ON FRACTIONAL VOLATILITY MODELLING

by

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submitted in accordance with the requirements for
the degree of

Doctor of Philosophy

in the subject

Operations Research

at the

University of South Africa

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August 2022

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On Fractional Volatility Modelling

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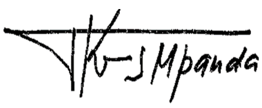

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Date

Abstract

In this thesis, we investigate the roughness feature within realised volatility for different financial markets by using the multifractal detrended fluctuation approach and microstructure noise index technique, and we confirm that the Hurst parameter $H \neq 1/2$. To include this feature in stochastic volatility modelling, we construct an arbitrage-free financial market model that consists of two assets, the risk-free and the risky assets. The price of a risk-free asset is described by an exponential function while the one for a risky asset is driven by a geometric Brownian motion with its stochastic volatility described as a function of fractional Cox-Ingersoll-Ross process defined by $Y_t = Z_t^2$, where the process $(Z_t)_{t\geq 0}$ satisfies a singular stochastic differential driven by fractional Brownian motion $(W_t^H)_{t\geq 0,\, H\in(0,1)}$. The stochastic process $(Z_t)_{t\geq 0}$ verifies $dZ_t = \left(f(t, Z_t)Z_t^{-1}dt + \sigma dW_t^H\right)/2$, with $f(t,z)$ being a continuous function on $\mathbb{R}_+^2$ that represents the drift of the stochastic process $(Y_t)_{t\geq 0}$. We show that the fractional volatility process is strictly positive for all $H \in (\frac{1}{2},1)$ and in the case where $H < 1/2$, we consider a sequence of increasing drift functions (f_n) and we prove that the probability of hitting zero tends to 0 as $n \to \infty$. We also show that both fractional volatility and stock price processes are Malliavin differentiable for all $H \in (0,1)$ and deduce an expression of the expected payoff function having different forms. Some simulations of option prices were performed.

Keywords: Fractional Brownian Motion, Hurst parameter, Malliavin calculus, Financial Market Model, Stock Price process, Fractional Volatility Process, Fractional Cox-Ingersoll-Ross process, Heston model, Option Pricing, Payoff function.

Research papers

Forthcoming research paper

Mpanda, M. M., Mukeru, S., & Mulaudzi, M. (2022). Generalisation of Fractional Cox-Ingersoll-Ross Process. Results in Applied Mathematics, forthcoming (arXiv preprint arXiv:2008.07798).

Manuscripts under review

Mpanda, M., Mukeru, S. and Mulaudzi, M. (2022). *Malliavin differentiability of fractional Heston-type model and applications to option pricing*, Submitted for publication. Decision Sciences, University of South Africa (arXiv preprint arXiv:2207.10709).

Mpanda, M., Mukeru, S. and Mulaudzi, M. (2022). *Analysing South African Financial Stock Markets Volatility*, Submitted for publication. Decision Sciences, University of South Africa.

Conference presentation

Mpanda, M.M., Mukeru, S. & Mulaudzi, M.P. (2022, June 13−17). *Generalised Fractional Cox-Ingersoll-Ross process.* Paper presented at the 11th World Congress of the Bachelier Finance Society, Hong Kong, China, page 252. (Book of abstracts)

Dedication

To my children Kevin, Cornelia and Camelia Mpanda.
To my wife Sandrine Mpanda
and my parents Cornelie Kapya Mwadi and Theodore Kalombo.

Acknowledgements

I wish to thank God Almighty for everything He has done for me throughout my studies; I am forever grateful for the strength and courage He gave me whenever I was weak and discouraged. Completing this thesis would probably not have become a reality if it was not for His grace, intelligence and wisdom.

I would also like to thank my supervisors, Professor Safari Mukeru and Professor Mmboniseni Mulaudzi for their excellent supervision and continuous encouragement, both scientific and human; Thank you to all my Colleagues in the Department of Decision Sciences and particularly to the Head of the Department Dr Masetshaba and the former Head of the Department Professor Basson for trusting and believing in me.

Special thanks to Professor Elisa Alos Alcalde and Professor Luhandjula for their valuable comments and suggestions. Thank you to sister Virginie Matondo and to my friends Belette Misakabu, Jeannot and Serge Mpiana with their respective wives who have also played an important role in the completion of this thesis.

Furthermore, I am grateful for the strong financial support I received through the MDSP grant during my studies as this was needed in my professional training to strengthen my knowledge in the field of quantitative finance through conferences and workshops.

Lastly, I wish to thank my wife, Sandrine Mpanda for her continuous and unconditional love and support in all of its kind, my kids Kevin Mpanda and Cornelia Mpanda, as well as my parents, Theodore Kalombo and Cornelie Kapya for everything they have done for me. I dedicate this Thesis to them.

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Chapter 1

Introduction

The field of quantitative finance has a long and interesting history. It all started with the thesis of Bachelier (1900) who assumed that the changes in stock prices could be considered as independent and identically distributed normal random variables. This implies that the price process has a *"lack of memory"* or *"Markov property"*. The model proposed by Bachelier was in fact a standard Brownian motion.

Bachelier's work can therefore be regarded as the first route to modern option pricing theory and the first application of Brownian motion in finance. Broadly speaking, Brownian motion describes the chaotic movements of microscopic particles in a fluid resulting from their collision with atoms and molecules in the fluid. This phenomenon was observed for the first time by the biologist Robert Brown in 1827 and was extensively studied by Einstein in 1905 and Wiener (1923).

As discussed by Jarrow and Protter (2004), the modern option pricing theory was sharped by the development of the theory of stochastic processes. This was influenced by the work of Itô who, in particular, analysed the infinitesimal behavior of a Markovian particle. This analysis gave birth to the theory of stochastic differential equations (SDE). Specifically, an SDE is an equation of the form

$$dX_t = \eta(t,X_t)\,dt + \sigma(t,X_t)\,dW_t,$$

where $(W_t)_{t\geq 0}$ represents the standard Brownian motion (or Wiener process), and the parameters $\eta(t,X_t)$ and $\sigma(t,X_t)$ are adapted processes that represent the drift and volatility of $(X_t)_{t\geq 0}$ respectively. In particular, Samuelson (1964) observed that standard Brownian motion can take negative values, and consequently it is not suitable to model the dynamics of stock prices. He proposed replacing the standard Brownian motion by a non-negative variation of Brownian motion called *"geometric Brownian motion"*, which is a stochastic process that satisfies the following differential equation:

$$dX_t = \eta\, X_t\, dt + \sigma\, X_t\, dW_t, \tag{1.1}$$

where the parameters η and σ are positive constants that are considered as the drift and volatility of the infinitesimal return process $R_t := \log X_t$. By combining some previous studies in quantitative finance and with the help of Itô's calculus, Black and Scholes (1973) considered a financial market model that consists of two assets: a risk-free asset whose prices satisfy the differential equation $dA_t = rA_t dt$, where r is a constant interest rate, and risky asset defined by the geometric Brownian motion (1.1). They proposed a model commonly known as the *"Black-Scholes model"* for pricing European call and/or put options written on a stock.

Since then, the *"Black-Scholes model"* has become the cornerstone model for both practitioners and researchers. The model was set up within the arbitrage-free framework, it was rapidly adapted by almost all financial markets and has been used for pricing and hedging both vanilla and exotic options. See e.g. Fouque et al. (2011) for a summary.

The Black-Scholes model comes with strong assumptions, one of them is by restricting the volatility to be constant. Recall that the volatility is an important indicator in financial market sectors. It is one of the measures used by investors to gauge the risk related to fluctuation of a security or market index within a chosen period.

This was the main motivation of Hull and White (1987) and Heston (1993) to replace the constant volatility σ by a stochastic process driven by a standard Brownian motion. For example, Heston (1993) considered a financial market model defined by

$$\begin{cases} dA_t = rA_t dt, \\ dX_t = \eta X_t dt + \sqrt{Y_t} X_t dB_t, \\ dY_t = \theta(\mu - Y_t)dt + \nu\sqrt{Y_t} d\tilde{B}(t) \end{cases} \tag{1.2}$$

where $(Y_t)_{t\geq 0}$ is a stochastic process that represents the instantaneous variance of the infinitesimal return dX_t/X_t. The process $(\tilde{B}_t)_{t\geq 0}$ is a standard Brownian motion that represents the randomness of $(Y_t)_{t\geq 0}$ and the parameter $\theta > 0$ represents the speed of reversion of the stochastic variance process $(Y_t)_{t\geq 0}$ towards its long-run mean $\mu > 0$ and the parameter $\nu > 0$ is the volatility of $(Y_t)_{t\geq 0}$.

The stochastic volatility process $(Y_t)_{t\geq 0}$ is commonly known as the *"Cox-Ingersoll-Ross (CIR) process"* and was initially introduced by Cox et al. (1985) to model the dynamics of interest rates. This process is popular due to several interesting properties which include positiveness provided that the (Feller) condition $2\theta\mu > \nu^2$ holds, mean reversion in the sense that the process is pulled towards its long-run mean μ when it goes higher or lower than μ. Moreover, the *CIR* process admits a stationary distribution and it is ergodic. For more details, see e.g. Göing-Jaeschke et al. (2003), Chou and Lin (2006) and Guo (2008) with references therein.

Since the standard *CIR* process is driven by a Brownian motion $(\tilde{B}_t)_{t\geq 0}$, then it does not display memory. However, it was shown that historical volatility time-series have a dependency structure. For example, volatility may display long-range dependency (See e.g. Comte and Renault (1998) and Chronopoulou and Viens (2010)), or short-range dependency known as *"rough volatility"* as recently demonstrated by Gatheral et al. (2018) and Livieri et al. (2018) with references therein. The dependency can be measured by using the so-called Hurst parameter $H \in (0,1)$ initially introduced by Hurst (1951). This was a main motivation of replacing the standard Brownian motion in (1.2) by a fractional Brownian motion denoted by $(W_t^H)_{t\geq 0}$ as a source of randomness.

Roughly speaking, the fractional Brownian motion (*fBm* for short) is a stochastic process initially introduced by Kolmogorov (1940) and later by Mandelbrot and Van Ness (1968) as a centered Gaussian process characterized by its covariance defined by

$$\mathbb{E}\big[W_t^H W_s^H\big] = \frac{1}{2}\Big(t^{2H} + s^{2H} - \mid t - s \mid^{2H}\Big), \quad \forall s,t \geq 0. \tag{1.3}$$

The *fBm* is a generalisation of standard Brownian motion which coincides with the last when the Hurst parameter $H = 1/2$. This process displays a certain range of dependency and presents several important properties which include self-similarity, stationarity of increments and time inversion.

Replacing the standard Brownian motion by a *fBm* on the stochastic volatility process in the market model (1.2) yields the so-called *fractional Cox-Ingersoll-Ross (fCIR) process*. A classical definition of *fCIR* process was previously introduced by Mishura and Yurchenko-Tytarenko (2018) as a square of the stochastic process driven by an additive *fBm*. In other words, under a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, set a stochastic process as

$$dZ_t = \frac{1}{2}\Big(\big(\mu - \theta Z_t^2\big) Z_t^{-1} dt + \sigma dW_t^H\Big). \tag{1.4}$$

Then the *fCIR* process $(Y_t)_{t\geq 0}$ can be defined by

$$Y_t(\omega) = Z_t^2(\omega)\mathbf{1}_{[0,\tau(\omega))}, \qquad \forall t \geq 0, \quad \omega \in \Omega, \tag{1.5}$$

where τ is the first time the stochastic process $(Z_t)_{t\geq 0}$ hits zero. The corresponding financial market model initially defined by (1.2) now takes the following form:

$$
\begin{cases}
dA_t = rA_t dt, \\
dX_t = \eta X_t dt + \sigma Y_t X_t dB_t, \\
Y_t(\omega) = Z_t^2(\omega)\mathbf{1}_{[0,\tau(\omega))} \\
dZ_t = \frac{1}{2}\Big(\big(\mu - \theta Z_t^2\big) Z_t^{-1} dt + \nu dW_t^H\Big)
\end{cases}
\tag{1.6}
$$

The parameters of this financial market model can be obtained via calibration. The model (1.6) with $H > 1/2$ was previously investigated by Alòs and Yang (2017), Bezborodov et al. (2019) with $\mu = 0$ and Mishura and Yurchenko-Tytarenko (2020), and was proved to be free of arbitrage.

The parameters θ and μ being constant, perfect calibration may not be possible. For example, Benhamou et al. (2010) show that the calibration error is reduced sensibly when using time-dependent parameters in the standard Heston model. This was our main motivation for replacing the drift of volatility by a continuous function.

In this thesis, we extend the idea of Mishura and Yurchenko-Tytarenko (2018) and construct the fractional volatility process as a generalisation of *fCIR* process defined by $Y_t(\omega) = Z_t^2(\omega)\mathbf{1}_{[0,\tau(\omega))}$ as previously but with the process $(Z_t)_{t\geq 0}$ given by

$$
dZ_t = \frac{1}{2} f(t,Z_t) Z_t^{-1} dt + \frac{1}{2}\sigma dW_t^H, \quad Z_0 > 0,
\tag{1.7}
$$

where $f : [0,\infty) \times [0,\infty) \to (-\infty,\infty)$, $(t,x) \mapsto f(t,x)$, is a continuous function and the stochastic process $(W_t^H)_{t\geq 0}$ is a *fBm* with Hurst parameter $H \in (0,1)$ that represents the randomness of $(Z_t)_{t\geq 0}$. We shall refer to the corresponding financial market model as the *"generalised fractional Heston-type (fHt) model"*. This model is free of arbitrage and incomplete since it has more than one source of randomness given by $(B_t)_{t\geq 0}$ and $(W_t^H)_{t\geq 0}$.

As a special case, the time-dependent fractional Heston model can be constructed with the stochastic volatility process $(Y_t)_{t\geq 0}$ defined by (1.5)-(1.7) where the drift function is $f(t,z) = \theta_t(\mu_t - z^2)$. This model can be considered as a generalisation of the time-dependent Heston model previously discussed by Benhamou et al. (2010) with its volatility of infinitesimal return process driven by a standard Brownian motion.

To ensure the existence and uniqueness of the stochastic differential equations driven by an additive *fBm* of the form (1.7), Hu et al. (2008) proved that the drift function $g(t,z) := f(t,z)/z$ must satisfy the following conditions for $H > 1/2$:

(c_1) $g : [0,\infty) \times (0,\infty) \to [0,\infty)$ is a nonnegative continuous function which has a continuous partial derivative $\partial g(t,z)/\partial z \leq 0$, $\forall(t,z) \in (0,\infty) \times (0,\infty)$.

(c_2) There exist $z_1 > 0$, $a > \frac{1}{H} - 1$ and a continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for all $t > 0$ such that $g(t,z) \geq \varphi(t)z^{-a}$, $\forall t \geq 0$ and $0 < z < z_1$.

Under the above assumptions, the stochastic differential Equation (1.7) has a strictly positive solution $(Z_t)_{t\geq 0}$ that is, almost surely $Z_t > 0$ for all $t > 0$. (See Theorem 2.1 and Theorem 3.1 in Hu et al. (2008)). In addition, they also showed that under the following condition:

(c_3) there exists a function $h : [0,\infty) \to [0,\infty)$ which is nonnegative and locally bounded such that $g(t,z) \leq h(t)(1 + 1/z)$ for all $t \geq 0$ and $z > 0$,

then the solution $(Z_t)_{t\geq 0}$ is such that for any fixed $T > 0$,

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |Z_t|^p\right) < \infty, \quad \forall p > 0.$$

The first objective of this study is to investigate the existence and positiveness of the stochastic process $(Z)_{t\geq 0}$ satisfying (1.7) for all Hurst parameters

$H \in (0,1)$ under conditions weaker than (c_1) and (c_2). We shall consider the following conditions:

(d_1) The function $g : [0,\infty) \times (0,\infty) \to (-\infty,\infty)$ defined by $g(t,z) = f(t,z)/(2z)$ is continuous and admits a continuous partial derivative with respect to z on $(0,\infty)$. In addition, there exists a number $z^* > 0$ such that for every $z > z^*$, $g(t,z) < 0$, for all $t \geq 0$.

(d_2) for any $T > 0$, there exists $z_T > 0$ such that $f(t,z) > 0$, for all $0 < t \leq T$ and $0 \leq z \leq z_T$.

Condition (d_1) is given to ensure that the first time the solution reaches zero is strictly positive. Intuitively, when the stochastic process $(Z_t)_{t\geq 0}$ increases beyond the threshold z^*, the drift function becomes negative and as a consequence $(Z_t)_{t\geq 0}$ tends to revert back to previous values. This is an interesting property in finance. Condition (d_2) implies that for all $S > 0$ and $T > 0$, there exists $z_T > 0$ such that $\inf\{f(t,z) : S \leq t \leq T,\ 0 \leq z \leq z_T\} > 0$.

Firstly, we will show that under conditions (d_1) and (d_2) given above, the stochastic differential Equation (1.7) has a unique solution (Z_t) which is continuous and positive up to the time of first visit to zero. We will also show that the square stochastic process $(Y_t)_{t\geq 0}$ (which is also defined up to the first time (Z_t) it hits zero) satisfies the stochastic differential equation

$$dY_t = f(t,\sqrt{Y_t})dt + \sigma\sqrt{Y_t} \circ dW_t^H, \quad Y_0 > 0, \ H \in (0,1).$$

We shall also prove that in the case where $H > 1/2$, the solution to Equation (1.7) is not only positive up to the time of the first visit to zero but it is strictly positive everywhere. In other words, it never hits zero on the whole line $[0,\infty)$ almost surely. It is remarkable that this result is true under mild conditions (d_1) and (d_2).

In the case where $H < 1/2$, we have obtained that the probability of the process $(Y_t)_{t\geq 0}$ hitting zero is small if the drift function f is sufficiently large.

More precisely, if $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of continuous drift functions f_n defined on $[0,\infty)\times[0,\infty)$ taking values in $\mathbb{R}$ and satisfying conditions (d_1) and (d_2) such that $\lim_{n\to\infty} f_n = \infty$ and (Y_t^n) is the solution to Equation (1.7) corresponding to f_n (up to the first time it hits zero), then the probability of (Y_t^n) hitting zero converges to 0 as $n \to \infty$. We provide some illustrating examples using simulations.

Note that Kubilius (2020) recently studied the stochastic differential equation of the form

$$dX_t = g(X_t)dt + \sigma X_t^{\beta} dW_t^H \tag{1.8}$$

for $1/2 < H < 1$, $1/2 \leq \beta < 1$ and where the function g is such that there exists a continuously differentiable function f defined on $(0,\infty)$ such that: (1) $g(x) = x^{\beta} f(x^{1-\beta})$, (2) there exist $a > 0$ and $\alpha \geq 0$ such that $f(x) > ax^{-(1+\alpha)}$ for sufficiently small x and (3) there exists $K \in \mathbb{R}$ such that $f'(x) \leq K$. Under these conditions, it is proven that Equation (1.8) has a unique and positive solution and derived an important estimator of the H for the solution. In some sense our model (1.7) extends (1.8). It would be interesting to carry out an analysis of the H parameter of the solution to Equation (1.7) as in Kubilius (2020).

Our second objective is to show that both stock price process $(X_t)_{t\geq 0}$ defined in (1.6) and fractional volatility process $(Y_t)_{t\geq 0}$ given by (1.5) and (1.7) are Malliavin differentiable for all $H \in$ (0,1). To achieve this, we construct an approximating sequence $(Z_t^{\epsilon})_{t\geq 0,\epsilon>0}$ of $(Z_t)_{t\geq 0}$ in the light of Alòs and Ewald (2008) and its corresponding sequence $(X_t^{\epsilon})_{t\geq 0,\,\epsilon>0}$ of price process $(X_t)_{t\geq 0}$. We prove that Z_t^{ϵ} and X_t^{ϵ} convergence to Z_t and X_t respectively in $L^p(\Omega)$ for all $p \geq 1$ and $H \in$ (0,1). This allowed us to find the expression of the Malliavin derivatives. As a straight consequence, the expected payoff function can be derived following Altmayer and Neuenkirch (2015). This result will open doors to several other applications of Malliavin calculus in finance previously discussed under standard stochastic volatility models framework (Alòs and Lorite; 2021).

This thesis is an open-door for researchers and practitioners in financial modelling who want to use this fractional volatility model for different purposes, not only because of its roughness at all levels, but also its positiveness, its differentiability and its ability to fit different financial market conditions.

The results in this thesis are summarized in three manuscript papers that are under review namely: (1) Generalisation of Fractional Cox-Ingersoll-Ross Process (also available on arXiv:2008.07798), (2) Malliavin differentiability of fractional Heston-type model and applications to option pricing (arXiv:2207.10709), and (3) Analysing South African Financial Stock Markets Volatility.

This thesis is structured as follows. Chapter 2 introduces some important properties of fractional Brownian motion (*fBm*). Chapter 3 is devoted to the stochastic analysis for *fBm* by using tools in Malliavin calculus. Chapter 4 introduces the Black-Scholes model and one of its major extension commonly known as the standard Heston model. Chapter 5 discusses the roughness and multifractality property of volatility time series. Chapter 6 introduces the fractional Heston model and proposes its general form. The positiveness and Malliavin differentiability of both stock price and generalised fractional Cox-Ingersoll-Ross processes are discussed in Chapter 7. As an application to option pricing, Chapter 8 discusses the expected payoff function for a special exotic option constructed as a combination of vanilla and standard exotic options. Some simulations of option prices are also provided.

Chapter 2

Fractional Brownian Motion

It has been shown that most of financial time series, particularly instantaneous volatility, have random behavior and carry a dependency structure. See e.g., Comte and Renault (1998) or Gatheral et al. (2018) with references therein. The fractional Brownian motion, shortly written as *fBm*, is a potential candidate that can be used to model such situations.

Broadly speaking, *fBm* is a stochastic process that belongs to the class of centered Gaussian processes. This process is considered to be a generalisation of standard Brownian motion and was initially introduced by Kolmogorov (1940) as a realisation of a Wiener spiral in the Hilbert space and further developed by Mandelbrot and Van Ness (1968). The sample paths of *fBm* depend on a parameter $H \in (0,1)$ known as *"Hurst parameter"* or *"Hurst index"* that was initially discussed by the hydrologist Hurst (1951).

2.1 Definitions and existence of *fBm*

2.1.1 Some useful definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where Ω is a sample space, $\mathcal{F}$ is a sigma-algebra of subsets of Ω and $\mathbb{P}$ a probability measure on a measurable space $(\Omega, \mathcal{F})$. We recall the following definitions:

Definition 2.1. A random variable is a measurable function $X : \Omega \to \mathbb{R}$ with respect to the σ-algebra $\mathcal{F}$ and the Borel σ-algebra on $\mathbb{R}$. A random variable X is said to be a Gaussian random variable if its characteristic function given by $\mathbb{E}\left[e^{itX}\right]$ takes the following form:

$$\mathbb{E}\left[e^{itX}\right] = \exp\left(it\mu - \frac{t^2\sigma^2}{2}\right), \quad \forall t \in \mathbb{R}.$$

where μ and σ are parameters that represent respectively the mean and standard deviation of the Gaussian distribution. We write $X \sim N(\mu, \sigma^2)$.

Definition 2.2. A stochastic process denoted by $(X_t)_{t\geq 0}$ is a family of random variables defined on the probability space $(\Omega,\mathcal{F}, \mathbb{P})$.

Definition 2.3. A filtration is an increasing family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ that represents information available at the time $t \geq 0$. This means that for every $s, t \geq 0$ with $s < t$, $\quad \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$. A quadruple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a filtered probability space.

Definition 2.4. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A stochastic process $(X_t)_{t\geq 0}$ is said to be adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t -$ measurable.

Definition 2.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A standard Brownian motion, denoted by $(B_t)_{t\geq 0}$, is a stochastic process satisfying the following conditions:

- (c_1) $\forall \omega \in \Omega, \quad B_0(\omega) = 0 \quad \mathbb{P} - a.s.$ and $\mathbb{E}[B_t] = 0, \quad \forall t \geq 0.$
- (c_2) The sample path $t \mapsto B_t$ is continuous a.s.
- (c_3) The process $(B_t)_{t\geq 0}$ has independent increments.
- (c_4) For any $0 \leq s \leq t$, the random variable $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.

Definition 2.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $H \in (0,1)$ a real constant parameter. A Gaussian process $(W_t^H)_{t\geq 0}$ is called fractional Brownian motion (*fBm*) defined on $(\Omega, \mathcal{F}, \mathbb{P})$ if the following conditions are satisfied.

- (c_1) $\forall \omega \in \Omega$, $W_0^H(\omega) = 0 \;\; \mathbb{P} - a.s.$ and $\mathbb{E}[W_t^H] = 0,$

(c_2) $\forall s,t > 0$,

$$\mathbb{E}\big[W_t^H W_s^H\big] = \frac{1}{2}\Big(t^{2H} + s^{2H} - \mid t-s \mid^{2H}\Big). \tag{2.1}$$

The constant parameter H is well-known as *"Hurst parameter"*. For $H = 1/2$, the *fBm* $(W_t^H)_{t\geq 0}$ coincides with the standard Brownian motion $(B_t)_{t\geq 0}$, that is, $(B_t)_{t\geq 0} \equiv (W_t^{\frac{1}{2}})_{t\geq 0}$.

The definition above means that fractional Brownian motion is a Gaussian process with mean 0 and variance t^{2H}, that is $W_t \sim N(0,t^{2H})$. From this definition, we may deduce the following intrinsic properties of *fBm*.

Proposition 2.1. *Let* $(W_t^H)_{t\geq 0}$ *be a fBm with Hurst parameter* $H \in (0,1)$ *defined on* $(\Omega,\mathcal{F},\mathbb{P})$. *Then the following properties hold.*

(p_1) $(W_t^H)_{t\geq 0}$ *has homogeneous increments, i.e.,* $\forall s,t \geq 0$,

$$W_{t+s}^H - W_s^H \sim W_t^H.$$

(p_2) $\forall \omega \in \Omega$, *the sample paths* $t \mapsto W_t^H(\omega)$ *are continuous* $\mathbb{P} - a.s.$

(p_3) *The increments of* $(W_t^H)_{t\geq 0}$ *are not independent for all* $H \neq \frac{1}{2}$.

(p_4) $(W_t^H)_{t\geq 0}$ *is self-similar process with Hurst parameter* H. *This means that for any non-random constant* $c > 0$, $W_{ct}^H \sim c^H W_t^H$.

(p_5) *The sample paths* $t \mapsto W_t^H(\omega)$ *are almost surely Hölder continuous of order strictly less than Hurst parameter* H.

2.1.2 Existence of *fBm*

Several approaches to proving the existence of *fBm* exist in the literature. We refer the readers to Mandelbrot and Van Ness (1968), Decreusefond et al. (1999) and Nourdin (2012). One possible way is through the Kolmogorov theorem.

Definition 2.7. Let $\psi : \mathbb{R}_+^2 \to \mathbb{R}$ be a symmetric function in the sense that $\forall s,t \in \mathbb{R},\ \psi(s,t) = \psi(t,s)$. Then ψ is said to be of positive type if for any $t_i \in \mathbb{R}$ and constants $c_j,\ i,j = 1,...,n \in \mathbb{N}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \psi(t_i,t_j) \geq 0.$$

The existence of a centered Gaussian process can be shown by using an extension of the Kolmogorov theorem in connection with symmetric functions known as *"Daniell-Kolmogorov theorem"* given below.

Theorem 2.2. *Let $\psi : \mathbb{R}_+^2 \to \mathbb{R}$ be a symmetric function of positive type. Then there exists a centered Gaussian process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance function $\psi(s,t) = \mathbb{E}[X_s X_t]$.*

Proposition 2.3. *The fBm $(W_t^H)_{t\geq 0}$ with covariance function $\psi(s,t) = \mathbb{E}[W_s^H W_t^H]$ given by (2.1) exists for all Hurst parameters $H \in (0,1)$.*

The proof of this proposition relies on showing that the covariance function (2.1) of *fBm* is symmetric of positive type. Consequently from Theorem 2.2, *fBm* exists for all Hurst parameters $H \in (0,1)$. For a detailed proof, we refer the reader to Nourdin (2012, Proposition 1.6).

2.2 Basic representations of *fBm*

Several representations of *fBm* can be found in the literature (See e.g. Marinucci and Robinson (1999) with references therein). In terms of stochastic integral with respect to Brownian motion, the *fBm* can be represented in at least three different ways as summarised in the following proposition.

Proposition 2.4. *Let $H \in (0,1)$ be the Hurst parameter and $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then the stochastic processes $(\hat{W}_t^H)_{t\geq 0}$, $(\tilde{W}_t^H)_{t\geq 0}$ and $(W_t^H)_{t\geq 0}$ defined respectively by*

$$\hat{W}_t^H = \frac{1}{c_H}\left(\int_{\mathbb{R}} \left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right) dB_u\right), \tag{2.2}$$

$$\tilde{W}_t^H = \frac{1}{d_H}\left(\int_{-\infty}^{0}\frac{1-\cos(ut)}{|u|^{H+\frac{1}{2}}}dB_u + \int_0^{\infty}\frac{\sin(ut)}{|u|^{H+\frac{1}{2}}}dB_u\right) \tag{2.3}$$

and

$$W_t^H = \int_0^t \kappa_H(s,t)dB_s \tag{2.4}$$

are fractional Brownian motions with Hurst parameter $H \in (0,1)$. *In the representations* (2.2) *and* (2.3)*, the parameters* c_H *and* d_H *are constants and are given by*

$$c_H = \left(\frac{1}{2H} + \int_0^{\infty}\left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 du\right)^{\frac{1}{2}},$$

and

$$d_H = \left(2\int_0^{\infty}\frac{1-\cos(u)}{u^{2H+1}}du\right)^{\frac{1}{2}}.$$

The function $\kappa_H(s,t)$ *in the representation* (2.4) *is a square integrable kernel given by*

$$\kappa_H(s,t) = \begin{cases} M_1(H)s^{\frac{1}{2}-H}\displaystyle\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du, & \text{if } H > \frac{1}{2} \\ M_2(H)\left(\left(\dfrac{t}{s}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{1}{2}} \right. \\ \qquad \left. -\left(H-\dfrac{1}{2}\right)s^{H-\frac{1}{2}}\displaystyle\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right), & \text{if } H < \frac{1}{2}. \end{cases} \tag{2.5}$$

where $M_1(H)$ *and* $M_2(H)$ *are constants given by*

$$M_1(H) = \left(\frac{H(2H-1)}{\beta(H-\frac{1}{2}, 2-2H)}\right)^{\frac{1}{2}},$$

and

$$M_2(H) = \left(\frac{2H}{(1-2H)\beta(H+\frac{1}{2}, 1-2H)}\right)^{\frac{1}{2}}.$$

with $\beta(\cdot,\cdot)$ *the Beta function defined by* $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$, $\forall p,q > 0$, A detailed proof of this proposition is given in Nualart (2003). The representation $(\hat{W}_t^H)_{t\geq 0}$ is known as *"moving average representation"* of *fBm* and was introduced by Mandelbrot and Van Ness (1968). The stochastic process $(\tilde{W}_t^H)_{t\geq 0}$ is called *"spectral or harmonisable representation"* and the process $(W_t^H)_{t\geq 0}$ is called *"interval representation"* or *"Volterra representation"* and was previously discussed by Norros et al. (1999).

Throughout this work, we shall use the Volterra representation of *fBm* defined by (2.4). Note that the square integrable kernel $\kappa_H(s,t)$ has been investigated in the literature. One of its representations is given by

$$\kappa_H(s,t) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\, {}_2\mathbf{F}_1\Big(H-\frac{1}{2}; \frac{1}{2}-H; H+\frac{1}{2}; 1-\frac{t}{s}\Big)\mathbf{1}_{[0,t]}(s), \qquad (2.6)$$

for all Hurst parameters $H \in (0,1)$, $s \in [0,t]$. Here $\Gamma(\cdot)$ is a Gamma function and ${}_2\mathbf{F}_1(a,b,c;z)$ is the Gauss hypergeometric function. Particularly, for Hurst parameter $H = 1/2$, the kernel is reduced to an indicator function, that is,

$$\kappa_H(s,t) = \mathbf{1}_{[0,t]}(s).$$

See Hult (2003) for more details.

2.3 Covariance function of *fBm*

In this subsection, we discuss some important results of the covariance function of *fBm* beyond its natural definition given by (2.1). The main references that were used are Decreusefond et al. (1999) and Neuman and Rosenbaum (2018) with references therein.

2.3.1 Representation of covariance function

Several representations of the covariance function of *fBm* exist. One of them can be deduced from the interval representation (2.4). Before we discuss the topic, we have to note that the covariance function shares in general the same properties with the inner product denoted by $\langle\cdot,\cdot\rangle$.

Consider a vector space $\mathcal{H}$ of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite second moment. Clearly, $\mathcal{H}$ is a Hilbert space with the inner product $\langle X, Y\rangle = \mathbb{E}\big[XY\big]$. The covariance function of *fBm* is the function $\psi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by

$$\psi(s,t) = \mathbb{E}[W_s^H W_t^H] = \langle W_s^H, W_t^H\rangle = \frac{1}{2}\Big(s^{2H} + t^{2H} - \big|t-s\big|^{2H}\Big).$$

The following proposition gives a representation of the covariance function in terms of the kernel defined by (2.5).

Proposition 2.5. *For any Hurst parameter $H \in (0,1)$, the covariance of fBm can be represented by the following expression:*

$$\psi(s,t) = \int_0^{s\wedge t} \kappa_H(s,r)\kappa_H(r,t)dr, \tag{2.7}$$

where κ_H is the kernel defined by (2.5).

For the proof, we refer to Norros et al. (1999) or Nualart (2003).

2.3.2 Covariance function of *fBm* for small Hurst parameters

Some findings show that the volatility is rough, that is, can be modeled with *fBm* with small Hurst parameters (See Gatheral et al. (2018) and Livieri et al. (2018) with references therein). In this case, the covariance function can be determined through the normalised *fBm*. For more details, see e.g. Neuman and Rosenbaum (2018) and references therein.

2.4 Fundamental properties of *fBm*

2.4.1 Long-range and short-range dependency

Definition 2.8. Let $L : (0,\infty) \longrightarrow \mathbb{R}$ be a Borel function. Then L is said to be slowly varying at infinity if for any constant $c > 0$,

$$\lim_{t\to+\infty} \frac{L(ct)}{L(t)} = 1.$$

Definition 2.9. Let $(X_t)_{t\geq 0}$ be a stationary stochastic process (*i.e.* $X_t - X_s \sim X_{t-s}, \quad \forall s,t \geq 0, \ s < t$) with the autocovariance function $\psi(\delta) = cov(X_t,X_{t+\delta}), \ \delta > 0$. Then $(X_t)_{t\geq 0}$ is said to display long-range dependence or long memory if there exists $\alpha \in (0,\frac{1}{2})$ such that

$$\psi(\delta) = L(\delta)\delta^{2\alpha-1}, \quad \text{as} \quad \delta \to \infty,$$

where L is a slowly varying function at infinity. In discrete time, let $(X_n)_{n\in\mathbb{N}}$ be a stationary process with autocovariance function $\psi(n) = cov(X_n,X_{n+1})$. Then the stochastic process $(X_n)_{n\in\mathbb{N}}$ is said to display long-range dependence if $\sum_{n=0}^{\infty} \mid \psi(n) \mid = \infty$ and short-range dependence if $\sum_{n=0}^{\infty} \mid \psi(n) \mid < \infty$.

Proposition 2.6. *The fBm $(W_t^H)_{t\geq 0}$ displays long-range dependency property for $H \in (\frac{1}{2}, 1)$ and short-range for $H \in (0, \frac{1}{2})$.*

Proof. The Taylor expansion of the autocovariance function $\psi(n)$ defined by (2.1) yields

$$\psi(n) \sim H(2H-1)n^{2H-2}, \quad as \quad n \longrightarrow \infty.$$

It follows that $\sum_{n=0}^{\infty} \mid \psi(n) \mid = \infty$ when $H > 1/2$ and $\sum_{n=0}^{\infty} \mid \psi(n) \mid < \infty$ for $H < 1/2$. □

2.4.2 Semimartingality

Semimartingality of a stochastic process is an important property that need to be discussed carefully. Roughly speaking, a stochastic process is said to be semimartingale if it can be decomposed as a local martingale and cadlag (*"Continue-A-Droite et Limit-A-Gauche"*) processes. An alternative definition of semimartingale process in terms of quadratic variations can also be used. Below we refer to Rogers (1997) to show that *fBm* is not semimartingale except for Hurst parameter $H = 1/2$.

Definition 2.10. Let $(X_t)_{t\in[0,T]}$ be a stochastic process with sample paths defined on the interval $[0,T]$ and $p > 0$. Consider a partition $\Pi = \{t_0, \cdots, t_m\}$ of $[0,T]$ with $t_0 = 0$ and $t_m = T$, and the corresponding sum $\sum_{i=0}^{m} \left|X_{t_i} - X_{t_{i-1}}\right|^p$. Then the $p-$variation of $(X_t)_{t\geq 0}$ denoted by $\langle X \rangle_p$ is the supremum of these sums for all possible partitions.

If $\langle X\rangle_p$ is finite almost surely, then the sample paths of $(X_t)_{t\in[0,T]}$ are said to have bounded $p-$variations. In particular, if $p=2$, then $p-$variation is said to be quadratic, denoted by $\langle X\rangle$.

Definition 2.11. The stochastic process $(X_t)_{t\in[0,T]}$ is said to be a semimartingale process if $\langle X\rangle<\infty$ and $\langle X\rangle\neq 0$. When $\langle X\rangle=0$, the sample paths of $(X_t)_{t\geq 0}$ must have bounded variations.

Proposition 2.7. *Let $(W_t^H)_{t\in[0,T]}$ be a fBm with $H\in(0{,}1)$. Fix $m=2^n$ and $t_i=\frac{iT}{2^n}$ for all $i=1,2,...,2^n$. For any $p>0$, the limit below holds with probability one.*

$$\lim_{n\to\infty}\sum_{i=1}^{2^n}\left|W^H_{\frac{iT}{2^n}}-W^H_{\frac{(i-1)T}{2^n}}\right|^p=\begin{cases}0 & \text{if} \quad pH>1\\ \infty & \text{if} \quad pH<1\\ T & \text{if} \quad pH=1.\end{cases}$$

See Rogers (1997).

Proposition 2.8. *Let $(W_t^H)_{t\geq 0}$ be a fBm with Hurst parameter $H\in(0{,}1)$. Then $(W_t^H)_{t\geq 0}$ is not semimartingale for all $H\neq 1/2$.*

Proof. This is a straightforward application of Proposition 2.7 by setting $p=2$ and by using Definition 2.11. For $H=1/2$, that is the case of standard Brownian motion, the quadratic variation is finite and the stochastic process is semimartingale. For $H<1/2$, the quadratic variation is infinite and cannot be a semimartingale. Finally, for $H>1/2$, the quadratic variation is null but

$$\sup_{m\geq 0}\sum_{i=1}^{m}\left|X_{t_i}-X_{t_{i-1}}\right|$$

is infinite. To prove that, let $p\in(1,\frac{1}{H})$ and by Proposition 2.7,

$$\lim_{n\to\infty}\sum_{i=1}^{2^n}T^{pH}\left(2^n\right)^{1-pH}=\infty.$$

On the other hand,

$$\sum_{i=1}^{2^n}\left|X_{\frac{iT}{2^n}}-X_{\frac{(i-1)T}{2^n}}\right|^p\leq\sup_{1\leq i\leq 2^n}\left|X_{\frac{iT}{2^n}}-X_{\frac{(i-1)T}{2^n}}\right|^{p-1}\times\sum_{i=1}^{2^n}\left|X_{\frac{iT}{2^n}}-X_{\frac{(i-1)T}{2^n}}\right|.$$

The sample paths of *fBm* being continuous almost surely (Proposition 1.1, p_2), then it follows that

$$\sup_{1 \leq i \leq 2^n} \left| X_{\frac{iT}{2^n}} - X_{\frac{(i-1)T}{2^n}} \right|^{p-1}$$

tends towards zero as $n \to \infty$. We deduce that

$$\lim_{n \to \infty} \sum_{i=1}^{2^n} \left| X_{\frac{iT}{2^n}} - X_{\frac{(i-1)T}{2^n}} \right| = \infty.$$

It follows that *fBm* can only be semimartingale when $H = 1/2$. □

This proposition shows clearly that the stochastic analysis of semimartingales is not applicable to *fBm*. However, Cheridito et al. (2001) showed that a linear combination of standard Brownian motion and *fBm* known as *"Mixed fBm"*, yield semimartingality property for only $H \in (3/4, 1)$. This finding is very surprising. However, the main criticism brought on mixed *fBm* is that it possibly cannot be used in financial modeling as a remedy of Brownian motion shortfalls. This is simply because empirical observations for asset prices or historical volatilities show that the probability that the Hurst parameter H lies between 3/4 and 1 tends towards zero. See Cajueiro and Tabak (2005), Cajueiro and Tabak (2008) and Livieri et al. (2018) as practical examples among many other results.

2.4.3 Markov Property

Definition 2.12. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Let $(X_t)_{t\geq 0}$ be a stochastic process adapted to the filtration $(\mathcal{F}_t)$. Then $(X_t)_{t\geq 0}$ is said to be a Markov process if for all Borel set $B \subset \mathbb{R}$ and for $s \leq t$,

$$\mathbb{P}\left[X_t \in B \mid \mathcal{F}_s\right] = \mathbb{P}\left[X_t \in B \mid X_s\right], \quad s \geq 0.$$

As shown in Kallenberg (1998, Proposition 11.7), if $(X_t)_{t\geq 0}$ is a centered Gaussian process with covariance function $\psi(s,t)$, then the stochastic process $(X_t)_{t\geq 0}$ is Markovian if and only if

$$\psi(s,t) = \frac{\psi(s,u)\psi(t,t)}{\psi(t,u)}, \quad s,t,u \geq 0, \quad u < s. \tag{2.8}$$

This may be used to prove the non-Markov property of *fBm* given in the following proposition.

Proposition 2.9. *The fBm $(W_t^H)_{t\geq 0}$ with $H \in (0,1)$ is not a Markov process for all $H \neq 1/2$.*

2.4.4 Hölder Continuity

Definition 2.13. Let $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$ be two stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The $(X_t)_{t\geq 0}$ is said to be a modification of $(\tilde{X}_t)_{t\geq 0}$ if $\mathbb{P}(X_t = \tilde{X}_t) = 1, \ \forall t \geq 0$.

Definition 2.14. (Hölder Continuity). Let $(X_t)_{t\geq 0}$ be a stochastic process and $\alpha \in (0,1]$ be a constant. A sample path $t \mapsto X_t(\omega)$ is said to be Hölder continuous of order α if there exists a positive random constant $c = c(\omega)$ such that for all $s, t \geq 0$,

$$\left|X_t - X_s\right| \leq c\left|t - s\right|^{\alpha}; \quad a.s.$$

Theorem 2.10. (Kolmogorov's Continuity Criterion). *Let $(X_t)_{t\in[0,T]}$, $T > 0$ be a real-valued stochastic process. If there exist positive constants p, c and β such that*

$$\mathbb{E}\left[\left|X_t - X_s\right|^p\right] \leq c|t - s|^{1+\beta}, \quad \forall s \geq 0, \ \ t \leq T,$$

then the stochastic process $(X_t)_{t\in[0,T]}$ admits a modification that is $\alpha-$Hölder continuous almost surely for any $\alpha \in (0,\beta/p)$.

Hence, the following proposition follows from Theorem 2.10 above.

Proposition 2.11. *The sample paths of fBm are Hölder continuous with order strictly less than H. That is, in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\exists \Omega' \subset \Omega, \ \mathbb{P}(\Omega') = 1$, such that $\forall \omega \in \Omega'$, $\forall 0 \leq s \leq t$ and $\forall \alpha > 0, \ \exists c = c(\omega, \alpha)$:*

$$\left|W_t^H(\omega) - W_s^H(\omega)\right| \leq c\left|t - s\right|^{H-\alpha}. \tag{2.9}$$

The proof of this proposition follows from the Kolmogorov's Continuity Criterion 2.10. For more details, see Mishura (2008).

2.5 Supremum of *fBm*

The exact distribution of the supremum of *fBm* $(W_t)_{t\in[0,T]}$ on $[0,T]$ is still an open problem. Here, we give some related results discussed by Molchan (1999) and Aurzada (2011).

Theorem 2.12. *Let $(W_t^H)_{t\geq 0}$ be a fBm with $H \in (0,1)$. Then the following expressions hold:*

(a) $\forall r \geq 1,$
$$\mathbb{E}\Big[\Big(\sup_{s\in[0,t]} W_s^H\Big)^r\Big] < \infty. \tag{2.10}$$

(b) $\forall x > 0,$
$$\lim_{x\to 0^+} \frac{1}{\log x}\mathbb{P}\Big(\sup_{s\in[0,1]} W_s^H \leq x\Big) = -1 + \frac{1}{H}. \tag{2.11}$$

For (a), see Mishura (2008). The Assertion (b) is rooted in the following results discussed by Aurzada (2011) and Nourdin (2012):

$$\begin{cases} \mathbb{E}\left[\left(\int_0^t \exp(W_s^H)ds\right)^{-1}\right] \sim Ht^{H-1}\mathbb{E}\Big[\sup_{s\in[0,1]} W_s^H\Big] \quad \text{as} \quad t\to\infty \\ \liminf_{x\to 0^+}\left[\frac{1}{\log x}\log\mathbb{P}\Big(\sup_{s\in[0,1]} W_s^H \leq x\Big)\right] \geq -1 + \frac{1}{H} \\ \limsup_{x\to 0^+}\left[\frac{1}{\log x}\log\mathbb{P}\Big(\sup_{s\in[0,1]} W_s^H \leq x\Big)\right] \leq -1 + \frac{1}{H}. \\ \lim_{x\to\infty} \frac{1}{x^2}\log\mathbb{P}\Big(\sup_{s\in[0,1]} W_s^H \geq x\Big) = -\frac{1}{2}. \end{cases}$$

Chapter 3

Tools in Malliavin Calculus for finance

Malliavin calculus is a field of stochastic analysis that deals with derivatives and integration with respect to white noise. It has been widely used in quantitative finance since it fits different diffusion processes, especially those driven by *fBm*. In this chapter, we present some important tools in Malliavin calculus with applications to stochastic processes used in finance. The main references of this chapter are Decreusefond et al. (1999), Norros et al. (1999), Nualart (2003), Nualart (2006) and Biagini et al. (2008). The following section introduces some preliminaries on Malliavian calculus.

3.1 Preliminaries

3.1.1 Malliavin Derivative

Let $\mathcal{H} = L^2([0,T])$. Then $\mathcal{H}$ is a real separable Hilbert space. In addition, let $(B_t)_{t\in[0,T]}$ be a Brownian motion and define $B(\phi) = \int_0^T \phi(t)dB_s,\ \phi \in \mathcal{H}$. Then from the theory of stochastic calculus,

$$\mathbb{E}\big[B(\phi_1)B(\phi_2)\big] = \int_0^T \phi_1(t)\phi_2(t)dt, \quad \forall\phi_1,\, \phi_1 \in \mathcal{H}.$$

Obviously, the expected value above is an inner product (usually denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$). A natural exercise is to find the derivative, let's say $\mathcal{D}$, such that $\mathcal{D}B(\phi) = \phi$. This is what the Malliavin derivative can do for any White noise. Recall the following definitions:

Definition 3.1. Let $\mathcal{H}$ be a real separable Hilbert space induced with the inner product $\langle \cdot,\cdot \rangle_{\mathcal{H}}$. A stochastic process $(X_\phi)_{\phi\in\mathcal{H}}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called isonormal Gaussian process if the following condition holds

$$\mathbb{E}\big[X(\phi_1)X(\phi_2)\big] = \langle \phi_1, \phi_2 \rangle_{\mathcal{H}}, \quad \forall \phi_1,\, \phi_2 \in \mathcal{H}. \tag{3.1}$$

The *fBm* $(W_t^H)_{t\geq 0}$ (see Definition 2.6) with covariance function defined by (2.1) or (2.7) which are the inner products in the Hilbert space $\mathcal{H}$ is a special example of isonormal Gaussian process. Let ξ be the set of real-valued step functions on the interval $[0,T]$ and define the Hilbert space $\mathcal{H} = \overline{(\xi, \langle \cdot,\cdot \rangle_{\mathcal{H}})}$ (that is the closer of ξ). Then

$$\psi(s,t) = \big\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \big\rangle_{\mathcal{H}}, \quad s,t \geq 0, \tag{3.2}$$

where $\mathbf{1}_{[0,\cdot]}$ is an indicator function.

Definition 3.2. Let $C^\infty(\mathbb{R}^n)$ be the set of infinitely differentiable functions. A random variable G taking the form

$$G = g(X(\phi_1), \cdots, X(\phi_n)), \quad \phi_1, \cdots, \phi_n \in \mathcal{H}, \quad g \in C^\infty(\mathbb{R}^n),$$

is said to be *"smooth"*. The set of all smooth random variables shall be denoted by $\mathcal{S}$.

Definition 3.3. Let $G = g(X(\phi_1), \cdots, X(\phi_n)) \in \mathcal{S}$. The Malliavin (or stochastic) derivative $\mathcal{D}$ of G with respect to $x_i = X(\phi_i)$ is defined by

$$\mathcal{D}^{x_i} G = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\big(X(\phi_1), \cdots, X(\phi_n)\big)\phi_i. \tag{3.3}$$

The superscript x_i is often omitted on the Malliavin derivative when the smooth random variable G depends only on one random variable.

Example 3.1.

1. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion, with $B_t = B(\mathbf{1}_{[0,t]})$. Then $\mathcal{D}_s B_t = \mathbf{1}_{[0,t]}(s)$.

2. Let $(X_t)_{[0,T]}$ be a geometric Brownian motion that verifies the following differential equation:

$$dX_t = \eta X_t dt + \sigma X_t dB_t,$$

where η and σ are positive constants. Then $\mathcal{D}_s X_t = \sigma X_t \mathbf{1}_{[0,t]}(s)$. Note that in this example, the volatility σ is a constant. If the volatility $\sigma = \sigma(Y_t)$ is stochastic depending on the stochastic process $(Y_t)_{[0,T]}$, some approximations will be needed although the Malliavin derivative is similar. This will be further discussed in Chapter 6.

Example 3.2. Let $(W_t^H)_{t\geq 0}$ be a *fBm* taking the Volterra representation (2.4), that is $W_t^H = \int_0^t \kappa_H(s,t)dB_s$ with $(B_t)_{t\geq 0}$ the standard Brownian motion. Then the Malliavin derivative with respect to B_t of W_t^H at time $s \geq 0$, denoted by $\mathcal{D}_s^B W_t^H$ is given by

$$\mathcal{D}_s^B W_t^H = \kappa_H(s,t)\mathbf{1}_{[0,t]}(s),$$

and for any standard Brownian motion $(\tilde{B}_t)$ that is independent to $(B_t)_{t\geq 0}$, the Malliavin derivative with respect to $\tilde{B}_t$ of W_t^H is

$$\mathcal{D}_s^{\tilde{B}} W_t^H = 0.$$

Definition 3.4. Let $\phi \in \mathcal{H}$ and define the inner product $\langle \mathcal{D}G, \phi\rangle_{\mathcal{H}}$ by

$$\langle \mathcal{D}G, \phi\rangle_{\mathcal{H}} = \lim_{\epsilon\to 0} \frac{\Big[g\big(X_{\phi_1} + \epsilon\langle\phi_1,\phi\rangle_{\mathcal{H}}, \cdots, X(\phi_n) + \epsilon\langle\phi_n,\phi\rangle_{\mathcal{H}}\big) - g(X_{\phi_1}, \cdots, X(\phi_n))\Big]}{\epsilon}. \tag{3.4}$$

Then $\langle \mathcal{D}G, \phi\rangle_{\mathcal{H}}$ is called directional Malliavin derivative of the random variable G at $\epsilon = 0$.

From the above definition, we may introduce the integration-by-part formula discussed in detail by Nualart (2006).

Proposition 3.1. *Let* $G = g(X(\phi_1), \cdots, X(\phi_n)) \in \mathcal{S}$, $\phi_i \in \mathcal{H}$. *Then*

$$\mathbb{E}\big[\langle \mathcal{D}G, \phi\rangle_{\mathcal{H}}\big] = \mathbb{E}\big[GX(\phi)\big], \quad \phi \in \mathcal{H}. \tag{3.5}$$

Moreover, let $G_1, G_2 \in \mathcal{S}$*. Then*

$$\mathbb{E}\big[\langle \mathcal{D}G_1G_2, \phi\rangle_{\mathcal{H}}\big] = \mathbb{E}\big[G_1G_2X(\phi) - G_1\langle \mathcal{D}G_2, \phi\rangle_{\mathcal{H}}\big]. \tag{3.6}$$

Remark. The relation (3.6) represents the Malliavin derivative of a product of two smooth random variables.

Definition 3.5. Let $f : \Omega \to \mathcal{H}$ be a strongly measurable function (i.e., f can be represented as the pointwise limit of simple functions of the form $\sum_k x_k \mathbf{1}_{T_k}$). Then f is called p-integrable if for any $p \geq 1$,

$$\int_\Omega ||f||^p d\mathbb{P} < \infty.$$

The set of all p-integrable strongly measurable functions shall be denoted by $L^p(\Omega; \mathcal{H})$ and the set of all p-integrable measurable functions by $L^p(\Omega)$. For $p = 2$, the space $L^2(\Omega; \mathcal{H})$ is a Hilbert space induced by the inner product $\langle f_1, f_2\rangle = \int_\Omega \langle f_1, f_2\rangle_{\mathcal{H}} d\mathbb{P}$.

Proposition 3.2. *The Maliavian derivative operator* $\mathcal{D}$ *is closable from* $L^p(\Omega)$ *to* $L^p(\Omega; \mathcal{H})$.

See Nualart (2006).

Note that the domain of Malliavian derivative operator $\mathcal{D}$ in the space $L^p(\Omega)$ denoted by $\mathbb{D}^{1,p}$ is the closure of the space of smooth random variables $\mathcal{S}$ equipped with the norm

$$\left\|G\right\|_{1,p} = \left(\mathbb{E}[|G|]^p + \mathbb{E}\left\|\mathcal{D}G\right\|^p_{\mathcal{H}}\right)^{\frac{1}{p}} < \infty.$$

More generally, denote $\mathcal{D}^k$, $k > 0$, the k^{th} iteration of the Malliavian derivative and $\mathbb{D}^{k,p}$ its domain. Then $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ equipped with the norm:

$$\left\|G\right\|_{k,p} = \left(\mathbb{E}[|G|]^p + \sum_{i=1}^{k} \mathbb{E}\left\|\mathcal{D}^i G\right\|^p_{\mathcal{H}^{\otimes i}}\right)^{\frac{1}{p}} < \infty, \tag{3.7}$$

where $\mathcal{H}^{\otimes i}$ is the i^{th} tensor power of the separable Hilbert space $\mathcal{H}$. Fix $p = 2$, $k \geq 1$ and let G_1,$G_2 \in \mathcal{S}$. Then the space $\mathbb{D}^{1,2}$ is an Hilbert space equipped with the inner product

$$\langle G_1, G_2\rangle_{k,2} = \mathbb{E}\big[G_1G_2\big] + \sum_{i=1}^{k}\mathbb{E}\big[\langle \mathcal{D}^i G_1, \mathcal{D}^i G_2\rangle_{\mathcal{H}^{\otimes i}}\big].$$

The following proposition establishes the chain rule formula for Malliavin derivative that was previously discussed by Nualart (2006, Proposition 1.2.3).

Proposition 3.3. *Let $G = (G_1, \cdots, G_n)$ to be a vector of smooth random variables $G_i \in \mathbb{D}^{1,p}$, $i = 1, \cdots, n$ for a fixed $p \geq 1$. Define $F : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Then $F(G) \in \mathbb{D}^{1,p}$ and*

$$\mathcal{D}F(G) = \sum_{i=1}^{n}\frac{\partial F(G)}{\partial x_i}\mathcal{D}G_i.$$

This proposition can be extended to the case where G verifies the Lipschitz condition as stated in the following lemma:

Lemma 3.4. *Let G be a random variable whose law is absolutely continuous with respect to the Lebesgue measure $\mathbb{R}$ and let $F : \mathbb{R} \to \mathbb{R}$ be a function that verifies the Lipschitz condition, that is, for every $x,y \in \mathbb{R}$, there exists a constant K such that*

$$\big|F(x) - F(y)\big| \leq K\big|x - y\big|.$$

Then $F(G)$ is Malliavin differentiable and

$$\mathcal{D}F(G) = F'(G)\mathcal{D}G.$$

This is a straight consequence of Nualart (2006, Proposition 1.2.4).

Remark. The above lemma is also applicable to $G = (G_1, \cdots, G_n)$ with each $G_i \in \mathbb{D}^{1,p}$, $i = 1, \cdots, n$.

3.1.2 Divergence operator

Loosely speaking, the divergence operator is adjoint of Malliavin operator $\mathcal{D}$ with its domain that consists of square integrable random processes defined in the space $\mathcal{H}$. The following definition is more formal.

Definition 3.6. Let $\mathcal{D}$ be the Malliavin derivative operator and consider the linear mapping $\delta : L^2(\Omega, \mathcal{H}) \to L^2(\Omega)$ such that for any $U \in L^2(\Omega, \mathcal{H})$, there exists a square integrable function $\delta(U)$ that verifies the following equality

$$\mathbb{E}\big[\langle \mathcal{D}G, U\rangle_{\mathcal{H}}\big] = \mathbb{E}\big[G\delta(U)\big],$$

for all $G \in \mathbb{D}^{1,2}$. Then δ is called *"Skorokhod"* or *"divergence operator"*.

Remarks

(a) The domain of the operator δ is given by $\operatorname{Dom}\delta = \{U \in L^2(\Omega, \mathcal{H}) : \big|\mathbb{E}\big[\langle \mathcal{D}G, U\rangle_{\mathcal{H}}\big]\big| \leq c||G||_{L^2(\Omega)}\}$, where $c = c(U)$ is a random constant.

(b) The divergence operator can be interpreted as a stochastic integral with respect to a Gaussian process known as a divergence (or Skorokhod) integral.

(c) By taking G as a constant, then $\mathbb{E}[\delta(U)] = 0$.

(d) If $G \in \mathbb{D}^{1,2},\ U \in \operatorname{Dom}\delta$ and $GU \in L^2(\Omega, \mathcal{H})$, then

$$\delta(GU) = G\delta(U) - \langle \mathcal{D}G, U\rangle. \tag{3.8}$$

(e) Fix $n > 0$ and let $G_i = g(X(\phi_i)) \in \mathcal{S},\ \phi_i \in \mathcal{H},\ \ i = 1, \cdots, n$. Assume that

$$U = \sum_{i=1}^{n} G_i\phi_i.$$

From (3.6), we may deduce that

$$\delta(U) = \sum_{i=1}^{n} G_iX(\phi_i) - \sum_{i=1}^{n}\langle \mathcal{D}G_i, \phi_i\rangle_{\mathcal{H}}. \tag{3.9}$$

Remarks. Assume that the Hilbert space $\mathcal{H}$ takes the form $\mathcal{H} = L^2(\mathbb{B}, \mathcal{B}, \mu^*)$ where $(\mathbb{B}, \mathcal{B}, \mu^*)$ is a measure space with μ^* that represents a non-atomic measure on the measurable space $(\mathbb{B}, \mathcal{B})$. Then $\mathrm{Dom}\,\delta \subset L^2(\mathbb{B} \times \Omega)$ and the Skorohod integral $\delta(U)$ with respect to the Gaussian process $(X(\phi))_{\phi \in L^2}$ takes the following form

$$\delta(U) = \int_{\mathbb{B}} U dX(\phi). \tag{3.10}$$

Under the above settings, we may deduce the following lemma.

Lemma 3.5. *Let $(U_t)_{t\in[0,T]} \in \mathbb{D}^{1,2}(L^2([0,T]))$ and assume that the stochastic process $(\mathcal{D}_t U_s)_{s,t\in[0,T]}$ is integrable. Then*

$$\mathcal{D}_t(\delta(U)) = U_t + \int_0^T \mathcal{D}_t U_s dX_s. \tag{3.11}$$

3.2 Malliavian Calculus for *fBm* with $H > 1/2$

3.2.1 The divergence operator for *fBm* with $H > 1/2$

Recall from our first chapter that the Volterra representation of *fBm* is given by (2.4), that is, $W_t^H = \int_0^t \kappa_H(s,t) dB_s$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion, and where $\kappa_H(s,t)$ is a square integrable kernel given by

$$\kappa_H(s,t) = \kappa_1(H) s^{\frac{1}{2}-H} \int_s^t (u-t)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du. \tag{3.12}$$

where $\kappa_1(H)$ is a constant defined by

$$\kappa_1(H) = \left(\frac{H(2H-1)}{\beta(H-\frac{1}{2}, 2-2H)} \right)^{\frac{1}{2}}.$$

Now from (3.12), we can deduce that

$$\frac{\partial \kappa_H}{\partial t}(s,t) = \kappa_1(H) \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \tag{3.13}$$

As previously, let ξ be the set of real-valued step functions on the interval $[0,T]$. We define a linear operator $\kappa^* : \xi \to L^2([0,T])$ such that

$$\left(\kappa_H^* \phi\right)(s) = \int_s^T \phi \frac{\partial \kappa_H}{\partial t}(s,t) dt. \tag{3.14}$$

It was shown in Norros et al. (1999) that if $\phi = \mathbf{1}_{[0,T]}$, then $B_t \equiv W_t^H \left((\kappa^*)^{-1} \mathbf{1}_{[0,T]}\right)$ is indeed a standard Brownian motion. Moreover, the image of κ_H^* coincides with the space $L^2[0,T]$, that is

$$\mathcal{H} = (\kappa_H^*)^{-1} L^2[0,T]. \tag{3.15}$$

Consequently, the domain of Malliavin derivative with respect to *fBm* (denoted by $\mathbb{D}_H^{1,2}$) is then given by

$$\mathbb{D}_H^{1,2} = (\kappa_H^*)^{-1} \left(\mathbb{D}^{1,2}(L^2(0,T))\right). \tag{3.16}$$

Proposition 3.6. *Let $\mathcal{D}^{W^H}$ and $\mathcal{D}^B$ be respectively the Malliavin derivatives with respect to fBm $(W_t^H)_{t\geq 0}$ and the standard Brownian motion $(B_t)_{t\geq 0}$. Then for any $G = g(W_t^H) \in \mathbb{D}_H^{1,2}$,*

$$\kappa_H^* \left(\mathcal{D}^{W^H} G\right) = \mathcal{D}^B G. \tag{3.17}$$

Proof. We closely follow Alos et al. (2001). For any $G = g(W_t^H) \in \mathbb{D}_H^{1,2}$, we have

$$\begin{aligned}
\mathbb{E}\left[\langle U, \mathcal{D}^{W^H} G\rangle_{\mathcal{H}}\right] &= \mathbb{E}\left[\langle U, \mathcal{D}^{W^H} g(W_t^H)\rangle_{\mathcal{H}}\right] \\
&= \mathbb{E}\left[\langle \kappa_H^* U, g'(W_t^H) \kappa_H^* \mathbf{1}_{[0,t]}\rangle_{L^2(0,T)}\right] \\
&= \mathbb{E}\left[\langle \kappa_H^* U, g'(W_t^H) \kappa_H(t,r) \mathbf{1}_{[0,t]}(r)\rangle_{L^2(0,T)}\right] \\
&= \mathbb{E}\left[\langle \kappa_H^* U, \mathcal{D}^B g(W_t^H)\rangle_{L^2(0,T)}\right].
\end{aligned}$$

As

$$\mathbb{E}\left[\langle U, \mathcal{D}^{W^H} G\rangle_{\mathcal{H}}\right] = \mathbb{E}\left[\langle \kappa_H^* U, \kappa_H^* \mathcal{D}^{W^H} G\rangle_{L^2(0,T)}\right],$$

then

$$\mathbb{E}\left[\langle\kappa_H^* U, \kappa_H^* \mathcal{D}^{W^H} G\rangle_{L^2(0,T)}\right] = \mathbb{E}\left[\langle\kappa_H^* U, \mathcal{D}^B g(W_t^H)\rangle_{L^2(0,T)}\right],$$

which concludes the proof. □

Remarks.

As a straight consequence of Proposition 3.6, let $\delta_H(U)$ be the divergence integral with respect to *fBm* $(W_t^H)_{t\geq 0}$ of $U \in L^2(\Omega, \mathcal{H})$ and denote $\delta_{1/2}$ the divergence operator for Brownian motion. Then the following hold:

(1) $\delta_H(U) = \delta_{1/2}(\kappa_H^* U) = \int_0^T \kappa_H^* U_s \delta_{1/2} dB_s.$

(2) $\mathrm{Dom}\,\delta_H = (\kappa_H^*)^{-1}\,\mathrm{Dom}\,\delta_{1/2},$

(3) The divergence integral can be represented in terms of pathwise integral with respect to *fBm* $(W_t^H)_{t\geq 0}$ as follows

$$\delta(U_H \mathbf{1}_{[0,T]}) = \int_0^t U_s dW_s^H. \tag{3.18}$$

3.2.2 Connection to Stratonovich integral

There exists a connection between the divergence Stratonovich integrals with respect to *fBm*. The Stratonovich integral belongs to the class of pathwise integrals defined as follows.

Definition 3.7. Let $(U_t)_{t\in[0,T]}$ be a stochastic process and $(W_t^H)_{t\in[0,T]}$ a *fBm*. The pathwise Stratonovich integral with respect to *fBm* denoted by $\int_0^T U_s \circ dW_s^H$ is defined as a pathwise limit (when it exists) given by

$$\int_0^T U_s \circ dW_s^H = \lim_{n\to\infty} \sum_{i=1}^n \frac{U_{t_{i+1}} + U_{t_i}}{2} \left(W_{t_i}^H - W_{t_{i-1}}^H\right), \tag{3.19}$$

where $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ is a partition of the interval $[0, T]$ such that

$$\sup_{0\leq i\leq n} |t_i - t_{i-1}| \to 0 \quad \text{as} \quad n \to \infty.$$

Proposition 3.7. *Fix* $T > 0$. *Let* $(U_t)_{t\in[0,T]} \in \mathbb{D}^{1,2}$ *and* $(W_t^H)_{t\in[0,T]}$ *be a fBm with Hurst parameter* $H > 1/2$. *Then the Stratonovich integral* $\int_0^T U_s \circ dW_s^H$ *can be represented by*

$$\int_0^T U_s \circ dW_s^H = \delta(U) + H(2H-1)\int_0^T\int_0^T \mathcal{D}U_t|t-s|^{2H-2}dsdt, \tag{3.20}$$

provided that the double integral is well defined, that is,

$$\int_0^T\int_0^T |\mathcal{D}U_t||t-s|^{2H-2}dsdt < \infty, \tag{3.21}$$

or equivalently, for $p > \frac{1}{2H-1}$, *the following condition must hold*

$$\int_0^T \left(\int_0^T |\mathcal{D}U_t|^p dt\right)^{\frac{1}{p}} < \infty. \tag{3.22}$$

See e.g. Nualart (2006).

Remark. By using (3.18), the representation (3.20) can be written as

$$\int_0^T U_s \circ dW_s^H = \int_0^t U_s dW_s^H + H(2H-1)\int_0^T\int_0^T \mathcal{D}U_t|t-s|^{2H-2}dsdt. \tag{3.23}$$

3.2.3 Itô Formula with respect to *fBm*

Proposition 3.8. *Consider* $(X_t)_{t\in[0,T]}$ *be a stochastic process with continuous sample paths defined by*

$$X_t = \int_0^t U_s dW_s^H,$$

where $(U)_{t\in[0,T]} \in \mathbb{D}^{1,2}$. *Let* $f \in \mathcal{C}^2(\mathbb{R})$. *Then*

$$\begin{aligned} f(X_t) =& f(0) + \int_0^t f'(X_s)U_s dW_s^H \\ &+ H(2H-1)\int_0^t f''(X_s)U_s\left(\int_0^T \left|s-r\right|^{2H-2}\left(\int_0^s \mathcal{D}_r U_r dW_r^H\right)dr\right)ds \\ &+ H(2H-1)\int_0^t f''(X_s)U_s\left(\int_0^s U_r\left(s-r\right)^{2H-2}dr\right)ds. \end{aligned} \tag{3.24}$$

Remark. Since $\lim_{H\to\frac{1}{2}} \frac{2H-1}{s^{2H-1}}(s-r)^{2H-2}\mathbf{1}_{[0,s]} = 1$, then (3.20) coincides with the classical Itô formula when $H \to 1/2$ as given in Fouque et al. (2011, Section 1.1.4).

3.3 Malliavian calculus for *fBm* with $H < 1/2$

3.3.1 Divergence operator for *fBm* with $H < 1/2$

Definition 3.8. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a locally integrable function. The Riemann−Liouville fractional integral of order $\alpha \in (0,1)$ is defined as

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}f(r)dr,$$

where $\Gamma(\,\cdot\,)$ is the Gamma function. The corresponding fractional derivatives of order $\alpha \in (0,1)$ are given by

$$D^{\alpha}_{0+}f = \frac{d}{dt}I^{1-\alpha}_{0+}f,$$

provided that the derivatives above exist.

Proposition 3.9. *Consider an interval* $(a,b) \subset [0,T]$ *and let* $(U_t)_{t\in(a,b]}$ *be a stochastic process defined on the space* $\mathcal{H}$ *by* $U_t = W^H_t\mathbf{1}_{(a,b]}$. *Then for any* $H \in (\frac{1}{4},\frac{1}{2})$,

$$\mathbb{P}\big[U \in \mathcal{H}\big] = 1 \tag{3.25}$$

and for any $H \in (0,\frac{1}{4}]$,

$$\mathbb{P}\big[U \in \mathcal{H}\big] = 0. \tag{3.26}$$

Proof. We follow the lines of Cheridito and Nualart (2005). To prove (3.25), we first recall from Aurzada (2011) that for any $H \in (\frac{1}{4},\frac{1}{2})$, there exists a random constant $c = c(\omega)$, $\omega \in \Omega_1 \subset \Omega$, such that

$$\sup_{t\in(a,b]}\big|W^H_t(\omega)\big| \le c$$

and

$$\sup_{s,t\in(a,b]} \frac{\left|W_t^H(\omega) - W_s^H(\omega)\right|}{|t-s|^{\frac{1}{4}}} \leq c, \quad \forall s \neq t.$$

For any $\omega \in \Omega_1$, we define:

$$\Psi(t) = U_t(\omega) = W_t^H(\omega)\mathbf{1}_{(a,b]}$$

and for $\alpha = \frac{1}{2} - H$, we set

$$\tilde{c} = \frac{\alpha}{\Gamma(1-\alpha)} c(\omega).$$

Then for $\epsilon > 0$, we have:

(a) For $t \in (-\infty, a)$, $D_{\epsilon+}^{\alpha}\Psi(t) = 0$.

(b) For $t \in (a, b]$,

$$\begin{aligned}
\left|D_{\epsilon+}^{\alpha}\Psi(t)\right| &\leq \frac{\alpha}{\Gamma(1-\alpha)} \left(\mathbf{1}_{\{t-a>\epsilon\}} \int_{\epsilon}^{t-a} \left|\frac{\Psi(t) - \Psi(t-r)}{r^{1+\alpha}}\right| dr \right. \\
&\qquad \left. + \left|\Psi(t)\right| \int_{\min(t-a,\epsilon)}^{T} r^{-(1+\alpha)} dr \right) \\
&\leq \tilde{c} \left(\mathbf{1}_{\{t-a>\epsilon\}} \int_{\epsilon}^{T} r^{-(\frac{3}{4}+\alpha)} dr + \int_{\min(t-a,\epsilon)}^{T} r^{-(1+\alpha)} dr \right) \\
&\leq \tilde{c} \left[\frac{1}{\frac{1}{4}-\alpha}(t-a)^{\frac{1}{4}-\alpha} + \frac{1}{\alpha}\,(t-a)^{\alpha}\right].
\end{aligned}$$

(c) For $t \in (b, \infty)$,

$$\begin{aligned}
\left|D_{\epsilon+}^{\alpha}\Psi(t)\right| &\leq \frac{\alpha}{\Gamma(1-\alpha)} \int_{t-b}^{t-a} \frac{\left|\Psi(t-r)\right|}{r^{1+\alpha}} dr \\
&\leq \tilde{c} \left(\int_{t-b}^{t-a} r^{-(1+\alpha)} dr\right) \\
&= \frac{\tilde{c}}{\alpha}\Big((t-b)^{-\alpha} - (t-a)^{-\alpha}\Big).
\end{aligned}$$

Hence, we may conclude that

$$D^{\alpha}_{\epsilon+}\Psi(t) = \tilde{\Psi}(t)$$

where

$$\tilde{\Psi}(t) = \begin{cases} 0, & \text{for } \ t \in (-\infty, a) \\ \tilde{c}\left[\frac{1}{\frac{1}{4}-\alpha}(t-a)^{\frac{1}{4}-\alpha} + \frac{1}{\alpha}\,(t-a)^{\alpha}\right], & \text{for } \ t \in (a, b] \\ \frac{\tilde{c}}{\alpha}\Big((t-b)^{-\alpha} - (t-a)^{-\alpha}\Big), & \text{for } \ t \in (b, \infty) \end{cases}$$

We may easily observe that $\Psi(t) \in L^2(\mathbb{R})$. It follows from Samko et al. (1993) that $\Psi(t) \in \mathcal{H}$. Next step now is to show (3.23) and it will be done by contradiction. If $U \in \mathcal{H}$, then from Samko et al. (1993), there exists $\omega \in \Omega \subset \Omega_1$ such that the sample path $U(\omega)$ satisfies

$$\int_0^T \big[U_{t+r}(\omega) - U_r(\omega)\big]dr = \mathcal{O}(t^2\alpha) \ \text{ as } \ t \to 0. \tag{3.27}$$

On the other hand, it easy to check that the stochastic process $\hat{W}^H_t = W^H_{t+a} - W^H_a$ is also a *fBm* with Hurst parameter $H \in (0,1)$ and, by self-similarity property (See Proposition 2.1),

$$\begin{aligned} t^{-2H}\int_0^{b-a-t} \big[\hat{W}^H_{r+t} - \hat{W}^H_r\big]^2 dr &\sim \int_0^{b-a-t} \big[\hat{W}^H_{\frac{r+t}{t}} - \hat{W}^H_{\frac{r}{t}}\big]^2 dr \\ &= t\int_0^{\frac{b-a}{t-1}} \big[\hat{W}^H_{v+1} - \hat{W}^H_v\big]^2 dv \\ &= \frac{(t-1)(b-a-t)}{(b-a)}\int_0^{\frac{b-a}{t-1}} \big[\hat{W}^H_{v+1} - \hat{W}^H_v\big]^2 dv \end{aligned} \tag{3.28}$$

By the Birkhoff ergodic theorem, (3.28) converges almost surely to

$$(b-a)\mathbb{E}\big[\big(\hat{W}^H_1\big)^2\big] \ \text{ as } \ t \to 0.$$

This implies that there exists a sequence $(t_i)_{i\in\mathbb{N}}$ converging to zero such that for all $\omega \in \Omega \subset \Omega_1$,

$$\int_0^T \big[U_{r+t_i}(\omega) - U_r(\omega)\big]^2 dr \geq \int_a^{b-t_i} \big[\hat{W}^H_{r+t_i}(\omega) - \hat{W}^H_r(\omega)\big]^2 dr$$
$$= \int_0^{b-a-t_i} \big[\hat{W}^H_{r+t_i}(\omega) - \hat{W}^H_r(\omega)\big]^2 dr \tag{3.29}$$
$$= \frac{b-a}{2} t_i^{2H} \mathbb{E}\big[\big(\hat{W}^H_1\big)^2\big].$$

This contradicts (3.26) since the expressions (3.27) and (3.28) are both satisfied only when $H > \alpha$, that is, for $H > \frac{1}{4}$. Therefore, the probability (3.25) holds. □

This proposition shows clearly that for $H \leq \frac{1}{4}$ the stochastic process $U_t = W^H_t \mathbf{1}_{(a,b]}$ does not belong to the domain $\mathrm{Dom}_H\,\delta_H$. Therefore, this domain shall be extended to a larger one, denoted by $\mathrm{Dom}^{\diamond}_H\,\delta_H$ that will contain U_t for all $H \in (0,1)$. Again, we construct this domain by following closely Cheridito and Nualart (2005). As previously, let us define a linear operator $\kappa^{\diamond}_H : \xi \to L^2(\Omega)$ such that

$$\big(\kappa^{\diamond}_H \phi\big)(s) = \kappa_H(T,s)\phi(s) - \int_s^T \big(\phi(t) - \phi(s)\big)\frac{\partial \kappa_H}{\partial t}(s,t)dt, \tag{3.30}$$

where $\kappa_H(t,s)$ is a kernel defined by (2.5). In terms of fractional derivatives (See Definition 3.8), the operator $\kappa^{\diamond}_H$ is defined by

$$(\kappa^{\diamond}_H \phi)(s) = K_2(H)\Gamma\left(H + \frac{1}{2}\right) s^{\frac{1}{2}} \Big(D^{\frac{1}{2}-H} u^{H-\frac{1}{2}}\phi(u)\Big)(s).$$

We may observe from (3.30) that if $\phi(s) = \mathbf{1}_{[0,t]}(s)$, then

$$(\kappa^{\diamond}_H \mathbf{1}_{[0,t]})(s) = \kappa_H(t,s)\mathbf{1}_{[0,t]}(s).$$

Let $\hat{\kappa}^{\diamond}_H$ be the adjoint operator of $\kappa^{\diamond}_H$ and define the space $\mathcal{H}^{\diamond}$ as

$$\mathcal{H}^{\diamond} = \big(\kappa^{\diamond}_H\big)^{-1}\big(\hat{\kappa}^{\diamond}_H\big)^{-1} L^2(\mathbb{R}).$$

Let $G = g(W^H_{\phi_1}, \cdots, W^H_{\phi_n})$ be a smooth random variable with $\phi_i \in \mathcal{H}^{\diamond}$ and

$g \in C^\infty(\mathbb{R}^n)$. Then we have the following definition:

Definition 3.9. Let $\mathcal{D}$ a Malliavin derivative operator and consider a linear mapping $\delta_H : L^2(\Omega, \mathcal{H}^\diamond) \to L^2(\Omega)$ such that for any stochastic process $U \in L^2(\Omega, \mathcal{H}^\diamond)$ and $G \in \mathbb{D}^{1,2}$, $\delta_H(U)$ verifies the following equality

$$\mathbb{E}\big[\delta(U)G\big] = \int_0^T \mathbb{E}[U_t \hat{\kappa}_H^\diamond \kappa_H^\diamond \mathcal{D}G].$$

The domain of this operator δ_H is given by $\mathrm{Dom}_H^\diamond\,\delta_H = \{U \in L^2(\Omega, \mathcal{H}^\diamond) : \big|\mathbb{E}\big[\delta(U)G\big]\big| < \infty\}$.

Remarks

(a) $Dom_H\,\delta \subset \mathrm{Dom}_H^\diamond\,\delta$. Moreover, $Dom_H\,\delta_H = \mathrm{Dom}_H^\diamond\,\delta_H \bigcap \big[\bigcup_{p>1} L^p(\Omega; \mathcal{H}^\diamond)\big]$ (See Cheridito and Nualart (2005, Proposition 3.5)).

(b) For any $U \in \mathrm{Dom}_H^\diamond\,\delta_H$ such that $\mathbb{E}[U] \in L^2(\mathbb{R})$. Then it follows that $\mathbb{E}[U] \in \mathcal{H}^\diamond$ (See Cheridito and Nualart (2005, Proposition 3.6)).

3.3.2 Itô Formula with respect to *fBm* with $H \in (0,1/2)$

After the above settings, we may discuss the Itô formula with respect to *fBm* with $H \in (0,1/2)$ within a larger domain $\mathrm{Dom}_H^\diamond\,\delta_H$. The following version of Itô Formula was discussed by Cheridito and Nualart (2005).

Theorem 3.10. *Let $G \in C^2(\mathbb{R})$ be a continuous function with continuous first and second derivatives such that on $[0,T]$, there exist two positive constants c_1 and $c_2 < \frac{1}{4}T^{-2H}$ such that $\max\{|G(t)|, |G'(t)|, |G''(t)|\} \leq c_1 e^{c_2 t^2}$, $\forall t \in [0,T]$. Then $G'(W_t^H)\mathbf{1}_{(a,b]}(t) \in Dom_H^\diamond\,\delta_H$ and*

$$G(W_t^H) = G(0) + \int_0^t G'(W_s^H)dW_s^H + H\int_0^t G''(W_s^H)s^{2H-1}ds.$$

3.3.3 Connection to symmetric integral

Definition 3.10. Let $(U_t)_{t\in[0,T]}$ be a stochastic process with integrable paths. The symmetric integral of U_t with respect to *fBm* $(W_t^H)_{t\in[0,T]}$, denoted by

$\int_0^T U_s * dW_s^H$ is defined by

$$\int_0^T U_s * dW_s^H = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^T U_s \left(W_{s+\epsilon}^H - W_{s-\epsilon}^H \right) ds, \tag{3.31}$$

provided the limit exists.

The following proposition can be regarded as an alternative to Proposition 3.7 when $H < 1/2$.

Proposition 3.11. *Fix $T > 0$. Let $U = (U_t)_{t \in [0,T]} \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ be a stochastic process and let $(W_t^H)_{t \in [0,T]}$ be a fBm with Hurst parameter $H < 1/2$. Then the symmetric integral $\int_0^T U_s * dW_s^H$ can be represented by*

$$\int_0^T U_s * dW_s^H = \delta(U) + Tr\mathbb{D}U, \tag{3.32}$$

where $Tr\mathbb{D}U$ is the trace of the Malliavin derivative $\mathcal{D}U$ and it is defined by

$$Tr\,\mathcal{D}U = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^T \left\langle \mathcal{D}U_s, \mathbf{1}_{[s-\epsilon, s+\epsilon] \bigcap [0,T]} \right\rangle,$$

provided that the limit exists.

See e.g. Nualart (2006).

Chapter 4

Stochastic volatility modelling under Brownian motion

This chapter introduces the standard Black-Scholes model and one of its major extensions commonly known as the Heston model under the standard Brownian motion.

4.1 Black-Scholes model and beyond

The modern derivative pricing theory started growing with the work of Black and Scholes (1973), commonly known as the *"Black-Scholes model"* and was considered as one of the best models for option pricing. This model gives the formula of the fair option price obtained from a portfolio that consists of a risk-free asset and a risky asset. This yields a partial differential equation with a boundary condition given as a payoff function. For each type of option, a solution can be found.

Roughly speaking, an option in finance is an agreement that gives its holder the right to buy (for call option) or to sell (for put option) a fixed amount at a specified future time. There exist two classes of options, vanilla options which include European and American options, and exotic options which are any option except vanilla options. Barrier options, Lookback options, Asian options, etc. are typical examples of well-known exotic options (See e.g. Fouque et al. (2011) with references therein for more details).

4.1.1 Black-Scholes formula

In the history of quantitative finance, the Black-Scholes formula is viewed as a benchmark for option pricing on assets. Several approaches of deriving this formula exist in the literature; here we first use the partial differential approach by following closely the idea of Wilmott (2013). In the sense of Black and Scholes (1973), the financial market model consists of a risk-free asset whose price, denoted by $(A_t)_{t\in[0,T]}$, verifies the following ordinary differential equation:

$$dA_t = rA_t dt, \quad A_0 = 1, \tag{4.1}$$

where r is a positive constant interest rate. The solution to (4.1) is given by $A_t = e^{rt}$. The process $(A_t)_{t\in[0,T]}$ is also commonly known in finance as *"money in the bank"*. The second model describes the dynamics of risky assets $(X_t)_{t\in[0,T]}$ defined by the following geometric Brownian motion:

$$dX_t = \eta X_t dt + \sigma X_t dB_t, \tag{4.2}$$

where $(B_t)_{t\in[0,T]}$ is a standard Brownian motion that represents the source of randomness of the risky asset defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, η and σ are positive constants that represent respectively the drift and volatility of the infinitesimal return dX_t/X_t. Summarizing the financial market model under the Black-Scholes settings, we have

$$\begin{cases} dA_t = rA_t dt, \\ dX_t = \eta X_t dt + \sigma X_t dB_t. \end{cases} \tag{4.3}$$

Definition 4.1. *A portfolio* or *trading strategy* is a pair (φ_t^0, φ_t), where φ_t^0 and φ_t are adapted processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that represent the amount of assets A_t and X_t respectively owned by an investor at time t. The portfolio value denoted by Π_t can be expressed as

$$\Pi_t = \varphi_t^0 A_t + \varphi_t X_t.$$

This portfolio is said to be admissible if its value Π_t is bounded below almost surely. In addition, a portfolio is said to be self-financing if the change of the value of the portfolio depends on the change of risk-free and the risky assets only, that is

$$d\Pi_t = \varphi_t^0 dA_t + \varphi_t dX_t.$$

Definition 4.2. Let Π_t be the value of the portfolio (φ_t^0,φ_t) at time $t \in [0,T]$, $T > 0$. There is an arbitrage opportunity in a financial market if the following conditions hold:

(c_1) $\Pi_0 = 0$

(c_2) $\Pi_T \geq 0$ and $\mathbb{P}\big[\omega : \Pi_T(\omega) > 0\big] > 0, \quad$ a.s.

We may note that the non-arbitrage principle requires that the riskless portfolio must grow exponentially at the risk-free rate $r > 0$, that is, must verify the following ordinary differential equation:

$$d\Pi_t = r\Pi_t dt. \tag{4.4}$$

Now to find the Black-Scholes equation, let $P = (P(t, X_t))_{t\in[0,T]}$ be the price at time t of an European-style option written on a stock which expires at the maturity date T. Call options and the underlying asset price X_t are positively correlated; put options and X_t are negatively correlated. One may use this phenomena to construct a special portfolio that consists of the option price and the short underlying asset position (given by $-\varphi_t X_t$) as follows:

$$\Pi_t = P(t,X_t) - \varphi_t X_t \tag{4.5}$$

and by self-financing, we have

$$d\Pi_t = dP(t,X_t) - \varphi_t dX_t.$$

After applying the standard Itô's formula (See e.g. Fouque et al. (2011, Section 1.1.4)) on the option price process $P = P(t, X_t)$, we get the following stochastic differential equation

$$dP = \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial X_t}dX_t + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial X_t^2}dt.$$

Consequently

$$d\Pi_t = \frac{\partial P}{\partial t}dt + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial X_t^2}dt + \left(\frac{\partial P}{\partial X_t} - \varphi_t\right) dX_t. \tag{4.6}$$

The next step is to eliminate risk in the portfolio, commonly known in quantitative finance as *"dynamic hedging"* as well explained by Kassouf and Thorp (1967). This can be done by choosing the quantity φ_t in (4.6) as

$$\varphi_t = \frac{\partial P}{\partial X_t}, \tag{4.7}$$

which obviously yields a risk-free portfolio whose dynamics verifies the following differential equation:

$$d\Pi_t = \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial X_t^2}\right) dt. \tag{4.8}$$

Since the dynamics of portfolio process must satisfy the differential equation (4.4) to avoid arbitrage, then by plugging (4.5) and (4.8) into Equation (4.4) yields the following parabolic partial differential equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial X_t^2} + rX_t \frac{\partial P}{\partial X_t} - rP = 0. \tag{4.9}$$

Equation (4.9) is well-known as the *"Black-Scholes partial differential equation"* and can shortly be written as

$$\mathcal{A}_0 P(t,X_t) = 0, \tag{4.10}$$

where $\mathcal{A}_0$ is the differential operator defined by

$$\mathcal{A}_0 = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2}{\partial X_t^2} + rX_t \frac{\partial}{\partial X_t} - r\,\cdot\,. \tag{4.11}$$

By associating a payoff function $h(T,X_T)$ to Equation (4.10) at the maturity time T yields the Black-Scholes terminal value problem. For example, when the payoff function is $h(T,X_T) = (X_T - S)^+$ (that is the case of European call option), where S is a strike price, then the Black-Scholes terminal problem is given by

$$\begin{aligned} &\mathcal{A}_0 P(t,X_t) = 0 \\ &h(T,X_T) = (X_T - S)^+ \end{aligned} \tag{4.12}$$

that yields the following solution

$$P(t,X_t) = X_t \mathcal{N}(d_+(t,T,X_t)) - Se^{-r(T-t)} \mathcal{N}(d_-(t,T,X_t)),$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution and

$$d_+(t,T,X_t) = \frac{\ln(\frac{X_t}{S}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_-(t,T,X_t) = d_+(t,T,X_t) - \sigma\sqrt{T-t}.$$

See e.g. Black and Scholes (1973) for more details and Fouque et al. (2011) for derivations of Black-Scholes boundary value problems for different vanilla and exotic options.

4.1.2 Limitation of the Black-Scholes formula

The Black-Scholes formula was recognised as an excellent model by both practitioners and researchers for pricing and hedging derivatives, and has marked the history of quantitative finance. It was also rapidly adapted to different options and financial market models. The main drawback of the model is by assuming the log-return volatility to be constant. This assumption was proven to be unrealistic and inconsistent with data. An example of this can be observed in historical and implied volatility which are not constants. See e.g. Dupire (1994), Derman and Kani (1994) with references therein.

A solution to this problem was to replace the constant volatility with a stochastic process resulting in what is now known as *"stochastic volatility modelling"*. See e.g. Hull and White (1987) and Heston (1993) for more details. The Heston model is one of popular stochastic volatility processes used in finance. Its popularity is due to the positiveness of volatility among several other features.

4.2 Stochastic Volatility Modelling

Several stochastic volatility models were suggested in the literature. The Heston model is more popular due to its positiveness among many other properties. This section introduces a general form of Heston-type model under the standard Brownian motion.

4.2.1 Generalisation of the standard Heston-type model

Let $(Y_t)_{t\geq 0}$ be the solution the stochastic differential equation:

$$dY_t = f(t,Y_t)dt + \nu\sqrt{Y_t}d\tilde{B}_t \tag{4.13}$$

where $(\tilde{B}_t)_{t\geq 0}$ is a standard Brownian motion and $f(t,y)$ is the drift function. For the stochastic process $(Y_t)_{t\geq 0}$ to exist, the drift $f(t,y)$ must satisfy the Lipschitz condition, that is, for all $y_1,\ y_2 \in \mathbb{R}$, there exists a constant K such that

$$\big|f(t,y_1) - f(t,y_2)\big| \leq K\big|y_1 - y_2\big|.$$

The positiveness of stochastic process $(Y_t)_{t\geq 0}$ can be discussed by referring to Hu et al. (2008, Theorem 2.1). With this, one may construct a financial market model that consists of a risk-free asset $(A_t)_{t\geq 0}$ that satisfies (4.1) and risky asset $(X_t)_{t\geq 0}$ that verifies the following geometric Brownian motion:

$$dX_t = \eta X_t dt + \sigma(Y_t)X_t dB_t, \tag{4.14}$$

where $\eta > 0$ and $\sigma(Y_t)$ are respectively the positive constant drift and stochastic volatility of the infinitesimal return dX_t/X_t. The stochastic process $(Y_t)_{t\geq 0}$ satisfies (4.13). For the stochastic process $(X_t)_{t\geq 0}$ to be well-defined, the following conditions must hold:

$$\int_0^t \mathbb{E}[\sigma^2(Y_s)]ds < \infty \quad \text{and} \quad \int_0^t \mathbb{E}\left[(\sigma(Y_s)X_s)^2\right] ds < \infty.$$

In addition, the Brownian motions $(B_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ are assumed to be correlated, that is, there exists $\rho \in [-1,1]$ such that $\mathbb{E}\big[B_t\tilde{B}_t\big] = \rho t$. This holds if there exists an independent Brownian motion $(\tilde{V}_t)_{t\geq 0}$ such that

$$B_t = \rho\tilde{B}_t + \sqrt{1-\rho^2}\tilde{V}_t. \tag{4.15}$$

The financial market model can be summarised as

$$\begin{cases} dA_t = rA_t dt, \\ dX_t = \eta X_t dt + \sigma(Y_t) X_t dB_t, \\ dY_t = f(t,Y_t)dt + \nu\sqrt{Y_t} d\tilde{B}_t \\ B_t = \rho \tilde{B}_t + \sqrt{1-\rho^2}\tilde{V}_t \end{cases} \tag{4.16}$$

4.2.2 Option pricing under stochastic volatility

Partial Differential Equation Approach

In this section, we closely follow Wilmott (2013) and Fouque et al. (2011) to extend the Black-Scholes formula under the stochastic volatility model (4.13). Due to the fact that there are two sources of randomness, it follows that hedging must be done on both stock price and volatility processes.

Let $P_1 = P_1(t,T_1,X_t,Y_t)$ and $P_2 = P_2(t,T_2,X_t,Y_t)$ be the option price processes written on a stock at maturity dates T_1 and T_2 respectively. The option price process $P_2(t,T_2,X_t,Y_t)$ will enable hedging of risk on volatility. We construct a self-financing portfolio that consists of a triplet $(\varphi_t^1, -\varphi_t^2, -\varphi_t)$ with its portfolio value Π_t that verifies

$$d\Pi_t = \varphi_t^1 dP_1 - \varphi_t dX - \varphi_t^2 dP_2.$$

To avoid cumbersome notations, we set $X_t = x$ and $Y_t = y$. Then by applying the bi-dimensional Itô formula (see Fouque et al. (2011, Section 1.9.1)) to the process Π_t, one may obtain

$$\begin{aligned} d\Pi_t = &\left(\varphi_t^1 \left[\frac{\partial}{\partial t} + \mathcal{A}_{(x,y)}\right] P_1 - \varphi\eta x - \varphi_t^2 \left[\frac{\partial}{\partial t} + \mathcal{A}_{(x,y)}\right] P_2\right) dt \\ &+ \left(x\sigma(y)\left[\varphi_t^1 \frac{\partial P_1}{\partial x} - \varphi_t^2 \frac{\partial P_2}{\partial x} - \varphi_t\right]\right. \\ &\qquad \left. +\rho\nu\sqrt{y}\left[\varphi_t^1\frac{\partial P_1}{\partial y} - \varphi_t^2 \frac{\partial P_2}{\partial y}\right]\right) dB_t \\ &+ \nu\sqrt{(1-\rho^2)y}\left[\varphi_t^1 \frac{\partial P_1}{\partial y} - \varphi_t^2\frac{\partial P_2}{\partial y}\right] d\tilde{V}_t, \end{aligned} \tag{4.17}$$

where $\mathcal{A}_{(x,y)}$ is the differential operator defined by

$$\mathcal{A}_{(x,y)} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \rho\nu\sqrt{y}\sigma(y)x\frac{\partial^2}{\partial x \partial y} + \frac{\nu^2}{2}y\frac{\partial^2}{\partial y^2} + \eta x\frac{\partial}{\partial x} + f(y)\frac{\partial}{\partial y}.$$

By performing dynamic hedging from the sources of randomness B_t and $\tilde{V}_t$, we may choose φ and φ_t^1 as follows:

$$\begin{cases} \varphi = \varphi_t^1 \dfrac{\partial P_1}{\partial x} - \varphi_t^2 \dfrac{\partial P_2}{\partial x} \\ \varphi_t^1 = \varphi_t^2 \dfrac{\partial P_2}{\partial y}\left(\dfrac{\partial P_1}{\partial y}\right)^{-1}, \end{cases}$$

provided that $\frac{\partial P_1}{\partial y} \neq 0$. On the other hand, the dynamics of the riskless portfolio must satisfy $d\Pi_t = r\Pi_t dt$ to avoid arbitrage as mentioned earlier. This yields the following:

$$\begin{aligned} &\frac{\dfrac{\partial P_1}{\partial t} + \mathcal{A}_{(x,y)}P_1 - (\eta - r)x\dfrac{\partial P_1}{\partial x} - rP_1}{\dfrac{\partial P_1}{\partial y}} \\ &= \frac{\dfrac{\partial P_2}{\partial t} + \mathcal{A}_{(x,y)}P_2 - (\eta - r)x\dfrac{\partial P_2}{\partial x} - rP_2}{\dfrac{\partial P_2}{\partial y}}. \end{aligned} \tag{4.18}$$

This equation has two unknowns, P_1 and P_2. As the price process P_1 and P_2 have two different maturity dates, then there exists an option price $P = P(t,T,X_t)$ that does not depend on none of maturity dates T_1 and T_2 which is equal to a function, let us say $P^* = P^*(t,x,y)$, that does not depend neither on T_1 nor T_2. That means

$$\frac{\dfrac{\partial P}{\partial t} + \mathcal{A}_{(x,y)}P - (\eta - r)x\dfrac{\partial P}{\partial x} - rP}{\dfrac{\partial P}{\partial y}} = P^*. \tag{4.19}$$

The function $P^* = P^*(t,x,y)$ can be chosen as (See Wilmott (2013) for the explanation about this choice)

$$P^*(t,x,y) = -f(t,y) + \nu\sqrt{y}q(t,x,y), \tag{4.20}$$

where the function $q(t,x,y)$ can be viewed as a total risk premium and it is given by

$$q(t,x,y) = \rho\frac{\eta - r}{\sigma(y)} + \gamma(t,x,y)\sqrt{1-\rho^2},$$

with $\gamma(t,x,y)$ being an arbitrary function that can be considered as a volatility risk premium. Plugging the expression (4.20) into (4.19) yields the following partial differential equation:

$$\begin{aligned}\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2(y)x^2\frac{\partial^2 P}{\partial x^2} + rx\frac{\partial p}{\partial x} - rP + \rho\nu\sqrt{y}x\sigma(y)\frac{\partial^2 P}{\partial x\partial y}\\ + \frac{\nu^2}{2}y\frac{\partial^2 P}{\partial y^2} + f(t,y)\frac{\partial P}{\partial y} - \nu\sqrt{y}q\frac{\partial P}{\partial y} = 0.\end{aligned} \tag{4.21}$$

The partial differential equation (4.21) can be grouped into four terms as:

$$\mathcal{A}_0P + \mathcal{A}_yP + \rho\nu\sqrt{y}x\sigma(y)\frac{\partial^2 P}{\partial x\partial y} - \nu\sqrt{y}q\frac{\partial P}{\partial y} = 0, \tag{4.22}$$

where $\mathcal{A}_0P$ is the standard Black-Scholes equation with volatility $\sigma(Y_t)$, the second term $\mathcal{A}_yP$ is an infinitesimal generator of the stochastic volatility $(Y_t)_{t\geq 0}$, the third term represents the correlation and the last term may be viewed as a premium. In general, the partial differential equation (4.21) or (4.22) can be written shortly as

$$\mathcal{A}P = 0, \tag{4.23}$$

where $\mathcal{A}$ is an operator defined by

$$\begin{aligned}\mathcal{A} = \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} - \nu\sqrt{y}q\frac{\partial}{\partial y} + \rho\nu\sqrt{y}x\sigma(y)\frac{\partial^2}{\partial x\partial y}\\ + \frac{1}{2}\sigma^2(y)x^2\frac{\partial^2}{\partial x^2} + \frac{\nu^2}{2}y\frac{\partial^2}{\partial y^2} + f(t,y)\frac{\partial}{\partial y} - r\cdot\end{aligned} \tag{4.24}$$

Associating Equation (4.23) to a payoff function given as a final condition $P(T,x,y) = h(x)$, one may obtain a terminal value problem to which the solution can be found. The analytical solution to (4.23) is not always easy to find. In general, numerical techniques are used to solve the problem.

Risk-Neutral Approach

This approach consists of finding an equivalent martingale measure $\mathbb{Q}$ for which the discounted stock price process $(X_t^*)_{t\geq 0}$ defined by $X_t^* = A_t^{-1} X_t$ is martingale. The option price process $P(t,T,X_t)$ is then given by

$$P(t,T,X_t) = \mathbb{E}^*\left[e^{-r(T-t)}h(X_T)\middle|\mathcal{F}_t\right], \tag{4.25}$$

where $\mathbb{E}^*[\,\cdot\,]$ denotes the expectation taken under the risk neutral probability measure $\mathbb{Q}$. To find the probability $\mathbb{Q}$, we first need to observe that the discounted price process $(X_t^*)_{t\geq 0}$ is martingale when it is driven by a standard Brownian motion $(B_t^*)_{t\geq 0}$ defined by

$$B_t^\star = B_t + \int_0^t \frac{\eta - r}{\sigma(Y_s)} ds, \tag{4.26}$$

where $\frac{\eta - r}{\sigma(Y_t)}$ is well known in finance as the *"stochastic Sharp ratio"* or the *"market price of risk"*. For (4.26) to exist and to avoid outliers in the discounted process $(\tilde{X}_t)_{t\geq 0}$, it is natural that the volatility $\sigma(Y_s)$ must not be null. This shall be set as the following generic assumption.

Assumption 4.1. There exists a minimum value of the volatility $\sigma_{\min}$ such that for all $y > 0$, $\sigma(y) > \sigma_{\min} > 0$.

We may observe that the discounted stock price process will remain martingale although the independent Brownian motion is shifted by the expression $\int_0^t \gamma(s,X_s,Y_s)ds$. This means there exists a standard Brownian motion $(\tilde{V}_t^\star)_{t\geq 0}$ defined by

$$\tilde{V}_t^\star = \tilde{V}_t + \int_0^t \gamma(s,X_s,Y_s)ds,$$

that keeps the discounted stock prices process martingale. The process $\gamma(s,X_s,Y_s)$ is also known in finance as the *"volatility risk premium"* and the total risk premium is the quantity $\rho\frac{\eta - r}{\sigma(Y_t)} + \sqrt{1-\rho^2}\gamma(t,X_t,Y_t)$.

On the other hand, from the classical Girsanov's theorem (see e.g. Fouque et al. (2011, Section 1.4.1)), the Brownian motions $(B_t^\star)_{t\geq 0}$ and $(\tilde{V}_t^\star)_{t\geq 0}$ are independent under the risk neutral probability measure $\mathbb{Q}$ that verifies the following differential equation:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Bigg(-\frac{1}{2}\int_0^T \left(\frac{(\eta-r)^2}{\sigma^2(Y_s)} + \gamma^2(s,X_s,Y_s)\right) ds \\ - \int_0^T \frac{\eta-r}{\sigma(Y_s)} dB_s^* - \int_0^T \gamma(s,X_s,Y_s) d\tilde{V}_s^\star\Bigg). \tag{4.27}$$

4.2.3 Free-arbitrage property

From the well-known *"fundamental theorem of option pricing"*, a financial market defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is free of arbitrage if there exists an equivalent martingale measure $\mathbb{Q}$ for which the discounted price is martingale. Moreover, the market model is complete if and only if the measure $\mathbb{Q}$ is unique (Fouque et al.; 2011). This theorem yields the following result.

Proposition 4.1. *The standard Heston-type model (4.16) is free of arbitrage.*

Proof. The equivalent martingale measure $\mathbb{Q}$ exists indeed and it is given in the differential equations (4.27). See also Bezborodov et al. (2019, Theorem 4) for additional comments. □

4.3 An Example: Standard Heston Model

4.3.1 Standard Heston model

The standard Heston model corresponds to the market model (4.16) with stochastic volatility $\sigma(Y_t) = \sqrt{Y_t}$ and the drift $f(t,Y_t) = \theta(\mu - Y_t)$. Hence the stochastic process $(Y_t)_{t\geq 0}$ is called *"instantaneous variance"* and verifies the following differential equation:

$$dY_t = \theta(\mu - Y_t)dt + \nu\sqrt{Y_t}d\tilde{B}_t, \tag{4.28}$$

where θ is a positive parameter that represents the speed of reversion of the stochastic process $(Y_t)_{t\geq 0}$ towards its long-run mean $\mu > 0$, the parameter $\nu > 0$ is the volatility of the stochastic process $(Y_t)_{t\geq 0}$, and $(\tilde{B})_{t\geq 0}$ is the standard Brownian motion. The stochastic process $(Y_t)_{t\geq 0}$ is commonly known in financial mathematics as the *"Cox-Ingersoll-Ross process"* and was initially introduced by Cox et al. (1985) to model the dynamics of interest rates. This process is strictly positive provided that the Feller condition $\nu^2 \leq 2\theta\mu$ holds, it is mean reverting, stationary and ergodic. For more details, see e.g. Chou and Lin (2006) and Guo (2008) with references therein. Hence, the financial market model reads

$$\begin{cases} dA_t = rA_t dt, \\ dX_t = \eta X_t dt + \sqrt{Y_t} X_t dB_t, \\ dY_t = \theta(\mu - Y_t)dt + \nu\sqrt{Y_t} d\tilde{B}_t \\ B_t = \rho\tilde{B}_t + \sqrt{1-\rho^2}\tilde{V}_t. \end{cases} \tag{4.29}$$

4.3.2 Option pricing formula

We follow Fouque et al. (2011) for the derivation of the option price formula. The partial differential equation (4.21) under the Heston model is given by

$$\begin{aligned} \frac{\partial P}{\partial t} + rx\frac{\partial P}{\partial x} + \theta(\mu - y)\frac{\partial P}{\partial y} + \rho\nu yx\frac{\partial^2 P}{\partial x \partial y} \\ + \frac{1}{2}yx^2\frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\nu^2 y\frac{\partial^2 P}{\partial y^2} - rP = 0. \end{aligned} \tag{4.30}$$

The next step is to write the above parabolic partial differential equation in terms of a Green function in order to solve the parabolic partial differential equation (4.30). For this, we need to start with the following changes of variables:

$$\begin{aligned} \tau(t) &= T - t \\ z(t,x) &= r\tau(t) + \log x \\ y &= y, \end{aligned} \tag{4.31}$$

then it follows that

$$P(t,x,y) = P(T - \tau, \exp(z(t,x) - r\tau(t)), y), \tag{4.32}$$

and at the maturity date T, the payoff function is

$$P(T,x,y) = P(T, \exp(z(T,x)),y) = h(\exp(z(t,x))). \tag{4.33}$$

In what follows, we use z for $z(t,x)$ and τ for $\tau(t)$ to simplify the notations. Assume now that the option price is proportional to a factor $\exp(-r\tau)$, that is, there exist an option price $\tilde{P}(\tau, z, y)$ such that

$$P(t,x,y) = \exp(-r\tau)\tilde{P}(\tau, z, y),$$

with $P(T,x,y) = \tilde{P}(0, z, y) = h(\exp(z)) = \tilde{h}(z)$. Then the following holds:

$$\begin{aligned}
\frac{\partial P}{\partial t} &= \exp(-r\tau)\left[\frac{\partial \tilde{P}}{\partial \tau} - r\tilde{P}\right] \\
\frac{\partial P}{\partial x} &= \exp(-r\tau)\frac{1}{x}\frac{\partial \tilde{P}}{\partial z} \\
\frac{\partial^2 P}{\partial x^2} &= -\exp(-r\tau)\frac{1}{x^2}\left[\frac{\partial \tilde{P}}{\partial z} - \frac{\partial \tilde{P}^2}{\partial z^2}\right] \\
\frac{\partial^2 P}{\partial x \partial y} &= \exp(-r\tau)\frac{1}{x}\frac{\partial^2 \tilde{P}}{\partial z \partial y} \\
\frac{\partial P}{\partial y} &= \exp(-r\tau)\frac{\partial \tilde{P}}{\partial y} \\
\frac{\partial^2 P}{\partial y^2} &= \exp(-r\tau)\frac{\partial^2 \tilde{P}}{\partial y^2}.
\end{aligned} \tag{4.34}$$

Substituting equations (4.34) into (4.30) yield

$$\begin{aligned}
-\frac{\partial \tilde{P}}{\partial \tau} + \frac{1}{2}y\left(\frac{\partial^2 \tilde{P}}{\partial z^2} - \frac{\partial \tilde{P}}{\partial z}\right) + \theta(\mu - y)\frac{\partial \tilde{P}}{\partial y} \\
+ \frac{1}{2}\nu^2 y\frac{\partial^2 \tilde{P}}{\partial y^2} + \rho\nu y\frac{\partial^2 \tilde{P}}{\partial z \partial y} = 0,
\end{aligned} \tag{4.35}$$

and the pricing problem can be written shortly as

$$\begin{aligned} \mathcal{A}\tilde{P}(\tau, z, y) &= 0 \\ \tilde{P}(0,z,y) &= \tilde{h}(z) \end{aligned} \tag{4.36}$$

where $\mathcal{A}$ is the differential operator defined by

$$\begin{aligned} \mathcal{A} = -\frac{\partial}{\partial \tau} + \frac{1}{2}y\left(\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\right) + \theta(\mu - y)\frac{\partial}{\partial y} \\ + \frac{1}{2}\nu^2 y \frac{\partial^2}{\partial y^2} + \rho\nu y \frac{\partial^2}{\partial z \partial y}. \end{aligned} \tag{4.37}$$

To solve the option pricing problem (4.36), we may now use the Green's function as a tool. We first need to note the following definition.

Definition 4.3. Let $\mathcal{A}$ be a differential operator, linear in the variable z and let $G = G(\tau,z,y;\varphi)$ be a function in variables τ, z, y and φ. Then G is called a Green's function for $\mathcal{A}$ if the following equation is satisfied

$$(\mathcal{A}^{-1})_{z,\tilde{z}} = G(\tau,z,y;\tilde{z}),$$

where $\mathcal{A}^{-1}$ is the inverse operator of $\mathcal{A}$; the subscripts on $\mathcal{A}^{-1}$ means that $\mathcal{A}^{-1}$ acts on z and $\tilde{z}$. In addition, the function $G(\tau,z,y;\tilde{z})$ verifies

$$\mathcal{A}G = \varsigma(z - \tilde{z}),$$

with $\varsigma(\cdot)$ is the dirac delta function.

Therefore, the option price $\tilde{P}(\tau(t), z, y)$ can be represented as

$$\tilde{P}(t,z,y) = \exp(r\tau)\int_{\mathbb{R}} G(\tau,z-\tilde{z},y)\tilde{h}(\tilde{z})d\tilde{z} \tag{4.38}$$

and the option price problem (4.36) can be reformulated as

$$\begin{aligned} \mathcal{A}G(\tau,z,y) &= 0 \\ G(0,z,y) &= \varsigma(z - \tilde{z}). \end{aligned} \tag{4.39}$$

By applying the Fourier transform to the option price problem (4.39), we obtain the following result

$$\begin{aligned} \widehat{\mathcal{A}}\ \widehat{G}(\tau,z,y) &= 0 \\ \widehat{G}(0,z,y) &= 1, \end{aligned} \tag{4.40}$$

where $\widehat{\mathcal{A}}$ and $\widehat{G}(\tau,z,y)$ are Fourier transforms of $\mathcal{A}$ and $G = G(\tau,z,y)$ respectively. The operator $\widehat{\mathcal{A}}$ is given by

$$\widehat{\mathcal{A}} = -\frac{\partial}{\partial \tau} + \frac{1}{2}(-k^2 + ik)y + \frac{1}{2}\nu^2 y \frac{\partial^2}{\partial y^2} + (\theta\mu - (\theta + \rho\nu ik)y)\frac{\partial}{\partial y}, \tag{4.41}$$

where we have used the fact that the Fourier transform of $\tilde{P}$ is $\widehat{\tilde{P}}$ defined by

$$\widehat{\tilde{P}}(\tau,z,y) = \frac{\exp(r\tau)}{2\pi}\int_{\mathbb{R}} \exp(-ikz)\widehat{G}(\tau,k,y)\widehat{\tilde{h}}(k)dk. \tag{4.42}$$

Assume that the Fourier transform of the Green's function $G(\tau,k,y)$, denoted by $\widehat{G}(\tau,k,y)$, takes the following form:

$$\widehat{G}(\tau,k,y) = \exp\left[A_1(\tau,k) + yA_2(\tau,k)\right]. \tag{4.43}$$

Then the problem (4.40) is transformed into the following system of two ordinary differential equations

$$\begin{cases} \dfrac{dA_1}{d\tau}(\tau,k) = \theta\mu A_2(\tau,k) \\ \quad A_1(0,k) = 0, \\ \dfrac{dA_2}{d\tau}(\tau,k) = \dfrac{1}{2}\nu^2 A_2^2(\tau,k) - (\theta + \rho\nu ik)A_2(\tau,k) + \dfrac{1}{2}(-k^2 + ik) \\ \quad A_2(0,k) = 0. \end{cases} \tag{4.44}$$

The second equation of the system (4.44) is a Riccati equation (See Rouah (2015) for more details). The solution for $A_1(\tau,k)$ can be obtained by integration on both side of the first equation of (4.44). After tedious computations, an analytical solution can be found and is given by

$$\begin{cases} A_1(\tau,k) = \dfrac{\theta\mu}{\nu^2}\left((\theta+\rho ik\nu+a_1(k))\tau - 2\log\left(\dfrac{1-a_2(k)\exp(\tau a_1(k))}{1-a_2(k)}\right)\right) \\ A_2(\tau,k) = \dfrac{\theta+\rho ik\nu+a_1(k)}{\nu^2}\left(\dfrac{1-\exp(\tau a_1(k))}{1-a_2(k)\exp(\tau a_1(k))}\right), \end{cases} \tag{4.45}$$

where

$$\begin{aligned} a_1(k) &= \sqrt{\nu^2(k^2-ik)+(\theta+\rho ik\nu)^2} \\ a_2(k) &= \frac{\nu+\rho ik\nu+a_1(k)}{\nu+\rho ik\nu-a_1(k)}. \end{aligned} \tag{4.46}$$

The following proposition gives a summary of the option price formula under the Heston model.

Proposition 4.2. *Consider the financial market model (4.28). Then the option price $P = P(t,x,y)$ written on a stock $X_t = x$ is given by*

$$P(t,x,y) = \frac{1}{2\pi A(\tau)}\int_{\mathbb{R}} \exp\left(A_1(\tau,k)+yA_2(\tau,k)-ikz\right)\widehat{\tilde{h}}(k)dk, \tag{4.47}$$

where

$$\begin{aligned} &A(\tau) = \exp(r\tau) \\ &\tau = T-t \\ &z = r\tau - \log x \\ &P(t,x,y) = \tilde{P}(t,z,y)A^{-1}(\tau) \end{aligned} \tag{4.48}$$

and

$$A_1(\tau,k) = \frac{\theta\mu}{\nu^2}\left((\theta+\rho ik\nu+a_1(k))\tau - 2\log\left(\frac{1-a_2(k)\exp(\tau a_1(k))}{1-a_2(k)}\right)\right)$$

$$A_2(\tau,k) = \frac{\theta+\rho ik\nu+a_1(k)}{\nu}\left(\frac{1-\exp(\tau a_1(k))}{1-a_2(k)\exp(\tau a_1(k))}\right).$$

The process $A(\tau)$ is the risk-free asset price. Note that some extra conditions must be taken into consideration for the integral in Equation (4.42) or (4.47) to be well-defined and to facilitate numerical derivations. For an

European call option, Kahl and Jäckel (2005) find an explicit formula which is a consequence of Proposition 4.2 given as follows.

Proposition 4.3. *Consider the Heston model (4.47) and let $p(t,X_t,Y_t)$ be the option price written on the stock $X_t = x$ with the payoff function being given by $h(x) = (x - S)^+$, where S is a positive constant that represents the strike price. Then the option price $P(t,x,y)$ under the Heston model is given by*

$$P(t,x,y) = \frac{A(\tau)}{2\pi} \int_{\mathbb{R}} (uc_\infty)^{-1} \mathfrak{I}\left(\frac{-\log u + c_\infty i k_{im}}{c_\infty}\right) du, \tag{4.49}$$

where

$$\begin{aligned}
\mathfrak{I}(k) &= \frac{S^{1+ik} \exp\left(A_1(\tau,k) + vA_2(\tau,k) - izk\right)}{ik - k^2} \\
k &= k_{re} + ik_{im} \\
c_\infty &= \frac{v - \theta\mu\tau\sqrt{1-\rho^2}}{\nu} \\
k_{re} &= \frac{-\log x}{c_\infty}
\end{aligned} \tag{4.50}$$

Although the standard Heston model is well accepted by both practitioners and researchers, the exact option pricing formula (4.47) is not easy to deal with even for the simplest options such as European option that is given by (4.49). This is a motivation for practitioners and researchers to go either for approximations (See e.g. Alòs and Ewald (2008)) or to use risk neutral approach given by (4.25).

On the other hand, the standard Heston model does not capture dependency features within the volatility time series. The observation from log-returns on a given security or option prices suggest roughness of volatility time series. This will be discussed in our next chapter 5.

Chapter 5

Roughness and multifractality properties of volatility time series

Standard stochastic volatility models (including the Heston model discussed previously) are developed with the assumption that their random parts are driven by a standard Brownian motion. This means that the Hurst parameter $H = 0.5$ and the volatility time series do not display memory. Throughout this chapter, we demonstrate that this is not always the case within volatility time series.

Firstly, we consider selected major stock market indices since 2012 and estimate their realised volatility. We apply the multifractal detrended fluctuation analysis (**MF-DFA**) technique and we find that the Hurst parameter is of order 0.5 to 0.8. In this case, we may say that the volatility displays long-range dependence. Similar results were found by Comte and Renault (1998), Cajueiro and Tabak (2008), Chronopoulou and Viens (2010), Power and Turvey (2010), Abuzayed et al. (2018), Cont and Das (2022) with references therein. In addition, we found that the volatility displays the multifractal property in general. The source of this multifractality is mostly due to broad distributions of the volatility time series.

When using the microstructure noise index (**MNI**) approach, we find that the log-volatility are rough with Hurst parameter of order 0.2 to 0.3. Similar results were discussed by Gatheral et al. (2018), Livieri et al. (2018) with references therein.

5.1 Multifractality and roughness of realised volatility

5.1.1 Multifractal detrended fluctuation analysis

First of all, recall that the volatility time series is not observable; it has to be estimated from the stock price return or option prices. Throughout this chapter, we use the realised volatility (Andersen et al.; 2003) as a proxy of volatility. Denote $(X_{t_i})_{i=0,\cdots,N}$ the daily observed stock prices, and defined the log-return (r_{t_i}) as

$$r_{t_i} = \log X_{t_i} - \log X_{t_{i-1}}.$$

The (daily) realised volatility $(\sigma_{t_i})_{i=0,\cdots,N}$ is given by

$$\sigma_{t_i} = \left(\sum_{i=0}^{N} r_{t_i}^2 \right)^{\frac{1}{2}}.$$

The multifractality properties of realised volatility can be analysed by using the so-called *"generalised Hurst exponent"* which will be discussed later in this section. We use the multifractal detrended fluctuation analysis (MF-DFA) method previously introduced by Kantelhardt et al. (2002) as an extension of the standard detrended fluctuation technique. For this, we consider the realised volatility time series for N trading days on the time interval $[0,T]$ given by $(\sigma_{t_i})_{i=0,\cdots,N}$, with $\sigma_{t_N} = \sigma_T$ and denote $\sigma_{t_i} = \sigma_i$ for simplicity. The MF-DFA technique consists of the following steps.

1. Determine the time series $(V_i)_{i=0,\cdots,N}$ as

$$V_i = \sum_{k=0}^{i} \big(\sigma_k - \mu_i\big), \tag{5.1}$$

where $\mu_i = \frac{1}{i} \sum_{k=1}^{i} \sigma_k$. This step can be done twice to capture time series with strong anti-correlation as observed by Kantelhardt et al. (2002). This means that one may determine again another time series as previously:

$$\tilde{V}_i = \sum_{k=0}^{i} \big(V_k - \tilde{\mu}_i\big), \tag{5.2}$$

where $(V_k)_{k=0,\cdots,N}$ is given by (5.1) and $\tilde{\mu}_i = \frac{1}{i}\sum_{k=1}^{i} V_k$.

2. Set $N_\delta = \lfloor \frac{N}{\delta} \rfloor$ with δ being different lags chosen such that $\delta < N/4$, and divide the time series $(V_i)_{i=0,\cdots,N}$ into N_δ equal segments of length δ that do not overlap. To cover the whole time interval $[0,T]$, we may start from t_0 to t_{N_s} (which is not necessarily equal to t_N) and from t_N to $t_{N-\delta N_\delta}$. This yields $2N_\delta$ segments.

3. For each segment $j = 0, \cdots, N_\delta$ and $j = N_\delta + 1, \cdots, 2N_\delta$ with $j = \delta i$, eliminate the trend of the volatility time series by fitting the polynomial $P_j^m(i)$ of order m, and calculate the variance as follows:

$$F^2(j,\delta) = \begin{cases} \frac{1}{\delta}\displaystyle\sum_{i=0}^{\delta} \Big(V_n - P_j^m(i)\Big)^2, & j = 0, \cdots, N_\delta, \\ \frac{1}{\delta}\displaystyle\sum_{i=0}^{\delta} \Big(V_{n'} - P_j^m(i)\Big)^2, & j = N_\delta + 1, \cdots, 2N_\delta \end{cases} \tag{5.3}$$

where $n = (j-1)\delta + i$ and $n' = N - (j - N_\delta)\delta + i$. The time series (V_n) and $(V_{n'})$ are determined using (5.1) or (5.2).

4. Next, calculate the fluctuation function $F_q(\delta)$ of order q as the Hölder mean (or generalised mean) of $F^2(j,\delta)$ with exponent $q \neq 0$ or as a logarithmic mean for $q = 0$ as given below

$$F_q(\delta) = \begin{cases} \left(\frac{1}{2N_\delta}\displaystyle\sum_{j=0}^{2N_\delta} \Big(F^2(j,\delta)^{\frac{q}{2}}\Big) \right)^{\frac{1}{q}} & \forall q \neq 0, \\ \exp\left[\frac{1}{4N_\delta}\displaystyle\sum_{j=0}^{2N_\delta} \log\big(F^2(j,\delta)\big) \right] & \forall q \to 0. \end{cases} \tag{5.4}$$

For $q = 2$, one may recover the classical detrended fluctuation analysis previously discussed by Peng et al. (1994).

5. Finally, determine the inherent scaling behavior of the fluctuation function $F_q(\delta)$. If the time series (σ_i) display long or short range power-law correlations, then the fluctuation function can be approximated as
$$F_q(\delta) \sim \delta^{\tilde{h}(q)}, \tag{5.5}$$
where $\tilde{h}(q) = 1 + h(q)$. The function $h(q)$ is commonly known as the *"generalised Hurst exponent"*. If $h(q)$ depends on q, then the original volatility time series $(\sigma_{t_i})_{i=0,\cdots,N}$ is said to be multifractal and if $h(q) :=$ H is constant, then $(\sigma_{t_i})_{i=0,\cdots,N}$ is said to be monofractal. In this later case, the variance $F^2(j,\delta)$ stays identical for all lags δ. If $q = 2$, then $h(2)$ is simply the Hurst parameter H initially introduced by Hurst (1951). If $0 < h(2) < 1/2$, the time series (σ_i) displays short-range dependence (or) and if $1/2 < h(2) < 1$, (σ_i) displays long-range dependence (or long memory) and if $h(q) = 1/2$, there is no dependency (or memory) at all. See Section 2.4.1 for definitions.

5.1.2 Sources of multifractality

There exists two main types of sources of multifractality (Kantelhardt et al.; 2002): the broad (or fat-tailed) probability density function (Type I) and temporal correlations of small and large fluctuations (Type II). The easiest way to recognise this, is by performing random shuffles of the original time series to eliminate correlations of type II. In this case, type I should be the only source of multifractality and the generalised Hurst exponent from the shuffled time series (denoted by $h_{\text{shuf}}(q)$) coincides with the original generalised Hurst exponent $h(q)$, that is $h_{\text{shuf}}(q) \sim h(q)$. If multifractality is only generated from the source of type II, then the generalised Hurst exponent is constant with $h_{\text{shuf}}(q) \sim 1/2$.

5.1.3 Microstructure noise index

When using the microstructure noise index (MNI) approach, Gatheral et al. (2018) defined the fluctuation function slightly differently for stochastic volatility as

$$F_q(\delta) = \frac{1}{N_\delta}\sum_{j=1}^{N_\delta}\Big|\log\sigma_j - \log\sigma_{j-1}\Big|^q. \tag{5.6}$$

Under the assumption that for some functions $h(q) > 0$ and $c(q) > 0$ depending on q, the limit $N_\delta^{qh(q)} F_q(\delta) \to c(q)$ as $\delta \to 0$ holds. In this case, $h(q)$ can be viewed as the regularity of Y. Particularly, if $(\log\sigma_j)$ behaves as a *fBm*, then $(\log\sigma_j)$ is monofractal and $h(q)$ is a constant. Now, by the law of large numbers,

$$F_q(\delta) \sim \mathbb{E}\Big[\Big|\log\sigma_\delta - \log\sigma_0\Big|^q\Big] = c(q)\delta^{\zeta(q)} \quad \text{as} \quad \delta \to \infty, \tag{5.7}$$

where $\zeta(q) = qh(q)$, and provided that the volatility time series (σ_{t_i}) satisfies the stationarity property. Consequently,

$$\log F_q(\delta) \sim \tilde{c}(q) + \zeta(q)\log\delta, \tag{5.8}$$

where $\tilde{c}(q) = \log c(q)$. Therefore, to find the Hurst parameter H one has to compute $F_q(\delta)$ for each arbitrary lag δ and analyse the log-log plot of $F_q(\delta)$ versus δ. Fit a straight line with slope qH, and deduce the value of H, since q is known. For more details, see Rosenbaum (2011) and Gatheral et al. (2018).

5.2 Financial data

To estimate the Hurst parameters of volatility, we retrieve from Yahoo finance daily adjusted closing prices for selected major world stock market indices (including some market indices of emerging economy countries) as shown in Table 5.1 from 10 February 2012 to 10 August 2022. We use the realised volatility of log-returns as our proxy[1] as shown in figures 5.1. The descriptive statistics of log-returns and realised volatility are summarised in tables 5.2

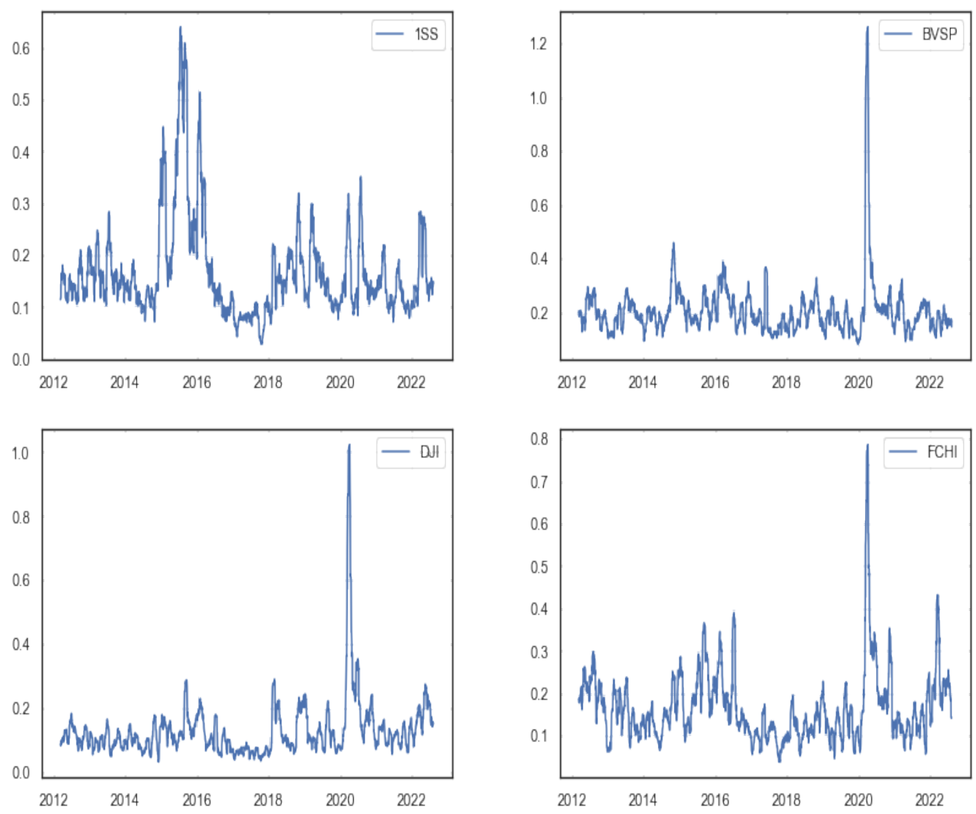
[1] Similarly, Gatheral et al. (2018) used the realised kernel, while Livieri et al. (2018) the implied volatility as proxies of volatility to estimate the Hurst parameter.

and 5.3 respectively.

Table 5.1: Selected world stock market indices

Code	Index Name	Code	Index Name
GSPC	S&P 500	BVSP	IBOVESPA
DJI	DOW JONES INDUSTRIAL AVERAGE	HSI	HANG SENG INDEX
IXIC	NASDAQ Composite	GDAXI	DAX PERFORMANCE-INDEX
FCHI	CAC 40	000001.SS	SSE COMPOSITE INDEX
FTSE	FTSE Index	J203	FTSE/JSE SA ALL SHARE INDEX

Figure 5.1: Realised Volatility of different stock market indices.



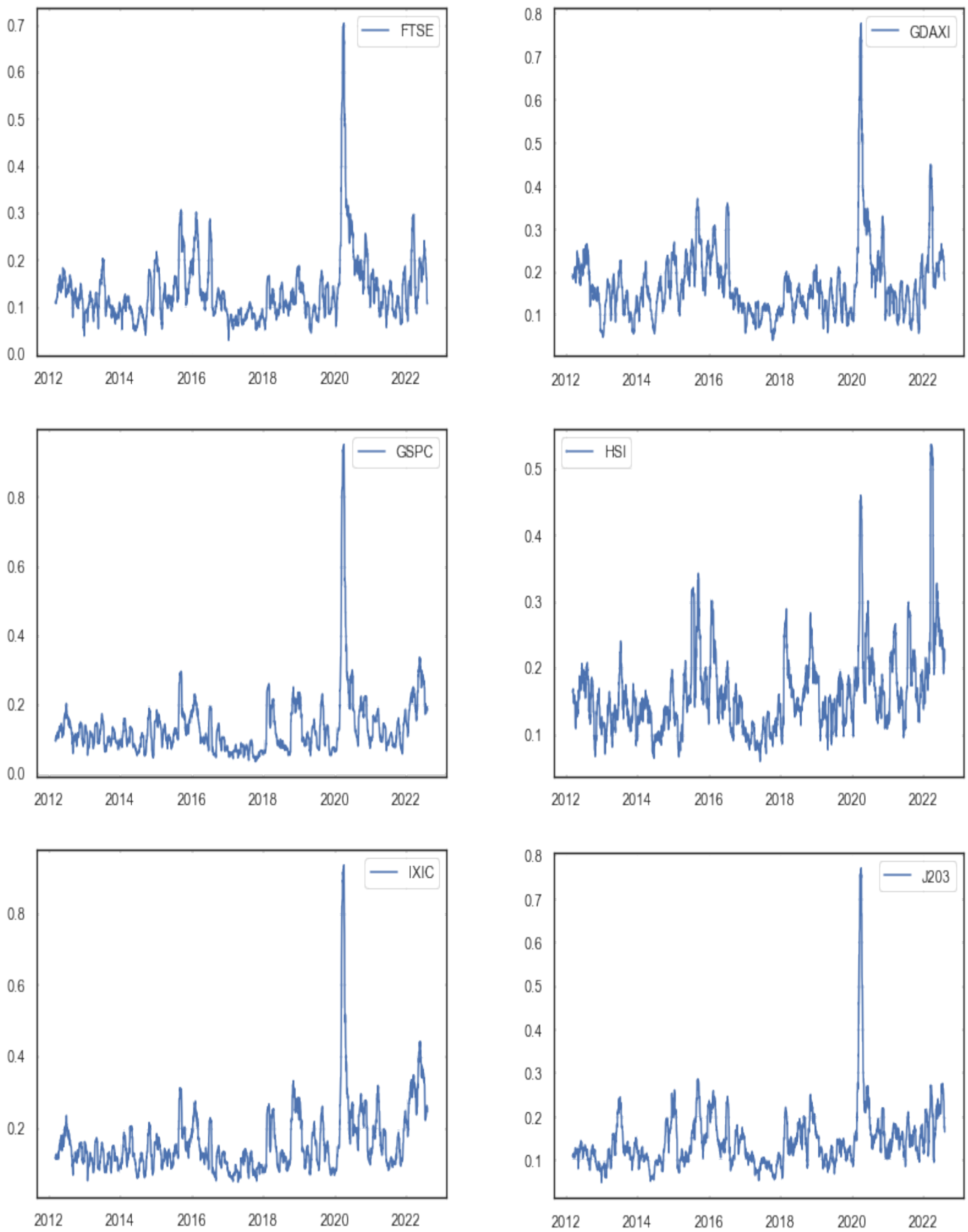

FTSE
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
2012
2014
2016
2018
2020
2022
GDAXI
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
2012
2014
2016
2018
2020
2022
GSPC
0.0
0.2
0.4
0.6
0.8
2012
2014
2016
2018
2020
2022
HSI
0.1
0.2
0.3
0.4
0.5
2012
2014
2016
2018
2020
2022
IXIC
0.2
0.4
0.6
0.8
2012
2014
2016
2018
2020
2022
J203
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
2012
2014
2016
2018
2020
2022

Table 5.2: Descriptive statistics of log-returns time series

	1SS	BVSP	DJI	FCHI	FTSE	GDAXI	GSPC	HSI	IXIC	J203.JO
Obs. (N)	2730	2730	2730	2730	2730	2730	2730	2730	2730	2730
Mean	0.000098	0.000153	0.000341	0.000238	0.000084	0.000227	0.000386	0.000040	0.000478	0.000222
SD	0.012452	0.014984	0.010362	0.012037	0.009727	0.012071	0.010381	0.011232	0.012123	0.010392
Minimum	−0.088732	−0.159930	−0.138418	−0.130983	−0.115117	−0.130549	−0.127652	−0.060183	−0.131492	−0.102268
Quartile 1	−0.004727	−0.007157	−0.003082	−0.005073	−0.004329	−0.004787	−0.003087	−0.005088	−0.003823	−0.004502
Median	0.000000	0.000000	0.000135	0.000322	0.000054	0.000214	0.000135	0.000000	0.000371	0.000000
Quartile 3	0.005527	0.007951	0.004725	0.006093	0.004702	0.006238	0.004908	0.005571	0.006403	0.005839
Maximum	0.060399	0.130223	0.107643	0.080561	0.086664	0.104143	0.089683	0.086928	0.089347	0.072615

Table 5.3: Descriptive statistics of realised volatility time series

	1SS	BVSP	DJI	FCHI	FTSE	GDAXI	GSPC	HSI	IXIC	J203.JO
Obs. (N)	2730	2730	2730	2730	2730	2730	2730	2730	2730	2730
Mean	0.167682	0.205616	0.129067	0.166250	0.133165	0.168208	0.132225	0.161986	0.161465	0.144931
SD	0.095204	0.108859	0.097148	0.086236	0.072547	0.082893	0.093203	0.064620	0.097745	0.072377
Minimum	0.028405	0.081198	0.031084	0.038256	0.028913	0.040708	0.033852	0.059390	0.049705	0.048198
Quartile 1	0.111049	0.154822	0.078266	0.110839	0.089989	0.115032	0.080299	0.119805	0.103349	0.103653
Median	0.141285	0.187444	0.106714	0.148921	0.116644	0.154365	0.109996	0.150051	0.133233	0.129049
Quartile 3	0.191033	0.230664	0.149400	0.200453	0.154843	0.200439	0.157668	0.185886	0.193088	0.168583
Maximum	0.639929	1.261800	1.023685	0.786337	0.702989	0.776279	0.952041	0.536219	0.933666	0.769343

5.3 Empirical results

In the process of applying the **MF-DFA** technique, we consider two main periods: **Period 1** from 10 February 2012 to 10 August 2022 and **Period 0**[2] prior the Covid-19 pandemic, that is, from 10 February 2012 to 10 March 2020. We choose different lags $\delta = 1, \cdots, 251$[3] and we eliminate the trend of the realised volatility time series by fitting $(V_i)_{i=0,\cdots,N}$ or $(\tilde{V}_i)_{i=0,\cdots,N}$ defined respectively by (5.1) and (5.2) to a second order polynomial $P_j^2(i),\ i = 0, \cdots, N$ and $j = \delta i$. The obtained fluctuation functions $F_q(\delta)$ and the generalised Hurst exponents defined respectively by (5.4) and (5.5) of all stock market indices depend on q as shown in figures 5.2, 5.4, 5.6, 5.8, 5.10, 5.12, 5.14, 5.16, 5.18 and 5.20. Thus, this shows that the volatility time series enjoy the multifractality property in general.

To determine the source of this multifractality, we analyse the mean of 1000 shuffled time series of the original realised volatility and deduce the gener-

[2]On 11 March 2020, the Word Health Organisation declared the Covid-19 as a global pandemic with over 80950 confirmed cases in over 110 countries.

[3]For $\delta > N/4$, the fluctuation function becomes unreliable.

alised Hurst exponent $h_{\text{shuf}}(q)$. We observe the following:

(a) For some financial markets such as **HSI** (See Figure 5.16, **Period 1**), $h_{\text{shuf}}(q)$ coincides with $h(q)$ almost everywhere. In this case, source of Type I is mostly dominated.

(b) Financial markets such as **FCHI** (See Figure 5.8, **Period 1**), $h_{\text{shuf}}(q)$ and $h(q)$ are different but depend on q. However, we observe from most of financial markets that $h_{\text{shuf}}(2) \sim 0.5$. In this case, the multifractility is due to Type I and Type II.

(c) Financial markets such as **FTSE** (See Figure 5.10, **Period 0**), $h_{\text{shuf}}(q)$ coincides with $h(q)$ for all $q > 0$ and $h_{\text{shuf}}(2) \sim 0.5$. In this case, Type I is dominated but the presence of Type II cannot be neglected.

We may conclude that the source of multifractality is mostly due to sources of Type I. However, Type II should not be neglected.

In addition, the volatility time series display long range dependency with Hurst exponent $h(2) := H$ of order 0.5 to 0.7 during **Period 0**. These values are higher during **Period 1** which includes the Covid-19 pandemic. See tables 5.3 to 5.12 for a summary of selected values of $h(q)$. It follows that the fluctuation function and generalised Hurst exponent also depend on the "event timelines".

When using the microstructure noise index (MNI) approach, the log-log plots of fluctuation functions $F_q(\delta)$ defined by (5.7) versus q were performed and yielded the estimated Hurst exponent obtained from $\zeta(q) \sim qH$ (See figures 5.3, 5.5, 5.7, 5.9, 5.11, 5.13, 5.15, 5.17, 5.19 and 5.21 for **Period 1** only). We find that the log-volatility are rough with Hurst exponent of order 0.2 to 0.3 prior and during the covid-19 pandemic as shown in tables 5.3 to 5.12. In this context, we may say that the volatility is rough as previously observed by Gatheral et al. (2018) and subsequent results.

Figure 5.2: **1SS** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q)$, $q \in [-15, 15]$ (**MF-DFA**).

Period 0 **Period 1**

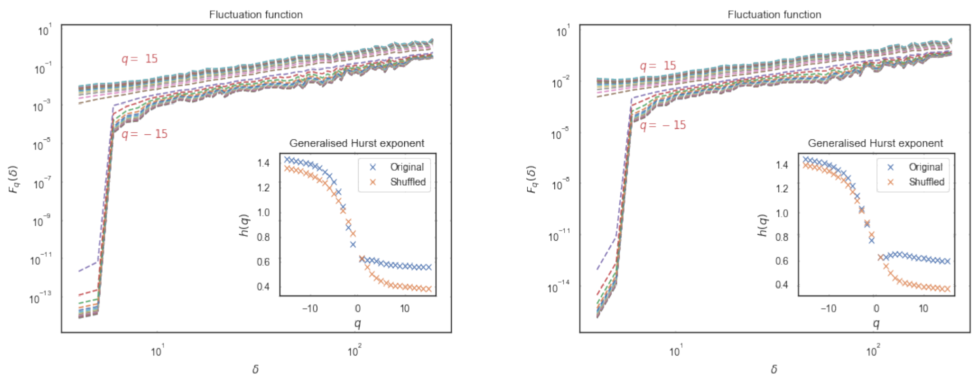


Figure 5.3: **1SS** Fluctuation Function $F_q(\delta)$ and $\zeta(q)$, $q \in [0.5, 3]$ (**MNI**).

(A) $F_q(\delta)$ **(B)** $\zeta(q) \sim qH$

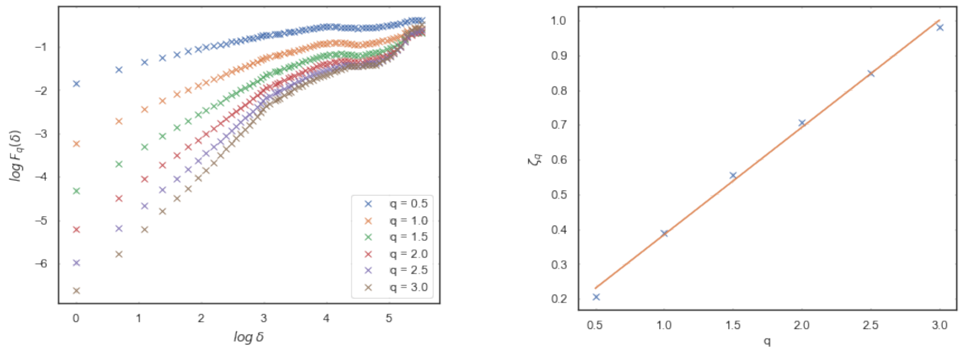


Table 5.4: **1SS** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1** (10/02/2012 to 10/08/2022)		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	1.421591	1.349413	–	1.442607	1.379425	–
$h(-10)$	1.386503	1.307803	–	1.401762	1.334572	–
$h(-6)$	1.292496	1.205044	–	1.288448	1.220748	–
$h(-2)$	0.878993	0.926703	–	0.905894	0.918216	–
$h(2) := H$	0.617345	0.557615	0.369539	0.632000	0.555546	0.354025
$h(6)$	0.584704	0.431628	–	0.647181	0.413202	–
$h(10)$	0.565686	0.404372	–	0.617552	0.379965	–
$h(14)$	0.557758	0.390742	–	0.598700	0.365185	–

Figure 5.4: **BVSP** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q),\ q \in [-15, 15]$ (**MF-DFA**)

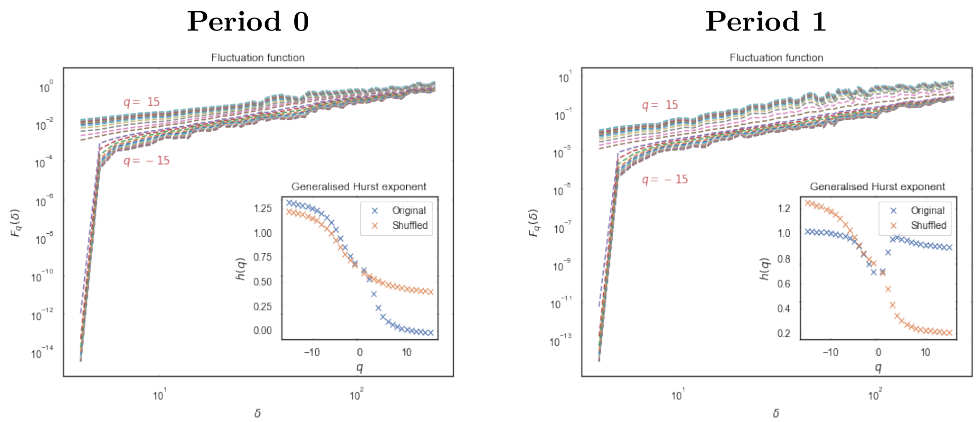


Figure 5.5: **BVSP** Fluctuation Function $F_q(\delta)$ and $\zeta(q),\ q \in [0.5, 3]$ (**MNI**).

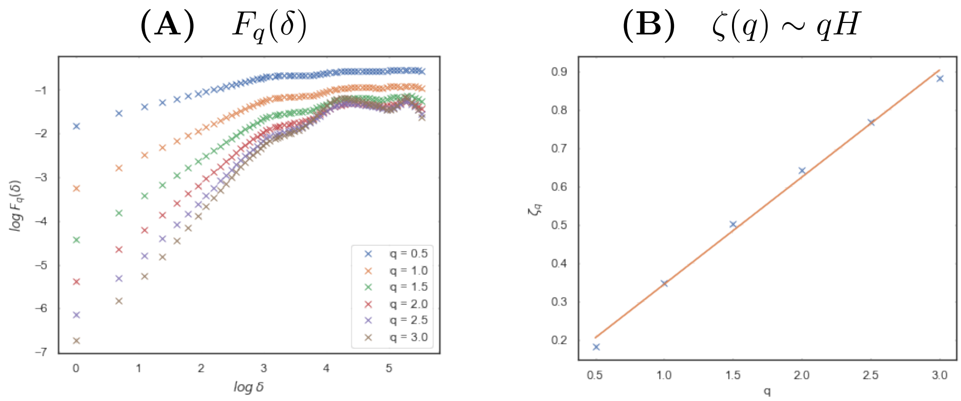


Table 5.5: **BVSP** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1**(10/02/2012 to 10/08/2022)		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	1.302556	1.221180	–	1.016113	1.238036	–
$h(-10)$	1.250648	1.157005	–	1.001396	1.175942	–
$h(-6)$	1.110142	0.998925	–	0.961745	1.027427	–
$h(-2)$	0.761120	0.722730	–	0.758341	0.796255	–
$h(2) := H$	0.532515	0.556492	0.276002	0.853822	0.553865	0.320254
$h(6)$	0.092457	0.470652	–	0.944826	0.268599	–
$h(10)$	0.008754	0.427830	–	0.910372	0.221613	–
$h(14)$	0.004833	0.403873	–	0.891654	0.205730	–

Figure 5.6: **DJI** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q),\ q \in [-15, 15]$ (**MF-DFA**)

Period 0 **Period 1**

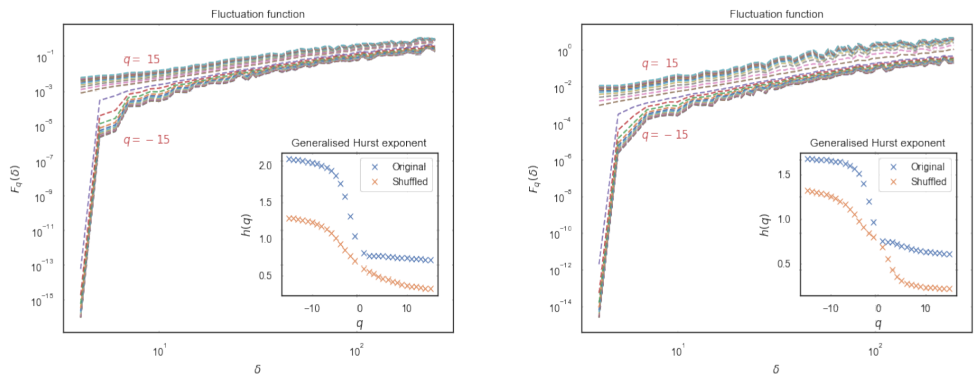


Figure 5.7: **DJI** Fluctuation Function $F_q(\delta)$ and $\zeta(q),\ q \in [0.5, 3]$ (**MNI**).

(A) $F_q(\delta)$ **(B)** $\zeta(q) \sim qH$

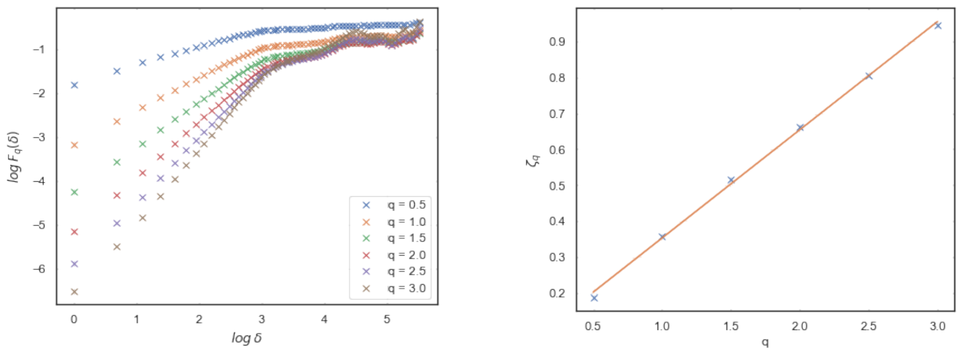


Table 5.6: **DJI** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1**(10/02/2012 to 10/08/2022)		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	2.088692	1.275498	–	1.672211	1.308202	–
$h(-10)$	2.045217	1.220277	–	1.655236	1.252919	–
$h(-6)$	1.929241	1.076383	–	1.601587	1.112991	–
$h(-2)$	1.309514	0.771193	–	1.196371	0.846375	–
$h(2) := H$	0.776157	0.555106	0.285657	0.748110	0.554983	0.331512
$h(6)$	0.757996	0.439847	–	0.684569	0.280341	–
$h(10)$	0.736079	0.368481	–	0.637779	0.235104	–
$h(14)$	0.718593	0.328915	–	0.614068	0.220327	–

Figure 5.8: **FCHI** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q)$, $q \in [-15, 15]$ (**MF-DFA**)

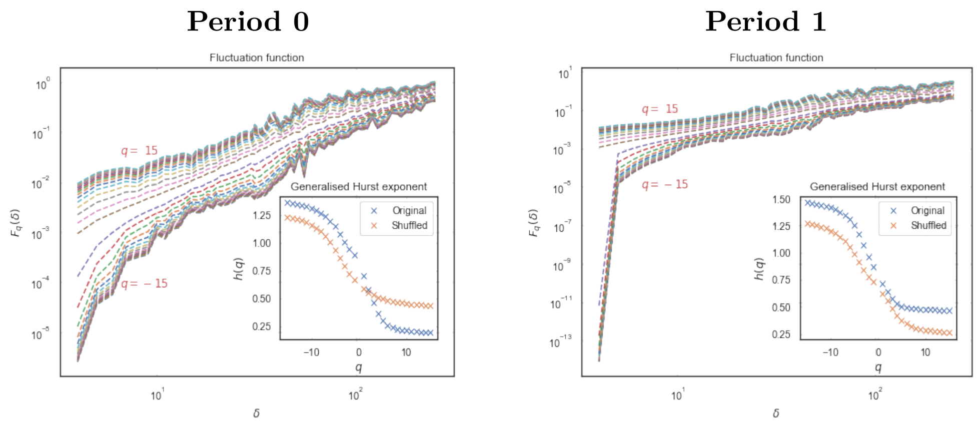


Figure 5.9: **FCHI** Fluctuation Function $F_q(\delta)$ and $\zeta(q)$, $q \in [0.5, 3]$ (**MNI**).

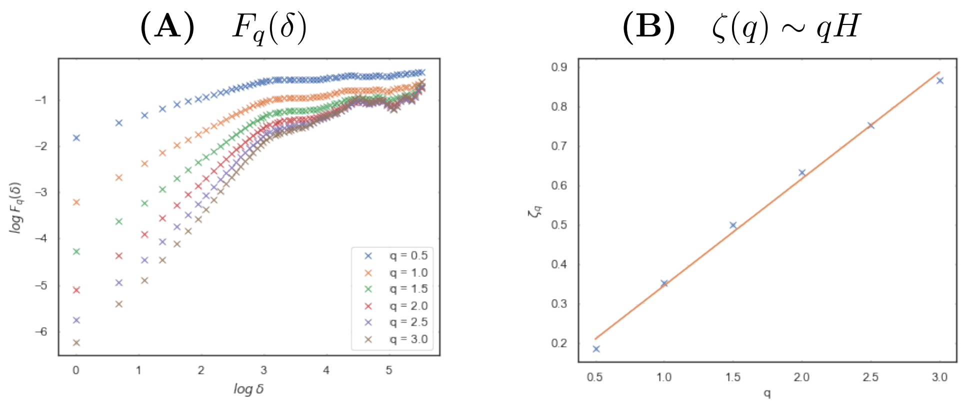


Table 5.7: **FCHI** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1**(10/02/2012 to 10/08/2022)		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	1.359729	1.239441	–	1.460154	1.259297	–
$h(-10)$	1.313247	1.174652	–	1.417552	1.199516	–
$h(-6)$	1.202345	1.012883	–	1.309917	1.050754	–
$h(-2)$	0.950623	0.725444	–	0.963263	0.782130	–
$h(2) := H$	0.580979	0.556520	0.280993	0.638468	0.556255	0.316489
$h(6)$	0.260257	0.481881	–	0.492475	0.343460	–
$h(10)$	0.208931	0.451810	–	0.475211	0.285767	–
$h(14)$	0.195218	0.435451	–	0.469807	0.266785	–

Figure 5.10: **FTSE** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q),\ q \in [-15, 15]$ (**MF-DFA**)

Period 0 **Period 1**

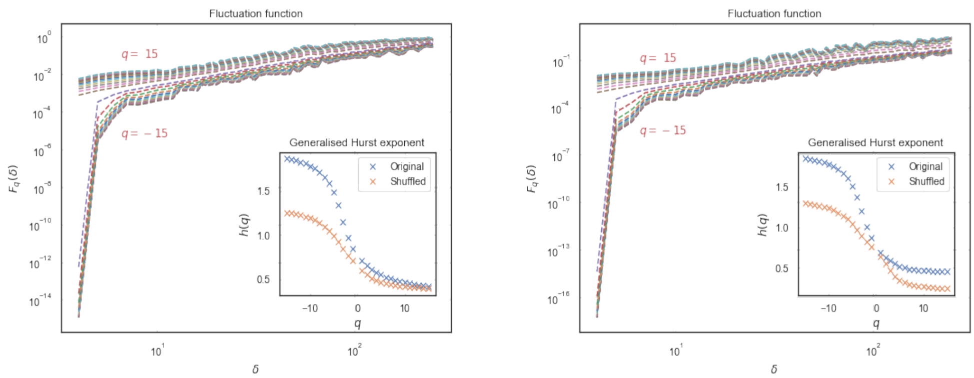


Figure 5.11: **FTSE** Fluctuation Function $F_q(\delta)$ and $\zeta(q),\ q \in [0.5, 3]$ (**MNI**).

(A) $F_q(\delta)$ **(B)** $\zeta(q) \sim qH$

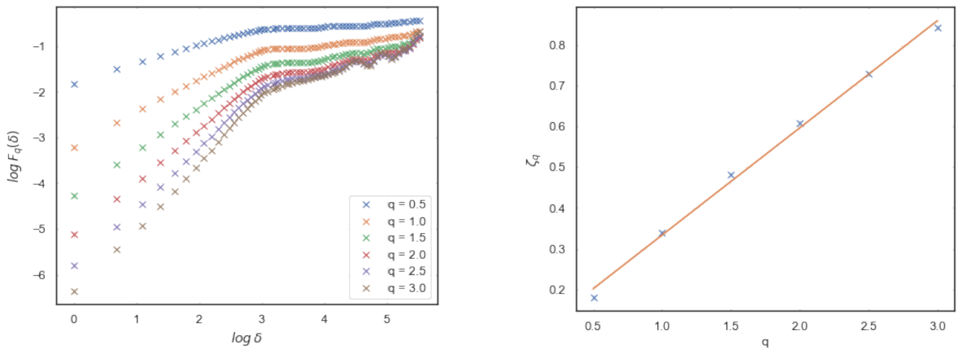


Table 5.8: **FTSE** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1**		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	1.852467	1.257613	–	1.844968	1.288874	–
$h(-10)$	1.780357	1.200827	–	1.783604	1.235304	–
$h(-6)$	1.584935	1.059545	–	1.610959	1.101096	–
$h(-2)$	0.970782	0.776609	–	1.014557	0.830161	–
$h(2) := H$	0.656075	0.555795	0.272705	0.633030	0.555509	0.304612
$h(6)$	0.528627	0.458362	–	0.500261	0.319560	–
$h(10)$	0.463959	0.425628	–	0.467171	0.267337	–
$h(14)$	0.432069	0.408090	–	0.458427	0.250053	–

Figure 5.12: **GDAXI** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q)$, $q \in [-15, 15]$ (**MF-DFA**)

Period 0 **Period 1**

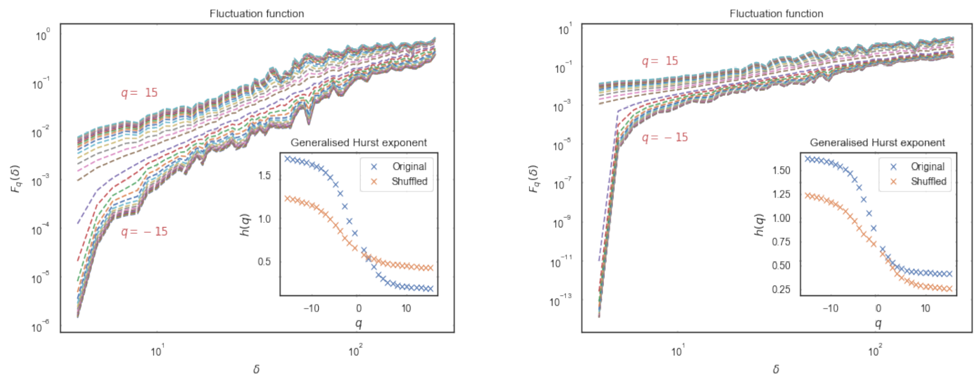


Figure 5.13: **GDAXI** Fluctuation Function $F_q(\delta)$ and $\zeta(q)$, $q \in [0.5, 3]$ (**MNI**).

(A) $F_q(\delta)$ **(B)** $\zeta(q) \sim qH$

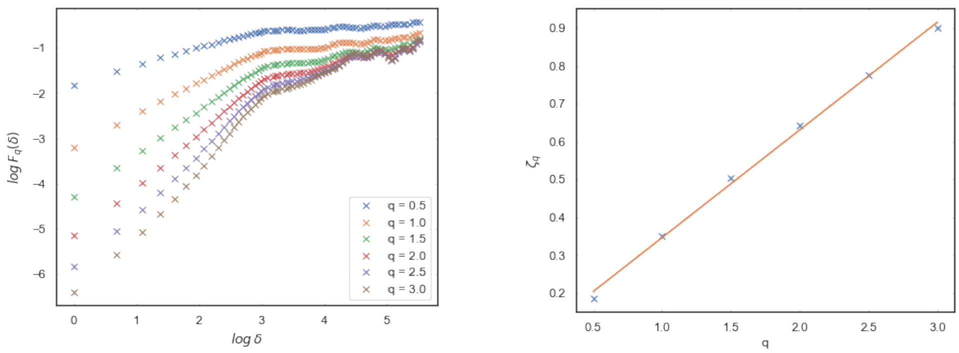


Table 5.9: **GDAXI** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1**		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	1.678776	1.215541	–	1.613181	1.235344	–
$h(-10)$	1.623902	1.150479	–	1.574803	1.174310	–
$h(-6)$	1.473403	0.989872	–	1.462481	1.027462	–
$h(-2)$	0.974964	0.713174	–	1.045414	0.778809	–
$h(2) := H$	0.539951	0.555377	0.295113	0.589171	0.557152	0.321595
$h(6)$	0.267405	0.483850	–	0.450309	0.343581	–
$h(10)$	0.214509	0.455224	–	0.425196	0.285758	–
$h(14)$	0.196061	0.440086	–	0.414179	0.266188	–

Figure 5.14: **GSPC** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q)$, $q \in [-15, 15]$ (**MF-DFA**)

Period 0 **Period 1**

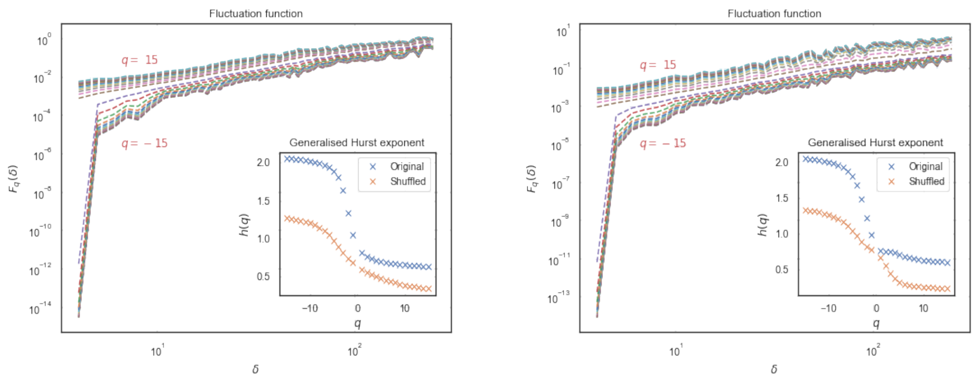


Figure 5.15: **GSPC** Fluctuation Function $F_q(\delta)$ and $\zeta(q)$, $q \in [0.5, 3]$ (**MNI**).

(A) $F_q(\delta)$ **(B)** $\zeta(q) \sim qH$

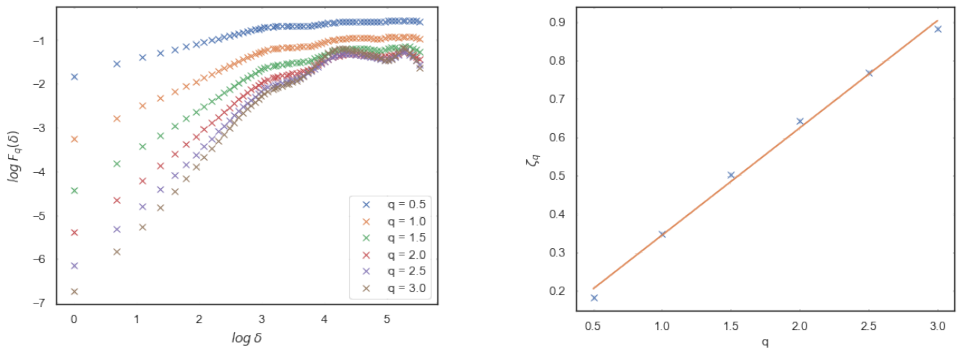


Table 5.10: **GSPC** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1**		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	2.047344	1.264491	–	2.040143	1.320792	–
$h(-10)$	2.018016	1.139041	–	1.998172	1.261406	–
$h(-6)$	1.936343	0.992953	–	1.874105	1.108772	–
$h(-2)$	1.339908	0.739310	–	1.215638	0.821894	–
$h(2) := H$	0.762391	0.555491	0.280501	0.744185	0.557539	0.330803
$h(6)$	0.686119	0.464776	–	0.681556	0.293461	–
$h(10)$	0.656698	0.432597	–	0.627063	0.248721	–
$h(14)$	0.639907	0.416626	–	0.601072	0.234418	–

Figure 5.16: **HSI** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q)$, $q \in [-15, 15]$ (**MF-DFA**)

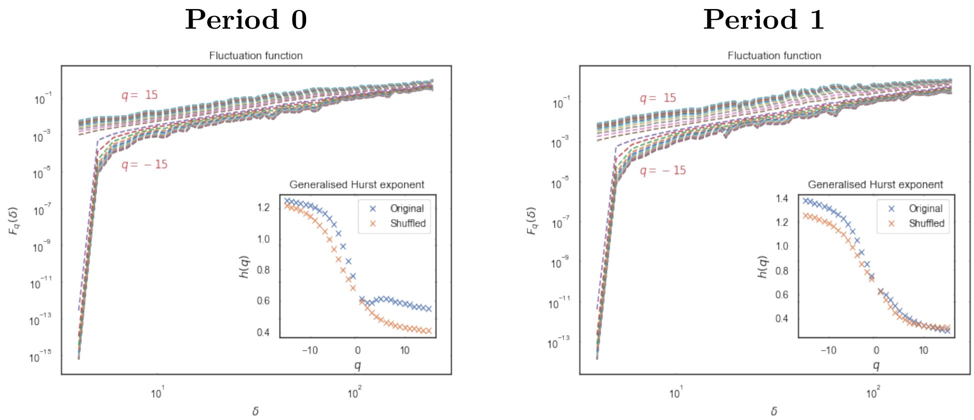


Figure 5.17: **HSI** Fluctuation Function $F_q(\delta)$ and $\zeta(q)$, $q \in [0.5, 3]$ (**MNI**).

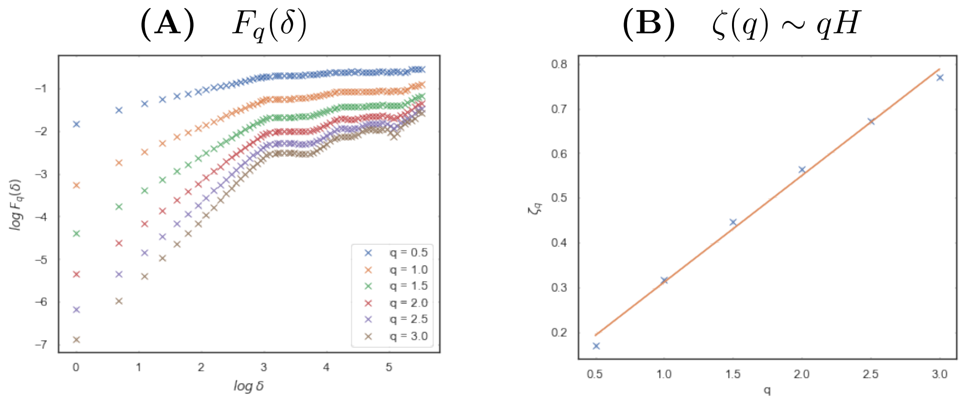


Table 5.11: **HSI** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1**		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	1.241045	1.200398	–	1.371597	1.254418	–
$h(-10)$	1.212088	1.139041	–	1.315450	1.196501	–
$h(-6)$	1.133941	0.992953	–	1.183382	1.053445	–
$h(-2)$	0.857404	0.739310	–	0.849666	0.788817	–
$h(2) := H$	0.586586	0.555491	0.278996	0.596510	0.555450	0.282349
$h(6)$	0.612770	0.464776	–	0.424257	0.384711	–
$h(10)$	0.580548	0.432597	–	0.342621	0.336305	–
$h(14)$	0.555460	0.416626	–	0.304832	0.318986	–

Figure 5.18: **IXIC** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q),\ q \in [-15, 15]$ (**MF-DFA**)

Period 0 **Period 1**

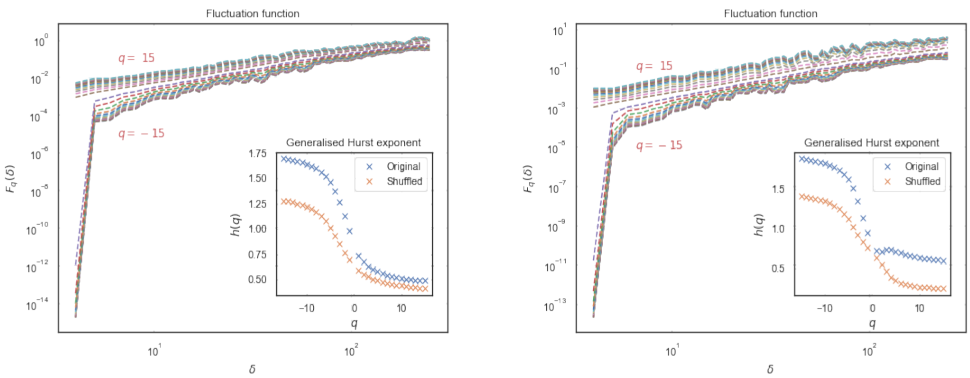


Figure 5.19: **IXIC** Fluctuation Function $F_q(\delta)$ and $\zeta(q),\ q \in [0.5, 3]$ (**MNI**).

(A) $F_q(\delta)$ **(B)** $\zeta(q) \sim qH$

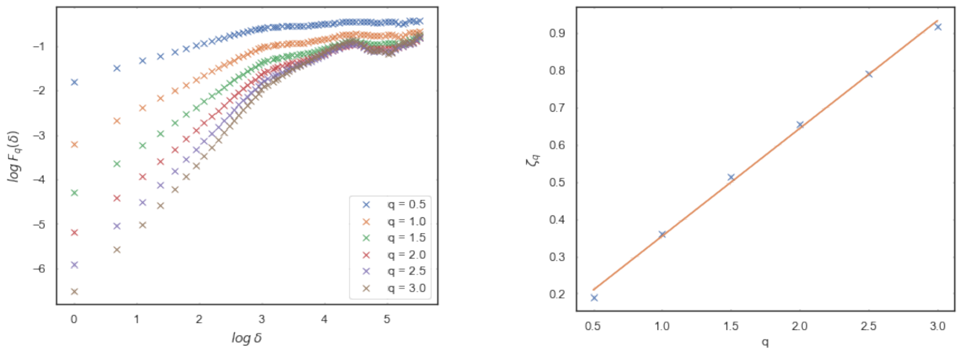


Table 5.12: **IXIC** values of generalised Hurst exponent $h(q)$

	Period 0 (Prior the Covid-19 pandemic)			**Period 1**		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	1.687229	1.269386	–	1.878549	1.396210	–
$h(-10)$	1.639742	1.216256	–	1.826700	1.340509	–
$h(-6)$	1.518275	1.076540	–	1.693917	1.187641	–
$h(-2)$	1.126494	0.768008	–	1.139335	0.836749	–
$h(2) := H$	0.675043	0.555486	0.272079	0.717800	0.556574	0.328282
$h(6)$	0.559124	0.476950	–	0.693269	0.317135	–
$h(10)$	0.512165	0.440926	–	0.636601	0.268927	–
$h(14)$	0.491122	0.418427	–	0.609397	0.253728	–

Figure 5.20: **J203** Volatility Fluctuation Function $F_q(\delta)$ and the generalised Hurst exponent $h(q)$, $q \in [-15, 15]$ (**MF-DFA**)

Period 0 **Period 1**

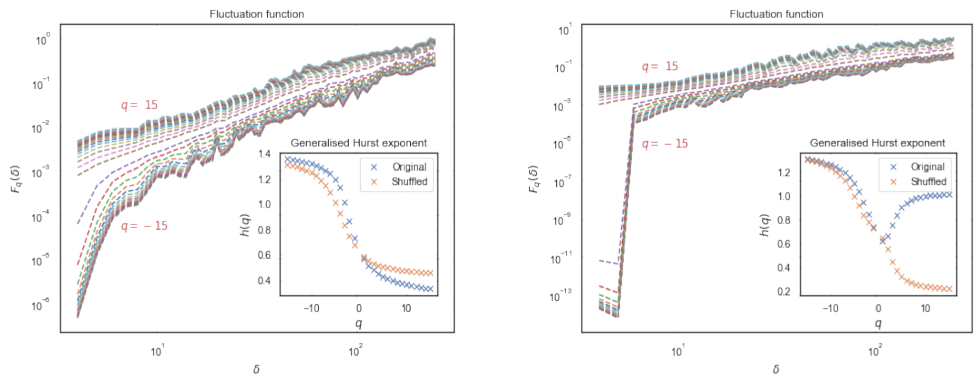


Figure 5.21: **J203** Fluctuation Function $F_q(\delta)$ and $\zeta(q)$, $q \in [0.5, 3]$ (**MNI**).

(A) $F_q(\delta)$ **(B)** $\zeta(q) \sim qH$

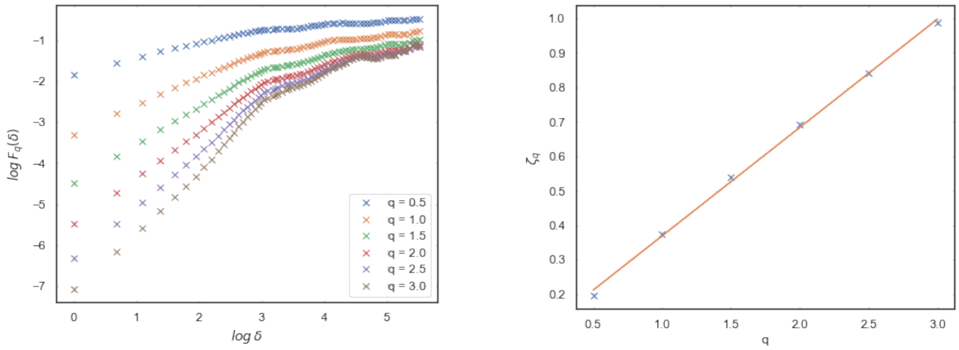


Table 5.13: **J203** values of generalised Hurst exponent $h(q)$

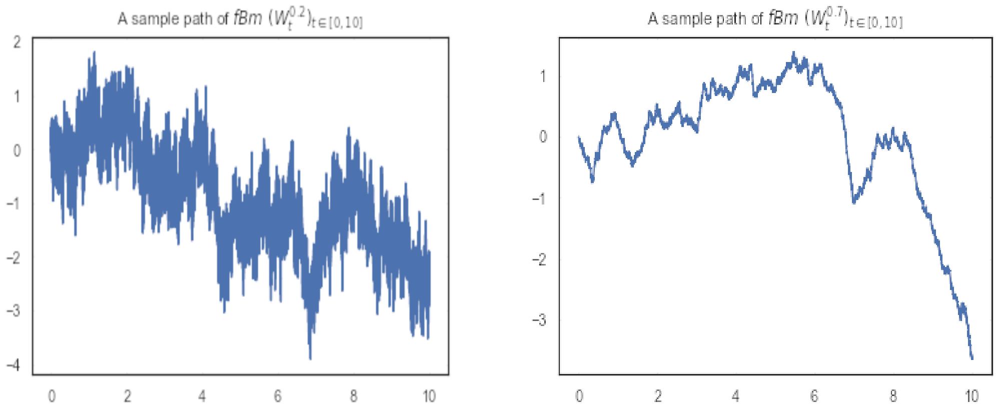
	Period 0 (Prior the Covid-19 pandemic)			**Period 1**		
$h(q)$	**MF-DFA**	**Shuffled**	**MNI**	**MF-DFA**	**Shuffled**	**MNI**
$h(-14)$	1.353418	1.314675	–	1.301887	1.2881311	–
$h(-10)$	1.318935	1.256170	–	1.258195	1.224682	–
$h(-6)$	1.240393	1.096257	–	1.155945	1.061922	–
$h(-2)$	0.864210	0.748953	–	0.845457	0.776682	–
$h(2) := H$	0.518455	0.555196	0.340130	0.663775	0.556908	0.346055
$h(6)$	0.410350	0.497614	–	0.938929	0.294917	–
$h(10)$	0.363307	0.476313	–	0.995972	0.243772	–
$h(14)$	0.339515	0.462623	–	1.007934	0.226535	–

5.4 More implementations

5.4.1 Implementation on *fBm*

We would like to apply the previous approaches to test the roughness of *fBm* $(W_t^H)_{t\geq 0,\, H\in(0,1)}$ obtained through simulations. Note that *fBm* is a monofractal process where the generalised Hurst exponent $h(q) := H$ is a constant. Recall that techniques for simulation of *fBm* are divided into two categories: the exact and approximate techniques. Exact techniques include the Hosking method (Hosking; 1984), Cholesky method (Asmussen; 1998) and Davies-Harte method (Davies and Harte; 1987). On the other hand, approximate techniques rely heavily on representations and properties of *fBm*. A summary of these techniques was discussed by Dieker (2004).

Figure 5.22: A sample path of *fBms* $(W^H)_{t\in[0,10]}$, $H = 0.2$ and $H = 0.7$.



Consider a *fBm* $(W_t^H)_{t\in[0,10]}$ with two different Hurst parameters $H = 0.2$ and $H = 0.7$ as show on Figure 5.22. Next, we apply the MF-DFA approach to analyse the roughness and to recover those Hurst parameters. The fluctuation function and the generalised Hurst function are given in figures 5.23 and 5.24 for $H = 0.2$ and $H = 0.7$ respectively.

Figure 5.23: $F_q(\delta)$ and $h(q)$, $q \in [0, 15]$ of *fBm* $(W_t^H)_{t\in[0,1],\, H=0.2}$.

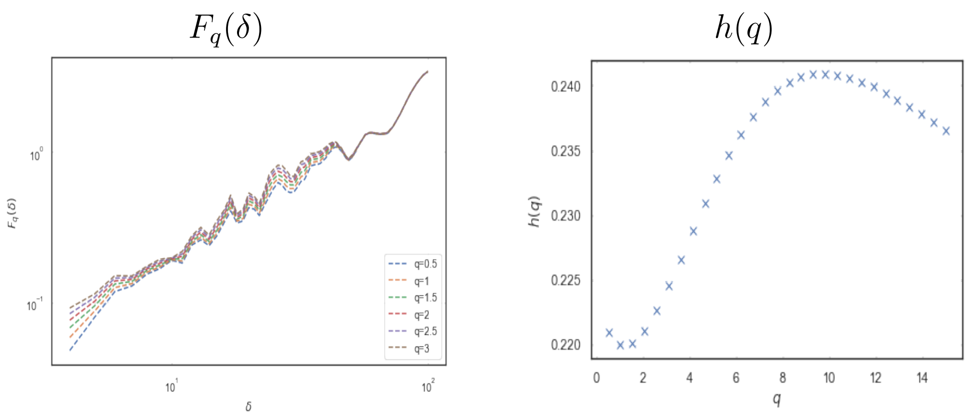
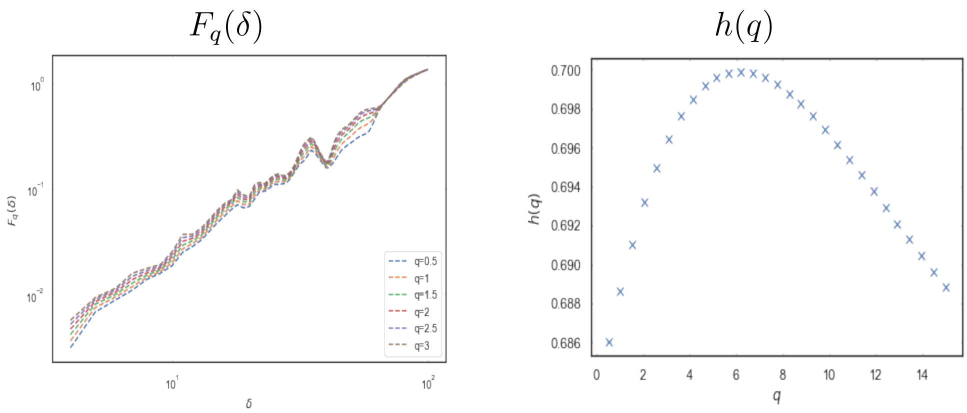


Figure 5.24: $F_q(\delta)$ and $h(q)$, $q \in [0, 15]$ of *fBm* $(W_t^H)_{t\in[0,1],\, H=0.7}$.



We may observe that the values of Hurst parameters for *fBm* (W_t^H) with $H = 0.2$ and $H = 0.7$ are recovered with an error of ± 0.02 as shown in Table 5.14 below.

Table 5.14: Selected values of $h(q)$ for *fBms* $(W_t^H)_{t\in[0,1],\, H=0.2,\, H=0.7}$

$h(q)$	$h(0.5)$	$h(1)$	$h(1.5)$	$h(2)$	$h(2.5)$	$h(3)$
$(W_t^H)_{t\in[0,1],\, H=0.2}$	0.221258	0.220367	0.220629	0.221740	0.223442	0.225521
$(W_t^H)_{t\in[0,1],\, H=0.7}$	0.686006	0.688613	0.691043	0.693179	0.694979	0.696447

5.4.2 Implementation on stochastic process with an additive *fBm*

The stochastic process $(Y_t)_{t\geq 0}$ with additive *fBm* verifies the following differential equation:

$$dY_t = f(t,Y_t)dt + \nu dW_t^H, \tag{5.9}$$

where $f(t,Y_t)$ and ν are respectively the adapted drift process and positive constant volatility of $(Y_t)_{t\geq 0}$. This process was previously discussed by Nualart and Ouknine (2002) and Hu et al. (2008) for $H > 1/2$. Particularly, Nualart and Ouknine (2002) showed that (5.9) exists and is unique if the drift function $f(t,y)$ satisfies the linear growth condition (that is, for a constant $c > 0$, $|f(t,y)| \leq c(1+|z|))$ for $H < 1/2$, and the Hölder continuity condition for $H > 1/2$ (that is, for $c > 0$, $y_1, y_2 \in \mathbb{R}$, $\frac{2-H}{2} < \alpha < 1$ and $\beta > H - \frac{1}{2}$, $|f(t,y_1) - f(t,y)| \leq c(|y_1 - y_2|^\alpha + |t_1 - t_2|^\beta))$. This kind of process is important in this work as it will be used as a stochastic model of instantaneous (or spot) volatility in upcoming chapters. Further analysis such as positiveness and differentiability will also be examined.

To generate sample paths of (5.9), we choose the drift function $f(t,y) = \theta_t(\mu_t - y)$[4], where $\theta_t = \theta > 0$ and $\mu_t = c + \frac{\nu^2}{2\theta}\left(1 - e^{-2\theta t}\right)$, where $c > 0$ is a constant. This yields the following drift function:

$$f(t,y) = \frac{\nu^2}{2\theta}\left(1 - e^{-2\theta t}\right) + (c - \theta y^2), \tag{5.10}$$

We shall then simulate the corresponding process $(Y_t)_{t\in[0,T]}$ on a compact interval $[0,T]$ using the Euler method (See e.g. Higham et al. (2002) for more details about the method). Subdivide the interval $[0,T]$ into N subintervals of equal length $\delta t = T/N$ with end points $0 = t_0, t_1, t_2, \ldots, t_N = T$. The corresponding discrete version of the process $(Y_t)_{t\geq 0}$ is given by

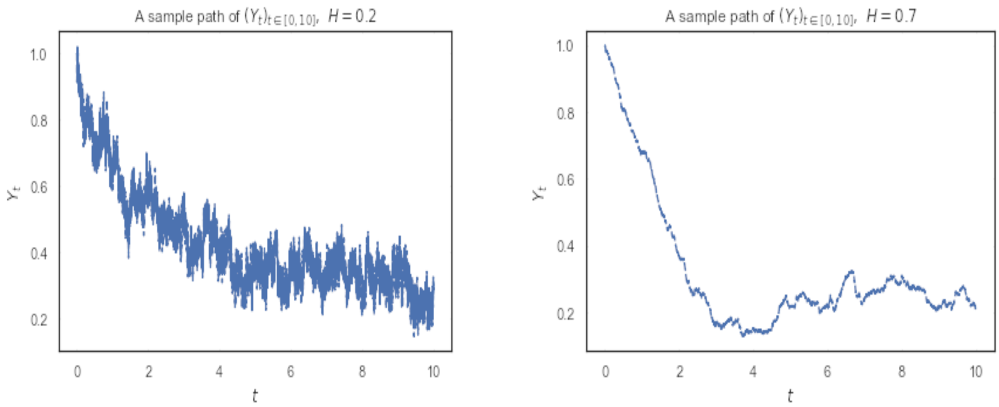
$$\hat{Y}_{t_i} = \hat{Y}_{t_{i-1}} + f(t_{i-1},\hat{Y}_{t_{i-1}})\delta t + \nu\delta W_{t_i}^H \tag{5.11}$$

with $\delta W_{t_i}^H = W_{t_i}^H - W_{t_{i-1}}^H$. Set all the parameters as follows: the constant

[4] In this case the stochastic process $(Y_t)_{t\geq 0}$ is mean-reverting.

volatility $\nu = 0.2$, initial value $Y_0 = 1$, time-step $\delta t = 0.001$, $\theta = 1$ and $c = 0$. A sample path of $(Y_t)_{t\geq 0}$ for $H = 0.2$ and $H = 0.7$ are given respectively in Figure 5.25.

Figure 5.25: A sample path of $(Y_t)_{t\in[0,10]}$, $H = 0.2$ and $H = 0.7$.



The fluctuation function $F_q(\delta)$ and the generalised Hurst exponent $h(q)$ for the stochastic process $(Y_t)_{t\geq 0}$ with $H = 0.2$ and $H = 0.7$ are given in figures 5.26 and 5.27 respectively.

Figure 5.26: $F_q(\delta)$ and $h(q)$ of $(Y_t)_{t\in[0,10]}$ with $H = 0.2$.

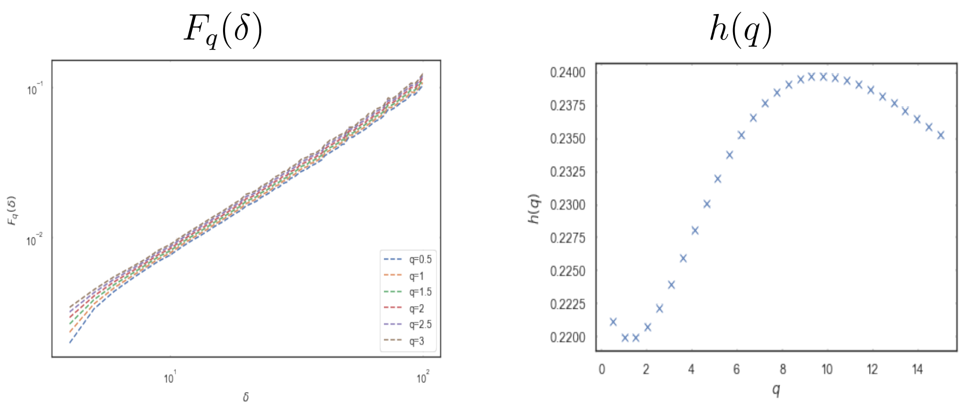
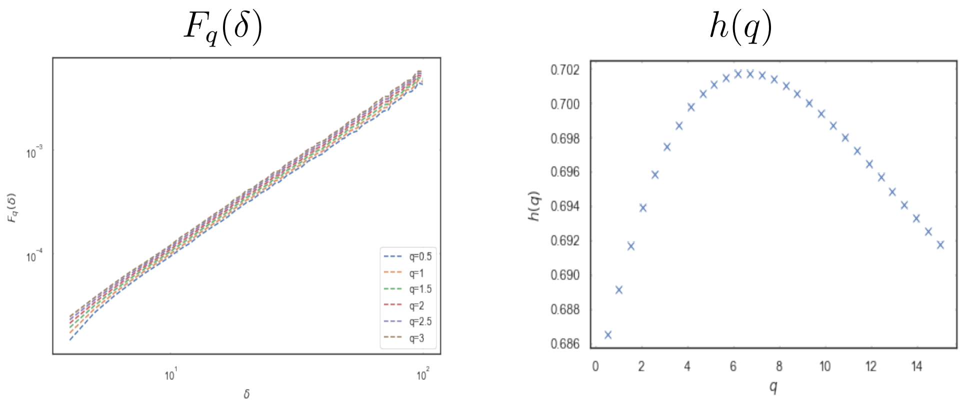


Figure 5.27: $F_q(\delta)$ and $h(q)$, $q \in [0, 15]$ of $(Y_t)_{t\in[0,10]}$ with $H = 0.7$.



As previously, we recovered the values of Hurst parameters $H = 0.2$ and $H = 0.7$ with an error of ± 0.02 as shown in Table 5.14. This also shows that $h(q)$ depends only on the random part of the stochastic process $(Y_t)_{t\geq 0}$.

Table 5.15: Selected values of $h(q)$ for $(Y_t)_{t\in[0,10], H=0.2, H=0.7}$

$h(q)$	$h(0.5)$	$h(1)$	$h(1.5)$	$h(2)$	$h(2.5)$	$h(3)$
$(Z_t)_{t\in[0,1]}$, $H = 0.2$	0.221099	0.219903	0.219874	0.220703	0.222130	0.223936
$(Z_t)_{t\in[0,1]}$, $H = 0.7$	0.686486	0.689173	0.691698	0.693941	0.695855	0.697444

5.5 Conclusion

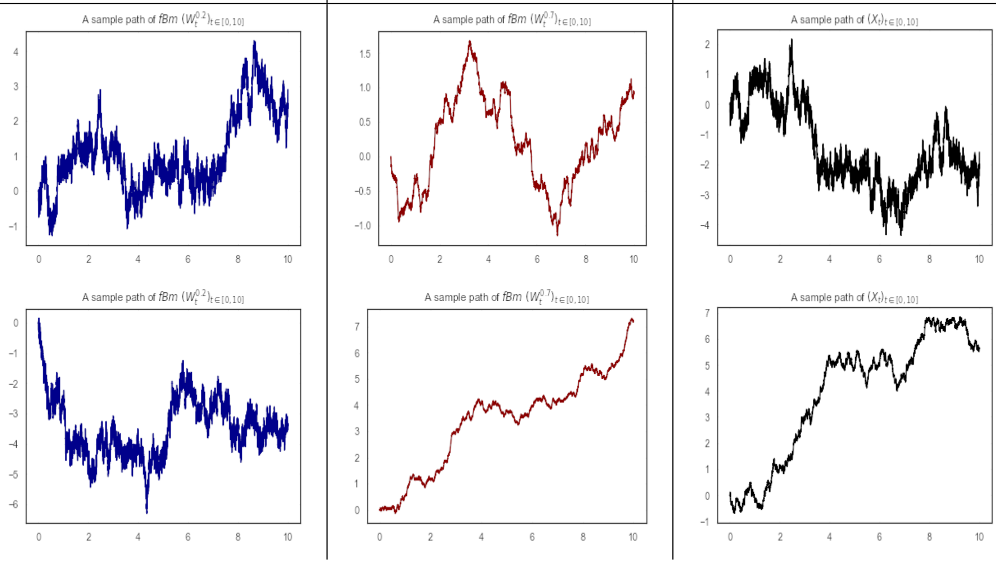
In this chapter, we use multifractal detrended fluctuation analysis to demonstrate that the volatility displays the multifractality property through realised volatility time series estimated from prices of major stock market log-return indices from 10 February 2012 to 10 August 2022. This multifractality takes its origin from the broad probability density function as well as temporal correlations. We also show that the volatility display long-range dependence with Hurst parameter of order 0.5 to 0.7. These values are even higher on the period from 08 February 2012 to December 2020, when the Covid-19 pandemic was on its peak, with Hurst parameters of order 0.6 to 0.8. Similar results were found in Comte and Renault (1998), Cajueiro and Tabak (2008), Chronopoulou and Viens (2010), Power and Turvey (2010), Abuzayed et al. (2018) and Cont and Das (2022).

When using the microstructure noise index technique, we find that the log-volatility time series is rough for all realised volatility, that is, it displays short-range dependence with Hurst parameter of order 0.2 to 0.3. These results are in line with some findings in Gatheral et al. (2018), Livieri et al. (2018), Bayer et al. (2016) and Takaishi (2020). One may conclude that there are some contradictions, and this is the subject of further investigations. However, in both cases, the Hurst parameter is not always half unlike stochastic volatility models under standard Brownian such as the Heston model.

Regarding the volatility roughness, Cont and Das (2022) believe that the *"origin of the roughness observed in realized volatility time series lies in the microstructure noise rather than the volatility process itself"*. On the other hand, Alòs and Lorite (2021) think that short and long range volatility processes are somehow compatible by illustrating the following example. Consider two *fBms* $(W_t^{H=0.2})_{t\geq 0}$ and $(W_t^{H=0.7})_{t\geq 0}$. Then the random variable $X_t := W_t^{H=0.2} + W_t^{H=0.7}$ behaves either as a short or long range dependent process as shown in Figure 5.28.

Figure 5.28: Illustration of Long-range versus Short-range dependence of different fractional Brownian motions.

$(W_t^{H=0,2})_{t\in[0,10]}$	$(W_t^{H=0,7})_{t\in[0,10]}$	$X_t := W_t^{H=0,2} + W_t^{H=0,7}$



5.5.1 Modelling volatility

Section 5.4.2 shows that the roughness of a given stochastic process depends only on its random component. Therefore, it will make sense to replace the standard Brownian motion of the stochastic volatility in Heston model by a *fBm* discussed in chapter 2. The corresponding financial market model, namely *"fractional Heston model"*, is then given by:

$$\begin{cases} dA_t = rA_t dt, \\ dX_t = \eta X_t dt + \sqrt{Y_t} X_t dB_t, \\ dY_t = \theta(\mu - Y_t)dt + \nu\sqrt{Y_t} dW_t^H \end{cases} \tag{5.12}$$

where the stochastic processes $(B_t)_{t\geq 0}$ and $(W_t^H)_{t\geq 0}$ are indeed correlated. The stochastic process $(Y_t)_{t\geq 0}$ is called *"fractional Cox-Ingersoll-Ross (fCIR)"* process and can be considered as a generalisation of the standard *CIR* process. The correlation between $(B_t)_{t\geq 0}$ and $(W_t^H)_{t\geq 0}$ can be determined by representing the *fBm* (W_t^H) in terms of a standard Brownian motion as introduced in Proposition 2.4. In the remainder of this thesis, we will use the Volterra (or interval) representation defined by (2.4) and (2.5).

We may also note that the *fCIR* process $(Y_t)_{t\geq 0}$ cannot be used directly to model volatility time series since its adapted volatility process $\nu\sqrt{Y_t}$ does not verify the Lipschitz condition to guarantee its existence and uniqueness. For this reason, the stochastic process $(Y_t)_{t\geq 0}$ can be decomposed as a square of a stochastic process with additive *fBm* of the form (5.9). This will be further discussed in chapters 6 and 7.

Chapter 6

Generalisation of Fractional Heston-Type Model

The previous chapter shows clearly that the Hurst parameter is not necessarily equal to half as in the case of stochastic volatility models driven by standard Brownian motions (See e.g. Heston (1993) and subsequent results). In this chapter, we discuss financial markets of the form (5.12) known as the fractional Heston-type model and propose a general form of this model where the adapted drift of the volatility is replaced by a continuous function. Our analysis considers all values of Hurst parameters.

6.1 Fractional Heston Model

6.1.1 The financial market model

In the fractional Heston model, the standard Brownian motion $(\tilde{B}_t)_{t\geq 0}$ in (4.28) is replaced by a *fBm* $(W_t^H)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$. See e.g. Bayer et al. (2016), Alòs and Yang (2017), Livieri et al. (2018), Bezborodov et al. (2019), Fallah et al. (2019), El Euch et al. (2019) and Mishura and Yurchenko-Tytarenko (2020) with references therein. The resulting stochastic volatility model is called *"fractional Cox-Ingersoll-Ross (fCIR) process"*.

The *fCIR* process has been defined in different ways. Alòs and Yang (2017) defined the *fCIR* process for $H > 1/2$ in terms of fractional integral and derived the approximated option price formula for an European option. On the other hand, Mishura and Yurchenko-Tytarenko (2018) used the following definition:

$$Y_t(\omega) = Z_t^2(\omega)\mathbf{1}_{[0,\tau(\omega))}, \quad \forall t \geq 0, \quad \omega \in \Omega, \tag{6.1}$$

where $(Z_t)_{t\geq 0}$ is a singular stochastic differential equation driven by an additive *fBm* $(W_t^H)_{t\geq 0,\, H\in(0,1)}$ and taking the following form:

$$dZ_t = \frac{1}{2}\left(\frac{\mu}{Z_t} - \theta Z_t\right)dt + \frac{\nu}{2}dW_t^H, \tag{6.2}$$

with μ and θ defined as previously, ν is also a positive parameter and the *fBm* $(W_t^H)_{t\geq 0}$ represents the source of randomness of both processes $(Z_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$. In addition, the random variable τ in (6.1) represents the first time the process $(Z_t)_{t\geq 0}$ hits zero. It is explicitly defined by

$$\tau(\omega) = \inf\left\{t > 0 : Z_t(\omega) = 0\right\}. \tag{6.3}$$

The numerical scheme of *fCIR* process defined by (6.1) and (6.2) was investigated by Hong et al. (2019). The financial market model can be summarised as follows:

$$\begin{cases} dA_t = rA_t dt, \\ dX_t = \eta X_t dt + \sigma(Y_t)X_t dB_t, \\ Y_t = Z_t^2\mathbf{1}_{[0,\tau(\omega)]} \\ dZ_t = \frac{1}{2}\left(\frac{\mu}{Z_t} - \theta Z_t\right)dt + \frac{\nu}{2}dW_t^H. \end{cases} \tag{6.4}$$

The financial market model (6.4) was previously used for pricing derivatives by Bezborodov et al. (2019) for $\mu = 0$ and Mishura and Yurchenko-Tytarenko (2020) for any $\mu \geq 0$ in the case of long-range dependency volatility time-series, that is where $H > 1/2$. For the case of rough volatility that is where $H < 1/2$, some studies are currently being investigated but not in the form of our financial market model.

For example, El Euch and Rosenbaum (2018) and El Euch et al. (2019) investigated the rough Heston model where the fractional volatility has a special form which coincides with the standard Heston model when $H \to 1/2$. Some results were discussed in Bayer et al. (2016) where rough volatility takes the form of the rough Bergomi model.

6.1.2 Option pricing

The risk-neutral approach (4.25) has mostly been used since the analytical solution does not exist under fractional volatility modelling. Several investigations are rather focused on computing the expected payoff function which is not necessary continuous in general. For example, Bezborodov et al. (2019) assumed that the stock price is driven by a geometric Brownian motion as in (6.4) and the dynamics of instantaneous volatility $(Y_t)_{t\geq>0}$ is described by a fractional Ornstein-Uhlenbeck process that satisfies the following stochastic differential equation:

$$dY_t = -\theta Y_t dt + dW_t^H,$$

where θ is positive parameter and $(W_t^H)_{t\geq 0}$ a *fBm* with $H > 1/2$. They discussed the expected payoff in two representations: The first is similar to Altmayer and Neuenkirch (2015) and is given by

$$\mathbb{E}\left[h(X_T)\right] = \mathbb{E}\left[\frac{L(X_T)}{X_T}\Big(1 + \frac{I_T}{T}\Big)\right]$$

where

$$L(X_T) = \int_0^{X_T} h(x)dx \quad \text{and} \quad I_T = \int_0^T \frac{1}{\sigma(Y_t)} dB_t,$$

with $\sigma(Y_t)$ the volatility of the infinitesimal return that is differentiable and satisfies the polynomial growth condition. The second representation of the expected value is proposed as:

$$\mathbb{E}\left[g(R_T)\right] = \frac{1}{\sqrt{2\pi}}\mathbb{E}\left[\frac{1}{\tilde{I_T}}\int_{-\infty}^{+\infty} G\Big(\big(s + R_0 + \eta T - \frac{1}{2}\tilde{I_T}^2\big)\tilde{I_T}\Big)e^{-\frac{s^2}{2}} sds\right]$$

where

$$R_t := \log X_t,\ g(x) := h(e^x),\ G(x) := \int_0^x h(s)ds \text{ and } \tilde{I_T}^2 := \int_0^T \sigma^2(Y_s)ds.$$

On the other hand, Alòs and Yang (2017) approximated the option price formula in terms of the Black-Scholes formula that works only for European option and for $H > 1/2$. They constructed the fractional volatility process from the standard *CIR* process $\tilde{Y}_t = \tilde{Z}_t^2$, where $(\tilde{Z}_t^2)_{t\geq 0}$ satisfies

$$\tilde{Z}_t^2 = \theta + (\tilde{Z}_0^2 - \theta)e^{-\kappa t} + \nu \int_0^t e^{-\kappa(t-s)}\tilde{Z}_s dW_t,$$

where $\theta,\ \kappa$ and ν are positive parameters, and where $(W_t)_{t\geq 0}$ represents the standard Brownian motion. Setting

$$C_1(t, \tilde{Z}_0) = \theta + (\tilde{Z}_0^2 - \theta)e^{-\kappa t}$$

and

$$Z_t^W = \int_0^t e^{-\kappa(t-s)}\tilde{Z}_s dW_t,$$

the fractional volatility process $\tilde{Y}_t = \tilde{Z}_t^2$ can be constructed as follows

$$\tilde{Z}_t^2 = C_1(t, \tilde{Z}_0) + a_1\nu Z_t^W + a_2\nu I_+^{H-\frac{1}{2}} Z_t^W$$

with a_1 and a_2 non-random positive parameters and $I_+^x f(t)$ the fractional Riemann-Liouville integral defined by

$$I_+^x = \frac{1}{\Gamma(x)} \int_0^t (t-s)^{x-1} f(s)ds.$$

Let $P^{BS}(t,X_t)$ be the option price under the classical Black-Scholes model and set

$$M_t = \int_t^T \mathbb{E}\big[\tilde{Z}_s^2\big]ds.$$

Then the option price $P(t,X_t)$ under fractional volatility model can be approximated for the European option (that is when $h(X_T) = (X_T - S)_+$) as follows:

$$P(t,R_t) = P^{BS}(0,R_0) + \frac{1}{2}\int_0^T e^{-rs}\mathcal{J}_1(s,R_s)\tilde{Z}_t d\langle M,R\rangle_s + \frac{1}{8}\int_0^T e^{-rs}\mathcal{J}_2(s,R_s)\tilde{Z}_t d\langle M,R\rangle_s.$$

where

$$\mathcal{J}_1(s,R_s) = \left(\frac{\partial^3}{\partial s^3} - \frac{\partial^2}{\partial s^2}\right)P^{BS}(s,R_s)$$

and

$$\mathcal{J}_2(s,R_s) = \left(\frac{\partial^4}{\partial s^4} - 2\frac{\partial^2}{\partial s^3} + \frac{\partial^2}{\partial s^2}\right)P^{BS}(s,R_s).$$

For more details, see Alòs and Yang (2017).

6.2 Generalisation of Fractional Heston Model

Although the stochastic volatility process in (6.4) presents several important features, its drift function that consists of the reversion speed and long-run mean are assumed to be constant. This is not always consistent with the volatility time-series and perfect calibration of the parameters may not be possible. For example, Benhamou et al. (2010) shows that calibrating with time-varying parameters of the drift function minimises the calibration error. El Euch et al. (2019) have noted similar observations for rough volatility models.

To overcome this, we leave a window of flexibility of the drift taking a general form and satisfying some weak assumptions that will be discussed later. We shall now introduce a fractional Heston-type model where the volatility follows a generalised *fCIR* process defined as a square of a stochastic process driven by an additive *fBm* with $H \in (0,1)$. In the next section, we shall discuss the existence and uniqueness of such a stochastic process.

6.2.1 The generalised *fCIR* process

Definition 6.1. Let $(Z_t)_{t\geq 0}$ be a stochastic process that satisfies the following stochastic differential equation

$$dZ_t = \frac{1}{2} f(t,Z_t) Z_t^{-1} dt + \frac{\nu}{2} dW_t^H, \tag{6.5}$$

where $f : [0,\infty) \times [0,\infty) \to (-\infty,\infty)$, $(t,z) \mapsto f(t,z)$ is a continuous function. Then the generalised *fCIR* process $(Y_t)_{t\geq 0}$ is defined by

$$Y_t = Z_t^2 \mathbf{1}_{[0,\tau)}(t). \tag{6.6}$$

This kind of stochastic process was previously introduced by Hu et al. (2008). To ensure the existence of the solution to (6.5), we need to impose the following conditions on the drift function $f(t,z)$ given in the assumption below.

Assumption 6.1.

(i) The drift function $g : [0,\infty) \times (0,\infty) \to (-\infty,\infty)$ defined by $g(t,z) = f(t,z)/z$ is continuous and admits a continuous partial derivative with respect to z on $(0,\infty)$. In addition, there exists a number $z^* > 0$ such that for every $z > z^*$, $g(t,z) < 0$, for all $t \geq 0$.

(ii) for any $T > 0$, there exists $z_T > 0$ such that $f(t,z) > 0$ for all $0 < t \leq T$ and $0 \leq z \leq z_T$.

Theorem 6.1. *If the drift function $f(t,z)$ satisfies Assumption 6.1, then for all $H \in (0,1)$, equation (6.5) has a unique solution $(Z_t)_{t\geq 0}$ which is continuous and positive up to time of the first visit to zero.*

Proof. Let $\ell > 0$ be a small number such that $\ell < Z_0$. For fixed $T > 0$, consider the sequence of processes $(Z_n(t))$ defined on $[0,T]$ by

$$Z_0(t) = Z_0,$$

for all $t \in [0,T]$ and for all $n \in \mathbb{N}$,

$$Z_{n+1}(t) = \begin{cases} Z_0 + \int_0^t g(s, Z_n(s))ds + \frac{\sigma}{2} W_t^H, & \text{if} \quad t \leq \tau_{n,\ell} \\ \ell & \text{otherwise} \end{cases}$$

where $g(t,z) = f(t,z)/(2z)$ and $\tau_{n,\ell} = \inf\{0 \le t \le T : Z_n(t) = \ell\}$ where $\tau_{n,\ell}$ is the first time that the process $(Z_n(t))$ reaches the level ℓ with $\inf(\emptyset) = +\infty$. Clearly, if $Z_n(t)$ does not reach the level ℓ on $[0,T]$, then Z_{n+1} is defined by

$$Z_{n+1}(t) = Z_0 + \int_0^t g(s, Z_n(s))ds + \frac{\sigma}{2}W_t^H, \quad t \in [0,T].$$

For instance

$$Z_1(t) = Z_0 + \int_0^t g(s, Z_0)ds + \frac{\sigma}{2}W_t^H, \quad t \in [0,T].$$

We want to show that there exists a number $\eta > 0$ independent of n and such that $\tau_{n,\ell} \ge \eta$ for all n. It is clear that $\tau_{n,\ell} \ge \tau_{n+1,\ell}$ because $Z_{n+1}(t) = \ell$ for all $t \ge \tau_{n,\ell}$. The function $t \mapsto g(t, Z_n(t))$ is bounded on $t \in [0, \tau_{n,\ell}]$. Indeed, for every $t \in [0, \tau_{n,\ell}]$, write $[0,t] = I_1 \cup I_2$ where I_1 is the union of sub-intervals of $[0,t]$ where $Z_n \le z^*$ and I_2 is the union of sub-intervals of $[0,t]$ where $Z_n > z^*$. Then

$$\int_0^t g(s, Z_n(s))ds = \int_{I_1} g(s, Z_n(s))ds + \int_{I_2} g(s, Z_n(s))ds \le \int_{I_1} g(s, Z_n(s))ds$$

because $g(s, Z_n(s)) < 0$ for $s \in I_2$ by Assumption 6.1(**i**). Therefore

$$\begin{aligned} Z_{n+1}(t) &= Z_0 + \int_0^t g(s, Z_n(s))ds + \frac{\sigma}{2}W_t^H \\ &\le Z_0 + \int_{I_1} g(s, Z_n(s))ds + \frac{\sigma}{2}W_t^H. \end{aligned}$$

Let

$$A = \sup(\{|g(s,z)| : s \in [0,T] \text{ and } z \in [\ell, z^*]\}).$$

Clearly $A < \infty$ because g is continuous on $[0,+\infty) \times (0,+\infty)$. Because for $s \in I_1$, $Z_n(s) < z^*$, then

$$Z_{n+1}(t) \le Z_0 + At + \frac{\sigma}{2}W_t^H \le B$$

where

$$B = Z_0 + AT + \frac{\sigma}{2} \sup_{0 \le t \le T} |W_t^H|.$$

(Here the bound B is independent of n). Therefore for all $t \in [0, \tau_{n,\ell}]$, we have that $Z_{n+1}(t) \in [\ell, B]$ for all $n \in \mathbb{N}$. Since $\tau_{n+1,\ell} \le \tau_{n,\ell}$, it follows in particular that

$$Z_{n+1}(t) \in [\ell, B] \text{ for all } 0 \le t \le \tau_{n+1,\ell}.$$

Moreover, since by definition,

$$Z_{n+1}(t) = Z_0 + \int_0^t g(s, Z_n(s))ds + \frac{\sigma}{2} W_t^H,$$

taking $t = \tau_{n+1,\ell}$ yields

$$\ell = Z_0 + \int_0^{\tau_{n+1,\ell}} g(s, Z_n(s))ds + \frac{\sigma}{2} W_{\tau_{n+1,\ell}}^H.$$

Set

$$K = \sup(\{|g(s,z)| : s \in [0,T] \text{ and } z \in [\ell, B]\}),$$

then

$$\ell \ge Z_0 - K\tau_{n+1,\ell} + \frac{\sigma}{2} W_{\tau_{n+1,\ell}}^H.$$

Equivalently

$$\frac{\sigma}{2} W_{\tau_{n+1,\ell}}^H \le \ell - Z_0 + K\tau_{n+1,\ell}$$

which implies that

$$\tau_{n+1,\ell} \ge \inf\{t \ge 0 : \frac{\sigma}{2} W_t^H \le \ell - Z_0 + Kt\}.$$

Set

$$\eta = \inf\{t \ge 0 : \frac{\sigma}{2} W_t^H \le \ell - Z_0 + Kt\}.$$

Clearly $\eta > 0$ because obviously the fractional Brownian motion (W_t^H) starts at 0, that is, $W_0^H = 0$ and $\ell < Z_0$. Hence, $\tau_{n+1,\ell} \ge \eta > 0$ uniformly for n (and η is independent of n).

Let $\tau_\ell = \inf_{n\geq 0} \tau_{n,\ell}$, then $\tau_\ell \geq \eta > 0$. We will then show that the problem has a positive solution on the interval $[0,\tau_\ell]$. For all n and all $t \in [0,\tau_\ell]$, $Z_n(t) \geq \ell$ and $Z_n(t) \leq B$.

Since the function $g(t,x)$ admits a partial derivative with respect to x on $(0,\infty)$, then in particular for fixed t, the function $(t,x) \mapsto g(t,x)$ is uniformly Lipschitz for x in a bounded closed interval away from 0. In addition, since for all $t \in [0,\tau_\ell]$, $Z_n(t) \in [\ell, B]$, then there exists $C > 0$ such that

$$\big|g(t,Z_n(t)) - g(t,Z_{n-1}(t))\big| \leq C\big|Z_n(t) - Z_{n-1}(t)\big|, \quad t \in [0,\tau_\ell].$$

Therefore,

$$\begin{aligned}\big|Z_{n+1}(t) - Z_n(t)\big| &\leq \int_0^t \big|(g(s,Z_n(s)) - g(s,Z_{n-1}(s)))\big| ds \\ &\leq C\int_0^t \big|Z_n(s) - Z_{n-1}(s)\big| ds.\end{aligned}$$

Then an application of Grönwall's lemma implies that the sequence $(Z_n(t))$ converges uniformly on the interval $[0,\tau_\ell]$ and hence its limit is a positive continuous solution to (6.5) on $[0,\tau_\ell]$. Therefore, equation (6.5) admits a positive solution up to the first time it hits the level ℓ. For the uniqueness of the solution, if (Z_t) and $(\tilde{Z}_t)$ are two solutions on some interval $[0,\tau_\ell)$ starting at the same point Z_0, then for any $t < \tau_\ell$,

$$|Z_t - \tilde{Z}_t| \leq \int_0^t |(g(s,Z_s) - g(s,\tilde{Z}_s))| ds \leq C\int_0^t |Z_s - \tilde{Z}_s| ds.$$

Again Grönwall's lemma implies that $Z_t = \tilde{Z}_t$ everywhere in $[0,\tau_\ell)$. Since $\ell > 0$ can be taken arbitrarily small, this implies the existence of a solution up to the first time it hits 0. □

In the next proposition, we prove that the generalised *fCIR* process can be represented in a standard form of *fCIR* process in terms of the Stratonovich integral (See Definition (3.7), (3.19)).

Proposition 6.2. *The generalised fCIR process* $(Y_t)_{t\geq 0}$ *defined by (6.6) up to the first time it hits zero, satisfies the following stochastic differential equation:*

$$dY_t = f(t, \sqrt{Y_t})dt + \nu\sqrt{Y_t} \circ dW_t^H. \tag{6.7}$$

Proof. For $\tau := \inf\{s > 0 : Z_s = 0\}$ and $t \in [0,\tau)$ fixed, we have from equations (6.5) and (6.6) that

$$Y_t = Z_t^2 = \left(Z_0 + \frac{1}{2}\int_0^t f(s,Z_s)Z_s^{-1}ds + \frac{\nu}{2}dW_t^H\right)^2,$$

where Z_0 is an initial value of the stochastic process $(Z_t)_{t\in[0,\tau)}$. In discrete time, assume that the interval $[0,t]$ is subdivided into N equal subintervals with $0 < t_1 < \cdots < t_N = t$, the time-steps $\Delta t = t/N$, and $t_i = i\Delta t$, $i = 0,\cdots,N$. Then it follows that

$$\begin{aligned}
Y_t &= Y_0 + \sum_{i=1}^{N}(Y_{t_i} - Y_{t_{i-1}}) \\
&= Y_0 + \sum_{i=1}^{N}\left(\left[Z_0 + \int_0^{t_i}\frac{1}{2}f(s,Z_s)Z_s^{-1}ds + \frac{\nu}{2}dW_{t_i}^H\right]^2\right. \\
&\qquad \left. - \left[Z_0 + \frac{1}{2}\int_0^{t_{i-1}} f(s,Z_s)Z_s^{-1}ds + \frac{\nu}{2}W_{t_{i-1}}^H\right]^2\right) \\
&= Y_0 + \sum_{i=1}^{N}\left[\frac{1}{2}\int_{t_{i-1}}^{t_i} f(s,Z_s)Z_s^{-1}ds + \frac{\nu}{2}\left(W_{t_i}^H - W_{t_{i-1}}^H\right)\right] \\
&\qquad\qquad \times \left[2Z_0 + \frac{1}{2}\left(\int_0^{t_i} f(s,Z_s)Z_s^{-1}ds + \int_0^{t_{i-1}} f(s,Z_s)Z_s^{-1}ds\right)\right. \\
&\qquad\qquad\qquad \left. + \frac{\nu}{2}\left(W_{t_i}^H + W_{t_{i-1}}^H\right)\right].
\end{aligned}$$

The last equation above is obtained by factorising the difference of two squares. After some expansions, we obtain that

$$
\begin{aligned}
Y_t = Y_0 &+ Z_0 \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(s,Z_s) Z_s^{-1} ds \\
&+ \frac{1}{4} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(s,Z_s) Z_s^{-1} ds \left(\int_0^{t_i} f(s,Z_s) Z_s^{-1} ds + \int_0^{t_{i-1}} f(s,Z_s) Z_s^{-1} ds \right) \\
&+ \frac{\nu}{4} \sum_{i=1}^{N} \left(W_{t_i}^H + W_{t_{i-1}}^H \right) \int_{t_{i-1}}^{t_i} f(s,Z_s) Z_s^{-1} ds \\
&+ \nu Z_0 \sum_{i=1}^{N} \left(W_{t_i}^H - W_{t_{i-1}}^H \right) \\
&+ \frac{\nu}{4} \sum_{i=1}^{N} \left(\int_0^{t_i} f(s,Z_s) Z_s^{-1} ds + \int_0^{t_{i-1}} f(s,Z_s) Z_s^{-1} ds \right) \left(W_{t_i}^H - W_{t_{i-1}}^H \right) \\
&+ \frac{\nu^2}{4} \sum_{i=1}^{N} \left(W_{t_i}^H + W_{t_{i-1}}^H \right) \left(W_{t_i}^H - W_{t_{i-1}}^H \right).
\end{aligned}
$$

Let

$$
Y_t = Y_0 + \sum_{k=1}^{6} \mathfrak{I}_k(N,t,Z_t)
$$

where

$$
\begin{aligned}
\mathfrak{I}_1(N,t_i,Z_{t_i}) &= Z_0 \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(s,Z_s) Z_s^{-1} ds \\
\mathfrak{I}_2(N,t_i,Z_{t_i}) &= \frac{1}{4} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(s,Z_s) Z_s^{-1} ds \left(\int_0^{t_i} f(s,Z_s) Z_s^{-1} ds + \int_0^{t_{i-1}} f(s,Z_s) Z_s^{-1} ds \right) \\
\mathfrak{I}_3(N,t_i,Z_{t_i}) &= \frac{\nu}{4} \sum_{i=1}^{N} \left(W_{t_i}^H + W_{t_{i-1}}^H \right) \int_{t_{i-1}}^{t_i} f(s,Z_s) Z_s^{-1} ds \\
\mathfrak{I}_4(t_i,Z_{t_i}) &= \nu Z_0 \sum_{i=1}^{N} \left(W_{t_i}^H - W_{t_{i-1}}^H \right) \\
\mathfrak{I}_5(N,t_i,Z_{t_i}) &= \frac{\nu}{4} \sum_{i=1}^{N} \left(\int_0^{t_i} f(s,Z_s) Z_s^{-1} ds + \int_0^{t_{i-1}} f(s,Z_s) Z_s^{-1} ds \right) \left(W_{t_i}^H - W_{t_{i-1}}^H \right)
\end{aligned}
$$

$$\mathcal{I}_6(N, t_i, Z_{t_i}) = \frac{\nu^2}{4}\sum_{i=1}^{N}\left(W_{t_i}^H + W_{t_{i-1}}^H\right)\left(W_{t_i}^H - W_{t_{i-1}}^H\right).$$

Set

$$\mathcal{I}(t) = \int_0^t f(s, Z_s) Z_s^{-1} ds.$$

Then it follows that

$$\sum_{k=1}^{3} \mathcal{I}_k(N, t_i, Z_{t_i}) = \Big(\mathcal{I}(t_i) - \mathcal{I}(t_{i-1})\Big) Z_0 + \sum_{i=1}^{N}\Big(\mathcal{I}(t_i) - \mathcal{I}(t_{i-1})\Big)\Big(\frac{(\mathcal{I}(t_i) - \mathcal{I}(t_{i-1})}{4} + \frac{\nu(W_{t_i}^H + W_{t_{i-1}}^H)}{4}\Big).$$

After making use of the definition of Stratonovich integral 3.7, we get the following

$$\lim_{N\to\infty}\sum_{k=1}^{3} \mathcal{I}_k(N, t_i, Z_{t_i}) = Z_0\mathcal{I}(t) + \frac{1}{2}\int_0^t \left(\mathcal{I}(s) + \nu W_s^H\right) \circ d\mathcal{I}(s).$$

Since $\mathcal{I}(s)$ is differentiable, then it follows that

$$\begin{aligned}
\lim_{N\to\infty}\sum_{k=1}^{3} \mathcal{I}_k(N, t_i, Z_{t_i}) &= Z_0\mathcal{I}(t) + \frac{1}{2}\int_0^t \left(\mathcal{I}(s) + \nu W_{t_s}^H\right) dI(s) \\
&= \left(\int_0^t f(s, Z_s) Z_s^{-1} ds\right) Z_0 \\
&\quad + \frac{1}{2}\int_0^t \left(\left(\int_0^s f(u, Z_u) Z_u^{-1} du\right) + \nu W_s^H\right) f(s, Z_s) Z_s^{-1} ds \\
&= \int_0^t f(s, Z_s) Z_s^{-1}\left(Z_0 + \frac{1}{2}\int_0^s f(u, Z_u) Z_u^{-1} du + \frac{\nu}{2} W_s^H\right) ds \\
&= \int_0^t f(s, Z_s) Z_s^{-1} Z_s ds = \int_0^t f(s, Z_s) ds.
\end{aligned}$$

On the other hand

$$\sum_{k=4}^{6}\mathcal{I}_k(N,t_i,Z_{t_i}) = \nu Z_0\sum_{i=0}^{N}\left(W^H_{t_i}-W^H_{t_{i-1}}\right) + \frac{\nu}{2}\sum_{i=0}^{N}\frac{\mathcal{I}(t_i)+\mathcal{I}(t_{i-1})}{2}\left(W^H_{t_i}-W^H_{t_{i-1}}\right) + \frac{\nu^2}{4}\sum_{i=0}^{N}\left(W^H_{t_i}+W^H_{t_{i-1}}\right)\left(W^H_{t_i}-W^H_{t_{i-1}}\right).$$

Once more, we get from Definition 3.7 the following result

$$\begin{aligned}\lim_{N\to\infty}\sum_{k=4}^{6}\mathcal{I}_k(N,t_i,Z_{t_i}) &= \nu Z_0 W^H_t + \frac{\nu}{2}\int_0^t I(s)\circ dW^H_s + \frac{\nu^2}{2}\int_0^t W^H_s\circ dW^H_s \\ &= \nu Z_0 W^H_t + \frac{\nu}{2}\int_0^t\Big(\int_0^s f(u,Z_u)Z_u^{-1}du\Big)\circ dW^H_s + \frac{\nu^2}{2}\int_0^t W^H_s\circ dW^H_s \\ &= \nu Z_0 W^H_t + \frac{\nu}{2}\int_0^t\left(2Z_s-2Z_0-\nu W^H_s\right)\circ dW^H_s + \frac{\nu^2}{2}\int_0^t W^H_s\circ dW^H_s \\ &= \nu\int_0^t Z_s\circ dW^H_s.\end{aligned}$$

The second equality holds since $\int_0^s f(u,Z_u)Z_u^{-1}du = 2Z_s - 2Z_0 - \nu W^H_s$. Now when $N\to\infty$, that is when $\delta t\to 0$, we have

$$\begin{aligned}\lim_{N\to\infty} Y_{\delta t N} &= Y_0 + \lim_{N\to\infty}\sum_{k=1}^{6}\mathcal{I}_k(N,t,Z_t) \\ &= Y_0 + \int_0^t f(s,Z_s)ds + \nu\int_0^t Z_s\circ dW^H_s \\ &= Y_0 + \int_0^t f\big(s,\sqrt{Y_s}\big)ds + \nu\int_0^t \sqrt{Y_s}\circ dW^H_s.\end{aligned}$$

It follows that

$$dY_t = f(t,\sqrt{Y_t})dt + \nu\sqrt{Y_t} \circ dW_t^H,$$

which concludes the proof of this proposition. □

Remark 6.1. The function $f(t,\sqrt{Y_t})$ represents the drift of the generalised *fCIR* process $(Y_t)_{t\geq 0}$.

6.2.2 The generalised fractional Heston-Type model

We may now construct the fractional Heston model under the generalised *fCIR* process that shall be called *"Generalised fractional Heston-type (fHt) model"*. The *fBm* $(W_t^H)_{t\geq 0}$ shall be represented by

$$W_t^H = \int_0^t \kappa_H(s,t)dV_t, \tag{6.8}$$

where $(V_t)_{t\in[0,T]}$ is a standard Brownian motion and where $\kappa_H(s,t)$ is a square integrable kernel given by (2.5). The Brownian motion $(B_t)_{t\in[0,T]}$ (which represents the source of randomness of the stock price process $(X_t)_{t\in[0,T]}$) and Brownian motion $(V_t)_{t\in[0,T]}$ are assumed to be correlated. That is, there exists a constant $\rho \in [-1, 1]$ such that

$$\mathbb{E}\big[B_t V_t\big] = \rho t. \tag{6.9}$$

The relation (6.9) means that there exists a Brownian motion $(\tilde{V}_t)_{t\in[0,T]}$ independent of $(V_t)_{t\in[0,T]}$, that is $\mathbb{E}\big[V_t,\tilde{V}_t\big] = 0$, such that

$$B_t = \rho V_t + \sqrt{1-\rho^2}\tilde{V}_t. \tag{6.10}$$

Therefore, all components of the financial market model under *fCIR* process are summarised as follows:

$$\begin{cases} dA_t = rA_t dt, \\ dX_t = \eta X_t dt + \sigma(Y_t) X_t dB_t, \\ Y_t = Z_t^2 \mathbf{1}_{[0,\tau(\omega)]} \\ dZ_t = \frac{1}{2} f(t,Z_t) Z_t^{-1} dt + \frac{1}{2} \nu dW_t^H \\ W_t^H = \displaystyle\int_0^t \kappa_H(s,t) dV_t \\ B_t = \rho V_t + \sqrt{1-\rho^2} \tilde{V}_t. \end{cases} \tag{6.11}$$

Remark 6.2. For $H > 1/2$ and when $f(t,z) = (\mu - \theta z^2)$ where θ and μ are constants, the generalised *fCIR* process $(Y_t)_{t \geq 0}$ coincides with the *fCIR* process given by Mishura and Yurchenko-Tytarenko (2018). When $f(t,z) = -\theta z^2$, then $(Y_t)_{t \geq 0}$ coincides with the one defined in Mishura et al. (2018).

In addition, when the speed of reversion or the long-run mean are time dependent, that is $\theta = \theta_t$ or $\mu = \mu_t$ with $f(t,z) = (\mu_t - \theta_t z^2)$, the process $(Y_t)_{t \geq 0}$ can be regarded as time-dependent *fCIR* process and the corresponding market shall be called *"time-dependent fHt model"* and shall be considered as an extension of time-dependent Heston model previously discussed by Benhamou et al. (2010). This kind of model has not been investigated thus far to the best of our knowledge.

Taking into consideration Proposition 6.2, the financial market model can be written as

$$\begin{cases} dA_t = rA_t dt \\ dX_t = \eta X_t dt + \sigma(Y_t) X_t dB_t \\ dY_t = f(t, \sqrt{Y_t}) dt + \nu \sqrt{Y_t} \circ dW_t^H \\ W_t^H = \displaystyle\int_0^t \kappa_H(s,t) dV_t \\ B_t = \rho V_t + \sqrt{1-\rho^2} \tilde{V}_t. \end{cases} \tag{6.12}$$

Remark 6.3.

(1) This financial market model is applicable for all $H \in (0,1)$ unlike previous models which only work for $H \geq 1/2$.

(2) The stock price process $(X_t)_{t\geq 0}$ is given by

$$X_t = X_0 \exp\left[\eta t + \int_0^t \sigma(Y_s)dB_s - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds\right]. \tag{6.13}$$

Since the stock price is driven by a standard Brownian motion, it is easy to show from Itô calculus that $(X_t)_{t\geq 0}$ and its infinitesimal return $dR_t := dX_t/X_t$ exist and are unique. This is due to the fact that

$$\int_0^t \mathbb{E}[\sigma^2(Y_s)]ds < \infty \quad \text{and} \quad \int_0^t \mathbb{E}\left[(\sigma(Y_s)X_s)^2\right]ds < \infty.$$

6.2.3 No arbitrage properties

The arbitrage-free property can be determined by the existence of an equivalent martingale measure as discussed previously. Since the stock price process is still driven by the standard Brownian motion, one may follow the idea of Bezborodov et al. (2019, Theorem 4) to prove that the probability measure $\mathbb{Q}$ exists and can be deduced from the following equation:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{(\eta-r)^2}{2}\int_0^T \frac{1}{\sigma^2(Y_s)}ds \right.$$
$$\left. +(\eta-r)\rho\int_0^T \frac{1}{\sigma(Y_s)}dV_s + \sqrt{1-\rho^2}\,(\eta-r)\int_0^T \frac{1}{\sigma(Y_s)}d\tilde{V}_s\right). \tag{6.14}$$

Remark 6.4. The equation (6.14) is well-defined if the Assumption 4.1 is satisfied. In general, the sample paths of the stochastic volatility process must always be strictly positive almost surely. This property is crucial and will be discussed in our next chapter.

Chapter 7

Positiveness and Differentiability of The Generalised Fractional Heston-Type Model

In this chapter, we discuss positiveness of the generalised *fCIR* process of the fractional Heston-type model defined by $Y_t(\omega) = Z_t^2(\omega)\mathbf{1}_{[0,\tau(\omega))}$, where the stochastic process $(Z_t)_{t\geq 0}$ satisfies a singular stochastic differential equation driven by an additive *fBm* given by $dZ_t = \frac{1}{2}\left(f(t, Z_t)Z_t^{-1}dt + \sigma dW_t^H\right)$ as previously. We also show that both the fractional volatility and stock price processes are Malliavin differentiable. This last property is an open-door to further applications of Malliavin calculus in quantitative finance and all results of standard Heston model case (See e.g. Alòs and Lorite (2021) for a summary) can be extended to fractional Heston models.

7.1 Positiveness of the generalised *fCIR* process

Positiveness is an important property that deserves particular attention. Recall that the standard Cox-Ingersoll-Ross process is positive when the Feller condition $2\theta\mu > \nu^2$ holds. For the generalised *fCIR* process, we firstly consider the Assumption 6.1 given again by

(i) The drift function $g : [0, \infty) \times (0, \infty) \to (-\infty, \infty)$ defined by $g(t,z) = f(t,z)/z$ is continuous and admits a continuous partial derivative with respect to z on $(0, \infty)$. In addition, there exists a number $z^* > 0$ such that for every $z > z^*$, $g(t,z) < 0$, for all $t \geq 0$, and

(ii) for any $T > 0$, there exists $z_T > 0$ such that $f(t, z) > 0$ for all $0 < t \leq T$ and $0 \leq z \leq z_T$.

7.1.1 Positiveness analysis of *fCIR* process for $H > 1/2$

Theorem 7.1. *Let $(Z_t)_{t \geq 0}$ be a stochastic process defined by*

$$dZ_t = \frac{f(t,Z_t)}{2Z_t}dt + \frac{\nu}{2}dW_t^H, \; Z_0 > 0, \tag{7.1}$$

where $\nu > 0$, $(W_t^H)_{t \geq 0}$ is a fBm with $H > \frac{1}{2}$ and $f : [0, \infty) \times [0, \infty)$ is a continuous function that satisfies Assumption 6.1. Then the sample paths $Z_t(\omega)$ are positive almost surely.

Proof. The existence and uniqueness of the stochastic process $(Z_t)_{t \geq 0}$ were discussed in Theorem 6.1. We shall now prove that under the same assumption, the process $(Z_t)_{t \geq 0}$ is positive and will never hit zero almost surely. To achieve that, we have to show that

$$\mathbb{P}(\omega \in \Omega : \tau = \infty) = 1,$$

where $\tau(\omega) = \inf\{t > 0 : Z_t(\omega) = 0\}$. We prove this by contradiction by assuming that $\mathbb{P}(\omega \in \Omega : \tau = \infty) < 1$ or equivalently $\mathbb{P}(\omega \in \Omega : \tau < T) > 0$, for any $T > 0$. As discussed in our first chapter, the sample paths of *fBm* $(W_t^H)_{t \geq 0}$ is locally Hölder continuous of order $H - \alpha$ for each small number $\alpha > 0$. Therefore, we can fix a subset Ω_1 of the underlying sample space Ω with $\mathbb{P}(\Omega_1) = 1$ such that for each $\omega \in \Omega_1$, $\alpha > 0$,

$$\left|W_t^H(\omega) - W_s^H(\omega)\right| \leq c|t - s|^{H-\alpha}, \; \forall s, t \in [0, T]$$

where $c = c(T, \omega, \alpha)$ is a random constant depending on T, ω and α. Our assumption $\mathbb{P}(\tau < T) > 0$ implies $\mathbb{P}(\tau < T) = \mathbb{P}\{\omega \in \Omega_1 : \tau(\omega) < T\} > 0$. Now choose $\omega \in \Omega_1$ with $\tau(\omega) < T$. It is given that the process (Z_t) starts at the point $Z_0 > 0$. Consider a small number ε such that $0 < \varepsilon < Z_0$ and let τ_ε be the last time the process $(Z_t(\omega))_{t\geq 0}$ hits ε before reaching zero, that is,

$$\tau_\varepsilon(\omega) = \sup\{t \in (0,\tau(\omega)) : Z_t(\omega) = \varepsilon\}. \tag{7.2}$$

Since

$$Z_t = Z_0 + \frac{1}{2}\int_0^t f(s,Z_s)Z_s^{-1}ds + \frac{\nu}{2}W_t^H,$$

then

$$Z_\tau - Z_{\tau_\varepsilon} = \frac{1}{2}\int_{\tau_\varepsilon}^{\tau} f(s,Z_s)Z_s^{-1}ds + \frac{\nu}{2}\left(W_\tau^H - W_{\tau_\epsilon}^H\right).$$

Since clearly, $Z_\tau = 0$ and $Z_{\tau_\varepsilon} = \varepsilon$, then

$$\frac{1}{2}\int_{\tau_\varepsilon}^{\tau} f(s,Z_s)Z_s^{-1}ds + \frac{\nu}{2}\left(W_\tau^H - W_{\tau_\epsilon}^H\right) = -\varepsilon$$

or equivalently,

$$\frac{\nu}{2}\left(W_\tau^H - W_{\tau_\epsilon}^H\right) = -\varepsilon - \frac{1}{2}\int_{\tau_\varepsilon}^{\tau} f(s,Z_s)Z_s^{-1}ds.$$

As $f(t,z)$ is a positive function and $Z_s > 0$ for all $s < \tau$, then clearly

$$\int_{\tau_\varepsilon}^{\tau} f(s,Z_s)Z_s^{-1}ds > 0.$$

This implies that

$$\frac{\nu}{2}\left|W_\tau^H - W_{\tau_\epsilon}^H\right| = \varepsilon + \frac{1}{2}\int_{\tau_\varepsilon}^{\tau} f(s,Z_s)Z_s^{-1}ds$$

or equivalently

$$\nu\left|W_\tau^H - W_{\tau_\epsilon}^H\right| = 2\varepsilon + \int_{\tau_\varepsilon}^{\tau} f(s,Z_s)Z_s^{-1}ds.$$

Since $\omega \in \Omega_1$, and $\tau_\varepsilon, \tau \in [0,T]$, then

$$\left|W_\tau^H - W_{\tau_\epsilon}^H\right| < c\left|\tau - \tau_\varepsilon\right|^{H-\alpha}.$$

Hence

$$2\varepsilon + \int_{\tau_\varepsilon}^{\tau} f(s,Z_s)Z_s^{-1}ds \le \nu c\big|\tau - \tau_\varepsilon\big|^{H-\alpha}.$$

On the other hand, for all $s \in [\tau_\varepsilon, \tau]$, it is the case that $Z_s \in [0,\varepsilon]$. Let

$$\tilde{f} = \inf\{f(t,z) : 0 \le t \le T, 0 \le z \le Z_0\}.$$

The infimum $\tilde{f}$ exists because f is continuous. Since $0 < \tau_\epsilon < \tau \le T$ and $0 < \varepsilon < Z_0$, then for all $s \in [\tau_\varepsilon, \tau]$, $f(s, Z_s) \ge \tilde{f}$. This implies $f(s,Z_s)Z_s^{-1} \ge \tilde{f}\varepsilon^{-1}$ that yields

$$\int_{\tau_\varepsilon}^{\tau} f(s,Z_s)Z_s^{-1}ds \ge \int_{\tau_\varepsilon}^{\tau} \tilde{f}\varepsilon^{-1}ds = \tilde{f}\varepsilon^{-1}(\tau - \tau_\varepsilon). \tag{7.3}$$

Therefore, $2\varepsilon + \tilde{f}\varepsilon^{-1}(\tau - \tau_\varepsilon) \le \nu c\big|\tau - \tau_\varepsilon\big|^{H-\alpha}$ from which it follows that

$$\tilde{f}\varepsilon^{-1}(\tau - \tau_\varepsilon) - c\nu\big|\tau - \tau_\varepsilon\big|^{H-\alpha} + 2\varepsilon \le 0. \tag{7.4}$$

Consider the function F_ε defined by

$$F_\varepsilon(x) = \tilde{f}\varepsilon^{-1}x - c\nu x^{H-\alpha} + 2\varepsilon,$$

that is, $F_\varepsilon(x)$ is obtained by replacing $\tau - \tau_\varepsilon$ with x. Then the inequality (7.4) yields

$$F_\varepsilon(\tau - \tau_\varepsilon) \le 0, \tag{7.5}$$

for all $\omega \in \Omega_1$ such that $\tau(\omega) < T$. The next step in this proof is to show that the inequality in (7.5) does not hold. First of all, it is clear that $F_\varepsilon(0) = 2\varepsilon > 0$. We shall indeed obtain that there exists a fixed number $\varepsilon^* > 0$, such that for all $0 < \varepsilon < \varepsilon^*$, it is the case that $F_\varepsilon(x) > 0$ for all $x > 0$. This will contradict (7.5) from which it will follow that $\mathbb{P}(\tau < T) = 0$ for a fixed time T. To show that $F_\varepsilon(x) > 0$ for all $x > 0$, we need to find all critical points of $F_\varepsilon(x)$. Clearly, the first and second derivatives with respect to x are respectively given by

$$F'_\varepsilon(x) = \tilde{f}\varepsilon^{-1} - c\nu(H - \alpha)x^{H-\alpha-1}$$

and

$$F_\varepsilon''(x) = -c\nu(H-\alpha)(H-\alpha-1)x^{H-\alpha-2}.$$

It is clear that $F_\varepsilon(x)$ is convex as $F_\varepsilon''(x) > 0$. Moreover, the critical point $\hat{x}$ of $F_\varepsilon(x)$ is given by

$$\hat{x} = \left(\frac{\tilde{f}\varepsilon^{-1}}{c\nu(H-\alpha)}\right)^{\frac{1}{H-\alpha-1}}.$$

Note that $\hat{x}$ is well defined since $\tilde{f} \geq 0$. Hence,

$$\begin{aligned}
F_\varepsilon(\hat{x}) &= \tilde{f}\varepsilon^{-1}\hat{x} - c\nu\hat{x}^{H-\alpha} + 2\varepsilon \\
&= \hat{x}\left(\tilde{f}\varepsilon^{-1} - c\nu\hat{x}^{H-\alpha-1}\right) + 2\varepsilon \\
&= \hat{x}\left(\tilde{f}\varepsilon^{-1} - \frac{\tilde{f}\varepsilon^{-1}}{H-\alpha}\right) + 2\varepsilon \\
&= \frac{\hat{x}\tilde{f}\varepsilon^{-1}(H-\alpha-1)}{H-\alpha} + 2\varepsilon \\
&= \left(\frac{\tilde{f}^{H-\alpha}}{c\nu(H-\alpha)^{2+\alpha-H}}\right)^{\frac{1}{H-\alpha-1}} \varepsilon^{\frac{H-\alpha}{1-H+\alpha}}(H-\alpha-1) + 2\varepsilon.
\end{aligned}$$

Since $H-\alpha-1 < 0$, then

$$F_\varepsilon(\hat{x}) \geq \left(\frac{\tilde{f}^{H-\alpha}}{c\nu(H-\alpha)^{2+\alpha-H}}\right)^{\frac{1}{H-\alpha-1}} \varepsilon^{\frac{H-\alpha}{1-H+\alpha}}(H-\alpha-1) + 2\varepsilon.$$

Set

$$\begin{aligned}
\kappa &= -\left(\frac{\tilde{f}^{H-\alpha}}{c\nu(H-\alpha)^{2+\alpha-H}}\right)^{\frac{1}{H-\alpha-1}} (H-\alpha-1) \\
q &= \frac{H-\alpha}{1-H+\alpha}.
\end{aligned}$$

Clearly, since $H > 1/2$, we can choose α so small that $H > \frac{1}{2} + \alpha$ and obtain that $q \geq 1$. Then it follows that

$$F_\varepsilon(\hat{x}) \geq -\kappa\varepsilon^q + 2\varepsilon.$$

It is now an easy matter to show that there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, it is the case that

$$F_\varepsilon(\hat{x}) \geq -\kappa\varepsilon^q + 2\varepsilon > 0.$$

Indeed, choosing $\varepsilon^* \leq \left(\frac{2}{\kappa}\right)^{\frac{1}{q-1}}$ yields $F_\varepsilon(\hat{x}) > 0$ for all $x > 0$. This concludes the proof of the theorem. □

To illustrate the above result of Theorem 7.1, we consider a generalisation of a time-dependent *CIR* process commonly known as the *"extended CIR"* process. Recall that the extended *CIR* process is defined by

$$dY_t = \theta_t(\mu_t - Y_t)dt + \nu\sqrt{Y_t}dW_t, \ Y_0 > 0 \tag{7.6}$$

where θ_t is the time-dependent speed of reversion towards its time-dependent long run mean μ_t of the process $(Y_t)_{t\geq 0}$ and ν a positive parameter. This model was initially introduced by Hull and White (1990) and it is widely used in both short interest rates and stochastic volatilities modelling as time-dependent Heston model (See e.g. Benhamou et al. (2010)). The choice of parameters θ_t and μ_t are done through market calibration. The general case where the Brownian motion is replaced with a *fBm* shall be called *"time-dependent fCIR process"* and takes the form

$$Y_t = Z_t^2 \mathbf{1}_{[0,\tau)}, \quad t \geq 0 \tag{7.7}$$

where

$$dZ_t = \frac{f(t,Z_t)}{2Z_t}dt + \frac{\nu}{2}dW_t^H, \quad Z_0 > 0. \tag{7.8}$$

with the drift function given by

$$f(t,z) = \theta_t\big(\mu_t - z^2\big). \tag{7.9}$$

We shall then simulate the corresponding process $(Y_t)_{t\in[0,T]}$ on a finite interval $[0,T]$ using the Euler method (See e.g. Higham et al. (2002) for more details about the method). Subdivide the interval $[0,T]$ into N subintervals of equal length $\delta t = T/N$ with end points $0 = t_0, t_1, t_2, \ldots, t_N = T$. The corresponding discrete version of the process $(Y_t)_{t\geq 0}$ is given by

$$\hat{X}_{t_i} = \hat{Z}^2_{t_i},$$

where $Z_0 > 0$ and for $i = 1,2,\ldots,N$,

$$\hat{Z}_{t_i} = \begin{cases} \hat{Z}_{ti-1} + \dfrac{f(t_{i-1},\hat{Z}_{t_{i-1}})}{2\hat{Z}_{t_{i-1}}}\delta t + \dfrac{\nu}{2}\delta W^H_{t_i} & \text{if } Z_{t_{i-1}} > 0, \\ 0 & \text{otherwise} \end{cases}$$

with $\delta W^H_{t_i} = W^H_{t_i} - W^H_{t_{i-1}}$.

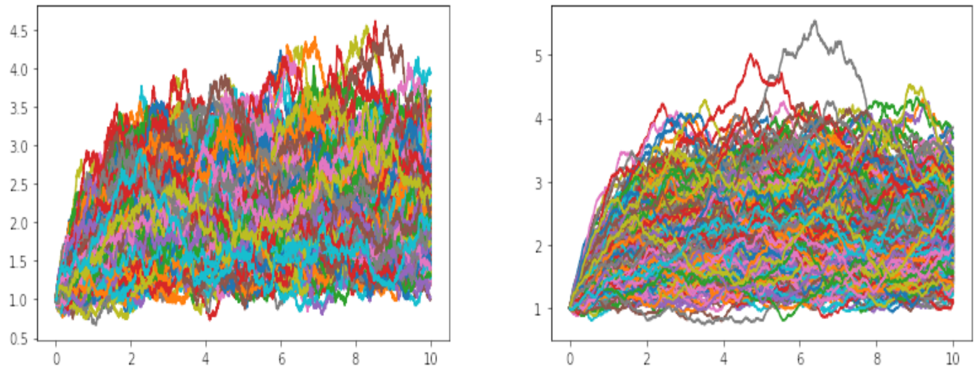
In what follows, we shall consider two different drift functions for simulation of the process (7.8). Firstly, let $\theta_t = \theta > 0$ and $\mu_t = c + \frac{\nu^2}{2\theta}\Big(1 - e^{-2\theta t}\Big)$, where $c > 0$ is a constant. This yields the drift function

$$f(t,z) = \frac{\nu^2}{2\theta}\Big(1 - e^{-2\theta t}\Big) + (c - \theta z^2), \tag{7.10}$$

It is clear that the function $f(t,z)$ satisfies Assumption 6.1.

We simulate 1000 sample paths of the process $(Y_t)_{t\in[0,T]}$ where $T = 10$, volatility $\nu = 0.4$ starting at $Z_0 = 1$ with time-step $\delta t = 0.001$ and the results are given in Figure 7.1 (with given parameters c, θ and H). All the sample paths in Figures 7.1 where $H > 0.5$ are strictly positive (do not hit zero) in line with Theorem 7.1.

Figure 7.1: **(A)** Sample paths of *fCIR* process for $H > 1/2$



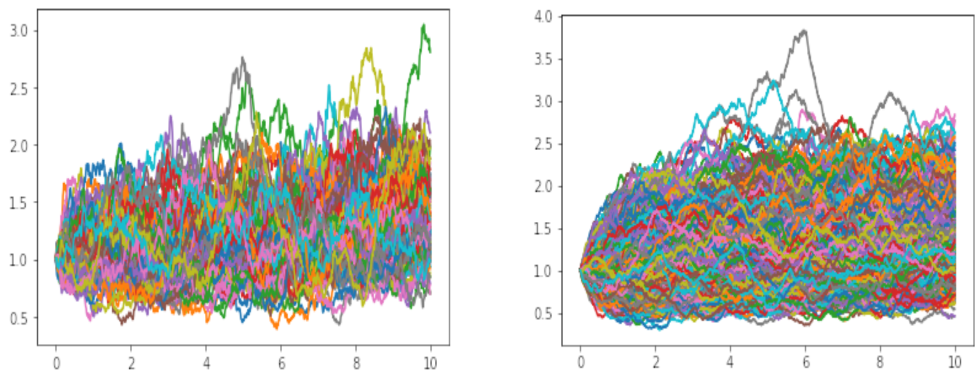
$\theta = 1,\ \ c = 2,\ \ H = 0.6$ $\qquad$ $\theta = 1,\ \ c = 2,\ \ H = 0.8$

For the second illustration, we consider again $\theta_t = \theta > 0$ and $\mu_t = \left(1 + \frac{c}{\theta}\right) e^{ct} + \frac{\nu^2}{2\theta}\left(1 - e^{-2\theta t}\right)$, where $c > 0$ is a constant. This yields the functions

$$f(t,z) = \Big(\theta + c\Big)e^{ct} + \frac{\nu^2}{2}\Big(1 - e^{-2\theta t}\Big) - \theta z^2, \qquad (7.11)$$

As previously, we considered 1 000 realisations of the sample paths of the stochastic process $(Y_t)_{t\in[0,10]}$ with volatility $\nu = 0.4$ starting at $v_0 = 1$ with time-step $\delta t = 0.001$. We have observed similar results and the output is given in Figure 7.2.

Figure 7.2: **(B)** Sample paths of *fCIR* process for $H > 1/2$



$\theta = 1,\ \ c = 0.02,\ \ H = 0.6$ $\qquad$ $\theta = 1,\ \ c = 0.02,\ \ H = 0.8$

7.1.2 Analysis of Positiveness of $(Y_t)_{t\geq 0}$ for $H < 1/2$

The proof of Theorem 7.1 relies heavily on the fact that $H > 1/2$ and in general the result is not true for $H < 1/2$. We shall consider a sequence of continuous functions

$$f_k(t, z) : [0, \infty) \times [0, \infty) \to (-\infty, +\infty), \;\; k \in \mathbb{N}$$

such that each function f_k satisfies the Assumption 6.1. Moreover for each point $(t,z) \in [0, \infty) \times [0, \infty)$, $f_k(t,z) \leq f_{k+1}(t,z)$ and $\lim_{k\to\infty} f_k(t,z) = \infty$. Now, consider for each k, the stochastic process $(Z_t^{(k)})_{t\geq 0}$ defined by

$$Z_t^{(k)} = \begin{cases} Z_0 + \displaystyle\int_0^t \frac{f_k(t,Z_s^{(k)})}{Z_s^{(k)}} ds + \frac{\nu}{2} W_t^H & \text{if } t < \tau^{(k)}(\omega) \\ 0 & \text{otherwise,} \end{cases}$$

where $\tau^{(k)}(\omega) = \inf\{t \geq 0 : Z_t^{(k)}(\omega) = 0\}$. We have the following result:

Theorem 7.2. *For any $T > 0$, let $(Z_t)_{t\in[0,T]}$ be a stochastic process defined by (7.1) driven by a fBm with Hurst parameter $H < 1/2$. Then*

$$\mathbb{P}(\omega \in \Omega : \tau^{(k)}(\omega) > T) \to 1 \quad \textit{as } k \to \infty.$$

Proof. The proof of this more general theorem is based on Mishura and Yurchenko-Tytarenko (2018). Firstly, we assume that there exists $T > 0$, an increasing sequence $(k_n)_{n>1}$ and $p > 0$ such that

$$\mathbb{P}(\tau^{(k_n)} \leq T) \to p, \quad k_n \to \infty. \tag{7.12}$$

As in the previous proof, for fixed $T > 0$, consider a point z_T small enough such that $0 < z_T < Z_0$ and $T_1 > 0$ is the first time the process $(Z_t)_{t\geq 0}$ hits the value z_T. Take $0 < \varepsilon < z_T$. Then uniformly for all $k \in \mathbb{N}$, $f_k(t, z) > 0$ for all $T_1 \leq t \leq T$ and $0 \leq z \leq \varepsilon$. Let $\tilde{f} = \inf\{f(t,z) : T_1 \leq t \leq T, 0 \leq z \leq z_T, \}$. Clearly $\tilde{f} > 0$. Also let $\tau_\varepsilon^{(k_n)} = \sup\{t \in (0,\tau) : Z_t^{(k_n)} = \varepsilon\}$ be the last hitting time of ε before reaching zero. Let

$$\tilde{f}_k = \inf\{f_k(t,z) : S \le t \le T, 0 \le z \le Z_0\},\ k > 0.$$

Moreover, for a small number $\alpha > 0$, the subspace Ω_1 of probability 1 such that for all $s, t \in [0,T]$, $|W_t^H(\omega) - W_s^H(\omega)| \le c|t-s|^{H-\alpha}$, where $c = c(T,\omega,\alpha)$ is a constant depending on T, ω and α. Let

$$\Omega_T^{(k_n)} = \left\{\omega \in \Omega_1 : \tau^{(k_n)} \le T\right\}. \tag{7.13}$$

Then, for all $\omega \in \Omega_T^{(k_n)}$, similar arguments as in the proof of Theorem 7.1, yields

$$Z^{(k_n)}_{\tau^{(k_n)}} - Z^{(k_n)}_{\tau_\varepsilon{}^{(k_n)}} = -\varepsilon = \frac{1}{2}\int_{\tau_\varepsilon{}^{(k_n)}}^{\tau^{(k_n)}} f_{k_n}(t, Z_s^{(k_n)}) \left(Z_s^{(k_n)}\right)^{-1} ds + \frac{\nu}{2}\left(W^H_{\tau^{(k_n)}} - W^H_{\tau_\varepsilon{}^{(k_n)}}\right)$$

In a similar way as in the previous proof, for all $s \in [\tau_\varepsilon{}^{(k_n)}, \tau^{(k_n)}]$, $f_{k_n}(t, Z_s^{(k_n)}) \ge \tilde{f}_{k_n}$ and therefore

$$f_{k_n}(t, Z_s^{(k_n)}) \left(Z_s^{(k_n)}\right)^{-1} \ge \tilde{f}_{k_n}\varepsilon^{-1}.$$

Since

$$\left|W^H_{\tau^{(k_n)}} - W^H_{\tau_\varepsilon^{(k_n)}}\right| \le c\left|\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\right|^{H-\alpha},$$

it follows (as in the previous proof) that

$$c\nu\left(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\right)^{H-\alpha} \ge \tilde{f}_{k_n}\varepsilon^{-1}(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}) + 2\varepsilon.$$

This implies in particular that

$$\begin{cases} c\nu\left(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\right)^{H-\alpha} \ge 2\varepsilon \\ c\nu\left(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\right)^{H-\alpha} \ge \tilde{f}_{k_n}(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)})\varepsilon^{-1}. \end{cases} \tag{7.14}$$

We shall show that the two inequalities are contradictory. Elementary calculations show that the second inequality in (7.14) is equivalent to

$$\left(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\right) \leq \left(\frac{1}{c\nu}\tilde{f}_{k_n}\varepsilon^{-1}\right)^{\frac{1}{H-\alpha-1}}.$$

Taking both side with power $H-\alpha$ and thereafter multiplying both sides by $c\nu$ yields

$$\begin{aligned} c\nu\left(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\right)^{H-\alpha} &\leq c\nu\left(\frac{1}{c\nu}\tilde{f}_{k_n}\varepsilon^{-1}\right)^{\frac{H-\alpha}{H-\alpha-1}} \\ &= \left(c^{\frac{1}{1-H+\alpha}}\right)\left(\nu^{\frac{1}{1-H+\alpha}}\right)\varepsilon^{\frac{H-\alpha}{1-H+\alpha}}\left(\tilde{f}_{k_n}\right)^{-\frac{H-\alpha}{1-H+\alpha}}. \end{aligned}$$

On the right-hand side, the Hölder constant $c = c(\omega)$ is random depending on the path ω of *fBm*. As in Mishura and Yurchenko-Tytarenko (2018), it is well-known that $c(\omega)$ is finite almost surely and hence as $\mathbb{P}\left(\bigcap_{n>1}\Omega_T^{(k_n)}\right) = p > 0$, then there exists a (non-random) constant M and a subset E of $\bigcap_{n>1}\Omega_T^{(k_n)}$ with $\mathbb{P}(E) > 0$ such that $c = c(\omega) \leq M$ for all $\omega \in E$. Therefore, everywhere in E,

$$c\nu\left(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\right)^{H-\alpha} \leq \left(M^{\frac{1}{1-H+\alpha}}\right)\left(\nu^{\frac{1}{1-H+\alpha}}\right)\varepsilon^{\frac{H-\alpha}{1-H+\alpha}}\left(\tilde{f}_{k_n}\right)^{-\frac{H-\alpha}{1-H+\alpha}}.$$

Clearly M and ν are constants. Moreover, since $f_n(t,z) \to \infty$ as $n \to \infty$ (for every (t,z)) then clearly also $\tilde{f}_{k_n} \to \infty$ for $k_n \to \infty$. Hence

$$\lim_{k_n\to\infty}\left(\tilde{f}_{k_n}\right)^{-\frac{H-\alpha}{1-H+\alpha}} = 0,$$

because $-\frac{H-\alpha}{1-H+\alpha} < 0$. Then clearly, for any given $\varepsilon > 0$, we can choose k_n very large (depending on ε) such that

$$\left(M^{\frac{1}{1-H+\alpha}}\right)\left(\nu^{\frac{1}{1-H+\alpha}}\right)\varepsilon^{\frac{H-\alpha}{1-H+\alpha}}\left(\tilde{f}_{k_n}\right)^{-\frac{H-\alpha}{1-H+\alpha}} < 2\varepsilon.$$

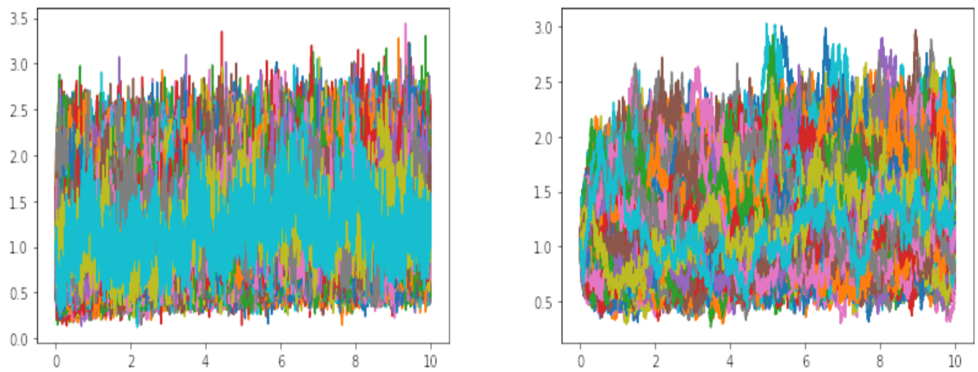
This yields

$$c\nu\left(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\right)^{H-\alpha} < 2\varepsilon,$$

which contradicts the first inequality in (7.14). This concludes the proof of the theorem. □

To illustrate this theorem, we consider again the extended *fCIR* process defined by (7.7), (7.8) and (7.10). We simulate 1000 sample paths of the stochastic volatility $(Y_t)_{t\in[0,T]}$ where $T = 10$, the volatility of volatility $\nu = 0.4$ starting at $Z_0 = 1$ with time-step $\delta t = 0.001$ and the results are given in Figure 7.3 (with given parameters c, θ and $H < 1/2$).

Figure 7.3: Sample paths of *fCIR* process for $H < 1/2$



$\theta = 1,\ c = 0.02,\ H = 0.1$ $\qquad$ $\theta = 1,\ c = 0.02,\ H = 0.4$

7.2 Malliavin Differentiability of $(Z_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$

7.2.1 Differentiability of $(Z_t)_{t\geq 0}$

In this section, we show that the generalised *fCIR* process is Malliavin differentiable. Before this, it is advisable to investigate previous results for the standard Heston model discussed by Alòs and Ewald (2008) who considered the Cox-Ingersoll-Ross process (4.3) of the form

$$dY_t = (\mu - \theta Y_t)dt + \nu\sqrt{Y_t}dB_t,$$

where μ, θ and ν are positive constants and $(B_t)_{t\geq 0}$ is a standard Brownian motion as discussed in our previous chapter. They introduced the square root process $Z_t := \sqrt{Y_t}$, where $(Z_t)_{t\geq 0}$ is stochastic process that satisfies

$$dZ_t = \left(\left(\frac{\mu}{2} - \frac{\nu^2}{8}\right)\frac{1}{Z_t} - \frac{\theta}{2}Z_t\right)dt + \frac{\nu}{2}dB_t. \tag{7.15}$$

This differential equation (7.15) was obtained by using the Itô formula. Firstly, they constructed the approximation processes $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ as solutions of the following stochastic differential equation:

$$dZ_t^\epsilon = \left(\left(\frac{\mu}{2} - \frac{\nu^2}{8}\right)\Lambda_\epsilon(Z_t^\epsilon) - \frac{\theta}{2}Z_t^\epsilon\right)dt + \frac{\nu}{2}dW_t, \tag{7.16}$$

where

$$\Lambda_\epsilon(z) = \Phi_\epsilon(z)z^{-1}, \quad \Lambda_\epsilon(0) = 0, \tag{7.17}$$

with $\Phi_\epsilon(z)$ a differentiable function defined by

$$\Phi_\epsilon(z) = \begin{cases} 1, & \text{if } \ z < \epsilon \\ 0, & \text{if } \ z \geq 2\epsilon \end{cases}$$

and $\Phi_\epsilon(z) \leq 1$ for all $z \in \mathbb{R}$. This kind of approximations (7.16) were also discussed by Altmayer and Neuenkirch (2015) and Mishura and Yurchenko-Tytarenko (2019). Alòs and Ewald (2008) proved that $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ converges to Z_t in $L^2(\Omega)$ and are both Malliavin differentiable. They also derived an approximation of option price formula using Malliavin calculus.

In light of these previous constructions, we need an additional weak assumption on the drift function $g(t,z)$ to construct our approximations processes $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ of $(Z_t)_{t\geq 0}$.

Assumption 7.1.

- For $H > 1/2$, the drift function $g(t,z) = f(t,z)/z$ is monotonic in z, that is for all $z_1 \in \mathbb{R}$ such that $z_1 \leq z$,

$$g(t,z_1) \leq g(t,z).$$

- For $H \leq 1/2$, the function $g(t,z)$ satisfies the linear growth condition, that is, for any positive constant c,

$$|g(t,z)| \leq c(1+|z|).$$

Under this assumption, we may construct the approximating sequence $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ that satisfies the following differential equation:

$$dZ_t^\epsilon = \frac{1}{2} f(t,Z_t^\epsilon)\Lambda_\epsilon(Z_t^\epsilon)dt + \frac{\sigma}{2} dW_t^H, \qquad Z_0^\epsilon = Z_0 > 0, \tag{7.18}$$

where the function $\Lambda_\epsilon(z)$ in (7.18) is defined by

$$\Lambda_\epsilon(z) = (z\mathbf{1}_{\{z>0\}} + \epsilon)^{-1}. \tag{7.19}$$

It is easy to verify that $\Lambda_\epsilon(z) > 0$ for all $\epsilon > 0$. As a straight consequence, the drift of $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ is also positive. In addition, $\lim_{z\to 0} \Lambda_\epsilon(z) = \epsilon^{-1}$, $\lim_{z\to\infty} \Lambda_\epsilon(z) = 0$ and

$$\Lambda_\epsilon'(z) = \begin{cases} 0, & \text{if } \ z \leq 0 \\ -\frac{1}{(z+\epsilon)^2}, & \text{if } \ z > 0 \end{cases} \tag{7.20}$$

In addition, the drift function of (7.17) given by $g(t,z) = f(t,z)\Lambda_\epsilon(z)$ verifies Assumption 6.1. Consequently from Theorem 6.1, the process $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ has a unique solution which is continuous and positive up to time of the first visit zero. The main motivation of constructing the process $Z_t^\epsilon(\omega)$ is to enable its sample paths to be strictly positive everywhere almost surely for all Hurst parameters $H \in (0,1)$. The next step is to show that for every $t \geq 0$, the sequence Z_t^ϵ converges to Z_t in L^p as $\epsilon \to 0$.

Proposition 7.3. *The sequence of estimated random variables Z_t^ϵ defined by (7.18) with its drift that verifies the Assumption 7.1, converges to Z_t in $L^p(\Omega)$ for all $p \geq 1$.*

Proof. We discuss the proof of this proposition in three separate steps in line with different Hurst parameters: $H = 1/2$, $H > 1/2$ and $H < 1/2$.

Case 1. $H = 1/2$. This case was discussed previously by Alòs and

Ewald (2008, Proposition 2.1) under a specific construction of the function $\Lambda_\epsilon(z)$ given by (7.17).

Case 2. For $H > 1/2$, the dominated convergence theorem shall be applied. Firstly, we need to show the pointwise convergence of the approximated stochastic process $(Z_t^\epsilon)_{t\geq 0}$ towards $(Z_t)_{t\geq 0}$ that satisfies (7.1), that is $\lim_{\epsilon\to 0} Z_t^\epsilon = Z_t$. For this, let $\tau_\epsilon(\omega) = \inf\{t \geq 0 : Z_t(\omega) \leq \epsilon\}$ be the first time the process $(Z_t)_{t\geq 0}$ hits ϵ . Since the sample paths of the stochastic process $(Z_t)_{t\geq 0}$ are positive everywhere almost surely as discussed in Theorem 7.1, then $\mathbb{P}(\omega \in \Omega : \tau_0 = \infty) = 1$ a.s. and consequently, $\lim_{\epsilon\to 0} \tau_\epsilon = \infty$ almost surely.

Now, denote $(Z_t^{\tau_\epsilon})_{t\in[0,\tau_\epsilon]}$ be the stochastic process $(Z_t)_{t\geq 0}$ up to the stopping time τ_ϵ. Then, for all $t \in [0,\tau_\epsilon]$ and using the definition of $\Lambda_\epsilon(z)$ given by (7.19), $Z_t^{\tau_\epsilon} = Z_t^\epsilon$ almost surely when $\epsilon \to 0$ since the drift function $f(t,z)$ verifies Assumption 7.1, that is monotonicity property.

Again, the positiveness of $(Z_t)_{t\geq 0}$ means that $\lim_{\epsilon\to 0} Z_t^{\tau_\epsilon} = Z_t$ *a.s.* We may conclude that $\lim_{\epsilon\to 0} Z_t^{\tau_\epsilon} = \lim_{\epsilon\to 0} Z_t^\epsilon = Z_t$ almost surely and for all $t \geq 0$.

On the other hand, the result from Hu et al. (2008, Theorem 3.1) shows that for a fixed $T > 0$ and for all $p \geq 1$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|Z_t\big|^p\Big] = C < \infty,$$

where $C = C(p,H,\gamma,\beta,T,Z_0)$ is a non-random constant taking the form

$$C = C_1(1 + Z_0)\exp\left[C_2\Big(1 + \big|\big|W^H\big|\big|^{\frac{\gamma}{\beta(\gamma-1)}}\Big)\right],$$

where $\beta \in (\frac{1}{2}, H)$, $\gamma > \frac{2\beta}{2\beta-1}$, $C_1 = C_1(\gamma,\beta,T)$ and $C_2 = C_2(\gamma,\beta,T)$ are nonrandom constants depending on parameters γ, β, T, and

$$||W^H|| = \sup_{s\geq 0,\, t\leq T}\left\{\frac{|W_s^H - W_t^H|}{|s-t|^\beta}\right\}.$$

This result also implies that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|Z_t^\epsilon\big|^p\Big] = C(p,H,\gamma,\beta,T,Z_0) < \infty.$$

It follows that $\sup_{t\in[0,T]}\big\{\big|Z_t^\epsilon(\omega)\big|\big\} \in L^p(\Omega)$ which yields the desired L^p convergence.

Case 3. For $H < 1/2$, we consider a sequence of an increasing drift functions $f_k(t,z),\ k \in \mathbb{N}$ and define the stochastic process $(Z_t^{(\epsilon,k)})_{t\geq 0}$ as follows:

$$Z_t^{(\epsilon,k)} = \begin{cases} Z_0 + \dfrac{1}{2}\displaystyle\int_0^t f_k\left(t,Z_s^{(\epsilon,k)}\right)\Lambda\left(Z_s^{(\epsilon,k)}\right)ds + \dfrac{\nu}{2}W_t^H & \text{if } t < \tau^{(k)}(\omega) \\ 0 & \text{otherwise,} \end{cases}$$

where the function $\Lambda(z)$ is defined by (7.18) and $\tau^{(k)}(\omega) = \inf\{t \geq 0 : Z_t^{(\epsilon,k)}(\omega) = 0\}$ is the first time that the stochastic process $(Z_t^{(\epsilon,k)})_{t\geq 0}$ hits zero. If we now define $\tau^{(\epsilon,k)}(\omega) = \inf\{t \geq 0 : Z_t^{(\epsilon,k)}(\omega) \leq \epsilon\}$ be the first time the process $(Z_t^{(\epsilon,k)})_{t\geq 0}$ hits ϵ, then from theorem 7.2, for any fixed $T > 0$, $\mathbb{P}(\omega \in \Omega : \tau^{(\epsilon,k)} > T) \to 1$ as $k \to \infty$. This implies that $\lim_{(\epsilon,k)\to(0,\infty)} \tau^{(\epsilon,k)} = \tilde{T} > T$ almost surely. This is because the process $(Z_t^{(\epsilon,k)})_{t\geq 0}$ remains positive up to time $\tilde{T}$ which is not necessary equal to infinity unlike the previous case.

After using similar arguments of **Case 2**, one may conclude that $\lim_{\epsilon\to 0} Z_t^{\tau_\epsilon} = \lim_{\epsilon\to 0} Z_t^\epsilon = Z_t$ for all $t \in [0,\tilde{T}]$. Next, we need to show that $\mathbb{E}\big[\sup_{t\in[0,T]}\big|Z_t\big|^p\big] < \infty$. To achieve this, we borrow some ideas from Mishura and Yurchenko-Tytarenko (2019).

Firstly, let $\tilde{Z}_0$ be a small positive value less than the initial value Z_0 such that $0 < \tilde{Z}_0 < Z_0$ and let $\tau_1 = \tau_1(\epsilon,\omega)$ be the last time the stochastic process $(Z_t^\epsilon)_{t\geq 0,\, \epsilon>0}$ hits (or before hits) $\tilde{Z}_0$, that is,

$$\tau_1(\epsilon,\omega) = \sup\{t \geq 0 : Z_t^\epsilon(\omega) \geq \tilde{Z}_0,\ \forall t \in [0,T]\}. \tag{7.21}$$

Technically, there exists a constant $M \geq 2$ such that $\tilde{Z}_0 = \frac{Z_0}{M}$. Now we can consider two cases: $t \in [0, \tau_1]$ and $t \in (\tau_1, T]$.

Case 3.1: $t \in [0, \tau_1]$. By triangle inequality, we have

$$\begin{aligned} |Z_t^\epsilon|^p =& \Big| Z_0 + \frac{1}{2}\int_0^t f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_s^\epsilon)ds + \frac{\nu}{2}W_t^H\Big|^p \\ \leq& \left(Z_0 + \frac{1}{2}\left|\int_0^t f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_s^\epsilon)ds\right| + \frac{\nu}{2}|W_t^H|\right)^p \\ \leq& \left(Z_0 + \frac{1}{2}\int_0^t \Big|f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_s^\epsilon)\Big|ds + \frac{\nu}{2}|W_t^H|\right)^p. \end{aligned} \tag{7.22}$$

By applying the Callebaut's inequality theorem which can be expressed as: $\forall a_i \geq 0$ and $\forall n,p \geq 1$;

$$\left(\sum_{i=1}^n a_i\right)^p \leq n^p \sum_{i=1}^n (a_i)^p,$$

it will be easy to show that for all $p \geq 1$,

$$\begin{aligned} &\left(Z_0 + \frac{1}{2}\int_0^t \Big|f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_s^\epsilon)\Big|ds + \frac{\nu}{2}|W_t^H|\right)^p \\ &\qquad \leq 3^p\left(Z_0^p + \left(\frac{1}{2}\int_0^t \Big|f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_s^\epsilon)\Big|ds\right)^p + \left(\frac{\nu}{2}|W_t^H|\right)^p\right). \end{aligned} \tag{7.23}$$

From (7.21), we may deduce that $Z_t^\epsilon \geq \tilde{Z}_0 > 0$, with t on $[0, \tau_1]$. This yields $\Lambda_\epsilon(Z_t^\epsilon) < MZ_0^{-1},\ M \geq 2$ and

$$\int_0^t \Big|f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_t^\epsilon)\Big|ds \leq \left(\frac{M}{Z_0}\right)\int_0^t \Big|f(s,Z_s^\epsilon)\Big|ds. \tag{7.24}$$

Since the drift function satisfies the linear growth condition, this means there exists a positive constant k such that $f(t,z) \leq k(1+|z|)$. It follows from (7.24) that

$$\int_0^t \Big|f(s,Z_s^\epsilon)\Big|ds \leq \int_0^t \Big|k(1+|Z_s^\epsilon|)\Big|ds \leq k\left(T+\int_0^t |Z_s^\epsilon|ds\right). \quad (7.25)$$

Inequalities (7.22), (7.23) and (7.25) yield the following:

$$|Z_t^\epsilon|^p \leq 3^p\left(Z_0^p + \left(\frac{kM}{2Z_0}\right)^p\left(T+\int_0^t |Z_s^\epsilon|ds\right)^p + \left(\frac{\nu}{2}\right)^p |W_t^H|^p\right).$$

On the other hand, recall from our first chapter, Theorem 2.12 that $|W_t^H| < \sup_{s\in[0,T]}|W_s^H| < \infty$ and since

$$\left(T+\int_0^t |Z_s^\epsilon|ds\right)^p \leq 2^p\left(T^p+\int_0^t |Z_s^\epsilon|^p ds\right),$$

then it follows that

$$\begin{aligned}
|Z_t^\epsilon|^p &\leq (3Z_0)^p + \left(\frac{3kMT}{Z_0}\right)^p + (3\nu)^p \sup_{s\in[0,T]}|W_s^H|^p + \left(\frac{3kM}{Z_0}\int_0^t |Z_s^\epsilon|ds\right)^p. \\
&\leq (3Z_0)^p + \left(\frac{3kT}{Z_0}\right)^p + (4\nu)^p \sup_{s\in[0,T]}|W_s^H|^p + \left(\frac{3k}{Z_0}\int_0^t |Z_s^\epsilon|ds\right)^p.
\end{aligned}$$

From the Grönwall-Bellman inequality theorem, we obtain

$$\begin{aligned}
|Z_t^\epsilon|^p &\leq \left((3Z_0)^p + \left(\frac{3kMT}{Z_0}\right)^p + (4\nu)^p \sup_{s\in[0,T]}|W_s^H|^p\right)\exp\left(\left(\frac{3kM}{Z_0}\right)^p t\right) \\
&\leq \left((3Z_0)^p + \left(\frac{3kMT}{Z_0}\right)^p\right)\exp\left(\left(\frac{3kM}{Z_0}\right)^p T\right) \\
&\qquad + \left((4\nu)^p \sup_{s\in[0,T]}|W_s^H|^p\right)\exp\left(\left(\frac{3kM}{Z_0}\right)^p T\right)
\end{aligned}$$

which can be shortly written as $|Z_t^\epsilon|^p \leq C$, where $C = C(r,k,T,Z_0,\nu,H)$ is a non-random constant in parameters r,k,T,Z_0,ν and H taking the following form

$$C \leq C_1 + C_2 \sup_{s\in[0,T]}|W_s^H|^p,$$

with $C_1 = C_1(p, k,T,Z_0)$ and $C_2 = C_2(p, k,T,Z_0,\nu)$ are non-random constants defined respectively by

$$C_1 = (3Z_0)^p\left(1+\left(\frac{kMT}{Z_0^2}\right)^p\right)\exp\left(\left(\frac{3kM}{Z_0}\right)^p T\right) \tag{7.26}$$

and

$$C_2 = (4\nu)^p \exp\left(\left(\frac{3kM}{Z_0}\right)^p T\right). \tag{7.27}$$

Case 3.2: $t \in (\tau_1, T]$, with $T > \tau_1 > 0$. Define

$$\tau_2 = \tau_2(\epsilon,\omega) = \sup\{s \in (\tau_1,t) : |Z_s^\epsilon(\omega)| < \tilde{Z}_0\}.$$

Then we have:

$$\begin{aligned}
|Z_t^\epsilon|^p &\leq |Z_t^\epsilon - Z_{\tau_2}^\epsilon|^p + |Z_{\tau_2}^\epsilon|^p \\
&\leq Z_0^p + |Z_t^\epsilon - Z_{\tau_2}^\epsilon|^p \\
&\leq Z_0^p + \left(\frac{1}{2}\right)^p \left|\int_{\tau_2}^t f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_t^\epsilon)ds + \nu\big(W_t^H - W_{\tau_2}^H\big)\right|^p \\
&\leq Z_0^p + \left(\int_{\tau_2}^t \Big|f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_t^\epsilon)\Big|ds\right)^p + (2\nu)^p\Big(\big|W_t^H\big|^p + \big|W_{\tau_2}^H\big|^p\Big).
\end{aligned} \tag{7.28}$$

As previously, the integral in the last inequality of (7.28) can be expressed as follows

$$\int_0^t \Big|f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_t^\epsilon)\Big|ds \leq \frac{k}{Z_0}\left(T + \int_0^t |Z_s^\epsilon|ds\right), \quad \forall t \in [0,T].$$

On the other hand, we may observe that

$$\big|W_t^H\big|^p + \big|W_{\tau_2}^H\big|^p \leq 2 \sup_{s\in[0,T]} |W_s^H|^p.$$

It follows that,

$$\begin{aligned}|Z_t^\epsilon|^p &\leq Z_0^p + \left(\frac{2kT}{Z_0}\right)^p + \left(\frac{2k}{Z_0}\int_0^t |Z_s^\epsilon|^r ds\right)^p + 2(2\nu)^p \sup_{s\in[0,T]} |W_s^H|^p \\ &\leq (3Z_0)^p + \left(\frac{3kMT}{Z_0}\right)^p + (4\nu)^p \sup_{s\in[0,T]} \left|W_s^H\right|^p + \left(\frac{3kM}{Z_0}\int_0^t |Z_s^\epsilon| ds\right)^p.\end{aligned}$$

From this expression, we may also conclude that $|Z_t^\epsilon|^p \leq C$, where $C = C(C_1,C_2)$ where C_1 and C_2 are non-random constants defined by (7.26) and (7.27) respectively. This shows that $\mathbb{E}\left[\sup_{t\in[0,T]} \left|Z_t^\epsilon\right|^p\right] < \infty$ and consequently, $\mathbb{E}\left[\sup_{t\in[0,T]} \left|Z_t\right|^p\right] < \infty$.

This concludes the proof of this proposition. □

Corollary 7.4. *Fix $p \geq 1$. Then*

$$\lim_{\epsilon\to 0} \mathbb{E}\left[\sup_{t\geq 0} \left|\sigma(Y_t^\epsilon) - \sigma(Y_t)\right|^p\right] = 0 \quad a.s.$$

Proof. This follows immediately from the previous proposition and the fact that $\sigma(y)$ satisfies the linear growth condition. □

In what follows, we show that the stochastic processes $(Z_t^\epsilon)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are Malliavin differentiable with respect to the Brownian motions $(V)_{t\geq 0}$, $(\tilde{V})_{t\geq 0}$ and *fBm* $(W_t^H)_{t\geq 0}$.

Proposition 7.5. *Let $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ be a stochastic process that verifies the stochastic differential equation (7.18) driven by a fBm $(W_t^H)_{t\in[0,T]}$ that takes the Volterra representation form given by*

$$W_t^H = \int_0^t \kappa_H(s,t) dB_s,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion and $\kappa_H(s,t)$ is a square integrable kernel given by (2.5). Assume that the drift function $f(t,z)$ is differentiable and define

$$F_\epsilon(t,z) = \frac{\partial f(t,z)}{\partial z}\Lambda_\epsilon(z) + f(t,z)\Lambda'_\epsilon(z),$$

where $\Lambda'_\epsilon(z)$ is defined by (7.20). Moreover, let $\mathcal{D}_u^B$ and $\mathcal{D}_u^W$ be the Malliavin derivatives at the time $u \in [0,T]$ with respect to $(B_t)_{t\geq 0}$ and $(W_t^H)_{t\geq 0}$ respectively. Then it follows that $Z_t^\epsilon \in \mathbb{D}^{1,p}$, $Y_t^\epsilon = (Z_t^\epsilon)^2 \in \mathbb{D}^{1,p}$,

$$\begin{cases} \mathcal{D}_u^B Y_t^\epsilon = 2Z_t^\epsilon \mathcal{D}_u^B Z_t^\epsilon \\ \mathcal{D}_u^W Y_t^\epsilon = 2Z_t^\epsilon \mathcal{D}_u^W Z_t^\epsilon, \end{cases} \tag{7.29}$$

where $\mathcal{D}_u^B Y_t^\epsilon$ and $\mathcal{D}_u^W Y_t^\epsilon$ are given respectively by

$$\mathcal{D}_u^B Z_t^\epsilon = \frac{\nu}{2}\left(\kappa_H(t,u) + \int_u^t \kappa_H(s,u) F_\epsilon(s,Z_s^\epsilon)\exp\Big(\int_s^t F_\epsilon(u,Z_u^\epsilon)du\Big)ds\right)\mathbf{1}_{[0,t]}(u) \tag{7.30}$$

and

$$\mathcal{D}_u^W Z_t^\epsilon = \frac{\nu}{2}\left(\exp\Big(\int_s^t F_\epsilon(u,Z_u^\epsilon)du\Big)\right)\mathbf{1}_{[0,t]}(u). \tag{7.31}$$

Proof. The expressions (7.29) are due to the chain rule of Malliavin derivatives. The Malliavin derivative $\mathcal{D}_u^B Z_t$ can be found as follows:

$$\begin{aligned}\mathcal{D}_u^B Z_t^\epsilon =& \frac{1}{2}\int_0^t \mathcal{D}_u^B\left(f(s,Z_s^\epsilon)\Lambda_\epsilon(Z_s^\epsilon)\right)ds + \frac{\nu}{2}\mathcal{D}_u^B W_t^H \\ =& \frac{1}{2}\int_0^t F_\epsilon(s,Z_s^\epsilon)\mathcal{D}_u^B Z_s^\epsilon ds + \frac{\nu}{2}\kappa_H(t,u)\mathbf{1}_{[0,t]}(u)\end{aligned}$$

The function $F_\epsilon(s,z)$ exists indeed since $\Lambda'_\epsilon(z)$ is well-defined for all $H \in (0,1)$. Next, by letting $D_t = \mathcal{D}_u^B Z_t^\epsilon$, we obtain a Volterra integral equation given by

$$D_t = \frac{1}{2}\int_0^t F_\epsilon(s,Z_s^\epsilon)D_s ds + \frac{\nu}{2}\kappa_H(t,u)\mathbf{1}_{[0,t]}(u),$$

to which a solution is given by

$$D_t = \frac{\nu}{2}\left(\kappa_H(t,u) + \int_u^t \kappa_H(s,u)F_\epsilon(s,Z_s^\epsilon)\exp\Big(\int_s^t F_u du\Big)ds\right)\mathbf{1}_{[0,t]}(u).$$

Since $D_t \in L^p(\Omega)$, then it follows that the stochastic process $Z_t^\epsilon \in \mathbb{D}^{1,p}$ from Nualart (2006). The proof of (7.31) can be deduced in a similar way or by following the idea of Hu et al. (2008, Theorem 3.3). □

Remark 7.1.

(1) This proposition holds for all $H \in (0,1)$. However, for $H > 1/2$ one may use the stochastic process (7.1) without going through its approximating sequence $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ since the sample paths of $(Z_t)_{t\geq 0}$ are strictly positive everywhere almost surely as shown in Theorem 7.1.

(2) As a straight consequence of Proposition 7.3, we have

$$\lim_{\epsilon\to 0} F_\epsilon(t,z) = F(t,z)$$

where

$$F(t,z) = \Big(\frac{\partial f(t,z)}{\partial z} - f(t,z)\Big) z^{-2}.$$

It follows from Proposition 7.5 that $Z_t \in \mathbb{D}^{1,p}$, and

$$\mathcal{D}_u^{\tilde{V}} Z_t = 0, \tag{7.32}$$

$$\mathcal{D}_u^{V} Z_t = \frac{\nu}{2}\Bigg(\kappa_H(t,u) + \int_u^t \kappa_H(s,u) F(s,Z_s) \exp\Big(\int_s^t F(u,Z_u)du\Big)ds\Bigg)\mathbf{1}_{[0,t]}(u) \tag{7.33}$$

and

$$\mathcal{D}_u^{W} Z_t = \frac{\nu}{2}\Bigg(\exp\Big(\int_s^t F(u,Z_u)du\Big)\Bigg)\mathbf{1}_{[0,t]}(u). \tag{7.34}$$

7.2.2 Differentiability of the stock price process $(X_t)_{t\geq 0}$

With $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$, let us construct the approximating sequence $(X_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ of the stock price process $(X_t)_{t\geq 0}$ defined by the following geometric Brownian motion:

$$dX_t^\epsilon = \eta X_t^\epsilon dt + \sigma(Y_t^\epsilon) X_t dB_t, \tag{7.35}$$

where

$$Y_t^\epsilon = (Z_t^\epsilon)^2,$$

with $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ the approximating sequence that satisfies (7.18). The solution to (7.35) is unique and can be found by using the standard Itô formula (Fouque et al.; 2011, Section 1.1.4). Next step is to prove that X_t^ϵ converges to X_t in L^p, $p \geq 1$ as given in the following proposition.

Proposition 7.6. *The sequence X_t^ϵ converges to X_t in $L^p(\Omega)$ for all $p \geq 1$.*

Proof. Consider the sequence of log-returns $R_t^\epsilon := \log X_t^\epsilon$ that satisfies:

$$R_t^\epsilon = R_0 + \eta t - \frac{1}{2}\int_0^t \sigma^2(Y_s^\epsilon)ds + \int_0^t \sigma(Y_s^\epsilon)dB_s,$$

where $R_0 := \log X_0$. Then for some non-random constant $C > 0$, one may have:

$$\mathbb{E}\left[\sup_{t\geq 0}\left|R_t^\epsilon - R_t\right|^p\right] \leq \frac{C}{2^p}\,\mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^t \left(\sigma^2(Y_s^\epsilon) - \sigma^2(Y_s)\right)ds\right|^p\right] + C\mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^t \left(\sigma(Y_s^\epsilon) - \sigma(Y_s)\right)dB_s\right|^p\right]$$

Set

$$\mathbb{T}_1 := \mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^t \left(\sigma^2(Y_s^\epsilon) - \sigma^2(Y_s)\right)ds\right|^p\right]$$

and

$$\mathbb{T}_2 := \mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^t \left(\sigma(Y_s^\epsilon) - \sigma(Y_s)\right)dB_s\right|^p\right].$$

Then it follows firstly that $\mathbb{T}_1 \to 0$ from Corollary 7.4. To analyse convergence of $\mathbb{T}_2$, we use the Burkholder - Davis - Gundy inequality (that is, for any martingale M, $\mathbb{E}\left[|\sup_{t\geq 0} M_t|^p\right] \leq c\mathbb{E}\left[\langle M\rangle_t^{\frac{p}{2}}\right]$ for some constant $c = c(p)$ depending on $p \geq 1$ and where $\langle\cdot\rangle$ represents the quadratic variation). One may deduce that

$$\mathbb{T}_2 \leq c(p)\mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^t \left(\sigma(Y_s^\epsilon) - \sigma(Y_s)\right)ds\right|^{\frac{p}{2}}\right],$$

which also converges to zero from Corollary 7.4. It follows that

$$\lim_{\epsilon\to 0}\sup_{t\geq 0}\left|R_t^\epsilon - R_t\right|^p = 0, \quad \forall p > 0$$

that implies the desired L^p convergence of X_t^ϵ to X_t. □

Proposition 7.7. *Assume that the volatility $\sigma(y)$ is Lipschitz and differentiable. Then $X_t^\epsilon, R_t^\epsilon \in \mathbb{D}^{1,p}$ and for all $u \leq t$, we have*

$$\mathcal{D}_u^B X_t^\epsilon = X_t^\epsilon \mathcal{D}_u^B R_t^\epsilon, \;\; \mathcal{D}_u^V X_t^\epsilon = X_t^\epsilon \mathcal{D}_u^B R_t^\epsilon \;\; \text{and} \;\; \mathcal{D}_u^{\tilde{V}} X_t^\epsilon = X_t^\epsilon \sqrt{1-\rho^2}\,\sigma(Y_t^\epsilon)\mathbf{1}_{[0,t]}(u), \tag{7.36}$$

where

$$\mathcal{D}_u^B R_t^\epsilon = \left(\int_u^t \sigma'(Y_s^\epsilon)\mathcal{D}_u^B Y_s^\epsilon dB_s - \int_u^t \sigma(Y_s^\epsilon)\sigma'(Y_s^\epsilon)\mathcal{D}_u^B Y_s^\epsilon ds\right)\mathbf{1}_{[0,t]}(u) \tag{7.37}$$

and

$$\begin{aligned}\mathcal{D}_u^V R_t^\epsilon = &\Bigg(\rho\int_u^t \sigma'(Y_s^\epsilon)\mathcal{D}_u^V Y_s^\epsilon dV_s + \sqrt{1-\rho^2}\int_u^t \sigma'(Y_s^\epsilon)\mathcal{D}_u^V Y_s^\epsilon d\tilde{V}_s \\ &- \int_u^t \sigma(Y_s)\sigma'(Y_s^\epsilon)\mathcal{D}_u^V \sigma(Y_s^\epsilon)ds\Bigg)\mathbf{1}_{[0,t]}(u)\end{aligned} \tag{7.38}$$

In addition,

$$\sup_{u,\,t\geq 0}\left|\mathcal{D}_u R_t^\epsilon - \mathcal{D}_u X_t\right| \to 0,$$

where $\mathcal{D}_u$ represents a Malliavin derivative with respect to B_t, V_t or $\tilde{V}$.

Proof. The equation (7.36) follows immediately from chain rule formula for Malliavin derivatives. Expressions of derivatives $\mathcal{D}_u^B R_t^\epsilon$ and $\mathcal{D}_u^V R_t^\epsilon$ are straight consequences of Nualart (2006, Theorem 1.2.4).

Corollary 7.8. *The laws of both stock price process $(X_t)_{t\geq 0}$ and its log return $(R_t)_{t\geq 0}$ are absolutely continuous.*

Proof. One may verify that $\left|\left|\mathcal{D}_u^B R_t\right|\right|_{L^2(\Omega)} > 0$ and $\left|\left|\mathcal{D}_u^B X_t\right|\right|_{L^2(\Omega)} > 0$ almost surely, then the absolutely continuity with respect to the Lebesgue measure on $\mathbb{R}$ follows immediately from Nualart (2006, Theorem 2.1.3). □

Remark 7.2.

(1) The Malliavin differentiability property of both stochastic volatility and stock price processes will be crucial for the derivation of the expected payoff function that will be discussed in the next chapter.

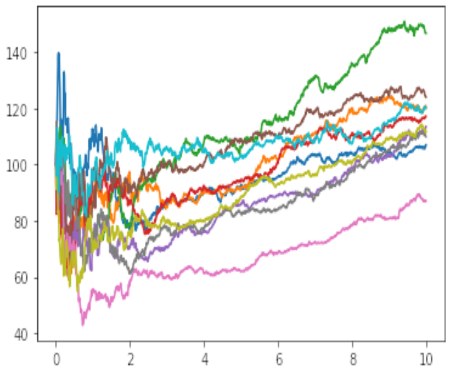
(2) The approximated stochastic volatility and stock price processes will be compulsory for $H \leq 1/2$ and optional for $H > 1/2$. However, for the sake of consistency, we shall use the approximated sequences (7.18) and (7.35) with $\epsilon = 0$ for $H > 1/2$ and with $\epsilon > 0$ for $H \leq 1/2$.

(3) For the simulations of stock price process, one may use the Euler-Maruyama approximation scheme as discussed previously. This can be done by considering the time interval $[0,T]$ that is subdivided into N sub-intervals of equal length such that $0 = t_0, t_1, \cdots, t_N = T$ with $t_i = iT/N$ and the lag $\Delta t = T/N$. The estimated stock price at time t_i denoted by $(\hat{X}_{t_i})_{i=1,\cdots,N}$ and the volatility $(\hat{Y}_{t_i})_{i=1,\cdots,N}$ are respectively given by

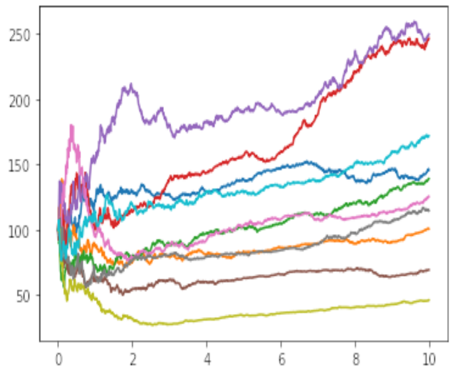
$$\begin{cases} \hat{X}_{t_{i+1}} = \hat{X}_{t_i}\left(1 + \eta\Delta t + \sigma(\hat{Y}_{t_i})\,\left(\rho\Delta V_{t_i} - \sqrt{1-\rho^2}\Delta\tilde{V}_{t_i}\right)\right) \\ \hat{Y}_{t_i} = \hat{Z}_{t_i}^2 1_{[0,\tau(\omega)]} \\ \hat{Z}_{t_{i+1}} = \hat{Z}_{t_i} + \dfrac{1}{2}\displaystyle\int_0^{t_{i+1}} f(s,\hat{Z}_s)\Lambda(\hat{Z}_s)ds + \tfrac{1}{2}\nu\Delta W_{t_{i+1}}^H. \end{cases} \tag{7.39}$$

where $\Delta V_{t_i} = V_{t_{i+1}} - V_{t_i}$, $\Delta\tilde{V}_{t_i} = \tilde{V}_{t_{i+1}} - \tilde{V}_{t_i}$ and $\Delta W_{t_{i+1}}^H = W_{t_{i+1}}^H - W_{t_i}^H$ are respectively the increment of Brownian motions $V_{t\in[0,10]}$, $\tilde{V}_{t\in[0,T]}$ and *fBm* $W_{t\in[0,T]}^H$. As an illustrative example, the following figures represent 10 sample paths of the stock price process on the interval $[0,T]$ with $N = 1000$, $\rho = 0.6$, $X_0 = 100$, $\eta = r = 0.05$, $\nu = 0.1$, and $\sigma(\hat{Y}_{t_i}) = 0.8\hat{Y}_{t_i} + 0.1$. The drift of the fractional volatility process is defined by (7.9) with $\theta = 1,\ c = 2$.

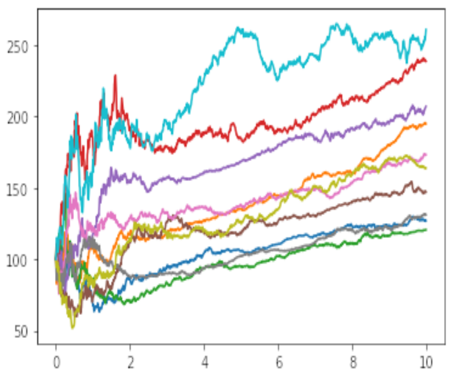
Figure 7.4: Ten (10) sample paths of stock price process



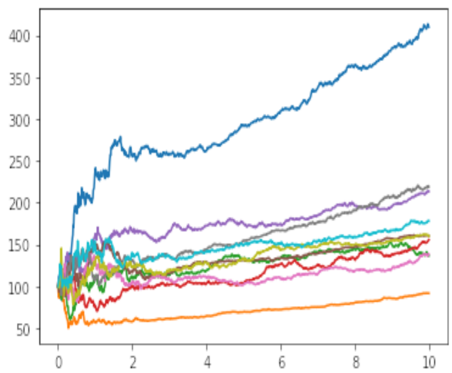
$H = 0.15,\ \epsilon = 0.001$



$H = 0.5,\ \epsilon = 0.001$



$H = 0.65,\ \epsilon = 0$



$H = 0.9,\ \epsilon = 0$

Chapter 8

An Application to Option Pricing

One common technique used in option pricing is the standard Monte Carlo Simulation. This method works perfectly when the payoff function converges in mean (See e.g. Fu and Hu (1995)). This is not always attainable for discontinuous payoff functions. A straight solution is to transform the expected payoff function $\mathbb{E}[h(X_t)]$ into an expectation of a continuous function, and this can be achieved by using the Malliavin calculus tools.

The aim of this chapter is to discuss option pricing under general settings where the volatility of the infinitesimal return is defined in terms of the square of the generalised *fCIR* process driven by *fBm* with Hurst parameters $H \in (0,1)$. We shall consider payoff functions that are not necessary continuous, such as a combination of vanilla and exotic options, and derive its option price.

Under the above settings, the standard Monte-Carlo technique produces high relative errors as observed by Altmayer and Neuenkirch (2015). We rely on some results of Malliavin calculus to discuss the expected payoff function since the volatility and stock price processes are Malliavin differentiable as discussed in our previous chapter. Throughout this chapter, we shall reconsider the financial market model of the form

$$\begin{cases} dA_t = rA_t dt, \\ dX_t = \eta X_t dt + \sigma(Y_t) X_t dB_t, \\ Y_t = Z_t^2 \mathbf{1}_{[0,\tau(\omega)]} \\ dZ_t = \frac{1}{2} f(t,Z_t) Z_t^{-1} dt + \frac{1}{2}\nu dW_t^H \\ W_t^H = \int_0^t \kappa_H(s,t) dV_t \\ B_t = \rho V_t + \sqrt{1-\rho^2}\tilde{V}_t, \end{cases} \tag{8.1}$$

where all components of the above financial market model were discussed in chapters 4 and 5. The main references here are Altmayer and Neuenkirch (2015), Bezborodov et al. (2019), Hong et al. (2019) and Mishura and Yurchenko-Tytarenko (2020).

8.1 The Expected Payoff function

We apply some results in Malliavin calculus to derive the expected value of the payoff function denoted by $\mathbb{E}[h(X_T)]$, where $h : \mathbb{R} \to \mathbb{R}$ represents the payoff function that satisfies the following assumption:

Assumption 8.1. The payoff function $h : \mathbb{R} \to \mathbb{R}$ and its antiderivative denoted by $L(x)$ satisfy the Lipschitz condition.

Proposition 8.1. $L(X_T) \in \mathbb{D}^{1,2}$.

Proof. Firstly, it is straightforward to check that $\mathbb{E}[L^2(X_T)] < \infty$ since $L(x)$ also verifies the linear growth condition and the sample paths of the stock price process $(X_t)_{t\in[0,T]}$ are bounded almost surely. On the other hand, since L verifies Assumption 8.1 and the laws of the stock price process $(X_t)_{t\in[0,T]}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ (See Corollary 7.8), then from the chain rule formula for Malliavin derivatives (See Lemma 3.4), we may deduce

$$\mathcal{D}^V L(X_T) = L'(X_T)\mathcal{D}^V X_T = h(X_T)\mathcal{D}^V X_T.$$

It follows that

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left(\mathcal{D}_s^V L(X_T)\right)^2 ds\right] &= \mathbb{E}\left[\int_0^T \left(h(X_T)\mathcal{D}_s^V X_T\right)^2 ds\right] \\ &= \mathbb{E}\left[h^2(X_T)\int_0^T \left(\mathcal{D}_s^V X_T\right)^2 ds\right] \\ &\leq \left(\mathbb{E}\left[h^4(X_T)\right]\int_0^T \mathbb{E}\left[\left(\mathcal{D}_s^V X_T\right)^4\right] ds\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

The first inequality is due to Hölder inequality and the finiteness of the last expression makes sense since $X_t \in \mathbb{D}^{1,2}$ as discussed in our fourth chapter. It follows from (3.7) that $||L||_{1,2} < \infty$ which concludes the proof. □

As now $L(X_T)$ is Malliavin differentiable, then the following lemma that discusses the expected payoff follows.

Lemma 8.2. *Let $h(x)$, $x \in \mathbb{R}$ be a payoff function that satisfies Assumption 8.1 and denote $h(e^x) := g(x)$ with its antiderivative $G(x)$ that also satisfies the Lipschitz condition. Set*

$$I_T := \frac{1}{T\sqrt{1-\rho^2}}\int_0^T \frac{1}{\sigma(Y_u)} d\tilde{V}. \tag{8.2}$$

Then

$$\mathbb{E}\left[g(R_T)\right] = \mathbb{E}\Big[G(R_T)I_T\Big], \tag{8.3}$$

and

$$\mathbb{E}\left[h(X_T)\right] = \mathbb{E}\left[\frac{L(X_T)}{X_T}\Big(1 + I_T\Big)\right]. \tag{8.4}$$

where $R_T := \log X_T$ and

$$L(X_T) = \int_0^{X_T} h(x)dx. \tag{8.5}$$

Proof. We follow the idea of Altmayer and Neuenkirch (2015). To establish the equality (8.3), we rewrite $\mathbb{E}[g(R_T)]$ as

$$\mathbb{E}[g(R_T)] = \mathbb{E}\left[\frac{1}{T}\int_0^T g(R_T)du\right] = \mathbb{E}\left[\frac{1}{T}\int_0^T g(R_T)\mathcal{D}_u^{\tilde{V}}R_T\frac{1}{\mathcal{D}_u^{\tilde{V}}R_T}du\right].$$

From Proposition 8.1, we may deduce that $G(R_T) \in \mathbb{D}^{1,2}$ and

$$\mathcal{D}^{\tilde{V}}G(R_T) = g(R_T)\mathcal{D}^{\tilde{V}}R_T.$$

We now obtain

$$\mathbb{E}[g(R_T)] = \mathbb{E}\left[\frac{1}{T}\int_0^T \mathcal{D}^{\tilde{V}}G(R_T)\frac{1}{\mathcal{D}_u^{\tilde{V}}R_T}du\right].$$

In addition, from Proposition 7.7,

$$\mathcal{D}_u^{\tilde{V}}R_T = \sqrt{1-\rho^2}\sigma(Y_u)\mathbf{1}_{[0,t]}(u)$$

and since the integral $\int_0^T \frac{1}{\sigma(Y_u)}du$ is well defined from Assumption 4.1, then we have:

$$\mathbb{E}[g(R_T)] = \mathbb{E}\left[\frac{G(R_T)}{T\sqrt{1-\rho^2}}\int_0^T \frac{1}{\sigma(Y_u)}d\tilde{V}_u\right],$$

and defining I_T by (8.2), we obtain (8.3). To establish (8.4), we rewrite the function $G(x)$ (which is the antiderivative of $g(x)$) as follows

$$G(x) = \int_0^x g(u)du + C,$$

where C is a constant taking the form $C = \int_0^1 h(u)du$ and by using the standard integration by part formula, one may obtain

$$G(x) = \frac{L(e^x)}{e^x} + \int_0^x \frac{L(e^u)}{e^u}du.$$

With this setting, we have

$$
\begin{aligned}
\mathbb{E}[h(X_T)] &= \mathbb{E}[g(R_T)] \\
&= \mathbb{E}\Big[G(R_T)I_T\Big] \\
&= \mathbb{E}\left[\left(\frac{L(X_T)}{X_T} + \int_0^{R_T} \frac{L(e^u)}{e^u}du\right) I_T\right] \\
&= \mathbb{E}\left[\frac{L(X_T)}{X_T}I_T\right] + \mathbb{E}\left[\left(\int_0^{R_T} \frac{L(e^u)}{e^u}du\right) I_T\right] \\
&= \mathbb{E}\left[\frac{L(X_T)}{X_T}I_T\right] + \mathbb{E}\left[\frac{L(X_T)}{X_T}\right] \\
&= \mathbb{E}\left[\frac{L(X_T)}{X_T}\Big(1 + I_T\Big)\right].
\end{aligned}
$$

□

Remark 8.1.

1. The expected value of the payoff function given by (8.3) and (8.4) excludes the case of perfect correlation between the stock price and stochastic volatility process, that is where $\rho = \pm 1$. This cannot be viewed as a drawback since perfect correlation is rare to happen in financial markets.

2. The exact formula (8.4) also holds for jump discontinuous payoff functions, that is, there exists a point $x_0 \in \mathbb{R}$ such that $\lim_{x \to x_0^+} h(x) \neq \lim_{x \to x_0^-} h(x)$.

8.2 Approximation of The Expected Payoff function

We may use again the Euler-Maruyama approximation scheme to compute the expected payoff numerically. We may use the following approximations:

$$\begin{cases} \hat{X}_{t_{i+1}} = \hat{X}_{t_i}\left(1 + \eta\Delta t + \sigma(\hat{Y}_{t_i})\left(\rho\Delta V_{t_i} - \sqrt{1-\rho^2}\Delta\tilde{V}_{t_i}\right)\right) \\ \hat{Y}_{t_i} = \hat{Z}_{t_i}^2 1_{[0,\tau(\omega)]} \\ \hat{Z}_{t_{i+1}} = \hat{Z}_{t_i} + \frac{1}{2}\int_0^{t_{i+1}} f(s,\hat{Z}_s)\Lambda(\hat{Z}_s)ds + \frac{1}{2}\nu\Delta W_{t_{i+1}}^H \\ \hat{I}_T = \frac{1}{T\sqrt{1-\rho^2}}\sum_{i=0}^{N}\frac{1}{\sigma(\hat{Y}_i)}\Delta\tilde{V}_{t_i}, \end{cases} \tag{8.6}$$

with $0 = t_0, t_1, \cdots, t_N = T$ with $t_i = iT/N$ and the lag $\Delta t = T/N$ as previously. The following proposition discusses the absolute error of the approximated expected payoff.

Proposition 8.3. *For any $r > 0$, $p \in (0,1]$, $q,\varepsilon,H \in (0,1)$ and $\Delta t < 1 - \varepsilon$ there exist finite and non-random constants $K = K(X_0, Y_0, H, T, \nu, q, r)$, $K_1 = K_1(X_0, Y_0, H, T, \nu, q, r)$ and $K_2 = K_2(r)$ such that*

$$\mathbb{E}\left[\left|\frac{L(X_T)}{X_T} - \frac{L(\hat{X}_T)}{\hat{X}_T}\right|^2\right] \leq K\Delta t^{2qH}. \tag{8.7}$$

Moreover, after setting

$$\hat{h}(\hat{X}_T) = \frac{L(\hat{X}_T)}{\hat{X}_T}\left(1 + \hat{I}_T\right),$$

then

$$\left|\mathbb{E}\left[h(X_T)\right] - \mathbb{E}\left[\hat{h}(\hat{X}_T)\right]\right| \leq K_1 + K_2\Delta t^{qH}. \tag{8.8}$$

The proof of this proposition can be done by following Hong et al. (2019, Theorem 4.1), Bezborodov et al. (2019, Lemma 14 and Theorem 15) and Mishura and Yurchenko-Tytarenko (2020).

8.3 Some simulations

In this section, we simulate option prices for different forms of drift functions available in the literature and different values of Hurst parameters. To give more credit to the exact formula (8.7), we use a special class of discontinuous payoff functions known in option trading as *"combination options"* constructed by combining standard options, strike prices or/and maturity dates under the same stock price process. For the sake of simplicity, we shall consider payoff functions given as a combination of vanilla and exotic options with the same strike price and same maturity date.

8.3.1 Pricing options with volatility taking the form of Ornstein-Uhlenbeck and standard *fCIR* process

Firstly, we consider the stochastic process $(Z_t)_{t\geq 0}$ defined as a Ornstein-Uhlenbeck process, that is with $f(t,z) = -\theta z^2$, where θ is a positive parameter, $\nu = 2$ and $H > 1/2$. Under these settings, one may recover the model discussed by Bezborodov et al. (2019) with $Y_t = Z_t^2$ instead. In this case, the volatility process will not necessarily be positive almost surely since it violates the Assumption 6.1 and consequently the Theorems 7.1 and 7.2 do not apply, and the probability of hitting zero is high. To compensate for this, the volatility function $\sigma(y)$ is chosen to be strictly positive.

In addition, we define the payoff function $h(x)$ as a combination of European and binary options with the same strike price S and time to maturity T, that is $h(X_T) = (X_T - S)_+ + \mathbf{1}_{\{X_T > S\}}$. It is easy to check that the strike price S is a removable discontinuity of the payoff function h. In addition, the expression of $L(X_T)$ can be deduced from (8.5) as

$$L(X_T) = \begin{cases} \frac{1}{2}\Big[\big(X_T - S\big)\big(X_T - S + 2\big)\Big] & \text{if } X_T \geq S \\ 0 & \text{otherwise.} \end{cases} \tag{8.9}$$

We use the same parameters ($\eta = r = 0.2$, $\theta = 0.6$, $T = 1, H = 0.6$) with different forms of volatility process $\sigma(Y_t)$ of the infinitesimal return process

dX_t/X_t as in Bezborodov et al. (2019). Since for this, we may not use equations (6.1) and (6.2), we consider the direct form of the stochastic volatility $(Y_t)_{t\geq 0}$ driven by a *fBm* represented by the Volterra stochastic integral (2.5) which can be discretised as follows:

$$W^H_{t_j} = \frac{N}{T}\sum_{i=0}^{j-1}\left(\int_{t_i}^{t_{i+1}} \kappa_H(t_j,s)ds\right)\delta V_i, \tag{8.10}$$

for all $j = 1,\cdots,N;\ i = 0,\cdots,j$ and where $\delta V_i = V_i - V_{i-1}$ is the increment of standard Brownian motion with $W^H_{t_0} = 0$. Here $\kappa_H(t_j,s)$ is a discretised square integrable kernel (2.6) given by

$$\kappa_H(t_j,s) = \frac{(t_j-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\,{}_2\mathbf{F}_1\left(H-\frac{1}{2};\frac{1}{2}-H;H+\frac{1}{2};1-\frac{t_j}{s}\right)\mathbf{1}_{[0,t_j]}(s),\ \forall s\in[0,t_j]. \tag{8.11}$$

In this case, we observe that the values of option prices are not remarkably different for $\rho = 0$ and $H \geq 1/2$. The option prices are increasing or decreasing when ρ is positive or negative respectively.

Recall that the financial market model used in Bezborodov et al. (2019) has several limitations which are not in line with what can be observed. Examples of these are no correlations between returns and volatility, possibility of negative volatility and zero long-run mean μ. Now, taking into account the standard *fCIR* process that describes the volatility, with $f(t,z) = \mu - \theta z^2$ and correlation ρ between infinitesimal returns and volatility, the option prices are simulated with $\rho = 0.5$ and $\mu = 0.1$.

We perform 100 trials for 1000 simulations and 500 time-steps on the time interval [0,1]. We get the mean of option prices (that is, expected payoff function discounted by the net present value) with their corresponding coefficient of variations. Table 8.1 corresponds to the formula (8.4) and Table 8.2 to direct estimation of expected payoff function also known as the standard Monte Carlo method.

Table 8.1: Option prices using Direct Estimations

H	**0.1**		**0.3**		**0.5**		**0.7**		**0.9**	
Mean/CV	Mean	CV	Mean	CV	Mean	CV	Mean	CV	Mean	CV
$\sigma(Y_t) = \sqrt{Y_t + 0.1}$	0.774185342	0.062159457	0.782211975	0.015363114	0.775305642	0.053605636	0.765667823	0.022561751	0.776062568	0.061121985
$\sigma(Y_t) = Y_t + 0.1$	0.932824188	0.023154477	0.959352477	0.019764205	0.946670803	0.008803027	0.952432308	0.016014640	0.948353316	0.008871172
$\sigma(Y_t) = \sqrt{Y_t^2 + 1}$	0.707885444	0.093317545	0.715438258	0.077237936	0.695277007	0.053520175	0.720631067	0.041407711	0.729078909	0.085659766

Table 8.2: Option prices using (8.4)

H	**0.1**		**0.3**		**0.5**		**0.7**		**0.9**	
Mean/CV	Mean	CV	Mean	CV	Mean	CV	Mean	CV	Mean	CV
$\sigma(Y_t) = \sqrt{Y_t + 0.1}$	0.79340973	0.07560649	0.81121348	0.04028921	0.78827183	0.11421244	0.76642501	0.08935762	0.7704734	0.13411309
$\sigma(Y_t) = Y_t + 0.1$	0.99910672	0.09628926	0.95410606	0.16524115	0.97622451	0.06896021	0.97074148	0.10076119	1.013755924	0.10492516
$\sigma(Y_t) = \sqrt{Y_t^2 + 1}$	0.67871381	0.08759139	0.69286223	0.09071164	0.66834204	0.10850252	0.69416225	0.09554705	0.707316469	0.07008638

Remark 8.2.

We note here that the standard Monte Carlo error is of order 0.04 while the exact error of the formula (8.4) can be deduced from (8.8) and needs further investigations. However, we observe that for all Hurst parameters H, the means of option prices become consistent for $N \geq 500$ for the standard Monte Carlo method while the means of option prices are consistent when using (8.8) from only when $N \geq 180$.

8.3.2 Pricing options with volatility taking the form of *fCIR* process with time varying parameters

In this section we perform some simulations of option prices under the fractional Heston model with time varying parameters. For this, the drift function is given by (7.9) in our previous chapter, that is,

$$f(t,z) = \mu_t - \theta_t z^2,$$

where $\theta_t = \theta > 0$ and $\mu_t = c + \frac{\nu^2}{2\theta}\left(1 - e^{-2\theta t}\right)$. It follows that

$$f(t,z) = \frac{\nu^2}{2\theta}\left(1 - e^{-2\theta t}\right) + (c - \theta z^2).$$

We shall use the same values of parameters as given in Chapter 6, that is, $Z_0 = 1,\ \nu = 0.4,\ c = 0.02,\ \theta = 1$. To keep positiveness of the stochastic process $(Z_t)_{t\geq 0}$ for all $H \in (0,1)$, we shall rather use its approximated stochastic process $(Z_t^\epsilon)_{t,\epsilon\geq 0}$ defined previously as

$$dZ_t^\epsilon = \frac{1}{2} f(t,Z_t^\epsilon)\Lambda_\epsilon(Z_t^\epsilon)dt + \frac{\sigma}{2}dW_t^H, \quad Z_0^\epsilon = Z_0 > 0,$$

where the function $\Lambda_\epsilon(z)$ is defined by

$$\Lambda_\epsilon(z) = (z\mathbf{1}_{\{z>0\}} + \epsilon)^{-1}$$

with $\epsilon = 0.01$ for $H < 1/2$ and $\epsilon = 0$ for $H \geq 1/2$. As previously, the *fBm* is simulated by using the formula (8.9) and (8.10). We again perform 100 trials for 1000 simulations and 500 time-steps on the time interval [0,1]. We get the mean of option prices with their corresponding coefficient of variations for different volatility functions $\sigma(y)$ under the European-Binary option as given in tables 8.3 and 8.4.

Note that Remark 8.2 is also true for this case. In addition, the formula (8.4) is mostly needed in this case because of heavy computations due to time varying parameters.

Table 8.3: Option prices using Direct Estimations

H	**0.1**		**0.3**		**0.5**		**0.7**		**0.9**	
Mean/CV	Mean	CV	Mean	CV	Mean	CV	Mean	CV	Mean	CV
$\sigma(Y_t) = \sqrt{Y_t + 0.1}$	0.757738549	0.048177774	0.769114549	0.057692257	0.756162793	0.045562288	0.756665572	0.051234111	0.763148888	0.043265712
$\sigma(Y_t) = Y_t + 0.1$	0.932035897	0.012595508	0.934337494	0.022642941	0.933212125	0.024487	0.928706032	0.014969569	0.929103212	0.01457107
$\sigma(Y_t) = \sqrt{Y_t^2 + 1}$	0.770104152	0.088196662	0.782432528	0.062946479	0.75433847	0.069371091	0.746931996	0.072156192	0.75975843	0.084981952

Table 8.4: Option prices using (8.4)

H	**0.1**		**0.3**		**0.5**		**0.7**		**0.9**	
Mean/CV	Mean	CV	Mean	CV	Mean	CV	Mean	CV	Mean	CV
$\sigma(Y_t) = \sqrt{Y_t + 0.1}$	0.769174923	0.159481951	0.79459017	0.136648616	0.781942914	0.157116756	0.747618003	0.12525256	0.755713234	0.06592363
$\sigma(Y_t) = Y_t + 0.1$	0.94650013	0.102404072	1.02769617	0.128530355	0.919334248	0.111971197	0.983793301	0.095406694	0.88152163	0.101523439
$\sigma(Y_t) = \sqrt{Y_t^2 + 1}$	0.803170587	0.273211512	0.793796973	0.205160841	0.756164588	0.210899491	0.742696383	0.203031148	0.759959966	0.198280955

Conclusion and Further Research

We have constructed an arbitrage-free financial market model that consists of a risk-free asset with prices A_t that verifies $dA_t = rA_t dt$ and the risky asset with price given as a geometric Brownian motion $dX_t = \eta X_t dt + \sigma(Y_t)X_t dB_t$. The volatility of infinitesimal return dX_t/X_t given by $\sigma(Y_t)$ is a function of the generalised *fCIR* process $(Y_t)_{t\geq 0}$ defined by $Y_t^2 = Z_t^2 \mathbf{1}_{[0,\tau)}$ with $dZ_t = \frac{1}{2}f(t,Z_t)Z_t^{-1}dt + \frac{1}{2}\sigma dW_t^H$, $Z_0 > 0$, where $f(t,x)$ is a continuous function on $\mathbb{R}_+^2$ that satisfies two mild conditions. We firstly show that the fractional volatility process $(Y_t)_{t\geq 0}$ satisfies the differential equation $dY_t = f(t,\sqrt{Y_t})dt + \sigma\sqrt{Y_t} \circ dW_t^H$.

We have also discussed positiveness of the volatility process. We proved that if the Hurst parameter $H > 1/2$, the process $(Y_t)_{t\geq 0}$ will never hit zero, that is, it remains strictly positive everywhere almost surely under some mild assumptions on the function $f(t,x)$. In the case where $H < 1/2$, we considered a sequence of increasing drift functions (f_n) that tends to infinity and we proved that the probability of hitting zero converges to zero as n goes to infinity. The positiveness can be kept for this last case by introducing an approximating sequence of $(Z_t)_{t\geq 0}$ defined by $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$ that satisfies the stochastic differential equation: $dZ_t^\epsilon = \frac{1}{2}f(t,Z_t^\epsilon)\Lambda_\epsilon(Z_t^\epsilon)dt + \frac{\sigma}{2}dW_t^H$, $Z_0^\epsilon = Z_0 > 0$. These results are illustrated with some simulations.

In addition, the stock price and volatility processes are proven to be Malliavin differentiable through the approximating sequence $(Z_t^\epsilon)_{t\geq 0,\,\epsilon>0}$. This property and the strictly positiveness enabled us to deduce an expression of the expected payoff function that may have discontinuities such a combination of vanilla and exotic options. Some simulations of option prices with different forms of volatility function $\sigma(Y_t)$ were performed.

The next step in this research is calibration of volatility parameters to real market data through analytical or by using machine learning techniques. The flexibility of the adapted drift process will be of great impact in improving calibration error.

In addition, the Malliavin differentiability of both stock price and volatility processes is an open door to several other applications in quantitative finance other than option pricing discussed in this thesis. For example, one may investigate the implied volatility surface by using Malliavin calculus tools.

Finally, in this thesis we assumed that the interest rate is a positive constant parameter. This limitation can be overcome by including standard stochastic interest rate models. For example, the short-interest rate may be described by a standard Cox-Ingersoll-Ross process as suggested by Hull and White (1990), and the corresponding financial market model that requires further investigation would have the following form:

$$\begin{cases} dA_t = r_t A_t dt, \\ dr_t = \tilde{\theta}(\tilde{\mu} - r_t)dt + \tilde{\sigma}\sqrt{r_t}d\tilde{B}_t \\ dX_t = \eta X_t dt + \sigma(Y_t)X_t dB_t, \\ Y_t = Z_t^2 \mathbf{1}_{[0,\tau(\omega)]} \\ dZ_t = \frac{1}{2}f(t,Z_t)Z_t^{-1}dt + \frac{1}{2}\nu dW_t^H \\ W_t^H = \displaystyle\int_0^t \kappa_H(s,t)dV_t, \end{cases}$$

where $\tilde{\theta}$ represents the speed of reversion of the stochastic interest rate process $(r_t)_{t\geq 0}$ towards its long run mean $\tilde{\mu}$ and $\tilde{\sigma}$ is the volatility. Attention should be paid to the correlations between Brownian motions $(B_t)_{t\geq 0}$, $(\tilde{B}_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ (or $(W_t^H)_{t\geq 0}$).

Bibliography

Abuzayed, B., Al-Fayoumi, N. and Charfeddine, L. (2018). Long range dependence in an emerging stock markets sectors: volatility modelling and VaR forecasting, *Applied Economics* **50**(23): 2569–2599.

Alòs, E. and Ewald, C.-O. (2008). Malliavin differentiability of the heston volatility and applications to option pricing, *Advances in Applied Probability* **40**(1): 144–162.

Alòs, E. and Lorite, D. G. (2021). *Malliavin Calculus in Finance: Theory and Practice*, CRC Press.

Alos, E., Mazet, O. and Nualart, D. (2001). Stochastic calculus with respect to Gaussian processes, *The Annals of Probability* **29**(2): 766–801.

Alòs, E. and Yang, Y. (2017). A fractional Heston model with $H > 1/2$, *Stochastics* **89**(1): 384–399.

Altmayer, M. and Neuenkirch, A. (2015). Multilevel Monte Carlo quadrature of discontinuous payoffs in the generalized Heston model using Malliavin integration by parts, *SIAM Journal on Financial Mathematics* **6**(1): 22–52.

Andersen, T. G., Bollerslev, T., Diebold, F. X. and Labys, P. (2003). Modeling and forecasting realized volatility, *Econometrica* **71**(2): 579–625.

Asmussen, S. (1998). *Stochastic simulation with a view towards stochastic processes*, University of Aarhus. Centre for Mathematical Physics and Stochastics.

Aurzada, F. (2011). On the one-sided exit problem for fractional brownian motion, *Electronic Communications in Probability* **16**: 392–404.

Bachelier, L. (1900). Théorie de la spéculation, *Annales scientifiques de l'École normale supérieure*, Vol. 17, pp. 21–86.

Bayer, C., Friz, P. and Gatheral, J. (2016). Pricing under rough volatility, *Quantitative Finance* **16**(6): 887–904.

Benhamou, E., Gobet, E. and Miri, M. (2010). Time dependent Heston model, *SIAM Journal on Financial Mathematics* **1**(1): 289–325.

Bezborodov, V., Di Persio, L. and Mishura, Y. (2019). Option pricing with fractional stochastic volatility and discontinuous payoff function of polynomial growth, *Methodology and Computing in Applied Probability* **21**(1): 331–366.

Biagini, F., Hu, Y., Øksendal, B. and Zhang, T. (2008). *Stochastic calculus for fractional Brownian motion and applications*, Springer Science & Business Media.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities, *Journal of political economy* **81**(3): 637–654.

Cajueiro, D. O. and Tabak, B. M. (2005). Possible causes of long-range dependence in the Brazilian stock market, *Physica A: Statistical Mechanics and its Applications* **345**(3-4): 635–645.

Cajueiro, D. O. and Tabak, B. M. (2008). Testing for long-range dependence in world stock markets, *Chaos, Solitons & Fractals* **37**(3): 918–927.

Cheridito, P. and Nualart, D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0,\frac{1}{2})$, *Annales de l'IHP Probabilités et statistiques*, Vol. 41, pp. 1049–1081.

Cheridito, P. et al. (2001). Mixed fractional Brownian motion, *Bernoulli* **7**(6): 913–934.

Chou, C.-S. and Lin, H.-J. (2006). Some properties of CIR processes, *Stochastic analysis and applications* **24**(4): 901–912.

Chronopoulou, A. and Viens, F. G. (2010). Hurst index estimation for self-similar processes with long-memory, *Recent Development in Stochastic Dynamics and Stochastic Analysis*, World Scientific, pp. 91–117.

Comte, F. and Renault, E. (1998). Long memory in continuous-time stochastic volatility models, *Mathematical finance* **8**(4): 291–323.

Cont, R. and Das, P. (2022). Rough volatility: fact or artefact?, *arXiv preprint arXiv:2203.13820* .

Cox, J. C., Ingersoll Jr, J. E. and Ross, S. A. (1985). A theory of the term structure of interest rates, *Theory of Valuation*, World Scientific, pp. 129–164.

Davies, R. B. and Harte, D. (1987). Tests for Hurst effect, *Biometrika* **74**(1): 95–101.

Decreusefond, L. et al. (1999). Stochastic analysis of the fractional Brownian motion, *Potential analysis* **10**(2): 177–214.

Derman, E. and Kani, I. (1994). Riding on a smile, *Risk* **7**(2): 32–39.

Dieker, T. (2004). *Simulation of fractional Brownian motion*, PhD thesis, Masters Thesis, Department of Mathematical Sciences, University of Twente, The Netherlands.

Dupire, B. (1994). Pricing with a smile, *Risk* **7**(1): 18–20.

El Euch, O., Gatheral, J. and Rosenbaum, M. (2019). Roughening heston, *Risk* pp. 84–89.

El Euch, O. and Rosenbaum, M. (2018). Perfect hedging in rough heston models, *The Annals of Applied Probability* **28**(6): 3813–3856.

Fallah, S., Najafi, A. R. and Mehrdoust, F. (2019). A fractional version of the Cox–Ingersoll–Ross interest rate model and pricing double barrier option with Hurst index, *Communications in Statistics-Theory and Methods* pp. 1–16.

Fouque, J.-P., Papanicolaou, G., Sircar, R. and Sølna, K. (2011). *Multiscale stochastic volatility for equity, interest rate, and credit derivatives*, Cambridge University Press.

Fu, M. C. and Hu, J.-Q. (1995). Sensitivity analysis for monte carlo simulation of option pricing, *Probability in the Engineering and Informational Sciences* **9**(3): 417–446.

Gatheral, J., Jaisson, T. and Rosenbaum, M. (2018). Volatility is rough, *Quantitative Finance* **18**(6): 933–949.

Göing-Jaeschke, A., Yor, M. et al. (2003). A survey and some generalizations of Bessel processes, *Bernoulli* **9**(2): 313–349.

Guo, Z. J. (2008). A note on the CIR process and the existence of equivalent martingale measures, *Statistics & Probability Letters* **78**(5): 481–487.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options, *The review of financial studies* **6**(2): 327–343.

Higham, D. J., Mao, X. and Stuart, A. M. (2002). Strong convergence of Euler-type methods for nonlinear stochastic differential equations, *SIAM Journal on Numerical Analysis* **40**(3): 1041–1063.

Hong, J., Huang, C., Kamrani, M. and Wang, X. (2019). Optimal strong convergence rate of a backward Euler type scheme for the Cox–Ingersoll–Ross model driven by fractional Brownian motion, *Stochastic Processes and their Applications* .

Hosking, J. R. (1984). Modeling persistence in hydrological time series using fractional differencing, *Water resources research* **20**(12): 1898–1908.

Hu, Y., Nualart, D. and Song, X. (2008). A singular stochastic differential equation driven by fractional Brownian motion, *Statistics & Probability Letters* **78**(14): 2075–2085.

Hull, J. and White, A. (1987). The pricing of options on assets with stochastic volatilities, *The journal of finance* **42**(2): 281–300.

Hull, J. and White, A. (1990). Pricing interest-rate-derivative securities, *The review of financial studies* **3**(4): 573–592.

Hult, H. (2003). Approximating some volterra type stochastic integrals with applications to parameter estimation, *Stochastic processes and their applications* **105**(1): 1–32.

Hurst, H. (1951). The long-term storage capacity of reservoirs, *Transactions of American Society Civil Engineer* **116**(1): 770–799.

Jarrow, R. and Protter, P. (2004). A short history of stochastic integration and mathematical finance: The early years, 1880–1970, *A festschrift for Herman Rubin*, Institute of Mathematical Statistics, pp. 75–91.

Kahl, C. and Jäckel, P. (2005). Not-so-complex logarithms in the heston model, *Wilmott magazine* **19**(9): 94–103.

Kallenberg, O. (1998). Components of the Strong Markov Property, *Stochastic Processes and Related Topics*, Springer, pp. 219–230.

Kantelhardt, J. W., Zschiegner, S. A., Koscielny-Bunde, E., Havlin, S., Bunde, A. and Stanley, H. E. (2002). Multifractal detrended fluctuation analysis of nonstationary time series, *Physica A: Statistical Mechanics and its Applications* **316**(1-4): 87–114.

Kassouf, S. T. and Thorp, E. O. (1967). Beat the market: A scientific stock market system.

Kolmogorov, A. N. (1940). The Wiener spiral and some other interesting curves in Hilbert space, *Dokl. Akad. Nauk SSSR*, Vol. 26, pp. 115–118.

Kubilius, K. (2020). Estimation of the hurst index of the solutions of fractional sde with locally lipschitz drift, *Nonlinear analysis: modelling and control* **25**(6): 1059–1078.

Livieri, G., Mouti, S., Pallavicini, A. and Rosenbaum, M. (2018). Rough volatility: evidence from option prices, *IISE Transactions* **50**(9): 767–776.

Mandelbrot, B. B. and Van Ness, J. W. (1968). Fractional Brownian motions, fractional noises and applications, *SIAM review* **10**(4): 422–437.

Marinucci, D. and Robinson, P. M. (1999). Alternative forms of fractional Brownian motion, *Journal of statistical planning and inference* **80**(1-2): 111–122.

Mishura, Y. (2008). *Stochastic calculus for fractional Brownian motion and related processes*, Vol. 1929, Springer Science & Business Media.

Mishura, Y., Piterbarg, V., Ralchenko, K. and Yurchenko-Tytarenko, A. (2018). Stochastic representation and path properties of a fractional Cox–Ingersoll–Ross process, *Theory of Probability and Mathematical Statistics* **97**: 167–182.

Mishura, Y. and Yurchenko-Tytarenko, A. (2018). Fractional Cox–Ingersoll–Ross process with non-zero «mean», *Modern Stochastics: Theory and Applications* **5**: 99–111.

Mishura, Y. and Yurchenko-Tytarenko, A. (2019). Fractional Cox–Ingersoll–Ross process with small Hurst indices, *Modern Stochastics: Theory and Applications* **6**(1): 13–39.

Mishura, Y. and Yurchenko-Tytarenko, A. (2020). Approximating Expected Value of an Option with Non-Lipschitz Payoff in Fractional Heston-Type Model, *International Journal of Theoretical and Applied Finance* .

Molchan, G. M. (1999). Maximum of a fractional Brownian motion: probabilities of small values, *Communications in mathematical physics* **205**(1): 97–111.

Neuman, E. and Rosenbaum, M. (2018). Fractional Brownian motion with zero Hurst parameter: a rough volatility viewpoint, *Electronic Communications in Probability* **23**.

Norros, I., Valkeila, E. and Virtamo, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions, *Bernoulli* **5**(4): 571–587.

Nourdin, I. (2012). *Selected aspects of fractional Brownian motion*, Vol. 4, Springer.

Nualart, D. (2003). Stochastic integration with respect to fractional Brownian motion and applications, *Contemporary Mathematics* **336**: 3–40.

Nualart, D. (2006). *The Malliavin calculus and related topics*, Vol. 1995, Springer.

Nualart, D. and Ouknine, Y. (2002). Regularization of differential equations by fractional noise, *Stochastic Processes and their Applications* **102**(1): 103–116.

Peng, C.-K., Buldyrev, S. V., Havlin, S., Simons, M., Stanley, H. E. and Goldberger, A. L. (1994). Mosaic organization of dna nucleotides, *Physical review e* **49**(2): 1685.

Power, G. J. and Turvey, C. G. (2010). Long-range dependence in the volatility of commodity futures prices: Wavelet-based evidence, *Physica A: Statistical Mechanics and its Applications* **389**(1): 79–90.

Rogers, L. C. G. (1997). Arbitrage with fractional Brownian motion, *Mathematical Finance* **7**(1): 95–105.

Rosenbaum, M. (2011). A new microstructure noise index, *Quantitative Finance* **11**(6): 883–899.

Rouah, F. D. (2015). *The Heston Model and Its Extensions in VBA*, John Wiley & Sons.

Samko, S. G., Kilbas, A. A. and Marichev, O. I. (1993). *Fractional integrals and derivatives*, Vol. 1, Gordon and Breach Science Publishers, Yverdon Yverdon-les-Bains, Switzerland.

Samuelson, P. A. (1964). Proof that properly discounted present values of assets vibrate randomly, *The Bell Journal of Economics and Management Science* pp. 369–374.

Takaishi, T. (2020). Rough volatility of bitcoin, *Finance Research Letters* **32**: 101379.

Wiener, N. (1923). Differential-space, *Journal of Mathematics and Physics* **2**(1-4): 131–174.

Wilmott, P. (2013). *Paul Wilmott on quantitative finance*, John Wiley & Sons.