THE BRING-JERRARD QUINTIC EQUATION, ITS SOLUTIONS AND A FORMULA FOR THE UNIVERSAL GRAVITATIONAL CONSTANT

by

EDWARD THABO MOTLOTLE

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DECLARATION BY CANDIDATE

I declare that *THE BRING-JERRARD QUINTIC EQUATION, ITS SOLUTIONS AND A FORMULA FOR THE UNIVERSAL GRAVITATIONAL CONSTANT* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

Dedication

I dedicate this dissertation to my family, my mother Dintlenyane Motlotle and to the memory of Kedikilwe Motlotle (my grandmother) and Albert Thupane (my father). Thank you for the encouragement and support.

Contents

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Abstract

In this research the Bring-Jerrard quintic polynomial equation is investigated for a formula. Firstly, an explanation given as to why *finding a formula* and *the equation being unsolvable by radicals* may appear contradictory when read out of context. Secondly, the reason why some mathematical software programs may fail to render a conclusive test of the formula, and how that can be corrected is explained. As an application, this formula is used to determine another formula that expresses the gravitational constant in terms of other known physical constants. It is also explained why up to now it has been impossible to determine this expression using the current underlying theoretical basis.

Introduction

As the title indicates, the objective of this contribution is to add to the research on subjects that have occupied researchers in mathematical and physical sciences for many centuries. The literature (e.g. Cajori [1] and Struik [2]) indicates that polynomials equations, of which the quintic equation is part, were first investigated more than four thousand years ago. However the gravitational constant is a more recent problem - about three hundred years old. Many researchers (e.g. Cauchy [3], Euler [4] and Lagrange [5]) seem to have given up resolving that the quintic cannot be solved and our knowledge of the gravitational constant is getting worse rather than better. We however have a different viewpoint.

The quintic equation

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{0.1}$$

with rational parameters $a_0, a_1, \cdots, a_4$, is well known for being difficult to solve. Abel and Ruffini have proved that it is impossible to *solve* it over a field of rational numbers, see Rosen [6] and Pesic [7]. The *appendage* 'over a field of rational numbers' is usually omitted in the literature.

Bring [8] and Jerrard [9], however have shown the equation that can be reduced to a simple form with only two parameters. That is,

$$x^5 + bx + c = 0. \tag{0.2}$$

Glashan [10], Young [11] and Runge [12] established that some forms of this equation (0.2), the Bring-Jerrard quintic equation as it is now called, can be *solved* in radicals. They have shown that all equations of the form

$$x^5 + \frac{5\mu^4(4\nu+3)}{\nu^2+1}x + \frac{4\mu^5(2\nu+1)(4\nu+3)}{\nu^2+1} = 0,$$

with rational μ and ν can be *solved*. A similar conclusion was arrived at by Spearman and Williams [13] for the equation

$$x^5 + \frac{5e^4(3 \pm 4c)}{c^2+1}x + \frac{-e^5(\pm 11 + 2c)}{c^2+1} = 0$$

with rational c and e.

There have been other contributions; for example, Ioakimidis and Papadakis [14], [15] and [16], as well as Chowla [17], Cockle [18], [19], Canfield and King [20], [21] and [22].

Of course, there are other ways of solving the quintic. Numerical analysis is one, but the results that follow from such approach would require countless iterations to convert into radical form, and are not as such helpful in the development of algebra and related fields. A direct route seems to be the only plausible approach.

In this contribution, we determine a general formula for expressing the roots of the Bring-Jerrard quintic equation (0.2) in radical form. This is regardless of the nature of such roots, be they rational or irrational, real or unreal.

Our theoretical basis and contribution cannot be well understood without first unpacking the word *solve* and its derivatives as used in abstract algebra, and its arbitrary day-to-day interpretation. This is done in chapter 1 .

Chapter 2 is dedicated to Newton sums, witch we use to solve the Bring-Jerrard quintic equation. Before we can do this however, we first have to present the equation in a form that is suitable for manipulations through these Newton sums. In chapter 3 we generalise the solutions obtained in chapter 2 to the various sub-cases of the general quintic equation. The chapter concludes with solutions to this general equation and a numerical experiment.

Chapter 4 discusses the application. It is argued that Einstein's gravitation theory does not succeed in establishing a formula for the universal gravitational constant in terms of known physical constants. This argument is then used to determine the formula. A brief historical background should naturally include studies on quadratic, cubic and quartic equations. Around 2000 BC Babylonians were among the first to solve the quadratic equation

$$x^2 + ax = b$$

where a and b are non-negative numbers.

The Indians and the Greeks also appear to have known about the cubic and quartic equations. However, the Italians, Spione del Ferro, Cardano and Niccolo Tartaglia, were the ones who provided some convincing solutions to these equations.

The word 'solvable' is used herein two ways for polynomials equations. The popular usage and a simplistic interpretation of the Abel-Ruffini theorem (see Pesic [7]) is that the word 'solve' means to determine a formula for finding roots to a polynomial in radicals. Quadratics, cubics and quartics are then examples of polynomials equations that can be solved, because formulas for determining their roots do exist. The second meaning is adopted from contemporary algebra: Let F be a field, and let $f(x) \in F(x)$. We say that $f(x)$ is solvable by radicals over F if $f(x)$ splits in some extension $F(a_1, a_2, \cdots, a_n)$ of F and there exist positive integers $k_1, k_2, \cdots, k_n$ such that ${a_1}^{k_1} \in F$ and ${a_i}^{k_i} \in F(a_1, a_2, \cdots, a_n)$ for $i = 1, 2, \cdots, n$, see [23]. According to this definition, all three polynomials equations mentioned above are in general not *solvable*, but have *solvable* cases. For example, the quadratic $x^2 + 2 = 0$ is not *solvable* in the field of real numbers while $x^2 - 4 = 0$ is. This can also be observed in cubics and quartics. Some are *solvable* while others are not, even though formulas exist to establish their roots in radical form. This is Abel and Ruffini's invention: The basis of group theory.

We shall continue to italise the word and its derivatives to indicate this abstract usage. The quintic is also not *solvable* in general as suggested by the Abel-Ruffini theorem(see [7]), but has two known *solvable* cases. The two cases are the ones mentioned in Glashan [10], Young [11], Runge [12] and Spearman and Williams [13]. There have been contributions to determining formulas for them, see Dummit [24] and independently Kobayashi and Nakagawa [25]. The formulas so determined solve only the two cases however. Here we are interested in a formula that solves the Bring-Jerrard Quintic equation.

The origin of the abstract interpretation of the word can justifiably be credited to the Abel-Ruffini theorem. It is quite possible that Abel and Ruffini may have started off by first trying to solve the fifth order polynomial equation, and ended up with the definition (see Pesic [7]).

The tendency to drop the *appendage* from the Abel-Ruffini theorem has often led to interesting interpretations. Some interpretations that still prevail today maintain that it is impossible to establish a formula for finding quintic roots in radicals, see Hamilton [26], Ibragimov [27] and Livio [28]. History has it that Erland Bring (1736-1798) disagreed (see Adamchik [29]). He tried determining the formula using the Tschirnhausian transformation

$$y = \mu x^4 + \beta x^3 + \lambda x^2 + \gamma x + \delta. \tag{0.3}$$

He would probably have succeeded if he had not omitted $\mu = 1$ the leading parameter. Tschirnhaus, in his papers [30] and [31], had earlier used the same assumption to solve the quartic polynomial. Our analysis shows that, in the quintic case, μ and δ depend on one another. To avoid the pitfall Bring himself encountered, we reduce (0.3) to a quadratic monomial by excluding terms containing μ, β and δ.

Our theoretical basis and contribution cannot be well understood without first unpacking the word *solve* and its derivatives as used in abstract algebra, and its arbitrary day-to-day interpretation, which is done in chapter 1.

In 1666, Newton (see Scheinerman and Mircea [32]) introduced what today are known as Newton's identities. They relate power sums to elementary symmetric polynomials. Tschirnhaus found transformations for the elimination of some of the intermediate terms in polynomials and Bring [8] used these to transform the general quintic polynomial to the simple form.

Of course, there are many ways of solving the quintic equation, but radical solutions are the ones that are still being sought. Some believe they are impossible to find, while others think they are simply hard to establish. The theoretical basis for this unsolvability is field theory, which is grounded on group theory.

The quintic equation can be solved numerically to any degree of accuracy one desires. For example, the MatLab code roots ($[1, 1, 1, 1, 1, 1]$) solve the polynomial equation

$x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$

Of all quintic solutions, one that is often confusing is the one resulting from differential equation solutions. This confusion revolves around the question of whether they are radical or not. For example, in Drociuk [33] converts a Bring-Jerrard equation into a Fuchian generalised hypergeometric differential equation, and then uses the hypergeometric functions to solve the polynomial equation.

In 1858, Hermite investigated a quintic of the form

$$x^5 - 3x + 2a = 0,$$

where $a = sin(\alpha)$. He found solutions of the form $x_1 = 2sin(\alpha/3), x_2 = 2sin((\alpha + 2\pi)/3)$ and $x_3 = 2sin((\alpha + 4\pi)/3)$. The restricting condition here is that $|sin(\alpha)|$ is always less than 1.

Most contemporary writings (i.e. Pesic [7], Dummit [34], Livio [28], Spearman and Williams [35]) hold that the quintic equation, be it the general case or the Bring-Jerrard, cannot be solved in radicals. It is maintained that a formula for solving this equation using radicals is impossible to establish. We agree with this and elaborate on it in section 1.0.5. In addition, we argue that this means that not all roots resulting from the formula, when found, will be radical numbers. Some could simply be rational with a possibility of others being algebraically transcendental.

It is alleged that Bring (see Adamchik [29] and Hamilton [26] disagreed with what the Abel-Ruffini theorem states, hence the attempt at solving the equation himself. To avoid the pitfalls and mistakes that he might have unknowingly committed, we employ differential forms to sort the parameters in our calculations. This is done in section 2.1.

There are numerous reports (e.g. Fateman [36] and Postel [37]) on the Internet that some very popular mathematical software has an error which its designers have failed to declare. Unfortunately, we did not know this at the beginning of our study. The formula is too complicated to test using a simple hand-held calculator. The faults encountered while trying to use this software are discussed in section 2.2.1.

One other formula that has mystified mathematical physicists for centuries is one for the gravitational constant. In the past numerous attempts at finding a formula that expresses this constant in terms of other known physical constants have not been successful. Our study shows that such a formula is connected to the quintic equation. As an application, discussed in section 3, we use the formula found in section 2.2 to establish an expression for the gravitational constant.

Chapter 1

Theoretical basis

The task at hand in this chapter is to demonstrate that the quintic equation may be unsolvable in radicals, but this does not mean that a formula for its roots does not exist. It may be difficult to find, but not impossible to establish. We demonstrate that the solution to this equation is a collection of rational and radical numbers, which is probably why its Galois group cannot have an extension field, thus rendering it impossible to solve. This conclusion follows from the contributions by Abel and Ruffini, Ioakimidis and Papadakis, and Dummit.

A theorem Stewart (see [38]; Gallian [23]) exists that states that, if a polynomial is solvable by radicals, then its Galois group is solvable. The theorem by Abel and Ruffini proves that the Galois group for the quintic polynomial is not solvable. Ioakimidis and Papadakis [[15] and [16]] established a formula for determining real roots from nonlinear equations, and carried out a demonstration on a Lagrange quintic equation. Dummit [34] proved a theorem which suggests that in the case of a sextic equation, the real root is essentially a rational number.

We will unpack this argument, draw a distinction syllogistically between radical numbers and a solvable group, and finally lead to the conclusion.

Radical solutions

Simply put, radical solutions are solutions expressed in radical numbers. These numbers represent a stage in the development of numbers, as people grappled with equations. We will briefly outline this development from a very simple equation up to the definition of rational numbers.

1.0.1 The linear equation

Consider the simple function

$$P_1 : x \to a_1 x + a_0,$$

with the coefficients a_0 and a_1 belonging to the set of *integers* $\mathbb{Z}$. In addition, the set $\mathbb{Z}$ is closed under addition $+$, subtraction $-$ and multiplication $\times$; division $/$ is the only binary operation under which integers are not closed. The root of the equation

$$P_1(x) = 0$$

is a *rational* number:

$$\alpha = -\frac{a_0}{a_1}.$$

This essentially defines a rational number as a any number that can be written in the form a_0/a_1.

All rational numbers $\mathbb{Q}$ except for zero, 0, on the other hand, are closed under all four binary operations. Unfortunately, at some point it was realised that the solutions to equations will not always be rational numbers.

1.0.2 Quadratic equation

Two types of number are introduced in this section, that is, *complex* and radical *numbers*. Consider the quadratic function

$$P_2 : x \to a_2 x^2 + a_1 x + a_0.$$

Note that it is immaterial whether the coefficients are integers or rational numbers; we can easily change from one system to the other. The equation

$$P_2(x) = 0$$

has the roots

$$\alpha_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

and

$$\alpha_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2a_0}}{2a_2}.$$

The roots α_1 and α_2 are *radical* numbers $\sqrt[n]{\mathbb{Q}}$, when $\triangle = a_1^2 - 4a_2a_1 > 0$ is not a perfect square. It is assumed that the reader understands that we chose the symbol $\sqrt[n]{\mathbb{Q}}$ conveniently here to represent the set of radical numbers. It was felt that, as the other numbers already have symbols assigned to them, we may as well invent one for radicals.

The roots are *complex* when $\triangle < 0$. Complex numbers $\mathbb{C}$, are not close under division.

1.0.3 Cubic equation

The cubic function

$$P_3 : x \to a_3x^3 + a_2x^2 + a_1x + a_0,$$

also has radical numbers in its zeros. In addition to radicals in $\surd$ or $1/2$, we now have them in the order of $1/3$. One of the three zeros is

$$\begin{aligned}\alpha_1 &= -\frac{b}{3} - \frac{2^{1/3}(-b^2+3c)}{3(-2b^3+9bc+\sqrt{4(-b^2+3c)^3+(-2b^3+9bc-27d)^2}-27d)^{1/3}} \\ &+ \frac{(-2b^3+9bc+\sqrt{4(-b^2+3c)^3+(-2b^3+9bc-27d)^2}-27d)^{1/3}}{3\ 2^{1/3}}.\end{aligned}$$

1.0.4 Quartic equation

The quartic function

$$P_4 : x \to a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

also has radical numbers as its roots. That is,

$$\begin{aligned}\alpha_1 &= -\frac{1}{2}\sqrt{\frac{4\times 2^{1/3}a_5}{\left(27a_4^2+\sqrt{729a_4^4-6912a_5^3}\right)^{1/3}}+\frac{\left(27a_4^2+\sqrt{729a_4^4-6912a_5^3}\right)^{1/3}}{3\times 2^{1/3}}} \\ &- \frac{1}{2}\sqrt{-\frac{4\times 2^{1/3}a_5}{\left(27a_4^2+\sqrt{729a_4^4-6912a_5^3}\right)^{1/3}}-\frac{\left(27a_4^2+\sqrt{729a_4^4-6912a_5^3}\right)^{1/3}}{3\times 2^{1/3}}+\xi},\end{aligned}$$

where

$$\xi = \frac{-a_5}{-\frac{1}{2}\sqrt{\frac{4\times 2^{1/3}a_5}{\left(27a_4^2+\sqrt{729a_4^4-6912a_5^3}\right)^{1/3}}+\frac{\left(27a_4^2+\sqrt{729a_4^4-6912a_5^3}\right)^{1/3}}{3\times 2^{1/3}}}}.$$

1.0.5 Quintic equation

The quintic function

$$P_5 : x \to a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

on the other hand, is said not to have any radical roots; its roots are simply *algebraic*. Algebraic numbers are those numbers that are solutions to polynomial with rational coefficients.

This leads to the theorem 1.1.

Theorem 1.1 *Every radical number is algebraic.*

The converse, however, is not true: not every algebraic number is radical. However this does not mean all real numbers $\mathbb{R}$ are algebraic. The gap is filled by what are called *transcendental* numbers.

This concludes our discussion on the types of number that would emerge from the formula, when found. A problem that will arise when this formula is found is what types of number follow from it.

1.1 Solvable groups

In the event that the quintic equation has no radical roots, it will have algebraic roots, which is what the contraposition of theorem 1.1 suggests. The Galois group associated with this equation will then be solvable over algebraic numbers.

Let us now hypothesise and assume that a formula does exist for solving the quintic equation and that the roots are to be found in the set

$$\mathbb{A} = \{\mathbb{Z}, \sqrt[n]{\mathbb{Q}}\},$$

naturally excluding transcendental numbers and some rational numbers. That is, let one or more of the roots be an integer, $\mathbb{Z}$, and the rest be radicals, $\sqrt[n]{\mathbb{Q}}$. The Galois group associated with the quintic in question will then not be solvable in $\mathbb{Z}$. This is because the *splitting* will be mixed, with some factors falling outside the designated area. It is also not solvable in $\sqrt[n]{\mathbb{Q}}$, and for the same reason, however it should be solvable over $\mathbb{A}$, which is impossible, as the building blocks of abstract algebra do not allow it.

In this section, we will trace the Abel-Ruffini theorem (see Pesic [7]) from the basic group definition to the Galois group. The aim will be to explore the definitions and theorems, examining what they have to say on number systems like integers, rational numbers and other known systems. In particular, we turn our attention to whether mixed sets like $\mathbb{A}$ are supported.

Definition 1.1 *Group*

A non-empty set G with a binary operation $\circ$ is called a group if all its elements $a_1, \cdots, a_n$ satisfy the following properties:

1. **Closure**. *If $a_i, a_j \in G$, then*

$$a_i \circ a_j \;=\; a_k \in G.$$

2. **Identity**. *There exists a unique element $a_0 \in G$ such that*

$$a_i \circ a_0 \;=\; a_0 \circ a_i \;=\; a_i.$$

The element a_0 is an identity in G.

3. **Inverses**. *For every $a_i \in G$, there exists $a_{-i} \in G$, such that*

$$a_i \circ a_{-i} \; = \; a_{-i} \circ a_i \; = \; a_0.$$

The element a_{-i} is the inverse of a_i.

4. **Associativity**. *For all elements a_i, a_j and a_k in G, we have*

$$(a_i \circ a_j) \circ a_k \; = \; a_i \circ (a_j \circ a_k).$$

This is called the associative property.

A group is said to be Abelian if the elements commute. That is,

$$a_i \circ a_j \; = \; a_j \circ a_i.$$

Definition 1.2 *Subgroup*

A subset of G, that is, $H \subset G$ is its subgroup if it is itself a group under the operation of G. In notation form, we say

$$H \leq G.$$

Definition 1.3 *Cyclic subgroup*

The set $< a >= \{a^n | n \in \mathbb{Z}\}$ is called a cyclic subgroup of G if $< a > \leq G$.

Definition 1.4 *Centre of group*

The set $Z(G) \subset G$ is the called the centre of G if it commutes with all elements of G.

Definition 1.5 *Permutation*

A permutation is a function that maps a set, say A, onto itself.

This will come in handy in chapter 4, when we test our formula for the Bring-Jerrard quinctic equation.

Theorem 1.2 *Product of disjoint cycles*

Every permutation of a finite set can be written as a cycle or as a product of cycles [23] page 89 .

Theorem 1.3 *Cayley's Theorem (see Gallian [23])*

Every group is isomorphic to a group of permutations.

Definition 1.6 *Solvable group*

A group G is solvable if there exist normal subgroups $G_0, G_1, ..., G_k$ such that

$$G_0 \leq G_1 \leq ... \leq G_k = G,$$

where G_0 is the identity of the group only.

Example 1.1 *The set of integers $\mathbb{Z}$ is a group under addition. It is abelian, but they are not a group under multiplication. This is because not all integers have inverses. The integer 2, for example, has 1/2 as its inverse, but 1/2 is not an integer.*

Example 1.2 *The set of rational numbers $\mathbb{Q}$ is an abelian group under addition.*

Example 1.3 *The set of real numbers $\mathbb{R}$ is an abelian group under addition.*

Example 1.4 *The set of complex numbers $\mathbb{C}$ is an abelian group under addition.*

Example 1.5 *Non-zero radicals form an abelian group under multiplication.*

Example 1.6 *The set $\mathbb{A}$, introduced earlier, is a group under addition, but not under multiplication. The number 9, for example, has 1/9 as its inverse, but this inverse is neither an integer nor a proper radical number and hence not in $\mathbb{A}$.*

1.1.1 Rings and fields

Definition 1.7 *Ring (see Dummit and Foote [34] page 225)*

(1) A ring R is a set together with two binary operations $+$ and $\times$ (called addition multiplication) satisfying the following axioms:

(i) $(R, +)$ *is an abelian group,*

(ii) $\times$ *is associative* $(a \times b) \times c = a \times (b \times c)$, *for all* $a, b, c \in R$

(iii) The distributive laws holds in R: for all $a, b, c \in R$

$$(a + b) \times c = (a \times c) + (b \times c) \quad \text{and} \quad a \times (b + c) = (a \times b) + (a \times c).$$

(2) The ring R is commutative if Multiplication is commutative

(3) The ring R is said to have an identity (or contain a 1) if there is an element $1 \in R$ *with*

$$1 \times a = a \times 1 = a \quad \text{for all} \quad a \in R$$

Definition 1.8 *Subring*

A subset of S*, that is,* $S \subset R$ *is its subring if it is itself a ring under the operations of* R*. In notation form, we say*

$$S \leq R.$$

Example 1.7 *The set integers* $\mathbb{Z}$ *is a ring.*

Example 1.8 *The set of rational numbers* $\mathbb{Q}$ *is a ring.*

Example 1.9 *The set of real numbers* $\mathbb{R}$ *is a ring.*

Example 1.10 *The set of complex numbers* $\mathbb{C}$ *is a ring.*

Example 1.11 *The set of radical numbers* $\sqrt[n]{\mathbb{Q}}$ *is a ring.*

Example 1.12 *The set* $\mathbb{A}$*, introduced earlier, is not a ring because it is not a group under addition.*

1.1.2 Integral domains

Definition 1.9 *Zero-divisors*

An element a in a commutative ring R is called a zero-divisor if there is a nonzero element b in R such that $ab = 0$.

Definition 1.10 *Integral domain*

A commutative ring with a unity is said to be an integral domain if it has non-zero-divisors.

Example 1.13 *The set of integers $\mathbb{Z}$ is not an integral domain, because it is not closed under division.*

Example 1.14 *The set of rational numbers $\mathbb{Q}$ is an integral domain.*

Example 1.15 *The set of real numbers $\mathbb{R}$ is an integral domain.*

Example 1.16 *The set of complex numbers $\mathbb{C}$ is an integral domain.*

Example 1.17 *The set of radical numbers $\sqrt[n]{\mathbb{Q}}$ is an integral domain.*

Example 1.18 *The set $\mathbb{A}$, introduced earlier, is not an integral domain as it is not a group under addition, and also because its integers are not closed under division.*

1.1.3 Fields

Definition 1.11 *Field*

A field is a commutative ring with a unity and multiplicicative inverses for nonzero elements.

The set of rational numbers qualifies as a field because it is closed under the four rational operations, namely addition, multiplication, subtraction and division. The set of integers on the other hand, is not a field because all the nonzero integers are not closed under division.

Definition 1.12 *Extension Field*

A field E is called an extension of a field F, if it is contained in F, and shares the same operations.

Definition 1.13 *Splitting field*

If E is an extension field of F and $f(x)$ is a polynomial in $F(x)$, then we say $f(x)$ splits in E if it can be factored into a product of linear factors in $E(x)$.

1.1.4 Polynomial rings

Definition 1.14 *Polynomial ring*

A polynomial ring over R is that commutative ring whose elements are polynomials with the coefficients in R. That is, $p(x) \in R(x)$ means

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0,$$

with $a_i \in R$ for $i = 1, 2, ..., n$.

Definition 1.15 *Irreducible polynomial*

A polynomial $f(x)$ from the polynomial ring $R(x)$ is said to be irreducible over D, an integral domain, if $f(x) = g(x)h(x)$ means either $g(x)$ or $h(x)$ is a unit in $R(x)$.

Definition 1.16 . *Solvable polynomial*

A polynomial $f(x)$ in a polynomial ring $R(x)$, is said to be solvable by radicals over the integral domain F if there exist some positive integers $k_1, ..., k_n$ such that $a_i^{k_i} \in F$ and $f(x)$ splits in F (see Gallian [23] page 503).

Example 1.19 *The set of integers $\mathbb{Z}$ is not a field, because it has no units.*

Example 1.20 *The set of rational numbers $\mathbb{Q}$ is a field.*

Example 1.21 *The set of real numbers $\mathbb{R}$ is a field.*

Example 1.22 *The set of complex numbers $\mathbb{C}$ is a field.*

Example 1.23 *The set of radical numbers $\sqrt[n]{\mathbb{Q}}$ is a field.*

Example 1.24 *The set $\mathbb{A}$, introduced earlier, is not a field as it is not a group under addition, and also because its integers have no units.*

1.1.5 Solvability

Theorem 1.4 *Solvable by radicals*
If $f(x)$ is solvable by radicals over a field F, then the Galois group $G(E/F)$ is solvable. (see Gallian [23])

We will now discuss a few examples to shed light on the fact that the converse of this theorem is not necessarily true.

The quadratic equation

Let $f(x)$ be a quadratic polynomial in the polynomial ring $D(x)$ over the integral domain F. That is,

$$f(x) = a_1x^2 + a_2x + a_3,$$

where $a_i \in F$ with $i = 1, 2, 3$. Also let α_1 and α_2 be the solutions for $f(x) = 0$. Therefore

$$E_\alpha = \{\alpha_1, \alpha_2\}$$

is in the extension E of F. However a formula for these elements does exist. These are solutions to the quadratic equation are in radical form. Can we then conclude that the Galois group arising from this is solvable? Not quite!

Solvability over a field of integers Z

Suppose the integral domain F is a field of integers Z. This means that if E is an extension of F, then it should also be constituted by integers. The set E_α will not be a subset of E when

$$a_2^2 - 4a_1a_3 < 0.$$

Therefore, the Galois group $G(F/Z)$ will not be solvable for integers satisfying this condition.

Solvability over a field of rational numbers Q

Suppose the integral domain F is a field of rational numbers Q. This means if E is an extension of F, then it should also be constituted by rational numbers. The set E_α will not be a subset of E when

$$a_2^2 - 4a_1a_3 = \sqrt{2}.$$

Therefore, the Galois group $G(F/Q)$ will not be solvable if $E_\alpha \subset E$.

Solvability over a field of real numbers R

Suppose the integral domain F is a field of real numbers $\mathbb{R}$. This means that if E is an extension of F, then it should also be constituted by real numbers. The set E_α will not be a subset of E when

$$a_2^2 - 4a_1a_3 < 0.$$

Therefore, the Galois group $G(F/R)$ will not be solvable if $E_\alpha \subset E$.

The cubic equation

Let $f(x)$ be a cubic polynomial in the polynomial ring $D(x)$ over the integral domain F. That is,

$$f(x) = a_1x^3 + a_2x^2 + a_3x + a_4, \tag{1.1.1}$$

where $a_i \in F$ with $i = 1, 2, 3, 4$. Also let α_1, α_2 and α_3 be the solutions for $f(x) = 0$. Therefore

$$E = \{\alpha_1, \alpha_2, \alpha_3\}$$

is in the extension of F. However a formula for these elements does exist. One such formula is the one for the cubic equation in radical form. Can we then conclude that the Galois group arising from this, is solvable? Not quite!

Solvability over a field of real numbers with a complex extension

Suppose the integral domain F is a field of real numbers, then the extension E will not necessarily be in F. this is the case when

$$4(-b^2+3c)^3+(-2b^3+9bc-27d)^2<0.$$

Therefore, the converse of the solvability theorem is not generally true for quadratics, as it does not generally follow that the Galois group $G(E/F)$ is solvable when $f(x)=0$ is solvable by radicals over a field F.

That is, it does not follow that if $f(x)=0$ is not solvable over some integral domain F, then it is not solvable by radicals.

Solvability over a field of integers with an irrational extension

Suppose the integral domain F is a field of integers, then the extension E will not necessarily be in F. This is the case when

$$4(-b^2+3c)^3+(-2b^3+9bc-27d)^2=2.$$

It is possible to have the parameters a_1, a_2 and a_3 assuming integer values, resulting in α_1 and α_2 being irrational.

Again, the converse of the solvability theorem does not hold

The quartic equation

Let $f(x)$ be a cubic polynomial in the polynomial ring $D(x)$ over the integral domain F. That is,

$$f(x)=a_1x^4+a_3x^2+a_2x+a_4x+a_5,$$

where $a_i \in F$ with $i=1,2,3,4,5$. Also let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be the solutions for $f(x)=0$. Therefore

$$E=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$$

is in the extension of F. However a formula for these elements does exist.

This is the one for the quartic equation in radical form. Can we then conclude that the Galois group arising from this is solvable? Not quite!

Solvability over a field of real numbers with a complex extension

Suppose the integral domain F is a field of real numbers, then the extension E will not necessarily be in F. In the case of the quartic equation, the roots can discriminated with

$$729a_4^4 - 6912a_5^3 < 0.$$

Therefore, the converse of the solvability theorem is not generally true for quartics. It does not generally follow that the Galois group $G(E/F)$ is solvable when $f(x) = 0$ is solvable by radicals over a field F.

That is, it does not follow that if $f(x) = 0$ is not solvable over some integral domain F, then it is not solvable by radicals.

Solvability over a field of integers with an irrational extension

Suppose the integral domain F is a field of integers, then the extension E will not necessarily be in F. This is the case when

$$729a_4^4 - 6912a_5^3 = 2.$$

It is possible to have the parameters a_1, a_2 and a_3 assuming integer values, resulting in α_1 and α_2 being irrational. *Again, the converse of the solvability theorem does not hold.*

1.2 Solvability by radicals

1.2.1 The Galois group

What the theorem proves, stems from the converse of a theorem which states that if a polynomial is solvable in radicals, then its Galois group is solvable (see Gallian

[23] and Stewart[38]). To understand what this means, we have to know what a Galois group is.

Definition 1.17 *Galois group*
According to Gallian [23], if we let E be an extension field of the field F, then an automorphism of E is a ring isomorphism from E to E. The automorphism group of E fixing F, $Gal(E/F)$, is the set of all automorphisms of E called the Galois group and takes every element of F to itself.

According Ioakimidis and Papadakis [15] a quintic equation has at least one real root. Since this real root is not transcendental, it therefore possible to choose rational parameters of the equation, such that the real root becomes an integer. When that is the case, then we have the roots contained in the set $\mathbb{A}$. Since this set, as has been demonstrated many times in this chapter, is not a field, ring or group, there is no way that an automorphism group of E fixing F can exist. Hence, the Galois group $Gal(E/F)$ can never exist.
There is, therefore, nothing to prove or disprove, because E, F and $Gal(E/F)$ will just be sets without any firm algebraic structure. We will determine the formula for these in the next chapter.
There seems to be a pattern that could be followed in choosing a form to use in solving polynomials above the order of the quadratic. This flows from the work of del Ferro (1465–1526), Tartaglia (1500–1557) and Ferrari (1522–1565) in their approach to the cubic and quartic equations. The form used for the cubic was

$$r^3 - K - L = 0,$$

while the quartic required

$$r^4 - K - L = 0.$$

Inductively, one could have then expected Abel to have opted for

$$r^5 - K - L = 0, \tag{1.2.1}$$

but he chose

$$r^5 - p_0^5 - p_1^5 K - p_2^5 K^2 - p_3^5 K^3 - p_4^5 K^4 = 0$$

instead. Fortunately, this form can be reduced to (1.2.1). To achieve this, we reintroduce Abel's expression with different symbols for the parameters:

$$r^5 - q_0^5 - q_1^5 L - q_2^5 L^2 - q_3^5 L^3 - q_4^5 L^4 = 0,$$

After numerous experiment we then let

$$p_0^5 + p_2^5 K^2 + p_3^5 K^3 + p_4^5 K^4 + q_0^5 + q_2 L^2 + q_3^5 L^3 + q_4^5 L^4 = 0$$

which lead us to the correct solution with $p_1 = q_1 = 2$. Having retrieved continuity, we proceed with the task of converting (1.2.1) into a differential equation. Hence, the polynomial is raised to the fifth algebraic order. That is,

$$\left(r - K^{1/5} - L^{1/5}\right)^5 = 0,$$

so that

$$r^5 - 5\rho r^2 + 5\rho^2 r - (K + L) = 0 \tag{1.2.2}$$

for some parameter ρ determined by K and L. Equating this equation to (0.2) leads to a contradiction: we get two disagreeing values for ρ. In one instance we have $\rho = 0$ in the other $\rho = \sqrt{b/5}$. This is clearly a contradiction. Nevertheless, we were able to overcome it using Newton's identities and Tschirnhaus' transformations. As already mentioned, this came at a cost. The resulting formula is messy.

Just so that (1.2.2) can resemble (0.2), we opt for the old parameters, b and c, and introduce a for the new term:

$$r^5 + ar^2 + br + c = 0. \tag{1.2.3}$$

1.2.2 Solvability by radicals and rationals

The conclusion we draw from this chapter is that the quintic equation is not solvable by radicals because its extension field is not truly a field. This set is of the type $\mathbb{A}$. The rational number among the roots can be traced to the work by Abel [7], Ioakimidis and Papadakis [16], and Dummit[24].

It was Abel [7], however, who introduced the theorem.

Theorem 1.5 *Let $R(x, y)$ be a rational function of x and y, i.e. a polynomial divided by another polynomial, and consider y as an algebraic function $y(x)$ of x, i.e. as a root of a the polynomial equation $f(x, y) = 0$. Take the sum of all the integrals $\int R(x, y(x))dx$ computed over an interval (a real curve on the algebraic curve f) starting at some special point and ending at $(x_i(t), y_i(t))$. Then this sum, as a function of the parameter t, is the sum of a rational function in t, and a logarithm of another such rational function in t.*

According to Ioakimidis and Papadakis [16], the Lagrange quintic equation, a fifth-order equation appearing in a stationary solution of the three-body problem in celestial mechanics, has one positive root.

Their results followed from studying transcendental functions. The number could not have been transcendental because it is algebraic, but it was not clear whether it was rational. This matter has been cleared up by Dummit [34] in the proof of the theorem.

Theorem 1.6 *The irreducible quintic*

$$f(x) = x^5 + px^3 + qx^2 + rx + s \in Q[x]$$

is solvable by radicals if and only if the sextic

$$x^6 + 8ax^5 + 40a2x^4 + 160a3x^3 + 400a4x^2 + (512a5 - 3125b4)x + (256a6 - 9375ab4) = 0$$

has a rational root. If this is the case, the sextic factors into the product of a linear polynomial $(x - \theta)$ and an irreducible quintic $g(x)$.

It may not be clear from Dummit's proof that $g(x) = f(x)$. But the argument in support of a real root for the quintic, can also be extended to the sextic. The real root in the quintic cannot therefore differ from the one in Dummit's theorem. The real root is hence rational. It is true then that the quintic cannot be solved. One reason for this is that the extension set cannot truly be a field. The formula for the roots, when found, should generate rational and radical numbers as roots and as solutions for the quintic. Thais is carried out in the next chapter.

Chapter 2

The solution by radicals

In the previous chapter it was demonstrated that the fact that the Galois group for the Bring-Jerrard quintic equation is unsolvable does not mean it cannot be solved in radicals. This chapter discusses the determination of the formula. There has been attempts by others in the past, and were not successful. To avoid the challenges most encountered, we subject the equation to a bit of mathematical rigor. The cubic is used to demonstrate the approach.

This chapter is predominantly focused on Newton sums, which we use to solve the Bring-Jerrard quintic equation. However before we can do so, we first have to present the equation in a form suitable for manipulations through these identities. This format is similar to that which the Italians deduced for cubics and quartics.

2.1 Differential forms

According to H Flanders, differential forms are the things found under integral signs. We will use these forms to find a suitable for the that Tschirnhaus transformations that can be used to solve the quintic equation.

According to Olver [39] the differential forms can be defined as a smooth map $F : M \to N$ between manifolds will map smooth curves on M to smooth curves on N, and thus induce a map between their vectors. The result is a linear map

$dF : TM|_x \to TN_{F(x)}$ between the tangent spaces of the two manifolds, called the differential of F. More specifically, if the parametrized curve $\phi(t)$ has a tangent vector $\mathbf{v}|_x = \phi'(t)$ at $x = \phi(t)$, then the image curve $\psi(t) = F[\phi(t)]$ will have a tangent vector $\mathbf{w}|y = dF(\mathbf{v}|x) = \psi'(t)$ at the image point $y = F(x)$. Alternatively, if we regard tangent vectors as derivations, then we can define the differential by chain rule formula

$$dF(\mathbf{v}|_x)[h(y)] = \mathbf{v}[h \circ F(x)] \quad \text{for any} \quad h : N \to \mathbb{R}.$$

In terms of local coordinates,

$$dF(\mathbf{v}|_x) = dF\Big(\sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x^i}\Big) = \sum_{i=1}^{n} \Big(\sum_{i=1}^{m} \xi^i \frac{\partial F^j}{\partial x^i}\Big) \frac{\partial}{\partial y^j}.$$

The algebra of forms

Forms can be added:

$$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3.$$

The commutation rule takes the form

$$\omega_1 \wedge \omega_2 = (-1)^{ab} \omega_2 \wedge \omega_1,$$

where a is the rank of ω_1 and b that of ω_1. The wedge product is totally antisymmetric. That is,

$$dx^1 \wedge dx^2 \wedge dx^3 \wedge ... = \sum_{\pi} (-1)^{\pi} \pi \left(dx^1 \otimes dx^2 \otimes dx^3 \otimes ...\right)$$

The symbol π represents a permutation of the dx^i.

The calculus of forms

The exterior derivative d is a map from $p-$forms to $(p+1)-$forms. If

$$\phi = \phi_0 dx^1 \wedge dx^2 \wedge dx^3 \wedge ...,$$

then

$$d\phi = (d\phi_0)dx^1 \wedge dx^2 \wedge dx^3 \wedge ....$$

Differentiation is linear. That is,

$$d(\phi + \psi) = d\phi + d\psi.$$

Leibniz rule:

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^a \phi \wedge d\psi,$$

where a is the rank of ϕ. Closure:

$$d(d\phi) = 0.$$

To apply these to the quintic equation, we first assume a solution of the form

$$y = K^{1/5} + L^{1/5} + M^{1/5}.$$

To convert this algebraic equation into a differential form, we first have to convert it into a differential equation. For that, we introduce a function

$$Y = f(z).$$

In general, the function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable in some open convex set $\mathbf{D} \subset \mathbb{R}^n$. The functions $f(z)$ are at least five times continuously differentiable. In this contribution, we consider the case

$$z = (y, \lambda) \in \mathbf{D},$$

with

$$\frac{\partial Y}{\partial \lambda} = K^{1/5} + L^{1/5} + M^{1/5}.$$

Next, we introduce a differential 1-form ω such that

$$\omega = -dY + \xi dy + \eta d\lambda.$$

Here the quantity (Y, y, λ) constitutes a point in M, a differential manifold with a tangent vector space $TM|_{(Y,y,\lambda)}$ at that point. The 1-form ω, then, is simply a smooth valued function

$$\omega : TM|_{(Y,y,\lambda)} \to \mathbb{R}.$$

The parameters ξ and η are new dependent variables. Sectioning gives

$$\omega = -\left(\frac{\partial Y}{\partial y}dy + \frac{\partial Y}{\partial \lambda}d\lambda\right) + \xi dy + \eta d\lambda.$$

That is,

$$\omega = \left(\xi - \frac{\partial Y}{\partial y}\right) dy + \left(\eta - \frac{\partial Y}{\partial \lambda}\right) d\lambda.$$

Annulling this defines ξ and η as

$$\xi = \frac{\partial Y}{\partial y}, \quad \eta = \frac{\partial Y}{\partial \lambda}.$$

N.B. *The purpose of annulling is to eliminate the risk for logical mistakes. For example, suppose some function $f(x)$ is given by*

$$f(x) = ax + b.$$

If it also known that the same function can be expressed as

$$f(x) = \xi x + \eta,$$

then is easy to deduce that $\xi = a$ and $\eta = b$. But a complicated expression like the one under investigation, is not as trivial. It also serves to conserve consistency in dimensions, see Buckingham Pi-theorem in Bluman and Stephen [40]. Besides, the solution to this equation has eluded some of the greatest mathematicians over the centuries, including Cauchy, Newton, Galois, Ruffini, Lagrange and others. So, we were bound to repeat their mistakes if we had chosen to follow in their foot steps verbatimly.

Now to continue, it follows that

$$d\omega = d\xi \wedge dy + d\eta \wedge d\lambda.$$

Sectioning gives

$$d\omega = \left(\frac{\partial \xi}{\partial y}dy + \frac{\partial \xi}{\partial \lambda}d\lambda\right) \wedge dy + \left(\frac{\partial \eta}{\partial y}dy + \frac{\partial \eta}{\partial \lambda}d\lambda\right) \wedge d\lambda.$$

That is,

$$d\omega = \frac{\partial \xi}{\partial y}dy \wedge dy + \frac{\partial \eta}{\partial \lambda}d\lambda \wedge d\lambda + \frac{\partial \xi}{\partial \lambda}d\lambda \wedge dy + \frac{\partial \eta}{\partial y}dy \wedge d\lambda.$$

Since $dy \wedge dy = d\lambda \wedge d\lambda = 0$ and $dy \wedge d\lambda = -d\lambda \wedge dy$, then

$$d\omega = \left(\frac{\partial \xi}{\partial \lambda} - \frac{\partial \eta}{\partial y}\right) dy \wedge d\lambda.$$

Hence,

$$\frac{\partial \xi}{\partial \lambda} = \frac{\partial \eta}{\partial y}.$$

These are the tools that are going to be useful for simplifying

$$\left(-K^{1/5} - L^{1/5} - M^{1/5} + y\right)^5 = 0. \tag{2.1.1}$$

When you convert the polynomial using the transformation we adopted, we end up with a differential equation hence the derivative $\frac{\partial Y}{\partial \lambda}$. The reason for seeking a differential equation is so that we could use differential forms. The differential forms is a more secure approach for obtaining a fault free solution to the polynomial.

In expanded form, with $y^{(i)} = \partial^i Y/\partial \lambda^i,\ i = 1, 2, 3, 4, 5$, this equation (2.1.1) assumes the form

$$
\begin{aligned}
&-K - 5K^{4/5}L^{1/5} - 10K^{3/5}L^{2/5} - 10K^{2/5}L^{3/5} \\
&-5K^{1/5}L^{4/5} - L - 5K^{4/5}M^{1/5} - 20K^{3/5}L^{1/5}M^{1/5} - 30K^{2/5}L^{2/5}M^{1/5} \\
&-20K^{1/5}L^{3/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 30K^{2/5}L^{1/5} \\
&M^{2/5} - 30K^{1/5}L^{2/5}M^{2/5} - 10L^{3/5}M^{2/5} - 10K^{2/5}M^{3/5} - 20K^{1/5}L^{1/5}M^{3/5} \\
&-10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M + (5K^{4/5} + 20K^{3/5}L^{1/5} \\
&+30K^{2/5}L^{2/5} + 20K^{1/5}L^{3/5} + 5L^{4/5} + 20K^{3/5}M^{1/5} + 60K^{2/5}L^{1/5}M^{1/5} \\
&+60K^{1/5}L^{2/5}M^{1/5} + 20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5} + 60K^{1/5}L^{1/5}M^{2/5} \\
&+30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5} + 5M^{4/5})\frac{\partial Y}{\partial \lambda} \\
&+(-10K^{3/5} - 30K^{2/5}L^{1/5} - 30K^{1/5}L^{2/5} - 10L^{3/5} - 30K^{2/5}M^{1/5} \\
&-60K^{1/5}L^{1/5}M^{1/5} - 30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} \\
&-10M^{3/5})\frac{\partial^2 Y}{\partial \lambda^2} + (10K^{2/5} + 20K^{1/5}L^{1/5} + 10L^{2/5} \\
&+20K^{1/5}M^{1/5} + 20L^{1/5}M^{1/5} + 10M^{2/5})\frac{\partial^3 Y}{\partial \lambda^3} + (-5K^{1/5} - 5L^{1/5} \\
&-5M^{1/5})\frac{\partial^4 Y}{\partial \lambda^4} + \frac{\partial^5 Y}{\partial \lambda^5} = 0. \qquad (2.1.2)
\end{aligned}
$$

The fourth order partial derivative, $\partial^4 Y/\partial\lambda^4$, follows from

$$\left(-K^{1/5} - L^{1/5} - M^{1/5} + y\right)^4 = 0,$$

the third order partial derivative, $\partial^3 Y/\partial\lambda^3$, from

$$\left(-K^{1/5} - L^{1/5} - M^{1/5} + y\right)^3 = 0$$

and the second order partial derivative, $\partial^2 Y/\partial\lambda^2$, from

$$\left(-K^{1/5} - L^{1/5} - M^{1/5} + y\right)^2 = 0.$$

We can eliminate the derivative $\partial^4 Y/\partial\lambda^4$ from the equation (2.1.2) we end with:

$$\begin{aligned}
&-K - 5K^{4/5}L^{1/5} - 10K^{3/5}L^{2/5} - 10K^{2/5}L^{3/5} - 5K^{1/5}L^{4/5} - L - 5K^{4/5}M^{1/5} \\
&-20K^{3/5}L^{1/5}M^{1/5} - 30K^{2/5}L^{2/5}M^{1/5} - 20K^{1/5}L^{3/5}M^{1/5} - 5L^{4/5}M^{1/5} - \\
&10K^{3/5}M^{2/5} - 30K^{2/5}L^{1/5}M^{2/5} - 30K^{1/5}L^{2/5}M^{2/5} - 10L^{3/5}M^{2/5} \\
&-10K^{2/5}M^{3/5} - 20K^{1/5}L^{1/5}M^{3/5} - 10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} \\
&-M + (5K^{4/5} + 20K^{3/5}L^{1/5} + 30K^{2/5}L^{2/5} + 20K^{1/5}L^{3/5} + 5L^{4/5} + 20K^{3/5}M^{1/5} \\
&+60K^{2/5}L^{1/5}M^{1/5} + 60K^{1/5}L^{2/5}M^{1/5} + 20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5} \\
&+60K^{1/5}L^{1/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5} \\
&+5M^{4/5})\frac{\partial Y}{\partial\lambda} + (-10K^{3/5} - 30K^{2/5}L^{1/5} - 30K^{1/5}L^{2/5} \\
&-10L^{3/5} - 30K^{2/5}M^{1/5} - 60K^{1/5}L^{1/5}M^{1/5} - 30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} \\
&-30L^{1/5}M^{2/5} - 10M^{3/5})\frac{\partial^2 Y}{\partial\lambda^2} + (10K^{2/5} + 20K^{1/5}L^{1/5} + 10L^{2/5} + 20K^{1/5}M^{1/5} \\
&+20L^{1/5}M^{1/5} + 10M^{2/5})\frac{\partial^3 Y}{\partial\lambda^3} - 4\frac{\partial^5 Y}{\partial\lambda^5} = 0. \qquad (2.1.3)
\end{aligned}$$

The products $(KL)^{1/5}, (KM)^{1/5}$ and $(ML)^{1/5}$ occur quite frequently in the equation

(2.1.3), so we set them to $\rho KL, \rho KM$ and ρLM, respectively. Hence,

$$-K - 10K^{3/5}L^{2/5} - 10K^{2/5}L^{3/5} - 5K^{1/5}L^{4/5} - L - 5K^{4/5}M^{1/5}$$
$$-20K^{3/5}L^{1/5}M^{1/5} - 30K^{2/5}L^{2/5}M^{1/5} - 20K^{1/5}L^{3/5}M^{1/5} - 5L^{4/5}M^{1/5}$$
$$-10K^{3/5}M^{2/5} - 30K^{2/5}L^{1/5}M^{2/5} - 30K^{1/5}L^{2/5}M^{2/5} - 10L^{3/5}M^{2/5}$$
$$-10K^{2/5}M^{3/5} - 20K^{1/5}L^{1/5}M^{3/5} - 10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5}$$
$$-M + (5K^{4/5} + 20K^{3/5}L^{1/5} + 30K^{2/5}L^{2/5} + 20K^{1/5}L^{3/5} + 5L^{4/5} + 20K^{3/5}M^{1/5}$$
$$+60K^{2/5}L^{1/5}M^{1/5} + 60K^{1/5}L^{2/5}M^{1/5} + 20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5}$$
$$+60K^{1/5}L^{1/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5}$$
$$+5M^{4/5})\frac{\partial Y}{\partial \lambda} + (-10K^{3/5} - 30K^{2/5}L^{1/5} - 30K^{1/5}L^{2/5} - 10L^{3/5}$$
$$-30K^{2/5}M^{1/5} - 60K^{1/5}L^{1/5}M^{1/5} - 30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5}$$
$$-10M^{3/5})\frac{\partial^2 Y}{\partial \lambda^2} + (10K^{2/5} + 20K^{1/5}L^{1/5} + 10L^{2/5} + 20K^{1/5}M^{1/5} + 20L^{1/5}M^{1/5}$$
$$+10M^{2/5})\frac{\partial^3 Y}{\partial \lambda^3} - 4\frac{\partial^5 Y}{\partial \lambda^5} - 5K^{3/5}\rho \mathrm{kl} = 0. \tag{2.1.4}$$

In the above equation (2.1.4) substituting $(-nK^{1/n} - nL^{1/n} - nM^{1/n})$ with $-ny$ and $n = 5$, we have the following equation:

$$-K - 10K^{2/5}L^{3/5} - 5K^{1/5}L^{4/5} - L - 5K^{4/5}M^{1/5} - 20K^{3/5}L^{1/5}M^{1/5}$$
$$-30K^{2/5}L^{2/5}M^{1/5} - 20K^{1/5}L^{3/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5}$$
$$-30K^{2/5}L^{1/5}M^{2/5} - 30K^{1/5}L^{2/5}M^{2/5} - 10L^{3/5}M^{2/5} - 10K^{2/5}M^{3/5}$$
$$-20K^{1/5}L^{1/5}M^{3/5} - 10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5}$$
$$-M + (5K^{4/5} + 20K^{3/5}L^{1/5} + 30K^{2/5}L^{2/5} + 20K^{1/5}L^{3/5} + 5L^{4/5}$$
$$+20K^{3/5}M^{1/5} + 60K^{2/5}L^{1/5}M^{1/5} + 60K^{1/5}L^{2/5}M^{1/5} + 20L^{3/5}M^{1/5}$$
$$+30K^{2/5}M^{2/5} + 60K^{1/5}L^{1/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5}$$
$$+5M^{4/5})\frac{\partial Y}{\partial \lambda} + (-10K^{3/5} - 30K^{2/5}L^{1/5} - 30K^{1/5}L^{2/5} - 10L^{3/5}$$
$$-30K^{2/5}M^{1/5} - 60K^{1/5}L^{1/5}M^{1/5} - 30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5}$$
$$-10M^{3/5})\frac{\partial^2 Y}{\partial \lambda^2} + (10K^{2/5} + 20K^{1/5}L^{1/5} + 10L^{2/5} + 20K^{1/5}M^{1/5}$$
$$+20L^{1/5}M^{1/5} + 10M^{2/5})\frac{\partial^3 Y}{\partial \lambda^3} - 4\frac{\partial^5 Y}{\partial \lambda^5} - 5K^{3/5}\rho \mathrm{kl} - 10K^{1/5}\rho \mathrm{kl}^2 = 0. \tag{2.1.5}$$

Replacing $K^{4/n}L^{1/n}$ from the equation (2.1.5) with $K^{3/n}\,\rho\mathrm{kl}$ and $n = 5$, we get the following equation:

$$
\begin{aligned}
&-K - 5K^{1/5}L^{4/5} - L - 5K^{4/5}M^{1/5} - 20K^{3/5}L^{1/5}M^{1/5} - 30K^{2/5}L^{2/5}M^{1/5} \\
&-20K^{1/5}L^{3/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 30K^{2/5}L^{1/5}M^{2/5} \\
&-30K^{1/5}L^{2/5}M^{2/5} - 10L^{3/5}M^{2/5} - 10K^{2/5}M^{3/5} - 20K^{1/5}L^{1/5}M^{3/5} \\
&-10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M + (5K^{4/5} + 20K^{3/5}L^{1/5} \\
&+30K^{2/5}L^{2/5} + 20K^{1/5}L^{3/5} + 5L^{4/5} + 20K^{3/5}M^{1/5} + 60K^{2/5}L^{1/5}M^{1/5} \\
&+60K^{1/5}L^{2/5}M^{1/5} + 20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5} + 60K^{1/5}L^{1/5}M^{2/5} \\
&+30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5} + 5M^{4/5})\frac{\partial Y}{\partial \lambda} \\
&+(-10K^{3/5} - 30K^{2/5}L^{1/5} - 30K^{1/5}L^{2/5} - 10L^{3/5} - 30K^{2/5}M^{1/5} \\
&-60K^{1/5}L^{1/5}M^{1/5} - 30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} \\
&-10M^{3/5})\frac{\partial^2 Y}{\partial \lambda^2} + (10K^{2/5} + 20K^{1/5}L^{1/5} + 10L^{2/5} + 20K^{1/5}M^{1/5} \\
&+20L^{1/5}M^{1/5} + 10M^{2/5})\frac{\partial^3 Y}{\partial \lambda^3} - 4\frac{\partial^5 Y}{\partial \lambda^5} - 5K^{3/5}\rho\mathrm{kl} \\
&-10K^{1/5}\rho\mathrm{kl}^2 - 10L^{1/5}\rho\mathrm{kl}^2 = 0. \qquad (2.1.6)
\end{aligned}
$$

Replacing $K^{3/n}L^{2/n}$ in the equation (2.1.6) with $K^{1/n}\rho\mathrm{kl}^2$ and $n = 5$, we end with the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 20K^{3/5}L^{1/5}M^{1/5} - 30K^{2/5}L^{2/5}M^{1/5} \\
&-20K^{1/5}L^{3/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 30K^{2/5}L^{1/5}M^{2/5} \\
&-30K^{1/5}L^{2/5}M^{2/5} - 10L^{3/5}M^{2/5} - 10K^{2/5}M^{3/5} - 20K^{1/5}L^{1/5}M^{3/5} \\
&-10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M + (5K^{4/5} + 20K^{3/5}L^{1/5} \\
&+30K^{2/5}L^{2/5} + 20K^{1/5}L^{3/5} + 5L^{4/5} + 20K^{3/5}M^{1/5} + 60K^{2/5}L^{1/5}M^{1/5} \\
&+60K^{1/5}L^{2/5}M^{1/5} + 20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5} + 60K^{1/5}L^{1/5}M^{2/5} \\
&+30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5} + 5M^{4/5})\frac{\partial Y}{\partial \lambda}
\end{aligned}
$$

$$
\begin{aligned}
&+(-10K^{3/5}-30K^{2/5}L^{1/5}-30K^{1/5}L^{2/5}-10L^{3/5}\\
&-30K^{2/5}M^{1/5}-60K^{1/5}L^{1/5}M^{1/5}-30L^{2/5}M^{1/5}-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}\\
&-10M^{3/5})\frac{\partial^2 Y}{\partial\lambda^2}+(10K^{2/5}+20K^{1/5}L^{1/5}+10L^{2/5}+20K^{1/5}M^{1/5}+20L^{1/5}M^{1/5}\\
&+10M^{2/5})\frac{\partial^3 Y}{\partial\lambda^3}-4\frac{\partial^5 Y}{\partial\lambda^5}-5K^{3/5}\rho\mathrm{kl}-5L^{3/5}\rho\mathrm{kl}\\
&-10K^{1/5}\rho\mathrm{kl}^2-10L^{1/5}\rho\mathrm{kl}^2=0. \qquad (2.1.7)
\end{aligned}
$$

If we set $K^{2/n}L^{3/n}$ for $L^{1/n}\rho\mathrm{kl}^2$ in the equation (2.1.7) and $n=5$, we end with the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-30K^{2/5}L^{2/5}M^{1/5}-20K^{1/5}L^{3/5}M^{1/5}-5L^{4/5}M^{1/5}\\
&-10K^{3/5}M^{2/5}-30K^{2/5}L^{1/5}M^{2/5}-30K^{1/5}L^{2/5}M^{2/5}-10L^{3/5}M^{2/5}\\
&-10K^{2/5}M^{3/5}-20K^{1/5}L^{1/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}\\
&-M+(-10K^{3/5}-30K^{2/5}L^{1/5}-30K^{1/5}L^{2/5}-10L^{3/5}-30K^{2/5}M^{1/5}\\
&-60K^{1/5}L^{1/5}M^{1/5}-30L^{2/5}M^{1/5}-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}\\
&-10M^{3/5})\frac{\partial^2 Y}{\partial\lambda^2}+(10K^{2/5}+20K^{1/5}L^{1/5}+10L^{2/5}\\
&+20K^{1/5}M^{1/5}+20L^{1/5}M^{1/5}+10M^{2/5})\frac{\partial^3 Y}{\partial\lambda^3}-4\frac{\partial^5 Y}{\partial\lambda^5}-5K^{3/5}\rho\mathrm{kl}-5L^{3/5}\rho\mathrm{kl}\\
&-20K^{2/5}M^{1/5}\rho\mathrm{kl}-10K^{1/5}\rho\mathrm{kl}^2-10L^{1/5}\rho\mathrm{kl}^2+\frac{\partial Y}{\partial\lambda}(5K^{4/5}\\
&+30K^{2/5}L^{2/5}+20K^{1/5}L^{3/5}+5L^{4/5}+20K^{3/5}M^{1/5}+60K^{2/5}L^{1/5}M^{1/5}\\
&+60K^{1/5}L^{2/5}M^{1/5}+20L^{3/5}M^{1/5}+30K^{2/5}M^{2/5}+60K^{1/5}L^{1/5}M^{2/5}\\
&+30L^{2/5}M^{2/5}+20K^{1/5}M^{3/5}+20L^{1/5}M^{3/5}+5M^{4/5}+20K^{2/5}\rho\mathrm{kl})=0. \qquad (2.1.8)
\end{aligned}
$$

Placing $K^{1/n}L^{4/n}$ for $L^{3/n}\rho\mathrm{kl}$ in the equation (2.1.8) and $n=5$, we achieve the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-20K^{1/5}L^{3/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}\\
&-30K^{2/5}L^{1/5}M^{2/5}-30K^{1/5}L^{2/5}M^{2/5}-10L^{3/5}M^{2/5}-10K^{2/5}M^{3/5}\\
&-20K^{1/5}L^{1/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M+(-10K^{3/5}\\
&-30K^{2/5}L^{1/5}-30K^{1/5}L^{2/5}-10L^{3/5}-30K^{2/5}M^{1/5}-60K^{1/5}L^{1/5}M^{1/5}\\
&-30L^{2/5}M^{1/5}-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}-10M^{3/5})\frac{\partial^2 Y}{\partial\lambda^2}\\
&+(10K^{2/5}+20K^{1/5}L^{1/5}+10L^{2/5}+20K^{1/5}M^{1/5}+20L^{1/5}M^{1/5}\\
&+10M^{2/5})\frac{\partial^3 Y}{\partial\lambda^3}-4\frac{\partial^5 Y}{\partial\lambda^5}-5K^{3/5}\rho\mathrm{kl}-5L^{3/5}\rho\mathrm{kl}-20K^{2/5}M^{1/5}\rho\mathrm{kl}-10K^{1/5}\rho\mathrm{kl}^2\\
&-10L^{1/5}\rho\mathrm{kl}^2-30M^{1/5}\rho\mathrm{kl}^2+\frac{\partial Y}{\partial\lambda}(5K^{4/5}+20K^{1/5}L^{3/5}+5L^{4/5}+20K^{3/5}M^{1/5}\\
&+60K^{2/5}L^{1/5}M^{1/5}+60K^{1/5}L^{2/5}M^{1/5}+20L^{3/5}M^{1/5}+30K^{2/5}M^{2/5}\\
&+60K^{1/5}L^{1/5}M^{2/5}+30L^{2/5}M^{2/5}+20K^{1/5}M^{3/5}+20L^{1/5}M^{3/5}+5M^{4/5}\\
&+20K^{2/5}\rho\mathrm{kl}+30\rho\mathrm{kl}^2)=0. \qquad (2.1.9)
\end{aligned}
$$

If we exchange $K^{3/n}L^{1/n}$ for $K^{2/n}\rho\mathrm{kl}$ in the equation (2.1.9) and $n=5$, we result obtain the following:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}-30K^{2/5}L^{1/5}M^{2/5}\\
&-30K^{1/5}L^{2/5}M^{2/5}-10L^{3/5}M^{2/5}-10K^{2/5}M^{3/5}-20K^{1/5}L^{1/5}M^{3/5}\\
&-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M+(-10K^{3/5}-30K^{2/5}L^{1/5}\\
&-30K^{1/5}L^{2/5}-10L^{3/5}-30K^{2/5}M^{1/5}-60K^{1/5}L^{1/5}M^{1/5}-30L^{2/5}M^{1/5}\\
&-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}-10M^{3/5})\frac{\partial^2 Y}{\partial\lambda^2}+(10K^{2/5}+20K^{1/5}L^{1/5}\\
&+10L^{2/5}+20K^{1/5}M^{1/5}+20L^{1/5}M^{1/5}+10M^{2/5})\frac{\partial^3 Y}{\partial\lambda^3}-4\frac{\partial^5 Y}{\partial\lambda^5}\\
&-5K^{3/5}\rho\mathrm{kl}-5L^{3/5}\rho\mathrm{kl}-20K^{2/5}M^{1/5}\rho\mathrm{kl}-20L^{2/5}M^{1/5}\rho\mathrm{kl}
\end{aligned}
$$

$$-10K^{1/5}\rho\text{kl}^2 - 10L^{1/5}\rho\text{kl}^2 - 30M^{1/5}\rho\text{kl}^2 + \frac{\partial Y}{\partial \lambda}(5K^{4/5}$$
$$+5L^{4/5} + 20K^{3/5}M^{1/5} + 60K^{2/5}L^{1/5}M^{1/5} + 60K^{1/5}L^{2/5}M^{1/5} + 20L^{3/5}M^{1/5}$$
$$+30K^{2/5}M^{2/5} + 60K^{1/5}L^{1/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5}$$
$$+20L^{1/5}M^{3/5} + 5M^{4/5} + 20K^{2/5}\rho\text{kl} + 20L^{2/5}\rho\text{kl} + 30\rho\text{kl}^2) = 0. \tag{2.1.10}$$

If we put $K^{1/n}L^{3/n}$ for $L^{2/n}\rho\text{kl}$ in the above eqation (2.1.10) and $n = 5$, we get the following equation:

$$-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 30K^{1/5}L^{2/5}M^{2/5}$$
$$-10L^{3/5}M^{2/5} - 10K^{2/5}M^{3/5} - 20K^{1/5}L^{1/5}M^{3/5} - 10L^{2/5}M^{3/5}$$
$$-5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M + (10K^{2/5} + 20K^{1/5}L^{1/5} + 10L^{2/5}$$
$$+20K^{1/5}M^{1/5} + 20L^{1/5}M^{1/5} + 10M^{2/5})\frac{\partial^3 Y}{\partial \lambda^3} - 4\frac{\partial^5 Y}{\partial \lambda^5} - 5K^{3/5}\rho\text{kl}$$
$$-5L^{3/5}\rho\text{kl} - 20K^{2/5}M^{1/5}\rho\text{kl} - 20L^{2/5}M^{1/5}\rho\text{kl} - 30K^{1/5}M^{2/5}\rho\text{kl}$$
$$-10K^{1/5}\rho\text{kl}^2 - 10L^{1/5}\rho\text{kl}^2 - 30M^{1/5}\rho\text{kl}^2 + \frac{\partial^2 Y}{\partial \lambda^2}(-10K^{3/5}$$
$$-30K^{1/5}L^{2/5} - 10L^{3/5} - 30K^{2/5}M^{1/5} - 60K^{1/5}L^{1/5}M^{1/5} - 30L^{2/5}M^{1/5}$$
$$-30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} - 10M^{3/5} - 30K^{1/5}\rho\text{kl}) + \frac{\partial Y}{\partial \lambda}(5K^{4/5}$$
$$+5L^{4/5} + 20K^{3/5}M^{1/5} + 60K^{1/5}L^{2/5}M^{1/5} + 20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5}$$
$$+60K^{1/5}L^{1/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5} + 5M^{4/5}$$
$$+20K^{2/5}\rho\text{kl} + 20L^{2/5}\rho\text{kl} + 60K^{1/5}M^{1/5}\rho\text{kl} + 30\rho\text{kl}^2) = 0. \tag{2.1.11}$$

If we set $K^{2/n}L^{1/n}$ for $K^{1/n}\rho\text{kl}$ in the equation (2.1.11) and $n = 5$, the outcome is the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}-10L^{3/5}M^{2/5}\\
&-10K^{2/5}M^{3/5}20K^{1/5}L^{1/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M\\
&+(10K^{2/5}+20K^{1/5}L^{1/5}+10L^{2/5}+20K^{1/5}M^{1/5}+20L^{1/5}M^{1/5}+10M^{2/5})\frac{\partial^3 Y}{\partial\lambda^3}\\
&-4\frac{\partial^5 Y}{\partial\lambda^5}-5K^{3/5}\rho\mathrm{kl}-5L^{3/5}\rho\mathrm{kl}-20K^{2/5}M^{1/5}\rho\mathrm{kl}-20L^{2/5}M^{1/5}\rho\mathrm{kl}\\
&-30K^{1/5}M^{2/5}\rho\mathrm{kl}-30L^{1/5}M^{2/5}\rho\mathrm{kl}-10K^{1/5}\rho\mathrm{kl}^2-10L^{1/5}\rho\mathrm{kl}^2-30M^{1/5}\rho\mathrm{kl}^2\\
&+\frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5}-10L^{3/5}-30K^{2/5}M^{1/5}-60K^{1/5}L^{1/5}M^{1/5}-30L^{2/5}M^{1/5}\\
&-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}-10M^{3/5}-30K^{1/5}\rho\mathrm{kl}-30L^{1/5}\rho\mathrm{kl})\\
&+\frac{\partial Y}{\partial\lambda}(5K^{4/5}+5L^{4/5}+20K^{3/5}M^{1/5}+20L^{3/5}M^{1/5}+30K^{2/5}M^{2/5}\\
&+60K^{1/5}L^{1/5}M^{2/5}+30L^{2/5}M^{2/5}+20K^{1/5}M^{3/5}+20L^{1/5}M^{3/5}+5M^{4/5}\\
&+20K^{2/5}\rho\mathrm{kl}+20L^{2/5}\rho\mathrm{kl}+60K^{1/5}M^{1/5}\rho\mathrm{kl}+60L^{1/5}M^{1/5}\rho\mathrm{kl}\\
&+30\rho\mathrm{kl}^2)=0. \qquad (2.1.12)
\end{aligned}
$$

If we replace $K^{1/n}L^{2/n}$ with $L^{1/n}\rho\mathrm{kl}$ in the above equation (2.1.12) and $n=5$, it will lead to the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}-10L^{3/5}M^{2/5}\\
&-10K^{2/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M-4\frac{\partial^5 Y}{\partial\lambda^5}\\
&-5K^{3/5}\rho\mathrm{kl}-5L^{3/5}\rho\mathrm{kl}-20K^{2/5}M^{1/5}\rho\mathrm{kl}-20L^{2/5}M^{1/5}\rho\mathrm{kl}\\
&-30K^{1/5}M^{2/5}\rho\mathrm{kl}-30L^{1/5}M^{2/5}\rho\mathrm{kl}-20M^{3/5}\rho\mathrm{kl}-10K^{1/5}\rho\mathrm{kl}^2\\
&-10L^{1/5}\rho\mathrm{kl}^2-30M^{1/5}\rho\mathrm{kl}^2+\frac{\partial^3 Y}{\partial\lambda^3}(10K^{2/5}+10L^{2/5}+20K^{1/5}M^{1/5}\\
&+20L^{1/5}M^{1/5}+10M^{2/5}+20\rho\mathrm{kl})+\frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5}-10L^{3/5}\\
&-30K^{2/5}M^{1/5}-30L^{2/5}M^{1/5}-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}-10M^{3/5}
\end{aligned}
$$

$$-30K^{1/5}\rho\mathrm{kl} - 30L^{1/5}\rho\mathrm{kl} - 60M^{1/5}\rho\mathrm{kl}) + \frac{\partial Y}{\partial\lambda}(5K^{4/5} + 5L^{4/5}$$

$$+20K^{3/5}M^{1/5} + 20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5}$$

$$+20L^{1/5}M^{3/5} + 5M^{4/5} + 20K^{2/5}\rho\mathrm{kl} + 20L^{2/5}\rho\mathrm{kl} + 60K^{1/5}M^{1/5}\rho\mathrm{kl}$$

$$+60L^{1/5}M^{1/5}\rho\mathrm{kl} + 60M^{2/5}\rho\mathrm{kl} + 30\rho\mathrm{kl}^2) = 0. \quad (2.1.13)$$

If we exchange $K^{1/n}L^{1/n}$ with ρkl in the equation (2.1.13) and $n = 5$, we end up with the following equation:

$$-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10L^{3/5}M^{2/5} - 10K^{2/5}M^{3/5}$$

$$-10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M - 4\frac{\partial^5 Y}{\partial\lambda^5}$$

$$-20K^{2/5}M^{1/5}\rho\mathrm{kl} - 20L^{2/5}M^{1/5}\rho\mathrm{kl} - 30K^{1/5}M^{2/5}\rho\mathrm{kl}$$

$$-30L^{1/5}M^{2/5}\rho\mathrm{kl} - 20M^{3/5}\rho\mathrm{kl} - 5(K^{3/5} - (K^{1/5} + L^{1/5} + M^{1/5})^3$$

$$+\frac{\partial^3 Y}{\partial\lambda^3})\rho\mathrm{kl} - 10K^{1/5}\rho\mathrm{kl}^2 - 10L^{1/5}\rho\mathrm{kl}^2 - 30M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial^3 Y}{\partial\lambda^3}(10K^{2/5}$$

$$+10L^{2/5} + 20K^{1/5}M^{1/5} + 20L^{1/5}M^{1/5} + 10M^{2/5} + 20\rho\mathrm{kl}) + \frac{\partial^2 Y}{\partial\lambda^2}(-10L^{3/5}$$

$$-30K^{2/5}M^{1/5} - 30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} - 10M^{3/5}$$

$$-10(K^{3/5} - (K^{1/5} + L^{1/5} + M^{1/5})^3 + \frac{\partial^3 Y}{\partial\lambda^3}) - 30K^{1/5}\rho\mathrm{kl} - 30L^{1/5}\rho\mathrm{kl}$$

$$-60M^{1/5}\rho\mathrm{kl}) + \frac{\partial Y}{\partial\lambda}(5K^{4/5} + 5L^{4/5} + 20L^{3/5}M^{1/5}$$

$$+30K^{2/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5}$$

$$+5M^{4/5} + 20M^{1/5}(K^{3/5} - (K^{1/5} + L^{1/5} + M^{1/5})^3 + \frac{\partial^3 Y}{\partial\lambda^3}) + 20K^{2/5}\rho\mathrm{kl}$$

$$+20L^{2/5}\rho\mathrm{kl} + 60K^{1/5}M^{1/5}\rho\mathrm{kl} + 60L^{1/5}M^{1/5}\rho\mathrm{kl} + 60M^{2/5}\rho\mathrm{kl}$$

$$+30\rho\mathrm{kl}^2) = 0. \quad (2.1.14)$$

If we substitute $y \to K^{(1/5)} + L^{(1/5)} + M^{(1/5)}$, $\rho\mathrm{kl} \to (KL)^{(1/5)}$, $\rho\mathrm{km} \to (KM)^{(1/5)}$, $\rho\mathrm{lm} \to (LM)^{(1/5)}$ and $\rho \to (KLM)^{(1/5)}$, in the the equation (2.1.14) we end up with the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 10L^{3/5}M^{2/5} \\
&-10K^{2/5}M^{3/5} - 10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M - 4\frac{\partial^5 Y}{\partial \lambda^5} \\
&-5K^{3/5}\rho\mathrm{kl} - 5L^{3/5}\rho\mathrm{kl} - 20K^{2/5}M^{1/5}\rho\mathrm{kl} - 20L^{2/5}M^{1/5}\rho\mathrm{kl} \\
&-30K^{1/5}M^{2/5}\rho\mathrm{kl} - 30L^{1/5}M^{2/5}\rho\mathrm{kl} - 20M^{3/5}\rho\mathrm{kl} - 20M^{1/5}\rho\mathrm{kl}^2 \\
&-10\frac{\partial Y}{\partial \lambda}\rho\mathrm{kl}^2 + \frac{\partial^3 Y}{\partial \lambda^3}(10K^{2/5} + 10L^{2/5} + 20K^{1/5}M^{1/5} + 20L^{1/5}M^{1/5} \\
&+10M^{2/5} + 20\rho\mathrm{kl}) + \frac{\partial^2 Y}{\partial \lambda^2}(-10K^{3/5} - 10L^{3/5} - 30K^{2/5}M^{1/5} - 30L^{2/5}M^{1/5} \\
&-30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} - 10M^{3/5} - 30K^{1/5}\rho\mathrm{kl} - 30L^{1/5}\rho\mathrm{kl} \\
&-60M^{1/5}\rho\mathrm{kl}) + \frac{\partial Y}{\partial \lambda}(5K^{4/5} + 5L^{4/5} + 20K^{3/5}M^{1/5} + 20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5} \\
&+30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5} + 5M^{4/5} + 20K^{2/5}\rho\mathrm{kl} + 20L^{2/5}\rho\mathrm{kl} \\
&+60K^{1/5}M^{1/5}\rho\mathrm{kl} + 60L^{1/5}M^{1/5}\rho\mathrm{kl} + 60M^{2/5}\rho\mathrm{kl} + 30\rho\mathrm{kl}^2) = 0. \qquad (2.1.15)
\end{aligned}
$$

Replacing $K^{r/n}$ *with* $y^r - (K^{1/n} + L^{1/n} + M^{1/n})^r - K^{r/n}]$ in the equation (2.1.15) and $n = 5$, then simplifying we end up with the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 10L^{3/5}M^{2/5} \\
&-10K^{2/5}M^{3/5} - 10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M - 4\frac{\partial^5 Y}{\partial \lambda^5} \\
&-5L^{3/5}\rho\mathrm{kl} + 5(3K^{2/5}L^{1/5} + 3K^{1/5}L^{2/5} + L^{3/5} + 3K^{2/5}M^{1/5} \\
&+6K^{1/5}L^{1/5}M^{1/5} + 3L^{2/5}M^{1/5} + 3K^{1/5}M^{2/5} + 3L^{1/5}M^{2/5} + M^{3/5})\rho\mathrm{kl} \\
&-20K^{2/5}M^{1/5}\rho\mathrm{kl} - 20L^{2/5}M^{1/5}\rho\mathrm{kl} - 30K^{1/5}M^{2/5}\rho\mathrm{kl} - 30L^{1/5}M^{2/5}\rho\mathrm{kl} \\
&-20M^{3/5}\rho\mathrm{kl} - 5\frac{\partial^3 Y}{\partial \lambda^3}\rho\mathrm{kl} - 20M^{1/5}\rho\mathrm{kl}^2 - 10\frac{\partial Y}{\partial \lambda}\rho\mathrm{kl}^2 \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(10K^{2/5} + 10L^{2/5} + 20K^{1/5}M^{1/5} + 20L^{1/5}M^{1/5} + 10M^{2/5}
\end{aligned}
$$

$$+20\rho\text{kl}) + \frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5} - 10L^{3/5} - 30K^{2/5}M^{1/5} - 30L^{2/5}M^{1/5}$$
$$-30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} - 10M^{3/5} - 30K^{1/5}\rho\text{kl} - 30L^{1/5}\rho\text{kl}$$
$$-60M^{1/5}\rho\text{kl}) + \frac{\partial Y}{\partial\lambda}(5K^{4/5} + 5L^{4/5} + 20K^{3/5}M^{1/5} + 20L^{3/5}M^{1/5}$$
$$+30K^{2/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5}$$
$$+5M^{4/5} + 20K^{2/5}\rho\text{kl} + 20L^{2/5}\rho\text{kl} + 60K^{1/5}M^{1/5}\rho\text{kl} + 60L^{1/5}M^{1/5}\rho\text{kl}$$
$$+60M^{2/5}\rho\text{kl} + 30\rho\text{kl}^2) = 0. \tag{2.1.16}$$

If we change $-10K^{1/5}\rho\text{kl}^2 - 10L^{1/5}\rho\text{kl}^2$ with $-10y\rho\text{kl}^2 + 10M^{1/5}\rho\text{kl}\hat{}2$ in the equation (2.1.16) and $n = 5$, we get the following:

$$-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 10L^{3/5}M^{2/5}$$
$$-10K^{2/5}M^{3/5} - 10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M - 4\frac{\partial^5 Y}{\partial\lambda^5}$$
$$-5L^{3/5}\rho\text{kl} + 5(3K^{2/5}L^{1/5} + 3K^{1/5}L^{2/5} + L^{3/5} + 3K^{2/5}M^{1/5} + 6K^{1/5}L^{1/5}M^{1/5}$$
$$+3L^{2/5}M^{1/5} + 3K^{1/5}M^{2/5} + 3L^{1/5}M^{2/5} + M^{3/5})\rho\text{kl} - 20K^{2/5}M^{1/5}\rho\text{kl}$$
$$-20L^{2/5}M^{1/5}\rho\text{kl} - 30K^{1/5}M^{2/5}\rho\text{kl} - 30L^{1/5}M^{2/5}\rho\text{kl} - 20M^{3/5}\rho\text{kl}$$
$$-20M^{1/5}\rho\text{kl}^2 + \frac{\partial^3 Y}{\partial\lambda^3}(10K^{2/5} + 10L^{2/5} + 20K^{1/5}M^{1/5} + 20L^{1/5}M^{1/5}$$
$$+10M^{2/5} + 15\rho\text{kl}) + \frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5} - 10L^{3/5} - 30K^{2/5}M^{1/5}$$
$$-30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} - 10M^{3/5} - 30K^{1/5}\rho\text{kl}$$
$$-30L^{1/5}\rho\text{kl} - 60M^{1/5}\rho\text{kl}) + \frac{\partial Y}{\partial\lambda}(5K^{4/5} + 5L^{4/5} + 20K^{3/5}M^{1/5}$$
$$+20L^{3/5}M^{1/5} + 30K^{2/5}M^{2/5} + 30L^{2/5}M^{2/5} + 20K^{1/5}M^{3/5} + 20L^{1/5}M^{3/5}$$
$$+5M^{4/5} + 20K^{2/5}\rho\text{kl} + 20L^{2/5}\rho\text{kl} + 60K^{1/5}M^{1/5}\rho\text{kl} + 60L^{1/5}M^{1/5}\rho\text{kl}$$
$$+60M^{2/5}\rho\text{kl} + 20\rho\text{kl}^2) = 0. \tag{2.1.17}$$

If we replace $-5K^{3/5}\rho\text{kl}$ with the command $-5\rho\text{kl}y\hat{}3 + 5\rho\text{klExpandAll}[(K^{1/n} + L^{1/n} + M^{1/n})\hat{}3 - K^{3/n}$ in the equation (2.1.17) , we end with the following equation:

$$-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}-10L^{3/5}M^{2/5}$$
$$-10K^{2/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M-4\frac{\partial^5 Y}{\partial\lambda^5}$$
$$-5L^{3/5}\rho\mathrm{kl}-20K^{2/5}M^{1/5}\rho\mathrm{kl}-20L^{2/5}M^{1/5}\rho\mathrm{kl}-30K^{1/5}M^{2/5}\rho\mathrm{kl}$$
$$-30L^{1/5}M^{2/5}\rho\mathrm{kl}-20M^{3/5}\rho\mathrm{kl}-20M^{1/5}\rho\mathrm{kl}^2$$
$$+\frac{\partial^3 Y}{\partial\lambda^3}(10K^{2/5}+10L^{2/5}+20K^{1/5}M^{1/5}+20L^{1/5}M^{1/5}+10M^{2/5}+15\rho\mathrm{kl})$$
$$+5\rho\mathrm{kl}(L^{3/5}+3K^{2/5}M^{1/5}+6K^{1/5}L^{1/5}M^{1/5}$$
$$+3L^{2/5}M^{1/5}+3K^{1/5}M^{2/5}+3L^{1/5}M^{2/5}+M^{3/5}+3K^{1/5}\rho\mathrm{kl}$$
$$+3L^{1/5}\rho\mathrm{kl})+\frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5}-10L^{3/5}$$
$$-30K^{2/5}M^{1/5}-30L^{2/5}M^{1/5}-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}-10M^{3/5}$$
$$-30K^{1/5}\rho\mathrm{kl}-30L^{1/5}\rho\mathrm{kl}-60M^{1/5}\rho\mathrm{kl})+\frac{\partial Y}{\partial\lambda}(5K^{4/5}+5L^{4/5}+20K^{3/5}M^{1/5}$$
$$+20L^{3/5}M^{1/5}+30K^{2/5}M^{2/5}+30L^{2/5}M^{2/5}+20K^{1/5}M^{3/5}+20L^{1/5}M^{3/5}$$
$$+5M^{4/5}+20K^{2/5}\rho\mathrm{kl}+20L^{2/5}\rho\mathrm{kl}+60K^{1/5}M^{1/5}\rho\mathrm{kl}$$
$$+60L^{1/5}M^{1/5}\rho\mathrm{kl}+60M^{2/5}\rho\mathrm{kl}+20\rho\mathrm{kl}^2)=0. \qquad (2.1.18)$$

If we put $K^{2/5}L^{1/5}$-> $K^{1/5}\rho$kl, $K^{1/5}L^{2/5}$->$L^{1/5}$ ρkl in the above (2.1.18), we attain the following equation:

$$-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}-10L^{3/5}M^{2/5}$$
$$-10K^{2/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M$$
$$+6\frac{\partial^5 Y}{\partial\lambda^5}-5K^{2/5}M^{1/5}\rho\mathrm{kl}+30K^{1/5}L^{1/5}M^{1/5}\rho\mathrm{kl}-5L^{2/5}M^{1/5}$$
$$\rho\mathrm{kl}-15K^{1/5}M^{2/5}\rho\mathrm{kl}-15L^{1/5}M^{2/5}\rho\mathrm{kl}-15M^{3/5}\rho\mathrm{kl}$$
$$+15K^{1/5}\rho\mathrm{kl}^2+15L^{1/5}\rho\mathrm{kl}^2-20M^{1/5}\rho\mathrm{kl}^2+\frac{\partial^3 Y}{\partial\lambda^3}(-20K^{1/5}L^{1/5}+15\rho\mathrm{kl})$$

$$+\frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5}-10L^{3/5}-30K^{2/5}M^{1/5}-30L^{2/5}M^{1/5}-30K^{1/5}M^{2/5}$$
$$-30L^{1/5}M^{2/5}-10M^{3/5}-30K^{1/5}\rho\mathrm{kl}-30L^{1/5}\rho\mathrm{kl}-60M^{1/5}\rho\mathrm{kl})+\frac{\partial Y}{\partial\lambda}(5K^{4/5}$$
$$+5L^{4/5}+20K^{3/5}M^{1/5}+20L^{3/5}M^{1/5}+30K^{2/5}M^{2/5}+30L^{2/5}M^{2/5}$$
$$+20K^{1/5}M^{3/5}+20L^{1/5}M^{3/5}+5M^{4/5}+20K^{2/5}\rho\mathrm{kl}+20L^{2/5}\rho\mathrm{kl}$$
$$+60K^{1/5}M^{1/5}\rho\mathrm{kl}+60L^{1/5}M^{1/5}\rho\mathrm{kl}+60M^{2/5}\rho\mathrm{kl}+20\rho\mathrm{kl}^2)=0. \quad (2.1.19)$$

If we replace $10K^{2/5}$ with $10y^2-10(K^{1/n}+L^{1/n}+M^{1/n})^2-K^{2/n}$ in the equation (2.1.19) and $n=5$, we end with the following equation:

$$-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}-10L^{3/5}M^{2/5}$$
$$-10K^{2/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M+6\frac{\partial^5 Y}{\partial\lambda^5}$$
$$-5K^{2/5}M^{1/5}\rho\mathrm{kl}+30K^{1/5}L^{1/5}M^{1/5}\rho\mathrm{kl}$$
$$-5L^{2/5}M^{1/5}\rho\mathrm{kl}-15K^{1/5}M^{2/5}\rho\mathrm{kl}-15L^{1/5}M^{2/5}\rho\mathrm{kl}$$
$$-15M^{3/5}\rho\mathrm{kl}-35M^{1/5}\rho\mathrm{kl}^2+15\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2$$
$$+\frac{\partial^3 Y}{\partial\lambda^3}(-20K^{1/5}L^{1/5}+15\rho\mathrm{kl})$$
$$+\frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5}-10L^{3/5}-30K^{2/5}M^{1/5}-30L^{2/5}M^{1/5}$$
$$-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}-10M^{3/5}-30K^{1/5}\rho\mathrm{kl}-30L^{1/5}\rho\mathrm{kl}$$
$$-60M^{1/5}\rho\mathrm{kl})+\frac{\partial Y}{\partial\lambda}(5K^{4/5}+5L^{4/5}+20K^{3/5}M^{1/5}$$
$$+20L^{3/5}M^{1/5}+30K^{2/5}M^{2/5}+30L^{2/5}M^{2/5}+20K^{1/5}M^{3/5}+20L^{1/5}M^{3/5}$$
$$+5M^{4/5}+20K^{2/5}\rho\mathrm{kl}+20L^{2/5}\rho\mathrm{kl}+60K^{1/5}M^{1/5}\rho\mathrm{kl}$$
$$+60L^{1/5}M^{1/5}\rho\mathrm{kl}+60M^{2/5}\rho\mathrm{kl}+20\rho\mathrm{kl}^2)=0. \quad (2.1.20)$$

If we place $15K^{1/5}\rho\mathrm{kl}^2+15L^{1/5}\rho\mathrm{kl}^2$->$15y\rho\mathrm{kl}^2-15M^{1/5}\rho\mathrm{kl}^2$ in the above equation (2.1.20) and $n=5$, we obtain the equation:

$$-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}-10L^{3/5}M^{2/5}$$
$$-10K^{2/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M+6\frac{\partial^5 Y}{\partial\lambda^5}$$
$$-5K^{2/5}M^{1/5}\rho\text{kl}-5L^{2/5}M^{1/5}\rho\text{kl}-15K^{1/5}M^{2/5}\rho\text{kl}$$
$$-15L^{1/5}M^{2/5}\rho\text{kl}-15M^{3/5}\rho\text{kl}-5\frac{\partial^3 Y}{\partial\lambda^3}\rho\text{kl}$$
$$-5M^{1/5}\rho\text{kl}^2+15\frac{\partial Y}{\partial\lambda}\rho\text{kl}^2$$
$$+\frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5}-10L^{3/5}-30K^{2/5}M^{1/5}-30L^{2/5}M^{1/5}$$
$$-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}-10M^{3/5}-30K^{1/5}\rho\text{kl}-30L^{1/5}\rho\text{kl}$$
$$-60M^{1/5}\rho\text{kl})+\frac{\partial Y}{\partial\lambda}(5K^{4/5}+5L^{4/5}+20K^{3/5}M^{1/5}$$
$$+20L^{3/5}M^{1/5}+30K^{2/5}M^{2/5}+30L^{2/5}M^{2/5}+20K^{1/5}M^{3/5}+20L^{1/5}M^{3/5}$$
$$+5M^{4/5}+20K^{2/5}\rho\text{kl}+20L^{2/5}\rho\text{kl}+60K^{1/5}M^{1/5}\rho\text{kl}$$
$$+60L^{1/5}M^{1/5}\rho\text{kl}+60M^{2/5}\rho\text{kl}+20\rho\text{kl}^2)=0. \qquad (2.1.21)$$

If we set $K^{1/5}L^{1/5}$->ρkl in the equation (2.1.21), we achieve the following equation:

$$-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}-10K^{3/5}M^{2/5}-10L^{3/5}M^{2/5}$$
$$-10K^{2/5}M^{3/5}-10L^{2/5}M^{3/5}-5K^{1/5}M^{4/5}-5L^{1/5}M^{4/5}-M+6\frac{\partial^5 Y}{\partial\lambda^5}$$
$$-5K^{2/5}M^{1/5}\rho\text{kl}-5L^{2/5}M^{1/5}\rho\text{kl}-15K^{1/5}M^{2/5}\rho\text{kl}-15L^{1/5}M^{2/5}\rho\text{kl}$$
$$-15M^{3/5}\rho\text{kl}-35\frac{\partial^3 Y}{\partial\lambda^3}\rho\text{kl}-5M^{1/5}\rho\text{kl}^2+\frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5}-10L^{3/5}$$
$$-30K^{2/5}M^{1/5}-30L^{2/5}M^{1/5}-30K^{1/5}M^{2/5}-30L^{1/5}M^{2/5}-10M^{3/5}$$
$$-30M^{1/5}\rho\text{kl})+\frac{\partial Y}{\partial\lambda}(5K^{4/5}+5L^{4/5}+20K^{3/5}M^{1/5}+20L^{3/5}M^{1/5}$$
$$+30K^{2/5}M^{2/5}+30L^{2/5}M^{2/5}+20K^{1/5}M^{3/5}+20L^{1/5}M^{3/5}+5M^{4/5}$$
$$+20K^{2/5}\rho\text{kl}+20L^{2/5}\rho\text{kl}+60K^{1/5}M^{1/5}\rho\text{kl}+60L^{1/5}M^{1/5}\rho\text{kl}$$
$$+60M^{2/5}\rho\text{kl}+35\rho\text{kl}^2)=0. \qquad (2.1.22)$$

If we put $-30K^{1/5}\rho$ kl $-30L^{1/5}\rho$kl->$-30y\rho\text{kl}+30M^{1/5}\rho\text{kl}$ in the equation (2.1.22) , we end up with the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 10L^{3/5}M^{2/5}\\
&-10K^{2/5}M^{3/5} - 10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M + 11\frac{\partial^5 Y}{\partial\lambda^5}\\
&-5K^{2/5}M^{1/5}\rho\mathrm{kl} - 5L^{2/5}M^{1/5}\rho\mathrm{kl} - 15K^{1/5}M^{2/5}\rho\mathrm{kl} - 15L^{1/5}M^{2/5}\rho\mathrm{kl}\\
&-15M^{3/5}\rho\mathrm{kl} - 35\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5} - 10L^{3/5}\\
&-30K^{2/5}M^{1/5} - 30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} - 10M^{3/5}\\
&-30M^{1/5}\rho\mathrm{kl}) + \frac{\partial Y}{\partial\lambda}(-20K^{3/5}L^{1/5} - 30K^{2/5}L^{2/5} - 20K^{1/5}L^{3/5}\\
&-60K^{2/5}L^{1/5}M^{1/5} - 60K^{1/5}L^{2/5}M^{1/5} - 60K^{1/5}L^{1/5}M^{2/5} + 20K^{2/5}\rho\mathrm{kl}\\
&+20L^{2/5}\rho\mathrm{kl} + 60K^{1/5}M^{1/5}\rho\mathrm{kl} + 60L^{1/5}M^{1/5}\rho\mathrm{kl} + 60M^{2/5}\rho\mathrm{kl}\\
&+35\rho\mathrm{kl}^2) = 0. \qquad (2.1.23)
\end{aligned}
$$

If we substitute $5K^{4/5}$ with $5y^4 - 5(K^{1/n} + L^{1/n} + M^{1/n})^4 - K^{4/n}]$ in the above equation (2.1.23) and $n = 5$, we achieve the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} - 10K^{3/5}M^{2/5} - 10L^{3/5}M^{2/5}\\
&-10K^{2/5}M^{3/5} - 10L^{2/5}M^{3/5} - 5K^{1/5}M^{4/5} - 5L^{1/5}M^{4/5} - M + 11\frac{\partial^5 Y}{\partial\lambda^5}\\
&-5K^{2/5}M^{1/5}\rho\mathrm{kl} - 5L^{2/5}M^{1/5}\rho\mathrm{kl} - 15K^{1/5}M^{2/5}\rho\mathrm{kl}\\
&-15L^{1/5}M^{2/5}\rho\mathrm{kl} - 15M^{3/5}\rho\mathrm{kl} - 35\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{kl}\\
&-5M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial^2 Y}{\partial\lambda^2}(-10K^{3/5} - 10L^{3/5}\\
&-30K^{2/5}M^{1/5} - 30L^{2/5}M^{1/5} - 30K^{1/5}M^{2/5} - 30L^{1/5}M^{2/5} - 10M^{3/5}\\
&-30M^{1/5}\rho\mathrm{kl}) + \frac{\partial Y}{\partial\lambda}(-60K^{2/5}L^{1/5}M^{1/5}\\
&-60K^{1/5}L^{2/5}M^{1/5} - 60K^{1/5}L^{1/5}M^{2/5} + 60K^{1/5}M^{1/5}\rho\mathrm{kl}\\
&+60L^{1/5}M^{1/5}\rho\mathrm{kl} + 60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) = 0. \qquad (2.1.24)
\end{aligned}
$$

If we replace $K^{3/5}L^{1/5}$->$K^{2/5}\rho\mathrm{kl}$, $K^{2/5}L^{2/5}$->$\rho\mathrm{kl}^2$ and $K^{1/5}L^{3/5}$->$L^{2/5}\rho\mathrm{kl}$ in the equation (2.1.24), we end up with the following:

$$-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}+30K^{2/5}L^{1/5}M^{2/5}+30K^{1/5}L^{2/5}M^{2/5}$$
$$+20K^{2/5}M^{3/5}+60K^{1/5}L^{1/5}M^{3/5}+20L^{2/5}M^{3/5}+25K^{1/5}M^{4/5}+25L^{1/5}M^{4/5}$$
$$+9M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})$$
$$-5K^{2/5}M^{1/5}\rho\mathrm{kl}-5L^{2/5}M^{1/5}\rho\mathrm{kl}$$
$$-15K^{1/5}M^{2/5}\rho\mathrm{kl}-15L^{1/5}M^{2/5}\rho\mathrm{kl}-15M^{3/5}\rho\mathrm{kl}$$
$$-5M^{1/5}\rho\mathrm{kl}^2+\frac{\partial^2 Y}{\partial\lambda^2}(30K^{2/5}L^{1/5}+30K^{1/5}L^{2/5}$$
$$+60K^{1/5}L^{1/5}M^{1/5}-30M^{1/5}\rho\mathrm{kl})$$
$$+\frac{\partial Y}{\partial\lambda}(-60K^{2/5}L^{1/5}M^{1/5}-60K^{1/5}L^{2/5}M^{1/5}$$
$$-60K^{1/5}L^{1/5}M^{2/5}+60K^{1/5}M^{1/5}\rho\mathrm{kl}+60L^{1/5}M^{1/5}\rho\mathrm{kl}$$
$$+60M^{2/5}\rho\mathrm{kl}+5\rho\mathrm{kl}^2)=0. \tag{2.1.25}$$

If we replace $K^{3/5}$ with the command $y^3-(K^{1/n}+L^{1/n}+M^{1/n})^3-K^{3/n}]$ in the above equation (2.1.25) and $n=5$, we achieve the following equation:

$$-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+60K^{1/5}L^{1/5}M^{3/5}$$
$$+20L^{2/5}M^{3/5}+25K^{1/5}M^{4/5}+25L^{1/5}M^{4/5}+9M+\frac{\partial^5 Y}{\partial\lambda^5}$$
$$+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})-5K^{2/5}M^{1/5}\rho\mathrm{kl}$$
$$-5L^{2/5}M^{1/5}\rho\mathrm{kl}+15K^{1/5}M^{2/5}\rho\mathrm{kl}$$
$$+15L^{1/5}M^{2/5}\rho\mathrm{kl}-15M^{3/5}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2$$
$$+\frac{\partial^2 Y}{\partial\lambda^2}(60K^{1/5}L^{1/5}M^{1/5}+30K^{1/5}\rho\mathrm{kl}$$
$$+30L^{1/5}\rho\mathrm{kl}-30M^{1/5}\rho\mathrm{kl})$$
$$+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}+60M^{2/5}\rho\mathrm{kl}$$
$$+5\rho\mathrm{kl}^2)=0. \tag{2.1.26}$$

If we put $K^{2/5}L^{1/5}$->$K^{1/5}\rho$kl and $K^{1/5}L^{2/5}$->$L^{1/5}\rho$kl in the above equation (2.1.26), we end up with the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+60K^{1/5}L^{1/5}M^{3/5}\\
&+20L^{2/5}M^{3/5}+25K^{1/5}M^{4/5}+25L^{1/5}M^{4/5}+9M+\frac{\partial^5 Y}{\partial\lambda^5}\\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})\\
&-5K^{2/5}M^{1/5}\rho\mathrm{kl}-5L^{2/5}M^{1/5}\rho\mathrm{kl}+15K^{1/5}M^{2/5}\rho\mathrm{kl}\\
&+15L^{1/5}M^{2/5}\rho\mathrm{kl}-15M^{3/5}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2\\
&+\frac{\partial^2 Y}{\partial\lambda^2}(60K^{1/5}L^{1/5}M^{1/5}-60M^{1/5}\rho\mathrm{kl}\\
&+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl})+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}\\
&+60M^{2/5}\rho\mathrm{kl}+5\rho\mathrm{kl}^2)=0. \qquad (2.1.27)
\end{aligned}
$$

If we put $30K^{1/5}\rho\mathrm{kl}+30L^{1/5}\rho\mathrm{kl}\text{-}>30y\rho\mathrm{kl}-30M^{1/5}\rho\mathrm{kl}$ in the equation (2.1.27), we achieve the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+60K^{1/5}L^{1/5}M^{3/5}\\
&+20L^{2/5}M^{3/5}+25K^{1/5}M^{4/5}+25L^{1/5}M^{4/5}+9M+\frac{\partial^5 Y}{\partial\lambda^5}\\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})\\
&-5K^{2/5}M^{1/5}\rho\mathrm{kl}-5L^{2/5}M^{1/5}\rho\mathrm{kl}\\
&+15K^{1/5}M^{2/5}\rho\mathrm{kl}+15L^{1/5}M^{2/5}\rho\mathrm{kl}\\
&-15M^{3/5}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2\\
&+\frac{\partial^2 Y}{\partial\lambda^2}(60K^{1/5}L^{1/5}M^{1/5}-60M^{1/5}\rho\mathrm{kl}\\
&+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl})+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}\\
&+60M^{2/5}\rho\mathrm{kl}+5\rho\mathrm{kl}^2)=0. \qquad (2.1.28)
\end{aligned}
$$

If we place $y\rightarrow K\hat{}(1/5)+L\hat{}(1/5)+M\hat{}(1/5)$, $\rho\mathrm{kl}\rightarrow(KL)\hat{}(1/5)$, $\rho\mathrm{km}\rightarrow(KM)\hat{}(1/5)$, $\rho\mathrm{lm}\rightarrow(LM)\hat{}(1/5)$ and $\rho\rightarrow(KLM)\hat{}(1/5)$ in the equation (2.1.28), we end up with the following equation:

$$-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5}$$
$$+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5}$$
$$+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl}$$
$$-5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl}$$
$$+15L^{1/5}M^{2/5}\rho\mathrm{kl} - 15M^{3/5}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2$$
$$+\frac{\partial^2 Y}{\partial\lambda^2}(60K^{1/5}L^{1/5}M^{1/5} - 60M^{1/5}\rho\mathrm{kl}$$
$$+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}) + \frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}$$
$$+60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) = 0. \tag{2.1.29}$$

If we set $(-nK^{1/n} - nL^{1/n} - nM^{1/n}) \to -ny$ in the equation (2.1.29) and $n = 5$, we obtain with the following equation:

$$-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5}$$
$$+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5}$$
$$+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl}$$
$$-5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl}$$
$$-15M^{3/5}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial^2 Y}{\partial\lambda^2}(60K^{1/5}L^{1/5}M^{1/5}$$
$$-60M^{1/5}\rho\mathrm{kl} + 30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl})$$
$$+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5} + 60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) = 0. \tag{2.1.30}$$

If we substitute $K^{4/n}L^{1/n} \to K^{3/n}\ \rho\mathrm{kl}$ in the equation (2.1.30) and $n = 5$, we obtain the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} \\
&+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} \\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl} \\
&-5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} \\
&+15L^{1/5}M^{2/5}\rho\mathrm{kl} - 15M^{3/5}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 \\
&+\frac{\partial^2 Y}{\partial\lambda^2}(60K^{1/5}L^{1/5}M^{1/5} - 60M^{1/5}\rho\mathrm{kl} \\
&+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}) + \frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5} + \\
&60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) = 0. \qquad (2.1.31)
\end{aligned}
$$

If we replace $K^{3/n}L^{2/n} \to K^{1/n}\rho\mathrm{kl}\verb|^|2$ in the equation (2.1.31) and $n = 5$, we end up with the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} \\
&+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} \\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) \\
&-5K^{2/5}M^{1/5}\rho\mathrm{kl} - 5L^{2/5}M^{1/5}\rho\mathrm{kl} \\
&+15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl} - 15M^{3/5}\rho\mathrm{kl} \\
&-5M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial^2 Y}{\partial\lambda^2}(60K^{1/5}L^{1/5}M^{1/5} \\
&-60M^{1/5}\rho\mathrm{kl} + 30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}) \\
&+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5} \\
&+60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) = 0. \qquad (2.1.32)
\end{aligned}
$$

If we put $K^{2/n}L^{3/n} \to L^{1/n}\rho\mathrm{kl}\verb|^|2$ in the equation (2.1.32) and $n = 5$, we obtain with the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} \\
&+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl} \\
&-5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl} \\
&-15M^{3/5}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial^2 Y}{\partial \lambda^2}(60K^{1/5}L^{1/5}M^{1/5} \\
&-60M^{1/5}\rho\mathrm{kl} + 30\frac{\partial Y}{\partial \lambda}\rho\mathrm{kl}) \\
&+\frac{\partial Y}{\partial \lambda}(-60K^{1/5}L^{1/5}M^{2/5} + 60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) = 0. \qquad (2.1.33)
\end{aligned}
$$

If we set $K^{1/n}L^{4/n} \to L^{3/n}\rho\mathrm{kl}$ in the equation (2.1.33) and $n = 5$, we remain with the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} \\
&+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl} \\
&-5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl} \\
&-15M^{3/5}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial^2 Y}{\partial \lambda^2}(60K^{1/5}L^{1/5}M^{1/5} \\
&-60M^{1/5}\rho\mathrm{kl} + 30\frac{\partial Y}{\partial \lambda}\rho\mathrm{kl}) + \frac{\partial Y}{\partial \lambda}(-60K^{1/5}L^{1/5} \\
&M^{2/5} + 60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) = 0. \qquad (2.1.34)
\end{aligned}
$$

If we place $K^{3/n}L^{1/n} \to K^{2/n}\rho\mathrm{kl}$ in the equation (2.1.34) and $n = 5$, we end up with following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} \\
&+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} \\
&-5L^{2/5}M^{1/5}\rho\text{kl} + 15K^{1/5}M^{2/5}\rho\text{kl} + 15L^{1/5}M^{2/5}\rho\text{kl} \\
&-15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 \\
&+\frac{\partial^2 Y}{\partial \lambda^2}(60K^{1/5}L^{1/5}M^{1/5} - 60M^{1/5}\rho\text{kl} \\
&+30\frac{\partial Y}{\partial \lambda}\rho\text{kl}) + \frac{\partial Y}{\partial \lambda}(-60K^{1/5}L^{1/5}M^{2/5} \\
&+60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2) = 0. \qquad (2.1.35)
\end{aligned}
$$

If we set $K^{2/n}L^{2/n} \rightarrow \rho\text{kl}^2$ in the equation (2.1.35) and $n = 5$, we achieve the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} \\
&+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} \\
&-5L^{2/5}M^{1/5}\rho\text{kl} + 15K^{1/5}M^{2/5}\rho\text{kl} + 15L^{1/5}M^{2/5}\rho\text{kl} \\
&-15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 \\
&+\frac{\partial^2 Y}{\partial \lambda^2}(60K^{1/5}L^{1/5}M^{1/5} - 60M^{1/5}\rho\text{kl} \\
&+30\frac{\partial Y}{\partial \lambda}\rho\text{kl}) + \frac{\partial Y}{\partial \lambda}(-60K^{1/5}L^{1/5}M^{2/5} \\
&+60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2) = 0. \qquad (2.1.36)
\end{aligned}
$$

If we replace $K^{1/n}L^{3/n} \rightarrow L^{2/n}\rho\text{kl}$ in the equation (2.1.36) and $n = 5$, we attain the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+60K^{1/5}L^{1/5}M^{3/5}\\
&+20L^{2/5}M^{3/5}+25K^{1/5}M^{4/5}+25L^{1/5}M^{4/5}+9M+\frac{\partial^5 Y}{\partial\lambda^5}\\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})-5K^{2/5}M^{1/5}\rho\mathrm{kl}\\
&-5L^{2/5}M^{1/5}\rho\mathrm{kl}+15K^{1/5}M^{2/5}\rho\mathrm{kl}+15L^{1/5}M^{2/5}\rho\mathrm{kl}\\
&-15M^{3/5}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2\\
&+\frac{\partial^2 Y}{\partial\lambda^2}(60K^{1/5}L^{1/5}M^{1/5}-60M^{1/5}\rho\mathrm{kl}\\
&+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl})\\
&+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}+60M^{2/5}\rho\mathrm{kl}+5\rho\mathrm{kl}^2)=0. \qquad (2.1.37)
\end{aligned}
$$

If we substitute $K^{2/n}L^{1/n} \to K^{1/n}\rho\mathrm{kl}$ into the equation (2.1.37) and $n = 5$, we obtain the following g equation

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+20L^{2/5}M^{3/5}+25K^{1/5}M^{4/5}\\
&+25L^{1/5}M^{4/5}+9M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}\\
&-35\rho\mathrm{kl})-5K^{2/5}M^{1/5}\rho\mathrm{kl}-5L^{2/5}M^{1/5}\rho\mathrm{kl}\\
&+15K^{1/5}M^{2/5}\rho\mathrm{kl}+15L^{1/5}M^{2/5}\rho\mathrm{kl}+45M^{3/5}\rho\mathrm{kl}\\
&+30\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2\\
&+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2=0. \qquad (2.1.38)
\end{aligned}
$$

If we place $K^{1/n}L^{2/n} \to L^{1/n}\rho\mathrm{kl}$ in the equation (2.1.38) and $n = 5$, we obtain the following equation:

$$-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5}$$
$$+25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl}$$
$$-5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl} + 45M^{3/5}\rho\mathrm{kl} + 30\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{kl}$$
$$-5M^{1/5}\rho\mathrm{kl}^2 + 5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2 = 0. \tag{2.1.39}$$

If we substitute $K^{1/n}L^{1/n} \to \rho\mathrm{kl}$ in the above equation (2.1.39), we end up with the following equation:

$$-K - L - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5}$$
$$+25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl}$$
$$-5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl} + 45M^{3/5}\rho\mathrm{kl} + 30\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{kl}$$
$$-5M^{1/5}\rho\mathrm{kl}^2 + 5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2 - 5K^{3/5}\rho\mathrm{km} = 0. \tag{2.1.40}$$

If we replace $y \to K\hat{}(1/5) + L\hat{}(1/5) + M\hat{}(1/5)$, $\rho\mathrm{kl} \to (KL)\hat{}(1/5)$ $\rho\mathrm{km} \to (KM)\hat{}(1/5)$, $\rho\mathrm{lm} \to (LM)\hat{}(1/5)$ and $\rho \to (KLM)^(1/5)$ in the equation (2.1.40), we obtain the following equation:

$$-K - L - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5}$$
$$+25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl})$$
$$-5K^{2/5}M^{1/5}\rho\mathrm{kl} - 5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl}$$
$$+45M^{3/5}\rho\mathrm{kl} + 30\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2$$
$$+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2 - 5K^{3/5}\rho\mathrm{km} = 0. \tag{2.1.41}$$

If we put $(-nK^{1/n} - nL^{1/n} - nM^{1/n}) \to -ny$ in the equation (2.1.41) and $n = 5$, we end up with the following equation:

$$
\begin{aligned}
&-K - L - 5L^{4/5}M^{1/5} + 20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M \\
&+\frac{\partial^5 Y}{\partial \lambda^5} + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} - 5L^{2/5}M^{1/5}\rho\text{kl} \\
&+15K^{1/5}M^{2/5}\rho\text{kl} + 15L^{1/5}M^{2/5}\rho\text{kl} + 45M^{3/5}\rho\text{kl} \\
&+30\frac{\partial^3 Y}{\partial \lambda^3}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 + 5\frac{\partial Y}{\partial \lambda}\rho\text{kl}^2 \\
&-5K^{3/5}\rho\text{km} + 20M^{1/5}\rho\text{km}^2 = 0. \qquad (2.1.42)
\end{aligned}
$$

Substituting $K^{4/n}M^{1/n} \to K^{3/n}$ ρkm in the equation (2.1.42) and $n = 5$, we obtain the following equation:

$$
\begin{aligned}
&-K - L - 5L^{4/5}M^{1/5} + 20L^{2/5}M^{3/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} - 5L^{2/5}M^{1/5}\rho\text{kl} \\
&+15K^{1/5}M^{2/5}\rho\text{kl} + 15L^{1/5}M^{2/5}\rho\text{kl} + 45M^{3/5}\rho\text{kl} + 30\frac{\partial^3 Y}{\partial \lambda^3}\rho\text{kl} \\
&-5M^{1/5}\rho\text{kl}^2 + 5\frac{\partial Y}{\partial \lambda}\rho\text{kl}^2 - 5K^{3/5}\rho\text{km} \\
&+25M^{3/5}\rho\text{km} + 20M^{1/5}\rho\text{km}^2 = 0. \qquad (2.1.43)
\end{aligned}
$$

If we put $K^{3/n}M^{2/n} \to K^{1/n}\rho\text{km\^{}2}$ in the equation (2.1.43) and $n = 5$, we reach the following equation:

$$
\begin{aligned}
&-K - L - 5L^{4/5}M^{1/5} + 20L^{2/5}M^{3/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} \\
&-5L^{2/5}M^{1/5}\rho\text{kl} + 15K^{1/5}M^{2/5}\rho\text{kl} + 15L^{1/5}M^{2/5}\rho\text{kl} \\
&+45M^{3/5}\rho\text{kl} + 30\frac{\partial^3 Y}{\partial \lambda^3}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 \\
&+5\frac{\partial Y}{\partial \lambda}\rho\text{kl}^2 - 5K^{3/5}\rho\text{km} + 25M^{3/5}\rho\text{km} + 20M^{1/5}\rho\text{km}^2 = 0. \qquad (2.1.44)
\end{aligned}
$$

If we place $K^{2/n}M^{3/n} \to M^{1/n}\rho\text{km\^{}2}$ in the equation (2.1.44) and $n = 5$, we achieve the following equation:

$$
\begin{aligned}
&-K - L - 5L^{4/5}M^{1/5} + 20L^{2/5}M^{3/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl} \\
&-5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} \\
&+15L^{1/5}M^{2/5}\rho\mathrm{kl} + 45M^{3/5}\rho\mathrm{kl} \\
&+30\frac{\partial^3 Y}{\partial \lambda^3}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 \\
&+5\frac{\partial Y}{\partial \lambda}\rho\mathrm{kl}^2 - 5K^{3/5}\rho\mathrm{km} \\
&+25M^{3/5}\rho\mathrm{km} + 20M^{1/5}\rho\mathrm{km}^2 = 0. \qquad (2.1.45)
\end{aligned}
$$

If we put $K^{1/n}M^{4/n} \to M^{3/n}\rho\mathrm{km}$ in the equation (2.1.45) and $n = 5$, we obtain the following equation:

$$
\begin{aligned}
&-K - L - 5L^{4/5}M^{1/5} + 20L^{2/5}M^{3/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) \\
&-5K^{2/5}M^{1/5}\rho\mathrm{kl} - 5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} \\
&+15L^{1/5}M^{2/5}\rho\mathrm{kl} + 45M^{3/5}\rho\mathrm{kl} \\
&+30\frac{\partial^3 Y}{\partial \lambda^3}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 \\
&+5\frac{\partial Y}{\partial \lambda}\rho\mathrm{kl}^2 - 5K^{3/5}\rho\mathrm{km} \\
&+25M^{3/5}\rho\mathrm{km} + 20M^{1/5}\rho\mathrm{km}^2 = 0. \qquad (2.1.46)
\end{aligned}
$$

If we place $K^{3/n}M^{1/n} \to K^{2/n}\rho\mathrm{km}$ in the equation (2.1.46) and $n = 5$, we get the following equation:

$$
\begin{aligned}
&-K - L - 5L^{4/5}M^{1/5} + 20L^{2/5}M^{3/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5L^{2/5}M^{1/5}\rho\mathrm{kl} \\
&+15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl} + 45M^{3/5}\rho\mathrm{kl} \\
&+30\frac{\partial^3 Y}{\partial \lambda^3}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 \\
&+5\frac{\partial Y}{\partial \lambda}\rho\mathrm{kl}^2 - 5K^{3/5}\rho\mathrm{km} + 25M^{3/5}\rho\mathrm{km} \\
&-5K^{1/5}\rho\mathrm{kl}\rho\mathrm{km} + 20M^{1/5}\rho\mathrm{km}^2 = 0. \qquad (2.1.47)
\end{aligned}
$$

If we replace $K^{2/n}M^{2/n} \to \rho\mathrm{km}\hat{}2$ in the equation (2.1.47) and $n = 5$, we attain the following equation:

$$
\begin{aligned}
&-K - L - 5L^{4/5}M^{1/5} + 20L^{2/5}M^{3/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) - 5L^{2/5}M^{1/5}\rho\mathrm{kl} \\
&+15L^{1/5}M^{2/5}\rho\mathrm{kl} + 45M^{3/5}\rho\mathrm{kl} \\
&+30\frac{\partial^3 Y}{\partial \lambda^3}\rho\mathrm{kl} \\
&-5M^{1/5}\rho\mathrm{kl}^2 + 5\frac{\partial Y}{\partial \lambda}\rho\mathrm{kl}^2 - 5K^{3/5}\rho\mathrm{km} \\
&+25M^{3/5}\rho\mathrm{km} - 5K^{1/5}\rho\mathrm{kl}\rho\mathrm{km} \\
&+15M^{1/5}\rho\mathrm{kl}\rho\mathrm{km} + 20M^{1/5}\rho\mathrm{km}^2 = 0. \qquad (2.1.48)
\end{aligned}
$$

If we put $K^{1/n}M^{3/n} \to M^{2/n}\rho\mathrm{km}$ in the equation (2.1.48) and $n = 5$, we achieve the following equation:

$$
\begin{aligned}
&-K-L-5L^{4/5}M^{1/5}+20L^{2/5}M^{3/5}+25L^{1/5}M^{4/5}+9M\\
&+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})-5L^{2/5}M^{1/5}\rho\mathrm{kl}+15L^{1/5}M^{2/5}\rho\mathrm{kl}\\
&+45M^{3/5}\rho\mathrm{kl}+30\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2-5K^{3/5}\rho\mathrm{km}\\
&+25M^{3/5}\rho\mathrm{km}-5K^{1/5}\rho\mathrm{kl}\rho\mathrm{km}+15M^{1/5}\rho\mathrm{kl}\rho\mathrm{km}\\
&+20M^{1/5}\rho\mathrm{km}^2=0. \qquad (2.1.49)
\end{aligned}
$$

If we substitute $K^{2/n}M^{1/n}\rightarrow K^{1/n}\rho\mathrm{km}$ into the equation (2.1.49) and $n=5$, we get the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}-5L^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+60K^{1/5}L^{1/5}M^{3/5}\\
&+20L^{2/5}M^{3/5}+25K^{1/5}M^{4/5}+25L^{1/5}M^{4/5}+9M+\frac{\partial^5 Y}{\partial\lambda^5}\\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})-5K^{2/5}M^{1/5}\rho\mathrm{kl}\\
&-5L^{2/5}M^{1/5}\rho\mathrm{kl}+15K^{1/5}M^{2/5}\rho\mathrm{kl}+15L^{1/5}M^{2/5}\rho\mathrm{kl}\\
&-15M^{3/5}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}\\
&+60M^{2/5}\rho\mathrm{kl}+5\rho\mathrm{kl}^2)+\frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\mathrm{kl}\\
&+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}+60L^{1/5}\rho\mathrm{km})=0. \qquad (2.1.50)
\end{aligned}
$$

If we place $K^{1/n}M^{2/n}\rightarrow M^{1/n}\rho\mathrm{km}$ in the equation (2.1.50) and $n=5$, we get the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} - 5L^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} \\
&+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} \\
&-35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl} - 5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} \\
&+15L^{1/5}M^{2/5}\rho\mathrm{kl} - 15M^{3/5}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5} \\
&+60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) + \frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\mathrm{kl} + 30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl} \\
&+60L^{1/5}\rho\mathrm{km}) = 0. \qquad (2.1.51)
\end{aligned}
$$

If we place $K^{1/n}M^{1/n} \rightarrow \rho\mathrm{km}$ in the equation (2.1.51) and $n = 5$, we achieve the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} \\
&+20L^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} \\
&-35\rho\mathrm{kl}) - 5K^{2/5}M^{1/5}\rho\mathrm{kl} - 5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} \\
&+15L^{1/5}M^{2/5}\rho\mathrm{kl} - 15M^{3/5}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 \\
&+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5} + 60M^{2/5}\rho\mathrm{kl} + 5\rho\mathrm{kl}^2) + \frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\mathrm{kl} \\
&+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl} + 60L^{1/5}\rho\mathrm{km}) - 5L^{3/5}\rho\mathrm{lm} = 0. \qquad (2.1.52)
\end{aligned}
$$

If we put $K^{1/n}M^{1/n} \rightarrow \rho\mathrm{km}$ into the equation (2.1.52) and $n = 5$, we remain with the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} + 20L^{2/5}M^{3/5} \\
&+25K^{1/5}M^{4/5} + 25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\mathrm{kl}) \\
&-5K^{2/5}M^{1/5}\rho\mathrm{kl} - 5L^{2/5}M^{1/5}\rho\mathrm{kl} + 15K^{1/5}M^{2/5}\rho\mathrm{kl} + 15L^{1/5}M^{2/5}\rho\mathrm{kl} \\
&-15M^{3/5}\rho\mathrm{kl} - 5M^{1/5}\rho\mathrm{kl}^2 + \frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5} + 60M^{2/5}\rho\mathrm{kl} \\
&+5\rho\mathrm{kl}^2) + \frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\mathrm{kl} + 30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl} + 60L^{1/5}\rho\mathrm{km}) \\
&-5L^{3/5}\rho\mathrm{lm} = 0. \qquad (2.1.53)
\end{aligned}
$$

From the equation (2.1.53) we exchange $-nK^{1/n} - nL^{1/n} - nM^{1/n}$ with $-ny$ and $n = 5$, we obtain:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} + 25K^{1/5}M^{4/5} \\
&+25L^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} \\
&-5L^{2/5}M^{1/5}\rho\text{kl} + 15K^{1/5}M^{2/5}\rho\text{kl} + 15L^{1/5}M^{2/5}\rho\text{kl} \\
&-15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 + \frac{\partial Y}{\partial \lambda}(-60K^{1/5}L^{1/5}M^{2/5} \\
&+60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2) + \frac{\partial^2 Y}{\partial \lambda^2}(-60M^{1/5}\rho\text{kl} + 30\frac{\partial Y}{\partial \lambda}\rho\text{kl} + 60L^{1/5}\rho\text{km}) \\
&-5L^{3/5}\rho\text{lm} + 20M^{1/5}\rho\text{lm}^2 = 0. \qquad (2.1.54)
\end{aligned}
$$

If we replace $L^{4/n}M^{1/n}$ with $L^{3/n}$ ρlm in the above equation (2.1.54) and $n = 5$, we attain the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 60K^{1/5}L^{1/5}M^{3/5} + 25K^{1/5}M^{4/5} \\
&+9M + \frac{\partial^5 Y}{\partial \lambda^5} + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} - 5L^{2/5}M^{1/5}\rho\text{kl} \\
&+15K^{1/5}M^{2/5}\rho\text{kl} + 15L^{1/5}M^{2/5}\rho\text{kl} - 15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 \\
&+\frac{\partial Y}{\partial \lambda}(-60K^{1/5}L^{1/5}M^{2/5} + 60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2) + \frac{\partial^2 Y}{\partial \lambda^2}(-60M^{1/5}\rho\text{kl} \\
&+30\frac{\partial Y}{\partial \lambda}\rho\text{kl} + 60L^{1/5}\rho\text{km}) - 5L^{3/5}\rho\text{lm} + 25M^{3/5}\rho\text{lm} \\
&+20M^{1/5}\rho\text{lm}^2 = 0. \qquad (2.1.55)
\end{aligned}
$$

If we put $L^{3/n}M^{2/n}$ for $L^{1/n}\rho\text{lm}$^2 in the equation (2.1.55), we reach the following:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+60K^{1/5}L^{1/5}M^{3/5}+25K^{1/5}M^{4/5}\\
&+9M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})-5K^{2/5}M^{1/5}\rho\mathrm{kl}-5L^{2/5}M^{1/5}\rho\mathrm{kl}\\
&+15K^{1/5}M^{2/5}\rho\mathrm{kl}+15L^{1/5}M^{2/5}\rho\mathrm{kl}-15M^{3/5}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2\\
&+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}+60M^{2/5}\rho\mathrm{kl}+5\rho\mathrm{kl}^2)+\frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\mathrm{kl}\\
&+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}+60L^{1/5}\rho\mathrm{km})-5L^{3/5}\rho\mathrm{lm}+25M^{3/5}\rho\mathrm{lm}\\
&+20M^{1/5}\rho\mathrm{lm}^2=0. \qquad (2.1.56)
\end{aligned}
$$

If we substitute $L^{2/n}M^{3/n}\to M^{1/n}$ with ρlm^2 in the equation (2.1.56), we remain with the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+60K^{1/5}L^{1/5}M^{3/5}+25K^{1/5}M^{4/5}\\
&+9M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})-5K^{2/5}M^{1/5}\rho\mathrm{kl}\\
&-5L^{2/5}M^{1/5}\rho\mathrm{kl}+15K^{1/5}M^{2/5}\rho\mathrm{kl}+15L^{1/5}M^{2/5}\rho\mathrm{kl}-15M^{3/5}\rho\mathrm{kl}\\
&-5M^{1/5}\rho\mathrm{kl}^2+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}+60M^{2/5}\rho\mathrm{kl}+5\rho\mathrm{kl}^2)\\
&+\frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\mathrm{kl}+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}+60L^{1/5}\rho\mathrm{km})-5L^{3/5}\rho\mathrm{lm}\\
&+25M^{3/5}\rho\mathrm{lm}+20M^{1/5}\rho\mathrm{lm}^2=0. \qquad (2.1.57)
\end{aligned}
$$

If we swap $L^{1/n}M^{4/n}$ with $M^{3/n}\rho$lm in the equation (2.1.57), we result with the following equation:

$$
\begin{aligned}
&-K-L-5K^{4/5}M^{1/5}+20K^{2/5}M^{3/5}+25K^{1/5}M^{4/5}+9M+\frac{\partial^5 Y}{\partial\lambda^5}\\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-35\rho\mathrm{kl})-5K^{2/5}M^{1/5}\rho\mathrm{kl}-5L^{2/5}M^{1/5}\rho\mathrm{kl}\\
&+15K^{1/5}M^{2/5}\rho\mathrm{kl}+15L^{1/5}M^{2/5}\rho\mathrm{kl}-15M^{3/5}\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2\\
&+\frac{\partial Y}{\partial\lambda}(-60K^{1/5}L^{1/5}M^{2/5}+60M^{2/5}\rho\mathrm{kl}+5\rho\mathrm{kl}^2)+\frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\mathrm{kl}\\
&+30\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}+60L^{1/5}\rho\mathrm{km})-5L^{3/5}\rho\mathrm{lm}+60K^{1/5}M^{2/5}\rho\mathrm{lm}\\
&+25M^{3/5}\rho\mathrm{lm}+20M^{1/5}\rho\mathrm{lm}^2=0. \qquad (2.1.58)
\end{aligned}
$$

If we replace $L^{3/n}M^{1/n}$ with $L^{2/n}\rho\text{lm}$ form the above equation (2.1.58), we get the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 9M \\
&+\frac{\partial^5 Y}{\partial \lambda^5} + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} + 15K^{1/5}M^{2/5}\rho\text{kl} \\
&+15L^{1/5}M^{2/5}\rho\text{kl} - 15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 + \frac{\partial Y}{\partial \lambda}(-60K^{1/5}L^{1/5}M^{2/5} \\
&+60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2) + \frac{\partial^2 Y}{\partial \lambda^2}(-60M^{1/5}\rho\text{kl} + 30\frac{\partial Y}{\partial \lambda}\rho\text{kl} + 60L^{1/5}\rho\text{km}) \\
&-5L^{3/5}\rho\text{lm} + 60K^{1/5}M^{2/5}\rho\text{lm} + 25M^{3/5}\rho\text{lm} - 5L^{1/5}\rho\text{kl}\rho\text{lm} \\
&+20M^{1/5}\rho\text{lm}^2 = 0. \qquad (2.1.59)
\end{aligned}
$$

If we exchange $L^{1/n}M^{3/n}$ for $M^{2/n}\rho\text{lm}$ into the equation (2.1.59), we attain the following equation:

$$
\begin{aligned}
&-K - L - 5K^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial \lambda^5} \\
&+\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} + 15K^{1/5}M^{2/5}\rho\text{kl} - 15M^{3/5}\rho\text{kl} \\
&-5M^{1/5}\rho\text{kl}^2 + \frac{\partial^2 Y}{\partial \lambda^2}(-60M^{1/5}\rho\text{kl} + 30\frac{\partial Y}{\partial \lambda}\rho\text{kl} + 60L^{1/5}\rho\text{km}) \\
&-5L^{3/5}\rho\text{lm} + 60K^{1/5}M^{2/5}\rho\text{lm} + 25M^{3/5}\rho\text{lm} - 5L^{1/5}\rho\text{kl}\rho\text{lm} \\
&+15M^{1/5}\rho\text{kl}\rho\text{lm} + 20M^{1/5}\rho\text{lm}^2 + \frac{\partial Y}{\partial \lambda}(60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2 \\
&-60K^{1/5}M^{1/5}\rho\text{lm}) = 0. \qquad (2.1.60)
\end{aligned}
$$

If we usurp $L^{1/n}M^{3/n}$ with $M^{2/n}\rho\text{lm}$ in the equation (2.1.60) , we achieve the following equation:

$$-K - L - 5K^{4/5}M^{1/5} + 20K^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}$$
$$(-10M^{2/5} - 35\rho\text{kl}) - 5K^{2/5}M^{1/5}\rho\text{kl} + 15K^{1/5}M^{2/5}\rho\text{kl} - 15M^{3/5}\rho\text{kl}$$
$$-5M^{1/5}\rho\text{kl}^2 + \frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\text{kl} + 30\frac{\partial Y}{\partial\lambda}\rho\text{kl} + 60L^{1/5}\rho\text{km}) - 5L^{3/5}\rho\text{lm}$$
$$+60K^{1/5}M^{2/5}\rho\text{lm} + 25M^{3/5}\rho\text{lm} - 5L^{1/5}\rho\text{kl}\rho\text{lm} + 15M^{1/5}\rho\text{kl}\rho\text{lm}$$
$$+20M^{1/5}\rho\text{lm}^2 + \frac{\partial Y}{\partial\lambda}(60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2 - 60K^{1/5}M^{1/5}\rho\text{lm}) = 0 \qquad (2.1.61)$$

If we set $L^{2/n}M^{1/n}$ as $L^{1/n}\rho\text{lm}$ into the equation (2.1.61), we get the following equation:

$$-K - L + 20K^{2/5}M^{3/5} + 25K^{1/5}M^{4/5} + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\text{kl})$$
$$-5K^{2/5}M^{1/5}\rho\text{kl} + 15K^{1/5}M^{2/5}\rho\text{kl} - 15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 - 5K^{3/5}\rho\text{km}$$
$$+\frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\text{kl} + 30\frac{\partial Y}{\partial\lambda}\rho\text{kl} + 60L^{1/5}\rho\text{km}) - 5L^{3/5}\rho\text{lm} + 60K^{1/5}M^{2/5}\rho\text{lm}$$
$$+25M^{3/5}\rho\text{lm} - 5L^{1/5}\rho\text{kl}\rho\text{lm} + 15M^{1/5}\rho\text{kl}\rho\text{lm} + 20M^{1/5}\rho\text{lm}^2 + \frac{\partial Y}{\partial\lambda}(60M^{2/5}\rho\text{kl}$$
$$+5\rho\text{kl}^2 - 60K^{1/5}M^{1/5}\rho\text{lm}) = 0. \qquad (2.1.62)$$

If we change $L^{1/n}M^{2/n} \to M^{1/n}\rho\text{lm}$ into the equation (2.1.62), the remaining equation is:

$$-K - L + 9M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2$$
$$-5K^{3/5}\rho\text{km} + 25M^{3/5}\rho\text{km} - 5K^{1/5}\rho\text{kl}\rho\text{km} + 15M^{1/5}\rho\text{kl}\rho\text{km} + 20M^{1/5}\rho\text{km}^2$$
$$+\frac{\partial^2 Y}{\partial\lambda^2}(-60M^{1/5}\rho\text{kl} + 30\frac{\partial Y}{\partial\lambda}\rho\text{kl} + 60L^{1/5}\rho\text{km}) - 5L^{3/5}\rho\text{lm} + 25M^{3/5}\rho\text{lm}$$
$$-5L^{1/5}\rho\text{kl}\rho\text{lm} + 15M^{1/5}\rho\text{kl}\rho\text{lm} + 60M^{1/5}\rho\text{km}\rho\text{lm} + 20M^{1/5}\rho\text{lm}^2$$
$$+\frac{\partial Y}{\partial\lambda}(60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2 - 60\rho\text{km}\rho\text{lm}) = 0. \qquad (2.1.63)$$

If we replace $L^{1/n}M^{1/n} \to \rho\text{lm}$ into the equation (2.1.63), the outcome is:

$$-K - L + 9M + \frac{\partial^5 Y}{\partial \lambda^5} + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 35\rho\text{kl}) - 15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2$$
$$-5K^{3/5}\rho\text{km} + (-25(K^{3/5} + 3K^{2/5}L^{1/5} + 3K^{1/5}L^{2/5} + L^{3/5} + 3K^{2/5}M^{1/5}$$
$$+6K^{1/5}L^{1/5}M^{1/5} + 3L^{2/5}M^{1/5} + 3K^{1/5}M^{2/5} + 3L^{1/5}M^{2/5}) + 25\frac{\partial^3 Y}{\partial \lambda^3})\rho\text{km}$$
$$-5K^{1/5}\rho\text{kl}\rho\text{km} + 15M^{1/5}\rho\text{kl}\rho\text{km} + 20M^{1/5}\rho\text{km}^2 + \frac{\partial^2 Y}{\partial \lambda^2}(-60M^{1/5}\rho\text{kl}$$
$$+30\frac{\partial Y}{\partial \lambda}\rho\text{kl} + 60L^{1/5}\rho\text{km}) - 5L^{3/5}\rho\text{lm} + (-25(K^{3/5} + 3K^{2/5}L^{1/5}$$
$$+3K^{1/5}L^{2/5} + L^{3/5} + 3K^{2/5}M^{1/5} + 6K^{1/5}L^{1/5}M^{1/5} + 3L^{2/5}M^{1/5}$$
$$+3K^{1/5}M^{2/5} + 3L^{1/5}M^{2/5}) + 25\frac{\partial^3 Y}{\partial \lambda^3})\rho\text{lm} - 5L^{1/5}\rho\text{kl}\rho\text{lm} + 15M^{1/5}\rho\text{kl}\rho\text{lm}$$
$$+60M^{1/5}\rho\text{km}\rho\text{lm} + 20M^{1/5}\rho\text{lm}^2 + \frac{\partial Y}{\partial \lambda}(60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2$$
$$-60\rho\text{km}\rho\text{lm}) = 0 \tag{2.1.64}$$

If we place $K^{4/n}M^{1/n} \to K^{3/n}$ ρkm into the equation (2.1.64), the remaining equation is:

$$-K - L + 9M + \frac{\partial^5 Y}{\partial \lambda^5} - 15M^{3/5}\rho\text{kl} - 5M^{1/5}\rho\text{kl}^2 - 5K^{3/5}\rho\text{km} - 25(K^{3/5}$$
$$+3K^{2/5}L^{1/5} + 3K^{1/5}L^{2/5} + L^{3/5} + 3K^{2/5}M^{1/5} + 6K^{1/5}L^{1/5}M^{1/5} + 3L^{2/5}M^{1/5}$$
$$+3K^{1/5}M^{2/5} + 3L^{1/5}M^{2/5})\rho\text{km} - 5K^{1/5}\rho\text{kl}\rho\text{km} + 15M^{1/5}\rho\text{kl}\rho\text{km}$$
$$+20M^{1/5}\rho\text{km}^2 + \frac{\partial^2 Y}{\partial \lambda^2}(-0M^{1/5}\rho\text{kl} + 60L^{1/5}\rho\text{km}) - 5L^{3/5}\rho\text{lm} - 25(K^{3/5}$$
$$+3K^{2/5}L^{1/5} + 3K^{1/5}L^{2/5} + L^{3/5} + 3K^{2/5}M^{1/5} + 6K^{1/5}L^{1/5}M^{1/5} + 3L^{2/5}M^{1/5}$$
$$+3K^{1/5}M^{2/5} + 3L^{1/5}M^{2/5})\rho\text{lm} - 5L^{1/5}\rho\text{kl}\rho\text{lm} + 15M^{1/5}\rho\text{kl}\rho\text{lm}$$
$$+60M^{1/5}\rho\text{km}\rho\text{lm} + 20M^{1/5}\rho\text{lm}^2 + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5}$$
$$-5\rho\text{kl} + 25\rho\text{km} + 25\rho\text{lm}) + \frac{\partial Y}{\partial \lambda}(60M^{2/5}\rho\text{kl} + 5\rho\text{kl}^2$$
$$-60\rho\text{km}\rho\text{lm}) = 0. \tag{2.1.65}$$

From the equation (2.1.65) we replace the $K^{2/5}M^{3/5} \to M^{1/5}$ ρkm^2, $K^{1/5}M^{4/5} \to M^{3/5}$ ρkm,

$K^{2/5}M^{1/5}$->$K^{1/5}\rho$ km ,$K^{1/5}M^{2/5}$->$M^{1/5}\rho$km, and $K^{1/5}M^{1/5}$->ρkm, we obtain the following equation:

$$-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}-15M^{3/5}\rho\text{kl}-5M^{1/5}\rho\text{kl}^2+5\frac{\partial Y}{\partial\lambda}\rho\text{kl}^2-5K^{3/5}\rho\text{km}$$
$$-25(K^{3/5}+3K^{2/5}L^{1/5}+3K^{1/5}L^{2/5}+L^{3/5}+3K^{2/5}M^{1/5}+6K^{1/5}L^{1/5}M^{1/5}$$
$$+3L^{2/5}M^{1/5}+3K^{1/5}M^{2/5}+3L^{1/5}M^{2/5})\rho\text{km}+15\rho\rho\text{km}-5K^{1/5}\rho\text{kl}\rho\text{km}$$
$$+20M^{1/5}\rho\text{km}^2-5L^{3/5}\rho\text{lm}-25(K^{3/5}+3K^{2/5}L^{1/5}+3K^{1/5}L^{2/5}+L^{3/5}$$
$$+3K^{2/5}M^{1/5}+6K^{1/5}L^{1/5}M^{1/5}+3L^{2/5}M^{1/5}+3K^{1/5}M^{2/5}+3L^{1/5}M^{2/5})\rho\text{lm}$$
$$+15\rho\rho\text{lm}-5L^{1/5}\rho\text{kl}\rho\text{lm}+60M^{1/5}\rho\text{km}\rho\text{lm}+20M^{1/5}\rho\text{lm}^2+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}$$
$$-5\rho\text{kl}+25\rho\text{km}+25\rho\text{lm})=0. \tag{2.1.66}$$

If we replace $25M^{3/5}$ with $25y^3-25(K^{1/n}+L^{1/n}+M^{1/n})^3-M^{3/n}$ into the equation (2.1.66), we get to the following equation:

$$-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}-15M^{2/5}\rho-5M^{1/5}\rho\text{kl}^2+5\frac{\partial Y}{\partial\lambda}\rho\text{kl}^2-5(-3K^{2/5}L^{1/5}$$
$$-3K^{1/5}L^{2/5}-L^{3/5}-3K^{2/5}M^{1/5}-6K^{1/5}L^{1/5}M^{1/5}-3L^{2/5}M^{1/5}-3K^{1/5}M^{2/5}$$
$$-3L^{1/5}M^{2/5}-M^{3/5}+\frac{\partial^3 Y}{\partial\lambda^3})\rho\text{km}+15\rho\rho\text{km}-5K^{1/5}\rho\text{kl}\rho\text{km}+20M^{1/5}\rho\text{km}^2$$
$$-5L^{3/5}\rho\text{lm}+15\rho\rho\text{lm}-5L^{1/5}\rho\text{kl}\rho\text{lm}+60M^{1/5}\rho\text{km}\rho\text{lm}+20M^{1/5}\rho\text{lm}^2$$
$$+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-5\rho\text{kl}+25\rho\text{km}+25\rho\text{lm})-25\rho\text{km}(-3K^{2/5}L^{1/5}-3K^{1/5}L^{2/5}$$
$$-3K^{2/5}M^{1/5}-6K^{1/5}L^{1/5}M^{1/5}-3L^{2/5}M^{1/5}-3K^{1/5}M^{2/5}-3L^{1/5}M^{2/5}-M^{3/5}$$
$$+\frac{\partial^3 Y}{\partial\lambda^3}+6\rho+3K^{1/5}\rho\text{kl}+3L^{1/5}\rho\text{kl}+3K^{1/5}\rho\text{km}+3M^{1/5}\rho\text{km}+3L^{1/5}\rho\text{lm}$$
$$+3M^{1/5}\rho\text{lm})-25\rho\text{lm}(-3K^{2/5}L^{1/5}-3K^{1/5}L^{2/5}-3K^{2/5}M^{1/5}-6K^{1/5}L^{1/5}M^{1/5}$$
$$-3L^{2/5}M^{1/5}-3K^{1/5}M^{2/5}-3L^{1/5}M^{2/5}-M^{3/5}+\frac{\partial^3 Y}{\partial\lambda^3}+6\rho+3K^{1/5}\rho\text{kl}$$
$$+3L^{1/5}\rho\text{kl}+3K^{1/5}\rho\text{km}+3M^{1/5}\rho\text{km}+3L^{1/5}\rho\text{lm}+3M^{1/5}\rho\text{lm})=0. \tag{2.1.67}$$

If we replace the $(-60M^{1/5}\rho\text{kl}+60L^{1/5}\rho\text{km})\to 0$, $(60M^{2/5}\rho\text{kl}+5\rho\text{kl}^2$
$-60\rho\text{km}\rho$ lm)->$5\rho\text{kl}^2$ and $M^{1/5}\rho\text{kl}\to\rho$ into the equation (2.1.67), we achieve the

following equation:

$$-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}-15M^{2/5}\rho-5M^{1/5}\rho\mathrm{kl}^2+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2$$
$$+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-5\rho\mathrm{kl}-5\rho\mathrm{km})+90K^{2/5}L^{1/5}\rho\mathrm{km}$$
$$+90K^{1/5}L^{2/5}\rho\mathrm{km}+5L^{3/5}\rho\mathrm{km}+90K^{2/5}M^{1/5}\rho\mathrm{km}+180K^{1/5}L^{1/5}M^{1/5}\rho\mathrm{km}$$
$$+90L^{2/5}M^{1/5}\rho\mathrm{km}+90K^{1/5}M^{2/5}\rho\mathrm{km}+90L^{1/5}M^{2/5}\rho\mathrm{km}+30M^{3/5}\rho\mathrm{km}$$
$$-135\rho\rho\mathrm{km}-80K^{1/5}\rho\mathrm{kl}\rho\mathrm{km}-75L^{1/5}\rho\mathrm{kl}\rho\mathrm{km}$$
$$-75K^{1/5}\rho\mathrm{km}^2-55M^{1/5}\rho\mathrm{km}^2+75K^{2/5}L^{1/5}\rho\mathrm{lm}+75K^{1/5}L^{2/5}\rho\mathrm{lm}$$
$$-5L^{3/5}\rho\mathrm{lm}+75K^{2/5}M^{1/5}\rho\mathrm{lm}+150K^{1/5}L^{1/5}M^{1/5}\rho\mathrm{lm}+75L^{2/5}M^{1/5}\rho\mathrm{lm}$$
$$+75K^{1/5}M^{2/5}\rho\mathrm{lm}+75L^{1/5}M^{2/5}\rho\mathrm{lm}$$
$$+25M^{3/5}\rho\mathrm{lm}-135\rho\rho\mathrm{lm}-75K^{1/5}\rho\mathrm{kl}\rho\mathrm{lm}-80L^{1/5}\rho\mathrm{kl}\rho\mathrm{lm}$$
$$-75K^{1/5}\rho\mathrm{km}\rho\mathrm{lm}-75L^{1/5}\rho\mathrm{km}\rho\mathrm{lm}-90M^{1/5}\rho\mathrm{km}\rho\mathrm{lm}$$
$$-75L^{1/5}\rho\mathrm{lm}^2-55M^{1/5}\rho\mathrm{lm}^2=0. \tag{2.1.68}$$

If we replace $M^{3/5}\rho$kl->$M^{2/5}\rho$, $K^{2/5}L^{1/5}$->$K^{1/5}\rho$ kl, $K^{1/5}L^{2/5}$->
$L^{1/5}\rho$kl ,$K^{2/5}M^{1/5}$->$K^{1/5}\rho$km , $K^{1/5}L^{1/5}M^{1/5}\to\rho, L^{2/5}M^{1/5}\to L^{1/5}\rho\mathrm{lm}$
,$K^{1/5}M^{2/5}$->$M^{1/5}\rho$km, $L^{1/5}M^{2/5}$->$M^{1/5}\rho$ lm and $K^{3/5}$ with
y^3 $-$ $(K^{1/n}+L^{1/n}+M^{1/n})$^3 $-$ $K^{3/n}$ in the above equation (2.1.68),
we achieve the following equation:

$$-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}-15M^{2/5}\rho-5M^{1/5}\rho\mathrm{kl}^2+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2+$$
$$\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-5\rho\mathrm{kl}-5\rho\mathrm{km})+5L^{3/5}\rho\mathrm{km}+180K^{1/5}L^{1/5}M^{1/5}\rho\mathrm{km}$$
$$+90L^{2/5}M^{1/5}\rho\mathrm{km}+90K^{1/5}M^{2/5}\rho\mathrm{km}+90L^{1/5}M^{2/5}\rho\mathrm{km}+30M^{3/5}\rho\mathrm{km}$$
$$-135\rho\rho\mathrm{km}+10K^{1/5}\rho\mathrm{kl}\rho\mathrm{km}+15L^{1/5}\rho\mathrm{kl}\rho\mathrm{km}+15K^{1/5}\rho\mathrm{km}^2$$

$$-55M^{1/5}\rho\text{km}^2 + 75K^{2/5}L^{1/5}\rho\text{lm} + 75K^{1/5}L^{2/5}\rho\text{lm} - 5L^{3/5}\rho\text{lm}$$

$$+150K^{1/5}L^{1/5}M^{1/5}\rho\text{lm} + 75L^{2/5}M^{1/5}\rho\text{lm} + 75K^{1/5}M^{2/5}\rho\text{lm} + 75L^{1/5}M^{2/5}\rho\text{lm}$$

$$+25M^{3/5}\rho\text{lm} - 135\rho\rho\text{lm} - 75K^{1/5}\rho\text{kl}\rho\text{lm} - 80L^{1/5}\rho\text{kl}\rho\text{lm} - 75L^{1/5}\rho\text{km}\rho\text{lm}$$

$$-90M^{1/5}\rho\text{km}\rho\text{lm} - 75L^{1/5}\rho\text{lm}^2 - 55M^{1/5}\rho\text{lm}^2 = 0. \tag{2.1.69}$$

If we replace $+90K^{2/5}L^{1/5} \to +90K^{1/5}\rho\text{kl}$, $90K^{1/5}L^{2/5} \to 90L^{1/5}\rho$ kl with $K^{2/5}M^{1/5} \to K^{1/5}\rho\text{km}$ into the equation (2.1.69), we achieve the following equation:

$$-K - L + 9M + \frac{\partial^5 Y}{\partial \lambda^5} - 15M^{2/5}\rho - 5M^{1/5}\rho\text{kl}^2 + 5\frac{\partial Y}{\partial \lambda}\rho\text{kl}^2 +$$

$$\frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 5\rho\text{kl} - 5\rho\text{km}) + 5L^{3/5}\rho\text{km}$$

$$+90L^{2/5}M^{1/5}\rho\text{km} + 90K^{1/5}M^{2/5}\rho\text{km} + 90L^{1/5}M^{2/5}\rho\text{km}$$

$$+30M^{3/5}\rho\text{km} + 45\rho\rho\text{km} + 10K^{1/5}\rho\text{kl}\rho\text{km} + 15L^{1/5}\rho\text{kl}\rho\text{km}$$

$$+15K^{1/5}\rho\text{km}^2 - 55M^{1/5}\rho\text{km}^2 + 75K^{2/5}L^{1/5}\rho\text{lm} + 75K^{1/5}L^{2/5}\rho\text{lm}$$

$$-5L^{3/5}\rho\text{lm} + 75L^{2/5}M^{1/5}\rho\text{lm} + 75K^{1/5}M^{2/5}\rho\text{lm} + 75L^{1/5}M^{2/5}\rho\text{lm}$$

$$+25M^{3/5}\rho\text{lm} + 15\rho\rho\text{lm} - 75K^{1/5}\rho\text{kl}\rho\text{lm} - 80L^{1/5}\rho\text{kl}\rho\text{lm} - 75L^{1/5}\rho\text{km}\rho\text{lm}$$

$$-90M^{1/5}\rho\text{km}\rho\text{lm} - 75L^{1/5}\rho\text{lm}^2 - 55M^{1/5}\rho\text{lm}^2 = 0. \tag{2.1.70}$$

If we replace $K^{1/5}L^{1/5}M^{1/5}$ with ρ in the equation (2.1.70), we end up with the following equation:

$$
\begin{aligned}
&-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}-15M^{2/5}\rho-5M^{1/5}\rho\mathrm{kl}^2+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}\\
&-5\rho\mathrm{kl}-5\rho\mathrm{km})+5L^{3/5}\rho\mathrm{km}+30M^{3/5}\rho\mathrm{km}+45\rho\rho\mathrm{km}+10K^{1/5}\rho\mathrm{kl}\rho\mathrm{km}\\
&+15L^{1/5}\rho\mathrm{kl}\rho\mathrm{km}+15K^{1/5}\rho\mathrm{km}^2+35M^{1/5}\rho\mathrm{km}^2+75K^{2/5}L^{1/5}\rho\mathrm{lm}\\
&+75K^{1/5}L^{2/5}\rho\mathrm{lm}-5L^{3/5}\rho\mathrm{lm}+25M^{3/5}\rho\mathrm{lm}+15\rho\rho\mathrm{lm}-75K^{1/5}\rho\mathrm{kl}\rho\mathrm{lm}\\
&-80L^{1/5}\rho\mathrm{kl}\rho\mathrm{lm}+15L^{1/5}\rho\mathrm{km}\rho\mathrm{lm}+75M^{1/5}\rho\mathrm{km}\rho\mathrm{lm}\\
&+20M^{1/5}\rho\mathrm{lm}^2=0. \qquad (2.1.71)
\end{aligned}
$$

If we substitute $L^{2/5}M^{1/5}\to L^{1/5}\ \rho\mathrm{lm}$, $K^{1/5}M^{2/5}$->$M^{1/5}\rho\mathrm{km}$, $L^{1/5}M^{2/5}$ $\to M^{1/5}\rho\mathrm{lm}$ into the equation (2.1.71), we end with the following equation:

$$
\begin{aligned}
&-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}+10K^{2/5}\rho-75L^{2/5}\rho+60M^{2/5}\rho+15\rho\rho\mathrm{kl}-5M^{1/5}\rho\mathrm{kl}^2\\
&+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-5\rho\mathrm{kl}-5\rho\mathrm{km})+30M^{3/5}\rho\mathrm{km}+45\rho\rho\mathrm{km}\\
&+15K^{1/5}\rho\mathrm{km}^2+35M^{1/5}\rho\mathrm{km}^2+75K^{2/5}L^{1/5}\rho\mathrm{lm}+75K^{1/5}L^{2/5}\rho\mathrm{lm}\\
&-5L^{3/5}\rho\mathrm{lm}+25M^{3/5}\rho\mathrm{lm}+30\rho\rho\mathrm{lm}-75K^{1/5}\rho\mathrm{kl}\rho\mathrm{lm}\\
&+20M^{1/5}\rho\mathrm{lm}^2=0. \qquad (2.1.72)
\end{aligned}
$$

If we place $L^{3/5}\rho\mathrm{km}$->$L^{2/5}\rho$, $K^{1/5}\rho\mathrm{kl}\rho\mathrm{km}$->$K^{2/5}\rho$, $L^{1/5}\rho\mathrm{kl}\rho\mathrm{km}\to\rho\mathrm{kl}\rho$, $L^{1/5}\rho\mathrm{kl}\rho\mathrm{lm}$- >$L^{2/5}\rho$, $L^{1/5}\rho\mathrm{km}\rho\mathrm{lm}$->$\rho\mathrm{lm}\rho$, $M^{1/5}\rho\mathrm{km}\rho\mathrm{lm}$- >$M^{2/5}\rho$ into the equation (2.1.72),

we attain the following equation:

$$
\begin{aligned}
&-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}-5M^{1/5}\rho\mathrm{kl}^2+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}\\
&-5\rho\mathrm{kl}-5\rho\mathrm{km})+30M^{3/5}\rho\mathrm{km}+15K^{1/5}\rho\mathrm{km}^2+35M^{1/5}\rho\mathrm{km}^2\\
&+75K^{2/5}L^{1/5}\rho\mathrm{lm}+75K^{1/5}L^{2/5}\rho\mathrm{lm}-5L^{3/5}\rho\mathrm{lm}+25M^{3/5}\rho\mathrm{lm}\\
&-75K^{1/5}\rho\mathrm{kl}\rho\mathrm{lm}+20M^{1/5}\rho\mathrm{lm}^2+\rho(10K^{2/5}-75L^{2/5}+60M^{2/5}\\
&+15\rho\mathrm{kl}+45\rho\mathrm{km}+30\rho\mathrm{lm})=0. \qquad (2.1.73)
\end{aligned}
$$

If we replace $K^{2/5}L^{1/5}$->$K^{1/5}\rho$kl, $K^{1/5}L^{2/5} \rightarrow L^{1/5}\rho$kl above equation (2.1.73), we achieve the following equation:

$$-K - L + 9M + \frac{\partial^5 Y}{\partial \lambda^5} - 5M^{1/5}\rho\text{kl}^2 + 5\frac{\partial Y}{\partial \lambda}\rho\text{kl}^2 + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 5\rho\text{kl}$$
$$-5\rho\text{km}) + 30M^{3/5}\rho\text{km} + 15K^{1/5}\rho\text{km}^2 + 35M^{1/5}\rho\text{km}^2 - 5L^{3/5}\rho\text{lm}$$
$$+25M^{3/5}\rho\text{lm} + 75L^{1/5}\rho\text{kl}\rho\text{lm} + 20M^{1/5}\rho\text{lm}^2$$
$$+\rho(10K^{2/5} - 75L^{2/5} + 60M^{2/5} + 15\rho\text{kl} + 45\rho\text{km} + 30\rho\text{lm}) = 0. \qquad (2.1.74)$$

Substituting $M^{1/5}\rho\text{kl}^2$->$\rho\text{kl}\rho$,$L^{1/5}\rho\text{kl}\rho\text{lm}$->$L^{2/5}\rho$,$y\rho\text{kl}^2$ with
$(K\hat{}(1/n) + L\hat{}(1/n) + M\hat{}(1/n))$ ρkl^2 in the equation (2.1.74), we achieve the following equation:

$$-K - L + 9M + \frac{\partial^5 Y}{\partial \lambda^5} + 75L^{2/5}\rho - 5\rho\rho\text{kl} + 5(K^{1/5}\rho\text{kl}^2 + L^{1/5}\rho\text{kl}^2$$
$$+M^{1/5}\rho\text{kl}^2) + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 5\rho\text{kl} - 5\rho\text{km}) + 30M^{3/5}\rho\text{km} + 15K^{1/5}\rho\text{km}^2$$
$$+35M^{1/5}\rho\text{km}^2 - 5L^{3/5}\rho\text{lm} + 25M^{3/5}\rho\text{lm}$$
$$+20M^{1/5}\rho\text{lm}^2 + \rho(10K^{2/5} - 75L^{2/5} + 60M^{2/5} + 15\rho\text{kl}$$
$$+45\rho\text{km} + 30\rho\text{lm}) = 0. \qquad (2.1.75)$$

If we replace $M^{1/5}\rho\text{kl}^2$->$\rho\text{kl}\rho$ into the above equation (2.1.75) , we get the following equation:

$$-K - L + 9M + \frac{\partial^5 Y}{\partial \lambda^5} + 5K^{1/5}\rho\text{kl}^2 + 5L^{1/5}\rho\text{kl}^2 + \frac{\partial^3 Y}{\partial \lambda^3}(-10M^{2/5} - 5\rho\text{kl}$$
$$-5\rho\text{km}) + 30M^{3/5}\rho\text{km} + 15K^{1/5}\rho\text{km}^2 + 35M^{1/5}\rho\text{km}^2 - 5L^{3/5}\rho\text{lm}$$
$$+25M^{3/5}\rho\text{lm} + 20M^{1/5}\rho\text{lm}^2 + \rho(10K^{2/5} + 60M^{2/5} + 15\rho\text{kl}$$
$$+45\rho\text{km} + 30\rho\text{lm}) = 0. \qquad (2.1.76)$$

If we replace $5K^{1/5}\rho$ $\text{kl}^2 + 5L^{1/5}\rho\text{kl}^2 \rightarrow 5y\rho\text{kl}^2 - 5M^{1/5}\rho\text{kl}^2$, $M^{1/5}\rho\text{kl}^2$->ρ $\text{kl}\rho$ into the above equation (2.1.76), we end up with the equation:

$$-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}-5M^{1/5}\rho\mathrm{kl}^2+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-5\rho\mathrm{kl}$$
$$-5\rho\mathrm{km})+30M^{3/5}\rho\mathrm{km}+15K^{1/5}\rho\mathrm{km}^2+35M^{1/5}\rho\mathrm{km}^2-5L^{3/5}\rho\mathrm{lm}$$
$$+25M^{3/5}\rho\mathrm{lm}+20M^{1/5}\rho\mathrm{lm}^2+\rho(10K^{2/5}+60M^{2/5}+15\rho\mathrm{kl}$$
$$+45\rho\mathrm{km}+30\rho\mathrm{lm})=0. \tag{2.1.77}$$

If we substitute $M^{1/5}\rho\mathrm{kl}^2$->$\rho\mathrm{kl}\rho$ into the equation (2.1.77) we end with the following equation:

$$-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-5\rho\mathrm{kl}-5\rho\mathrm{km})$$
$$+30M^{3/5}\rho\mathrm{km}+15K^{1/5}\rho\mathrm{km}^2+35M^{1/5}\rho\mathrm{km}^2-5L^{3/5}\rho\mathrm{lm}+25M^{3/5}\rho\mathrm{lm}$$
$$+20M^{1/5}\rho\mathrm{lm}^2+\rho(10K^{2/5}+60M^{2/5}+10\rho\mathrm{kl}+45\rho\mathrm{km}$$
$$+30\rho\mathrm{lm})=0. \tag{2.1.78}$$

If we replace $30M^{3/5}$ with $30y$^3 − 30 ($K^{1/n}+L^{1/n}+M^{1/n}$)^3 − $M^{3/n}$ into the equation (2.1.78) , we attain the following equation:

$$-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-5\rho\mathrm{kl}-5\rho\mathrm{km})$$
$$+(-30(K^{3/5}+3K^{2/5}L^{1/5}+3K^{1/5}L^{2/5}+L^{3/5}+3K^{2/5}M^{1/5}$$
$$+6K^{1/5}L^{1/5}M^{1/5}+3L^{2/5}M^{1/5}+3K^{1/5}M^{2/5}+3L^{1/5}M^{2/5})$$
$$+30\frac{\partial^3 Y}{\partial\lambda^3})\rho\mathrm{km}+15K^{1/5}\rho\mathrm{km}^2+35M^{1/5}\rho\mathrm{km}^2-5L^{3/5}\rho\mathrm{lm}$$
$$+25M^{3/5}\rho\mathrm{lm}+20M^{1/5}\rho\mathrm{lm}^2+\rho(10K^{2/5}+60M^{2/5}+10\rho\mathrm{kl}$$
$$+45\rho\mathrm{km}+30\rho\mathrm{lm})=0. \tag{2.1.79}$$

If we substitute; $K^{1/5}L^{1/5}M^{1/5}$->ρ, $K^{2/5}M^{1/5}$->$K^{1/5}\rho\mathrm{km}$, $K^{2/5}L^{1/5}$ ->$K^{1/5}\rho\mathrm{kl}$,$K^{1/5}L^{2/5}$->$L^{1/5}\rho\mathrm{kl}$, $L^{2/5}M^{1/5}$->$L^{1/5}\rho\mathrm{lm}$, $K^{1/5}M^{2/5}$->$M^{1/5}\rho\mathrm{km}$, $L^{1/5}M^{2/5}$->$M^{1/5}$ $\rho\mathrm{lm}$ in the equation (2.1.79), we result in the following equation:

$$
\begin{aligned}
&-K-L+9M+\frac{\partial^5 Y}{\partial\lambda^5}+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2-30K^{3/5}\rho\mathrm{km}-30L^{3/5}\rho\mathrm{km}\\
&-90K^{1/5}\rho\mathrm{kl}\rho\mathrm{km}-90L^{1/5}\rho\mathrm{kl}\rho\mathrm{km}-75K^{1/5}\rho\mathrm{km}^2-55M^{1/5}\rho\mathrm{km}^2\\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-10M^{2/5}-5\rho\mathrm{kl}+25\rho\mathrm{km})-5L^{3/5}\rho\mathrm{lm}+25M^{3/5}\rho\mathrm{lm}\\
&-90L^{1/5}\rho\mathrm{km}\rho\mathrm{lm}-90M^{1/5}\rho\mathrm{km}\rho\mathrm{lm}+20M^{1/5}\rho\mathrm{lm}^2\\
&+\rho(10K^{2/5}+60M^{2/5}+10\rho\mathrm{kl}-135\rho\mathrm{km}+30\rho\mathrm{lm})=0. \qquad (2.1.80)
\end{aligned}
$$

If we substitute $K^{1/5}\rho\mathrm{kl}\rho\mathrm{km}\text{->}\ K^{2/5}\rho$, $L^{1/5}\rho\mathrm{kl}\rho\mathrm{km}\text{->}\rho\mathrm{kl}\rho$, $L^{1/5}$ $\rho\mathrm{km}\rho\mathrm{lm}\text{->}\rho\mathrm{lm}\rho$, $M^{1/5}\rho\mathrm{km}\rho\mathrm{lm}\text{->}M^{2/5}\rho$ in the above equation (2.1.80), we end up with the following equation:

$$
\begin{aligned}
&-K-L+9M-10M^{2/5}\frac{\partial^3 Y}{\partial\lambda^3}+\frac{\partial^5 Y}{\partial\lambda^5}-80K^{2/5}\rho-30M^{2/5}\rho-5\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{kl}\\
&-80\rho\rho\mathrm{kl}+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2-30K^{3/5}\rho\mathrm{km}-30L^{3/5}\rho\mathrm{km}+25\frac{\partial^3 Y}{\partial\lambda^3}\rho\mathrm{km}-135\rho\rho\mathrm{km}\\
&-75K^{1/5}\rho\mathrm{km}^2-55M^{1/5}\rho\mathrm{km}^2-5L^{3/5}\rho\mathrm{lm}+25M^{3/5}\rho\mathrm{lm}-60\rho\rho\mathrm{lm}\\
&+20M^{1/5}\rho\mathrm{lm}^2=0. \qquad (2.1.81)
\end{aligned}
$$

If we place $M^{2/5}y^3\text{->}M^{2/5}(K\string^(1/n)+L\string^(1/n)+M\string^(1/n))\string^3$ into the equation (2.1.81), we reach the following equation:

$$
\begin{aligned}
&-K-L-10K^{3/5}M^{2/5}-30K^{2/5}L^{1/5}M^{2/5}-30K^{1/5}L^{2/5}M^{2/5}\\
&-10L^{3/5}M^{2/5}-30K^{2/5}M^{3/5}-60K^{1/5}L^{1/5}M^{3/5}-30L^{2/5}M^{3/5}-30K^{1/5}M^{4/5}\\
&-30L^{1/5}M^{4/5}-M+\frac{\partial^5 Y}{\partial\lambda^5}+5\frac{\partial Y}{\partial\lambda}\rho\mathrm{kl}^2-30K^{3/5}\rho\mathrm{km}\\
&-30L^{3/5}\rho\mathrm{km}-75K^{1/5}\rho\mathrm{km}^2-55M^{1/5}\rho\mathrm{km}^2\\
&+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\mathrm{kl}+25\rho\mathrm{km})+\rho(-80K^{2/5}-30M^{2/5}-80\rho\mathrm{kl}-135\rho\mathrm{km}-60\rho\mathrm{lm})\\
&-5L^{3/5}\rho\mathrm{lm}+25M^{3/5}\rho\mathrm{lm}+20M^{1/5}\rho\mathrm{lm}^2=0. \qquad (2.1.82)
\end{aligned}
$$

If we substitute$K^{3/5}M^{2/5}\text{->}K^{1/5}\rho\mathrm{km}\string^2$, $K^{2/5}L^{1/5}M^{2/5}\text{->}\rho\mathrm{km}\rho$, $K^{1/5}L^{2/5}M^{2/5}\text{- >}$ $\rho\mathrm{lm}\rho$, $L^{3/5}M^{2/5}\text{->}L^{1/5}\ \rho\mathrm{lm}\string^2$, $K^{2/5}M^{3/5}\text{->}M^{1/5}\rho\ \mathrm{km}\string^2$, $K^{1/5}L^{1/5}M^{3/5}\text{->}M^{2/5}\ \rho$,

$L^{2/5}M^{3/5}$->$M^{1/5}$ ρlm^2, $K^{1/5}M^{4/5}$->$M^{3/5}\rho$km, $L^{1/5}M^{4/5} \to M^{3/5}\rho$lm into the equation (2.1.82), we obtain the following equation:

$$-K - L - M + \frac{\partial^5 Y}{\partial \lambda^5} + 5\frac{\partial Y}{\partial \lambda}\rho \text{kl}^2 - 30K^{3/5}\rho \text{km} - 30L^{3/5}\rho \text{km}$$
$$-30M^{3/5}\rho \text{km} - 85K^{1/5}\rho \text{km}^2 - 85M^{1/5}\rho \text{km}^2 + \frac{\partial^3 Y}{\partial \lambda^3}(-5\rho \text{kl}$$
$$+25\rho \text{km}) + \rho(-80K^{2/5} - 90M^{2/5} - 80\rho \text{kl} - 165\rho \text{km} - 90\rho \text{lm}) - 5L^{3/5}\rho \text{lm}$$
$$-5M^{3/5}\rho \text{lm} - 10L^{1/5}\rho \text{lm}^2 - 10M^{1/5}\rho \text{lm}^2 = 0. \tag{2.1.83}$$

If we replace $K^{3/5}$ with y^3 $- (K^{1/n} + L^{1/n} + M^{1/n})$^3 $- K^{3/n}]$,
and $K^{2/5}L^{1/5}\rho$km-> $K^{2/5}$ ρ, $K^{1/5}L^{2/5}\rho$km->ρkl ρ, $K^{2/5}M^{1/5}\rho \text{km} \to K^{1/5}\rho$km^2, $K^{1/5}L^{1/5}M^{1/5}\rho \text{km} \to \rho\rho \text{km}$, $L^{2/5}M^{1/5}\rho$km->ρlmρ, $K^{1/5}M^{2/5}\rho \text{km} \to M^{1/5}\rho$km^2 and $L^{1/5}M^{2/5}\rho$km-> $M^{2/5}\rho$ into the equation (2.1.83), we end up with the following equation:

$$-K - L - M + \frac{\partial^5 Y}{\partial \lambda^5} + 90K^{2/5}\rho + 90M^{2/5}\rho + 90\rho\rho \text{kl} + 5\frac{\partial Y}{\partial \lambda}\rho \text{kl}^2 + \frac{\partial^3 Y}{\partial \lambda^3}(-5\rho \text{kl}$$
$$-5\rho \text{km}) + 180\rho\rho \text{km} + 5K^{1/5}\rho \text{km}^2 + 5M^{1/5}\rho \text{km}^2 + \rho(-80K^{2/5} - 90M^{2/5}$$
$$-80\rho \text{kl} - 165\rho \text{km} - 90\rho \text{lm}) - 5L^{3/5}\rho \text{lm} - 5M^{3/5}\rho \text{lm} + 90\rho\rho \text{lm}$$
$$-10L^{1/5}\rho \text{lm}^2 - 10M^{1/5}\rho \text{lm}^2 = 0. \tag{2.1.84}$$

If we substitute $5K^{1/5}\rho \text{km}^2$ with $5y\rho \text{km}^2 - 5$ ρkm^2 $(K^{1/n} + L^{1/n} + M^{1/n})$^1 $- K^{1/n}$ into the equation (2.1.84), we end up with the following equation:

$$-K - L - M + \frac{\partial^5 Y}{\partial \lambda^5} + 5\frac{\partial Y}{\partial \lambda}\rho \text{kl}^2 + \frac{\partial^3 Y}{\partial \lambda^3}(-5\rho \text{kl} - 5\rho \text{km}) + 5K^{1/5}\rho \text{km}^2$$
$$+5M^{1/5}\rho \text{km}^2 + \rho(10K^{2/5} + 10\rho \text{kl} + 15\rho \text{km}) - 5L^{3/5}\rho \text{lm} - 5M^{3/5}\rho \text{lm}$$
$$-10L^{1/5}\rho \text{lm}^2 - 10M^{1/5}\rho \text{lm}^2 = 0. \tag{2.1.85}$$

If we put $L^{1/5}\rho \text{km}^2$ into ρ ρkm into the equation (2.1.85), we reach the following equation:

$$-K-L-M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\text{kl}-5\rho\text{km})+(-5(L^{1/5}+M^{1/5})$$
$$+5M^{1/5})\rho\text{km}^2+\rho(10K^{2/5}+10\rho\text{kl}+15\rho\text{km})+\frac{\partial Y}{\partial\lambda}(5\rho\text{kl}^2+5\rho\text{km}^2)$$
$$-5L^{3/5}\rho\text{lm}-5M^{3/5}\rho\text{lm}-10L^{1/5}\rho\text{lm}^2-10M^{1/5}\rho\text{lm}^2=0. \qquad (2.1.86)$$

If we exchange $M^{1/5}\rho\text{lm}^2$ with $\rho\text{lm}^2\ y-\rho\text{lm}^2\ (K^{1/n}+L^{1/n}+M^{1/n})$^1 $-\ M^{1/n}$ into the above equation (2.1.86), we obtain the following:

$$-K-L-M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\text{kl}-5\rho\text{km})-5L^{1/5}\rho\text{km}^2+\rho(10K^{2/5}+10\rho\text{kl}$$
$$+15\rho\text{km})+\frac{\partial Y}{\partial\lambda}(5\rho\text{kl}^2+5\rho\text{km}^2)-5L^{3/5}\rho\text{lm}-5M^{3/5}\rho\text{lm}$$
$$-10L^{1/5}\rho\text{lm}^2-10M^{1/5}\rho\text{lm}^2=0. \qquad (2.1.87)$$

From the equation (2.1.87) we substitute $K^{1/5}\rho$ lm^2->$\rho\text{lm}\rho$ and if with $M^{3/5}\rho\text{lm}$- >$\rho\text{lm}\ y$^3 $-\ \rho\text{lm}\ (K^{1/n}+L^{1/n}+M^{1/n})$^3 $-\ M^{3/n}]$, we end up with the following equation:

$$-K-L-M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\text{kl}-5\rho\text{km})+\rho(10K^{2/5}+10\rho\text{kl}+10\rho\text{km})$$
$$+\frac{\partial Y}{\partial\lambda}(5\rho\text{kl}^2+5\rho\text{km}^2)-5L^{3/5}\rho\text{lm}-5M^{3/5}\rho\text{lm}-10L^{1/5}\rho\text{lm}^2$$
$$-10M^{1/5}\rho\text{lm}^2=0. \qquad (2.1.88)$$

If we put $K^{3/5}\rho\text{lm}$->$K^{2/5}\rho$, $K^{2/5}L^{1/5}\rho\text{lm}$->$\rho\text{kl}\rho$, $K^{1/5}L^{2/5}\rho\text{lm}$->$L^{2/5}\rho$, $K^{2/5}M^{1/5}\rho\text{lm}$->$\rho\text{km}\rho$, $K^{1/5}L^{1/5}M^{1/5}\rho\text{lm}$->$\rho\rho\text{lm}$, $L^{2/5}M^{1/5}\rho\text{lm}$->$L^{1/5}$ ρlm^2, $K^{1/5}M^{2/5}\rho\text{lm}$-> $M^{2/5}\rho$ into the equation (2.1.88), we result in the following equation:

$$-K-L-M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\text{kl}-5\rho\text{km})+\rho(10K^{2/5}+10\rho\text{kl}+10\rho\text{km})$$
$$-5L^{3/5}\rho\text{lm}-5M^{3/5}\rho\text{lm}+10K^{1/5}\rho\text{lm}^2+\frac{\partial Y}{\partial\lambda}(5\rho\text{kl}^2+5\rho\text{km}^2$$
$$-10\rho\text{lm}^2)=0. \qquad (2.1.89)$$

If we use equation (2.1.89) $+ y15\rho\mathrm{lm}^2 - (K^{1/n} + L^{1/n} + M^{1/n})$ $15\rho\mathrm{lm}^2$, we end up with the following equation:

$$
\begin{aligned}
&-K - L - M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\mathrm{kl} - 5\rho\mathrm{km} - 5\rho\mathrm{lm}) + 5K^{3/5}\rho\mathrm{lm} \\
&+15K^{2/5}L^{1/5}\rho\mathrm{lm} + 15K^{1/5}L^{2/5}\rho\mathrm{lm} + 15K^{2/5}M^{1/5}\rho\mathrm{lm} \\
&+30K^{1/5}L^{1/5}M^{1/5}\rho\mathrm{lm} + 15L^{2/5}M^{1/5}\rho\mathrm{lm} + 15K^{1/5}M^{2/5}\rho\mathrm{lm} \\
&15L^{1/5}M^{2/5}\rho\mathrm{lm} + \rho(10K^{2/5} + 10\rho\mathrm{kl} + 10\rho\mathrm{km} + 10\rho\mathrm{lm}) \\
&+\frac{\partial Y}{\partial\lambda}(5\rho\mathrm{kl}^2 + 5\rho\mathrm{km}^2 - 10\rho\mathrm{lm}^2) = 0. \qquad (2.1.90)
\end{aligned}
$$

If we change $K^{1/5}\rho\mathrm{lm}^2$->$\rho\mathrm{lm}\rho$, $L^{1/5}M^{2/5}\rho\mathrm{lm}$->$M^{1/5}\rho\mathrm{lm}^2$ in the above equation (2.1.90) , we obtain the following equation:

$$
\begin{aligned}
&-K - L - M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\mathrm{kl} - 5\rho\mathrm{km} - 5\rho\mathrm{lm}) + 15L^{1/5}M^{2/5}\rho\mathrm{lm} \\
&+15L^{1/5}\rho\mathrm{lm}^2 + \rho(15K^{2/5} + 15L^{2/5} + 15M^{2/5} + 25\rho\mathrm{kl} + 25\rho\mathrm{km} + 40\rho\mathrm{lm}) \\
&+\frac{\partial Y}{\partial\lambda}(5\rho\mathrm{kl}^2 + 5\rho\mathrm{km}^2 - 10\rho\mathrm{lm}^2) = 0. \qquad (2.1.91)
\end{aligned}
$$

If we replace $K^{2/5}$ with y^2 $- (K^{1/n} + L^{1/n} + M^{1/n})$^2 $- K^{2/n}]$, $K^{1/5}L^{1/5} \to \rho\mathrm{kl}]$ into the equation (2.1.91),
and $n = 5$, we end up with the following equation:

$$
\begin{aligned}
&-K - L - M + \frac{\partial^5 Y}{\partial\lambda^5} + \frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\mathrm{kl} - 5\rho\mathrm{km} - 5\rho\mathrm{lm}) + 15L^{1/5}M^{2/5}\rho\mathrm{lm} \\
&-15K^{1/5}\rho\mathrm{lm}^2 - 15M^{1/5}\rho\mathrm{lm}^2 + \rho(15K^{2/5} + 15L^{2/5} + 15M^{2/5} + 25\rho\mathrm{kl} \\
&+25\rho\mathrm{km} + 40\rho\mathrm{lm}) + \frac{\partial Y}{\partial\lambda}(5\rho\mathrm{kl}^2 + 5\rho\mathrm{km}^2 + 5\rho\mathrm{lm}^2) = 0. \qquad (2.1.92)
\end{aligned}
$$

If we substitute $K^{1/5}L^{1/5} \to \rho\mathrm{kl}$, $K^{1/5}M^{1/5} \to \rho\mathrm{km}$, $L^{1/5}M^{1/5} \to \rho\mathrm{lm}]$ into the equation (2.1.92), end up with the following equation:

$$-K-L-M+\frac{\partial^5 Y}{\partial\lambda^5}+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\mathrm{kl}-5\rho\mathrm{km}-5\rho\mathrm{lm})+\rho(15K^{2/5}$$
$$+15L^{2/5}+15M^{2/5}$$
$$+25\rho\mathrm{kl}+25\rho\mathrm{km}+25\rho\mathrm{lm})+\frac{\partial Y}{\partial\lambda}(5\rho\mathrm{kl}^2+5\rho\mathrm{km}^2$$
$$+5\rho\mathrm{lm}^2)=0. \tag{2.1.93}$$

If replace $(5\ \rho\mathrm{kl}^2+5\rho\mathrm{km}^2+5\rho\mathrm{lm}^2)\rightarrow 5(\rho\mathrm{kl}+\rho\mathrm{km}+\rho\mathrm{lm})\hat{}2-5(2y\rho)$ into the equation (2.1.93), we end up with the following equation:

$$-K-L-M+\frac{\partial^5 Y}{\partial\lambda^5}+15\frac{\partial^2 Y}{\partial\lambda^2}\rho+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\mathrm{kl}-5\rho\mathrm{km}-5\rho\mathrm{lm})$$
$$+\rho(-30K^{1/5}L^{1/5}-30K^{1/5}M^{1/5}-30L^{1/5}M^{1/5}+25\rho\mathrm{kl}+25\rho\mathrm{km}+25\rho\mathrm{lm})$$
$$+\frac{\partial Y}{\partial\lambda}(5\rho\mathrm{kl}^2+5\rho\mathrm{km}^2+5\rho\mathrm{lm}^2)=0. \tag{2.1.94}$$

If we set $-30K^{1/5}L^{1/5}-30K^{1/5}M^{1/5}-30L^{1/5}M^{1/5}$ to equal 0 and $25\rho\mathrm{kl}+25\rho\mathrm{km}+25\rho\mathrm{lm}$ to be $-5\rho\mathrm{kl}-5\rho\mathrm{km}-5\rho\mathrm{lm}$ into the above equation (2.1.94), the resulting equation is:

$$-K-L-M+\frac{\partial^5 Y}{\partial\lambda^5}+15\frac{\partial^2 Y}{\partial\lambda^2}\rho+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\mathrm{kl}-5\rho\mathrm{km}-5\rho\mathrm{lm})+\rho(-5\rho\mathrm{kl}$$
$$-5\rho\mathrm{km}-5\rho\mathrm{lm})+\frac{\partial Y}{\partial\lambda}(5\rho\mathrm{kl}^2+5\rho\mathrm{km}^2+5\rho\mathrm{lm}^2)=0. \tag{2.1.95}$$

If we set $5\rho\mathrm{kl}^2+5\rho\mathrm{km}^2+5\rho\mathrm{lm}^2$ to equal $-10\frac{\partial Y}{\partial\lambda}\rho+5(\rho\mathrm{kl}+\rho\mathrm{km}+\rho\mathrm{lm})^2$ into the above equation (2.1.95), the resulting equation is:

$$-K-L-M+\frac{\partial^5 Y}{\partial\lambda^5}+15\frac{\partial^2 Y}{\partial\lambda^2}\rho+\frac{\partial^3 Y}{\partial\lambda^3}(-5\rho\mathrm{kl}-5\rho\mathrm{km}-5\rho\mathrm{lm})+\rho(-5\rho\mathrm{kl}$$
$$-5\rho\mathrm{km}-5\rho\mathrm{lm})+\frac{\partial Y}{\partial\lambda}(-10\frac{\partial Y}{\partial\lambda}\rho+5(\rho\mathrm{kl}+\rho\mathrm{km}+\rho\mathrm{lm})^2)=0. \tag{2.1.96}$$

If we replace $(\rho\mathrm{kl}+\rho\mathrm{km}+\rho\ \mathrm{lm})\rightarrow r, (-5\rho\mathrm{kl}-5\rho\mathrm{km}-5\rho\mathrm{lm})\rightarrow -5r]$ in the equation 2.1.96 , end up with the following equation:

$$-K - L - M + 5r^2\frac{\partial Y}{\partial\lambda} - 5r\frac{\partial^3 Y}{\partial\lambda^3} + \frac{\partial^5 Y}{\partial\lambda^5} - 5r\rho + 5\frac{\partial^2 Y}{\partial\lambda^2}\rho = 0. \quad (2.1.97)$$

To convert (2.1.97) into a differential form, we let $y^i = \partial^i Y(\lambda, y)/\partial\lambda^i$ and define a differential form ω_5:

$$\omega_5 = -d\left(\frac{\partial^5 Y}{\partial\lambda^5}\right) + \mu d\lambda + \theta_0 dY + \theta_1 dY' + \theta_2 dY'' + \theta_3 dY^{(3)}.$$

The parameters $\mu, \theta_0, \theta_1, \theta_2$ and θ_3 are new dependent variables. Sectioning gives

$$\begin{aligned}\omega_5 \;=\; & -\frac{\partial^6 Y}{\partial\lambda^6}d\lambda - \frac{\partial}{\partial Y}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right)dY \\ & -\frac{\partial}{\partial Y'}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right)dY' \\ & -\frac{\partial}{\partial Y''}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right)dY'' - \frac{\partial}{\partial Y^{(3)}}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right)dY^{(3)} \\ & +\mu d\lambda + \theta_0 dY + \theta_1 dY' + \theta_2 dY'' + \theta_3 dY^{(3)}. \quad (2.1.98)\end{aligned}$$

That is,

$$\begin{aligned}\omega_5 \;=\; & \left(\mu - \frac{\partial^6 Y}{\partial\lambda^6}\right)d\lambda + \left(\theta_0 - \frac{\partial}{\partial Y}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right)\right)dY + \left(\theta_1 - \frac{\partial}{\partial Y'}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right)\right)dY' \\ & + \left(\theta_2 - \frac{\partial}{\partial Y''}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right)\right)dY'' + \left(\theta_3 - \frac{\partial}{\partial Y^{(3)}}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right)\right)dY^{(3)}. \quad (2.1.99)\end{aligned}$$

Annulling: Since $dY^{(i)} \wedge dY^{(i)} = d\lambda \wedge d\lambda = 0$ and $dY^{(i)} \wedge d\lambda = -d\lambda \wedge dY^{(i)}$, $dY^{(i)} \wedge dY^{(j)} = -dY^{(j)} \wedge dY^{(i)}$ for i and $j = 0, 1, 2, 3$ and $Y^{(0)} = Y$, then

$$\begin{aligned}\mu &= \frac{\partial^5 Y}{\partial\lambda^5}, \\ \theta_0 &= \frac{\partial}{\partial Y}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right), \\ \theta_1 &= \frac{\partial}{\partial Y'}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right), \\ \theta_2 &= \frac{\partial}{\partial Y''}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right), \\ \theta_3 &= \frac{\partial}{\partial Y^{(3)}}\left(\frac{\partial^5 Y}{\partial\lambda^5}\right).\end{aligned}$$

2.2 Tschirnhausian transformations, Newton sums and the solution

Trying to solve the quintic by determining the parameters K, L and M in (2.1.97) from (0.2) leads to an unsolvable system, because there are more equations than unknowns. To balance the system we employ Tschirnhausian transformations and Newton sums.

In 1666, Newton (see Scheinerman and Mircea [32]) published a set of identities that became known as Newton identities. These relate the roots of an equation an to its coefficients.

Following Newton, if the roots of the polynomial

$$y^m \; + \; a_{m-1}y^{m-1} \; + \; \cdots \; + \; a_1 y \; + \; a_0 \; = \; 0$$

are denoted by $x_i, \; i = 1, 2, \cdots, m$, then the sum of their nth powers is

$$S_n(y) = S_n(y_k) = \sum_{k=1}^{m} y_k^n, \quad \text{for} \quad n \in N$$

Where y_k is a root of the Bring-Jerrard equation (2.1.97). This can be rephrased as

$$S_n(y) = -na_{m-n} - \sum_{j=1}^{n-1} S_{n-j}a_{m-j}(y),$$

with $a_j = 0$ for $j < 0$. For our purpose here, we let the symbol $\aleph_\Sigma$ be the power sum generator in accordance with Newton's criterion for roots, then

$$\aleph_\Sigma \left(y^n\right) \; = \; S_n(y).$$

As indicated above, the parameters K, L and M are not sufficient to solve the problem. To circumvent this, we introduce new parameters $C_i, \; i = 0, 1, 2, 3, 4, 5,$ and a new variable z that we transform to, that is,

$$C_5 z^5 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0 = 0. \tag{2.2.1}$$

Differential forms show that

$$\theta_i = C_i/C_0 \quad \text{for} \ \ i = 1, 2, 3, 4, 5.$$

This determines $m = K^{1/5}$, $l = L^{1/5}$ and $m = M^{1/5}$. How this actually happens require that we first transform (2.1.97) from the variable y to the new variable z using Tschirnhausian transformations. According to Admchik and Jeffrey [29], a Tschirnhaus quadratic substitution $y_k = z_k^2 + \alpha z + \beta$, can be used to transform the general quintic to the principal quintic.
For our transformation, we use the simpler relation

$$y_k = z_k^4 + \alpha z_k^3 + \beta z_k^2 + \gamma z_k + \delta \tag{2.2.2}$$

a quartic transformation, where y_k is a root of the Bring-Jerrard equation (2.1.97), and z_k is root of the transformed quintic equation (2.2.1).
Applying the Newton identity operator $\aleph_\Sigma$ then, we have

$$\aleph_\Sigma \left[(y_k)^s - (z_k^2 + \alpha z + \beta)^s \right] = 0, \quad s = 0, 1, 2, 3, 4, 5. \tag{2.2.3}$$

From the equation (2.2.3) we now end with the following equations:

$$\frac{4a_1}{5} - \frac{C_4}{5} - \delta = 0, \tag{2.2.4}$$

$$-\frac{4{a_1}^2}{5} + \frac{1}{5}\left(-2C_3 + {C_4}^2\right) + 2a_0\alpha\beta + \frac{4a_1\beta^2}{5} + 2a_0\gamma + \frac{8a_1\alpha\gamma}{5} + \frac{8a_1\delta}{5} - \delta^2 = 0 \tag{2.2.5}$$

$$
\begin{aligned}
&-\frac{4{a_1}^4}{5}+\frac{1}{5}\left(-4C_1+2{C_3}^2+4C_2C_4-4C_3{C_4}^2+{C_4}^4\right)+4{a_0}^3\alpha+\frac{84}{5}{a_0}^2a_1\alpha^2\\
&+\frac{52}{5}a_0{a_1}^2\alpha^3+\frac{4{a_1}^3\alpha^4}{5}+\frac{56}{5}{a_0}^2a_1\beta+\frac{156}{5}a_0{a_1}^2\alpha\beta+\frac{48}{5}{a_1}^3\alpha^2\beta+\frac{24{a_1}^3\beta^2}{5}\\
&-6{a_0}^2\alpha^2\beta^2-4{a_0}^2\beta^3-\frac{36}{5}a_0a_1\alpha\beta^3-\frac{4{a_1}^2\beta^4}{5}+\frac{52}{5}a_0{a_1}^2\gamma+\frac{48}{5}{a_1}^3\alpha\gamma\\
&-4{a_0}^2\alpha^3\gamma-24{a_0}^2\alpha\beta\gamma-\frac{108}{5}a_0a_1\alpha^2\beta\gamma-\frac{108}{5}a_0a_1\beta^2\gamma-\frac{48}{5}{a_1}^2\alpha\beta^2\gamma-6{a_0}^2\gamma^2\\
&-\frac{108}{5}a_0a_1\alpha\gamma^2-\frac{24}{5}{a_1}^2\alpha^2\gamma^2-\frac{48}{5}{a_1}^2\beta\gamma^2+4a_0\beta\gamma^3+\frac{4a_1\gamma^4}{5}\\
&+\frac{16{a_1}^3\delta}{5}-12{a_0}^2\alpha^2\delta-\frac{36}{5}a_0a_1\alpha^3\delta-12{a_0}^2\beta\delta-\frac{216}{5}a_0a_1\alpha\beta\delta-\frac{48}{5}{a_1}^2\alpha^2\beta\delta\\
&-\frac{48}{5}{a_1}^2\beta^2\delta-\frac{108}{5}a_0a_1\gamma\delta-\frac{96}{5}{a_1}^2\alpha\gamma\delta+12a_0\beta^2\gamma\delta+12a_0\alpha\gamma^2\delta+\frac{48}{5}a_1\beta\gamma^2\delta\\
&-\frac{24{a_1}^2\delta^2}{5}+12a_0\alpha\beta\delta^2+\frac{24}{5}a_1\beta^2\delta^2+12a_0\gamma\delta^2+\frac{48}{5}a_1\alpha\gamma\delta^2\\
&+\frac{16a_1\delta^3}{5}-\delta^4=0, \qquad (2.2.6)
\end{aligned}
$$

$$
\begin{aligned}
&-((4a_1^4)/5)+1/5(-4C_1+2C_3^2+4C_2C_4-4C_3C_4^2+C_4^4)+4a_0^3\alpha+84/5a_0^2a_1\alpha^2\\
&+52/5a_0a_1^2\alpha^3+(4a_1^3\alpha^4)/5+56/5a_0^2a_1\beta+156/5a_0a_1^2\alpha\beta+48/5a_1^3\alpha^2\beta\\
&+(24a_1^3\beta^2)/5-6a_0^2\alpha^2\beta^2-4a_0^2\beta^3-36/5a_0a_1\alpha\beta^3-(4a_1^2\beta^4)/5+52/5a_0a_1^2\gamma\\
&+48/5a_1^3\alpha\gamma-4a_0^2\alpha^3\gamma-24a_0^2\alpha\beta\gamma-108/5a_0a_1\alpha^2\beta\gamma-108/5a_0a_1\beta^2\gamma\\
&-48/5a_1^2\alpha\beta^2\gamma-6a_0^2\gamma^2-108/5a_0a_1\alpha\gamma^2-24/5a_1^2\alpha^2\gamma^2-48/5a_1^2\beta\gamma^2\\
&+4a_0\beta\gamma^3+(4a_1\gamma^4)/5+(16a_1^3\delta)/5-12a_0^2\alpha^2\delta-36/5a_0a_1\alpha^3\delta-12a_0^2\beta\delta\\
&-216/5a_0a_1\alpha\beta\delta-48/5a_1^2\alpha^2\beta\delta-48/5a_1^2\beta^2\delta-108/5a_0a_1\gamma\delta-96/5a_1^2\alpha\gamma\delta\\
&+12a_0\beta^2\gamma\delta+12a_0\alpha\gamma^2\delta+48/5a_1\beta\gamma^2\delta-(24a_1^2\delta^2)/5+12a_0\alpha\beta\delta^2+24/5a_1\beta^2\delta^2\\
&+12a_0\gamma\delta^2+48/5a_1\alpha\gamma\delta^2+(16a_1\delta^3)/5-\delta^4=0, \qquad (2.2.7)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{5}(-5{a_0}^4+4{a_1}^5)-C_0+\frac{C_1C_4}{5}-\frac{1}{5}C_2(-2C_3+{C_4}^2)-\frac{1}{5}C_3(-3C_2+3C_3C_4\\
&-{C_4}^3)-\frac{1}{5}C_4(-4C_1+2{C_3}^2+4C_2C_4-4C_3{C_4}^2+{C_4}^4)-19{a_0}^3a_1\alpha\\
&+\frac{27}{5}{a_0}^2{a_1}^2(-10\alpha^2-5\beta)+\frac{17}{5}a_0{a_1}^3(-10\alpha^3-20\alpha\beta-5\gamma)+\frac{4}{5}{a_1}^4(-5\alpha^4\\
&-30\alpha^2\beta-10\beta^2-20\alpha\gamma-5\delta)-\delta^5-{a_0}^3(-\alpha^5-20\alpha^3\beta-30\alpha\beta^2-30\alpha^2\gamma\\
&-20\beta\gamma-20\alpha\delta)-\frac{14}{5}{a_0}^2a_1(-5\alpha^4\beta-30\alpha^2\beta^2-10\beta^3-20\alpha^3\gamma-60\alpha\beta\gamma\\
&-10\gamma^2-30\alpha^2\delta-20\beta\delta)-\frac{13}{5}a_0{a_1}^2(-10\alpha^3\beta^2-20\alpha\beta^3-5\alpha^4\gamma\\
&-60\alpha^2\beta\gamma-30\beta^2\gamma-30\alpha\gamma^2-20\alpha^3\delta-60\alpha\beta\delta-20\gamma\delta)-\frac{4}{5}{a_1}^3(-10\alpha^2\beta^3\\
&-5\beta^4-20\alpha^3\beta\gamma-60\alpha\beta^2\gamma-30\alpha^2\gamma^2-30\beta\gamma^2-5\alpha^4\delta-60\alpha^2\beta\delta-30\beta^2\delta\\
&-60\alpha\gamma\delta-10\delta^2)+{a_0}^2(-\beta^5-20\alpha\beta^3\gamma-30\alpha^2\beta\gamma^2-30\beta^2\gamma^2-20\alpha\gamma^3\\
&-30\alpha^2\beta^2\delta-20\beta^3\delta-20\alpha^3\gamma\delta-120\alpha\beta\gamma\delta-30\gamma^2\delta-30\alpha^2\delta^2-30\beta\delta^2)\\
&+\frac{9}{5}a_0a_1(-5\beta^4\gamma-30\alpha\beta^2\gamma^2-10\alpha^2\gamma^3-20\beta\gamma^3-20\alpha\beta^3\delta-60\alpha^2\beta\gamma\delta\\
&-60\beta^2\gamma\delta-60\alpha\gamma^2\delta-10\alpha^3\delta^2-60\alpha\beta\delta^2-30\gamma\delta^2)+\frac{4}{5}{a_1}^2(-10\beta^3\gamma\\
&-20\alpha\beta\gamma^3-5\gamma^4-5\beta^4\delta-60\alpha\beta^2\gamma\delta-30\alpha^2\gamma^2\delta-60\beta\gamma^2\delta-30\alpha^2\beta\delta^2\\
&-30\beta^2\delta^2-60\alpha\gamma\delta^2-10\delta^3)-a_0(-\gamma^5-20\beta\gamma^3\delta-30\beta^2\gamma\delta^2-30\alpha\gamma^2\delta^2\\
&-20\alpha\beta\delta^3-20\gamma\delta^3)-\frac{4}{5}a_1(-5\gamma^4\delta-30\beta\gamma^2\delta^2-10\beta^2\delta^3\\
&-20\alpha\gamma\delta^3-5\delta^4)=0 \qquad (2.2.8)
\end{aligned}
$$

After a very lengthy analysis, we get

$$
\begin{aligned}
-m^5 \quad = \quad & \frac{2}{15}\mathrm{b}(15\mathrm{c}(-\frac{5\mathrm{c}}{4\mathrm{b}}+\frac{1}{2}\surd(\frac{25\mathrm{c}^2}{4\mathrm{b}^2}+(4(\frac{2}{15})^{1/3}\mathrm{b}^2)/(1125\mathrm{c}^2\mathrm{b}^2-\sqrt{15}\xi)^{1/3}\\
&+(1125\mathrm{c}^2\mathrm{b}^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}\mathrm{b}))-\frac{1}{2}\surd(\frac{25\mathrm{c}^2}{2\mathrm{b}^2}\\
&-(4(\frac{2}{15})^{1/3}\mathrm{b}^2)/(1125\mathrm{c}^2\mathrm{b}^2-\sqrt{15}\xi)^{1/3}-(1125\mathrm{c}^2\mathrm{b}^2\\
&-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}\mathrm{b})-(125\mathrm{c}^3)/(4\mathrm{b}^3\surd(\frac{25\mathrm{c}^2}{4\mathrm{b}^2}+(4(\frac{2}{15})^{1/3}\mathrm{b}^2)\\
&/(1125\mathrm{c}^2\mathrm{b}^2-\sqrt{15}\xi)^{1/3}+(1125\mathrm{c}^2\mathrm{b}^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}\mathrm{b})))))\\
&+12\mathrm{b}(-\frac{5\mathrm{c}}{4\mathrm{b}}+\frac{1}{2}\surd(\frac{25\mathrm{c}^2}{4\mathrm{b}^2}+(4(\frac{2}{15})^{1/3}\mathrm{b}^2)/(1125\mathrm{c}^2b^2-\sqrt{15}\xi)^{1/3}\\
&+(1125\mathrm{c}^2\mathrm{b}^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}\mathrm{b}))\\
&-\frac{1}{2}\surd(\frac{25\mathrm{c}^2}{2\mathrm{b}^2}-(4(\frac{2}{15})^{1/3}\mathrm{b}^2)/(1125\mathrm{c}^2\mathrm{b}^2\\
&-\sqrt{15}\xi)^{1/3}-(1125\mathrm{c}^2\mathrm{b}^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}\mathrm{b})\\
&-(125\mathrm{c}^3)/(4\mathrm{b}^3\surd(\frac{25\mathrm{c}^2}{4\mathrm{b}^2}+(4(\frac{2}{15})^{1/3}\mathrm{b}^2)/(1125\mathrm{c}^2\mathrm{b}^2\\
&-\sqrt{15}\xi)^{1/3}+(1125\mathrm{c}^2\mathrm{b}^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}\mathrm{b})))))^2),
\end{aligned}
\tag{2.2.9}
$$

$$
\begin{aligned}
l \;=\;& -\frac{1}{2^{1/3}5^{4/15}}(3^{1/15}((-\frac{1}{3^{2/5}}(25^{3/5}(15c(-\frac{5c}{4b}+\frac{1}{2}\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)\\
&/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\sqrt{84375c^4b^4-256b^9})^{1/3}\\
&/(2^{1/3}15^{2/3}b))-\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}+(1125c^2\ tb^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))+12b(-\frac{5c}{4b}+\frac{1}{2}\surd(\frac{25c^2}{4b^2}\\
&+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}\ +(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))\\
&-\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}+(1125c^2b^2\ \ -\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)))))^2)^{2/5}(-\frac{1}{15^{1/5}}((15c(-\frac{5c}{4b}\\
&+\frac{1}{2}\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}\\
&/(2^{1/3}15^{2/3}b))-\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))+12b(-\frac{5c}{4b}+\frac{1}{2}\\
&\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}\\
&/(2^{1/3}15^{2/3}b))-\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))^2)^{1/5})\ -\frac{2bm}{5}))\\
&+\frac{1}{m^6}((-\frac{1}{15^{1/5}}((15c(-\frac{5c}{4b}+\frac{1}{2}\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}\\
&+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))-\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2\\
&-\sqrt{15}\xi)^{1/3}-(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}\\
&+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}\\
&+(1125c^2b^2-\sqrt{15}xi)^{1/3}/(2^{1/3}15^{2/3}b)))))\ +12b(-\frac{5c}{4b}+\frac{1}{2}\surd(\frac{25c^2}{4b^2}\\
&+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))
\end{aligned}
$$

$$-\frac{1}{2}\sqrt{(}\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}\ -(1125c^2b^2-\sqrt{15}\xi)^{1/3}/$$

$$(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\ \sqrt{(}\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}$$

$$+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))^2)^{1/5})-\frac{2bm}{5})^5)\ -\sqrt{(}-\frac{20}{3}(15c(-\frac{5c}{4b}$$

$$+\frac{1}{2}\sqrt{(}\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}$$

$$/(2^{1/3}15^{2/3}b))-\frac{1}{2}\sqrt{(}\frac{25c^2}{2b^2}$$

$$-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)$$

$$-(125c^3)/(4b^3\sqrt{(}\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))+12b(-\frac{5c}{4b}+\frac{1}{2}\sqrt{(}\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))\ -\frac{1}{2}\sqrt{(}\frac{25c^2}{2b^2}$$

$$-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)$$

$$-(125c^3)/(4b^3\sqrt{(}\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))^2)m^3+(\frac{1}{3^{2/5}}(25^{3/5}(15c\ (-\frac{5c}{4b}+\frac{1}{2}\sqrt{(}\frac{25c^2}{4b^2}$$

$$+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}\ /(2^{1/3}15^{2/3}b))$$

$$-\frac{1}{2}\sqrt{(}\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2-\sqrt{15}\xi)^{1/3}$$

$$/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\sqrt{(}\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))+12b(-\frac{5c}{4b}+\frac{1}{2}\sqrt{(}\frac{25c^2}{4b^2}$$

$$+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))$$

$$-\frac{1}{2}\sqrt{(}\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2\ -\sqrt{15}\xi)^{1/3}$$

$$/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\sqrt{(}\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}$$

$$+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))^2)^{2/5}(-\frac{1}{15^{1/5}}((15c(-\frac{5c}{4b}+\frac{1}{2}\sqrt{(}\frac{25c^2}{4b^2}$$

$$+(4(\frac{2}{15})^{1/3}b^2)\ /(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))$$

$$-\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)\ /(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}\ +(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)))))+12b(-\frac{5c}{4b}$$

$$+\frac{1}{2}\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}$$

$$/(2^{1/3}15^{2/3}b))\ \ -\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}$$

$$/(2^{1/3}15^{2/3}b))))))^2)^{1/5})-\frac{2bm}{5}))-\frac{1}{m^6}((-\frac{1}{15^{1/5}}((15c(-\frac{5c}{4b}+\frac{1}{2}\surd(\frac{25c^2}{4b^2}$$

$$+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))$$

$$-\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)))))+12b(-\frac{5c}{4b}+\frac{1}{2}\surd(\frac{25c^2}{4b^2}$$

$$+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))$$

$$-\frac{1}{2}\surd(\frac{25c^2}{2b^2}-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}-(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}$$

$$+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))))))^2)^{1/5})$$

$$-\frac{2bm}{5})^5))^2))/((15c(-\frac{5c}{4b}+\frac{1}{2}\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}$$

$$+(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b))-\frac{1}{2}\surd(\frac{25c^2}{2b^2}$$

$$-(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}-(1125c^2b^2-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)$$

$$-(125c^3)/(4b^3\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2-\sqrt{15}\xi)^{1/3}+(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)))))\ \ +12b(-\frac{5c}{4b}+\frac{1}{2}\surd(\frac{25c^2}{4b^2}+(4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2$$

$$-\sqrt{15}\xi)^{1/3} + (1125c^2b^2 - \sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)) - \frac{1}{2}\sqrt{(}\frac{25c^2}{2b^2} - (4(\frac{2}{15})^{1/3}b^2)$$
$$/(1125c^2b^2 - \sqrt{15}\xi)^{1/3} - (1125c^2b^2 - \sqrt{15}\xi)^{1/3}$$
$$/(2^{1/3}15^{2/3}b) - (125c^3)/(4b^3\sqrt{(}\frac{25c^2}{4b^2} + (4(\frac{2}{15})^{1/3}b^2)/(1125c^2b^2$$
$$-\sqrt{15}\xi)^{1/3} + (1125c^2b^2 - \sqrt{15}\xi)^{1/3}/(2^{1/3}15^{2/3}b)))))^2)^{1/5}m^3))^{1/3}),$$

where

$$\xi = \sqrt{84375c^4b^4 - 256b^9}.$$

The parameter k follows from

$$k^5 + l^5 \;=\; 0.$$

$$\zeta \;=\; -\frac{5\,c}{4\,b} + \frac{1}{2}\sqrt{\left(\frac{25\,c^2}{4\,b^2} + \frac{4\,\left(\frac{2}{15}\right)^{1/3}\,b^2}{\eta^{1/3}} + \frac{\eta^{1/3}}{2^{1/3}\,15^{2/3}\,b}\right)} -$$
$$\frac{1}{2}\sqrt{\left(\frac{25\,c^2}{2\,b^2} - \frac{4\,\left(\frac{2}{15}\right)^{1/3}\,b^2}{\eta^{1/3}} - \frac{\eta^{1/3}}{2^{1/3}\,15^{2/3}\,b} - \right.}$$
$$\left.(125\,c^3)\Big/\left(4\,b^3\,\sqrt{\left(\frac{25\,c^2}{4\,b^2} + \left(4\,\left(\frac{2}{15}\right)^{1/3}\,b^2\right)\Big/\eta^{1/3} + \frac{1}{2^{1/3}\,15^{2/3}\,b}\eta\right)}\right)\right)$$

with

$$\eta = 1125\,c^2\,b^2 - \sqrt{15}\,\sqrt{84375\,c^4\,b^4 - 256\,b^9}$$

and

$$C_0 \;=\; -c^2 + cb\zeta + c\zeta^5.$$

2.2.1 A criterion for avoiding bugs in the symbolic software: Mathematica

Apparently, the engineers of the software Mathematica adopted series expansions to compute formulae not recognized by their product(see Fateman [36]). This we have observed in their versions 5 to 7. To avoid this bug, we first note that if x_0 is a solution for

the equation $x^5 + ax + b = 0$, then $s^n x_0$ turns out to be a solution for the equation $(s^n x)^5 + as^{5n-1}(s^n x) + bs^{5n} = 0$, for all any value of n. We can then avoid the bug by using a Cauchy property on convergent sequences, that is, according the to ratio test, the series $\{a_n\}$ converges if $|\frac{a_{m+1}}{a_m}| < 1$. We choose $n < 1$, which guarantees convergence.
That is, *If x_0 is the solution of*

$$x^5 + ax + b = 0; \tag{2.2.10}$$

then $s^n x_0$ is the solution of

$$(s^n x)^5 + as^{5n-1}(s^n x) + bs^{5n} = 0. \tag{2.2.11}$$

Where a, b, s and n are rational numbers.
To prove this, multiply 2.2.10 with s^{5n}. That is,

$$s^{5n}(x^5 + ax + b) = 0.$$

This can also be expressed in the form

$$(s^n x)^5 + as^{5n-1}(s^n x) + bs^{5n} = 0,$$

which concludes the proof.
If we choose $s = 10$ and n to be an integer, we note that the introduction of s^{5n} only serves to shift the position of the decimal.

2.3 A numerical experiment

The equations (2.2.9) and (2.2.10) led to Table 2.1. The precision in the table can be improved. For example, the parameters $(b, c) = (e \times 10^{-5000000} i, \pi \times 10^{-4000000})$, led to the five roots:

$$\begin{aligned}
y_1 &= -1.1616230521691030 \times 10^{-800000} - 3.7743420924792644 \times 10^{-800001} i, \\
y_2 &= -7.179225283558979 \times 10^{-800001} + 9.8813558832801133 \times 10^{-800001} i, \\
y_3 &= 1.1616230521691030 \times 10^{-800000} - 3.7743420924792644 \times 10^{-800001} i, \\
y_4 &= 7.179225283558979 \times 10^{-800001} + 9.8813558832801133 \times 10^{-800001} i, \\
y_5 &= -1.221402758160169 \times 10^{-800000} i.
\end{aligned}$$

These can be improved even further. For example, y_1 can be improved to

$$
\begin{aligned}
y_1 \quad = \quad & -1.16162305216910305916547585075202137071579627034672402253 2\\
& 542761619734886208810145197513450833674851868145568078060987\\
& 641071490992831597141615930458904504542410568400651947449935\\
& 463906096698454276862526244948403500685312299670713724062635\\
& 359978310843158634782692203727661964959698780770402176366314\\
& 373550490855313535848013285040294295463884335140511658263349\\
& 324750055792860491007895320676657728 04 \times 10^{-800000}\\
- \quad & 3.7743420924792646403539237695053267201218163270448333865582 8\\
& 457559247309747499055330498796790960290564944330474670477182 9\\
& 088489952503493034323699541210797474127343220046351619514237 4\\
& 696937420986127587831876358334384081541856762513473834077362 77\\
& 626196536044320807528254957970033266336766582843802700181536 06\\
& 395051627286538697996006432497090427916063520178053953794132 45\\
& 87569599145186868109116289492 \times 10^{-800001} i. \qquad (2.3.1)
\end{aligned}
$$

The computer generated solutions:

$$
\begin{aligned}
y_1 &= -1.1616230521691030 \times 10^{-800000} - \; 3.7743420924792647 \times 10^{-800001} i,\\
y_2 &= -7.179225283558979 \times 10^{-800001} + \; 9.8813558832801138 \times 10^{-800001} i,\\
y_3 &= 1.1616230521691030 \times 10^{-800000} - \; 3.7743420924792647 \times 10^{-800001} ii,\\
y_4 &= 7.179225283558979 \times 10^{-800001} + \; 9.8813558832801138 \times 10^{-800001} i,\\
y_5 &= -5.54405555626329 \times 10^{-800026} - 1.2214027581601698 \times 10^{-800000} i.
\end{aligned}
$$

In comparison with the y_1 in (2.3.1), a number that has more significant numbers, the computer generated

$$
\begin{aligned}
y_1 \;=\;& -1.16162305216910305916547585075202137071579627034672402 25\\
& 3254276161973488620881014519751345083367485186814556807806\\
& 0987641071490992831597141615930458904504542410568400651947\\
& 4499354639060966984542768625262449484035006853122996707137\\
& 2406263535997831084315863478269220372766196495969878077040\\
& 2176366314373550490855313535848013285040294295463884335140\\
& 5116582633493247500557928604910078953206766577280385 \times 10^{-800001}\\
-\;& 3.774342092479264640353923769505326720121816327044833\\
& 3865582845755924730974749905533049879679096029056494 4\\
& 3304746704771829088489952503493034323699541210797474 1\\
& 2734322004635161951423746969374209861275878318763583 3\\
& 4384081541856762513473834077362776261965360443208075 2\\
& 8254957970033266336766582843802700181536063950516272 8\\
& 6538697996006432497090427916063520178053953794132458 7\\
& 5695991451868681091162895 \times 10^{-800001} i.
\end{aligned}
$$

This is the reason we choose $s^n = 10^{-800000}$ this guarantees convergence. This may seem to reduce our equation to zero, which of course is not the case, as demonstrated by example that follow. For example, the equation $x^5 + x + 1 = 0$ in MatLab this can be written as $p1 = [1, 0, 0, 0, 1, 1]$ the roots follow from the command roots(p1). One of the five roots is

$$x_0 = 0.87744 + 0.7786i \tag{2.3.2}$$

Using our convention, we can write equation 2.2.10 in the form

$$(10^{-1}x) + 10^{-4}.(10^{-1}x) + 10^{-5} = 0$$

to avoid confusion, let $r = 10^{-1}x$ therefore

$$r^5 + 10^{-4}r + 10^-5 = 0$$

in MatLab, this assumes the form $P2 = [1, 0, 0, 0, 0.0001, 0.00001]$ one root resulting from $p2$ is

$$r_0 = 0.087744 + 0.074486i. \tag{2.3.3}$$

Now compare the result in 2.3.2 and 2.3.3 that is

$$\begin{aligned} x_0 &= 0.87744 + 0.7786i, \\ r_0 &= 0.087744 + 0.074486 \end{aligned}$$

Table 2.1: A numerical experiment.

$c \times 10^{80}$	$b \times 10^{90}$	Analytic	Numerical
$10^5 i$	$10^5 i$	$-1.0000 \times 10^{-17} + 2.0000 \times 10^{-25} i$	$-1.0000 \times 10^{-17} + 2.0000 \times 10^{-25}$
$1000i$	1000	$-3.9810 \times 10^{-18} + 3.1697 \times 10^{-26} i$	$-3.9810 \times 10^{-18} + 3.1697 \times 10^{-26} i$
$100i$	100	$-2.5118 \times 10^{-18} + 1.2619 \times 10^{-26} i$	$-2.5118 \times 10^{-18} + 1.2619 \times 10^{-26} i$
$10i$	10	$-1.5848 \times 10^{-18} + 5.0237 \times 10^{-27} i$	$-1.5848 \times 10^{-18} + 5.0237 \times 10^{-27} i$
$9i$	9	$-1.5518 \times 10^{-18} + 4.8164 \times 10^{-27} i$	$-1.5518 \times 10^{-18} + 4.8164 \times 10^{-27} i$
$8i$	8	$-1.5157 \times 10^{-18} + 4.5947 \times 10^{-27} i$	$-1.5157 \times 10^{-18} + 4.5947 \times 10^{-27} i$
$7i$	7	$-1.4757 \times 10^{-18} + 4.3558 \times 10^{-27} i$	$-1.4757 \times 10^{-18} + 4.3558 \times 10^{-27} i$
$6i$	6	$-1.4309 \times 10^{-18} + 4.0953 \times 10^{-27} i$	$-1.4309 \times 10^{-18} + 4.0953 \times 10^{-27} i$
-5	$5i/3$	$-1.0533 \times 10^{-18} - 3.4225 \times 10^{-19} i$	$-1.0533 \times 10^{-18} - 3.4225 \times 10^{-19} i$
$5i$	5	$-1.3797 \times 10^{-18} - 3.8073 \times 10^{-27} i$	$-1.3797 \times 10^{-18} - 3.8073 \times 10^{-27} i$
4	$4i$	$-1.2549 \times 10^{-18} + 4.0774 \times 10^{-19} i$	$-1.2549 \times 10^{-18} - 4.0775 \times 10^{-19} i$
$4i$	4	$-1.3195 \times 10^{-18} + 3.4822 \times 10^{-27} i$	$-1.3195 \times 10^{-18} + 3.4822 \times 10^{-27} i$
$3i$	3	$-1.2457 \times 10^{-18} + 3.1036 \times 10^{-27} i$	$-1.2457 \times 10^{-18} + 3.1036 \times 10^{-27} i$
$2i$	2	$-1.1486 \times 10^{-18} + 2.6390 \times 10^{-27} i$	$-1.1486 \times 10^{-18} + 2.6390 \times 10^{-27} i$
$1i$	1	$-1.0000 \times 10^{-18} + 2.0000 \times 10^{-27} i$	$-1.0000 \times 10^{-18} + 2.0000 \times 10^{-27}$

Chapter 3

Application: *A formula for the gravitational constant*

The universal gravitational constant, also known as the “big G", has confounded many for a long time. As Schwarzschild puts it [41], ‘our knowledge of G gets worse, rather than better. Another states: ‘Since Cavendish first measured this constant 200 years ago, it remains one of the most elusive constants in physics’. See the following: Goddard [42], Gillies[43], Gunlach[[44], [45], [46]], Kuroda[47], Fujii[48], Roy and Datta[49], Duval[50], Luther[51], Schlamminger[52] and Laporesi[53] . Our approach is hinges on several assumptions that regard the source of a planetory’s magnetosphere.

It is believed that since magnetic fields surround electric currents, it can be surmised that circulating electric currents in the Earth’s molten metallic core are the origin of the magnetic field. Our approach requires that we assume that every particle constituting the planet contributes to the magnetosphere, otherwise it would be impossible to derive it.

According to the theory of general relativity, gravitational waves are those waves caused by the motion of matter as predicted in the theory and propagating at the speed of light. The theory holds that an object in motion should have length contraction. Our simple view is that gravitational waves are those electromagnetic waves that are responsible for gravitational forces between objects and are generated by the relative motion of the positive charges to the negative charges in an atom, neutron or any electrically neutral subatomic particle as the objects they constitute move. Our moving object should not only have length contractions, but the contractions should alternate with expansions. The periods for these alternations should be long enough to allow a gravitational wave to pass through or for one to be generated.

In 1934 Fritz Zwicky postulated that the presence of dark matter explains some of the motion galaxies undergo. While investigating electrodynamic equations for the "big G" formula, we encountered a complex force whose real component is the well known gravitational acceleration. With the advent of dark matter and dark energy, it is possible to find room for the imaginary component.

Our theoretical basis cannot be explained well without going back to the question Aristotle posed more than two thousand years ago. He (384 BC-322 BC) asked the question as to why an arrow moves when shot from a bow. What agent acts on it all the time to cause its motion? Galileo (1564-1642) disagreed with the answer Aristotle suggested. It is not clear however Galileo also disagreed with the question itself.

The equations used here include Newton's second law of motion

$$m \, \frac{d^2 \mathbf{r}}{dt^2} \; = \; \mathbf{F}. \tag{3.0.1}$$

This equation describes the position $\mathbf{r} = (x, y, z)$ in time t of an object with mass m subjected to a force $\mathbf{F}$.

The other equation is taken from gravitational law, which states that there is a force $\mathbf{F}$ between any two bodies of mass m_1 and m_2 described by

$$\mathbf{F} \; = \; G \, \frac{m_1 \, m_2}{r^2} \, \hat{\mathbf{r}}.$$

We also need Cauchy(1789-1857)'s first law of motion. That is,

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \; = \; \frac{1}{\tilde{\rho}} \left(b_i + \frac{\partial \tau_{ij}}{\partial x_j} \right), \quad i, j = 1, 2, 3. \tag{3.0.2}$$

The parameter $\tilde{\rho}$ is the mass density, $\mathbf{u} = (u_1, u_2, u_3)$ ($u_i = dx_i/dt$) being the spatial velocity (velocity of material elements within the object) and τ_{ij} being the stress tensor. The parameter b_i is the barotropic force.

Cauchy's law answers Aristotle's question for us, and this is what we use. Our understanding is that there are material displacement within a body when that body is in motion, which can be explained through this law. There are, therefore, displacements within bodies corresponding to the universal motion. Linking these displacements to electrodynamics has two consequences. Firstly, we are able to deduce the gravitational law and hence the corresponding constant. Secondly, an expression for the universal momentum can also be deduced.

The electron-proton interaction is governed by the columbic potential

$$V = -K_e \frac{q}{r},$$

where K_e is Coulomb's constant, q is the charge and r the separating distance. The link is through magnetic and electric fields described in Maxwell's (1831-1879)'s equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \text{ Gauss,} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday,} \\ \nabla \cdot \mathbf{B} &= 0 \text{ Gauss,} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{Ampere.} \end{aligned}$$

The symbol **E** is the electric field, **B** is the magnetic field, $\nabla \cdot$ is the divergence operator, $\nabla \times$ is the curl operator, **J** is the current density, μ is permeability of free space constant, ϵ is permittivity of free space and ρ is the charge density.

For small particles that are very close to one another, Coulomb's potential is best replaced by the Yukawa potential

$$V = -K_e \frac{q}{r} e^{r/r_0},$$

for some parameter r_0.

3.0.1 The vector space $\{\mathbf{E}, \mathbf{B}, \mathbf{H}\}$

One other equation necessary in our investigation is a wave equation. To introduce it, we first note that a wave equation for electric magnetic fields **E** and **B** can be deduced from Maxwell's equation. If **H** is the third field associated with gravitational waves, it follows then that the set $\{\mathbf{E}, \mathbf{B}, \mathbf{H}\}$ is an orthogonal group $O(3, F)$ of degree 3 over a *field* F. The third field **H** is subsequently defined by

$$\mathbf{H} = \mathbf{E} \times \mathbf{B},$$

so that

$$
\begin{aligned}
\frac{\nabla \cdot \mathbf{H}}{h_0} &= \nabla \cdot (\mathbf{E} \times \mathbf{B}) \\
&= (\nabla \times \mathbf{B}) \cdot -\mathbf{E}(\nabla \times \mathbf{B}) \\
&= -\mathbf{B} \cdot \{-\frac{\partial \mathbf{B}}{\partial \mathbf{t}}\} - \mathbf{E} \cdot \{\mu_{\mathbf{0}}\mathbf{J} + \mu_{\mathbf{0}}\epsilon_{\mathbf{0}}\frac{\partial \mathbf{E}}{\partial \mathbf{t}}\} \\
&= -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial \mathbf{t}} - \mu_{\mathbf{0}}\mathbf{E} \cdot \mathbf{J} - \mu_{\mathbf{0}}\epsilon_{\mathbf{0}}\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial \mathbf{t}}
\end{aligned}
\tag{3.0.3}
$$

$$
\nabla \cdot \mathbf{H} = -\mu_0 \mathbf{E} \cdot \mathbf{J} - \mu_0 \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \tag{3.0.4}
$$

and

$$
\begin{aligned}
\frac{\nabla \times \mathbf{H}}{h_0} &= \nabla(\mathbf{E} \times \mathbf{B}) \\
&= \{\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla\}\mathbf{E} - \{\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla\}\mathbf{B} \\
&= \{0 + \mathbf{B} \cdot \nabla\}\mathbf{E} - \{\frac{\rho}{\epsilon_{\mathbf{0}}} + \mathbf{E} \cdot \nabla\}\mathbf{B} \\
&= (\mathbf{B} \cdot \nabla)\,\mathbf{E}\frac{\rho}{\epsilon_0}\mathbf{B} - (\mathbf{E} \cdot \nabla)\,\mathbf{B}.
\end{aligned}
$$

In space , where $\rho = 0$, we get from the above equations:

$$
\begin{aligned}
\nabla \times (\nabla \times \mathbf{H}) &= \mu_0\epsilon_0(\nabla \times \mathbf{E_t}) \times \mathbf{E} + \mu_{\mathbf{0}}\epsilon_{\mathbf{0}}\mathbf{E_t} \times (\nabla \times \mathbf{E}) - (\nabla \times \mathbf{B}) \times \mathbf{B_t} \\
&\quad -\mathbf{B} \times (\nabla \times \mathbf{B_t}) \\
&= \mu_0\epsilon_0(-\mathbf{B_t})_{\mathbf{t}} \times \mathbf{E} + \mu_{\mathbf{0}}\epsilon_{\mathbf{0}}.\mathbf{E_t} \times (-\mathbf{B_t}) - (\mu_{\mathbf{0}}\epsilon_{\mathbf{0}}.\mathbf{E_t}) \times \mathbf{B_t} \\
&\quad -\mathbf{B} \times (\mu_{\mathbf{0}}\epsilon_{\mathbf{0}}.\mathbf{E_t})_{\mathbf{t}} \\
&= \mu_0\epsilon_0\{-\mathbf{B_{tt}} \times \mathbf{E} - \mathbf{B} \times \mathbf{E_{tt}}\} + \mu_{\mathbf{0}}\epsilon_{\mathbf{0}}\{-\mathbf{E_t} \times \mathbf{B_t} - \mathbf{E_t} \times \mathbf{B_t}\} \\
&= \mu_0\epsilon_0\{-\mathbf{B_{tt}} \times \mathbf{E} - \mathbf{B} \times \mathbf{E_{tt}} - \mathbf{2E_t} \times \mathbf{B_t}\} \\
&= \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}(\mathbf{B} \times \mathbf{E}) \\
&= \mu_0\epsilon_0.\mathbf{H_{tt}}
\end{aligned}
\tag{3.0.5}
$$

a wave equation in **H**.

As a consequence, and after a very lengthy analysis, we get a tenth-order polynomial

$$
\alpha_5 U^{10} + \alpha_4 U^8 + \alpha_3 U^6 + \alpha_2 U^4 + \alpha_1 U^2 + \alpha_0 = 0, \tag{3.0.6}
$$

with the parameters

$$\begin{aligned}\alpha_0 &= -4\mu^4, \quad \alpha_1 = 8r_0\mu^3, \quad \alpha_2 = -8c^2 r_0\mu^3, \\ \alpha_3 &= 12c^2 r_0^2\mu^2, \quad \alpha_4 = -6c^2 r_0^3\mu, \quad \alpha_5 = c^2 r_0^4\mu,\end{aligned}$$

and $\mu = -K_e(e^-)^2/9$.

The universal constant is subsequently given by

$$\begin{aligned}\tilde{G} &= \frac{i}{u\ Z_H\ \left(-K_e\frac{(e^-)^2}{9}\right)^2}\left(2K_v\frac{(e^-)^2}{9}\left(\frac{U^2}{2}-\frac{\left(-K_e\frac{(e^-)^2}{9}\right)}{r_0}\right)^3\right)\times \\ &\left(\frac{U^2\left(-K_e\frac{(e^-)^2}{9}\right)^2}{2\left(\frac{U^2}{2}-\frac{\left(-K_e\frac{(e^-)^2}{9}\right)}{r_0}\right)^3}+\left(r_0+\frac{e^{\frac{i\pi}{3}}\left(-K_e\frac{(e^-)^2}{9}\right)}{\frac{U^2}{2}-\frac{\left(-K_e\frac{(e^-)^2}{9}\right)}{r_0}}\right)^2\right). \qquad (3.0.7)\end{aligned}$$

The parameter u is the atomic mass unit and Z_H is the atomic number of hydrogen. The parameter $\tilde{G}$ has two components. That is,

$$\tilde{G} = G + iG_m.$$

The real component G is the gravitational constant we seek. The other component G_m can be used in Cauchy's law to determine the stress tensor τ_{ij}. The tensor, together with a suitable constitutive law, should lead to the universal momentum. The momentum in turn leads to the current density $\mathbf{J}$ appearing in Maxwell's equations. This current density brings us back to (3.0.6) and (3.0.7). This cycle begs the question as to whether it is the momentum that is being sustained by the gravitational waves, or the other way around.

A closer look at equation (3.0.6) reveals the general quintic equation (0.1). This results from letting

$$x \ = \ U^2,$$

with $a_i = \alpha_1/\alpha_5,\ i = 1, 2, 3, 4$ vice versa

3.0.2 A formula for the big 'G'

From the quintic solution x then followes the relative velocity $U = \sqrt{x}$. The physical parameters needed to evaluate this velocity are

$$\begin{aligned}
G &= 6.67259 \times 10^{-11} Nm^2/\mathrm{kg}^2, \quad \text{gravitational constant from experiment,} \\
r_0 &= \frac{1}{2}(0.52917720859)\mathring{\mathrm{A}}, \quad \text{Bohr's atom size,} \\
r_0 &= \frac{1}{2}(0.528)\mathring{\mathrm{A}}, \quad \text{hypothetical atom size,} \\
e^- &= 1.602176487 \times 10^{-19}\mathrm{C}, \quad \text{charge of an electron,} \\
K_v &= \frac{\mu_0}{4\pi} = 1 \times 10^{-7}\mathrm{Wb/(A\ m)}, \qquad \text{permeability of free space,} \\
K_e &= 8.987551787 \times 10^{9}\mathrm{N/C^2}, \qquad \text{permittivity of free space,} \\
c &= 2.99792458 \times 10^{8}\mathrm{m/s}, \quad \text{speed of light in vacuum,} \\
Z_H &= 1.000794, \quad \text{mass number of a hydrogen atom,} \\
u &= 1.660538782 \times 10^{-27}\mathrm{kg}, \quad \text{atomic mass unit.}
\end{aligned}$$

The actual numerical value corresponding to Bohr's atom size 0.529$\mathring{\mathrm{A}}$ is $U = \pm 6.89696\, i \times 10^{-10} m/s$, from which the gravitational constant was found to be

$$G = 6.656 \times 10^{-11}\mathrm{Nm^2/kg^2}. \tag{3.0.8}$$

Different texts give different values for atomic sizes. The best results resulted from the atomic size 0.528$\mathring{\mathrm{A}}$, the hypothetical hydrogen atomic size (any value above or below veers away from the experimental result), and it led to the velocity $U = \pm 6.90451\, i \times 10^{-10} m/s$, from which the gravitational constant was found to be

$$G = 6.671 \times 10^{-11}\mathrm{Nm^2/kg^2}. \tag{3.0.9}$$

3.1 Conclusion

We have achieved the objectives that were set . A formula for solving the quintic polynomial in radicals has been determined and a formula for expressing the gravitational constants in terms of other known physical constant was determined. In addition, we have confirmed suspicions that the computer based *Symbolic package* does use approximation techniques.

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