \dots + c_n^{\mu, m}(\alpha) \quad (2.2.11)$$

yields the following assertion.

**Corollary 2.2.5.** *Under the assumptions of previous lemma, for any  $h \in L_2(M, \tau)$ , the sequence  $c_n^\mu(\alpha)h$  converges in  $L_2(M, \tau)$  norm as  $n \rightarrow \infty$ .*

Corollary 2.2.5 proves the convergence of time averages in Theorem 2.1.2.

Similarly, the following results are valid:

**Theorem 2.2.6.** *Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ . Let  $(M, \tau)$  be a non-commutative probability space and let  $p > 1$ . Suppose that all operators*

$$\alpha_1, \dots, \alpha_m : L_p(M, \tau) \longmapsto L_p(M, \tau)$$

are contractions. Then, for any

$$v \in L_p(M, \tau)$$

and  $i \in \{1, \dots, m\}$ , the sequence

$$\left(\frac{1}{p_i}\right) c_n^{\mu, i}(\alpha)v \rightarrow \bar{v}_i \quad (2.2.12)$$

in  $L_p(M, \tau)$  as  $n \rightarrow \infty$ , where the operator

$$\bar{v}_i \in L_p(M, \tau),$$

and

$$\alpha_\mu(\bar{v}_1, \dots, \bar{v}_m) = (\bar{v}_1, \dots, \bar{v}_m). \quad (2.2.13)$$

*Proof.* If  $\alpha_1, \dots, \alpha_m$  are contractions on  $L_p(M, \tau)$ , then  $\alpha_\mu$  is a contraction on

$$(L_p(M, \tau))^m.$$

The result follows from Corollary 2.2.3 and Lorch's Ergodic Theorem applied to the contraction  $\alpha_\mu$  (see [11] or [44], Theorem 1.2 on p. 73).  $\square$

**Corollary 2.2.7.** *Under assumptions of the previous Theorem, for any*

$$v \in L_p(M, \tau),$$

*the sequence  $c_n^\mu(\alpha)v$  converges in  $L_p(M, \tau)$  as  $n \rightarrow \infty$ .*

Now let  $(M, \tau)$  be a non-commutative probability space as above, and

$$\alpha_1, \dots, \alpha_m : L_1(M, \tau) \longmapsto L_1(M, \tau)$$

be linear operators.

Now we specialize the construction of  $\alpha_\mu$  from 2.2.5 to the case of  $L_1(M, \tau)$

Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$  with initial distribution

$$p = (p_1, \dots, p_m)$$

and transition probability matrix

$$P = (p_{ij}),$$

and let

$$\alpha_\mu : (L_1(M, \tau))^m \mapsto (L_1(M, \tau))^m$$

be the operator defined as before.

The space

$$(L_1(M, \tau))^m$$

can be identified with the space

$$L_1(M \times \{1, \dots, m\}, \tau \times p),$$

where  $\tau \times p$  is the product of the state  $\tau$  on the algebra  $A$  and the probability distribution

$$p = (p_1, \dots, p_m)$$

on  $\{1, \dots, m\}$ . Now the operator  $\alpha_\mu$  becomes an operator on the space

$$L_1(M \times \{1, \dots, m\}, \tau \times p).$$

It is clear that, if  $\alpha_1, \dots, \alpha_m$  are positive, then so is  $\alpha_\mu$ ; if  $\alpha_1, \dots, \alpha_m$  are  $L_1(A, \tau)$ -contractions, then so is  $\alpha_\mu$ ; if  $\alpha_1, \dots, \alpha_m$  are  $L_\infty(M, \tau)$ -contractions then so is  $\alpha_\mu$ ; if  $\alpha_1, \dots, \alpha_m$  preserve the state  $\tau$ , then  $\alpha_\mu$  preserves the measure  $\tau \times p$ .

**Lemma 2.2.8.** *Let  $(M, \tau)$  be a non-commutative probability space and let  $\alpha_1, \dots, \alpha_m$  be positive kernels. Then, for any operator*

$$\varphi \in L_1(M, \tau)$$

*and  $i = 1, \dots, m$ , sequence  $c_n^{\mu, i}(\alpha)\varphi$  converges as  $n \rightarrow \infty$  both bilateral almost everywhere and in  $L_1(M, \tau)$ . If*

$$\bar{\varphi}_i = \lim_{n \rightarrow \infty} \left(\frac{1}{p_i}\right) c_n^{\mu, i}(\alpha)\varphi, \quad (2.2.14)$$

*then*

$$\alpha_\mu(\bar{\varphi}_1, \dots, \bar{\varphi}_m) = (\bar{\varphi}_1, \dots, \bar{\varphi}_m). \quad (2.2.15)$$

To prove the lemma, we use the following known fact [20]:

**Theorem 2.2.9.** *If  $\alpha$  is a positive kernel on the non-commutative probability space  $(M, \tau)$ , then, for any*

$$\varphi \in L_1(M, \tau),$$

*there exists an operator*

$$\bar{\varphi} \in L_1(M, \tau)$$

*such that*

$$\frac{1}{n}(\varphi + \alpha\varphi + \dots + \alpha^{n-1}\varphi) \rightarrow \bar{\varphi} \quad (2.2.16)$$

*as  $n \rightarrow \infty$  both bilateral almost everywhere and in  $L_1(M, \tau)$ . The operator  $\bar{\varphi}$  satisfies the relation*

$$\alpha\bar{\varphi} = \bar{\varphi}. \quad (2.2.17)$$

*Proof.* (of Lemma 2.2.8) Applying Theorem 2.2.9 to the operator  $\alpha_\mu$ , and using Corollary 2.2.3, we obtain statement of the lemma.  $\square$

The bilateral almost everywhere convergence in the Theorem above also holds for spaces of infinite measure; therefore, we have the following lemma.

**Lemma 2.2.10.** *Let  $(M, \tau)$  be a von Neumann algebra with faithful normal semifinite trace  $\tau$  and let  $\alpha_1, \dots, \alpha_m$  be positive kernels. Then for any operator*

$$\varphi \in L_1(M, \tau)$$

and  $i = 1, \dots, m$ , the sequence  $c_n^{\mu, i}(\alpha)\varphi$  converges bilateral almost everywhere as  $n \rightarrow \infty$ .

Now consider contractions on  $L_p(M, \tau)$ , for  $p > 1$ .

**Lemma 2.2.11.** *Let  $(M, \tau)$  be a non-commutative probability space, let  $p > 1$ , and let  $\alpha_1, \dots, \alpha_m$  be positive  $L_p(M, \tau)$ -contractions. For any operator*

$$\varphi \in L_p(M, \tau)$$

and  $i = 1, \dots, m$ , the sequence

$$\left(\frac{1}{p_i}\right)c_n^{\mu, i}(\alpha)\varphi \tag{2.2.18}$$

converges as  $n \rightarrow \infty$  in  $L_p(M, \tau)$  to an operator

$$\bar{\varphi}_i \in L_p(M, \tau).$$

We have

$$\alpha_\mu(\bar{\varphi}_1, \dots, \bar{\varphi}_m) = (\bar{\varphi}_1, \dots, \bar{\varphi}_m). \tag{2.2.19}$$

*Proof.* If  $\alpha_1, \dots, \alpha_m$  are contractions, then so is  $\alpha_\mu$ . Applying Theorem 2 from [74] (see also [11] or [44], p.73) to the operator  $\alpha_\mu$  and using Corollary 2.2.3, we obtain the result. □

**Corollary 2.2.12.** *Under the assumptions of the previous Lemma, for any operator*

$$\varphi \in L_p(M, \tau),$$

*the sequence  $c_n^\mu(\alpha)\varphi$  converges in  $L_p(M, \tau)$  norm.*

Corollary 2.2.12 completes the proof of the convergence of time averages in Theorem 2.1.3.

## 2.3 Invariance of the Limit

In this section we establish the invariance of the limit in Theorems 2.1.1-2.1.3 and complete the proof of these theorems.

The following theorem allows us to conclude invariance of the limit in Theorem 2.1.2 from the Lemma 2.2.8 and as a consequence invariance of the limit in Theorem 2.1.1.

**Theorem 2.3.1.** *Let  $(M, \tau)$  be a non-commutative probability space and let  $\alpha_1, \dots, \alpha_m$  be positive kernels on  $L_1(M, \tau)$ . Let  $\mu$  be a strictly  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ . Suppose that the operators*

$$\varphi_1, \dots, \varphi_m \in L_1(M, \tau) \tag{2.3.1}$$

*satisfy the condition*

$$\alpha_\mu(\varphi_1, \dots, \varphi_m) = (\varphi_1, \dots, \varphi_m). \tag{2.3.2}$$

*Then*

$$\varphi_1 = \dots = \varphi_m = \varphi, \tag{2.3.3}$$

and

$$\alpha_i \varphi = \varphi \tag{2.3.4}$$

for all  $i = 1, \dots, m$ .

In order to prove Theorem 2.3.1 we first establish a similar result for contractions on the Hilbert space

$$H = L_2(M, \tau).$$

**Theorem 2.3.2.** *Let  $(M, \tau)$  be a non-commutative probability space, and let the linear operators*

$$\alpha_1, \dots, \alpha_m : L_2(M, \tau) \longmapsto L_2(M, \tau)$$

*be contractions. Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ , and let*

$$h_1, \dots, h_m \in L_2(M, \tau)$$

*be such that*

$$\alpha_\mu(h_1, \dots, h_m) = (h_1, \dots, h_m). \tag{2.3.5}$$

*If measure  $\mu$  is strictly irreducible, then*

$$h_1 = \dots = h_m = h, \tag{2.3.6}$$

and

$$\alpha_i h = h \tag{2.3.7}$$

for each  $i = 1, \dots, m$ .

The main idea of the proof is just this: if  $v_1, v_2$ , and  $v_3$  are operators from  $L_2(M, \tau)$  such that

$$\|v_1\| = \|v_2\| = \|v_3\| \tag{2.3.8}$$

and

$$v_1 = \frac{(v_2 + v_3)}{2}, \quad (2.3.9)$$

then

$$v_1 = v_2 = v_3. \quad (2.3.10)$$

Y. Guivarc'h used this observation in [33] to prove the invariance of the limit function in his Ergodic Theorem.

*Proof.* Let  $(p_1, \dots, p_m)$  be initial distribution of the measure  $\mu$ , and let  $P = (p_{ij})$  be the transition probability matrix of  $\mu$ . For any  $i, j \in \{1, \dots, m\}$  and  $n \in \mathbb{N}$ , denote by  $p_{ij}^{(n)}$  the  $n$ -step transition probability from  $i$  to  $j$  (in other words,  $p_{ij}^{(n)} = (P^n)_{ij}$ ).

We partition proof of Theorem 2.3.2 into series of steps.

**Proposition 2.3.3.** *Let  $(M, \tau)$  be a non-commutative probability space, and let linear operators*

$$\alpha_1, \dots, \alpha_m : L_2(M, \tau) \longmapsto L_2(M, \tau)$$

*be contractions. Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ , such that the corresponding Markov chain is irreducible. Suppose that operators*

$$h_1, \dots, h_m \in L_2(M, \tau)$$

*satisfy the relation*

$$\alpha_\mu(h_1, \dots, h_m) = (h_1, \dots, h_m). \quad (2.3.11)$$

*Then there is an  $r \in \mathbb{R}$ , such that*

$$\|h_1\| = \dots = \|h_m\| = r \quad (2.3.12)$$

*and, if*

$$p_{ij} > 0, \quad (2.3.13)$$

then

$$\|\alpha_j h_i\| = r. \quad (2.3.14)$$

*Proof.* Assume that

$$\|h_1\| \geq \|h_i\|$$

for any  $i = 1, \dots, m$ . Since equality 2.3.11 implies

$$h_1 = \sum_{i=1}^m \left( \frac{p_i p_{i1}}{p_1} \right) \alpha_1 h_i \quad (2.3.15)$$

and invariance of initial vector  $(p_1, \dots, p_m)$  with respect to transition matrix  $P^T$  implies

$$1 = \sum_{i=1}^m \frac{p_i p_{i1}}{p_1}. \quad (2.3.16)$$

It follows from the triangle inequality that

$$\|h_1\| = \|\alpha_1 h_i\| = \|h_i\| \quad (2.3.17)$$

if

$$p_{i1} > 0. \quad (2.3.18)$$

Similarly,

$$\|h_1\| = \|h_i\| \quad (2.3.19)$$

for any  $i$  such that

$$p_{i1}^{(2)} > 0, \quad (2.3.20)$$

and so on. The Markov chain corresponding to the measure  $\mu$  is irreducible; hence,

$$\|h_1\| = \dots = \|h_m\|, \quad (2.3.21)$$

and

$$\|h_j\| = \|\alpha_j h_i\| \quad (2.3.22)$$

if

$$p_{ij} > 0.$$

□

**Proposition 2.3.4.** *Suppose that*

$$h_1, \dots, h_n, h \in L_2(M, \tau)$$

*satisfy the condition*

$$\|h_1\| = \|h_2\| = \dots = \|h_n\| = \|h\|. \quad (2.3.23)$$

*Let*

$$h = c_1 h_1 + \dots + c_n h_n \quad (2.3.24)$$

*for some*

$$c_1 > 0, \dots, c_n > 0$$

*such that*

$$c_1 + \dots + c_n = 1. \quad (2.3.25)$$

*Then*

$$h_1 = h_2 = \dots = h_n = h. \quad (2.3.26)$$

*Proof.* This immediately follows from equality condition for the Cauchy-Bunyakowsky-Schwartz inequality in the Hilbert space. □

**Proposition 2.3.5.** *Let  $(M, \tau)$  be a non-commutative probability space,*

$$\alpha : L_2(M, \tau) \longmapsto L_2(M, \tau)$$

*be a contraction, and let operators*

$$h_1, h_2 \in L_2(M, \tau)$$

satisfy the relations

$$\|h_1\| = \|h_2\| = \|\alpha h_1\| = \|\alpha h_2\|. \quad (2.3.27)$$

Then

$$\alpha h_1 = \alpha h_2 \quad (2.3.28)$$

implies

$$h_1 = h_2. \quad (2.3.29)$$

*Proof.* Indeed, if

$$h_1 \neq h_2,$$

then

$$\left\| \frac{h_1 + h_2}{2} \right\| < \|h_1\| \quad (2.3.30)$$

by Proposition 2.3.4. Since

$$\left\| \alpha \left( \frac{h_1 + h_2}{2} \right) \right\| = \|h_1\| \quad (2.3.31)$$

and  $\alpha$  is a contraction, we arrive at a contradiction.  $\square$

In what follows,

$$(PP^T)_{ij}$$

stands for the  $(i, j)$ -entry of the matrix  $PP^T$ .

**Proposition 2.3.6.** *Let  $(M, \tau)$  be a non-commutative probability space, and let linear operators*

$$\alpha_1, \dots, \alpha_m : L_2(M, \tau) \longmapsto L_2(M, \tau)$$

*be contractions. Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ , and let*

$$h_1, \dots, h_m \in L_2(M, \tau)$$

be such that

$$\alpha_\mu(h_1, \dots, h_m) = (h_1, \dots, h_m). \quad (2.3.32)$$

Let transition matrix  $P$  of  $\mu$  be irreducible. Then

$$(PP^T)_{ij} > 0$$

implies

$$h_i = h_j. \quad (2.3.33)$$

*Proof.* By Proposition 2.3.3, if  $P$  is irreducible, then

$$\|h_1\| = \dots = \|h_m\|. \quad (2.3.34)$$

Note that

$$(PP^T)_{ij} > 0$$

if and only if there is a  $k$  for which

$$p_{ik} > 0$$

and

$$p_{jk} > 0.$$

Since

$$h_k = \sum_{l=1}^m \left( \frac{p_l p_{lk}}{p_k} \right) \alpha_k h_l, \quad (2.3.35)$$

it follows from Proposition 2.3.4 and 2.4.16 that

$$h_k = \alpha_k h_i = \alpha_k h_j, \quad (2.3.36)$$

and

$$\|h_k\| = \|h_i\| = \|h_j\| \quad (2.3.37)$$

by Proposition 2.3.3. By Proposition 2.3.5 this yields

$$h_i = h_j, \quad (2.3.38)$$

which completes the proof.  $\square$

Combination of statements of the Propositions 2.3.3-2.3.6 finishes the proof of Theorem 2.3.2.  $\square$

Let us return to the proof of Theorem 2.3.1.

Suppose that there exist  $\varphi_i$  and  $\varphi_j \in L_1(M, \tau)$  with  $i, j \in \{1, \dots, m\}$  and  $\|\varphi_i - \varphi_j\|_{L_1} \geq \epsilon > 0$  and satisfying equality 2.3.2. Since  $L_2(M, \tau)$  is dense in the  $L_1(M, \tau)$  in  $L_1(M, \tau)$  norm, we can find  $h_j \in L_2(M, \tau)$  satisfying  $\|h_j - \varphi_j\|_{L_1} < \epsilon/3$  for each  $j \in \{1, \dots, m\}$ . Let

$$(\overline{h_1}, \dots, \overline{h_m}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha_\mu^n(h_1, \dots, h_m). \quad (2.3.39)$$

The limit in equation 2.3.39 exists in  $L_1$  and  $L_2$  norm. Hence,  $\overline{h_l} \in L_2(M, \tau)$ . Since  $\alpha_\mu$  is contraction in  $L_1(M, \tau)$ , then  $\|\overline{h_l} - \varphi_l\|_{L_1} \leq \epsilon/3$ . In addition, the following equality holds

$$\alpha_\mu(\overline{h_1}, \dots, \overline{h_m}) = (\overline{h_1}, \dots, \overline{h_m}).$$

Hence, from Theorem 2.3.2 the following equality holds:

$$\overline{h_1} = \overline{h_2} = \dots = \overline{h_m}.$$

The latter equality implies that

$$\epsilon \leq \|\varphi_i - \varphi_j\|_{L_1} \leq \|\varphi_i - \overline{h_i}\|_{L_1} + \|\overline{h_j} - \varphi_j\|_{L_1} \leq 2\epsilon/3.$$

We came to contradiction with the suggestion that  $\epsilon \leq \|\varphi_i - \varphi_j\|_{L_1}$ . Theorem 2.3.1 is established.

**Theorem 2.3.7.** *Let  $(M, \tau)$  be a non-commutative probability space, let  $p > 1$  and let*

$$\alpha_1, \dots, \alpha_m : L_p(M, \tau) \longmapsto L_p(M, \tau)$$

*be contractions. Let  $\mu$  be a  $\sigma_m$ -invariant Markov measure on  $\Omega_m$ , and let operators*

$$\varphi_1, \dots, \varphi_m \in L_p(M, \tau)$$

*be such that*

$$\alpha_\mu(\varphi_1, \dots, \varphi_m) = (\varphi_1, \dots, \varphi_m). \quad (2.3.40)$$

*If the measure  $\mu$  is strictly irreducible, then*

$$\varphi_1 = \dots = \varphi_m = \varphi \quad (2.3.41)$$

*and*

$$\alpha_i \varphi = \varphi \quad (2.3.42)$$

*for any  $i = 1, \dots, m$ .*

*Proof.* The proof of the latter Theorem reproduces that of Theorem 2.3.2 above. The key observation is that Proposition 2.3.4 holds for the space  $L_p(M, \tau)$  since  $L_p(M, \tau)$  is a strictly convex space (see for example [56]).  $\square$

Theorems 2.3.7 and 2.3.1 imply Theorem 2.1.3.

## 2.4 Ergodic Type Theorem and Skew Product Transformations

In this section we prove Skew Ergodic Theorem (originated by S.Kakutani). Next, we apply Theorem 2.1.1 to the proof of the Ergodic Theorem for a finitely generated

locally free semigroup acting as a contraction  $T$  on a tracial von Neumann algebra  $M$ . These semigroups were introduced by Vershik in [67].

Let  $\alpha_1, \dots, \alpha_m$  be positive kernels.

For each  $\omega \in W_m$  denote by  $T_\omega$  the operator defined as

$$T_\omega : M \longmapsto M, T_\omega = \alpha_{\omega_1} \circ \dots \circ \alpha_{\omega_n} \quad (2.4.1)$$

For the sake of convenience, we set  $T_\omega$  with  $|\omega| = 0$  equal to the identity operator on  $M$ .

Note that the order of actions of the operators is reversed compared to the definition in equation (2.1.5). To stress this we use notation  $T_\omega$  instead of  $\alpha_\omega$ .

For any operator  $\varphi$  in  $L_1(M, \tau)$  we denote

$$S_j^\mu(T)(\varphi) = \sum_{|\omega|=j} \mu(\omega) T_\omega(\varphi). \quad (2.4.2)$$

Here  $T$  is just a notational device to remind the reader that this operator is built from the  $T_\omega$ s. We denote by

$$C_n^\mu(T)(\varphi) = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\mu(T)(\varphi) \quad (2.4.3)$$

Česaro average of the  $S_j^\mu(T)(\varphi)$  over  $j$ .

**Theorem 2.4.1.** *Let  $\varphi$  be an operator affiliated to a tracial von Neumann algebra  $M$  with separable predual  $\varphi \in L_1(M, \tau)$ . Then there exists an operator  $\bar{\varphi} \in L_1(M, \tau)$  such that*

$$C_n^\mu(T)(\varphi) \longmapsto \bar{\varphi} \text{ in } L_1(M, \tau) \text{ norm,} \quad (2.4.4)$$

and if  $\alpha_1, \dots, \alpha_m$  preserve the trace  $\tau$

$$\tau(\varphi) = \tau(\bar{\varphi}). \quad (2.4.5)$$

If, in addition,  $\varphi \in L_{1+\epsilon}(M, \tau)$ , then

$$C_n^\mu(T)(\varphi) \longmapsto \bar{\varphi} \text{ bilateral almost everywhere .} \quad (2.4.6)$$

The rest of the section is devoted to the proof of Theorem 2.4.1.

For any  $\omega \in W_m$ ,  $\omega = \omega_1 \dots \omega_n$  denote the expression  $\omega_n \dots \omega_1$  by  $\omega^*$ . If  $\mu$  is  $\sigma_m$ -invariant Borel probability measure on  $\Omega_m$ , then formula  $\mu^*(\omega) = \mu(\omega^*)$  defines a  $\sigma_m$ -invariant Borel probability measure with the property  $\mu^{**} = \mu$ . The latter implies that

$$C_n^\mu(T) = c_n^{\mu^*}(T), \quad (2.4.7)$$

here  $c_n^{\mu^*}(T)$  is defined in equality (2.2.9).

Let  $\Psi = L_\infty(\Omega_m, \mu) \overline{\otimes} M$  be a tensor product of von Neumann algebras  $L_\infty(\Omega_m, \mu)$  and  $M$ . Algebra  $\Psi$  may be considered as a von Neumann algebra  $L_\infty(\Omega_m, \mu, M)$  of essentially bounded ultra-weakly measurable functions  $\psi : \Omega_m \mapsto M$ , with the norm  $\|\psi\| = \text{ess sup}_{\omega \in \Omega_m} \|\psi(\omega)\|_\infty$ . A trace for the algebra  $\Psi$  is given by the formula

$$\nu(\psi) = \int_{\Omega_m} \tau(\psi(\omega)) d\mu(\omega), \text{ for } \psi \in \Psi \quad (2.4.8)$$

Consider an operator  $\Phi$  defined by

$$\Phi(\psi)(\omega) = \alpha_{\omega_1}(\psi(\sigma(\omega))) \text{ for } \psi \in \Psi \quad (2.4.9)$$

It is easy to see that  $\Phi : \Psi \mapsto \Psi$ , and  $\Phi$  is a positive kernel on  $\Psi$ .

Extension of the operator  $\Phi$  to  $L_1(\Omega_m, \mu, M_*)$ , pre-conjugate of the algebra  $L_\infty(\Omega_m, \mu, M) = (L_1(\Omega_m, \mu, M_*))^*$  with norm  $\|\cdot\|_1$  we denote also by  $\Phi$ .

For every  $x \in M$  denote by  $x^e$  embedding of  $x$  into  $\Psi$  as an function having value  $x$  for each  $\omega \in \Omega_m$  or  $x^e = \mathbb{I} \otimes x$ . This embedding extends by norm  $\|\cdot\|_1$  into embedding

of  $L_1(M, \tau)$  into  $L_1(\Omega_m, \mu, M_*)$ . We denote the images under the embedding by  $M^e$  and  $L_1(M^e)$ .

**Lemma 2.4.2.** *For the  $\omega \in W_m$  with  $|\omega| = n$  and  $\varphi \in L_1(M, \tau)$ , the  $n$ -th iteration of  $\Phi$  satisfies*

$$\Phi^{(n)}(\psi)(\omega) = \alpha_{\omega_1} \cdots \alpha_{\omega_n}(\psi(\sigma_m^{(n)}\omega)) \text{ for } \psi \in \Psi \quad (2.4.10)$$

and the following equality holds for  $\Phi$

$$\Phi^{(n)}(\varphi^e) = T_\omega(\varphi) \text{ for } \varphi \in L_1(M, \tau) \quad (2.4.11)$$

*Proof.* The statement of the lemma follows directly from the definition of the action  $\Phi$  and imbedding  $M^e$ .  $\square$

**Lemma 2.4.3.** *For  $\varphi \in L_1(M, \tau)$  the following equality holds:*

$$\int_{\Omega_m} \Phi^j(\varphi^e) d\mu(\omega) = \sum_{|\omega|=j} \mu(\omega) T_\omega(\varphi) \quad (2.4.12)$$

Denote by

$$\varphi_n^e = \frac{1}{n} \sum_{j=0}^{n-1} \Phi^j(\varphi^e) \quad (2.4.13)$$

the  $n$ -th Česaro average of  $\varphi^e$ .

**Lemma 2.4.4.** *For  $\varphi \in L_1(M, \tau)$  the following equality holds:*

$$\int_{\Omega_m} \varphi_n^e d\mu(\omega) = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{|\omega|=j} \mu(\omega) T_\omega(\varphi) \quad (2.4.14)$$

and

$$C_n^\mu(T)(\varphi) = \int_{\Omega_m} \varphi_n^e d\mu(\omega) \quad (2.4.15)$$

Denote by  $Proj(M)$  the set of all orthogonal projection operators in  $M$ . Let  $E_e$  be a conditional expectation of the  $\Psi$  onto subalgebra  $M^e$  defined as

$$E_e(\psi(\omega)) = \int_{\Omega_m} \psi(\omega) d\mu(\omega) \quad (2.4.16)$$

**Lemma 2.4.5.** *Sequence  $\varphi_n^e$  converges in norm of  $\|\cdot\|_1$ .*

*Proof.* Follows from the Mean Ergodic Theorem, see for example [44], or for the version of the Mean Ergodic Theorem for symmetric spaces see [73]. For convenience of the reader we give the precise formulation of Theorem 1.5, Chapter 9 from the book [44] following the book's notation.

**Theorem 2.4.6** (Theorem 1.5, Chapter 9, [44]). *Let  $\Phi$  be a faithful family of normal states of a von Neumann algebra  $\mathfrak{A}$  and let  $\mathfrak{S} = \{T_u : u \in \mathfrak{V}\}$  be a  $d$ - parametric semigroup of  $w^*$  - continuous positive contractions in  $\mathfrak{A}$  with  $T_{u*}\phi \leq \phi$  for all  $u \in \mathfrak{V}$  and  $\phi \in \Phi$ . Then*

- i)  $(A_n)_*\psi$  converges for  $n \rightarrow \infty$  in norm to  $P_*\psi$  for all  $\psi \in \mathfrak{A}_*$ , where  $P_*$  is a positive contraction in  $\mathfrak{A}_*$ ;*
- ii)  $\sigma - \lim_{n \rightarrow \infty} A_n x = Px$  for all  $x \in \mathfrak{A}$ , where  $P$  is the adjoint of  $P_*$ ;*
- iii) any  $x \in \mathfrak{A}$  is a  $s^*$  - limit of a norm - bounded net  $(x_\gamma)$  such that each  $x_\gamma - Px$  is a finite convex combination of elements of the form  $(I - T_u)x$ .*

Precise definition of the  $d$ -dimensional Cesaro average  $(A_n)$  may be found in section 6.1 of the book [44], p. 195, which explains what it means when the multi-index tends to infinity. We use this theorem in the case when there is only one faithful normal state and dimension  $d = 1$ . In this case  $A_n$  coincides with the regular Cesaro average.

Statement of the lemma follows from the 2.4.6 applied to the operator  $\Phi$  acting in the space  $L_1(\Omega_m, \mu, M_*)$  □

**Lemma 2.4.7.** *Suppose that  $\varphi \in L_{1+\epsilon}(M, \tau)$ . Then there exists a sequence of positive operators  $B_n$  from  $L_1(\Omega_m, \mu, M)$  decreasing to 0 such that for positive  $\varphi$*

$$-B_n \leq \varphi_n^e - \bar{\varphi} \leq B_n, \text{ for all } n \quad (2.4.17)$$

*Proof.* Follows from the theorem about majorant convergence , see [21]. □

Since a conditional expectation is a contraction in the norm  $\|\cdot\|_1$ , the latest lemma and (2.3.13)-(2.3.16) imply  $\|\cdot\|_1$  convergence in Theorem 2.4.1.

**Lemma 2.4.8.** *Let  $E$  be a conditional expectation of von Neumann algebra  $\mathfrak{M}$  onto a subalgebra  $\mathfrak{B} \subset \mathfrak{M}$ . If a sequence  $\{X_n\}_{n=1}^\infty$  of self-adjoint operators from the algebra  $\mathfrak{M}$  is such that its majorant converges to 0, then  $\{E(X_n)\}_{n=1}^\infty$  majorant converges to 0.*

*Proof.* Follows from application of expectation  $E$  to the majorant convergence inequality (see [21], Theorem 3.1). □

The proof of Theorem 2.4.1 follows from the lemmas above and the fact that majorant convergence implies bilateral almost everywhere convergence (see for example [21], Section 3).

## 2.5 Ergodic Type Theorem for the Action of Finitely Generated Locally Free Semigroups

**Definition 2.5.1.** A locally free semigroup (see [68] and references there)  $\mathcal{LFS}_{m+1}$  with  $m$  generators is defined as a semigroup determined by generators satisfying the

following relations:

$$\mathcal{LFS}_{m+1} = \{g_1, \dots, g_m \mid g_i g_j = g_j g_i; i, j \in \{1, \dots, m\}, |i - j| > 1\} \quad (2.5.1)$$

Semigroup  $\mathcal{LFS}_{m+1}$  is associated with a topological Markov chain with states  $\{1, \dots, m\}$  and transition matrix

$$m = (m_{i,j}), \quad m_{i,j} = \begin{cases} 1, & \text{if } |i - j| \leq 1 \text{ or } i \leq j; \\ 0, & \text{otherwise.} \end{cases} \quad (2.5.2)$$

The set of admissible words in the chain corresponds to the  $W_m$ ; the set of admissible one-sided sequences corresponds to  $\Omega_m$ , and left shift  $\sigma_m$  corresponds to shift on  $\Omega_m$ . Each word  $\omega_1 \dots \omega_n$  corresponds to  $g_\omega = g_{\omega_1} \dots g_{\omega_n}$ .

The correspondence  $\omega \mapsto g_\omega$  defines a bijection between  $W_m$  and  $\mathcal{LFS}_{m+1}$ , and from (4.2) it follows that system  $(\Omega_m, \sigma_m)$  mixes topologically, hence ergodic measure has a positive measure on cylinders corresponding to the words  $W_m$ .

Now we assume that semigroup  $\mathcal{LFS}_{m+1}$  acts as a semigroup with generators  $g_i$  mapped to the kernels  $\alpha_i$  acting on a tracial von Neumann Algebra  $(M, \tau)$ . Applying Theorem 2.1.1, we obtain an Ergodic Theorem for the action of  $\mathcal{LFS}_{m+1}$ .

# Chapter 3

## Stochastic Banach Principle and Some Applications for Semi-finite von Neumann Algebras

### 3.1 Preliminaries

**Definition 3.1.1.** Let  $x$  be a measurable operator from  $S(M)$  and  $t > 0$ . The **t-th singular number of  $x$**  is defined as

$$\mu_t(x) = \inf\{\|xe\| \text{ where } e \text{ is a projection in } P(M) \text{ with } \tau(\mathbb{I} - e) < t\}. \quad (3.1.1)$$

*Remark 3.1.1.* Note that **measure topology** is defined in Fack and Kosaki's [18] as linear topology with fundamental system of neighborhoods around 0 given by  $V(\epsilon, \delta) = \{x \in S(M) \text{ such that there exists a projection } e(x, \epsilon, \delta) \text{ with } \|xe\| < \epsilon \text{ and } \tau(\mathbb{I} - e) < \delta\}$ .

**Definition 3.1.2.** Denote by  $\lambda_t(x)$  **the distribution function of  $x$**  defined as

$$\lambda_t(x) = \tau(E_{(t, \infty)}(|x|)), \quad t \geq 0, \quad (3.1.2)$$

here  $E_{(t, \infty)}(|x|)$  is a spectral projection of  $x$  corresponding to interval  $(t, \infty)$ .

*Remark 3.1.2.* For the measurable operator  $x$ , we have  $\lambda_t(x) < \infty$  for large enough  $t$  and  $\lim_{t \rightarrow \infty} \lambda_t(x) = 0$ . Moreover, the map  $\mathbb{R} \ni t \rightarrow \lambda_t(x)$ , is non-increasing and continuous from the right (because  $\tau$  is normal and  $\{|x| > t_n\} \uparrow \{|x| > t\}$  (and hence in strong operator topology) as  $t_n \downarrow t$ ). The distribution  $\lambda_t(x)$  is a non-commutative analogue of the distribution function in classical analysis, (see. [18] p. 272 or [64]).

We would need the following statement about properties of the  $\mu_t(x)$  (see for example proposition 2.4 [70], or lemma 2.5 [18]):

**Lemma 3.1.1.** *Let  $x, y \in S(M)$  be measurable operators.*

*i) Map  $\mathbb{R} \ni t \rightarrow \mu_t(x)$  is non-decreasing and continuous from the right.*

*Moreover  $\lim_{t \downarrow 0} \mu_t(x) = \|x\|_\infty \in [0, \infty]$ ,*

*ii)  $\mu_t(x) = \mu_t(|x|) = \mu_t(x^*)$  and  $\mu_t(\alpha x) = |\alpha| \mu_t(x)$  for  $\alpha \in \mathbb{C}$ ,  $t > 0$ ,*

*iii)  $\mu_t(x) \leq \mu_t(y)$  for  $0 \leq x \leq y$ ,  $t > 0$ ,*

*iv)  $\mu_{t+s}(x + y) \leq \mu_t(x) + \mu_s(y)$  for  $t, s > 0$ ,*

*v)  $\mu_t(yxz) \leq \|y\|_\infty \|z\|_\infty \mu_t(x)$ , for  $y, z \in M$ ,  $t > 0$ ,*

*vi)  $\mu_{t+s}(yx) \leq \mu_t(x) \mu_s(y)$  for  $t, s > 0$ .*

## 3.2 Stochastic Banach Principle

We start the section with the description of some conditions equivalent to stochastic convergence (cmp. with lemma 3.1 [18]).

**Lemma 3.2.1.** *Let  $M$ ,  $\tau$  be as before. Consider following conditions:*

i) Sequence  $\{x_n\}_{n=1}^{\infty}$  converges to 0 in measure,

ii) For every  $\epsilon > 0, \delta > 0$  there exist a positive real  $0 < \delta' < \delta$  and an integer  $N_0$  such that for  $n \geq N_0$

$$\mu_{\delta'}(x_n) < \epsilon,$$

iii) For every  $\epsilon > 0, \delta > 0, p \in P(M)$  with  $\tau(p) < \infty$  there exists an integer  $N_0$  and a sequence of projections  $\{e'_n\}_{n \geq N_0} \subset P(M)$ ,  $e'_n \leq p$  such that

$$\|x_n e'_n\|_{\infty} < \epsilon \text{ and } \tau(p - e'_n) < \delta \text{ for } n \geq N_0.$$

The following relations take place:  $i) \Leftrightarrow ii) \Rightarrow iii)$ . If  $\tau$  is finite then  $iii) \Rightarrow i)$ .

*Proof.* Implication  $ii) \Rightarrow i)$  follows from the fact that condition  $\mu_{\delta'}(x_n) < \epsilon$  implies that for some sequence of projections  $\{e_n\}_{n=1}^{\infty}$ , holds  $\|x_n e_n\| \leq 2\epsilon$  and  $\tau(\mathbb{I} - e_n) \leq \delta'$ .

Implication  $i) \Rightarrow ii)$  follows from the definition of measure convergence 0.3.5.

Implication  $i), ii) \Rightarrow iii)$  follows from the inequality

$\tau(p - p \wedge q) = \tau(p \vee q - q) \leq \tau(\mathbb{I} - q)$ , hence sequence  $\{e'_n = e_n \wedge p\}_{n=1}^{\infty}$  satisfies iii) (here projections  $e_n$  are defined in the Proof  $ii) \Rightarrow i)$ ).

The case when  $\tau$  is finite follows immediately since  $\tau(\mathbb{I}) < \infty$ . □

We need the following technical statement which is interesting by itself:

**Lemma 3.2.2.** *Let  $x, y$  be self-adjoint measurable operators from  $S(M)$ ,  $t, s$  be positive real. Then*

$$\lambda_{t+s}(x + y) \leq \lambda_t(x) + \lambda_s(y) \tag{3.2.1}$$

*Proof.* Indeed,

$$\begin{aligned}
& \| |(x+y)|(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| = \\
& \| (x+y)(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| \leq \\
& \leq \| x(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| + \| y(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| = \\
& = \| |x|(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| + \| |y|(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| \leq \\
& \leq \| |x|(\mathbb{I} - \{|x| > t\}) \| + \| |y|(\mathbb{I} - \{|y| > s\}) \| \leq \\
& t + s.
\end{aligned} \tag{3.2.2}$$

Here the first and the second equality follows from the equality  $\| |z| u_z^* u_z |z| \| = \| |z|^2 \| = \| z^* z \|$ , where  $z \in M_h$ ,  $u_z$  is a partial isometry from  $M$  such that  $z = u_z |z|$ , and

$$u_z^* u_z = l(z), \quad u_z u_z^* = r(z), \tag{3.2.3}$$

where  $l(z)(r(z))$  is a left (right) support of  $z$ . Inequality 3.2.2 means that

$$\mu_{\lambda_t(x) + \lambda_s(y)}(x+y) \leq t + s. \tag{3.2.4}$$

Let  $\xi$  be a vector from Hilbert space  $\mathfrak{H}$  and suppose that

$$\xi \in \{|x+y| > s+t\} \mathfrak{H} \cap (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \mathfrak{H}. \tag{3.2.5}$$

Then

$$((t+s)\|\xi\|)^2 < (|x+y|\xi, |x+y|\xi) = ((x+y)\xi, (x+y)\xi) \leq ((t+s)\|\xi\|)^2, \tag{3.2.6}$$

here the first inequality follows from inclusion  $\xi \in \{|x+y| > s+t\} \mathfrak{H}$ , the equality follows from the spectral decomposition 3.2.3, the second inequality follows from inclusion  $\xi \in (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \mathfrak{H}$ .

Inequality 3.2.6 implies that  $\|\xi\| = 0$  or, in other words,

$$\{|x + y| > s + t\} \wedge ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) = 0.$$

Hence,

$$\begin{aligned} \{|x + y| > t + s\} &= \{|x + y| > t + s\} - \\ &\{|x + y| > t + s\} \wedge ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) \sim \\ &\sim \{|x + y| > t + s\} \vee ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) - \\ &((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) \leq \\ &\leq \mathbb{I} - ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) = \{|x| > t\} \vee \{|y| > s\}. \end{aligned} \tag{3.2.7}$$

Here  $\sim$  means projection equivalence. Since trace  $\tau$  is invariant on equivalent projections,

$$\tau(\{|x + y| > t + s\}) \leq \tau(\{|x| > t\} \vee \{|y| > s\}) \leq \tau(\{|x| > t\}) + \tau(\{|y| > s\}) \tag{3.2.8}$$

and, hence, the inequality (3.2.1) takes place.  $\square$

**Theorem 3.2.3.** *Let  $(B, \|\cdot\|)$  be a Banach space. Let  $\Sigma = \{A_n, n \in \mathbb{N}\}$  be a set of linear operators  $A_n : B \rightarrow S(M)$ .*

*i) Suppose that there exists a function  $C(\lambda) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\lim_{\lambda \rightarrow \infty} C(\lambda) = 0$ , and such that*

$$\sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\|\}) \leq C(\lambda) \tag{3.2.9}$$

*holds for every  $b \in B$ ,  $\lambda \in \mathbb{R}_+$ .*

*Then the subset  $\tilde{B}$  of  $B$  where  $A_n(b)$  converges in measure (stochastically) is closed in  $B$ .*

ii) Conversely, if  $A_n$  is a set of continuous in measure maps from  $B$  into  $S(M)$  and for each  $b \in B, \lambda \in \mathbb{R}_+$

$$\lim_{\lambda \rightarrow \infty} \sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda\}) = 0, \quad (3.2.10)$$

then there exists a function  $C(\lambda) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\lim_{\lambda \rightarrow \infty} C(\lambda) = 0$ , and

$$\sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\|\}) \leq C(\lambda) \quad (3.2.11)$$

Part i) of the theorem 3.2.3 means that under the condition of the linear uniform boundedness (3.2.9), the set of the stochastical convergence is closed.

Part ii) of the theorem 3.2.3 means that if the set of uniform boundedness is closed, then linear uniform boundedness takes place.

Note that even though the condition in the part ii) looks more restrictive, it is similar in nature to the condition of part i), since we can restrict everything to the closed linear subspace  $B_1$  of Banach space  $B$  ( $B_1$  is also Banach space).

*Proof.* Part i) We first show that condition 3.2.9 implies continuity of the set  $\Sigma$  of operators . Let  $B \supset \{b_k\}_{k=1}^\infty$  be a sequence in  $B - \{b\}$  converging to  $b \in B$ . Then for  $\lambda, \epsilon \in \mathbb{R}_+$  with  $2\lambda \sup_{k \geq n} \|b - b_k\| < \epsilon$

$$\begin{aligned} \tau(\{|A_n(b_k) - A_n(b)| > \epsilon\}) &\leq \tau(\{|A_n(b_k - b)| > \lambda \|b_k - b\|\}) \leq \\ &C(\lambda \|b_k - b\|^{-1}) \xrightarrow{b_k \rightarrow b} 0, \end{aligned} \quad (3.2.12)$$

for  $k \geq n$ , and, hence  $A_n$  continuous. Note that the inequality follows from the fact that right part of 3.2.9 does not depend on the norm of  $b$ .

Suppose now that the sequence  $A_n(b_k)$  converges when  $n \rightarrow \infty$  and, in addition, sequence  $b_k \rightarrow b$  for  $k \rightarrow \infty$ . There exists a subsequence  $b_{k_j}$  of  $b_k$  such that sequence

$x_j = \lim_{n \rightarrow \infty} A_n(b_{k_j})$  converges stochastically. To show this we choose a sequence of  $\{k'_i\}_{i=1}^{\infty}$  base on the inequality 3.2.9 in such a manner that

$$\tau(\{|A_n(b_{k'_j} - b_{k'_{j+l}})| > 2^{-j}\}) \leq 2^{-j} \text{ for all } n \quad (3.2.13)$$

and

$$\tau(\{|A_n(b_{k'_j} - b)| > 2^{-j}\}) \leq 2^{-j} \text{ for all } n \quad (3.2.14)$$

This may be done since  $b_n \xrightarrow{n \rightarrow \infty} b$ , and  $C(\lambda) \xrightarrow{\lambda \rightarrow \infty} 0$ . It is sufficient to choose a sequence  $\{\lambda_j\}_{j=1}^{\infty}$  in such a way that  $C(\lambda_j) < 2^{-j}$  and  $\|b_{k_j} - b\| < \lambda_j^{-1} 2^{-2j}$ .

Choose  $n_j$  in such a manner that for  $N > n_j$  holds

$$\tau(\{|A_N(b_{k_j}) - x_j| > 2^{-j}\}) < 2^{-j}. \quad (3.2.15)$$

This is possible since  $A_n(b_{k_j})$  converges stochastically to  $x_j$ .

Then for  $j, i \in \mathbb{N}$  and  $n > n_{i+j}$

$$\begin{aligned} \tau(\{|x_j - x_{j+i}| > 3 \cdot 2^{-j}\}) &= \tau(\{|(x_j - A_n(b_{k_j})) + \\ &\quad (A_n(b_{k_j}) - A_n(b_{k_{j+i}})) + (A_n(b_{k_{j+i}}) - x_{j+i})| > 3 \cdot 2^{-j}\}) \leq \\ &\tau(\{|A_n(b_{k_j}) - x_j| > 2^{-j}\}) + \tau(\{|A_n(b_{k_j}) - A_n(b_{k_{j+i}})| > 2^{-j}\}) + \\ &\tau(\{|A_n(b_{k_{j+i}}) - x_{j+i}| > 2^{-(j+i)}\}) \leq 3 \cdot 2^{-j}. \end{aligned} \quad (3.2.16)$$

Here the first inequality follows from 3.2.1.

Denote the stochastic limit of  $\{x_j\}_{j=1}^{\infty}$  by  $x_0$ . If necessary by taking a subsequence of  $\{x_j\}$  and reindexing, we suppose that

$$\tau(\{|x_j - x_0| > 2^{-j}\}) \leq 2^{-j}. \quad (3.2.17)$$

Sequence  $\{A_n(b)\}_{n=1}^{\infty}$  converges to  $x_0$  stochastically. Indeed, for  $n > n_j$  the following

inequality holds

$$\begin{aligned} \tau(\{|A_n(b) - x_0| > 3 \cdot 2^{-j}\}) &= \tau(\{|(A_n(b) - A_n(b_{k_j})) + \\ & (A_n(b_{k_j}) - x_j) + (x_j - x_0)| > 3 \cdot 2^{-j}\}) \leq \tau(\{|A_n(b) - A_n(b_{k_j})| > 2^{-j}\}) + \quad (3.2.18) \\ & \tau(\{|A_n(b_{k_j}) - x_j| > 2^{-j}\}) + \tau(\{|x_j - x_0| > 2^{-j}\}) \leq 3 \cdot 2^{-j}. \end{aligned}$$

Here the first inequality follows from 3.2.1, and the second inequality follows by noting that the first part follows from 3.2.14, the second part follows from 3.2.15 and choice of  $n$ , and the third part follows from 3.2.17.

Part i) is established.

Part ii) Suppose that for every  $b \in B$  and  $\lambda \in \mathbb{R}_+$  holds

$$\sup_n \tau(\{|A_n(b)| > \lambda\}) \xrightarrow{\lambda \rightarrow \infty} 0. \quad (3.2.19)$$

For fixed  $\epsilon > 0$  and  $\lambda \in \mathbb{N}$  define  $B_\lambda = \{b \in B \mid \sup_n \tau(\{|A_n(b)| > \lambda\}) \leq \epsilon\}$ . Then from 3.2.19 it follows that

$$B = \bigcup_{\lambda \in \mathbb{N}} B_\lambda \quad (3.2.20)$$

Let  $B_{\lambda,k}$  be a set defined as  $\{b \in B \mid \sup_{n \geq k} \tau(\{|A_n(b)| > \lambda\}) \leq \epsilon\}$ . Then

$$B_\lambda = \bigcap_{k \in \mathbb{N}} B_{\lambda,k} \quad (3.2.21)$$

Sets  $B_{\lambda,k}$  are closed. Indeed, let  $B_{\lambda,k} \supset \{b_j\}_{j=1}^\infty$  converge to  $b \in B$ . Then

$$\begin{aligned} \tau(\{|A_n(b)| > \lambda + \gamma\}) &= \tau(\{|A_n(b_j) - (A_n(b_j) - A_n(b))| > \lambda + \gamma\}) \leq \\ & \tau(\{|A_n(b_j)| > \lambda\}) + \tau(\{|(A_n(b_j) - A_n(b))| > \gamma\}) \leq \epsilon + \epsilon_j \end{aligned} \quad (3.2.22)$$

Here the first inequality follows from 3.2.1, the first estimate follows from the definition of  $B_{\lambda,k}$ . From continuity of the  $A_n$  in measure and the convergence of  $b_j$  to  $b$ , it follows that  $\epsilon_j$  can be chosen so that it decreases to 0 as  $j$  increases. Free choice of  $j$  implies that  $\tau(\{|A_n(b)| > \lambda + \gamma\}) \leq \epsilon$ .

Since  $\lambda_t(x)$  is continuous from the right (3.1.2), then

$$\tau(\{|A_n(b)| > \lambda\}) = \lim_{m \rightarrow \infty} \tau(\{|A_n(b)| > \lambda + \gamma_m\}) \leq \epsilon, \quad (3.2.23)$$

where  $\gamma_m \xrightarrow{m \rightarrow \infty} 0$ . Hence,  $b \in B_{\lambda,k}$ , or  $B_{\lambda,k}$  is closed. Set  $B_\lambda$  is closed as an intersection of closed sets (3.2.21).

It follows from the Baire category principle that there exists  $\lambda$  such that set  $B_\lambda$  has non empty interior. Let  $B(b_0, r) = \{b \in B \mid \|b - b_0\| \leq r\}$  be contained in the  $B_\lambda$ .

Then

$$\tau(\{|A_n(b)| > \lambda\}) \leq \epsilon \text{ for every } b \in B(b_0, r). \quad (3.2.24)$$

Moreover, for  $b = b_0 - r \cdot c \in B(b_0, r)$  with  $c \in B$ ,  $\|c\| \leq 1$ , holds

$$\begin{aligned} \tau(\{|A_n(r \cdot c)| > 2 \cdot \lambda\}) &= \tau(\{|A_n(r \cdot c - b_0) + A_n(b_0)| > 2 \cdot \lambda\}) \leq \\ &\tau(\{|A_n(r \cdot c - b_0)| > \lambda\}) + \tau(\{|A_n(b_0)| > \lambda\}) \leq 2 \cdot \epsilon. \end{aligned} \quad (3.2.25)$$

Let  $\gamma \geq 2 \cdot \lambda/r$ . From 3.2.25 it follows that  $\tau(\{|A_n(c)| > \gamma\}) \leq 2 \cdot \epsilon$ , for every  $c \in B$ ,  $\|c\| \leq 1$ .

Let  $C(\gamma) = \sup_{c \in B, \|c\| \leq 1} \tau(\{|A_n(c)| > \gamma\}) \leq 2 \cdot \epsilon$ . Free choice of  $\epsilon$  implies that

$$\lim_{\gamma \rightarrow \infty} C(\gamma) = 0, \quad (3.2.26)$$

hence 3.2.9 is valid. □

For the application of theorem 3.2.3 it is convenient to combine both parts i) and ii).

**Theorem 3.2.4.** *Let  $(B, \|\cdot\|)$  be a Banach space. Let  $A_n$  be a set of continuous in measure linear maps from  $B$  into  $S(M)$ , let  $\lambda \in R_+$ , and for each  $b \in B$  holds*

$$\lim_{\lambda \rightarrow \infty} \sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda\}) = 0. \quad (3.2.27)$$

*Then subset  $\tilde{B}$  of  $B$  where  $A_n(b)$  converges in measure (stochastically) is closed in  $B$ .*

*Proof.* Follows immediately from applying consecutively Theorem 3.2.3 part ii) then part i).  $\square$

Let  $e$  be a projection in  $M$ , let  $M_e$  be von Neumann algebra consisting of operators of form  $exe$ ,  $x \in M$ . If  $\tau$  is a semifinite normal faithful trace on  $M$  then  $\tau_e = \tau|_{M_e}$  is a semifinite (possibly finite) faithful normal trace on  $M_e$ . Indeed, tracial property, semifiniteness, normalness and faithfulness of  $\tau_e$  follows directly from similar properties of  $\tau$ . Space  $S(M_e, \tau_e)$  is isomorphic to the  $S(M, \tau)_e$  since both these spaces are closures of the  $(M_{\tau\text{-finitesupport}})_e = (M_e)_{\tau_e\text{-finitesupport}}$ .

**Proposition 3.2.5.** *Let  $B_n$  be a sequence of continuous in measure operators on  $S(M, \tau)$ . Let  $e_i \in P(M)$ ,  $i = 1, 2$ ,  $\mathbb{I} = e_1 + e_2$  be projections in  $M$ . Suppose that relation  $e_i(B_n(x)) = B_n(x_{e_i}) = (B_n(x))e_i$  holds for every  $n \in \mathbb{N}$  and  $x \in S(M, \tau)$ , or, in other words,  $e_i$  commutes with  $B_n$ . Suppose also following relations hold*

$$\lim_{\lambda \rightarrow \infty} \sup_{n \in \mathbb{N}} \tau(\{|B_n(x_{e_i})| > \lambda\}) = 0, \quad (3.2.28)$$

for  $i = 1, 2$  and every  $x \in S(M, \tau)$ . Then the following equality is valid:

$$\lim_{\lambda \rightarrow \infty} \sup_{n \in \mathbb{N}} \tau(\{|B_n(x)| > \lambda\}) = 0. \quad (3.2.29)$$

*Proof.* The following relations are valid:

$$\tau(\{|B_n(x_{e_i})| > \lambda\}) = \tau(e_i\{|B_n(x)| > \lambda\}). \quad (3.2.30)$$

In order to show inequality 3.2.30, we establish equality  $\{|Y_e| > \lambda\}e = \{|Y| > \lambda\}e$  for  $Y \in S(M, \tau)$  and projection  $e \in M$  satisfying equality  $Y_e = eYe = Ye$ . Indeed,  $Y_e = eYe = Ye$  implies equality  $Y_{(\mathbb{I}-e)} = (\mathbb{I} - e)Y(\mathbb{I} - e) = Y(\mathbb{I} - e)$ . Let  $H = H(M, \tau(\cdot, \cdot))$  be a standard representation of algebra  $M$  based on fsn trace  $\tau$  (see

for example Takesaki, [65], chapter V, Theorem 2.22). Since  $Y$  is self-adjoint, then operators  $(i\mathbb{I} \pm Y)$  are reversible, and  $(i\mathbb{I} \pm Y)D(Y) = H$ , (see for example Reed, Simon [57], Theorem 8.3), here  $D(Y)$  is a domain of  $Y$ . Since  $Y_e = Ye$  and  $Y_{(\mathbb{I}-e)} = Y(\mathbb{I}-e)$ , then  $(i\mathbb{I} \pm Y)eD(Y) = eH$ , or  $((i\mathbb{I} \pm Y)|_{eH})^{-1} = ((ie \pm Y_e)|_{eH})^{-1} = (ie \pm Y_e)^{-1}e$ .

For every polynomial  $P(q, s)$  holds following equality:

$$P((i\mathbb{I} + Y)^{-1}, (i\mathbb{I} - Y)^{-1})e = P((ie + Y_e)^{-1}e, (ie - Y_e)^{-1}e).$$

By Stone-Weierstrass theorem (see for example Reed, Simon [57], Theorem 8.20) polynomials without constant part of  $(r \pm i)^{-1}$  are dense in the  $C_\infty(\mathbb{R})$  - algebra of all continuous complex valued functions on  $\mathbb{R}$  vanishing on infinity. Hence all polynomial of  $(r \pm i)^{-1}$  are dense in the algebra of all continuous complex valued functions on  $\mathbb{R}$  having the same limit on  $\pm$  infinity.

Hence for any continuous function  $f$  on  $\mathbb{R}$  having the same limit on  $\pm$  infinity the following equality holds

$$f(Y)e = f(Y_e)e.$$

For the case when  $f$  is a Borel function see for example Corollary 5.6.31 of Kadison and Ringrose, Vol 1 [41]. We gave the proof of the above equality for the sake of completeness.

Define for  $\lambda > 0$  and  $n \geq 2\lambda^{-1}$  a sequence of continuous functions  $\{f_n\}$  as

$$f_n(r) = \begin{cases} 1, & \text{for } |r| > \lambda; \\ 0, & \text{for } |r| < \lambda - 1/n; \\ \text{linear,} & \text{for } \lambda - 1/n \leq |r| \leq \lambda. \end{cases}$$

Note that sequence  $\{f_n\}$  converges pointwise to function  $\chi_{|r|>\lambda}$ , functions  $f_n$  are continuous, and they have the same limit on  $\pm$  infinity and are uniformly bounded.

By [57], Theorem 8.5 (d),

$$\{|Y_e| > \lambda\} e = so - \lim f_n(Y_e)e = so - \lim f_n(Y)e = \{|Y| > \lambda\} e,$$

here  $so - \lim$  means limit in strong topology on  $M \subset B(H)$ .

Condition  $\lambda > 0$  implies that  $Ker(e) \subset Ker(\{|Y_e| > \lambda\})$ , hence  $\{|Y_e| > \lambda\} = \{|Y_e| > \lambda\} e$ , here by  $Ker(Q)$  we denote null set (kernel) of linear operator  $Q \in B(H)$ .

That means that equality  $\{|Y_e| > \lambda\} = \{|Y| > \lambda\} e$  is established. Equality 3.2.30 follows from applying trace  $\tau$  to both parts of the equality  $\{|Y_e| > \lambda\} = \{|Y| > \lambda\} e$  for  $Y = B_n(x)$

Statement 3.2.29 follows now from the fact that (it follows from  $B_n$  commuting with  $e_i$  and 3.2.1)

$$\begin{aligned} \tau(\{|B_n(x)| > \lambda_1 + \lambda_2\}) &= \tau(\{|(e_1 + e_2)B_n(x)(e_1 + e_2)| > \lambda_1 + \lambda_2\}) = \\ &= \tau(\{|(e_1B_n(x)e_1 + e_2B_n(x)e_2)| > \lambda_1 + \lambda_2\}) \leq \\ &= \tau(\{|B_n(x_{e_1})| > \lambda_1\}) + \tau(\{|B_n(x_{e_2})| > \lambda_2\}). \end{aligned} \tag{3.2.31}$$

□

*Remark 3.2.1.* We are going to use 3.2.27 in the next section when dealing with Stochastic Ergodic Theorem, since under the conditions of Stochastic Ergodic Theorem estimate 3.2.27 takes place.

### 3.3 Stochastic ergodic theorems

In this section we establish stochastic convergence of the bounded Besicovitch sequences, and show stochastic ergodic theorems for uniform subsequences.

In this section we use following assumptions:  $M$  is a von Neumann algebra with faithful normal tracial state  $\tau$ , and  $\alpha$  is an  $*$ -automorphism of algebra  $M$ . Denote by  $A_n(x) = \frac{1}{n} \sum_{l=0}^{n-1} \alpha^l(x)$ , for  $x \in M$ . Define  $\alpha'$  as a linear map on  $L_1(M, \tau)$  satisfying  $\tau(x \cdot \alpha(y)) = \tau(\alpha'(x)y)$  for  $x \in L_1(M, \tau), y \in M$ , and  $A'_n(x) = \frac{1}{n} \sum_{l=0}^{n-1} \alpha'^l(x)$ , for  $x \in L_1(M, \tau)$ .

Let us recall some definitions from Grabarnik and Katz [30] and Chilin, Litvinov and Skalski [13].

**Definition 3.3.1.** A positive operator  $h \in M_+$  is called **weakly wandering** if

$$\|A_n(h)\|_\infty \xrightarrow{n \rightarrow \infty} 0 \quad (3.3.1)$$

The following definition is due to Ryll-Nardzewski [59].

**Definition 3.3.2.** Let  $\mathbb{C}_1$  denote the unit circle in  $\mathbb{C}$ . A **trigonometric polynomial** is a map  $P_k(n) : \mathbb{N} \mapsto \mathbb{C}$ , where  $P_k(n) = \sum_{j=0}^{k-1} b_j \cdot \lambda_j^n$  for  $\{\lambda_j\}_{j=0}^{k-1} \subset \mathbb{C}_1$ .

Bounded Besicovitch sequences are bounded sequences from the  $l_1$ -average closure of the trigonometric polynomials.

More precisely,

**Definition 3.3.3.** A sequence  $\beta_n$  of complex numbers is called a **Bounded Besicovitch sequence** (BB-sequence) if

- (i)  $|\beta_n| \leq C < \infty$  for every  $n \in \mathbb{N}$  and
- (ii) For every  $\epsilon > 0$ , there exists a trigonometric polynomial  $P_k$  such that

$$\limsup_n \frac{1}{n} \sum_{j=1}^{n-1} |\beta_j - P_k(j)| < \epsilon \quad (3.3.2)$$

Let  $\mu$  be the normalized Lebesgue measure (Radon measure) on  $\mathbb{C}_1$ . Let  $\tilde{M}$  be the von Neumann algebra of all essentially bounded ultra-weakly measurable functions  $f : (\mathbb{C}_1, \mu) \rightarrow M$ . Algebra  $\tilde{M}$  is isomorphic to  $L_\infty(\mathbb{C}_1, \mu) \overline{\otimes} M$  -which is a  $W^*$  tensor product of  $L_\infty(\mathbb{C}_1, \mu)$  and  $M$ ,  $\tilde{M}$  is a dual to the space  $L_1(\mathbb{C}_1, \mu) \overline{\otimes} M_*$  ( the definition of  $W^*$  tensor product and form of the predual space of the  $W^*$  tensor product could be found for example in Takesaki, [65], Theorem IV.7.17). The space  $L_1(\mathbb{C}_1, \mu) \overline{\otimes} M_*$  may be considered as a set of  $L_1$  functions on  $(\mathbb{C}_1, \mu)$  with values in  $M_*$ . Algebra  $\tilde{M}$  has a natural trace  $\tilde{\tau}(f) = \int_{\mathbb{C}_1} \tau(f(z))d\mu(z)$ , and  $\tilde{M}_*$  is isomorphic to  $L_1(\tilde{M}, \tilde{\tau})$ .

Let  $\sigma$  be an automorphism of  $(\mathbb{C}_1, \mu)$  as a Lebesgue space with measure. We define automorphism  $\alpha \otimes \sigma$  of  $(\tilde{M}, \tilde{\tau})$  as a closure of the linear extension of automorphism acting on  $(\tilde{M}, \tilde{\tau}) \ni x(z)$  as  $\alpha \otimes \sigma(x(z)) = \alpha(x(\sigma(z)))$ .

*Example 3.3.1.* An example of such an automorphism is  $\tilde{\alpha}_\lambda(x(z)) = \alpha(x(\lambda \cdot z))$ , for  $\lambda \in \mathbb{C}_1$ .

In this case

$$A_n(x) = \frac{1}{n} \sum_{l=1}^{n-1} \tilde{\alpha}_\lambda^l(x) = \frac{1}{n} \sum_{l=1}^{n-1} \alpha^l(x(\lambda^l \cdot z)). \quad (3.3.3)$$

In particular, if  $x(z) \equiv z \cdot x$  for  $x \in M$  then

$$A_n(x \cdot z) = z \cdot \frac{1}{n} \sum_{l=1}^{n-1} \lambda^l \cdot \alpha^l(x). \quad (3.3.4)$$

The following lemma connects stochastic convergence in  $L_1(\tilde{M}, \tilde{\tau})$  with pointwise convergence on  $C_1$  and stochastic convergence in  $M$  (cmp. with [13]).

**Lemma 3.3.2.** *i) If  $L_1(\tilde{M}, \tilde{\tau}) \ni x_n \xrightarrow{n \rightarrow \infty} x_0 \in L_1(\tilde{M}, \tilde{\tau})$  b.a.u. , then  $x_n(z) \xrightarrow{n \rightarrow \infty} x_0(z)$  stochastically for almost every  $z \in \mathbb{C}_1$*

*ii) Suppose that  $h$  is a weakly wandering operator with support  $\text{supp}(h) = \mathbb{I}$  for sequence  $A_n$ . Then  $A'_n(x)$  converges to 0 stochastically for every  $x \in L_1(M, \tau)$ .*

iii) Let algebra  $\mathcal{N} = (M, \tau) \overline{\otimes} L_\infty(X, \mu)$ , (here  $X$  is a separable Hausdorff compact set, and  $\mu$  is Lebesgue measure),  $\alpha$  is an automorphism of  $M$ , and  $\sigma$  is an automorphism of  $L_\infty(X, \mu)$ . Then  $\alpha \otimes \sigma$  is an automorphism of  $\mathcal{N}$ . Suppose that  $h$  is a weakly wandering operator with support  $\text{supp}(h) = \mathbb{I}$  for sequence  $A_n$  corresponding to automorphism  $\alpha \otimes \sigma$ . Then  $A'_n(x(z))$  converges to 0 stochastically for almost every  $z \in \mathbb{C}_1$ .

*Proof.* Part i) follows from [13], Lemma 4.1 which states that under the hypothesis of part i) exists b.a.u. convergence of  $x_n(z)$  to  $x_0(z)$  for almost every  $z$  in  $\mathbb{C}_1$ , (hence doubleside stochastic convergence), and the fact that double side stochastic convergence is equivalent to (one sided) stochastic convergence (see [13], Theorem 2.2).

Part ii) We suppose that  $x \in L_1(M, \tau)_+$  and  $A'_n(x)$  is a sequence satisfying

$$\tau(A'_n(x)h) \rightarrow 0 \text{ for } n \rightarrow \infty. \quad (3.3.5)$$

The following inequality is valid:

$$ts \cdot \tau(\{A'_n(x) > t\} \wedge \{h > s\}) \leq \tau(A'_n(x)h). \quad (3.3.6)$$

Indeed, for projections  $e_1, e_2 \in P(M)$ , we have  $e_1 e_2 e_1 \geq e_1 \wedge e_2$ . To see this, note that since  $e_1 \wedge e_2$  commutes with  $e_1, e_2$ , we have  $(\mathbb{I} - e_1 \wedge e_2) e_1 e_2 e_1 (e_1 \wedge e_2) = 0$ , and, hence  $e_1 e_2 e_1 = (\mathbb{I} - e_1 \wedge e_2) e_1 e_2 e_1 (\mathbb{I} - e_1 \wedge e_2) + (e_1 \wedge e_2) e_1 e_2 e_1 (e_1 \wedge e_2) = (\mathbb{I} - e_1 \wedge e_2) e_1 e_2 e_1 (\mathbb{I} - e_1 \wedge e_2) + (e_1 \wedge e_2)$ .

Then,

$$\begin{aligned} ts \cdot \tau(\{A'_n(x) > t\} \wedge \{h > s\}) &\leq t\tau(\{A'_n(x) > t\} s \{h > s\} \{A'_n(x) > t\}) \leq \\ &t\tau(\{A'_n(x) > t\} h \{A'_n(x) > t\}) = t\tau(\{A'_n(x) > t\} h) \leq \tau(A'_n(x)h). \end{aligned} \quad (3.3.7)$$

Hence, 3.3.6 is valid.

Furthermore,

$$\tau(\{A'_n(x) > t\}) \leq \frac{1}{ts} \tau(A'_n(x)h) + \tau(\mathbb{I} - \{h > s\}). \quad (3.3.8)$$

The latter inequality follows from 3.3.6, and the fact that  $\tau(e_1) \leq \tau(e_1 \wedge e_2) + \tau(\mathbb{I} - e_2)$ .

Indeed,

$$\begin{aligned} \tau(e_1 - e_1 \wedge e_2) &= \tau((\mathbb{I} - e_1 \wedge e_2)e_1(\mathbb{I} - e_1 \wedge e_2)) = \tau(e_1(\mathbb{I} - e_1 \wedge e_2)e_1) \leq \\ &\tau(e_1(\mathbb{I} - e_2)e_1) = \tau(e_1(\mathbb{I} - e_2)) \leq \tau(\mathbb{I} - e_2). \end{aligned} \quad (3.3.9)$$

Hence 3.3.8 is valid.

Note that inequality 3.3.8 with the fact that  $\tau(A'_n(x)h) \xrightarrow{n \rightarrow \infty} 0$  implies that  $\sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\| \}) \leq C(\lambda)$ . Indeed, sequence  $\{\tau(A'_n(x)h)\}_{n=1}^\infty$  is bounded by the constant  $C_0$  as a converging sequence. Choose a monotonically decreasing sequence of  $\{s_j\}_{j=1}^\infty \subset \mathbb{R}_+$  such that  $\tau(\mathbb{I} - \{h > s_j\}) < 2^{-j}$ , and  $t_j = 2^j s_j^{-1}$ . Then

$$\tau(\{A'_n(x) > t_j\}) \leq \frac{1}{t_j s_j} C_0 + 2^{-j} = (C_0 + 1)2^{-j}. \quad (3.3.10)$$

Hence the condition of theorem 3.2.4 is satisfied. For the dense subset in  $L_1(M, \tau)_+$  of view  $x \in M_+ \cap L_1(M, \tau)$  we have (consider only  $x$  with  $\|x\|_\infty \leq 1$ )  $\tau(A'_n(x)h) \leq \tau(A'_n(\mathbb{I})h)$ . Convergence  $\tau(A'_n(\mathbb{I})h) \rightarrow 0$  implies that we can choose sequences  $\{s_j\} \rightarrow 0$  and  $\{t_j\} \rightarrow 0$  in 3.3.8 with  $\tau(\{A'_n(x) > t_j\}) \rightarrow 0$ . Hence, on a dense subset of  $L_1(M, \tau)$  we have stochastic convergence. From theorem 3.2.4 follows stochastic convergence of the  $A'_n(x)$ .

Note that condition ii) does not imply convergence in  $L_1(M, \tau)$ .

Part iii). The idea of the proof is similar to one of the proof for ii). We provide only necessary modifications. Let  $E_1$  be a conditional expectation with respect to

trace  $\tau \otimes \mu$  of  $(M, \tau) \overline{\otimes} L_\infty(X, \mu)$  onto  $(M, \tau) \overline{\otimes} \text{Const}(X, \mu)$ , and  $E_2$  be a conditional expectation with respect to trace  $\tau \otimes \mu$  of  $(M, \tau) \overline{\otimes} L_\infty(X, \mu)$  onto  $\mathbb{C} \cdot \mathbb{I} \overline{\otimes} L_\infty(X, \mu)$ , (the definition of conditional expectation with respect to trace  $\tau \otimes \mu$  and its existence could be found in [65]). Due to the form of the  $\alpha \otimes \sigma$ , both  $E_j$ 's commute with  $A_n$ , for  $j = 1, 2$ .

Since

$$\|A_n(h)\|_\infty \geq \|E_1 A_n(h)\|_\infty = \|A_n(E_1 h)\|_\infty, \quad (3.3.11)$$

and  $\text{supp}(h) \leq \text{supp}(E_1 h)$  is valid, it follows that  $\text{supp}(E_1 h) = \mathbb{I}$ . Indeed,  $x \geq 0$ ,  $x \neq 0$  implies  $\tau(E_1 x) = \tau(x) > 0$ . Hence  $0 < \tau((E_1 a)h) = \tau(a(E_1 h))$  and  $\text{supp}(E_1 h) = \mathbb{I}$  for every  $a \in M$ .

Hence  $E_1(h)$  is a weakly wandering operator.

For positive  $x(z) \in L_1(M, \tau) \overline{\otimes} L_1(X, \mu)$  holds

$$\|x\|_1 = \int_X \|x(z)\|_1 \cdot d\mu(z), \quad (3.3.12)$$

hence  $\|x(z)\|_1$  is an  $L_1(X, \mu)$  function. Applying classical Hopf inequality (see for example [44], Theorem 2.1, p. 8) we get

$$\mu(\sup_n \{\|A'_n(x)(z)\|_1 > \lambda\}) \leq \frac{\text{Const}}{\lambda} \int_X \|x(z)\|_1 \cdot d\mu(z), \quad (3.3.13)$$

or, outside of a set  $X_0 \subset X$  of arbitrary small measure the value of  $\|A'_n(x)(z)\|_1$  is uniformly bounded. Proceeding like in the part ii) applied for every  $z \in X_0$ , we get stochastic convergence for every  $z \in X_0$ .  $\square$

**Theorem 3.3.3** (Neveu Decomposition for the special case of tensor product of von Neumann algebras). *Let algebra  $\mathcal{N} = (M, \tau) \overline{\otimes} L_\infty(X, \mu)$ , (here  $X$  is a Hausdorff separable compact set, and  $\mu$  is Lebesgue measure),  $\alpha$  is an automorphism of  $M$ ,*

and  $\sigma$  is an automorphism of  $L_\infty(X, \mu)$ . Then  $\tilde{\alpha} = \alpha \otimes \sigma$  is an automorphism of  $\mathcal{N}$ . Suppose that in addition automorphism  $\sigma$  is ergodic. Then there exists an  $\tilde{\alpha}$  invariant projection in  $\mathcal{N}$  of view  $e_1 = e_{11} \otimes \mathbb{I}$ ,  $e_2 = \mathbb{I} - e_1$  with  $e_1(z) = e_M$  for almost every  $z \in X$  such that

- i) There exists a normal state  $\rho$  on  $\mathcal{N}$  with  $\text{supp}(\rho) = e_1$  and for almost each  $z \in X$ ,  $\rho(z)$  is invariant with respect to automorphism  $\alpha'$ ;
- ii) There exists a weakly wandering operator  $h \in \mathcal{N}$  with  $\text{supp}(h) = e_2$  and for almost each  $z \in X$ ,  $h(z)$  is a weakly wandering operator in  $M$ .

*Proof.* Corollary 1.1 of [30] implies existence of a projection  $\tilde{e}_1$  in  $\mathcal{N}$  such that i) there exists  $\tilde{\alpha}'$  invariant normal state  $\rho$  with support  $\text{supp}(\rho) = \tilde{e}_1$  and ii) there exists a weakly wandering operator  $h \in \mathcal{N}$  with support  $\mathbb{I} - \tilde{e}_1$ . Our goal is to show that similar statements are valid for almost every  $z \in X$ .

Since  $\sigma$  is ergodic, then for every  $x \in M \otimes \text{Const}(X, \mu)$  (constant function on  $X$  with values in  $M$ ) holds

$$\begin{aligned} \rho(z)(x(z)) &= (\tilde{\alpha}'\rho(z))(x(z)) = \rho(z)(\alpha(x(\sigma(z)))) = \\ &= \rho(z)(\alpha(x(z))) = (\alpha'(\rho(z)))(x(z)), \end{aligned} \quad (3.3.14)$$

or  $\rho(z)$  is  $\alpha'$  invariant. Suppose that function  $z \rightarrow \rho(z)$  is not constant or  $z \rightarrow \rho(z)$  is such that there exists real  $r_0 \in \mathbb{R}_+$  and  $x(z) \equiv x_0 \in M_+$  with  $\mu(\{z \in X | \rho(z)(x(z)) \leq r_0\}) > 0$  and  $\mu(\{z \in X | \rho(z)(x(z)) < r_0\}) > 0$ . Since  $\sigma$  is ergodic, there exists  $n \in \mathbb{N}$  such that

$$\mu(\sigma^{-n}(\{z \in X | \rho(z)(x(z)) \leq r_0\}) \cap \{z \in X | \rho(z)(x(z)) < r_0\}) > 0. \quad (3.3.15)$$

Hence,

$$\begin{aligned} \rho(z)(x(z)) &= (\tilde{\alpha}'^n \rho(z))(x(z)) = (\alpha')^n(\rho(z))(x(\sigma^n(z))) = \\ &= \rho(z)(x(\sigma^n(z))) = \rho(\sigma^{-n}z)(x((z))), \end{aligned} \quad (3.3.16)$$

or  $r_0 \geq \rho(z)(x_0) = \rho(\sigma^{-n}z)(x_0) < r_0$ . Contradiction shows that function  $z \rightarrow \rho(z)$  is constant.

This implies that  $\text{supp}(\rho) = \text{supp}(\rho(z)) = \tilde{e}_1(z)$  is constant.

Part ii) follows directly arguments of Proof 3.3.2 iii) (recall that existence of the weakly wandering operator  $h$  follows from Corollary 1.1 of [30]).  $\square$

**Theorem 3.3.4.** *Let algebra  $\mathcal{N} = (M, \tau) \overline{\otimes} L_\infty(X, \mu)$ , (here  $X$  is a separable Hausdorff compact set, and  $\mu$  is normalized Lebesgue measure),  $\alpha$  is an automorphism of  $M$ , and  $\sigma$  is an automorphism of  $L_\infty(X, \mu)$ . Then  $\tilde{\alpha} = \alpha \otimes \sigma$  is an automorphism of  $\mathcal{N}$ . Suppose that in addition automorphism  $\sigma$  is ergodic. Then for almost every  $z \in X$  the averages  $A'_n(x(z))$  converge stochastically.*

*Proof.* Proof of the theorem follows directly from 3.3.3 and 3.3.2 which applied to the part of partition where there exists a weakly wandering operator, and from the regular individual ergodic theorem [70] applied to the part where an invariant normal state exists, and Proposition 3.2.5.  $\square$

Now we are in a position to prove stochastic convergence of the bounded Besicovitch sequences.

**Theorem 3.3.5** (Stochastic Ergodic Theorem for bounded Besicovitch sequences). *Let  $\{\beta_j\}_{j=1}^\infty$  be a bounded Besicovitch sequence. Let  $M$  be a von Neumann algebra*

with finite faithful normal tracial state  $\tau$ . Let  $\alpha$  be an automorphism of  $M$ . Then the sequence

$$\tilde{A}_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} \beta_j \alpha'^j(x)$$

converges stochastically for  $x \in L_1(M, \tau)$ .

*Proof.* The statement of the theorem is valid if bounded Besicovitch sequence  $\{\beta_j\}_{j=1}^{\infty}$  is a trigonometric polynomial  $P_k(j)$ .

Indeed, choosing  $\tilde{\alpha}$  as in example 3.3.1 we get from theorem 3.2.4 and the fact that irrational rotation on the  $\mathbb{C}_1$  is ergodic (Equidistribution Kronecker-Weyl Theorem, see for example [38] p. 146) that

$$A_n(x \cdot z) = z \cdot \frac{1}{n} \sum_{l=1}^{n-1} \lambda^l \cdot \alpha^l(x), \quad (3.3.17)$$

hence

$$\frac{1}{n} \sum_{l=1}^{n-1} \lambda^l \cdot \alpha^l(x)$$

converges stochastically for irrational  $\lambda$ .

For the rational  $\lambda$  convergence follows from the fact that it is a finite combination of averages of the  $\alpha'^m$ , where  $m$  is denominator.

Taking linear combinations of terms as in 3.3.17 implies the statement for trigonometric polynomials.

Statement of the theorem is valid for the  $x \in M \cap S(M)$ . Indeed, using approximation of the BB sequence by trigonometric polynomials as in 3.3.2 one gets for

$$A_n(k, x) = \frac{1}{n} \sum_{l=1}^{n-1} P_k(l) \cdot \alpha^l(x)$$

$$\|\tilde{A}_n(x) - A_n(k, x)\|_{\infty} \leq \frac{1}{n} \left( \sum_{l=0}^{n-1} |\beta_l - P_k(l)| \right) \cdot \|x\|_{\infty} \quad (3.3.18)$$

and hence, stochastic convergence.

Note also that for every  $x \in L_1(M, \tau)$

$$\|\tilde{A}_n(x) - A_n(k, x)\|_1 \leq \frac{1}{n} \left( \sum_{l=0}^{n-1} |\beta_l - P_k(l)| \right) \cdot \|x\|_1. \quad (3.3.19)$$

Hence by remark 3.2.1 averages  $\tilde{A}_n(x)$  are uniformly bounded in the sense of 3.2.9.

Result of the theorem follows from the Stochastic Banach Principle, Theorem 3.2.4 and density of  $M \cap S(M)$  in  $L_1(M, \tau)$ .  $\square$

The following theorem is implied by the Stochastic Ergodic Theorem for bounded Besicovitch sequences. (cmp. [46])

For the following definitions see, for example, [44], p. 260.

Let  $\sigma$  be a homeomorphism of a compact metric space  $X$  with metric  $\varrho$  such that all powers of  $\sigma^l$  are equicontinuous. Assume also that there exists  $z \in X$  with dense orbit  $\sigma^l(z)$  in  $X$ . Then there exists a unique (hence ergodic)  $\sigma$  invariant measure  $\nu$  on the  $\sigma$  algebra of Borel sets  $\mathfrak{B}$ . Each non-empty open set has a positive  $\nu$  measure.

A sequence  $u_j$  is called **uniform** if there exists such a dynamical system  $(X, \mathfrak{B}, \nu, \sigma)$  and a set  $Y \in \mathfrak{B}$  with  $\nu(\partial Y) = 0$  and  $\nu(Y) > 0$  and point  $y \in X$  with  $u_j = j^{\text{th}}$  entry time of orbit of  $y$  into  $Y$ .

**Theorem 3.3.6.** *Let  $M, \tau, \alpha$  be as in previous theorem,  $\{u_j\}_{j \geq 0}$  be a uniform sequence. Then the averages*

$$\frac{1}{n} \sum_{j=0}^{n-1} \alpha^{u_j} x$$

*converge stochastically for  $x \in L_1(M, \tau)$ .*

*Proof.* Follows from the previous theorem and the fact (see [59]) that any uniform sequence is a bounded Besicovitch sequence.  $\square$

# Bibliography

- [1] **Abdullaev, R.Z.**,  $L_p$ -spaces for Jordan algebras with semifinite trace. (Russian), preprint *VINITI*, No. **1875-83**, 1983, 19 pp.
- [2] **Abdullaev, R. Z.**, Nonassociative  $L_p$ -spaces.(Russian), *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk.*, **6**, 1983, pp. 3–5.
- [3] **Akcoglu, M. A.**, A pointwise ergodic theorem in  $L_p$ -spaces. *Canad. J. Math.*, **27** (1975), no. 5, pp. 1075–1082.
- [4] **Alaoglu, L.; Birkhoff, G.**, General ergodic theorems, *Ann. of Math.* **41** (1940) pp. 293–309.
- [5] **Ayupov, Sh.A.**, *Classification and representation of ordered Jordan algebras.* (Russian), Tashkent: Fan., 1986, 124 pp.
- [6] **Ayupov, Sh.A.; Rakhimov, A.A.; Usmanov, Sh.M.**, *Jordan, real and Lie structures in operator algebras.* (English), Mathematics and its Applications (Dordrecht)., Dordrecht: Kluwer Academic Publishers., 1997, 225 pp.
- [7] **F. Benatti**, *Deterministic chaos in infinite quantum systems*, Springer Verlag, 1993
- [8] **Berdikulov, M.A.**, Spaces  $L_1$  and  $L_2$  for semifinite  $JBW$ -algebras. (Russian), *Dokl. Akad. Nauk UzSSR 1982*, **6**, 1982, pp. 3-4 .

- [9] **Bratteli, O.; Robinson, D.**, *Operator Algebras and Quantum Statistical Mechanics.*, V I, Texts and Monographs in Physics, Springer-Verlag, New York-Heidelberg, 1979, 500 pp.
- [10] **Bufetov, A. I.**, Ergodic theorems for actions of several mappings. (Russian) *Uspekhi Mat. Nauk* **54** (1999), no. 4 (328), pp. 159–160; translation in Russian Math. Surveys **54** (1999), no. 4, pp. 835–836.
- [11] **Bufetov, A. I.**, Operator ergodic theorems for actions of free semigroups and groups. (Russian) *Funktsional. Anal. i Prilozhen.* **34** (2000), no. 4, pp. 1–17; translation in *Funct. Anal. Appl.* **34** (2000), no. 4, pp. 239–251
- [12] **Connes, A.** *Noncommutative geometry* Academic Press, San Diego, CA, 1994.
- [13] **Chilin, V.I.; Litvinov, S.; Skalski, A.** A few remarks in non-commutative ergodic theory., *J. Operator Theory* **53:2**( 2005) pp. 331-350.
- [14] **Dixmier J.**, Formes lineaires sur une anneau d'operateurs., *Bull. Soc. Math. France*, **81**, 1953, pp. 9–39.
- [15] **Dixmier, J.**, *Von Neumann Algebras.*, North-Holland Mathematical Library, V. 27, North-Holland Publishing Co., Amsterdam-New York, 1981, 437 pp.
- [16] **Dixmier, J.**, *Les algebres d'operateurs dans l'espace hilbertien (algebres de von Neumann)* , Gauthier-Villar, Paris, 1960.
- [17] **Dunford, N.; Schwartz, J.T.**, *Linear Operators I.*, Pure and Applied Mathematics, V. 7, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958, 858 pp.
- [18] **Fack, T.; Kosaki, H.**, Generalized  $s$ -numbers of  $\tau$ -measurable operators, *Pacific Journal of Mathematics*, V. **123**, No. 2, 1986

- [19] **Garsia, A.** *Topics in Almost Everywhere Convergence.* , Chicago, Markham Publ. Co., 1970.
- [20] **Goldstein, M.S.**, Theorems of Almost Everywhere Convergence in von Neumann Algebras. (Russian), *J. Oper. Theory*, V. **6**, 1981, pp. 233-311.
- [21] **Goldstein, M.S.; Grabarnik, G.Ya.**, Almost Sure Convergence Theorems in von Neumann Algebras., *Israel J. Math.*, V. **76**, 1991, No. 1-2, pp. 161–182.
- [22] **Goldstein, S.; Lindsay, J.M.**, KMS symmetric Markov semigroups, *Math Z.* **219**.(1995), pp. 591–608.
- [23] **Goldstein, M.; Litvinov, S.**, Banach Principle in the space of  $\tau$ -measurable operators, *Studia Math.* **143**.(2000), pp. 33–41.
- [24] **Grabarnik G.Ya; Shwartz, L.** Stochastic Banach Principle In Operator Algebras, Accepted for publication in *Studia Mathematica*, (2007), pp. 1-15.
- [25] **Grabarnik, G.Ya.; Katz, A.A. and Shwartz, L.**, On noncommutative ergodic type theorems for free finitely generated semi- groups., accepted for publication in *Vladikavkaz Mathematical Journal* pp. 1-16, (2007)
- [26] **Grabarnik, G.Ya.; Katz, A.A. and Shwartz, L.**, Ergodic Type Theorems for Actions of Finitely Generated Semigroups on von Neumann Algebras I , *Proceedings of the 3rd Annual Hawaii International Conference on Statistics, Mathematics and Related Fields, Honolulu, Hawaii, USA.*, (2004) pp. 1-15
- [27] **Grabarnik, G.Ya.; Katz, A.A. and Shwartz, L.**, Ergodic Type Theorems for Actions of Finitely Generated Semigroups on von Neumann Algebras II , *Proceedings of the 4th Annual Hawaii International Conference on Statistics, Mathematics and Related Fields, Honolulu, Hawaii, USA.*, (2005), pp. 1-10

- [28] **Grabarnik G.Ya; Katz, A.; Shwartz, L.** On weak convergence of iterates in quantum  $L_p$ -spaces, Accepted for publication in *International Journal of Mathematics and Mathematical Sciences*, (2005), pp 1-12
- [29] **Grabarnik G.Ya; Katz, A.; Shwartz, L.** Ergodic Type Theorems for Actions of Finitely Generated Semigroups on von Neumann Algebras III, poster in *International Conference on Operator Algebras and Quantum Probability*, Tashkent, (2005)
- [30] **Grabarnik, G.Ya. and Katz, A.A.**, Ergodic type theorems for finite von Neumann algebras , *Israel Journal of Mathematics*, **90** (1995), pp. 403-422.
- [31] **Grigorchuk, R.I.**, Individual ergodic theorem for actions of free groups, In: *Proceedings of the Tambov Workshop in the Theory of Functions*, (1986).
- [32] **Grigorchuk, R.I.**, Ergodic theorems for actions of free groups and free semi-groups. (Russian, English) *Math. Notes* **65**, No.5, pp. 654-657 (1999); translation from *Mat. Zametki* **65**, No.5, pp. 779-783 (1999)
- [33] **Guivarc'h, Y.**, Généralisation d'un théorème de von Neumann. (French) *C. R. Acad. Sci. Paris Sér. A-B*, **268** (1969), pp. A1020–A1023.
- [34] **Hanche-Olsen, H.; Størmer, E.**, *Jordan operator algebras*. (English), Monographs and Studies in Mathematics, 21. Boston - London - Melbourne: Pitman Advanced Publishing Program. VIII, 1984, 183 pp.
- [35] **Hajian, A. and Kakutani, S.**, Weakly wandering sets and invariant measures, *Transactions of the American Mathematical Society*, **110** (1964), pp. 131-151.
- [36] **Jajte, R.**, Strong limit theorem in non-commutative probability, *Lecture Notes in Mathematics*, **1110**, Springer-Verlag, Berlin, 1985, pp. 162.

- [37] **Iochum, B.**, Non-associative  $L_p$ -spaces. (English), *Pac. J. Math.*, **122**, 1986, pp. 417-433.
- [38] **Katok, A.; Hasselblatt B.** *Introduction to the Modern Ergodic Theory of Dynamical Systems* Cambridge, 1995.
- [39] **Kakutani, S.**, Random ergodic theorems and Markoff processes with a stable distribution, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950, pp. 247–261. University of California Press, Berkeley and Los Angeles, 1951.
- [40] **Katz, A.A.**, On the weak convergence of operator iterations in von Neumann algebras, *Vladikavkaz Math. J.***5**(3)(2003), pp. 39-45.
- [41] **Kadison, R V.; Ringrose, J. R.** *Fundamentals of the Theory of Operator Algebras, volume I* Elementary Theory of Pure and Applied Mathematics. Academic Press, New York, London, 1983.
- [42] **Kingman, J.F.C.**, Subadditive ergodic theory, *Annals of Probability*, **1** (1973), pp. 883–909.
- [43] **Kovacs, I. and Szücs, J.**, Ergodic type theorem in von Neumann algebras , *Acta Scientiarum Mathematicarum (Szeged)*, **27** (1966), pp. 233–246.
- [44] **Krengel, U.**, *Ergodic Theorems*, de Greuter, Berlin, 1985.
- [45] **Narnhofer,H.; Thirring,W.; Wiklicky,H.**, Transitivity and ergodicity of quantum systems, *J. of Stat. Phys.*, **52**, No. 3, 1988, pp. 1097–1112.
- [46] **Litvinov, S.; Abdulaev M**, On individual ergodic theorems in von Neumann algebras , *Studia Mathematica*, **145**(1) 2001, pp. 55–63.
- [47] **Lorch, E.R.**, Means of iterated transformations in reflexive vector spaces. *Bull. Amer. Math. Soc.* **45**, (1939). pp. 945–947.

- [48] **Lance, E. C.**, Ergodic theorems for convex sets and operator algebras., *Invent. Math.* V. **37**, 1976, No. 3, pp. 201–214.
- [49] **Li, B.**, *Real operator algebras*. (English), River Edge, NJ: World Scientific, 2003, 241 pp.
- [50] **Nelson, E.**, Notes on non-commutative integration, *J. Funct. Anal.*, **15** (1974), pp. 103-116.
- [51] **Nevo, A.**, Harmonic analysis and pointwise ergodic theorems for noncommuting transformations., *J. Amer. Math. Soc.* **7** (1994), no. 4, pp. 875–902.
- [52] **Nevo, A. and Stein, E.M.**, A generalization of Birkhoff’s pointwise ergodic theorem. *Acta Math.* **173** (1994), no. 1, pp.135–154.
- [53] **Oseledec, V. I.**, Markov chains, skew products and ergodic theorems for ”general” dynamic systems(Russian) *Teor. Veroyatnost. i Primenen.*, **10** (1965), pp. 551–557.
- [54] **Pedersen, G.K.**, *C\*-algebras and their automorphism groups.*, London Mathematical Society Monographs, V. 14, Academic Press, Inc., London-New York, 1979, 416 pp.
- [55] **Petz, D.**, Ergodic theorems in von Neumann algebras.,*Acta Sci. Math. (Szeged)*, V. **46**, 1983, No. 1-4, pp. 329–343.
- [56] **Pisier, G.; Xu, Q.** *Non-Commutative  $L_p$ -Spaces, Handbook of the Geometry of Banach Spaces Vol. 2*, pp. 1459-1517, North-Holland, Amsterdam, 2003.
- [57] **Reed, M.; Simon, B.** *Methods of Modern Mathematical Physics Vol. 1*, Academic Press, New York, 1972.
- [58] **Rosenblatt, J.; Wierdl, M.** *Pointwise ergodic theorems via harmonic analysis. Ergodic theory and its connections with harmonic analysis* (Alexandria,

- 1993), 3151, London Math. Soc. Lecture Note Ser., 205, Cambridge Univ. Press, Cambridge, 1995.
- [59] **Ryll-Nardzewski, C.**, Topics in ergodic theory ., *Lecture Notes in Math.* , V. **472**, 1975, pp. 131–157.
- [60] **Sakai, S.**, *C\*-algebras and W\*-algebras.*, Ergebnisse der Mathematik und ihrer Grenzgebiete, V. 60, Springer-Verlag, New York-Heidelberg, 1971, 253 pp.
- [61] **Segal, I. E.**, A non-commutative extension of abstract integration., *Ann. of Math.*, V. **57**, 1952, pp. 401–457.
- [62] **Sato, R.**, Operator averages for subsequences., *Math. J. Okayama Univ.* **22** (1980), pp. 161-168.
- [63] **Sinai, Ja. G.; Anshelevich, V. V.**, Some questions on noncommutative ergodic theory.(Russian), *Uspehi Mat. Nauk*, V. **31**, 1976, No. 4, 190, pp. 151–167.
- [64] **Stein, E., and Weiss, G.**, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.
- [65] **Takesaki, M.**, *Theory of Operator Algebras, I.*, Springer-Verlag, New York-Heidelberg, 1979, 415 pp.
- [66] **Trunov, N.V.; Sherstnev, A.N.**, Introduction to the theory of noncommutative integration.(Russian, English), *J. Sov. Math.* **37**, 1987, pp. 1504-1523; trans. from *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat.* **27**, 1985, pp. 167-190.
- [67] **Vershik, A.M.**, Numerical characteristics of groups and relations between them , *Zap. Nauchn. Semin. POMI*, **256**(1999), pp. 5-18.
- [68] **Viennot, G.X.**, *Ann. N. Y. Ac. Sci.*, **576**(1989), p. 542.

- [69] **Weber, M.**, *Entropie metrique et convergence presque partout.*, Travaux en Cours, 58. Hermann, Paris, 1998
- [70] **Yeadon, F. J.**, Non-commutative  $L_p$ -spaces., *Math. Proc. Camb. Phil. Soc.*, V. **77**, 1975, pp. 91–102.
- [71] **Yeadon, F.J.; Kopp, P.E.**, Inequalities for non-commutative  $L_p$ -spaces and an application. (English), *J. Lond. Math. Soc.*, II. Ser. **19**, 1979, pp. 123-128.
- [72] **Yeadon, F.J.**, Ergodic theorems for semi-finite von Neumann algebras, II , *Mathematical Proceedings of the Cambridge Philosophical Society*, **88** (1980), pp. 135-147.
- [73] **Yeadon, F.J.**, Ergodic theorems for semi-finite von Neumann algebras, I, *Journal of the London Mathematical Society*, **16** (1977), pp. 326-332.
- [74] **Yeadon, F.J.**, Ergodic theorems for semi-finite von Neumann algebras, II , *Mathematical Proceedings of the Cambridge Philosophical Society*, **88** (1980), pp. 135-147.