FINDING AN EFFECTIVE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL APPROACH FOR CIRCLE GEOMETRY

by

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ABSTRACT

This research study carried out an investigation into finding a contemporary problem- solving instructional approach that will be effective for teaching and learning of mathematics in South African schools, with specific focus on circle geometry. Prior to conducting this study, a retrospection was done into the mathematical practices implemented, in schools in South Africa, by researchers, educational practitioners and stakeholders such as Non-Governmental Organisations. The aforementioned unanimously identified the instructional approaches for teaching and learning of mathematics, particularly, the traditional teaching and learning approach, as problematic and counter-productive, and this might be contributing to poor learners' performances. In a bid to replace the obsolete traditional approach, the researcher in this study recommended: "teaching thinking skills" and "teaching effective problem-solving instructional approaches" as more appropriate. With regards to teaching thinking skills, the infusion approach (teaching thinking skills, along with content instructions), was highlighted. For teaching effective problem-solving, Polya's Problem-Solving Model, was investigated. To ensure an effective design and implementation of the proposed problem-solving instructional approach, the APOS theory (ACE teaching cycle) was adopted. Also, the teaching and learning of circle geometry was carried out in a collaborative classroom setting. This proposed instructional approach was tentatively, labelled as "IPAC mathematics problem-solving instructional model" or simply, the "IPAC model". This was an acronym for the four elements of this new approach, namely - the infusion approach, Polya's approach, and APOS theory in a collaborative learning classroom. Two groups of Grade 11 mathematics learners served as participants for this study: group 1 - 11A had 30 learners (the control group) and group 2- 11B had 32 learners (the experimental group). Data collected methods for this study were: observations of participants in their natural classroom settings, recorded videos, questionnaires, photograph of participants' work (classwork/homework and standardized tests). This study followed a mixed-method research design, hence, both quantitative and qualitative data analyses procedures were implemented. The quantitative data was analysed by implementing inferential statistics and descriptive statistics, while the

APOS theory analysis was used to analyse the qualitative facet of the collected data. During the APOS theory analysis, content analysis was done on participants' written responses to each of the four standardized tests' data. The content analysis was carried out on the written responses of participants, from both the control and the experimental groups. The research findings that emanated from this study were the following: that this new method of teaching and learning is valid, practical and effective; there was a statistically significant improvement in the test scores of participants who were taught by the new instructional approach; participants' conceptual understanding, procedural fluency, strategic competence and mathematical reasoning skills were enhanced; participants' problem-solving competence improved, during and after the intervention; the IPAC model guided the majority of the participants to operate at the object and schema levels in relation to the APOS theory mental conceptions. Lastly, the ACE teaching instructional approach significantly guided and enhanced participants' cognitive engagement and development, which ultimately, optimized their problem-solving competence. Based on these research findings, the researcher recommended among others, that the new instructional approach - the IPAC model, should be implemented for teaching and learning of circle geometry in South African schools. The researcher also recommended that cultivation of thinking skills and implementation of effective problem-solving instructional approaches should be prioritized in mathematics classrooms in South Africa. The researcher established from this study that the developed IPAC model will serve as an effective and a reliable pedagogical tool which can address some of the teaching and learning challenges teachers and learners encounter in mathematics classrooms.

OPSOMMING

Hierdie navorsingstudie het 'n ondersoek gedoen na die vind van 'n kontemporêre probleemoplossende onderrigbenadering wat effektief sal wees vir onderrig en leer van wiskunde in Suid-Afrikaanse skole, met spesifieke fokus op sirkelmeetkunde. Voor die uitvoering van hierdie studie is 'n terugblik gedoen na die wiskundige praktyke wat in skole in Suid-Afrika geïmplementeer is deur navorsers, opvoedkundige praktisyns en belanghebbendes soos nie-regeringsorganisasies. Die instruksionele benaderings vir onderrig en leer van wiskunde, veral die tradisionele onderrig-en-leerbenadering, is eenparig geïdentifiseer as problematies en teenproduktief, en dit kan dalk bydra tot swak leerders se prestasies. In 'n poging om die uitgediende tradisionele benadering te vervang, het die navorser in hierdie studie aanbeveel: "onderrig van denkvaardighede" en "onderrig van effektiewe probleemoplossende onderrigbenaderings" as meer gepas. Met betrekking tot die onderrig van denkvaardig hede, is die infusiebenadering (onderrig van denkvaardighede, tesame met inhoudsinstruksies), uitgelig. Vir die onderrig van effektiewe probleemoplossing is Polya se probleemoplossingsmodel ondersoek. Om 'n effektiewe ontwerp en implementering van die voorgestelde probleemoplossende onderrigbenadering te verseker, is die APOS-teorie (GOS-onderrigsiklus) aanvaar. Die onderrig en leer van sirkelmeetkunde is ook in 'n samewerkende klaskameropset uitgevoer. Hierdie voorgestelde onderrigbenadering is voorlopig, gemerk as "IPAC wiskunde probleemoplossing instruksionele model" of eenvoudig die "IPAC model". Dit was 'n akroniem vir die vier elemente van hierdie nuwe benadering, naamlik - die infusiebenadering, Polya se benadering en APOS-teorie in 'n samewerkende leerklaskamer. Twee groepe graad 11-wiskunde-leerders het as deelnemers vir hierdie studie gedien: groep 1 - 11A het 30 leerders (die kontrolegroep) en groep 2-11B het 32 leerders (die eksperimentele groep). Data wat ingesamel is metodes vir hierdie studie was: waarnemings van deelnemers in hul natuurlike klaskamerinstellings, opgeneemde video's, vraelyste, foto van deelnemers se werk (klaswerk/huiswerk en gestandaardiseerde toetse). Hierdie studie het 'n gemengde-metode navorsingsontwerp gevolg, dus is beide kwantitatiewe en kwalitatiewe data-ontledingsprosedures geïmplementeer. Die kwantitatiewe data is ontleed deur inferensiële statistiek en beskrywende statistiek te implementeer, terwyl die APOS

teorie-analise gebruik is om te analiseer die kwalitatiewe faset van die versamelde data. Tydens die APOS-teorie-analise is inhoudsontleding gedoen op deelnemers se geskrewe antwoorde op elk van die vier gestandaardiseerde toetse se data. Die inhoudsanalise is uitgevoer op die geskrewe reaksie van deelnemers, van beide die kontrole- en die eksperimentele groepe. Die navorsingsbevindinge wat uit hierdie studie voortgespruit het, was die volgende: dat hierdie nuwe metode van onderrig en leer geldig, prakties en effektief is; daar was 'n statisties beduidende verbetering in die toetstellings van deelnemers wat deur die nuwe onderrigbenadering onderrig is; deelnemers se konseptuele begrip, prosedurele vlotheid, strategiese bevoegdheid en wiskundige redenasievaardighede is verbeter; deelnemers se probleemoplossingsbevoegdheid het verbeter, tydens en na die intervensie; die IPAC-model het die meerderheid van die deelnemers gelei om op die objek- en skemavlakke te werk in verhouding tot die APOS-teorie se verstandelike opvattings. Laastens het die GOS-onderrigbenadering die deelnemers se kognitiewe betrokkenheid en ontwikkeling aansienlik gelei en verbeter, wat uiteindelik hul probleemoplossingsbevoegdheid geoptimaliseer het. Op grond van hierdie navorsingsbevindinge het die navorser onder andere aanbeveel dat die nuwe onderrigbenadering - die IPAC-model, geïmplementeer moet word vir onderrig en leer van sirkelmeetkunde in Suid-Afrikaanse skole. Die navorser het ook aanbeveel dat die kweek van denkvaardighede en implementering van effektiewe probleemoplossende onderrigbenaderings in wiskundeklaskamers in Suid-Afrika geprioritiseer moet word. Die navorser het uit hierdie studie vasgestel dat die ontwikkelde IPAC-model sal dien as 'n effektiewe en betroubare pedagogiese hulpmiddel wat sommige van die onderrig- en leeruitdagings wat onderwysers en leerders in wiskundeklaskamers ondervind, kan aanspreek.

I-ABSTRACT

Lolu cwaningo luqukethe uphenyo mayelana nokuthola ikhambi elingaxazulula ekutholeni indlela eqondile engaletha imiphumela ewusizo ekufundiseni nasekufundeni kwezibalo ezikoleni zaseMzansi Africa, ezophinde ibhekane ngqo ne circle Geometry. Ngaphambi kokuba kuqale lolu cwaningo, kube nolunye ucwaningo olunzulu olwenziwe ngezinye izindlela esezivele zikhona mayelana nezibalo, ezikoleni zaseMzansi Africa, lwenziwa ngabacwaningi, izifundiswa ezingo ncweti Kanye nezinhlangano ezizimele. Inhlangano ebizwa nge okushiwo ngenhla luhlonze indlela eqondile yokufundisa nokufunda izibalo, ikakhulukazi, indlela ejwayelekile yokwenza, njengezindlela eziyinkinga nezingahambisani, futhi lokhu ngungaba yimbangela ekungenzini kahle kwabafundi. Emkhankasweni wokushintsha lolu hlelo oludala lokwenza olungasasizi, uMhlaziyi kulolu cwaningo uncome ukuthi: "ikhono elufundisa ukuzicabangela" Kanye "nekhono lokufundisa elisebenzayo ukuzixazululela izinkinga" njengendlela okuyiyo efanele. Mayelana nekhono elifundisa ukuzicabangela, indlela eyiqophelo (ikhono elifundisa ukuzicabangela, elihambisana nemigomo equkethwe), luthintiwe. Mayelana nohlelo oluwusizo ekuxazululeni izinkinga, uhlelo luka Polya lokuxazulula izinkinga luphenyiwe. Ukuqinisekisa ukuthi uhlelo olusebenzayo futhi oluzosentsenziswa ekuphakamiseni indlela eqondile enemigomo ekuxazululeni izinkinga yokwenza, i APOS theory (ACE teaching cycle) iyona ekhethiwe. Okunye, uhlelo lokufundisa nokufunda i circle geometry lukhishiwe endleleni ehlanganisayo yokuhlala egunjini lokufunda. Okwamanje Lolu hlelo oluphakanyisiwe lokufundisa, lubekwe njenge "IPAC indlela yezibalo eqondile yokuxazulula izinkinga enemigomo" . Lokhu kuyigama elifinqiwe elakhiwe izinhlamvu ezine kule ndlela entsha ebizwa nge infusion approach, Polya's approach, Kanye ne APOS theory egunjini lokufunda elihlanganisile. Amaqembu amabili ebanga le shuminanye labafundi bezibalo basentshenzisiwe ukubamba iqhaza kulolu cwaningo: iqembu lokuqala ibanga 11A ebelinabafundi abangu 30 (iqembu labaqondisi) bese iqembu lesibili ibanga 11B ebelinabafundi abangu 32 (iqembu elenzayo). Ucwaningo oluqoqiwe lwalendlela lube kanje: imibono yalaba ebekade bebambe iqhaza egunjini lokufunda obuhleliwe, baqophe amavidiyo, babhala imibuzo, bathatha izithombe zalaba ekade bembambe iqhaza lwalomsebenzi wokubamba iqhaza. (imisebenzi yasegunjini

lokufunda/imisebenzi yasekhaya Kanye nokwenza uvivinyo). Lolu phenyo lulandele uhlelo oluxubile okuwuhlelo lokuphenya, yingakho zombili lezi zinhlelo zokuqukethwe nokuseZingeni zokuqoqa uphenyo olwenziwe zisentshenzisiwe. Uhlelo lokuqukethwe lemininingwane lusentshenzisiwe ukuhlaziya ngokusebenzisa uhlelo lokuqoqa okutholakele Kanye nohlelo lokwenza okutholakele, futhi kube kwenziwa ne APOS theory analysis ukuhlaziya okusezingeni eliphezulu zigxenye zonke lwemininingwane eqoqiwe. Ngesikhathi se APOS theory analysis, ukuhlaziywa kokuqukethwe okwenziwe ababambe iqhaza babhale okwenzekile ngesikhathi benza lezi zivivinyo ezine ezibekiwe. Uhlelo lokuhlaziya okuqukethwe lwenziwe labhalwa yilaba kade bebambe iqhaza, kuwo womabili amaqembu , elokuqondisa nelokwenza. Uphenyo olutholakele kulolu hlelo lunje: lolu hlelo lokufundisa nokufunda luyasebenza, luyenzeka, futhi lunomehluko: ngokwezibalo kube nomehluko omkhulu oncono ezibalweni zalabo ekade bebambe iqhaza besebenzisa indlela entsha yemigomo: bonke ekade bebambe iqhaza bathole ithuba lokuthi kuthuthuke amakhono abo ekwazini ukuqonda ukuzicabangela, ekwazini ukwenza izinto ezinomehluko eyinqubomgomo, ukumelana nezindlela eziningi eziphumelelisayo Kanye nekhono lokuqonda izibalo; ikhono lalabo ekade bebambe iqhaza ekuxazululeni izinkinga ngokusezingeni lithuthukile, ngesikhathi nangemuva kokwenza ucwaningo; I IPAC model ukwenzisa abaningi balaba ekade bebambe iqhaza kalula umsebenzi ngokuhlukana kwamazinga kusentsenziswa i APOS theory. Ekugcineni, indlela yokwenza ebizwa nge ACE teaching ikwazile okwenzisa kahle ngokusezingeni eliphezulu futhi yakhuphula labo ebekade bebambe iqhaza yaphinde yabathuthukisa, lokhu okwenze bakwazi ukuba sezingeni lokuphumelela ukuxazulula izinkinga. Ngenxa yalokhu okutholakale kucwaningo, umcwaningi uncome ukuthi kokunye, indlela entsha yokwenza ngemigomo – i-IPAC, kumele isentshenziswe ekufundiseni nasekufundeni i circle geometry ezikoleni zaseMzansi Africa. Umcwaningi uphinde waphakamisa ukuthi ukuthuthukisa ikhono lokuzicabangela nokwenziwa kwezindlela ezisebenzayo zokuxazulula izinkinga kumele zibekwe phambili emagunjini okufunda izibalo eMzansi Africa. Umcwaningi ubeke indlela eseqophelweni eliphezulu eyisisekelo kusukela kwisifundo esenziwe yokuthi i IPAC model iyona esebenza njenge ndlela eyithuluzi elibonakalayo futhi elinemiphumela emihle ethembekile, engakwazi ukubhekana nezinkinga futhi ixazulule izinqinamba zokufundisa nokufunda ezikoleni, lezi othisha nabafundi ababhekana nazo egunjini lokufundela izibalo.

Key terms

- APOS theory
- Demystifying mathematics
- Heuristics
- Infusion Approach
- Instructional Approaches
- Metacognition
- Polya's Approach
- Teaching thinking
- Teaching problem-solving strategies
- Thinking mathematically

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DECLARATION

Name: Fitzgerald Abakah

Student number: 57576009

Degree: Doctor of Philosophy (Mathematics Education)

Title: Finding an effective problem-solving heuristic instructional approach for circle geometry.

I declare that the above thesis is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

I further declare that I submitted the thesis to originality checking software and that it falls within the accepted requirements for originality.

I further declare that I have not previously submitted this work, or part of it, for examination at Unisa for another qualification or at any other higher education institution.

30-12-2022

(SIGNATURE) DATE

(FITZGERALD ABAKAH)

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DEDICATION

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LIST OF ABBREVIATIONS

- CAPs: Curriculum and Assessment Policy Statement
- CE: Class Exercises
- DDP: District Development Programme
- DoBE: Department of Basic Education
- FET: Further Education and Training
- FSS: Fresh Start Schools
- GD: Genetic Decomposition
- GET: General Education and Training
- HOD- Head of Department for Mathematics
- IPAC: Infusion approach, Polya's approach, and APOS theory in a collaborative learning classroom
- MGD: Modified Genetic Decomposition
- MTLF: Mathematics Teaching and Learning Framework
- NCDoE: Northern Cape Department of Education
- NCS: National Curriculum Statement
- NCTM: National Council of Teachers of Mathematics
- NDP: National Development Plan
- NECT: National Education Collaboration Trust

- NGO: Non-Governmental Organization
- NSC: National Senior Certificate
- PGD: Preliminary Genetic Decomposition
- TBL: Thinking-Based Learning
- TIMSS: Trends in Mathematics and Science Study
- TLMs: Teaching and Learning Materials
- T-R: Teacher-Researcher
- TVET: Technical and Vocational Education and Training

CHAPTER ONE

BACKGROUND OF THE STUDY

1.1 Overview of this Chapter

This chapter is aimed at orientating the reader, highlighting the problem that led to the study, what has been done previously to address the problem and what the researcher had set out to achieve in the study. This chapter introduces the reader to effective problem-solving heuristic instructional approach for circle geometry, the concept of thinking, teaching thinking, and the rationale of the study as well as the significance of the study, the problem statement, aims and objectives of the study. Included in the discussions are - the research questions, hypothesis of the study, definitions of key terms and variables, scope of the study and lastly, a summary of the study.

1.1.1 Introduction

This study sought to find an effective, problem-solving heuristic instructional approach for mathematics, with a focus on circle geometry. Problem-solving is an integral instructional approach (Syarifuddin & Atweh, 2022). Mathematics teaching and learning activities are now structured, in a bid to enable learners acquire problem-solving competences. In view of this, a lot of contemporary Mathematics curricula are now focused on problem-solving since it has become the modern medium for the development of mathematical knowledge (Taplin, 2010; Rahman, Ghazali, & Ismail, 2003). Contemporary Mathematics curricula require students to be able to build new mathematical knowledge through solving problems that arise in mathematics and other context. In addition, students are expected to adopt and apply a variety of appropriate strategies to achieve this and are also mandated to monitor and reflect on the process of mathematical problem-solving. This presupposes that students need to learn how to solve problems; their efforts can be promising and productive if effective problem-solving instructional approaches are implemented in mathematics classrooms (Phuntsho & Dema, 2019).

Finding effective problem-solving instructional approaches for teaching and learning of mathematics has taken center stage for researchers, for more than a century. Such approaches are expected to serve modern demands of applying mathematical ideas to problem-solving, not only in mathematics but also, in real life situations. That is the reason problem-solving has become the focal point of most contemporary mathematics curricula, in which the world is perceived as gearing towards the fourth industrial revolution; a curriculum, hence, which teach students reasoning and thinking skills is paramount. This will demand students to engage in explicit organized thinking about mathematical content and enable them to facilitate their reflection on problem-solving, thereby, giving students the opportunity to engage in more practice. This can be achieved by employing any thinking skills they might have acquired in the mathematics classroom (Swartz & Reagan, 1998).

The mathematics curriculum in Singapore, for instance, is designed around a framework with "Mathematics Problem-solving" as its primary aim. This framework is expected to guide teachers to develop students into critical thinkers and good problem-solvers by inculcating creative and critical-thinking ideas in their mathematics lessons. Mathematics teachers, therefore, are expected to blend mathematics content with teaching thinking skills (Ministry of Education, 2001). In South Africa as well, the mathematics curriculum, clearly stipulated in the Curriculum and Assessment Policy Statement (CAPs) document, was designed to incorporate mathematical modeling; this is a key component of problem-solving (Department of Basic Education [DoBE], 2010). This was incorporated to enable learners to connect mathematical ideas to real life contexts, however, its implementation in classrooms, according to reports and workshops organized by the DoBE, has been highly less efficacious. Mathematical knowledge is tentative, intuitive, subjective and dynamic, hence, South Africa needs to critically review its unfulfilling mathematical practices, so that the country may fulfill contemporary global mathematical demands (Cuoco, 2000; DoBE, 2010; DoBE, 2018).

1.2 Background and purpose of the study

"Umalusi has observed a worrying trend in mathematics, where the subject does not seem to be progressing in tandem with other cognate subjects in terms of learner performance. Mathematics is not showing any signs of improvement, hence, Mathematics needs to be taught differently if pupils' marks are going to improve. Mathematics teachers 'must teach differently'……." (Volmink, 2020, p.2).

The Council for Quality Assurance in General and Further Education and Training (Umalusi), prior to the release of the 2019 Grade 12 matric results in January 2020, made the above comment after the poor matric results, in Mathematics. This shows the concerns about the teaching of the subject in schools, indicating that there is a need for mathematics teachers to approach the subject differently. The chairperson of Umalusi, was obviously ominous about the current situation's implications on posterity; a clear indication that the way mathematics is taught in South African schools is unsatisfactory and below national and global expectations. This has led to learners mystifying mathematics which, have impacted negatively on learners' achievements in the subject across all levels of the South Africa's educational hierarchy; he has, therefore, labelled mathematics as a difficult-to-teach subject for mathematics teachers in South Africa (Umalusi, 2020). This reiterates that there is a need for South African mathematics teachers to switch to a new teaching paradigm, however, the mammoth and contentious question is, *"Which teaching paradigm could that be?"* To identify a remedy to the above conundrum, the chairperson of Umalusi issued "a clarion call" to all and sundry involved in teaching mathematics. From this, it is hoped that a more proactive and concerted effort can be orchestrated to initiate the much-needed action. The results, can, greatly influence how mathematics can be taught in South African schools to improve the teaching and learning of the subject.

Circle geometry is an integral content under Euclidean geometry in the mathematics curriculum in South Africa. It needs to be mentioned that Euclidean geometry covers about 50 marks out of the total marks allocation of 150, for the NSC Paper 2 mathematics examinations. This represents more than 30% of the total mark allocation for Paper 2, therefore, low marks obtained by learners in Euclidean geometry will inevitably affect their overall performance in mathematics. As a mathematics teacher, I always spend substantial time, motivating and encouraging learners about mathematics' relevance in the world and the fact that Euclidean geometry carries a significant part of the mark allocation in examinations, as illustrated in Figure 1.1 below (DoBE, 2018; Abakah, 2019).

Weighting of Content Areas			
Description	**Grade 10**	**Grade 11**	**Grade. 12**
PAPER 1 (Grades 12:bookwork: maximum 6 marks)			
Algebra and Equations (and inequalities)	30 ± 3	45 ± 3	25 ± 3
Patterns and Sequences	15 ± 3	25 ± 3	25 ± 3
Finance and Growth	10 ± 3		
Finance, growth and decay		15 ± 3	15 ± 3
Functions and Graphs	30 ± 3	45 ± 3	35 ± 3
Differential Calculus			35 ± 3
Probability	15 ± 3	20 ± 3	15 ± 3
TOTAL	**100**	**150**	**150**
PAPER 2: Grades 11 and 12: theorems and/or trigonometric proofs: maximum 12 marks			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Statistics	15 ± 3	20 ± 3	20 ± 3
Analytical Geometry	15 ± 3	30 ± 3	40 ± 3
Trigonometry	40 ± 3	50 ± 3	40 ± 3
Euclidean Geometry and Measurement	30 ± 3	50 ± 3	50 ± 3
TOTAL	**100**	**150**	**150**

Figure 1.1: Weighting of Content Areas (CAPS, 2010, p.10)

The low performance of mathematics learners in relation to circle geometry questions, has placed a sharp focus on how circle geometry instructions are carried out in the classrooms. Another reason may be attributed to the fact that high school mathematics learners have not yet developed appropriate mental structures at the process, object and schema levels of the APOS Theory, especially at the schema level. For clarification: the process level demands learners to be able to recall from memory and apply appropriate circle geometry concepts and routine procedures; the object level demands learners to be able to reflect on internalised ideas to perform higher-order reasoning and creative thinking around circle geometry concepts and the schema level demands learners to be able to solve non-routine circle geometry problems through the applications of advanced geometric and reflective thinking (Brijlall, 2020). Some of their incorrect responses to questions on circle geometry indicate that they lack appropriate problem-solving skills, since some of the modus operandi for teaching circle geometry in many classrooms have proved to be a source of bewilderment. An overwhelming majority of South African teachers are still addicted to the traditional method of teaching, however, a vast number of countries such as Singapore and Netherlands are immersed deeply in constructivism teaching methodologies (Thompson, 2014) which have proven to promote active and efficient teaching and learning in mathematics classrooms (DoBE, 2018;Thompson, 2014).

Review of relevant documents testify to a continuous trend of poor performances of learners in mathematics, locally and internationally. On the local front, the diagnostic report released by the Department of Basic Education, annually, on the Grade 12 matric candidates' performance is very informative. Detailed in this document include a question-by-question analysis of the question paper, content areas of questions that need improvement, misconceptions and common errors of candidates and suggestions for improvement. The aforementioned shows that candidates are in dire need of effective problem-solving instructional approaches with Euclidean geometry being consistently and consecutively the most talked about content area. Many misconceptions and errors identified from candidates' responses, the most difficult content area when compared to other content areas, and Euclidean geometry as a content area that needs a lot of improvement have been recorded. All these point to the fact that the abysmal performance of candidates in Euclidean geometry might have contributed to the high failure rates in mathematics, and circle geometry is an integral part of Euclidean geometry, highlighting the need for this study (DoBE Diagnostic reports, 2017, 2018, 2019, 2020).

Internationally, South African learners' participation in Trends in Mathematics and Science Study (TIMSS), shows an unacceptable performance which displays a need for intervention. TIMSS has served as an external benchmark from 1995; it was conducted every four years, until 2019. It continuously showed how badly South African learners perform, when their mathematics scores are juxtaposed with those of other participating countries (Reddy, 2013). South Africa's Minister of Basic Education, Mrs Angelina Matsie Motshekga, on the 8th of December 2020, announced the results of South African learners' participation in TIMSS, for the year 2019, and these were partly, her words:

"TIMSS surveys pupils from 64 countries every four years in mathematics and science. In South Africa, the survey looked at Grade 5 and Grade 9 pupils. 519 schools and 20829 pupils participated in the survey. From the 2019 survey, 6 out of 10 pupils in South Africa lack basic mathematics and science knowledge. 4 out of 10 pupils demonstrated basic mathematics and science knowledge. The mathematics and science ability levels increased from 11% in 2013 to 41% in 2019. While the country had recorded significant progress, it was far behind its peers and competitors." (DoBE, 2020).

There are two points of interest from the Minister of Basic Education's 2019 TIMSS report and they are - (1) 4 out of 10 pupils had the ability to demonstrate basic mathematics and science knowledge; this was perceived to be a significant progress; (2) while the country had recorded significant progress, it was far behind its peers and competitors. The above, clearly demonstrate that South Africa's woes in the mathematics subject still persist, and are in need of urgent attention. As posited by Khanum (2006) learners who lack good educational foundation would inevitably encounter difficulties in the subsequent grades; this is exemplified in South African schools.

An analysis of the matric examinations for the past five years - 2016, 2017, 2018, 2019, 2020 (with a pass percentage at 30% and above) were 51,1 %; 51,9 %; 58,0 %, 54,6 %, and 53,8 %, respectively. For the same years, the pass percentage for mathematics at 40% and above, were 33,5 %; 35,1 %; 37,1 %; 35,0 % and 35,6 %, respectively. This implies that on the average, about half (50%) of the mathematics candidates end up not passing the subject, even at 30% pass rate (DoBE Diagnostic Report, 2020). With reference to (DoBE, 2010) - 25% of the task is allocated to knowledge procedure questions; 30% is demarcated for routine procedure questions, while 30% and 15% are allocated to complex procedures and problem-solving questions respectively, which are higher order questions. In total, therefore, 55% of a task are allocated for knowledge and routine procedure questions. These are low-order questions, yet, about half of mathematics candidates are not able to pass the mathematics paper even at 30% pass mark. This is an indication that previous mathematics education research and recommendations and other implemented intervention programs, obviously, have not had the expected impact on candidates' achievements.

My classroom experiences, in collaboration with my experiences as a Department of Education marker for the National Senior Certificate (NSC) Examinations for Grade 12 mathematics paper 2, concur with the diagnostic report and the TIMSS results (Abakah, 2019). Every piece of evidence, hence, highlights the underperformances of mathematics learners. This in my view, may be due to the instructional approaches that have been and are still being implemented in South African schools, as mentioned by Umalusi earlier. Ineffective - teaching, learning and assessment methods - implemented in mathematics classrooms are obvious contributory factors to students' underperformance in Mathematics (Umugiraneza, Bansilal & North, 2017). The researcher holds the view that poor classroom mathematical practices, have greatly contributed to learners' underperformance and for some learners to reconsider their decision to study mathematics further as shown below.

1.3 Trend of mathematics learners across the FET phase

"The decrease in candidate numbers in mathematics and the concomitant increase in numbers offering mathematical literacy remain a matter of concern. It's not getting any worse and it's not getting any better - it's staying in the same place." (Volmink, 2020, p.2).

Mathematics is a compulsory subject from Grade R to Grade 9 in South Africa, however, at the Further Education and Training phase (FET), that is, from Grades 10- 12, learners have the option to either study Mathematics, Mathematical Literacy or Technical Mathematics; emphasis was placed on mathematics learners at the FET Phase since this research study is aimed at them. Ten schools in an education district of the Northern Cape Department of Education (NCDoE), were randomly selected. The number of mathematics learners from Grade 10 to Grade 12 of each school was recorded, as shown in the Table below. For the purpose of ensuring anonymity, the names of the education district and the ten schools have not been provided.

Table 1.1: Trend of mathematics learners across the FET phase (NCDoE, 2021).

School	No. of Grade 10	No. of Grade 11	No. of Grade 12
1	132	96	62
2	26	12	3
3	36	21	12
4	52	37	27
5	145	94	68
6	117	88	61
7	38	29	18
8	41	34	22
9	27	16	9
10	23	19	11
TOTAL	**637**	**446**	**293**

It can be observed from the Table above that for each of the ten randomly-selected schools, the numbers of mathematics learners decline across the FET phase: Grades 10-12. It can be observed that for some of the selected schools, not even half of the learners who started Mathematics in Grade 10, continued with the subject to Grade 12. Most of those who discontinued, it can be assumed, opted for Mathematical literacy. This may be in accordance with the idea that learners have developed a negative attitude towards mathematics (Ndlovu, 2017). The reason for learners discontinuing with Mathematics as a subject, either in Grades 10, 11 or Grade 12, is really worrisome, and needs to be investigated further (NCDoE, 2021). About 50% of the few learners who study Mathematics up to Grade 12 end up not passing the subject, even at 30% pass mark, as mentioned earlier.

From my experience as a mathematics teacher in South Africa, prior to the 2020 academic year, not all learners who study Mathematics up to Grade 12 write the mathematics NSC Examination. Some of the learners may, either willingly, decide to modulate the subject or the school authorities and/or subject teachers can decide not to allow the learner(s) to write the subject during the NSC Examination. This is probably, because the teachers were not convinced about the chances of the said learner(s) passing the subject due to their low competence level; such a situation will affect the pass rate of the subject in particular, as well as the pass rate of the school in general.

It needs to be explained here that, to modulate a subject means that subject will not be written by the candidate during the November/December NSC Examination year. The candidate will write the subject at a later examination period, known as the Supplementary examinations, which is normally conducted in May/June every year. Some Grade 12 learners who modulate the NSC Examination mathematics paper, do not avail themselves sometimes for the writing of the supplementary examination as some may have dropped out in Grade 12, which is a cause for concern. This indicates that the dropout from the mathematics subject affects learners from Grades 10 -12 (NCDoE, 2021).

South Africa's mathematics troubles have also been recorded in the junior grades (DoBE, 2018), although, the situation with the high school phase, inevitably, negatively affect studies at the tertiary levels - universities and TVET colleges (DoBE, 2018). At tertiary level, students' high failure rate in mathematics prevents their enrolment in mathematics-related courses, such as Engineering, Actuarial Science, Accounting, Economics, Mathematics and Statistics. Sometimes, even if these students do register for such courses, there is a significant dropout rate as they quit from these mathematics-related courses. Tertiary records indicate that there is the high students' attrition rate from engineering courses, at the third year of study, in the following courses: civil engineering, mechanical engineering, aerospace/aeronautical engineering, electrical engineering and electronic engineering, among others. TVET college students in South Africa are also not spared from the mathematics failure picture; a lot of these students are unable to complete engineering courses, from N1-N6 levels. More failure and drop-out rates are recorded from N3-N6 levels (DoBE, 2018; NCDoE, 2021).

The narrations above, reveal that South Africa's mathematics struggles cut across all levels of her educational hierarchy. The considerable failure rate at the tertiary level, has also raised serious concerns about 'what' and `how' mathematics is taught at high schools, forcing lecturers and authorities at the tertiary level to have serious reservations about the teaching and learning of the subject, in South Africa, at pre-tertiary levels (DoBE, 2018).

The researcher, supports the assertion that ineffective problem-solving instructional approaches that have been employed in mathematics classrooms, contribute to learners' underperformance in Mathematics. These ineffective approaches also, contribute greatly to the inability of mathematics instructors to make the subject attractive and interesting in schools. This inability of the teachers, hence, results in students' poor performances, as well as high attrition and dropout rates in connection with the subject. In this situation, tertiary institutions may be blaming FET teachers, FET teachers may also be blaming primary school teachers and so on. The country, obviously, cannot continue with this blaming game. A collaborative and more concerted effort is needed by all and sundry in the teaching of mathematics across all levels of South Africa's educational hierarchy. This can be achieved by adopting and implementing effective and efficient instructional approaches in mathematics classrooms to improve the teaching and learning of mathematics in classrooms, particularly, from the school's base level. This will also contribute immensely in making mathematics attractive and an interesting subject to be studied in schools (DoBE, 2018).

1.4 What has been done by DoBE previously to address the problem?

The Department of Basic Education in South Africa have not stayed unperturbed about the nation's mathematics struggles since 1994, when South Africa became a recognised democratic country. DoBE has constantly admitted and recognised all difficulties encountered with mathematics as a subject (DoBE, 2010; DoBE, 2018). This is with regards to the teaching and learning of the subject in schools, which has resulted in abysmal performances of learners in examinations. DoBE in conjunction with mathematics education researchers in the country, have constantly experimented with a plethora of initiatives (Brijlall, 2015). These have centred on how teaching and learning of mathematics can be effectively carried out in schools in South Africa. Most of such initiatives have focused on learner-centred approaches (DoBE, 2018; Umugiraneza, Bansilal, & North, 2017) and these have resulted in the formation of the National Education Collaboration Trust (NECT) and the formulation of the Mathematics Teaching and Learning Framework (MTLF). All these bodies are discussed in detail below.

1.4.1 National Education Collaboration Trust (NECT)

This body is an amalgamation of stakeholders in education - teacher unions, businesses, religious groups, trusts, foundations and non-Governmental organizations (NGOs). This body was born in the year 2015 to help DoBE to improve education, to support the agenda of the National Development Plan (NDP) after it was inaugurated in the year 2012. The NDP envisions that by 2030, schools will provide all learners with quality education, especially in Literacy, Mathematics and Science. The satisfactory performance level to be achieved by the year 2030 has been determined and set to at least 50%. Also, the target of learners and schools performing at this level has also been determined and set at 80%. Furthermore, performance targets for Grade 6 and Grade 8 have been set at 600 and 500 in

SACMEQ and TIMSS respectively; it is estimated that 450 000 learners should be eligible to study Mathematics at university by 2030 (DoBE, 2018).

The DoBE has mandated the NECT to conduct learning and training programmes on its behalf. This was with the objective of providing teachers with more skills, methodologies and content knowledge. This would enable them to be more efficacious in classrooms, with regards to the teaching of the said subjects which includes mathematics (NECT, 2015). The NECT, in its quest in improving education came up with the District Development Programme (DDP). The DDP developed curriculum learning programmes for Mathematics, Science and Language teachers and involved district officials, principals, teachers, parents and learners. It began on an ad-hoc and pilot basis in the year 2015, with a small groups of schools, which were termed as 'Fresh Start Schools' (FSS). The FSS received training and support on the implementation of the curriculum programmes which were introduced by the DDP, before these learning programmes could be implemented in the rest of the schools (NECT, 2015).

As the researcher, I acknowledge that this NECT initiative is innovative and promising, thus, my hope to contribute my quota and ensure the agenda of the NECT comes into realisation and fulfilment. In line with the findings of UMALUSI and to be able to meet the NDP's goal by 2030, teachers need to teach differently. This can only be made possible by employing effective problem-solving instructional approaches in mathematics classrooms, particularly, circle geometry, which this study sets out to investigate (DoBE, 2018).

1.4.2 Mathematics Teaching and Learning Framework (MTLF)

In her foreword message to the South Africa's Mathematics Teaching and Learning Framework (MTLF) document, Basic Education Minister Mrs Angelina Matsie Motshekga, made the following remarks:

"During the 2016 Mathematics Indaba, I called for the overhauling of the South African pedagogical-content knowledge outlook in Mathematics. I said that we needed to reinvigorate the teaching of mathematics in its entirety – classroom learning practices, content, teaching and assessments. I emphasised the urgent need to pay particular attention to the development of a new curriculum for initial teacher education, induction and continuing professional development. This Mathematics Teaching and Learning Framework presented here is a first step towards achieving exactly that. The Framework seeks to and succeeds in laying a firm foundation for a new manner in which mathematics is taught thus changing the way it's learned" (DoBE, 2018, p.3).

This teaching and learning framework (as illustrated in Figure 2 below) was introduced to South African mathematics teachers to enable them employ appropriate, relevant and efficient teaching and learning strategies and approaches in mathematics classrooms. This was a follow-up to CAPS to ensure its effective implementation, in mathematical practices in South African schools. The focus of this framework is - to guide teachers to teach mathematics to learners effectively; to enable learners develop conceptual understanding, procedural fluency, strategic competence; to develop learners' ability to formulate, present, and decide on appropriate strategies to solve mathematical problems, mathematical reasoning skills, and to promote a learner-centred classroom (DoBE, 2018).

The researcher reckons that these expected learning outcomes are appropriate in promoting mathematical proficiency, however, 'how' the above learning outcomes can be practically and realistically achieved in mathematics classrooms, is what the researcher finds inadequately articulated in the document. The document, therefore, lacks appropriate instructional approaches to be implemented in the mathematics classrooms for the attainment of the above desired learning outcomes. This is what the researcher sought to address in this research study.

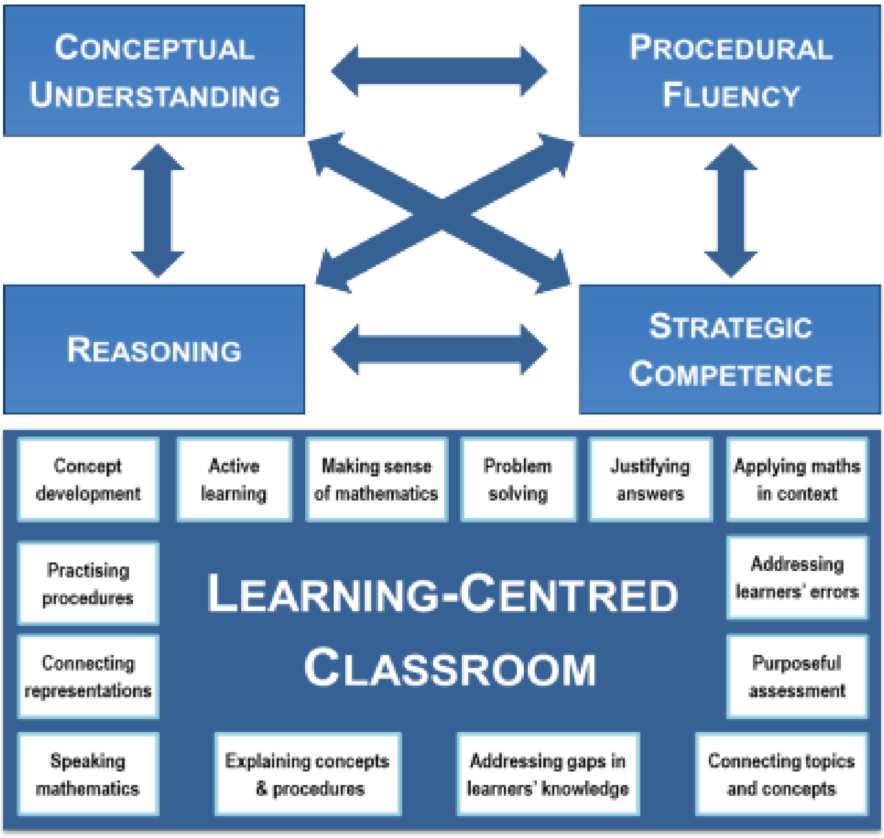


Figure 1.2: Mathematics teaching and learning framework for South Africa (DoBE, 2018, p.9).

1.5 What the researcher sets out to do differently in this study, to address the problem

As informed by Jinwen and Bikai (2007),"problem-solving can be viewed as both an instructional goal and as an instructional approach". In this study, the researcher carried out an investigation into an effective, problem-solving and heuristic instructional approach to improve the teaching and learning of circle geometry. In the process, the researcher investigated two factors which he regarded as seriously lacking in South Africa's Mathematics classroom instructions - (1) teaching thinking skills, (2) teaching effective problem-solving instructional approaches; these were regarded as germane/central to this study. These were anticipated as being helpful in demystifying mathematics, hence, they will serve as the fulcrum, in building an effective problem-solving instructional approach. These two factors are also in line with the CAPS curriculum for mathematics, which aims to develop learners into efficient problem- solvers who apply critical and creative thinking skills, both with mathematical and general problems. How these can be achieved was what this study sought to address (DoBE, 2010; DoBE, 2018).

The researcher purposefully perused learner-centred approaches and it's more advanced form - Thinking-Based Learning (TBL) approach - was considered to be ideal for this study (Swartz, Costa, Beyer, Reagan & Kallick, 2010). According to Swartz *et al.*, (2010), the goals of Thinking-Based-Learning are in three facets: (1) during schooling era, students thinking will become better; (2) students will become better with content learning; (3) after schooling era, students use of good thinking will not end, but rather, they will carry on with it by applying it in their every-day lives and professional work. These goals reiterates the need to inculcate teaching of thinking skills in South Africa's mathematics classrooms.

Walsh and Paul (1988) established that creative and critical thinking are not the same as intelligence and do not develop with maturity but they must be taught. Teaching thinking is mandatory to enable and to guide an individual learner to reach his or her potential zone of proximal development (Swartz & McGuiness, 2014). Teaching of thinking as a skill is necessary for all students whether they are of weak, average or high intelligence (De Bono, 1992); this means that it is not for only the weak or average learners but also highly intelligent students. This is based on the assertion that, high intelligent individuals need to improve their thinking skills so that their high-level intelligence can be utilized if not, much of the potential of high intelligence will not be realised (De Bono, 1992, Swartz *et al.,* 2010). According to De Bono (1992), "thinking is an operating skill that can be improved by training, by practice, and through learning how to do it better. Thinking is no different from any other skill and we can get better at the skill of thinking if we have the will to do so". This supports the assertion that thinking is a skill that needs to be taught and thoroughly practiced.

Teaching of thinking was considered ideal for this study, based on the premise that 'learning to think' and 'thinking to learn' promote deep and lasting learning

(Ritchhart & Perkins, 2004). As opined by Polya (1945, p.6), "problem-solving skills are not an inborn talent but rather, they need to be learnt and practiced". Thinking skills is a vital component of effective problem-solving, especially in mathematics, thus, cannot be dissociated from solving problems in mathematics classrooms. This presupposes that teaching thinking skills is as important as teaching content knowledge, hence, the former needs to be learnt and adequately practiced by learners. Teaching thinking skills has become relevant due to the inability of learners to apply appropriate reasoning and thinking skills both in classroom tasks and in examinations. This is made very evident as a greater percentage of learners, normally, abandon questions which demand high-order reasoning and advanced mathematical thinking. In many instances, the few learners who attempt to solve those supposedly 'difficult' questions end up giving irrelevant responses to them. This demonstrates that thinking skills are crucial problem-solving skills, thus, learners need to learn and sufficiently practice them, just as other problem-solving skills (DoBE Diagnostic report, 2020).

In teaching thinking skills, the researcher adopted the infusion approach (Swartz, 1992). That is teaching thinking skills, along with content instructions in the proposed instructional approach which focused on: 'teaching of thinking' and 'teaching for thinking' (Zulkpli, Abdullah, Kohar & Ibrahim, 2017). Laborde (2005) is of the view that students are unable to use theoretical statements in deductive reasoning. Groth, 2005; Herbst, Gonzalez & Macke (2005) also informs us that students encounter difficulties related to measurement, deductive proofs and linking chains of reasoning, when solving geometry problems. The diagnostic report by the (DoBE, 2017, p.6) says,

"Although certain subjects have registered a decline in learner performance, it was noted that in a number of schools the quality of learners' responses had improved. However, it is a cause for concern that in many schools, learners had a mediocre understanding of the subject matter and this translated into poor quality responses and misconceptions. It was also evident that candidates performed well in questions that required lower-order thinking skills. However, many learners performed poorly

in questions that demanded analytical, evaluative and problem-solving skills and candidates were severely disadvantaged by their lack of these cardinal skills".

The above assessment indicates that students substantially lack thinking skills, which is one of the competence skills expected for solving circle geometry problems. The researcher has realised that lack of thinking skills is worrisome and a serious cause of concern. This deprives students of the ability to solve circle geometry problems well, hence, thinking skills need to be taught (Swartz & Reagan, 1998). In the light of the above, the infusion approach was introduced in an effort to teach students thinking skills in the circle geometry classroom. This required the prescribed content material to be restructured so that the teaching of thinking skills can be appropriately integrated into the conventional instruction (Aizikovitsh & Amita, 2010).

Schweiger (2003) posits that, in as much as there is increasing advocacy for teaching problem-solving, yet, its precise meaning and how it can be taught, remain a challenge to mathematics instructors. In a bid to teach students how to understand a problem and how to effectively approach a problem, the researcher adopted Polya's problem-solving model. Polya's instructional approach consists of: understanding the problem, devising a plan, carrying out the plan, and reviewing the steps that will guide the classroom discussion phase of the Activities, Classroom Discussion and Exercises (ACE) teaching cycle, associated with the APOS theory. The APOS theory was, as well, implemented to monitor learners' mental constructions.

Under this proposed approach, lessons were carried out in a collaborative learning environment, which has been proven to promote improved higher order learning abilities (Brijlall, 2015).This proposed problem-solving instructional approach is known as, "IPAC mathematics problem-solving instructional model" or simply, the "IPAC model". This is an acronym for the four elements of this new approach - the infusion approach, Polya approach, and APOS theory - in a collaborative learning classroom.The main elements which the researcher incorporated into this problem-solving instructional model, belonged to two distinct facets: one pertaining to teaching (the pedagogical dimension) and one pertaining to learning (the cognitive development dimension), as elaborated below.

(a) **The pedagogical dimension:**

The ACE teaching approach associated with the APOS theory, Polya's problem-solving regime and Infusion as a technique of cognitive development of thinking skills, constitute the pedagogical dimension. The above are all action-driven theories or techniques. They are to be implemented by the teacher in the classroom while teaching circle geometry, which would be the main product of the study – a designed/crafted model of how to teach circle geometry, effectively.

(b) **The learning dimension**

The sub-constructs propelling this model are interiorization and encapsulation or reification towards the mathematical object, circle geometry; this constitutes the learning dimension of the proposed IPAC model. For clarification: interiorization requires the individual to be able to perform tasks internally in his/her mind without following step-wise procedures and encapsulation or reification demands the individual to be able to reflect on and apply internalised ideas through advanced mathematical thinking wholly in the individual's mind by applying his/her intuitions or imaginations (Brijlall, 2020). These sub-constructs can be viewed as the indispensable structural cognitive modification processes that need to happen in order to cope with circle geometry. These sub-constructs are those theories pertaining to the development of higher-order thinking skills in learners. The APOS theory is a progression monitoring tool: the genetic deconstruction of the topic circle geometry forms the basis of the progression one would want to effect in learners, and the APOS levels are the systematic goals on the pathway towards achieving mathematical proficiency. The teacher keeps track of progress through a reflection of attained levels as demonstrated in a learner's work and the assessment and its instrument designed to gauge the success of the applications of ACE, Polya and infusion in the pedagogical model. This instructional approach, it is hoped, can effectively improve the teaching and learning of mathematics in general and circle geometry in particular by improving mathematics learners' conceptual understanding, mathematical reasoning and thinking skills (DoBE, 2018).

1.6 The concept of thinking

In this section the traits of a good thinker, types of thinking and teaching thinking skills are elaborated to inform readers about thinking and thinking skills – what they are, how they can be taught to students and their instructional objectives.

1.6.1 What is good thinking?

Thinking skills include creative, critical, logical, reflective, and metacognitive thinking. As students get acquainted and exposed to unrelated and unfamiliar problems/questions/tasks, these thinking skills are triggered, directly and indirectly (King, Goodson, & Rohani, 2013). In developing thinking, students' prior content knowledge is paramount (Lai, 2011; King, Goodson, & Rohani, 2013). According to Ritchhart and Perkins (2004), good thinkers, simply 'think well when they think'. They do so at the appropriate moment; when they are individually motivated and committed, taking relevant decisions and appropriate problem-solving paths. Good thinkers are able to think/reason systematically, solve problems well, think logically, employ good vocabulary and they are, as well, able to utilize adequate pool of information, efficiently. Furthermore, they stay on track on their goals, and they are able to showcase their intelligence, practically and academically (Ritchhart & Perkins, 2004).

1.6.2 Types of thinking

According to Swartz & Reagan (1998, p.11), *"three important types of thinking exist and they are: (1) Generating ideas (alternative possibilities): multiplicity of ideas, varied ideas, new ideas, detailed ideas, et cetera. (2) Clarifying ideas: this means analysing ideas by comparing or contrasting, classification or definition / analysing arguments by finding conclusions or reasons and uncovering assumptions. (3) Assessing the reasonableness of ideas: (i) Supporting basic information by determining accurate observation and determining reliable secondary sources, (ii) making inferences by using evidence: causal explanation, prediction, generalization, reasoning by analogy and by making deductions through conditional reasoning and categorical reasoning."*

In other words, Swartz and Reagan (1998, p.11) classified the above three types of thinking skills as - understanding and retaining ideas, generating original ideas (creative thinking), and assessing the reasonableness of ideas (critical thinking). They termed the above as the 'three domains of thinking'. These researchers emphasized that decision-making and problem-solving are complex, specialized processes that drive the agenda of Infusing Critical and Creative Thinking (ICCT) into content instruction. They reiterated that for these three domains of thinking to be demonstrated, 'decision- making' and 'problem-solving', inevitably come into play. The authors also explained that skilful decision-making, is promoted by giving answers to the critical questions, such as - (1) What makes a decision necessary? (2) What are my options? (3) What are the likely consequences of these options? (4) How important are these consequences? (5) What is the best option in the light of the consequences?

Stylianides (2007) cited in Zulkpli, Abdullah, Kohar & Ibrahim (2017) aver that a thinking strategy may include: generalising, applying, analogising, explaining, and finding evidence and examples; Caram and Davis (2005) had also stated 'representing the subject in a new way' as another thinking strategy. Crawford, (2001), opines that there are five thinking strategies - experiencing, relating, cooperating, applying, and transferring. He established that these are the skills a problem-solver is expected to portray in the classroom when the teacher is presenting a lesson on `thinking skills'.

1.6.3 Teaching thinking skills

Costa (2001) cited in Zulkpli, Abdullah, Kohar & Ibrahim (2017) suggested that teaching thinking skills comprises of three components: - (i) teaching of thinking, (which is about direct teaching skills in non-circular context and comprises of creative thinking skills, critical thinking skills, COGAFF, CSE, and others); (ii) teaching for thinking (which is about, thinking on circular context, such as cognitive acceleration) and (iii) infusion (which is about, restructuring content lessons to develop and focus on thinking skills and infusing thinking into instruction). Fogarty (2009) also cited in Zulkpli, Abdullah, Kohar & Ibrahim (2017) made a similar point, that the teaching of higher-order thinking involves four-dimensional framework - teaching for thinking, teaching of thinking, teaching about thinking and teaching with thinking; he referred to these four-dimensional framework as the "four corners" of thinking. In teaching thinking skills the IERT Model, comprising of four structures of infusion of lessons in teaching and learning, namely, Introduction, Engagement, Reflection and Transfer, can be mentioned (Swartz & Parks, 1994). In teaching thinking high-order questioning is paramount (Swartz & Parks, 1994).

In addition to the above, Swartz & Perkins (1992, p. 55-68) came up with nine basics of teaching thinking: (1) Why improve thinking? (because, by default, everyday human thinking tends to be hasty, narrow, fuzzy, and sprawling. Learners can be guided to improve their thinking without resorting to any technical concept of good thinking); (2) better thinking depends on better organization. (To improve thinking, we need to cultivate explicit use of the verbal and graphic organizers learners already know and introduce them to some they do not know); (3) infusion means direct and explicit attention to teaching thinking within content area instruction. (Efforts to teach thinking should include infusion); (4) the development of students' thinking calls for cultivating their (a) skills, (b) processes and (c) dispositions concerning better thinking. (A well-rounded approach addresses a diversity of all three); (5) explicit attention to thinking during content instruction (via verbal and graphic thinking organizers, discussion, reflection, and so on) makes a crucial difference between just giving students experiences of better thinking and empowering them to be users of better thinking practices across diverse settings; (6) developing learners' metacognition is an important facet of teaching thinking; (7) teaching for transfer of better ways of thinking is an important part of teaching of thinking; (8) attention to thinking in content instruction needs to be fairly frequent to (a) build students' skills and insights and (b) systematically deepen content understanding, as well as noting that (9) the infusion of thinking into content area instruction is not a "quick fix" accomplishable by a single workshop. Schools and teachers need to commit themselves to a continuing process of thorough staff development (much of which may be handled internally by the school). These nine basics of teaching thinking, comprehensively, serve as guidelines and technicalities to emphasize and to adhere to, when teaching thinking, especially by novice teachers who wish to incorporate it into their mathematics instructions. Systematically, these inform teachers of what teaching of thinking entails.

In endorsing the above, Swartz *et al.* (2010) suggested some factors which make teaching of thinking skills effective; they are dispositions, knowledge and skills. They referred to them as 'the three essential elements of teaching thinking skills'. Students are expected to develop certain intelligent dispositions in the classroom. These dispositions may include - persistence, learning to overcome obstacles in problem-solving and to be receptive to other people's point of views (Swartz & Parks, 1994; Ong & Borich, 2006; Swartz, Costa, Beyer, Reagen & Kallick, 2010; Swartz, 2012; Swartz & McGuiness, 2014); cited in (Zulkpli, Abdullah, Kohar & Ibrahim, 2017).
Swartz and Perkins (1992), emphasized that in teaching thinking, teachers need to go beyond developing skills and processes, by urgently helping students develop the dispositions they need, to think well. That is by: (1) giving thinking more time, (2) making thinking adventurous and broad, (3) making thinking clear and careful and (4) making thinking organized. The authors termed these four elements of developing dispositions as the "four keys to thinking dispositions".

1.7 Heuristics as a problem-solving instructional approach

As suggested by Bay (2000), problem-solving is the teaching of strategies, or heuristics, to solve problems. Heuristics seek to discover, invent and apply appropriate strategies and techniques required for obtaining an appropriate solution to a problem. This implies that heuristics can never be dissociated from problem-solving in mathematics due to its usefulness and the pivotal role it plays in mathematics problem-solving. Problem-solving in mathematics cannot be effective by downplaying the roles of heuristics.

Higgins (1971) cited in Ofori-Kusi (2017), highlights four characteristics of heuristics; it: (1) approaches content through problems; (2) reflects on problem-solving techniques in the logical construction of instructional procedures; (3) demands flexibility when uncertain and requires alternative approaches; and (4) seeks to maximize learners' actions and participation in the learning process. The key words from these characteristics are: 'problem-solving techniques' and 'instructional procedures'. As said earlier, problem-solving can serve as an instructional goal and as an instructional approach (Jinwen & Bikai, 2007). This presupposes that the new proposed instructional approach in this study, can appropriately be used as a heuristic for circle geometry's problem-solving. The new approach, hence, can be used, both as a problem-solving technique and as an instructional procedure, due to its heuristic nature.

1.8 Rationale of the study

The clarion call, issued by the chairperson of South Africa's Council for Quality Assurance in General and Further Education and Training (Umalusi, 2020) is the rationale for this study. Umalusi raised concerns about how mathematics instructions are carried out in classrooms as this was assumed to be directly responsible for learners' abysmal performances in examinations. Umalusi placed a responsibility on

South African teachers to implement effective instructional methods in mathematics classrooms, thereby, advocating for South African teachers to instruct differently.

This research study is a follow-up to my previous research study for my master's degree: *"Exploring Mathematics Learners' Problem-Solving Skills in Circle Geometry in South African Schools: A case study of a high school in the Northern Cape Province".* High school learners' ineptitude to perform well in Mathematics, circle geometry in particular, triggered a need to conduct my previous research. For my previous research, learners were observed in their natural classroom setting over a two-year period at a high school. Polya's problem-solving instructional approach was coupled with social constructivist instructional approach as an intervention in a case in which the teacher had been teaching mathematics using the traditional approach. The outcomes showed that, the adopted Polya-social constructivism as an intervention instructional approach motivated learners to perform better in circle geometry, improved classroom dynamics and learners' problem-solving skills. The need to advance the research into Polya-social constructivism instructional approaches and to find a contemporary instructional approach to improve the teaching and learning of circle geometry, necessitated the conducting of the current study.

Justification for merging infusion approach, Polya's approach, APOS theory, and Collaborative approach to instructions for developing the proposed problem-solving instructional approach

As mentioned earlier, the proposed problem-solving instructional approach was designed by merging three frameworks: Infusion approach, Polya's problem-solving instructional approach, APOS theory, and Collaborative approach to instructions. What necessitated the adopting of these theories/ approaches as constituents of the proposed problem-solving instructional approach are elaborated below:

Infusion Approach

There are different ways of teaching thinking, however, the researcher considered the infusion approach as ideal for the South African context. This is based on the assertion that when thinking skills are taught as a set of processes, aside from content, learning is incapacitated and handicapped (Smart-Morstad, 2008). Silva, 2008; Case, 2005; Pithers & Soden (2000) cited in Lai (2011) support the assertion that thinking skills are better taught along with content instructions, not as separate entities. They established that knowledge and thinking have to be taught simultaneously.

Euclidean geometry by its nature, requires rigorous reasoning and thinking, which, although they are essential problem-solving skills, have continuously been ignored by mathematics instructors. Teaching of thinking skills as a problem-solving skill, requires teachers to ask high-order and/or structural problem-solving skills' questions that can motivate learners to use existing knowledge to acquire new knowledge (Swartz, 2012). This is the main rationale for adopting and implementing the infusion approach at the questioning stages of the proposed problem-solving instructional approach (discussion, activities and exercise phases), so that thinking skills can be effectively incorporated into content instructions. The infusion approach promotes great learning improvements, more classroom participation, better quality student responses and more enthusiasm for learning (Swartz, 2012). The more teaching of thinking is integrated into content instruction, the more students will think about their own learning (Swartz, 2003).

Polya instructional approach

The Polya's problem-solving instructional approach has been proven to be efficacious in mathematics problem-solving and this research study hopes to further confirm this. The researcher realised that incorporating it as part of this study's intervention instructional approach can contribute to its effectiveness in mathematics classrooms.

The approach can serve a dual purpose - firstly, it can be used as a problem-solving teaching guide and secondly, it can assist learners to follow teaching steps, independently, during their individual practices.

It can systematically enable teachers to demonstrate to learners how a mathematical problem is expected to be solved; this can be done with the aid of problems or activities appropriately selected and sequenced to serve desired objectives of mathematics lessons. Logoglu (2017) claims that students' success in the process of solving mathematics problem is improved by applying Polya's steps accurately. As posited by Yuan (2013) Polya's approach has influenced educators at all levels and it helps educators to guide students effectively. Lee (2017) goes further to claim that using Polya's method could significantly improve the effectiveness of mathematics lessons.

Secondly, Polya's approach as mentioned above, serves as a problem-solving guide for learners to follow the processes independently, during their individual practices. The implementation of Polya's approach in teaching and learning has resulted in a significant improvement of students' performances in problem-solving, thus, the rationale to incorporate it as part of this action-driven intervention.

APOS theory

The APOS Theory as a constructivist approach, has either, wholly or partially, been adopted in contemporary mathematics education research studies; it has mainly been used by mathematics education researchers for investigating or developing pedagogical instructional approaches and/or as a research paradigm. The APOS Theory's application in contemporary curriculum development in mathematics education has taken centre stage in modern mathematics education research and it has so far, proved to be efficacious.

This theory has been adopted and implemented effectively in mathematics teaching and learning, for different mathematical concepts, although only a handful of such studies focused on geometry. To this end, adopting and implementing it in this research study to improve teaching and learning of circle geometry, is justified. This is due to the acknowledgements this instructional approach has received from studies that employed the APOS theory (Dubinsky, 2010), hence, it is hoped the current study will achieve the same level of efficacy by adopting and implementing the theory.

Collaborative approach to instructions

Collaborative instructional approach grew out of the concept of social interaction (Woolfolk, 2014). It has received much advocacy by researchers who agree with Piaget's and Vygotsky's notion of what teaching and learning is and how they should be conducted in classrooms; it is centred on cognitive development (Woolfolk, 2014). Due to its much-heralded efficacy in mathematics classrooms, collaborative or cooperative instructional approach has been recommended by numerous creative and critical-thinking advocate researchers, for teaching thinking skills (Swartz & Regan, 1998; Heyman, 2008; Thayer-Bacon, 2000).

Collaborative instructional approach is an integral and inherent component for conducting infusion lessons and APOS theory pedagogy (Swartz, 2012; Chagwiza, Maharaj & Brijlall, 2020). The positive impacts of collaborative instruction on learner achievements did not go unnoticed by researchers such as Swartz (2012) and Chagwiza, Maharaj & Brijlall (2020), who are advocating for its inculcation in mathematics lessons. They have proved beyond reasonable doubt that collaborative learning if used in infused lessons yields both improvements in thinking and enhanced content knowledge. Swartz (2012) believes that collaborative learning can create an overall conducive atmosphere for thinking in the classroom. Jailani and Retnawati (2016) also agree that collaborative learning, assists metacognition development, hence, the researcher's belief that collaborative instructional approach is appropriate for this study.

1.9 Problem statement

Poor learners' achievements in mathematics, has raised concerns about how teaching and learning of the subject occur in schools in South Africa; this has led to multifaceted concerns among mathematics education researchers. This has also necessitated them carrying out investigations to: (1) identify the problems and challenges in mathematics classrooms and (2) ascertain which instructional approaches can be effective to improve the teaching and learning of mathematics in South African schools (Umalusi, 2020, DoBE, 2018). As an FET mathematics educator in South Africa for close to a decade spanning across four provinces: Limpopo, Gauteng, North-west, Northern Cape - and as a researcher, I have followed the plethora of happenings and practices in mathematics classrooms; the intention was to ascertain how teachers teach mathematics, how learners learn mathematics, and many other related issues.

In most schools, one notices with concern how ineffectively mathematics is taught to learners in classrooms, with many teachers, still glued to the traditional ways of teaching. This situation and with reference to literature, motivated the idea to develop a more proactive and action-driven instructional approach, which will be effective for teaching and learning of mathematics in schools in South Africa. To this end, an approach that focused on geometry was nurtured. This was based on the premise that geometry has some of the most tenacious, troublesome and problematic mathematical concepts; it is seen as a difficult-to-teach and difficult-to-learn content area in schools in South Africa. This section of mathematics seems to make learners, mathematically incapacitated and helpless (Brijlall, 2015). The researcher strongly concur with French (2004) that an instructional approach which can address the teaching and learning challenges with regards to Euclidean geometry, can simultaneously address the teaching and learning challenges of other content areas of the FET mathematics curriculum. This propelled the need to focus this research on circle geometry, which is an indispensable component of Euclidean geometry's content areas.

1.10 Aim(s) of the study

Investigating the use of instructional approaches as a problem-solving heuristic strategy, to improve the teaching and learning of circle geometry, was the primary aim of this research study.

1.11 Objectives of the study

In a bid to accomplish the aim of this study, the following objectives were listed to be achieved:

(1) To develop an effective problem-solving heuristic instructional approach which can be used to improve the teaching and learning of circle geometry at Grade 11 level.

(2) To determine if the proposed problem-solving instructional approach, has any effect on learners' achievement in circle geometry at Grade 11 level.

1.12 Research questions

In order to realize the objectives of the study, the following research questions will be answered:

Main research question

What are the characteristics of an effective problem-solving heuristic instructional approach for circle geometry content at Grade 11?

This gave a leeway to describe the proposed model, its application and effects. To obtain a detailed description of how the proposed instructional approach can influence Grade 11 learners' achievement in circle geometry, the following sub-questions were formulated:

(1) How can the problem-solving heuristic instructional approach be developed and implemented for the teaching and learning of circle geometry to Grade 11 learners?

(2) What is the impact of the problem-solving heuristic instructional approach on learners' achievements in circle geometry at Grade 11 level?

(3) How do the participants' level of mental construction affect their abilities in their problem-solving techniques?

1.13 Hypothesis of the study

The hypothesis and the null hypothesis of the study were formulated as follows:

Hypothesis (H_A): There is a statistically significant improvement in the circle geometry test scores of learners who used the proposed instructional approach as a problem- solving heuristic strategy.

Null hypothesis (H_0): There is no statistically significant improvement in the circle geometry test scores of the learners who used the proposed instructional approach as a problem-solving heuristic strategy.

1.14 Significance of the study

The main motivation for conducting this research was to improve the teaching and learning of mathematics so as to better solve problems relating to circle geometry. It is envisaged that this research study, can contribute enormously in eradicating the difficulties encountered by teachers and learners in relation to the teaching and learning of circle geometry in South African schools. This is due to the anticipation that the results will bring forward new ideas that are in line with global and current trends in the teaching and learning of geometry in particular, and mathematics in general. This is because the proposed approach is in contrast to the ineffective traditional teaching and learning approach, which is still predominant in South

African schools. The findings of this research study can also positively influence how future mathematics and/or geometry curricula will be formulated to ensure that there is effective teaching and learning of mathematics, particularly circle geometry. In addition, it can serve as a reference tool for mathematics students, educators, mathematics teacher educators, subject specialists, educationists, and policy makers when finding an effective and a reliable pedagogical approach for mathematics in general and circle geometry in particular.

1.15 The scope of the study

Euclidean geometry is an integral component of the mathematics curriculum (DoBE, 2010) and is partitioned into two broad sub-topics: 'space and shape geometry' and 'measurements'. The content areas under each sub-topic have been sequenced from the junior grades (Grades R-10), before the introduction of circle geometry in Grade 11(DoBE, 2010). The understanding of circle geometry is also required for similarity and proportionality, which is the Euclidean geometry topic in Grade 12. When learners are taught circle geometry, they are expected to apply all the concepts of Euclidean geometry they had acquired from the earlier grades. These concepts include amongst others: types of angles, properties of quadrilaterals, properties of different types of triangles, areas and perimeters of 2D and 3D shapes and Pythagoras theorem.

'What' to teach learners is comprehensively explained in the curriculum, however, the 'how' part of teaching the content of circle geometry in the classrooms has never been thoroughly explained since the introduction of Euclidean geometry in the South African mathematics curriculum. Various research studies have been conducted, both locally and internationally, to address the teaching and learning difficulties related to circle geometry, however, the high failure rates of learners in the content area still prevails. This has been of great concern for mathematics education researchers. This led to the realisation that teaching thinking skills along with content instructions (infusion approach) was crucial. This strategy also goes with teaching learners effective problem-solving (Polya's approach), monitoring learners' mental constructions (APOS theory) and carrying out lessons in a collaborative classroom setting. This it is anticipated would effectively help to address the teaching and learning difficulties related to mathematics in general and circle geometry, in particular. Polya's approach to problem-solving has been implemented in some research works, however, the APOS theory has scarcely been tried for research studies related to geometry. In addition, the infusion approach has hardly been implemented in South Africa's context, hence, the researcher also sought to investigate its relevance in the proposed strategy.

This study focused on Grade 11 learners who were doing mathematics and was carried out in the learners' natural classroom environment. Two Grade 11 mathematics classes were used for this study. One class served as the control group (they were taught circle geometry using the traditional approach), while the other group served as the experimental group (they were taught circle geometry by applying the proposed problem-solving instructional approach). The two classes comprised of learners who had been promoted from Grade 10, although, a few were repeating Grade 11. The selection of learners to serve as participants for the two classes was devoid of gender criteria, race criteria, culture, ethnicity and any other elements of prejudice, discrimination and bias.

1.16 Definitions of key terminologies and variables

Problem-solving instructional approaches - these are ways teachers employ to teach students a specified content. This should, in turn, guide students to learn that particular content effectively - individually or when they are in groups.

Teaching thinking- the act of guiding learners to think creatively and critically, thereby, enabling students to interrogate their own thinking when solving problems.

APOS theory- this is a mathematics education curriculum development tool. It has its theoretical, pedagogical or methodological characteristics, for mental construction of mathematical knowledge through four mental construction stages, which hierarchically are: Action, Process, Object and Schema mental constructions (Dubinsky, 1991).

Reflective abstraction- this is a concept introduced by Piaget (1977). He asserts that, when a new problem is confronted, the individual can go beyond the observables and put them into relationship, resulting in logico-mathematical knowing. This refers to how the evolution of concepts from actions-processes-objects-schemas comes into effect. These processes are: interiorization, encapsulation and thematization, sequentially. The concept of reflective abstraction has become a powerful tool in the study of advanced mathematical thinking (Brijlall & Ndlazi, 2019; Chagwiza, Maharaj & Brijlall, 2020).

Interiorization – As the individual becomes aware in totality of content procedures, the individual is able to perform the tasks, devoid of prompts and step-by-step procedures. The individual is said to have interiorized the said action into a process (Dubinsky, 2000; Tziritas, 2011).

Encapsulation – As the individual realises and performs a transformation in totality, from his/her imagination and intuition, s/he can be said to have encapsulated the process into a cognitive object (Maharaj, 2010; Brijlall & Ndlazi, 2019; Chagwiza, Maharaj & Brijlall, 2020).

Thematization - this refers to the interconnection, and linkage of objects, processes and actions, to form a meaningful coherent whole, that is, a schema (Tziritas, 2011; Brijlall, 2020).

Genetic decomposition of a concept - this refers to the structured/hypothesized mental constructs which can describe how the concept can develop, intrinsically, in an individual (Dubinsky, 2001).

1.17 Overview of this study

This thesis is partitioned into ten distinct chapters, a list of references used for the study, as well as list of appendices. The composition of each of the ten chapters are delineated below:

Chapter One

This chapter gives the following details: reasons for the need to conduct this study, introduction, background and purpose of the study, the concept of thinking, teaching thinking and heuristics as a problem-solving instructional approach. It also elaborates on the rationale of the study, significance of the study, the problem statement, aims and objectives of the study, research questions and hypothesis of the study, definitions of key terms, and lastly, the organisation of the thesis.

Chapter Two

Under this chapter, relevant concepts germane to this study are discussed. This chapter also includes literature, relevant to this study - infusion approach, Polya's approach, and APOS theory. These were reviewed and presented for the readers' comprehension.

Chapter Three

The theoretical frameworks for this study are elaborated in this chapter, which, thus, is dominated by teaching and learning theories. Also detailed are how these theories relate to the established research constructs, and the proposed instructional approach.

Chapter Four

The research paradigm, the research design, research procedures, instrumentations, development of instruments, pilot studies procedures (conducting prototypes), data collection procedures, population and sampling, validity and reliability of the study, are all presented in this chapter. The final sections elaborated on the data analysis' procedures.

Chapter Five

This chapter presents the analysis and discussion of data emanating from the conducted lessons. In relation to the above, pre-intervention observations, main-intervention observations and post-intervention observations sessions that were carried out during this study are sequentially presented and discussed.

Chapter Six

This chapter presents a discussion and analysis of the data captured from the standardised tests' instruments. For this procedure, participants' scores for each of the four standardized tests as well as, the composite results are presented and analysed. Furthermore, in this chapter, a hypothesis test is carried out on participants' composite scores to determine if they are statistically significant.

Chapter Seven

In this chapter, the mental constructions the participants demonstrated in relation to each level of APOS theory's mental conception are presented and discussed. How the research findings had affected the initial genetic decomposition are also explained in detail in this chapter.

Chapter Eight

This chapter presents the analysis and discussions of the questionnaire data extracted from - participants, teacher as an observer and HOD as an observer.

Chapter Nine

In this chapter, the findings that emanated from conducting this research study, as well as discussion of the research findings, are presented in accordance with each research question.

Chapter Ten

The summary of the study, implications of the research findings for teaching, learning and policy, limitations of this study, conclusion and recommendations that emanated from this study, are all elaborated in this chapter.

1.18 Conclusion

This chapter has given a comprehensive account of what necessitated this research. For this purpose, the predicaments and concerns of DoBE, as well as, other stakeholders in mathematics education, were elaborated in this chapter. Efforts made by DoBE to address some of the teaching and learning challenges in mathematics were explicitly stated. Outlined also was what the researcher plans to do to address the teaching and learning difficulties that have seriously curtailed South African learners' mathematics progress. Furthermore, the aims, objectives, significance and scope of the study were clearly discussed to orientate the readers. The next chapter examines concepts and literature that are relevant and essential to this study.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Under this chapter, problem-solving as a higher-order thinking skill is discussed. The discussions go further to look at problem-solving as an instructional approach, cognitive demands of problem-solving and how problem-solving can characterize creativity of mathematical thoughts. Included also are instructional methods/strategies; circle geometry in the CAPS curriculum and how Euclidean Geometry can enhance learning of mathematics. Circle geometry as a troublesome content area in FET, students' difficulties and misconceptions in circle geometry are all discussed. This chapter, as well, pays attention to how students' difficulties and misconceptions in circle geometry can be managed, metacognition in problem-solving and assessment. In addition, an assessment instrument designed to gauge the success of the applications of ACE, Polya and infusion in the pedagogical model are all detailed.

2.2 Problem-solving as a higher-order thinking skill

In this section problem-solving as an instructional approach, cognitive demands of problem-solving and how problem-solving can characterize creativity of mathematical thoughts are discussed for readers to understand the meaning and expectations of problem-solving in mathematics context.

2.2.1 Problem-solving as an instructional approach

Teaching and learning of mathematics has proven to be a mammoth task, comprising of multifaceted dimensions (Gono & Pacoy, 2021). Some of these dimensions include: how to solve problems, relevant problem-solving instructional approaches and teaching students how to think mathematically. Most contemporary mathematics curricula expect students to learn to 'think mathematically' and they must 'think mathematically to learn' (National Research Council, 2012). Problem-solving, metacognition, and monitoring ones cognition, according to William and Maat (2020), greatly enable students to think mathematically.

Problem-solving is one of the critical skills in learning mathematics, hence contemporary mathematics curricula, put emphasis on teaching problem-solving. Problem-solving activities allow learners to facilitate connections between related ideas, to consolidate mathematical knowledge and to think creatively. That is the reason it is used as an instructional approach (Jinwen & Bikai, 2007).

Instructional approaches are mediums, which can improve students' confidence in thinking mathematically (Dhlamini, 2012). As mentioned earlier, problem-solving can be considered as an instructional goal and as an instructional approach (Jinwen & Bikai, 2007). Globally-recognised researchers who 'might' have made meaningful impact in developing effective problem-solving instructional approaches to accommodate modern demands include: Piaget, De Bono, Vygotsky, Polya, and Van Hiele. 'Might' is used here based on the researcher's evaluation that although their ideas have proven to be significant over the years, some of their processes in arriving at their conclusions, as well as some of their submissions are still in contention by other research groups which share different epistemological ideas (Moon & Blackman, 2017). Their research works commenced in the previous century and their ideas have contributed immensely to contemporary problem-solving instructional approaches and mathematics education research studies.

The research works of Piaget, De Bono and Vygotsky, are connected to instructions in the education field in general (Woolfolk, 2014), however, I agree that to a large extent their problem-solving approaches that emanated from their research works have a strong link to the teaching and learning of mathematics. This is in relation to their assertions that mathematical processes are purely cognitive processes, characterised by reasoning and thinking skills (Ekawati *et al.*, 2019). For instance Piaget, De Bono and Vygotsky's research works put great emphasis on the role of cognition in problem-solving, hence, they considered learning as a cognitive activity. Piaget (1980) proposed that meaningful learning takes place as a result of mental construct and that the minds of learners go through some series of thinking and reasoning procedures. These procedures go through various stages of development of learners, moving gradually from infancy to a later stage of the learners' development where he/she is expected to solve abstract problems.

De Bono in an effort to find an effective problem-solving instructional approach specifically conjectured a thinking procedure called the "Six Thinking Hats". He devised this to help problem-solvers to improve: decision making, focus thinking and create awareness that there are multiple perspectives to an issue. This gives the indication that a mathematical problem can be solved using different ways. The Six Thinking Hats, separates mathematical thinking into six clear functions and roles. Each thinking role is identified with a coloured symbolic "thinking hat". The first Thinking Hat (white) symbolizes information known or needed, that is, the needed facts. The second Thinking Hat (yellow) symbolizes brightness and optimism, for which the individual explores the positives and probe for value and benefit. The third Thinking Hat (Black) is used for making judgments or why something may or may not work. This hat is used to spot the difficulties and dangers in a solution path. The fourth Thinking Hat (red) signifies feelings, hunches and intuition. When using this hat, one can express emotions and feelings and share fears, likes and dislikes, loves and hates. This is in accordance with the findings of Tugba and Bulent (2016), as they specified that belief and personal factors affect mathematics problem-solving abilities. The fifth Thinking Hat (green) symbolizes creativity, possibilities, alternatives and new ideas. The sixth Thinking Hat (blue) is used to manage the thinking process. It serves as the control mechanism that ensures the effectiveness of the other Thinking Hats (De Bono, 1985).

The author emphasized that thinking is a skill which needs to be taught; he defined "thinking as an operating skill through which intelligence acts upon experience". According to De Bono (1992), "an individual who practices poor thinking for years will become an extremely skilled poor thinker". He averred that intelligence and thinking skills are not particularly correlated, hence, "high intelligent individuals are not necessarily good thinkers…… many high intelligent individuals' minds are trapped in poor ideas because they can defend them so well". This is what he termed as 'intelligence trap', which can be remedied through pedagogy. According to him, high intelligent individuals need to enhance their thinking skills so that their high intelligence status can be optimized.

Vygotsky on his part specified that, there is a huge gap between the problem-solver and the expected solution to the problem, which he termed "zone of proximal development" (ZPD), (Daniels, 2001). According to him, the gap to be filled during the problem-solving process involves a cognitive procedure, which is characterized by reasoning and thinking skills. In support of the assertion that problem-solving is a thinking process, Swartz and Perkins (1990), developed the 'Map of the Thinking domain'. They characterized the thinking processes into five domains: creative thinking, retention and use of information, decision making, problem solving, and critical thinking. They also termed, linking of thinking skills to the thinking processes as metacognition, which involves higher-order thinking processes.

Polya's contribution to finding an effective problem-solving instructional approach, came with the proposition of four distinct steps to be followed as a problem-solving heuristic. He noted that a problem-solver, first, needs to be able to understand the problem, before coming up with a plan to solve the problem (devise a plan). Afterwards, the problem-solver then must carry out the envisaged plan and finally, needs to review and to reflect on all steps and procedures adapted as the solution path for the problem under consideration. Polya (1945) asserted that problem-solving skills are not an inborn talent but, rather, they need to be learned and sufficiently practised. Polya's approach has proven to be an effective problem-solving heuristic over the years. For instance, Yuan, (2013) as cited in Wickramasinghe & Valles (2015) maintains that, it helps educators to guide students effectively to solve problems on their own. Germane to this study is the second stage (devising a plan), since the problem-solver is required to think rigorously and endlessly, until a meaningful solution to the problem/question to be solved is reached. This is the main aim of conducting this research study.

The much-recognised research study that was purely connected to geometry was done by Van Hiele. The study involved steps, referred to as 'geometric thinking steps' required by a problem-solver who is expected to fall under one of the five steps of geometric thinking. Van Hiele categorized problem-solvers as either those who can recognize shapes by their appearances, cannot emphatically identify properties of shapes although they may have the ability to recognize characteristics, and they may also lack the ability to differentiate shapes, as "Level 0", which he termed as **visualization**. Van Hiele categorized problem-solvers who can identify properties of shapes and learn to use appropriate vocabulary related to properties of shapes as "Level 1", which he termed as **analysis**. Problem-solvers in this category, may lack the ability to make connections between different shapes and their properties. Those who can recognize relationships among the properties of shapes or classes of shapes and are able to follow logical arguments using such properties, were categorized as "Level 2", which he termed as **informal deduction**. "Level 3", which he termed **Deduction,** demands problem-solvers to go beyond just identifying the characteristics of shapes. Under this categorization, problem-solvers are required to be able to construct proofs using postulates or axioms, as well as giving interpretations to contextual problems. **Rigor** is "Level 4", the highest level of geometric thought according to Van Hiele. This mandates problem-solvers to work in different geometric or axiomatic systems, which require advanced mathematical thinking. Problem-solvers are also expected to apply concepts comprehensively and to make models (Oladosu, 2014).

To corroborate the levels of geometric thinking elaborated above, Van Hiele presented an instructional approach, he termed 'phases of learning to be followed in the geometry classroom`. Phase 1 (Information) expects the problem-solver to become familiar with the working domain. Phase 2 (Guided Orientation) expects the problem- solver to become conscious of the relations, try to express them in words, and learn technical language which accompanies the subject matter. Phase 3 (Explicitation) expects the problem-solver to express their discoveries in words using technical terminology. Phase 4 (Free Orientation) expects the problem-solver to be given the opportunity to experience a variety of problems about the learning concepts. Phase 5 (Integration) expects the problem-solver to summarize the learning concepts and integrate them with previous learning (Reed, 1996).

To collaborate the above, Driscoll (2016) in the work 'Fostering Geometric Thinking', provides a framework of productive mental habits geared specifically towards geometric thinking. These are reasoning with relationships, generalizing geometric ideas, investigating invariants and reflection. Driscol, Nikula and DePiper (2016) outlined four principles for designing instructions that create good mathematical discussions in the classroom - challenging tasks, multimodal representations, development of mathematics communication and repeated structured practice.

As asserted by Fujita and Ding (2006) an effective problem-solving instructional approach in geometry should encourage students to engage in investigative activities, demonstrative creativity, and make discoveries in geometrical contexts so that students develop powers of spatial thinking, visualization and geometrical reasoning. The authors maintain that lessons in geometry should focus on the development and application of spatial concepts through which students learn to make sense and represent the world. This is aided by an effective problem-solving instructional approach.

The implementation of an effective problem-solving instructional approach can help students' to develop their spatial visualization awareness, spatial orientation awareness, and spatial relation awareness (DoBE, 2010; Pittalis & Christou, 2010). In support of the above, there are principles and standards for school mathematics that can be followed by problem-solvers, to enable them to solve problems well. These include - being able to analyse characteristics and properties of two- and three- dimensional geometric shapes; developing mathematical arguments about geometric relationships, to specify locations and describe spatial relationships using coordinate geometry and other representational systems; having the ability to apply transformations and use symmetry to analyse mathematical situations and cultivating the ability to use visualization, spatial reasoning and geometric modelling, (National Council of Teachers of Mathematics [NCTM], 2000). In addition to the above Yıldırım and Ersözlü (2013), argue that there is a positive correlation between students' metacognitive awareness levels and their problem- solving levels regarding routine and non-routine problems. The authors emphasized that, metacognitive awareness significantly predicted problem-solving levels, hence, metacognition plays significant roles in problem-solving.

According to Kilpatrick, Swafford and Findell (2001), any problem-solving instructional approach to be employed in a geometry classroom must promote students' conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition reasoning and sense making. The authors referred to these as the five strands of mathematical proficiency and they regarded them as essential to the development of mathematical proficiency.

In developing students' mathematical proficiency skills as part of an effective problem-solving instructional approach, Moschkovich (2002) unequivocally stated that communication in the mathematics class is key in the teaching and learning of mathematics. This presupposes that student-student interactions and student-teacher interactions are healthy mathematics education practices which cannot be overlooked in geometry classrooms. In support of the above Sfard (2008) avers that analysing students' and teachers' discourse in mathematics classrooms is relevant to better understand the learning and teaching phenomena in geometry.

As emphasised by Vygotsky, students can improve their level of intelligence with the help of a more competent peer (student-student interactions). In view of this, social constructivism instructional approach may be ideal in the circle geometry classroom (Abakah, 2019). It may enlighten mathematics educators and teacher-educators of mathematics, of the expected levels of interaction in the mathematics teaching and learning continuum. Driscol (2016) in his research work, `Mathematical Thinking and Communication`, informs us about how gestures, drawings, manipulative instructional materials, and technology can be used as tools for reasoning and communication.

Communication, a key element of social constructivism as an instructional approach advocates for learners to be in groups during teaching and learning sessions (Brijlall, 2015). It also advocates for teachers to move freely in the classrooms, so that each groups' discuss can be effectively monitored by the teacher. This motivated Polya's instructional approach and an updated, modernised constructivism instructional approach (APOS Theory) to be adopted together as part of the proposed problem-solving instructional approach, designed in this study. Polya's instructional approach served as the problem-solving guide during classroom discussions. Learners followed these steps in the problem-solving situations, thereby, they served as a problem-solving path. In lieu of the above, the APOS theory guided the design and implementation of the proposed problem-solving instructional approach; this gave the expected level of interaction in the circle geometry classroom. Engaging students in problem-solving in circle geometry, which is an integral part of South Africa's mathematics curriculum, demands that students should think logically, critically, creatively, systematically and innovatively. All these enable students to develop appropriate mathematical skills (Alex & Mammen, 2016).

2.2.2 Cognitive demands of problem-solving

As mentioned earlier, finding effective problem-solving instructional approaches, including cognition and thinking skills have been the focal point of modern research studies. William and Maat (2020) makes us to understand that mathematics is about seeking for patterns through observations, study and experimentations. They also mention that the tools for mathematics are abstraction, symbolic representation, and symbolic manipulation. The authors emphasized that being trained in the use of these tools does not mean that one thinks mathematically, but that, learning to think mathematically means: developing a mathematical point of view through the processes of mathematization, abstraction, sense-making and having a predilection to apply them, which presupposes that thinking skills and cognition are vital problem-solving tools.

Stein, Smith, Henningsen and Silver (2000) defined cognitive demand as, "the kind and level of thinking required of students in order to successfully engage with and solve problems". They emphasized that a problem-solver's ability to find solutions to problems requires an organised thought process, which demands the problem-solver to analyse information accurately, precisely, comprehensively and without bias. Stein *et al.* (2000) suggested that both problem-solving and critical thinking involve gathering of information, defining the issues and breaking them down into smaller sections, through to an outcome or solution.

Cognitive or thinking skills may include: critical thinking, problem-solving, attention, concentration and memory, organisation and planning (Jenicek & Hitchcock, 2005). These researchers aver that critical thinking plays a significant role in problem-solving. They also identified seven components of the critical thinking process which play a critical role in problem-solving, namely, problem identification and analysis, clarification of meaning, gathering the evidence, assessing the evidence , inferring conclusions, considering other relevant information and making an overall judgment.

Evans and Swan (2014), in cognisance of the above, elaborated how Bloom's Taxonomy (knowledge, comprehension, application, analysis, synthesis and evaluation) can be used to increase cognitive demands in geometry. Firstly, the instructional objectives of every lesson can be written so as to demand high levels of cognitive demands. Teachers can write assessment and instructional/learning activities or tasks that align to the proposed instructional objectives, thereby, increasing learners' cognitive demands. Stein and Smith (1998) emphasised that high cognitive demand tasks can be separated into two sub-categories: 'Procedures with Connections' and 'Doing Math' and that teachers can use the Taxonomy's levels to reflect on their own instruction.

According to Driscol, Nikula and DePiper (2016) the best way to promote communication and build proficiency in mathematical reasoning in the classroom is to 'lighten the cognitive demand'. This means learners are given the opportunity to develop their own mathematical conjectures, prove and disprove each other's mathematical constructions, postulates and arrive at solutions to questions. As learners are able to connect their mathematical ideas to solving real life problems, a scene of a fascinating learning environment is created; this results in learners gaining confidence and developing the desire to do more. This enables learners to develop meaning and understanding about the mathematical concepts under consideration; this in turn promotes their ability to make sense of how mathematical concepts and techniques can be used to solve real life problems. In this context, learners are freely and unlimitedly allowed to develop their own mathematical ideas, hence, they are able to develop their own patterns and relationships. They are also able to develop model(s) which can be used to solve related real-life situations. These outcomes demonstrate the value of mathematics, as it contributes to problem-solving in real-life situations and cognitive skills, serving as essential component of the problem-solving process.

2.2.3 How problem-solving can characterize creativity of mathematical thought

As mentioned earlier, this study aimed at finding an effective problem-solving instructional approach for circle geometry, and creativity is indisputably, a crucial component of problem-solving in mathematics (DOE, 2018; Plucker, Beghetto & Dow, 2004). Mathematics as a discipline is governed by rules and principles, however, Mehdi, Narges, Yaftian and Shahrnazer (2012) point out that mathematics is not all about following rules and procedures but it dwells much on logic and creativity. One of the pivotal tasks for mathematics educators is to develop mathematical creativity. Creativity is a mathematical activity (Sequera, 2007), hence mathematicians rely on creativity to build meaningful relationships among several concepts, which is also a characteristic imbedded in problem-solving (Vianney & Navarro, 2011).

In support of the above Kim (2009) maintains that creativity in mathematics is generally related to problem-solving, which depends on the nature of the problems posed to learners; this is one of the most essential aspects of creative problem-solving. Brinkmann (2004) argues that solving such challenging mathematics problems could encourage students to be creative in doing mathematics and enable students to think as mathematicians do. This means that students are encouraged to reflect on their own ideas. He stressed that only creative teachers can train creative students.

For Idrisa, Mohd and Nor (2010), creativity in mathematics guides students to make sense of the world, hence, students need to perform investigations and activities that stimulate their curiosity and awaken their desire for mathematical creativity. Creative acts in mathematics could consist of designing new fruitful mathematical concepts, discovering an unknown relation and reorganizing the structure of a mathematical theory (Sriraman, 2004; Liljedahl & Sriraman, 2006). Bharath and Sriraman (2004), explain that social interaction, imagery, heuristics, intuition and proofs are the common traits of mathematical creativity.

Chamberline and Moon (2005) explain that giving strict action-guides in problem-solving impedes the development of creativity. They stressed that it is necessary to foster creativity of students as this will improve mathematical teaching; students must learn how to think, do and undo mathematical conjectures and free their imagination. This will guide them to generate new ideas. In addition, Posamentier, Smith and Stepelman (2010), note that solving mathematical problems and identifying their meaning favour the search and development of reasoning that will lead to creativity. This enables students to offer several solutions to problems, hence, through creativity, different ways in which a particular problem could be solved can be identified by students. Haylock (2012) emphasized that the onus lie on teachers to develop mathematical strategies and instruments favouring creative learning in mathematics classrooms.

It is argued by Lester and Kehle (2003) that mathematical problems with several solutions allow flexibility in individual's mathematical thinking by encouraging the switching from one mental operation to another. They continue that solving problems in different ways characterizes creativity in mathematical thought, although, some solutions may be more creative than others. Mathematical knowledge is considered tentative, intuitive, subjective and dynamic (Cuoco, 2000).

Warwick, Alcock and Simpson (2001) reiterated that in exploiting creativity, conceptual understanding of mathematical concepts are prioritized and this highlights the power of creativity and logic. It also enables problem-solvers to ascertain the authenticity and truthfulness of mathematical processes. They concluded that if the solution to a problem is obtained consistently, then the credibility of the process is justified.

Learners understand mathematical concepts from observations, experimentations and abstractions using senses (Cuoco, 2000). A learner is viewed as an active constructer of knowledge, and solving-problems in different ways characterizes creativity of mathematical thought. To collaborate the above, the NCTM (2001) asserted that new mathematical knowledge can be formulated through applying and adapting a variety of appropriate strategies to solve problems. These strategies include, recognizing reasoning and proof as fundamental aspects of mathematics; making and investigating mathematical conjectures; developing and evaluating mathematical arguments and proofs. The ability to select processes using various types of reasoning and methods of proofs; organizing and consolidating mathematical thinking by communicating mathematical thinking coherently and clearly to peers, teachers, and others, are some of the strategies to be considered. Additionally, analysing and evaluating mathematical thinking and strategies of others; using the language of mathematics to express mathematical ideas precisely; recognizing and using connections among mathematical ideas are other strategies that can be used. Others include, understanding how mathematical ideas interconnect and build on one another to produce a coherent whole and recognizing and applying mathematics in the context of problem-solving situations.

In collaboration, Lesh and Zawojewski (2007) citing Al Cuoco (1998) in his article "Habits of the Mind" stressed the need for students: to think about mathematics the way mathematicians do; to be pattern sniffers; to be experimenters; to be describers; to be "thinkerers"; to have the ability of taking ideas apart and putting them back together; to be inventors; to be conjecturers; to be guessers; and most relevantly, to be visualizers (ability of constructing mental pictures and manipulating the pictures in various dimensions). These characteristics can only be attained if the problem-solver has mastered the relevant content knowledge of the mathematical concept under consideration (Mason, 2006).

Watson (2005) and Mason (2006) maintain that Mathematics is a cumulative subject in which learners have to understand lower-order concepts before they can learn higher – order ones. The quest for problem-solving skills reveals the value and the power of mathematics. The skill also acts as a teacher since problem-solving processes in varied contexts, makes us more knowledgeable. Additionally, a problem-solving process which did not partly or wholly bring about a solution to a problem in one situation can be used in another instance to achieve success. In support of the above, Cooper (2011), informs that competent problem-solvers usually envisage their chances of being successful at getting a solution to a mathematical problem. They would look at various strategies and decide on an initial plan, carry out the plan and modify it when necessary; these all rely on creativity.

2.3 Instructional methods/strategies

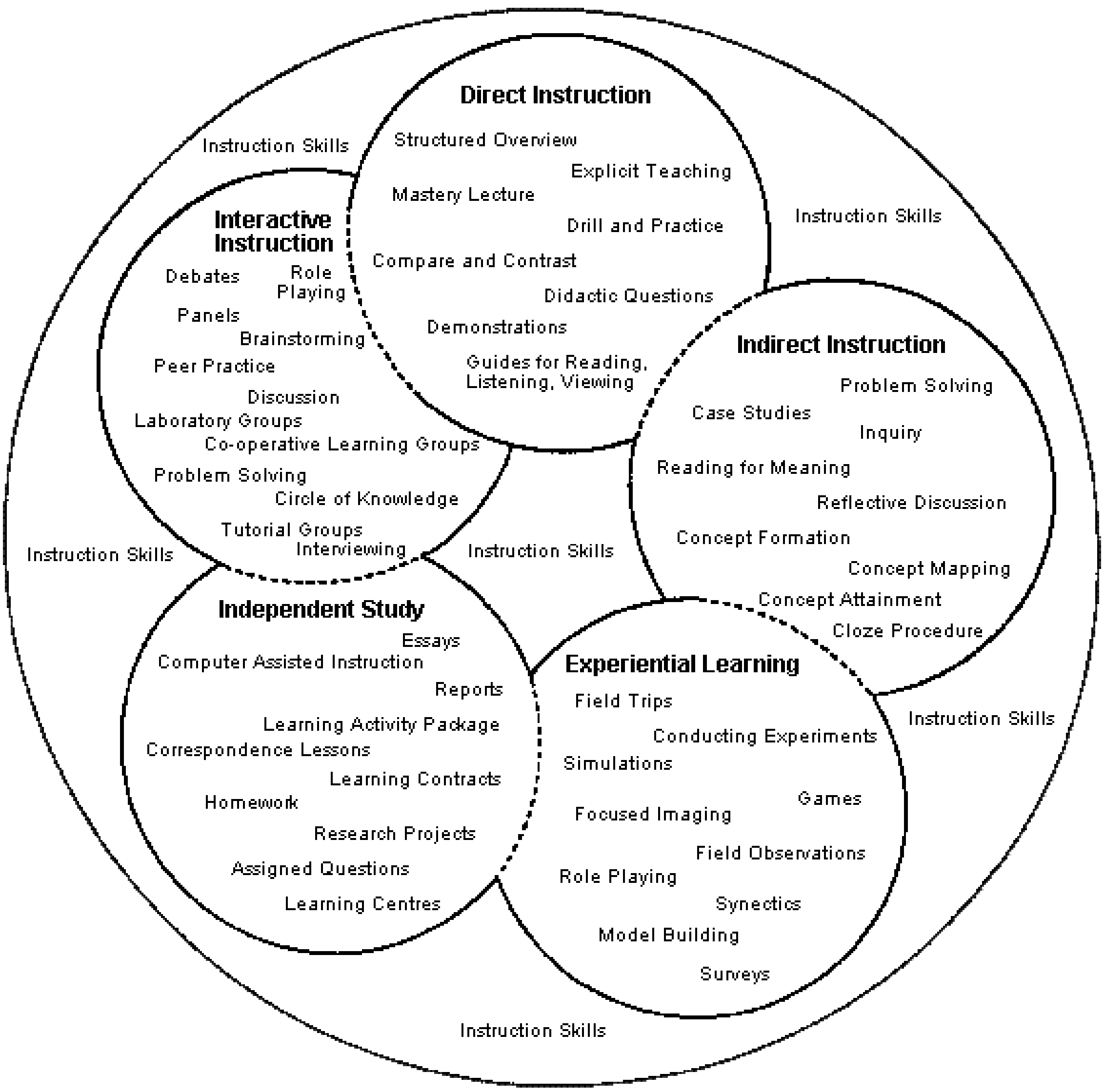


Figure 2.1: Instructional methods/strategies

The Figure above, which was developed by Joyce and Weil (1986) informs us about possible instructional approaches, namely, interactive instruction, direct instruction, indirect instruction, independent study and experiential learning. What are entailed under each of these instructional strategies are explained in the figure above. In relation to this research study, the aspects of relevance are: interactive instruction, indirect instruction and independent study; these three facets of instructional approaches, hence, are integrated in this study. In view of this, teaching and learning was conducted in a collaborative classroom setting, full of learners sitting in groups as they engage in activities like - reflective discussions, brainstorming, problem solving, peer practice, concept formation and making inquiries. As averred by Swartz and Reagan (1998) and Heyman (2008), discussions among group members enhance effective and efficient learning. They promote active learning in classrooms and optimize learning from others' viewpoints. Essentially, discussions serve as a good platform from which learners can develop and optimize their creative and critical thinking skills. After conducting group works, the collaborative groups were disintegrated and learners were then instructed to work independently, in which they were engaged with homework and standardized tests.

2.4 Metacognition in problem-solving

Metacognition is characterized as "thinking about thinking"; it aims to inspire learners to think about their thinking processes (reflection of their thinking), which promotes transfer of learning (Du Toit & Du Toit, 2013). According to Swartz and Reagan (1998), "if students are asked direct questions about their thinking that prompts them to reflect on what kind of thinking they did, how they did it, and how effective it was done". In view of this, developing metacognitive awareness and enhancing learners' metacognitive development, is pivotal in acquiring mathematical proficiency. They opined that metacognition promotes advanced mathematical thinking, thereby, playing an apropos role in problem-solving (Schoenfeld, 2007).

Likewise, Davidson, Deuser and Sternberg (2012), inform us that metacognition plays a vital role in problem-solving stating that there are four metacognitive processes which contribute positively to problem solving - (1) identifying and defining the problem, (2) mentally representing the problem, (3) planning how to proceed, and (4) evaluating what you know about your performance or focusing on individual differences in the use of these processes. According to Davidson et al. (2012), these four processes serve as a cognitive problem-solving path, which enable the problem-solver to think divergently while promoting focus thinking. This increases the problem-solvers' chances of getting an appropriate solution to the problem under consideration.

In the context of circle geometry, learners are directed to reflect on their thinking when solving circle geometry problems and this will greatly contribute to developing their mathematical proficiency so that they become independent thinkers and effective mathematics problem-solvers. To this end, learners are encouraged to make geometric decisions with proper justifications. When problem-solvers are encouraged to think about their own thinking, they will be intuitively guided to independently ascertain whether their proposed solution is appropriate or out of context. It will as well, enable the problem-solver to individually assess his/ her own thinking, whether it is meaningful, explicit and organized (Swartz & Reagan, 1998).

2.5 Euclidean Geometry

2.5.1 Nature of Euclidean Geometry

Euclid is a Greek mathematician who is credited as being the originator of Euclidean Geometry, described in his book, "The Elements". Euclidean Geometry encompasses applying logical reasoning, being intuitive and making deductions; these sharpen students' mathematical thinking skills through them applying postulates, axioms, theorems, to perform geometric proofs. It also entails awareness of areas and perimeters of 2D shapes, surface area and volume of 3D objects, geometry of straight lines, transformation geometry, and construction of geometric figures (Driscoll, 2010). According to Zeeman (2001,p.1) "geometry comprises of those branches of mathematics that exploit visual intuitions, which is the most dominant of our senses, to remember theorems, understand proofs, inspire conjectures, perceive reality and give global insight". In collaboration, the report by The Royal Society (2001), notes that geometry is useful in the following ways - the development of spatial awareness, geometrical intuition and the ability to visualise as well as the development of deductive reasoning and proofs. Battista (1999), notes that Euclidean Geometry accord students more mathematical insights which optimize their understanding of mathematical concepts.

2.5.2 How Euclidean Geometry can enhance learning of mathematics

According to Jones (2002, p.125) "geometry is intimately connected with the development of mathematics. The study of geometry contributes to helping students develop the skills of visualisation, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argument and proofs". In support of the above, French (2004) gave three reasons why geometry is included in the mathematics curriculum - (1) it extends spatial awareness, (2) it develops the skills of reasoning and (3) it informs challenges and stimulations. French (2004) continues that students' level of geometric competence stimulates their interest and conceptual understanding in other areas of mathematics, hence, if students are able to solve geometric problems accurately, it will enable them to develop adequate mathematical skills. This is due to the reason that solving problems in geometry enhances students' mathematical thinking capacity, thereby, promoting advanced mathematical thinking.

Gonzalez and Herbst (2006) cited in Ndlovu and Mji (2012) suggested more reasons to establish the necessity to integrate geometry in the curriculum: (1) geometry provides an opportunity for students to learn logic, transferable to other domains;

(2) geometry allows connections to the real world if students' experiences are matched with the demands of their current situation and future careers; (3) geometric proofs afford students variety of experience that resemble the activities of mathematicians; (4) geometry provides students with a unique opportunity to apply the intuition from geometric objects in describing the world. These are some factors which has established geometry as an indispensable and integral component of global contemporary mathematics curricula, hence, we cannot talk about mathematics without mentioning geometry.

2.6 Circle Geometry

2.6.1 Circle geometry in the CAPS curriculum

It can be seen in Figure 2.1 below, that circle geometry theorems have to be taught for a period of three weeks: week 6 to week 8. Teachers are required to guide Grade 11 mathematics learners to depend on the knowledge of geometry they had learnt in earlier grades, to guide their conception of new mathematical knowledge, that is, circle geometry. In Figure 2.2 below, can be seen that, how the teacher is expected to guide learners to understand and apply the knowledge of circle geometry, is conspicuously missing. Individual teachers, hence, may have to use their - discretion, knowledge, pedagogical approach, and experience. From the assertion that a vast number of teachers usually teach the way they were taught, (Cox, 2014), then undoubtedly, these teachers may have no knowledge or pedagogical competence about circle geometry. This is because, circle geometry was not a main or optional topic in the mathematics curriculum, prior to the inception of the CAPS curriculum. Those few teachers who might have been exposed to it in school might have learnt it by the traditional teaching and learning approach (Jansen & Dardagan, 2014; DOE, 2018). This is what this research study, therefore, sought to address.

TERM 1	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8	Week 9	Week 10
	(3-DAYS)	(5-DAYS)	(5-DAYS)	(5-DAYS)	(5-DAYS)	(5-DAYS)	(5-DAYS)	(5-DAYS)	(4-DAYS)	(3-DAYS)
Topics	Exponents and surds		Equations and inequalities			Euclidean Geometry			Trigonometry (reduction formulae, graphs, equations)	
	1. Simplify expressions and solve equations using the laws of exponents for rational exponents where $x^{\frac{p}{q}} = \sqrt[q]{x^p}; x > 0 ; q > 0$ 2. Add, subtract, multiply and divide simple surds. 3. Solve simple equations involving surds.		1. Complete the square 2. Solve Quadratic equations (by factorization and by using the quadratic formula) 3. Solve Quadratic inequalities in one unknown (Interpret solutions graphically.) **NB**:It is recommended that the solving of equations in two unknowns is important to be used in other equations like hyperbola-straight line as this is normal in the case of graphs 4. Equations in two unknowns, one of which is linear and the other quadratic 5. Nature of roots			Accept results established in earlier grades as axioms and also that a tangent to a circle is perpendicular to the radius, drawn to the point of contact. Then investigate and prove the theorems of the geometry of circles: • The line drawn from the centre of a circle perpendicular to a chord bisects the chord; • The perpendicular bisector of a chord passes through the centre of the circle; • The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre); • Angles subtended by a chord of the circle, on the same side of the chord, are equal; • The opposite angles of a cyclic quadrilateral are supplementary; • Two tangents drawn to a circle from the same point outside the circle are equal in length; • The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment. Use the above theorems and their converses, where they exist, to solve riders.			1. Derive and use the identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\theta \neq k.90^{\circ}, k$ an odd integer; and $sin^2\theta + cos^2\theta = 1$. 2. Derive and use reduction formulae to simplify the following expressions: 2.1. $\sin(90^{\circ} \pm \theta); \cos(90^{\circ} \pm \theta)$; 2.2. $\sin(180^{\circ} \pm \theta); \cos(180^{\circ} \pm \theta)$ and $\tan(180^{\circ} \pm \theta)$; 2.3. $\sin(360^{\circ} \pm \theta); \cos(360^{\circ} \pm \theta)$ and $\tan(360^{\circ} \pm \theta)$; 2.4. $\sin(-\theta); \cos(-\theta)$ and $\tan(-\theta)$; 3.Determine for which values of a variable an identity holds.	
SBA	Investigation or project						Test			

Figure 2.2 NCDOE Grade 11 mathematics work schedule

2.6.2 Circle Geometry theorems

The circle geometry theorems are as follows:

The tangent to a circle is perpendicular to the radius of the circle at the point of contact (tan ⊥ radius); if a line is drawn perpendicular to a radius at the point where the radius meets the circle, then the line is a tangent to the circle (converse of tan ⊥ radius) ; the line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord (line from centre to midpoint of chord); the line drawn from the centre of a circle perpendicular to a chord bisects the chord (line from centre ⊥ to chord) ; The perpendicular bisector of a chord passes through the centre of the circle; The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended by the same arc at the circumference (∠ at centre = 2 ×∠ at circumference) ; The angle subtended by the diameter at the circumference of the circle is 90° (∠s in semi-circle/diameter subtends right angle/If the angle subtended by a chord at the circumference of the circle is 90°, then the *chord is a diameter); Angles subtended by a chord of the circle, on the same side of the chord, are equal (∠s in the same segment); If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are con-cyclic; Equal chords subtend equal angles at the circumference of a circle.*

The diagrammatical representation of these theorems are presented in Figure 4.4, Chapter 4.

2.6.3 Circle Geometry as a troublesome content area in FET

Just the mention of Euclidean geometry in general and circle geometry, in particular, makes a lot of learners uncomfortable. What makes most of them confused are the geometry diagrams, which they have termed as being 'scary and non-hilarious'. The sight of those diagrams makes them disturbed and causes them to panic. This gives a clear indication of very negative attitude they might have developed towards geometry. This learners' very negative tendency towards geometry, may serve as a psychological barricade in their quest to solve circle geometry problems (Abakah, 2019).

Learners learn circle geometry theorems/concepts as sets of rules or principles to be followed, but circle geometry problems go beyond merely being able to memorize rules or principles (Ndlovu & Mji, 2012). Competence requires learners to link the ideas from the circle geometry theorems to form a holistic productive thought. It also requires learners to link the chains of their reasoning effectively to form a meaningful solution path, which will be helpful in solving a problem. Learners' mere ability to memorize the circle geometry theorems or concepts impedes their ability to be competent in geometric proofs and solving high-order geometric problems. They do not learn circle geometry as a content which requires rigorous mathematical thinking, analysing, being creative, conjecturing and linking chains of reasoning together to provide a meaningful solution path. They also do not learn circle geometry as a content which demands patience, persistence, perseverance and a 'never give up' attitude which will cultivate in them, the willingness to try, try and try again until an appropriate solution is reached (DoBE, 2018; Driscoll, 2010).

2.6.4 Students' difficulties and misconceptions in Circle Geometry

Inability of teachers to teach learners relevant and explicit problem-solving instructional approaches and to develop in them appropriate dispositions, attitudes, habits of mind, and others, may produce a lot of misconceptions and difficulties in students' minds. This may have devastating effects on developing their mathematical proficiency. Ndlovu and Mji (2012) for instance, maintain that students' misconceptions in circle geometry classrooms include inability to prove geometrically. Students inappropriately either list properties of geometric shapes as proofs and/or rewrite the known variables in a question as proofs. Ndlovu and Mji (2012) averred that the ineptitude of students to organise information in a logical chain of reasoning and arguments is another difficulty mathematics learners face; this develops into a misconception.

In addition, Özerem (2012) asserts that wrong usage of vocabulary to describe geometric statements and their relationships, inability to assess the validity of geometric arguments, and inability of students to know and apply appropriate formulae, theorems, postulates and axioms, might be responsible for students developing a lot of misconceptions in the geometry classroom. Oladosu, in her study of 2014, summarised some of the identified difficulties students face when solving geometry problems, these included - learners' lack of coordination in their views of three-dimensional objects (Battista & Clements, 1996); inability of learners to use theoretical statements in deductive reasoning and to recognise visually-relevant geometrical properties (Laborde, 2005); challenges in learning the appropriate language required for understanding and discussing geometric principles (Swindal, 2000); issues in relation to how students extract information from objects and form both natural and formal concepts (Battista, 2009); challenges related to measurement and deductive proof and linking chains of reasoning and understanding definitions in geometry (Chazan, 1993; Groth, 2005; Herbst, Gonzalez, & Macke, 2005). The conclusion was that "these difficulties centre on the meanings that students develop in relation to the learning they experience in and out of the geometry classroom" (Oladosu, 2014).

2.6.5 Managing students' difficulties and misconceptions in Circle Geometry

As asserted earlier, teaching students relevant instructional approaches and guiding them to develop the right dispositions, attitudes and habits of mind, in the geometry classroom, is key in managing students' misconceptions and difficulties (Swartz, 2012). For instance, Schwieger (2003), suggested the following ways for dealing with students' attitudes and misconceptions about problem-solving - asking interesting and real-life problems which students can understand and demonstrating to students that they do have the 8 problem-solving skills. These skills are - abilities to classify, deduce, estimate, generate patterns, hypothesize, translate, try, modify, and verify. In addition, the teacher is expected to give problem-solving examples illustrating the application of these skills; give practice that results in students sharpening these skills; demonstrate the necessity of implementing these problem-solving strategies; demonstrate that since multiple strategies are available, problem-solving is not necessarily impeded because a particular mathematical tool is unavailable; show that trials which do not lead to solution usually, also provide useful information to guide re-trials, that errors have not been committed, and that trial information should not be destroyed until after solutions have been reached; remind students that reaching solutions often take time and that experimentation is to be expected; remind students that there are no algorithms for true problems so they should not waste effort in trying to remember 'how we did this one the last time'; teach students that careful reading and comprehending the problem statement or situation are necessary and a search for 'key words' is likely to be counterproductive and give students practice with 'multiple' or 'conditional' solution problems.

In addition, Özerem (2012), suggested solutions to manage students' misconceptions. He established that it is necessary for teachers to - (1) use relevant vocabulary to describe geometric statements and their relationships; (2) apply logic to assess the validity of geometric arguments and (3) help students to memorize formulae easily. Ndlovu and Mji (2012), state that students' difficulties and misconceptions have pedagogical implications as this involves the teacher adjusting pedagogical strategies to deal with the particular difficulty/misconception identified. The implication is that, teachers need to adopt and implement appropriate and relevant instructional approaches (such as the proposed new instructional approach) in mathematics classrooms. This can help eradicate or reduce to the barest minimum, students' difficulties and misconceptions, the whole purpose of this research.

2.7 Assessment and assessment instruments designed to gauge the success of the applications of Infusion, Polya's model, ACE (APOS theory), and Collaborative instructional procedure in the pedagogical model

Assessment and evaluation play pivotal roles in supporting teaching and learning (Carl, 2012). In view of this, the introduction of a new instructional approach (the IPAC model), cannot be meaningfully implemented without formulating an effective assessment approach, that will go along with its implementation. The formulated assessment approach is made up of three facets: (1) generating and collecting evidence of achievement; (2) evaluating evidence and recording findings and (3) using findings to improve teaching and learning (DoBE, 2010). In measuring learners' geometric competences, two main modes of assessment techniques were implemented in the circle geometry classroom: (1) Interactive didactic assessment technique and (2) summative assessment technique (Blândul, 2009). They were implemented for gathering, recording, analysing, interpreting and documenting, in measurable terms, learners' geometric knowledge and to monitor learners' thinking abilities and mental constructions, in line with this research's focus (Brown & Knight, 1994). These assessment modes were implemented at different moments and each assessment method was implemented to serve a distinct purpose. For instance, interactive didactic assessment technique was implemented during teaching and learning sessions (assessment for learning). The summative assessment technique, however, was implemented to measure the overall outcomes of the teaching and learning sessions (assessment of learning), (DoBE, 2010).

As indicated, the interactive didactic assessment (self-assessment/inter-assessment), was employed to support teaching and learning of circle geometry. Blândul (2009), states "the interactive didactic assessment is a kind of control and examination activity in which the pupil/student has the possibility to appreciate both his/her knowledge and the knowledge of his colleagues, while the teacher's intervention is more like a prompt and systematic support given to the students in order to impose a formative character to the process". This was applied, with the aim of helping and inspiring each student to be at the centre of his/her own learning, not necessarily to rate students in accordance to their level of academic competence (Blândul, 2009).

As the collaborative instructional procedure is interactive in nature, the researcher realised that implementing this mode of assessment, in this study, would be appropriate. In view of this paper-and-pencil tasks: homework, class works, class tests, and investigations were implemented (See Table 2.1). Firstly, each learner will mark his/her own work (self-marking). After which, group members reshuffle their work randomly among themselves to be marked again (mutual-marking) then finally, the teacher will mark each individual learner's work. This means that three individuals will be responsible for monitoring each learner's mathematics learning and development: the learner him/herself, the learner's peer, and the teacher. The disparities in marks are then discussed, firstly, among group members and secondly, with the teacher. This will enable learners to identify their own misconceptions, difficulties or errors either individually or collaboratively. This mode of assessment gives an opportunity for learners to be responsible for their own learning, independently and collaboratively, which can improve their thinking skills. From this, significant students' academic improvements and increase in learners' confidence can be achieved. The process also encourages learners in a collaborative group to recognise and value their "scholar status" and that of their peers. When conducting summative assessment, the teacher, however, solitarily marked each learner's script (Blândul, 2009; Van den Heuvel-Panhuizen, 1996).

In addition to the above, the procedure, will as well, enable both teachers and learners to determine which learning objectives have been fully achieved, partially achieved or not achieved. This will enable alterations to instructional approaches and/or learning- support strategies to be considered when necessary, and most importantly, to address each individual's learning needs and difficulties. On the other hand, the summative assessment technique was implemented after teaching and learning had taken place. This was done to measure the end product of each learner's achievement (Ottevanger, 2001; DoBE, 2010).

Table 2.1: Assessment and assessment instruments

Type of assessment	**Assessment Activity**	**Assessment Instrument**	**How assessment may be conducted**
Interactive didactic	Class works (Group) Home work (Individual) Class tests (individual) Investigations(Group)	Memorandum Memorandum Memorandum Rubric/ Memorandum	Individual work and group work
Summative	Examinations (individual)	Memorandum	Individual work

This formulated assessment approach is in line with best assessment practices advocating that assessment should be of high-quality and meaningful to efficiently support teaching and learning. In view of this, the results of assessments should be timely, transparent, and readily available for interpretation and analysis purposes, (Council of Chief State School Officers [CCSSO], 2015). With this approach, as a mathematics teacher in South Africa, the traditional practice of teachers always asking their learners if they "understand" after a lesson, which includes circle geometry lessons, may not be necessary. I say this from my personal experience with learners that their acclamation having understood a mathematical lesson does not in any way collaborate with their classwork, homework or class test scores. They usually claim that they have understood a mathematical lesson, while their written responses to problems show the opposite. Hence, from now on in my teaching career, I will use individual learner's responses, classwork, homework or class test scores to judge if they have, in fact, understood a lesson. This is an indication that the traditional practice of asking learners if they have understood a mathematics lesson, as a judge of the efficacy of a teaching approach employed for mathematics lessons are counter-productive and misleading.

Programme of assessment

This assessment programme (presented in Table 2.2 below), was formulated in line with the mental construction lessons, advocated in this research study. They are - circle geometry action mental construction (GD 1), circle geometry process mental construction (GD 2); circle geometry object mental construction (GD 3); and circle geometry schema mental construction (GD 4). This is a thinking-based model, which entails more application and transfer of knowledge, hence, the researcher asserts that 60% of items of a task should be application oriented (GD 3 & GD 4) while 40% of items may be allocated to measure the level of individual learner's knowledge (GD 1 & GD 2). This will serve as the foundation on which learners can develop new mathematical ideas, through active mathematical thinking. This was based on the assertion of Marzano (1997) that assessment should focus on students' use of knowledge and complex reasoning rather than on their recall of low-level information. From this programme of assessment, mathematics learners will understand that mastering the knowledge of a concept is woefully inadequate in achieving mathematical proficiency. Proper mathematics learning, instead, is centred on intuitive, inductive and deductive application and transfer of knowledge to solve mathematical problems, practically or theoretically.

As posited by King, Goodson and Rohani (2013), valid assessment of thinking skills demands students to be exposed to unfamiliar questions and tasks by relying on their prior knowledge of the mathematical content under consideration. Dewey, (1933) cited in King, Goodson and Rohani (2013), reiterated that thinking does not occur spontaneously but must be "evoked" by "problems and questions" or by "some perplexity, confusion or doubt." Lai (2011) posits that thinking-skills assessment questions should be open-ended, in real-world contexts, accommodate meaningful varied ways of solving problems, trigger students to go an extra mile in solving problems, not by merely remembering and concentrating on previously-learned information. Lai (2011) established that such assessment tasks should make student reasoning visible by requiring students to provide evidence or logical arguments in support of judgments, choices, claims, or assertions.

Table 2.2 - Programme of assessment (CAPS, 2010: p. 53)

Mental construction level	**Skills to be assessed**	**Exemplar**
GD 1 **20%**	• Straight recall • Direct use of circle geometry theorems	**Appendix A**
GD 2 **20%**	• Direct applications of the circle geometry theorems to solve problems	**Appendix B**

GD 3 **30%**	• higher order geometric reasoning and creative thinking is required	**Appendix C**
GD 4 **30%**	• Non-routine problems which requires higher order reasoning, creative, critical and reflective thinking	**Appendix D**

In the light of the above, a modern day mathematics teacher is expected to "understand what thinking skills involves, i.e. understand strategies that can be used to teach thinking skills, gain facility in using the skills that are being taught, identify a range of appropriate contexts in applying these thinking skills, develop infused lessons in their instructional areas and implementing appropriate instructional strategies in the classroom to promote skilful thinking", (Swartz & Reagan, 1998). The primary aim of all the elaborations above is to develop mathematics students into good thinkers and competent problem-solvers, not only in mathematics, but most importantly, for them to be able to contribute significantly to solving real life problems, using mathematical ideas.

2.8 Review of Literatures

Research studies which are in line with the key elements of this proposed instructional approach - circle geometry, APOS theory, infusion approach, Polya's approach and how technology can be integrated with geometry teaching and learning, are all discussed here. This section, however, is non-inclusive of similar research studies that incorporate collaborative classroom procedures as the researcher believes that it might be considered redundant or a repetition of facts already captured by the current research.

2.8.1 Circle geometry

Circle geometry as a field of research is gradually aiding the mathematics education industry to ensure an effective instruction in classrooms. This is made possible as these research studies tend to highlight the teaching and learning difficulties related to circle geometry and to suggest useful recommendations to address them. Predominantly, these studies have centred on how students approach solving problems in circle geometry and misconceptions associated with the topic, how students' thinking affect their abilities in solving circle geometry problems, as well as their solving techniques. Some of these studies, germane to this study are presented in the discussions below.

Mwelese and Wanjala (2014) investigated the effects of problem-solving strategies on secondary school students' achievement in circle geometry. The research aimed to determine whether there is any significant difference in students' achievement when taught using a problem-solving strategy and when taught using conventional methods and to determine their attitudes towards mathematics when taught using a problem-solving strategy and when taught using conventional methods. The study utilised an experimental research design based on Solomon- Four-Fold Design. The findings from this research were that, there was no significant difference in the general performance between the two groups on pre-test, however, after the intervention there was a significant difference in the achievement on tests between the control and the experimental groups. The results, therefore, confirmed that the students who were taught mathematics using problem-solving strategy performed significantly better than those taught using the conventional methods. This is an indication that the problem-solving strategy has a positive impact on students' achievements in circle geometry.

Another research study by Oladosu (2014) - "Secondary School Students' Meaning and Learning of Circle Geometry" - established that meanings held for circle geometry concepts are contextually based. It was evident from the findings that participants sometimes hold more than one meaning for a concept depending on the context and structure of questions asked. It also ascertained that students' understanding of knowledge construction in learning circle geometry, depends on their sense-making of related real-world experiences, spatial awareness and their process of interpretation of geometry task; that it was necessary to pay attention to details and try to understand the relationships between concepts used in a task. The identified meanings confirmed the role of spatial reasoning in learning circle geometry, as participants constructed meanings that relied on size, shape, location, direction, and orientation.

Phadiela (2011) also conducted a research on "Problem-solving in geometry in collaborative small group settings on how learners apply or utilise mathematical tools while working in small groups". This research established that small-groups enables an atmosphere for interaction, which further enhanced learners' problem-solving skills. From this research, it was recommended that collaborative group work should form an integral part of mathematics learning in general and geometry learning in particular.

Tabat (2016) as well, conducted a research on circle geometry. He investigated - "Secondary school students' understanding of circle geometry in a computer environment". The researcher sought to examine the limit at which students can discover properties of circle geometry while working in a computer environment; how the Van Hiele Model can be used to describe students' learning of circle geometry while working in a computer setting, and the role computers play in helping students to visualize concepts in circle geometry. From the findings, it was realised that students learn and understand circle geometry effectively when working in a computer environment and that students can be at different levels of understanding with different geometric concepts. Most significantly, it was established that instructional materials can make a positive impact when they are used for circle geometry teaching and learning by encouraging and promoting higher levels of students thinking.

A study by Tay and Mensah-Wonkyi (2018) investigated the effects of using GeoGebra on senior high school students' performance in circle geometry. Two high schools were used - one school had a group of 24 learners in a class and served as the control group, while a group of 25 learners in a class in another school served as the experimental group. These schools and learners were purposefully sampled to serve as the research field. This study adopted and implemented a quasi-experimental design, using non-equivalent quasi-experimental design. The study implemented an achievement test for pre-test and post-test measures. The experimental group were taught by using GeoGebra, while the control group were taught through the traditional approach. Paired samples t-test and analysis of covariance were employed for the analysis of the collected data from the pre-test and post-test. The findings showed a statistically significant positive effect for students who learnt circle geometry through GeoGebra as students in the experimental group, who were taught circle geometry with GeoGebra performed better, in comparison to the achievement of the control group. The two groups demonstrated that GeoGebra lessons are more interesting, practical and easy to understand, in contrast to lessons from the traditional approach, hence, the researchers recommended that GeoGebra needs to be incorporated in teaching and learning of circle geometry.

2.8.2 APOS theory

The APOS theory has extensively and effectively been used in a plethora of mathematics education research studies. Some of these studies that are relevant to this study are elaborated below.

Syarifuddin and Atweh (2022) adopted and implemented the APOS theory in an action research study, to establish how students' engagement in learning elementary linear algebra can be enhanced. During the study, participants were taught an elementary linear algebra module using the ACE teaching cycle approach. In this study, the constructivist theory served as the theoretical framework and data collection and analysis followed the mixed-methods approach. Focused-group discussions, classroom observations and questionnaire, served as the main methods of data collection for this study. The implementation of the ACE teaching cycle in this study were three-fold: (1) the concept-maps' activity, to elicit participants' prior knowledge, (2) classroom discussions and (3), exercises given to participants as homework. This study established that the ACE instructional teaching approach did not only enhance the study participants' engagements cognitively and affectively, it also enhanced their behavioural engagement; this is of much relevance to the current study. It serves as a strong literature underpinning the philosophical bases for this current study since almost the same research approaches were implemented in both studies - the ACE teaching cycle as a teaching methodology, the mixed-methods approach for data collection and analysis and the constructivism approach as the research paradigm. The researcher, therefore, posits that the same level of success and efficiency can be envisaged for this current study. The only disparities with the two studies being - the mathematical content under consideration (circle geometry in contrast to algebra) and the context of conduct of these two studies (South Africa instead of Indonesia).

Researchers, Borji, Alamolhodaei and Radmehr, in 2018, adopted the APOS Theory to explore the teaching and learning of derivatives with focus on its graphical understanding. In the said study, a Genetic Decomposition was developed using the outcomes of previous studies and/or by the researchers' personal teaching experiences. A Maple software was used to design an ACE teaching cycle, which was then implemented on an experimental group while the traditional approach was implemented on a control group. The achievements of the students in the two groups were then compared. The findings showed that students who formed the experimental group exhibited better understanding of the topic under consideration (derivatives), in comparison with the students who formed the control group. The researchers, hence, advocated for the APOS theory (ACE teaching cycle), to be implemented in mathematics classrooms for the teaching and learning of derivatives.

Another research study which made use of the APOS theory was carried out by Tziritas (2011). The APOS theory was employed in this study to serve two purposes - as a research approach, and as a teaching and learning approach. The APOS theory was used to develop and test a teaching cycle for the improvement of students' early conceptual understanding of "related rates problems". The genetic decomposition was used to construct an ACE teaching cycle which was then tested on two groups of students. Tziritas (2011) observed that the students responded well to the ACE cycle in terms of completing the problems, but they were completely unable to solve the related rates problems. The researcher realised that the students encountered several difficulties when they used the APOS theory to study the conceptual steps of related rates problems. From these findings Tziritas (2011) asserted that the APOS theory is useful in a web of concepts, such as function, variable, derivative, and others, but related rates problems, however, involve modelling and problem-solving abilities. The action, process, object, mental constructions, therefore, did not seem to be an appropriate framework for studying and fostering the development of those abilities. He concluded by saying that a combination of frameworks may clarify students' difficulties with related rates problems.

Arnawa and Yerizon (2019) conducted a pre-experiment one-shot case study to explore how students' level of proof ability in abstract algebra could be improved through the APOS theory's approach. In this study, abstract algebra lessons were conducted by implementing the ACE cycle and the theory was used to determine students' level of understanding of the concept of abstract algebra. University students who had registered for the abstract algebra course served as participants for the study. The levels of students' ability in proofs were grouped into three categories - level 1 (sin tactic), level 2 (concrete semantics), and level 3 (abstract semantics). The findings were that - most students experience difficulties in learning abstract algebra, that there is a gender difference in students' ability to proof in abstract algebra, although, statistically not significant. The researchers discovered that students with a background in teaching and learning based on APOS theory

approach could perform better with regards to abstract algebra proofs and that their level of thinking was also comparably higher.
A case study which involved the APOS theory in investigating junior high school students' quantitative reasoning process in solving covariant problems was conducted by Syarifuddin, Nusantara, Qohar and Muksar in 2019. In this study, data was collected through a co-variation problem task and task-based interviews with students who performed different processes to determine their existing mental processes. From this research, the researcher discovered two different students' quantitative reasoning processes for co-variation problems - inductive quantitative reasoning process and the deductive quantitative reasoning process. This study revealed that the APOS theory enhanced students understanding and quantitative reasoning processes when solving covariate problems.

Malahlela (2017) carried out a qualitative error analysis intervention study by using errors and misconceptions as a resource to teach functions to Grade 11 learners, involving the APOS theory. Pre-test and post-test measures were also used. This study sought to determine the errors and misconceptions Grade 11 learners show on functions, learning affordances and constraints that can be created if teaching is directed at learners' errors and misconceptions, and the extent that learners' achievements on functions can be boosted if teaching is directed at learners' errors and misconceptions. The findings from this research, relevant to this study, were that, using the APOS theory for analysis, enabled the researcher to identify learners' errors and misconceptions when teaching functions; in addition, learners' achievements in functions were found to have improved beyond the average level.

Another research study, which employed the APOS theory was carried out by Voskoglou (2015). In this study, the researcher compared the performance of two student groups - an experimental group and a control group - on their understanding of real numbers in general and of irrational numbers, in particular. In the study, the control group was taught through the traditional approach, while the experimental group was taught by APOS teaching and learning approach. The findings revealed that students who were taught by the APOS theory's teaching and learning approach (the experimental group) performed better as they exhibited better understanding in comparison to the control group.

Moon (2019), incorporated the APOS theory to explore alternative approaches to the solution test for graphs of algebraic inequalities in two variables. The study used the framework of action and process conceptions of the APOS theory. The suggested alternatives offered a relational understanding for graphs of inequalities in two variables by incorporating the critical concept of the variable; this could be used as the medium of instruction, transitioning from graphs of one-variable functions to graphs of two variable functions. In conclusion, the action and process conceptions which were employed as the framework for this study enabled the study participants to conjecture more alternative approaches.

A similar study was carried out by Maharaj (2010); he investigated university students' understanding of limits of functions using the APOS theory. Firstly, he conducted lessons on limits of functions with specific focus on "content knowledge of functions". Lessons were conducted with undergraduate university science students. The findings established that, students find difficulty in understanding the limits concept, due to the fact that many students do not have appropriate mental structures at the process, object and schema levels of the APOS theory.

Maharaj, in 2013, implemented the APOS theory in another research into university students' understanding of derivatives and their applications. The relevant rules for finding derivatives and their applications were taught to undergraduate science students. This study established that those students had difficulty in applying the rules for derivatives and this was possibly the result of many students without the appropriate mental structures at the process, object and schema levels.

Ofori-Kusi, in his 2017 research, explored the effects of a problem-solving heuristic instructional method on Grade 6 learners' achievements in algebra. The APOS theory was coupled with a modelling perspective for developing an instructional approach; this served as a guide when developing modelling-eliciting activities (MEAs), while the APOS theory served as a guide when sequencing the activities used to develop the Grade 6 learners' conceptions in algebra. This study ascertained that the developed instructional approach, improved the scores of Grade 6 learners when solving algebra problems. The researcher deduced from this study that implementing problem-solving heuristics, can have a positive impact on Grade 6 learners' achievements in algebra.

The APOS theory was also used by Ndlovu (2015) to explore pre-service teachers' mental constructions of matrix algebra concepts, in a case study, conducted in South Africa. The study was underpinned by the assertion that understanding the mental constructions the pre-service teachers made when learning mathematical concepts would lead to improved instructional methods. In this study, groups of first- and second-year university students were exposed to teaching and learning of some of matrix algebra concepts, these students were expected to learn. Firstly, the concepts were taught to students, then they were expected to express their thinking through solving matrix algebra-related problems during tutorials, supplemented by interviews. The finding from this study, relevant to this current study, was that many pre- service teachers were mainly operating at an action and process stages, although, a few were operating at an object stage. Ndlovu (2015) concluded by asserting that the introduction of a modified, itemised, genetic decomposition in the study would help in the teaching and learning of matrix algebra concepts.

A pre-test and post-test measures examining university students' achievements in Abstract Algebra proofs in a quasi-experimental, nonrandomized research study was conducted by Arnawa, Sumarno, Kartasasmita and Baskoro (2007). Two groups were used for the study - experimental and control. In the study, 180 students from two different universities, that is, two mathematics classes from each university, served as the research participants for the study. The experimental group was exposed to the APOS theory's instructional approach, while the control group was taught using the traditional approach. It was deduced from the analysed collected data that the proof ability of the experimental group was significantly better than those in the controlled group, hence, it was suggested that the APOS theory have to be applied in the teaching and learning of abstract algebra courses.

Tokgöz (2015) investigated undergraduate and graduate engineering and mathematics students' ability to transform algebraic functions to their geometric representations. This study was conducted on the basis that success in many engineering and mathematics courses is tied to a well-developed calculus knowledge, such as limits of functions, first derivative, second derivative, asymptote, and others. In view of this, participants were either enrolled or recently enrolled in a two-week Numerical Methods/Analysis course. The APOS theory was used to analyse video-recorded data and also, data from students' written responses to 'graphing a quotient function' - a sub-topic under Numerical Analysis. In the task, participants were asked to sketch the graph of a given quotient function after calculating its limiting values, first derivative, second derivative and asymptotes. The results of the analysis of the collected qualitative and quantitative data revealed that higher success rate in Mathematics was recorded among all the participants, hence, advocating for the APOS theory to be incorporated in Numerical Methods/Analysis lessons.

Bansilal, Brijlall and Mkhwanazi (2014) investigated the mathematics content knowledge of teachers in South Africa. This study was conducted on the bases that, many research studies have highlighted the problem of poor content knowledge of mathematics teachers in South Africa. For this study, 253 mathematics teachers served as participants and an abridged form of a previous Grade 12 Mathematics Paper One was administered to them as a standardized test and served as the main source of data for the study. An analysis of their written responses to the standardized test indicated that the teachers obtained an average of 57% in the test. An APOS theory's analysis of the responses revealed that many teachers who were working at an action level of concepts would require help and scaffolding to move to process or object levels of understanding of those concepts. The analysis further revealed that on average, teachers obtained 29% on higher-order level questions. These results caused the researchers some concern as these teachers were expected to teach tasks that are set at high cognitive levels, with their Grade 12 learners. This explains why an overwhelming majority of mathematics learners abandon high/higher order level of questions in examinations.

A case study to explore the errors that are displayed by students when learning derivatives of trigonometric functions in an extended curriculum programme was undertaken by Siyepu in 2012. The study sought to identify errors that were displayed by students in their solutions based on the APOS theory and to address these errors by using the two principles of Vygotsky's socio-cultural theory of learning, namely, "the zone of proximal development" and "more knowledgeable others". A group of university mathematics-registered students served as participants for this study. Data were analysed through categorising the errors in the students' written work; data were collected through finding common themes and patterns in audio and video recordings as well as from in-depth interviews. The findings of this study revealed that students committed interpretation, arbitrary, procedural, linear extrapolation and conceptual errors. These findings were found to be consistent with the literature as they confirmed that, errors are based on students' poor prior knowledge; this forced students to over-generalise certain mathematical procedures, algorithms and rules of differentiation in their solutions.

Drlik (2015) carried out an investigation into students' understanding of Functions and Calculus. This study sought to determine if there is a correlation between students' success in calculus and students' understanding of functions. Students' understanding of functions was measured using two questionnaires. One was a modification of an existing measure based on APOS theory, while the other was developed by the researcher by adopting and implementing items from the concept image literature. The questionnaires were administered to 116 students who were enrolled in a first-year calculus course. Participants' responses to the items of the questionnaires were juxtaposed on their exam scores. The data analysis recorded a positive correlation between understanding of functions and success in calculus. This demonstrates that students can be successful in a first-year calculus course without demonstrating a process level understanding of functions at the beginning of the course.

Davis and Martin (2015) incorporated the APOS theory in a research study to investigate if students know 'what a Function is'. The researcher conducted an experiment with university students whose major subject was mathematics. The researcher conducted an APOS theory analysis on students' small group presentations where they were asked specific questions about the nature of functions. Students presented their understanding of functions in groups of two or three, which were recorded and then transcribed. The researchers identified some overall trends in students' understanding, their common misconceptions and difficulties. They labelled the common identified misconceptions as 'obstacles to success in a variety of undergraduate courses'. They then concluded that, the numerous common misconceptions and the difficulties students encountered during the study indicated that students do not really know 'what a Function is'.

Voskoglou (2001) carried out an experimental study into students' understanding of the concept of infinity by implementing the APOS theory. The researcher studied the effects of an instruction into the basic philosophical/epistemological aspects of the concept of infinity on students' abilities to deal with situations involving directly or indirectly the concept of infinity. Voskoglou (2001), deduced from the results that an instruction to the basic philosophical/epistemological aspects of infinity could improve students' skills to deal successfully with the topic. Another deduction the researcher made, relevant to this study, is that, the APOS theory gives an adequate, modern explanation of some difficulties students face while learning the infinity concept.

2.8.3 Polya Approach

In'am (2014), implemented Polya's method in solving Euclidean geometry problems, in a mixed-method study; 85 university students in the Department of Mathematics Education, served as participants for the study. Instruments were used to ascertain how students' responded to Polya's method and to ascertain how they were able to solve the two given Euclidean geometry problems. The research findings indicated that the majority of students were able to understand the problems given to them, hence, their performance level was judged as good. These students could effectively plan the solution for the given problems, and they could as well, carry out their solution plan, efficaciously, however, some students were not able to make any review on the 'look back' component of Polya's method.

Brijlall (2015) conducted a case study to explore the stages of Polya's problem-solving model during collaborative learning with Grade Ten learners by focusing on fractions. Participants were from two classes; some worked in groups in one class, while those in the other class worked individually. Social constructivism was adopted as a theoretical framework and the stages advocated by Polya were explored when analysing learners' responses to the problems they were expected to solve. The findings established that learners who worked in groups, could effectively explore most of the stages of Polya's problem-solving model. The findings also guided the researcher to pick out stages of Polya's model that stimulated effective problem-solving.

Another study which implemented Polya's problem-solving method was conducted by Valles and Wickramasingh (2015). The method was utilised in one of the two introductory-level statistics classes taught by the same instructor, and a comparison was made between the performances in the two classes. The results indicated that there was a significant improvement of students' performance in the class in which Polya's method was implemented.

Abakah (2019) examined problem-solving skills in circle geometry concepts in a two-year intervention case study involving Grade 11 Mathematics learners in a high school. Polya's problem-solving instructional approach was coupled with social constructivist instructional approach as the intervention approach for this study. The research findings that emanated from this study were: (1) the research intervention evoked learners' desire and interest to learn circle geometry and (2) it enhanced the study participants' performance and problem solving-skills in circle geometry concepts. The researcher, hence, recommended this intervention instructional approach for circle geometry instructions in South Africa, based on its efficacy.

An experimental study involving pre-test and post-test measures to investigate the effects of Polya's Problem-Solving Model (PPSM) on learning achievement and analysing ability in mathematics of fourth grade students was undertaken by Phuntsho and Dema (2019). An achievement test and time-series record were used to collect data. The data analysis' results revealed an improvement in students' mathematical academic achievement using PPSM, hence, the study recommended this approach as an alternative method for teaching and learning of mathematical problem-solving.

Another researcher, Gray (2018), implemented Polya's problem-solving method in a study with students. He investigated the impact of applying the first two steps of Polya's four problem-solving steps in an advanced mathematics course in a high school classroom. The process of problem-solving was introduced to the students to determine the impact of the problem-solving approach. The researcher reported that the introduction of the problem-solving approach enabled students to gain new awareness about problem-solving, therefore, the approach can be said to have had significantly, positive influence on teaching and learning of mathematics problem-solving in an advanced mathematics classroom.

Kousar (2010), incorporated Polya's problem-solving heuristic steps as part of a problem-solving approach on the academic achievement of students in mathematics at secondary school level. This study involved 48 mathematics learners who were partitioned equally into two groups - the control and the experimental - based on their pre-test scores. After the treatment, a post-test was used to ascertain the effects of the treatment. A two-tailed t-test was used to analyse the data, which revealed that both the experimental and control groups were almost at the same level with regards to their mathematics scores, at the beginning of the experiment. The experimental group, however, outscored the control group significantly on the post-test. This is an indication that the teaching strategy which incorporated Polya's problem-solving heuristic steps, had a positive impact on the experimental group's performance.

A research which focused on Updating, Modernizing, and Testing Polya's Theory of Mathematical Problem-Solving, based on the Cognitive, Affective, and Information Processing Theories of Learning, Emotions, and Complex Performances, was carried out by Carifio (2015). The disruptive influence of emotions (both positive and negative) of Polya's problem-solving model and the oscillation of emotions during mathematical problem-solving were investigated. The results revealed that, positive emotions energize, organize, focus, and improved performance, although, negative emotions also provided highly valuable information for problem-solvers in general and for sophisticated problem-solvers, in particular.

Mehmood (2014) conducted an experimental study which investigated the effects of Polya's problem-solving method of teaching on the revised Bloom's Taxonomy, in mathematics at elementary level. Three groups were formed through proportionate random sampling, prior to conducting a pre-test. The experimental group was taught by the researcher implementing Polya's problem-solving method, and the other two groups were taught using the conventional method by the same teacher. The items on the pre-test were reshuffled and formed the post-test. Forty lessons were taught during the 8-week experiment, to all three groups. The pre-test and post-test scores were analysed on SPSS. The findings from this study were that students performed better on the Revised Bloom's Taxonomy when Polya's method was used to teach them, as compared to the conventional method. From this research, Mehmood

(2014) concluded that Polya's method is more effective than the conventional method for teaching Mathematics, hence, the method was recommended for teaching Mathematics at elementary level, also that it should be added to the teachers training programmes as well.

A non-equivalent pre-test and post-test quasi-experimental research was performed by Lee and Chen (2015) based on Polya's approach, using question prompts and multimedia demonstration on the topic, "The Effects of Polya Questioning Instruction for Geometry Reasoning in Junior High Schools". In the study, two classes of Grade 7 students were randomly selected as the experimental group and were taught based on Polya questioning. Two other groups were selected to constitute the control group and they were taught based on direct presentation. The research findings revealed that the students who were taught by employing Polya questioning, performed better and exhibited stronger sense of participation, in comparison to students who received direct presentation, based on the post-test scores.

Hayyulbathin, Winarni and Murwaningsih (2014) conducted a study under the topic "Modifying Polya's Step to Solve Maths Story Problems". The modified Polya's five steps consisted of: (1) understand the problem, (2) devise a plan, (3) carry out the plan, (4) look back and (5) decide a conclusion. These modified steps proved effective for solving 'Maths Story Problems', hence, the authors advocated for its implementation in classroom lessons.

In a research employing a semi-experimental design, in collaboration with pre-test and post-test measures, Loğoğlu (2017) investigated the effect of mathematics teaching, using Polya's problem-solving steps, involving 4th grade learners. These learners who were studying at two state elementary schools were divided into two groups - experimental and control. Loğoğlu (2017) concluded that learners' success in the process of solving mathematics problems was improved by applying Polya's steps accurately.

A research study, entitled "Comprehensive Monitoring and Polya's heuristics as tools for problem-solving" carried out by Schurter (2001), made use of three different groups; all three groups were taught using different methodologies. Group 1- served as the control group and they were taught using the traditional method; Group 2 served as the experimental group and they were taught through comprehensive monitoring alone, while Group 3 served as another experimental group but they were taught using comprehensive monitoring, in conjunction with Polya's 4-step method. Pre-test, post-test, interview and questionnaires were used for data collection and the collected data were analysed using t-test and ANOVA. The research established that, the two groups who were exposed to either comprehensive monitoring alone or in conjunction with Polya's heuristics both performed better in mathematical problem-solving, than those who did not receive either type of instruction.

2.8.4 Infusion Approach

Solving mathematical problems is a cognitive act, which requires extensive thinking and reasoning (Swartz, 1992). Thinking skills cannot be dissociated from mathematics, especially in geometry. Limited studies have incorporated the infusion approach in the Mathematics Education field; some are reviewed below.

A study titled "Evaluating an infusion approach to the teaching of critical thinking skills through mathematics in secondary schools" investigated the viability and consequences of developing critical thinking in probability, by employing the infusion approach (Aizikovitsh & Amita, 2010). An ANOVA test showed that the experimental group considerably improved their critical thinking abilities and disposition. The researchers used California Critical Thinking Dispositions Inventory (CCTDI) to measure the achievement of students in probability learning. The 'dispositions' referred to - truth seeking, encouraging open mindedness, mental flexibility, analyticity, reactions to difficulties encountered, systematization, maturity and confidence. The infusion approach was found to be a credible approach to be employed to improve students' achievements. This finding establishes that thinking skills must be taught along with relevant dispositions and processes. This also justified the current research adopting and implementing the infusion approach, based on the assumption that the strategy will improve students achievements in circle geometry, since both probability and geometry are mathematical concepts.

Zulkpli, Abdullah, Kohar and Ibrahim (2017) carried out a review on the infusion approach in teaching thinking by evaluating its advantages and impacts. They specifically investigated whether the approach yields positive or negative outcomes on students' thinking; ten articles and thesis were selected for the review. Their findings showed that the infusion approach improved and optimized students' knowledge, attitudes and values, thus, positive impact from the adoption of the infusion approach was recorded. The authors explained that the use of infusion approach in teaching may be two-folds - either it benefits teachers in terms of their quality of teaching or it benefits the students in terms of improving their thinking skills. They further mentioned that the infusion approach could be applied in teaching thinking in any subject. This is based on the emphasis placed on thinking skill elements and the crucial roles of teachers in enhancing students' higher order thinking, which can encourage active learning among students, in all subjects.

2.8.5 Integration of technology in teaching geometry

Integration of technology into teaching and learning of mathematical concepts, like those in geometry, is receiving a lot of attention from researchers, especially now that the world is viewed as gearing towards the fourth industrial revolution. In this regard, the application of technology is envisaged as having the potential to affect human daily life, beyond humans' imagination (DoBE, 2018). In support of the above, Driscol (2016), asserts that technology can promote mathematical reasoning and thinking. Additionally, Driscol, Nikula and DePiper (2016) aver that teachers are expected to incorporate visual representations into mathematics instruction, and this can be achieved by integrating technology into geometry teaching and learning.

The introduction of technology in the mathematics classroom can make teaching and learning of difficult-to-learn concepts such as those in geometry, easier (Ruthven & Hennessey, 2002). These researchers, note that GeoGebra, Geocadabra and Geometers' sketch pads are the most common technological software available for teaching and learning of geometry. According to (NCTM, 2000), "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning". Hosein (2009), asserts that technological tools make a positive impact on mathematics teaching and learning as they promote high quality thinking around mathematical ideas. This is characterized by critical thinking abilities, formulation of patterns and relationships, relating concrete entities to the abstract domain of conceptualizations, intuitions and making deductions from abstract constructions. Haddad and Draxler (2002), had asserted that technological tools have an influence on learners' achievements, by establishing that the use of technological tools promotes learners' conceptual understanding and achievements. Fonkert (2010), also established that the use of technology influences how students interact in classrooms, as well as, how they learn; this should guide how technology may be incorporated in classrooms.

To determine how learners can use the Geometer's sketchpad, Hannafin and Scott (2001) conducted a research; the study was conducted in a student-centred environment. The study suggested that the Geometers sketchpad is very useful in the geometry classroom, however, it was recorded that some participants had some difficulties in using it and that some were not willing to use the sketchpad. In reaction to these findings, Pierce (2007), suggested that students' mathematical skills and confidence to tackle problems is to some extent, associated with their willingness to use technology.

Moila (2006) studied the use of educational technology in Mathematics teaching and learning in a secondary school. It was established that there was inadequate educational technological tools and no proper planning and insufficient teacher training, with regards to their usage. He concluded that the handful of technological tools which were available, in sampled school were not being used for their rightful purposes.

An investigation, by Adenubi (2016), into the effect of "Animated Computer 3-D Figures Illustration (ACTDFI)" on the learning of polyhedron in geometry, involved four Grade 9 classes, in four different schools. The study employed a mixed-method research design. The quantitative component involved a quasi-experimental approach, by using the pre-test and post-test designs, while the qualitative component involved the inquiry approach, through classroom observations. ACTDFI was used as an intervention for two weeks in three experimental groups, while in the control group, the traditional teaching approach was adopted. It was concluded that the use of ACTDFI facilitated the learning of polyhedron during the intervention, therefore, improved academic achievement of the learners.

In determining if there could be any impact in 'infusing technology into a mathematics methods course' Li (2010), examined how the integration of multimedia and online discussion into a mathematics methods course could affect student teachers' beliefs about geometry and their attitudes toward educational technology. The study established that the incorporation of technology changed the student teachers' attitudes and it positively affected their attitudes toward geometry and its teaching.

2.9 Implications of the literature review

With reference to the variety of literature reviewed in relation to this research, the effectiveness and potential of incorporating each of the research constructs: APOS theory, Polya's approach and infusion approach, have been established and justified. The review also provided details as to how each research construct can be applied, in different contexts to yield positive results. To this end, the literature reviewed served as guidelines and reference tool in relation to the conducting of this research study. It directed the researcher on how each of the research constructs can be implemented effectively. By so doing, it directed the researcher on how to appropriately integrate the three research constructs in a single study, as this research sought to establish.

2.10 Conclusion

The reviewed literature established that there were limited studies that had incorporated the APOS theory and the infusion approach, relevant to this study. This is because only a handful of the identified literature, under the APOS theory and the infusion approach, involved circle geometry. The reviewed literature also established that there is limited research that partially or wholly integrated any two or all the three research constructs, as this study sought to examine. The process also enabled the researcher to identify gaps in literature, which made him to conclude that the current study will adequately add to the existing body of knowledge in the field of mathematics education, thus, provided advocacy to substantiate the fact that this study was worthwhile to be conducted. In the next chapter, the theoretical frameworks that underpin this study are comprehensively explored and delineated.

CHAPTER THREE
THEORETICAL CONSIDERATIONS

3.1 Introduction

This chapter informs readers about theories which are related and germane to this study namely, teaching and learning theories - traditional perspective and constructivism perspective of learning, cognitive learning theory, infusion as an instructional approach to teaching thinking, the didactic triangle, Robert Marzano's teaching with dimensions of learning, the APOS theory and its processes, interiorization, encapsulation as instructional goals and preliminary genetic decomposition of circle geometry mental conception. In addition to the above, Richard Paul's wheel of reasoning, Howard Gardner's theory of multiple intelligence and descriptions of reasoning habits through the GHOM perspective are all included in this chapter. Lastly, how the research constructs of this study collaborates with the theoretical frameworks are also delineated in this chapter.

3.2 Teaching and learning theories

3.2.1 Traditional perspective of teaching and learning

In a traditional classroom, a task is often used by the teacher to introduce a new technique, then students practice the technique using similar tasks. This is what some refer to as 'Triple X' teaching - 'exposition, examples, and exercises' (Evans & Swan, 2014). By exposition, the students are exposed to problem-solving mathematical procedures and students are expected to follow these mathematical procedures without much input. Students are then given examples of the mathematical problems on the particular content under consideration; then related problems are presented to students as class exercises or homework. This implies that non-related problems cannot be conceptualised by the students. This limits the students` thinking and reasoning skills; this procedure does not promote advanced mathematical thinking.

The field or context in which this study was conducted, is well-grounded in the traditional instructional approach, hence, mathematical concepts were being taught as a set of rules and formulae to be followed. Most teachers were involved in these traditional teaching practices which show the tradition notions of mathematical literacy. These practices did not accommodate strides made in mathematical literacy, hence, this study conducted at the school level, sought to expose teachers to the, demands and expectations of contemporary mathematical practices.

3.2.2 Constructivism perspective of teaching and learning

In lieu of the traditional instructional approach, demands of an effective problem-solving instructional approach to serve modern curricular needs, led to the discovery of the constructivism perspective of learning; this emanated from a series of research which commenced, about a century ago. In the constructivism paradigm, by using observations, experimentations, individual and corporative learning, learners are given the opportunity to make their own mathematical constructions and to defend them. Brainstorming, among others are key features of the constructivism epistemology; these serve as inherent techniques, which are necessary for teaching and learning of problem-solving (NCTM, 2001).

The constructivism epistemology arguably, best supports problem-solving in mathematics. Selecting the appropriate learning content, which makes room for experimentations, activities, projects, and analyse, encourage learners to give their opinion; these are all healthy practices which promote the agenda of problem-solving. The constructivism instructional approach affords students more opportunities to - relate the concepts they learnt, in the classrooms, to real life situations; develop their computational skills, communicate mathematically and use and interpret mathematical symbols and expressions appropriately. This approach also guides students to develop the correct use of mathematical language, while also guiding them to solve questions or problems in a variety of ways. It further uses mathematical processes, such as making conjectures, proving assertions and modelling situations as well as being able to use available technology to optimize teaching and learning (DoBE, 2010; NCS, 2004).

To throw more light into the need for the constructivists' perspective on teaching and learning, (NCTM, 2001), articulated five goals of learning through this approach. They are - children learn to 'value mathematics, become confident in their ability to do mathematics, become effective mathematical problem-solvers, learn to communicate and reason, mathematically. In the mathematics classroom, students have to be exposed to numerous and varied interrelated experiences. They are also encouraged to value the mathematical enterprise, develop mathematical habits of mind, and understand and appreciate the role of mathematics in real-life. Learners need to be encouraged to create, explore, guess and even make and correct errors so that they gain confidence in their ability to solve complex problems.

Lerman (2012) avers that teaching mathematics based on the constructivist perspective makes mathematical knowledge tentative, intuitive, subjective and dynamic, hence, mathematical knowledge is not just somewhere, pending discovery, but it is intuitively constructed by learners using the resources in their environment, enabled by an effective mathematics curriculum (Whitebread, 2010). To elucidate further, Chiu (2010) gave the following characteristics of a constructivist mathematics curriculum - it guarantees meaningful learning and understanding, it promotes creative thinking, reasoning and exploration, it provides students with an experimenting learning environment (which enables them to guess, conjecture and test hypotheses), it makes for independent learning (which accords students the freedom to meaningfully construct their mathematical ideas). This approach also allows students to ponder on their mathematical construction and social interaction, which promotes effective communication between pedagogues and students, when they are alone, in groups or in class.

Jones, Fujita and Ding (2006) maintain that "Applying geometry through modelling, deductive reasoning, development and use of conjecture, problem-solving in a range of contexts should be encouraged when teaching geometry". They are also of the view that creating awareness of the historical and cultural heritage of geometry in society, as well as the contemporary applications of geometry should be encouraged when learning geometry. Instructions in geometry should encourage students to engage in investigative activities, demonstrative creativity, and make discoveries in geometric contexts so that students develop their powers of spatial thinking, visualization and geometrical reasoning. The researcher posits that the above instructional objectives of geometry can be best accomplished, by implementing the constructivists' perspective on teaching and learning.

Constructivism instructional approach advocates for student-centred teaching methods. This approach gives learners the opportunity to investigate, explore, experiment; it encourages learners to depend on their prior knowledge of mathematical concepts to construct new mathematical knowledge, under the guidance of an effective teacher (Noyce, 2001). Faulkenberry (2006) reiterated that implementing constructivism in mathematics classrooms gives students the leeway to construct their own mathematical knowledge through "self-modification of cognitive structures" which is "largely unconscious, yet a goal-directed process by which the student reacts to a cognitive disturbance by changing how he or she thinks about a concept to accommodate the novel piece of information, thus relieving the cognitive disturbance". He further averred that constructivism as a teaching practice is a complex mode of pedagogy which places students at the centre of the learning process-instead of the teacher, thus, the emphasis migrates from teacher-centred instructional approach to a learner-centred one.

To elucidate the constructivists' approach (NCTM, 2001) gave details about the role of teachers in a constructivist classroom. The approach expects teachers to form a challenging and supportive classroom learning environment; teachers are mandated to establish and nurture an environment conducive for learning mathematics through the decisions they make, the conversations they orchestrate and the physical setting they create. Gibson, Ivancevich, Donnely and Konopask (2003) also support the notion that the teacher is responsible for creating an intellectual environment where advanced mathematical thinking is developed. Teachers' actions are what encourage students to think, ask questions, solve problems, and discuss their ideas, strategies, and solutions. To summarize the above, Gore (2001), pronounced that the constructivist teacher's role is that of a coach and facilitator.

3.2.3 Cognitive Learning Theory

Cognitive learning theory explains how internal and external factors influence an individual's mental processes to supplement learning. It promotes understanding, depending on students' prior knowledge to acquire new knowledge. In the cognitive learning theory, students apply their knowledge and skills to solve problems in real life situations and this improves problem-solving skills. This learning theory employs metacognition to examine how an individual's learning is influenced by his/her taught processes. Jerome Bruner, Jean Piaget (theory of cognitive development) are the recognized proponents of cognitive learning theory. This learning theory is categorized into two - Social Cognitive Theory (the influence of the environment on learning) and Cognitive Behavioural Theory (mental processes of learning). Cognitive Learning Strategies are learner-centred strategies (Jean Piaget), Meaningful Learning strategy (David Ausubel), and Learning Through Discovery strategy (Jerome Bruner). Cognitive Learning Modes include Explicit Learning, Implicit Learning, Meaningful Learning, Discovery Learning, Receptive Learning, Experiential Learning, and Observation Learning (Pajares, 2002).

3.3 Infusion as an instructional approach to teaching thinking

Another emanation from the constructivist instructional approach is the Infusion approach (Swartz, 1992), which aids the teaching of thinking into content instruction. The infusion approach specifically educates us on how thinking skills can be incorporated in any content lesson.

3.3.1 Preparing infusion lessons

According to (Swartz & Reagan, 1998, p.2):

"Infusion requires restructuring content area lessons. This does not mean however, that teachers are trained to teach pre-existent lessons from a ready- made packet of materials. Rather, teachers learn how to design infused lessons themselves and to teach them to their students. This involves engaging students in explicit organized thinking about a topic in the regular curriculum, facilitating their reflective understanding of such skilful thinking, giving students additional practice in deliberately using this kind of thinking so that they can engage in it independently, when appropriate without prompting. Standard textbooks do not commonly offer lessons that expound on instruction in thinking skills and processes, teachers must restructure their lessons to accomplish the goals of infusion".

This extract highlights the role and relevance of the teacher when teaching thinking skills using the infusion approach. This means that teachers needs to be self-motivated and knowledgeable in infusion approach methodology so as to skilfully restructure their lessons to accommodate the inculcation of thinking skills, along with their content instruction. This requires adequate planning on the side of the teacher. The details above also highlight the need for teachers to organize their lessons systematically with the relevant assessment approach, for example, questioning (orally or written), which will aid students to learn and to improve their thinking skills, since the available textbooks may not have been structured to serve the purpose of conducting infusion lessons.

Swartz and Reagan (1998, p.53), advised that when formulating an infusion lesson, the following essentials needs to be considered and/or thoroughly discussed so that appropriate decisions can be taken on them:

"the grade level and subject area of the lesson, the content objectives of the lesson; the thinking skills objectives of the lesson, how the thinking skills and their importance will be introduced to the students, how the content emphasis will be

introduced to the students, which specific strategies will be used in the introduction to make the thinking strategy that are being taught explicit, how the lesson will engage the students in the active use of this thinking strategy in connection with the content being taught, what kind of graphic organizer will be used in the lesson and how it/they will be used. Of vital importance is also what other ways the thinking strategy will be kept explicit as the students go through the thinking activity, what strategies will be used in the lesson to prompt the students to reflect on their thinking, how the teacher intends to reinforce the thinking skill through transfer activities".

Swartz and Reagan (1998, p.57) also aver that in developing an infused lesson, the following critical questions need to be considered:

- What kind of thinking does the lesson seek to improve, and what improvements are sought?
- How is the lesson design geared to students' active involvement in the kind of thinking the lesson is designed to improve?
- How do the instructional strategies make explicit what students can do to improve their thinking?
- To what degree are students prompted to think about their thinking in this lesson? How is that accomplished?
- What varied opportunities are provided in this lesson for students to practice the kind of thinking taught?
- What methods are used in the lesson to enhance student thinking?
- How is the lesson designed so that instruction in the thinking process improves students' understanding of the content?

To collaborate the above, Ong (2000), presented the steps to be followed when teaching thinking, that is, how content knowledge can be taught to promote thinking skills - Step (1) identify relevant learning outcomes; Step (2) write/review the learning objectives to incorporate these types of thinking as writing higher order objectives is an integral part of instructional planning; Step (3) select teaching strategies used to teach these thinking skills such as cooperative learning, role play, thinking maps, Socratic dialogue, higher order questioning and problem posing, teacher modelling, the language of thinking, graphic organisers and reflective logs or learning journals; Step (4) develop authentic learning tasks that will facilitate transfer of thinking skills in authentic learning contexts by employing well-constructed and managed learning tasks that reflect real world activities with the aim of solving problems; Step (5) develop assessments to measure the thinking skills which must be highly relevant to the content. Ong (2000) further explained that the teaching of thinking skills in the curriculum requires the implementation of the above five systematic steps which he referred to as the 'curriculum approach' to teaching thinking. He emphasized that there is a need for a curriculum which promotes thinking to afford students enough opportunity to think and develop thinking dispositions.

Infusion lessons play three key roles and according to Swartz and Reagan (1998, p.40), they are - (1)"engaging students in active thinking structured by explicit organizing, and focussing prompts; (2) helping students reflect about their thinking and (3) giving students a variety of opportunities to practice these habits of thought while the teacher gradually phases out of the process". These researchers emphasised that collaborative learning, which creates an overall atmosphere for thinking in the classroom, can best promote infused lessons. Collaborative teaching and learning practices include learners sitting in groups in class, learners given opportunities to discuss/ interact among themselves so as to make them responsible for their own mathematical constructions; these can be done by them creating their own mathematical ideas; developing their own conjectures and mathematical models; identifying their own mathematical misconceptions and effectively correcting them and making them more positive minded and confident in the mathematics classroom. These will, hopefully, promote the development and cultivation of thinking skills. This was the basis on which the researcher in the current study experimented the proposed IPAC model in a collaborative classroom

3.3.2 Components of infusion lessons

An infusion lesson comprises of four components (as illustrated in Figure 3.1 below) namely, introduction (teacher educates students about thinking and thinking skills), thinking actively (activities prompting students to articulate, evaluate, and plan their thinking), thinking about thinking and applying thinking. After the essential preliminary / introduction stage, the teacher guides students on how they can think actively. This is endorsed by the following activities, which are: teacher guiding students on how to use thinking skills to solve problems; the teacher guiding students on how they can reach solutions to thinking tasks and using thinking skills when they are in "collaborative thinking groups" (Swartz & Reagan, 1998; National Centre for Thinking, 1996). The teacher is the one directing questions to students to guide them to reflect on their own thinking (thinking about thinking). In this regard, the teacher gives students guidance during the lesson. Lastly, the teacher guides students on how they can apply their thinking to solve problems by applying their thinking skills to promote transfer. The expectations are that students can master these problem-solving habits (Swartz & Reagan, 1998; National Centre for Thinking, 1996).

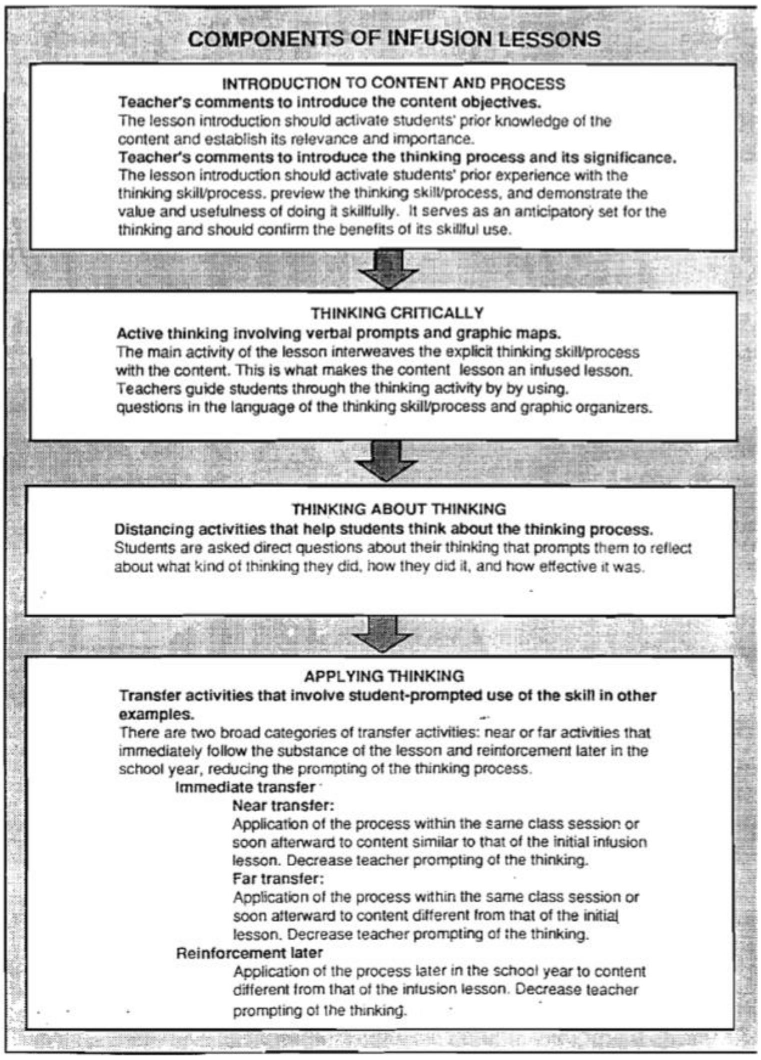


Figure 3.1 components of infusion lesson (National Centre for Teaching Thinking, 1996, p.43)

3.3.3 Teaching to internalize a thinking skill

With reference to Figure 3.2 below, for a thinking skill to be taught to achieve internalization on students' minds, the following 3-step procedure must be adhered to: teaching the thinking skill, practicing the skill and using the thinking skill independently. During a session of teaching a thinking skill, the teacher sequentially organizes questions to guide students' thinking. The teacher also inspires students to develop metacognition through organized questioning, and guides students to apply the thinking skills they might have learnt to solve familiar problems or new contexts. During sessions of practicing the skill and using the thinking skill independently, students use meaningful, organized and reflective questioning to develop metacognition and transfer of learning. This is done rapidly, when practicing the skill and when using the thinking skill independently (Swartz & Reagan, 1998; National Centre for Thinking, 1996).

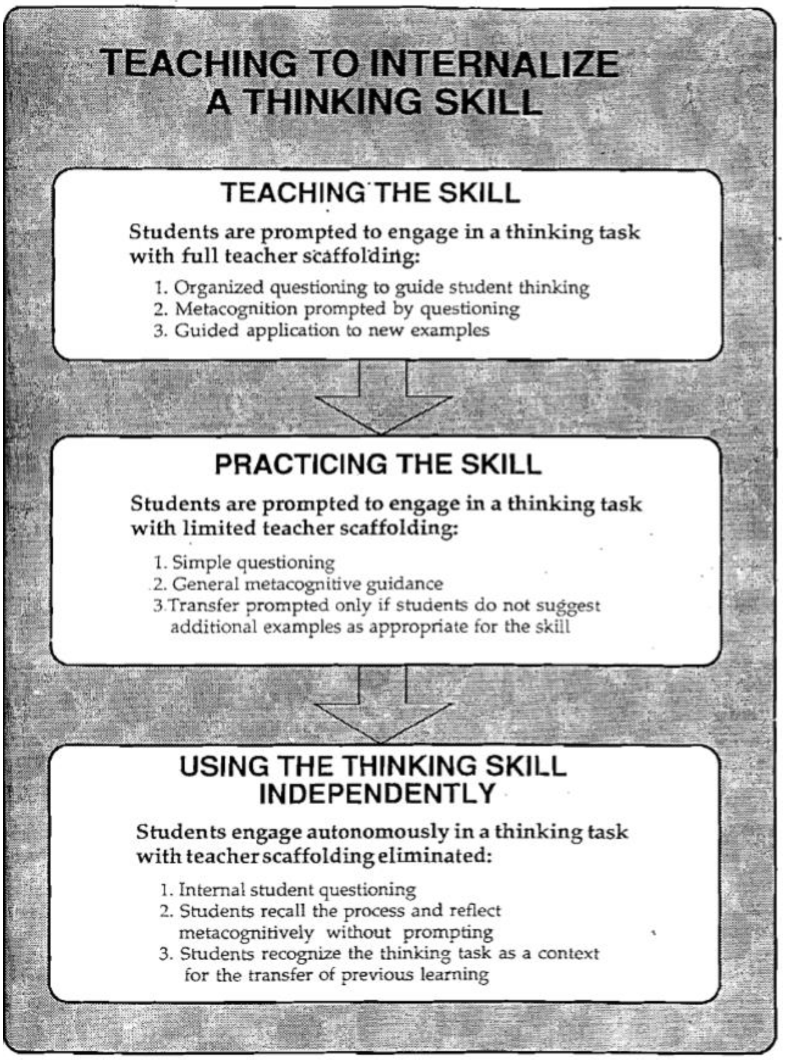


Figure 3.2 Teaching to internalize a thinking skill (National Centre for Teaching Thinking, 1996, p. 44).

3.3.4 How the infusion approach was implemented in this study

In accordance with the components of conducting an infused lesson, the four components were adhered to in this study, however, "thinking about thinking" and "applying thinking", that is, components 3 and 4, were integrated as one process. The researcher agrees with the fact that components (3) and (4) can, practically, be separate processes, however, he believes that an individual who is reflecting on his or her thinking can as well, consider how the thinking can be applied simultaneously, hence, consolidated them as one. In view of the above, the infusion approach adapted in this study is as follows:-

(1) Introduction: the teacher educates students about thinking skills and their relevance; the thinking skills are taught to students. The teacher gives students instructions on thinking skills and the content objectives. The thinking skills considered for this study were: (1) understanding and retention of ideas, (2) generating ideas (creative thinking), (3) assessing reasonability of ideas (critical thinking), and (4) engaging students in metacognitive reflections. How each of these thinking skill was implemented in this study is delineated under the section on explication and implementation of the proposed new instructional approach, in Chapter 4 (Swartz, 1992; Swartz et al., 2010; Swartz, 2012).

(2) The teacher guides students to think actively. At this stage students get the opportunity to practice the thinking skills they were taught during the introductory session. During this time, the teacher guides them to plan their thinking, using questioning (orally or paper-and-pencil thinking tasks). The teacher organises students into "collaborative thinking groups" and guides them to solve thinking tasks. Group members discuss their ideas among themselves, while the teacher gives them feedback on their discussions, on whether their ideas are absolutely correct, absolutely wrong or moderate. Through these feedbacks given to learners, they enable them to identify their mistakes and that alone, is a powerful learning opportunity for them (Boaler, 2016). In addition to the above, King, Goodson and Rohani (2013) aver that group activities, namely,

student discussions, peer tutoring and cooperative learning can be effective in the development of thinking skills. These researchers established that activities should involve - challenging tasks, teacher encouraging students not to give up until a solution is reached, as well as prompt and continuous feedback about group progress (Swartz, 1992; Swartz *et al.*, 2010; Swartz, 2012; King, Goodson & Rohani, 2013).

(3) The teacher now focus on students thinking about thinking and applying thinking, hence, practicing thinking skills is continued at this stage in groups. The teacher directs questions to students to guide them to reflect on their own thinking (thinking about thinking) by (1) students identifying the kind of thinking they just engaged in, (2) describing how they did it (3) asking whether it was a good way to do this kind of thinking (4) how they can turn their ideas into an explicit plan for doing the same type of thinking again. The above is monitored by providing students with simple graphics that serve as reflection and recording devices for their thinking. The teacher also gave directions to students when they were engaged in the thinking map for skilled decision-making. Lastly, the teacher guides students on how they can apply their thinking to solve problems by applying their thinking skills to promote transfer (Swartz, 1992; Swartz *et al.*, 2010; Swartz, 2012).

(4) The students now use the thinking skills independently. At this juncture, the "collaborative thinking groups" were disbanded and each student is made to work individually. Under this session, standardized tests and examinations, where necessary, may be conducted. Each student's responses will enable the researcher to measure and to ascertain the impact the new instructional approach had on study participants' achievements (Swartz, 1992; Swartz *et al.*, 2010; Swartz, 2012).

3.4 Robert Marzano's teaching with dimensions of learning

In his quest in creating "a different kind of classroom", Robert Marzano came up with five dimensions of learning/thinking - (1) developing positive attitudes and perceptions about learning (through learning climate and classroom tasks); (2) acquiring and integrating knowledge (through interactive learning); (3) extending and refining knowledge (through the following cognitive activities - comparing, classifying, inducing, deducing , analysing errors, constructing supported evidence, abstracting, analysing viewpoint); (4) using knowledge meaningfully (through transfer of knowledge) and (5) developing productive habits of mind (through the development of mental skills). The above were based on the following six basic assumptions - (1) instruction must reflect the best of what we know about how learning occurs; (2) learning involves a complex system of interactive processes that include the five dimensions of learning; (3) what we know about learning indicates that instructions focusing on interdisciplinary curricular themes are the most effective ways to promote learning; (4) the K-12 curriculum should include explicit teaching of higher-level attitudes, perceptions and mental habits that facilitate learning; (5) a comprehensive approach to instruction includes at least two distinct types of instruction - one that is more teacher-directed and another that is more student-directed; (6) assessment should focus on students' use of knowledge and complex reasoning rather than on their recall of low-level information (Marzano, 1997).

Marzano (1997) also elaborated on classroom conducts (decisions to be taken by the teacher in the classroom), and the mode of assessment that goes along with the five dimensions. The Dimensions of Learning Training Manual contains guidelines for conducting comprehensive training and staff development in the dimensions program. Learning videotapes implementing dimensions of learning explain the different ways the program can be used in a school and they discuss the various factors that must be considered when deciding which approach to use. Marzano

(1997) also stressed on the need to implement cooperative teaching approach during instructions.

Robert Marzano's five dimensions of learning were introduced to educators to enable them to have a better understanding of cognition and learning, to improve curriculum, instruction and assessment. This model for teaching and learning was aimed at developing students' critical thinking skills. Marzano (1997) stated that the learning process contains and requires the interaction of five thinking styles which he called, "Learning Dimensions". These five dimensions are from thinking styles which explain the way the mind works during learning (Marzano, 1997).

3.5 The didactic triangle

The didactic instruction triangle, details that there are three dimensions to a mathematical instruction - student, teacher, and content - as shown in Figure 3.3 below. This may be applied for conceptualizing teaching and learning in mathematics classrooms as a heuristic device, focusing on developmental activities and the analysis of developmental events; for situating and contextualizing each element in relation to the other and as a 'mediating artefact' used by the teacher with the intention of leading students to acquire new knowledge (Goodchild & Sriraman, 2012).

In relation to this study, the content taught to learners was circle geometry; the proposed IPAC model served as the pedagogical and didactic tool, which was used for teaching and learning of circle geometry, respectively. This didactic triangle sought to apprise the researcher that in conducting circle geometry lessons, there are three dimensions to be considered - (1) 'teacher-content' dimension (the teacher as the facilitator must have adequate content knowledge of circle geometry); (2) 'teacher-learner' dimension (how circle geometry concepts will be taught to learners, which requires the teacher implementing an effective problem-solving instructional approach); (3) 'learner-content dimension' (how learners can be guided to be responsible for their own learning to aid them to be effective geometry problem-solvers). In view of this, the teacher as the facilitator is mandated to have adequate knowledge of these three instructional dimensions and most importantly, how the teacher can be an effective mediator of these three instructional dimensions to optimize teaching and learning of mathematical concepts, which is circle geometry in this case.

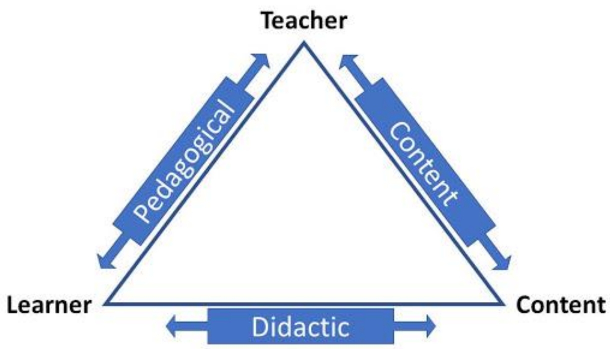


Figure 3.3: The didactic triangle (Adopted from Goodchild & Sriraman, 2012)

3.6 APOS theory and the associated ACE teaching approach

This theory emanated from the constructivist instructional approach, a build-up of Piaget's concept of reflective abstraction (Piaget, 1978). The APOS Theory has Action, Process, Object and Schema as its components (Brijlall, & Ndlazi, 2019). It elaborates on how mathematical concepts can be learnt and focuses on what might be going through the mind of a learner as he or she tries to learn a mathematical concept (Arnon, Cottrill, Dubinsky, Oktac, Fuenstes, & Trigueros, 2014). The theory can also be used for developing possible pedagogical strategies for teaching and learning a particular mathematical concept. Piaget termed this, 'genetic decomposition', which he hypothesized theoretically and tested empirically to ascertain its efficacy (Maharaj, 2010; Ofori-Kusi, 2017; Dubinsky, 2001). Data is gathered in the process to either validate the teaching pedagogy or to call for amendments to it (Mathews & Thomas, 1996; Dubinsky, 2001). As explained by Arnon *et al.* (2014), the APOS Theory varies from other mathematics education theories in the following ways - its theoretical approach, methodology and types of results offered and approaches implemented, as its constituent parts are highly correlated. It supplies open-ended questions to be answered by researchers and it has proven to be effective in providing solutions to instructional difficulties pertaining to a lot of mathematical concepts. That is one of the motivation for a lot of mathematics education researchers, as they are enticed to implement it in their diverse research studies.

This theory considers mathematics learning as a cognitive activity and it emphasizes that learning and understanding any mathematical concept starts with manipulating previously-constructed mental or physical objects to form actions; actions are then internalized to form processes which are then encapsulated into objects. The processes and objects are then organized into schemas. That is the reason the APOS theory has four hierarchical procedures - action, process, object and schema (Dubinsky, 2001). Each procedure is discussed in detail below:

Action-conception stage

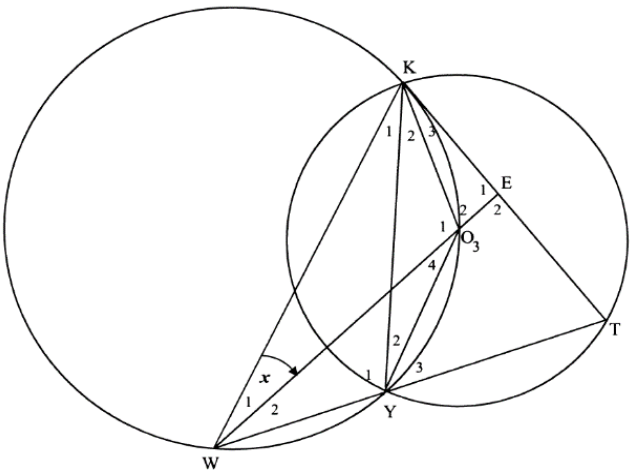
A reaction to stimuli which an individual perceives as external, is first conceived as an action. It is characterized by the individual following step-by-step instructional procedures explicitly, as he/she is made to understand, in detail, what he/she is expected to do and the need to perform each step of the transformation as required.

Question 12.1 (Figure 3.4 below), typifies the action stage of geometry's mental construction. Students are required to give four other angles equal to x. Students' ability to do this depends on them being able to develop explicit expression/stimuli to mainly identify and recognise appropriate relevant geometric properties, axioms and theorems. Students are required to recall the circle geometry theorems, so that they will be able to identify and recognise the specific theorems which will be applicable, to provide a solution to the question under consideration. Students at this stage may also be required to state any of the circle geometry theorems and/or converse of any of them. They can also be asked to complete a statement about a theorem and/or converse of it by filling in a dotted line to make the statement about a theorem valid. Students who are able to perform this task can be judged to be at the action stage of geometry mental construction (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010).

QUESTION 12

In the diagram below, two circles intersect at K and Y. The larger circle passes through O, the centre of the smaller circle. T is a point on the smaller circle such that KT is a tangent to the larger circle. TY produced meets the larger circle at W.
WO produced meets KT at E.

Let $\hat{W}_1 = x$



12.1	Determine FOUR other angles, each equal to x.	(8)
12.2	Prove that $\hat{T} = 90° - x$.	(3)
12.3	Prove that KE = ET.	(3)
12.4	Prove that $KE^2 = OE.WE$	(6)
		[20]

Figure 3.4- Exemplar (Mental construction task)

Process-conception stage

The reiteration and reflection of an action, interiorize the action into a mental process (becomes part of the mind of the individual). The mental structure of the process stage demands that the individual can perform the transformation imaginatively, without going through the specified steps/ procedures. For instance, Question 12.2 (Figure 3.4 above) is the typical example of the process stage of geometry mental construction. Students who are able to prove that $\widehat{T} = 90^0 - x$, may be at the process stage of geometry conception, since they are expected to apply their knowledge of geometry, to interpret the geometric diagram without necessarily following any step-by-step procedures. Students are required to reflect on the action process of "finding other angles equal to x, which is interiorised by the students, and this enables them to reverse the action processes (converse of theorems), when necessary (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010).

Object-conception stage

As the individual reflects on the process conception repetitively, he/she becomes totally aware of the processes, concepts and procedures. This is made possible by means of a mental act known as 'encapsulation'. Dubinsky (2001) emphasized that finding an effective instructional approach to enable the individual to reach the encapsulation mental act is not an easy venture, although, just a handful of instructional approaches can guide the individual to reach this level. The individual can be judged to have encapsulated the process into a cognitive object, if the individual is wholly able to realise, construct and explicitly or imaginatively perform a transformation, automatically. Encapsulation may be an antithesis of the original procedure (de-encapsulation), so that the individual can get back to the previous stage (process stage), when necessary (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010).

For instance, in question 12.3 (Figure 3.4 above), students who could prove that KE = ET, geometrically, can be judged to be at the object stage of solving geometry problems. Students are required to find, on their own, appropriate techniques and skills in solving geometry problems intuitively. Students are required to have encapsulated geometry theorems, converses of them and their effective applications into a cognitive object. The action process of giving four other angles equal to x, and the process stage of proving that $\widehat{T} = 90^0 - x$, are expected to be applied, cognitively.

Schema-conception stage

This requires the individual to construct, interconnect, organise and link actions, processes and objects into a coherent framework known as a 'schema'. The individual may have to decide when and where the schema will be applicable. For instance, from Figure 3.4 above, students who are capable of providing appropriate solution to question 12.4, are required to connect the action process of finding other angles that are equal to x; that is the process stage of proving that $\widehat{T} = 90^0 - x$, and the object stage of proving that KE = ET forms a coherent framework (schema). Making effective connections across geometry concepts to form a meaningful solution path is to form a schema; to solve more complex problems is what is required to prove that $KE^2 = OE.WE$, as shown in question 12.4 (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010).

In summary, as illustrated in Figure 3.5 below, in geometric conceptualization, circle geometry is regarded as a mathematical object, as a newly formed mental construct which consists of the successful merge of actions and processes (doings). This transition to an object, happens firstly through interiorization where the learner attaches meaning to the idea of the circle, then through reification or encapsulation of the actions and processes. Now it is no longer continuous steps of actions and processes, but the learner can now take that object (the phenomenon of circles and their properties) and freely work with it, manipulate it, and reason about it (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010).

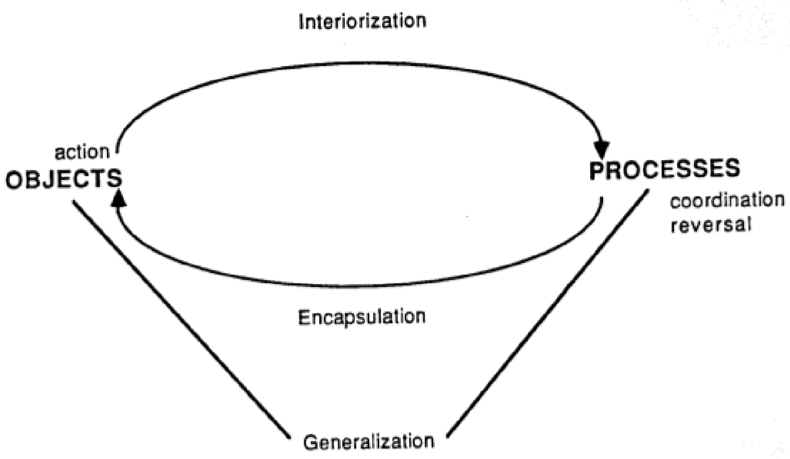


Figure 3.5 Schemas and their construction (Dubinsky, 1991; p. 33)

3.6.1 APOS theory as a teaching and learning approach

The APOS theory has received considerable acceptance since its inception, from researchers who have engaged it in their research work (Brijlall, 2020). Its unique characteristics make it stand out from other research and or teaching and learning theories. For example, Dubinsky and McDonald, (2001), characterized the APOS theory as follows: (1) mathematical knowledge is constructed through mental construction: actions, processes, objects, and organizing these in schemas, (2) using computer, (3) using cooperative learning groups, and (4) using ACE teaching cycle (activities, class discussion, and exercise).

In Arnon *et al.* (2014), the following characteristics of the APOS Theory were given: (1) it differs from most mathematics education research in its theoretical approach, methodology, and types of results offered; (2) it contains theoretical, methodological, and pedagogical components that are closely linked together; (3) it continues to attract researchers who find it useful to answer questions related to the teaching and learning of numerous mathematical concepts, and (4) it continues to supply open-ended questions to be resolved by researchers. From the above, Arnon *et al.* (2014), suggested that the APOS theory can be classified as a paradigm.

Viewing the above traits of the APOS theory, the ones that are germane to this study are - construction of circle geometry knowledge and understanding through actions, processes, objects, and schemas mental constructions and using ACE teaching cycle (activities, class discussion, and exercise). The ACE teaching cycle is based on two general hypotheses/assumptions: (1) assumption on mathematical knowledge (finding solutions to mathematical problems by applying appropriate mental structures and by approaching them from a social context) and (2) hypothesis on learning (mathematical concepts are not learnt directly but by applying appropriate mental structures to make meaning from them) (Piaget, 1978).

From the above postulates, the ACE teaching cycle seeks to find appropriate and effective strategies - which can guide the individual to develop appropriate mental structures; which may be applicable in any given problem-solving situation; which are aimed at guiding students to develop detailed understanding of mathematical concepts, by applying the mental structures to seek for new mathematical knowledge. Teaching through the APOS theory is aimed at providing strategies for helping students build appropriate mental structures, and guiding them to apply these structures to construct their understanding of mathematical concepts (Maharaj, 2010).

Activities, classroom discussion and exercises are the three components of the ACE teaching approach which is also the pedagogical component of the APOS theory; these three components are repeated in a circular pattern. Mathematical problems are presented to learners to enable them to develop appropriate mental structures during the activities stage. Learners are, thereafter, given the opportunity to reflect on the work done at the activities stage during the class discussion stage. Exercises in relation to the learnt concepts during the activities and class discussion stages are then given to learners; this stage is characterized by transfer and application of knowledge. These repeated processes are meant for guiding students so that they will be able to develop appropriate mental structures, for solving mathematical problems (Maharaj, 2010).

3.6.2 Interiorization and encapsulation as instructional goals/objectives

According to Bixler (2007), "instructional goals can be described as broad, generalized statements, which gives an indication of the target to be reached at the end of an instructional session, that is, what the learner is expected to be able to do after an instructional session, while instructional objectives specify what has to be done to accomplish instructional goals". This can be achieved by providing a focus for instruction, providing guidelines for learning, providing targets for formative and summative assessment, conveying instructional intent to others and providing for evaluation of instruction (Mager, 1997; Gronlund & Brookhart, 2009), cited in Brumfield and Carrigan (2011). Brumfield & Carrigan, (2011) described instructional objectives as "the foundation of instructional effectiveness"; Moore, (2001) stated that pedagogues are aimed at establishing objectives; teachers can instruct towards the objectives and evaluate them afterwards.

Instructional objectives can either be in the psychomotor domain (physical skills), affective domain (attitudes and emotions) and cognitive domain (knowledge, understanding, recall, application and transfer of knowledge), (Bixler, 2007). While the main instructional goal for this study is to provide an opportunity for students to develop their thinking and problem-solving skills, the instructional objectives to achieve this aim may not be either psychometric, or affective, but they rather fall under the cognitive domain, which is germane to this study; that is, developing learners into active, creative, and critical thinkers, hence, effective problem solvers.

The aforementioned processes (interiorization and encapsulation) are indispensable mental processes that drive the mental construction stages of learning mathematical concepts (Dubinsky, 1991). In identifying a mathematical object, that is circle geometry in this case, the transition from the process stage to the object stage is termed as 'interiorization' and the transition from the object stage to the schema stage is termed 'encapsulation'. These are processes which occur wholly in the minds of the individual and require active application and transfer of conceptual understanding, in this case, of circle geometry knowledge and geometric concepts.

While these processes may be new to South African teachers/learners, especially in the context of geometry, they are urged to be 'adapting' and 'adaptive' to these mental processes, since in the long run, they will be helpful to optimize teaching and learning, in the following aspects - (1) they serve as specific/measurable student competence descriptors, (2) they serve as tools, used as a benchmark to ensure teachers and learners reach their goals, (3) they ensure teaching/learning is focused (Bixler, 2007; Maharaj, 2010; Brijlall & Ndlazi, 2019; Chagwiza, Maharaj & Brijlall, 2020).

As mentioned by Dubinsky (2001), for learners to get to the object stage and schema mental construction levels, is extremely difficult, yet, these are the expected end products/goals which they must attain in order for them to be proficiently creative, critical thinkers and proficient mathematics problem-solvers (by developing and applying appropriate mental structures). These two mental processes direct teachers and learners on what they are required to do/achieve during each level, hence, they serve as specific competence traits learners need to attain. They serve as a yardstick for determining learners' competence, therefore, they ensure classroom instructions are well directed.

Interiorization and encapsulation can be described as instructional goals/objectives which can promote mathematical proficiency (Dubinsky, 2001), thus, teachers are being urged to lead potential mathematicians (students), to cross the bridge of interiorization and encapsulation. This will, additionally, help in closing the gap between what learners know and what they may be able to do mathematically, (the zone of proximal development), so that their mathematical competences may be realised.

For further discussions on this aspect, readers of this report can refer to GD3- Object stage of circle geometry mental construction lesson and GD4- Schema stage of circle geometry mental construction lesson, in Chapter 4 on the explication and implementation of the proposed problem-solving instructional approach on the experimental group. Under this section, both interiorization and encapsulation, as mental processes are discussed in detail - what they mean, what is expected to be done/achieved at each level, the role they play in problem-solving, an exemplar to explain each process, as well as the instructional approach for each process.

3.6.3 Preliminary genetic decomposition

A genetic decomposition postulates or hypothesizes that particular actions, processes, and objects play a role in the construction of a mental schema for dealing with a given mathematical situation (Maharaj, 2010; Brijlall & Ndlazi, 2019; Chagwiza, Maharaj & Brijlall, 2020). Hierarchically, learners' circle geometry mental constructions were developed by adhering to the conjectured initial genetic decomposition as presented below. This guided the researcher to implement what was expected to be done at each level of mental construction. Figure 3.6, as well, will inform readers of this report how circle geometry mental constructions were conjectured and carried out.

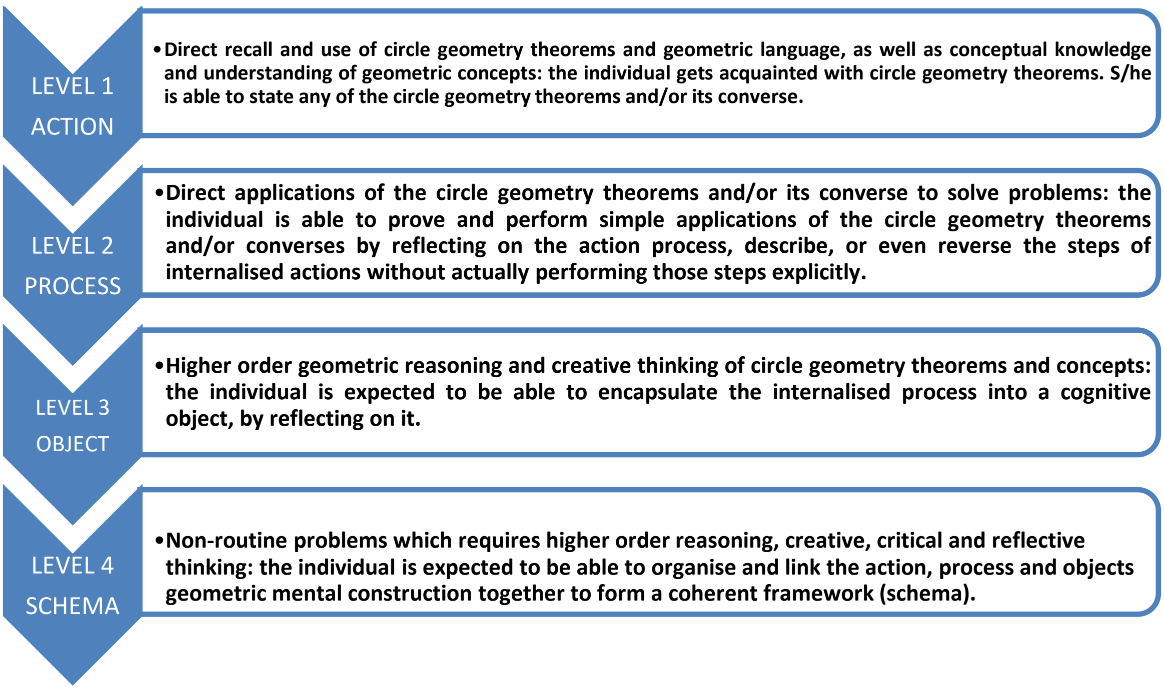


Figure 3.6: Preliminary genetic decomposition of circle geometry

3.7 Richard Paul's wheel of reasoning

Richard Paul in 1992, came up with the eight elements of productive thinking which served as a thinking model and logical reasoning tool, in a bid to promote creative and critical thinking skills. The eight elements are - issue, purpose, point of view, assumptions, concepts, evidence, inferences, and implications or consequences. The problem-solver is expected to consider these eight elements as a process, and to meaningfully, consider each of the elements. He emphasized that reasoning is a crucial skill to be learnt by students. The relevance of Richard Paul's wheel of reasoning to this study is the emphasis it placed on thinking and reasoning as a crucial skill for students, and his assertion that thinking and reasoning are processes, which need to be meaningfully formed from one element to the other. This is what this research study sought to investigate further.

3.8 Howard Gardner's theory of multiple intelligence

Howard Gardner in 1983, came up with the theory of multiple intelligence, on the bases that humans possess different types of intelligence - verbal-linguistic, logical-mathematical, visual-spatial, musical, naturalistic, bodily-kinaesthetic, interpersonal and intrapersonal. He emphasized that people are not born with all of the intelligence they will ever have.

The one which relates to mathematics is Logical-Mathematical Intelligence (number reasoning) which is just one out of the eight categories. The individual who possesses this type of intelligence will have the ability to analyse problems logically, carry out mathematical operations, and investigate issues scientifically. This indicates that all minds are not the same, therefore, the mental structures of an individual might not be appropriate for mathematical concepts naturally, hence, we may not be able to compel its development. The implication of this theory to this study is that, since not every individual's mind is accustomed to mathematical reasoning, learners who have not attained the first level of mental construction, may not be moved to the next level. After multiple efforts to get a learner to achieve at a particular mental construction level fails, then the learner may be referred to other options (other forms of intelligence). Maybe that is where his/her competence lies, not in mathematics.

3.9 Descriptions of Reasoning Habits through GHOM Perspective

Teaching thinking skills also involves dispositions - attitudes or habits of mind, open- and fair-mindedness, curiosity, pliability, a proclivity to demand explanations to know and understand the reason for taking any decision, readiness to learn, and being receptive to varied opinions (Lai, 2011). Classroom instructions which incorporate the conception of habits of mind tries to close the gap between what the users and makers of mathematics do and what they say. The quest to encourage students to be able to "think about mathematics the way mathematicians do, brought about the concept - "mathematical habits of mind" (Cuoco, Goldenberg & Mark, 1996); this has been extended to geometry - "geometric habits of mind". Geometry is one of the integral content areas of mathematics (DoBE, 2018). "Geometric habits of mind" serve as an instructional tool in the development of geometric thinking; they aim to deepen students thinking in geometric properties, postulates and axioms, so that they can meaningfully and appropriately find solutions to geometric problems (Driscol, DiMatteo, Nikula, & Egan, 2007). The Geometric thinking habits, listed in Figure 3.3, are elaborated below.

Geometric Thinking Habits	Indicators of the GHoM Habits	Student Indicators
Reasoning with relationships	Focus on relationships among separate figures	Determines the relationship between the properties of geometric shapes
	Focus on relationships among the pieces in a single figure	Identifies/classifies the properties of shapes
	Use special reasoning skills to focus on relationships	Associates more geometric shapes with proportional reasoning (congruence-similarity)
Generalizing geometric ideas	Seek solutions from familiar cases or known solutions	Makes generalizations from the special case to explain the problem situation
	Seek a range of solutions using assumed simplifying conditions	Adapts a general situation in a problem for the special case
	Seek complete solution sets or general rules	Can think of all possible situations based on the data in the problem
Investigating invariants	Use dynamic thinking and searching	When a geometric shape is transformed in any way or enlarged/reduced in a specific rate, solves the problem by determining which features of the shape have changed and which ones remained fixed
	Check evidence of effects	Can imagine the geometrical structure as mobile so as not to disturb the conditions of the problem and can explain the emergent effect
Balancing exploration and reflection	Put exploration in the foreground	Can make additional drawings to help solve the problem / Can develop different strategies for solving the problem
	Put end goals in the foreground	Asks questions related to retro-metacognitive capacity in problem solving / Can make explanations through mathematical language for correctness of problem solving

Figure 3.7 Geometric thinking habits: adopted from Driscoll et al., (2008).

Reasoning with relationships

Any of the circle geometry theorems by their nature, are laden with relationships. For instance radius ⊥ tangent, tangent – chord theorem, angle in a semi-circle, and others, all exhibit the relationships that exist between a radius and a tangent, a tangent and its attached chord, a diameter in a circle, respectively. We cannot, hence, talk about geometric problems without discussing the relationships that exist between the geometric figures, lines, angles, and many others. In principle, a standard circle geometry question incorporates at least three theorems and/or converse of theorems. A problem-solver's ability to know and recognise the relationships among the incorporated circle geometry theorems, will greatly enable him/her to interpret and understand the problem to be solved better. This will as well, enable the problem-solver to plan on how to solve the problem better, therefore, enable him or her to obtain the right solution to the problem. A problem-solver, focusing on relationships among the pieces in a single figure, and being able to employ special reasoning skills to focus on these relationships, would greatly guide him/her to develop an appropriate problem-solving path; this would increase the possibilities of reaching a meaningful, appropriate and multifaceted solutions to the geometric problem under consideration (Driscol, DiMatteo, Nikula, & Egan, 2007).

Generalizing geometric ideas, investigating invariants and balancing exploration and reflection

Geometric axioms, theorems, postulates and ideas, are not tentative. They are invariants (constants), definite and certain principles which can be applied to a variety of geometric problems. This presupposes that known solutions from similar cases may be partly or wholly applied to reach solutions to other problems which might be either in the same or different contexts. For instance, the geometric knowledge that radius ⊥ tangent, likewise other circle geometry theorems, does not change, irrespective of the nature of the given problem can be generalised to provide solutions to other geometric problems. Investigating invariants entails using dynamic thinking to search for solutions to geometric problems. The researcher of the current study holds the assertion that investigations and explorations through activities and experimentations will lead learners to generate new geometric ideas, hence, the basis of integrating the three thinking habits in this section to eschew tautological narrations. In the same reasoning, balancing exploration and reflection are synonymous to 'the role of metacognition in problem-solving', presented in Chapter two, hence, it was not further discussed in this chapter (Driscol, DiMatteo, Nikula, & Egan, 2007).

3.10 An exposition of how I conceptualized the theoretical framework based on idealised research constructs for this study

As mentioned earlier, the research constructs for this study consists of the amalgamation of three recognised theories. These theories are - the infusion approach (an approach of teaching thinking integrated with content instructions); Polya's Problem Solving Model (a strategy for teaching an effective problem-solving instructional approach) and the APOS theory, to guide the design and implementation of the proposed problem-solving instructional approach. The APOS theory was also used to monitor learners' mental constructions. In addition, instructions were carried out in a collaborative classroom setting. According to Lai, (2011), "In theory all people can be taught to think critically. It is for this reason that instructors are urged to provide explicit instructions in critical thinking to teach how to transfer new contexts. Instructors are also urged to use cooperative or collaborative learning methods and constructivist approaches that are learner-centred". This is what this study sought to establish.

The details of each of these research constructs have been explained in detail in this chapter. The explanations focused on - what they mean and what they stand for, the meaning of their constituent parts, the role each part plays in each theory/approach and how they were implemented in this study. Other empirically-established instructional approaches/theories - Robert Marzano's notion of teaching with dimensions of learning and the didactic triangle, Richard Paul's wheel of reasoning, Howard Gardner's theory of multiple intelligence and descriptions of reasoning habits through GHOM perspective by Driscoll *et al.* (2008) - were all presented in this chapter.

Robert Marzano's idea of teaching with dimensions of learning is a well-established thinking-based instructional approach. It served as an exemplar for the design and implementation of the proposed IPAC model, and as a thinking-based model. Focused upon was - how it was formulated, tried and tested by research, as well as how it was effectively implemented in classrooms. Its implementation and level of efficacy recorded in academia, served as the theoretical basis to the proposed IPAC model, which is under investigation in this study. The didactic triangle informs the reader that there are three facets of instruction - student, teacher, and content. The relationship among these facets are illustrated in the triangle; this aims at guiding teachers to effectively lead students to acquire new mathematical knowledge (circle geometry) - the main purpose of developing the proposed IPAC model. Furthermore, Richard Paul's wheel of reasoning served as a theoretical basis of teaching thinking, in particular, the systematic approach in teaching thinking. Descriptions of reasoning habits through the GHOM perspective by Driscoll *et al.* (2008), also served as the theoretical basis of how learners' geometrical thinking can be optimized by detailing teaching and learning dispositions needed to enable learners think geometrically. In addition, Howard Gardner's theory of multiple intelligence served as a theoretical basis for the researcher to assert that not every learner, naturally, have mathematically inclined mental structures for him or her to be effective mathematical thinkers and problem-solvers, hence, a learner who has not attained basic mathematical knowledge, may not be eligible to be moved into a higher class, where he/she is expected to learn higher-order knowledge. The researcher acknowledges that the learning environment, learners' attitudes and other factors can retrogressively hinder learners' mathematical development; all efforts, hence must be made to address all issues which may possibly affect the development of learners' mathematical knowledge. If every effort fails, then in relation to Howard Gardner's theory of multiple intelligence, the learner may belong to other domains of intelligence, not mathematics. This implies that, the teacher as a professional, must be able to evaluate the learner and help him/her to know and to discover his true domain of intelligence, thereby, not making the learner feel like 'a bad learner', just because he/she is not competent, mathematically.

Additionally, the proposed IPAC model as a thinking-based model requires lessons to be conducted in a collaborative classroom. This presupposes that integrating the infusion approach, APOS theory and Polya's approach (devising a plan stage) may serve a good purpose. This is because all these theories, involve the inculcation of a collaborative instructional approach to optimize learners' thinking. To this end, although each of these three theories are unique, they are not dissimilar. For instance, the four suggested characteristics of the APOS theory by Dubinsky and McDonald (2001), established how cooperative learning, as a form of collaborative learning is used to teach learners to develop their thinking, just like the infusion and Polya's approaches (devising a plan stage). These characteristics are - mathematical knowledge is constructed through mental construction (actions, processes, objects, and schemas); using computer; using cooperative learning groups and using the ACE teaching cycle.

The relevant part of the APOS theory which motivated the researcher to integrate it as part of the proposed model is its ability to be employed as a diagnostic tool, to monitor learners' mental constructions; this is its major attraction for its application in this study. In circle geometry, for example, learners could either get stuck on the action level or proceed to the process level in proofs or solving Euclidean geometry problems, but they would be rigid in their processes, just following "steps" as taught/drilled. This diagnosis then precedes the design of an appropriate intervention, to at least effect relational understanding at the integrated object level. Diagnostically, one may find that very few learners ever reach the schema level, which requires fully-fledged and mature higher-order thinking.

The schema level of thinking, more than any other and Polya's approach (devising a plan stage), requires a culmination of the infusion approach; in fact, the schema level cannot be reached without active infusion of thinking skills into the content area, in this case, circle geometry. To this end, I would say that the teacher modelling reasoning (in a circle geometry problem) at the schema level, would be crucial within the infusion process. In collaboration of the above, the assertion by Mudrikah (2016), adds more theoretical support for integrating the infusion approach and the APOS theory in this research study. Mudrikah (2016) asserts that Problem-Based Learning in association with the Action-Process-Object-Schema levels of the APOS theory, can be incorporated in lessons to enhance students' high-order mathematical thinking abilities. This is what the current researcher sought to ascertain in this study, hence, it can be concluded that the theoretical frameworks for this study are in harmony with the research constructs for this study.

3.11 Conclusion

This chapter presented details of teaching and learning theories, which are germane to this study. It also provided details of each component of the research construct, which were amalgamated to form the proposed instructional approach, for this study. How each component of the research construct was implemented is, as well, extensively delineated in this chapter to enable readers of this research report to have thorough understanding of each of the research constructs, and how they were integrated to form the proposed instructional approach. Lastly, how the researcher conceptualized the relevant theoretical frameworks for this study, are also explored. This was done to enable readers to understand how the theoretical framework for this study correlates with the research construct of this study. In the next chapter, the research methods that were implemented in this study are presented. The presentation includes - the justification for implementing each section of the research method and details of how the lessons were carried out in both groups (the control and the experimental).

CHAPTER FOUR

RESEARCH METHODOLOGY

4.1 Introduction

It needs to be emphasised that the proposed problem-solving instructional approach, which was under investigation in this study, was an entirely new approach which was being developed and tested, hence, extensive practical thinking, experience and expert assistance were required to fulfil the desired aim of this study. In this context, emic issues (observer's perspective) and etic issues, (issues outside the perspective of the observer), such as, a review of relevant documents on previously implemented instructional approaches in South Africa, were required. This study, hence, was pragmatic in nature, so that evidence, deemed crucial, can be obtained to answer the research questions.

McMillan and Schumacher (2014), inform that a research which places emphasis on common sense and practical thinking in addition to scientific methods to implement a desired and appropriate approach for the purpose of a study, needs to be pragmatic in nature. The authors maintain that scientific methods, by itself, may not be sufficient to obtain solutions to the research questions. The design and implementation of the proposed problem-solving instructional approach was pivotal to this study. Also, measuring the effects/influence it had on the targeted study participants' achievements/performance was essential to this study as this helped the researcher, to judge the degree of efficacy of the proposed problem-solving instructional approach in mathematics classrooms; this was the essence of this study.

4.2 Critical research questions

As a reminder, this research study sought to find an effective problem-solving instructional approach that can be used as a heuristic to improve the teaching and learning of circle geometry. This research, therefore, was conducted to obtain answers to the following research questions:

Main research question:

What are the characteristics of an effective problem-solving heuristic for circle geometry problem solving at Grade 11?

To obtain a comprehensive description of how the proposed instructional approach can influence Grade 11 learners' achievements in circle geometry, the following sub-questions were formulated:

(1) *How can the problem-solving heuristic instructional approach be developed and implemented for the teaching and learning of circle geometry by Grade 11 learners?*

(2) *What is the impact of the problem-solving heuristic instructional approach on learners' achievements in circle geometry at Grade 11 level?*

(3) *How do the participants' level of mental construction affect their abilities in their problem-solving techniques?*

4.3 Research paradigm

A research study is underpinned by a research paradigm (Hughes, 2010). This means that the research study is backed and directed by the assumptions, beliefs, norms and values, which goes along with the paradigm selected for the study; these also guide the researcher in considering an appropriate methodology (Kivunja & Kuyini, 2017). Constructivism, served as the paradigm of this research study, hence, it followed the constructivism's epistemology, assumptions, beliefs, norms and values (Kamal, 2019). The constructivism paradigm, arguably, best supports problem-solving in mathematics. This instructional approach affords students more opportunities for them to - relate the concepts they learnt to real life situations; develop their computational skills; communicate mathematically and to recognise, interpret and apply mathematical terminologies, expressions and notations. The approach, thus, accords students, the opportunity to develop the correct use of mathematical language, and to solve questions or problems in a variety of ways. Furthermore, constructivism enables students to competently use mathematical processes, such as - making conjectures, proving and disproving assertions, modelling situations, and using available technology (DoBE, 2010; DoBE, 2004). To support the need for the constructivist paradigm, the National Council of Teachers of Mathematics (NCTM, 2001) articulates five goals of learning by the constructivist approach - it gives learners more mathematical insights, which gives them reason to value and appreciate mathematics; it helps learners to become confident in their ability to do mathematics; learners become effective mathematical problem-solvers; learners acquire the ability to communicate mathematically and it assists learners to develop their mathematical reasoning and thinking skills; as this research aims to accomplish.

4.4 Research approach: mixed methods

A mixed-method research design approach was employed in this study. The researcher realised that collecting qualitative and quantitative data about the same time would be necessary; this would give enough valid data to answer the research questions comprehensively.

Justification for employing mixed-method research design for this study

Creswell (2012) defined a mixed-methods research as "a procedure for collecting, analysing, and 'mixing' both quantitative and qualitative research and methods in a single study to understand a research problem". He continued that a mixed methods research becomes relevant when a researcher realises that quantitative or qualitative data alone, will be inadequate to address the research problem. Similarly, Leech and Onwuegbuzie (2008) explained that although a mixed-method research couples both qualitative and quantitative methods, both methods are used, simultaneously, to study “the same underlying phenomenon”. Kyne (2021), assert that it is a research method much used by contemporary researchers.

To establish the effects the proposed problem-solving instructional approach would have on the study participants (experimental group), both numerical and interpretive presentations were required, hence, the need to employ a mixed-method research design. The quantitative part was required to provide numerical presentations of the study participants’ achievements at the research site, while the qualitative data would provide detailed narrations and interpretations of events and activities at the site. The researcher, thus, realised that both quantitative and qualitative research techniques were required to comprehensively answer the research questions of this study. The mixed-method approach, therefore, was deemed appropriate to address the issues of:

> how the proposed instructional approach to be used as a problem-solving heuristic can be developed and implemented in the circle geometry classroom; how the proposed instructional approach can influence the study participants’ learning of circle geometry; how the proposed instructional approach can influence the study participants’ problem-solving skills in the learning of circle geometry and how the proposed instructional approach can influence the study participants’ performance in the learning of the concepts of circle geometry.

All these issues required intensive and extensive narration and interpretation of occurrences and/or activities at the research field. To this end, adopting a qualitative research design was best suited for providing insights into these research questions.
The researcher realised that quantitative research design was also appropriate to address the research main question - how the proposed instructional approach can influence the study participants’ performance and achievement in the learning of the concepts of circle geometry. The research questions demanded both quantitative

and qualitative data, therefore, the researcher realised that adopting a mixed-method approach for this study was most appropriate.

4.5 Research design: Educational Design Research (EDR)

(Easterday, Lewis & Gerber, 2018, p.22), defined educational design research as:

"a meta-methodology conducted by education researchers to create practical interventions and theoretical design models through a design process of focusing, understanding, defining, conceiving, building, testing, and presenting, that recursively nests other research processes to iteratively search for empirical solutions to practical problems of human learning. EDR recognises that neither theory nor interventions alone are sufficient; theory and interventions drive each other in complex, iterative ways. Interventions unguided by theory are likely to be incremental and haphazard. Theory derives its purpose from application and application derives its power from theory".

After detailed consideration of a number of research designs, the researcher, with reference to literature and from expert guidance, settled on Educational Design Research (EDR), simply termed as the Design Research, as the research design for this research study; this is a design emanating from Freudenthal in 1991. The researcher opted for EDR due to the unique nature of this study - to develop knowledge and solutions. This study is about development and implementation of a proposed problem-solving instructional approach, which is to be used as an intervention, to improve the teaching and learning of circle geometry concepts. This is an entirely new instructional approach which is to be designed, developed, tried and tested, before it may be declared as efficacious to be implemented for circle geometry teaching and learning.

In the opinion of Van den Akker and Plomp (1993), design research is characterized by its twofold purpose: (1) development of prototypical products (curriculum documents and materials), including empirical evidence of their quality and (2) generating methodological directions for the design and evaluation of such products. EDR seeks to design and develop an intervention (teaching-learning strategies), in the case for this study, as a solution to a complex educational problem in South Africa. The researcher, envisaged that the proposed IPAC model, can be implemented as a long-term teaching and learning approach.

In relation to the purpose of EDR, two types can be identified - development studies (design and evaluation of educational interventions which seeks to find solutions for complex problems in educational practice by research) and validation studies (development or validation of a theory such as interventions, learning processes, learning environments, among others). Due to the unique nature of this research study - to develop and implement a proposed problem-solving instructional approach - a new instructional strategy, which will serve as an intervention instrument, needs to be developed and tested, before its full scale implementation in the field. From the above, this study can be designated as being both developmental and validation design study (van den Akker, Bannan, Kelly, Nieveen & Plomp, 2007; Fauzan, Plomp, & Gravemeijer, 2013).

This study can be classified as a developmental design study. This is because it seeks to design and evaluate an educational intervention - the IPAC model - which seeks to find an effective problem-solving instructional approach to improve the teaching and learning of circle geometry which has been declared as a difficult-to-teach and a difficult-to-learn content in South Africa (DoBE, 2018). This study can also be classified as validation design study, since it seeks to validate a teaching and learning theory, to be used as an educational intervention. This may influence teaching and learning processes of the concept of circle geometry, as well as the learning environment (for example, a collaborative classroom setting, as compared to the traditional classroom setting).

4.5.1 Phases of EDR

EDR evolves through three main phases: analysis, design, and evaluation, iteratively in a circular form. Its systematic approach to research is presented in Figure 3.1 below and its circular format is illustrated in Figure 3.2 below (Van den Akker, Bannan, Kelly, Nieveen & Plomp, 2007).

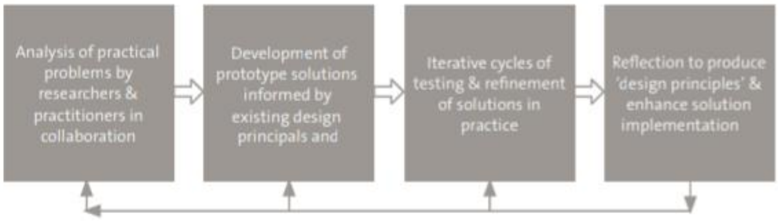


Figure 4.1 Systematic approach to EDR (Reeves, 2000) cited in Van den Akker, et al., 2007, pg. 18).

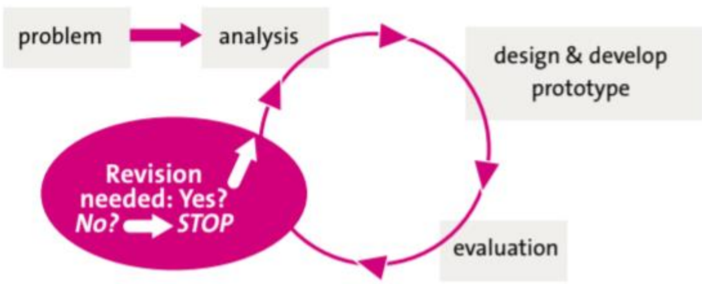


Figure 4.2 Circular nature of EDR phases (Van den Akker, et al., 2007, p.17)

4.5.2 Implementation of the phases of EDR for this study

In relation to the three phases of EDR, this research study, was structured hierarchically: front-end analysis stage, prototyping stage and assessment stage (Fauzan, Plomp, & Gravemeijer, 2013).

(1) Front-end analysis stage

This stage aims to establish where this study will begin and end. In doing so, relevant DoBE curriculum documents: diagnostic reports, subject reports, and others were analysed, expert advice was obtained, stakeholders (teachers, students, educational practitioners) inputs were obtained, relevant literature in relation to this study were reviewed and the context and problem analyses of this study were done. From this, the researcher understood and focused the problem, defined goals, conceived the outline of a solution, built the solution, tested and presented the solution. In designing the proposed IPAC model, a team of mathematics curriculum experts reviewed the content and structure of the proposed model, developed a prototype, tested the prototype, and evaluated the result after conducting the prototype. The team of mathematics curriculum experts revised both the content and structure of the IPAC model, but the ideas/assumptions on which it was built, remained unchanged.

During the testing stage of the proposed IPAC model, a pool of high schools in the Northern Cape Province of the Republic of South Africa were involved. Three high schools - one high-performing school, one average-performing school and one low-performing school, were purposefully selected, so that the result of the study can be judged as a fair reflection of the other schools which were not involved in this study. The results from these can then be generalised as a true reflection about the efficacy of the proposed IPAC model in schools in South Africa, as a whole. The research activities have been elaborated at the prototype stage. The ACE teaching cycle was implemented as the pedagogical approach for this study, with reference to expert guidance, literature review of similar-conducted research studies, which incorporated the ACE teaching cycle (Van den Akker, Bannan, Kelly, Nieveen & Plomp, 2007).

(2) Prototyping stage

The prototype stage entailed an in-depth process of the mixed-method approach which allowed for multiple - theoretical stance, types of data collection instruments, and methods of data analysis (Bannan, 2007). The developed prototypes were prefatory versions of the proposed IPAC model, which guided the design of the end product (the IPAC model), before it can be fully implemented in accordance with the views of Van den Akker, et al. (2007). In this study, three prototypes of the proposed IPAC model were developed - 1, 2 and 3. Prototype 3 was developed, in relation to events in prototype 2; the details of each prototype are elaborated below:

Prototype 1

The research question which addressed prototype 1 is as follows:

(1) How can the proposed instructional approach, to be used as a problem-solving heuristic, be developed and implemented in the circle geometry classroom?

In developing the proposed IPAC model, the initial draft model was reviewed by a team of mathematics education curriculum experts - three FET mathematics subject specialists and two university lecturers - who specialize in mathematics education in line with cognition, problem-solving and instructional approaches. They were to ascertain if the proposed IPAC model would be efficacious with regards to its content, structure, and whether applicable to the South African context. After the model was reviewed and revised, it was tested in a high school, which was randomly, sampled. This technique was used for this prototype, since all schools in the district were homogeneous in nature, hence, they exhibited similar characteristics. For this reason, the researcher trusted that any of the schools would duly serve the desired purpose and would also yield similar results. To achieve the purpose of prototype 1, a case study research design was employed.

Evaluation of prototype 1

The focus, purpose, activities and instruments for evaluating prototype 1 are elaborated below:

(i) Focus and purpose of the evaluation

The focus of this evaluation was to determine what the prototype seeks to measure: the development and implementation of the proposed IPAC model, as well as, the validity of the proposed IPAC model; the purpose was to clarify the reasons the prototype was developed - to ascertain if the proposed IPAC model was meaningfully constructed; can be effectively implemented in mathematics classrooms, and if it can be applied to South Africa's context.

(ii) Evaluation activities and instruments

This section elaborates on how the purpose of developing the prototype can be achieved. For this process, the following methods were adopted and implemented - review of the proposed IPAC model by a team of curriculum experts; classroom experiments; classroom observations in high schools; interviews with Grade 11 mathematics students and teachers from randomly selected high school and analysing learners' portfolios. Also, the tools (instruments) needed for the collection of data as well as the measurement and analyses of the data, are presented. These tools were needed when implementing activities for the developed prototype: questionnaires, standardized tests, and validity form.

Prototype 2

This prototype was developed and implemented to measure the effects/influence the proposed IPAC model had had on participants, with regards to the teaching and learning of circle geometry. How it can influence grade 11 learners' achievements; how it can influence the study participants' learning of circle geometry; how it can influence the study participants' problem-solving skills in the learning of circle geometry, and others were the issues considered. This prototype was developed by employing a true experimental research design.

The research questions for this stage were:

(a) How does the proposed instructional approach influence the study participants' learning of circle geometry?

(b) How does the proposed instructional approach to be used as a problem-solving heuristic influence Grade 11 learners' achievements in circle geometry?

*(c) How does the pro*posed instructional approach influence the study participants' problem-solving skills in the learning of circle geometry?*

(d) How do the participants approach solving problems in circle geometry?

(e) How do the participants' level of mental construction affect their abilities in their problem-solving techniques?

Evaluation of prototype 2

The focus, purpose, activities and instruments for evaluating prototype 2 are presented as follows. The focus of the evaluation of this prototype was aimed at investigating the effects/influence the proposed IPAC model had had on learners, with regards to the teaching and learning of circle geometry; the purpose was to ascertain if the proposed IPAC model produced the desired effects and if it influenced learners positively as expected. The evaluation activities were - classroom experiments; classroom observations in a high school randomly sampled; questionnaires to be answered by Grade 11 mathematics students and teachers from the sampled school and analysing learners' portfolios. The data collection instruments were: questionnaires and standardized tests.

Prototype 3

This prototype was a follow up to some recommendations from implementing prototype 2, about the efficacy of the proposed IPAC model, (an iteration of prototype 2). It was also aimed at enabling the researcher to obtain more evidence about the practicality and efficacy of the proposed IPAC model (its application and effects), hence, more schools, mathematics learners and teachers, were introduced to the model. This prototype also enabled the researcher, to further observe how well, the proposed IPAC model, can be implemented in mathematics classrooms. Just like prototype 2, a true experimental research design was employed for prototype 3.

The research question for this stage of the study was:

(a) *What are the characteristics of an effective problem-solving heuristics for circle geometry problem solving at Grade 11?*

Evaluation of prototype 3

The focus, purpose, activities and instruments for evaluating prototype 3, were - the focus aimed at investigating the reliability and efficacy of the proposed IPAC model; the purpose was to ascertain if the proposed IPAC model would yield the desired result and the same results (consistency of results) when applied in different learning environments (reliability). The evaluation activities were: classroom experiments and observations in three high schools, as well as analysing learners' portfolios. The data collection instruments were: questionnaires and standardized tests. This prototype's evaluation prompted a slight adjustment to be made in the implementation of the proposed IPAC model.

Justification for adopting a true experimental research design for prototypes 2 and 3

These prototypes focused on two groups - control and experimental. The experimental group was subjected to an intervention (the proposed IPAC model). The model was developed to investigate the cause-and-effect of the intervention; participants were randomly allocated to groups, as seen in a collaborative classroom setting. How the experimental group responded to the intervention, in contrast to the control group needed to be investigated. This enabled the researcher to ascertain the efficacy of the intervention - its effect and influence on the experimental group. This was necessary so that the intervention may be implemented on a whole scale, when its efficacy had been proven, based on how the experimental group responded to the intervention. The key descriptors in the process were - random allocation to groups, measuring cause-and-effect of an intervention and testing the efficacy of an intervention - are the inherent traits of a true experimental design; these were required to be implemented at this stage of this research (McMillan & Schumacher, 2014).

(3) The assessment stage

After the prototype stage, the assessment stage, the final phase was next. At this juncture, the final version of the proposed IPAC model was tried in three different schools from different districts in the Northern Cape Department of Education. For this process, one high performing school, one average performing school and one poor performing school were used. This assessment stage exposed more mathematics teachers and learners, from the 3 schools, to the proposed IPAC model. It also gave the researcher, the opportunity to assess the efficacy of the final version of the proposed IPAC model by examining its implementation and practicality in the classrooms. To achieve this purpose, a case study research design was employed.

Justification for adopting case study research design for the assessment stage and prototype 1

This research study had various intentions – it aimed to measure the effects the proposed problem-solving instructional approach will have on the study participants' learning of circle geometry; how the proposed instructional approach can influence the study participants' problem-solving skills in circle geometry and how the proposed instructional approach can influence the study participants' achievement/performance in circle geometry. These points require the participants to be observed extensively, over an adequate and in-depth period of time in their natural, school habitat. From the words of Stakes (1995), this will greatly enable the researcher to clearly obtain a good record of events to provide a detailed description of events at the research site, for further analysis and/or reporting. According to Stakes (1995, p.62), "the researcher should let the occasion tell its story: the situation, the problem, the resolution or the irresolution of the problem".

A case study research approach can enable a researcher to extensively, reflect on his/her eye-witness account of proceedings at the research field. It will, as well, enable the researcher to – have a complete overview of events and proceedings at the research field; to judge what can lead to significant understanding of events and activities; recognise good sources of data; either consciously or unconsciously, test the veracity of his/her eyewitness account as well as the robustness of the interpretations of the accounts. These require sensitivity and objectivity, which should inevitably be guided by the research questions, according to Stakes (1995). A case study approach, in addition to the above characteristics, will accord the researcher the chance to give full details about the nature of the research field, making all necessary resources available for readers and communicating with stakeholders, hence, it will aid the researcher to revise, disseminate materials and reports about the study to all relevant stakeholders comprehensively; Stakes (1995, p.53) referred to the above as "providing audience the opportunity for understanding". In light this context, a case study research approach, was considered ideal for prototype 1 and the assessment stage.

Evaluation of the assessment stage

The focus of the evaluation of the assessment stage was to evaluate the final version of the proposed IPAC model. The purpose of this evaluation was to ascertain if the final version of the proposed IPAC model is efficacious, with respect to its practicality and how well it can be implemented in classrooms. The evaluation activities included - a review of the proposed IPAC model by the team of curriculum experts, classroom experiments, classroom observations in the three high schools, questionnaires which were filled in by Grade 11 mathematics students and teachers from the three schools and analysing of learners' portfolios. The tools (instruments) needed for collection, measurement and analyses of data were - observation scheme and attestation form. It needs to be noted that the last of the three schools that will be used to evaluate the final version of the proposed instructional approach, will be described in this report as the 'main research field' for this study.

4.5.3 How an Educational Design Research (EDR) is appropriate for this study

The unique nature of this research study: to develop knowledge, and solutions (interventions), motivated the use of EDR as the research design for this study. EDR addresses research questions, reviews reference literature, produces theoretical claims, seeks to generalise; it also entails systematic evaluation, including formative data collection, documentation, and analysis necessary for reproducing research. The above traits of EDR, motivated its application for this study (Edelson, 2002; Bannan, 2007; Van den Akker, Bannan, Kelly, Nieveen & Plomp, 2007).

Fauzan, Plomp and Gravemeijer (2013) carried out an EDR study (development and validation studies) to construct a high-quality Realistic Mathematics Education (RME)-based geometry course that was suitable for teaching and learning the topic - Area and Perimeter - at Grade 4 in an Indonesian primary schools. Prior to conducting this research study, mathematics education in Indonesia at the basic level, was considered poor and mathematics teaching and learning was mostly centred on the traditional approach. The researcher observed that the poor performances recorded and most mathematics teachers going by the traditional teaching and learning approach in Indonesia before the EDR study was conducted, are the same concerns reported in South Africa. The current researcher, hence, trusts that a similar form of this EDR study can adequately be applied in South Africa's context, to find an efficient and productive problem-solving instructional approach, as was done in Indonesia since the contexts prior to the implementation of the EDR study in these two countries, appear to be similar.

4.6 Description of the research field

As mentioned earlier, the last of the three schools that were used to evaluate the final version of the proposed instructional approach, will be regarded as the main research field for the study. This is a high school in the Northern Cape Province which offers education and training in the following subject areas - Sciences (Life Science, Physical Science, Mathematics, Computer Applications Technology (CAT), Agricultural Science); Commerce (Accounting, Economics, Business Studies); and Humanities (Geography, Consumer Studies, Tourism). English as a first additional language, is the medium of instruction of the school, although Setswana is the home language of the community in which the school is situated. It is one of the schools with the highest enrolment of learners, especially, for Mathematics and Physical Science in the district and is classified as a quintile one school. The school lacks basic facilities such as: stationery, water, well-resourced library, sports field, classrooms and offices for teachers, however, its computer centre is in a good condition with the appropriate resources for CAT instruction; despite all these factors which can hinder the effective functioning of a school, it is one of the best performing schools in the district.

The school has - 27 classrooms of standard size in addition to three movable classrooms, learners' desks, teachers' tables and chairs, black boards, cupboards and some learners' textbooks for the different subjects offered. These constitute the available Learning, Teaching and Stationery Materials, commonly referred to as LTSMs, in the school, however, they are not enough to accommodate all learners in the school. It also has an enrolment of 1252 learners, a principal, 2 deputy principals, 4 heads of departments, 30 teachers and five non-teaching staff. Each teacher at the research site was a class-teacher; each of the sampled class was, therefore, managed by a teacher who sits in his/her class as the learners rotate to go for lessons in his/her class.

The Mathematics Department of the school has six teachers, in addition to the head of the department, who has a Bachelor of Education in Mathematics Education qualification and has been teaching mathematics for over three decades. The mathematics teacher involved in this study has Honours degree in mathematics education and has been teaching Mathematics (Grades 8-12) at the research field for fifteen consecutive years. This presupposes that he is a duly qualified, experienced mathematics teacher. Easy accessibility to the school, the knowledge that the school offers mathematics as a subject, the knowledge that the school has adequate learners taking mathematics as a subject, and the availability of qualified, experienced mathematics teachers and head of department in the school, are some of the factors which motivated adopting the school as the research site.

4.7 Classroom settings

The participants in the experimental group were seated in groups of three, to which they were randomly assigned. Enough spaces were left between the groups to ensure easy mobility for the researcher/teacher in the classroom. This enabled the researcher/teacher to attend to any of the groups when necessary and promoted effective communication/interaction in the classroom - student-student, student-teacher or student-researcher. This ensured that a positive teaching and learning environment was maintained, encouraging teaching and learning of new mathematical concepts, during the teaching and learning continuum. On the other hand, the participants in the control group were not directed to sit in groups. They were lined up in rows and columns as seen in a traditional classroom setting, where spaces are left between the columns as the teacher stands in front of the students, directing them on what to do.

4.8 Research procedure

The study was carried out in over one-year period at the research fields with two Grade 11 mathematics classes: 11A and 11B. One class (11A) served as the control group while the other class (11B) served as the experimental group. As mentioned earlier, the last of the three schools that were used to evaluate the final version of the proposed instructional approach, was regarded as the main research field for the study, hence, the research procedure and results at this school is what is presented in this section.

The main research was conducted in term one of the 2022 academic year; it commenced on the 14th of February, 2022 and ended on the 25th of March, 2022. The normal time (duration), for mathematics teaching and learning at the FET level is 4.5 hours per week and circle geometry is a topic to be taught in Grade 11. It is to be taught for a duration of three weeks in term 1, in accordance with Northern Cape Department of Education's FET mathematics work schedule.

Two weeks prior to the commencement of this research study, the researcher visited the research field so that he can be introduced to the study participants by the school authorities, including their mathematics teacher. At this juncture, the aim of the research, classroom sitting arrangements, expected classroom teaching procedures and the expected classroom ethics (expected classroom conduct, the do's and don'ts) were communicated to the learners. Also, the academic benefits for participating in this research were communicated to the learners.

After the introductory procedure, pre-intervention classroom observations proceeded, for a duration of about one week. The researcher patiently observed the teacher and learners as they went about their normal teaching and learning duties in the classroom. The researcher observed that teaching and learning was conducted by the traditional instructional approach. After conducting the pre-intervention classroom observations, conducting the main study, was next on the agenda.

Classroom observations were conducted, which were dominated by conducting circle geometry lessons in the participants' natural classroom environment.

The researcher was always present at the substantive mathematics teacher's classroom as the learners rotated to attend various subjects; 11A class (control group) was taught by the researcher, while the substantive teacher and head of department for mathematics at the school, observed proceedings. The 11B class (experimental group) were, as well, taught by the researcher. The classroom settings, the teaching approach, how learners approached problems and any other relevant happenings were noted and recorded in the note pad. Photographs and video recordings of any relevant happenings were also taken.

The research procedure for the experimental group is elaborated in the next sub-section. The focus of this study, as mentioned earlier, is to test the effects of the proposed problem-solving instructional approach on the achievements of the participants, hence, much attention was placed on the experimental group. The standardized tests that were administered to the control group, were administered also to the experimental group; the disparities in the participants' responses from the two groups were then noted. This enabled the researcher to measure the differences in the study participants' achievements both numerically/statistically, that is, quantitatively (scores after marking) and qualitatively (how they solved the problems).

The post-intervention observations were conducted with the experimental group. This enabled the researcher to know how the study participants approached problem-solving after the intervention and to make a valid and meaningful conclusion about the effects of the proposed problem-solving instructional approach on the experimental group. Lastly, a questionnaire was administered to the substantive mathematics teacher, and the head of department for mathematics at the school which served as the research field. This was done to determine their stance on each of the two instructional strategies, implemented on each of the two groups.

4.9 Observations at the research field

According to Bogdan and Biklen (1997) "Observation is a research data collection technique in which data is collected in a natural environment to elicit naturally-occurring behaviour". As a participant observer the researcher made sure he organised classroom observation sessions for both experimental and the control groups. I was present at the research field from the time the school started to the time it closed - from 07:45 am to 14:15 pm - each school day, throughout the duration of the research study. All that were personally observed were recorded in my note pad, which formed part of my field notes while anything that I observed but could not explain its details, immediately, were video recorded due to time constraints; all relevant activities and events at the research field were also video recorded. The video-recorded data were later transcribed, while all classroom instructional sessions were summarized and appropriately documented.

During the conduct of the research study, 5 pre-intervention observations, 35 main study observations and 6 post-intervention observations were conducted. An in-depth observation of activities, happenings, and most importantly, how teaching and learning of circle geometry was conducted in both experimental group and the control group were noted and documented. To obtain enough evidence, first hand, unabridged information and deep understanding of the proceedings at the research field were ensured, so that the research questions can be adequately answered. All forms of communications, interactions, discussions, teaching and learning sessions, activities, informal tests, class / homework and standardized tests, which provided useful data for the study were documented through video recordings, note pads, photograph of learners work and others. During the observation sessions, particular attention was placed on: how the proposed instructional approach influenced the study participants' learning of circle geometry; how the proposed instructional approach influenced the participants' problem-solving skills in the learning of circle geometry; how the study participants' approached solving-problems in circle geometry; how the study participants' level of mental construction affect their abilities in their problem-solving techniques; and how the proposed instructional approach influenced the participants' performance in the learning of the concepts of circle geometry.

4.10 Study population and sampling

The study population was Grade 11 learners in South African high schools, while the sample population was Grade 11 learners in the Northern Cape Province high schools. This research study was focused on Grade 11 mathematics because, circle geometry is a topic to be taught in Grade 11, in accordance with FET CAPs mathematics curriculum. South Africa is demarcated into nine provinces. The researcher chose the Northern Cape Province due to proximity logistics. In the Northern Cape Province, there are 139 high schools divided into five education districts. Education District Four, was randomly selected from the five, as the research site for this study. Simple random sampling technique was employed for the selection of the education district, since all the five education districts in the Northern Cape Province have the same traits, characteristics and structures, hence, they can be judged to be homogenous or similar in nature.

According to Taherdoost (2016), simple random sampling does not give any one in the group, any assurance of being selected.This presupposes that any of the five education districts could have been used for the study, however District 4 was randomly selected. In Education District 4, there are 52 high schools and purposive sampling method was used to choose the schools that served as the research fields based on selection factors, such as - research consent from the school, financial considerations, suitability of the school for the research study, a school which had adequate mathematics learners, easy accessibility to the school, among others. In support of the above, Palys (2008) states that in purposive sampling, the researcher is mandated to make some specific choices relevant to his/her study. He continued that these choices should be able to tell - "with whom, where, and how one does one's research". Additionally, Tongco (2007) explains that purposive sampling can serve as informant selection tool; Engel and Schutt (2009) posit that in purposive sampling, "the research elements are chosen for a precise purpose". It needs to be mentioned that due to the specialised nature of this research, only a research field with some desired characteristics can be used for the study, which justified the employment of purposive sampling technique to select the research fields.

4.11 Study participants

From the overall number of learners at the research field, only Grade 11 mathematics learners, were targeted to serve as participants for this study, since this study is aimed at them. Out of the total of Grade 11 mathematics learners, only those who willingly agreed to serve as participants for the study were considered; luckily, all of them agreed to serve as participants in the study. They were then divided into two groups: one group served as the control group, while the other served as the experimental one; each group attended the mathematics lessons at different times. The researcher served as the teacher for both groups; the control group were taught through the traditional approach, while the experimental group were taught using the intervention approach- the proposed IPAC model. The two groups comprised of learners, from the same school who have been learning under the same teacher and had been exposed to the same learning conditions, hence, the two groups were homogeneous or shared enough similar characteristics to be considered appropriate for the study. Group 11A had 30 learners, comprising of 17 females and 13 males, while 11B had 32 learners, comprising of 18 females and 14 males. Learners in the two classes were between the ages of 15-18 years, from different ethnic and social backgrounds.

4.12 DATA COLLECTION

4.12.1 Data collection procedure

As mentioned above, this study followed a mixed-method research approach, thus, both qualitative and quantitative data collection techniques were applied for data collection to answer the research questions. Data were collected over a period of 13 months. The final part of the study was carried out at a research field, where teachers conduct mathematics lessons by the traditional instructional method (from the pre-intervention observations). The same set of data for each group (the experimental and control) were collected. The researcher obtained permission from the District and the School Governing Body of the schools in which the research study was conducted, prior to data collection.

The sources of the data collected were - observations of students in their natural classroom settings, recorded videos, questionnaires, photographs of participants' work (classwork/homework, standardized tests), and field notes. Circle geometry lessons were conducted in four stages - lessons 1 to 4 for each group and each lesson was structured to cover 6 periods (45 minutes per period equalling 4.5 hours per week) - the total number of periods allocated for teaching mathematics in the CAPs curriculum. At the end of conducting each of the four lessons, the same standardized test was administered to each group. After administering the last standardized test (lesson four's standardized test), a questionnaire was also administered to the participants in the experimental group; the researcher gave them three days to complete the questionnaire. The responses to each of the four standardized tests, were then compared. This enabled the researcher to measure the effects the proposed problem-solving instructional approach, had had on the participants' achievements (experimental group) in circle geometry problem-solving (McMillan & Schumacher, 2014). Additional data were gathered (post-intervention observations) while replication and triangulation of data were done to validate data (Stake, 1995).

4.12.2 Data collection plan

Upon receipt of the ethical clearance letter from the University, for conducting this study, the selection of sites for the pilot test, reliability test and the main study commenced, in the educational Districts of the Northern Cape and North West Departments of Education, thereafter, the main research study proceeded, as summarized in the table below:

Table 4.1: Data collection plan

Period	Research activity
August 2021	Prototype 1
September 2021	Prototype 2
October 2021	Prototype 3
November 2021- August 2022	Pre-intervention observations The assessment stage -Main intervention observations Research field 1 Research field 2 Research field 3- main research filed
September 2022	Post-intervention observations and issuing questionnaire to teacher and HOD.

4.13 Instrumentation

Abawi (2013, p.2) asserts that "Accurate and systematic data collection is critical to conducting scientific research"; that data collection enables the researcher to gather relevant data to address the research problem. He explains that data collection strategies include - documents review, observation, questioning, measuring, or a combination of these different methods. In this study, both qualitative and quantitative research instruments were used for data collection, these included - video recorder for relevant events at the research fields, camera for taking photographs of participants' work at the research field, note pads for taking notes relevant to the study, class exercises/homework, Standardized tests (ST) on circle geometry, observation schedule to guide the classroom observations and questionnaires.

4.13.1 Development of instruments

Video camera, photo camera and note pads were ensured to be in good condition before usage. The proposed problem-solving instructional approach, as well as, the standardized tests instruments and questionnaires were developed, as elaborated below.

(1) The proposed IPAC model

Three distinct theories/approaches inspired the design of this proposed instructional approach: the APOS theory, Polya's problem solving instructional approach, and the infusion approach. Also, lessons were conducted in a collaborative classroom setting. As noted earlier, the pedagogical component of the APOS theory is the ACE teaching cycle. The main purpose that the APOS theory served was to guide the design and implementation of the proposed problem-solving instructional approach; it was also used to monitor learners' mental constructions. Polya's problem-solving instructional approach was only employed to guide the classroom discussion-phase of the ACE teaching cycle. The infusion approach was implemented at the second stage of Polya's problem-solving instructional approach (devising a plan stage), characterized by brainstorming, problem-solving and decision-making (Swartz & Reagan, 1998). In addition, the infusion approach was adopted so that relevant tasks that will incorporate thinking skills into circle geometry, as a content field will be selected and incorporated appropriately. According to the National Council of Teachers of Mathematics (NCTM, 2000), teachers can ask questions (problem-posing) which will require students to be critical thinkers, hence, teachers need to select relevant and meaningful tasks during lessons. Problem-posing, problem-solving, and conjecturing are three essential mathematical activities, while problem-based instructions provide opportunities for students to develop their reasoning, sense-making skills and meaning (NCTM, 2000, 2009).

In addition to the above, Zulkpli, Abdullah, Kohar and Ibrahim (2017), Stylianides (2007) and Caram & Davis (2005), suggest that mathematics teachers need to use a variety of teaching methods, such as questioning skills and strategies. These will allow teachers to ask questions that challenge students' cognitive ability; guide students to apply different thinking strategies such as generalising, applications, analogising, explaining, finding evidence and examples as well as presenting the subject in new ways. This is because questioning strategies can intrigue, arouse curiosity, stimulate interest and intrinsic motivation for students to obtain new information (Caram & Davis, 2005; Nafisah *et al.,* 2011).

The infusion approach was implemented during the "Discussion phase", "Activities phase" and the "Exercise phase" of the ACE teaching cycle, also when formulating questions that constituted the standardized tests. This is because during these stages, questions or problems/activities /tasks can be selected by the researcher. As elucidated by Mudrikah (2016), problem-based learning is appropriate for improving students' high-order mathematical thinking ability since it can encourage reflective abstraction-related mental actions, mental processes, mental objects and schemas in students.

Systematically, the following steps were followed when designing/developing the proposed problem-solving instructional approach:

(1) Formulation of theoretical analysis of circle geometry concept(s): The researcher achieved this by depending on his knowledge and experiences of teaching circle geometry, although, expert assistance/advice was requested, where necessary from specialists.

(2) Generation of genetic decomposition for circle geometry concept(s) from the theoretical analysis of the circle geometry concepts(s): Circle geometry

lessons were based on the mental constructions that learners require at that stage of genetic decomposition (Tziritas, 2011).

(3) Class discussion phase of the ACE teaching cycle: This was guided by the Polya problem-solving instructional approach, consisting of - understanding the problem, devising a plan, carrying out the plan, and reviewing the steps.

(4) Activities phase of the ACE teaching circle: Activities or tasks which promote thinking skills were selected, so that the thinking skills to be taught/learnt can be incorporated in the content instruction (infusion approach).

(5) Exercise phase of the ACE teaching circle: Activities or tasks which will promote thinking skills were selected, so that the thinking skills to be taught/learnt can be incorporated in the content instruction (infusion approach).

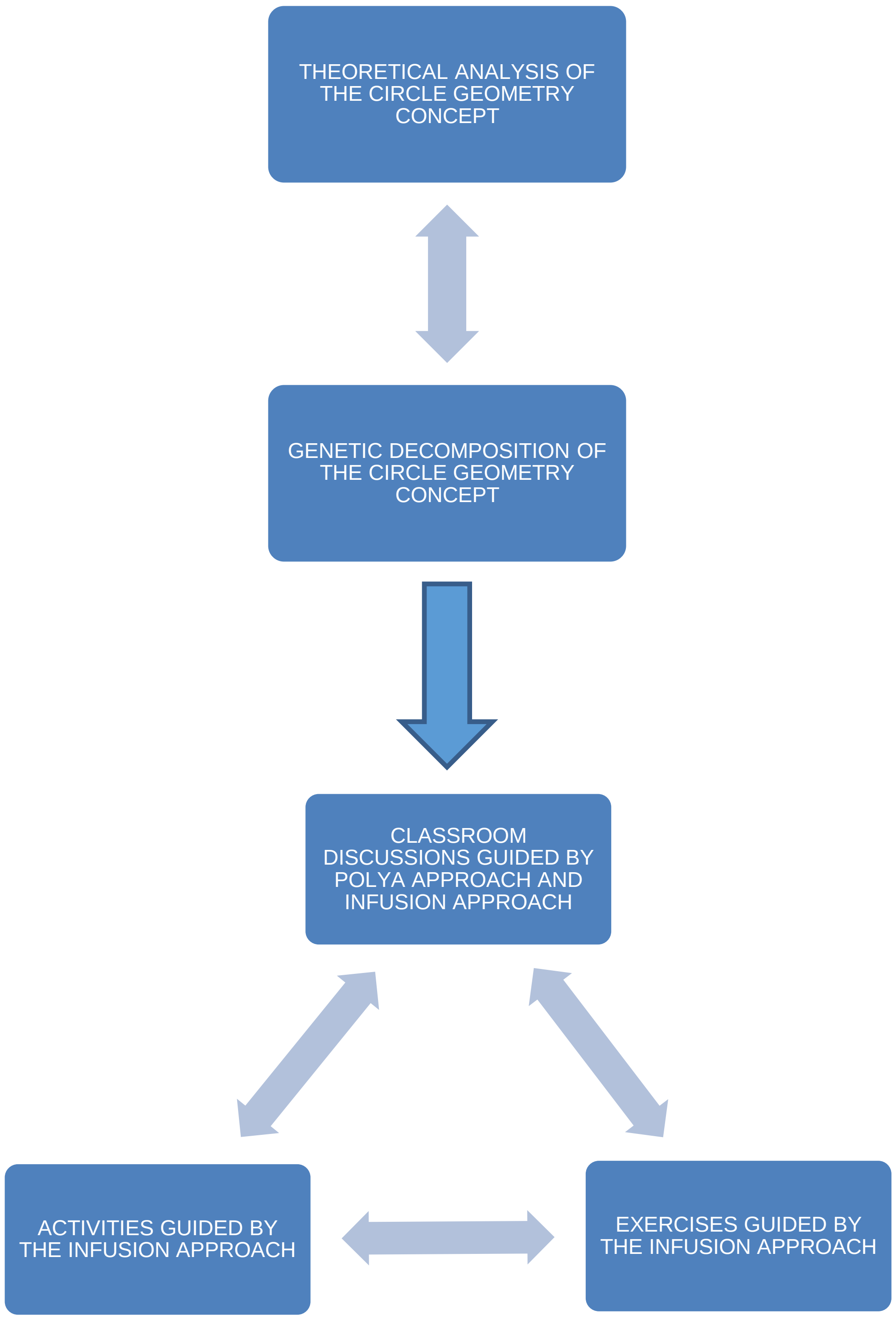


Figure 4.3: diagrammatic representation of the proposed approach

(2) Standardized tests

Malimu (2017) maintains that a standardized test is an effective data-collection tool, for measuring personality, achievement and performance. It can be used wholly as a data-collection instrument or partly to complement other measures, especially in a mixed-method research; they can be used to measure specific knowledge, skills, behaviour or cognitive activity under study. Malimu (2017) also suggested that the content of a standardized test should be selected, sequenced and scaffolded appropriately so that the instrument can be used to measure what it is intended for. Standardized tests facilitate collection of data as results can be easily analysed and they save time. For the current study, the items on the standardized test were carefully selected from the question bank of the NSC examinations, from the Department of Basic Education database, validated and reliability checked as required.

Similarly, Morgan and Harmon (2001) had stated that one participant's score can be compared with scores of other participants, when using the standardized test, therefore, conducting a standardized test on both the experimental and the control group enabled the researcher to measure the effects that the proposed problem-solving instruction approach had on the study participants' achievements in circle geometry problem- solving ability. The same set of questions were given to both groups on the standardized tests.

(3) Questionnaires

Roopa and Rani (2012) define a questionnaire as: "a series of questions asked to individuals to obtain statistically useful information about a given topic"; they can be used for data collection in any type of research. These researchers continue that a questionnaire is a potent data collection tool in the following ways: data collection success rate is faster and less expensive; obtaining objective responses from participants is comparably high and it can be used to obtain required information from a large number of respondents. The duo further averred that in developing a questionnaire, the researcher must ensure that relevant items are selected, ordered logically, scaled correctly, tested to ascertain their efficacy, they are faultless in content and structure to enable the obtaining of relevant information from study participants.

Questionnaires can be easy to construct; they are portable; they can be used to gather large datasets relatively easily, either through direct contact, mail, or online via the web or email (Young, 2016). Collected data can be analysed easily, compared to spoken data which must be recorded and transcribed before analysis. Young (2016) further claims that questionnaires are normally used to collect information pertaining to - study participants' background and demographic details, behaviours, attitudes, as well as opinions and knowledge about a subject under consideration.

Burgess (2001) had suggested that the questions of a questionnaire should be relevant, concise and efficient. The author recommends partitioning and designing of a questionnaire to accommodate 3 elements: (1) determine the questions to be asked, (2) select the question type for each question and specify the wording, and (3) design the question sequence and overall questionnaire layout. On the same topic, Abawi (2013), explains that a questionnaire consists of "series of questions and other prompts for the purpose of gathering information from respondents". Abawi (2013), adds that a questionnaire design and administration consists of the following 6 steps: (1) defining the objectives of the study; (2) identifying the target respondents and methods to reach them; (3) designing the questionnaire; (4) pilot-testing; (5) administrating the questionnaire and (6) interpreting the results. Questionnaires are either self-administered or researcher-administered, and they can be used by researchers to obtain standardized data (Keinath, Neuner, 2007). In this research study, two separate questionnaires were developed to ascertain the efficacy of the proposed IPAC model: one was administered to teachers/HODs at the research field, while the other one was answered by student participants. How they were developed is elaborated below:

(i) Questionnaire to be completed by Teachers/HODs

This questionnaire was designed to enable the researcher to know how the new teaching approach was implemented and how well the conducted lessons were understood. The items of this research questionnaire were divided into four parts: part A: how the teaching of thinking skills was conducted; part B: mode of presentation of lesson; part C: how well the conducted lessons were understood, and part D: any other comments/remarks. The details of each of the four parts of this questionnaire, are presented in Appendix H.

(ii) Questionnaire to be completed by study participants

This questionnaire was designed to measure the effects/influence the proposed problem-solving instructional approach had had on learners, in the teaching and learning of circle geometry. The items of this research questionnaire were divided into three parts: part A: how the new instructional approach can influence the study participants' learning of circle geometry; part B: how it can influence the study participants' problem-solving skills, involving circle geometry problems, and part C: any other comments/remarks. Details of each part of this questionnaire can be found in Appendix I.

4.13.2 Explication and implementation of the proposed IPAC model

Sequentially, the procedures below, in relation to the IPAC model as implemented on the experimental group, are explained.

(1) Generation of genetic decomposition of circle geometry concepts from the theoretical analysis of them: Circle geometry lessons were based on the mental constructions that learners require at that stage of genetic decomposition (Tziritas, 2011). This was sub-divided into four mental

construction lessons: GD1 - action stage of circle geometry mental construction lesson; GD2 - process stage of circle geometry mental construction lesson; GD3- object stage of circle geometry mental construction lesson, and GD4 - schema stage of circle geometry mental construction lesson. The circle geometry lessons were delivered in four lessons stages, in accordance with the mental construction level of the learners. Learners who had not attained GD1 mental construction level, may not be moved on to GD2 mental construction level; this rule was applied to progression within all the levels. The details of each of the four stages/levels are given below.

GD1- Action stage of circle geometry mental construction lesson

Learners are expected to solve problems by following detailed step by step knowledge procedures; this may require specific teaching, and the need to perform each step clearly. Straight recall and use of circle geometry theorems and geometric language, as well as conceptual knowledge and understanding of geometric concepts are required at this stage. Questions which are relevant to this stage were administered to learners, during the discussion phase, activities phase and exercise phase to incorporate thinking skills into the conceptual understanding of circle geometry concepts (infusion approach). As mentioned earlier, developing students thinking skills requires students to be exposed to unfamiliar questions and tasks based on their previous knowledge. In view of the above, this GD1 lesson, served as the prior knowledge that needs to be obtained by students, which will guide them in the development of higher thinking skills (GD2, GD3 & GD4 lessons), (Maharaj, 2010; Swartz & Reagan, 1998; Tziritas, 2011; CAPs, 2012; King, Goodson & Rohani, 2013).

GD2- Process stage of circle geometry mental construction lesson

Learners at the process level are expected to reflect on the action process and describe, or even reverse the steps of previously learned objects without actually performing those steps. They are expected to appropriately distinguish between the different geometry theorems and to know when and how to use each theorem in a given problem-solving situation. Learners are also expected to prove and perform simple applications of the circle geometry theorems they learnt during the action stage. As learners continuously repeat and reflect on an action, it may become interiorised into a mental process. Questions which are relevant to this stage were administered to learners, during the discussion phase, the activities phase and the exercise phase to incorporate thinking skills into the conceptual understanding of circle geometry concepts (infusion approach), (Swartz & Reagan, 1998; Maharaj, 2010; Tziritas, 2011; CAPs, 2012).

GD3- Object stage of circle geometry mental construction lesson

Learners at the object level are expected to reflect on a particular set of processes until they are able to perform encapsulations on mathematical concepts. Learners at this stage can, therefore, be said to have encapsulated the process into a cognitive object. At this stage, higher-order reasoning is required since there is no obvious route to the solution to the problem which could not involve making significant connections between different geometric concepts which demands adequate conceptual understanding and application of geometric concepts. Questions which are relevant to this stage were administered to learners, during the discussion phase, activities phase and exercise phase to incorporate thinking skills into the conceptual understanding of circle geometry concepts (infusion approach), (Swartz & Reagan, 1998; Maharaj, 2010; Tziritas, 2011; CAPs, 2012)

GD4- Schema stage of circle geometry mental construction lesson

At this level - actions, objects and processes of a mathematical concept are interconnected in the learners' minds to construct schemas. Learners are expected to be able to organise and link these stages together to form a coherent framework (schema). This stage is characterized by non-routine problems which require higher-order reasoning and processes. Breaking the problem into its constituent parts in order to reach the solution to the problem may be performed at this stage. The schema level of thinking, more than any other, requires a culmination of the infusion approach; in fact, the schema level cannot be reached without active infusion of thinking skills into the content area, in this case, circle geometry. The teacher him/herself modelling reasoning (in a circle geometry problem) at the schema level, is crucial within the infusion process. Lastly, questions which are relevant to this stage were administered to learners, during the discussion, activities and exercise phases to incorporate thinking skills into the conceptual understanding of circle geometry concepts (infusion approach), (Swartz & Reagan, 1998; Maharaj, 2010; Tziritas, 2011; CAPs, 2012).

(2) Class discussion phase of the ACE teaching cycle was guided by the Polya problem-solving instructional approach steps: understanding the problem, devising a plan, carrying out the plan, and reviewing the steps. It was also guided by the infusion approach. Questioning skills that can improve students thinking skills, as advocated by the infusion approach were ensured. The procedures for the class discussion phase are elaborated below (adapted from Abakah, 2019).

(a) The teacher gives a leading question (in the case of a new concept) or writes a problem to be solved on the board (in the case of continuation of a previous concept).

(b) The study participants start to discuss the solution in line with the Polya problem-solving approach steps enumerated earlier, characterized by problem-posing, problem-solving, and conjecturing. What learners are expected to do at each step is delineated below:

(i) Step 1. Understanding the problem

Learners are expected to carefully read and understand the problem to be solved, to paraphrase the problem in their own words, if necessary, to emphasize what they understood, and to determine what the problem asks them to solve, that is, to determine the unknown.

(ii) Step 2. Devising a plan (thinking stage).
Learners are mandated to think rigorously and endlessly, until a reasonable solution to the question/problem is reached.

(iii) Step 3. Carrying out the plan.
Learners are expected to implement the strategy/thought they have devised in the previous step by performing necessary actions or computations.

(iv) Step 4. Looking back
Learners are taught to check the validity of the final solution they had come up with. They are, then asked to interpret the result they found and to determine whether the solution makes sense and is reasonable in the context of the problem, (i-iv adapted from: Valles & Wickramasingh, 2015).

(c) The teacher goes round each group to moderate or correct the groups' discussions.

(d) The teacher stops the discussion and allow the study participants to present their solutions and allow the groups to criticise/support each other's solutions.

(e) The teacher finalises the solution by accepting or correcting the solution proposed by the study participants and give more detailed explanation on the problem(s) before introducing another problem to be solved to the study participants.

(3) Activities phase of the ACE teaching circle is implemented. Activities or tasks which will promote thinking skills are selected appropriately, so that the thinking skills to be taught/learnt can be incorporated in the content instruction (infusion approach).

(4) Exercise phase of the ACE teaching circle is implemented by giving learners class works; at the end of implementing the proposed problem-solving instructional approach, standardized tests were conducted. Activities or tasks which will promote thinking skills were selected, so that the thinking skills to

be taught/learnt can be incorporated in the content instruction (infusion approach).

4.13.2.1 Thinking skills, processes and dispositions to be taught under each genetic decomposition (GD) lesson

The development of students' thinking calls for cultivating their skills, processes, and dispositions concerning better thinking (Swartz & Perkins, 1998). In view of this, the researcher found it relevant to present how each GD lesson was aligned to the thinking skills, processes and dispositions to be taught in the circle geometry classroom. They are sequentially presented in the Table below, however, how the thinking skills, together with their corresponding processes, and dispositions were implemented in this study, is elaborated under the section on details of how lessons were conducted with the experimental group.

Table 4.2: Thinking skills, processes and dispositions to be taught under each GD lesson

GD lesson	Thinking skills to be taught	Process which enhances thinking skills	Disposition to promote lesson	Questioning approach to be used for lesson
GD1 **(Introductory lesson)**	Understanding and retention of ideas	Decision making	Making thinking 'clear and careful'	Recalling of content knowledge by asking direct procedure questions
GD2	Generating ideas (Creative thinking)	Decision making and problem solving	Making thinking 'clear and careful'	Asking indirect procedure questions, which requires application of GD 1 knowledge

GD3	Assessing reasonability of ideas (Critical thinking).	Decision making and problem solving	Making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time.	Asking higher order questions
GD4	Blending generating of ideas in GD2, with assessing reasonability of ideas in GD 3, and engaging students in metacognitive reflections	Decision making and problem solving	Making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time.	Asking higher order questions

(Swartz & Perkins, 1990; Swartz & Reagan, 1998)

4.13.2.2 How students overall achievements in circle geometry were assessed and evaluated

Table 4.3: Assessment and evaluation methods

GD lesson	Type of assessment	Mode of questioning	Descriptors to be applied	Purpose of descriptors to be applied
GD1	Didactic Summative	Direct recall, direct procedure questions	Statements	Understanding
GD2	Didactic Summative	Indirect procedure questions, which requires simple application of knowledge	Statements, reasons	Understanding, transfer and application of knowledge from GD1

GD3	Didactic Summative	Higher order questions	Statements, reasons, explanations of reasons	Transfer and application of knowledge, managing ones cognition (Kuhn, 1989).
GD4	Didactic Summative	Higher order questions	Statements, reasons, explanations of reasons	Transfer and application of knowledge, managing ones cognition (Kuhn, 1989).

(i) Assessing and evaluating GD1 & GD2 tasks

As presented in Table 3.2 above, direct procedure questioning approach is to be implemented in a GD1 lesson. Simple responses (geometric statements) are required as solutions to GD1 tasks, which enable pedagogues to measure learners' geometric understanding. The researcher opines that the use of only one descriptor - geometric statements - is appropriate to evaluate learners' geometric understanding at GD1 level, since only direct recall of geometric concepts and/or theorems are mainly the learning objectives of a GD1 lesson (DoBE, 2010).

As learners commence to intuitively perceive circle geometry theorems and concepts as a mental object, internalization of circle geometry theorems takes place, in the mind of learners. To this end, indirect procedure questioning approach which requires simple application of circle geometry theorems and related geometric concepts are demanded; this requires learners to provide reasons for every geometric statement they make, and which may be appropriate in measuring learners' geometric competences at the GD2 level, as elaborated in Table 3.2 above. The researcher considers the two descriptors used at this level (statement and corresponding reason) to be appropriate for determining learners' geometric competence since the learning objectives of a GD2 lesson primarily require learners to acquire geometric understanding, transfer and application of knowledge from GD1.

(ii) Assessing and evaluating GD3 & GD4 tasks

With reference to Table 3.2 above, higher-order questioning approach are required for GD3 and GD4 tasks and problems. GD3 and GD4 tasks mandate learners to give reasons for each geometric statement made, and to briefly, give explanations for each reason given. The researcher, based on the explanations given for each stated reason, was able to determine learners' thinking efficiency, that is, if their thinking is meaningful, explicit and organized. The researcher believes that learners been tasked to give reasons for their geometric statements, (which is currently observed for assessing mathematics learners' competence in Euclidean geometry by the Department of Basic Education in South Africa) is helpful (DoBE, 2010). Requesting for two descriptors alone - statement and reason – I, however, find to be inadequate in addressing the gap between curriculum expectations and what learners are expected to achieve (the zone of proximal development), (Vygotsky, 1978). In addition to giving reasons for geometric statements, the researcher recommends that also giving explanations to reasons provided for geometric statements, would be greatly beneficial. It will promote students geometric awareness and understanding, as well as enable them to manage their cognition better by reflecting on their own thinking (William & Maat, 2020).

4.13.2.3 How lessons were conducted in each group: the control group and the experimental group

As mentioned earlier, circle geometry lessons were conducted in four stages: lessons 1 - 4, for both the control and the experimental groups. It needs to be mentioned that the two classes - 11A and 11B - do not attend mathematics lessons together. Each class has its own time, within a day to attend mathematics lessons; there are 6 periods per week for each class and each period lasts 45 periods (4.5 hours per week). Prior to the commencement of each lesson, the teacher makes sure that expected classroom settings are adhered to by participants: control group members were made to sit individually in rows and columns while participants in the experimental group were made to sit in groups of three. At the end of conducting each of the four lessons, the same four standardized tests were administered to each group. In the Table 4.4, are explanations of how each of the four lessons was conducted. Also, differences in teacher's role and participants' role for the experimental group and the control group was elaborated in Table 4.5.

Table 4.4: Overview of how lessons were conducted on each group

Lesson	How lessons were conducted on the control group (part 1)	How lessons were conducted on the experimental group (part 2)
1	Teaching participants to know and to recognise circle theorems, converse of theorems, and geometric properties.	GD- 1 lesson, together with corresponding thinking skills to be taught, process and dispositions which enhance these thinking skills and questioning approach as presented in Table 3.1 above.
2	Teaching participants to directly and routinely apply circle geometry theorems, converse of theorems, and geometric properties.	GD- 2 lesson, together with corresponding thinking skills to be taught, processes and dispositions which enhance these thinking skills and questioning approach as presented in Table 3.1 above
3	Teaching participants to prove circle geometry theorems, converse of theorems, and to apply their knowledge to solve non-routine problems and tasks which require creative and high order thinking.	GD- 3 lesson, together with corresponding thinking skills to be taught, processes and dispositions which enhances these thinking skills and questioning approach as presented in Table 3.1 above.
4	Teaching participants to apply their knowledge to solve non-routine problems and tasks which require critical, higher-order and advanced mathematical	GD- 4 lesson, together with corresponding thinking skills to be taught, processes and dispositions which enhance these thinking skills

	thinking.	and questioning approach as presented in Table 3.1 above.

Table 4.5: Differences in teacher's and participants' role for the experimental group and the control group

No.	Control group (Traditional approach)	Experimental group (Proposed instructional approach)
1	The objectives and aims of each of the four lessons are stated, however, students receive information passively. Mathematics ideas are given in a ready-made fashion. Students and teacher generally refer to text-books.	Contents are structured in relation to the mental construction steps- actions, processes, objects, and schemas. Students are involved actively in learning. Mathematics ideas (definitions, lemmas, and theorems) are discovered by students through fact-finding, through classroom activities.
2	The teacher serves the role of a knowledge transmitter - the teacher directly explains mathematical ideas.	The teacher serves the role of a facilitator. He provides guidelines and assistance to learners, in groups or entire class, through scaffolding, questioning and giving hints.
3	Learners working individually, is prioritized. Limited interaction among learners in class. One- or two-way interaction involving teacher.	Learners working in groups, is prioritized. Unlimited learner-to-learner, and teacher-to-learner interactions in class. Learners learn from peers through group-work and discussions.

Adapted from: (Arnawa; Sumarno, Kartasasmita & Baskoro, 2007, p.145)

Part 1: Details of how lessons were conducted with the control group

LESSON PLAN FOR LESSON 1		
TOPIC: Circle geometry	**Date**: 14-18/02/2022	**Class**: Grade 11 A
Subject: Mathematics	**Educator**: Abakah F	**Duration**: 6 periods (week one- 45 minutes per period)

SUB- TOPICS:

Each theorem was taught individually:

Period 1: Circle geometry theorems which were conjectured from the centre of the circle (Group I), Period 2: Circle geometry theorems which were not conjectured from the centre of the circle (Group II), Period 3: Circle geometry theorems which involve tangents (Group III), Period 4: Cyclic quadrilaterals group (Group IV), Periods 5 & 6: Combination and consolidation of all circle geometry theorems. Groups I, II, III & IV are all shown in Figure 3.4 below.

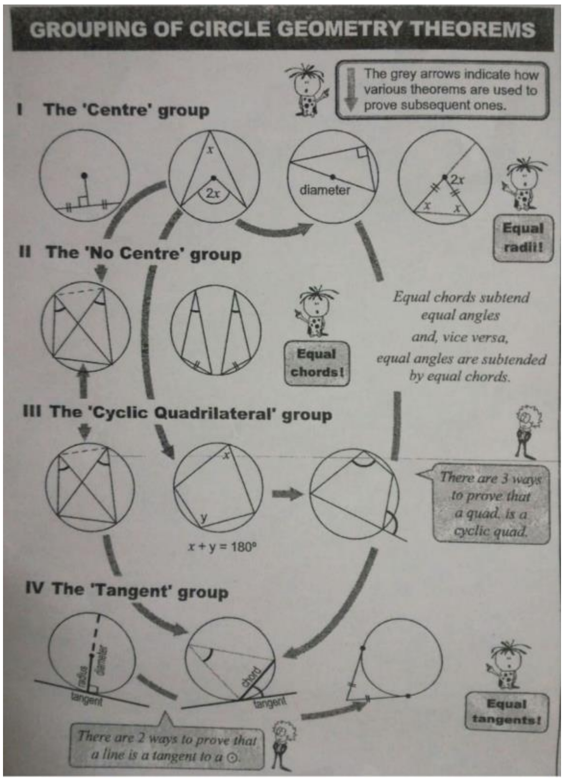


Figure 4.4 Circle geometry theorems (Eadie & Lampe, 2016, p. 94)

TEACHING AND LEARNING MATERIALS

- Grade 11 classroom mathematics textbook
- Grade 11 Platinum mathematics textbook.

RELEVANT PREVIOUS KNOWLEGDE

- The learner can draw and label parts of the circle: sector, segment, chord, among others.
- The learner can recognise that twice the radius is equal to the diameter.

LEARNING OBJECTIVES

By the end of the lesson, the learner will be able to:

- Know and to recognise circle geometry theorems, converse of theorems, and geometric properties.

INSTRUCTIONAL OBJECTIVES

The learner will be able to:

- Recognise and identify circle geometry theorems, converse of theorems, and geometric properties.
- Recognise similarities and differences that exist between circle geometry theorems.

DEVELOPMENT

Step 1- Introduction

The teacher revises with learners to:

- Draw and label parts of the circle.
- Recognise that twice the radius is equal to the diameter.

Step 2

Teacher illustrates circle geometry theorems to be learnt in each period, on the chalk board. S/he explains to learners the understanding behind each of the theorems.

Step 3

Teacher explains to learners the converse of each of the circle geometry theorems.

Step 4

Teacher explains to learners the similarities and differences that exist between circle geometry theorems.

Step 5

Teacher solves related examples with learners based on knowledge of circle geometry theorems, converse of theorems, similarities and differences between theorems.

Step 4 Assessment- Home work

With reference to examples solved in class, the teacher gives homework to learners:

Classroom mathematics textbook (Grade 11)

Memorandum: Teachers guide, classroom mathematics, Grade 11

Step 5

Remarks by the teacher on lesson conducted: ……………………………………..

LESSON PLAN FOR LESSON 2

TOPIC: Circle geometry	**Date**: 21-25/02/2022	**Class**: Grade 11 A
Subject: Mathematics	**Educator**: Abakah F	**Duration**: 6 periods (week two - 45 minutes per period)

TEACHING AND LEARNING MATERIALS

- Grade 11 Classroom mathematics textbook.
- Grade 11 Platinum mathematics textbook.

RELEVANT PREVIOUS KNOWLEGDE

The learner is able to:

- Recognise circle geometry theorems, converse of theorems, and geometric properties.
- Recognise similarities and differences that exist between circle geometry theorems.

LEARNING OBJECTIVES

By the end of the lesson, the learner will be able to:

- Directly and routinely apply circle geometry theorems, converse of theorems, and geometric properties.

INSTRUCTIONAL OBJECTIVES

The learner will be able to:

- Solve routine circle geometry problems which involve direct applications of circle geometry theorems, converse of theorems and geometric properties.

DEVELOPMENT

Step 1- Introduction

The teacher revises with learners on:

- Circle geometry theorems, converse of theorems, similarities and differences of circle geometry theorems. The teacher summarizes the work that was done in lesson one.

Step 2

Teacher solves related circle geometry problems on the chalkboard, which involve routine circle geometry problems and direct applications of circle geometry theorems, converse of theorems and geometric properties.

Step 3

Assessment- Class work

With reference to examples solved in class, the teacher gives class work to learners:

Classroom mathematics textbook (Grade 11)

Memorandum: Teachers guide, classroom mathematics, Grade 11.

Step 4

Remarks by the teacher on lesson conducted: ………………………………………….

LESSON PLAN FOR LESSON 3

TOPIC: Circle geometry	**Date**: 28/02/2022 to 4/3/2022	**Class**: Grade 11 A

Subject: Mathematics	**Educator**: Abakah F	**Duration**: 6 periods (week three - 45 minutes per period)

SUB- TOPICS:

Periods 1-3: proving circle geometry theorems

Periods 4-6: Solving non-routine problems and tasks which require creative and high-order thinking.

TEACHING AND LEARNING MATERIALS

- Grade 11 Classroom mathematics textbook
- Grade 11 Platinum mathematics textbook.
-

RELEVANT PREVIOUS KNOWLEGDE

- The learner is able to solve routine circle geometry problems which involve direct applications of circle geometry theorems, converse of theorems and geometric properties.

LEARNING OBJECTIVES

By the end of the lesson, the learner will be able to:

- Prove circle geometry theorems.
- Apply their knowledge to solve non-routine problems and tasks which require creative and high-order thinking.

INSTRUCTIONAL OBJECTIVES

The learner would be able to:

- Prove any circle geometry theorems.
- Solve non-routine problems and tasks which require creative and high-order thinking.

DEVELOPMENT

Step 1- Introduction

The teacher summarizes the work that was done in lesson two.

Step 2

Teacher teaches learners how to prove circle geometry theorems on the chalkboard.

Step 3

Assessment- Class work

Teacher solves related circle geometry problems on the chalkboard, involving non-routine problems and tasks which require creative and high-order thinking.

Step 4

Assessment- Home work

With reference to examples solved in class, the teacher gives homework to learners:

Platinum mathematics textbook (Grade 11).

Memorandum: Teachers guide, platinum mathematics, Grade 11.

Step 4

Remarks by the teacher on lesson conducted: ……………………………………………….

LESSON PLAN FOR LESSON 4

TOPIC: Circle geometry	**Date**: 7-11/03/2022	**Class**: Grade 11 A
Subject: Mathematics	**Educator**: Abakah F	**Duration**: 6 periods (week four - 45 minutes per period)

TEACHING AND LEARNING MATERIALS

- Grade 11 Classroom mathematics textbook
- Grade 11 Platinum mathematics textbook

RELEVANT PREVIOUS KNOWLEGDE

The learner is able to:

- Prove any of the circle geometry theorems.
- Apply their knowledge to solve non-routine problems and tasks which require creative and high-order thinking.

LEARNING OBJECTIVES

By the end of the lesson, the learner will be able to:

- Apply their knowledge to solve non-routine problems and tasks which require critical, high-order and advanced mathematical thinking.

INSTRUCTIONAL OBJECTIVES

The learner will be able to:

- Solve non-routine problems and tasks which require critical, higher-order and advanced mathematical thinking.

DEVELOPMENT

Step 1- Introduction

The teacher summarizes the work that was done in lesson three.

Step 2

Teacher solves non-routine problems and tasks which require critical, higher-order and advanced mathematical thinking on the chalkboard.

Step 3

Teacher solves more related problems on the chalkboard.

Step 4

Assessment- Class work

Teacher gives related circle geometry problems to learners to solve, involving non-routine problems and tasks which require creative and high-order thinking.

Step 5

Assessment- Home work

With reference to examples solved in class, the teacher gives homework to learners:

Platinum mathematics textbook (Grade 11)

Memorandum: Teachers guide, platinum mathematics, Grade 11.

Step 6

Remarks by the teacher on lesson conducted: ………………………………………….

Part 2: Details of how lessons were conducted with the experimental group

Lessons 1 & 2 on solving direct and routine questions, which merely require recalling and direct application of knowledge, were devoid of Polya's approach during the class discussion stages. It was regarded as superfluous at these initial two stages since conjecturing and generation of knowledge and new ideas were not required, however, lessons 3 & 4, regarded as the pinnacle of the study, fully incorporated Polya and the infusion approaches.

<table>
<tr><th colspan="3">LESSON PLAN FOR LESSON 1</th></tr>
<tr><td>TOPIC: Circle geometry</td><td>Date: 16-20/02/2022</td><td>Class: Grade 11 B</td></tr>
<tr><td>Subject: Mathematics</td><td>Educator: Abakah F</td><td>Duration: 6 periods
(week one- 45 minutes per period)</td></tr>
<tr><td colspan="3">SUB-TOPICS
Theorems were taught categorically, according to the features they share, to promote categorical reasoning.
Period 1: Circle geometry theorems which were conjectured from the centre of the circle, as shown in Figure 3.4 above.
Period 2: Circle geometry theorems which were not conjectured from the centre of the circle, as shown in Figure 3.4 above.
Period 3: Circle geometry theorems which involve tangents, as shown in Figure 3.4 above.
Period 4: Cyclic quadrilaterals group, as shown in Figure 3.4 above.
Periods 5 & 6: Combination and consolidation of all circle geometry theorems.
TEACHING AND LEARNING MATERIALS
- Grade 11 Classroom mathematics textbook
- Grade 11 Platinum mathematics textbook</td></tr>
</table>

RELEVANT PREVIOUS KNOWLEGDE

The learner is able to:

- Draw and label parts of the circle.
- Recognise that twice the radius is equal to a diameter.

LEARNING OBJECTIVES

By the end of the lesson, the learner will be able to:

- Attain GD 1 mental construction level, based on the generated preliminary GD1.
- Master the corresponding thinking skills at GD1 (understanding and retention of ideas), master processes (decision making) and dispositions (making thinking clear and careful) which enhances these thinking skills.

INSTRUCTIONAL OBJECTIVES

The learner will be able to:

- Recognise circle geometry theorems, converse of theorems, and geometric properties.
- Recognise similarities and differences that exist between circle geometry theorems.

DEVELOPMENT

Step 1- Introduction

The teacher instruct learners to:

- Draw and label parts of a circle.
- Find the relationship between the radius and diameter.

Step 2

Teacher introduces the thinking skills to be taught at this level - understanding and retention of ideas; corresponding processes (decision making); and dispositions (making thinking clear and careful), which enhance these thinking skills. The teacher educates learners about the need to master these thinking skills with the corresponding processes and dispositions.

Step 3

Teacher guide learners to know, understand, recognise and discover circle geometry theorems, the converse of each circle geometry theorem, as well as similarities and differences that exist between the circle geometry theorems, by using activity sheets, during class discussions.

Step 4 Teacher uses the questioning approach of recalling of content knowledge, by asking direct procedure questions, to guide learners to solve problems. **Step 5** Teacher guides learners to solve more related problems, at this level. **Step 6 Assessment- Home work** Teacher gives homework to learners using Classroom mathematics textbook (Grade 11). Memorandum: Teachers guide, classroom mathematics, Grade 11. Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work randomly among themselves to be marked again. Finally, the teacher marked each individual learner's work (Appendix G). The disparities in marks were then discussed, firstly, among group members and secondly, with the teacher (see section 2.7 of Chapter 2). **Step 7** Remarks by the teacher on lesson conducted: ………………………………………….

LESSON PLAN FOR LESSON 2		
TOPIC: Circle geometry	**Date**: 21-25/02/2022	**Class**: Grade 11 B
Subject: Mathematics	**Educator**: Abakah F	**Duration**: 6 periods (week two - 45 minutes per period)
TEACHING AND LEARNING MATERIALS - Grade 11 Classroom mathematics textbook. - Grade 11 Platinum mathematics textbook. **RELEVANT PREVIOUS KNOWLEGDE**		

The learner is able to:

- Solve problems at GD 1 mental construction level, based on the generated preliminary GD1
- Master the corresponding thinking skills at GD1 (understanding and retention of ideas), master processes (decision-making) and dispositions (making thinking clear and careful) which enhance these thinking skills.

LEARNING OBJECTIVES

By the end of the lesson, the learner will be able to:

- Attain GD 2 mental construction level based on the generated preliminary GD2.
- Master the corresponding thinking skills at GD2: generating ideas (creative thinking), corresponding processes (decision making and problem solving), and dispositions (making thinking clear and careful), which enhance these thinking skills.

INSTRUCTIONAL OBJECTIVES

The learner will be able to:

- Apply his/her knowledge to solve problems and tasks at GD 2 mental construction level based on the generated preliminary GD2, and corresponding thinking skills at GD2: generating ideas (creative thinking).

DEVELOPMENT

Step 1- Introduction

The teacher guides learners to revise the work done in lesson one, in a summary form.

Step 2

The teacher introduces the thinking skills to be taught at this level: Generating ideas (Creative thinking), corresponding processes (decision making and problem solving); and dispositions (making thinking clear and careful), which enhance these thinking skills. The teacher educates learners about the need to master these thinking skills with the corresponding processes, and dispositions. Also, the teacher educates learners on how these thinking processes and dispositions, can enhance their thinking.

Step 3

The teacher guides learners to depend on their knowledge of lesson one, to solve problems which require indirect application of circle geometry theorems, by using activity sheets during class discussions.

Step 4

The teacher implements the questioning approach at this level - asking indirect procedure questions, which require application of knowledge of lesson 1, to guide learners to solve problems at this level, in class.

Step 5

The teacher guides learners to solve more related problems at this level.

Step 6

Assessment- Class work

The teacher gives class work to learners:

Classroom mathematics textbook (Grade 11).

Memorandum: Teachers guide, classroom mathematics, Grade 11.

Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work randomly among themselves to be marked again. Finally, the teacher marked each individual learner's work (Appendix G). The disparities in marks were then discussed, firstly, among group members and secondly, with the teacher (see section 2.7 of Chapter 2).

Step 7

Remarks by the teacher on lesson conducted: ……………………………………….

LESSON PLAN FOR LESSON 3		
TOPIC: Circle geometry	**Date**: 28/02/2022 to 4/3/2022	**Class**: Grade 11 B
Subject: Mathematics	**Educator**: Abakah F	**Duration**: 6 periods (week three - 45 minutes per period)

SUB- TOPICS:

Period 1- Educating learners on the thinking skills, thinking processes and dispositions to be taught at this level.

Period 2- Educating learners on Polya's approach.

Periods 3, 4, 5 & 6- Solving problems relevant to this level, in line with the Polya's and infusion approaches.

TEACHING AND LEARNING MATERIALS

- Grade 11 Classroom mathematics textbook
- Grade 11 Platinum mathematics textbook

RELEVANT PREVIOUS KNOWLEGDE

The learner is able to:

- Solve problems at GD 2 mental construction level based on the generated preliminary GD2.
- Master the corresponding thinking skills at GD2: generating ideas (creative thinking), corresponding processes (decision making and problem solving) and dispositions (making thinking clear and careful).

LEARNING OBJECTIVES

By the end of the lesson, the learner will be able to:

- Attain GD 3 mental construction level, based on the generated preliminary GD3.
- Master the corresponding thinking skills at GD3: assessing reasonability of ideas (Critical thinking), corresponding processes (decision making and problem solving); and dispositions (making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time).

INSTRUCTIONAL OBJECTIVES

The learner will be able to:

- Apply his/her knowledge to solve problems and tasks at GD 3 mental construction level based on the generated preliminary GD3, and corresponding thinking skills at GD3: assessing reasonability of ideas (critical thinking).

DEVELOPMENT

Step 1- Introduction

The teacher guides learners to revise the work done in lesson two, in a summary form.

Step 2

The teacher introduces:

- The thinking skills to be taught at this level: assessing reasonability of ideas (critical thinking) via causal explanations, predictions, generalizations, reasoning by analogy and by making deductions through conditional reasoning and categorical reasoning (Swartz & Reagan, 1998).
- Corresponding processes: decision making and problem solving, and dispositions (making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time). Decision-making as a thinking process at this level is achieved by following the following prompts and question guidelines: (1) what are my options? (2) What are the likely consequences of these options? (3) What is the best option in light of the consequences? (Swartz & Reagan, 1998).
- The teacher educates learners about the need to master these thinking skills with the corresponding processes, and dispositions. In addition, the teacher educates learners on how these thinking processes and dispositions, can enhance their thinking.

Step 3

- The teacher educates learners about Polya's approach: understanding the problem, devising a plan, carrying out the plan, and reviewing the steps. The teacher explains to learners what each step means and its relevance in solving problems. What is expected to be done by the students at each level, is also communicated to learners.

Step 4

- The teacher guides learners to depend on their knowledge of lesson two, to solve problems which require indirect and non-routine application of circle geometry knowledge, during class discussions, in line with Polya's approach, which is characterized by problem-posing, problem-solving, and conjecturing. The procedure for the class discussion phase is elaborated as follows (adapted from Abakah, 2019):

- The teacher gives a leading question (in the case of a new concept) or write a problem to be solved on the board (in the case of continuation of a previous concept), by implementing the questioning approach at this level (asking higher-order questions).
- The study participants start to discuss the solution in line with the Polya problem-solving approach steps enumerated earlier, which are characterized by problem-posing, problem-solving, and conjecturing.
- The teacher goes round each group to moderate or correct the discussions.
- The teacher stops the discussions, allows the study participants to present their solutions and allows other groups to critique/support each other's solutions.
- The teacher finalises the discussions by accepting or correcting the solution proposed by the study participants and give more detailed explanation of the problem(s) before introducing another problem to be solved by the study participants.

Step 5

Teacher guides learners to solve more related problems at this level.

Step 6

Assessment- Home work

The teacher gives homework to learners:

Platinum mathematics textbook (Grade 11).

Memorandum: Teachers guide, platinum mathematics, Grade 11.

Learners were mandated to give reasons for each statement made, as well as, a brief explanation of every reason, given for every statement made.

Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work randomly among themselves, to be marked again. Finally, the teacher marked each individual learner's work (Appendix G). The disparities in marks were then discussed firstly, among group members and secondly, with the teacher (see section 2.7 of Chapter 2).

Step 7

Remarks by the teacher on lesson conducted: ………………………………………….

LESSON PLAN FOR LESSON 4

TOPIC: Circle geometry	**Date**: 7-11/03/2022	**Class**: Grade 11 B
Subject: Mathematics	**Educator**: Abakah F	**Duration**: 6 periods (week four - 45 minutes per period)

SUB- TOPICS:

Period 1- Educating learners on the thinking skills, thinking processes and dispositions to be taught at this level.

Periods 2, 3, 4, 5 & 6 - Solving problems relevant to this level, in line with Polya's and the infusion approaches.

TEACHING AND LEARNING MATERIALS

- Grade 11 Classroom mathematics textbook
- Grade 11 Platinum mathematics textbook

RELEVANT PREVIOUS KNOWLEGDE

The learner is able to:

- Solve problems at GD 3 mental construction level based on the generated preliminary GD3.
- Master the corresponding thinking skills at GD3, corresponding processes and dispositions which enhance these thinking skills.

LEARNING OBJECTIVES

By the end of the lesson, the learner will be able to:

- Attain GD 4 mental construction level based on the generated preliminary GD4
- Master the corresponding thinking skills at GD4: blending generating of ideas in GD2, with assessing reasonability of ideas in GD 3, an'd engaging students in metacognitive reflections, corresponding processes (decision-making and problem solving), and dispositions (making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time).

INSTRUCTIONAL OBJECTIVES

The learner will be able to:

- Apply his/her knowledge to solve problems and tasks at GD 4 mental construction level based on the generated preliminary GD4, and corresponding thinking skills at GD4.

DEVELOPMENT

Step 1- Introduction

The teacher guides learners to revise the work done in lesson three, in a summary form.

Step 2

The teacher introduces:

- The thinking skills to be taught at this level are: blending generating of ideas in GD2, with assessing reasonability of ideas in GD 3; engaging students in metacognitive reflections via causal explanations, predictions, generalizations, reasoning by analogy and by making deductions through conditional and categorical reasoning (Swartz & Reagan, 2018).
- Corresponding processes: decision making and problem solving; and dispositions (making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time). Decision-making as a thinking process at this level is achieved by adhering to the following prompts and question guidelines: (1) what are my options? (2) What are the likely consequences of these options? (3) What's the best option in the light of the consequences? (Swartz & Reagan, 2018).
- The teacher educates learners on the need to master these thinking skills with the corresponding processes and dispositions. In addition, the teacher educates learners on how these thinking processes and dispositions, can enhance their thinking.

Step 3

The teacher guides learners to depend on their knowledge of lesson two, to solve problems which require indirect and non-routine application of circle geometry knowledge, during class discussions, in line with Polya's approach, which is characterized by problem-posing, problem-solving, and conjecturing. The procedure for the class discussion phase is elaborated as follows (adapted from Abakah, 2019):

- The teacher gives a leading question (in the case of a new concept) or write a problem to be solved on the board (in the case of continuation of a previous concept), by implementing the questioning approach at this level: asking higher-order questions.
- The study participants start to discuss the solution in line with Polya's problem-solving approach steps enumerated earlier. The teacher goes round each group to moderate or correct each group's discussions.
- The teacher stops the discussion and allow the study participants to present their solutions and allow groups to critique each other's solutions.

- The teacher finalises the discussions by accepting or correcting the solution proposed by the study participants and gives more detailed explanation on the problem(s) before introducing another problem to be solved, to the study participants.

Step 4

The teacher guides learners to solve more related problems at this level.

Step 5

Assessment- Home work

The teacher gives homework to learners:

Platinum mathematics textbook (Grade 11).

Memorandum: Teachers guide, platinum mathematics, Grade 11.

Learners were mandated to give reasons for each statement made, as well as, a brief explanation for every reason, given for every statement made.

Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work randomly among themselves to be marked again. Finally, the teacher marked each individual learner's work (Appendix G). The disparities in marks were then discussed firstly, among group members and secondly, with the teacher (see section 2.7 of chapter 2).

Step 6

Remarks by the teacher on lesson conducted: ………………………………………….

4.14 Issues of validity and reliability

(1) Proposed problem-solving instructional approach

(i) Validity

Face validity test was conducted on the proposed problem-solving instructional approach. Strategies to ensure validity are normally conducted on a research work to test if the items of the instrument appear to be relevant, well structured, and mathematically correct. In view of the above, 3 mathematics education researchers: 2 MSc (Mathematics Education) graduates and 1 PhD (Mathematics Education) graduate whose research specializations were in line with cognition and problem-solving skills in geometry, served as judges for the validation process. The face validity form was issued to each of the 3 judges for this purpose. They were to rate the instrument as either: *poorly structured, moderately structured, well-structured* or *very well structured*. 2 of the judges rated the instrument as *very well structured,* while 1 rated the instrument as *well-structured.* His reservations about the instrument was noted and in collaboration with the two other judges, a concerted effort was made to amend the instrument accordingly, until all judges declared the instrument as valid enough to be used for the research study.

(ii) Reliability

The proposed problem-solving instructional approach was piloted to test its reliability (see prototype 3). This was relevant because the researcher wanted to measure the instrument's effectiveness before it could be used for the main study, hence, the final version of the proposed IPAC model was tested at three high schools. The unanimous consistent responses of the study participants from the three high schools were noted and the level of consistency of results were determined. The consistent results established the reliability of the proposed problem-solving instructional approach.

(2) Standardized test

(i) Validity of Standardized Test

Face validity and content validity tests were carried out on the standardized test instrument. As mentioned earlier, face validity is conducted to test if the items of an instrument appear to be relevant, well structured, and correct. Content validity was conducted to test if the items of the instrument are within the appropriate range of the learners to whom the instrument was administered to, that is, in terms of curriculum content area, grade level, age group, and others.

For this purpose, 6 mathematics instructors - 3 experienced FET mathematics teachers, 2 district subject specialists and 1 mathematics lecturer at a university did a content validity process on the instrument. Face validity and content validity forms were issued to each official to carry out the process. (Please see Appendix J for face validity and Appendix K for content validity). All responses from these officials were noted; the disparities in their responses were then discussed among themselves until a resolution was reached. This resulted in items of the instrument which were considered irrelevant being taken out and other items, considered germane to the study, but were not part of the instrument, to be included.

(ii) Reliability of Standardized Test

McMillan & Schumacher, (2014), define a reliability test as "'the consistency of measurement - the extent to which the results are similar over different forms of the same instrument or occasions of data collection". The duo added that it is the extent to which measures are free from error. This means that if an instrument has few errors, then it is reliable, and vice versa.

To ensure reliability of the standardized test, the test-retest reliability test technique was employed to test the consistency of scores over time (McMillan & Schumacher, 2014). It was applied on 25 Grade 12 mathematics learners who had studied circle geometry in schools which were not involved in the study prior to the commencement of the pilot studies. The standardized tests were administered to these learners and about a week afterwards, the re-test was conducted. The items of the original test were re-arranged to constitute the items of the re-test; the date for conducting the re-test was never made known to the learners. Learners' responses to the items of the instrument were compared to determine if consistent responses were given by the study participants for each item in both test. The regression line of best fit was drawn on a scatter diagram designed from the test-retest scores to determine the degree of strength between the two variables. Also, the Cronbach's alpha coefficient was calculated (0.72), which indicated that the two variables were reliable.

3. Questionnaires

(i) Validity

The developed rubric/questionnaires were subjected to face validity. Three university mathematics education lecturers, whose research interests are in line with cognition and problem solving were engaged to help in validating these instruments (see Appendix L). They were mandated to rate the instrument on how appropriate it was structured using the responses: *not structured appropriately*, *fairly structured appropriately*, and *adequately structured appropriately*. Two of the judges raised concerns about the questionnaire for Teachers/HODs, while one raised concerns about the questionnaire for study participants. This resulted in the researcher making relevant adjustments and corrections on each of the two instruments. The modified instruments were then returned to the judges for final moderation. The items on the instruments were unanimously approved and declared as appropriately structured, before they were used for this study.

(ii) Reliability

The developed questionnaires were checked for reliability as required. For this purpose, 5 mathematics instructors comprising of the three judges who were used for the validity test, one subject specialist, and one experienced mathematics educator, were engaged for this process. These 5 judges work at different locations and institutions, far apart from each other. They were mandated to establish if the two questionnaires can serve their desired purposes. Firstly, the consistency of responses from the judges established that the instruments can be implemented to achieve the desired objectives. Secondly, the unanimous approval by the judges gave a clear indication that the questionnaires were reliable enough to achieve the desired results.

4.15 Pilot studies

The prototypes, as elaborated earlier constituted the pilot studies for this research study. The purpose and procedures for conducting each prototype have been discussed earlier in this chapter. The researcher wish to remind readers of this research report that the research field for the main study was in an education district different from the education districts in which the other prototypes were conducted, in order not to compromise the study.

4.15.1 Pilot Studies Procedure

As mentioned earlier, the pilot studies were conducted in Grade 11 mathematics classes. The researcher presented the instructional approach, instead of the respective mathematics teacher in each school selected for each of the pilot studies. This was necessary as the researcher could not travel to each of the schools, prior to conducting the pilot study, to train each mathematics teacher at these schools on how to implement the instructional approach, due to time and financial constraints.

When conducting each pilot study, firstly, the expected classroom settings and expected classroom conducts were made known to the learners. Secondly, the researcher educated the learners about Polya's instructional approach. After which, the implementation of the proposed problem-solving instructional heuristic approach followed. All relevant data were collected for each of the pilot studies conducted and the analysed data of each of the pilot studies conducted, were compared with each other.

4.16 Data Analysis Procedure

As earlier stated, this research study followed a mixed-method research design, hence, the qualitative and the quantitative data were analysed separately. The data analysis from the quantitative and qualitative data were then consolidated as one; from this, more meaningful and valid conclusions were made from the analysed data.

The same set of data were collected from both the experimental and the control groups. The data collected from each group were analysed separately and the results from each group were then compared. This enabled the researcher to determine any similar/dissimilar data analysis results from the two groups. This enabled the researcher to search for patterns in data, determine the correlation that existed between occurrences of events, activities and outcomes of proceedings at the research sites. This, according to (Stakes, 1995) enables a researcher to make meaningful valid conclusions and recommendations. This, as well, assisted the researcher to judge the efficacy of the proposed problem-solving instructional approach, and the effects/influence it had on the experimental group.

Data which were homogeneous/similar were analysed together, hence, video recorded data and field notes were analysed together since both data recorded relevant happenings or occurrences or activities at the research fields. The standardized tests were marked by the researcher; they were then subjected to quantitative data analysis methods and qualitative data utilised the APOS theory analysis. They are all delineated below.

4.16.1 Quantitative data analysis procedure

Data analysis of the quantitative data was conducted two fold - inferential and descriptive statistics. Under inferential statistics, tools such as the generalized mixed effect model, was used to make inferences, generalizations, predictions and estimations on the research conducted. From this, an informed, meaningful and valid conclusions on the study were made. Under descriptive statistics, relevant tables were created, graphical representations, and statistical numerical calculations (mean, mode, median, standard deviation, among others) which were relevant to describe activities or events at the research fields, were done.

4.16.2 Qualitative data analysis procedure

Inductive analysis was applied to the qualitative data. Here the APOS theory was primarily used to analyse the mental constructions illustrated by learners' written responses to the standardized tests instruments (content analysis). Varied mental constructions, demonstrated by participants in both the experimental and control groups for each of the four standardized tests were thoroughly analysed and discussed, with reference to the preliminary genetic decomposition (PGD).

After content analysis was carried out on participants written responses, the aspects which were not considered by the researcher in the PGD were noted and consolidated to form emerging themes. These emerging themes informed the researcher to make meaningful changes to the PGD and guided its revision to form the modified genetic decomposition (MGD). The proposed MGD's were then presented as a recommendation from the conduct of this study, for future pedagogy for the concept of circle geometry.

4.17 Trustworthiness, authenticity and triangulation

4.17.1 Trustworthiness

In ensuring trustworthiness, the following variables were considered: credibility, dependability, conformability, transferability, authenticity of research data and activities (McMillan & Schumacher, 2014). These were carried out, as follows**:** (1)

credibility: to ensure credibility of the study, the researcher made sure that the research fields for pilot studies and the main study, were far apart from each other to avoid compromising the research as a result of cross research-field transfer of research information; (2) dependability: the research steps that led to the findings were spelt out and transparent; these can be located from Chapters 1 to 5 of this research report; (3) conformability: this study conformed to general research criteria. In addition, the researcher indicated procedures for checking and re-checking the data during the study. This will enable readers of this research report to ascertain and confirm the research results; (4) transferability: the researcher described the research field, study participants and the research context in detail so that it will be easy for any reader to apply or generalize the findings of this study in a similar context.

4.17.2 Authenticity

The researcher carefully perused relevant literature when conducting this research; these included - theses and dissertations conducted in the field of mathematics education on similar topics, diagnostic reports from the Department of Education and other departmental documents. Through this review, the researcher identified gaps in literatures on similar studies to the research topic, hence, an independent research was carried out, such that this study adds to the existing body of knowledge and it does not duplicate information.

4.17.3 Triangulation

In triangulating data, different strategies were used for data collection, namely, observations at research field, questionnaires, standardized tests, activity sheets, video recordings and audio recordings. Data were also triangulated by collecting them over a prolonged period of time (over one- year). To give more weight to the triangulation process, the same individuals were involved in the data collection in all aspects - different schools and different students, but same data collection procedures. An analysis of the differently collected data revealed the same trend of responses or feedbacks (McMillan & Schumacher, 2014).

4.18 Ethical considerations

4.18.1 Informed consent

From the words of Stakes (1995), "data gathering is mostly carried out at places under someone's/some people's authority, hence, etiquette rules demand that permission is requested for access to the research field to be granted". In fulfilment of the above, an ethical clearance letter from the University gave me permission to conduct this research at the selected sites (see Appendix F). Concomitantly, permission letters from the Provincial Department of Education (see Appendix P), District Department of Education, School Governing Bodies (SGBs) of schools in which this research was conducted, parents/guardians (see Appendix M), learners (see Appendix N), were all obtained.

4.18.2 Confidentiality

Anonymity was ensued by not requesting for any form of identification from the study participants. The identities of all persons who participated in this research study were not provided, hence, guaranteeing them confidentiality. Participants were assured that their responses were going to be used purely for academic purposes, and in line with the aims and objectives of the research study. They were promised that all the pictures, video recordings and audio recordings that were taken at the research fields will be kept safely by the researcher after analysis of data, over a considerable period of time in his personal archive room for reference purposes, and afterwards, they will all be destroyed.

4.18.3 Voluntary participation

As said earlier, a learner only participated after, he/she or the guardian had indicated on the learners' consent letter the willingness to participate in the research study. Learners who showed no interest of participating in this research study, therefore, were not forced to do so. This was made known to the participants verbally and on their consent forms and learners were assured that they will not be subjected to any form of punishment if they decide not to participate in this research. Participants were assured that all the tasks and standardized tests which will be conducted during this research study will not form part of their formal assessment tasks, hence, it will not negatively affect their term and/or year School Based Assessment (SBA) scores, if they decide not to partake in this research study. They were, as well, informed that they will not receive any form of payment and/or compensation if they decide to participate in this research study, but rather, they must focus on the academic benefits they will gain, if they decide to participate.

4.19 Conclusion

This Chapter elaborated on specific mechanisms that directed the conduct of this research study. It elaborated on dimensions such as: research paradigm, research design, research procedures, instrumentations, development of instruments, data collection procedures, population and sampling, validity and reliability issues that were followed in conducting this study. The researcher ensured that relevant research techniques in relation to the above parameters, were carefully considered and implemented in order to obtain data that were relevant, adequate, meaningful and valid, and could be used for data analyses. The next chapter deals with presentation of the data analysis procedures and findings of this study.

CHAPTER FIVE

ANALYSIS AND DISCUSSION OF DATA EMANATING FROM THE LESSONS

5.1 Introduction

In this chapter, relevant data for each lesson are presented. For generating data three observation sessions were carried out, during: pre-intervention, main-intervention and post-intervention stages. The classroom observations' data analyses of each of the three sessions are elaborated below. This process was necessary, as the researcher needed enough information and evidence on how mathematics teaching and learning had been, before, during and after the intervention. The researcher believes that these three domains of information will really assist to gather enough evidence to ascertain the effects the new problem-solving instructional approach, had had on the experimental group.

5.2 Pre-intervention classroom observations

Pre-intervention classroom observations were undertaken for a period of one week. This enabled the researcher to have good knowledge of how mathematics teaching and learning was carried out at the school as the researcher was always present in the substantive mathematics teacher's classroom. He continuously observed the teacher and learners as they went about their normal teaching and learning. The desks in the classroom were arranged in rows and columns with spaces left between them, with the teacher positioning himself in front of the white board and directing learners on what they should do. Below, is an exemplar of what was observed:

The teacher wrote the topic on the board. He explained the concept to the learners, solved three examples with them, and asked the learners to follow the examples to complete a class-exercise. This was captured in the researcher's observation notes as presented below -

Teacher- Did you understand the examples that we did?
Class: Yes sir
Teacher: Please follow the examples we did and do the class-work on the board
Class: Yes sir

The researcher observed that there was limited engagement and interaction between learners-and-learners and learners-and-teacher. The normal routine was that learners were first exposed to the concept; the teacher explains the mathematical concept to learners, with little or no contribution from the learners; then the teacher solved examples based on the concept introduced to them and after which exercises were given to learners as classwork and/or homework. Regurgitations of how problems were solved by the teacher were expected from the learners. Based on this evidence, the researcher ascertained that teaching and learning was usually conducted by the traditional instructional approach (Evans & Swan, 2014).

5.3 Main-intervention classroom observations

The presentations of observations for each lesson are detailed in this section. Classroom lessons were observed continuously throughout the duration of this study by the substantive mathematics teacher and the head of department for mathematics at the research field. All that they observed, were detailed in their observation notes pad and video recordings and served as a useful source of data for this study. As said earlier, lessons were conducted by adhering to the formulated lesson plans presented in chapter four. To eschew voluminous and repetitive narrations, only details of the relevant parts of these classroom observations (group presentations/interactions), which focused on the experimental group's participants are emphasized in this section, however, a synopsis of the events/activities that led to the group presentations/interactions are provided for readers' comprehension.

Finally, a discussion and analysis of classroom observations data for each lesson are presented, together with excerpts of proceedings in the classroom.

5.3.1 Presentation of observations for lesson one

Introduction

1. The researcher who played the role of the teacher mandated participants to draw and label parts of the circle. He also guided them to find the relationship between the radius and diameter.
2. The researcher introduced the thinking skills to be taught at this level: understanding and retention of ideas. This was done by reiterating to participants that: "they must not cease to learn" and "they must not learn to forget". The brain as the central processing unit in academia, requires knowledge to be stored in it, before how the knowledge can be applied to solve problems, may be accomplished. This justifies the urgency of understanding and retaining ideas.

Body of lesson

3. Teacher guided participants to know, understand and recognise circle geometry theorems, the converse of each circle geometry theorem, as well as similarities and differences that exist between the circle geometry theorems. This was done by using activity sheets during class discussions; exemplars of the activity sheets used are presented below.

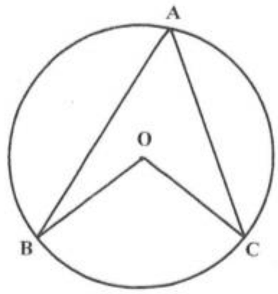
Vignette 5.1: Activity sheet 1

university of south africa

Work sheet

The relation between an angle at the centre of a circle and an angle on the circle subtended by the same arc (chord)

1. The diagram below shows a circle with centre O and A, B and C points on the circle. AB, AC, OB and OC were drawn



2. Use a protractor and measure the sizes of $B\hat{O}C$ and $B\hat{A}C$ and complete the first row of the table below.

Angle at the centre of circle vs an angle on the circle			
	$B\hat{O}C$	$B\hat{A}C$	Relation between $B\hat{O}C$ and $B\hat{A}C$
1	104°	52°	$B\hat{O}C = 2B\hat{A}C$
2	140°	70°	or
3	160°	80°	$\frac{1}{2}B\hat{O}C = B\hat{A}C$
4			
5			
6			

3. In your own words make a conjecture about the relation between an angle at the centre of a circle and an angle on the circle subtended by the same arc (chord):
Twice the angle at the circumference is equal to the angle at centre or half of angle at centre equals angle at circumference

4. Use the values of the angles shown in the presentation to complete the rest of the table.

5. Now complete the following statement:

"If an arc(a chord) of a circle subtend an angle at the centre of a circle and an angle on the circle, then twice the angle formed on the circle will be equal to the angle formed at the centre

Vignette 5.2: Activity sheet 2

university of south africa

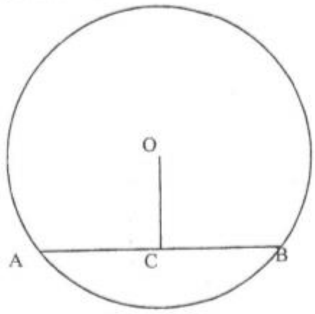
Work sheet

A line segment from the centre of a circle to the midpoint of a chord

1. Give the definition of a chord of a circle.

A chord is a line drawn from a point on the circumference of a circle to any other point on the circumference of the same circle

2. In the diagram below is a circle with centre O and a chord AB with midpoint C. Line segment OC is drawn.



3. Use a protractor, a compass and a ruler to measure the parts of the circle mentioned in the diagram below and complete the first row of the table.

Line segment from centre of circle to midpoint of chord				
	AC	BC	$O\hat{C}A$	$O\hat{C}B$
1	2,5 cm	2,5 cm	90°	90°
2	3cm	3 cm		
3	5 cm	5 cm		
4				
5				
6				

4. In your own words make a conjecture about the relation between the line segment from the centre of a circle to the midpoint of a chord of the circle:

The line segment from the circle of a circle is perpendicular to the chord at the midpoint of the chord

5. Use the values shown in the presentation to complete the rest of the table.

6. Now complete the following statement:

"If a line segment is drawn from the centre of a circle to the midpoint of a chord of the circle, then

The line segment becomes perpendicular to the chord

4. Teacher used the questioning approach for learners to recall content knowledge; this was done by asking direct procedure questions, to guide participants to solve problems.

5. Teacher guided participants to solve more related problems at this level.

Conclusion

6. Teacher gave classwork and/or homework to participants.

Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work randomly among themselves to be marked again. Finally, the teacher marked each individual learner's work (Appendix G-implementation of didactic assessment). The disparities in marks were then discussed, firstly, among group members and secondly, with the teacher (see section 2.7 of Chapter 2).

5.3.2 Presentation of observations for lesson two

Introduction

1. The teacher guided learners to revise the work done in lesson one, in a summary form.
2. The teacher introduced the thinking skills to be taught at this level: generating ideas (creative thinking). Participants were made to understand that there might be no one definite way of solving problems, hence, there is the need to broaden their thinking horizon so that different ways a problem can be solved may be conceptualised. At this stage, each group was encouraged not to only find one solution to a problem, but rather, they were mandated to find multiple ways a problem may be solved, with justifications.

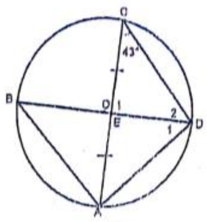
Body

3. Teacher implemented the questioning approach at this level by asking indirect procedure questions, which required application of knowledge of lesson one, to guide learners to solve problems at this level in class. This stage was characterized by the following procedures:
   - The teacher wrote a problem to be solved on the board.
   - The study participants started to discuss the solution among themselves.
   - The teacher went round each group to moderate or correct the discussions.
   - The teacher stopped the discussions and allowed the study participants to present their solutions and to criticise/support each other's solutions.

- The teacher finalised the discussions by accepting or correcting the solution proposed by the study participants; he also gave more detailed explanation on the solution before introducing another problem to be solved to the participants. One of the problems solved at this level is presented below:

Problem of the day

9.3 In the diagram, O is the centre of the circle. A, B, C and D are points on the circumference of the circle. Diameter BD bisects chord AC at E. Chords AB, CD and AD are drawn and $\hat{C} = 43°$.



9.3.1 Give a reason why $DE \perp AC$.

9.3.2 Calculate, giving reasons, the size of $\hat{B}$.

9.3.3 Prove that $\hat{E}_1 = B\hat{A}D$.

9.3.4 The length of the diameter of the circle is 28 units. Calculate the length of AB.

Figure 5.1 – Problem-solving diagram

"Boitshepo" (not her real name) from group five, did this presentation on behalf of her group.

"Eeeeee good morning class, I'm here to present the solution to "the problem of the day" for my group. Boitshepo wrote her group's solution on the board. She also explained the solution at the same time.

- *From the given question, we realised that $DE \perp AC$ based on the perpendicular bisector theorem (sub-question 9.3.1).*
- *For sub-question 9.3.2, we were asked to find the size of $\hat{B}$. From the diagram, $\widehat{B} = \hat{C}$, because these two angles are formed from the same chord AD. From the circle geometry theorems, angles from the same*

chord makes equal angles at the circumference of a circle. Based on this reason, we said that $\hat{B} = 43^0$*, since* $\hat{C} = 43^0$*.*

- *For sub-question 9.3.3, we were asked to prove that* $\widehat{E_1} = B\hat{A}D$*. From the given question, line BD is a diameter because it passes through the centre of the circle "O". Since BD is a diameter then* $B\hat{A}D = 90^0$*, based on "angle in a semi-circle theorem". Also from the diagram,* $\widehat{E_1} = 90^0$ *because* $DE \perp AC$ *based on the perpendicular bisector theorem (sub-question 9.3.1). From this point if* $B\hat{A}D = 90^0$ *and* $\widehat{E_1} = 90^0$*, then logically* $\widehat{E_1} = B\hat{A}D$.
- *For sub-question 9.3.4, we were asked to calculate the length of AB. From the given question, BD is the diameter and* $B\hat{A}D = 90^0$*. This implies that* ΔABD *is a right-angled triangle, so we can apply any of the trigonometric ratios on* ΔABD *to find the length of AB. Looking at the position of* $\hat{B}$ *on* ΔABD*, then AB is the adjacent side and BD is the hypotenuse side. So we realised that* $\cos \hat{B} = \frac{AB}{BD}$*, will be appropriate to find the length of AB. From sub-question 9.3.2,* $\hat{B} = 43^0$ *and it is given that BD=28 cm. Based on this, then* $\cos \widehat{43} = \frac{AB}{28}$*, therefore we cross-multiplied to get the length of AB= 20, 48 cm.*

This is my group's solution to the problem of the day. Thank you, class.

Boitshepo went back to her seat as she received cheers of applauds and a standing ovation for her good presentation. The teacher said to Boitshepo, "Well done Boitshepo, for the wonderful presentation."

4. The teacher guided the participants to solve more related problems, at this level.

Conclusion

5. Teacher gave classwork and/or homework to participants

Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work randomly among themselves to be marked again. Finally, the teacher marked each individual learner's work (Appendix G-implementation of didactic assessment). The disparities in marks were then discussed firstly, among group members and secondly, with the teacher (see section 2.7 of Chapter 2).

5.3.3 Presentation of observations for lesson three

Introduction

1. The teacher guided learners to revise the work done in lesson two, in a summary form

The teacher-researcher made an effort to elicit participants' prior knowledge of geometry by presenting the problem-solving activity below, to the participants.

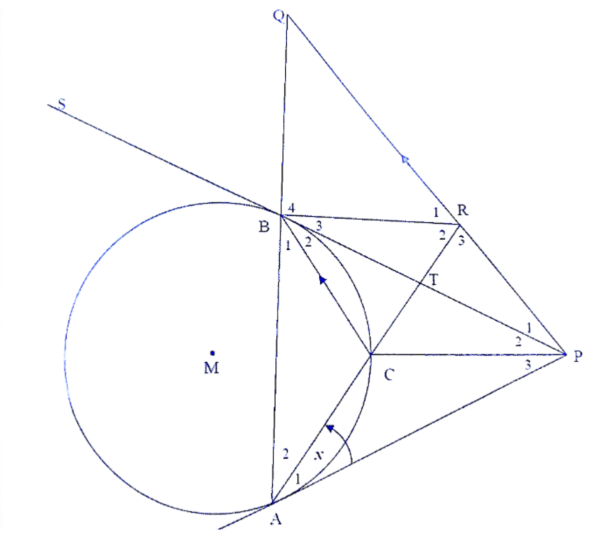


Figure 5.2: Problem-Solving diagram

Firstly, the teacher asked the participants to identify, with reasons, the lines on the diagram which are equal to each other. Some of the responses from the participants are presented below:

Participants' responses:

Mokwa : BP = AP (two tangents from the same exterior point outside a circle have the same lengths);
BC= BR & CP= RP(adjacent sides of kite BCPR are equal)
AB=AP & BR= RP(adjacent sides of kite ABRP are equal)
BT=TP & CT = TR (diagonals of parallelogram BCPR bisect each other)
BR=CP , BC=RP (opposite side lengths of parallelogram BCPR are equal)

Cynthia: BR=CP , BC=RP (opposite side lengths of parallelogram BCPR are equal)

Mpho: BP = AP (two tangents from the same exterior point outside a circle have the same lengths);
BC= BR & CP= RP(adjacent sides of kite BCPR are equal)
AB=AP & BR= RP(adjacent sides of kite ABRP are equal)
BT=TP & CT = TR (diagonals of parallelogram BCPR bisect each other)

Williams: $BR = CP\ , BC = RP\ (opposite\ side\ lengths\ of\ parallelogram\ BCPR\ are\ equal)$
AB=PA & RB= RP(adjacent sides of kite ABRP are equal)
Mabilo: $BR = CP\ , BC = RP\ (opposite\ side\ lengths\ of\ parallelogram\ BCPR\ are\ equal)$

Kars: BP = AP (two tangents from the same exterior point outside a circle have the same lengths);
BC= BR & CP= RP(adjacent sides of kite BCPR are equal)
AB=AP & BR= RP(adjacent sides of kite ABRP are equal)
BT=TP & CT = TR (diagonals of parallelogram BCPR bisect each other)
BR=CP , BC=RP (opposite side lengths of parallelogram BCPR are equal)

Boitumelo: PC = BR = BC = RP (Lengths of rhombus are equal)

Kamo: PB = AP (two tangents from the same exterior point outside a circle have the same lengths);
BC= RB & PC= PR(adjacent sides of kite BCPR are equal)
BT=TP & CT = TR (diagonals of parallelogram BCPR bisect each other)
BR=CP , BC=RP (opposite side lengths of parallelogram BCPR are equal)

And so on…….

Secondly, the participants were asked by the teacher to identify, with reasons, all the quadrilateral shapes from the diagram.

Participants' responses:

Mpho: BRPC – Parallelogram (Opposite sides are equal)

APRB - Kite (Adjacent sides are equal)

CPRB- Kite(Adjacent sides are equal)

Mokwa: $BRPC - kite\ (Adjacent\ sides\ are\ equal)$

BRPA – kite(Adjacent sides are equal)

BRPC- Rhombus(Opposite sides are equal)

CBQP- Trapezium(One pair of opposite sides , BC &QP , are parallel)

Thami: BRCP – kite (opposite sides are equal)

BRPA – kite(opposite sides are equal)

BRPC- Rhombus(Opposite sides are equal)

BRPC- Parallelogram (Opposite sides are equal)

Jessica: RBPA- Kite(no reason)

BRPC- Square(all sides are equal)

Chesslyn : $BRPC - kite\ (Adjacent\ sides\ are\ equal)$

BRAP – kite(Adjacent sides are equal)

CPRB- Rhombus(Opposite sides are equal)

Benson: BRPA – kite(Adjacent sides are equal)

BRCP- Rhombus(Opposite sides are equal)

CBQP- Trapezium(One pair of opposite sides are parallel)

Bongani: BRPC – Rectangle (Opposite sides are equal)

CQBP- kite (Opposite sides are equal)

Andile: CBQP- Trapezium(One pair of opposite sides , are parallel)

BRPA – kite(Adjacent sides are equal)

BRPC- Rhombus(Opposite sides are equal)

$BRCP - kite\ (Adjacent\ sides\ are\ equal)$

And so on …

Thirdly, the teacher asked the participants to write the special names of the following: BP, AB, AP, AR, SBP, QRP, ACR, "BC and QP".

Participants' responses:

Tiba:	BP - line	AB- arc	AP- line
	AR- line	SBP- line	QRP- line
	ACR- line	BC- arc	BR - line
Nthabiseng:	BP - tangent	AB- chord	AP- tangent
	AR- diagonal	SBP- tangent	QRP- parallel line
	ACR- diagonal	BC- chord	BR - tangent
Andrews:	BP - tangent	AB- chord	AP- tangent
	AR- line	SBP- tangent	QRP- line
	ACR- line	BC- chord	BR - tangent
Kamo: :	BP - line	AB- chord	AP- line
	AR- line	SBP- line	QRP- line
	ACR- line	BC- chord	BR - line
Lesego :	BP - line	AB- arc	AP- line
	AR- line	SBP- line	QRP- line
	ACR- diagonal	BC- chord	BR - line
Keseokile: :	BP - tangent	AB- chord	AP- tangent
	AR- diagonal	SBP- tangent	QRP- parallel line
	ACR- diagonal	BC- chord	BR - tangent
Alicia: :	BP - line	AB- arc	AP- line
	AR- line	SBP- line	QRP- line
	ACR- tangent	BC- arc	BR - tangent
Moeketsi :	BP - tangent	AB- chord	AP- tangent
	AR- chord	SBP- tangent	QRP- parallel line
	ACR- diagonal	BC- chord	BR - tangent

And so on …

Fourthly, the teacher asked the participants to work in groups of three, to deconstruct the diagram (bring the parts of the diagram apart) then to put the pieces together, to form the complete diagram again. The two most common ways employed by the groups are presented below:

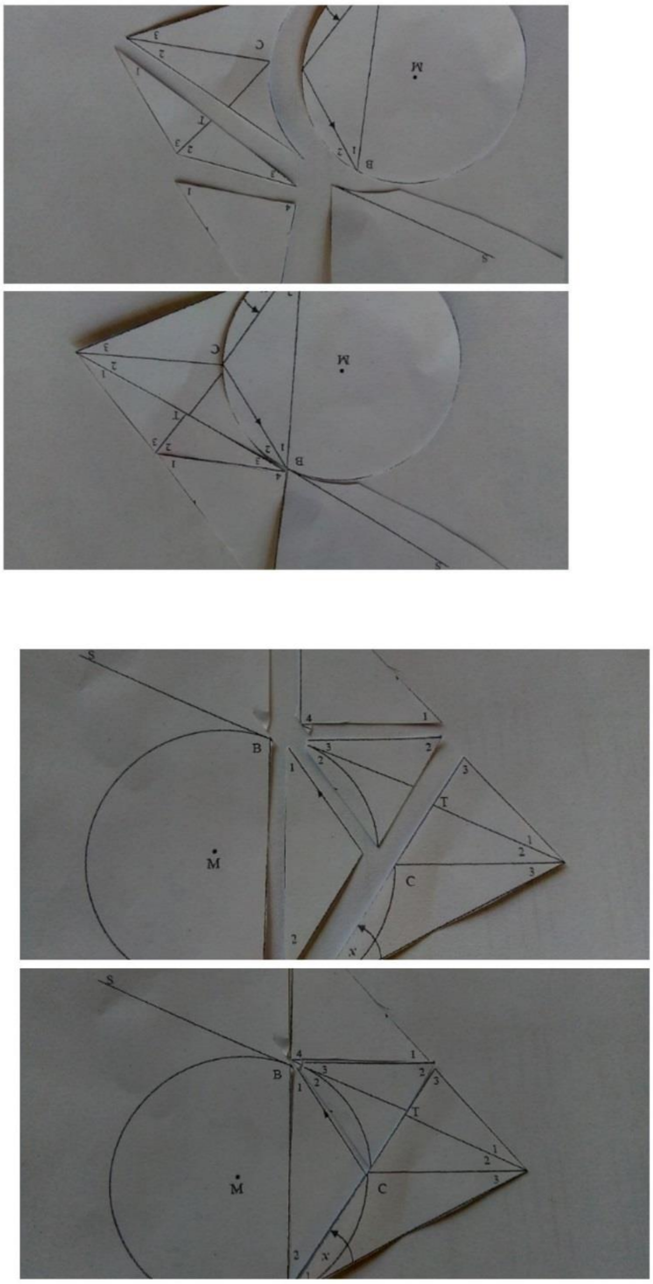


Figure 5.3: Deconstructing a diagram and reconstructing it again

2. The teacher introduced:

- The thinking skills to be taught at this level: assessing reasonability of ideas (critical thinking) via causal explanations, predictions, generalizations, reasoning by analogy and by making deductions through conditional reasoning and categorical reasoning (Swartz & Reagan, 1998).
- Corresponding processes: decision-making and problem-solving; and dispositions: making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time. Decision-making, as a thinking process at this level, is achieved by adhering to the following prompts and question guidelines: (1) What are my options? (2) What are the likely consequences of these options? (3) What is the best option in light of the consequences? (Swartz & Reagan, 1998).
- Participants were introduced to Polya's approach: understanding the problem, devising a plan, carrying out the plan, and reviewing the steps. What was expected to be done at each level, was also communicated to the learners.

Body

3. The teacher, during class discussions, guided the participants to depend on their knowledge of lesson two, to solve problems which required indirect and non-routine application of circle geometry knowledge. This was in line with Polya's approach, characterized by problem-posing, problem-solving, and conjecturing. The procedure for the class discussion phase is elaborated as follows (adapted from Abakah, 2019):

- The teacher gave a leading question by implementing the questioning approach at this level: asking higher-order questions.
- The study participants started to discuss the solution in line with Polya's problem-solving steps.
- The teacher went round each group to moderate or correct each groups' discussions.

- The teacher stopped the discussion and allowed the study participants to present their solutions and allowed groups to criticise/support each other's solutions.
- The teacher finalised the discussions by accepting or correcting the solution proposed by the participants. He then gave more detailed explanation of the problem before introducing another problem to be solved, to the participants. Exemplars of the presentations by the participants are elucidated below.

Lesson 3- presentation 1

Problem of the day

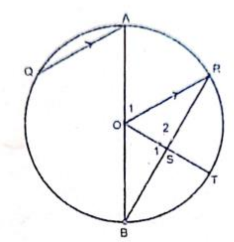
QUESTION 9

9.1 Complete the following sentence so that it is true:

A line drawn from the centre of the circle to the midpoint of a chord is….to the Chord.

9.2 In the diagram AB is a diameter of a circle with centre O. AQ and RB are two

Chords. OT is perpendicular to RB and QA∥ OR.



9.2.1 Prove that $\hat{A} = 2\hat{R}$ (5)

9.2.2 It is given that RB=8 cm and ST=2cm, determine the length of the radius(r). (5)

Figure 5.4: Problem-Solving diagram

"Lesego" (not the real name) (from group 2) and "Nthabiseng" (not the real name) (from group 3) were nominated by their group members, to make the presentation on their behalf.

Lesego commenced with the presentation; she presented the solution for question 9.2.1, while Nthabiseng presented the solution for question 9.2.2. They were both standing in front of the white board. The presentation by Lesego for question 9.2.1, is detailed below.

Lesego:

Good afternoon class. First of all, I would want to start with the problem-solving approach we used to solve the problem: understanding the problem, devising a plan, carrying out the plan, and look back on your solution.

(i) Understanding the problem

From the given problem, we understood that "O" is the centre of the circle, QA∥OR; Q, A, R, B, T are points on the circle, OA=OR=OE (equal radii).

(ii) Devising a plan

We brainstormed and shared ideas among ourselves to solve the problem by following the question prompts below:

(a) What are our options?

From the given diagram, we could only identify one option to solve this problem (9.2.1 & 9.2.2).

(b) What are the likely consequences of these options?

We realised that the option that we identified can meaningfully and appropriately solve the problem.

(c) What are the best option in the light of these consequences?

With reference to the diagram, we could identify one option which can best solve the problem (9.2.1 & 9.2.2).

(iii) Carrying out the plan

Lesego wrote the solution on the board before explaining

9.2.1 OR=OB (equal radii)

$\hat{R} = \hat{B}$ (Isosceles ΔORB)

$\Rightarrow \widehat{O_1} = 2R$ (Sum of two interior angles of a triangle= opposite exterior angle)

$\therefore \hat{A} = 2R$ (Alternate angles, QA∥OR)

Explanation

Lesego explained her group's solution to the class

"O" is the centre of the circle. So OR and OB will be the radius, which are equal. So ΔORB *will be isosceles triangle, since the lengths of two sides are equal, so* $\hat{R} = \hat{B}$*. From here, we had a reason to say that* $\widehat{O_1} = \hat{B} + \hat{R}$ $(\hat{B} = \hat{R})$*, that is sum of two interior angles of a triangle is equal to the opposite exterior angle). So* $\widehat{O_1} = \hat{R} + \hat{R}$*,* $\therefore \widehat{O_1} = 2\hat{R}$*. Since* $\widehat{O_1} = 2\hat{R}$*, then* $\hat{A} = 2\hat{R}$ *since* $\hat{A}$ *and* $\widehat{O_1}$ *are alternate angles which are equal, QA∥OR.*

After Lesego was done with the presentation for question 9.2.1., she handed over the white board marker to Nthabiseng to continue with the presentation for question 9.2.2. The presentation by Nthabiseng for question 9.2.2, is elaborated below.

Nthabiseng:

Hmmmmm good afternoon class, I will continue with the presentation for question 9.2.2. (Nthabiseng wrote the solution on the board before explaining)

From question 9.2.2, it is given that RB= 8cm, BS=SR=4cm (perpendicular bisector theorem)

Taking ΔOSR*, by Pythagoras theorem:*

$OR^2 = OS^2 + SR^2$............................(1)

$OT = OS + ST$

But $OT = OS + 2$

$\therefore OS = OT - 2$..........................(2)

Putting (2) into (1)

$OR^2 = (OT - 2)^2 + SR^2$

But $OR = OT = r$ *(equal radii)*

$r^2 = (r - 2)^2 + 4^2$

$r^2 = r^2 - 4r + 4 + 16$

$4r = 20$

$\therefore r = 5cm$

Explanation

Nthabiseng explained her group's solution to the class

From the given diagram, it can be observed that the problem is based on the "perpendicular bisector theorem", so $OT \perp BR$ *and BS=SR. so if BR=8cm, then BS=SR= 4cm. This implies that* ΔOBS *and* ΔOSR *are right-angled triangles, hence, we can apply the Pythagoras theorem to any of the two triangles, which are congruent. So by Pythagoras theorem,* $OR^2 = OS^2 + SR^2$*. It can be observed from the diagram that O, S and T, are on the same line, so we can say that* $OT = OS + ST$ *. It is given in the question that ST= 2 cm, so* $OT = OS + 2$ *. At this point, we made OS the subject, so* $OS = OT - 2$*. Putting this OS into the Pythagoras theorem above, then* $OR^2 = (OT - 2)^2 + SR^2$ *.But* $OR = OT = r$ *(radius of circle), so* $r^2 = (r - 2)^2 + 4^2$ *. From this point we expanded, simplified and solved for* $r = 5cm$.

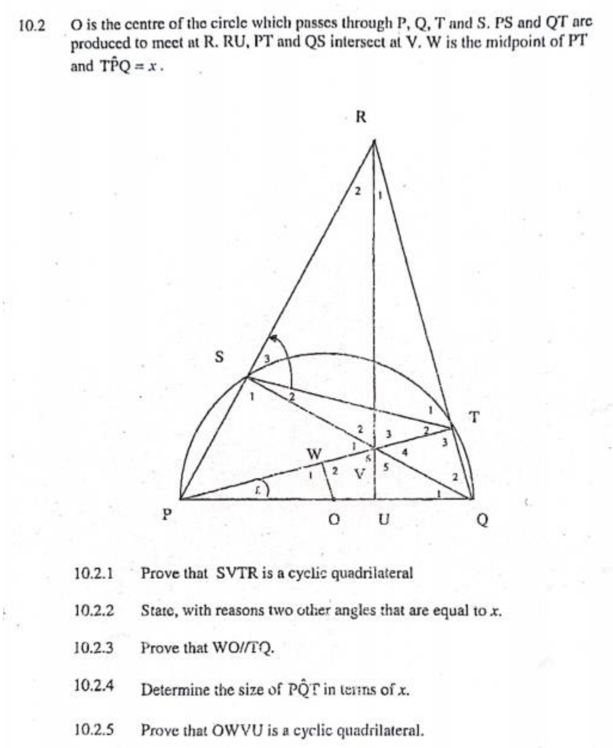
(iv) Looking back at the solution

For both questions 9.2.1 and 9.2.2, with reference to the diagram, we could observe that the solutions were meaningful and reasonable in the context of the given question. So there were no issues raised by any of the group members.

The teacher said to Lesego and Nthabiseng, "Thank you and well done for your good presentation", amid cheers of applaud as they returned to their seats.

Lesson 3 – presentation 2

<u>Problem of the day</u>


10.2 O is the centre of the circle which passes through P, Q, T and S. PS and QT are produced to meet at R. RU, PT and QS intersect at V. W is the midpoint of PT and $T\hat{P}Q = x$.

10.2.1 Prove that SVTR is a cyclic quadrilateral

10.2.2 State, with reasons two other angles that are equal to x.

10.2.3 Prove that WO//TQ.

10.2.4 Determine the size of $P\hat{Q}T$ in terms of x.

10.2.5 Prove that OWVU is a cyclic quadrilateral.

Figure 5.5: Problem-Solving diagram

This was presented by Keseokile (not her real name) from group six, and her presentation is given below.

Good afternoon class, our problem-solving approach is:

- *Understanding the problem*
- *Devising a plan*
- *Carrying out the plan*
- *Looking back at your solution*

(a) Understanding the problem

From this given question, we understood that "O" is the centre of the circle; P, S, T and Q are points on the circumference of the circle; SR and RT are tangents which are equal to each other.

(b) Devising a plan

As a group, we brainstormed and solved the given problem by following the question prompts below:

(i) What are my options?

Through our discussions, we identified 3 options for solving this question:

- *Option one requires us to prove that opposite interior angles of this quadrilateral are supplementary.*
- *Option two requires us to prove that a line in the quadrilateral subtends equal angles on its side.*
- *Option three requires us to prove that the interior angle of the quadrilateral is equal to the opposite exterior angle.*

(ii) What are the likely consequence of these options?

From two of the options, we could reasonably find a meaningful answer to the question under consideration.

(iii) What is the best option in the light of these consequences?

As a group, we realised that two out of the three identified options can best solve the problem.

(c) Carrying out the plan

Keseokile wrote the solution on the board before explaining

Option 1:

$\widehat{S_1} = 90^0$ *(angle in semi-circle)*

$\widehat{S_1} + R\hat{S}V = 180^0$ *(sum 0f angles on straight line)*

$90^0 \quad + R\hat{S}V = 180^0$

$\therefore \quad R\hat{S}V = 90^0$

Also, $\widehat{T_3} = 90^0$ *(angle in semi-circle)*

$\widehat{T_3} + R\hat{T}V = 180^0$ *(sum 0f angles on straight line)*

$90^0 \quad + R\hat{T}V = 180^0$

$\therefore \quad R\hat{T}V = 90^0$

$R\hat{S}V \; + \; R\hat{T}V = 90^0 + 90^0$

$= 180^0$

Since opposite interior angles of SVTR are supplementary, then SVTR is a cyclic quadrilateral.

Explanation

Keseokile explained her group's solution.

PQ is a diameter since it passes through the centre of the circle. But we know that a diameter makes an angle of 90^0 *at the circumference of the circle, so* $\widehat{S_1} = 90^0$. *It can be seen from the diagram that* $\widehat{S_1}$ *and* $R\hat{S}V$ *are angles on a straight line which are supplementary, hence,* $R\hat{S}V = 90^0$. *From the same reason above,* $\widehat{T_3} = 90^0$ *and* $R\hat{T}V = 90^0$. *From this point, it can be seen that* $R\hat{S}V + R\hat{T}V = 90^0 + 90^0 = 180^0$. *Based on this, we concluded that opposite interior angles of SVTR are supplementary, so this confirms that SVTR is a cyclic quadrilateral.*

Option 2

Keseokile wrote the solution on the board before explaining.

Let the point of intersection of RU and ST be 'M'.

Considering ΔSMV *and* ΔRMT

$S\hat{M}V = R\hat{M}T$

$\widehat{R_1} = \widehat{S_2}$

$\widehat{V_2} = \widehat{T_1}$

ΔSMV /// ΔRMT, $\angle\angle\angle$/*AAA*

Since $\widehat{R_1} = \widehat{S_2}$ *and they are both formed from the same line VT, then SVTR is a cyclic quadrilateral. Also,* $\widehat{V_2} = \widehat{T_1}$ *and they are both formed from the same line SR, so SVTR is a cyclic quadrilateral.*

Explanation

Keseokile explained her group's solution to the whole class.

Considering triangles Δ*SMV and* Δ*RMT, it can be observed that they are of the same shape and* ΔRMT *is an enlargement of* ΔSMV. $S\hat{M}V = R\hat{M}T$; $\widehat{R_1} = \widehat{S_2}$ *and* $\widehat{V_2} = \widehat{T_1}$. *So we can say that the two triangles are similar. It can also be observed that* $\widehat{R_1}$ and $\widehat{S_2}$ *come from the same line VT. Also,* $\widehat{V_2}$ *and* $\widehat{T_1}$ *come from the same line SR. From the above, we can say that SVTR is a cyclic quadrilateral since one of the three conditions for cyclic quadrilateral is satisfied.*

(iii) Looking back at our solution

By looking again to our solution, we realised that our solution is meaningful and appropriate in the context of the question. Also, simple mathematical errors were corrected.

The teacher, together with other study participants and the two observers (substantive mathematics teacher and HOD), all clapped hands for Keseokile in appreciation for her hard-work and good presentation, as she went back to sit down.

(4) Teacher guided participants to solve more related problems, at this level.

Conclusion

(5) Teacher gave classwork and/or homework to participants

Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work randomly among themselves to be marked again. Finally, the teacher marked each individual learner's work (see Appendix G - implementation of didactic assessment). The disparities in marks were then discussed, firstly, among group members and secondly, with the teacher (see section 2.7 of Chapter 2).

5.3.4 Presentation of observations for lesson four

Introduction

1. The teacher guided participants to revise the work done in lesson three, in a summary form.
2. The teacher introduced:

- The thinking skills to be taught at this level: blending generating of ideas in GD2, with assessing reasonability of ideas in GD 3, and engaging students in metacognitive reflections via causal explanations, predictions, generalizations, reasoning by analogy and by making deductions through conditional reasoning and categorical reasoning (Swartz & Reagan, 2018).
- Corresponding processes: decision-making and problem-solving; and dispositions: making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time. Decision-making, as a thinking process at this level was achieved by adhering to the following prompts and question guidelines: (1) What are my options? (2) What are

the likely consequences of these options? (3) What is the best option in the light of the consequences? (Swartz & Reagan, 2018).

Body

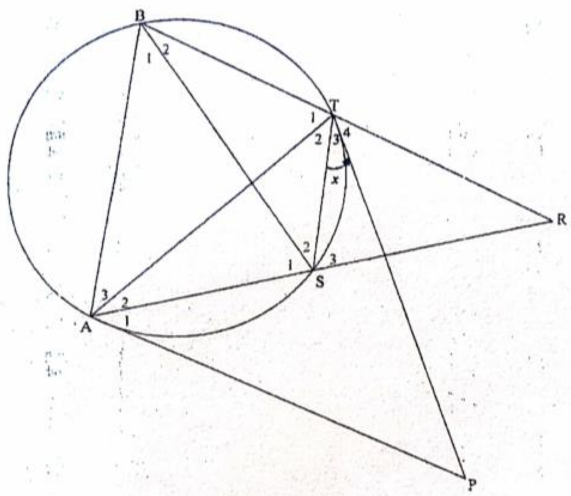
3. The teacher guided the participants to depend on their knowledge of lessons 2 &3, to solve problems which require indirect and non-routine application of circle geometry knowledge, during class discussions. This was done in line with Polya's approach, characterized by problem-posing, problem-solving, and conjecturing. The procedure for the class discussion phase is discussed in the next sections (adapted from Abakah, 2019).

- The teacher posed a leading question by implementing the questioning approach at this level: asking higher-order questions.
- The study participants started to discuss the solution in accordance with Polya's problem-solving approach steps enumerated earlier. The teacher went round the groups to moderate or correct their discussions.
- The teacher stopped the discussion and allowed the study participants to present their solutions and for the groups to criticise or support each other's solutions.
- The teacher finalised the solution by accepting or correcting the solution proposed by the study participants. He then gave a more detailed explanation of the problem before introducing others to be solved by the study participants. Exemplars of the presentations by the participants are detailed below.

<u>Problem of the day</u>

QUESTION 10

In the diagram, PA and PT are tangents to a circle at A and T respectively. B and S are points on the circle such that BT produced and AS produced meet at R and BR = AR. BS, AT and TS are drawn. $\hat{T}_3 = x$.



10.1 Give a reason why $\hat{T}_1 = \hat{A}_2 = x$.

10.2 Prove that:

10.2.1 AB || ST

10.2.2 $\hat{T}_4 = \hat{A}_1$

10.2.3 RTAP is a cyclic quadrilateral

Figure 5.6: Problem-Solving diagram

The teacher-researcher (T-R) observed that most of the groups were struggling to solve the problem of the day, hence, his role as a prompter/facilitator came to play here. He opted to discuss the given problem with the whole class, before he mandated the group discussions to proceed (Syarifuddin & Atweh, 2022). The interactions that ensued between the T-R and the participants in the classroom are presented below.

> *T-R: Class, as we learnt from Polya's problem-solving approach, understanding the question is paramount. So can you all please be attentive, concentrate on the diagram, meditate on the diagram and the questions individually for about five minutes.*
>
> *All participants become silent at this point. Each participant keenly paid attention to the diagram, trying to understand and interpret it. After five minutes had elapsed, the T-R proceeded with his interaction with the participants.*

T-R: From the given diagram, which lines are said to be equal?

Class: Line BR and line AR

T-R: Why is BR equal to AR?

Class: They are equal because it is stated in the question.

T-R: Which part of the question informs you about this?

Class: Here …..here…… here….here….here (pointing)

All participants indicated on the part of the question which says that the two lines are equal.

T-R: Good….good…good class. So if BR=AR, then what type of triangle is ΔBRA *?*

Class: Isosceles triangle

T-R: Good class. So if ΔBRA *is isosceles triangle, then what does it imply?*

Class became quiet. Each looking at the diagram carefully, before one of the study participants raised her hand ……..

T-R: Yes Keseokile……..

Keseokile: If two sides are equal in a triangle, it indicates that their opposite angles are also equal, so we can say that $\widehat{A_2} + \widehat{A_3} = \widehat{B_1} + \widehat{B_2}$.

T-R: Very good Keseokile, well done. Do you all understand that?

Class: Yes Sir.

T-R: Good. From the diagram, which angles can be said to be equal?

Class became quiet again as they studied the diagram again. After a minute, about five participants raised their hands. T-R instructed these five participants who had raised up their hands, to write their responses in their books. The T-R then went to each participant to check his/her response. The T-R saw that three of the participants had their responses right while the other two were wrong. T-R pointed at one of the participants who got it right. The interaction between them are as follows….

T-R: Alicia, which angles are equal?

Alicia: Eeeeee eeee eeeee (murmuring), $\hat{S_3} = \widehat{B_1} + \widehat{B_2}$.

T-R: Why do you say so Alicia?

Alicia: $\widehat{B_1} + \widehat{B_2}$ *is the interior angle and* $\hat{S_3}$ *is its exterior opposite angle. From one of the three conditions of cyclic quadrilateral, the interior angle is equal to the exterior opposite angle, so we can say that* $\widehat{S_3} = \widehat{B_1} + \widehat{B_2}$.

T-R: Excellent Alicia Excellent (he exclaimed). Let's clap our hands for her. Class can you all see that?

Class: No….No….Yes…No….Yes……………

Some participants said yes while others said no. The T-R then asked all participants who said "no" to raise their hands. They did. The T-R then went to each of them, one at a time. He explained to each of them individually until all of them understood. The T-R then proceeded…..

T-R: I hope it is clear now?

Class: Yes sir……Yes sir….Yes sir

T-R: Which other angles are equal?

Class became quiet once more. All looking at the diagram once more. After a few seconds, more than half of the study participants raised their hands. T-R then instructed the participants who had raised up their hands, to write their responses in their book. The T-R then went to each participant to check his/her response. The T-R saw most responses were correct, except for one. The T-R pointed at the participant who got it wrong, Williams, and one of the participants who got it right Lesego. The T-R asked Lesego to explain to Williams. The interaction between them was as follows.

Lesego: Williams can you see that TS is a chord and TP is tangent?

Williams: No I cannot see….

Lesego walks to Williams to show and explain to him. After Williams had understood what Lesego was saying, she then continued with her interaction with Williams………

Lesego: Looking at chord *TS and tangent TP, can you see that the angle between them is* $\hat{T}_3$?

Williams: Yes, I can see clearly

Lesego: Okay. So $\hat{T}_3$ is the angle between a tangent and a chord. Which interior angle is opposite to $\hat{T}_3$?

Williams became quiet for more than two minutes. Lesego realised that Williams was struggling, so she walked to him once again to show and explain. Lesego then continued…….

Lesego: can you now see that $\hat{T}_3$? *is opposite to the interior angle* $\widehat{A_2}$?

Williams: Yes

Lesego: Good. So we say that $\hat{T}_3 = \widehat{A_2}$. From one of the circle geometry theorems- the angle between a chord and a tangent is equal to the angle in the alternate segment. This is the tangent-chord theorem.

Williams: Yes, I remember that theorem. I understand now.

T-R: Thank you Lesego. Class I hope you all understand that?

Class: Yes sir…..Yes sir….Yes sir

T-R: That is good. You can all see that this question involves properties of isosceles triangles,

properties of angles on parallel lines with a transversal, the concept of cyclic quadrilateral and the concept of the tangent-chord theorem. In the future you may follow our class discussions we did today. You must be able to identify and recognise the concepts that were put together to form the question and you think and brainstorm around it, with reference to the question you are to answer. If necessary, you may have to break the diagram into its smaller constituent parts and you must be able to put the constituent parts together to form the original diagram again. All these will help you to understand and interpret the diagram well. I hope you all learn from this. Can you please continue with your group discussions?

The T-R gave participants about 15 minutes to conjecture and discuss their solution to the question. After which the group presentations followed as described below.

Lesson 4 –presentation 1

This presentation was done by Bakang (not his real name) from group 5, for sub-question 10.2.1. His presentation is elaborated below.

Bakang:

Good morning class, our problem-solving approach is:

- *Understanding the problem*
- *Devising a plan*
- *Carrying out the plan*
- *Looking back at your solution*

(a) *Understanding the problem*

We the group members made sure that we paid attention to every detail of the question and the corresponding diagram. This included geometric terms and symbols used. These enabled us to understand the question well before we started to work on finding the solution to the question.

(b) *Devising a plan*

We the group members brainstormed to solve the question with the guidance of the question prompts below:

(i) *What are my options?*

Five options were considered by the group members.

(ii) *What are the likely consequences of these options?*

Option 1- to prove that any pair of angles are corresponding angles.

Option 2- to prove that any pair of angles are alternate angles

Option 3- to prove that any pair of angles are co-interior angles

Option 4- to use the concept of cyclic quadrilateral

Option 5- to use the concept of similarity of triangles

(iii) *What is the best option in the light of the consequences?*

By considering the given diagram, options 4 & 5 were considered to be appropriate in answering the question by my group members. It was realised that options 1 & 2 can be achieved in the context of this question together with options 4 & 5.

(c) Carrying out the plan

Option 4: Bakang wrote the solution on the board before explaining.

$\widehat{A_2} + \widehat{A_3} = \widehat{B_1} + \widehat{B_2}$ *(BR= AR, ΔBRA is isosceles, <s opposite to equal sides are equal)*

$\widehat{S_3} = \widehat{B_1} + \widehat{B_2}$ *(Exterior <= opposite interior <)*

$\therefore \widehat{S_3} = \widehat{A_2} + \widehat{A_3}$

$\Rightarrow AB \parallel ST$ *(Corresponding <s)*

<u>Explanation</u>

<u>Bakang explained his group's solution to the whole class</u>

It is said in the given question that BR=AR. It implies that ΔBRA is an isosceles triangle and angles opposite to equal sides are equal, so we can say that $\widehat{A_2} + \widehat{A_3} = \widehat{B_1} + \widehat{B_2}$. Also, the exterior angle of a cyclic quadrilateral is equal to the opposite interior angle. From this, we say that $\widehat{S_3} = \widehat{B_1} + \widehat{B_2}$. So if $\widehat{B_1} + \widehat{B_2} = \widehat{A_2} + \widehat{A_3}$ and *the same* $\widehat{B_1} + \widehat{B_2} = \hat{S}_3$*, then we can say that* $\hat{S}_3 = \widehat{A_2} + \widehat{A_3}$*, which are corresponding angles. Therefore, we concluded that AB∥ST.*

Option 5- using the concept of similarity of triangles, Bakang wrote the solution on the board before explaining.

Taking ΔRTS *and* ΔRBA

$\hat{R}$ *is common*

$R\hat{B}A = R\hat{A}B$ *(BR = AR, isosceles* $\Delta\ ABR$*)*

$R\hat{T}S = R\hat{S}T$ *(TR=RS, Equal tangents)*

But $R\hat{T}S = R\hat{B}A$ *and* $R\hat{S}T = R\hat{A}B$

$\therefore \Delta RTS$ *///*ΔRBA*,* $\angle\angle\angle$*/AAA*

$R\hat{T}S\ \&\ R\hat{B}A$ *are corresponding <s*

$R\hat{S}T\ \&\ R\hat{A}B$ *are corresponding <s,* $\therefore$ *AB∥ST.*

Considering $\Delta RTS\ \&\ \Delta RBA$*, it can be observed that they are of the same shape and* ΔRBA *is an enlargement of* ΔRTS*. So we can say that the two triangles are similar, hence, their corresponding angles are the same. From this, it was determined that* $R\hat{T}S\ \&\ R\hat{B}A$ *are corresponding angles. Also,* $R\hat{S}T\ \&\ R\hat{A}B$ *are corresponding angles. This gave us a reason to conclude that AB∥ST.*

(d) Looking back our solution

We checked and made sure that our solution presented is meaningful and there are no mistakes, to the best of our knowledge in the solutions.

T-R: "Thank you very much Bakang.

We all clapped our hands for Bakang as he went back to his seat.

Lesson 4 –presentation 2

Problem of the day: the same problem of the day used for the first presentation of lesson four, was used for presentation 2, for sub-question 10.2.2.

This presentation was done by Alicia (not her real name) from group 7. Her presentation is detailed below.

Alicia:

Hello class, our problem-solving approach is: understanding the problem, devising a plan, carrying out the plan, and looking back at your solution.

(a) Understanding the problem

We ensured that we read and understood the question before we started finding the solution to the question; we paid attention to all the geometric symbols and terminologies used in the question.

(b) Devising a plan

The group members brainstormed to solve the problem by following the guidelines below:

(i) What are my options?

In all, two options were identified by the group members

(ii) What are the likely consequences of these options?

Option 1- by logically relating one geometric concept to the other, to reach a solution to the problem.

Option 2- to use the concept of similarity of triangles

(iii) What is the best option in the light of the consequences?

Although the group members agreed that option 2 was best, we also realised that option 1 can equally, and meaningfully serve the purpose too.

(c) Carrying out the plan

Option 1

Alicia wrote the solution on the board before explaining.

$\widehat{B_2} = x$ *(Tan-chord theorem)*

$x + \widehat{T_4} = \widehat{B_1} + \widehat{B_2}$ *(Corresponding* $\angle s$ *, AB∥ST)*

$x + \widehat{T_4} = \widehat{B_1} + x$

$\widehat{B_1} = \widehat{A_1}$ *(Tan-chord theorem)*

$\therefore \ \widehat{T_4} = \widehat{A_1}$ *as required.*

Explanation

Alicia explained her group's solution to the whole class.

The angle between a tangent (TP) and a chord (TS) is equal to the angle in opposite segment, so $\widehat{B_2} = x$*. With the same reason above, we say that* $\widehat{B_1} = \widehat{A_1}$ *(chord AS and tangent AP). From the given diagram, we can say that* $x + \widehat{T_4} = \widehat{B_1} + \widehat{B_2}$*. They are equal because they are corresponding angles. Since* $\widehat{B_2} = x$*, then* $\widehat{T_4} = \widehat{B_1}$*. Also, with the knowledge that* $\widehat{B_1} = \widehat{A_1}$*, then we can conclude that* $\widehat{T_4} = \widehat{A_1}$*, as expected.*

Option 2:

Alicia wrote the solution on the board before explaining.

Let the point of intersection of TP and AR be k

Taking ΔTKR *and* ΔAKP

$T\hat{R}K = K\hat{P}A$

$R\hat{K}T = P\hat{K}A$

$\widehat{T_4} = \widehat{A_1}$

$\therefore \Delta TKR \ /// \ \Delta AKP, \ \angle\angle\angle$*/AAA*

$\Rightarrow \widehat{T_4} = \widehat{A_1}$ *as required.*

Explanation

Alicia explained her group's solution to the whole class.

Considering ΔTKR *and* ΔAKP, *it can be observed that these two triangles have the same shape. It can also be observed that* ΔTKR *can be enlarged to form* ΔAKP. *From this reason, we can say that these two triangles are similar, and similar triangles have their corresponding angles to be equal. From this we had a reason to conclude that* $\widehat{T_4} = \widehat{A_1}$.

(d) Looking back at our solution

We went over our solution to ensure that we have made no mistakes.

T-R: Wonderful…………wonderful………wonderful, Alicia.

Amid cheers of applauds for her splendid presentation, Alicia then went back to her seat with her group members.

4. Teacher guided learners to solve more problems at this level.

Conclusion

5. Teacher gave classwork and/or homework to participants.

Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work randomly among themselves, to be marked again. Finally, the teacher marked each individual learner's work (Appendix G - implementation of didactic assessment). The disparities in marks were then discussed, firstly, among group members and secondly, with the teacher (see section 2.7 of Chapter 2).

5.4: Post-intervention classroom observations

The post-intervention classroom observations followed the main-intervention ones. This was necessary as the researcher wanted evidence on after-effects that the new approach had on the experimental group. The T-R observed that the participants, the substantive mathematics teacher and HOD, continued with this new approach for their mathematics lessons. Interestingly, the procedures embedded in the new approach was becoming "habit of the mind" (Driscol, DiMatteo, Nikula & Egan, 2007). The students and the regular teachers found no need to write the procedure down as they did during the intervention; they were becoming used to it. From this evidence, the researcher trusts that with time, the new approach will be a part of them and they will be able to work with it easily. Also, the lessons were more interactive, therefore, the participants enjoyed the lessons, showed understanding and purpose. The atmosphere in the classroom indicated that participants were ready to learn and do mathematics (Chapman, 2005).

5.5 ANALYSIS AND DISCUSSIONS OF CLASSROOM OBSERVATIONS

Paul and Elder (2005, p.1), defined critical thinking as:

" that mode of thinking — about any subject, content, or problem — in which the thinker improves the quality of his or her thinking by skilfully analysing, assessing, and reconstructing it. Critical thinking is self-directed, self-disciplined, self-monitored, and self-corrective thinking. It presupposes assent to rigorous standards of excellence and mindful command of their use. It entails effective communication and problem-solving abilities, as well as a commitment to overcome our native egocentrism and sociocentrism".

For lessons 1 and 2 classrooom observations, activities, interactions, group-discusions and engagements between participants, took place. These two lessons involved participants investigating and discovering circle geometry theorems on their own, using activity sheets, under the guidance of the T-R. As said earlier, direct procedure questioning approach was implemented to ascertain participants' geometric understanding, hence, "geometric statements", were used to evaluate participants' understanding; for lesson 1, only direct recall of geometric concepts and/or theorems took place, while lesson 2 required application of work done in lesson 1. Indirect procedure questioning - simple application of circle geometry theorems was used at this level, during which participants provided "geometric statements and corresponding reasons". All these culminated in the developing of the necessary fundamental and background knowledge of circle geometry - the main instructional objective for these two lessons. Participants were able to identify and use circle geometry terminologies appropriately and could also identify appropriate properties, axioms and theorems (Abakah, 2019); this was evident from the interactions between the T-R and participants and among learners. As averred by Watson (2005) and Mason (2006), developing learners' fundamental knowledge (lessons 1 & 2) is essential as this will enable them to prepare for higher-order concepts (lessons 3 & 4).

For lessons three and four classroom observations, higher-order questioning approach were required for GD3 and GD4 questions/activities. Participants were instructed to give geometric statements, reasons for each statement made and a brief explanation (justification) for each conjectured solution. This was based on the researcher's assertion that mandating learners to only give geometric statements and reasons for lessons 3 & 4 tasks is insufficient in achieving geometric proficiency (DoBE, 2018), hence, in this study, participants were tasked to give geometric statements, reason(s) for each geometric statement and a brief explanation of the conjectured solution(s).

As proposed earlier, participants being tasked to provide explanations to geometric statements and reasons enhanced their geometric awareness, geometric understanding, and guided them to manage their cognition (William & Maat, 2020). Evidence can be found from the presentations of lessons 3 & 4, that learners, briefly, provided explanations as to how they conjectured and arrived at their solution(s); thus they were able to justify how they arrived at their solutions through the explanations they provided. This according to William and Maat (2020), enables participants to think mathematically, to reflect on their thinking, and to monitor their own thinking.

5.5.1 Themes that emerged from the classroom observations

The analysis of the classroom observations data resulted in the emergence of the following themes - interactions and engagements, group discussions, group presentations, explaining conjectured solutions and questioning prompts/guidelines; These themes are essential as they create a good platform for participants to nurture their thinking skills (Lai, 2011; Whitebread, 2010; Chiu, 2010); they unlocked their mathematical competence and enabled them to communicate mathematically. These themes, therefore ensured effective communication during the conducted lessons as evidenced in the data presented above (Ekawati *et al.*, 2019). The effective communication that was prevalent during the conduct of the lessons demonstrated that mathematics is a cognitive act (Driscol, Nikula & DePiper, 2016; Ekawati *et al.*, 2019). This was made evident in the data presented - learners were placed at the centre of the instructional process and they conjectured their own solutions to the circle geometry tasks and activities. They also had the opportunity to present their conjectured solutions on the board, so that other groups could either support or criticize the solution presented, with justifications. All these, according to Driscol, Nikula and DePiper (2016), "lightened their cognitive demands". Effective communication during lessons proved to be pivotal in achieving mathematical proficiency (Moschkovich, 2002) and nurtured relevant mathematical discussions. It also gave participants the confidence to approach and solve challenging tasks (Driscol, Nikula & DePiper, 2016; NCTM, 2001), illustrating that mathematics problem-solving is laden with logic and creativity (Mehdi, Narges, Yaftian & Shahrnazer, 2012). Each of the themes that emerged are reviewed below.

(i) Classroom interactions and engagements

Classroom interactions and engagements were evident in the classroom observations data as learners freely interacted with their peers and with their teacher (Sfard, 2008; Syarifuddin & Atweh, 2022). At one point, participants from two different groups came together to make a presentation; this can be found under lesson 3-presentation 1: "Lesego" (from group 2) and "Nthabiseng" (from group 3) came together to make a presentation. This shows how participants were encouraged to freely interact with each other, over mathematical issues, in addition to other relevant issues, like, sharing activity sheets and instructional tools (compass, protractor, ruler, among others). This created a conducive atmosphere for teaching and learning to take place, and the teacher also had the freedom to engage with participants. Enough evidence of this can be found in the presented interactions in all the lessons, especially, under lessons 3 & 4; the T-R could go to any of the groups, to moderate and to monitor their mathematical conjectures and constructions as they sought solutions to circle geometry activity tasks and non-routine problems. They could ask any manner of question that impeded their understanding enabling them to learn circle geometry concepts with understanding and meaningfully, thereby broadening their thinking horizon (Driscol, 2016). According to Oladosu (2014) participants being given the freedom to construct their solutions and meanings for circle geometry tasks and activities, motivated participants to reason and to think mathematically.

Classroom interactions gave a leeway for the T-R to reach out to struggling groups; in this way, concepts that proved difficult to a particular group to comprehend were easily addressed by the teacher; this also indicates that, the T-R was able to address individual groups' difficulties at their own learning pace, depending on the nature of learners in a group. This means the exercises and activities for some groups were 'task-accelerated' while for others (struggling groups) they were 'task-synthesized' (Woolfolk, 2014).

In addition to the above, another dimension that promoted classroom interactions was the interactive didactic assessment method. From the classroom observations, at the end of each of the four lessons, an assessment was given - classwork/homework. Firstly, each individual learner marked his/her own work. Secondly, group members exchanged their work among themselves to be marked again. Finally, the teacher marked each individual learner's work (Appendix G - implementation of didactic assessment). The disparities in marks were then discussed, firstly, among group members and secondly, with the teacher (see section 2.7 of Chapter 2). The interactions and engagements that ensued among group members and groups and with the teacher greatly assisted the latter to adequately analyse, interpret and to correct each participant's misconceptions and incorrectly written responses (Carl, 2012).

(ii) Group-discussions

The next theme to be discussed is "group-discussions". As mentioned earlier, lessons were conducted in a collaborative classroom setting. According to Phadiela (2011), "small group work creates an atmosphere for interaction which further enhances their problem-solving skills". Evidence from the observation sessions informs that participants were mandated to sit in groups of three members, and to discuss, interact, engage and work with each other. These group discussions guided participants to conjecture different ways a problem could be solved which nurtured participants' mathematical creativity, required for solving non-routine problems (Posamentier, Smith & Stepelman, 2010). Swartz and Regan (1998), maintain that this strategy creates an overall atmosphere for thinking in the classroom - a point focused upon in this study. This proved to be essential as it nurtured participants' desire to learn mathematics; their confidence to do mathematics, and to communicate mathematically (DoBE, 2018). Also, these group-discussions enhanced participants' metacognitive awareness and development (Jailani & Retnawati, 2016).

(iii) Group-presentations

The group-presentations, specifically, placed participants at the centre of the instructional process (Ekawati *et al.*, 2019) as seen in the classroom observations' data. In some instances, some groups' solutions were rejected by other groups with justifications; they were able to explain to the group whose solutions were rejected, gave reasons why their mathematical constructions were wrong, in the context of the circle geometry diagram, under consideration. Groups then presented the correct solutions and explained their solutions to the whole class. This positioned participants to be responsible for their own learning as the T-R only served as a facilitator/prompter and only intervened when necessary (Ekawati et al., 2019).

These interactions promoted participants' confidence to communicate mathematically, not only with their group members, but also, with the entire class (Driscol, Nikula & DePiper, 2016). On other occasions, some participants who were unwilling to make a presentation on behalf of their group members were motivated and encouraged by the T-R to do so, and they obliged. They were made to understand that mistakes assist development; making an effort, making a mistake and learning from the corrections are all part of the learning process. Participants were, thus, cautioned against ridiculing, teasing, joking and laughing at fellow participants over mistakes they make; doing so was considered a violation and misconduct. Each participant was assured of being protected which propelled and encouraged many participants to avail themselves for group presentations.

(iv) Explaining conjectured solutions

According to (McClure, 2014, p.3):

"At NRICH we often say you can't do maths unless you talk maths. But the quality of the talk is important. It is not simply children sharing how they did a particular calculation, but describing why and how it worked, and how their method is the

same or different to those of others. In other words, it is about giving children opportunities to use those higher-level skills of comparing, explaining and justifying".

An important theme that came up for discussion was participants "explaining conjectured solutions". The proclivity to demand explanations to know and understand the reason for taking any decision (Lai, 2011), was prioritized by the researcher in this study, hence, participants were mandated to explain their conjectured solutions. Boaler (2016, p.28), notes, "Explaining your work is what, in mathematics, we call reasoning, and reasoning is central to the discipline of mathematics". Participants being given the opportunity to present and explain their conjectured solutions enabled them to communicate their mathematical ideas and reasoning to the class. The T-R realised that this act, will encourage and guide participants to reflect on their own problem-solving procedure, mathematical constructions and their conjectured solutions. All these culminated in developing participants' metacognitive awareness (William & Maat, 2020).

Metacognitive awareness promotes problem-solving - routine and non-routine problems - but especially, in the contexts of non-routine problems (Yıldırım & Ersözlü, 2013). From the available data, participants, being encouraged to explain their conjectured solutions, inculcated in them, the practice and the habit of being responsible for their own thinking and learning (Ekawati *et al.*, 2019). To achieve this, group members, among themselves, asked constructive questions about how they can solve a given problem, what strategies they can implement to effectively solve the given problem and how meaningful and reasonable their conjectured solutions were. This procedure was necessary because the participants knew that they had to openly present their conjectured solutions and they had to also justify them by explaining openly to the entire class, how they conjectured their solutions to the given problems (Swartz & Reagan, 1998). This problem-solving process, according to Du Toit & Du Toit (2013) promotes transfer of learning.

Participants' explanations, from the data presented, highlighted how they effectively applied knowledge of previous lessons to subsequent ones; how they were able to reason, brainstorm around circle geometry concepts as well as master the similarities, differences and relationships that exist between circle geometry concepts. Participants' explanations testified that they were able to generalize, transfer and apply these circle geometry concepts to varied problems, thereby reach meaningful solutions to them. These, Driscoll (2016), maintain fosters participants' geometric thinking.

(v) Questioning prompts/guidelines

At the classroom discussion stage, Polya's approach was integrated with the infusion approach. At the "devising a plan" stage of Polya's approach, questioning prompts/guidelines were implemented to invoke and nurture participants' thinking (for lessons 3 and 4). These were*: (1) What are my options? (2) What are the likely consequence of these options? (3) What is the best option in the light of these consequences?* These questioning prompts/guidelines are thinking and/or problem-solving decision-making processes, which are central to this study (Swartz & Reagan, 1998, p.15).

Firstly, the question, *"What are my options?",* guided participants to explore possible or alternative ways a problem can be solved. Participants brainstormed, tried-and-tested suggested possibilities and adequately considered how a given problem can be solved. Secondly, "*What are the likely consequence of these options?",* guided participants to explore and experiment the mathematical implications of the proposed options of solving a given problem. That is, what the proposed solutions/options of solving a given problem requires the problem-solver, to do mathematically, with reference to the question and the given geometric figure. Thirdly, *"What is the best option in the light of the consequence?"*, directed participants to consider which of the proposed solutions/options will be meaningful and correct, with justifications, in the context of the given question and the given geometric figure. This is to say that some proposed solutions/options might be meaningful and mathematically correct, but may not be applicable in the context of the given geometric figure, hence, these proposed solutions were discarded. Also, the given geometric figure might be interpreted correctly but some proposed solutions/options might not be applicable and meaningful in the context of the given question, hence, those proposed solutions/options were also discarded. If, however, the proposed solutions/options were meaningfully justified in the context of the given question and the given geometric figure, then such solutions were accepted (Swartz & Reagan, 1998).

These questioning prompts, guidance and classroom interactions with individual participants, group members as a unit or whole class, promote metacognition, enhances conceptual learning (Zepeda, Hlutkowsky, Partika & Nokes-Malach, 2018), as well as active, meaningful learning, integrated with metacognitive learning (Hartman, 2001). Teaching participants these questioning prompts/guidance is a valuable skill that metacognitively, guides them to become more self-directed learners (King, Goodson & Rohani, 2013).

5.6 Conclusion

All the relevant events, activities, classroom presentations, group discussions that the researcher observed were presented and discussed in detail, in this chapter. The relevant research question in relation to these are presented in Chapter 9. The next chapter - 6 – will dwell on the data analysis and discussions of the standardized tests.

CHAPTER SIX

PRESENTATION OF THE RESULTS OF THE STANDARDIZED TESTS

6.1 Introduction

In this chapter, the quantitative data analysis of the effects that the new problem-solving instructional approach had on the experimental group is examined. As said earlier, the standardized tests data constituted a useful source of information for this study. Discussions of data obtained from each of the four standardized tests for both the experimental and the control group participants, as well as the composite results of scores of the four standardized tests are presented in this chapter. Quantitative data analysis techniques were used to analyse and discuss the numerical facet of the standardized tests' data. Discussed in the following sections are both the descriptive and inferential data analysis strategies that were implemented.

6.2 Descriptive statistics data analysis results

Under this section, are presentations of results for each test - 1, 2, 3 and 4 - followed by the composite result of scores for all the four tests. In presenting each set of result, firstly, a Table was used for each standardized test for participants in both the experiment and the control groups. Secondly, a graphical representation of the scores is illustrated on a bar graph, to highlight the differences in scores obtained by participants in the experiment and the control groups, for the result of each test, as well as the composite scores. Thirdly, statistical numerical calculations-mean and standard deviation were done, to highlight the strength of the experiments.

6.2.1 Presentation of results for test one

(i) Tables-presentation of scores

Table 6.1: Lesson one standardized test results (out of 50 marks)

NO.	Experimental group		Control group	
	Participant	Marks obtained	Participant	Marks obtained
1.	E_1	41	C_1	27
2.	E_2	41	C_2	34
3.	E_3	31	C_3	31
4.	E_4	41	C_4	28
5.	E_5	40	C_5	26
6.	E_6	36	C_6	23
7.	E_7	41	C_7	27
8.	E_8	40	C_8	29
9.	E_9	40	C_9	26
10.	E_{10}	34	C_{10}	35
11.	E_{11}	40	C_{11}	24
12.	E_{12}	31	C_{12}	35
13.	E_{13}	34	C_{13}	29
14.	E_{14}	39	C_{14}	30
15.	E_{15}	30	C_{15}	34
16.	E_{16}	36	C_{16}	25
17.	E_{17}	38	C_{17}	25
18.	E_{18}	42	C_{18}	31
19.	E_{19}	19	C_{19}	33
20.	E_{20}	36	C_{20}	34
21.	E_{21}	36	C_{21}	30
22.	E_{22}	40	C_{22}	26
23.	E_{23}	38	C_{23}	04
24.	E_{24}	43	C_{24}	34
25.	E_{25}	15	C_{25}	30
26.	E_{26}	40	C_{26}	31
27.	E_{27}	39	C_{27}	32
28.	E_{28}	36	C_{28}	33
29.	E_{29}	36	C_{29}	38

30.	E_{30}	40	C_{30}	32
31.	E_{31}	37		
32.	E_{32}	37		

(ii) Graphical representations-Bar graph

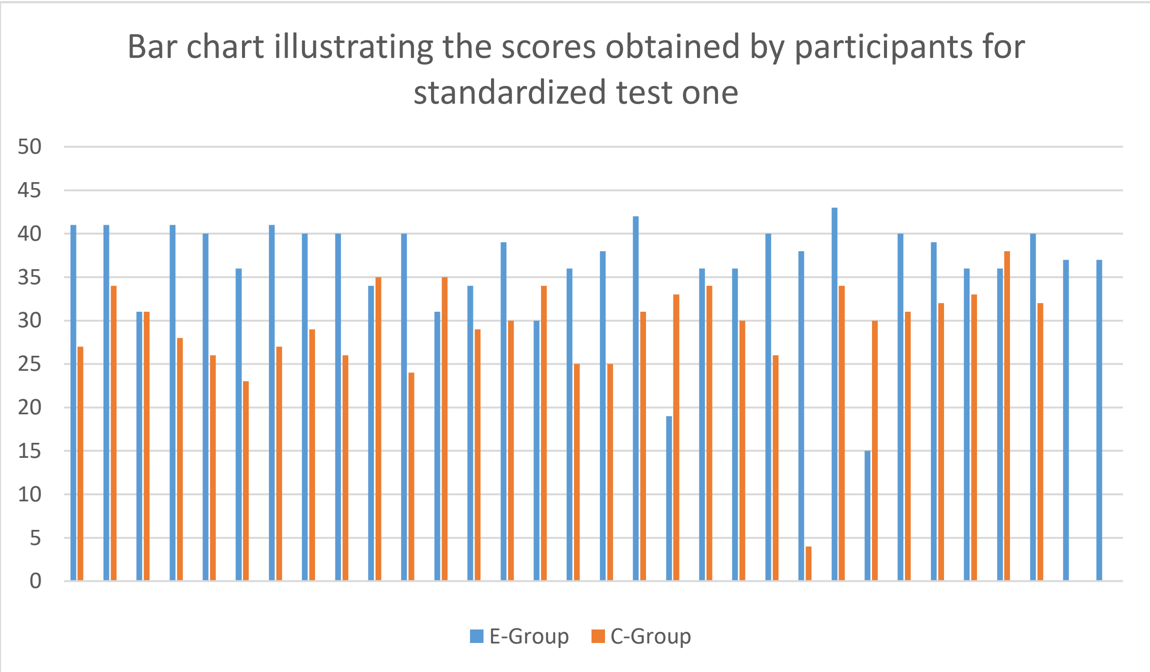


Figure 6.1: Bar graph illustrating lesson one standardized test results

(iii) Interpretation and analysis of results for test one

With reference to participants' scores for test one illustrated in Table 6.1 above, from the experimental group, the mean mark was 36, 47 and the standard deviation was 5,97. Also from the control group, the mean mark was 29, 20 and the standard deviation was 5, 97 showing that the mean mark of the experimental group was greater than the mean mark of the control group. This implied that the experimental group participants performed better than the control group participants for test one.

It was also observed that both groups recorded the same standard deviation of 5, 97, indicating that the marks obtained by participants in both groups were equally spread out from their respective calculated mean, hence, both groups had consistent scores.

6.2.2 Presentation of results for test two

(i) Tables-presentation of scores

Table 6.2: Lesson two standardized test results (out of 50 marks)

NO.	Experimental group		Control group	
	Participant	Marks obtained	Participant	Marks obtained
1.	E_1	47	C_1	47
2.	E_2	48	C_2	38
3.	E_3	47	C_3	23
4.	E_4	46	C_4	46
5.	E_5	38	C_5	37
6.	E_6	39	C_6	45
7.	E_7	47	C_7	42
8.	E_8	49	C_8	47
9.	E_9	49	C_9	46
10.	E_{10}	45	C_{10}	48
11.	E_{11}	49	C_{11}	45
12.	E_{12}	41	C_{12}	45
13.	E_{13}	44	C_{13}	24
14.	E_{14}	46	C_{14}	38
15.	E_{15}	35	C_{15}	42
16.	E_{16}	45	C_{16}	43
17.	E_{17}	49	C_{17}	42
18.	E_{18}	49	C_{18}	41

19.	E_{19}	34	C_{19}	36
20.	E_{20}	41	C_{20}	48
21.	E_{21}	39	C_{21}	45
22.	E_{22}	21	C_{22}	48
23.	E_{23}	48	C_{23}	01
24.	E_{24}	48	C_{24}	45
25.	E_{25}	35	C_{25}	42
26.	E_{26}	41	C_{26}	45
27.	E_{27}	46	C_{27}	41
28.	E_{28}	45	C_{28}	45
29.	E_{29}	43	C_{29}	40
30.	E_{30}	47	C_{30}	40
31.	E_{31}	47		
32.	E_{32}	47		

(ii) Graphical representations-Bar graph

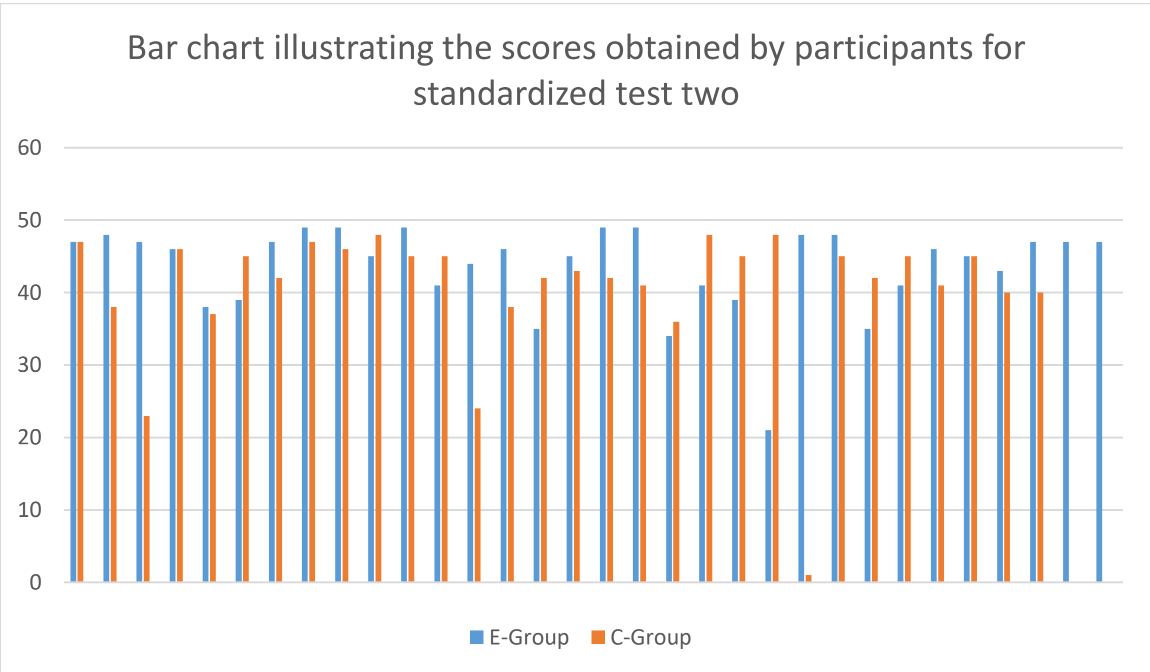


Figure 6.2: Bar graph illustrating lesson two standardized test results

(iii) Interpretation and analysis of results for test two

In relation to participants' scores for test two illustrated in Table 6.2 above, the experimental group participants had a mean mark of 43,59 and standard deviation of 5,97; participants in the control group had a mean mark of 40, 50 and standard deviation of 9, 41. This shows that the mean mark of the experimental group was greater than the mean mark of the control group, indicating that the experimental group participants performed better in test two. This was made more noticeable on the bar graph in Figure 6.2 above. The standard deviation of the experimental group was less than that of the control group. This implied that the marks obtained by participants in the experimental group were less spread out from the mean, while the marks in the control group were more spread out from the mean; thus, the marks obtained by participants in the experimental group were more consistent, than those in the control group.

6.2.3 Presentation of results for test three

(i) Tables-presentation of scores

Table 6.3: Lesson three standardized test results (out of 50 marks)

NO.	**Experimental group**		**Control group**	
	Participant	Marks obtained	Participant	Marks obtained
1.	E_1	49	C_1	04
2.	E_2	40	C_2	06
3.	E_3	36	C_3	27
4.	E_4	40	C_4	12

5.	E_5	06	C_5	16
6.	E_6	45	C_6	18
7.	E_7	43	C_7	34
8.	E_8	34	C_8	20
9.	E_9	43	C_9	-
10.	E_{10}	35	C_{10}	24
11.	E_{11}	43	C_{11}	-
12.	E_{12}	41	C_{12}	38
13.	E_{13}	36	C_{13}	22
14.	E_{14}	34	C_{14}	31
15.	E_{15}	38	C_{15}	26
16.	E_{16}	45	C_{16}	15
17.	E_{17}	40	C_{17}	-
18.	E_{18}	44	C_{18}	38
19.	E_{19}	06	C_{19}	23
20.	E_{20}	36	C_{20}	28
21.	E_{21}	40	C_{21}	-
22.	E_{22}	36	C_{22}	-
23.	E_{23}	41	C_{23}	07
24.	E_{24}	38	C_{24}	27
25.	E_{25}	10	C_{25}	-
26.	E_{26}	40	C_{26}	15
27.	E_{27}	40	C_{27}	16
28.	E_{28}	44	C_{28}	30
29.	E_{29}	35	C_{29}	28
30.	E_{30}	33	C_{30}	20
31.	E_{31}	41		
32.	E_{32}	19		

(ii) Graphical representations-Bar graph

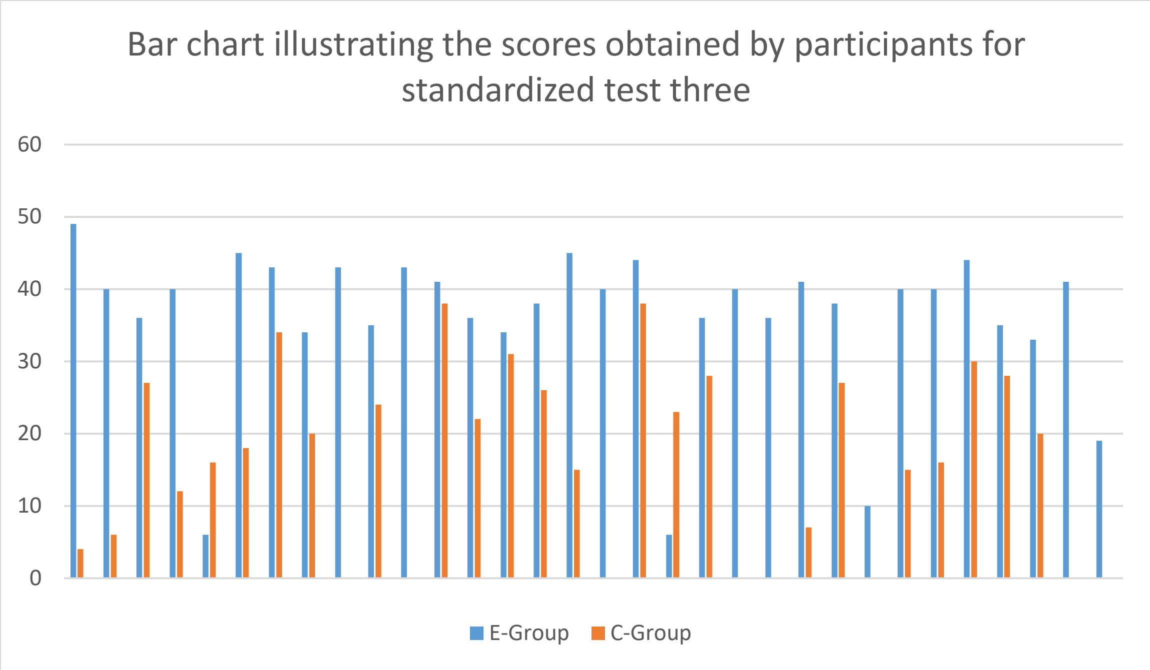


Figure 6.3: Bar graph illustrating lesson three standardized test results

(iii) Interpretation and analysis of results for test three

With reference to participants' scores for test three illustrated in Table 6.3 above, the mean mark of the experimental group was 35, 97 and standard deviation was 10, 56 and the mean mark of participants in the control group was 21, 88 and standard deviation was 9, 23. The mean mark of the experimental group was far greater than for the control group. This confirmed that the experimental group participants performed much better than the control group participants for test three according to Figure 6.3 above. The standard deviation of the control group was less than for the experimental group, implying that the marks obtained by participants in the control group were less spread out from the mean, while the opposite is true for the experimental group. The low marks obtained by participants in the control group, thus, were more consistent.

6.2.4 Presentation of results for test four

(i) Tables-presentation of scores

Table 6.4: Lesson four standardized test results (out of 50 marks)

NO.	Experimental group		Control group	
	Participant	Marks obtained	Participant	Marks obtained
1.	E_1	49	C_1	00
2.	E_2	43	C_2	-
3.	E_3	41	C_3	28
4.	E_4	44	C_4	14
5.	E_5	32	C_5	16
6.	E_6	42	C_6	14
7.	E_7	25	C_7	-
8.	E_8	45	C_8	13
9.	E_9	45	C_9	15
10.	E_{10}	43	C_{10}	21
11.	E_{11}	44	C_{11}	20
12.	E_{12}	19	C_{12}	28
13.	E_{13}	34	C_{13}	38
14.	E_{14}	43	C_{14}	15
15.	E_{15}	45	C_{15}	39
16.	E_{16}	31	C_{16}	12
17.	E_{17}	42	C_{17}	-
18.	E_{18}	47	C_{18}	37
19.	E_{19}	08	C_{19}	20
20.	E_{20}	43	C_{20}	41
21.	E_{21}	31	C_{21}	00
22.	E_{22}	20	C_{22}	21
23.	E_{23}	48	C_{23}	00
24.	E_{24}	44	C_{24}	36
25.	E_{25}	26	C_{25}	-

26.	E_{26}	45	C_{26}	37
27.	E_{27}	45	C_{27}	25
28.	E_{28}	44	C_{28}	23
29.	E_{29}	42	C_{29}	29
30.	E_{30}	40	C_{30}	19
31.	E_{31}	46		
32.	E_{32}	45		

(ii) Graphical representations-Bar graph

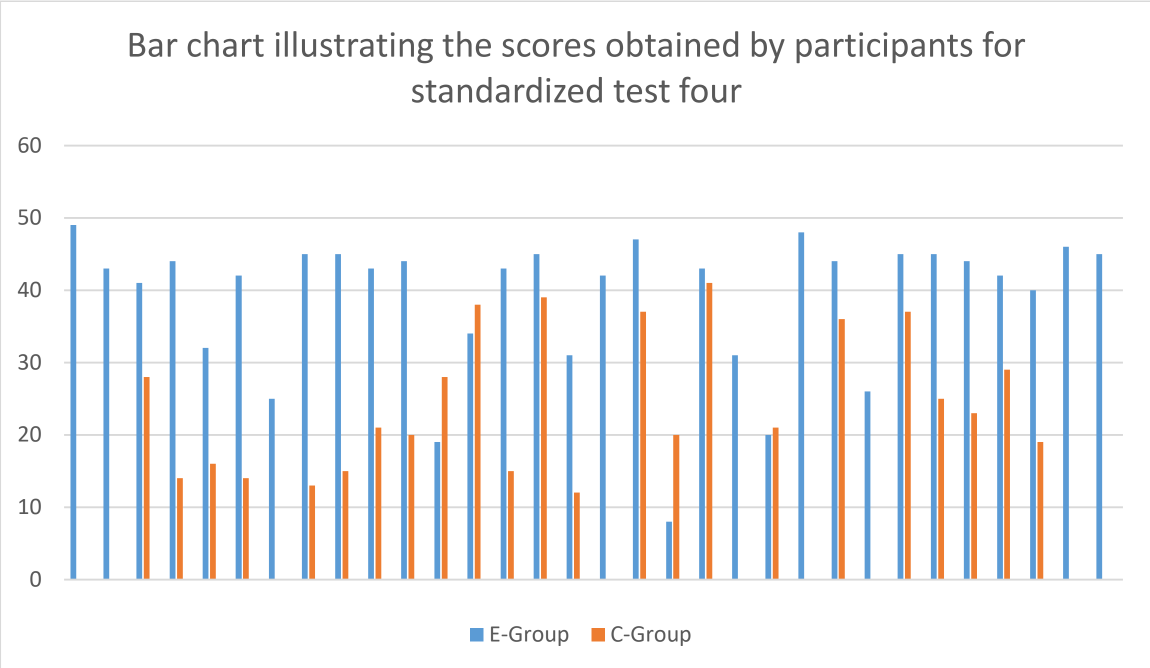


Figure 6.4: Bar graph illustrating lesson four standardized test results

(iii) Interpretation and analysis of results for test four

In relation to participants' scores for test four illustrated in Table 6.4 above, the experimental group participants' mean mark was 38, 78 and the standard deviation was 9, 67 while the control group's mean mark was 21, 58 and the standard deviation was 11, 75. The mean mark of the experimental group was far greater, as compared to the control group showing that the experimental group participants performed mch better for test four as can also be seen in the bar graph in Figure 6.4

above. The standard deviation of the experimental group was less than for the control group, indicating that the marks obtained by participants in the experimental group were less spread out from the recorded mean, while those in the control group were more spread out. This implied that the better marks obtained by participants in the experimental group were more consistent.

6.2.5 Presentation of composite result

(i) Tables-presentation of scores

Table 6.5: Composite result of the four standardized tests (out of 200 marks)

NO.	**Experimental group**		**Control group**	
	Participant	Marks obtained	Participant	Marks obtained
1.	E_1	186	C_1	78
2.	E_2	172	C_2	78
3.	E_3	155	C_3	109
4.	E_4	171	C_4	100
5.	E_5	116	C_5	95
6.	E_6	162	C_6	100
7.	E_7	156	C_7	103
8.	E_8	168	C_8	109
9.	E_9	177	C_9	87
10.	E_{10}	157	C_{10}	128
11.	E_{11}	176	C_{11}	89
12.	E_{12}	132	C_{12}	146
13.	E_{13}	148	C_{13}	113
14.	E_{14}	162	C_{14}	114
15.	E_{15}	148	C_{15}	141
16.	E_{16}	157	C_{16}	95
17.	E_{17}	169	C_{17}	67
18.	E_{18}	182	C_{18}	147

19.	E_{19}	67	C_{19}	112
20.	E_{20}	156	C_{20}	151
21.	E_{21}	146	C_{21}	75
22.	E_{22}	117	C_{22}	95
23.	E_{23}	175	C_{23}	12
24.	E_{24}	173	C_{24}	142
25.	E_{25}	86	C_{25}	72
26.	E_{26}	166	C_{26}	128
27.	E_{27}	170	C_{27}	114
28.	E_{28}	169	C_{28}	131
29.	E_{29}	156	C_{29}	135
30.	E_{30}	160	C_{30}	111
31.	E_{31}	171		
32.	E_{32}	148		

(ii) Graphical representations-Bar graph

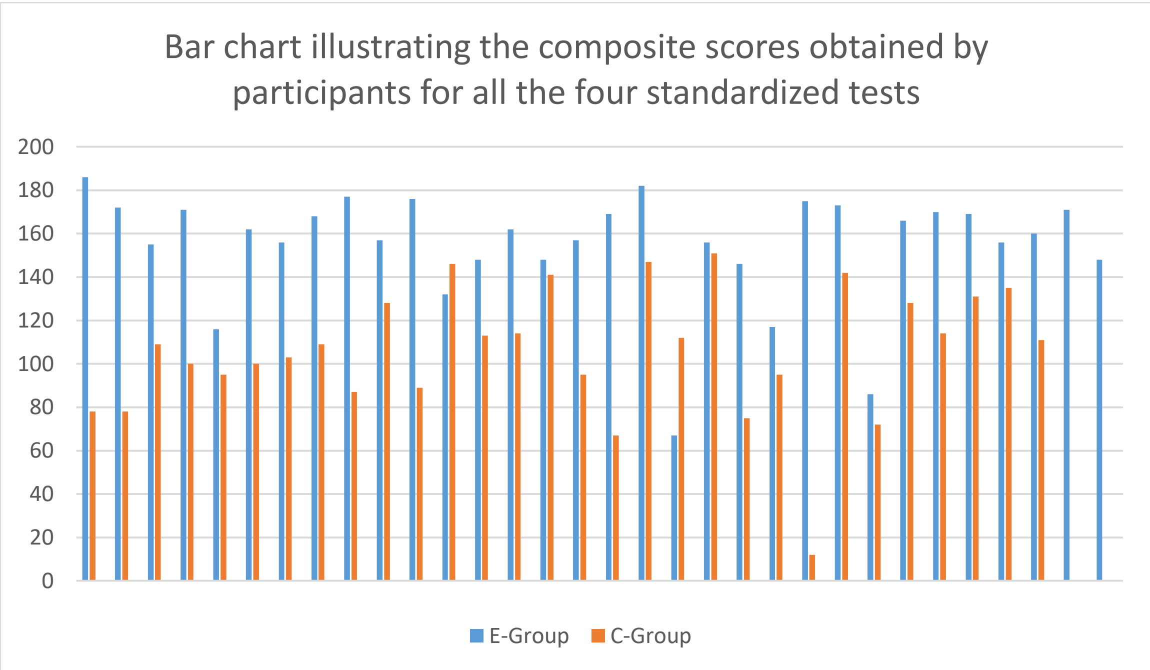


Figure 6.5: Bar graph illustrating participants' composite results

(iii) Interpretation and analysis of the composite results

With regards to participants' composite scores illustrated in Table 6.5 above, the experimental group participants recorded a mean mark of 154, 81 and standard deviation of 25, 86, while the control group recorded a mean mark of 105, 90 and standard deviation of 29.24. The mean mark of the experimental group was far greater than for the control group illustrating that the experimental group performed much better than the control group participants as shown in the bar graph in Figure 6.5 above. The standard deviation of the experimental groups was less, implying that their marks were less spread out from the recorded mean, than those obtained in the control group. The better marks obtained by participants in the experimental group were more consistent, in comparison to the marks obtained by the control group.

6.3 Inferential statistics data analysis results

The composite result presented above was used for the hypothesis test. This was done by adding the marks each participant in both the control and the experimental groups obtained in each of the four standardized tests. The researcher realised that using the composite scores for the hypothesis test might be better as the larger set of data will afford a degree of reliability to the findings, rather than using the scores for each of the four standardized tests.

6.3.1 Hypothesis test

Our main aim in this section was to apply statitsical methods to test the effectiveness of the new teaching method, hence, we sought to test the hypotheses that:

- H_o: The differences between the mean marks of treatment and controlled group is zero

- H_1: The differences between the mean marks of trearment and controlled group is not zero

In practice, for a large data set, due to the central limit theorem, marks are supposed to be, approximately, normally distributed, however, due to the aggregate nature of the marks obtained from participants' composite scores from this study, the assumption may not hold. In addition, the marks were recorded as count variables and the study utilised an independent measured controlled experiment. The result of this type of design is prone to differences between participants as a result of variations in learning abilities and barriers, thus, it was necessary to account for the effects of such nuisance factors (random effects) in the appropriate model. A statistical methodology which is able to test for the differences between two population means from a count distribution characterized by random variations is the generalized mixed effect model (Jiang, 2007). What, then, are generalized mixed effect models?

Generalized mixed effect models

Thiele and Markussen (2012) states that "generalized linear mixed models are regression models that allow researchers to choose among various distributions and link functions in order to model a wide range of types of dependent variables through linear combinations of one or multiple predictor variables (fixed effects)". Vital to these models are "random effects"; these, such as classrooms in an education study are tentative (Thiele & Markussen, 2012).
According to Stroup (2013) skewness and kurtosis values from educational research data generally deviate from the normal distribution. This is when the generalized linear mixed models become essential. This statistical model is efficient and conducive for analysing non-normally distributed data (continuous or discrete). These include - categorical data, count data, proportional response data, among others. Factors to be considered when using this model are -distribution of the data,

link function and nature of random effects, estimating the parameters and testing significance; this model may be complex in its implementation, however, it gives more detailed analysis (Stroup, 2013; Tango, 2017; Bono, Alarcón & Blanca, 2021).

Justification for using Negative Binomial Mixed Effect Model

This is an educational research involving learners in their natural classroom settings, hence, the study produced real data (quantitative), which are not normally distributed; standardized tests' data which were used for this analysis are count data, which belong to this category. Integral to count data is "over-dispersion", which generates standard errors. This implies that there exists, greater variance in the data on which this model will be applied (Stroup, 2013; Tango, 2017).

The researcher, therefore, opted for generalized mixed effect model - an appropriate statistical data analysis technique for analysing data that is not normally distributed. This was in contrast to exploring other avenues which are not directly applicable to the nature of the collected data and which will result in the data being transformed to obtain normality or using non-parametric analyses, so that classical approaches will be used (Bono, Alarcón & Blanca, 2021). The normality test was conducted on the count data to ascertain reasons if it is not a normally distributed data, hence, the mixed effect model was ideal for modelling this data.

Normality test

The generalized linear mixed effect model for count data assumes that the error distribution comes from a Poisson or the negative Binomial distribution, thus to determine the appropriateness of the data it is necessary that the possibility of the data coming from the normal distribution be ruled out, as this will invalidate the final results.

The normal quantile plot, the Jarque-Bera and the Shapiro-Wilk tests were, therefore, used in testing the normality or the non-normality of the data. From all these tests, the null hypotheses of normality is not rejected when the *p-values* < 0.005. The results of the tests are presented in Table 6.6 below, while the normal quantile plot was displayed in Figure 6.8 below.

In Table 6.6, the results of the two tests converge and support the presence of non-normality of the group marks. The Shapiro and the Jarque-Bera tests reject the null of normality (p-value<0.005). This result is confirmed by the clear deviations of the left sample quantile points against the theoretical quantile as indicated on Figure 6.8. It was, therefore, concluded that the marks are not normally distributed suggesting that a generalized linear model may be appropriate.

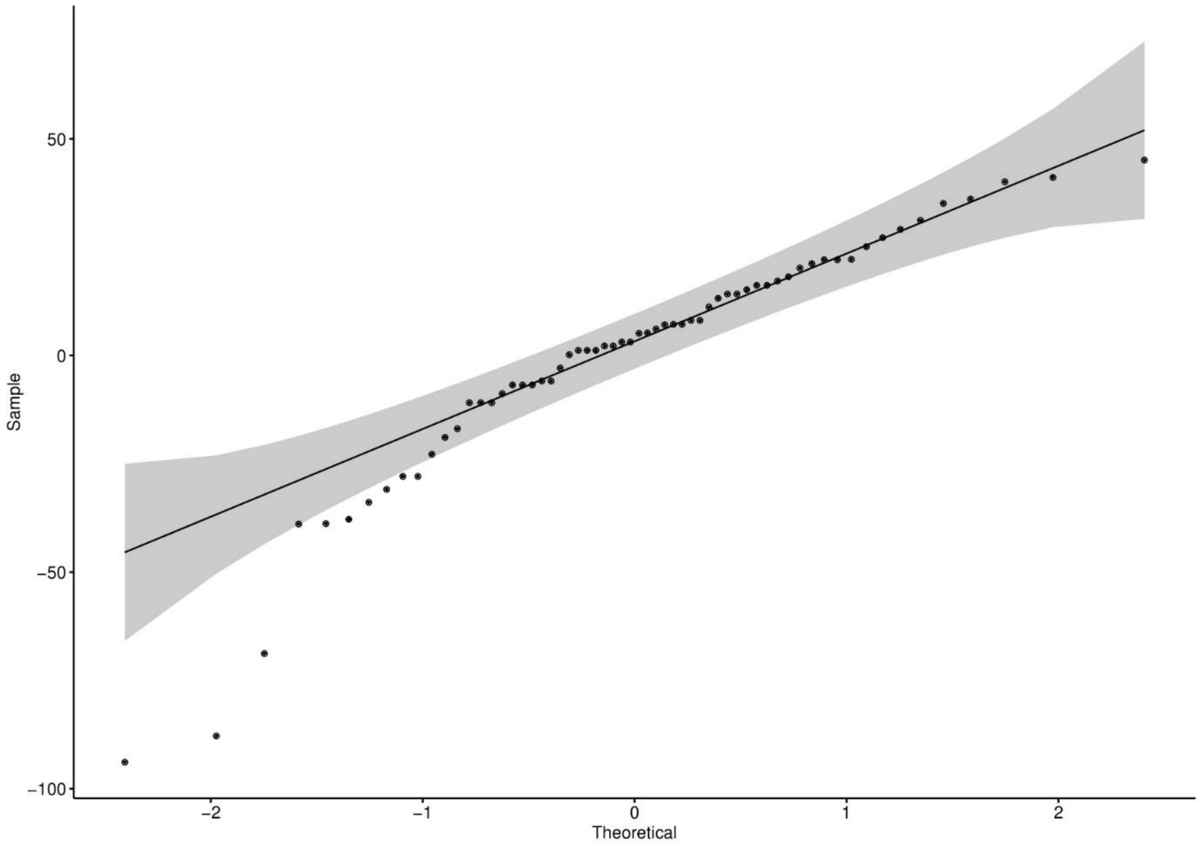


Figure 6.6: qq plot for control and treament groups

Table 6.6: Normality test

Test name	Test statistic	P-value
Shapiro-Wilk	0.909	0.000224
Jarque-BeraTest	60.512	0.000698

Count data can be modelled by either using the Poisson mixed effect model or negative binomial mixed effect model. These two mixed effect models share a lot of similarities, however, unlike the Poisson model, the negative binomial distribution has the scale parameter, which permits the variance to be larger/ smaller than the mean; this may decrease or solve over-dispersion (Stroup, 2013; Tango, 2017; Bono, Alarcón & Blanca, 2021). Count data in practice are usually overdispersed, hence, the overdisperson test was conducted to decide on whether to use Poisson or the negative binomial mixed effect model.

Over-dispersion Test

Sequence Identification numbers assigned to each of the participants was used to proxy the random effects. Johnson (2013) reconciled classical and Bayesian methods of significance testing for a large number of papers published in psychology journals and found that *p-values* of 0.005 and 0.001 correspond to 'strong' and 'very strong' evidence against the null hypothesis, while the *p-values* in the neighbourhood of 0.05 and 0.01 reflect only modest evidence. The level of significance in this study, therefore, was set at 0.005 (0.5%).

As mentioned earlier, there must be first a test for the nature of the overdispersion in the data before it can be decided to use either the mixed-effect Poisson or the mixed-effect negative binomial model in testing for the effectiveness of the new teaching method. If the data is overdispersed, the dispersion parameter should be greater than 1, thus we are interested in testing the following hypothesis:

H_0: dispersion=1

H_1: dispersion>1

We failed to reject the null hypothesis if the *p-value* > 0.005 as presented in Table 6.7 below.

Table 6.7:Over dispersion test

Alternative hypothesis	True dispersion is greater than 1
Dispersion parameter	6.125867
Z -statistic	3.0873
P-value	0.00101

It is evident from Table 6.7 above that the dispersion parameter is greater than one. The p-value of the test confirms this observation (P-value < 0.005), thus we reject the null hypothesis and conclude that the data is overdispersed. This conclusion indicates that the negative binomial mixed effect model is appropriate for modelling the data, hence, the negative binomial mixed effect model was used to test if the teaching method was effective using the MASS package of R programming language, to estimate the parameters of the model (see Appendix Q for the mathematical representation of the negative binomial mixed-effect distribution and regression model).

6.3.2 Results

6.3.2.1 Data visualization

From the box plots in Figure 6.6 below, it is clear that there is a difference between the mean marks of the control and experimental groups. The mean mark of the experimental group was clearly bigger than that of the control group. A look at the violin plot illustrated in Figure 6.7 below, indicates that the width of the treatment group as well as its mean is wider and bigger than those of the control group. These observations provide an indication that the new teaching method may be more effective.

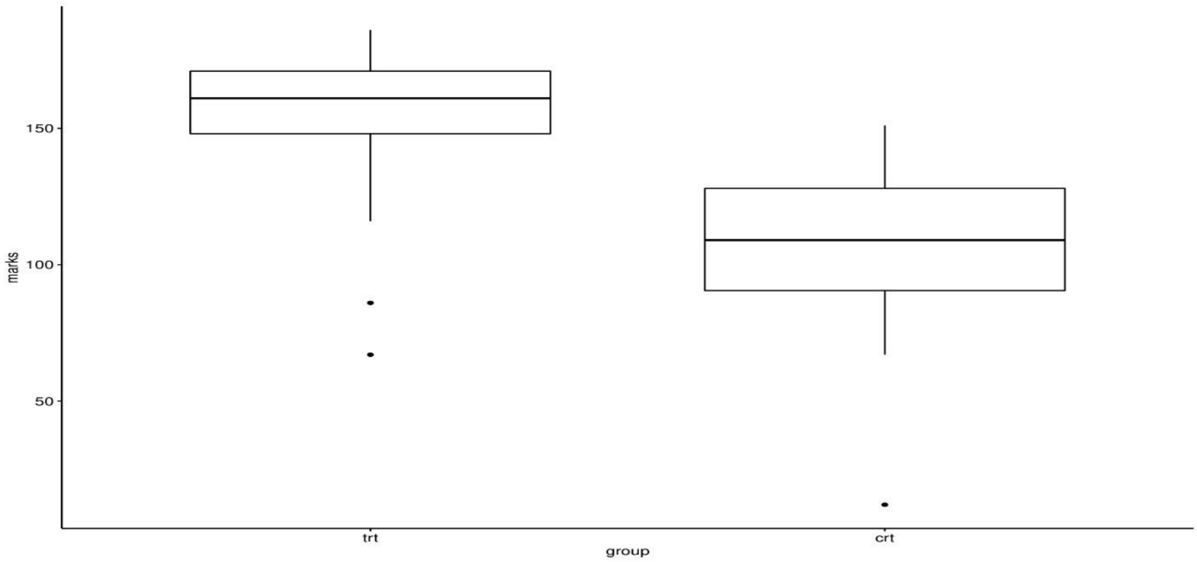


Figure 6.7: Box plot of aggregated marks

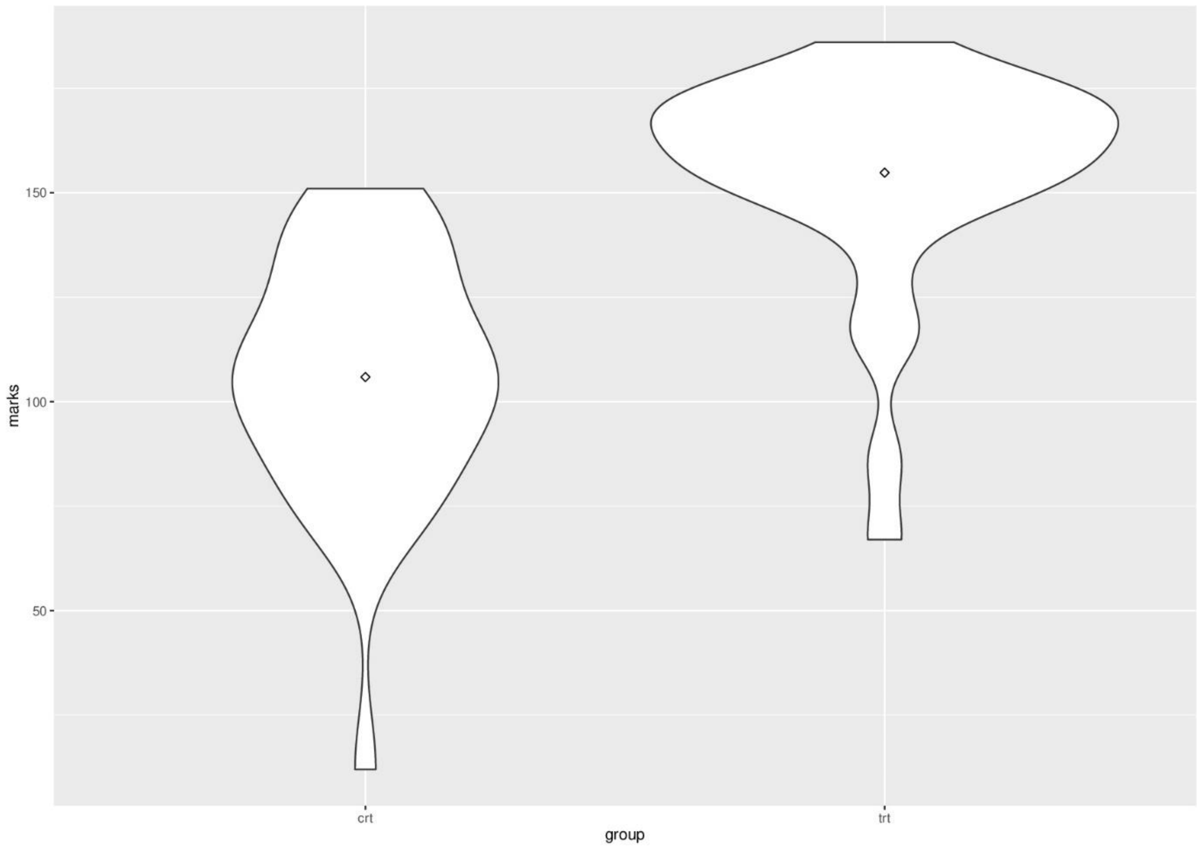


Figure 6.8: Violin Plot of aggregated marks

6.3.2.2 Model outputs

After fitting the data to the negative binomial mixed effect model, the estimated parameters obtained were summarized in Table 6.8 below. Valid conclusions and generalizations cannot be made based on an inadequate estimated model, thus before interpreting the estimated output and make inferences, it is imperative to assess the adequacy of the estimated model.The Chi-square test was used in this regard and the test results are presented in Table 6.9 below. This test reported on the residual deviance, the degrees of freedom (df) and the associated *p-value*.

The residual deviance is the difference between the deviance of the current model and the maximum deviance of the ideal model where the predicted values are identical to the observed ones. If the residual difference is small enough (*p-value*>0.002) and the goodness of fit test is not significant, this indicates that the model fits the data well and since the *p-value* > 0.002, we conclude that the model fits reasonably and it is adequate for making inferences about the two populations.

Table 6.8: Estimated parameters

Parameter	Estimate	Std. Error	Pr(>|z|)
Intercept	4.610054	0.185542	2e-16
Treatment	0.414302	0.134926	0.0021
Random effect	0.001097	0.003766	0.7709

Table 6.9: Chi-square Test

Residual deviance	Df	Pr(>|z|)
67.34626	59	0.2131441

The associated standard errors for the estimates are fairly small, suggesting that they are not far from the true population estimates. The differences between the means of the control group and the treatment group is 0.414302 and the corresponding *p-value* is 0.0021 which is less than the 0.5% level of significance, thus, there is a significant difference between the mean marks of the control and the treatment groups.

Before we can conclude that the treatment resulted in the improvement of marks after implementation, we must rule out the possibility of random effects by assesssing the significance of the random factor. It is clear from Table 6.7, that the random effect is not significant at 0.5% level of singnificance, therefore, the significant improvement in the marks of the treament group is solely due to the implementation of the new teaching method. We, therefore, conclude that, the new teaching method improved learners' performance in circle geometry.

6.4 Conclusion

In generic terms, the mean marks recorded from the experimental group's scores for each of the four standardized tests, as well as, from the composite scores were greater than the mean marks, recorded from the control group's scores, respectively. This established that the experimental group performed better than the control group. Also, the hypothesis test carried out in this chapter justified that there was a significant improvement in the marks of the experimental group and this was achieved, primarily, due to the implementation of the new teaching method - the IPAC model. The relevant research questions from the quantitative analysis and discussions of data captured from the standardized tests and detailed in this chapter, are further examined in Chapter Nine. The next chapter presents the analysis and discussions of learners' mental constructions, in relation to the APOS theory.

CHAPTER SEVEN

ANALYSIS AND DISCUSSIONS OF LEARNERS' MENTAL CONSTRUCTIONS

7.1 Introduction

APOS theory was used to inductively analyse the qualitative facet of the standardized tests data. The different mental constructions demonstrated by participants in both the experimental and control groups for each of the four standardized tests are focused upon and elucidated in this chapter. Additionally, each preliminary genetic decomposition and at which APOS conception, participants were operating are also explained in this chapter. Actual participants' written responses are also presented and discussed in this chapter, to highlight their level of circle geometry mental conception.

7.2 Composition of standardized tests instruments

Circle geometry lessons were partitioned into four distinct lessons - 1 to 4, - for both the control group and the experimental group, sequentially. After each lesson – 1 to 4 - was conducted, a standardized test was administered to the participants in both groups. Details of how each of the four lessons was conducted in both groups were presented in Table 4.3 (Chapter 4).

In the experimental group, lessons were delivered and the standardized tests (ST) were conducted in accordance with the level of mental constructions participants required at each stage of the genetic decomposition. This was sub-divided into four mental construction lessons: GD1- Action stage of circle geometry mental construction lesson, GD2- Process stage of circle geometry mental construction lesson, GD3- Object stage of circle geometry mental construction lesson, and GD4 - Schema stage of circle geometry mental construction lesson (Tziritas, 2011; Syarifuddin & Atweh, 2022). They are summarized in Table 7.1 below.

Table 7.1: Composition and analysis of standardized tests instruments

Lesson	ST	
1	Appendix A	Action level
2	Appendix B	Process level
3	Appendix C	Object level
4	Appendix D	Schema level

Participants' responses to the standardized tests were thoroughly analysed to ascertain the mental constructions they demonstrated. Varied mental constructions were portrayed by participants in the experimental and the control groups for each of the four standardized tests analysed. The analysis was done by implementing the following rating scale: (1) Level 0 - pre-action mental construction level, (2) Level 1- action mental construction level, (3) Level 2 - process mental construction, (4) Level 3- Object mental construction and (5) Level 4 - Schema mental construction. This rating scale was a modification of the evaluation codes by Asiala, Cottrill, Dubinsky & Schingendorf (1997); Chagwiza, Maharaj & Brijlall (2020). These are summarized in Table 7.2 below.

Table 7.2: Coding and rating scale to illustrate mental construction levels

Rating	GD level
0	Pre-action mental construction level
1	Action mental construction level
2	Process mental construction level
3	Object mental construction level

4	Schema mental construction level

Marks participants obtained in the standardized tests were not the only yardsticks for determining if a participant had attained a particular level of mental conception, much focus and priority were also placed on their written responses. Content analysis was done of participants' written responses in their scripts to determine if they had attained a particular mental construction level or not (Chagwiza, Maharaj & Brijlall, 2020). The number of participants who attained a particular mental construction level were coded as 'A', while those who did not, were coded as 'NA' as demonstrated in Table 7.3.

Table 7.3: Categorizing participants' mental construction levels

Declaration	Code
Attained	A
Not attained	NA

7.3 Analysis of participants' responses to each of the PGDs

Data were analysed in accordance with the generated preliminary genetic decomposition (PGD) of circle geometry mental conceptions (see Figure 3.5, in Chapter 3). To this end, data from these standardized tests, in response to each GD, were extracted, analysed and presented. The overall summary of the categorization of participants' mental constructions, according to APOS, on each of the four standardized tests' (ST) items on circle geometry, are presented in Table 7.4.

Table 7.4: Overall summary of APOS categorization of students' mental construction, on each of the four standardized tests' items on circle geometry

Participants	APOS categorization of students' mental constructions							
	Action (ST 1)		Process (ST 2)		Object (ST 3)		Schema (ST 4)	
	A	NA	A	NA	A	NA	A	NA
Control group	27	3	27	3	10	20	10	20
Experimental group	30	2	31	1	28	4	26	6

The first standardized test (ST 1) was conducted to determine participants' Action mental construction level. This level of mental construction measured direct recall and use of circle geometry theorems and geometric language, as well as conceptual knowledge and understanding of geometric concepts; this shows that an individual is acquainted with circle geometry theorems. Participants were expected to state any of the circle geometry theorems and/or its converse. In the Table above, 27 participants from the control group operated at the action level of the APOS theory in accordance with the preliminary genetic decomposition, while 3 participants did not attain this level; these 3 participants were judged as, operating at the "pre-action mental construction level". On the other hand, in the experimental group, 30 participants operated at the action level of the APOS theory in accordance with the preliminary genetic decomposition, while 2 participants did not attain the action level of mental construction, thus, these 2 participants are also operating at the "pre-action mental construction level" (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The second standardized test (ST 2) was conducted to determine participants' Process mental construction level. This level of mental construction measured direct applications of the circle geometry theorems and/or its converse to solve problems. Participants were expected to prove and perform simple applications of the circle geometry theorems and/or converses by reflecting on the action level, describe, or even reverse the steps of internalised actions without performing those steps explicitly. Table 7.4 shows that in the control group, 27 participants operated at the process level of the APOS theory in accordance with the preliminary genetic decomposition, while 3 participants did not attain this level. On the other hand, in the experimental group, 31 participants operated at the process level of the APOS theory in accordance with the preliminary genetic decomposition, while 1 participant did not attain the process level of mental construction (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The third standardized test (ST 3) was conducted to determine participants' Object mental construction level. This level of mental construction measured higher-order geometric reasoning and creative thinking around circle geometry theorems and concepts. Participants were expected to encapsulate the internalized process into a cognitive object, by reflecting on it. The results indicate that in the control group, only 10 participants operated at the object level of the APOS theory in accordance with the preliminary genetic decomposition, while 20 participants could not operate at this level. On the other hand, in the experimental group, 28 participants operated at the object level of the theory, in accordance with the preliminary genetic decomposition, while 4 participants could not (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The fourth standardized test (ST 4) was conducted to determine participants' Schema mental construction level. At this level participants should be able to solve non-routine problems; this level required higher-order reasoning, creativity, critical and reflective thinking. Participants were expected to organise and link the action, processes and objects' geometric mental construction together to form a coherent framework (schema). From Table 7.4, with the control group, only 10 participants operated at the schema level of the APOS theory in accordance with the preliminary genetic decomposition, while 20 participants did not operate at this level. On the other hand, in the experimental group, 26 participants operated at the schema level of the APOS theory in accordance with the preliminary genetic decomposition, while 6 participants did not operate at that level (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

7.4 The mental constructions illustrated by participants' written responses

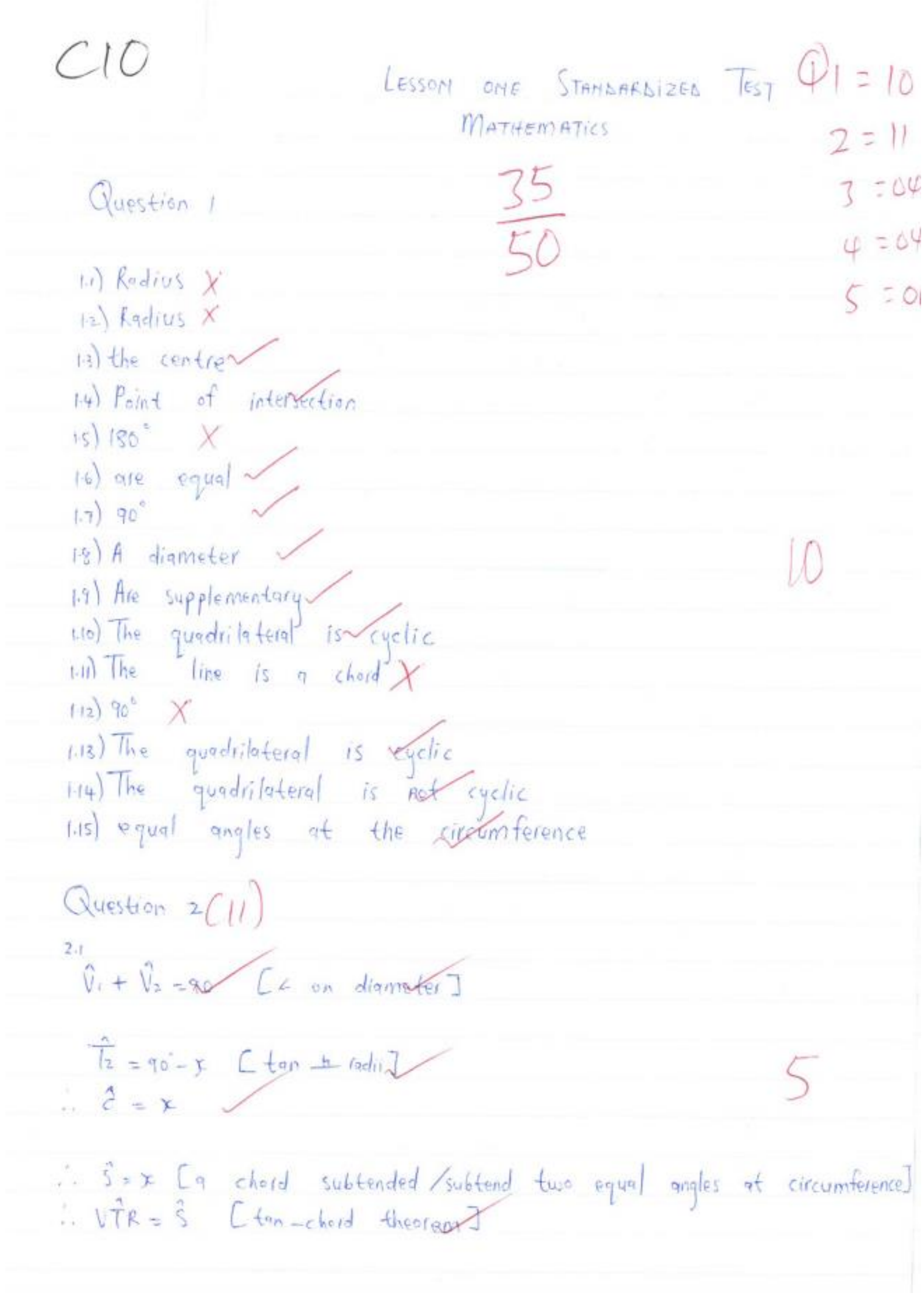
7.4.1 Participants' responses at Action level

Table 7.5: Participants' responses at action level

Action level	Control group		Experimental group	
	A	NA	A	NA
Participants	27	3	30	2

(i) Control groups' responses at action level

Vignette 7.1: Exemplar of control groups' written responses at action level


C10

LESSON ONE STANDARDIZED TEST

MATHEMATICS

Q1 = 10
2 = 11
3 = 04
4 = 04
5 = 01

35/50

Question 1

1.1) Radius
1.2) Radius
1.3) the centre
1.4) Point of intersection
1.5) 180°
1.6) are equal
1.7) 90°
1.8) A diameter
1.9) Are supplementary
1.10) The quadrilateral is cyclic
1.11) The line is a chord
1.12) 90°
1.13) The quadrilateral is cyclic
1.14) The quadrilateral is not cyclic
1.15) equal angles at the circumference

10

Question 2 (11)

2.1
$\hat{V}_1 + \hat{V}_2 = 90$ [∠ on diameter]

$\hat{T}_2 = 90° - x$ [tan ⊥ radii]

∴ $\hat{C} = x$

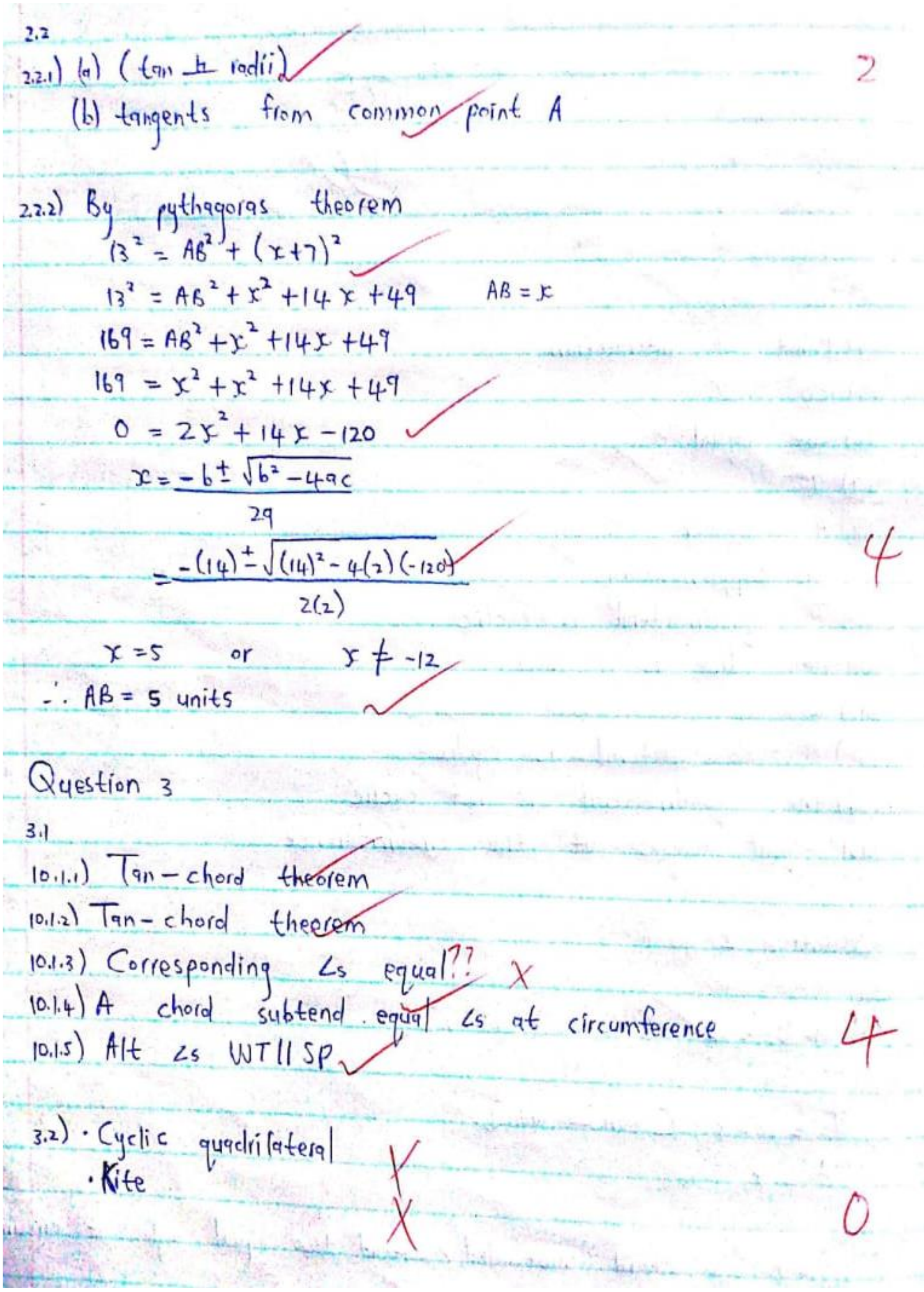
5

∴ $\hat{S} = x$ [a chord subtended/subtend two equal angles at circumference]

∴ $V\hat{T}R = \hat{S}$ [tan – chord theorem]

Scan 1: Written response of C_{10}

The written responses of participant C_{10} for question 1 and part of question 2 are displayed above (see scan 1). Questions 1 & 2 assessed participants' knowledge on circle geometry theorems and converse of theorems. It can be observed on scan 1 that he got most responses correct - an indicator of substantial understanding of circle geometry theorems and/or converse of theorems. The rest of the written response to question 2, is presented on the next scan (scan 2).



2.2

2.2.1) (a) (tan ⊥ radii) ✓ 2

(b) tangents from common point A ✓

2.2.2) By pythagoras theorem

$13^2 = AB^2 + (x+7)^2$ ✓

$13^2 = AB^2 + x^2 + 14x + 49$ AB = x

$169 = AB^2 + x^2 + 14x + 49$

$169 = x^2 + x^2 + 14x + 49$ ✓

$0 = 2x^2 + 14x - 120$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(14) \pm \sqrt{(14)^2 - 4(2)(-120)}}{2(2)}$ ✓ 4

$x = 5$ or $x \neq -12$

∴ AB = 5 units ✓

Question 3

3.1

10.1.1) Tan – chord theorem ✓

10.1.2) Tan – chord theorem ✓

10.1.3) Corresponding ∠s equal ?? ✗

10.1.4) A chord subtend equal ∠s at circumference ✓

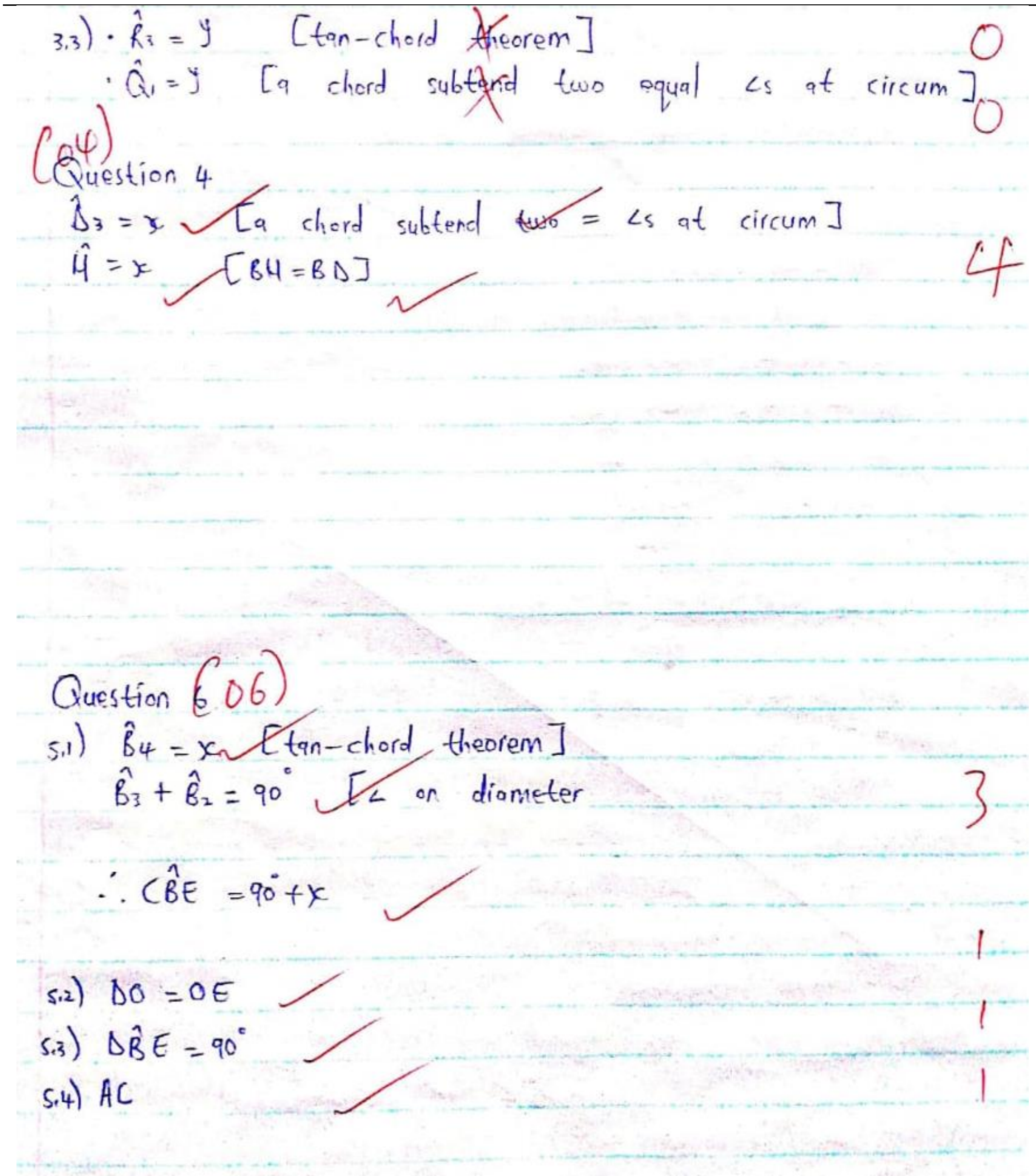
10.1.5) Alt ∠s WT ∥ SP ✓ 4

3.2) · Cyclic quadrilateral ✗

· Kite ✗ 0

Scan 2: Written response of C_{10}

The above illustration (scan 2) presents the written responses of C_{10} for the rest of question 2 - it assessed participants' knowledge on circle geometry theorems and conceptual knowledge and understanding of geometric concepts. Scan 2 also presents the written responses to question 3, although he got most responses correct, some few inappropriate responses on geometric terminologies were given by the participant. The next presentation, (scan 3) follows.


3.3) · $\hat{R}_1 = y$ [tan-chord theorem] 0
· $\hat{Q}_1 = y$ [a chord subtend two equal ∠s at circum] 0

(Q4) Question 4
$\hat{D}_3 = x$ ✓ [a chord subtend two = ∠s at circum]
$\hat{H} = x$ ✓ [BH = BD] ✓ 4

Question 5 (06)
5.1) $\hat{B}_4 = x$ ✓ [tan-chord theorem]
$\hat{B}_3 + \hat{B}_2 = 90°$ ✓ [∠ on diameter 3
∴ $C\hat{B}E = 90° + x$ ✓
5.2) DO = OE ✓ 1
5.3) $D\hat{B}E = 90°$ ✓ 1
5.4) AC ✓ 1

Scan 3: Written response of C_{10}

The rest of the written responses to question 3, 4 and 5 are presented above (scan 3). This part of question 3 also assessed participants' knowledge on circle geometry theorems, however, he provided wrong responses. Questions 4 and 5 assessed participants' knowledge on circle geometry theorems and geometric terminologies; it shows he got all the responses correct. The written responses on scans 1, 2 & 3, were collectively analysed and discussed below.

Analysis and discussion of participants' written responses

As said earlier, 27 out of the 30 (90%) of the control group participants were able to attain this level. The written responses (see scans 1, 2, 3) are exhibits of the work done by the control group at the action level to demonstrate their mental constructions. From question 1 (see scan 1), 10 out of 15 responses were correct. This question tested participants' knowledge and understanding of circle geometry theorems and converse of theorems. The 27 participants were able to directly recall from memory, the appropriate expected answers on circle geometry theorems, however, they had challenges with giving appropriate responses to questions on converse of theorems. This is an indication that they might have just memorized the theorems without conceptually understanding the geometric concepts involved (Chagwiza, Maharaj & Brijlall, 2020).

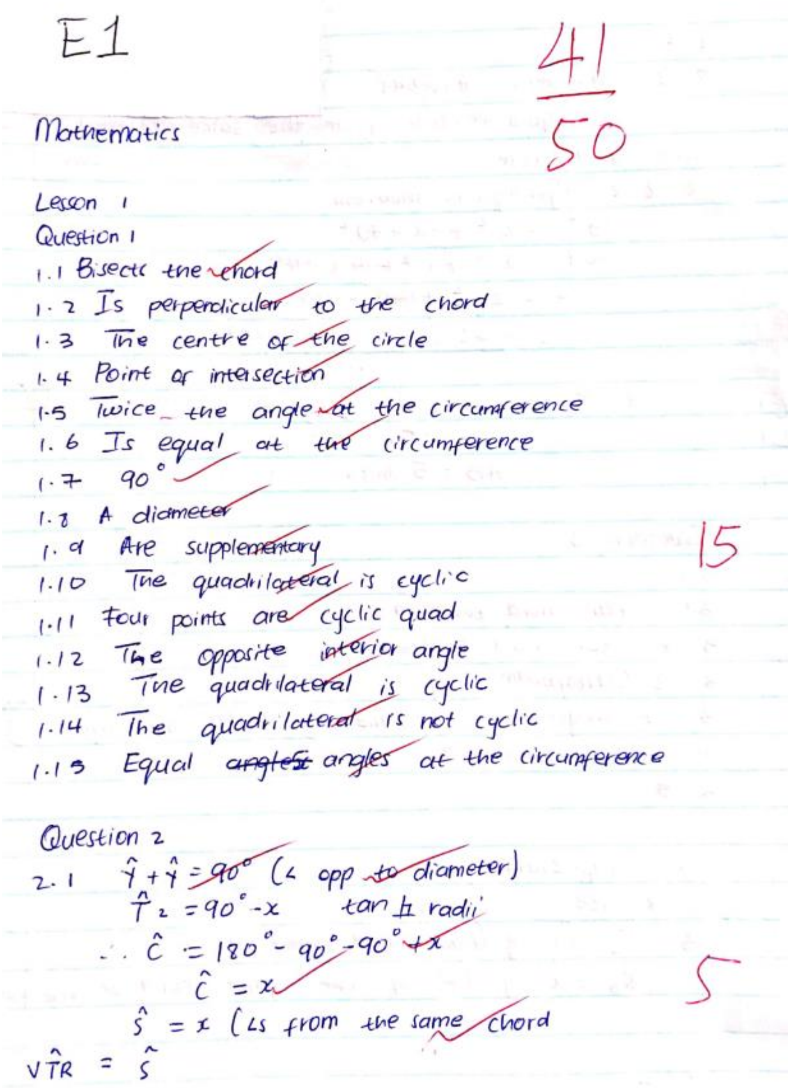
On scans 1 and 2, the written responses to question 2 can be seen. This participant was able to provide appropriate responses - correct statements and reasons, used geometric language and terminologies correctly and he could perform basic computations correctly by applying the Pythagoras theorem (see sub-question 2.2.2). The participant was able to recall from memory, the circle geometry theorems and identified and recognised the specific theorems which were applicable, to provide an appropriate solution to the question under consideration; he understood every detail of what was expected to be done and he was able to demonstrate that well. For instance, the correct substitutions made when applying the Pythagoras theorem, that is: $13^2 = AB^2+(x+7)^2$ is another indication that the participant was operating at the action level. After the correct substitutions in the Pythagoras theorem, he was able to expand, simplify, and use the quadratic formula, to solve for the value of x. The participant was able to recognise that x is a distance, thus, since negative distances do not exist, then the answer to this sub-question cannot be negative; based on this, he rejected the negative solution and accepted the positive value of x as the final solution to this sub-question. This testifies that he was able to follow step-by-step procedures efficiently. All these confirm that the participant was operating at the action level of circle geometry mental conception (Brijlall & Ndlazi, 2019; Brijlall, 2020).

For question 3, its written responses can be found on scans 2 and 3. It was observed that participants had challenges interpreting the geometric diagram. This resulted in participants in the control group giving some incorrect responses, however, they performed averagely in this question. In scan 3 the participant provided correct responses to questions 4 and 5 as he was able to recall all the appropriate expected answers from memory and their corresponding reasons. The participant demonstrated conceptual knowledge and understanding of geometric concepts, in line with the PGD at this level (Brijlall, 2020).

Even though their responses to the questions were not always correct, these 27 participants from the control group were able to recall the majority of the appropriate expected answers from memory, hence, the researcher concludes that these participants demonstrated enough knowledge on the conception of circle geometry concepts at the action mental construction level, in line with the PGD. The researcher, therefore, rated these 27 participants as having attained this level, however, 3 participants were unable to recall the majority of the appropriate expected answers; rather they either provided irrelevant responses mostly, or they provided no responses at all. The researcher concluded that they demonstrated little or no conception at this level and rated them as ”not attained”, (Wilson & Dubinsky, 2013) and as performing at the “pre-action mental construction level”.

(ii) Experimental groups’ responses at action level

Vignette 7.2: Exemplar of experimental groups’ written responses at action level


E1

41/50

Mathematics

Lesson 1

Question 1

1.1 Bisects the chord
1.2 Is perpendicular to the chord
1.3 The centre of the circle
1.4 Point of intersection
1.5 Twice the angle at the circumference
1.6 Is equal at the circumference
1.7 90°
1.8 A diameter
1.9 Are supplementary
1.10 The quadrilateral is cyclic
1.11 Four points are cyclic quad
1.12 The opposite interior angle
1.13 The quadrilateral is cyclic
1.14 The quadrilateral is not cyclic
1.15 Equal ~~angles~~ angles at the circumference

15

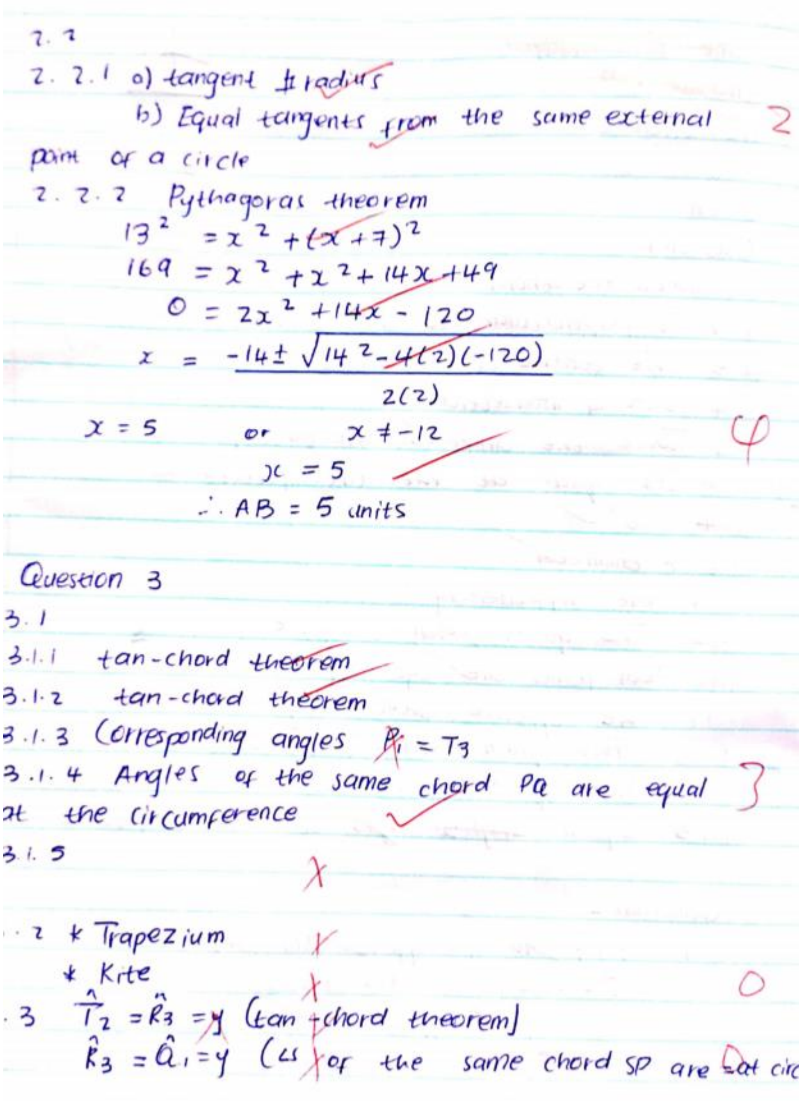
Question 2

2.1 $\hat{Y} + \hat{Y} = 90^\circ$ (∠ opp to diameter)
$\hat{T}_2 = 90^\circ - x$ tan ⊥ radii
$\therefore \hat{C} = 180^\circ - 90^\circ - 90^\circ + x$
$\hat{C} = x$
$\hat{S} = x$ (∠s from the same chord
$V\hat{T}R = \hat{S}$

5

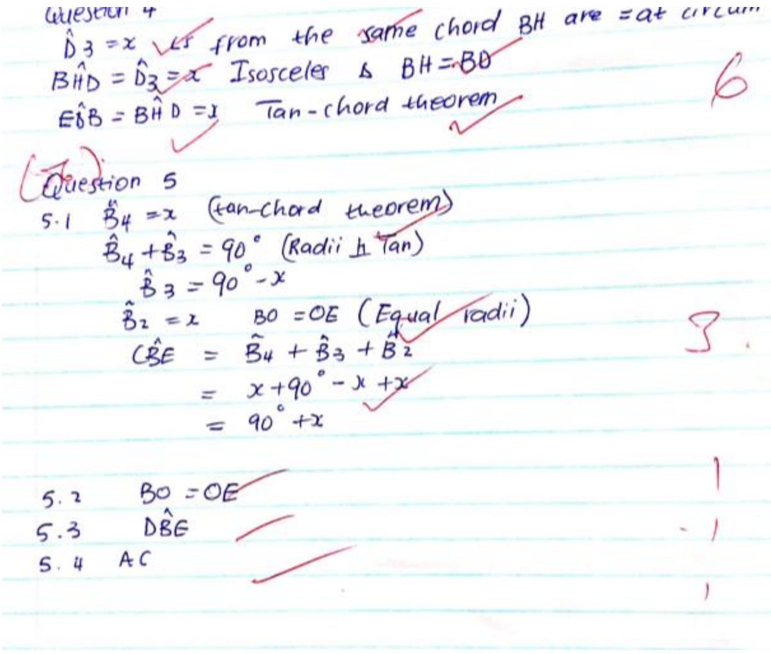
Scan 1: Written response of E_1

The written responses to question 1 and part of question 2 are presented above (see scan 1). Questions 1 & 2 assessed participants' knowledge on circle geometry theorems and converse of theorems. It can be observed that she could provide all the correct responses. The rest of the written responses to question 2, are presented in the next scan (scan 2).



Scan 2: Written response of E_1

The above illustration (scan 2) presents the rest of the written responses for question 2 - all written responses were correct. Scan 2 also reveal that the written responses to question 3- written responses were partly correct as some inappropriate responses on geometric terminologies were given by the participant. The other part of question 2 assessed participants’ knowledge on circle geometry theorems while question 3 assessed participants’ conceptual knowledge and understanding of geometric concepts. The next presentation (scan 3) follows.


Question 4
$\hat{D}_3 = x$ ∠s from the same chord BH are = at circum
$B\hat{H}D = \hat{D}_3 = x$ Isosceles △ BH = BD
$E\hat{B}B = B\hat{H}D = x$ Tan-chord theorem
6
Question 5
5.1 $\hat{B}_4 = x$ (tan-chord theorem)
$\hat{B}_4 + \hat{B}_3 = 90°$ (Radii ⊥ Tan)
$\hat{B}_3 = 90° - x$
$\hat{B}_2 = x$ BO = OE (Equal radii)
$C\hat{B}E = \hat{B}_4 + \hat{B}_3 + \hat{B}_2$
$= x + 90° - x + x$
$= 90° + x$
3
5.2 BO = OE
5.3 $D\hat{B}E$
5.4 AC

Scan 3: Written response of E_1

The written responses for questions 4 & 5 are presented above (see scan 3). These questions assessed participants' knowledge on circle geometry theorems and geometric terminologies. Impressively, she got all responses correct. The written responses (see scans 1, 2 & 3) were analysed and discussed below.

Analysis and discussion of participants' written responses

As a reminder, 30 out of the 32 (94%) of the experimental group participants were rated by the researcher as having attained the action level of circle geometry mental conception, based on their written responses, hence, they were judged to be in line with the proposed PGD at this level. The written responses (see scans 1, 2 and 3) reveal the work done by the experimental group, at the action level, to testify to their mental constructions. The researcher made this assertion, since in questions 1, 2, 4 and 5, E_1 could recall all the appropriate expected answers from memory.

A thorough perusal of the written responses by E_1 illustrate that in question 1 (see scan 1), all the 15 responses were correct as E_1 could recall all the correct answers from memory. This question tested participants' knowledge and understanding of circle geometry theorems and converse of theorems, as said earlier. These 30 participants were able to directly recall from memory the expected answers on circle geometry theorems while only a handful had challenges with giving appropriate responses to questions on converse of theorems. This testified that majority of participants in the experimental group, conceptually, understood the circle geometry concepts, terminologies and geometric language (Chagwiza, Maharaj & Brijlall, 2020).

With reference to scans 1 and 2, the unedited written responses by E_1 to question 2, testified that the participant was able to recall the expected responses to question 2. She provided correct statements and reasons, used geometric language and terminologies correctly and could also perform basic computations correctly by applying the Pythagoras theorem (see sub-question 2.2.2). The group was able to recall the circle geometry theorems from memory, and they identified and recognised the specific theorems which were applicable, to provide a solution to the question under consideration. They understood every detail of what they were expected to do and they were able to demonstrate that well, as required. The correct substitutions were, as well, made when applying the Pythagoras theorem, that is: $13^2 = AB^2 + (x+7)^2$. This was another indication that the participants were operating at the action level. Participant E_1 was able to expand, simplify, and used the quadratic formula effectively, to solve for the value of x. She rejected the negative value of x and accepted the positive value of x as the final answer; she demonstrated the above step-by-step procedures efficiently confirming that the participant was operating at the action level of circle geometry mental construction (Brijlall & Ndlazi, 2019; Brijlall, 2020).

The researcher examined the participant's written responses to question 3 (see scan 2). This conveyed that this participant from the experimental group also had challenges, interpreting the diagram well, but she responded better in comparison to the counterparts in the control group. In addition, the participant's written responses to questions 4 and 5 were examined (see scan 3); it showed that she was able to recall the expected answers from memory as she provided correct geometric statements and corresponding reasons. It can be observed from sub-question 5.1 above that E_1 correctly stated $\widehat{B_2} = x$, with the appropriate reason "Tan-chord theorem" and was also able to recognise that the "tangent is perpendicular to radius". Based on this, she correctly stated that $\widehat{B_4} + \widehat{B_3} = 90^0$. Furthermore, the participant demonstrated her conceptual understanding of circle geometry concepts when she indicated that $C\hat{B}E = \widehat{B_4} + \widehat{B_3} + \widehat{B_2}$ and correctly substituted the values of angles $\widehat{B_4}, \widehat{B_3}, \widehat{B_2}$ into the above equation. She then simplified to obtain the final answer of $C\hat{B}E = 90^0 - x$. All these provided enough evidence for the researcher to judge this participant as having attained the action level of mental construction, in accordance with the proposed PGD at this level (Brijlall, 2020).

The researcher used only the unedited written responses of E_1 as an evidence of the written responses provided by all the experimental group participants. It needs to be emphasized here that these 30 participants from the experimental group who were judged to have attained the action level of circle geometry mental construction, did not all provide 100% correct responses, however, they demonstrated enough conception of circle geometry concepts at the action mental construction level in line with the PGD, in their written responses. These participants could recall almost all the appropriate answers from memory, although, 2, were unable to do so; they either provided irrelevant responses, mainly, or they provided no responses at all (Chagwiza, Maharaj & Brijlall, 2020). Due to this, the researcher maintains that they have demonstrated little or no conception at this level, hence, they were rated as "not attained", (Wilson & Dubinsky, 2013) and to be operating at the "pre-action mental construction level".

7.4.2 Participants' responses at Process level

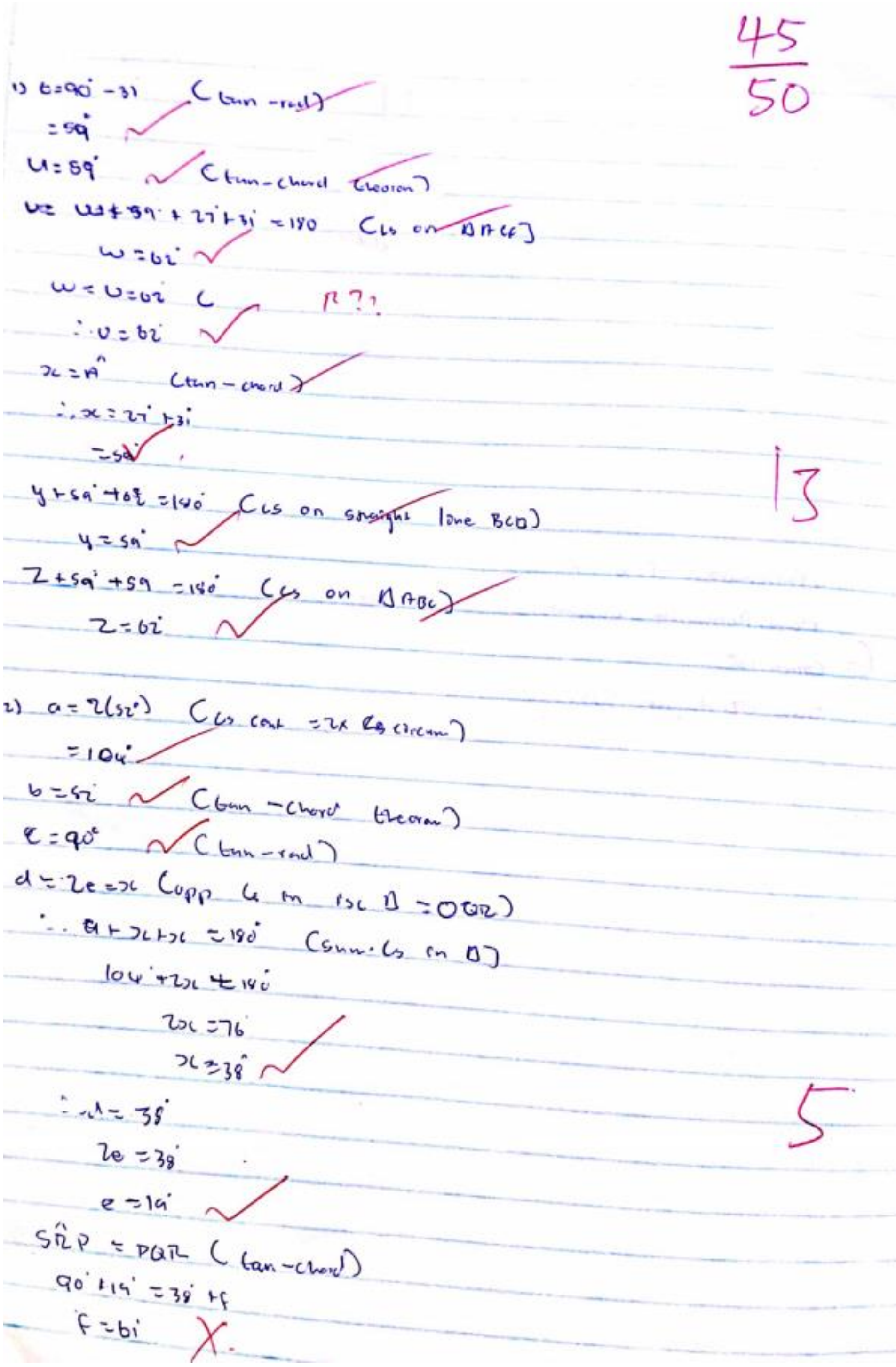
Table 7.6: Participants' responses at process level

Process level	Control group	Exerimental group

	A	NA	A	NA
Participants	27	3	31	1

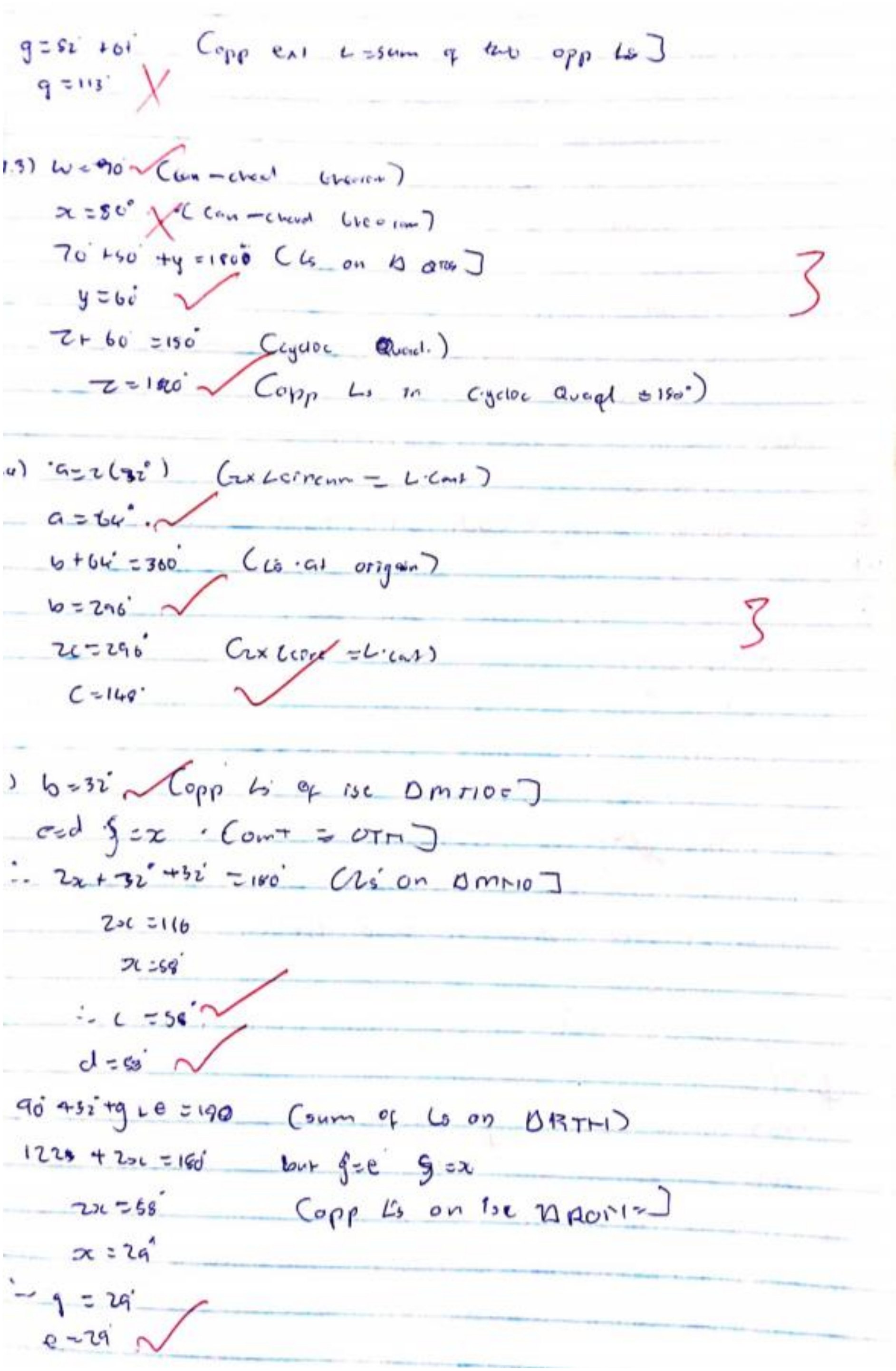
(i) Control groups' responses at process level

Vignette 7.3: Exemplar of control groups' written responses at process level



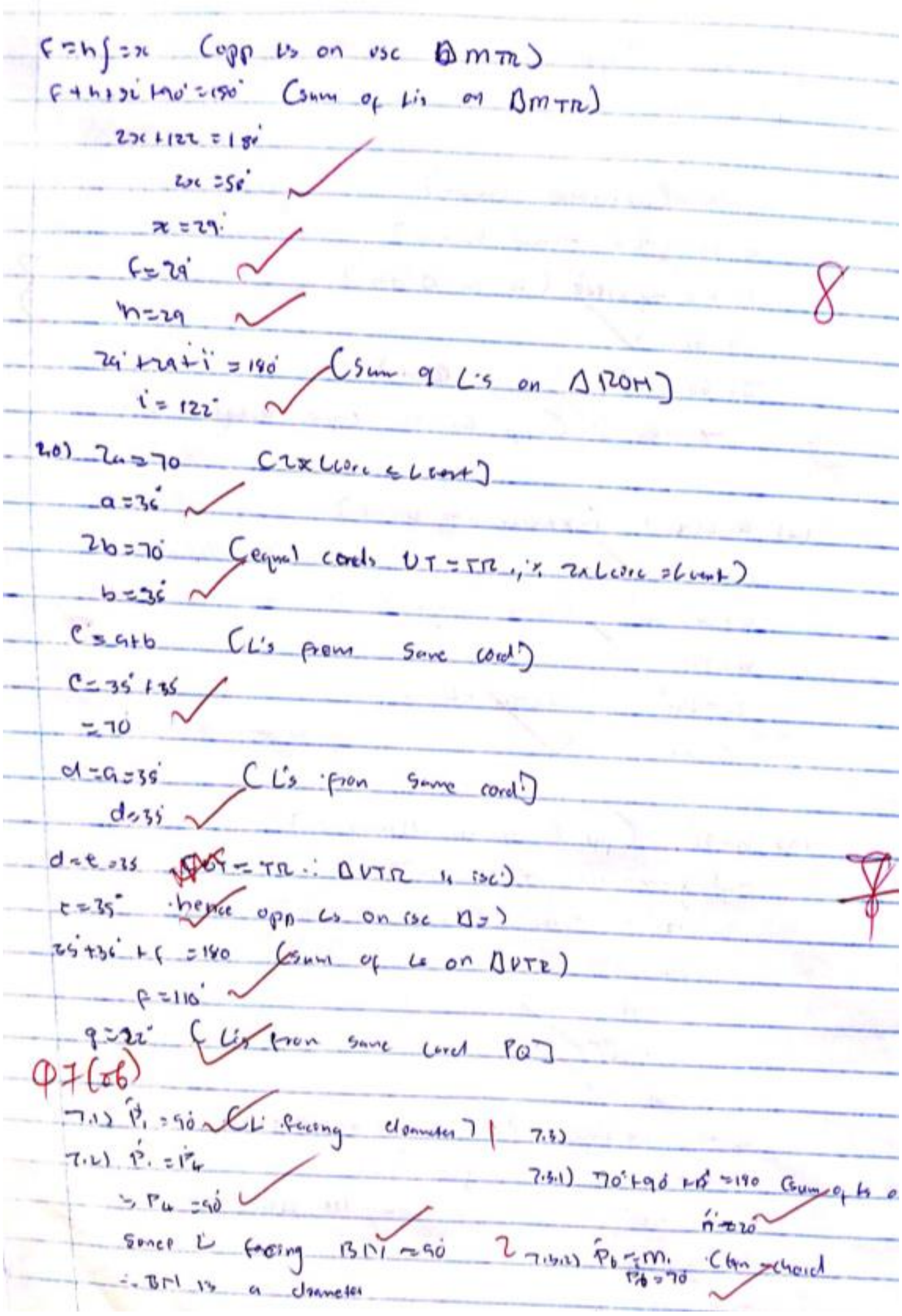
Scan 1: Written response of C_6

From the illustration above - scan 1 - it can be observed that he got almost all the written responses correct and was able to directly apply appropriate circle geometry theorems/converses to perform simple mathematical calculations to obtain correct answers to the unknown variables. Scan 2 is presented next.


g = 52° + 61° (Copp ext ∠ = sum of the opp ∠s]
g = 113°
1.3) w = 90° (tan-chord theorem)
x = 80° (tan-chord theorem]
70° + 50° + y = 180° (∠s on Δ]
y = 60°
z + 60° = 150° (cyclic quad.)
z = 120° (opp ∠s in cyclic quad = 180°)
a = 2(32°) (∠ at centre = ∠ at circ)
a = 64°
b + 64° = 360° (∠s at origin)
b = 296°
2c = 296° (∠ at centre = ∠ at circ)
c = 148°
b = 32° (opp ∠s of isc ΔMTO]
and f = x (OM = OT]
∴ 2x + 32° + 32° = 180° (∠s on ΔMTO]
2x = 116
x = 58°
∴ c = 58°
d = 58°
90° + 32° + g + e = 180 (sum of ∠s on ΔRTH)
122° + 2x = 180° but f = e g = x
2x = 58° (opp ∠s on isc ΔROM]
x = 29°
∴ g = 29°
e = 29°

Scan 2: Written response of C_6

The illustration above - scan 2 - also informs how well the participant was able to directly apply appropriate circle geometry theorems/converses to perform simple mathematical calculations to obtain correct answers to the unknown variables. The rest of the written responses to Test 2 are displayed on scan 3 below.


f = h, f = x (opp ∠s on isc ΔMTR)
f + h + x + 90° = 180° (Sum of ∠s on ΔMTR)
2x + 122 = 180°
2x = 58°
x = 29°
f = 29°
h = 29
29° + 29 + i = 180° (Sum of ∠s on ΔROH)
i = 122°
2.0) 2a = 70 (2 × ∠ circ = ∠ cent)
a = 35°
2b = 70° (equal cords UT = TR, ∴ 2x∠circ = ∠cent)
b = 35°
c = a + b (∠'s from same cord)
c = 35° + 35
= 70
d = a = 35 (∠'s from same cord)
d = 35
d = e = 35 (UT = TR ∴ ΔUTR is isc)
e = 35° hence opp ∠s on isc Δs)
35° + 35° + f = 180 (Sum of ∠s on ΔUTR)
f = 110°
g = 22° (∠'s from same cord PQ)
Q7 (26)
7.1) P̂₁ = 90° (∠ facing diameter) | 7.3)
7.2) P̂₁ = P̂₄
∴ P₄ = 90°
Since ∠ facing BM = 90
∴ BM is a diameter
7.3.1) 70° + 90 + n̂ = 180 (Sum of ∠s ...)
n̂ = 20°
7.3.2) P̂₆ = m̂ (tan chord)
P̂₆ = 70
8

Scan 3: Written response of C_6

The written responses illustrated above - scan 3 – confirm that the participant was able to directly apply appropriate circle geometry theorems/converses to obtain correct answers to the unknown variables, thus, was able to perform simple geometric proofs, correctly. The written responses on scans 1, 2, and 3, are analysed and discussed below.

Analysis and discussion of participants' written responses

A thorough review of participants written responses at the process stage of circle geometry mental conception was done by the researcher. It revealed that 27 participants from the control group demonstrated enough evidence of mental construction at the process level; this represented 90% of the participants. To illustrate the participants' mental construction at this level, the researcher presented the written responses (see scans 1, 2, 3) of the work done by the control group.

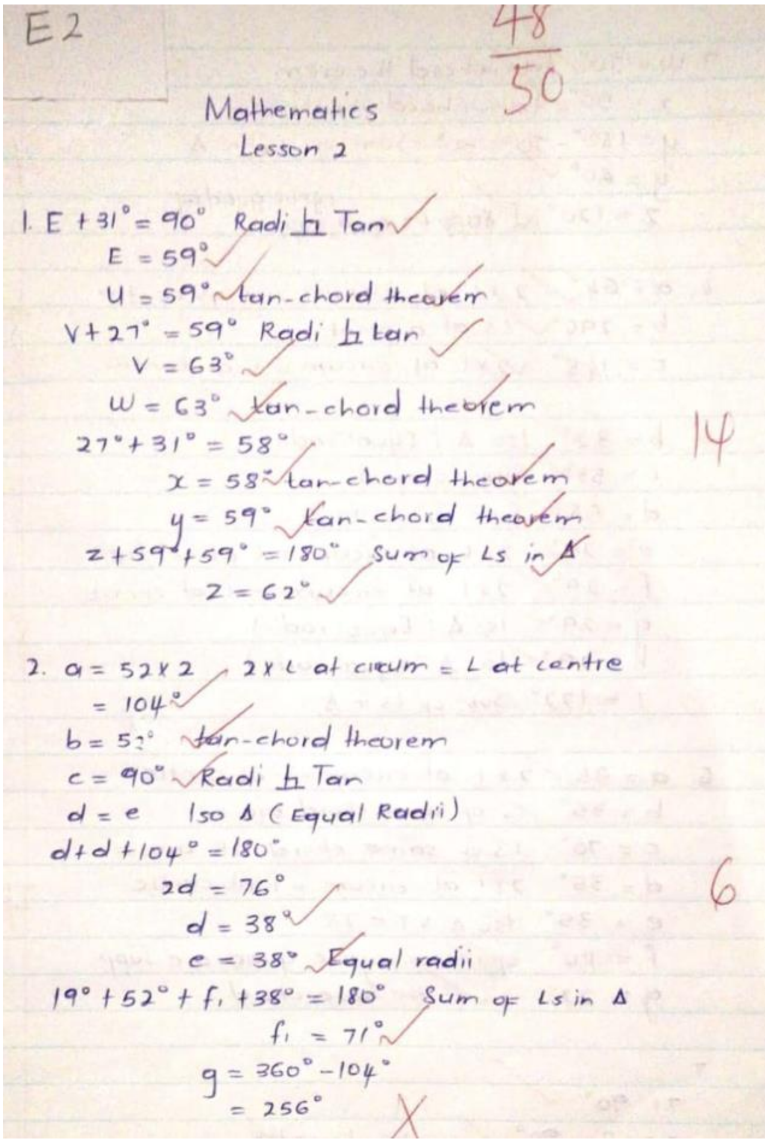
From questions 1-7 of the participant's written responses, enough evidence emerged that he was operating at the process level. It can be observed that C_6 efficiently applied circle geometry theorems and/or its converse to solve problems directly. This meant that C_6 was able to recall the correct statement of the theorems and then applied the theorems; this also implied that he had interiorised such memory into a process. In question 1 for instance, C_6 correctly stated that $t = 59^0$ and $u = 59^0$, both based on the tan-chord theorem; $w = 62^0$ and $z = 62^0$ by directly applying the basic mathematics concept that the sum of angles in a triangle are supplementary; and that $y = 59^0$ by directly applying the basic mathematics concept that the sum of angles on a straight line are supplementary. The correct process continued to question 2, where the participant rightly recalled and applied the appropriate circle geometry theorem - the angle at the centre is equal to twice the angle at the circumference. Based on this, he was able to obtain the value of $a = 104^0$. All the other sub-questions from question 2 to the other questions, followed the same trend hence, there was no need to mention all of them, in order to avoid repetitive narrations.

For sub-question 7.2, participants were asked to show that BN is the diameter of the smaller circle. From the written response C_6 rightly stated that $\widehat{P_4} = 90^0$ and since $\widehat{P_4}$ is opposite to the side BN, then, BN is the diameter of the circle. This is only valid because the angle opposite to BN is 90^0. This proves the application of the converse of the "angle in a semi-circle theorem", which the participant demonstrated correctly. This was also evidence that C_6 was able to recall from memory the appropriate expected theorems, and/or its converse and rightly applied them to obtain correct answers to the given problems (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The written responses by C_6, illustrated above, are representations of the work done by participants in the control group. This is also a sample of the responses of the 27 participants who were judged to have attained the process level of mental construction. Every evidence contained in the written responses by C_6 gave enough testimony to judge that these participants operated at the process level of circle geometry mental conception; this was in accordance with the PGD at this level. Only 2 participants demonstrated little or no conception at this level, hence, the researcher rated them as "not attained", based on their written responses (Wilson & Dubinsky, 2013); they either provided irrelevant responses or they provided no responses at all (Chagwiza, Maharaj & Brijlall, 2020).

(ii) Experimental groups' responses at process level

Vignette 7.4: Exemplar of experimental groups' written responses at process level



E2

48/50

Mathematics
Lesson 2

1. E + 31° = 90° Radi ⊥ Tan
E = 59°
U = 59° tan-chord theorem
V + 27° = 59° Radi ⊥ tan
V = 63°
W = 63° tan-chord theorem
27° + 31° = 58°
x = 58° tan-chord theorem
y = 59° tan-chord theorem
z + 59° + 59° = 180° Sum of ∠s in Δ
z = 62°

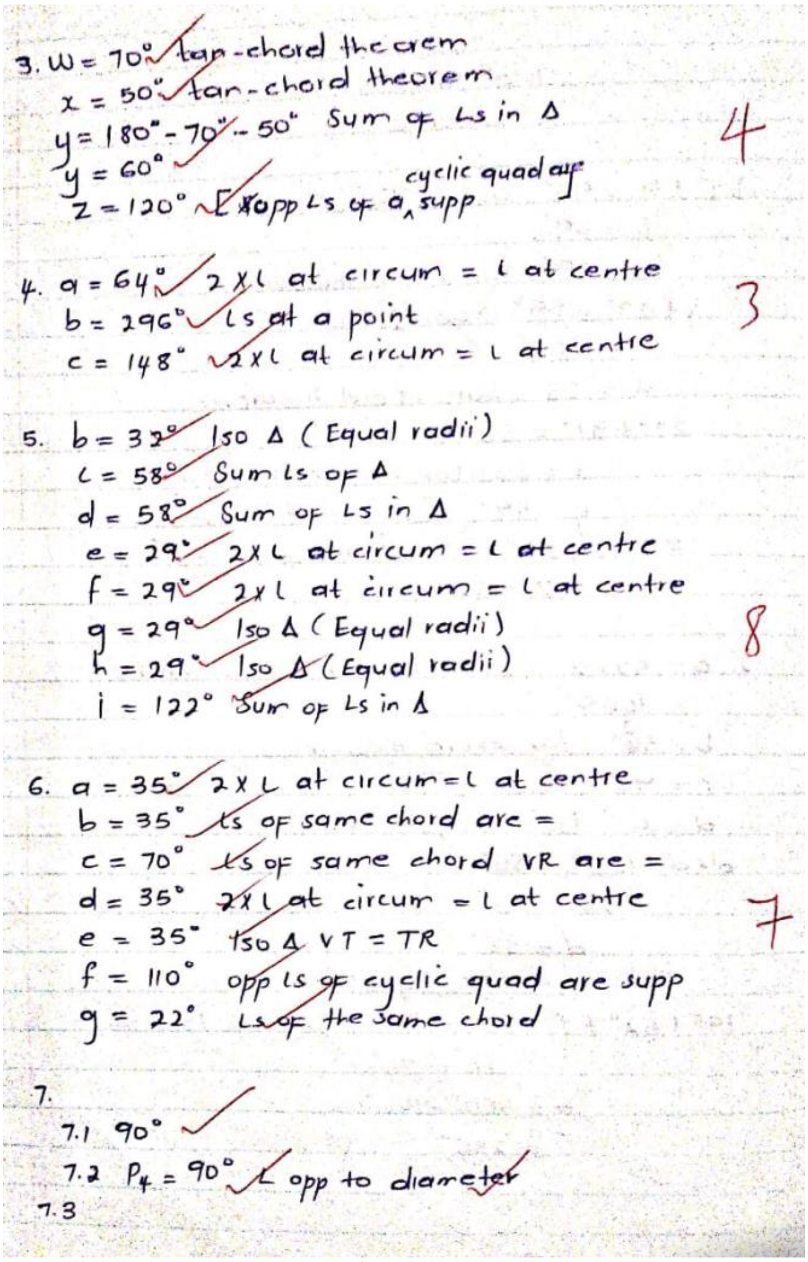
14

2. a = 52 × 2 2 × ∠ at circum = ∠ at centre
= 104°
b = 52° tan-chord theorem
c = 90° Radi ⊥ Tan
d = e Iso Δ (Equal Radii)
d + d + 104° = 180°
2d = 76°
d = 38°
e = 38° Equal radii
19° + 52° + f₁ + 38° = 180° Sum of ∠s in Δ
f₁ = 71°
g = 360° − 104°
= 256°

6

Scan 1: Written response of E_2

It can be observed that the participant got almost all responses correct (see scan 1). This confirmed that she was able to directly apply appropriate circle geometry theorems/converses to perform simple mathematical calculations, to obtain correct answers to the unknown variables; this is followed by scan 2.



3. w = 70° tan-chord theorem
x = 50° tan-chord theorem
y = 180° - 70° - 50° Sum of ∠s in Δ
y = 60°
z = 120° ∠ opp ∠s of a cyclic quad are supp

4. a = 64° 2 × ∠ at circum = ∠ at centre
b = 296° ∠s at a point
c = 148° 2 × ∠ at circum = ∠ at centre

5. b = 32° Iso Δ (Equal radii)
c = 58° Sum ∠s of Δ
d = 58° Sum of ∠s in Δ
e = 29° 2 × ∠ at circum = ∠ at centre
f = 29° 2 × ∠ at circum = ∠ at centre
g = 29° Iso Δ (Equal radii)
h = 29° Iso Δ (Equal radii)
i = 122° Sum of ∠s in Δ

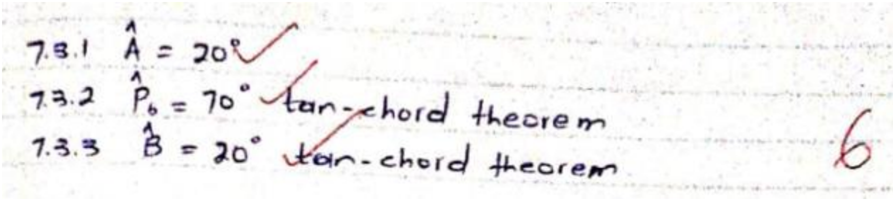
6. a = 35° 2 × ∠ at circum = ∠ at centre
b = 35° ∠s of same chord are =
c = 70° ∠s of same chord VR are =
d = 35° 2 × ∠ at circum = ∠ at centre
e = 35° Iso Δ VT = TR
f = 110° opp ∠s of cyclic quad are supp
g = 22° ∠s of the same chord

7.
7.1 90°
7.2 P_4 = 90° ∠ opp to diameter
7.3

4
3
8
7

Scan 2: Written response of E_2

Displayed above, is scan 2 of the written response of participant E_2. She directly applied appropriate circle geometry theorems/converses to obtain right answers to the unknown variables - all written responses were correct. Scan 3 is presented next.



Scan 3: Written response of E_2

The written response displayed above (see scan 3), as well, testified that the participant was able to apply appropriate circle geometry theorems to obtain correct solutions to the given problem and to perform simple geometric proofs correctly. Scans 1, 2 and 3 above, were analysed and discussed below.

Analysis and discussion of participants' written responses

As presented in Table 7.6 above, 31 out of the 32 participants from the experimental group operated at the process stage of circle geometry's mental conception as seen after the researcher thoroughly examined these participants' written responses. This informed that 97% of participants in the experimental group demonstrated enough evidence of mental construction at the process level. To justify this, the researcher has presented the unedited written responses (see scans 1, 2 and 3) by one of the participants in the experimental group. This is presented as evidence of the experimental groups' mental construction at this level. This meant that E_2 was able to recall the correct statement of circle geometry theorems and/or converses of circle geometry theorems and applied them effectively; this implied that she had interiorised such memory into a process.

Substantial proof from the written responses (see scans 1, 2 and 3) testified that she was operating at the process level. As illustrated above, E_2 rightly recalled the appropriate expected circle geometry theorems, then correctly applied the circle geometry theorems and/or converses of theorems to solve problems, directly. For example, from the written responses to question 1 (see scan 1): $t = 59^0$ (tan-chord theorem); $u = 59^0$ (tan-chord theorem); $w = 62^0$ (sum of angles in Δ is 180^0); $z = 62^0$

(sum of angles in Δ is 180^0); $y = 59^0$(sum of angles on straight line is 180^0). She provided the correct responses for the unknown values of $t, u, w, z \& y$ by recalling and applying circle geometry theorems and/or converses of theorems correctly. She also applied basic mathematics concepts – the sum of angles is a triangle/on a straight line, - which are action and process mental constructions. This demonstrates that E_2 was able to recall the correct statement of theorems and then applied them effectively, an indication that she had interiorised such memory into a process. This guided E_2 to obtain correct solutions to the unknown variables, as demanded by the question.

After the researcher examined the written responses by E_2 for question 1, he proceeded to thoroughly peruse the participant's written responses to question 2. Here, the researcher observed that E_2 also rightly, recalled correct statements of theorems/converses of theorems and their applications, confirming that she had interiorised such into a process. For instance, she correctly used the circle geometry theorem "the angle at the centre is equal to twice the angle at the circumference". It was observed that E_2 applied this theorem to correctly, obtain the value of $a = 104^0$. As stated earlier, all the other sections from questions 2 to 7 followed the same trend as the ones already mention above, in question 1 and sub-question 2.1 , thus, the discussions need not to be repeated.

The other question which was different from the rest of the sub-questions and needed attention was sub-question 7.2. For this question, participants were asked to show that BN is the diameter of the smaller circle. It can be observed from the written response by E_2 that $\widehat{P}_4 = 90^0$, and it can also be seen on the diagram for question 7 that $\widehat{P}_4$ is opposite to the side BN. The implication of this is that, then, BN is the diameter of the circle; this is because the angle opposite to BN is 90^0 and is the converse of the "angle in a semi-circle theorem". E_2 rightly applied this concept to perform this simple proof, which is a requirement in the PGD. This was evidence enough to confirm that E_2 was able to recall, prove and perform simple applications of the circle geometry theorems and/or converses. She did this by recalling the correct information on the converse of the theorem and then applied it effectively in her solution, thus, it can be concluded that she had interiorised such memory into a process (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The work done by E_2, is an exemplar of circle geometry mental constructions, participants in the experimental group demonstrated at the process level. Ample evidence contained in the unedited written responses by E_2, testified that she was operating at the process level, in line with the PGD, hence, the researcher rated E_2 as "attained". Table 7.6 portrays that only 1 participant, that is E_{22}, could not demonstrate enough evidence of mental construction at this level and in line with the PGD. In view of this, she was rated as "not attained", and even though E_{22}, provided correct responses in some instances, she got a lot of the mathematical constructions wrong. She could not recall from memory majority of the appropriate expected responses, nor talk of how she applied them to solve the circle geometry problems. Based on this, the researcher was convinced that E_{22} has little competence at the process level, hence, was rated as "not attained" (Wilson & Dubinsky, 2013).

7.4.3 Participants' responses at Object level

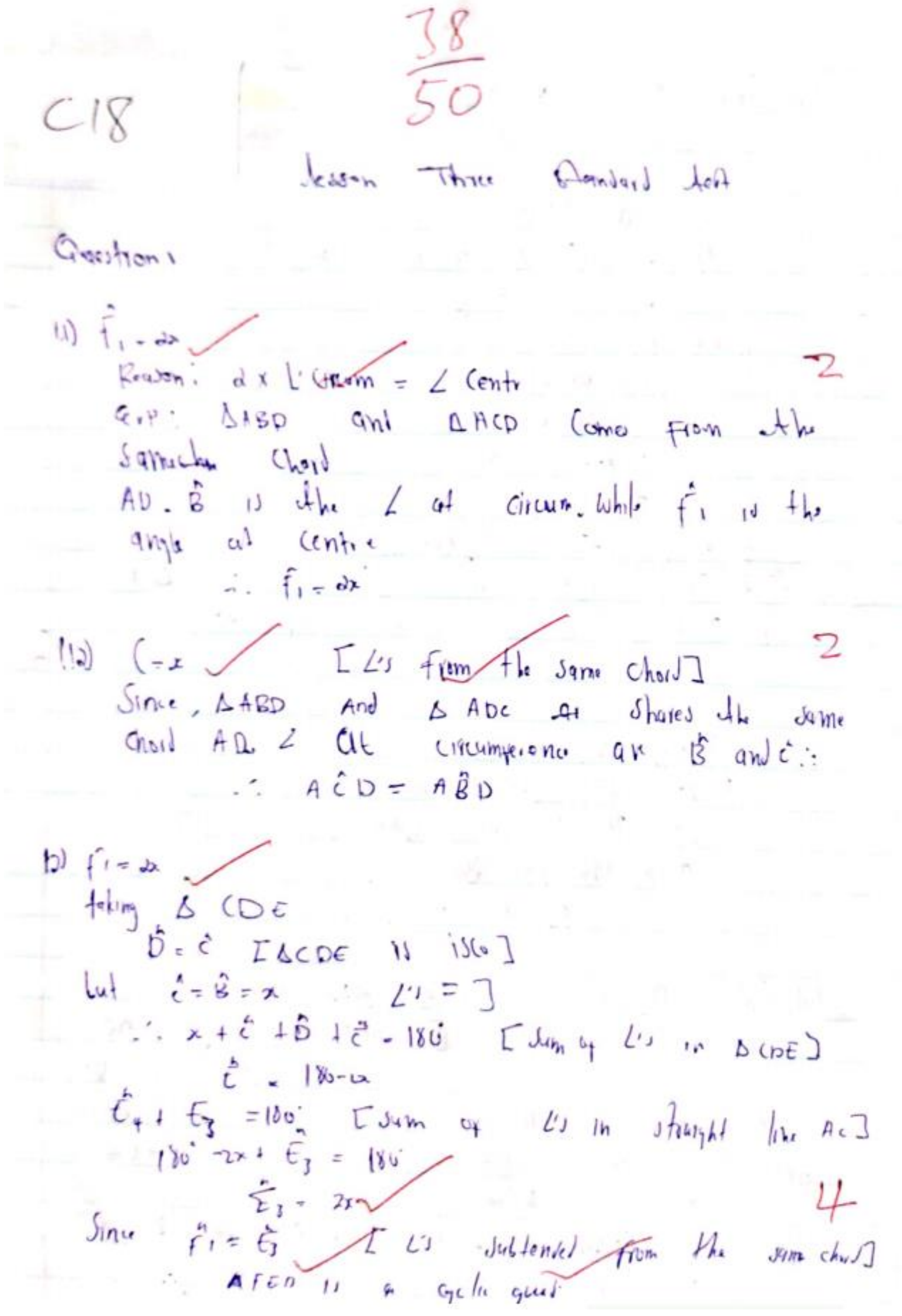
Table 7.7: Participants' responses at object level

Object level	Control group		Experimental group	
	A	NA	A	NA
Participants	10	20	28	4

(i) Control groups' responses at object level

(a) Participants who attained the object level

Vignette 7.5: Exemplar of control groups' written responses at object level


C18 38/50

Lesson Three Standard test

Question 1

1.1) $\hat{F}_1 = 2x$ 2

Reason: 2 x ∠ circum = ∠ centre

e.g.: ΔABD and ΔACD come from the same chord

AD. $\hat{B}$ is the ∠ at circum. While $\hat{F}_1$ is the angle at centre

∴ $\hat{F}_1 = 2x$

1.2) $\hat{C} = x$ [∠'s from the same chord] 2

Since, ΔABD and ΔADC shares the same chord AD ∠ at circumference are $\hat{B}$ and $\hat{C}$ ∴

∴ $A\hat{C}D = A\hat{B}D$

1.3) $\hat{F}_1 = 2x$

taking ΔCDE

$\hat{D} = \hat{C}$ [ΔCDE is isos]

but $\hat{C} = \hat{B} = x$ ∠'s =]

∴ $x + \hat{C} + \hat{D} + \hat{C} = 180°$ [Sum of ∠'s in ΔCDE]

$\hat{C} = 180 - 2x$

$\hat{C}_4 + \hat{E}_3 = 180°$ [Sum of ∠'s in straight line AC]

$180° - 2x + \hat{E}_3 = 180°$

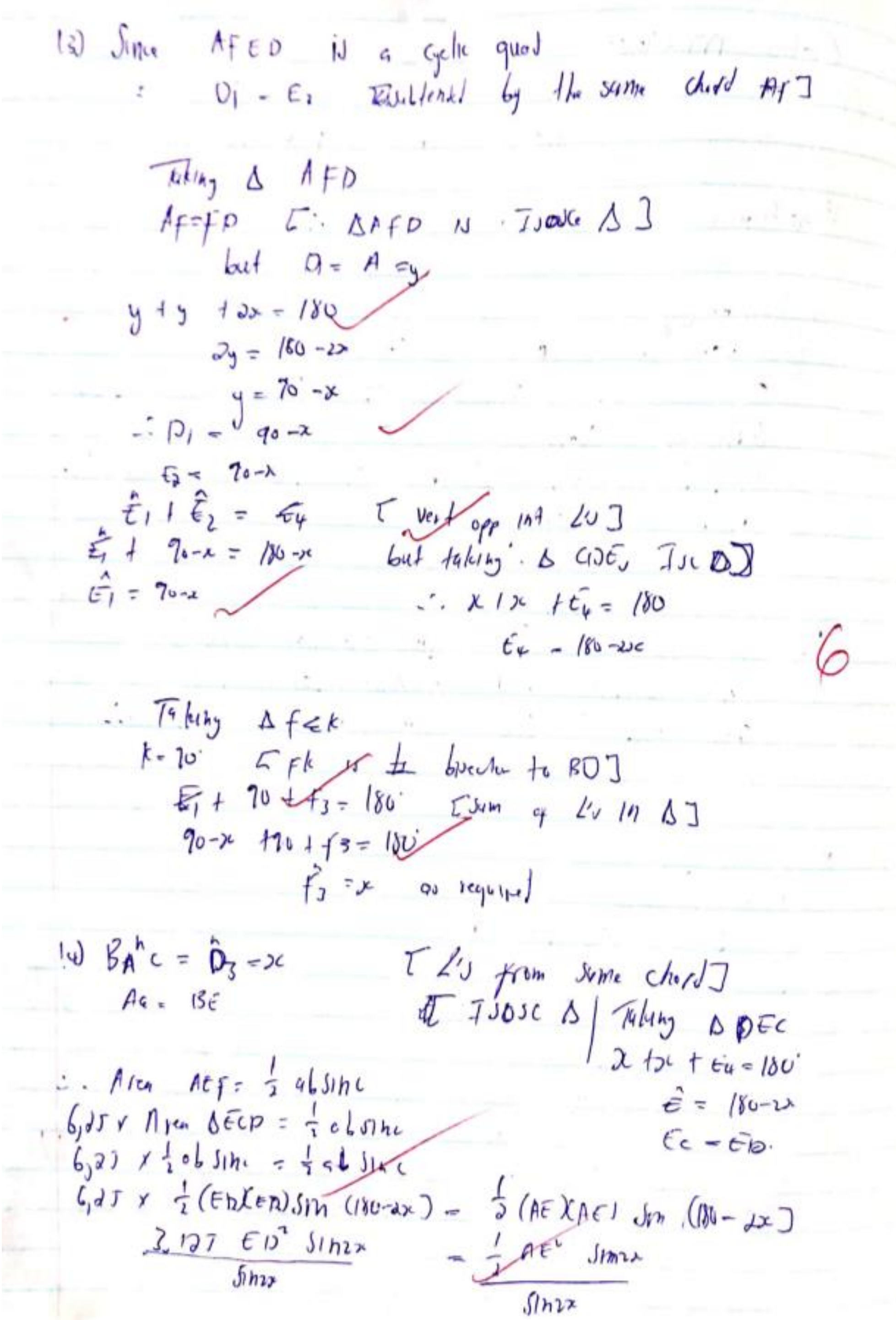
$\hat{E}_3 = 2x$ 4

Since $\hat{F}_1 = \hat{E}_3$ [∠'s subtended from the same chord]

∴ AFED is a cyclic quad

Scan 1: Written response of C_{18}

The above illustration (scan 1), illustrates that the participant was able to brainstorm, as well as logically and creatively think around appropriate circle geometry theorems and/or converses to conjecture meaningful solutions to the given problems. Additional written responses of participant C_{18} for test 3, are presented in scan 2 below.

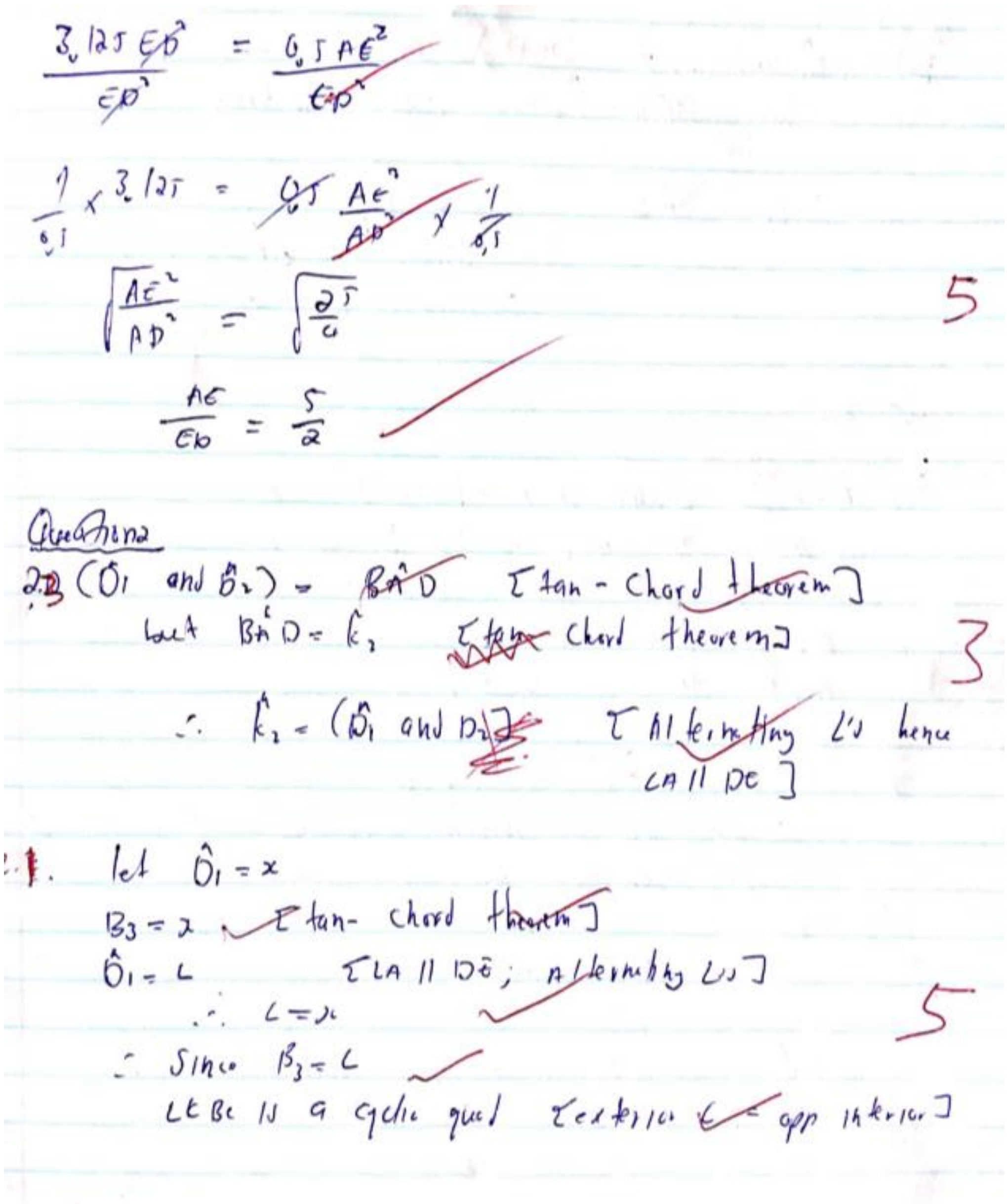

1.3) Since AFED is a cyclic quad
∴ D1 = E2 [subtended by the same chord AF]

Taking Δ AFD
AF=FD [∵ ΔAFD is isosc Δ]
but D1 = A = y
y + y + 2x = 180
2y = 180 − 2x
y = 90 − x
∴ D1 = 90 − x
E2 = 90 − x
Ê1 + Ê2 = E4 [vert opp int ∠s]
Ê1 + 90 − x = 180 − x
Ê1 = 90 − x
but taking Δ CDE, isc Δ]
∴ x + x + E4 = 180
E4 = 180 − 2x
6
∴ Taking Δ FEK
K = 90 [FK is ⊥ bisector to BD]
E1 + 90 + F3 = 180 [sum of ∠s in Δ]
90 − x + 90 + F3 = 180
F3 = x as required

1.4) BÂC = D̂3 = x [∠s from same chord]
AE = BE
[isosc Δ | Taking Δ DEC
x + x + E4 = 180
Ê = 180 − 2x
EC = ED.
∴ Area AEF = ½ ab sin c
6,25 × Area DECD = ½ cb sin c
6,25 × ½ ab sin c = ½ cb sin c
6,25 × ½ (ED)(ED) sin (180 − 2x) = ½ (AE)(AE) sin (180 − 2x)
3,127 ED² sin 2x / sin 2x = ½ AE² sin 2x / sin 2x

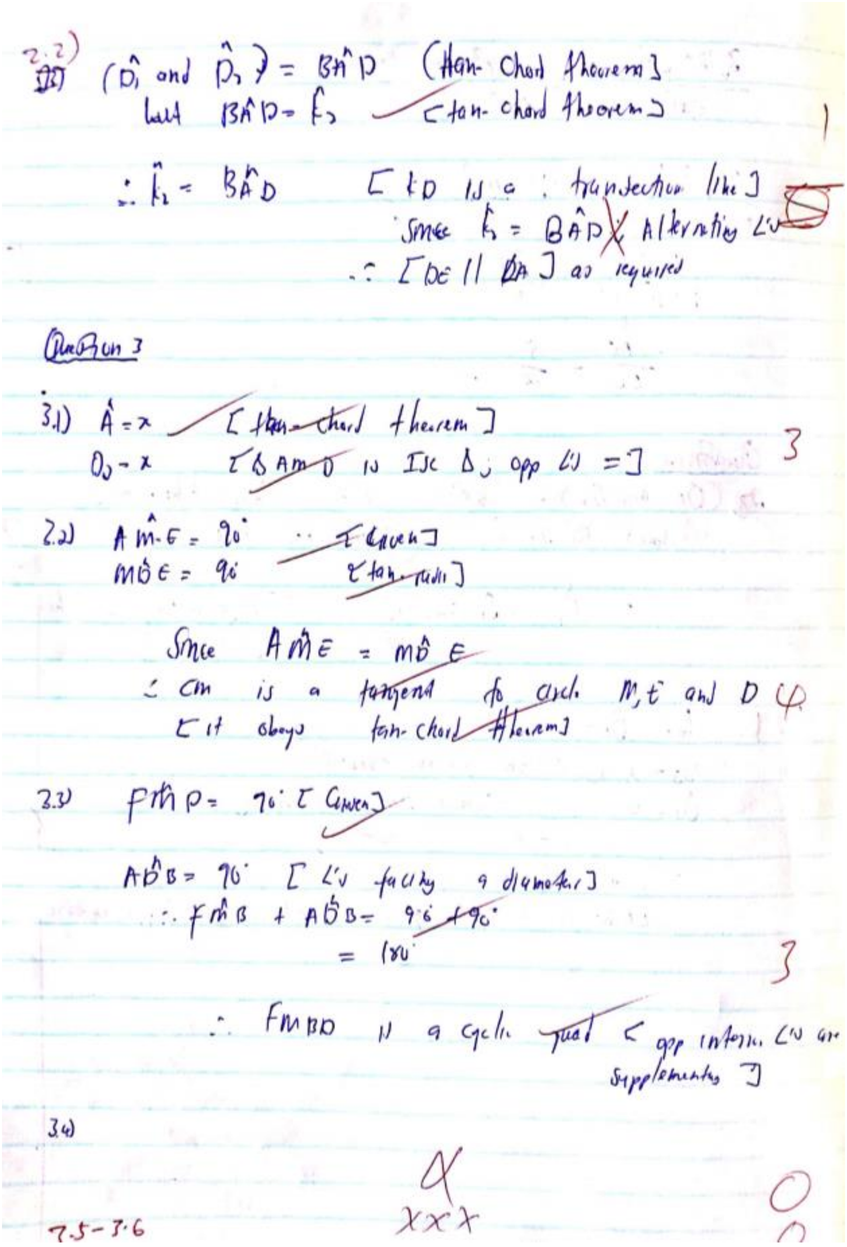
Scan 2: Written response of C_{18}

Participant C_{18} competence in logically applying circle geometry theorems and/or converses to conjecture meaningful solutions to the given problem was also exhibited on scan 2. The next illustration (scan 3) presents the participant's other written response for test 3.



Scan 3: Written response of C_{18}

The above illustration, (scan 3), depicts the participant's ability to brainstorm and creatively think around appropriate circle geometry theorems and geometric concepts so as to conjecture appropriate solutions to the given problems. The other responses of participant C_{18} for test 3, is presented on scan 4 below.


2.2) ($\hat{D}_1$ and $\hat{D}_2$) = $B\hat{A}D$ (tan-chord theorem)
but $B\hat{A}D = \hat{F}_2$ [tan-chord theorem]
∴ $\hat{F}_2 = B\hat{A}D$ [FD is a transection line]
Since $\hat{F}_2 = B\hat{A}D$ alternating ∠s
∴ [DE // BA] as required

Question 3

3.1) $\hat{A} = x$ [tan-chord theorem]
$\hat{D}_2 = x$ [∆AMD is Isc ∆; opp ∠s =]

3.2) $A\hat{M}E = 90°$ [Given]
$M\hat{D}E = 90°$ [tan-radii]
Since $A\hat{M}E = M\hat{D}E$
∴ CM is a tangent to circle M, E and D
[it obeys tan-chord theorem]

3.3) $F\hat{M}P = 90°$ [Given]
$A\hat{D}B = 90°$ [∠s facing a diameter]
∴ $F\hat{M}B + A\hat{D}B = 90° + 90°$
$= 180°$
∴ FMBD is a cyclic quad [opp interior ∠s are supplementary]

3.4)

3.5–3.6

Scan 4: Written response of C_{18}

The last part of the participant's written response for test 3 is illustrated above, on scan 4. It can be observed that he provided both correct and wrong solutions, therefore, he showed average performance. The written responses displayed on scan 4, together with those displayed on scans 1-3, are analysed and discussed below.

Analysis and discussion of participants' written responses

Table 7.7 above, summarized the number of participants who demonstrated enough evidence of circle geometry mental construction at the object level. As displayed in Table 7.7, only 10 out of the 30 participants operated at this level; this represented 33% of the participants in the control group, (see scans 1, 2, 3, and 4). The written response of C_{18}, is displayed as an illustration of the mental constructions, demonstrated by all the participants in this group. This analysis was done, question-by-question from 1 to 3.

Firstly, the researcher examined the written responses of question 1 (see scans 1, 2 & 3). It can be observed that C_{18} was, rightly, able to: (1) prove with reasons, that AFED is a cyclic quadrilateral (sub-question 1.2); (2) prove, with reasons, that $\widehat{F_3} = x$ (sub-question 1.3) and (3) to determine the value of $\frac{AE}{ED}$, given that area $\Delta AEB = 6,25 \times area\ \Delta DEC$ (sub-question 1.4). These are higher-order questions which required thorough geometric reasoning and creative thinking in relation to the applications of circle geometry theorems and concepts. From the written responses to these 3 sub-questions, it can be inferred that C_{18} was able to recall and apply correct statements of theorems and/or converses of theorems; this implied that he had interiorised such memory into a process. He then applied the internalised process by recognising and applying the appropriate circle geometry theorems and/or converses of theorems, imaginatively. The implication of this is that he had encapsulated the internalised process into a cognitive object (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

Secondly, the researcher examined the written responses to question 2 by C_{18} (see scans 3 & 4). C_{18} correctly proved that LKBC is a cyclic quadrilateral (sub-question 2.1) and he was also able to prove that DE ∥ LA (sub-question 2.3). However, he was not able to prove that $\widehat{B_2} = L\hat{A}D$ (sub-question 2.2), although he attempted to solve this sub-question, the researcher realised that his mathematical constructions were incorrect. With reference to the written responses to sub-questions 2.1 & 2.3, it can be seen that C_{18} was able to apply the internalised process by recognising and applying the appropriate circle geometry theorems and/or converses and the relevant geometric concepts - cyclic quadrilaterals and angle properties of parallel lines with a transversal (co-interior angles, corresponding angles, alternate angles and vertical opposite angles). C_{18} was able to recall and apply correct statements of theorems and/or converse of theorems. This implied that he had interiorised such memory into a process; he then applied the internalised process by recognising and applying the appropriate circle geometry theorems and/or converses of theorems imaginatively, thus, implying that he had encapsulated the internalised process into a cognitive object (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

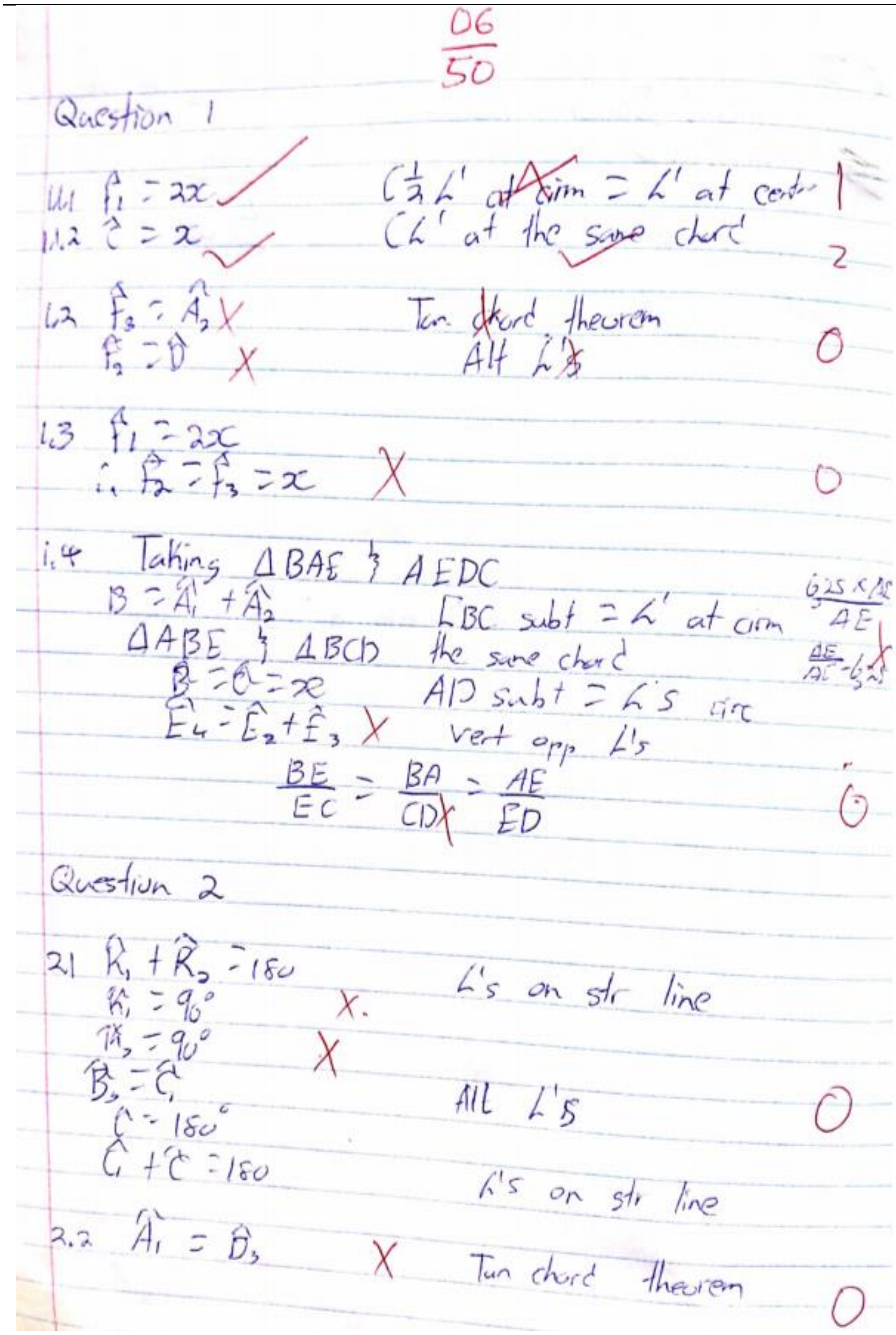
Thirdly, the researcher thoroughly examined the written responses by C_{18} to question 3 (see scan 4). C_{18} was able to prove that: CM is a tangent at M to the circle passing through M, E and D (sub-question 3.2); FMBD is a cyclic quadrilateral (sub-question 3.3); and $DC^2 = 5BC^2$ (sub-question 3.4). He, however, was unable to provide any responses to sub-questions 3.4, 3.5 & 3.6. With the written responses to sub-questions 3.1, 3.2 & 3.3, it can be seen that C_{18} was able to recall and apply the process conception by recognising and applying the appropriate circle geometry theorems and/or converses and the relevant geometric concepts. The implication of this is that he had encapsulated the internalised process into a cognitive object (Maharaj, 2010;

Brijlall & Ndlazi, 2019; Brijlall, 2020).

The analysis of the written responses by C_{18} above testified that indeed, he was operating at the object level of circle geometry mental conception, in line with the PGD. He could not provide appropriate responses to sub-questions 3.4, 3.5 & 3.6, however, he provided enough evidence of mental construction at the object level, for the majority of the questions, hence, the researcher rated C_{18} as having "attained" the object level of circle geometry mental construction.

(b) Participants who did not attain the object level

Vignette 7.6: Exemplar of control groups' written responses at object level


06/50

Question 1

1.1.1 $\hat{F}_1 = 2x$ C ½ ∠' at circ = ∠' at centre 1

1.1.2 $\hat{C} = x$ C∠' at the same chord 2

1.2 $\hat{F}_3 = \hat{A}_2$ X Tan chord theorem

$\hat{F}_2 = \hat{D}$ X Alt ∠'s 0

1.3 $\hat{F}_1 = 2x$

∴ $\hat{F}_2 = \hat{F}_3 = x$ X 0

1.4 Taking ΔBAE & ΔEDC

$B = \hat{A}_1 + \hat{A}_2$ EBC subt = ∠' at circ

ΔABE & ΔBCD the same chord

$\hat{B} = \hat{C} = x$ AD subt = ∠'s circ

$\hat{E}_4 = \hat{E}_2 + \hat{E}_3$ X vert opp ∠'s

$\frac{BE}{EC} = \frac{BA}{CD} = \frac{AE}{ED}$ X 0

Question 2

2.1 $\hat{R}_1 + \hat{R}_2 = 180$ ∠'s on str line

$\hat{R}_1 = 90°$ X.

$\hat{R}_2 = 90°$ X

$\hat{B}_3 = \hat{C}_1$ Alt ∠'s 0

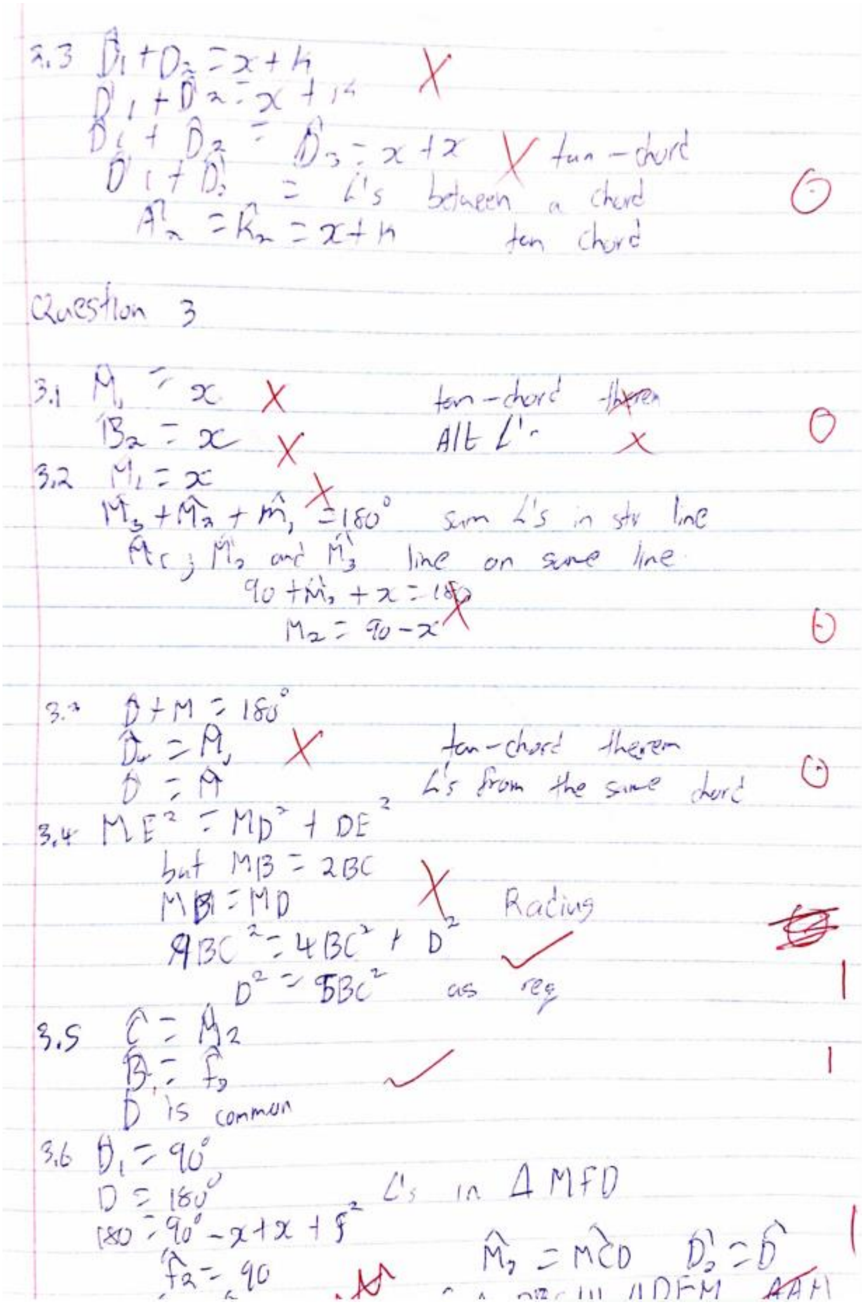
$\hat{C} = 180°$

$\hat{C}_1 + \hat{C} = 180$ ∠'s on str line

2.2 $\hat{A}_1 = \hat{B}_3$ X Tan chord theorem 0

Scan 1: Written response of C_2

The illustration above (scan 1) portrays a participant's written responses to the questions indicated. He provided mostly irrelevant responses and scored zero for most sub-questions. This reveals that he was unable to conjecture meaningful and appropriate solutions to the given problems. The next illustration (scan 2) presents the participant's written response to test 3.



Scan 2: Written response of C_2

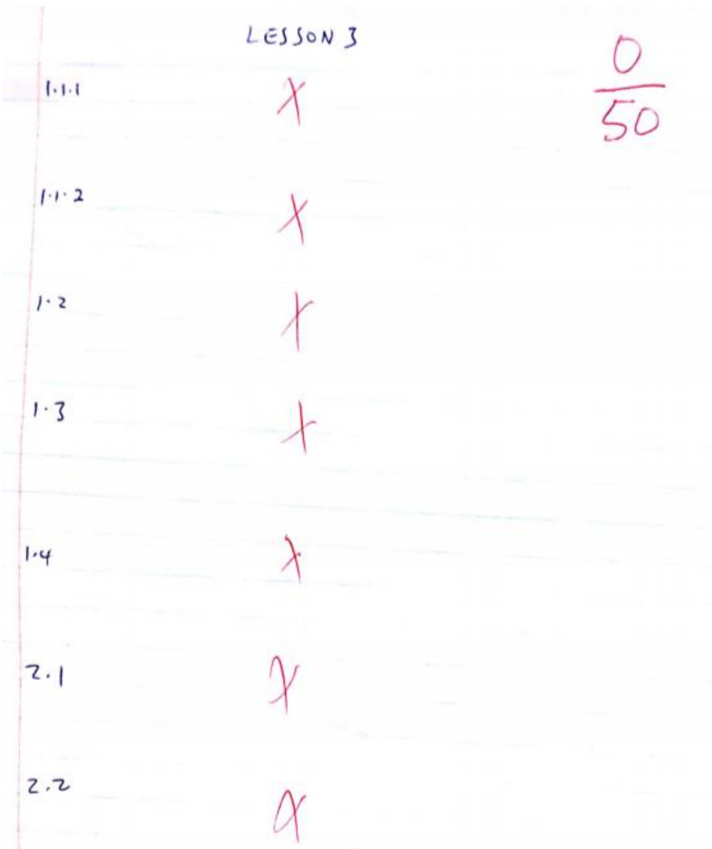
The participant's written response (see scan 2) also confirms what was observed on scan 1 - he provided mostly irrelevant responses, hence, he scored zero for most sub-questions. He demonstrated lack of understanding of circle geometry concepts as he could not logically and meaningfully conjecture appropriate responses to the given problems. The analysis and discussion of participant's written responses, displayed on scans 1 & 2 above are next.

Analysis and discussion of participants' written responses

Table 7.7 above, informs that 20 out of the 30 control group participants, could not attain the object level of circle geometry mental conception. This represented 67%, the majority, of the study participants in the control group. These participants demonstrated little or no conception and lack of understanding of circle geometry concepts at the object level. They either provided irrelevant responses or they provided no responses at all. They demonstrated no evidence of higher-order geometric reasoning and creative thinking of circle geometry theorems and concepts. The few of these participants could partly recall and apply the correct statement of theorems and/or converse of them. There were improper and non-meaningful connections in their written responses. They were out of line with reference to the expectations of the PGD at the object level. The written responses (see scans 1 & 2) were displayed as evidence to highlight their mental construction at the object level (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The written response by C_2 showed that he partly recognised the process level of mental conception by identifying and recognising the appropriate circle geometry theorem(s) required. For instance, in sub-question 1.1.1, C_2 stated that $\widehat{F_1} = 2x$, but, the corresponding reason was incorrect. He stated " $\frac{1}{2} < at\ circumference = < at\ centre''$ as the reason instead of stating that "$\frac{1}{2} < at\ centre = < at\ circumference''$. Also, in sub-question 1.1.2, he stated that $\widehat{c_1} = x$, with the corresponding reason," angles from the same chord", which is correct. This testified that C_2 could not encapsulate the internalised process into a cognitive object. Based on the above evidence, the researcher, judged that C_2 has demonstrated little competence of circle geometry concepts at the object level, hence, the researcher rated C_2 as "not attained". The written responses by the other 19 participants in the control group were similar to those of C_2, hence, there is no need for a repetition of the explanations. The researcher presented the written responses of C_2 as an exemplar of the written responses of the 20 participants who did not attain this level of circle geometry mental conception (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020). The second exemplar to highlight the control groups' written responses of participants who did not attain the object level, is presented next as Vignette 7.7.

Vignette 7.7: Exemplar of control groups' written responses at object level



Scan 1: Written response of C_{17}

The participant's answer sheet (see scan 1) show that she left all the questions unanswered. This characterizes lack of understanding, lack of confidence and technique to approach circle geometry problems. The above scan is analysed and discussed below.

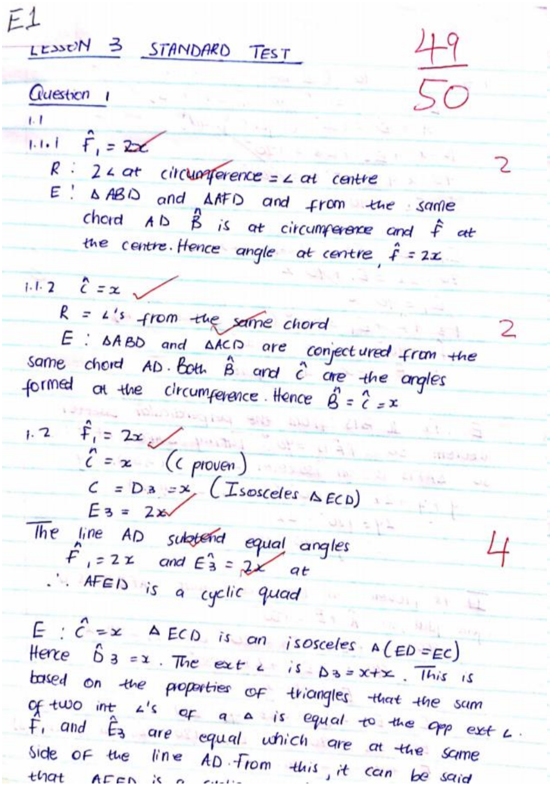
Analysis and discussion of participants' written responses

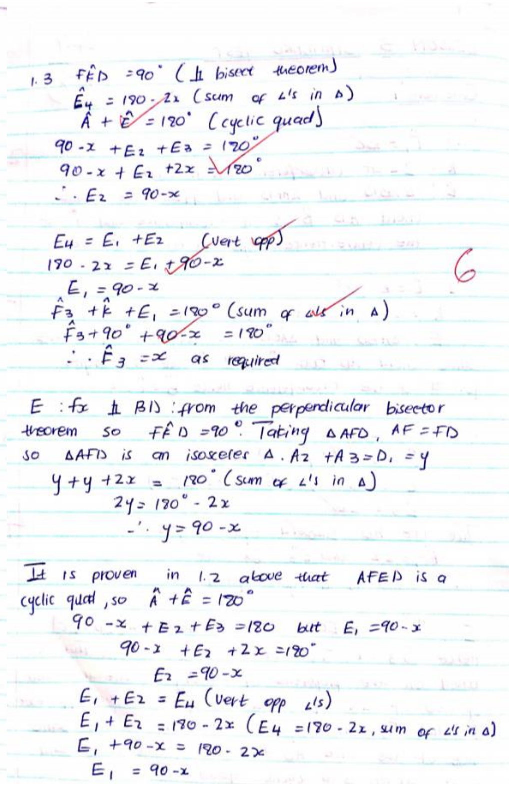
This participant scored zero for not providing any responses at all; scan 1 above serves as an exemplar of the work done by this participant and the rest of the group. This gave the researcher, a reason to conclude that C_{17} showed no understanding of the circle geometry concepts at the object level (Wilson & Dubinsky, 2013).

(ii) Experimental groups' responses at object level

(a) Participants who attained the object level

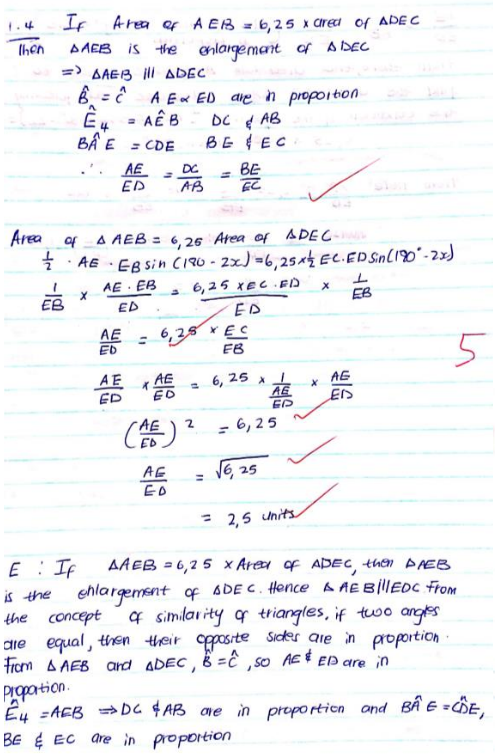
Vignette 7.8: Exemplar of experimental groups' written responses at object level





Scan 1: Written response of E_1 Scan 2: Written response of E_1

On scan 1 above, it can be observed that the participant was able to provide appropriate geometric statements, reasons and explanations to each question. This is an indication that she was able to brainstorm, logically and creatively think around appropriate circle geometry theorems and/or converses so as to conjecture meaningful solutions to the given problem. The written responses on scan 2 above, is a replica of scan 1, hence they share the same characteristics, as narrated above. The other written responses of participant E_1 to test 3 are presented on scans 3 and 4 below.

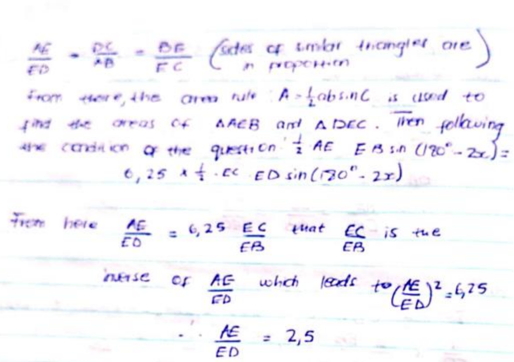


1.4 If Area of AEB = 6,25 x area of ΔDEC
Then ΔAEB is the enlargement of ΔDEC
⇒ ΔAEB ||| ΔDEC
$\hat{B} = \hat{C}$ AE & ED are in proportion
$\hat{E}_4 = A\hat{E}B$ DC & AB
$B\hat{A}E = CDE$ BE & EC
$\therefore \frac{AE}{ED} = \frac{DC}{AB} = \frac{BE}{EC}$

Area of ΔAEB = 6,25 Area of ΔDEC
$\frac{1}{2} \cdot AE \cdot EB \sin(180 - 2x) = 6,25 \times \frac{1}{2} EC \cdot ED \sin(180° - 2x)$
$\frac{1}{EB} \times \frac{AE \cdot EB}{ED} = \frac{6,25 \times EC \cdot ED}{ED} \times \frac{1}{EB}$
$\frac{AE}{ED} = 6,25 \times \frac{EC}{EB}$
$\frac{AE}{ED} \times \frac{AE}{ED} = 6,25 \times \frac{1}{\frac{AE}{ED}} \times \frac{AE}{ED}$
$\left(\frac{AE}{ED}\right)^2 = 6,25$
$\frac{AE}{ED} = \sqrt{6,25}$
= 2,5 units

5

E: If ΔAEB = 6,25 x Area of ΔDEC, then ΔAEB is the enlargement of ΔDEC. Hence ΔAEB|||EDC. From the concept of similarity of triangles, if two angles are equal, then their opposite sides are in proportion. From ΔAEB and ΔDEC, $\hat{B} = \hat{C}$, so AE & ED are in proportion. $\hat{E}_4$ = AEB ⇒ DC & AB are in proportion and $B\hat{A}E = C\hat{D}E$, BE & EC are in proportion

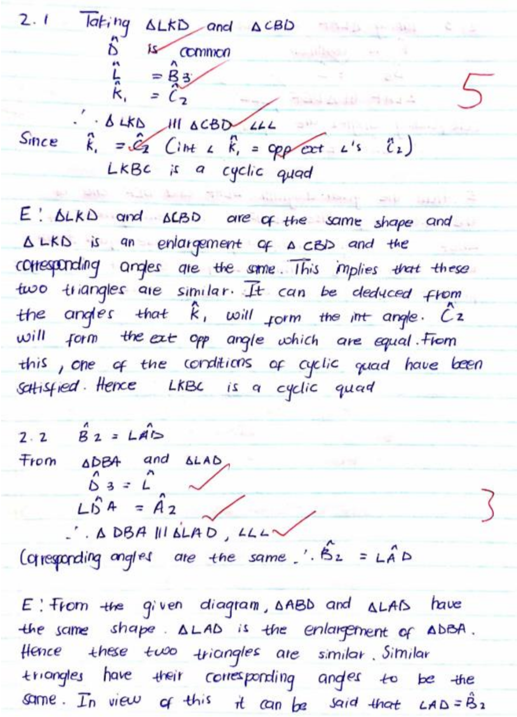


$\frac{AE}{ED} = \frac{DC}{AB} = \frac{BE}{EC}$ (sides of similar triangles are in proportion)
From here, the area rule $A = \frac{1}{2}ab\sin C$ is used to find the areas of ΔAEB and ΔDEC. Then following the condition of the question $\frac{1}{2} AE \; EB \sin(180° - 2x) = 6,25 \times \frac{1}{2} \cdot EC \; ED \sin(180° - 2x)$

From here $\frac{AE}{ED} = 6,25 \frac{EC}{EB}$ that $\frac{EC}{EB}$ is the inverse of $\frac{AE}{ED}$ which leads to $\left(\frac{AE}{ED}\right)^2 = 6,25$
$\therefore \frac{AE}{ED} = 2,5$

Scan 3: Written response of E_1 Scan 4: Written response of E_1

Likewise, on scan 3, it can be observed that the participant was able to provide appropriate geometric statements, reasons and explanations to each question. She was able to brainstorm around appropriate circle geometry theorems logically and creatively and/or conjecture appropriate solutions to the given problem. Scan 4 presented a continuation of the explanation as to how the participant conjectured an appropriate solution to sub-question 1.4 of test 3. The other written responses of participant E_1 to test 3 are presented on scans 5 and 6 below.

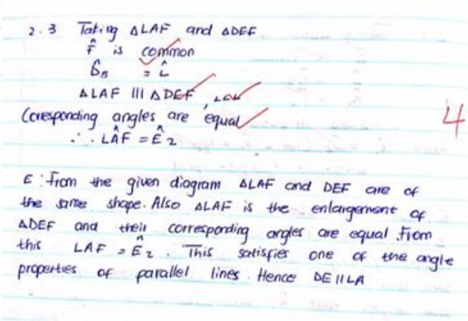


2.1 Taking ΔLKD and ΔCBD
$\hat{D}$ is common
$\hat{L} = \hat{B}_3$
$\hat{K}_1 = \hat{C}_2$
∴ ΔLKD ||| ΔCBD ∠∠∠
Since $\hat{K}_1 = \hat{C}_2$ (int ∠ $\hat{K}_1$ = opp ext ∠'s $\hat{C}_2$)
LKBC is a cyclic quad

E: ΔLKD and ΔCBD are of the same shape and ΔLKD is an enlargement of ΔCBD and the corresponding angles are the same. This implies that these two triangles are similar. It can be deduced from the angles that $\hat{K}_1$ will form the int angle. $\hat{C}_2$ will form the ext opp angle which are equal. From this, one of the conditions of cyclic quad have been satisfied. Hence LKBC is a cyclic quad

2.2 $\hat{B}_2 = L\hat{A}D$
From ΔDBA and ΔLAD
$\hat{D}_3 = \hat{L}$
$L\hat{D}A = \hat{A}_2$
∴ ΔDBA ||| ΔLAD, ∠∠∠
Corresponding angles are the same ∴ $\hat{B}_2 = L\hat{A}D$

E: From the given diagram, ΔABD and ΔLAD have the same shape. ΔLAD is the enlargement of ΔDBA. Hence these two triangles are similar. Similar triangles have their corresponding angles to be the same. In view of this it can be said that $LAD = \hat{B}_2$

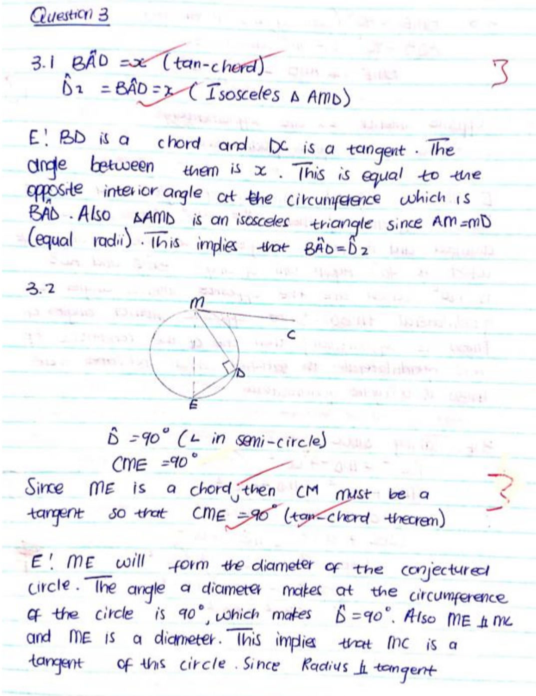


2.3 Taking ΔLAF and ΔDEF
$\hat{F}$ is common
$\hat{B}_5 = \hat{L}$
ΔLAF ||| ΔDEF, ∠∠∠
Corresponding angles are equal
∴ $L\hat{A}F = \hat{E}_2$

E: From the given diagram ΔLAF and DEF are of the same shape. Also ΔLAF is the enlargement of ΔDEF and their corresponding angles are equal. From this $LAF = \hat{E}_2$. This satisfies one of the angle properties of parallel lines. Hence DE ∥ LA

Scan 5: Written response of E_1 Scan 6: Written response of E_1

It is evident from scans 5 and 6 that the participant also provided appropriate circle geometry theorems, reasons and explanations to each sub-question. The other written responses of participant E_1 to test 3 are presented on scans 7 and 8 below.



Question 3

3.1 BAD = x (tan-chord)

D2 = BAD = x (Isosceles Δ AMD)

E: BD is a chord and DC is a tangent. The angle between them is x. This is equal to the opposite interior angle at the circumference which is BAD. Also ΔAMD is an isosceles triangle since AM = MD (equal radii). This implies that BAD = D2

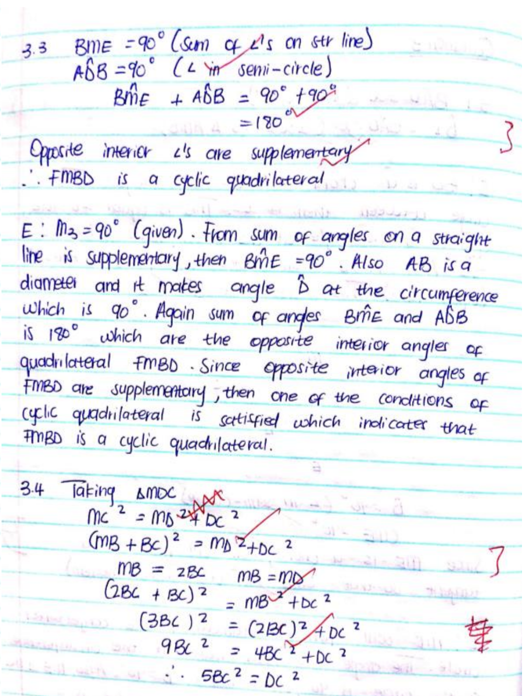
3.2

D = 90° (∠ in semi-circle)

CME = 90°

Since ME is a chord, then CM must be a tangent so that CME = 90° (tan-chord theorem)

E: ME will form the diameter of the conjectured circle. The angle a diameter makes at the circumference of the circle is 90°, which makes D = 90°. Also ME ⊥ MC and ME is a diameter. This implies that MC is a tangent of this circle. Since Radius ⊥ tangent



3.3 BME = 90° (Sum of ∠'s on str line)

ADB = 90° (∠ in semi-circle)

BME + ADB = 90° + 90°

= 180°

Opposite interior ∠'s are supplementary

∴ FMBD is a cyclic quadrilateral

E: M3 = 90° (given). From sum of angles on a straight line is supplementary, then BME = 90°. Also AB is a diameter and it makes angle D at the circumference which is 90°. Again sum of angles BME and ADB is 180° which are the opposite interior angles of quadrilateral FMBD. Since opposite interior angles of FMBD are supplementary, then one of the conditions of cyclic quadrilateral is satisfied which indicates that FMBD is a cyclic quadrilateral.

3.4 Taking ΔMDC

$MC^2 = MD^2 + DC^2$

$(MB + BC)^2 = MD^2 + DC^2$

$MB = 2BC$ $MB = MD$

$(2BC + BC)^2 = MB^2 + DC^2$

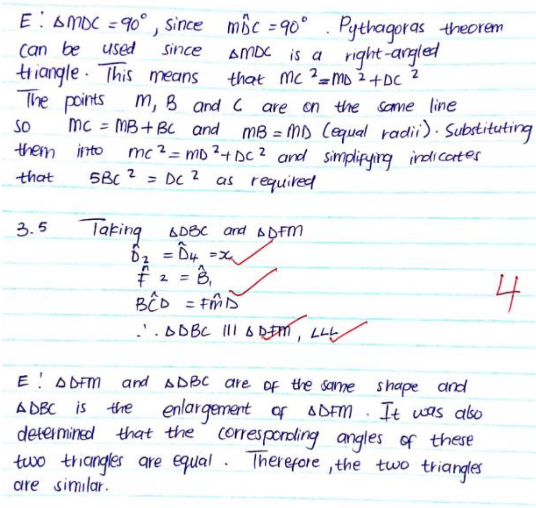
$(3BC)^2 = (2BC)^2 + DC^2$

$9BC^2 = 4BC^2 + DC^2$

$\therefore 5BC^2 = DC^2$

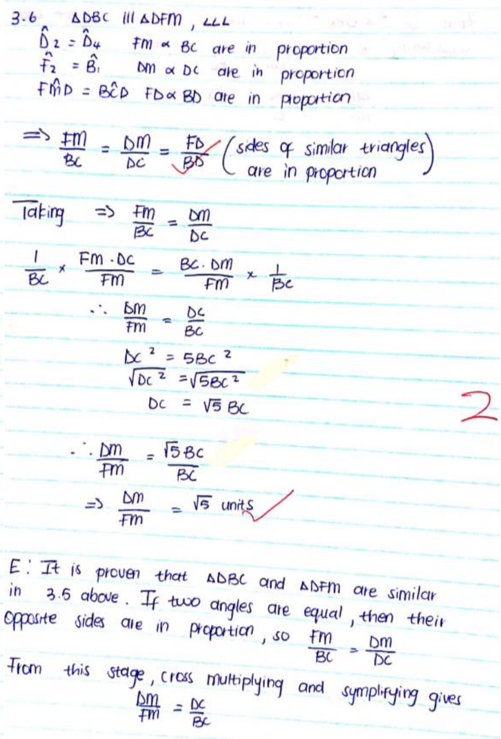
Scan 7: Written response of E_1 Scan 8: Written response of E_1

As said earlier, participant E_1 demonstrated adequate understanding of circle geometry concepts by applying relevant circle geometry theorems, reasons and explanations to each of the sub-questions (see scan 7). The same can be said about the written responses on scan 8. The other written responses of participant E_1 to test 3 are presented on scans 9 and 10 below.


E: $\Delta MDC = 90^\circ$, since $M\hat{D}C = 90^\circ$. Pythagoras theorem can be used since ΔMDC is a right-angled triangle. This means that $MC^2 = MD^2 + DC^2$
The points M, B and C are on the same line so $MC = MB + BC$ and $MB = MD$ (equal radii). Substituting them into $MC^2 = MD^2 + DC^2$ and simplifying indicates that $5BC^2 = DC^2$ as required

3.5 Taking ΔDBC and ΔDFM
$\hat{D}_2 = \hat{D}_4 = x$
$\hat{F}_2 = \hat{B}_1$
$B\hat{C}D = F\hat{M}D$
$\therefore \Delta DBC \parallel\!| \Delta DFM$, $\angle\angle\angle$
4

E: ΔDFM and ΔDBC are of the same shape and ΔDBC is the enlargement of ΔDFM. It was also determined that the corresponding angles of these two triangles are equal. Therefore, the two triangles are similar.


3.6 $\Delta DBC \parallel\!| \Delta DFM$, $\angle\angle\angle$
$\hat{D}_2 = \hat{D}_4$ FM ∝ BC are in proportion
$\hat{F}_2 = \hat{B}_1$ DM ∝ DC are in proportion
$F\hat{M}D = B\hat{C}D$ FD ∝ BD are in proportion

$\Rightarrow \frac{FM}{BC} = \frac{DM}{DC} = \frac{FD}{BD}$ (sides of similar triangles are in proportion)

Taking $\Rightarrow \frac{FM}{BC} = \frac{DM}{DC}$
$\frac{1}{BC} \times \frac{FM \cdot DC}{FM} = \frac{BC \cdot DM}{FM} \times \frac{1}{BC}$
$\therefore \frac{DM}{FM} = \frac{DC}{BC}$
$DC^2 = 5BC^2$
$\sqrt{DC^2} = \sqrt{5BC^2}$
$DC = \sqrt{5}\,BC$
2
$\therefore \frac{DM}{FM} = \frac{\sqrt{5}BC}{BC}$
$\Rightarrow \frac{DM}{FM} = \sqrt{5}$ units

E: It is proven that ΔDBC and ΔDFM are similar in 3.5 above. If two angles are equal, then their opposite sides are in proportion, so $\frac{FM}{BC} = \frac{DM}{DC}$
From this stage, cross multiplying and simplifying gives $\frac{DM}{FM} = \frac{DC}{BC}$

Scan 9: Written response of E_1 Scan 10: Written response of E_1

On scan 9, participant E_1 applied relevant circle geometry theorems, reasons and explanations to conjecture appropriate solutions to sub-question 3.5. This same approach was used to conjecture appropriate solutions to sub-question 3.6 (see scan 10). This highlighted the participant's substantial understanding of circle geometry concepts. This was followed by an analysis and discussion of the written responses of participant E_1 to test 3, illustrated on scans 1-10 above.

Analysis and discussion of participants' written responses

With reference to Table 7.7 above, 28 participants from the experimental group attained the object level of circle geometry mental conception; this represented 88% of participants in the experimental group. These participants were able to demonstrate enough evidence of circle geometry mental construction at the object level. As evidence, the researcher presented the written responses (see scans 1-10) as an exemplar of the mental constructions demonstrated by these 28 participants who were able to attain the object level of circle geometry mental conception. From the written responses by E_1, the researcher performed a question-by-question analysis. These are elucidated below.

First, the researcher rigorously perused participant's written responses to question 1. It can be observed that E_1 comprehensively and convincingly, provided appropriate written responses to this question. Each of the sub-questions in question 1 were solved by not only providing correct geometric statements and applications of basic mathematics concepts, but E_1 , as well, provided appropriate reasons in support of every geometric statement and every calculation made. For instance, in question 1, E_1 was able to prove with correct reasons and justifiable explanations, that AFED is a cyclic quadrilateral (sub-question 1.2); that $\widehat{F_3} = x$ (sub-question 1.3); and to determine the value of $\frac{AE}{ED}$, given that area ΔAEB = 6, 25 × $area\ \Delta DEC$ (sub-question 1.4). These sub-questions required higher-order geometric reasoning and creative thinking in relation to the applications of circle geometry theorems and concepts.

The above testified that she was able to recall from memory the correct statement of theorems and/or converse of theorems in her solutions. The implication of this is that she had encapsulated the internalised process into a cognitive object. In view of the above, E_1 was judged as operating at the object level (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

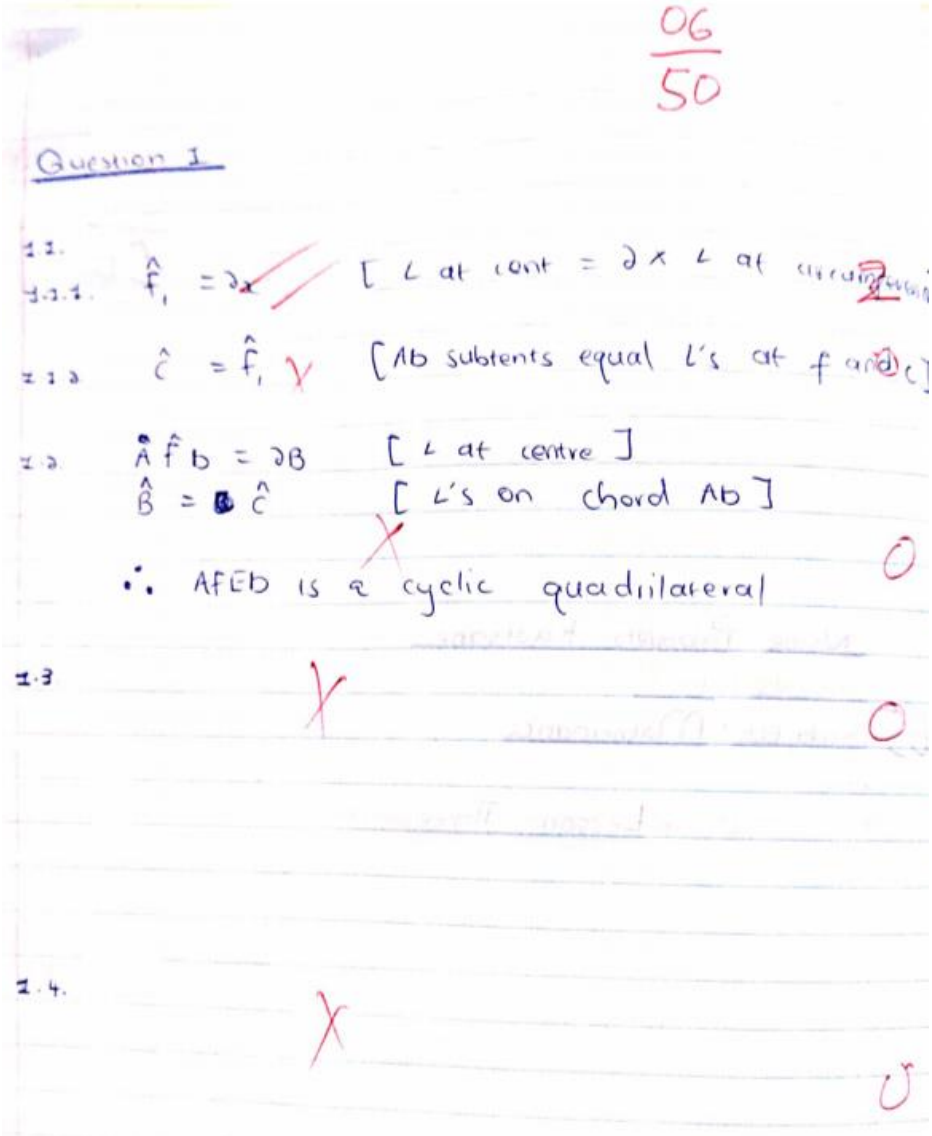
After the researcher analysed the participant's written responses to question 1, he continued to analyse the participant's written responses to question 2. E_1 also buttressed her solutions to all the sub-questions with appropriate reasons and explanations to justify her conjectured solutions. She correctly proved that: LKBC is a cyclic quadrilateral (sub-question 2.1); $\widehat{B_2} = L\hat{A}D$ (sub-question 2.2); DE ∥ LA (sub-question 2.3) among others. This demonstrates that she was able to recall from memory; had interiorised such memory into a process; and can apply the internalised process, imaginatively. The implication of this is that she had encapsulated the internalised process into a cognitive object, as E_1, concisely and meaningfully applied them to conjecture appropriate solutions to this question. This evidence also confirmed that E_1 had substantial understanding of the circle geometry concepts. Based on this evidence, the researcher concluded that E_1 was operating at the object level (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

After the analysis of question 2 was done, the researcher proceeded to analyse the written responses to the last question, that is, question 3. Also, by applying correct geometric statements, E_1 efficiently conjectured appropriate solutions to all the sub-questions in question 3 by supporting the conjectured solutions with relevant reasons and convincing explanations to justify her conjectured solution. Through this, she was able to prove that CM is a tangent at M to the circle passing through M, E and D (sub-question 3.2); to prove that FMBD is a cyclic quadrilateral (sub-question 3.3), $DC^2 = 5BC^2$ (sub-question 3.4), $\Delta DBC \, /// \, \Delta DFM$ (sub-question 3.5); and to determine the value of $\frac{DM}{FM}$ (sub-question 3.6).

Enough evidence from the written responses by E_1 demonstrated that she was operating at the object level, in accordance with the PGD. The written responses of E_1, displayed above, was only a sample of the mental constructions demonstrated by the 28 participants in the experimental group, who had attained the object level of circle geometry mental conception. That 88% of participants attained this level was contrary to the findings by Ndlovu and Brijlall (2015), who established that just a handful of participants operated at the object level.

(b) Participants who did not attain the object level

Vignette 7.9: Exemplar of experimental groups' written responses at object level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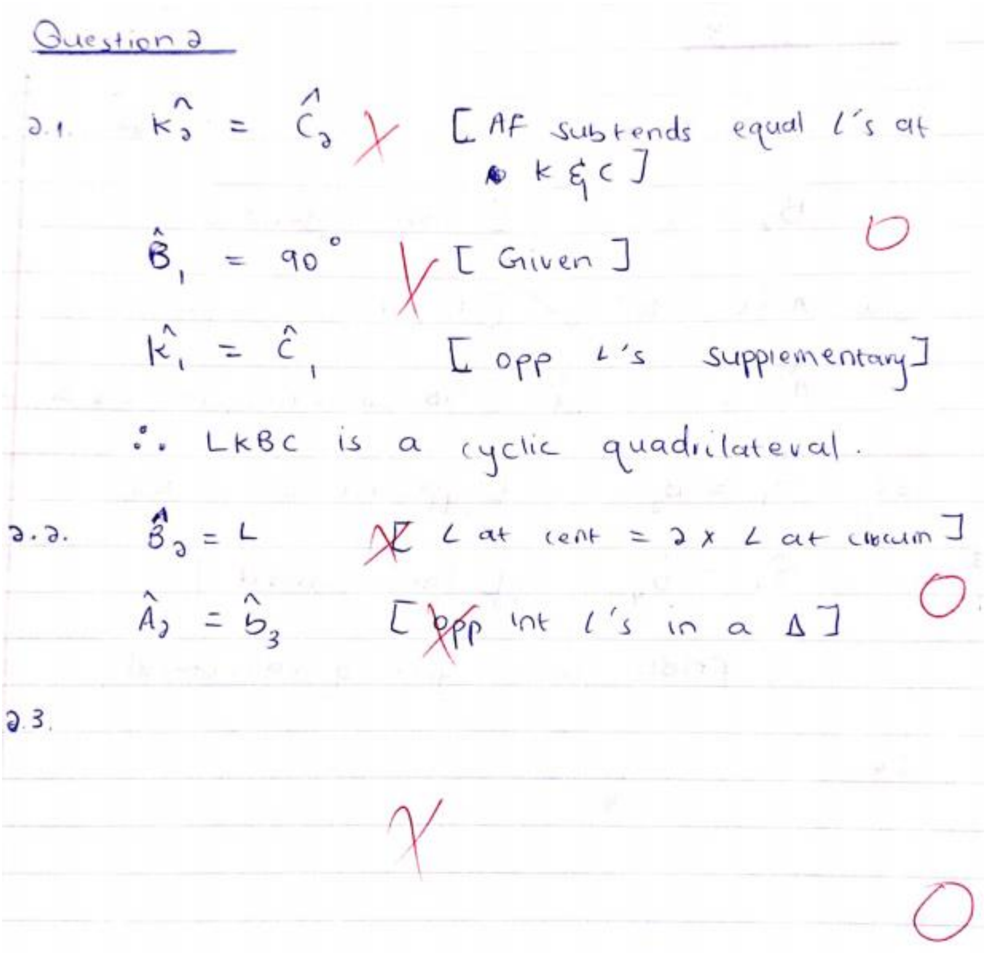
Question 1

1.1.
1.1.1. $\hat{f}_1 = 2x$ [∠ at cent = 2 × ∠ at circumference] 2
1.1.2 $\hat{c} = \hat{f}_1$ [Ab subtents equal ∠'s at f and c]
1.2. $\hat{A}\hat{f}b = 2B$ [∠ at centre]
$\hat{B} = \hat{c}$ [∠'s on chord Ab]
∴ AfEb is a cyclic quadrilateral 0
1.3 0
1.4. 0

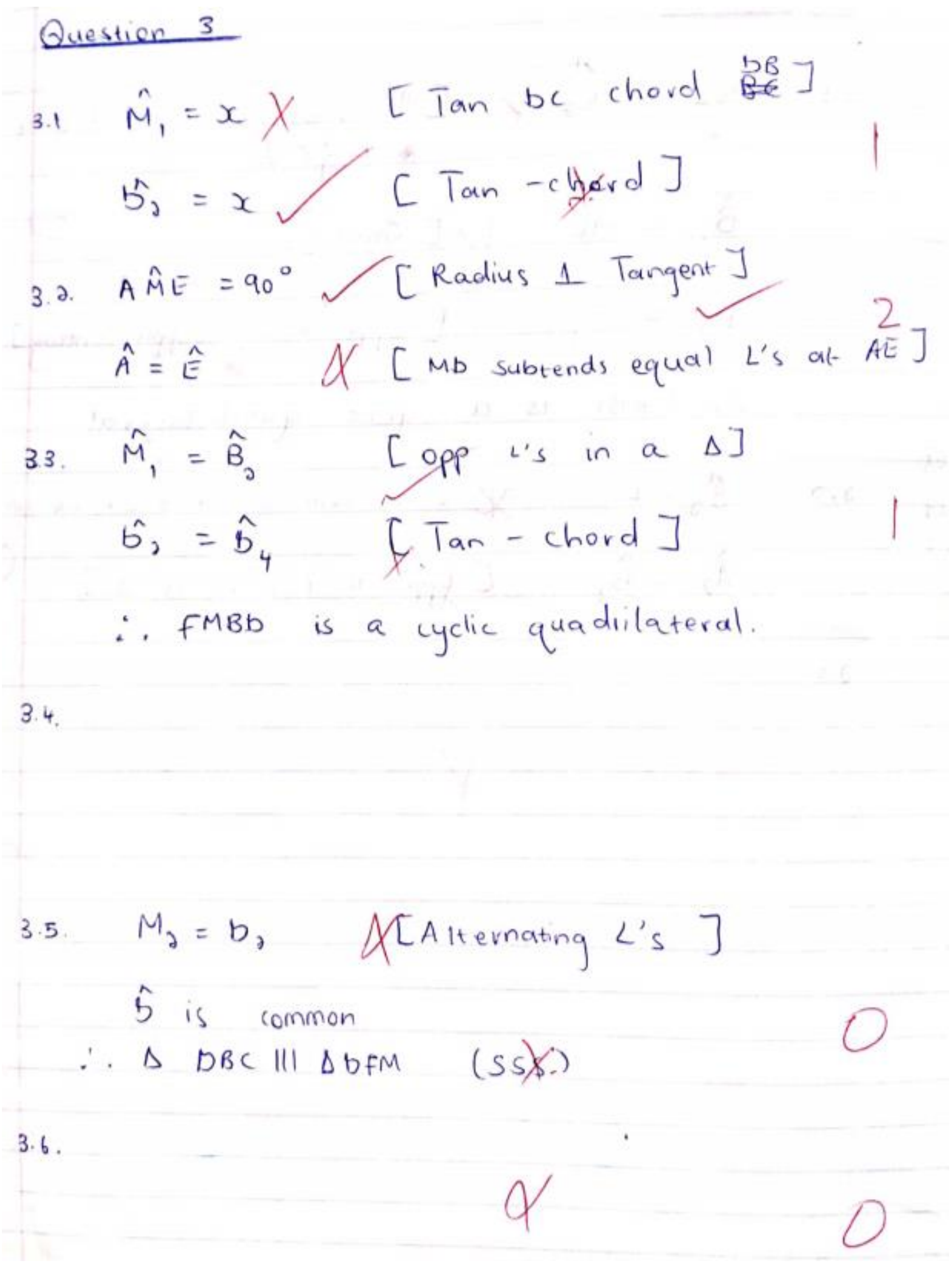
Scan 1: Written response of E_5

It can be observed that participant E_5 mostly provided irrelevant responses or he left the questions unanswered (see scan 1). He got the answer to only one sub-question indicating lack of understanding of circle geometry concepts. This was followed by a presentation of other written responses displayed on scan 2 below.



Scan 2: Written response of E_5

The written response of E_5, displayed on scan 2 above showed that he provided irrelevant responses to the given questions - an indicator of lack of understanding of circle geometry concepts. This is followed other wrong answers displayed on scan 3 below.



Question 3

3.1 $\hat{M}_1 = x$ [Tan bc chord DB]

$\hat{D}_2 = x$ [Tan-chord]

3.2. $A\hat{M}E = 90°$ [Radius ⊥ Tangent]

$\hat{A} = \hat{E}$ [MD subtends equal ∠'s at AE]

3.3. $\hat{M}_1 = \hat{B}_2$ [opp ∠'s in a Δ]

$\hat{D}_2 = \hat{D}_4$ [Tan - chord]

∴ FMBD is a cyclic quadiilateral.

3.4.

3.5. $M_2 = D_2$ [Alternating ∠'s]

$\hat{D}$ is common

∴ Δ DBC ||| Δ DFM (SS)

3.6.

Scan 3: Written response of E_5

On scan 3, participant E_5 either provided irrelevant responses to some sub-questions or did not answer them. He could only partly find appropriate solutions to very few sub-questions. The written responses on scans 1-3 above, are analysed and discussed below.

Analysis and discussion of participants' written responses

As illustrated in Table 7.7 above, 4 participants from the experimental group could not attain the object level of circle geometry mental conception. This was a representation of 12% of the participants who constituted the experimental group. Their written responses were devoid of substantial evidence, to demonstrate competence of circle geometry mental construction at the object level as can be seen in scans 1, 2, 3. These were some of the written responses by the 4 participants in the experimental group who demonstrated little competence of circle geometry mental construction and concepts at this level.

The researcher examined all the written responses by E_5. This established that E_5 made a lot of errors in his/her written response as most mathematical constructions were wrong; he got the responses of only one sub-question correct; that is, sub-question 1.1.1 where he rightly stated that $\widehat{F_1} = 2x$, with the corresponding reason: "$< at\ centre = 2 \times < at\ circumference$". For instance, in sub-question 3.1, he wrongly stated that $\widehat{M_1} = x$ and correctly stated that $\widehat{D_1} = x$ but the corresponding reason (Tan-chord) was wrong. In sub-question 3.2, he rightly stated that $A\widehat{M}E = 90^0$ and the corresponding reason was also correct, but he wrongly wrote that $\hat{A} = \hat{E}$. There were evidence of deficiencies in his geometric reasoning which showed that he could not think creatively and lacked the ability to apply the internalised process (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

Again, in sub-question 3.3, he wrote that $\widehat{M_1} = \widehat{B_2}$, based on the reason "opposite $<' s\ in\ isosceles \triangle$". Both the statement and the reason were correct in this case, but E_5 continued by wrongly stating that $\widehat{D_2} = \widehat{D_4}$ from the tan-chord theorem. In this instance, both the statement and the reason provided by E_5 were wrong. The written responses by E_5 were inadequate with reference to the requirements of the PGD. E_5 demonstrated little conception of circle geometry concepts at the object level, hence, E_5 was rated as "not attained" (Maharaj, 2010; Brijlall & Ndlazi, 2019; Brijlall, 2020).

7.4.4 Participants' responses at Schema level

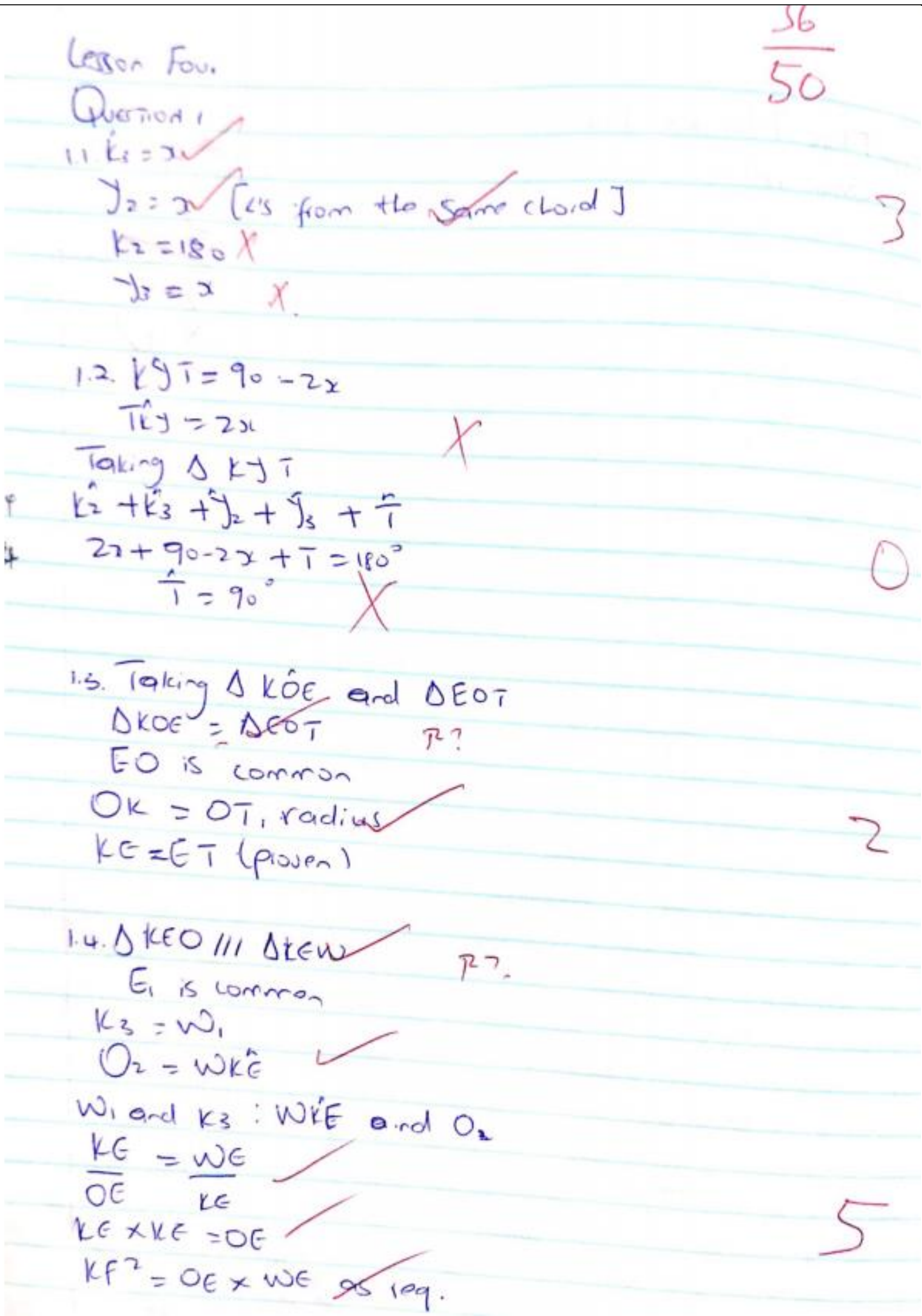
Table 7.8: Participants' responses at schema level

Schema level	Control group		Experimental group	
	A	NA	A	NA
Participants	10	20	26	6

(i) Control groups' responses at schema level

(a) Participants who attained the schema level

Vignette 7.10: Exemplar of control groups' written responses at schema level


Lesson Four.

Question 1

1.1 $\hat{K}_2 = x$

$\hat{Y}_2 = x$ [∠'s from the same chord]

$\hat{K}_2 = 180$

$\hat{Y}_3 = x$

1.2 $K\hat{Y}T = 90 - 2x$

$T\hat{K}Y = 2x$

Taking ΔKYT

$\hat{K}_2 + \hat{K}_3 + \hat{Y}_2 + \hat{Y}_3 + \hat{T}$

$2x + 90 - 2x + T = 180^\circ$

$\hat{T} = 90^\circ$

1.3 Taking $\Delta K\hat{O}E$ and ΔEOT

$\Delta KOE \equiv \Delta EOT$

EO is common

OK = OT, radius

KE = ET (proven)

1.4 $\Delta KEO \,|||\, \Delta KEW$

E_1 is common

$K_3 = W_1$

$O_2 = W\hat{K}E$

W_1 and K_3 : $W\hat{K}E$ and O_2

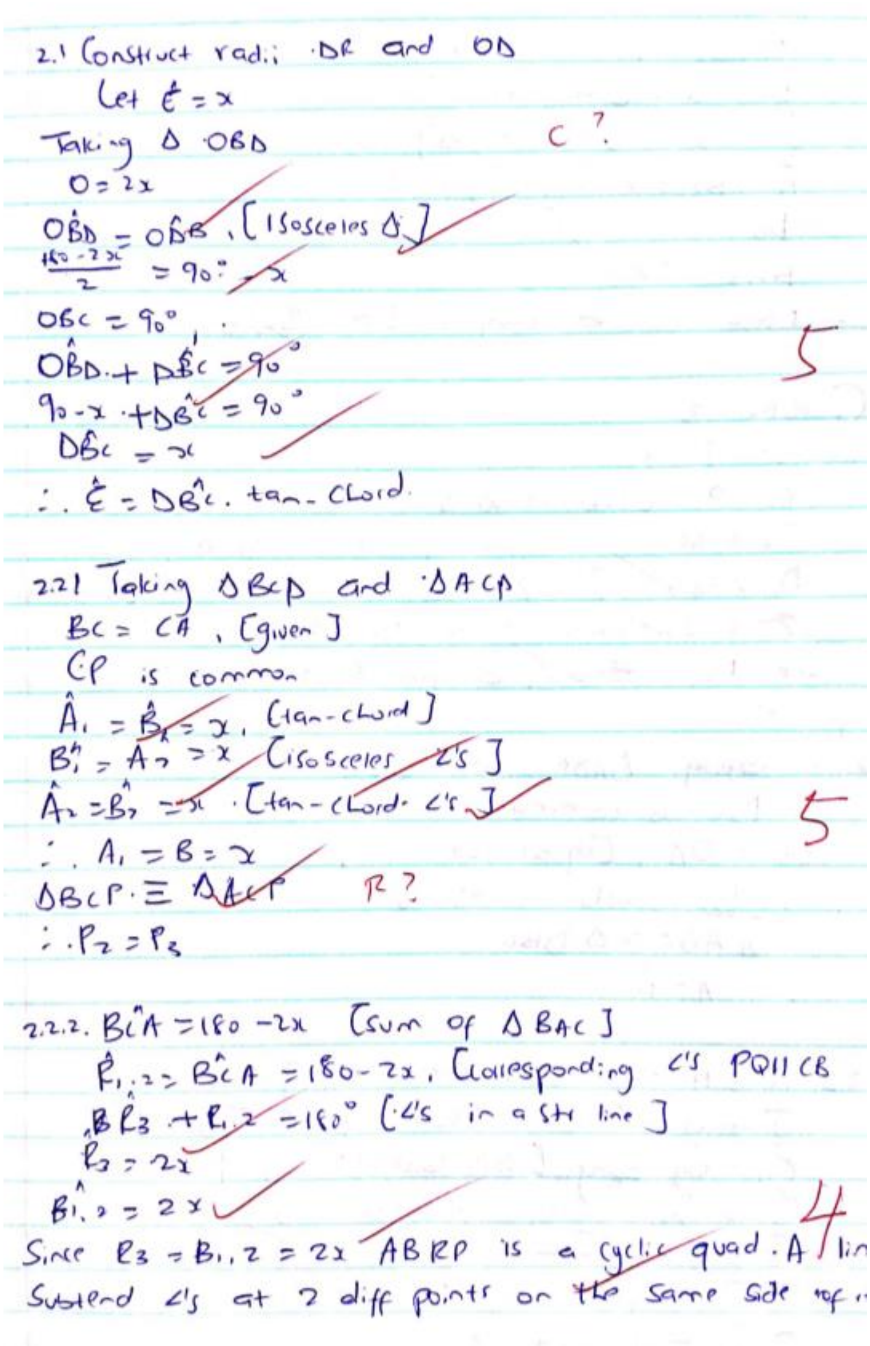
$\frac{KE}{OE} = \frac{WE}{KE}$

$KE \times KE = OE$

$KE^2 = OE \times WE$ as req.

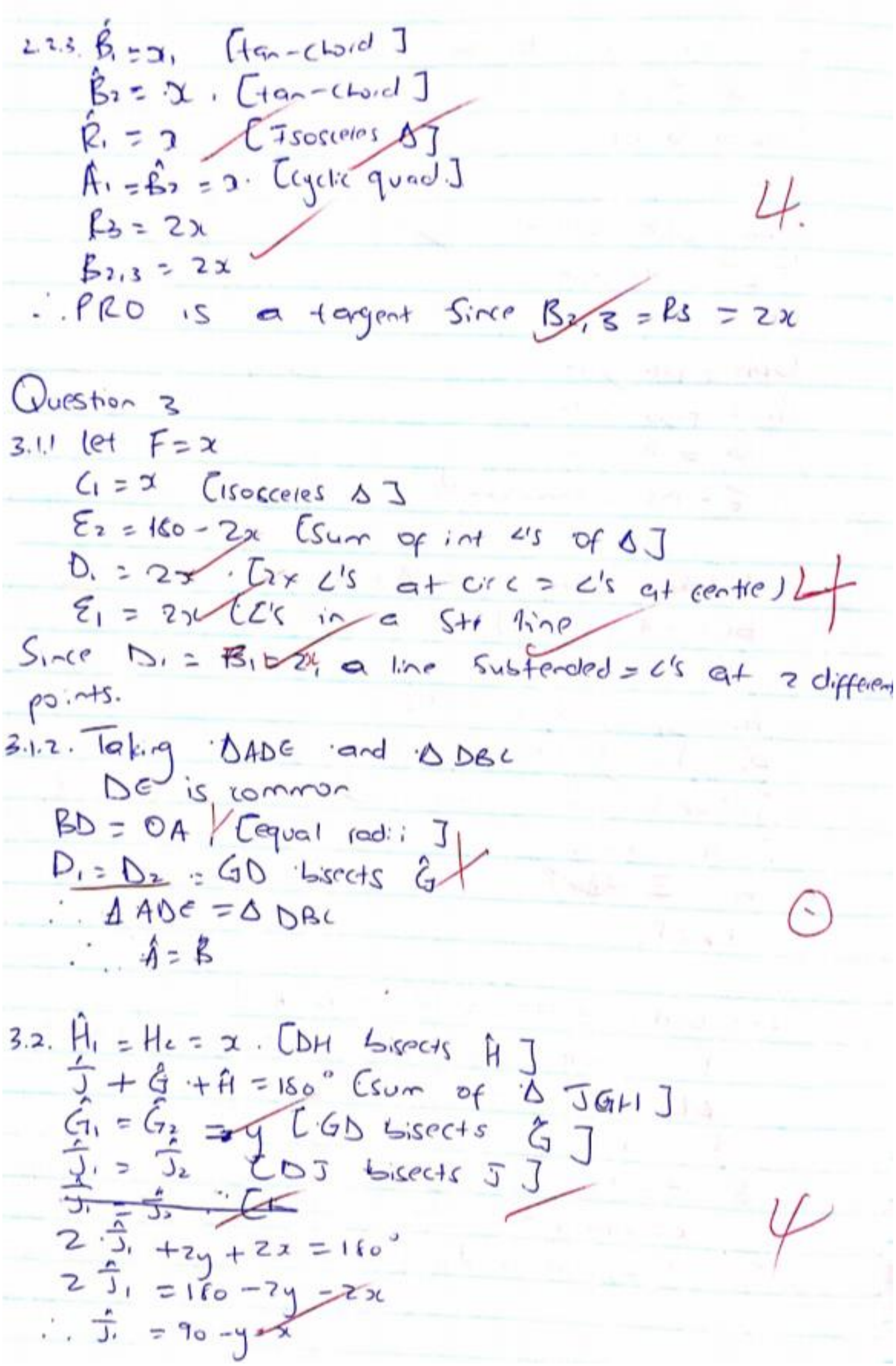
Scan 1: Written response of C_{24}

The above illustration (scan 1), shows that participant C_{24} provided many correct responses to the given questions, although, he also provided wrong solutions to some of the questions showing incorrect mathematical constructions and inability to apply appropriate circle geometry theorems. The other written responses to test 4 is presented next on scan 2.


2.1 Construct radii OR and OD
Let $\hat{E} = x$
Taking Δ OBD
O = 2x
$O\hat{B}D = O\hat{D}B$, [Isosceles Δ]
$\frac{180 - 2x}{2} = 90° - x$
$O\hat{B}C = 90°$
$O\hat{B}D + D\hat{B}C = 90°$
$90 - x + D\hat{B}C = 90°$
$D\hat{B}C = x$
∴ $\hat{E} = D\hat{B}C$. tan-chord
2.2.1 Taking Δ BCP and Δ ACP
BC = CA, [given]
CP is common
$\hat{A}_1 = \hat{B}_1 = x$, [tan-chord]
$\hat{B}_1 = \hat{A}_2 = x$ [isosceles ∠'s]
$\hat{A}_2 = \hat{B}_2 = x$ [tan-chord ∠'s]
∴ $A_1 = B = x$
ΔBCP ≡ ΔACP
∴ $P_2 = P_3$
2.2.2. $B\hat{C}A = 180 - 2x$ [sum of Δ BAC]
$\hat{R}_{1,2} = B\hat{C}A = 180 - 2x$, [corresponding ∠'s PQ ∥ CB]
$B\hat{R}_3 + \hat{R}_{1,2} = 180°$ [∠'s in a str line]
$\hat{R}_3 = 2x$
$\hat{B}_{1,2} = 2x$
Since $R_3 = B_{1,2} = 2x$ ABRP is a cyclic quad. A lin
Subtend ∠'s at 2 diff points on the same side of

Scan 2: Written response of C_{24}

The written response of C_{24} presented above (scan 2), testify that he conjectured appropriate solutions to all the questions, however, there were some few omissions, like not providing appropriate reasons/conditions for two of the sub-questions. The rest of the participant's written responses to test 4 is presented on scan 3 below.



2.2.3. $\hat{B}_1 = x$, [tan-chord]
$\hat{B}_2 = x$. [tan-chord]
$\hat{R}_1 = x$ [Isosceles Δ]
$\hat{A}_1 = \hat{B}_2 = x$. [cyclic quad.]
$R_3 = 2x$
$B_{2,3} = 2x$
∴ PRO is a tangent since $B_{2,3} = R_3 = 2x$

4.

Question 3
3.1.1 let $F = x$
$C_1 = x$ [Isosceles Δ]
$E_2 = 180 - 2x$ [sum of int ∠'s of Δ]
$D_1 = 2x$. [2× ∠'s at circ = ∠'s at centre]
$E_1 = 2x$ [∠'s in a str line
Since $D_1 = B_1 = 2x$, a line subtended = ∠'s at 2 different points.

4

3.1.2. Taking ΔADE and ΔDBC
DE is common
BD = OA [equal radii]
$D_1 = D_2$ ∴ GD bisects Ĝ
∴ ΔADE = ΔDBC
∴ Â = B

0

3.2. $\hat{H}_1 = H_2 = x$. [DH bisects Ĥ]
$\hat{J} + \hat{G} + \hat{H} = 180°$ [sum of Δ JGH]
$\hat{G}_1 = \hat{G}_2 = y$ [GD bisects Ĝ]
$\hat{J}_1 = \hat{J}_2$ [DJ bisects J]
~~$J_1 = J_2$ ∴ [~~
$2\hat{J}_1 + 2y + 2x = 180°$
$2\hat{J}_1 = 180 - 2y - 2x$
∴ $\hat{J}_1 = 90 - y - x$

4

Scan 3: Written response of C_{24}

Displayed above is scan 3 which reveals that participant C_{24} provided many correct responses to the given questions, hence, establishing that he has substantial understanding of circle geometry concepts; the written response of only one of the sub-questions (3.1.2) was wrong. The analysis and discussion of the written responses presented on scans 1-3 above are elaborated below.

Analysis and discussion of participants' written responses

At this juncture, the researcher critically analysed participants' mental constructions at the schema level. Part of the result of this analysis was that, 10 (33%) of participants from the control group demonstrated enough evidence of mental construction at the schema level of circle geometry mental conception. This information is presented in Table 7.8 above. The written responses in scans 1, 2, 3 is presented as evidence of the mental constructions, of these 10 participants who were able to operate at this level. The written responses of C_{24}, is displayed above as an exemplar.

First, the researcher thoroughly examined the responses to the questions, C_{24}, could not solve. It can be seen from above that C_{24} was not able to conjecture appropriate solutions to sub-questions 1.2 & 3.1.2 as he got the mathematical constructions to these two sub-questions wrong, thus, scored zero for these two sub-questions. This gave the researcher, a reason to conclude that C_{24} demonstrated no evidence of mental construction from the responses of these two sub-questions.

The researcher then proceeded to examine the written responses to the questions, C_{24}, could solve partly. The researcher realised that C_{24}, was partly able to conjecture the solutions for sub-questions 1.1, 1.3, 1.4, 2.1 & 2.2.1, although he omitted the relevant geometric reasons in support of the conjectured solutions. The researcher highlighted this by placing R? Or C? at the right spots in the solution steps, where these reasons/conditions should have been stated. Apart from this omission, it can be seen that C_{24} demonstrated enough evidence of mental construction, in line with the PGD, for example, C_{24}, was able to prove that $\widehat{P_2} = \widehat{P_3}$. He achieved this by relating one mathematical concept and construction to the other, until the desired solution could be reached. This enabled C_{24} to conjecture a meaningful solution to this sub-question. This was a non-routine problem, which required higher-order reasoning, creative, critical and reflective thinking (Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

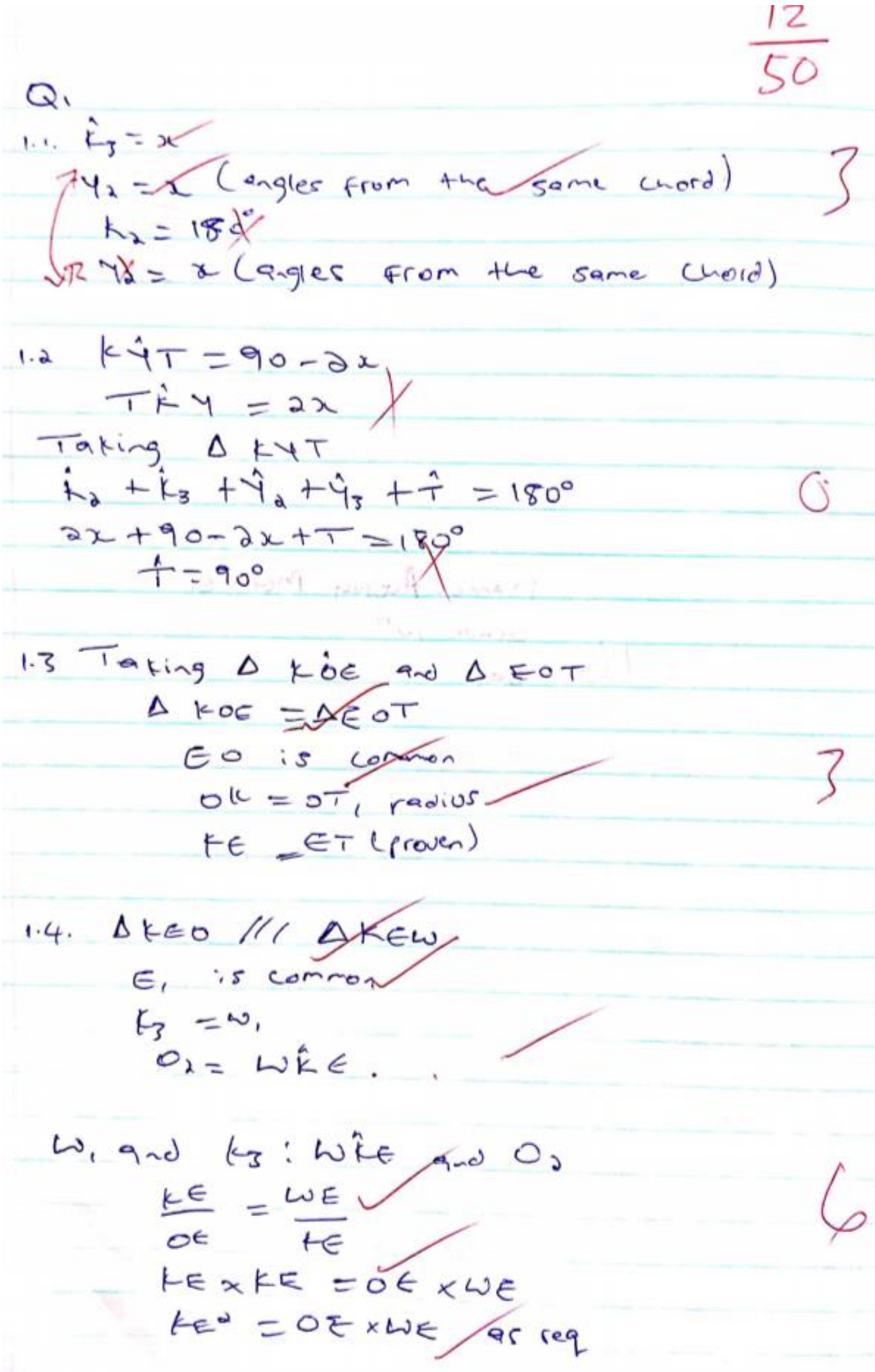
Lastly, the researcher examined the responses to the sub-questions, which C_{24}, could solve wholly; these were: 2.2.2, 2.2.3, 3.1.1 & 3.2. From the written responses he demonstrated enough evidence of circle geometry mental construction, in line with the PGD. For example, from the written responses to sub-question 2.2.3, C_{24}, wrote that $\widehat{B_1} = x$ based on the tan-chord theorem, that $\widehat{B_2} = x$, is also based on the tan-chord theorem and that $\widehat{R_1} = x$, based on angle properties of isosceles triangles. All these are action and process levels of mental constructions. He was able to interconnect, organise and link these actions and processes, to imaginatively form the object conception of $\widehat{A_1} = \widehat{B_2} = x$, $\widehat{R_3} = 2x$ and $\widehat{B_{2,3}} = 2x$ and to conclude that, $\widehat{R_3} = \widehat{B_{2,3}} = 2x$. The developed schema in relation to the geometric diagram, guided him to ascertain that PRQ is a tangent to circle BCR at R, as demanded by the question (Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The complete analysis of the participants' written responses above, enabled the researcher to establish that C_{24}, has demonstrated substantial evidence of circle geometry mental conception at the object level, hence, he was rated as "A", meaning he is operating in accordance with the expectations of the PGD. The above testified that he was able to recall from memory the correct statements since he had internalised process and encapsulated the internalised process into a cognitive object. From this point, he was able to put his thoughts together to obtain a meaningful solution to the given problem

(Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

(b) Participants who did not attain the schema level

Vignette 7.11: Exemplar of control groups' written responses at schema level


12/50

Q1

1.1. $\hat{K}_3 = x$

$Y_2 = x$ (angles from the same chord) 3

$K_2 = 180°$

$Y = x$ (angles from the same chord)

1.2 $K\hat{Y}T = 90 - 2x$

$T\hat{K}Y = 2x$

Taking ΔKYT

$\hat{K}_2 + \hat{K}_3 + \hat{Y}_2 + \hat{Y}_3 + \hat{T} = 180°$ 0

$2x + 90 - 2x + T = 180°$

$\hat{T} = 90°$

1.3 Taking $\Delta K\hat{O}E$ and ΔEOT

$\Delta KOE \equiv \Delta EOT$

EO is common 3

OK = OT, radius

KE = ET (proven)

1.4. $\Delta KEO ||| \Delta KEW$

E_1 is common

$K_3 = W_1$

$O_2 = W\hat{K}E$

W_1 and K_3 : $W\hat{K}E$ and O_2

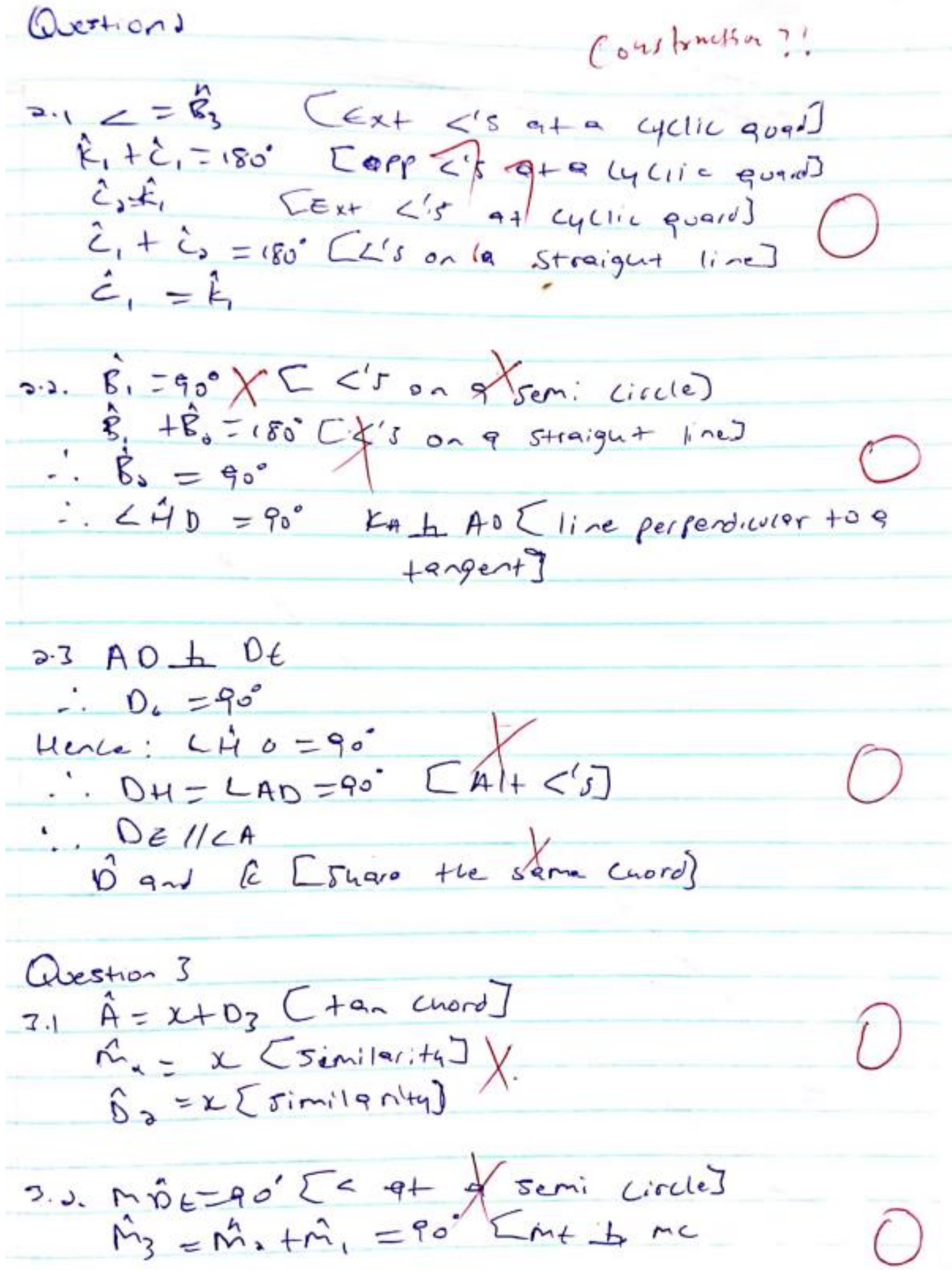
$\frac{KE}{OE} = \frac{WE}{KE}$ 6

$KE \times KE = OE \times WE$

$KE^2 = OE \times WE$ as req

Scan 1: Written response of C_{16}

Illustrated above is the written response of participant C_{16} (see scan 1) which shows that he provided many correct responses to the given questions, although, he also provided wrong solutions due to incorrect mathematical constructions. He was partly able to apply appropriate circle geometry theorems for sub-question 1.1. This participant's other written responses to test 4 is presented next, on scan 2.



Scan 2: Written response of C_{16}

The above illustration, (scan 2), informs that participant C_{24} provided totally incorrect responses to the given questions. He provided wrong mathematical constructions in most instances and showed no ability to apply appropriate circle geometry theorems in his solution. This is followed by a discussion of the participant's written responses as illustrated in scans 1 and 2 above.

Analysis and discussion of participants' written responses

After the researcher conducted a thorough analysis of the written responses by participants in the control group, it was established that 20 out of the 30 participants (the majority), demonstrated little or no evidence of circle geometry mental conception at the schema level (Ndlovu & Brijlall, 2015).This number represented 67%, which is majority of the participants in the control group. This is presented in Table 7.8 above. The written responses, (see scans 1 and 2), were evidence of the mental constructions these 20 participants from the control group demonstrated. This made the researcher to conclude that they exhibited little or no conception of circle geometry mental construction at this level. The researcher then wholly perused the written responses by C_{16}, as an exemplar of the work done by the 20 participants who could not operate at this level.

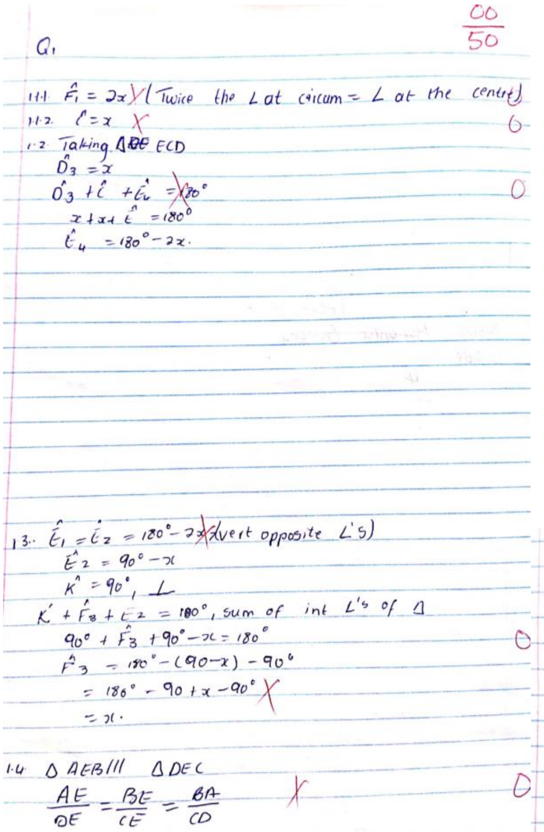
First, the researcher thoroughly examined the responses to the sub-questions that this participant could not completely solve at this level. From the written responses by C_{16} he could not conjecture appropriate expected solutions to sub-questions 1.2, 2.1, 2.2, 2.3, 3.1 & 3.2., so he scored zero for these sub-questions. C_{16} demonstrated no evidence of mental construction at this level, as contained in the PGD (Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

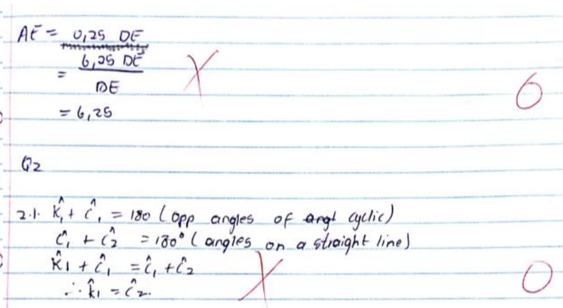
Secondly, the researcher examined the responses to the sub-questions, C_{16}, could solve partly. That is, sub-question 1.1., where he was unable to interpret the diagram well, hence, he could not determine all the four other angles, each equal to x, as demanded by the question. From the responses, two of the stated angles were correct, that is, $\widehat{K_3} = x$ and $\widehat{Y_2} = x$. The reason for $\widehat{K_3} = x$ was not stated, but he correctly provided the appropriate reason for $\widehat{Y_2} = x$ (angles from the same chord). C_{16} continued by providing two other wrong values of x, that is, $\widehat{K_2} = 180^0$ and $\widehat{Y_2} = x$. These were action and process mental conceptions, which C_{16} could not conceptualize, hence, he was unable to meaningfully and logically organise and link the action, process and object geometric mental constructions together, to solve the non-routine, higher-order questions in test 4. This contributed to C_{16} giving irrelevant responses to majority of the given questions at the schema level (Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

Finally, the sub-questions the participant could completely solve were thoroughly examined by the researcher as well; these sub-questions were: 1.3 and 1.4; C_{16} gave appropriate responses to these two sub-questions. He correctly recalled from memory and applied the appropriate geometric statement, geometric notations and geometric reasons to effectively, conjecture appropriate solutions to these sub-questions (Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

He was able to conjecture appropriate responses to these two sub-questions, however, provided incorrect mathematical constructions and irrelevant responses to majority of the sub-questions in test 4. The researcher concluded that C_{16} had demonstrated little conception of circle geometry mental construction at the schema level. This was because the requirements, contained in the PGD were not adequately demonstrated by C_{16}. Based on the above, the researcher rated C_{16} as "not attained". The second exemplar to highlight the control groups' responses- who did not attain the schema level- is presented next as Vignette 7.12.

Vignette 7.12: Exemplar of control groups' written responses at schema level





Scan 1: Written response of C_1 Scan 2: Written response of C_1

On scan 1, the written responses of participant C_1 can be found; these are dominated with totally incorrect answers similar to those on scan 2 above. The written responses on both scans 1 and 2 were collectively analysed and are discussed below.

Analysis and discussion of participants' written responses

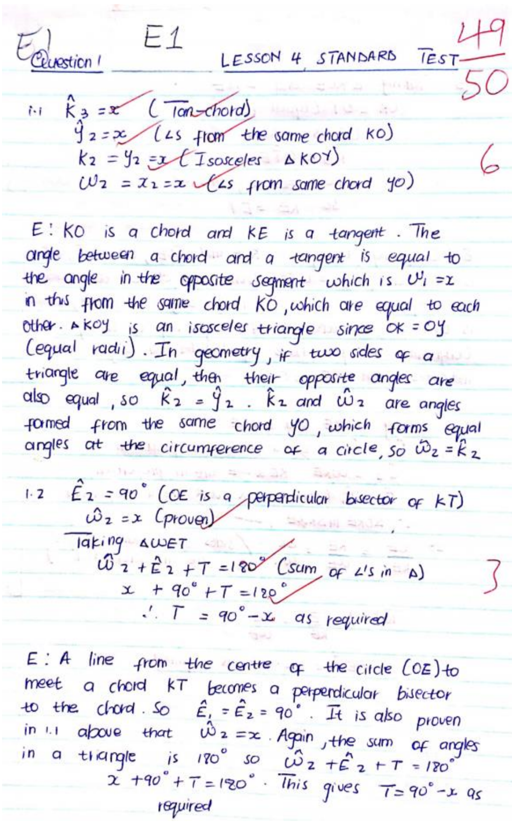
The written responses (see scans 1 and 2) were presented as evidence of the mental constructions these participants were able to demonstrate at the schema level. C_1 provided irrelevant responses as conjectured solutions to the given questions; all the mathematical constructions were wrong, hence, C_1 scored zero.

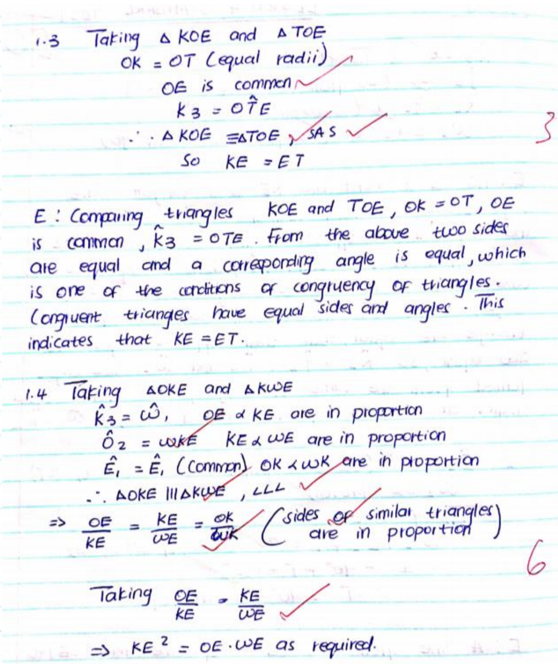
Based on the above the researcher, concluded that C_1 has demonstrated no conception at the object level, hence, C_1 was rated as "not attained".

(ii) Experimental groups' responses at schema level

(a) Participants who attained the schema level

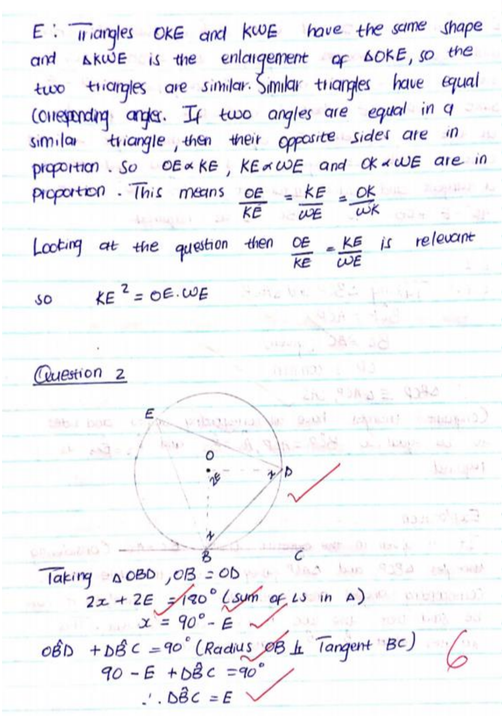
Vignette 7.13: Exemplar of experimental groups' written responses at schema level





Scan 1: Written response of E_1 Scan 2: Written response of E_1

Scan 1 is characterized by participant E_1 providing correct answers with justifications as she applied relevant circle geometry theorems, reasons and explanations for each sub-question. The same can be said about the written responses on scan 2. The other responses of participant E_1 to test 4 are presented in scans 3 and 4 below.



E : Triangles OKE and KWE have the same shape and ΔKWE is the enlargement of ΔOKE, so the two triangles are similar. Similar triangles have equal corresponding angles. If two angles are equal in a similar triangle, then their opposite sides are in proportion. So $OE \propto KE$, $KE \propto WE$ and $OK \propto WE$ are in proportion. This means $\frac{OE}{KE} = \frac{KE}{WE} = \frac{OK}{WK}$

Looking at the question then $\frac{OE}{KE} = \frac{KE}{WE}$ is relevant

so $KE^2 = OE.WE$

Question 2

Taking ΔOBD, OB = OD

$2x + 2E = 180°$ (sum of ∠s in Δ)

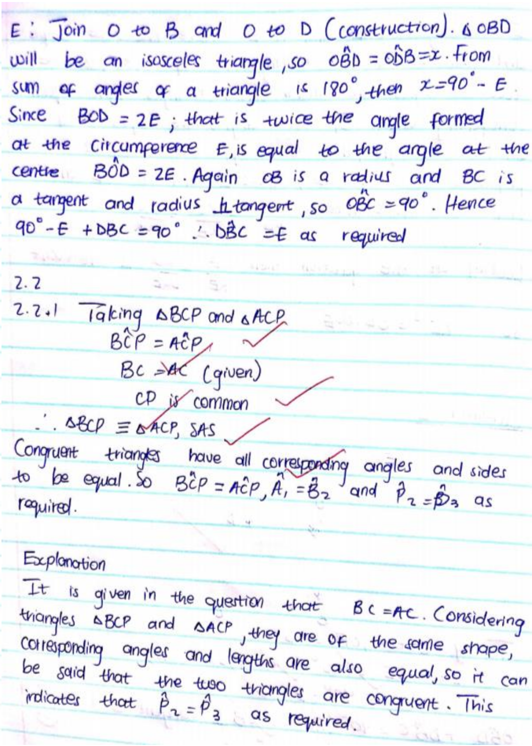
$x = 90° - E$

$O\hat{B}D + D\hat{B}C = 90°$ (Radius OB ⊥ Tangent BC)

$90 - E + D\hat{B}C = 90°$

$\therefore D\hat{B}C = E$

6



E: Join O to B and O to D (construction). ΔOBD will be an isosceles triangle, so $O\hat{B}D = O\hat{D}B = x$. From sum of angles of a triangle is 180°, then $x = 90° - E$. Since $BOD = 2E$; that is twice the angle formed at the circumference E, is equal to the angle at the centre $B\hat{O}D = 2E$. Again OB is a radius and BC is a tangent and radius ⊥ tangent, so $O\hat{B}C = 90°$. Hence $90° - E + DBC = 90°$ ∴ $D\hat{B}C = E$ as required

2.2

2.2.1 Taking ΔBCP and ΔACP

$B\hat{C}P = A\hat{C}P$

BC = AC (given)

CP is common

∴ ΔBCP ≡ ΔACP, SAS

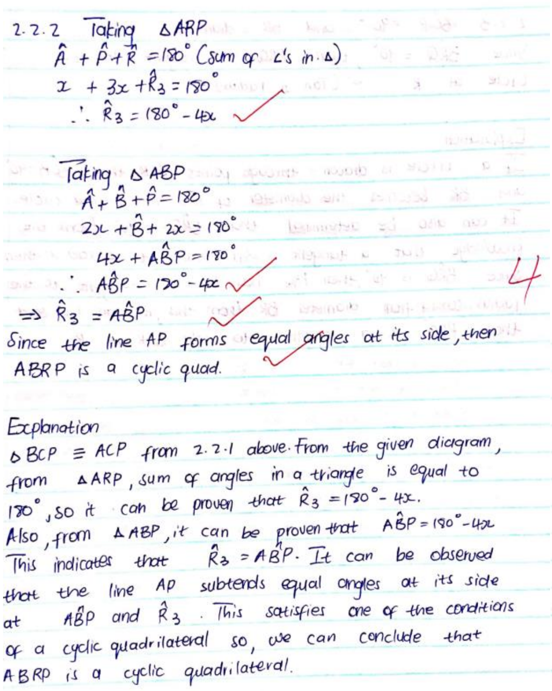
Congruent triangles have all corresponding angles and sides to be equal. So $B\hat{C}P = A\hat{C}P$, $\hat{A}_1 = \hat{B}_2$ and $\hat{P}_2 = \hat{P}_3$ as required.

Explanation

It is given in the question that BC = AC. Considering triangles ΔBCP and ΔACP, they are of the same shape, corresponding angles and lengths are also equal, so it can be said that the two triangles are congruent. This indicates that $\hat{P}_2 = \hat{P}_3$ as required.

Scan 3: Written response of E_1 Scan 4: Written response of E_1

Scan 3 illustrates that the participant was able to intuitively, prove the circle geometry theorem "tan-chord theorem" with the aid of a geometric diagram she had constructed; in addition, how the proof was conjectured was provided. For scan 4, the participant provided appropriate geometric statements, reasons and explanations to conjecture the solution for sub-question 2.2.1. The other written responses of participant E_1 to test 4 are presented on scans 5 and 6 below.



2.2.2 Taking ΔARP

$\hat{A} + \hat{P} + \hat{R} = 180°$ (sum of ∠'s in Δ)

$x + 3x + \hat{R}_3 = 180°$

$\therefore \hat{R}_3 = 180° - 4x$

Taking ΔABP

$\hat{A} + \hat{B} + \hat{P} = 180°$

$2x + \hat{B} + 2x = 180°$

$4x + A\hat{B}P = 180°$

$\therefore A\hat{B}P = 180° - 4x$

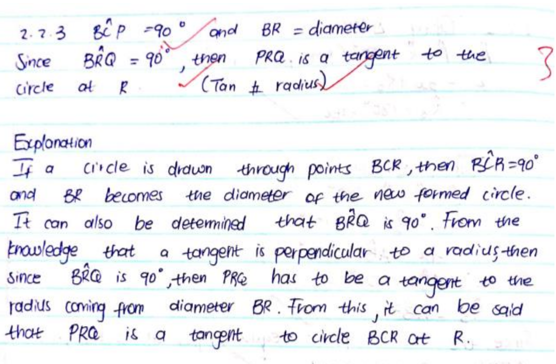
$\Rightarrow \hat{R}_3 = A\hat{B}P$

Since the line AP forms equal angles at its side, then ABRP is a cyclic quad.

4

Explanation

$\Delta BCP \equiv ACP$ from 2.2.1 above. From the given diagram, from ΔARP, sum of angles in a triangle is equal to 180°, so it can be proven that $\hat{R}_3 = 180° - 4x$. Also, from ΔABP, it can be proven that $A\hat{B}P = 180° - 4x$. This indicates that $\hat{R}_3 = A\hat{B}P$. It can be observed that the line AP subtends equal angles at its side at $A\hat{B}P$ and $\hat{R}_3$. This satisfies one of the conditions of a cyclic quadrilateral so, we can conclude that ABRP is a cyclic quadrilateral.



2.2.3 $B\hat{C}P = 90°$ and BR = diameter

Since $B\hat{R}Q = 90°$, then PRQ is a tangent to the circle at R (Tan ⊥ radius)

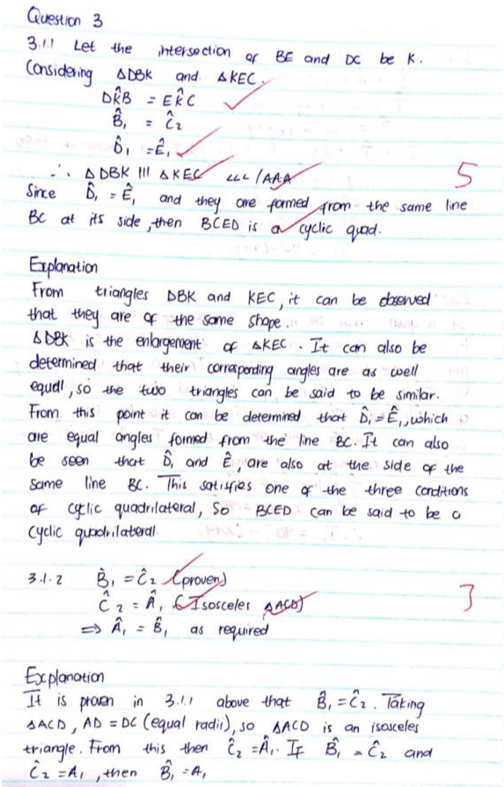
3

Explanation

If a circle is drawn through points BCR, then $B\hat{C}R = 90°$ and BR becomes the diameter of the new formed circle. It can also be determined that $B\hat{R}Q$ is 90°. From the knowledge that a tangent is perpendicular to a radius, then since $B\hat{R}Q$ is 90°, then PRQ has to be a tangent to the radius coming from diameter BR. From this, it can be said that PRQ is a tangent to circle BCR at R.

Scan 5: Written response of E_1 Scan 6: Written response of E_1

It can be seen on scans 5 and 6 above that participant E_1 provided appropriate circle geometry theorems, reasons and explanations to sub-question 2.2.2. The other written responses of participant E_1 to test 4 are presented on scans 7 and 8 below.



Question 3

3.1.1 Let the intersection of BE and DC be K.

Considering △DBK and △KEC

$D\hat{K}B = E\hat{K}C$

$\hat{B}_1 = \hat{C}_2$

$\hat{D}_1 = \hat{E}_1$

∴ △DBK ||| △KEC ∠∠∠ / AAA

Since $\hat{D}_1 = \hat{E}_1$ and they are formed from the same line BC at its side, then BCED is a cyclic quad.

5

Explanation

From triangles DBK and KEC, it can be observed that they are of the same shape. △DBK is the enlargement of △KEC. It can also be determined that their corresponding angles are as well equal, so the two triangles can be said to be similar. From this point it can be determined that $\hat{D}_1 = \hat{E}_1$, which are equal angles formed from the line BC. It can also be seen that $\hat{D}_1$ and $\hat{E}$, are also at the side of the same line BC. This satisfies one of the three conditions of cyclic quadrilateral, so BCED can be said to be a cyclic quadrilateral.

3.1.2 $\hat{B}_1 = \hat{C}_2$ (proven)

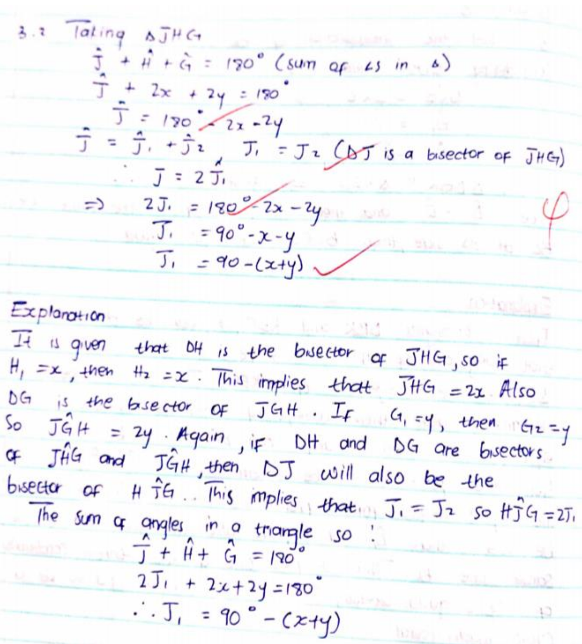
$\hat{C}_2 = \hat{A}_1$ (Isosceles △ACD)

$\Rightarrow \hat{A}_1 = \hat{B}_1$ as required

3

Explanation

It is proven in 3.1.1 above that $\hat{B}_1 = \hat{C}_2$. Taking △ACD, AD = DC (equal radii), so △ACD is an isosceles triangle. From this then $\hat{C}_2 = \hat{A}_1$. If $\hat{B}_1 = \hat{C}_2$ and $\hat{C}_2 = A_1$, then $\hat{B}_1 = A_1$



3.2 Taking △JHG

$\hat{J} + \hat{H} + \hat{G} = 180°$ (sum of ∠s in △)

$\hat{J} + 2x + 2y = 180°$

$\hat{J} = 180° - 2x - 2y$

$\hat{J} = \hat{J}_1 + \hat{J}_2$ $\quad J_1 = J_2$ (DJ is a bisector of JHG)

∴ $J = 2J_1$

$\Rightarrow 2J_1 = 180° - 2x - 2y$

$J_1 = 90° - x - y$

$J_1 = 90 - (x+y)$

Explanation

It is given that DH is the bisector of JHG, so if $H_1 = x$, then $H_2 = x$. This implies that $J\hat{H}G = 2x$. Also DG is the bisector of $J\hat{G}H$. If $G_1 = y$, then $G_2 = y$ So $J\hat{G}H = 2y$. Again, if DH and DG are bisectors of $J\hat{H}G$ and $J\hat{G}H$, then DJ will also be the bisector of $H\hat{J}G$. This implies that $J_1 = J_2$ so $H\hat{J}G = 2J_1$

The sum of angles in a triangle so:

$\hat{J} + \hat{H} + \hat{G} = 180°$

$2J_1 + 2x + 2y = 180°$

∴ $J_1 = 90° - (x+y)$

Scan 7: Written response of E_1 Scan 8: Written response of E_1

It can be observed on scan 7 that participant E_1 was able to apply relevant circle geometry theorems, reasons and explanations to conjecture meaningful solutions for each of the sub-questions; those on scan 8 show that she was able to brainstorm, creatively and logically think around relevant circle geometry theorems, with explanations to conjecture the correct solution. The written responses on scans 1-8 above were analysed and discussed below.

Analysis and discussion of participants' written responses

The researcher examined the experimental group's circle geometry mental constructions they demonstrated at the schema level by comprehensively performing content analysis of their responses. This analysis established that 26 (81%) of these participants operated at the schema level of circle geometry mental conception (see Table 7.8).

They were able to recall from memory the correct statement of theorems and/or converse of theorems and applied them in their solutions; this implied that they had interiorised such memory into a process. They then applied the internalised process by recognising and applying the appropriate circle geometry theorems and/or converses of theorems, imaginatively, thus implying that they had encapsulated the internalised process into a cognitive object. From this juncture, they were able to put their thoughts together to conjecture a meaningful and justifiable solution to the given problem (Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The written responses (see scans 1-8) were presented, as exemplars to illustrate the mental constructions these 26 participants demonstrated at the schema level. This was done for readers to ascertain what necessitated the researcher to judge the written responses of these participants as "operating at the schema level". The researcher analysed the written responses of E_1 comprehensively as detailed below.

The researcher thoroughly analysed the participant's written responses to the sub-questions she could not solve at all, those she could partly solve and those questions she solved completely. She left none of the sub-questions unanswered as she comprehensively provided correct geometric statements, correct and detailed mathematical constructions and appropriate reasons to buttress each geometric statement and each calculation made. In addition, she provided concise and meaningful explanations to justify how the conjectured solutions were obtained, confirming that E_1 had substantial understanding of circle geometry theorems and/or its converses. All the non-routine problems, which required higher-order reasoning, creative, critical and reflective thinking, as demanded by the PGD, were all solved with justifications by the participant (Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

For example, from the written responses to sub-question 2.2.2, it can be observed that she demonstrated enough evidence of circle geometry mental construction, in accordance with the PGD. She stated that $\hat{A} + \hat{P} + \hat{R} = 180^0$ and $\hat{A} + \hat{B} + \hat{P} = 180^0$(sum of angles in a triangle), which are all action and process levels of mental constructions. From here, she stated that: $\hat{A} = x$, $\hat{P} = 3x(\Delta APR)$ and $\hat{A} = 2x$, $\hat{P} = 2x(\Delta ABP)$, hence, was able to interconnect, organise and link these actions and processes, to imaginatively form the object conception of $\hat{P} = 3x(\Delta APR)$, $\hat{A} = 2x$, and $\hat{P} = 2x\ (\Delta ABP)$. This transition to the object level occurred through interiorization for she attached meaning to the idea of the circle, then through encapsulation of the actions and processes. This not only shows the steps of action and process, but the participant took that object, that is, the phenomenon of circles and their properties, and freely brainstormed around it; the participant thoroughly thought about it, and turned this object mental conceptions into a coherent framework, known as a schema. The developed schema in collaboration with the geometric diagram guided the participant to establish that $\widehat{R_3} = A\hat{B}P$, therefore, since a line AP forms equal angles at its side, then ABRP is a cyclic quadrilateral (Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

The rest of the sub-questions were answered by E_1 in the same manner as the written responses to sub-question 2.2.2, discussed above, hence, these narrations do not need any repetition. Based on this, the researcher established that E_1 demonstrated substantial evidence of circle geometry mental conception and operation at the schema level (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

(b) Participants who did not attain the schema level

Vignette 7.14: Exemplar of experimental groups' written responses at schema level

08/50

Question 1

1.1. $\hat{K}_3 = \angle\hat{O}_1 = x$ [tan chord] 2

$\hat{K}_3$ is an $\angle$

$\hat{W}_1$ [$\angle$ ko sub = $\angle$s circum

1.2. $\hat{T}_1 = 90°$

$180° = \hat{N}_2 + \hat{E}_2 + \hat{T}_2$ $\angle$sum $\triangle$s 3

$180 = x + 90° + T$ $\hat{T}_1 = 90 - x$

1.3 KO = TO [radius

OE = common

$\hat{E}_1 = \hat{E}_2 = 90°$ [OE pip bisect KT 2

$\triangle KOE \equiv \triangle TOE$ RHS

1.4. $\hat{E}$ is common

$\hat{K}_3 = \hat{W}_1$ (tan chord)

K_3 is a $\angle$ 1

$\hat{B}_1 = A_2$ [DEBA is an 100 sceles

$P_2 = 90 - 2x$

Question 2

2.1 0

2.2 1. BC = QR given

$\hat{P}_2 = \hat{B}_3$ (Alt $\angle$s)

$\hat{B}_2 = \hat{P}_1$ (Alt $\angle$s) 0

$\hat{P}_1 = P_3$

$\therefore \hat{P}_2 = B_3$

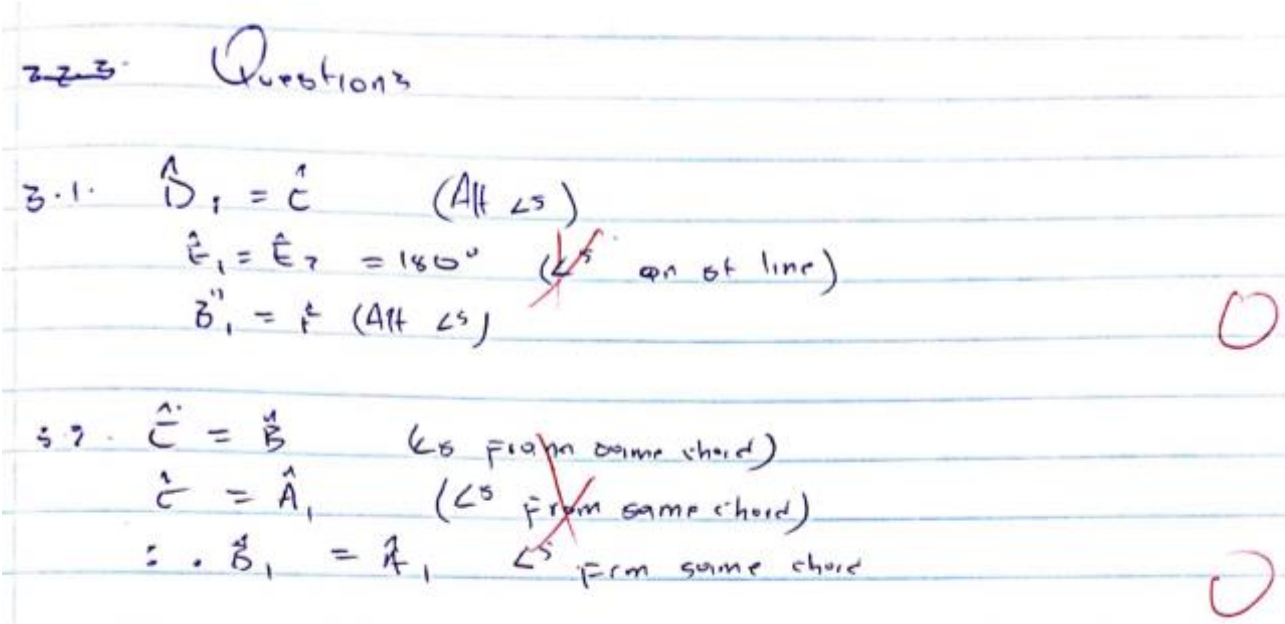
2.2.2. $\hat{A} = \hat{B}$ (tan chord theorem)

$\hat{A} = \hat{R}$

$\hat{A} + \hat{R} = 180$ 0

Scan 1: Written response of E_{19}

On scan 1 above, it can be observed that participant E_{19} was able to conjecture correct solutions to only some of the questions and left one of the sub-questions (2.1) unanswered. The rest of the responses of E_{19}, are presented on scan 2 below.



Scan 2: Written response of E_{19}

The last part of the responses of participant E_{19} are displayed above (scan 2). It is shown that he provided incorrect responses to all the given questions. He could not apply relevant circle geometry theorems, therefore, provided irrelevant responses and scored zero for all sub-questions on scan 2. This section is followed by an analysis and discussion of the participant's written responses as illustrated on scans 1 and 2 above.

Analysis and discussion of participants' written responses

As illustrated on Table 7.8 above, 6 out of the 32 participants in the experimental group could not operate at the schema level; this represented 19% of the participants. The researcher has highlighted the mental constructions demonstrated by these participants who could not operate at the schema level. He did so by presenting the written responses (see scans 1 and 2) as evidence.

It can be observed from the written responses displayed above that E_{19} was unable to conjecture appropriate responses to majority of the sub-questions - 2.1, 2.2.1, 2.2.2, 3.1 and 3.2 . E_{19} provided irrelevant responses to all the other sub-questions above, thus, demonstrating no conception at the schema level (Mathews & Thomas, 1996; Dubinsky, 2001; Maharaj, 2010; Maharaj, 2014; Brijlall & Ndlazi, 2019; Brijlall, 2020).

It can also be seen that, E_{19} was only able to conjecture the solution for sub-question 1.2 and partly conjectured the solutions for sub-questions 1.1, 1.3 and 1.4. He was able to recall from memory and applied appropriate theorems - actions and processes conceptions but had difficulties in organising and linking his thoughts together to obtain a meaningful solution to the given problem. There was no evidence of higher-order geometric reasoning, creative, critical and reflective thinking, as expected in the PGD. Based on these, the researcher concluded that E_{19} demonstrated little evidence of mental constructions at the schema level, as demanded by the PGD, hence, the researcher rated E_{19} as "not attained" at the schema level.

7.5 Modified genetic decomposition (MGD)

Conducted research studies, that implemented the APOS theory have resulted in the preliminary genetic decomposition (PGD) been meaningfully altered to form the modified genetic decomposition (MGD), (Maharaj, 2010; Maharaj, 2014; Ndlovu, 2015; Chagwiza, Maharaj & Brijlall, 2020; Brijlall, 2020). These studies modified the preliminary genetic decomposition, to address participants' difficulties and challenges, the researchers picked up, after a content analysis of participants' written responses were carried out. These participants might have demonstrated ample evidence of competence in mental construction at the other levels of circle geometry mental conception, however, the data analysis' results necessitated that some changes be made to the PGD at each level of conception. Similarly the current researcher proposed the MGDs which will add to the existing body of knowledge in the field of mathematics education at the basic education level (Grades 11 and 12), since this knowledge is presently lacking in available literature (Ndlovu & Brijlall, 2015; Chagwiza, Maharaj & Brijlall, 2020).

Themes in arriving at the MGDs

ACE teaching as an instructional approach, is characterized by careful and extensive planning, acting, observing, reflecting and creating a revised plan (Syarifuddin & Atweh, 2022). Whilst analysing the participants written responses, the researcher realised that some aspects were not considered in the preliminary genetic decomposition. These observations constituted the emerging themes that led the researcher to thoroughly reflect and revise the PGD to create the MGDs.

First, the APOS theory analysis was carried out on the standardized test - test 1. This standardized test was formulated to measure participants' mental construction at the action level of conception. It was ascertained that almost all the experimental group participants (30 out of 32) were able to demonstrate substantial evidence of mental construction at this level although they had challenges with some of the sub-questions in question 1, that measured participants' knowledge of circle geometry theorems and converse of theorems; this resulted in the refinement of the PGD at the action level. The written responses of some of the participants on converses of theorems were wrong, indicating that these participants had challenges differentiating between a theorem, its converse and when and how to apply the knowledge. The researcher, hence, proposed that a theorem and its converse should be taught simultaneously; this was the change made to the PGD at the action level, to form the MGD at that level. The researcher posits that if a theorem and its converse are taught to participants simultaneously, they will be able to differentiate between them well. It will as well, guide participants to know when and how to apply each of them, appropriately.

The second emerged theme concerned the standardized test 2; as said earlier, test 2 measured the mental constructions participants demonstrated at the process level of conception. The researcher deduced that some participants, even from the experimental group, were unable to give appropriate responses to sub-question 7.2 of test 2. This sub-question tested participants' knowledge of 'how' and 'when', they can apply their knowledge of the converse of a theorem but a lot of them failed in this regard. This informed the researcher to revise the PGD at the process level, hence, the researcher proposed that when problems on a theorem and its converse should be solved, teachers should highlight also 'when' and 'how' to apply each of them in solving problems. The researcher aver that this will go a long way to minimise the challenges participants encountered with sub-question 7.2 of test 2.

The third and fourth emerged themes arose from participants' responses to tests 3 and 4. These test instruments measured the mental constructions participants demonstrated at the object level of conception (test 3) and at the schema level of conception (test 4). The researcher observed that participants' written responses were promising and that the PGD at both levels were appropriate in nurturing participants' mental constructions at these levels of conceptions, although some participants made some errors in their mathematical constructions and in their reasoning, at both levels. The researcher realised from the examination of participants' written responses at these levels and from interactions with these participants that, their problem-solving competences at these levels will get better with time. This was based on the researcher's assertion that, higher-order geometric reasoning, creative, critical and reflective thinking is not a 'quick fix', which can be achieved over night, or within a short period of time (Swartz & Reagan, 1998); they develop over time. The researcher, hence, proposes that adequate varied problems should be solved at the object and schema levels. This will broaden participants' geometric awareness and reasoning capacity, thereby, make them effective and competent problem-solvers and good thinkers at these two crucial levels.

These themes that emerged at each level of mental construction were consolidated to form the modified genetic decomposition (Ndlovu & Brijlall, 2015; Chagwiza, Maharaj & Brijlall, 2020). This is presented in Figure 7.1 below. These changes are highlighted in red, on the modified genetic decomposition in Figure 7.1 below.

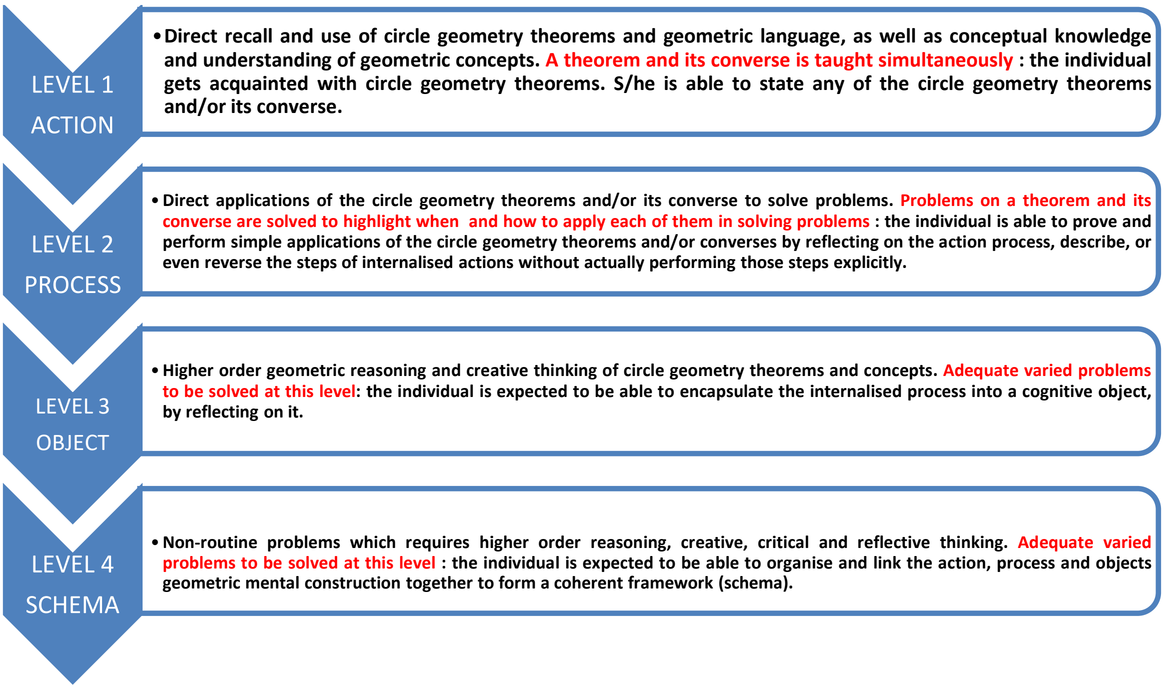


Figure 7.1: The modified genetic decomposition of circle geometry concepts

7.6 Conclusion

In this chapter how a participant was rated by the researcher as having “attained” or “not attained” each of the four levels of APOS conceptions were presented in accordance with the requirements of the PGD. As evidence to clarify the researcher’s ratings, actual written responses of some of the participants work were presented for readers’ perusal. In addition how the PGD was altered to form the MGD was also addressed to guide learners and teachers on the future pedagogy of circle geometry. Germane to this study is that, splendiferously, majority of participants from the experimental group operated at the object and schema levels: 88% of participants at the object level and 81% at the schema level. The implication of this is that the implementation of this new problem-solving instructional approach (the IPAC model) with the experimental group, guided the majority (more than 80%) of the participants from this group to effectively operate at these levels of circle geometry mental conception. The research question, in relation to each level of mental construction, demonstrated by participants, characterized by the APOS theory’s mental conceptions, is presented in Chapter 9.

CHAPTER EIGHT

DISCUSSION OF DATA EXTRACTED FROM THE QUESTIONNAIRES

8.1 Introduction

In this chapter are the data presentation and analysis of the questionnaires used for this study – teacher-observer questionnaire, the HOD as an observer questionnaire and participants' questionnaire. In this chapter, each question and its responses are summarised in a table for easy comprehension; also, a scan of the actual responses of the teachers, HODs and participants, are presented.

8.2 Teacher/HOD questionnaire

This questionnaire was designed to enable the researcher to know how the new instructional approach was implemented and how well lessons were understood. This questionnaire was answered by the substantive mathematics teachers and HODs for mathematics at the research fields. The items of this research questionnaire were divided into four parts. Part A: how teaching of thinking skills were conducted; Part B: mode of presentation of lesson; Part C: how well the lessons were understood; and Part D: any other comments/remarks. The responses to the questionnaire by the teachers are summarized and presented in the tables below:

8.2.1 Part A of the questionnaire

8.2.1.1 Presentation of responses of Part A

Before a discussion on the responses of Part A of the questionnaire, Table 8.1 below shows the data captured.

Table 8.1: Presentation of responses of Part A

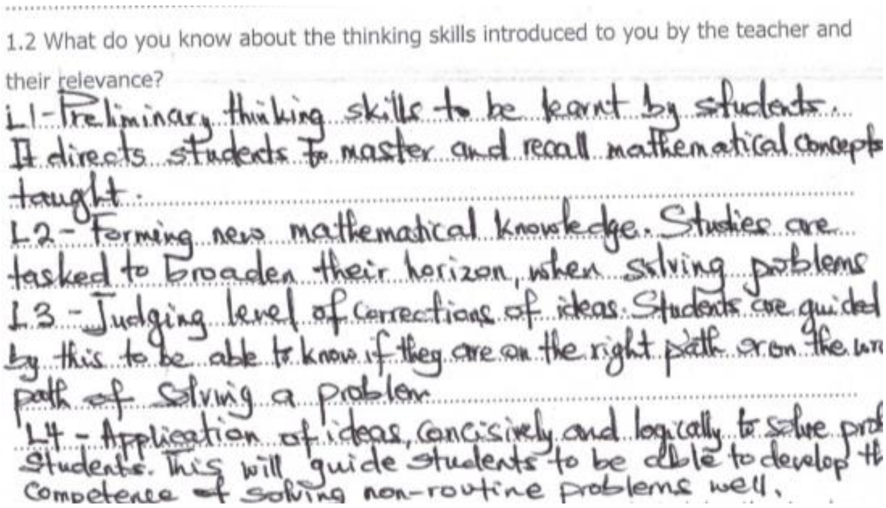
	PART A	
OBSERVER	QUESTION	EXTRACTED RESPONSE
TEACHER	1.1 Which thinking skills were introduced to you by the teacher	Thinking skills taught at Lessons 1, 2, 3 & 4 in line with lesson plans.
	1.2 What do you know about the thinking skills introduced to you by the teacher and their relevance?	To understand and retain ideas, brainstorm and apply ideas to solve problems meta-cognitively.
	1.3 What are the content objectives of the lesson conducted by the teacher	Learners to be good thinkers and good problem-solvers.
	2.1 How did the teacher guide students on how to use thinking skills to solve problems	Explanations, group-discussions and engagements.
	2.2 How did the teacher guide students on how they can reach solutions to thinking tasks, using thinking skills when they are in groups	Encouraging learners, giving thinking time, brainstorming and persistence.
	2.3 How did the teacher guide students on how they can reach solutions to thinking tasks, using thinking skills, individually	Encouraging individual learners, giving thinking time, brainstorming and persistence.
	3.1 How was the concept of "thinking about thinking" introduced to students by the teacher	Reflecting on their thinking.
	3.2 How did the teacher guide students to reflect on their own thinking	Question prompts/guidelines.
	4.1 How was the concept of "applying thinking" introduced to students by the teacher	Practical examples.
	4.2 How did the teacher guide students on how they can apply their thinking to solve problems by applying their thinking skills to promote transfer	Giving challenging and higher order questions to participants.
HOD	1.1 Which thinking skills were introduced to you by the teacher	Lessons 1, 2, 3 & 4 thinking skills as stated in lesson plans.
	1.2 What do you know about the thinking skills introduced to you by the teacher and their relevance?	To understand and retain ideas, brainstorm and apply ideas to solve problems meta-cognitively.
	1.3 What are the content objectives of the lesson conducted by the teacher	Learners to be good thinkers and good problem-solvers.
	2.1 How did the teacher guide students on how to use thinking skills to solve problems	Explanations, group-discussions and engagements.
	2.2 How did the teacher guide students on how they can reach solutions to thinking tasks, using thinking skills when they are in groups	Encouraging learners, giving thinking time, brainstorming and persistence.
	2.3 How did the teacher guide students on how they can reach solutions to thinking tasks, using thinking skills, individually	Encouraging individual learners, giving thinking time, brainstorming and persistence.
	3.1 How was the concept of "thinking about thinking" introduced to students by the teacher	Reflecting on their thinking.

	3.2 How did the teacher guide students to reflect on their own thinking	Using question prompts/guidelines.
	4.1 How was the concept of "applying thinking" introduced to students by the teacher	Real-life scenarios.
	4.2 How did the teacher guide students on how they can apply their thinking to solve problems by applying their thinking skills to promote transfer	Introducing challenging and higher order tasks and activities.

8.2.1.2 Analysis and discussion of "Part A" of Teacher/HOD questionnaire

As said before, "Part A" of this questionnaire sought to ascertain how teaching of thinking skills, essential to this study, were conducted. It was observed that both observers - teachers and HODs, - gave synonymous responses in this section, which is good for data validation and replication (McMillan & Schumacher, 2014). Responses from the two observers showed that the expected thinking skills were covered: lesson one - Understanding and retention of ideas; lesson two- Generating ideas (Creative thinking); lesson three - Assessing reasonability of ideas (Critical thinking) and lesson four- Blending generating of ideas in lesson 2, by assessing reasonability of ideas in lesson 3, and engaging students in metacognitive reflections (Swartz & Reagan, 1998). The relevance of these thinking skills, paramount to this study, are also stated. The responses from the two observers are presented below:

Teacher's response


1.2 What do you know about the thinking skills introduced to you by the teacher and their relevance?

L1- Preliminary thinking skills to be learnt by students. It directs students to master and recall mathematical concepts taught.

L2- Forming new mathematical knowledge. Studies are tasked to broaden their horizon, when solving problems

L3 - Judging level of correctness of ideas. Students are guided by this to be able to know if they are on the right path or on the wr path of solving a problem.

L4 - Application of ideas, concisely and logically to solve prob Students. This will guide students to be able to develop th competence of solving non-routine problems well.

HOD's response

1.2 What do you know about the thinking skills introduced to you by the teacher and their relevance?

L1- Foundation of thinking skills. It promotes conceptual understanding of the concept that was taught.
L2- Brainstorming. It enables new mathematical ideas about the concept taught to be conjectured.
L3- Ascertaining if ideas developed are meaningful. It enables the individual to self-determine if developed ideas are either correct, wrong or needs to be refine
L4- Transfer of ideas. It enables the indidual to think about his thinking.

It can be observed from the responses above that the thinking skills to be taught were appropriately scaffolded from the basic to complex thinking skills. This ensured that these thinking skills were taught strategically to the different levels of participants -the weak, the average and the strong learners. The role these four thinking skills played when solving problems were also stated for learners - to understand them; what they stand for; what they entail and how they can be introduced. Integral to this section was how the T-R guided participants to solve problems using these thinking skills. The responses of the two observers are presented below:

Teacher's response

2.1 How did the teacher guide students on how to use thinking skills to solve problems

Learners were placed at the centre of learning. They were required to discuss and communicate among themselves they brainstorm, look for multiple ways of answering a question, consider other alternatives provided by other group members. The teacher only served as a prompter. He went to each to guide them to know if they are on the right path or wrong path on some occasions. On other occasions he allows each group to present their solution on the board. This enabled other members to constructively criticize the solution provided or to support it.

HOD's response

2.1 How did the teacher guide students on how to use thinking skills to solve problems

For all lessons I observed, teaching and learning was conducted in a collaborative classroom setting - learners sitting, discussion and interacting in each other in groups of three. Thinking skills was purposefully implemented during the questioning stage of lesson, to guide learners to reach solutions to problems. After learners having understood the question, they are been nurtured to be able to brainstorm, conjecture, ideas, trial-and-error, consider more alternatives of solving the same question. They are urged to keep trying until a meaningful solution to a problem is reached.

The responses above assert that lessons were conducted in a collaborative classroom setting, which is apropos for nurturing learners' thinking and reasoning abilities (Swartz, 2012; Chagwiza, Maharaj & Brijlall, 2020; Brijlall, 2015). Also, participants were placed at the centre of their learning, a condition which is prioritized by the constructivists (Ekawati *et al.,* 2019). The above confirmed that an appropriate teaching strategy (constructivist approach) and an appropriate learning environment (collaborative classroom setting) were created during lessons, which enhanced participants' thinking (King, Goodson, & Rohani, 2013). Also, from the responses, participants were encouraged to find varied ways a particular problem can be solved. In addition, participants discussed and interacted among themselves when solving problems (Brijlall, 2015), thereby, nurtured participants to be creative thinkers (Swartz, 2012). Most integral to nurturing the thinking capacity of participants is the questioning asked during lessons - challenging and higher-order questions. This approach proved to be a powerful tool in developing participants' thinking competence. This intrigued them and pushed their thinking to the required limit, which assisted in reaching desired solutions to the given problem (Mudrikah, 2016; Nafisah *et al.,* 2011; Best, 2019).

8.2.2 Part B of the questionnaire

8.2.2.1 Presentation of responses of Part B

Table 8.2 below, presents the data captured on part B of the questionnaire. This precedes the discussions of Part B of the questionnaire.

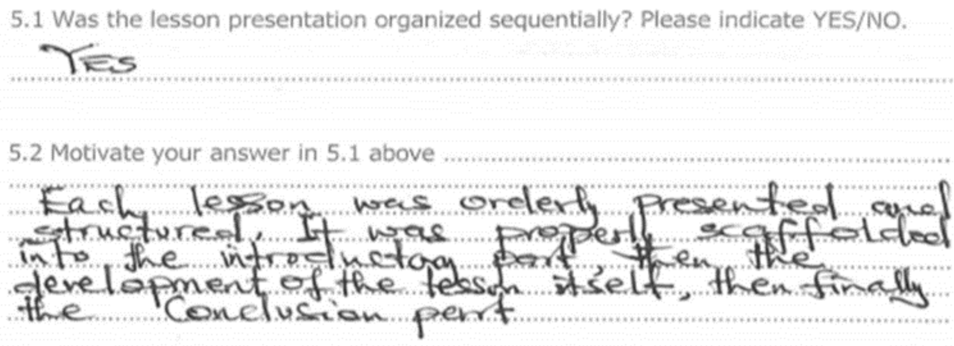
Table 8.2: Presentation of responses of Part B

	PART B	
OBSERVER	QUESTION	EXTRACTED RESPONSE
TEACHER	5.1 Was the lesson presentation organized sequentially? Please indicate YES/NO.	Yes
	5.2 Motivate your answer in 5.1 above	Lessons were well structured, scaffolded and presented.
	5.3 Was the lesson meaningfully and logically presented? Please indicate YES/NO.	Yes
	5.4 Motivate your answer in 5.3 above	Educating them on the new approach.
HOD	5.1 Was the lesson presentation organized sequentially? Please indicate YES/NO.	Yes
	5.2 Motivate your answer in 5.1 bove	Lessons were well structured.
	5.3 Was the lesson meaningfully and logically presented? Please indicate YES/NO.	Yes
	5.4 Motivate your answer in 5.3 above	Educating them on the new approach.

8.2.2.2 Analysis and discussion of "Part B" of Teacher/HOD questionnaire

This part of the questionnaire ascertained the mode of presentation of the lessons.

Teacher's response



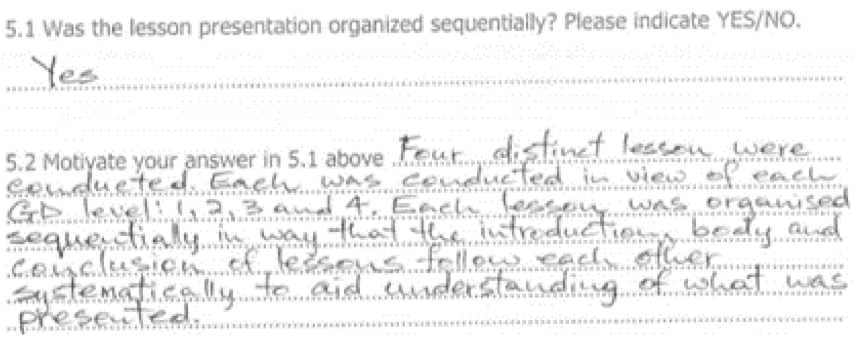
5.1 Was the lesson presentation organized sequentially? Please indicate YES/NO.

Yes

5.2 Motivate your answer in 5.1 above

Each lesson was orderly presented and structured. It was properly scaffolded into the introductory part then the development of the lesson itself, then finally the conclusion part

HOD's response



5.1 Was the lesson presentation organized sequentially? Please indicate YES/NO.

Yes

5.2 Motivate your answer in 5.1 above Four distinct lesson were conducted. Each was conducted in view of each GD level: 1, 2, 3 and 4. Each lesson was organised sequentially in way that the introduction, body and conclusion of lessons follow each other systematically to aid understanding of what was presented.

The responses above confirmed how lessons were conducted during the study. Both observers attested that lessons were organized sequentially, by responding 'Yes' to sub-question 5.1 above. Observers' justification for responding in the affirmative to sub-question 5.1, were also elaborated above. Their justification confirmed that lessons were orderly, well-structured and properly scaffolded, to enhance participants' understanding (NCTM, 2000).

8.2.3 Part C of the questionnaire

8.2.3.1 Presentation of responses of Part C

Prior to the discussion on the responses of Part C of the questionnaire, Table 8.3 below, presents the data captured.

Table 8.3: Presentation of responses of Part C

	PART C	
OBSERVER	QUESTION	EXTRACTED RESPONSE
TEACHER	6.1 How well did you understand the lesson? Specify if: not well understood, averagely understood and very well understood.	very well understood
	6.2 Motivate your answer in 6.1 above	Lessons were well sequenced and logically presented.
HOD	6.1 How well did you understand the lesson? Specify if: not well understood, averagely understood and very well understood.	very well understood
	6.2 Motivate your answer in 6.1 above	Lessons were sequenced properly and presented well.

8.2.3.2 Analysis and discussion of "Part C" of Teacher/HOD questionnaire

This part of the questionnaire sought to establish how well, lessons were understood. The responses on the Table 8.3 above attest that both observers inscribed 'very well understood' to question 6.1 above. Their reasons for this response to question 6.1 are that the logical, sequential presentation and effective delivery of lessons made them very well understood (NCTM, 2000).

8.2.4 Part D of the questionnaire

8.2.4.1 Presentation of responses of Part D

The data captured from the responses of Part D of the questionnaire, is presented on Table 8.4 below, before a discussion of the responses.

Table 8.4: Presentation of responses of Part D

	PART D	

OBSERVER	QUESTION	EXTRACTED RESPONSE
TEACHER	ANY OTHER COMMENTS/REMARKS	Advocating for the new approach.
HOD	ANY OTHER COMMENTS/REMARKS	Advocating for the new approach.

8.2.4.2 Analysis and discussion of "Part D" of Teacher/HOD questionnaire

This part of the questionnaire sought to find out if the two observers had other comments/remarks with regards to the conducted lessons using the new approach. Their responses to this are captured below.

Teacher's response

PART D: ANY OTHER COMMENTS/REMARKS

I learnt a lot during observations of these lessons. This novel teaching and learning approach promotes active participa-tion of learners.
Its implementation in mathematics classroom in South Africa, can go a long way to help address the teaching and learning challenges of mathematics in school in South Africa

HOD's response

PART D: ANY OTHER COMMENTS/REMARKS

The instructional approach is promising, innovative, dynamic and interactive. If used effectively for teaching and learning of circle geometry, it can greatly contribute in improving learner achievements in Mathematics.
Also it can serve as the medium through which learners can achieve mathematical proficiency.

The unedited final comments/remarks from the two observers confirmed that the new approach is helpful in teaching circle geometry, hence, the observers advocated for its implementation in all South Africa's mathematics classrooms. They made some interesting remarks about the new approach. These remarks were: the new instructional approach is promising, innovative, dynamic, interactive, and it can be used as a medium to achieve mathematical proficiency. This confirmed that this new approach is different and that it comes with new helpful ideas of teaching and learning mathematics and that this new approach is structured to accommodate 'the test of time'. This means that it can be updated in the future to serve long term mathematical needs of learners but still maintain its integral components (DoBE, 2018). Most importantly, the interactive nature of this new approach makes it interesting, it ensures effective interactions and engagements between participants and between participants and the teacher, which are integral in achieving mathematical proficiency (William & Maat, 2020; Kilpatrick, Swafford & Findell, 2001).

8.3 Participants questionnaire

This questionnaire was designed to measure the effects/influence the proposed problem-solving instructional approach had on learners, with regards to the teaching and learning of circle geometry. This questionnaire was answered by the study participants, individually. The items of this research questionnaire were divided into three parts. Part A: how the new instructional approach can influence participants' learning of circle geometry; Part B: how it can influence participants' problem-solving skills when solving circle geometry problems and Part C: any other comments/remarks. Participants' responses to the questionnaire are summarized on Table 8.5 below:

8.3.1 Part A of the questionnaire

8.3.1.1 Presentation of responses of Part A

Table 8.5: Presentation of responses of Part A

	PART A	
PARTICIPANTS	QUESTION	EXTRACTED RESPONSE
E_1	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yesa
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
- E_2	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_3	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes

	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_4	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_5	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_6	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_7	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively

	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_8	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_9	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{10}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes

	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{11}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{12}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{13}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes

	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{14}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{15}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{16}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{17}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively

	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{18}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{19}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Negatively
	2 Motivate your answer in question 1 above	It is complicated and confusing
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	No
	4 Motivate your answer in question 3 above	I don't understand the new approach.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	No
	6 Motivate your answer in question 5 above	I don't understand the new approach.
E_{20}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry

		problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{21}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{22}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{23}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.

E_{24}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{25}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Negatively
	2 Motivate your answer in question 1 above	It is complicated and confusing.
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	No
	4 Motivate your answer in question 3 above	I don't understand it.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	No
	6 Motivate your answer in question 5 above	I don't understand it.
E_{26}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{27}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry

		problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{28}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{29}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{30}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.

E_{31}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.
E_{32}	1 How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively	Positively
	2 Motivate your answer in question 1 above	Enhanced Confidence and how to approach questions
	3 From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO	Yes
	4 Motivate your answer in question 3 above	It taught me how I can learn and solve circle geometry problems.
	5 Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO	Yes
	6 Motivate your answer in question 5 above	It can help others in the same ways it has helped me.

8.3.1.2 Analysis and discussion of "Part A" of participants' questionnaire

This part of the questionnaire investigated how the new instructional approach influenced participants' learning of circle geometry. This was necessary as the researcher sought to know how effective the new instructional approach would be in mathematics classrooms when it is used as the medium of instruction. Some of the relevant responses from participants that attest to its effectiveness are presented below.

Participant's response 1

Question 1

How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively

positively

Question 2

Motivate your answer in question 1 above

Previously, I had difficulties with how i must go about a mathematics questio This new approach has shown me the way.

Participant's response 2

Question 1

How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively

Positively

Question 2

Motivate your answer in question 1 above

It gave me direction and reason for me to believe that i can also understand and solve mathematics

Participant's response 1

Question 3

From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO

yes

Question 4

Motivate your answer in question 3 above

this new approach shows me what i must do and how i must do it when solving a mathematics problem

Participant's response 2

Question 3

From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO

Yes

Question 4

Motivate your answer in question 3 above

It has assisted me greatly to learn maths meaningfully

Participant's response 1

Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO

Yes

Question 6

Motivate your answer in question 5 above

This is because it has directed me on how i must go about a mathematics question, so others will also benefit from it

Participant's response 2

Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO

Yes

Question 6

Motivate your answer in question 5 above

It is a good approach of learning mathematics and solving mathematics problems.

It can be observed from the Table 8.5 (for Part A) presented above that 30 out of the 32 participants (representing 94% of participants) had indicated "Yes" to two questions on this part of the questionnaire (questions 3 and 5). These participants indicated for questions 3 and 5 respectively that: from now on, they will use the new instructional approach to learn circle geometry, for it taught them how they can learn and solve circle geometry problems well; and they will recommend this new instructional approach, to any third party, for teaching and learning of circle geometry, for the reason that it can help others in the same ways it had helped them. The same 30 participants (representing 94% of participants) indicated for question 1 on the questionnaire that the new approach positively influenced how they learnt circle geometry, as it enhanced their confidence and it assisted them to approach circle geometry questions appropriately. Some of the unedited responses for questions 1, 2 & 3 are presented above for verification.

8.3.2 Part B of the questionnaire

8.3.2.1 Presentation of responses of Part B

Table 8.6 below, presents the data captured on Part B of the questionnaire; this is followed by a discussion of these responses.

Table 8.6: Presentation of responses of Part B

	PART B	
PARTICIPANTS	QUESTION	EXTRACTED RESPONSE
E_1	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either, positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new	Polya's approach

	instructional approach?	
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_2	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_3	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_4	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_5	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_6	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.

	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_7	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_8	1 How did the new instructional approach influence your problem solving skills when solving circle geometry problems? Please specify, either, positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_9	1 How did the new instructional approach influence your problem solving skills when solving circle geometry problems? Please specify, either, positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{10}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{11}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.

	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{12}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{13}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{14}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{15}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either, positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{16}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.

	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{17}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{18}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{19}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Negatively
	2 Motivate your answer in question 1 above	I don't understand it.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	I can't tell
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	I don't understand it.
E_{20}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{21}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new	Polya's approach

	instructional approach?	
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{22}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{23}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{24}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{25}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Negatively
	2 Motivate your answer in question 1 above	I don't understand it.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	I don't know.
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	I don't understand it.
E_{26}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach

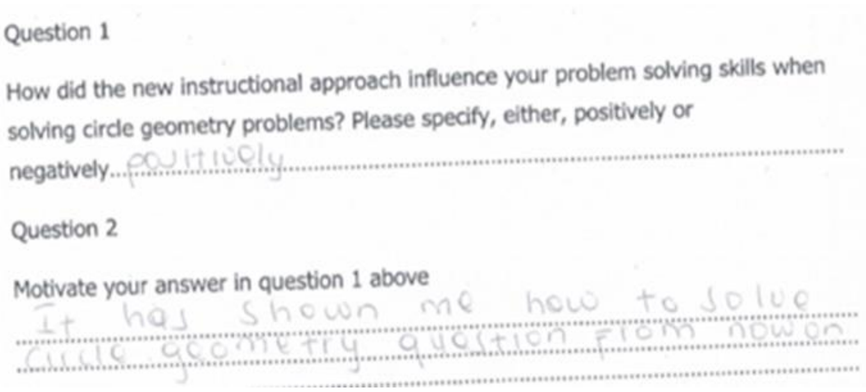
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{27}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{28}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either, positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{29}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{30}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{31}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach

	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.
E_{32}	1 How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively	Positively
	2 Motivate your answer in question 1 above	It guided me on how to solve problems.
	3 Which problem-solving skills did you learn when you were taught by the new instructional approach?	Polya's approach
	4 How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?	It taught me how to solve problems.

8.3.2.2 Analysis and discussion of "Part B" of participants' questionnaire

Part B of this questionnaire sought to investigate how the new approach can influence the study participants' problem-solving skills, with circle geometry problems; 30 out of the 32 participants (representing 94% of participants) indicated that an appropriate problem-solving instructional approach (Polya's approach) was implemented. These participants justified this by responding that this new approach influenced their problem-solving skills positively; it enabled them to solve circle geometry problems well. Some of the unedited responses from the participants which supported this are presented below.

Participant's response 1


Question 1

How did the new instructional approach influence your problem solving skills when solving circle geometry problems? Please specify, either, positively or negatively...positively

Question 2

Motivate your answer in question 1 above

It has shown me how to solve circle geometry question from now on

Participant's response 2

Question 1

How did the new instructional approach influence your problem solving skills when solving circle geometry problems? Please specify, either, positively or negatively........Positively........

Question 2

Motivate your answer in question 1 above

This new approach has taugh me the right path i will follow to solve circle geometry problems

8.3.3 Part C of the questionnaire

8.3.3.1 Presentation of responses of Part C

Prior to the discussion on the responses, Table 8.7 below shows the data captured.

Table 8.7: Presentation of responses of Part C

	PART C	
PARTICIPANTS	**QUESTION**	**EXTRACTED RESPONSE**
E_1	ANY OTHER COMMENTS/REMARKS	Advocating
E_2	ANY OTHER COMMENTS/REMARKS	Advocating
E_3	ANY OTHER COMMENTS/REMARKS	Advocating
E_4	ANY OTHER COMMENTS/REMARKS	Advocating
E_5	ANY OTHER COMMENTS/REMARKS	Advocating
E_6	ANY OTHER COMMENTS/REMARKS	Advocating

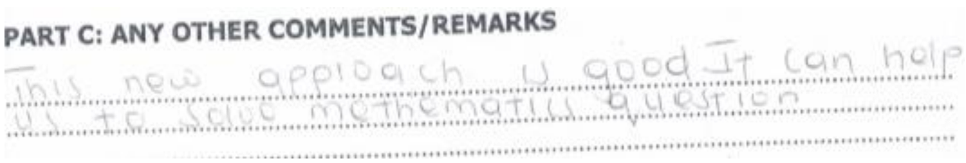
E_7	ANY OTHER COMMENTS/REMARKS	Advocating
E_8	ANY OTHER COMMENTS/REMARKS	Advocating
E_9	ANY OTHER COMMENTS/REMARKS	Advocating
E_{10}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{11}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{12}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{13}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{14}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{15}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{16}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{17}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{18}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{19}	ANY OTHER COMMENTS/REMARKS	Rejecting
E_{20}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{21}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{22}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{23}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{24}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{25}	ANY OTHER COMMENTS/REMARKS	Rejecting
E_{26}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{27}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{28}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{29}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{30}	ANY OTHER COMMENTS/REMARKS	Advocating
E_{31}	ANY OTHER COMMENTS/REMARKS	Advocating

E_{32}	ANY OTHER COMMENTS/REMARKS	Advocating

8.3.3.2 Analysis and discussion of "Part C" of participants' questionnaire

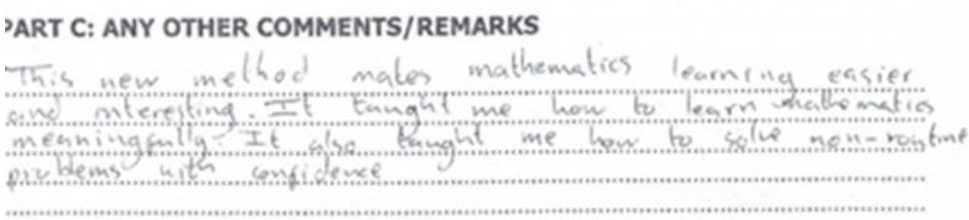
This part of the questionnaire investigated if the participants had other comments/remarks with regards to the conducted lessons which implemented the new problem-solving instructional approach. Some of their responses are captured below.

Participant's response 1


PART C: ANY OTHER COMMENTS/REMARKS

This new approach is good It can help us to solve methematics question

Participant's response 2


PART C: ANY OTHER COMMENTS/REMARKS

This new method makes mathematics learning easier and interesting. It taught me how to learn mathematics meaningfully. It also taught me how to solve non-routine problems with confidence

The unedited final comments/remarks from two of the participants (selected at random) ascertained that the new approach is helpful in learning circle geometry, hence, 30 out of the 32 participants (representing 94% of participants) advocated for its implementation in mathematics classrooms in South Africa. These participants indicated on the questionnaire that the new approach is relevant; it helped them to confidently solve non-routine circle geometry problems well; it made learning maths easier and interesting; it taught them how to learn maths meaningfully, among others. These remarks from the study participants prove that the new approach is helpful. The researcher was not surprised that these two participants - E_{19} and E_{25} - rejected the new approach. This is because they were not always present during lessons, hence, they were not able to follow proceedings well and systematically. This adversely affected their composite scores as they had the lowest scores.

8.4 Conclusion

The analysis of the questionnaire established that teaching circle geometry through the new approach went well. The introduction, classroom setting, the new ideas used during the lessons confirmed this. The responses by the two observers - teacher and HOD - attested that the relevant thinking skills were taught, lessons were sequentially, orderly and logically presented, hence, they advocated for its implementation for the teaching of mathematics in schools. In addition, the analysis of participants' questionnaire highlighted that the new approach helped them to learn and solve non-routine circle geometry problems as well.

The responses on the questionnaires (teacher/HOD responses) helped the researcher to establish that this new approach is effective for general teaching and when it is used as a medium to learn mathematics (participants' responses). Based on this, the researcher established that this new approach is useful, appropriate and efficacious for teaching and learning of circle geometry concepts. The relevant research question, linked to the analysis and discussions of the extracted responses from the questionnaires in this chapter, is presented in the next chapter.

CHAPTER 9
FINDINGS AND DISCUSSIONS

9.1 Introduction

As said earlier in sub-section 4.16, in chapter 4, the data analysis results from the quantitative data and qualitative data were consolidated as one. These analyses and results were presented in Chapters 5, 6, 7 and 8; based on these the research findings that emanated from the conduct of this study were deduced. They are presented and discussed in this chapter, in line with of each research question.

9.2 THE DEVELOPMENT AND IMPLEMENTATION OF THE NEW PROBLEM-SOLVING INSTRUCTIONAL APPROACH

Readers were informed that this study sought to find a distinct and an efficacious problem-solving instructional approach, for the teaching and learning of circle geometry, in South African schools. The research question, in relation to this was:

(1) How can the proposed instructional approach to be used as a problem-solving heuristic, be developed and implemented in the circle geometry classroom?

It was made known in Chapter 1 of this report, that the IPAC model, is a completely new and distinct instructional approach which was developed, tried and tested for this study. This investigation enabled the researcher to measure its degree of efficacy, appertaining to its validity, practicality and effectiveness (Nieveen, 1997; 1999) cited in (Fauzan, Plomp, & Gravemeijer, 2013). Nieveen, (1997; 1999) refered to the above three measurement descriptors as "the three quality criteria check" of an intervention. It was also mentioned earlier in Chapter 1 that this IPAC model, entails the integration of three key theories/approaches: Infusion approach, Polya's problem-solving approach and APOS theory, in a collaborative classroom setting. This interactive, action-driven and sophisticated problem-solving instructional approach was designed and developed with the assistance of expert advice and assistance; its implementation in mathematics classrooms was also assisted.

This study has established that this IPAC model, is promising, in leading mathematics learners on the pathway to achieving mathematical proficiency, in conformity with its validity, practicality and effectiveness (Fauzan, Plomp, & Gravemeijer, 2013). To establish the models adherence to these criteria, the researcher perused and analysed the collected data, and results that emanated. To this end, the information gathered informed that this IPAC model, can duly be developed and effectively implemented in mathematics classrooms, as this study sought to investigate. Each quality determinant criteria is discussed below.

9.2.1 Validity of the proposed IPAC model

Nieveen, (1997; 1999) defined 'validity' of an intervention approach as: "the extent that the design of the intervention include "state of the art knowledge" (content validity) and the various components of the intervention are consistently linked to each other (construct validity)". As mentioned in Chapter 4, this new problem-solving instructional approach was content and face validated by experts in mathematics education. During this procedure, utmost care was taken to ensure that relevant content knowledge - teaching thinking and effective problem-solving skills - was adopted to constitute this innovative instructional approach. Further perusal of literature into this knowledge construct pair, established that for the South African context, the infusion approach and Polya's problem-solving model, were appropriate in teaching thinking and how to become good problem-solvers, respectively (Swartz & Reagan, 1998; Carifio, 2015).

The researcher ensured that the knowledge construct, the structure, and the constituent parts of this new problem-solving instructional approach were properly and meaningfully linked and sequenced, during the development of this new teaching and learning approach, thus, it was well validated in view of its content and its structure. The components of this new approach were correctly correlated, sequenced and aligned with each other. These validity procedures were undertaken (see prototype 1 in Chapter four), before it was tried and tested and implementing Education Design Research as the research design for this study helped to establish the validity of this IPAC model. This enabled the researcher to conceptualize that the development and implementation of a well-proven and efficacious instructional approach is not a one-step procedure/project. To this end, the development and implementation of this new approach were iteratively checked, tried and tested (Easterday, Lewis & Gerber, 2018).

The constructivism paradigm, adopted as the research paradigm for this study aided the researcher greatly, in developing this IPAC model (Kamal, 2019). This guided and directed the researcher to employ the APOS theory and infusion approach, as constituent parts for this research-proven, problem-solving instructional approach. These two interactive learning approaches emanated from the constructivism paradigm. This adequately guided the researcher to effectively develop this IPAC model - a learner-centred, interactive and action-driven instructional approach. This paradigm provided the teaching and learning components that this new instructional approach needed. The original version of the IPAC model, changes made to it after the original version was tried, tested and reviewed, and the final version of the IPAC model are all thoroughly discussed below.

Original version of the IPAC model

The IPAC model is a multifaceted dimension problem-solving instructional approach (Gono & Pacoy, 2021). Details of the original version of the IPAC model were presented in Chapter 4, sub- section 4.13.1 - development of instruments. Under this sub-section, the role of each construct - infusion approach, Polya's approach and the APOS theory - in the development of this IPAC model were stated. The APOS theory instructional approach (ACE teaching cycle) has been verified empirically to be efficacious for teaching and learning a lot of mathematical concepts (Maharaj, 2010; Maharaj, 2013; Ndlovu, 2015; Voskoglou, 2015; Borji, Alamolhodaei &

Radmehr, 2018; Arnawa and Yerizon, 2019; Syarifuddin, Nusantara, Qohar & Muksar, 2019; Moon, 2019).

According to Tziritas (2011), solving "related rates problems" involve modelling and problem-solving abilities. Tziritas (2011) asserts that the action, process and object mental constructions, did not seem to be an appropriate framework for studying and fostering the development of those abilities. He then averred that a combination of frameworks may clarify students' difficulties with "related rates problems".

The researcher as an experienced mathematics teacher found the concept of "rate of change" to be synonymous to the concept highlighted in this study (circle geometry). This is so because circle geometry also involves modelling and problem-solving abilities, in association with logical, inductive and deductive reasoning. Based on the assertions made by Tziritas (2011), the researcher averred that the ACE teaching approach may not be wholly appropriate to address the teaching and learning difficulties of circle geometry, thus, based on the recommendations by Tziritas (2011), and others, the researcher adopted Polya's problem-solving approach and the infusion approach, which are potent instructional strategies, as other frameworks to support the ACE teaching cycle (Schurter, 2001; Aizikovitsh & Amita, 2010; Kousar, 2010; In'am, 2014; Mehmood, 2014; Hayyulbathin, Winarni & Murwaningsih, 2014; Brijlall, 2015; Carifio, 2015; Lee & Chen, 2015; Valles & Wickramasingh, 2015; Loğoğlu, 2017; Zulkpli, Abdullah, Kohar & Ibrahim, 2017; Gray, 2018; Abakah, 2019; Phuntsho & Dema, 2019).

Polya's problem-solving approach has proven to be an effective instructional approach, however, the researcher realised that it needed to be supplemented at the class-discussion stage of the IPAC model, so that the desired thinking and problem-solving competence of learners can be nurtured. To this end, the researcher modified Polya's problem-solving approach, to accommodate the infusion approach, so as to make the class-discussions more effective (Hayyulbathin, Winarni & Murwaningsih, 2014; Carifio, 2015). These three components are individually potent instructional strategies, but they complement and supplement each other, very effectively, hence, the researcher trusted that they will, collaboratively, elicit the expected problem-solving competence in circle geometry, as this study sought to accomplish. In the next sections, the researcher presents a diagrammatic representation of the original version of the IPAC model, also presented in Chapter 4, sub- section 4.13.1.

Changes made to the IPAC model

According to Syarifuddin & Atweh (2022), "the ACE teaching instructional approach requires adequate planning, acting, observing, reflecting and creating a revised plan". The investigation report of the preliminary stage of development and implementation of this new instructional approach, necessitated that some few adjustments, modifications and clarifications should be made. This was in view of the assertion that: "teachers are 'active modifiers' not 'passive acceptors' of a particular intervention" Rogers (2003) cited in Outhwaite, Gulliford & Pitchford (2020). These few adjustments were effected gradually, until it was established by mathematics education experts that, the different components of this new instructional approach were properly developed and well implemented in line with Durlak & DuPre (2008) cited in Outhwaite, Gulliford & Pitchford (2020).

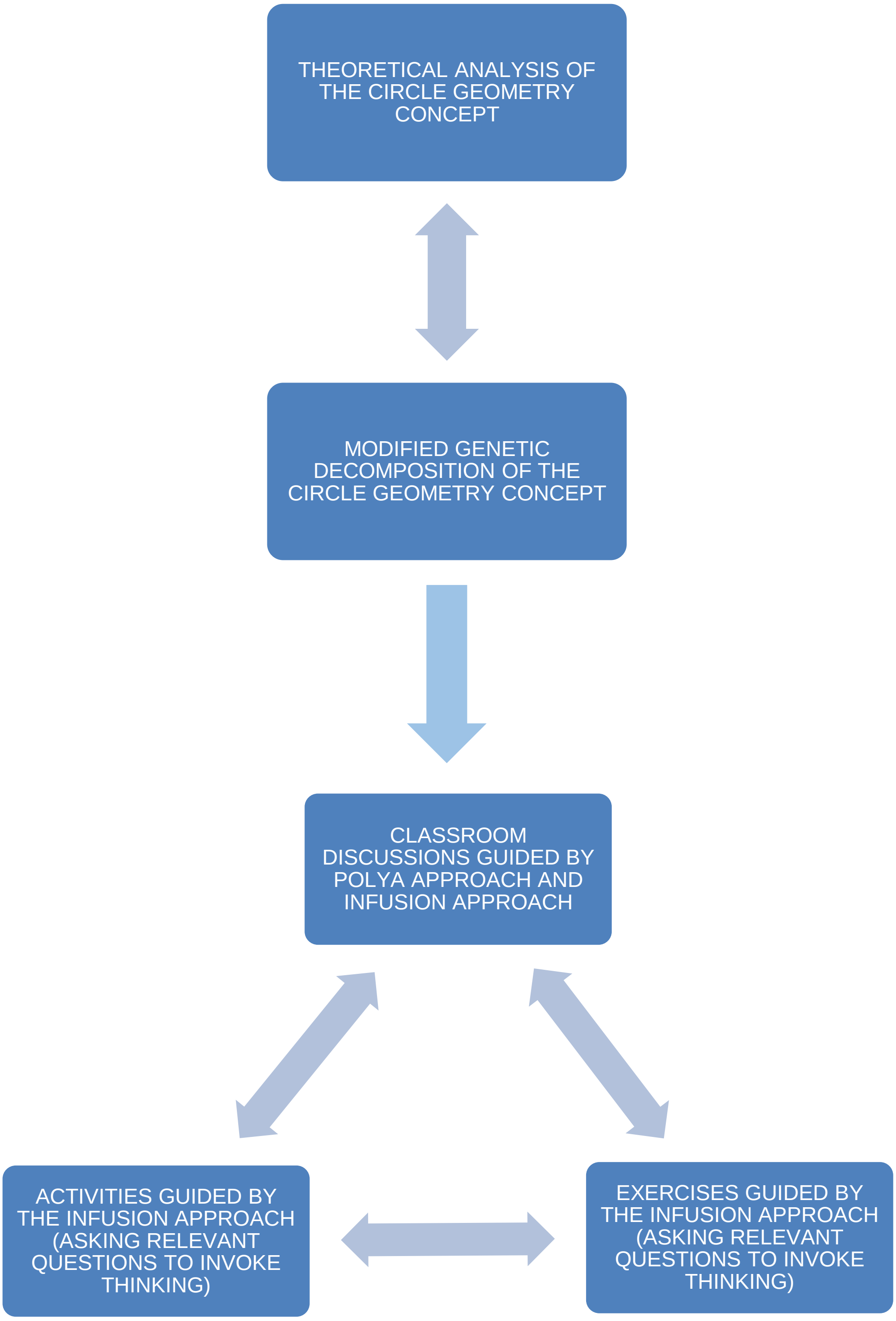
During the modification, the structure and content of the original version of the IPAC model remained constant. The alterations made on the original version were to provide clarifications on how the infusion approach would be integrated with Polya's problem-solving approach at the classroom discussion stage of the ACE teaching cycle. This became relevant as the original version did not provide detailed guidelines on how the infusion approach can practically be merged with Polya's problem-solving approach. This was picked up by the team of mathematics education curriculum experts when the IPAC model was tried and tested (during the prototype stage). To this end, the two said constructs were integrated together at two different stages of Polya's problem-solving approach: step 2 (devising a plan stage) and step 3(carrying out the plan stage).

The researcher and the team of mathematics education curriculum experts realised that "devising a plan" is a cognitive process, which require thorough understanding, application and transfer of knowledge, reflective, critical and creative thinking, just as the infusion approach. They were, thus, found to be synonymous, which necessitated integrating the two at step 2; practically, at this stage of Polya's approach, learners are guided to take meaningful decisions that can assist them to solve circle geometry problems, by adhering to the following 3 questioning prompts: (1) *What are my options? (2) What are the likely consequences of these options? (3) What is the best option in the light of these consequences?* (Swartz & Reagan, 1998). These prompts guided participants to apply and transfer their prior knowledge, when solving circle geometry problems (Zulkpli, Abdullah, Kohar & Ibrahim, 2017). It also guided them to think reflectively, critically and creatively without bounds, which are mandatory components of the infusion approach (Swartz & Reagan, 1998).

As said earlier, the infusion approach was also integrated with Polya's problem-solving approach at step 3 (carrying out the plan stage). This was because at this stage participants are mandated to put their thoughts into practice. The fusion of these two strategies at this stage was achieved by asking participants to explain (justifying) their conjectured solutions. This is to say that after participants presented their conjectured solutions, they were mandated to explain the solution they had presented (Mata-Pereira & da Ponte, 2017); this guided them to substantiate every decision they took. That is, what necessitated a decision to be taken when solving the problem, as participants were engaged in discussions and consultations among themselves. This guided participants to be responsible for their learning, to reflect on their own thinking, and to "think about their thinking", as they were engaged in group activities - an inherent component of the infusion approach (Ekawati *et al.*, 2019; Swartz & Reagan, 1998). This was the main rationale for integrating the two constructs at step 3 (carrying out the plan stage).

Final version of the IPAC model

The changes highlighted above, were effected on the original version of the IPAC model and constituted the final version of the IPAC model. A diagrammatic representation of the final version of the IPAC model, is illustrated below; the details of this IPAC model - how each research construct were applied and implemented, are presented on the said diagram. This is followed by discussions on the efficacy of this new problem-solving instructional approach.



NB: Details of the Classroom discussion stage- integrating the infusion approach and Polya's approach

1. Understanding the problem
2. Devising a plan

This is associated with taking meaningful decisions that will support learners to solve circle geometry problems efficiently. Learners are guided by the following questioning prompts/guidelines: (1) what are my options? (2) What are the likely consequence of these options? (3) What is the best option in the light of the consequence? (Swartz & Reagan, 1998).

3. Carrying out the plan

-Learners are required to present their solution, as well as a brief explanation to the solution.

4. Looking back

Figure 9.1: Final version of the IPAC model

9.2.2 Practicality of the proposed IPAC model

According to Nieveen (1997; 1999), the practicality of an intervention approach "refers to the extent that users (teachers and pupils) and other experts consider the intervention as appealing and usable in normal conditions". As mentioned earlier, the research design for this study - Education Design Research - effectively guided the researcher to repeatedly observe, monitor the development and implementation of this new instructional approach, in varied learning environments. This is so because all observations and findings about the implementation of this new approach, as it was tried and tested in different conditions were not dissimilar (Easterday, Lewis & Gerber, 2018; Outhwaite, Gulliford, & Pitchford, 2020).

The data analyses' results confirm that this proposed instructional approach is practically attainable, however, it has been established that the three-week duration allocated for circle geometry teaching and learning, is inadequate. The researcher realised that the above-mentioned timeframe for circle geometry is not enough to enable learners to achieve proper mastery of the content and its related application-oriented questions, not even among the highly intelligent learners. This implies that the average and week learners are very disadvantaged by this timeframe.

As mentioned in Chapter 1, the matric pass percentage for the past five years - 2016, 2017, 2018, 2019, 2020 - at 40% and above, were 33,5 %; 35,1 %; 37,1 %; 35,0 %; 35,6 %, respectively (Diagnostic Report, 2020). The researcher posits that even if this result is used as a yardstick to represent the percentage of performance for the high-intelligent category, then this would imply that about 60% of candidates belonging to the average-and-week learner category are disadvantaged by this timeframe, which is worrisome. During the conduct of this study, the researcher proposed that circle geometry lessons should be divided into 4 lessons - 1, 2, 3, and 4 - however, during the conduct of the research study, the researcher realised that more time will be needed, especially, for the average and weak learners to attain proper mastery.

The researcher is concerned that only 3 weeks are allocated for teaching and learning of circle geometry in schools in South Africa, although, DoBE is fully aware that majority of learners belong to the average and weak category, and that circle geometry has a high mark-allocation in examinations. This contributes to teachers, running through the geometry content to satisfy departmental timeline (DoBE, 2018).

The analyses established that the proposed IPAC model, can be implemented successfully, however, its implementation come along with some challenges - those of teaching thinking skills together with relevant dispositions, those of teaching using the Polya's approach, integrated with teaching thinking by the infusion approach, together with relevant dispositions, under the guidance of APOS theory. As mentioned earlier, "teaching of thinking is not a quick fix" (Swartz & Reagan, 1998), hence teaching and learning by this sophisticated approach, consisting of three components was not a smooth procedure. Through this study it has been established that the instructional aims and objectives of teaching and learning by this new instructional approach can be practically and realistically realised, however, utmost care must be taken to ensure that its implementation is carried out, well (Ritchhart & Perkins, 2004).

The analysis of the developed questionnaires, which were answered by teachers and Heads of Department for mathematics at the research fields, helped to ascertain the practicality of teaching circle geometry, by implementing the IPAC model. The questionnaire answers informed how teaching was conducted, mode of presentation of lesson, and others. Collaboratively, the analysis, helped to address the degree of practicality of learning by the IPAC model - how the new instructional approach can influence the study participants' problem-solving skills, with circle geometry questions. All responses by the observer teachers, HODs and participants were in the affirmative, showing their support for the model's practicality and how well it was implemented, hence, advocating for its usage for teaching and learning of circle geometry.

9.2.3 Effectiveness of the proposed IPAC model

Nieveen (1997; 1999) defined effectiveness of an intervention approach as "the extent that the experiences and outcomes from the intervention are consistent with the intended aims", hence, from the results of the analysis, implementing this new instructional approach as an intervention and achieving the instructional outcomes of this intervention were established (Outhwaite, Gulliford, & Pitchford, 2020). The analysis of participants' questionnaire helped to address the effectiveness of teaching circle geometry by using the IPAC model. They respondents indicated that lessons were well-understood; they also provided positive comments to motivate why they said lessons were well-delivered (see Chapter 8 for details).

The analysis of participants' questionnaire, helped to determine the level of effectiveness of learning circle geometry, using this IPAC model. In response to the questions - *From now on, will you use the new instructional approach to learn circle geometry? Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry?* - most participants indicated that the instructional approach influenced their learning of circle geometry positively; that from now on, they will use this instructional approach for teaching and learning of mathematical concepts; they will recommend this new instructional approach, to any third party, for teaching and learning of circle geometry. These responses helped to establish the effectiveness of this new problem-solving instructional approach and how well corresponding instructional aims and objectives can be well achieved (see chapter 8 for details).

9.3 HOW THE PROPOSED PROBLEM-SOLVING INSTRUCTIONAL APPROACH INFLUENCED PARTICIPANTS' LEARNING OF CIRCLE GEOMETRY

Teaching and learning of mathematics has proven to be a mammoth task, comprising of multifaceted dimensions for measuring learning (Gono, & Pacoy, 2021). In this section, how the implementation of this IPAC model influenced participants' achievements in circle geometry, learners' mental construction and how it influenced the way they solved circle geometry problems are detailed below.

9.3.1 LEARNERS' ACHIEVEMENTS IN CIRCLE GEOMETRY

In measuring participants' achievements in circle geometry concepts, the researcher marked and recorded each participant's written responses to each of the four standardized test instruments and carried out content analysis of each participant's written responses to each of these tests. This informed the researcher on how they solved the circle geometry problems, their problem-solving skills, the thinking skills they applied in their solutions, and the mental constructions they demonstrated in line with the APOS theory's mental conceptions. The research question in relation to the aforementioned was:

(2) How does the proposed instructional approach to be used as a problem-solving heuristic influence Grade 11 learners' achievements in circle geometry?

The findings that were deduced from this study are the following:

(i) There was a statistically significant improvement in the circle geometry test scores of participants who used the proposed instructional approach as a problem-solving heuristic.

Participants who were taught by the IPAC model performed much better, in comparison to participants who were taught by the traditional approach (see Chapter 6 - Tables 6.1, 6.2, 6.3, 6.4 & 6.5; and section 6.3 for details of Hypothesis test conducted). The individual test scores of study participants in the experimental group were better in comparison to the individual test scores of participants in the control group. The researcher, hence, deduced that the IPAC model had a positive impact on the experimental group's participants' achievements in circle geometry (Mwelese & Wanjala, 2014).

(ii) Participants' conceptual understanding, procedural fluency, strategic competence and mathematical reasoning skills were developed, as recommended by the Mathematics Teaching and Learning Framework (MTLF) for South Africa (Kilpatrick, Swafford, & Findell, 2001; DoBE, 2018). All these nurtures the "self-efficacy" of students in mathematics problem-solving (Shannon, 2008).

Conceptual understanding dimension

According to DoBE (2018), conceptual understanding entails: "comprehension of mathematical concepts, operations, and relations"; Crooks & Alibali (2014) define conceptual understanding as: "deep knowledge of the underlying concepts of mathematics and how they relate to one another". For Wiggins (2014):

"*Conceptual understanding in mathematics means that students understand which ideas are key (by being helped to draw inferences about those ideas) and that they grasp the heuristic value of those ideas. They are thus better able to use them strategically to solve problems – especially non-routine problems – and avoid common misunderstandings as well as inflexible knowledge and skill".*

From the assertion of Korn (2014), an effective instructional approach can be implemented in mathematics classrooms to nurture learners' conceptual understanding; this IPAC model proved to be one of such effective instructional approaches. Its implementation demanded participants to explain, describe and apply identical concepts in similar and/or dissimilar contexts (Korn, 2014).

Additionally, the model, used a 'Hands-On Approach', to develop participants' conceptual understanding of mathematical concepts. Hands-on activities, such as - activity sheets, investigation tasks and standardized tests, were implemented during the lessons with the experimental group. Participants were made responsible for their own learning; they were positioned at the centre of the learning sessions, and they were made to interact endlessly, among themselves, until a solution was reached. All these nurtured their conceptual understanding (Korn, 2014).

Implementing the IPAC model on the experimental group enabled both the mathematics teacher and participants to identify their mathematical knowledge which was necessary when solving non-routine problems, especially, 'how' and 'when' they should be applied to solve problems. This was evident due to how the mathematical ideas were communicated and justified through group members proving and disproving their mathematical ideas during classroom presentations. All these nurtured participants' conceptual understanding (Wiggins, 2014).

This IPAC model also served as a medium through which participants communicated mathematically, either in verbal or written form, which also nurtured participants' conceptual understanding. In addition, different indirect questioning approaches used during the implementation of the model promoted conceptual understanding, which developed participants' mathematical reasoning and thinking skills, especially, as participants were made to reflect on their own work by looking back at their own solutions as asserted by Hirschfeld-Cotton & Nebraska (2008). The researcher, in line with these authors, advocated for a transition from the traditional approach of teaching and learning mathematics to focusing on nurturing mathematics learners into good thinkers and good problem-solvers by promoting conceptual understanding of mathematical concepts.

This facet of problem-solving served as background and content knowledge for circle geometry concepts, during the conducted lessons. In view of this, during the study, lessons 1 and 2, were structured to develop conceptual understanding of circle geometry concepts. Lessons observations, group discussions, group presentations, explaining conjectured solutions, class activities, analysis of lesson 1 and 2 and the standardized tests, hence, established that the implementation of this IPAC model for mathematics instructions, effectively developed participants' conceptual understanding of mathematical concepts (see Chapter 5). This optimized retention of mathematical concepts (Kilpatrick, Swafford & Findell 2002; Wiggings, 2014; Stern, Ferraro, & Mohnkern, 2017; Malatjie & Machaba, 2019).

The class discussion phase of the ACE teaching cycle, guided by the Polya's and the infusion approaches, enhanced participants' conceptual understanding. This was particularly noticeable during the "understanding the problem" stage of Polya's approach. This was also evident from the thinking skills to be taught in lesson 1-understanding and retention of ideas. They all promoted conceptual understanding of circle geometry concepts by guiding participants to pay attention to every details, such as mathematical symbols, terminologies, and notations. This enabled participants to know what the question entailed and demanded, thereupon, they were expected to master and retain the circle geometry knowledge they had acquired. Learners then applied and transferred the acquired knowledge to solve non-routine problems and higher-order circle geometry concepts. This justified developing participants' conceptual understanding to become integral in achieving mathematical proficiency (Stern, Ferraro, & Mohnkern, 2017; DoBE, 2018).

This process enabled participants to have a fundamental knowledge of circle geometry concepts being taught and inculcate into participants that they must not cease to learn and they must not 'learn to forget'; this approach emphasizes that the era of learners operating on the notion of - 'chew', 'pour', and 'forget', - is obsolete. Rather, retention of mathematics knowledge, conceptual understanding and learning how to effectively transfer and apply fundamental knowledge to solve non-routine problems are essential (Wiggings, 2014; Stern, Ferraro, & Mohnkern, 2017). These were prioritized during the conducted lessons. According to Hiebert (2003), "We understand something if we see how it is related or connected to other things we know"; this aided the transition from "the known" to "the unknown", during lessons.

This establishes why developing conceptual understanding of mathematical concepts is pivotal in mathematics problem-solving classrooms (DoBE, 2018).

Procedural fluency dimension

This dimension comprises of "skills in carrying out procedures flexibly, accurately, efficiently, and appropriately" (DoBE, 2018). According to Foster (2013), "rather than constituting a threat to conceptual understanding, procedural fluency is its natural partner", hence, these two dimensions can be put together and used effectively in mathematics classrooms (Nahdi & Jatisunda, 2019). The above established that after learners have developed conceptual understanding, there is the need for mathematics teachers to nurture procedural fluency (NCTM, 2014).

The implementation of the IPAC model was in line with procedures, from one step to the other, following the ACE teaching approach - class discussion phase of the ACE teaching cycle guided by Polya's and infusion approaches, understanding the problem, devising a plan, carrying out the plan, and reviewing the steps. These are done along with teachers' guidance procedures - activities and exercise phases of the ACE teaching circle; all these were followed, accordingly, during this study. The proposed IPAC model, was effectively aligned, with well-sequenced and meaningful procedures (Foster, 2013; Nahdi & Jatisunda, 2019). The researcher, as the implementer of this instructional model abided by these procedures explicitly. This study substantiated that these procedures are flexible, accurate, efficient and appropriate (DoBE, 2018). Each procedure was meaningfully structured and linked with each other. This was logically done based on the full knowledge that not explicitly abiding by any of the well aligned procedures, might constitute a great void in teaching and learning mathematics when using the IPAC model as the medium. These are outlined under sub-section 4.13.2 - explication and implementation of the proposed IPAC model.

The ACE teaching approach, which served as the foundation for the development and implementation of this IPAC model, is characterized by activities, class discussions, and exercises. This afforded participants the opportunity to be exposed to different problems, embedded with different contexts. By so doing, participants were able to practise and solve different types of mathematical problems. In some instances, a problem that was initially solved in class was intentionally, again, given to participants to find out whether they had mastered the strategy of solving such domain of questions. This, according to Foster (2017), nurtured participants' procedural fluency competence.

In addition, the higher-order questioning approach, used to elicit and evoke participants thinking as required by the IPAC model, nurtured participants procedural fluency, which is vital for problem-solving and reasoning. This assisted participants to approach and solve higher- order problems, which require creative and critical thinking (Best, 2019).

Procedural Fluency entails having adequate knowledge of mathematical procedures, knowing when and how to implement these, correctly in similar or dissimilar contexts, and to acquire the appropriate skills to effectively implement them during problem-solving. If learners' procedural fluency are not nurtured, this will prevent them from developing their conceptual understanding, hence, they may not be able to solve problems, effectively (NCTM, 2014). This requires students to have adequate knowledge of mathematical procedures; be given the liberty and freedom to know which of these procedures will be approapriate to solve any given problem and for them to provide acceptable justifications for opting for a mathematical procedure, instead of others.

According to Ginsburg (2012) and NCTM (2014), procedural fluency is not about merely memorizing and recalling of mathematical knowledge but goes beyond this, as a lack of procedural fluency may contribute to a lack of conceptual understanding. Learners, thus, need to know their facts in order to use procedures, acceptably.

Strategic competence dimension

This entails, "the ability to formulate, represent, and decide on appropriate strategies to solve mathematical problems" (DoBE, 2018). From the above, this dimension can be partitioned into three categories: formulation of mathematical problems, solving of mathematical problems and selecting and implementing appropriate strategies and effective classroom practices.

Firstly, circle geometry problems and activities were appropriately formulated for the conducted lessons. The structured programme of assessment (presented in Table 2.2 in Chapter two), addressed this. This assessment programme was formulated in accordance with the mental construction lessons: GD 1, GD 2, GD 3 and GD 4. This mandates that 60% of items of a task should be application-oriented (GD 3 & GD 4), while 40% (GD 1 & GD 2) should nurture learners' prior knowledge. Also in Chapter 3, Table 3.2 summarized the assessment and evaluation methods used during the conducted lessons for this study. Here, the questioning approach that was implemented for each of the four lessons were presented. In a nutshell, the programme of assessment and the questioning approach for each of the four conducted lessons that goes along with the implementation of the IPAC model, guided the teacher to formulate appropriate problems, activities and tasks for each of the four lessons. These problems were appropriately selected, scaffolded and sequenced before they were implemented in the lessons.

Secondly, the IPAC model was constituted by fusing four potent problem-solving strategies; they were the ACE teaching cycle of the APOS theory, the infusion approach, Polya's approach and conducting lessons in a collaborative classroom setting. This IPAC model served as an effective medium, which guided the teacher and learners to learn the circle geometry concepts.

Thirdly, effective classroom practices were ensured by the teacher which enabled participants to develop their strategic competence. According to Özdemir & Pape (2012), four features of classroom practices show how strategic competence is supported in a classroom: (a) allowing autonomy and shared responsibility during the early stages of learning, (b) focusing on student understanding, (c) creating contexts for students to learn about strategic learning and to exercise strategic behaviour, and (d) helping students to personalise strategies by recognising their ideas and strategic behaviours. During the conducted lessons: (1) participants were made to be responsible for their own learning. They solved problems on their own and presented their solutions with justifications; (2) participants' conceptual understanding was prioritized; (3) contexts were created to guide participants to learn strategic learning and behaviour. The IPAC model enabled participants to learn strategically by adhering to the strategic learning procedures that come with their implementation; (4) participants' conjectured ideas and the strategic behaviours they had developed were recognised by the teacher. The implementation of the IPAC model goes along with appropriate problem-solving dispositions and thinking skills to be learnt for each lesson (see Table 3.1). Every effort participants made, either individually or collectively, was appreciated by the teacher - whether it was wrong or correct. The teacher applauded participants, in case of correct responses or the teacher corrected participants' wrong responses. These four classroom practices, which were followed during conducted lessons, according to Özdemir & Pape (2012), enabled participants to nurture their strategic competence.

Mathematical Reasoning dimension

This entails "the capacity for logical thought, reflection, explanation, and justification" (DoBE, 2018). These four elements were conjoined to form this IPAC model, hence, it showed prospects of teaching learners how to think and how they can, individually, monitor their thinking; this IPAC model proved to be an effective instructional approach. The ACE teaching cycle, where lessons were taught in order of the mental constructions of the APOS theory - action, process, object and schema - nurtured participants' mental constructions (Brijlall, 2020).

The infusion approach, implemented during the questioning stage - activities, exercises and class-room discussion - elicited and evoked participants' thinking (NCTM, 2014). The appropriate selection and sequencing of problems that were solved during the conducted lessons also assisted greatly in developing their reasoning capacity (Bozena & Konstantinos, 2019). The infusion approach, which was integrated with Polya's approach, during the classroom discussion stage, also gave participants the opportunity to nurture their reasoning and problem-solving competences (King, Goodson, & Rohani, 2013).

The implementation of the IPAC model, advocates for lessons to be conducted in a collaborative classroom setting. This creates a good learning environment and assist learners to nurture their thinking (Jailani & Retnawati, 2016). Participants were seated in small groups of three or four enabling them to discuss and come up with solutions among themselves. They were given the autonomy to make their own mathematical constructions, and to justify them. This resulted in participants from any of the groups presenting and explaining their conjectured solutions, on the board. These presenters gave reasons for their solutions when they were explaining their solutions on the board (King, Goodson, & Rohani, 2013). According to Boaler, (2016), "Explaining your work is what, in mathematics, we call 'reasoning', and reasoning is central to the discipline of mathematics". These oral explanations during the conducted lessons enabled these participants to think aloud (Mitsea & Drigas, 2019; Kaupp, Frank & Chen, 2014), therefore, they were able to reflect on their thinking, thereby, enhancing their cognitive awareness (Mitsea & Drigas, 2019). This also enabled participants to learn meta-cognitively where thinking, planning, goal setting, problem-solving, evaluating, informing and connecting education with real-life context are the norms (Fathima & Saravanakumar, 2012).

The effective communication that ensued during the conducted lessons, also promoted participants' thinking and enabled them to solve other problems (NCTM, 2014; King, Goodson & Rohani, 2013). This gave an opportunity to participants from the other groups to accept or reject, with justifications, the presented solutions on the board (Mata-Pereira & da Ponte, 2017). Their thinking competence enabled them to conjecture different ways a problem can be solved (Best, 2019).

(iii) Participants' problem-solving competence improved, during and after the intervention.

Polya, (1981), maintains that "to solve a problem means to find the action that is appropriate to achieve an aim". The implementation of the IPAC model elicited the much-needed action that guided participants to be good thinkers and effective problem-solvers. The IPAC model promoted active, meaningful learning, integrated with metacognitive learning (Gono & Pacoy, 2021; Hartman, 2001). Firstly, participants' conceptual understanding, procedural fluency, strategic competence and mathematical reasoning skills were developed, that enabled them to solve circle geometry problems (DoBE, 2018). These dimensions were all thoroughly discussed earlier.

Secondly, teaching and learning became more interactive, interesting and attractive. This was confirmed during the study when at one point some participants from the control group wanted to join the experimental group, however, the researcher objected to their request by reminding them about the purpose of conducting this research study; this was recorded in the observers' classroom observation report. This proved to be valuable as it proved the attractiveness of the intervention and contributed in improving participants' problem solving competence.

Lastly, the post-intervention observations enabled the researcher to adequately gather enough evidence to answer this research question. The implementation of the IPAC model optimized participants' confidence, desire and willingness to keep on trying until a solution is reached (Dhlamini, 2012). The results include - participants solved problems with a direction; they knew how to approach a problem; they were able to explore more alternative strategies, and they were able to explore more thinking skills. Also, they were much more comfortable working/ discussing with their peers and their classroom dynamics, optimized (Abakah, 2019). This IPAC model, showed prospects of developing learners into good thinkers which results in effective problem-solving, thus, the participants individually and collaboratively, became better problem-solvers (Gray, 2018; Phuntsho & Dema, 2019).

(iv) The IPAC model guided majority of the participants to operate at the object and schema levels in relation to the APOS theory's mental conceptions.

Finding an effective instructional approach to enable the individual to reach the encapsulation mental act is not an easy process, and just a handful of instructional approaches can guide an individual to reach this level (Dubinsky, 2000). The IPAC model, implemented with the experimental group proved to be one of such few instructional approaches that can guide participants to operate at these higher-mental conceptions - the object and schema levels. The evidence that testified to this was presented in Chapter 7, sub-section 7.4.3 - participants' responses at object level - and sub-section 7.4.4 - participants' responses at schema level. There was ample evidence of the acquisition of circle geometry's mental conceptions demonstrated by participants; these were presented and elaborated upon in the aforesaid sub-sections.

9.3.2 LEARNERS' MENTAL CONSTRUCTION LEVEL AND HOW IT INFLUENCED HOW THEY SOLVED CIRCLE GEOMETRY PROBLEMS

This study adequately addressed the mental constructions demonstrated by participants in both the control group and the experimental group. In doing so, the researcher thoroughly analysed participants' written responses, then performed content analysis on each response, of the four standardized test instruments. These are detailed in Chapter 7. The research question in relation to this was:

(3) *How do the study participants' level of mental construction affect their abilities in their problem-solving techniques?*

This study established that the ACE teaching instructional approach, which served as the foundation of the IPAC model, significantly guided and enhanced the study participants' cognitive engagements and development (Syarifuddin & Atweh, 2022); this, ultimately, optimized their problem-solving competence. The implementation of the IPAC model on the experimental group gave participants the freedom to engage themselves efficiently when solving circle geometry problems (Syarifuddin & Atweh, 2022). Details of the classroom engagements, discussions, group presentations, and explanations of conjectured solutions that were observed during the conducted lessons can be found in Chapter 5.

The above, was established at each of the four levels of mental construction, in relation to the APOS theory - action, process, object and schema levels (Brijlall, 2020). In chapter 7, the researcher presented the data analysis' results of the participants' mental constructions demonstrated at each level of circle geometry mental conception. The researcher deduced from these results that more that 90% of participants in the experimental group were able to operate at the action and process levels and more that 80% in this same group were able to operate at the object and schema levels. The above testified that the experimental group who were taught by the IPAC model demonstrated better understanding, better thinking and better problem-solving competence in circle geometry in comparison to their counterparts in the control group, who were taught by the traditional approach (Voskoglou, 2015; Tokgöz, 2015; Borji, Alamolhodaei, & Radmehr, 2018; Syarifuddin, Nusantara, Qohar & Muksar, 2019; Arnawa & Yerizon, 2019; Moon 2019).

The mental constructions they were able to demonstrate at each level of circle geometry mental conception, were synonymous with their problem-solving competence. This enabled them to give appropriate responses to a lot of the questions which constituted each of the four standardized test instruments. This resulted in participants in the experimental group getting high marks for each of the four conducted standardized tests in comparison with their compatriots in the control group. The results of the standardized tests, presented in Tables 6.1, 6.2, 6.3, 6.4 and 6.5, in Chapter 6, serve as testimonial to the above assertion established by this study (Borji, Alamolhodaei, & Radmehr, 2018; Syarifuddin, Nusantara, Qohar & Muksar, 2019; Arnawa & Yerizon, 2019; Moon, 2019).

According to (Mudrikah, 2016) problem-based mode of learning, in relation to action-process-object-schema mental constructions of the APOS theory, may elicit and optimize students' high-order mathematical thinking ability. This assertion by (Mudrikah, 2016), was confirmed by the data analysis results of this study (see Chapter 7) as majority of participants in the experimental group were able to operate at the object and schema levels. The action and process mental-conception lessons, rightly, served as an appropriate foundation to establish participants' prior knowledge upon which their object and schema mental conceptions were nurtured. This guided majority of participants in the experimental group to develop relevant problem-solving strategies, such as, how to approach and solve non-routine problems; their performance was better than that of their compatriots in the control group.

9.4 Conclusion

This chapter presented and thoroughly discussed the research findings that emanated from conducting this study, in accordance with each research question. In addition, relevant literature was presented that supported these research findings. These processes assisted the researcher to make meaningful and valid conclusions as well as recommendations, which are presented in the next chapter.

CHAPTER 10

SUMMARY OF THE STUDY AND RESEARCH FINDINGS, DISCUSSIONS, CONCLUSIONS, RECOMMENDATIONS AND LIMITATIONS OF THE STUDY

10.1 Introduction

This chapter serves as the epilogue of this research report. It presents the summary of this study and the findings that emanated from this study. This is necessary to enable readers to vividly comprehend the activities, happenings and practices implemented in this study, so as to understand the research findings. Conclusions that emanated from the findings, implications of the findings for teaching, learning and policy, recommendations, themes for future research, limitations of the study, are all presented in this chapter.

10.2 Summary of the study

Learners' consistent under-performances recorded in mathematics across all levels of South Africa's educational hierarchy, to the concerns of UMALUSI and all, necessitated the conduct of this study. The researcher acknowledges that some progress have been made by mathematics education researchers and DoBE to address the teaching and learning difficiencies in relation to mathematics as a subject, however, the researcher posited that there is still room for improvement, a huge gap in existing literature on the topic - teaching thinking and effective problem-solving instructional approaches; thus, the focus of this study. Teaching thinking skills (infusion approach) and teaching effective problem-solving instructional approaches (Polya's approach) were prioritized as the main teaching and learning strategies that could help address South Africa's teaching and learning difficulties. In view of this, the APOS theory (ACE teaching cycle) was adopted to guide the design and implementation of this proposed new method of teaching and learning mathematics in a collaborative classroom setting.

This study centred on Grade 11 mathematics learners. This was because circle geometry is a topic to be taught in Grade 11 in the South Africa's mathematics curriculum. At the main research field, two Grade 11 mathematics classes were used for the study - 11A (control group) comprised of 30 learners while 11B (experimental group) had 32 learners. The following data - classroom observations; recorded videos; questionnaires, photograph of study participants' work (classwork/homework and responses to standardized tests) and field notes - were collected for a duration of 13 months. Data analyis presentations can be found in Chapters 5,6,7 and 8; findings from the data analysis' results were presented and discussed in Chapter 9, while the recommendations and conclusion of this study are presented in this chapter.

10.3 Research findings

10.3.1 Summary of findings

With reference to each research question, the findings that emerged from this study are summarized below:

Table 10.1: Summary of research findings

Main research question: What are the characteristics of an effective problem-solving heuristic for circle geometry problem-solving at Grade 11?	
Sub-questions	**Findings**

(1) How can the proposed instructional approach to be used as a problem-solving heuristic be developed and implemented in the circle geometry classroom?	The validity, practicality and effectiveness of this new method of teaching and learning were established (Nieveen, 1997; 1999).
(2) How does the proposed instructional approach to be used as a problem-solving heuristic influence Grade 11 learners' achievements in circle geometry?	(i) There was a statistically significant improvement in the circle geometry test scores of learners who were subjected to the proposed instructional approach as a problem-solving heuristic. (ii) Participants' conceptual understanding, procedural fluency, strategic competence and mathematical reasoning skills were developed. (iii) Participants' problem-solving competence improved, during and after the intervention. (iv) The IPAC model guided majority of the participants to operate at the object and schema levels in relation to the APOS theory's mental conceptions.
(v) How do the participants' level of mental construction affect their abilities in their problem-solving techniques?	The ACE teaching instructional approach significantly guided and enhanced participants' cognitive engagements and development, which ultimately, optimized their problem-solving competence.

10.3.2 Discussions of research findings

In Chapter 9, the researcher thoroughly discussed the findings that emanated from the data analysis results individually, in line with each research sub-question. In process, the researcher discussed these findings collectively - how they relate with each other and in relation to the main research question: *What are the characteristics of an effective problem-solving heuristic for circle geometry problem-solving at Grade 11?*

The conduct of this study established what characterizes an effective problem-solving heuristic for circle geometry problem-solving at Grade 11. These were identified as - teaching of thinking skills and an effective problem-solving instructional approach. These created the foundation on which the new problem-solving instructional approach - the IPAC model - was constituted. As said earlier, the model was born from the amalgamation of the APOS theory, the infusion approach and Polya's approach, in a collaborative classroom setting. The teaching of thinking was specifically achieved by three components of this IPAC model - the ACE teaching cycle (APOS theory), the infusion approach, and lessons conducted in a collaborative classroom setting; in addition, teaching effective problem-solving was achieved by implementing Polya's problem-solving approach.

Substantial evidence from this study established that, the implementation of the ACE teaching cycle, the infusion approach and lessons conducted in a collaborative classroom setting, as components of this IPAC model, which enhanced participants' thinking. Germane to this study was the research design (EDR) which guided the researcher to establish the systematicity and profundity in this thinking-laden problem-solving instructional model (Fahim & Eslamdoost, 2014). This ultimately resulted in improved achievements for the experimental group - their understanding improved, their test scores improved, and their problem-solving competence improved - in comparison with participants in the control group. It enabled them to solve non-routine problems and to find different ways a particular problem can be solved, better than their control group counterparts. This is because the IPAC model, guided a lot of the participants to operate at the object and schema levels of the APOS theory's mental conception; this means that, the ACE teaching circle, which served as the foundation for the design and implementation of this IPAC model was worthwhile. It greatly assisted in developing this effective problem-solving instructioan approach, that this study sought to achieve (Voskoglou, 2015; Borji, Alamolhodaei & Radmehr, 2018; Arnawa and Yerizon, 2019; Syarifuddin, Nusantara, Qohar & Muksar, 2019; Moon, 2019; Syarifuddin & Atweh, 2022).

It was also established from the implementation of this IPAC model that lessons should be conducted in a collaborative classroom setting, therefore, participants working in small groups nurtured their thinking and problem-solving skills (Phadiela, 2011). This component of this IPAC model, afforded learners the opportunity to engage themselves efficiently when solving problems (Syarifuddin & Atweh, 2022).

Polya's problem-solving approach, which was integrated with the infusion approach, during the classroom discussions' stage of this IPAC model, as well, proved to be valuable. The Polya's approach, specifically, greatly assisted and guided the below-average and the average learners. It assisted them to effectively know how to approach a mathematical problem which was a serious challenge before the conducting of this study. The approach assisted them to know what mathematics is; how they can learn mathematics and how they can solve mathematics problems. From this, participants' problem-solving skills and dispositions improved and they became better problem-solvers. This resulted in experimental group's participants scoring higher marks in comparison to those in the control group (Kousar, 2010; Mehmood, 2014; Valles & Wickramasingh, 2015; Carifio, 2015; Lee & Chen, 2015; Gray, 2018; Phuntsho & Dema, 2019).

The drastic improvement in participants test scores was encouraging. These results were achieved by developing participants' conceptual understanding, procedural fluency, strategic competence and mathematical reasoning skills (DoBE, 2018). According to (NCTM, 2014), "procedural fluency builds on a foundation of conceptual understanding, strategic reasoning, and problem-solving". This presupposed that these three mathematical proficiency dimensions above, served as a good platform, on which participants developed their procedural fluency. According to Best (2019), "Fluency builds the foundations students use to tackle more complex, multi-step questions in problem-solving and reasoning activities, and it's crucial to their success". This enabled participants to solve varied challenging and non-routine problems such as, how to approach them and how to conjecture appropriate and meaningful solutions for them, with justifications.

The above research findings serve as evidence to assert that the implementation of this IPAC model, in mathematics classrooms, will add to, achieving "gamification in education" agenda; that is, making education creative, flexible and challenging (Mueller & Yankelewitz, 2014). The model assisted participants to learn circle geometry well thereby to solve circle geometry problems meaningfully. This is because the IPAC model make teachers and learners, active entities in mathematics classrooms. Learners collaborated and interacted well as they sought for solutions to challenging non-routine problems given to them by the teacher. The teacher also collaborated and interacted well with participants, especially the group that consisted of the average and below-average participants; the teacher was able to address their challenges, individually and wholly as a group (Mueller & Yankelewitz, 2014).

Another important dimension that came along with the implementation of this IPAC model was the assessment method - didactic assessment (see section 2.7 of Chapter 2). This served as a catalyst which propelled proper teaching and learning of circle geometry. As this assessment approach is interactive in nature, it adequately supported and promoted the collaborative instructional approach, embedded in the IPAC model. The main elements of this IPAC model - infusion approach, Polya's approach, ACE teaching circle, which all incorporate the collaborative classroom instructional approach and the assessment method (didactic assessment) are in harmony with each other, since they are all interactive in nature. All these cumulated in obtaining improved achievements from the experimental group participants, as against the control group participants. Enough evidence from the conduct of this study has established the afforesaid.

These research findings helped to ascertain that: - (1) teaching thinking Skills, (2) teaching effective problem-solving instructional approaches; which were investigated in this study, upon which the IPAC model was designed; were appropriate in demystifying mathematics, particularly circle geometry. This is because this study established that the IPAC model is an appropriate and effective problem-solving instructional approach - a medium to help learners to be good thinkers and good problem-solvers. These were accomplished by adopting and implementing constructivism as the research paradigm of this study. This contributed to participants' high achievements, from the implementation of this IPAC model on the experimental group (Syarifuddin & Atweh, 2022).

10.3.3 Conclusions from research findings

"Problem-solving skills must be developed and the confidence to deal with non-routine problems must be nurtured. It is something we as a country has to pay serious attention to. We may have to look at ways to help teachers teach mathematics differently" (Volmink, 2020, p.2).

As said earlier in Chapter 1, this study was conducted at Umalusi chairperson's behest. It was also conducted to address the methodological and pedagogical inadequacies in the CAPS mathematics curriculum and in the MTLF as Volmink, (2020) mentioned in his concluding remarks, while suggesting a new approach in addressing South Africa's mathematics debacle. This IPAC model is unprecedented, although enough evidence from this study has established that it can be used as a tool by mathematics teachers to enable them teach differently as the model incorporates new ways and ideas of teaching and learning mathematics, which had never been considered in the South African context; it can, thus, be used as a tool by learners to develop them into effective mathematics problem-solvers and good thinkers. This can lead them on the right path of achieving mathematical proficiency (DoBE, 2018). Enough evidence from this study has testified that this IPAC model, can be used as a medium through which Conceptual Understanding, Procedural Fluency, Strategic Competence, Mathematical Reasoning Skills and a Learner-centred Classroom, as inscribed in the MTLF, can be practically and realistically achieved (DoBE, 2018; Kilpatrick, Swafford, & Findell, 2001). These have been justified in Chapter 9 (see section 9.3.1).

Adequate evidence from this study has established that this IPAC model, can be used as a contributory teaching and learning resource, aimed at demystifying mathematics, thereby, making the subject understandable, interactive, interesting and a thinking-laden discipline; these, according to Gono & Pacoy (2021) promote meaningful mathematics learning. Enough evidence from this study has connoted that the appropriate usage of the IPAC model, will sufficiently improve learners' achievements in mathematics and teachers' desire to teach mathematics increased. Using this IPAC model supplemented and optimized learners' mathematics competence by incorporating active learning methods in mathematics classrooms, ensuring active participation by learners in mathematics classrooms, thereby, contributing in demystifying mathematics (Gono & Pacoy, 2021).

Additionally, the continuous implementation of this IPAC model will greatly nurture individual learners to be responsible for their own learning. It teaches students - firstly the skill of appropriately 'planning for their learning'; secondly, how they can meaningfully sequence and organize their learning; thirdly, about metacognitive awareness and how they can individually monitor their thinking and learning. All these come together in developing students into good learners, good thinkers and effective problem-solvers (William & Maat, 2020). According to Shannon (2008), all these factors will cumulate in students becoming effective self-directed learners.

This IPAC model will give mathematics learners and teachers, good reasons to completely minimise the implementation of the traditional instructional approach in mathematics classrooms. This might serve as a path that leads to saying farewell to rote learning (procedural knowledge), in favour of this new instructional approach which embraces logical, creative and critical thinking, metacognition, as well as conceptual understanding of circle geometry concepts (Hirschfeld-Cotton & Nebraska, 2008).

This IPAC model may not, per se serve as a complete panacea, to liberate South Africa from all students' mathematics difficulties, however, it may go a long way in addressing some of the challenges, especially, the pedagogical aspects which have been inadequately addressed in the CAPS mathematics curriculum and in the MTLF.

This is to say that this study has provided enough evidence to unequivocally assert that the four main elements of the MTLF for South Africa, as well as, the specific aims and skills inscribed in the CAPS mathematics curriculum, are justifiably attainable, by implementing this IPAC model for mathematics instructions. In view of the above, I strongly advocate that mathematics learners, teachers and instructors, locally and farther afield of South Africa, should exploit this IPAC model. As Albert Einstein (1921) once said: "education is not the learning of facts, but the training of the mind to think". Let us all unite and work towards achieving this goal, particularly, where mathematics is concerned.

10.3.4 Implications of the findings of the study for teaching, learning and policy

It can be deduced from the research findings that the use of this new instructional approach, will change the way mathematics is taught to learners and the way they learn mathematics (DoBE, 2018; NCTM, 2014). This implies that a new era for teaching and learning of mathematics has come, that is, teaching thinking skills and teaching effective problem-solving approaches. So let us all embrace it with alacrity. The researcher trusts that this new instructional approach, will greatly assist in addressing the mathematics teaching and learning difficulties encountered in South African schools.

The findings have reiterated that circle geometry teaching and learning is about seeking solutions, not just memorizing procedures (Schoenfeld, 2016). This is because, in the control group, some participants who could identify and use correct geometric terminology, as well as, identify appropriate relevant properties, axioms and theorems, lacked appropriate techniques and skills in solving problems. Most notably, they could not appropriately, make connections across the circle geometry concepts to solve more complex problems. Participants' progression from the knowledge dimension of circle geometry concepts to their application demand that these concepts be taught in an innovative, exploratory and experimental manner which can evoke learners' creativity, thinking skills and spatial awareness, which in all, enhance learners' problem-solving skills. This is what this new instructional approach brings to bear.

This contrasts with the traditional approach which primarily, demands learners to memorize procedures. This is not the case here, as the new instructional approach aroused the study participants' interest in the learning of circle geometry, improved the classroom dynamics during the mathematics lessons, and improved the study participants' individual problem-solving skills. The findings may inform policy makers and mathematics curriculum developers to formulate a mathematics curriculum which can guide mathematics instructors to teach in an innovative, exploratory and experimental manner, as characterized by this new instructional approach.

10.4 Recommendations

With reference to the results of this study, the researcher wishes to make the following recommendations:

1. This IPAC model should be implemented for teaching and learning of circle geometry in South African schools.
2. This research study has substantiated that, teaching thinking is paramount in mathematics classrooms, hence, teaching of thinking skills should be prioritized in mathematics classrooms in South Africa.
3. In addition to learners being tasked in Euclidean geometry to give statements and reasons, they must be required to give brief explanations to how they conjectured their solutions; this should be made mandatory for formative assessment tasks. Enough evidence from this study has established that,intuitively, this will enable learners to monitor their own thinking, guide their cognition and enable them to think about their own thinking effectively,

however, during summative assessment sessions, learners may not be required to do so.

4. Logics/logical reasoning may be introduced, either fully as a topic or as a sub-topic under the already alligned topics, in South Africa's mathematics curriculum, across all grades (R-12). I hold the view that this will optimize and further drive the agenda of teaching thinking in schools.
5. Teaching learners effective problem-solving approaches should be considered by mathematics instructors.
6. Giving thinking time (Swartz & Reagan, 1998) as said earlier, is one of dispositions of teaching thinking, hence, more time should be allocated for teaching and learning of circle geometry, in the CAPS mathematics curriculum. This research study has proven that, the current duration of three weeks allocated for teaching and learning of circle geometry in Grade 11, is inadequate in addressing the teaching and learning challenges, associated with circle geometry. If the three-week allocation can not be explicitly extended in the curriculum, the researcher proposes that the teacher must create more time, outside normal school hours, to put this agenda into fruition.
7. This research study has established that conducting mathematics lessons in collaborative classrooms is apropos, hence, I recommend that no matter which instrucional approach a teacher wishes to implement for any mathematics lesson, teaching and learning should be conducted in collaborative settings. This study has established that conducting mathematics lessons in this setting, by itself, is the starting point of achieving mathematics proficiency.
8. The modified genetic decomposition (MGD) proposed in this study may be used as a yardstick when designing circle geometry instructional lessons. This will be relevant for future pedagogy on the topic, so that the appropriate mental constructions can be elicited and nurtured.

10.5 Themes for future research

1. A possible area for investigation is finding how this IPAC model, can be integrated with technology, for teaching and learning of mathematical concepts. A literature review on the impact of technology on teaching and learning has informed the researcher that this IPAC model integrated with technology will be a good fit. Hence, Further research studies may be required to update this IPAC model for posterity. (Outhwaite, Gulliford, & Pitchford, 2020).
2. An investigation into finding how this IPAC model, can be used for teaching and learning of other mathematical concepts, other than circle geometry.
3. As earlier suggested by the researcher, teaching of thinking skills is paramount in enabling mathematics learners achieve mathematical proficiency, hence, how the infusion approach can wholly, be used for teaching and learning of mathematics, could be investigated further.
4. In chapter one, the researcher asserted that teaching learners effective problem-solving approaches, will greatly help mathematics learners to become better problem-solvers. This researcher only implemented Polya's problem-solving model, hence, the researcher proposes that further investigations should be carried out to devise other empirically-established effective problem-solving models, in place of Polya's problem-solving model, combined in this IPAC model.
5. The challenges of teaching and learning mathematical concepts by using this IPAC model, may be investigated further.
6. The challenges of teaching thinking by the infusion approach, in mathematics, in South Africa's context is another topic that can be researched.

10.6 Limitations of the Study

The researcher did not have the resources, to conduct this research at more research fields/sites. This research study was conducted during the resurgence of the covid-19 pandemic, hence, its protocols and restrictions, did not permit the researcher to conduct this research study at more research fields. A comparative analysis of similar happenings and proceedings, at several research fields could have helped to substantiate the research findings of this study with those from other research settings. Monetary demands and time considerations were contributory factors, which curtailed this research study to few research fields. In addition to the above limitations, this research study only focused on Grade 11 mathematics learners and strictly on the research topic, under consideration. A more open approach might invite further applications of the present research construct to future research topics, if generalization of the findings is to be pursued.

10.7 Conclusion

This chapter presented summaries of: how this study was conducted and the research findings; the conclusions that emanated from the findings were also presented. These may assist readers to ascertain the trustworthiness and authenticity of the study – if this study appropriately identified and addressed a relevant gap in literature in mathematics education. In addition, implications of the findings for teaching, learning and policy as well as recommendations are presented to highlight the relevance of the conducting of this study. Also, themes for future research and limitations of the study addressed how the IPAC model can be improved via follow-up relevant research studies to update the model to the benefit of posterity.

REFERENCES

Abakah, F. (2019). Exploring mathematics learners' problem-solving skills in circle geometry in South African schools: A case study of a high school in the Northern Cape Province. A dissertation submitted to the University of South Africa, South Africa.

Abawi, K. (2013). *Data Collection Instruments: Questionnaire and Interview*. STraining in Sexual and Reproductive Health Research Workshop, Geneva.

Aizikovitsh, E. and Amita, M. (2010). *Evaluating an infusion approach to the teaching of critical thinking skills through mathematics.* Ben Gurion University, Beer-Sheva, Israel.

Alcock, L. & Simpson, A. (2002). The Warwick Analysis Project: Practice and Theory. 10.1007/0-306-47231-7-10.

Al-Mutawah, Masooma Ali , Al-Mutawah & Thomas, Ruby & Eid, Abdulla & Mahmoud, Enaz & Fateel, Moosa. (2019). Conceptual Understanding, Procedural Knowledge and Problem-Solving Skills in Mathematics: High School Graduates Work Analysis and Standpoints. *International Journal of Education and Practice.* 7. 258-273. 10.18488/journal.61.2019.73.258.273.

Allendoerfer, C., (1969). The dilemma in geometry. *Mathematics Teacher* (62) 165-169.

Arnawa, I.M.; Sumarno, U; Kartasasmita, B and Baskoro, E.T. (2007). Applying the APOS Theory to improve students' ability to prove in elementary abstract algebra. *J. Indones Math. Soc.* (MIHMI) Vol. 13, No. 1 (2007), pp. 133–148.

Arnawa, M.I and Yerizon, S.N. (2019). Improving Students' Level of Proof Ability in Abstract Algebra Through the APOS Theory Approach. *International Journal of Scientific & Technology Research*, volume 8, issue 07, July 2019, ISSN 2277-8616 128.

Arnon, I., Cottril., Dubinsky,E.,Fuentes S.R.,Trigueros M.,& Weller, K. (2014). APOS Theory: A Framework for Research and Curriculum Develoment in Mathematics Education. New York: Springer.

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In Research in collegiate mathematics education II. *CBMS issues in mathematics education* (Vol. 6, pp. 1–32). Providence, RI: American Mathematical Society.

Bannan, B. (2007). The integrative learning design framework: An illustrated example from the domain of instructional technology. In T. Plomp & N. Nieveen (Eds.), An introduction to educational design research (pp. 53-73). Netherlands Institute for Curriculum Development.

Bansilal, S; Brijlall, D and Mkhwanazi, T (2014). An exploration of the common content knowledge of high school mathematics teachers. University of the Free State, South Africa. *Perspectives in Education*: 32(1), ISSN 0258-2236.

Battista, M. (2009). Highlights of research on learning school geometry. In T. V. Craine & R. Rubenstein (Eds.), Understanding Geometry for a Changing World: Seventy-first Yearbook (pp. 91-108). Reston VA The National Council of Teachers of Mathematics (NCTM).

Battista, M., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. Journal for Research in Mathematics Education, 27(3), 258-292.

Bautista, R. G. (2013). The Students' Procedural Fluency and Written-Mathematical Explanation on Constructed Response Tasks in Physics. *Journal of Technology and Science Education*. Natural Sciences and Mathematics Department AMA International University (Bahrain).

Bay, J. (2000). Linking Problem-Solving to Students Achievement in Mathematics *Issues and Outcomes* www.nca casi.org/JS1/2000V112/ Problem.solv3-32K search on 24-11-2006.

Best, J. (2019). Here's Why Mathematical Fluency is Critical for Problem-Solving and Reasoning. Blended teaching software powered by human support.

Bixler, A. (2007). Teaching evolution with the aid of science fiction. The American Biology Teacher, 69(6), 337-340.

Blândul, V.C. (2009). Applications of interactive didactic evaluation in pre-academic learning system. *Problems of education in the 21st century,* Volume 17, University of Oradea, Romania.

Boaler, J. (2016). *Mathematical mind-sets: Unleashing students' potential through creative Math, inspiring messages and innovative teaching.* San Francisco, CA: Jossey-Bass.

Bogdan, R., & Biklen, S. K. (1997). *Qualitative research for education*. Boston, MA: Allyn & Bacon.

Bozena, M. & Konstantinos, T. Z. (2019). Task characteristics that promote mathematical reasoning among young students: An exploratory case study.

Eleventh Congress of the European Society for Research in Mathematics Education (CERME11), Utrecht University, Feb, Utrecht, Netherlands. Ffhal-02414915.

Bradley, M. Campbell J., & McPetrie S. (2012). *Platinum mathematics learners book Grade 11*. Maskew Miller Longman (Pty) Ltd.

Brijlall, D. (2015). *Exploring The Stages of Polya's Problem-solving Model during Collaborative Learning: A Case of Fractions*. Department of Mathematics, Durban University of Technology, South Africa.

Brijlall, D. & Ndlazi, N.J. (2019). Analysing engineering students' understanding of integration A to propose a genetic decomposition. *Journal of Mathematical Behaviour*. https://doi.org/10.1016/j.jmathb.2019.01.006.

Brijlall, D. (2020). An APOS Analysis of engineering students' understanding of the nature of solutions of a system of linear equations. *Technology Reports of Kansai University*, 62(09), 5361-5374.

Brown, S. & Knight, P. (1994). Assessing Learners in Higher Education, London: Kogan Page.

Brumfield, T. & Carrigan, S. (2011). Instructional Objectives: The Foundation of Instructional Effectiveness. 1st Annual AALHE Conference: University of North Carolina, Greensboro.

Bono, R., Alarcón, R. & Blanca MJ (2021). Quality of Generalized Linear Mixed Models in Psychology: A Systematic Review. Volume 12, Article 666182. https://doi.org/10.3389/fpsyg.2021.666182.

Borji, V., Alamolhodaei, H. & Radmehr, F. (2018). Application of the APOS-ACE Theory to improve Students' Graphical Understanding of Derivative. EURASIA Journal of Mathematics, Science and Technology Education, 14(7), 2947-2967 ISSN: 1305-8223.

Bozena, M. & Konstantinos, T. Z. (2019). *Task characteristics that promote mathematical reasoning among young students: An exploratory case study.* Eleventh Congress of the European Society for Research in Mathematics Education (CERME11), Utrecht University, Feb, Utrecht, Netherlands. Ffhal-02414915.

Burgess, T.F. (2001). *A general introduction to the design of questionnaires for survey research*. Information Systems Services Guide to the Design of Questionnaires. University of Leeds, England.

Caram, C. A. & Davis, P. B. (2005). Inviting student engagement with questioning. Kappa Delta Pi Record, 42(1), 18-23.

Carifio, J. (2015). Updating, Modernizing, and Testing Polya's Theory of Mathematical Problem Solving in Terms of Current Cognitive, Affective, and Information Processing Theories of Learning, Emotions, and Complex Performances. Journal of Education and Human Development. *American Research Institute for Policy Development.* Vol. 4, No. 3, pp. 105-117 ISSN: 2334-296X (Print), 2334-2978 (Online).

Carl, A. (2012). *Teacher Empowerment through Curriculum Development. Theory into Practice,* Fourth Edition, Juta Legal and Academic Publishers.

Case, R. (2005). Moving critical thinking to the main stage. *Education Canada,* 45(2), 45–49.

Chapman, O. (2005). In Chick, H. L. & Vincent, J. L. (Eds.). Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, pp. 225-232. Melbourne: PME.

Chang, K. (2007). Developing geometry thinking through multimedia learning activities. *Computers in Human Behavior* .

Chagwiza, C.J., Maharaj, A., & Brijlall, D. (2020). University students' mental construction when learning the Convergence of a Series concept. *Pythagoras,* 41(1), a567. https://doi.org/10.4102/ pythagoras.v41i1.567.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. Educational Studies in Mathematics, 24(4), 359-387. doi: 10.1007/bf01273371.

Christou, C., Mousoulides, N., Pittalis M, Pitta-Pantazi, D. (2005). *Problem solving and problem-posing in a Dynamic Geometry.* The Montana Mathematics Enthusiast.

Caram, C. A. & Davis, P. B. (2005). Inviting Student Engagement with Questioning. Kappa Delta Pi Record, 42:1, 19-23, DOI: 10.1080/00228958.2005.10532080.

Clements, D. (2001). *Teaching and Learning Geometry*. In Jikil Parick(Ed), Research companion to the NCTM standards for mathematics, Reston, VA: NCTM.

Cooper, P. (2011). Creative Mathematics in the Primary Classroom Mathematics Teaching, n223 p21-22 Jul 2011. ERIC journal publications. ISSN: ISSN-0025-5785.

Council for Quality Assurance in General and Further Education and Training, (2020). Umalusi, Pretoria, South Africa. Retrieved from https://www.timeslive.co.za/news/south-africa/2020-01-03-maths-teachers-must-teach-differently-says-umalusi-after-poor-matric-results/#.

Council of Chief State School Officers. (2017). PSEL 2015 and Promoting Principal Leadership for the Success of Students with Disabilities. One Massachusetts Avenue, NW, Suite 700 Washington, DC 20001-1431. Voice: 202.336.7000.

Crawford, M. (2001). Teaching contextually: Research, rationale, and techniques for improving student motivation and achievement in Mathematics and Science. Texas: CORD.

Creswell, J. (2012). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research* (4thed.). Upper Saddle River, NJ: Pearson Education.

Crooks, N.M. & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review,* 34, 344–377.

Cuoco, A.A. Goldenberg, E.P & Mark,J. (1996). Habits of the mind: An organizing principle of mathematics curriculum. *Journal of Mathematics Behaviour,* 15:375-402.

Cuoco, A. A. (2000). Meta-Problems in Mathematics. The College Mathematics Journal. 31. 373-378. 10.1080/07468342.2000.11974176.

Daniel, H. (2001). *Vygotsky and Pedagogy*. University of Birmingham, School of Education, Edgbaston, Birmingham.

Davidson, J. E., Deuser, R. & Sternberg, R. J. (1995). The role of metacognition in problem solving. In J. Metcalfe & A. P. Shimamura (Ed.), Metacognition: Knowing about knowing (pp. 208-225). Cambridge, MA: MIT Press.

Davis, T.C. & Martin, G.L. (2015). *Do Students Really Know What a Function is? Applying APOS Analysis to Student Small Group Presentations*. Hawai'i Pacific University, USA.

De Bono, E. (1985). *Six thinking hats: An essential Approach to Business Management.* Little, Brown & company.

De Bono, E. (1992). *De Bono's thinking course*. BBC books. MICA management resources.

De Corte E, Verschaffel, L. & Masui, C. (2004). The CLIA – model: A framework for designing powerful learning environments for thinking and problem solving. *European Journal of Psychology of Education.*

Department of Education. (1996). *Problem solving approach to mathematics.* Singapore.

Department of Education. (2002). Introducing the mathematics learning area. South Africa.

Department of Basic Education. (2008). NCS Learning Programme Guidelines. Pretoria: DBE.

Department of Basic Education. (2010). Curriculum and Assessment Policy Statement (CAPS). Pretoria: DBE.

Department of Basic Education. (2012). Curriculum and Assessment Policy Statement. Mathematics. Grades 10-12. Pretoria: DBE.

Department of Basic Education. (2015). *Matric Diagnostic report,* South Africa.

Department of Basic Education. (2016). Matric Diagnostic report, South Africa.

Department of Basic Education. (2016). Matric Past Question Papers. Mathematics. Pretoria: DBE.

Department of Basic Education. (2017). Matric Diagnostic report, South Africa.

Department of Basic Education. (2018). Mathematics Teaching and Learning Framework for South Africa: Teaching Mathematics for Understanding. Pretoria: DBE.

De Villiers M. D. & Njisane R. M. (1987). *The development of geometric thinking among black high school pupils in Kwazulu Natal (Republic of South Africa).* Proceedings of the 11th Conference of the International Group for the Psychology of Mathematics Education, Montreal, Canada: In Bergeron, J.C, Herscovics, N. & Kieren, C. (Eds.).

De Villiers, M.D. (1997). The future of secondary school geometry. *Pythagoras,* 44, 37-54.

De Villiers, M.D. (2004). Using dynamic geometry to expand mathematics educators' understanding of proof. *International Journal of Mathematics education, Science and Technology.*

Dhlamini, J.J. (2012). *Investigating the effect of implementing a context-based problem-solving instruction on learners' performance*. Dissertation: University of

South Africa, submitted in accordance with the requirements for the degree of Doctor of philosophy in Mathematics, Science and Technology Education (Mathematics Education).

Driscoll, J. (1988). *Research within the reach: Secondary school Mathematics.* National Council of Teachers of Mathematics, Reston: VA.

Drlik, D.I. (2015). *Student Understanding of Function and Success in Calculus*. Thesis: Department of Mathematics Education, Boise State University, USA.

Dubinsky, E. D. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), Advanced mathematical thinking (pp. 95-123). Kluwer: Dordrecht. 62.

Dubinsky, E. D. (2001). Using a Theory of Learning in College Mathematics courses. MSOR Connections, 1(2), 10-15.

Du Toit, S.D. & Du Toit, G.F. (2013). Learner metacognition and mathematics achievement during problem-solving in a mathematics classroom. *The Journal for Trans-disciplinary Research in Southern Africa,* 9(3), Special edition, pp. 505-518.

Easterday,M.W. Lewis, D.G.R. & Gerber, E.M. (2018). The logic of design research. *Learning: Research and Practice,* 4:2, 131-160, DOI: 10.1080/23735082.2017.1286367.

Effandi, Z. & Siti Mistima, M. (2010). Analysis of Students' Error in Learning of Quadratic Equations. International Education Studies; Vol. 3, No. 3. Canadian Center of Science and Education ISSN 1913-9020 E-ISSN 1913-9039. Retrieved from www.ccsenet.org/ies.

Einstein, A. (1921). New York Times, Einstein Sees Boston; Fall on Edison Test; Asked to Tell Speed of Sound He Refers Questioner to Text Books (Special to The New York Times), page 15, New York.(Pro Quest).

Edelson, D. C. (2002). Design research: What we learn when we engage in design. The Journal of the Learning Sciences, 11(1), 105-121.

Ekawati, R., Kohar, A.W., Imah, E.M., Amin, S.M. & Fiangga, S. (2019). Students' Cognitive Processes in Solving Problem Related to the Concept of Area Conservation. *Journal on Mathematics Education*, 10(1), 21-36.

Engel, R. & Schutt, R. (2009). *Fundamentals of Social Work Research*. Thousand Oaks: SAGE.

Fahim, M. & Eslamdoost, S. (2014). Critical Thinking: Frameworks and Models for Teaching. English Language Teaching. *Canadian Centre of Science and Education,* 141-151.

Fathima, M.P. & Saravanakumar, A. R. (2012). Reflection on Metacognitive Strategies – Teaching Learning Perspective. IJSR - *International Journal of Scientific Research,* Volume 1, ISSN No. 2277 – 8179.

Fauzan, A., Plomp, T., & Gravemeijer, K. P. E. (2013). The development of an RME-based geometry course for Indonesian Primary schools. In T. Plomp, & N. Nieveen (Eds.), *Educational design research - Part B. Illustrative cases* (pp. 159-178). SLO: Netherlands institute for curriculum development.

Fennema, E. & Romberg, T. A. (1999). Mathematics classrooms that promote understanding. Mahwah, NJ: Lawrence Erlbaum.

Foster, C. (2013). Mathematical études: embedding opportunities for developing procedural fluency within rich mathematical contexts. *International Journal of Mathematical Education in Science and Technology*, 44(5), 765–774.

Foster, C. (2017). Developing mathematical fluency: comparing exercises and rich tasks. *Educational Studies in Mathematics,* (2018) 97:121–141 DOI 10.1007/s10649-017-9788-x.

French, D. (2004). *Teaching and learning geometry*. London: Continuum international publishing group.

Freudenthal, H. (1991). Revisiting Mathematics Education. China Lecturers. Dordrecht: Kluwer Academic Publishers.

Ginsburg, D. (2012). *Building Procedural Fluency and Conceptual Understanding in Mathematics.* Edutopia.org: University of Phoenix.

Gray, C.A. (2018).*The Impact of Applying the First Two Steps of Polya 's Four Problem Solving Steps in an Advanced Mathematics Classroom*. University of South Carolina Scholar Commons, Theses and Dissertations.

Gono, E. R. & Pacoy, E. P. (2021). Redefinition of the Parameters of Meaningful Mathematics Learning. *Turkish Journal of Computer and Mathematics Education* Vol.12 No.13 6524 – 6542.

Gronlund, N. E. & Brookhart, S. M. (2009). *Gronlund's writing instructional objectives* (8th ed.). Upper Saddle River, NJ: Pearson Education, Inc.

Groth, R. E. (2005). Linking theory and practice in teaching geometry. Mathematics Teacher, 99(1), 27-30.

Hartman, H.J. (2001). Metacognition in Science Teaching and Learning. In H. J. Hartman (Ed.) *2001 Metacognition in Learning and Instruction: Theory, Research, and Practice.* Chapter 9 Dordrecht, The Netherlands: Kluwer Academic Publishers. 173-201.

Hayyulbathin,I. ,Winarni, R. & Murwaningsih, T. (2014). *Modification of Polya's Step to Solve Math Story Problem.* 5 th ICRIEMS Proceedings Published by Faculty of Mathematics And Natural Sciences Yogyakarta State University, ISBN 978-602-74529-3-0 ME-119.

Hiebert, J. (2003). Signposts for Teaching Mathematics through Problem Solving. In F. Lester & R. Charles, *Teaching Mathematics Through Problem Solving, Grades Pre K-6.* NCTM: Reston, VA, 2003.

Hirschfeld-Cotton, K. & Nebraska, O. (2008). *Mathematical Communication, Conceptual Understanding, and Students' Attitudes Toward Mathematics. Action Research Projects 4.* University of Nebraska – Lincoln, https://digitalcommons.unl.edu/mathmidactionresearch/4.

Hitchcock, D. (2011). *Critical thinking as an educational ideal.* Conference paper, Manchester University.

Hoffer, A. (1983). *Geometry is more than proof.* Mathematics Teacher. 11- 18.

Howie, J.S. (2003). *Conditions of schooling in South Africa and the effects on mathematics achievement.* Centre for Evaluation and Assessment, Faculty of Education, University of Pretoria, South Africa.

Hughes, P. (2010). Paradigms, methods and knowledge. In G. MacNaughton, S. Rolfe and I. Siraj-Blatchford (Eds.), *Doing Early Childhood Research*, (2nd ed.)', Maidenhead: Open University Press.

In'am, A. (2014). *The Implementation of the Polya Method in Solving Euclidean Geometry Problems.* Mathematics Department, University of Muhammadiyah, Malang, Indonesia Mathematics.

Jansen, L. & Dardagan, C. (2014). *Experts warn on changes to matric maths curriculum.* The Star Late Edition.

Jailani, J. & Retnawati, H. (2016). The effectivity of problem teaching set in problem-based learning to improve students' higher order thinking skills and character. *Jurnal Pendidikan dan Pembelajaran,* 23(2), 111-123.

Jiang, J. (2007). *Linear and Generalized Linear Mixed Models and Their Applications.* New York, Springer, 2007.

Johnson, V. E. (2013). Revised standards for statistical evidence, Proceedings of the National Academy of Sciences, www.pnas.org/cgi/doi/10.1073/pnas.1313476110

Jones, K. (2002), Issues in the Teaching and Learning of Geometry. In: Linda Haggarty (Ed), *Aspects of Teaching Secondary Mathematics: perspectives on practice*. London: Routledge Falmer. Chapter 8, pp 121-139. ISBN: 0-415-26641-6.

Kamal, S. S. L. B. A., (2019). Research Paradigm and the Philosophical Foundations of a Qualitative Study. PEOPLE: *International Journal of Social Sciences,* 4(3), 1386-1394.

Kaupp, J., Frank, B. & Chen, A. (2014). *Evaluating Critical Thinking and Problem-Solving in Large Classes: Model Eliciting Activities for Critical Thinking Development.* Toronto: Higher Education Quality Council of Ontario.

Karen E.R. (1996). *Grade 11 students understanding of circle geometry in a computer environment*, Faculty of Education, Simon Fraser University.

Kent, G. & Foster, C. (2015). *Re-conceptualising conceptual understanding in mathematics.* In Proceedings of the Ninth Congress of European Research in Mathematics Education (CERME9): Thematic Working Group 17 (pp. 98–107). Charles University, Prague.

Khanum, S. (2006). *Problem solving in primary school children* (unpublished MPhil thesis). National Institute of Psychology Quaid-e-Azam University Islamabad.P-227.

Kilpatrick, J., Swafford, J. & Findell, B. (2001). *Adding It Up: Helping Children Learn Mathematics.* Washington, DC: National Academy Press.

Kim, K. H. (2009). Creative Problem Solving. In B. Kerr (Ed). Encyclopedia of Giftedness, Creativity and Talent. Sage Publications. . pp. 188- 191.

King, F.J., Goodson, L. & Rohani, F. (2013). Higher Order Thinking Skills: Definition, Teaching Strategies, Assessment. *Educational Services Program publications* (Centre for Advancement of Learning and Assessment). www.cala.fsu.edu.

Kivunja, C. & Kuyini, A.B. (2017). Understanding and Applying Research Paradigms in Educational Contexts. *International Journal of Higher Education,* 6 (5). https://doi.org/10.5430/ijhe.v6n5p26.

Korn, J. (2014). *Teaching Conceptual Understanding of Mathematics via a Hands-On Approach*. A Senior Thesis submitted in partial fulfilment of the requirements for graduation in the Honours Program Liberty University.

Kuhn, D. (1989). Children and adults as initiative scientists. *Psychological Review*, 96, 674-689.

Kyne, D. (2021). Mixed methods is the most important research skillset of the 2020s. Retrieved fromhttps://bootcamp.uxdesign.cc/mixed-methods-is-the-most-important-research-skillset-of-the-2020s-7085943e2f38.

Laborde, C. (2005). The hidden role of diagrams in pupils' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, O. Skovsmose & P. Valero (Eds.), Meaning in Mathematics Education (pp. 159-179). New York Springer.

Lai, E.R. (2011). Critical Thinking: A Literature Review. *Pearson's Research Report. Pearson's research report series* http://www.pearsonassessments.com/.

Lee, C. I. (2017). An Appropriate Prompts System Based on the Polya Method for Mathematical Problem Solving. EURASIA Journal of Mathematics Science and Technology Education, 13(3), 893-910. https://doi.org/10.12973/eurasia.2017.00649a.

Lee, C. & Chen, M. (2015). Effects of Polya Questioning Instruction for Geometry Reasoning in Junior High School National. Taipei University, TAIWAN. *Eurasia Journal of Mathematics, Science & Technology Education*, 2015, 11(6), 1547-1561.

Leech N. & Onwuegbuzie A. (2008). A typology of mixed methods research designs. *Quality and Quantity,* 43(2), March, pp. 265-275.

Lesh, R. & Zawojewski, J.S. (2007). Problem Solving and Modeling. In: Lester, F., Ed., Second Handbook of Research on Mathematics Teaching and Learning, Information Age Publishing, Greenwich, CT, 763-802.

Li, Q. (2010).Infusing technology into a mathematics methods course: any impact? *Journal of Educational Research*. Volume 47.

Loğoğlu, P.K. & Üredi, L. (2017) The Effect of Mathematics Teaching Through Polya's Problem Solving steps upon 4th grade students' success of solving mathematic problem. *European Journal of Education Studies* ISSN: 2501 - 1111 ISSN-L: 2501 - 1111 Available on-line at: www.oapub.org/edu Copyright © The Author(s). Open Access Publishing Group 195, Volume 3.

Maharaj, A. (2010). *An APOS Analysis of Students' Understanding of the Concept of a Limit of a Function*. School of Mathematical Sciences, University of Kwazulu-Natal, South Africa.

Maharaj, A. (2013*). An APOS Analysis of Students' Understanding of the Concept of a Limit of a Function*. School of Mathematical Sciences, University of Kwazulu-Natal, South Africa.

Maharaj, A. (2013). An APOS analysis of natural science students' understanding of derivatives. Aneshkumar School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, South Africa. *South African Journal of Education;* 2013; 33(1).

Mager, R. F. (1997). *Preparing instructional objectives* (3rd ed.). Atlanta, GA: The Center for Effective Performance, Inc.

Malatjie, F & Machaba, F (2019). Exploring Mathematics Learners' Conceptual Understanding of Coordinates and Transformation Geometry through Concept Mapping. *EURASIA Journal of Mathematics, Science and Technology Education,* 15(12), https://doi.org/10.29333/ejmste/110784.

Mwalimu, (2017). *Data collection methods.* Retrieved from https://www.slideshare.net ›.

Marzano, R. (1997). A different kind of classroom Teaching with dimensions of Learning U.S, Association for Supervision and curriculum development. 1250.N.Pitt.St. Alexandria, Virginia, VA22314.

Mata-Pereira J. & da Ponte J.P. (2017). Enhancing students' mathematical reasoning in the classroom: teacher actions facilitating generalization and justification. Article in Educational Studies in Mathematics.

Malahlela, M.V. (2017). *Using errors and misconceptions as a resource to teach functions to Grade 11 learners*. Thesis: Wits School of Education and the Faculty of Science, University of the Witwatersrand, South Africa.

McMillan J & Schumacher, S (2014). *Research in Education: Evidence-based inquiry.* Pearsons New International Edition.

McClure, J. (2014). *Developing Number Fluency - What, Why and How.* Article: The NRICH Project, as part of The Millennium Mathematics Project. University of Cambridge.

Mehdi, N., Narges, Y., Yaftian & Shahrnazer, B. (2012). Mathematical creativity: some definitions and characteristics. Procedia - Social and Behavioral Sciences 31 (2012) 285 – 291. Retrived from. www.sciencedirect.com.

Mehmood, S.T. (2014). *Effect of Polya's Problem Solving method of teaching on achievement of revised bloom's taxonomy in mathematics at elementary level.* Department of Education, Faculty of Social Sciences, International Islamic University, Islamabad, Pakistan.

Mitsea, E., Drigas, A. (2019). Learners' academic achievement, self-confidence and raise self- awareness. *Journey into the Metacognitive Learning Strategies.* IJOE – Vol. 15, No. 14.https://doi.org/10.3991/ijoe.v15i14.11379.

Moon, K. (2019). "Graphs of Two Variable Inequalities: Alternate Approaches to the Solution Test". *The Mathematics Enthusiast:* Vol. 16: No. 1, Article 7.

Moon, K., & Blackman, D. (2017). A guide to ontology, epistemology and philosophical perspectives for interdisciplinary researchers. *Conservation Biology,* 28: 1167-1177.

Morehouse, K. E. (2007). *"Building Conceptual Understanding and Algebraic Reasoning in Algebra".* Education and Human Development Master's Thesis. 433.

Morgan, G.A. & Harmon, R.J. (2001). Data Collection Techniques. *Journal of the American Academy of Child and Adolescent Psychiatry,* Lippincott Williams & Wilkins, a Wolters Kluwer Company.

Moore K. D. (2001). *Classroom teaching skills* (5th ed.). McGraw-hill. New York.

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning,* 4: 189-212.

Mudrikah, A. (2016). Problem-based learning associated by action-process-object-schema (APOS) theory to enhance students' high order mathematical thinking ability. *International Journal of Research in Education and Science* (IJRES), 2(1), 125- 135.

Mueller, M. & Yankelewitz, D. (2014). Teachers Promoting Student Mathematical Reasoning. Investigations in Mathematics Learning. *The Research Council on Mathematics Learning,* Winter Edition, Volume 7, Number 2.

Murray, H., Oliver, A., & Human, P. (1998). *Learning through problem solving.* In A. Oliver & K. New Stead (Eds.). Proceedings of the Twenty-second International conference for the psychology of Mathematics Education: vol. 1(pp. 169-185). Stellenbosch University, South Africa.

Mwelese, J.K. & Wanjala, M.M. (2014). Effect of Problem Solving Strategy on Secondary School Students' Achievement in Circle Geometry in Emuhaya District of Vihiga County.

Nafisah, K. K., Wan Mohd Rashid, W. A., Zulkarnain, A. & Maizam, A. (2011). A Study of The Effectiveness of the Contextual Approach to Teaching and Learning Statistics at the Universiti Tun Hussein Onn Malaysia (UTHM).

Nahdi, D. S. & Jatisunda, M. G. (2019). Conceptual Understanding and Procedural Knowledge: A Case Study on Learning Mathematics of Fractional Material in Elementary School. ICComSET 2019 *Journal of Physics.* Conference Series 1477 (2020) 042037IOP Publishing doi:10.1088/1742-6596/1477/4/042037.

National Council of Teachers of Mathematics (1985). The Secondary School Mathematics Curriculum. Year book of the NCTM. Reston, VA.

National Council of Teachers of Mathematics (2000). Principles to Actions: Ensuring Mathematical Success for All. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2009). Principles to Actions: Ensuring Mathematical Success for All. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2014). Principles to Actions: Ensuring Mathematical Success for All. Reston, VA: NCTM.

National Research Council. (2012). Improving Mathematical Education. Washington: National Academy Press. OEDC, (2014). PISA 2012 Result: What Students Knows and Can Do, Volume 1, OECD.

National Centre for Thinking (1996). Infusing The Teaching of Critical and Creative Thinking into Content Instruction: staff development manual, infusion program.

NCSS (2022). Negative Binomial Regression: Chapter 326. Retrieved from content/themes/ncss/pdf/Procedures/NCSS/Negative_Binomial_Regression.pdf https://ncss-wpengine.netdna-ssl.com/wp.

Ndlovu, Z.A. (2015). *Exploring pre-service teachers' mental constructions of matrix algebra concepts: A South African case study.* Thesis: Department of Mathematics Education, School of Education, Faculty of Humanities, University of KwaZulu-Natal.

Ndlovu, M. & Mji, A. (2012). Pedagogical implications of students' misconceptions about deductive geometric proof. *Acta Academica* 2012 44(3): 175-205 ISSN 0587-2405 © UV/UFS. https://www.researchgate.net/publication/235906806.

Ndlovu, V. (2017). Grade 10 – 12 Learners' attitude towards mathematics and how the attitudes affect performance. A research report submitted to the school of Education, University of Witwatersrand, in partial fulfilment for the requirements for the degree of Master of Education by coursework and research report.

Nieveen, N. (1997). *Computer support for curriculum developer: A study on the potential of computer support in the domain of formative curriculum evaluation.* Doctoral dissertation. Enschede, the Netherlands: University of Twente.

Nieveen, N. (1999). Prototyping to reach product quality. In J. van den Akker, R. Branch, K. Gustafson, N. Nieveen, & T. Plomp (Eds). *Design approaches and tools in education and training*. Dordrecht: Kluwer Academic Publisher.

Ofori-Kusi, D. (2017). An investigation into the use of problem-solving heuristics to improve the teaching and learning of mathematics. Thesis submitted to University of South Africa in accordance with the requirements for the degree of Doctor of Philosophy in Mathematics Education.

Ong, A.C. (2000). *Infusing thinking into curriculum content.* National Institute of Education, Nanyang Technological University. A paper presented at the AARE annual conference, Sydney.

Ong, A. C. & Borich, G. D. (Eds.). (2006). Teaching Strategies That Promote Thinking: Models And Curriculum Approaches. McGraw-Hill.

Oladosu L.O. (2014). *Secondary School students' meaning and learning of Circle Geometry*. University of Calgary, Alberta.

Ottevanger, W. (2001). Teacher support materials as a catalyst for science curriculum implementation in Namibia. Unpublished doctoral dissertation, University of Twente, Enschede.

Outhwaite, L. A., Gulliford, A. & Pitchford, N. J. (2020). A new methodological approach for evaluating the impact of educational intervention implementation on learning outcomes, *International Journal of Research & Method in Education,* 43:3, 225-242, DOI: 10.1080/1743727X.2019.1657081.

Özerem, A. (2012). *Misconceptions In Geometry And Suggested Solutions For Seventh Grade Students*. International conference on new horizons in education. Faculty of Education, Near East University.

Özdemir, İ & Pape, S. (2012). Supporting students' strategic competence: A case of a sixth-grade mathematics classroom. *Mathematics Education Research Journal,* 24. 10.1007/s13394-012-0033-8.

Paul, R. W., & Elder, L. (2006). Critical thinking: The nature of critical and creative thought. *Journal of Developmental Education,* 30(2), 34–35.

Palys, T. (2008). Purposive sampling. In L. M. Given (Ed.) *The Sage Encyclopaedia of Qualitative Research Methods.* (Vol.2). Sage: Los Angeles, pp. 697-8.

Perveen, K. (2010). Effect of the Problem-Solving Approach on Academic Achievement of Students In Mathematics At The Secondary Level. *Contemporary Issues In Education Research,* Volume 3, Number 3. Kohat University of Science Technology, Pakistan.

Phadiela C. (2011). *Problem-solving in geometry in collaborative small group settings: How learners appropriate mathematical tools while working in small groups.* University of Western Cape, South Africa.

Phuntsho, U. & Dema, Y. (2019). Examining the Effects of Using Polya's Problem-solving Model on Mathematical Academic Achievement and Analyzing Ability of the Fourth Grade Students. *Asian Journal of Education and Social Studies.* 5(2): 1-8, 2019; Article No.AJESS.51542, ISSN: 2581-6268.

Piaget, J. (1978). Success and Understanding. London and Henley: Routledge and Kegan Paul.

Pierce, R., & Ball, L. (2009). Perceptions that may affect teachers' intention to use technology in secondary mathematics classes. *Educational Studies in Mathematics, 71(3), 299-317.*

Pithers, R. & Soden, R. (2000). Critical Thinking in Education: A review. Educational Research. 42. 237-249. 10.1080/001318800440579.

Plucker, J., Beghetto, R. & Dow, G. (2004). Why Isn't Creativity More Important to Educational Psychologists? Potentials, Pitfalls, and Future Directions in Creativity Research. Educational Psychologist. 39. 83-96. 10.1207/s15326985ep3902_1.

Poloskaia, E., Savard, A., & Freiman, V. (2015). Duality of mathematical thinking when making sense of simple word problems: Theoretical easy. *Eurosia Journal of Mathematics, Science and Technology Education.*

Polya, G. (1945). *How to solve it: a new aspect of mathematical method*. Princeton, NJ, US.

Polya, G. (1945). The effects of Polya's heuristics and diary writing on children's problem solving. *Mathematics Education Research Journal.*

Posamentier, A.S., Smith, B., & Stepelman, J. (2010). Teaching Secondary Mathematics: Techniques and Enrichment Units (8h ed.). Allyn & Bacon. ISBN: 0135000033.

Rahman, S. A., Ghazali, M., & Ismail, Z. (2003). *Integrating ICT In Mathematics Teaching Methods Course: How Has ICT Changed Student Teachers' Perception About Problem Solving?* The Mathematics Education into the 21st Century Project Proceedings of the International Conference. The Decidable and the Undecidable in Mathematics Education Brno, Czech Republic.

Reddy, V. (2013). The good, the bad and the potential: Unpacking TIMSS 2011. *Human Sciences Research Council Review*, 11(2):15-16.

Ritchhart, R. & Perkins, D.N. (2004). *Learning to Think: The Challenges of Teaching Thinking.* The Cambridge handbook of thinking and reasoning.

Rose, C.M & Arline, C.B (2008). Uncovering Student Thinking in Mathematics, Grades 6-12: 30. Formative Assessment Probes for the secondary classroom. Thinking in Mathematics Resources. First edition, Corwin publishers.

Roopa, S. & Rani, M.S. (2012). Questionnaire Designing for a Survey. *The Journal of Indian Orthodontic Society*, 46 (4):273-277.

Roux, A. (2003). *The impact of language proficiency on geometrical thinking*. *Proceedings of the Ninth National Congress of the Association for Mathematics Education of South Africa.* Cape Town: Association for Mathematics Education of South Africa.

Royal Society/Joint Mathematics Council (2001). *Teaching and Learning Geometry*. London: Royal Society/ Joint Mathematics Council.

Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, Metacognition and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research in Mathematics Teaching and Learning*. New York: MacMillan.

Schoenfeld, A.H. (2016). Learning to Think Mathematically: Problem Solving, Metacognition and Sense Making in Mathematics (Reprint). Reprinted with permission from Handbook of Research in Mathematics Teaching and Learning, copyright 1992, by the National Council of Teachers of Mathematics. *Journal of Education*, volume 196, number 2.

Schoenfeld, A.H. (2007). What is Mathematical Proficiency and How Can It Be Assessed? In Schoenfeld A.H. (Ed.). *Assessing Mathematical Proficiency.* Cambridge: Cambridge University Press.

Schurter, W.A. (2001). *Comprehension monitoring and Polya's heuristics as tools for problem solving by developmental mathematics students.* University of the Ncarnate Word.

Schwieger, R. (2003). Why is teaching problem solving so difficult? Proceedings of the American Society for Engineering Education, USA, 6071-6077.

Sequera, E. (2007). Creativity and teacher professional development in mathematics for primary education. Thesis: Doctor of philosophy, University of Barcelona, Spain.

Shannon, S.V. (2008). Using Metacognitive Strategies and Learning Styles to Create Self-Directed Learners. *Institute for Learning Styles Journal*, Volume 1, Fall 2008, Pages 14 -28. Wayne State College.

Silva, E. (2008). Measuring Skills for the 21st Century. Washington, DC: Education Sector. Retrieved from http://www.educationsector.org/usr_doc/MeasuringSkills.pdf.

Smart-Morstad, K. (2008) Thinking-based Learning: Promoting Quality Student Achievement in the 21st Century. Critical Questions in Education Volume 3:1. New York: Teachers College Press, 2008. 288 pp.

Siyepu, S. W. (2012). *Analysis of Errors in Derivatives of Trigonometric Functions: a case study in an extended curriculum programme*. Thesis: school of Science and Mathematics Education. University of the Western Cape, South Africa.

Stakes, R.E. (1995). The Art Of Case Study Research. SAGE publications. Thousand Oaks: London.

Stern, J, Ferraro, K & Mohnkern, J. (2017). Tools for Teaching Conceptual Understanding, Secondary: Designing Lessons and Assessments for Deep Learning. Publisher: Corwin. Retrieved from https://sk.sagepub.com/books/tools-for-teaching-conceptual-understanding.

Stroup, W. W. (2013). *Generalized Linear Mixed Models: Modern Concepts, Methods and Applications*. Abingdon: Taylor and Francis.

Stylianides, G. J., Stylianides, A. J., and Philippou, G. N. (2007). Preservice teachers' knowledge.

Suh, J.; Seshaiyer, P.; Leong, K., Freeman, P. Corcoran, Meints, K., & Wills, T. (2012). *"Fostering Strategic Competence for Teachers through Modeling Rational Numbers Problem Tasks."* In Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology Mathematics Education, edited by L. R. Van Zoerst, J. J. Lo, and J. L. Kratky, pp. 47181. Kalamazoo, Mich.: Western Michigan University.

Swartz, R. (2012). Infusing instruction in thinking into content instruction: what do we know about its success? *Sri Lanka journal of educational research*.

Swartz, R. & Reagan, R. (1998). *Staff development training for teacher trainees.* The first Singapore summer institute on teaching thinking, Singapore.

Swartz, R.J. & Perkins, D.N. (1990). Teaching Thinking: Issues Approaches. *Psychology library editions: cognitive science.*

Swartz, R.J. & Parks, S. (1994). *Infusing The Teaching of Critical and Creative Thinking into Elementary Instruction.* Pacific Grove, USA: Critical Thinking Press & Software.

Swartz, R. J. & McGuiness, C. (2014). *Developing and Assessing Thinking Skills. Final Report of Thinking Skills Project.* Boston & Northern Ireland: The International Baccalaureate Organisation.

Swartz, R.J. Costa A.L., Beyer, B.K. Reagan, R. & Kallick, R. (2010). *Thinking-based learning: Promoting Quality Student Achievement in the 21st century*. New York: Teachers College Press.

Swindal, D. N. (2000). Learning geometry and a new language. Teaching Children Mathematics 7(4), 246-250. Retrieved from http://www.jstor.org/stable/41197577.

Syarifuddin, H., Nusantara, T., Qohar, A. & Muksar,M. (2019). Quantitative Reasoning Process in Mathematics Problem Solving: A Case on Co-variation Problems Reviewed from APOS Theory. *Universal Journal of Educational Research*, 7(10): 2133-2142.

Syarifuddin, H., & Atweh, B. (2022). The Use of Activity, Classroom Discussion, and Exercise (ACE) Teaching Cycle for Improving Students' Engagement in Learning Elementary Linear Algebra. *European Journal of Science and Mathematics Education,* 10(1), 104-138. https://doi.org/10.30935/scimath/11405.

Taherdoost, (2016). Sampling Methods in Research Methodology; How to Choose a Sampling Technique for Research. *International journal of academic research in management,* 18-27. 2139/ssrn.3205035.

Tango, T. (2017). *Repeated Measures Design with Generalized Linear Mixed Models for Randomized Controlled Trials*. London: Chapman & Hall.

Taplin, M. (2010). Mathematics through problem solving. Institute of Sathya Sai Education.Retrievedfrom:http://www.mathgoodies.com/articles/problemsolving.html.

Tay, M. K. & Mensah-Wonkyi T. (2018). Effect of using Geogebra on senior high school students' performance in circle theorems. *African Journal of Educational Studies in Mathematics and Sciences* Vol. 14, 2018.

Thayer-Bacon, B. J. (2000). *Transforming critical thinking: Thinking constructively.* New York, NY: Teachers College Press.

Thiele, J. & Markussen, B. (2012). Potential of GLMM in modelling invasive spread. *CAB Reviews* 7, 1–10. doi: 10.1079/PAVSNNR20127016.

Thompson, P.W. (2014). Constructivism in Mathematics Education. In: Lerman, S. (eds) Encyclopedia of Mathematics Education. Springer, Dordrecht. https://doi.org/10.1007/978-94-007-4978-8_31.

Tokgöz, E. (2015). *Analysis of STEM Majors' Calculus Knowledge by Using APOS Theory on a Quotient Function Graphing Problem*. Department of Engineering, School of Business & Engineering, Quinnipiac University, Hamden, CT, 06518. American Society for Engineering Education.

Tongco, M. D. C. (2007). Purposive Sampling as a Tool for Informant Selection. Department of Botany, University of Hawai`i at Manoa, 3190 Maile Way, Honolulu, HI, 96822 U.S.A. and Institute of Biology, University of the Philippines, Diliman, Philippines. *Ethno-botany Research & Applications* 5:147-158.

Trend in International Mathematics and Science Study, (2011). *Classrooms around the world*. Fifth series, International Association for the Evaluation of Educational Achievement. Lynch School of Education, Boston College.

Tziritas, M. (2011). *APOS Theory as a Framework to Study the Conceptual Stages of Related Rates Problems*. A Thesis in the Department of Mathematics and Statistics. Concordia University, Montreal, Canada.

Umugiraneza, O., Bansilal, S. & North, D. (2017). Exploring teachers' practices in teaching Mathematics and Statistics in KwaZulu-Natal schools. University of KwaZulu-Natal, South Africa. *African Journal of Education,* Volume 37, Number 2, doi: 10.15700/saje.v37n2a1306.

Van den Akker, J. & Plomp, T. (1993). Developmental research in curriculum: Propositions and experiences. Paper presented at the annual meeting of the American Educational Research Association (AERA), Atlanta, April 12-16.

Van Hiele, P.M. (1999). Developing Geometric Thinking Through Activities that Begin with Play. *Teaching Children Mathematics*, vol.5, 310 – 316.

Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education.* Utrecht: CD-β Press. Also publ. as thesis Universiteit Utrecht, 1996. - With ref. - With summary in Dutch. ISBN 90-73346-27-4 Subject headings: primary school education / mathematics education; assessment.

Vianney, J. P. & Navarro, A. (2011). Solutions of problems, creativity and invasion. Retrieved from http://www. buenastareas.com/ensayos/Solucion-De-Problemas-Creatividad-eInnovacion/3096644.html.

Volmink, J. (2020). Council for Quality Assurance in General and Further Education and Training, Umalusi, Pretoria, South Africa. Retrieved from https://www.timeslive.co.za/news/south-africa/2020-01-03-maths-teachers-must-teach-differently-says-umalusi- after-poor-matric-results/#.

Voskoglou, M.G. (2001). *On Student Understanding of the Concept of Infinity.* Graduate Technological Educational Institute, Western Greece.

Voskoglou, M.G. (2015). Fuzzy Logic in the APOS/ACE Instructional Treatment for Mathematics. *American Journal of Educational Research,* Vol. 3, No. 3.Science and Education Publishing. School of Technological Applications, Graduate Technological Educational Institute, Western Greece, Patras, Greece.

Vygotsky, L.S. (1978). Mind in society: the development of higher psychological processes; Cambridge, MA: Harvard University Press.

Wash, D. & Paul, R. (1988). *The goal of critical thinking: From educational ideal to educational reality.* Washington D.C.: American Federation of Teachers. Retrieved from http://files.eric.ed.gov/fulltext/ED295916.pdf.

Watson, A. & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. Mathematical Thinking and Learning, 8(2), 91–111.

Wickramasinghe, I. & Valles, J. (2015). *Can We Use Polya's Method to Improve Students' Performance in the Statistics Classes*? Advancing Education in Quantitative Literacy, Article 12, volume 8, scholar commons, University of south Florida.

Wiggins, G. (2014). *Conceptual Understanding in Mathematics. Thoughts on education.* Conceptual Understanding in Mathematics. (from: https://grantwiggins.wordpress.com/2014/04/23/conceptual understanding-in-mathematics/ retrieved on 06/05/2017).

William, S. K. & Maat, S. M. (2020). Understanding Students' Metacognition in AMathematics Problem Solving: A Systematic Review. *International Journal of Academic Research in Progressive Education and Development*, 9(3), 115–127.

Wilson, R. & Dubinsky, E. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior*, 32, 83- 101.

Woolfolk, A. (2014). *Educational Psychology* (12th ed). Upper saddle River, NJ: Pearson Education Inc.

Yıldırım, S. & Ersözlü, Z. N. (2013). The Relationship Between Students' Metacognitive Awareness and their Solutions to Similar Types of Mathematical Problems. Eurasia Journal of Mathematics, Science and Technology Education, 9(4), 411-415. https://doi.org/10.12973/eurasia.2013.946a.

Young T.J. (2016). Questionnaires and Surveys. In Zhu Hua, Ed. *Research Methods in Intercultural Communication: A Practical Guide.* Oxford: Wiley, pp.165-180.

Yuan, S. (2013). Incorporating Polya's problem solving method in remedial math. *Journal of Humanistic Mathematics* 3(1): 96-107.

Zepeda, C. D., Hlutkowsky, C. O., Partika, A. C., & Nokes-Malach, T. J. (2018). Identifying Teachers' Supports of Metacognition Through Classroom Talk and Its Relation to Growth in Conceptual Learning. Journal of Educational Psychology. Advance online publication. http://dx.doi.org/10.1037/edu0000300.

Zulkpli, Z., Abdullah, H., Kohar, A. & Ibrahim, N.H. (2017). *A review research on infusion approach in teaching thinking: advantages and impacts*. Universiti Teknologi Malaysia, Man in India, 97 (12): 289-298, Serials Publications.

APPENDIX A

TOTAL MARKS: 50 **LESSON ONE STANDARDIZED TEST** **DURATION: 1 HOUR**

QUESTION 1

Complete the following statements by filling in the missing words:

1.1 The line drawn from the centre of a circle, perpendicular to the chord …….

.

1.2 The line drawn from the centre of a circle to the midpoint of a chord …….

1.3 If PQ is a perpendicular bisector of chord AB, then PQ passes through …….

1.4 If PQ and JK are the perpendicular bisectors of any two non-parallel chords on the same circle, then PQ and JK will intersect each other and the centre of that circle will lie on their …….

1.5 The angle subtended by a chord at the centre of the circle is …….

1.6The angles subtended by a chord in the same segment of the circle …….

1.7 The angle subtended by a diameter on the circumference of a circle is always equal to ………

1.8 If a chord subtends a right angle on the circumference of a circle, then the chord is ………

1.9 The opposite angles of a cyclic quadrilateral …………………………………………………………………..

1.10 If the opposite angles of a quadrilateral are supplementary, then ……………………………………

1.11 If a line subtends equal angles at two points on the same side of itself, then …….

1.12 The angle between a chord and a tangent is equal to …………………………………………………………

1.13 If the exterior angle of a quadrilateral is equal to the interior opposite angle, then …….

1.14 If the exterior angle of a quadrilateral is not equal to the interior opposite angle, then

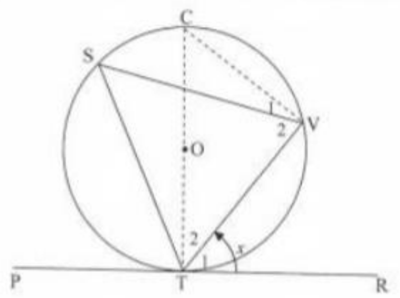
………

1.15 Equal chords subtends ………………………………………………………………………………………..................

(1 ×15 = 15)

QUESTION 2

2.1 In the diagram below, the circle with centre O passes through points S, T, and V. PR is a tangent to the circle at T. VS, ST and VT are joined.



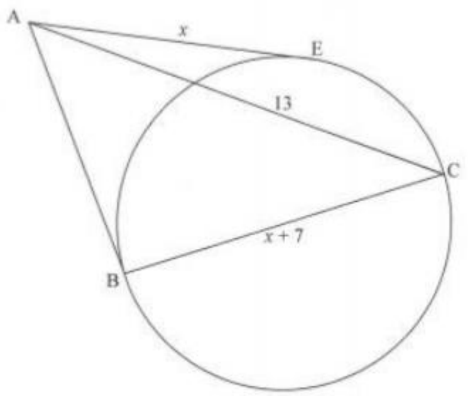
Given below is the partially completed proof of the theorem that states that $V\hat{T}R = \hat{S}$. Using the above diagram, complete the proof of the theorem.

Construction: Draw diameter TC and join CV.

Statement	Reason
Let: $V\hat{T}R = \hat{T}_1 = x$	
$\hat{V}_1 + \hat{V}_2 =$ ………	………………………………
$\hat{T}_2 = 90° - x$	………………………………
$\therefore \hat{C} =$ ………	Sum of the angles of a triangle
$\therefore \hat{S} = x$	………………………………
$\therefore V\hat{T}R = \hat{S}$	

(5)

2.2 In the diagram, AB and AE are tangents to the circle at B and E respectively. BC is a diameter of the circle. AC= 13 units, AE= x and BC = $x + 7$.



2.2.1 Give reasons for the statements below, by completing the table below:

	Statement	Reason
(a)	$A\hat{B}C = 90°$	
(b)	$AB = x$	

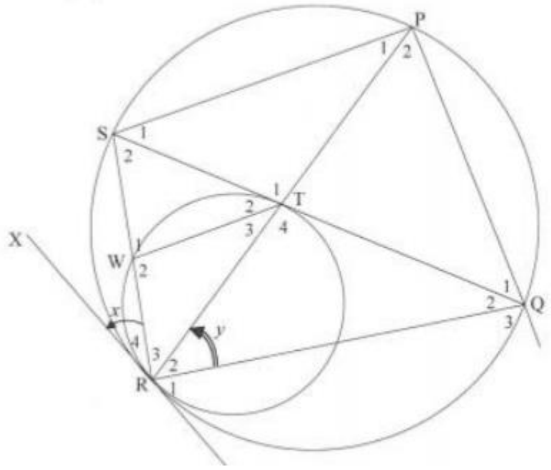
(2)

2.2.2 What is the length of AB? (4)

[11]

QUESTION 3

The two circles in the diagram have a common tangent XRY at R. W is any point on the small circle. The straight line RWS meets the large circle at S. The chord STQ is a tangent to the small circle, where T is the point of contact. Chord RTP is drawn. Let $\hat{R}_4 = x$ and $\hat{R}_2 = y$.



3.1 Give reasons for the statements below, by completing the diagram below.

Let $\hat{R}_4 = x$ and $\hat{R}_2 = y$		
	Statement	**Reason**
10.1.1	$\hat{T}_3 = x$	
10.1.2	$\hat{P}_1 = x$	
10.1.3	WT || SP	
10.1.4	$\hat{S}_1 = y$	
10.1.5	$\hat{T}_2 = y$	

(5)

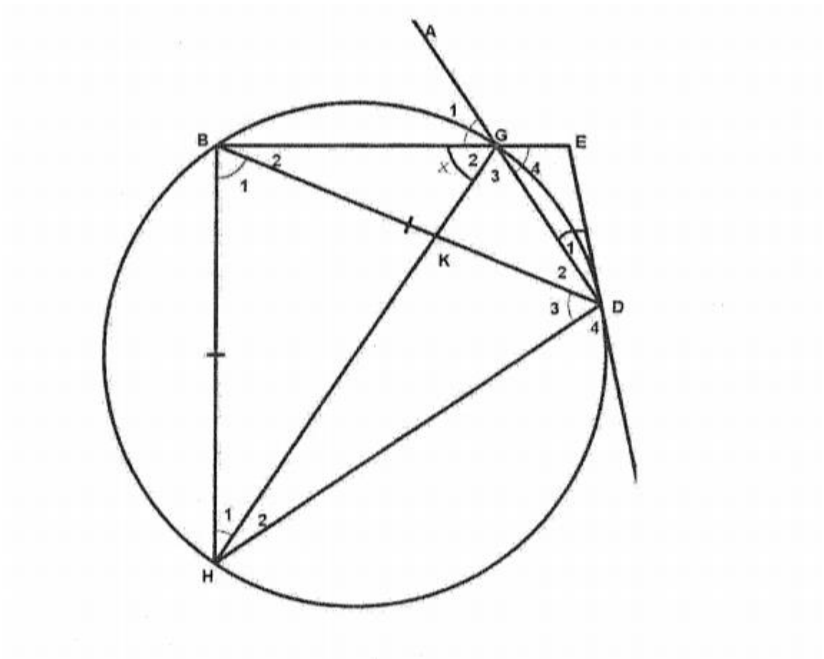
3.2 Name any two quadrilateral shapes from the diagram above. (2)

3.3 identify, with reasons, another TWO angles equal to y. (4)

$[\mathbf{11}]$

QUESTION 4

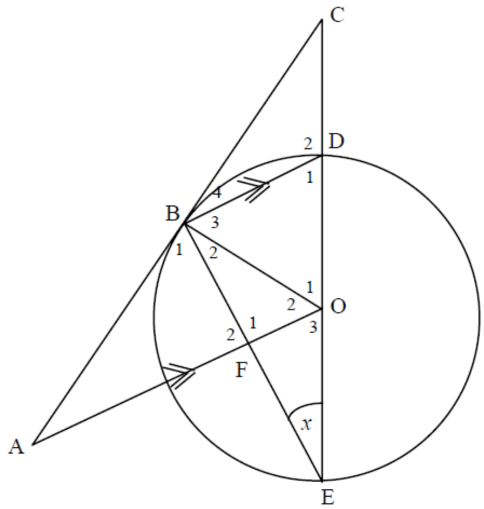
In the diagram below, BGDH is a cyclic quadrilateral. ED is a tangent to the circle at D. Chord GH and BD intersect at K. BG is produced to E and DG is produced to A. BH= BD and $\hat{G}_2 = x$.



Name, with reasons, 4 other angles equal to x. (7) $[\mathbf{7}]$

QUESTION 5

ED is a diameter of the circle, with centre O. ED is extended to C. CA is a tangent to the circle at B. AO intersects BE at F. BD || AO. $\hat{E} = x$.



5.1 Determine, with reasons, $C\hat{B}E$ in terms of x. (3)

5.2 Which lines are equal? (1)

5.3 Which angle is equal to 90^0 ? (1)

5.4 Which line can be considered as a tangent? (1)

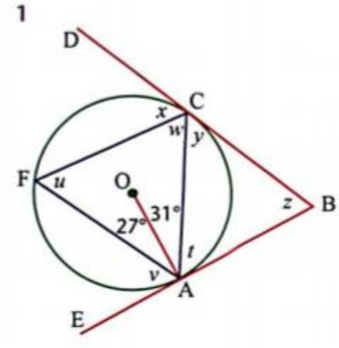
[6]

APPENDIX B

TOTAL MARKS: 50 **LESSON TWO STANDARDIZED TEST** **DURATION: 1 HOUR**

O is the centre of the circle in each diagram, for questions 1-6. In each case determine the values of a, b, c, etc. clearly state your reasons.

1



t = ...

u = ...

v = ...

w = ...

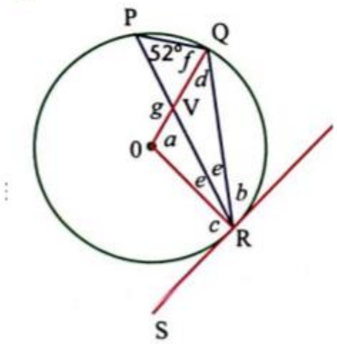
x = ...

y = ...

z = ...

14 MARKS (7 X 2)

2



a = ...

b = ...

c = ...

d = ...

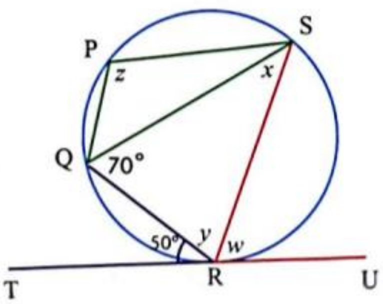
e = ...

f = ...

g = ...

7 MARKS (7 X1)

3



w = ...

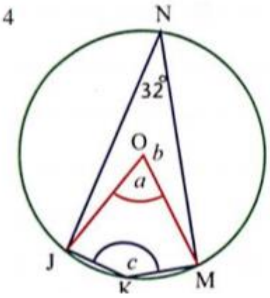
x = ...

y = ...

z = ...

4 MARKS (4 X 1)

4



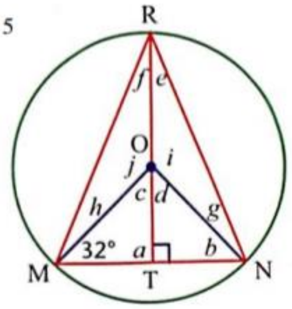
a = ...

b = ...

c = ...

3 MARKS- (3 X1)

5



b = ...

c = ...

d = ...

e = ...

f = ...

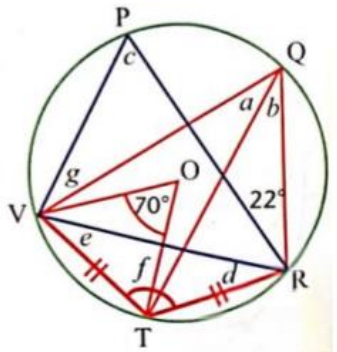
g = ...

h= ....

i = ...

8 MARKS- (8 X1)

6



a = ...

b = ...

c = ...

d = ...

e = ...

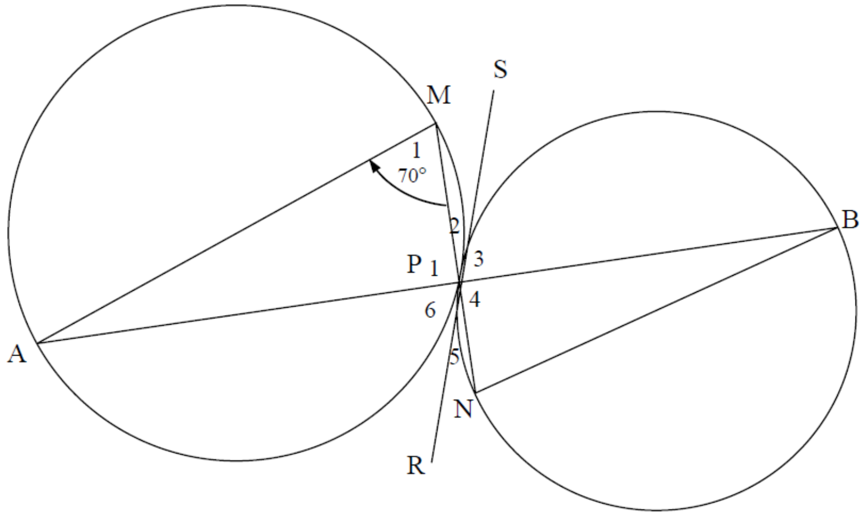
f = ...

g = ...

7 MARKS- (7 X1)

7

In the diagram below, AM is the diameter of the bigger circle AMP. RPS is a common tangent to both circles at P. APB and MPN are straight lines.



7.1 State the size of $\widehat{P}_1$. (1)

7.2 Hence, show that BN is the diameter of the smaller circle. (2)

7.3 If $\widehat{M}_1 = 70^0$, calculate the size of each of the following angles:

7.3.1 $\hat{A}$ (1)

7.3.2 $\widehat{P}_6$ (1)

7.3.3 $\hat{B}$ (2)

7 MARKS

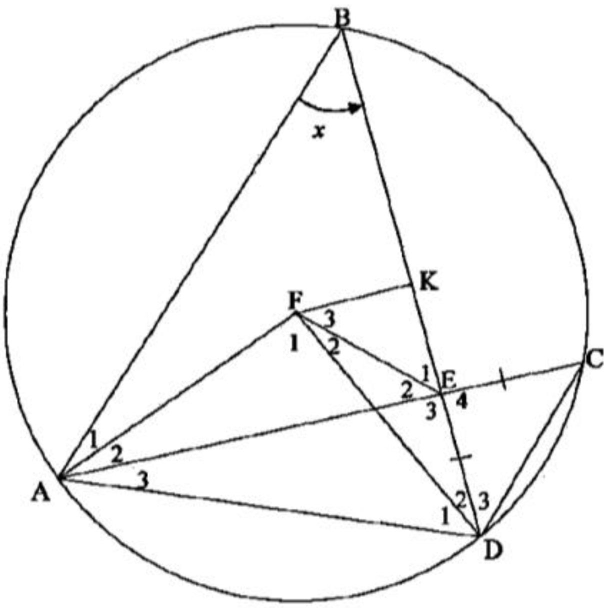
APPENDIX C

TOTAL MARKS: 50 LESSON THREE STANDARDIZED TEST DURATION: 2 HOURS

Instructions: Students are expected to give reasons, and explanations to reasons, they give to each statement they make, for each question in this section.

QUESTION 1

In the diagram, the circle with centre F is drawn. Points A,B,C and D lie on the circle. Chords AC and BD intersect at E such that EC= ED. K is the midpoint of chord BD. FK, AB, CD, AF, FE and FD are drawn. Let $\hat{B} = x$.



1.1 Determine, with reasons, the size of EACH of the following in terms of x:

1.1.1 $\hat{F}_1$ (2)

1.1.2 $\hat{C}$ (2)

1.2 Prove, with reasons, that AFED is a cyclic quadrilateral. (4)

1.3 Prove, with reasons, that $\hat{F}_3 = x$. (6)

1.4 If area $\Delta AEB = 6{,}25 \times area\, \Delta DEC$, calculate $\frac{AE}{ED}$. (5)

[**19**]

QUESTION 2

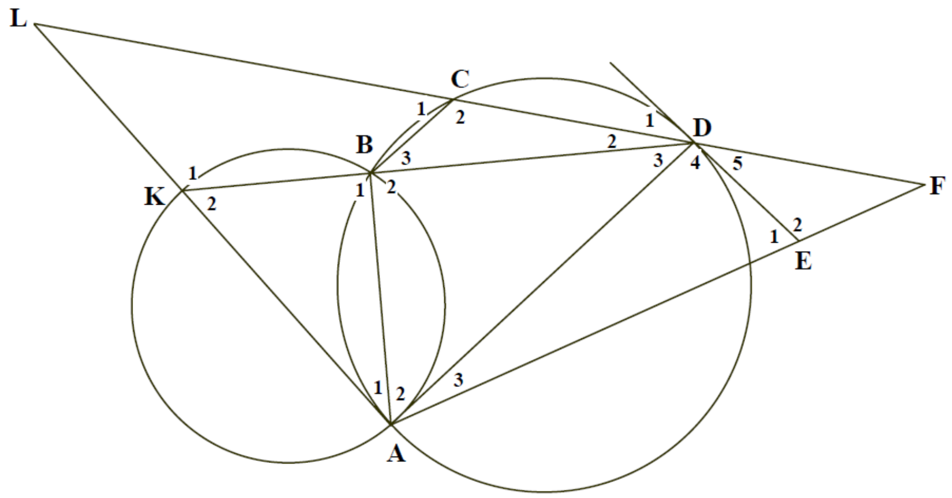
In the diagram below, two circles intersect each other at A and B.

ED is a tangent to circle ABCD.

DA is a tangent to circle AKB. DBK is a straight line.

AK and DC are produced to meet at L.

LCD and AE are produced to meet at F. CD = DF



Prove that:

2.1 LKBC is a cyclic quadrilateral. (5)

2.2 $\hat{B}_2 = L\hat{A}D$ (3)

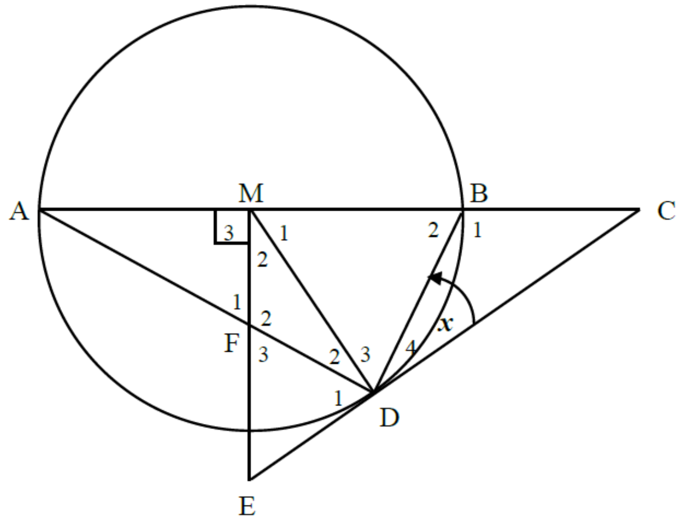
2.3 DE ∥ LA (4)

[**12**]

QUESTION 3

In the diagram, M is the centre of the circle and diameter AB is produced to C. ME is drawn perpendicular to AC such that CDE is a tangent to the circle at D. ME and chord AD intersect at F.

MB = 2BC.



3.1 If $\hat{D}_4 = x$, write down, with reasons, TWO other angles each equal to x. (3)

3.2 Prove that CM is a tangent at M to the circle passing through M, E and D. (4)

3.3 Prove that FMBD is a cyclic quadrilateral. (3)

3.4 Prove that $DC^2 = 5BC^2$. (3)

3.5 Prove that $\Delta DBC \;|||\; \Delta DFM$. (4)

3.6 Hence, determine the value of $\frac{DM}{FM}$. (2)

[19]

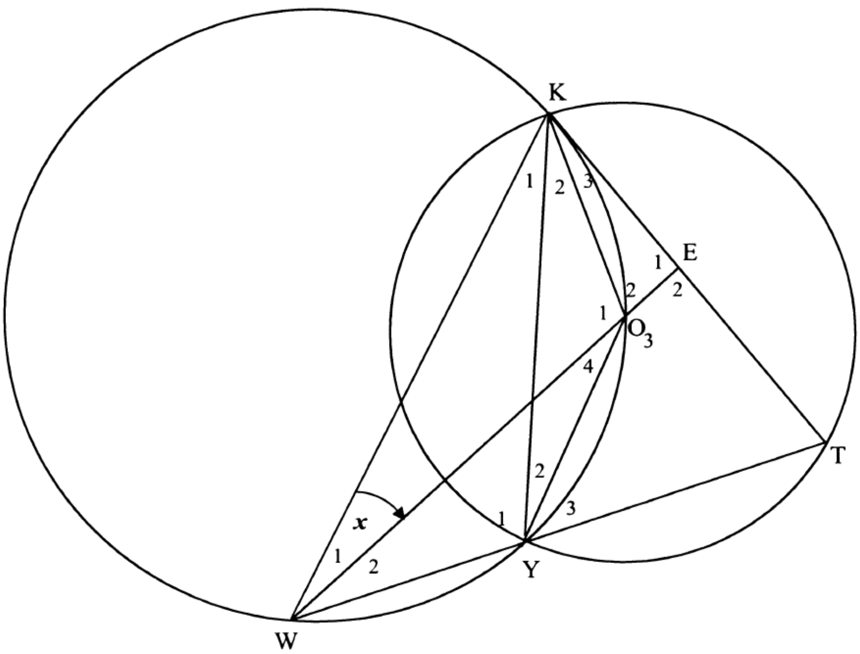
APPENDIX D

TOTAL MARKS: 50 LESSON FOUR STANDARDIZED TEST DURATION: 2 HOURS

Instructions: Students are expected to give reasons, and explanations to the reasons, they give to each statement they make, for each question in this section.

QUESTION 1

In the diagram below, two circles intersect at K and Y. The larger circle passes through O, the centre of the smaller circle. T is a point on the smaller circle such that KT is a tangent to the larger circle. TY produced meets the larger circle at W. WO produced meets KT at E. Let $\widehat{W}_1 = x$.



1.1 Determine FOUR other angles, each equal to x. (6)

1.2 Prove that $\widehat{T} = 90^0 - x$. (3)

1.3 Prove that KE = ET. (3)

1.4 Prove that $KE^2 = OE.WE$ (6)

$[\mathbf{18}]$

QUESTION 2

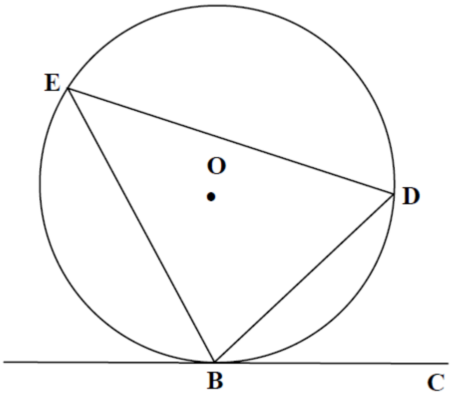
2.1

In the diagram alongside, O is the centre of circle EBD.

Use the diagram on DIAGRAM SHEET 2 or redraw the diagram in your ANSWER BOOK to prove the theorem which states that:

If BC is a tangent to the circle, then

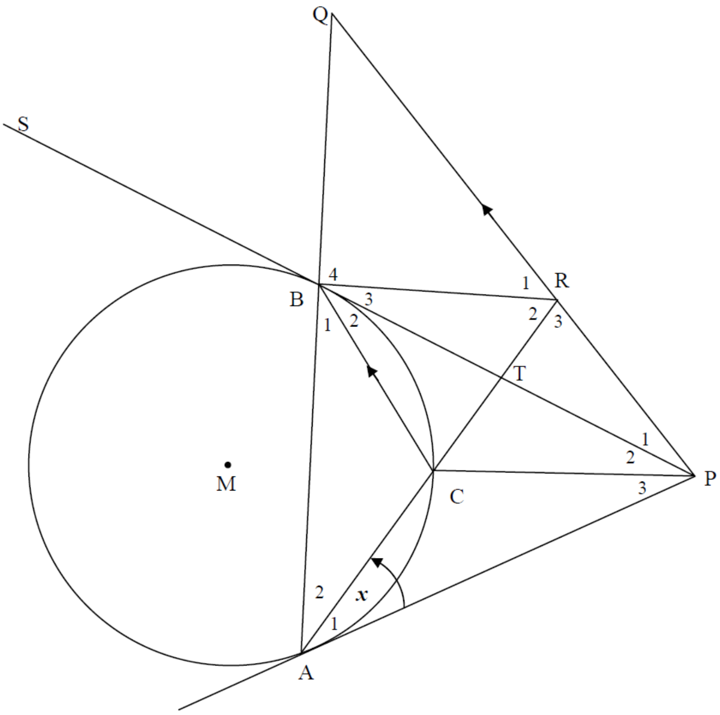
$D\hat{B}C = \hat{E}$



(6)

2.2

In the diagram below, PA and PBS are tangents to circle BCA with centre M.
AC is produced to R.
AB and PR are produced to Q.
BC = AC and PQ || CB.
PC is drawn.
Let $\hat{A}_1 = x$



Prove that:

2.2.1 $\hat{P}_2 = \hat{P}_3$ (6)

2.2.2 ABRP is a cyclic quadrilateral (4)

2.2.3 PRQ is a tangent to circle BCR at R (4)

[20]

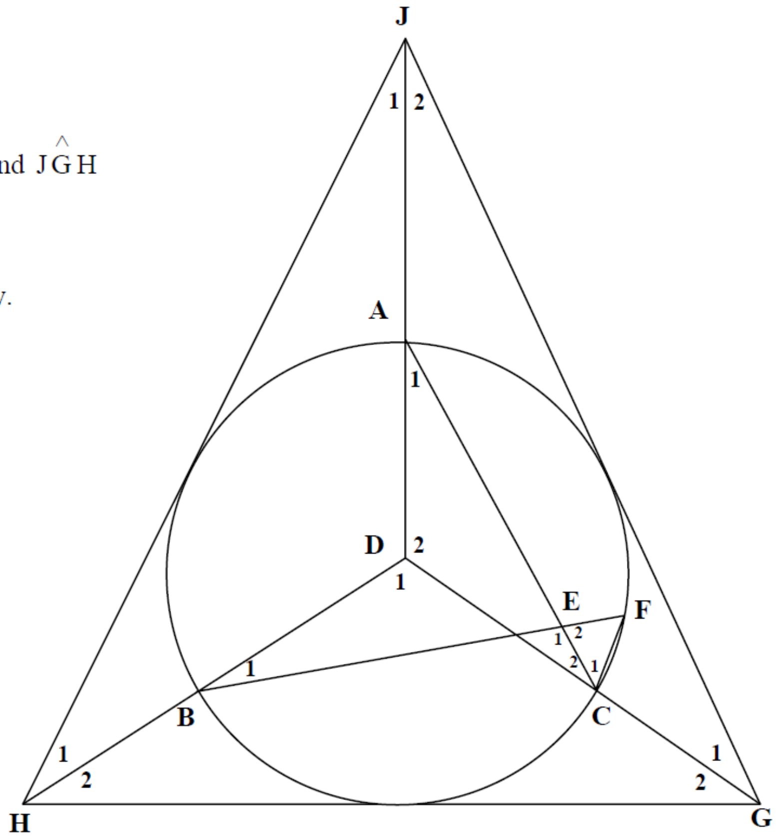
QUESTION 3

In the diagram alongside, circle BCFA with centre D is inscribed in Δ JHG.

The bisectors of angles $J\hat{H}G$ and $J\hat{G}H$ intersect at D.

JD, HD and GD intersect the circle at A, B and C respectively.

Chord AC intersects chord BF at E such that EF = EC



3.1 Prove that:

3.1.1 BCED is a cyclic quadrilateral (5)

3.1.2 $\hat{B}_1 = \hat{A}_1$ (3)

3.2 If $\hat{H}_1 = x$ and $\hat{G}_1 = y$, express $\hat{J}_1$ in terms of x and y. (4)

[12]

APPENDIX E

INFUSION LESSON PLAN

INFUSION LESSON PLAN

INFUSION LESSON
Introduction
Think Critically
Thinking about Thinking
Applying your Thinking

TITLE:

SUBJECT: GRADE :

OBJECTIVES

CONTENT

THINKING SKILL/PROCESS

METHODS AND MATERIALS

LESSON

INTRODUCTION TO CONTENT AND THINKING SKILL/PROCESS

INTRODUCTION
1. Importance of the thinking 2. How do you do the thinking 3. Importance of the content

THINKING CRITICALLY

SKILLFUL DECISION MAKING
1. Why necessary? 2. Options? 3. Consequences? 4. Importance? 5. Choice?

INFUSION LESSON PLAN

THINKING ABOUT THINKING

THINKING ABOUT THINKING

1. Kind of thinking?
2. How did you do it?
3. Is it effective?

APPLYING THINKING

APPLYING YOUR THINKING

1. Immediate transfer
   a. Near transfer
   b. Far transfer
2. Reinforcement Later

EXTENSION ACTIVITY (optional)

ASSESSING STUDENT THINKING

Adopted from *(National centre for thinking, 1996, p. 110).*

APPENDIX F

ETHICAL CLEARANCE CERTIFICATE

UNISA university of south africa

UNISA COLLEGE OF EDUCATION ETHICS REVIEW COMMITTEE

Date: 2021/08/11

Ref: **2021/08/11/57576009/30/AM**
Name: Mr F ABAKAH
Student No.:57576009

Dear Mr F ABAKAH

Decision: Ethics Approval from 2021/08/11 to 2026/08/11

Researcher(s): Name: Mr F ABAKAH
E-mail address: fitzgerald202020@gmail.com
Telephone: 0745521935

Supervisor(s): Name: PROF D BRIJLALL
E-mail address: deonarainb@dut.ac.za
Telephone: 0313732126

Title of research:

AN INVESTIGATION INTO FINDING AN EFFECTIVE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL APPROACH FOR CIRCLE GEOMETRY

Qualification: PhD Mathematics Education

Thank you for the application for research ethics clearance by the UNISA College of Education Ethics Review Committee for the above mentioned research. Ethics approval is granted for the period 2021/08/11 to 2026/08/11.

*The **medium risk** application was reviewed by the Ethics Review Committee on 2021/08/11 in compliance with the UNISA Policy on Research Ethics and the Standard Operating Procedure on Research Ethics Risk Assessment.*

The proposed research may now commence with the provisions that:

1. The researcher will ensure that the research project adheres to the relevant guidelines set out in the Unisa Covid-19 position statement on research ethics attached.
2. The researcher(s) will ensure that the research project adheres to the values and principles expressed in the UNISA Policy on Research Ethics.

University of South Africa
Preller Street, Muckleneuk Ridge, City of Tshwane
PO Box 392 UNISA 0003 South Africa
Telephone: +27 12 429 3111 Facsimile: +27 12 429 4150
www.unisa.ac.za

3. Any adverse circumstance arising in the undertaking of the research project that is relevant to the ethicality of the study should be communicated in writing to the UNISA College of Education Ethics Review Committee.
4. The researcher(s) will conduct the study according to the methods and procedures set out in the approved application.
5. Any changes that can affect the study-related risks for the research participants, particularly in terms of assurances made with regards to the protection of participants' privacy and the confidentiality of the data, should be reported to the Committee in writing.
6. The researcher will ensure that the research project adheres to any applicable national legislation, professional codes of conduct, institutional guidelines and scientific standards relevant to the specific field of study. Adherence to the following South African legislation is important, if applicable: Protection of Personal Information Act, no 4 of 2013; Children's act no 38 of 2005 and the National Health Act, no 61 of 2003.
7. Only de-identified research data may be used for secondary research purposes in future on condition that the research objectives are similar to those of the original research. Secondary use of identifiable human research data requires additional ethics clearance.
8. No field work activities may continue after the expiry date **2026/08/11**. Submission of a completed research ethics progress report will constitute an application for renewal of Ethics Research Committee approval.

Note:

The reference number **2021/08/11/57576009/30/AM** *should be clearly indicated on all forms of communication with the intended research participants, as well as with the Committee.*

Kind regards,

Prof AT Motlhabane
CHAIRPERSON: CEDU RERC
motlhat@unisa.ac.za

Prof PM Sebate
EXECUTIVE DEAN
Sebatpm@unisa.ac.za

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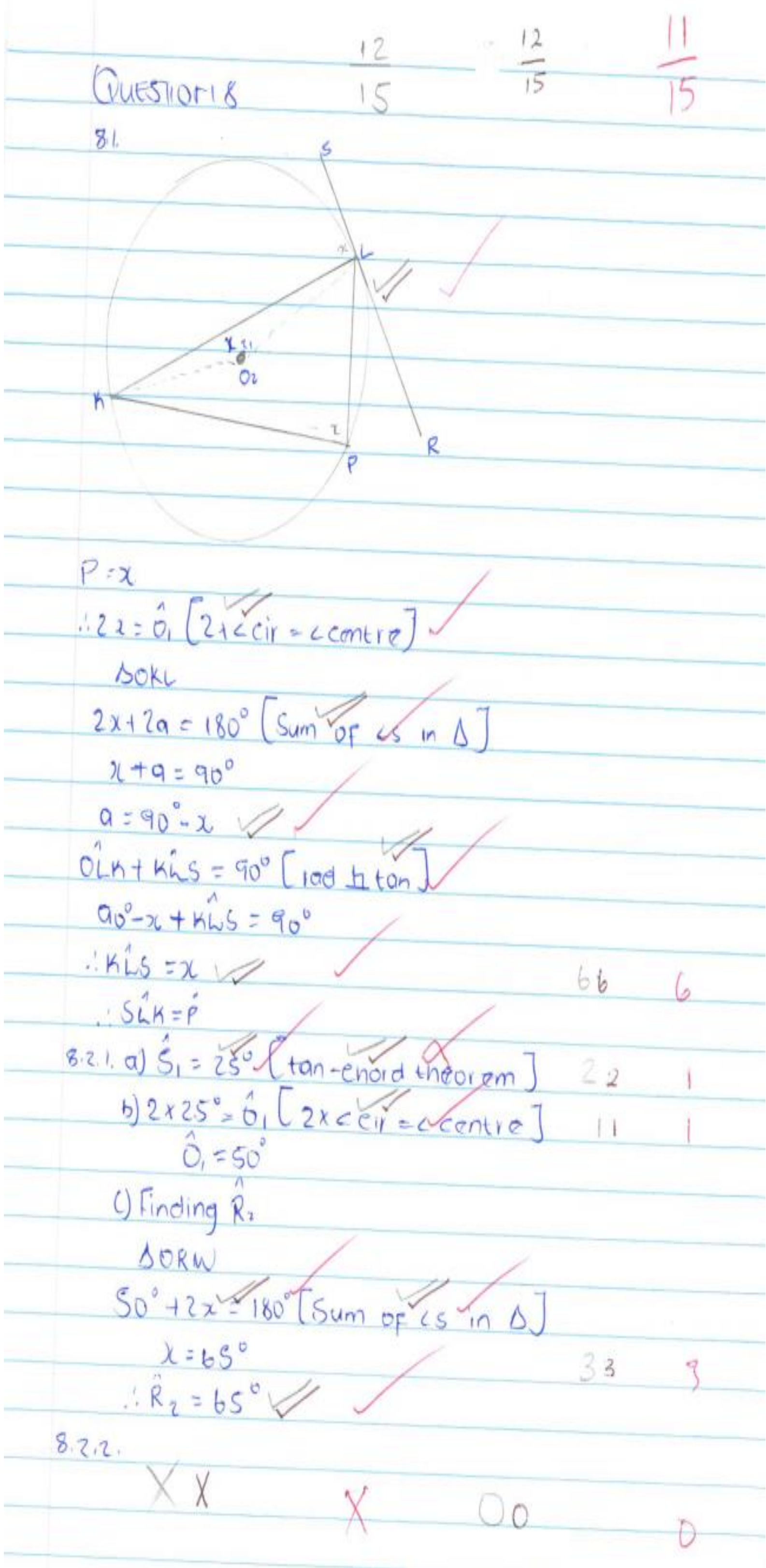
APPENDIX G

EXEMPLAR OF HOW THE ASSESSMENT METHOD WAS IMPLEMENTED



QUESTION 8 — 12/15 — 12/15 — 11/15

8.1.

P = x

∴ 2x = Ô₁ [2 × ∠ cir = ∠ centre]

ΔOKL

2x + 2a = 180° [Sum of ∠s in Δ]

x + a = 90°

a = 90° − x

OL̂K + KL̂S = 90° [rad ⊥ tan]

90° − x + KL̂S = 90°

∴ KL̂S = x — 6 6 — 6

∴ SL̂K = P̂

8.2.1. a) Ŝ₁ = 25° [tan-chord theorem] — 2 2 — 1

b) 2 × 25° = Ô₁ [2 × ∠ cir = ∠ centre] — 1 1 — 1

Ô₁ = 50°

c) Finding R̂₂

ΔORW

50° + 2x = 180° [Sum of ∠s in Δ]

x = 65°

∴ R̂₂ = 65° — 3 3 — 3

8.2.2.

X X — X — 0 0 — 0

Individual learner's marking- Pencil marks **Peer's Marking- Brown marks** **Teacher's marking- Red marks**

APPENDIX H

UNIVERSITY OF SOUTH AFRICA

DEPARTMENT OF MATHEMATICS EDUCATION

QUESTIONNAIRE TO BE COMPLETED BY TEACHER OBSERVER

TOPIC: AN INVESTIGATION INTO FINDING AN EFFECTIVE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL APPROACH FOR CIRCLE GEOMETRY

INTRODUCTION

Dear Respondent, this is an academic research on the above topic. The information received will be used for only academic purposes. This questionnaire is designed to enable the researcher to know how the new teaching approach was implemented and how well you understood the lesson. This questionnaire is to be answered by the mathematics teachers and HODs for mathematics at the research fields. The items of this research questionnaire are divided into four parts. Part A: how teaching of thinking skills was conducted, Part B: mode of presentation of lesson; Part C: how well you understood the lesson; and Part D: any other comments/remarks. We hope that you will provide the needed responses to the items of this research questionnaire, so that the desired purpose of this research study, can be accomplished. Your co-operation in this regard will be highly appreciated. Thank you.

PART A: HOW TEACHING OF THINKING SKILLS WAS CONDUCTED

QUESTION 1- INTRODUCTION OF LESSON

1.1 Which thinking skills were introduced to you by the teacher

L1 - Understanding and retention of ideas
L2 - Generating ideas (creating thinking)
L3 - Assessing reasonability of ideas (critical thinking)
L4 - Blending L1, L2, L3 and engaging studies in metacognitive reflections

1.2 What do you know about the thinking skills introduced to you by the teacher and their relevance?

L1 - Preliminary thinking skills to be learnt by students. It directs students to master and recall mathematical concepts taught.
L2 - Forming new mathematical knowledge. Studies are tasked to broaden their horizon, when solving problems
L3 - Judging level of corrections of ideas. Students are guided by this to be able to know if they are on the right path or on the wrong path of solving a problem
L4 - Application of ideas, concisively and logically to solve problems Students. This will guide students to be able to develop the competence of solving non-routine problems well.

1.3 What are the content objectives of the lesson conducted by the teacher

- Teaching learners to be good thinkers
- Guiding learners to be good problem-solvers
- Guiding learners to be responsible for their own learning

university of south africa

QUESTION 2- THINKING ACTIVELY

2.1 How did the teacher guide students on how to use thinking skills to solve problems

Learners were placed at the centre of learning. They were required to discuss and communicate among themselves they brainstorm, look for multiple ways of answering a question, consider other alternatives provided by other group members. The teacher only served as a prompter. He went to each to guide them to know if they are on the right path or wrong path on some occasions. On other occasions he allows each group to present their solution on the board. This enabled other members to constructively criticize the solution provided or to support it.

2.2 How did the teacher guide students on how they can reach solutions to thinking tasks, using thinking skills when they are in groups

Teacher encourages groups members to transform for them to respect alternatives viewpoints of other group members; find alternatives ways of solving the same problem from them to consider how a technique learnt can be used to solve similar unfamiliar problems

2.3 How did the teacher guide students on how they can reach solutions to thinking tasks, using thinking skills, individually?

Encouraging individual learners to be persistent and res until they are able to get a reasonable solutions to problems

QUESTION 3- THINKING ABOUT THINKING

3.1 How was the concept of "thinking about thinking" introduced to students by the teacher

Learners where necessary were made to understand that this concepts to nurture their ability to be able to reflect on their thought process.
The teacher used practical scenarios to explain this concept to them.

3.2 How did the teacher guide students to reflect on their own thinking

Learners were made to consider
- what kind of thinking they did?
- How they did it?
- How effective it was done?

Also the teacher provided with prompts or questions guidelines to enable them to assess the consequence of any decision they might make. That is what makes a decision necessary, what are the available options etc.

QUESTION 4- APPLYING THINKING

4.1 How was the concept of "applying thinking" introduced to students by the teacher

Teacher used practical examples to enable learners to understand how they can apply their knowledge and their thinking to solve higher-order questions.

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4.2 How did the teacher guide students on how they can apply their thinking to solve problems by applying their thinking skills to promote transfer

Teacher encouraged learners to think divergently and to be open minded. Teacher guided learner to solve high-order question by making them know how to approach them, how to break geometric diagrams into their separate parts and to re-assemble them again by applying their knowledge and skills in one context to solve problems for related or different context.

PART B: MODE OF PRESENTATION OF LESSON

QUESTION 5

5.1 Was the lesson presentation organized sequentially? Please indicate YES/NO.

YES

5.2 Motivate your answer in 5.1 above

Each lesson was orderly presented and structured. It was properly scaffolded into the introductory part, then the development of the lesson itself, then finally the conclusion part

5.3 Was the lesson meaningfully and logically presented? Please indicate YES/NO.

YES

5.4 Motivate your answer in 5.3 above The teacher ensured that each key construct of the IPHC model was well explained to learners. Also, how each construct was used during presentation of lesson was well explained. Furthermore, the teacher ensured that each construct was used at the right moment during each lesson

PART C: HOW WELL YOU UNDERSTOOD THE LESSON

QUESTION 6

6.1 How well did you understand the lesson? Specify if: not well understood, averagely understood and very well understood.

6.2 Motivate your answer in 6.1 above

Lesson was well sequenced. Also, it was orderly and logically presented

PART D: ANY OTHER COMMENTS/REMARKS

I learnt a lot during observations of these lessons. This novel teaching and learning approach promotes active participation of learners.

Its implementation in mathematics classroom in South Africa, can go a long way to help address the teaching and learning challenges of mathematics in school in South Africa

APPENDIX H

UNIVERSITY OF SOUTH AFRICA

DEPARTMENT OF MATHEMATICS EDUCATION

QUESTIONNAIRE TO BE COMPLETED BY HOD AS OBSERVER

TOPIC: AN INVESTIGATION INTO FINDING AN EFFECTIVE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL APPROACH FOR CIRCLE GEOMETRY

INTRODUCTION

Dear Respondent, this is an academic research on the above topic. The information received will be used for only academic purposes. This questionnaire is designed to enable the researcher to know how the new teaching approach was implemented and how well you understood the lesson. This questionnaire is to be answered by the mathematics teachers and HODs for mathematics at the research fields. The items of this research questionnaire are divided into four parts. Part A: how teaching of thinking skills was conducted, Part B: mode of presentation of lesson; Part C: how well you understood the lesson; and Part D: any other comments/remarks. We hope that you will provide the needed responses to the items of this research questionnaire, so that the desired purpose of this research study, can be accomplished. Your co-operation in this regard will be highly appreciated. Thank you.

PART A: HOW TEACHING OF THINKING SKILLS WAS CONDUCTED

QUESTION 1- INTRODUCTION OF LESSON

1.1 Which thinking skills were introduced to you by the teacher

L1 - Understanding and retention of ideas
L2 - Generating ideas (creative thinking)
L3 - Assessing reasonability of ideas (critical thinking)
L4 - Blending L1, L2, L3 and engaging students in metacognitive reflections

1.2 What do you know about the thinking skills introduced to you by the teacher and their relevance?

L1 - Foundation of thinking skills. It promotes conceptual understanding of the concept that was taught.
L2 - Brainstorming. It enables new mathematical ideas about the concept taught to be conjectured.
L3 - Ascertaining if ideas developed are meaningful. It enables the individual to self-determine if developed ideas are either correct, wrong or needs to be refined.
L4 - Transfer of ideas. It enables the indidual to think about his thinking.

1.3 What are the content objectives of the lesson conducted by the teacher

- Developing learners into creative and critical thinkers
- Developing learners into effective problem solvers
- Nurturing learners to be able to solve non-routine problems effectively
- Improving mathematical achievement of learners by introducing a different teaching and learning pedagogical approach to them.

QUESTION 2- THINKING ACTIVELY

2.1 How did the teacher guide students on how to use thinking skills to solve problems

For all lessons I observed, teaching and learning was conducted in a collaborative classroom setting - learners sitting, discussion and interacting in each other in groups of three. Thinking skills was purposefully implemented during the questioning stage of lesson, to guide learners to reach solutions to problems. After learners having understood the question, they are been nurtured to be able to brainstorm, conjecture, ideas, trial-and-error, consider more alternatives of solving the same question etc. They are urged to keep trying until a meaningful solution to a problem is reached.

2.2 How did the teacher guide students on how they can reach solutions to thinking tasks, using thinking skills when they are in groups

After each group are done with their solution to a given problem they were required to present their solution on the board. This will enable other group members to either support the presented solution by a group or to provide a counter solution to it. The teacher allows discussions among groups. He only intervened where necessary, either to correct a misconception or to provide prompts or clues when all learners appear to be out of ideas

2.3 How did the teacher guide students on how they can reach solutions to thinking tasks, using thinking skills, individually?

During classroom settings teaching and learning is conducted in collaboratively. However, homework and standardized tests were conducted individually. Teacher allows individual learners to brainstorm, conjecture ideas which will enable him/her to obtain

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a reasonable solution to given problem.

QUESTION 3- THINKING ABOUT THINKING

3.1 How was the concept of "thinking about thinking" introduced to students by the teacher

Teacher explained to learners that the concept "thinking about thinking" entails them developing the skills of reflecting on their thinking processes. It is also about them considering the consequence of any decision they take during the problem solving process

3.2 How did the teacher guide students to reflect on their own thinking

The teacher guided them to use the prompts below

1. What makes a decision necessary?
2. What are the available options?
3. What are the likely consequences of these options?
4. How important are these consequences?
5. What is the best option in light of the consequences?

QUESTION 4- APPLYING THINKING

4.1 How was the concept of "applying thinking" introduced to students by the teacher

This concept is about transfer of knowledge and thought processes to solve non-routine problems.

4.2 How did the teacher guide students on how they can apply their thinking to solve problems by applying their thinking skills to promote transfer

Teacher guides learners to apply their knowledge and thinking of familiar problems, familiar problems which have been put in other ways/context, or totally different concepts which will require learners to thoroughly brainstorm, for them to think inductively and deductively to obtain solutions to non-routine problems.

PART B: MODE OF PRESENTATION OF LESSON

QUESTION 5

5.1 Was the lesson presentation organized sequentially? Please indicate YES/NO.

Yes

5.2 Motivate your answer in 5.1 above Four distinct lesson were conducted. Each was conducted in view of each GD level: 1, 2, 3 and 4. Each lesson was organised sequentially in way that the introduction, body and conclusion of lessons follow each other systematically to aid understanding of what was presented.

5.3 Was the lesson meaningfully and logically presented? Please indicate YES/NO.

Yes

5.4 Motivate your answer in 5.3 above

This instructional approach under investigation is made up of three main key constructs; Apos theory, polya's problem solving model and infusion as an approach of teaching thinking. Hence, maximum care was taken to ensure that the presentation of each lesson was sequential, meaningful and logical. In view of of this, the role of each key construct in the instructional approach was clearly spelt out. Most importantly, when and how each key construct was used in the instructional approach was explicitly made know

PART C: HOW WELL YOU UNDERSTOOD THE LESSON

QUESTION 6

6.1 How well did you understand the lesson? Specify if: not well understood, averagely understood and very well understood. Very well understood

6.2 Motivate your answer in 6.1 above

Lesson was systematically, meaningfully and logically presented to promote understanding and eschew ambiguity of what was presented.

PART D: ANY OTHER COMMENTS/REMARKS

The instructional approach is promising, innovative, dynamic and interactive. If used effectively for teaching and learning of circle geometry, it can greatly contribute in improving learner achievements in mathematics.
Also it can serve as the medium through which learners can achieve mathematical proficiency.

APPENDIX I

UNIVERSITY OF SOUTH AFRICA

DEPARTMENT OF MATHEMATICS EDUCATION

QUESTIONNAIRE TO BE COMPLETED BY STUDY PARTICIPANTS

TOPIC: AN INVESTIGATION INTO FINDING AN EFFECTIVE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL APPROACH FOR CIRCLE GEOMETRY

INTRODUCTION

Dear Respondent, this is an academic research on the above topic. The information received will be used for only academic purposes. This questionnaire is designed to measure the effects/influence the proposed problem-solving instructional approach had on learners, with regards to the teaching and learning of circle geometry. This questionnaire is to be answered by the study participants. The items of this research questionnaire are divided into three parts. Part A: how the new instructional approach can influence the study participants' learning of circle geometry, Part B: how it can influence the study participants' problem-solving skills, when solving circle geometry problems and Part C: any other comments/remarks. We hope that you will provide the needed responses to the items of this research questionnaire, so that the desired purpose of this research study, can be accomplished. Your co-operation in this regard will be highly appreciated. Thank you.

PART A- How the new instructional approach can influence the study participants' learning of circle geometry

Question 1

How did the new instructional approach influence how you learn circle geometry? Please indicate YES or NO

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Question 2

Motivate your answer in question 1 above………………………………………………………………………………………………

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Question 3

From now on, will you use the new instructional approach to learn circle geometry? Please indicate either YES or NO

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Question 4

Motivate your answer in question 3 above

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Question 5

Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please indicate either YES or NO

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Question 6

Motivate your answer to question 5 above

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PART B: How the new instructional approach can influence the study participants' problem-solving skills, when solving circle geometry problems.

Question 1

How did the new instructional approach influence your problem-solving skills when solving circle geometry problems? Please specify, either positively or negatively...

Question 2

Motivate your answer in question 1 above……………………………………………………………………………………………

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Question 3

Which problem-solving skills did you learn when you were taught by the new instructional approach?

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Question 4

How did the problem-solving skills you stated in question 3 above, enable you to reach desired solutions, when solving problems in circle geometry?

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PART C: ANY OTHER COMMENTS/REMARKS

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APPENDIX J

FACE VALIDATION FORM FOR STANDARDIZED TESTS INSTRUMENT

The research instrument under consideration is designed to determine the influence/effect the proposed IPAC model had on study participants. The researcher in ensuring that appropriate items are served on standardized tests research instruments, means that this face validation process becomes necessary. Your assistance in this regard will be highly appreciated. Please rate each item on the EIT instrument on how well, the items are structured, using the scale below:

1= Not structured appropriately

2= fairly structured

3= adequately structured

Further comment(s) if any

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…………………………………………………………..

Personal information of Evaluator

Qualification: ……………………………… Status: …………………………………

Signature: …………………………………. Date: ……………………………….....

APPENDIX K

CONTENT VALIDATION FORM FOR STANDARDIZED TESTS INSTRUMENT

The research instrument under consideration is designed to determine the influence/effect the proposed IPAC model had on study participants. The researcher in ensuring that appropriate items are served on standardized tests research instruments, means that this face validation process becomes necessary. Your assistance in this regard will be highly appreciated. Please judge each item on standardized tests instruments on its level of relevance and level of appropriateness to Grade 11 learners as a test in circle geometry. In addition, please judge the instrument on how well it covers the Grade 11 circle geometry content. You are required to complete the table below:

Variable	1= Low	2= Fair	3= very high
Level of Relevance			
Level of Appropriateness			
Level of content covered			

Further comments(s) if any ……………………………………………………………

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………………………………………………………………………….

Personal information of Evaluator:

Qualification: ……………………….   Status: ………………………………

Signature: ……………………………   Date: ……………………………….

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APPENDIX L

VALIDITY FORM FOR QUESTIONNAIRES

The research instrument under consideration was designed to determine the influence/effect the proposed IPAC model had on study participants circle geometry learning and problem-solving. To answer the research questions, two research questionnaires-one for participants and the other for Teacher/HOD as observers, were developed. Your assistance is required to ensure that each item indicated on the questionnaires can serve its desired purpose. Please judge each item on the questionnaires on how well the items are structured by using the scale below:

1= Not structured appropriately
2= fairly structured
3= adequately structured

Further comment(s) if any

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Personal information of Evaluator

Qualification: ………………………………… Status: ………………………………………..

Signature: …………………………………. Date: ……………………………………..

APPENDIX M

PARENT'S CONSENT LETTER

Title of study: "**AN INVESTIGATION INTO FINDING AN EFFECTIVE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL APPROACH FOR CIRCLE GEOMETRY**".

Dear Parent/Guardian,

I am FITZGERALD ABAKAH. I am conducting a research study under the supervision of Professor Deonarain Brijlall, in the Department of Mathematics Education towards the degree of Doctor of Education (Mathematics Education), at the University of South Africa. My research study is entitled: "**AN INVESTIGATION INTO FINDING AN EFFECTIVE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL APPROACH FOR CIRCLE GEOMETRY**". The aim of the study is to find an efficacious instructional approach for teaching and learning of circle geometry.

This research study is mainly for academic purposes and all forms of information regarding your identification would be handled confidentially. We therefore request you to allow your child/dependant to participate in this research study, as it would, in the long run, contribute significantly in his / her understanding of mathematics. Please kindly note that, you are not forced in any way to permit your child/dependant to take part in this research study.

TO BE COMPLETTED BY PARENT/GUARDIAN

I ………………………………… willingly agree/disagree to permit my child/dependant to serve as a participant for this research study. I understand that this is an academic research and thus, I would not hold the researcher or authorities of the school responsible for any damages or unforeseen circumstances that may occur. As I have willingly accepted to allow my child/dependant to take part in this research, I pledge to encourage my child/dependent to be of good behaviour and cooperate fully to achieve the desired outcomes of this research study.

…………………………………………….   ……………………………….

(Signature of parent/guardian)   Date

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APPENDIX N

CONSENT AND ASSENT OF PARTICIPANT IN THE STUDY

I, __________________ (participant name), confirm that the person asking my consent to take part in this research has told me about the nature, procedure, potential benefits and anticipated inconvenience of participation. I have read (or had it explained to me) and I have understood every procedure and information of the study, as explained on the information sheet. I have had sufficient opportunity to ask questions and am prepared to participate in the study.

I understand that my participation is voluntary and that I am free to withdraw at any time without penalty. I am aware that the findings of this study will be processed into a research report, journal publications and/or conference proceedings, but that my participation will be kept confidential unless otherwise specified. I agree to the recording of the responses to the questionnaire and the formal test I will write. I have received a signed copy of the informed consent agreement.

TO BE COMPLETTED BY RESPONDENTS

I ……………………………………………..willingly agree/disagree to serve as a participant for this research study. I understand that this is an academic research and thus, I voluntarily participate in this research at my own risk, without any form of indemnity involved, hence I would not hold the researcher or authorities of the school responsible for any damages or unforeseen circumstances that may occur. As I have willingly accepted to take part in this research, I pledge to be of good behaviour and cooperate fully to achieve the desired outcomes of this research study.

Participant's Name & Surname (please print) ____________________________________

____________________________ ___________________________________

Participant's Signature Date

Researcher's Name & Surname (please print) ____________________________________

_____________________________ __________________________________

Researcher's signature Date

APPENDIX O

PERMISSION FROM 'THE ANSWER SERIES' PUBLISHER TO USE A PAGE IN THEIR MATHEMATICS TEXTBOOK IN THIS THESIS

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To fitzgerald202020@gmail.com

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Wishing you the very best with your studies!

Kind regards,

Alice

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APPENDIX P

PERMISSION TO CONDUCT RESEARCH IN NORTHERN CAPE DEPARTMENT OF EDUCATION SCHOOLS

IK Nkoane Education House

156 Barkly Road
Homestead
Kimberley 8301

Private Bag X5029
Kimberley 8300
Republic of South Africa

Tel: 053 839 6500
Fax: 053 839 6580/1
Web:ncdoe.ncpg.gov.za

DEPARTMENT OF EDUCATION

Enquiries: J.N. Horne
Contact No.: (053) 839 6757
Reference: L2.10.2.4.3
Date: 10 September 2021

Mr. Abakah
3 Kerk Street
Warrenton
8530

APPROVAL TO CONDUCT RESEARCH

The aforesaid matter with our Ref: L2.10.2.4.3 bears reference.

This letter serves to indicate that approval is granted to conduct research for the PhD dissertation titled: **"An investigation into finding an effective problem-solving Heuristic Instructional Approach for Circle Geometry"**

The onus rests with you as the researcher to organise appropriate and relevant time schedules with the schools and educators in order to conduct the research. Copies of this approval letter must be presented to the schools (Principal and SGB) and to the District Director as proof that permission for the research has been granted.

The following conditions must be strictly applied to conduct your research in the Northern Cape Department of Education. Approval may be withdrawn should any of the conditions listed below be flouted.

Criteria for approval	Comment
Value of the proposed research	The Research Topic covers a policy position of the Basic Education Sector and thus has value.
Policy and Strategic Alignment	The Research is aligned to Policy and to the Medium Term Strategic Framework as teacher development falls within the ambit of the Six Priorities of the Basic Education Sector, as well as Outcome Two of Priority Three of the MTSF.
Potential benefits to the NCDoE	The NCDoE would benefit from the research as empirical evidence would be available that is derived from the Northern Cape experiences. The findings of this research

2030

	experiences. The findings of this research study can positively influence how future Mathematics and/or Geometry curricula will be formulated to ensure the effective perpetuity of the teaching and learning of Mathematics, particularly, Circle Geometry.
Contribution to the knowledge base and literature in the Basic Education Sector	The Research Report findings will contribute to the knowledge base and literature of the Basic Education Sector after undergoing the necessary scrutiny for rigour and other tests. The study outcomes may serve as a reference tool for *inter alia* Mathematics students and teachers in relation to how problem solving in Circle Geometry can be improved.
Appropriateness of the methodology adopted	The methodology identified is appropriate for the study as the views of learners and teachers will be examined through the observation of Mathematics lessons, audio and video recordings as well as semi-structured interviews.
Ethical Considerations	Ethical clearance was granted by the Research Ethics Committee of the University of South Africa.
Accountability	The accountability is as per NCDoE Research Guidelines.
Conditions to be agreed by the applicant	The conditions to be adhered to are as per NCDoE Research Guidelines.
Types of Research	Qualitative Research will be used which will include observation of Mathematics lessons, questionnaires and semi-structured interviews to collect data.
Datasets	No datasets were requested by the researcher.
Covid -19 Considerations	COVID-19 health protocols *viz* wearing a mask, regular sanitizing and social distancing will be adhered to.

The Northern Cape Education Department wishes you well in this important undertaking and is looking forward to examine the findings of your Research Study.

Kind regards

MS. M. MARAIS
HEAD OF DEPARTMENT

10.09.2021
DATE

APPENDIX Q

NEGATIVE BINOMIAL MIXED EFFECT DISTRIBUTION AND REGRESSION MODEL

The Negative Binomial Distribution

The Poisson distribution may be generalized by including a gamma noise variable which has a mean of 1 and a scale parameter of ν. The Poisson-gamma mixture (negative binomial) distribution that results is

$$\Pr(Y = y_i|\mu_i, \alpha) = \frac{\Gamma(y_i + \alpha^{-1})}{\Gamma(y_i + 1)\Gamma(\alpha^{-1})}\left(\frac{\alpha^{-1}}{\alpha^{-1} + \mu_i}\right)^{\alpha^{-1}}\left(\frac{\mu_i}{\alpha^{-1} + \mu_i}\right)^{y_i}$$

where

$$\mu_i = t_i\mu$$

$$\alpha = \frac{1}{\nu}$$

The parameter μ is the mean incidence rate of y per unit of exposure. Exposure may be time, space, distance, area, volume, or population size. Because exposure is often a period of time, we use the symbol t_i to represent the exposure for a particular observation. When no exposure given, it is assumed to be one.

The parameter μ may be interpreted as the risk of a new occurrence of the event during a specified exposure period, t.

The results below make use of the following relationship derived from the definition of the gamma function

$$\ln\left(\frac{\Gamma(y_i + \alpha^{-1})}{\Gamma(\alpha^{-1})}\right) = \sum_{j=0}^{y_i-1}\ln(j + \alpha^{-1})$$

The Negative Binomial Regression Model

In negative binomial regression, the mean of y is determined by the exposure time t and a set of k regressor variables (the x's). The expression relating these quantities is

$$\mu_i = exp(\ln(t_i) + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_k x_{ki})$$

Often, $x_1 \equiv 1$, in which case β_1 is called the *intercept*. The regression coefficients $\beta_1, \beta_2, \ldots, \beta_k$ are unknown parameters that are estimated from a set of data. Their estimates are symbolized as $b_1, b_2, \ldots, b_k$.
Using this notation, the fundamental negative binomial regression model for an observation i is written as

$$\Pr(Y = y_i|\mu_i, \alpha) = \frac{\Gamma(y_i + \alpha^{-1})}{\Gamma(\alpha^{-1})\Gamma(y_i + 1)}\left(\frac{1}{1 + \alpha\mu_i}\right)^{\alpha^{-1}}\left(\frac{\alpha\mu_i}{1 + \alpha\mu_i}\right)^{y_i}$$

Adopted from (NCSS, 2022, p. 1-2).

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APPENDIX R

TURNITIN REPORT

PhD Thesis

ORIGINALITY REPORT

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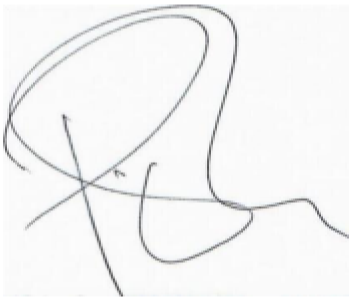
APPENDIX S

EDITING CERTIFICATE

PROOF OF EDITING

30 December, 2022

This is to certify that I, Dr P Kaburise, have proofread the thesis titled - **FINDING AN EFFECTIVE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL APPROACH FOR CIRCLE GEOMETRY** - by Fitzgerald Abakah (student number: 57576009). I have indicated some amendments which the student has undertaken to effect before the final thesis is submitted.



Dr P Kaburise (0794927451, 0637348805) email: phyllis.kaburise@gmail.com)

Dr P Kaburise: BA (Hons) University of Ghana (Legon, Ghana); MEd University of East Anglia (Cambridge/East Anglia, United Kingdom); Cert. Teaching English as a Foreign Language (Cambridge University, United Kingdom); Cert. English Second Language Teaching, (Wellington, New Zealand); PhD University of Pretoria (South Africa)