AN EXPLORATION OF LEARNING DIFFICULTIES EXPERIENCED BY GRADE 12 LEARNERS IN EUCLIDEAN GEOMETRY: A CASE OF NGAKA MODIRI MOLEMA DISTRICT

by

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I declare that the above dissertation is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

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I further declare that I have not previously submitted this work, or part of it, for examination at Unisa for another qualification or at any other higher education institution.

20/08/2022

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DEDICATION

I dedicate this dissertation to my family, my wife Mrs Mudhefi, and our three sons for always being there for me through this academic journey. For which without their encouragement and moral support my studies would have been more difficult. I also dedicate this piece of work to my younger Brother, Isaih Mudhefi, for assisting materially.

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ABSTRACT

The purpose of this study was to investigate the learning difficulties experienced by Grade 12 learners in Euclidean geometry. Despite the efforts exerted in terms of time, material and human resources in the teaching and learning of Euclidean geometry, learners' performance in Euclidean geometry remains unsatisfactory. As a result, this study sought to answer the research question: What are the learning difficulties that the Grade 12 learners experience in the learning of Euclidean geometry? Van Hiele levels of geometric thinking were a used as a framework to explore Grade 12 learners' learning difficulties. The study adopted a sequential exploratory mixed method design. The participants in the study were mathematics educators, learners, head of departments (HOD) from a sample of six schools in Ngaka Modiri Molema district, North West Province. As part of data collection procedure, Euclidean Geometry test developed in accordance with Van Hiele levels was administered to 60 Grade 12 learners who were randomly sampled from six schools which were purposively sampled. In addition, two HODs', four educators and 12 learners completed questionnaires. Furthermore, semi-structured interviews were purposively conducted with 12 learners and four educators based on learners' test results.

Data analysis involved the identification of misconceptions and errors committed at each Van Hiele level and were interpreted as learners learning difficulties. The test analysis involved categorising learner performance in the test by frequency distributions of those who achieved and those who did not achieve 50% and above in the question(s) set at each Van Hiele level. In addition, questionnaires were analysed by determining the frequency distribution of respondents' levels of agreement and disagreement to sets of predetermined questions at different Van Hiele levels. The findings of the study revealed that Grade 12 learners experience difficulties across Van Hiele levels of geometric thinking. These included amongst others learners' difficulties in identifying and naming angles between parallel lines and a transversal (Visualisation level); using analytical skills and correct geometry terminology to describe components of a circle (Analysis level); completing proofs of circle theorems and problem solving involving short deductions and multistep geometric riders (Informal deductive level); presenting a series of deductive steps leading to the desired geometric solution (Formal deductive level).

Based on these findings of the study, the recommendations were that educators should give learners the opportunity to discover geometry concepts through hands-on activities with geometric shapes to ground their understanding of geometry concepts. Furthermore, it was recommended that educators should use the Van Hiele model as a framework for teaching

Euclidean geometry by introducing learners to geometry concepts in a hierarchical manner from first developing among learners' visualisation skills up to formal deductive reasoning.

Key concepts: euclidean geometry, errors and misconceptions, geometry learning difficulties; spatial visualisation, deductive reasoning; van hiele model; pedagogical content knowledge; geometry content knowledge

ABBREVIATIONS

ACE	Advanced Certificate in Education
BSc	Bachelor of Sciences
CAPS	Curriculum and Assessment Policy Statement
CDASSG	Cognitive Development and Achievement in Secondary school Geometry
DBE	Department of Basic Education
FET	Further Education and Training Band
GSP	Geometer's Sketchpad
HOD	Head of Department
HSRC	Human Sciences Research Council
NCTM	National Council of Teachers of Mathematics
NSC	National Senior Certificate
NSTF	National Science and Technology Forum
OBE	Outcome-Based Education
PCK	Pedagogical Content Knowledge
PGCE	Post-graduate Certificate in Education
PME	Pre-service Mathematics Education
SP	Senior Phase
STEM	Science, Technology, Engineering and Mathematics
TKC	Tradition Knowledge-based Curriculum
UDE	University Diploma in Education
ZDP	Zone of Proximal Development

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CHAPTER 1: INTRODUCTION AND OVERVIEW OF THE STUDY

1.1 Introduction

This chapter started by presenting a general overview of the study. The chapter briefly captured a wide variety of aspects that are core of the investigation of learning difficulties as experienced by Grade 12 in Euclidean geometry. The aspects outlined include the background to the study, reform in the geometry curriculum in South Africa over the years, a review of geometry performance as reported by Department of Basic Education (DBE) over the past five years, the problem statement, the purpose of the study, research questions, research design and methodology. In addition to the above, the significance and rationale for the study were also outlined. The chapter concluded with a brief description of ethical consideration and preliminary chapter outline for the study.

1.2 An overview of the study

Euclidean geometry is an essential field of study in mathematics as it develops the learners' visual, intuitive and aesthetics senses (Alex & Mammen, 2016). According to Couto and Vale (2014), the development of geometrical thought is an important auxiliary to solve problems in learners' daily lives. Therefore, these views suggest that a well-grounded geometry conceptualisation among learners helps them to be better problem solvers. In the same vein, the National Council of Teachers of Mathematics (NCTM, 2002) emphasises that the major goal of secondary school geometry is to develop mathematical reasoning abilities and to promote a deeper understanding of the real world. Similarly, one of the aims of the mathematics curriculum in South Africa is to develop spatial skills, describe properties of shapes and objects, and identify, interpret and solve problems critically and creatively (DBE, 2011). Based on these views, the researcher concluded that the knowledge of geometry should not be underestimated as it promotes critical thinking and helps learners to develop a better understanding of the world around them. Regardless of the importance of geometry, data from the National Senior Certificate (NSC) diagnostic reports of the DBE from 2016-2020 show that learners' performance in geometry is exceedingly poor when compared to other topics in the Grade 12 mathematics papers. That persistent poor performance in Euclidean geometry prompted the researcher to undertake this study.

1.2.1 The background of geometry as a topic in school mathematics

The South African education system has been characterised by on-going curriculum changes since its democratic inception in 1994. Constituting those changes were policy revisions, modifications and reformations, like National Curriculum Statement 2001, Revised

National Curriculum Statement 2002, Curriculum 2005 and the Revised National Curriculum Statement implemented in the Further Education and Training Band (FET) in 2006, which resulted in Euclidean geometry being removed as a compulsory topic in mathematics curriculum (Alex & Mammen, 2014). As a result, high school geometry became an optional section in Paper 3 mathematics, which meant it was not compulsory for teachers to teach it (Mabotja, 2017; Van Putten et al., 2010). In other words, the educators could decide whether they wanted to teach it or not. Consequently, in 2008, only 3.8% (12 466) of the Grade 12 mathematics learners in South Africa wrote the optional Paper 3, and almost half of those learners (6 155) scored less than (30%) (DBE, 2009). These statistics are probably an indication that learners are struggling to understand Euclidean geometry concepts.

Moreover, research further revealed that one of the challenges brought about by geometry exclusion from the compulsory curriculum was a lack of consistency in the study of shape and space. For example, Siyepu and Mtonjeni (2014) argue that the exclusion of geometry posed challenges to students registering for engineering courses at university. Consequently, Euclidean geometry was re-introduced as a compulsory content area in the mathematics Curriculum and Assessment Policy Statement (CAPS) by the Department of Basic Education. To facilitate that re-introduction of geometry as a compulsory section in the CAPS curriculum, the DBE used a phased approach, starting with grade 10 in 2012, followed by grade 11 in 2013 and then Grade 12 in 2014. As a result, the class of 2014 became the first cohort of matric learners who wrote geometry as a compulsory section of mathematics Paper 2 in exit level examinations.

In support of the re-introduction of geometry as a compulsory section of mathematics Paper 2, one of the aims of the National Curriculum Statement Grade R-12 is the need for educators to produce learners who can effectively communicate using visual, symbolic or language skills in various modes and demonstrate learning with understanding (CAPS, 2011a). That aim links well with the importance of geometry, in particular Euclidean geometry, in that it helps learners to use various modes of representations and logical reasoning to solve geometric problems.

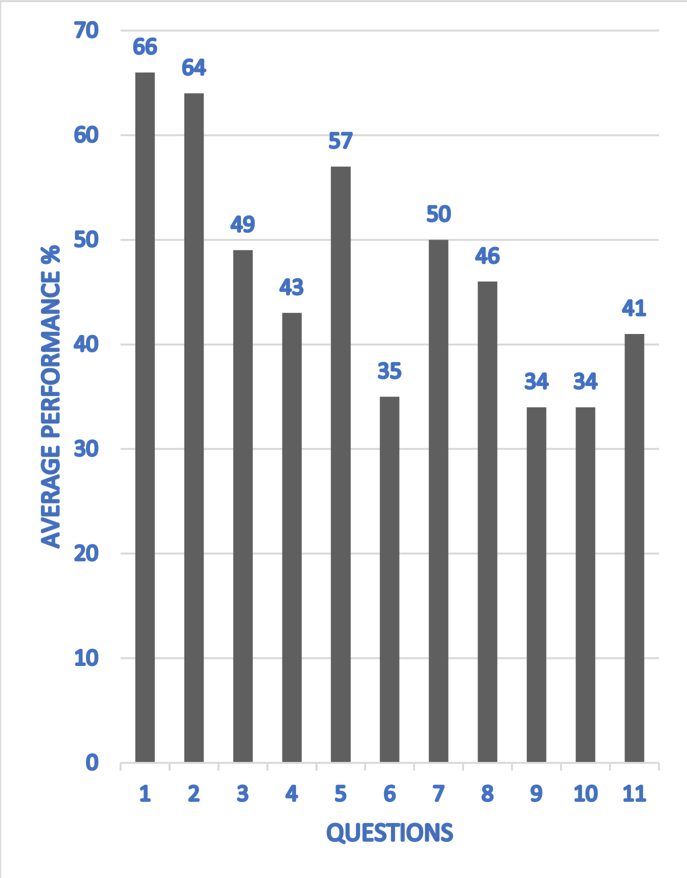
However, learning with understanding, particularly in the domain of Euclidean geometry, has been problematic as many learners are found struggling with the development of skills to recognise, analyse and think about spatial objects and images (Couto & Vale, 2014; Mabotja, 2017). In view of that, the re-introduction of geometry through CAPS as a compulsory section of Paper 2 mathematics was a welcome development, because the optional Paper 3 deprived most learners of the critical, logical, analytical and rigorous thinking needed to improve learner performance in mathematics and other disciplines since most the learners opted not to write the Paper 3 (Gunhan, 2014a). However, in the current curriculum (CAPS), mathematics educators and learners alike are compelled to do geometry as a section of Paper 2. Although the re-introduction of Euclidean geometry was a welcome development, Siyepu (2014) emphasises that geometry remains a threat to both learners and educators; hence, the need to undertake a study on the learning difficulties experienced by Grade 12 learners in Euclidean geometry.

In this regard, the above-mentioned South African education curriculum reformations and their impact on geometry learning were explained in detail in sub-section 2.3 of chapter 2.

1.2.2 Poor mathematics performance linked to challenges in Euclidean geometry

Euclidean geometry contributes 50±3 marks of the Grade 12 Mathematics Paper 2, DBE, CAPS (2011a) and research points to the fact that very few learners attempt these questions and those that attempt them; perform badly (DBE, 2018). For this reason, studies carried out earlier in South African secondary schools indicated that learners have weak knowledge of geometry as indicated by Grade 12 yearly examination reports (Baiduri, 2020; Bowie, 2009; Bonnie, 2016). In addition, Amazigo (2000) highlighted that learners' performance in Euclidean geometry in both internal and external examination remains consistently poor. The same notion was also articulated by the Third International Mathematics and Science Study (2006) report on the performance of learners in geometry, which indicated that, of the 50 countries that participated, learners from South Africa fared the worst of all in mathematics, and the section that showed the weakest learner performance was Euclidean geometry (Reddy, 2005). To this end, the NSC results bear testimony to the assertions above: of the 1 207 996 learners who entered Grade 1 in 2006, only 89 119 passed mathematics at the end of 2017 in the NSC, of which only 13 511 managed to obtain a mark of at least (50%) (DBE, 2018). Findings from those studies also indicated that geometry was probably a section of Paper 2 that accounted for learners' poor performance.

In support of the above, the figure 1.1 below is an extract from the 2017 diagnostic report, showing the average performance (%) per question in geometry, as compared to other questions in Paper 2. The general trend as indicated by the average performance per question suggested that learners are doing well in other Paper 2 topics, when compared to their performance in Euclidean geometry. Moreover, figure 1.1 also shows that in 2017, the average performance per question was below 50% for geometry questions when compared to questions like data handling, analytical geometry and trigonometry where the average performance was well above 50%. Deducing from the figures as indicated on the same graph, indications were that learners have challenges with geometric understanding, hence the poor performance. Those figures also supported Seroto's (2006) assertion that performance in the geometry section of Paper 2 is much lower than other sections of the mathematics paper in Grades 10, 11 and 12.



Q1	Data Handling
Q2	Data Handling
Q3	Analytical Geometry
Q4	Analytical Geometry
Q5	Trigonometry
Q6	Trigonometry
Q7	Trigonometry
Q8	Euclidean Geometry
Q9	Euclidean Geometry
Q10	Euclidean Geometry
Q11	Euclidean Geometry

Figure 1.1

Average performance % per question for Mathematics Paper 2 (DBE, 2017)

To further elaborate on what is articulated by Figure 1.1 above, the researcher also compiled average learner performances (%) for Euclidean geometry per question and the average geometry performance (%) per year over a period of five years from DBE diagnostic reports (2016-2020). Figure 1.2 below shows that over the past five years, geometry performance has been generally poor. In that regard, the trend as shown by the graph below was that, for the past five years, learners have been performing slightly above 50% in only question 8, which is the first geometry question for each of the papers, but from question 9 to question 11, learners performed well below 50%, with some averages well in the 30% range. Average performance for question 8 was slightly above 50% in the five years, most probably because it is a lower level (visual and analysis level) according to Van Hiele's levels of geometrical thinking.

However, the trend per question as indicated below was that, as the demand for geometric thinking increased according to the Van Hiele levels, learner achievement decreased, hence lower averages percentages were recorded from question 9 to question 11, where formal deductive reasoning is required. That trend probably point to the fact that a few learners in Grade 12 were operating above level 2 of Van Hiele's hierarchy of geometrical thought. In addition, the overall yearly average performance as indicated in figure 1.2 has been generally poor for the geometry section of the papers, since it was never above 50% in the five years (2016 to 2020). The highest annual average (%) was 47% in 2020 and the lowest was 38% in 2017, which is a worrisome phenomenon in terms of learner achievement.

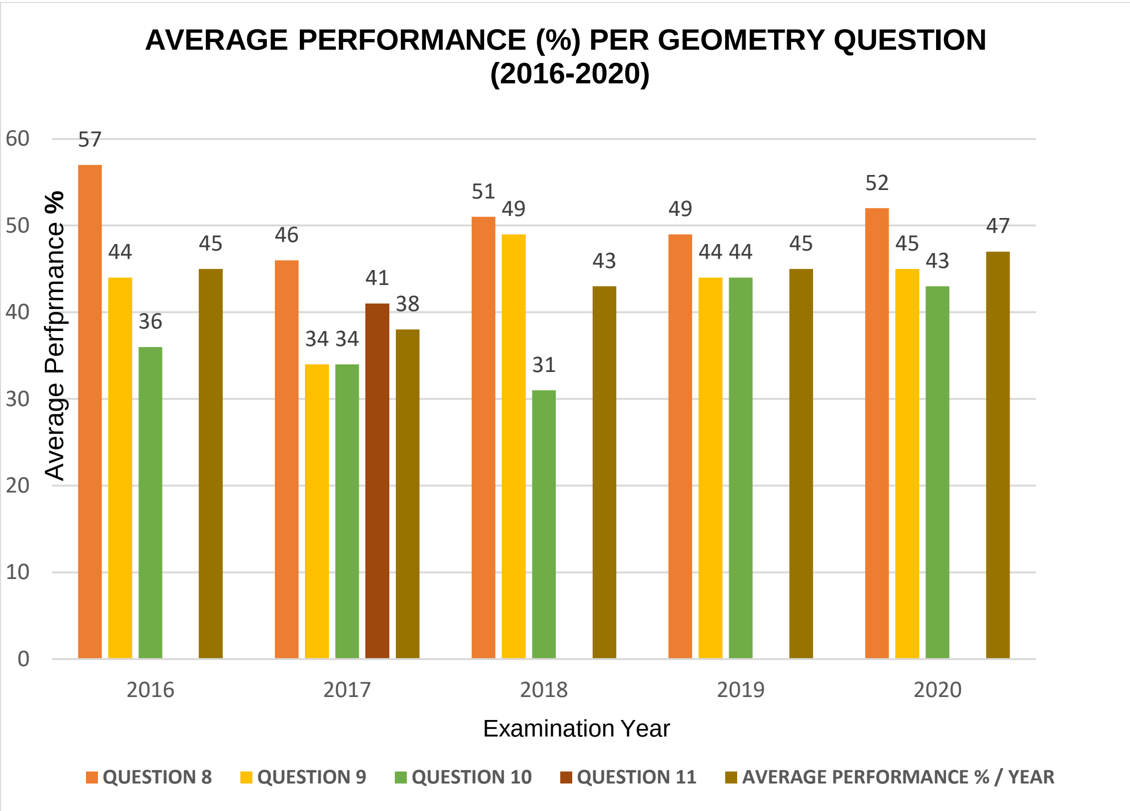


Figure 1.2

Average performance % for geometry mathematics Paper 2 (DBE, 2016-2020)

The above-mentioned statistics as indicated in figures 1.1 and 1.2 shows that geometrical knowledge is generally a problem in terms of what learners conceptualise and how they retain and integrate that knowledge. That is evidenced by the lower average performance percentages for Euclidean geometry questions when compared to performance in other mathematics strands like data handling and analytical geometry. Besides, figure 1.2 indicated persistently low overall yearly average performance in geometry over the past five years. These figures might as well support Van Putten et al. (2010) who argue in their research findings that, learners exiting secondary schools in South Africa lack in-depth an understanding of geometric concepts. Hence, there was a need to undertake a research to investigate the learning difficulties Grade 12 learners encounter in Euclidean geometry, with the hope of sheding light onto the reasons behind the overall poor performance in mathematics.

1.2.3 Challenges in Euclidean geometry linked to educator levels of mastery

Through the researcher's own experience as a mathematics teacher, the current trend in South African secondary schools is that there is an exodus of learners from mathematics to mathematical literacy. This problem seems to be fuelled by a phobia among the learners, which is a result of poor performance in mathematics, with a particular reference to Euclidean geometry. In support of this, studies have also shown that there is a meaningful correlation between attitude towards geometry and achievement, where attitude can be positive, like enjoyment and interest in geometry, or the opposite; and the worst is geometry phobia (Ahmed & Bora, 2018). Similarly, studies by researchers such as Hanna and De Villiers (2012) indicated that learners have difficulties with Euclidean geometry and one of the reasons for those challenges is the general learner anxiety associated with geometry problem solving. To further support the idea of geometry phobia among learners, Van Putten et al. (2010) argue that fear of geometry further obstructs achievement of knowledge as learners reject the notion of understanding emanating from that fear. As a result, this is a cause for concern to educators and other education stakeholders.

Furthermore, numerous resources have been put into use to enhance the learning and teaching of mathematics, Euclidean Geometry in particular, in the form of material and intellectual resources, through professional support forums and content training workshops for teachers, as well as extra and holiday lessons for learners. Despite all of that, learners are still performing badly in geometry. In addition, the researcher also realised through participating in content training workshops that most of the current crop of mathematics educators face difficulties with Euclidean geometry, most probably because they did not do it in high school and even those who did, still have content knowledge gaps. That could point to the poor performance by learners in geometry in Grade 12. Similarly, that anomaly about most educators' mastery levels of geometry as identified by the researcher concurred well with Bowie (2009) fears regarding the introduction of Euclidean geometry as a compulsory topic in Senior Phase (SP) and Further Education Training Phase (FET), when he alluded that probably the main reason why Euclidean geometry was considered as optional Paper 3 in South Africa, was that most educators were not familiar with the content.

1.2.4 Challenges in Euclidean geometry linked to Van Hiele's levels of geometric thinking

An analysis by Luneta (2015) showed that most Grade 12 learners performed badly because of conceptual errors they made when answering questions in geometry. Similarly, Alex and Mammen (2016) found that Grade 12 learners were still operating at visualisation level as indicated by Van Hiele's model of geometric thinking. Furthermore, there is compelling evidence from research showing that South African learners, especially Grade 12 learners, are operating below their levels of expectation (Luneta, 2015; Ngirishi & Bansilal, 2019; Siyepu, 2014). To be more precise, research has shown that learners are mostly at the concrete visualisation level instead of the abstract level in geometry, which demands higher levels of geometrical reasoning (Alex & Mammen, 2016; Luneta, 2015).

Generally, these researchers concured that poor performance by learners in Euclidean geometry might be attributed to a mismatch between learners' grade level of geometrical reasoning and the expected level of geometrical development, as proposed by Van Hiele's model. Furthermore, findings from the same studies also indicated that learners are experiencing difficulties in Euclidean geometry and those challenges probably impede meaningful learning, resulting in poor performance by learners. As result, this study intended to adopt the Van Hiele's model of geometrical thinking to help explore the learning difficulties that learners encountered in Euclidean geometry, mainly because it was used successfully in similar studies by previous researchers.

1.2.5 The gap identified in existing research

Even though there is general agreement on the relevance of Van Hiele's hierarchical levels in understanding geometry, it seems that previous studies did not focus on establishing the core learning difficulties that learners encounter in Euclidean geometry. Hence, the need for this investigation was to explore those learning challenges with the aim of alleviating the problems associated with poor learner performance in geometry. In the same vein, through the researcher's own experience in the learning and teaching of geometry, as well as evidence gathered from geometry literature, it is evident that learners do experience learning difficulties in Euclidean geometry.

In support of the above assertion, researchers such as Luneta and Makonye (2010) and Ozkan et al. (2018) underscore that Grade 12 learners have challenges in geometry due to a lack of basic skills in mathematics, and little has been done to explore those challenges. In addition, Mackle (2017) further argues that those geometry learning difficulties cause misconceptions and errors and prevent the generation of new knowledge by learners from occurring. Those more encompassing findings bear testimony to the existence of challenges in the teaching and learning of geometry, but from the researcher's point of view, those studies focused more on factors contributing to poor performances in geometry, and little research was done to explore the actual learning difficulties learners encountered with the understanding of Euclidean geometry.

Thus, the gap that existed in previous studies as identified by the researcher was that further research was still needed in addition to the work covered so far in exploring the challenges responsible for poor achievement in Euclidean geometry. In my view, one of those areas was to investigate the actual learning difficulties experienced by learners during their encounters with geometrical problems.

In view of the above-mentioned shortfalls from previous studies, this paper explored the actual challenges responsible for learners' persistent poor performance in the Euclidean geometry section of Paper 2 in Grade 12. Therefore, undertaking this investigation was warranted because it was hoped that by exploring the learners' learning difficulties, educators would be able to intervene more meaningfully in the teaching and learning of Euclidean geometry and help learners to improve their performance in Grade 12 tests and examinations.

1.3 Statement of the problem

Euclidean geometry, in particular circle geometry, is viewed as the most useful component of mathematics that trains one's eye to identify things much quicker, and one's mind to be steadier in doing things; thus any challenges associated with the teaching and learning of geometry deprive learners of these essential skills. As a result, Jones (2002) emphasises that understanding of geometry enables more learners to achieve high marks in mathematics. In addition, Couto and Vale (2014) argue that the development of geometrical thought is an important auxiliary to solve problems in learners' daily lives.

Regardless of the importance of geometry, learners' poor performance in this content area has been of much concern to mathematics educators, parents and the government (Adolphus, 2011). For example, the chief examiner's annual reports in mathematics in the National Senior Certificate (NSC) conducted by the DBE (2019) found that Grade 12 learners could not demonstrate conceptual understanding, interpret geometric diagrams and provide justification for their solutions, as demanded by the questions.

Furthermore, research indicated that learners' challenges and the resultant poor performance in Euclidean geometry were attributed to various factors such as geometry misconceptions that emanate from poor teaching and learning strategies (Luneta, 2014; Mukamba & Makamure, 2020; Ngirishi, 2019). Even though the phenomenon has been extensively researched from various viewpoints, there is limited literature on the actual learning difficulties experienced by Grade 12 learners. As a result, this study sought to explore learning challenges in geometry as experienced by learners in Grade 12.

1.4 Purpose of the study

The purpose of this study was to investigate the learning difficulties in Euclidean geometry as experienced by Grade 12 learners, together with the possible factors contributing to the identified challenges. In addition, the investigation focused on analysing learner competences in the domains of lines, angles, properties of triangles and quadrilaterals, circle theorems and geometric proofs in a bid to establish the actual challenges they have with Euclidean geometry. As a result, the research intended to fulfil this purpose buy answering the following question(s) through research.

1.5 The research questions

The proposed study was to answer the following question:

"What are the learning difficulties that the Grade 12 learners experience in the learning of Euclidean geometry?"

The researcher narrowed the research question further to specific sub-questions that focused on the learning difficulties and possible factors by asking questions that solicited stakeholders' views (learners, mathematics teachers and school mathematics HoDs)) on the learning difficulties encountered in Euclidean geometry in Grade 12. These included:

1. What are the Grade 12 learners' learning difficulties related to visualisation in Euclidean geometry?
2. What are the Grade 12 learners' learning difficulties related to analysis level in Euclidean geometry?
3. What are the Grade 12 learners' learning difficulties related to informal deduction level in Euclidean Geometry?
4. What are the Grade 12 learners' learning difficulties related to formal deduction level in Euclidean geometry?

1.5.1 Aims and objectives of the study

This study aimed to:

- Identify learning difficulties learners display when learning Euclidean geometry.
- Identify possible factors contributing to those challenges.

Research objectives

By embarking on the study, the researcher intended to explore:

- Grade 12 learners' challenges with geometric proofs as well as conjecturing abilities when solving circle geometry problems.
- Grade 12 learners' requisite visual skills in diagram analysis since diagrams have the potential to develop learners' visual access to the whole system of quantifiable relationships in circle geometry problems.
- Grade 12 learners' difficulties with proving and solving circle geometry problems that require providing reasons for statements.

Therefore, the study ultimately aimed to equip both educators and learners with the knowledge of possible geometry challenges and in that way, making them competent to deal with learning difficulties associated with the understanding of Euclidean geometry and hence improve learner performance. Given this, the study aimed to provide resource material by adding to the literature of geometry teaching for other studies in the field of mathematics, in particular geometry instruction, and supplement the studies by those seeking effective approaches to Euclidean geometry teaching as well as building learner confidence in solving circle geometric problems.

1.6 Relevance and need for the study

The study is relevant in the South African context because South Africa is experiencing a serious skills gap mainly in science, technology, engineering and mathematics (STEM) careers. Evidence of that shortage was found in a report by the National Science and Technology Forum (NSTF) (2018), which reported that South Africa is experiencing challenges that relate to its school education system, especially STEM subjects. As a result, it was also established that only 20% of the students are graduating in STEM-related courses; hence failing to close the gap for high-level skills in science-related fields. In addition, when commenting on the importance of geometry, White (2014) further emphasises that STEM subjects, especially mathematics, afford learners the opportunity to improve their problem-solving and critical thinking skills. In view of the above, this study was worth pursuing, not only because of the visual ability and logical reasoning that is required in Euclidean geometry as an essential discipline of mathematics, but also because of its value in life itself.

In view of the above, Sutiarso and Coesamin (2018) use media scaffolding to describe logical thinking as the development of spatial intuition about the world, reading and interpreting mathematical arguments as some of the major goals of teaching geometry. Thus, issues like the amount of geometric experience learners have, their attitude towards geometry, their application of dimensional deconstruction, educators' instructional experience and the influence of geometry teaching at elementary level featured as areas of concern in this study, although they only featured in the background, because the focus of the study was on exploring the learning challenges that learners encountered in Euclidean geometry.

Overall, research on this topic was of utmost importance because literature and experience indicated that Euclidean geometry is difficult for both educators and learners, as evidenced by poor performance in Grade 12 tests and examinations. The findings of the study and the associated recommendations aimed to improve learner performance in mathematics and to better inform future practice in learning Euclidean geometry.

1.7 Rationale for the study

The South African situation in terms of geometry performance has been described extensively in the previous paragraphs and reference was made to research findings confirming the situation. In more general terms, Luneta (2014) and Patkin and Lavenberg (2007) argue that geometry is regarded as the most difficult section of the high school mathematics curriculum because learners often think it does not relate to their daily life. However, that perception emanated partly from the traditional way in which geometry is taught in schools, together with a current shortage of qualified educators in this field (Alexander et al., 2014). Similarly, Bosman and Schulze (2018) and De Villiers (1997) argue that South African learners have been performing worse in Euclidean geometry than other sections of the mathematics curriculum such as algebra. Those findings were consistent with the general trend in the learners' annual performance, as presented in figure 1.2, where the overall performance across geometry questions was below 50%.

In addition to the views raised above, the rationale for this study was echoed by Makonye (2011) who argues that learners' poor geometric reasoning and thinking need to be explored and made clear to both educators and learners in terms the of concepts involved if meaningful and positive development was to be observed; otherwise, teaching would be diverted from the core challenges that learners encounter in Euclidean geometry. Therefore, the findings from previous researchers suggest that both the learners and the educators lack comprehensive understanding of geometry concepts. That is why it was important to explore geometry learning difficulties.

Furthermore, the complexity of the topic, the poor performance and the identified flawed geometrical reasoning and thinking pointed out in previous research findings prompted the need to explore the learning difficulties in Euclidean geometry, together with the contributing factors to those challenges. It was noted through the overall poor performance in mathematics paper 2 in internal and external mathematics examinations that Grade 12 learners had learning difficulties in Euclidean geometry. Deducing from the various geometry perspectives held by previous researchers, it could be argued that the consensus was; learners were not doing well in geometry. As result, the researcher considered it fit to investigate those challenges in view of poor instruction (teacher effect), poor comprehension (learner effect), and curriculum design, together with other possible factors responsible for those problems.

Another view was that learning difficulties in Euclidean geometry occured because of the lack of ability by learners to form mental images of geometrical figures and concepts (Mackle, 2017; Zilkova, 2019). In support of that, Ping and Kean Hua (2016) argue that teachers need to overcome misconceptions that happen to be the main cause for learners' learning difficulties in with geometry.

Overall, through exploring the learning difficulties that learners experience in Euclidean geometry, it was hoped that the study would contribute to the body of knowledge in the teaching and learning of geometry in its broadest sense by providing data that previous studies may not have investigated, as well as refining already identified challenges, with the aim of enhancing learner performance. Moreover, it was hoped that the identification of learning difficulties would enable educators to use strategies that effectively improve learners' understanding of geometric concepts and thereby improve learner performance.

1.8 Definition of terms

The following concepts are significant in the study:

Euclidean geometry: "a branch of geometry that deals with space and shape using a system of logical deductions" (Siyavula & volunteers, 2012, p. 349).

Inductive reasoning: "the process of observing data, recognizing patterns, and making generalization from those patterns" (Serra, 2002, p. 96).

Mixed methods design: "a study that brings together qualitative and quantitative techniques and/or data analysis within different phases of the research process" (McMillan, 2014, p. 4).

Visuospatial reasoning: "the activity of imagining static or dynamic objects and acting on them mentally" (Rivera, 2011).

Action research: “the study of a social situation with the view to improving the quality of the action within it” (Elliot, 1991).
Diagram analysis: “the analysis of a diagram to uncover all the information that is imbedded in the diagram” (Vorster, 2012, p. 9).
Professional development: “an opportunity given to educators to develop their knowledge, skills, approaches to improve their effectiveness in their classroom and organizations” (Loucks-Horsley, 1996).
Geometric Rider: “is simply a non-routine geometry problem” (Machisi, 2019, p. 3).

1.9 Research design and methodology

Creswell (2009) defines a research design as a set of guidelines and instruments to be followed in answering the research problem. In addition, the research design summaries the model(s) employed in the study. Furthermore, research methodology comprises different techniques, methods and a variety of procedures that researchers employ in implementing the research design or research methods (Creswell, 2014). This study followed a mixed-methods design, which integrates both qualitative and quantitative data collection methods. For this study, qualitative data were collected through interviews while, on the other hand, quantitative data were collected through tests and questionnaires. In view of this, Alex and Mammen (2018) and Jupp (2006) describe a questionnaire as a set of carefully designed questions administered in the same form to a group of people to gather data about the researcher’s topic of interest.

To this end, chapter 3 comprehensively detailed the research methodology used in this investigation. It gave a detailed account of the research design and the associated research methods that this study employed – both quantitative and qualitative methods.
This chapter described the population, sample(s), sampling, data collection procedures and the appropriate research instruments.

1.10 Issues of reliability and validity and trustworthiness

Validity is regarded as the degree to which inferences made regarding numerical scores are suitable, meaningful and useful to the sample (McMillan & Schumacher, 2014). In this investigation, both construct and content validity were taken into consideration to determine whether the test, questionnaires and interview questions really measured the concepts they were assumed to be measuring. This study used construct validity as alluded to by McMillan (2014) by ensuring the level at which interventions and measured variables represented intended, theoretical, underlying psychological constructs and elements. That was upheld in this study by ensuring that there was a balance between the subjects, the instruments used, and the procedure used to collect data.

Furthermore, to ensure content validity of the research instrument items, the researcher requested senior members in the mathematics department (HoDs and subject specialists) at the Department of Education of the Ngaka Modiri Molema District to assist with instrument validation prior administration to participants. Furthermore, trustworthiness was ensured by triangulation, member checking and consultation with experts. Therefore, mathematics experts assisted the researcher by examining the questionnaires and the test items to ensure content validity.

On the other hand, Cohen et al. (2007) argue that reliability is the relationship between the researcher's recorded data and what really happened in the natural setting where the research was conducted. Thus, in this study, reliability was ensured by triangulating data captured from written responses and correlating it with interview responses. Moreover, the researcher ensured credibility through triangulation where similar questions were asked to different participants (learners and teachers) to extract data from a variety of sources using different methods to answer the same researcher question(s). In this regard, the researcher also used member checking, where interview responses were taken back to participants for confirmation and validation of the data supplied. Thus, credibility of the qualitative aspects of the study was ensured by allowing interviewees to double-check their statements and close gaps left during earlier interviews (Dye et al., 2000). To ensure validity and reliability of data collected from learner performance in the geometry test, the researcher drafted questions from externally set Grade 12 papers to ensure good quality items.

1.11 Data analysis

Tavakoli (2012) views data analysis as a process of scaling down large volumes of collected data to a reasonable size by looking at themes, patterns, trends and performing statistical analyses that enable the development of summaries. The use of the mixed methods design enabled the analysis of both qualitative and qualitative data in a sequential manner, which was later triangulated to give a concrete argument in answering the research question(s). Qualitative data were analysed using thematic analysis and qualitative data through statistical analysis, using tables and graphical representations.

1.12 Ethical considerations

Research is viewed as a "scientific human endeavour that is organized according to a range of protocols, methods, guidelines and legislation" (Gerrish & Lacey, 2010, p. 15). For that reason, ethical considerations were relevant in situations where face-to-face interviews were conducted with a vulnerable group of respondents like learners. As a result, the position of the researcher was that of inquiry and confidentiality, thus he signed a code of ethical conduct. In support of the researcher's position, Neuman (2003, p. 124) argues that the researcher bears both moral and professional imperatives to be ethical, even under circumstances where the participants are not well informed or do not care about the importance of ethical considerations. Thus, informed consent must was sought from participants, after explaining the associated risks and discomforts, if any, as well as complete assurance of anonymity and confidentiality of records. To fulfil this purpose, confidentiality and anonymity of all participants was ensured in terms of test results, questionnaires and interviews**.**

In addition, participants were briefed of the importance of the investigation in the academic circles and were requested to give their cooperation. Thus, researcher sought permission to carry out the study from the principals of the sampled schools and participants signed consent forms. In addition, parents were informed of the investigation and its purpose. In the same way, they were requested to sign consent letters for their children voluntarily since some of the learners were still minors. Based on that, informed consent and voluntary participation were considered important, as both helped the participants to understand the information and allowed them the freedom of choice to decide whether to participate or decline (Polit & Beck, 2014).

Furthermore, participants were informed that their participation in the study was voluntarily and they were free to withdraw their participation at any stage of the study if they felt it necessary without being asked to give reasons for their withdrawal. The researcher assured the participants that their names and identities would not be revealed in data collection, data analysis and reporting of the results; hence, anonymity and confidentiality of participants' responses were guaranteed. The researcher asked for permission to use participants' responses in the research and their identity would be protected by using codes instead of their names.

1.13 Scope and limitations

The research took into consideration limitations associated with the use of too small samples since the results were to be generalised to the larger learner population. The study was methodologically limited in that the mixed methods approach used suited larger samples of participants in order to effectively collect both quantitative and qualitative data simultaneously. That was likely to be a challenge to the study although its effects were countered by the fact that the learner participants involved came from diverse family and school backgrounds. The other limitation was inherent in self-reporting instruments like the interviews, where the researcher relied on the faithfulness of participants in their responses. Most of the previous findings were based on pre- and post-tests prepared by the researchers themselves, hence there was a possibility of biased results; to curb a biased setting, the test items were extracted from past examination papers to ensure quality.

The other limitation was that the data were collected from learners' interview responses and a written test without going into the classroom. A realistic technique would have been that of a lesson observation where the researcher was more likely to have a clearer picture of the real learning difficulties as experienced by learners in a learning situation. However, the researcher tried to counter that by allowing educators to give their perspectives to the learning difficulties through administering educator questionnaire and interviews.

1.14 Preliminary chapter outline

CHAPTER 1: INTRODUCTION AND OVERVIEW OF THE STUDY

Chapter 1 outlined the overview of the study to include the background information on why there is a need to conduct a study of this nature and the need for further studies to be conducted in future. A brief overview was given of Euclidean geometry as a topic in Paper 2 school mathematics, together with a background to challenges linked to poor performance, educator levels of mastery and the Van Hiele levels of geometric thinking. The rational and significance of the study were outlined, together with an explanation of the need to carry out an investigation into the learning difficulties learners encounter with Euclidean geometry. The assumptions made in the study as well as the inherent limitations to the study were also outlined in chapter 1.

CHAPTER 2: LITERATURE REVIEW

The focus of chapter 2 was the review of literature on learning difficulties and the related possible factors responsible for those challenges, as identified by other researchers in Euclidean geometry. Furthermore, research details were provided on the reason for including Euclidean geometry in the high school curriculum, together with the inherent challenges to learners in terms of geometric understanding. Details were also provided on the core aspects of Euclidean geometry from research as potential sources of learning difficulties. Literature was explored to identify the possible learning difficulties with reference to specific geometrical reasoning models like the Van Hiele theory, Hoffer's model and Piaget's learning theory to authenticate the reality of poor performance in geometry by Grade 12 learners.

In the same vein, the theoretical framework that guided this research (the Van Hiele model of geometrical thinking) was also highlighted in detail. The researcher indicated shortcomings of previous research in addressing learning difficulties that Grade 12 learners encounter in Euclidean geometry. In chapter 2, the researcher indicated gaps in previous studies and then proposed possible ways through investigation to overcome those shortfalls; thereby bringing new dimensions to the research.

CHAPTER 3: THE RESEARCH DESIGN AND METHODOLOGY

Chapter 3 focused on the population, sample, sampling procedures, research instruments and data collection procedures. It discussed the actual undertaking of the study, provides motivation for the decision to select participants randomly from identified schools and discusses the quantitative and qualitative data obtained from the participants using various data collection methods. The first phase involved a pilot study where a sample of the participants was given various research instruments as a way of testing their feasibility in terms of validity, reliability and checking compatibility of the language used. Necessary adjustments were made to the research tools before they were administered to the whole group of participants. Research questions were restated, instrumentation put to effect and data analysis steps outlined and undertaken with specific reference to the research question(s).

CHAPTER 4: DATA PRESENTATION AND ANALYSIS

Chapter 4 was devoted to the presentation of research results, where data from learner performance in the test, semi-structure interviews and questionnaires were presented in detail and in summaries. The researcher employed various statistical techniques to correlate the data collected and establish trends behind those learning difficulties. That was done ensuring that data collected from educators and mathematics departmental heads correlated with that collected from learners to build a solid argument.

CHAPTER 5: RESEARCH FINDINGS, CONCLUSION AND RECOMMENDATIONS

Chapter 5 constituted the last chapter of the study where the findings of the research were spelt out, conclusions were made and recommendations given. The researcher highlighted the major issues depicted by the results, by clearly stating the learning difficulties identified as well as the factors responsible for the identified challenges. The findings were explained in terms of whether the research question(s) were answered and the researcher was at liberty to highlight those issues not addressed in this study for further study by other researchers. The chapter concluded with an overview on the implications of the findings to all the participants (learners, educators and education administrators and curriculum developers). The contribution of the study to improving the teaching and learning of Euclidean geometry was evaluated by establishing whether the findings answered the research question(s). The recommendations included suggestions on how the findings could be generalised to the larger population and recommendations were made to inform future practices and to ensure improved learner performance in Euclidean geometry.

1.15 Conclusion

In line with enabling the researcher to explore and probably identify geometry challenges with specific focus on the research questions; this research was conducted with Grade 12 mathematics learners in the Ngaka Modiri Molema district. The study was conducted using the Van Hiele model, together with associated geometry models to investigate and descry the learning difficulties that learners experience in Euclidean geometry with reference to circle geometry and geometric proofs. The summary to the study involved exploring issues related to the background and overview of the study, the relevance of the study, the research questions and the rationale for carrying out the study. As a result, the next chapter presented the review of literature related to learning difficulties in Euclidean geometry.

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

This chapter consists of a review of literature that explored learning difficulties experienced by Grade 12 learners in Euclidean geometry. The following key aspects are described in this review: The perspectives held by different researchers on what Euclidean geometry entails in mathematics education, the importance of the Euclidean geometry mathematics education, for example in the CAPS curriculum, with specific reference to geometry themes like socio-economic development, spatial visualisation and reasoning, geometric reasoning and proof, and scaling of argumentation level. The chapter also examines the status of Euclidean geometry in the South African school curriculum with a major focus on curriculum reformations from its democratic inception in 1994 to date. In addition, factors responsible for Grade 12 learners' poor performance in Euclidean geometry are also discussed. In this regard, some of the factors considered in this chapter include educators' inadequate geometry content knowledge, use of traditional teaching approaches, poor learner motivation and a lack of a proper support systems, just to mention a few.

Furthermore, descriptions of theoretical frameworks guiding this study, together with their relevance, are also presented in this chapter. The theoretical framework on which the study is anchored is the constructivist theory, with specific reference to cognitive and social constructivism as the underpinning theories. This is followed by the geometric thinking model of Van Hiele (1986) and the associated geometric reasoning skills by Hoffer (1981), as these models also affiliate to the constructivist principles. The chapter concludes by an overview of the integration of the above-mentioned theoretical frameworks in geometry learning and teaching in terms of Piaget's (1969) broader constructivist theory.

Furthermore, in the researcher's experience as a mathematics teacher, he has realised that, in most instances, learners develop geometry misconceptions which result in them making errors. These two interrelated concepts are elaborated on in the sections below since they were found to have a bearing on the challenges that learners experience in Euclidean geometry.

2.2 Euclidean geometry and its role mathematics education

Euclidean geometry is one of the areas of study that has been extensively researched in mathematics education. Thus, Euclidean geometry is generally referred to as the study of plane and solid shapes, with specific reference to axioms and theorems (Mamali, 2015; Benno, 2020). It involves, amongst others, relationships between angles, lines, surfaces and solids (Brannan et al., 2012). In Gűven and Okumus's (2011) point of view, these relationships are based on excellent reasoning abilities and deductive thinking (Mamali, 2015), which are essential for developing geometrical knowledge. Similarly, Tabak (2004) argues that geometry involves learning about geometric properties of a shape/figure (points, lines, planes, angles, different shapes, dimensions) that do not change when the figure is revolved or transformed. These views on Euclidean geometry seem to agree that it is an axiomatic and deductive study of mathematics which is linked to critical thinking and logical reasoning. The contention is that Euclidean geometry involves the application of geometric properties and relationships in a problem-solving context (Budi, 2010; Mabotja, 2017; Mabotja et al., 2018). Therefore, Euclidean geometry can be understood as an essential area of mathematics that equips learners with the necessary skills to develop and evaluate deductive arguments about figures and their properties. Thus, Euclidean geometry is an all-encompassing branch of mathematics that involves understanding of shapes and their properties as well as the use of deductive reasoning to prove geometry theorems and solve real-life problems.

2.2.1 The importance of geometry

The importance of geometry has been widely documented and valuable insights on its applications in socio-economic development (Jones, 2002; Chambers, 2008; Knight, 2006; Ozerem, 2012). Furthermore, other studies extensively focused on the role of geometry in enhancing learners' spatial virtualisation (e.g., Diezmann & Lowrie, 2011; Dimmel & Herbst, 2015; Jones & Tzekaki, 2016; Kalogirou et al., 2013; Sack et al., 2010; Whiteley et al., 2015), geometric reasoning and proof (e.g., Battista, 2007; Bayuningsih et al., 2018; Bronkhorst et al., 2021; Gunhan, 2014b; Mabotja, 2017; Mabotja et al., 2018) and development of logical thinking skills and scaling of argumentation level (DBE, 2014; Siyepu & Mtonjeni, 2014).

These constructs are discussed below:

2.2.1.1 Socio-economic development

According to Knight (2006), geometry has different applications in science and technology, which include the construction industry, design and architecture. Moreover, geometry also supports spiritual and cultural development, in addition to providing a rich context for doing mathematics. The contention is that geometry is a necessity in every society's socio-economic development. Thus, the geometrical knowledge and skills that learners acquire are essential in various sectors of development. From a developmental perspective, this means that geometry principles are applied in computer-aided design and geometric modelling, where simulation and testing of objects are done using computers before using physical models. Furthermore, another important factor of geometry is in the field of robotics through geometric modelling of the environment, in medical imaging through fields like geometric tomography, in computer animation and visual presentations through generation of images that can fool human perceptions to static and moving objects using geometric modelling (Jones, 2002). These examples demonstrate that the importance of geometry to socio-economic development cannot be underestimated.

A similar perspective to geometry is also held by Bayuningsih et al. (2018) who describe geometry as an exploratory field of mathematics that has links with the real world. This supports a widely held view that much of our cultural life is visual and it involves geometric principles (symmetry, perspective, orientation, etc.) that are embedded in our aesthetic appreciation of art, architecture and design as suggested above. From the above-mentioned, it can be argued beyond reasonable doubt that there is a link between geometry and everyday human activities. In addition, one can say it is worth learning geometry because the concepts involved capture learners' attention, promote critical thinking and have real-life applications. The same view is held by Ozerem (2012) who argues that learners may get a deeper understanding of the world if they learn geometry, as it improves their reasoning capacity and relates well with other branches of mathematics.

From the above-mentioned, one can deduce that geometry has both social and economic applications in the real world. Thus, there is a whole range of reasons why Euclidean geometry should be a major component of the learning experiences at all levels of the mathematics curriculum and hence the associated learning difficulties are worth exploring.

2.2.1.2 Spatial virtualisation and reasoning

Geometry is essential in developing learners' spatial virtualisation and reasoning. According to Kalogirou et al. (2013), spatial visualisation is the production of semiotic representations and a complete apprehension of any relations through training to be able to handle both the whole configuration of relations and the figure as a geometric object. Similarly, Mabotja (2017) and Sack et al. (2010) view spatial virtualisation and reasoning as the learner's ability to remember, reason and understand the spatial relations among objects or space. In addition, it can be viewed as the learner's ability to pose, generate, transform, communicate, document and reflect on visual information (Gunhan, 2014b). In NCTM's (2002) point of view, spatial visualisation builds learners' mental images of their surroundings and objects in them. Jones and Tzekaki (2016) also emphasise that learners have in-born abilities to understand the outside world; hence, geometry instruction should capitalise on that to enable learners to process information and reason with it through geometrical representations in their minds. In other words, visualisation involves an enhancement of learners' ability to form and transform visual representations, images and transfer abstract relationships into visual representations. Therefore, spatial virtualisation and reasoning are characterised by learners' problem-solving skills related to spatial problems using the applications of geometry shapes and relationships.

Spatial virtualisation and reasoning are often considered as a vital ingredient to learners' successful geometrical reasoning and problem solving. More precisely, studies by Kalogirou et al. (2013) underscored that there is positive correlation between spatial capabilities of learners and their geometric thinking and reasoning skills. Similarly, Davis and the Spatial Reasoning Study Group (2015), in their study of connections between spatial capabilities and geometric thinking of high school learners, found that there is a significant relationship between spatial capabilities, perspective taking as well as their apprehension of geometrical figures. Results from these studies also revealed that learners' spatial-knowledge development and their spatial reasoning could be enhanced by spatial tasks involving two- and three-dimensional geometric figures, with the help of technological tools (Jones & Tzekaki, 2016). In consideration of these views, one can conclude that spatial virtualisation and reasoning are pre-requisites for geometric achievement and problem solving. Thus, improved spatial reasoning abilities by learners translate into improved geometry problem-solving skills.

In addition, Jones and Tzekaki (2016) emphasise the importance of visual representations by arguing that diagrams are worth a million words and reading from diagrams helps learners to understand geometry. Put differently, Sollervall (2012) argues that whether geometrical diagrams are presented with or without accompanying words, they are the most powerful implements to effective, critical and creative thinking, not only because they enhance quick understanding of geometrical concepts, but also because they assist learners in making generalisations more easily than when using numerical examples. Consequently, the ideas put forward by these researchers suggest that learners really need to develop visual and reasoning skills, especially in analysing geometrical diagrams to understand and correctly interpret information that is communicated through diagrams.

The above-mentioned importance of spatial visualisation and reasoning aligns well with the views held by Bronkhorst et al. (2021) who emphasise that the development of visualisation skills in Euclidean geometry as an important component of high school mathematics helps to develop learners' reasoning skills at three representative levels of visuality, which are enactive, iconic and symbolic. When these views are examined closely, one can conclude that geometry teaching and learning need to ensure that learners acquire these three layers of visuality, as they are the basis for high-level abstraction that is required at college level. The three levels of visualisation necessary for geometrical reasoning are diagrammatically represented below:

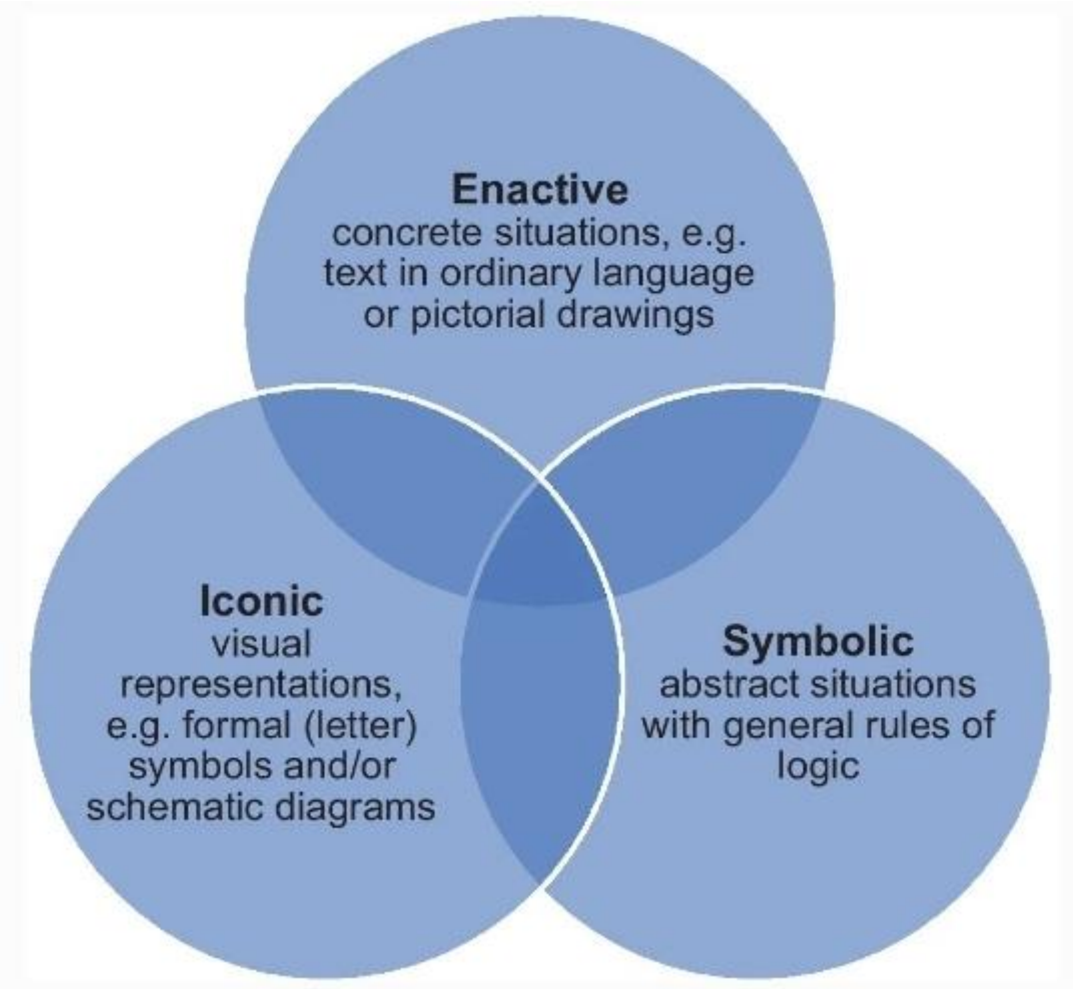


Figure 2.1

Learners' development of logical reasoning through different modes of visual and formal representations (Bronkhorst et al., 2021:4)

Thus, figure 2.1 above emphasises the importance of visualisation by enabling learners to translate reasoning used in these three layers to the concrete world or vice versa. In view of this, Chu et al. (2017) conclude that diagrams provide a means for learners to apply rules of logic to everyday situations, hence the need to develop all three the modes of visual representations in their reasoning.

In view of the above, as facilitators of learning, educators confess that diagrams are essential tools for problem-solving in Euclidean geometry and that they aid in grounding the study of abstract geometric objects in specific realisations that are available to learners (Dimmel & Herbst, 2015). It is therefore apparent from the afore-mentioned studies, that development of visualisation skills gives learners both direct access to geometric diagrams and their complete comprehension, as earlier alluded to. Alternatively, some mathematics educators suggested the inclusion of more visual activities during instruction to assist learners to understand geometric concepts (Suhre et al., 2020). As a result, the skills gained from these visual activities can enhance their visual interpretations and geometric concept development since many geometry questions are presented in a two-dimensional format on paper. In the same vein, Mabotja (2017) argues that enhancing learners' ability to visualise concepts eventually becomes a tool through they could use when solving geometric reason mathematical activities. In so doing, visualisation and reasoning becomes the key to successful geometry learning and understanding, as perceived by Makina (2010).

Regardless of the importance of spatial virtualisation and reasoning as an essential component of learners' actions and thought processes, researchers like Whiteley et al. (2015) argue that the schooling system has not been supporting it. This assertion is the brainchild of findings from their research indicating that learners retain a vague conception of dimension or space, resulting in their spatial capabilities tending to be rather non-elaborate. Similarly, Lowrie et al. (2011) indicate that learners struggle to locate information that is not provided in direct instruction, thus learners struggle to develop skills related to spatial orientation, spatial relation, understanding dimensions and localities; all of which are ingredients for spatial visualisation and reasoning. In other words, learners have limited spatial skills and visual thinking skills; hence they experience difficulties in their approach to geometry problems (Ferrara & Mammana, 2014).

Furthermore, Barut and Retnawati (2020) adopted the Van Hiele framework to 12th grade vocational high school learners' levels of geometrical thinking and the associated poor achievement in geometry. The results of their study revealed that learners have difficulties such as the inability to properly recognise geometry shapes using their formal definition, they lack the visualisation ability and, most importantly, some demonstrate insufficiency in providing proper visual reasoning capabilities in the interpretation of geometric diagrams. In the same vein, Winarti (2018) argues that learners' challenges with spatial reasoning and expressing their geometric thinking might be due to insufficient pre-knowledge of 'shape and space' and under-development of mental capabilities. Thus, these studies concur with the existence of visual and reasoning obstacles in learners' different recognition processes, especially with diagrams where, in some cases, the visualisation challenges might even distract their mental or relevant theoretical geometrical knowledge generation.

In view of the importance of visualisation in geometry learning, this study sought to explore learning difficulties related to spatial visualisation and reasoning experienced by learners and how they impact on their learning. It also explored whether poor visualisation by learners could be contributing to the challenges they have in geometry. Furthermore, this study looked at whether learners in Grade 12 have developed the requisite visual and reasoning skills in diagram analysis since diagrams have the potential to provide learners with visual access to the whole system of quantifiable relationships defined in a geometric problem.

2.2.1.3 Geometric reasoning and proof

Geometric reasoning constitutes the application of geometric properties and relationships in geometrical problems (Bronkhorst et al., 2020b; Budi, 2010; Mabotja, 2017). Gunhan (2014b) describes geometric reasoning as an ability to think logically and coherently by drawing conclusions from geometric facts. In addition, Battista (2007) views geometric reasoning as the use of conceptual systems to explore shape and space based on concepts such as length, angle measure, congruence and parallelism. In other words, learners must use properties and relationships of geometric shapes to reason and solve geometrical problems.

In view of the above, indicators of geometric reasoning include the ability to present geometric statements orally or in writing, performing geometric manipulations, checking validity of geometric statements and drawing conclusions from geometric statements. As a result, Mabotja (2017) emphasises that geometric reasoning is the basis on which learners can develop abilities for logical explanations and argumentations and hence produce authentic solutions to geometric problems. The above-mentioned researchers seem to hold a similar view that geometric reasoning improves learners' conceptual understanding of shapes and their properties which can be arrived at by inspections or logical deduction, thereby enhancing learners' abilities to apply those properties to solve geometric problems in a variety of situations.

Alternatively, Even and Tirosh (2008) argue that geometric reasoning acts as the basis on which learners are prepared for geometric proof; therefore, proof is only possible with geometric reasoning. In the same way, De Villiers and Heideman (2014) describe geometric proof as a form of geometric reasoning that uses an explicit sequence of established rules of deduction. Another set of recent literature revealed that proof is the foundation of mathematics, which helps to solve real-life problems and requires logical and critical thinking

(Bayuningsih et al., 2018). In this regard, geometric proofs help learners to adopt as their own that precise form of reasoning, which is more structurally specific than general, resulting in them being more logical in explaining and justifying their conclusions.

Despite the importance of reasoning and proof in the development of geometric understanding and problem-solving skills as alluded to above, numerous studies have documented that these critical thinking skills are lacking among Grade 12 learners (De Villiers, 2012; Mwadzaangati, 2017a; Ronda & Alder, 2016). In addition, evidence from numerous research studies clearly indicated that most learners have difficulties with geometric reasoning (Barut et al., 2020; Buyuningsih et al., 2018; Gunhan, 2014b; Mabotja et al., 2018). Similarly, Usiskin (1982, p. 427) points out that "If proof were a new idea with which we were experimenting, too few would experience success to make the idea last". More recently, Dreyfus (2014) argued through research that this lack of success in understanding proof by learners is all too often because they are asked to prove things that are obvious to them; hence, they fail to see the need for proof. He further argues that learners fail to differentiate between the forms of mathematical reasoning such as heuristic or argument, explanation, verification and proof. These assertions are a clear indication that learners have challenges with geometric reasoning and proof, thus the importance of this concept cannot be overemphasised. Therefore, there was a need for the reported study to explore those challenges.

Furthermore, De Villiers (2012) argues in favour of the above by emphasising that learners struggle to distinguish by intuition the given from what must be proven, but if learners develop skills to do empirical exploration tasks beforehand, they can enhance their skills not only to verify a theorem, but also to discover new knowledge and new ways of problem solving. Therefore, challenges associated with geometric proofs were further echoed by research findings from a large-scale survey in the United States, which revealed that only 30% of learners taking full-year geometry courses on proofs managed a 75% mastery level in proof writing.

In addition, Mwadzaangati (2017a) suggests that the two major learner challenges with geometric proof development are that it is a complex domain in itself, and it uses inappropriate teaching and learning strategies to prolong understanding of geometric proofs. Ronda and Alder (2016) further argue that educators should introduce learners to various functions of proof, including communication, discovery, intellectual challenge, verification and systematisation in that sequence, if geometry proofs and proving activities are to be meaningful to learners. Employing such strategies helps learners to refrain from seeing proof as simply an accumulation through explanation of empirically discovered facts as presented in textbooks and rather seeing them as the basis for mathematical argumentation.

Although extensive research has been done on factors contributing to poor performance in geometry (Robertson & Graven, 2019; Van Putten et al., 2010), the identified gap in those studies is that little research has so far that been done by these researchers to explore the actual learning difficulties that are linked to learners' geometric proof construction abilities and their proficiency in conjecturing and proving activities in the domain of circle geometry. Instead, research rather focused on contributing factors and strategies to improve learners' proving capabilities at the expense of exploring the real challenges learners experience with geometry proofs. In addition, despite the validation of the Van Hiele model by supporting researchers like Corley (1990) and Hoffer (1981), when they explain the requisite levels of geometric thinking by learners, these researchers fall short of the adequate descriptions of how learners solve problems in Euclidean geometry and their perceived difficulties. Furthermore, a major gap that still exists in research literature with regard to geometric proofs is what is known about how to support learners so that they shift from the belief "because it looks right" or "because it works in these cases" to arguments that are applicable in general; hence, this study sought to explore those challenges as experienced by Grade 12 learners with regard to geometric proofs as well as their conjecturing abilities.

2.2.1.4 Scaling of argumentation level

The CAPS aspires to make learners respond positively to real-world problems. In respect of this, the Euclidean geometry section of the mathematics (CAPS) curriculum aims to improve learners' thinking abilities by imparting skills to question, examine, conjecture and experiment with geometric figures (DBE, 2014). Thus, the geometry curriculum at high school level is designed with the intention of inculcating in learners the idea that geometry brings in the aspect of mathematical argumentation based on logical reasoning, as opposed to viewing mathematics as just calculations involving numbers algorithmically. In other words, in its efforts to promote mathematical argumentation, geometry helps learners solve problems in ways that stimulate curiosity and encourages exploration and insightful geometrical thinking. This idea is supported by Benno (2020) who suggests that geometry helps learners to develop insightful thinking about objects and clever ways of using theorems to given geometrical problems which enhances inductive thinking, as well as the importance of proof.

Furthermore, Bono's assertion supported the importance of geometry as earlier suggested by Thompson, cited in Yackel and Hanna (2003), and later confirmed by Bayuningsih et al. (2018) who argue that knowledge is built by a cognisant individual, where argumentation and justification should be the focus of high school geometry instruction. Thus, the general view by some researchers is that Euclidean geometry is a vital aid for mathematical communication which helps learners to develop logical thinking abilities and prepares them for mathematical argumentation required in higher learning (French, 2004; Siyepu & Mtonjeni, 2014).

2.3 Euclidean geometry in the South African school mathematics curriculum

Euclidean geometry has undergone various reforms in the history of South Africa's mathematics education curriculum. Such reforms include the introduction of Outcome-Based Education (OBE), Curriculum 2005 and Curriculum Assessment and Policy Statement (CAPS). The reforms were part of a raft of changes effected across all disciplines of the post-apartheid South African school curriculum landscape, which Froneman (2019) distinguished in terms of outbound Grade 12 cohorts, as follows:

- The Tradition Knowledge-based Curriculum (TKC), which was also documented as Curriculum 2005; a skills-based curriculum for all Grades 10 to 12 learners who matriculated in the years up to 2007.
- The constructivist curriculum implemented through Outcomes-Base Education (OBE), called the National Senior Certificated (NSC), for matriculants of 2008 to 2013.
- The revision of the (OBE) which was documented as the Curriculum and Assessment Policy Statement (CAPS) for Grade 12 cohorts that matriculated between 2014 and the present.

Through the above-mentioned curriculum reformations, Euclidean geometry as a content area in the mathematics curriculum was not spared. For example, Euclidean geometry was excluded from the mathematics school curriculum in 2008 (NSC, 2008); however, it was assessed in Paper 3, which was an optional paper. The background to the exclusion of Euclidean geometry from the compulsory mathematics curriculum since 2008 and its subsequent assessment in the optional paper 3 was linked to the persistent poor performance by learners. The research into this attributed to poor geometry content knowledge of educators (Bowie, 2009). Arguing from a similar perspective, Machisi (2020) points out that the exclusion of Euclidean geometry from the mainstream mathematics curriculum from 2008 was the result of a series of poor results in the Grade 12 mathematics. It can be concluded from these views that the reason for removing Euclidean geometry from the compulsory mathematics curriculum was that educators were not familiar with the content.

Similarly, Cunningham and Roberts (2010) point out those educators did not possess the level of geometric understanding necessary to teach it at a level recommended by the Nation

Council of Teachers of Mathematics (NCTM). In this regard, Froneman (2019) argues that the exclusion of geometry from the compulsory mathematics curriculum was meant to give educators enough time to close the content gap that had resulted in the poor performance by learners. In other words, the exclusion was meant to allow educators the time to develop their capacity to teach geometry content more effectively.

However, the exclusion of this significant area in the mathematics curriculum was widely criticised. For instance, Siyepu and Mtonjeni (2014) point out that its exclusion from the secondary school curriculum disadvantaged learners' pursuing engineering courses at university. Similarly, Kearsley, cited in ASSAF (2009), states that the decision negatively affects learners' success with further studies at tertiary level in health sciences and mathematical engineering. These earlier claims were supported by evidence from studies conducted later, whose findings were that university level learners who had not done geometry in high school had weaker mathematics skills than their counterparts who had a geometry background (Tachie, 2020). Furthermore, arguments put forward by the universities against geometry exclusion were that there was a lack of coherence in the study of shape and space, together with a diminished opportunity for learners to work with proof (Bowie, 2009).

To shed more light on the above issue, geometry exclusion did not only negatively affect mathematics departments at university level, but it also brought with it many inconsistences in high school geometry instruction. In this respect, Van Putten et al. (2010) argue that geometry exclusion resulted in both learning and teaching of the content area being voluntary since learners had the choice write Paper 3 or not, while, at the same time, educators had to use their discretion regarding whether to teach it or not (Mabotja, 2017). That also compromised the depth with which Euclidean geometry as a branch of mathematics was dealt. In addition, one of the main consequences of this move was that as Euclidean geometry was an optional extra, the enrolment for Mathematics Paper 3 dropped significantly with only 3,8% of the Grade 12 mathematics learners writing the paper in 2008 (Van Putten et al., 2010). One of the reasons behind the sudden decrease in enrolment was that many schools who did not have the necessary resources and capacity to teach geometry decided not to offer it. Hence, such a sudden decrease in learners enrolled for mathematics Paper 3 did not sit well with education stakeholders.

Furthermore, these inconsistencies in the teaching and learning of Euclidean geometry resulted in learners with an interest in in the content area not being able to study it as most of them were denied the opportunity to acquire the reasoning and logical skills inherent in geometry learning. This situation prevailed despite the National Council of Teachers of Mathematics (NCTM) (2002) having earlier indicated that geometrical reasoning enhances knowledge organisation and conceptual development, and that the exclusion of geometry deprived capable learners of the skills necessary to interpret geometric situations. According to Ngirishi and Bansilal (2019), when geometry was made optional, many learners chose not to study this section and did not gain access to a particular type of geometric reasoning encountered in geometry.

The criticism directed to the exclusion of Euclidean geometry from compulsory mathematics curriculum created, led to its re-introduction into CAPS in 2012 (DBE, 2011). The re-introduction of geometry in the compulsory CAPS curriculum meant that it was given the same attention as other mathematics content areas and, therefore, the teaching of it was mandatory.

Regardless of its re-introduction in the CAPS mathematics curriculum, research has shown that it is still characterised by teaching and learning challenges (Mabotja, 2017). Although the re-introduction of Euclidean geometry in the CAPS curriculum was applauded by South Africans, it brought anxiety to educators and learners (Govender, 2014; Ngirishi & Bansilal, 2019) because it is still problematic for both learners and educators. In other words, the challenges that led to its exclusion from the previous mathematics curriculum have still not yet been fully addressed (Ndlovu et al, 2014). Thus, the apparent convergence of findings from these studies is that the geometry challenges that learners continue to experience are probably a result of educators having inadequate knowledge of Euclidean geometry content, together with a lack of pedagogical content knowledge (PCK) for effective geometry instruction.

Furthermore, Tachie (2020) echoes that the in-service geometry training which educators received for the CAPS curriculum Euclidean geometry was not enough to prepare them for the challenges they would encounter in the classroom, which resulted in them not being comfortable with the topic. In the same vein, Govender (2014), Naidoo and Kapofu (2020) indicate that in South Africa, there are many educators in schools who did not study geometry in high school, college or university and are now expected to teach Euclidean geometry in CAPS; hence, the challenges associated with geometry conceptualisation by learners. Furthermore, Luneta (2014) underscores that those learners still have challenges in developing geometry concepts and skills.

However, the contention is that the past curriculum changes still have a negative impact on current discourse in Euclidean geometry performance, regardless of numerous curriculum reforms since 1994; therefore, learner performance is still poor. This research study sought to explore learning difficulties experienced by Grade 12 learners.

2.4 Factoring linked to learners' poor performance in geometry

Even though factors contributing to poor performance in Euclidean geometry play a background role in this study, as alluded to earlier, a brief discussion of some of these factors suffices, as these factors have a direct bearing on the challenges that Grade 12 learners experience with geometry. In the South African context and internationally, extensive research has so far been done on factors contributing to the poor performance in geometry. Previous studies conducted by Abdullah Zakaria (2013); Bramlet and Drake (2013); Dube (2016); John (2012); Robertson and Graven (2019); Siyepu (2014); Van Putten et al. (2010) suggest that some of the factors contributing to the poor performance in Grade 12 can be associated with a lack of learner motivation, poor geometrical knowledge of educators, a lack of a clearly stipulated learner support system, geometry language, misconceptions and errors, and learner attitude and abilities, to mention just a few. A brief discussion of some of these contributing factors is given below

2.4.1 Poor learner motivation

Research indicates that not listening to learners' voices and their contributions during geometry lessons results in them developing low self-efficacy and decreased levels of motivation (Department of Education & Training, 2018). In the same vein, poor learner motivation may result from a tendency by educators to teach geometry by informing learners of the properties of geometrical diagrams, with the educator requiring them to learn those properties and complete exercises to show that they have learnt the facts (Jones, 2014). Such an approach demotivates learners, as they are not given the chance to make logical connections, explain their reasoning and be proficient in geometry. In that way, learners end up seeing geometry as a form of mathematics where they are solely exposed to rules with no real-life relevance. It also results in learners failing to recognise connections between different ways of representing geometric ideas and struggling to solve geometry problems. That probably demotivates learners, resulting in their poor performance.

2.4.2 Educators' inadequate geometry content knowledge

From the researcher's point of view, it stands to reason that with inadequate geometry content knowledge, many educators enter the teaching profession while ill-equipped to teach geometry. Thus, it is an incontestable fact that no one can teach beyond their level of understanding. Therefore, poor performance in Euclidean geometry by learners is probably a result of the fact that tertiary education does not prepare educators well enough to effectively teach geometry in schools. For example, in studies carried out by Olivier (2014) and Dube (2016) on the level of the geometry content knowledge of practising educators, 60% of the educators indicated that they could not teach geometry with confidence. On the other hand, results from the same research revealed that CAPS training facilitators who were tasked with closing the content gap also seemed to lack the knowledge and skills needed to assist in-service educators to improve their geometry instruction (Ndlovu, 2013; Olivier, 2014).

In South Africa, Linda Chisholm Van made a presentation to the minister of Basic Education in 2012, which highlighted educators lacked both content and pedagogical content knowledge in subjects like mathematics, which geometry is part of (DBE, 2012). In the same way, Tachie (2020) states that many educators find it difficult to teach the concept of geometry in South Africa due to their lack of the necessary geometry content knowledge and lack of the necessary cognitive skills. In addition, Van Putten et al. (2010) argue that there seems to be a mismatch between what prospective educators learn in tertiary institutions and the actual needs and classroom expectations. Based on these ideas, a lack of geometry content knowledge by educators is blamed for the poor performance by learners, probably because it limits them in terms of explaining geometry concepts explicitly to the learners.

2.4.3 Lack of proper support systems

Articulated from the researcher's own experience as a mathematics educator, the challenges that learners experience with Euclidean geometry indicate that they were not exposed to a conceptual foundation and real-world relevance of geometry information in lower grades. Thus, the Human Science Research Council (HSRC) (2008) argues in favour of enhancing learner performance in geometry by giving them the opportunity to socially construct their own knowledge through educators playing a 'facilitating' role rather than a 'classroom expert' role. As a result, educators were warned against using inflexible approaches to support learners and were urged to involve learners so that they would understand the relevance of geometry teaching and learning. Similarly, research indicates that, with proper materials and intellectual support, learners can do well in geometry (HSRC, 2008).

Furthermore, the lack of educational technology in most South Africa schools has a negative impact on geometric understanding by learners in secondary schools. In this vein, John (2012) suggests that educational tools like iPads can give learners access to geometry sketch programmes, geometry vocabulary programmes, self-passed geometry lessons and online lectures that can enhance their performance.

In trying to resolve some of the challenges related to a lack of learner support materials, Jones and Tzekaki (2016) argue in favour of using different forms of digital technology to improve geometry reasoning and proving. However, the major challenges with Jones's argument are that the current situation in South Africa does not allow for a total shift to digital technology in resolving the difficulties in Euclidean geometry because a larger number of its schools are poorly resourced.

2.4.4 Traditional teaching approaches to geometry

Another factor that is viewed as contributing to learning difficulties is the persistent use of educator-centred approaches in the teaching of Euclidean geometry in secondary schools. More precisely, the use of traditional educator-centred approaches is also blamed for the difficulties that learners have with many geometry concepts, for example geometric proofs. The use of educator-centred methods in this section of geometry and other concepts has been characterised by numerous challenges because the approaches present situations where learners are viewed as mere receptors of mathematical facts, formulas, principles and theorems through rote memorisation (Armah et al., 2018). This results in learners not having the opportunity to participate in their own geometrical knowledge construction.

Alternatively, research also points to the fact that geometry underachievement in Grade 12 results from the failure by educators to implement unconventional teaching approaches such as Van Hiele theory-based instruction in teaching Euclidean geometry (Abdullah Zakaria, 2013). The same studies further argue that this underachievement stems from educators moving fast in lower grades to cover the syllabus, as geometry is normally taught towards the end of the academic year, resulting in learners being left behind. This confirms what was reported by Bramlet and Drake (2013) and Siyepu (2014) that educators struggle to teach geometry in ways that make geometry lessons ‘fun and beneficial’; for example, the use of the Geometer’s Sketchpad (GSP) to explore geometry concepts.

On the other hand, to alleviate the impact of employing inappropriate teaching approaches in geometry, Chimuka (2017) also states that there is a need to move away from educator-centred approaches to learner-centred approaches. For example, Bosman and Schulze (2018) describe learner-centred approaches as differentiated learning activities that are meant to create knowledge based on learners’ thinking and experiences. However, despite the recent calls for the application of constructivist teaching approaches, educators continued to revert to traditional teaching methods, resulting in geometry deficiencies such as a compromised conceptual understanding. Thus, this study sought to explore the associated geometry learning difficulties.

2.4.5 Geometry language and geometric definitions

These two aspects are a cause for concern in the learning of Euclidean geometry. Language that is used in geometry, especially the understanding of geometric terms, plays a significant role in the learning and grasping of geometrical knowledge (Khoo & Clements, 2001). Their study indicated that learners may be able to name shapes but may not be able to explain exact meanings or properties involving perimeter, area, angle and tangents, mainly due to the specific nature of geometry terminology.

Similarly, according to Robertson and Graven (2019, p. 77), language can "either include or exclude certain groups of learners from genuine opportunities for mathematical sense-making". As a result, by not teaching learners the terminology of Euclidean geometry, educators exclude them from understanding geometry concepts. In support of that, the Van Hiele theory emphasises the importance of developing the language of Euclidean geometry, without which learners fail to understand geometry concepts (see for example the work by Govender, 2014; Naziev, 2018; Robertson & Graven, 2019). As a mathematics teacher, the researcher has encountered similar challenges with learners at Grade 12 level.

On the other hand, in a study by Ng (2014) on the relationship between dragging diagrams, language and gestures by 12th grade bilingual learners (Grade12 equivalence in South Africa) in mathematical communication in calculus (geometric dynamic sketches), bilingual learners were found to utilise different resources, language, gestures and visual indicators in their communication of geometric information. The results of their study indicated a positive correlation between bilingual learners' language proficiency and their geometry achievement. However, this is not the case in the South African context where most learners from rural and townships schools are not proficient in English and, worse, are not conversant with Euclidean geometry terminology.

Similarly, in their study of pre-service teachers relating to the evaluation of information that relates to two-dimensional objects, Mulligan et al. (2018) discovered that they have a relatively negative evaluation of geometrical descriptions and topological descriptions. This may suggest that these educators carried the challenges from their high school years with them and passed them on to their learners during the teaching and learning of Euclidean geometry. As a result, these findings affirm the notion held by the above-mentioned studies that poor development of geometry language by learners and educators has a negative impact on learner performance. In other words, geometric language deficiencies may feed directly into poor performance by learners, as it is a critical ingredient to learner success.

Based on the above, this study sought to explore if geometry language forms part of the challenges that contribute to poor learner achievement in Euclidean Geometry.

2.4.6 Misconceptions and errors in Euclidean geometry

Numerous studies revealed that most learners have misconceptions in their geometrical thinking, which they exhibit as errors in their solutions to geometry problems. Some of the researchers who echoed these sentiments include Padhila and Rully (2019) who emphasise that, usually, these misconceptions are a result of learners and the educator not understanding each other during the learning and teaching process. As a result, learners develop their own geometric understanding, which is incorrect and inaccurate. In the same vein, Rofii et al. (2018) argue that misconceptions are a result of the failure by learners to understand the content, which may prevent them from understanding the subject matter. Similar views were articulated by Ozkan and Bal (2017) who argue that geometry misconceptions are conceptual representations constructed by a learner that make sense in relation to his/her current knowledge, but that are not aligned with conventional mathematics knowledge. These views suggest that misconceptions are probably a result of learners' limited understanding of geometry concepts during instruction. Therefore, those misconceptions are an impediment to meaningful learning because they result in learners making errors in geometry problem solving.

2.4.6.1 Errors in Euclidean geometry

An error is viewed as a deviation from accuracy or correctness (Happer, 2010). Similarly, Luneta (2013) views an error as a simple symptom of the challenges a learner is encountering during a learning experience, in that case difficulties experienced in Euclidean geometry. Hence, a researcher in geometry needs to analyse those errors to establish geometry learning difficulties. Errors are visible in learners' artefacts such as written text (Luneta & Makonye, 2010). Therefore, it was necessary for this study to use a written test as one of the research instruments.

The diagram below gives a summary of the learning process required to correct learners' geometrical errors as and when they arise.

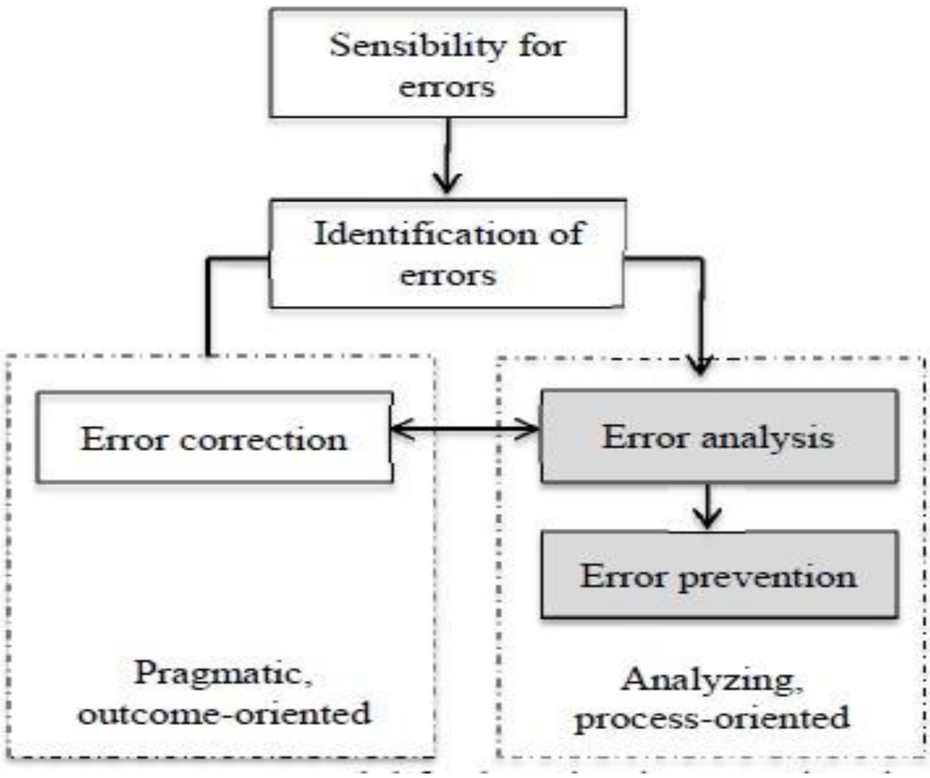


Figure 2.2

Process model for learning in error situations (Ranch et al., 2013)

Error analysis is an assessment approach that enables educators to establish whether learners are consistently making the same mistakes when performing basic geometric manipulations, and this enables the educators to directly teach the correct procedure for solving that specific problem (Van der Sandt, 2007). As summarised above, Ranch et al. (2013) categorise error analysis into two arms: pragmatic outcome oriented and analysing process oriented. In their error pattern analysis, the pragmatic approaches move directly from error identification to error correction, whereas the analysing approach includes an analysis of the type of error, how it comes about as well as with possible prevention strategies. The right arm of the diagram above is preferable in rectifying Euclidean geometry errors, as it provides more detail to learners' geometry challenges.

The researcher is of the view that some of the errors that learners make in Euclidean geometry are an indication of the challenges they have with geometric understanding. Research revealed that error analysis should involve an evaluation of the learner's areas of weakness and then identify the most frequent mistakes he/she makes, together with reasons why the learner has such challenges, with the aim of improving learner performance (Luneta, 2013). Furthermore, Angraini and Prahmana (2019) and Michael (2001) argue that geometry errors are probably a result of reasoning or conceptual difficulties that hinder learners' mastery of given Van Hiele level expectations. Thus, this study also investigates if learners experience challenges in the form of geometric errors by referring to Watson's error analysis model.

Furthermore, Watson (1980) gave a detailed classification of these errors as illustrated below:

2.4.6.2 Watson's classification of errors

The analysis of learner errors was conducted using Watson's (1980) classification of errors which uses Newman's model and states that "all errors could be place in the following categories":

Table 2.1

Watson's classification of errors (adapted from Watson, 1980, p. 322-323)

Category	**Type of error**	**Explanation**
1	Reading ability	Can the pupil read the question?
2	Comprehension	Can the pupil understand the question?
3	Transformation	Can the pupil select the mathematical process that is required to obtain the solution?
4	Process skills	Can the pupil perform the mathematical operations necessary for the task?
5	Encoding	Can the pupil write the answer in acceptable form?
6	Motivation	The pupil could have correctly solved the problem had he tried.
7	Carelessness	The pupil could not do all the steps but made a careless error, which is unlikely to be repeated.
8	Question Form	The pupil makes an error because of the way the problem has been presented.

Both learners and educators need to understand the nature of the error to enable the review of those errors. Hence, Watson (1980) conducted a study using the Newman model which enabled him to put all errors into eight different categories. The model was used to categorise the challenges Grade 12 learners' encounter in solving Euclidean geometry problems. Watson's model assisted the research to determine the prevalence of each type of error in learners' test responses; hence, assisting the researcher to explore the difficulties they have in Euclidean geometry.

2.4.6.3 Geometry misconceptions

Misconceptions are viewed as a lack of understanding of content that may prevent a learner from comprehending the subject matter (Michael, 2001). Alternatively, Ngirishi (2019) describes geometry misconceptions as conceptual or reasoning difficulties that hinder learners from constructing concepts in a sound or mathematically endorsed manner. Even though related to errors, misconceptions have more to do with the wrong application of concepts or formulae, which is an indication of a limited understanding of a particular situation; in this case, geometric concepts. From the afore mentioned, it can be argued that geometry misconceptions are seen as arising from an incorrect application of a rule, or when a learner overgeneralises a rule, under-generalise a rule or demonstrates an alternative conception of the situation.

Furthermore, this view is supported by Luneta (2015) who argues that constructivists identify a misconception when a relatively stable and functional set of beliefs held by a learner, conflicts with a different stance held by a group of scholars, education experts and educators. In addition, Makhubele (2014) argues that if learners have misconceptions and do make errors, a variety of bad emotions are triggered, like anxiety, fear, frustration and rage, which negatively affects their performance in geometry. As a classroom practitioner, the researcher agreed with Makhubele's (2014) findings because, in most instances, the mathematics learners were not comfortable with Euclidean geometry problems.

2.4.7 Geometry concept building

Research on errors and misconceptions as alluded-to above can be the result of faulty construction of geometrical ideas, mainly the way in which learners acquire both conceptual and procedural knowledge during learning and teaching. Schneider and Stern (2010) view conceptual knowledge as bringing together pieces of geometric information by linking existing knowledge to new knowledge or creating relationships. Their research found that learners struggled to create the link between lines, angles and shapes, together with logical arrangement of geometrical ideas. In support of this and Luneta (2014); Zakaria and Zaini (2009) suggest that for educators to ground their teaching of conceptual knowledge and understanding of Euclidean geometry, they need to teach through investigation, discussion, exploration and sharing of geometric ideas, since most of the learners made conceptual errors in geometry questions.

Acquisition of conceptual knowledge should then be accompanied by procedural knowledge, which allows learners to follow certain steps to solve a problem (Schneider & Stern, 2010).

Research indicates that most learners are found wanting in their demonstration of conceptual and procedural fluency, which is the learner's ability to recall and accurately execute geometrical procedures mainly in the domain of geometric proof and riders (Alex & Mannen, 2018). Hence, their emphasis was on educators teaching Euclidean geometry for conceptual understanding because it enables learners to develop their own correct understanding and articulation of geometrical concepts, which can improve the formation of concept images applicable in different geometry situations.

2.5 Theoretical frameworks

Every educational research should be firmly grounded on a well-defined framework for its findings to be more meaningful, credible and suitable for generalisation. A theoretical framework is generally viewed as the blueprint or guide for research (Grant & Osanloo, 2014). Therefore, it guides the research process by ensuring that the researcher does not deviate from the limits of accepted theory and ensuring that the research remains academic (Brondizio et al., 2014; Fulton & Krainnovich-Miller, 2010). In other words, selecting an appropriate theoretical framework in educational research has various benefits. In view of this, researchers such as Grant and Osanloo (2014), and Ravitch and Carl (2016) argue that an appropriate theoretical framework helps researchers to contextualise established theories in their studies, thereby helping them define their studies philosophically, epistemologically and methodologically. Based on these views, it is evident that a theoretical framework provides a lens through which the researcher authenticates his/her thoughts about the research problem.

Furthermore, a theoretical framework gives direction to the research process by guiding the researcher's choice of aspects like the research approach, the tools to be used and the procedures to be followed in data collection and analysis (Akintoye, 2015; Lester, 2005). In support of this, one can conclude that a theoretical framework ties the whole research process together, from the statement of the problem, through literature exploration and data analysis, up to the reporting of the research findings and conclusions. Similarly, Maxwell (2004) and Simon and Goes (2011) aver that the appropriate theoretical framework strengthens and deepens the essence of that study. As a result, this research is anchored on the widely accepted theory of constructivism through cognitive and social constructivism, which the researcher used to explore learning difficulties in Euclidean geometry. In the same vein, employing this worldview as the basis for exploring learner challenges in geometry enabled the researcher to apply two other theories that resonate well with the focus of this study – the Van Hiele model of geometric thinking (1986) and Hoffer's model of geometric skills (1981), as they affiliate to the anchor theory of constructivism.

2.5.1 The constructivism theory

The current research study adopted constructivism as a theoretical framework. Constructivism is viewed as a broad learning theory that describes how learners construct meaning in the learning environment (Llewellyn, 2005; Van de Walle, 2016). In addition, Machaba (2014, p. 2016) points out that constructivism is against the idea that learners are "blank slates", by arguing that learners come to school with a certain level knowledge, and they are not empty vessels on which to build. Thus, it considers learners' prior knowledge as an essential aspect of their learning. The contention is that learners should construct their own knowledge, building on their prior knowledge (knowledge they already have) so that they can understand new geometry information. In other words, the learners' understanding of geometry could be constrained or enhanced by the prior geometrical knowledge. The implication of learners' prior knowledge is that geometrical knowledge cannot be imposed on the learner by the teacher, but it should be developed by the learners themselves through restructuring of their cognitive frameworks to accommodate geometric information in situations where their existing schema cannot readily identify with the new information.

Based on the above, constructivism was adopted in this study as the relevant theoretical framework, because of its emphasis on learners constructing their own knowledge through active engagement with geometry content.

Furthermore, the researcher is of the opinion that the challenges that learners encounter in Euclidean geometry are probably a result of how they construct geometrical knowledge; hence, the choice of a constructivist paradigm. A learner's existing knowledge acts as the basis of the meaning these learners give to incoming information. Thus, constructivism has implications for this study in that active engagement, knowledge construction and communication were found to be the basis for understanding Euclidean geometry. Learners lacked these basic constructivist principles; therefore, their geometry conceptualisation is compromised, resulting in them having learning difficulties.

Furthermore, adopting the constructivist paradigm gave the researcher the opportunity to explore the difficulties learners experienced in Euclidean geometry and make sense of the perspectives held by learners, educators and mathematics departmental heads on those challenges, which the researcher believes are guided by the philosophy of social constructivism, as alluded to by Taylor and Medina (2013). Since learners were performing poorly in Euclidean geometry in Grade 12, through this paradigm, Bryman cited in Grix (2004), and Bosman and Schulze (2018), concurs that constructivism/interpretivism helps to gain a better understanding of reasons, insights and meanings attached to human action.

This forms the basis on which the researcher identifies and understands the learning difficulties.

The two basic concepts of the constructivist framework and their relevance to this study are explained below:

2.5.2 Cognitive constructivism: Piaget's theory

Geometry learning depends on the two processes of assimilation and accommodation. Piaget (1970) argues that new information can either be integrated into an existing schema (assimilation) or the existing schema can be modified to accommodate new ideas (accommodation). That means, if the learner's prior knowledge as discussed in the previous paragraph cannot identify with the incoming information, he/she will alter the existing knowledge structures to accommodate new content. Therefore, in cognitive constructivism, prior knowledge is the basis on which a learner acquires new information. This supports the notion that knowledge generation through cognitive constructivism is cumulative (builds on the existing knowledge).

Similarly, Brau (2018) and Llewellyn (2005) further argue that both assimilation and accommodation result in new knowledge being constructed from a learner's experiences, either by their schema giving meaning to new ideas/experiences or by the schema adapting to accommodate new experiences. In the case of Euclidean geometry, learners are required to match new ideas and concepts to those they already have (prior knowledge) by seeing and matching similarities through assimilation.

Alternatively, if learners encounter experiences that contradict what they already know in their mental structures, they should change their thinking so that the new information is fitted in by the process of accommodation (Brau, 2018). This suggests that the existing knowledge held by the learner should be organised, structured and restructured in order to give meaning to incoming information. These two processes are necessary in Euclidean geometry because learners are required to interact with geometric diagrams, which demands mental manipulation of the incoming information and the subsequent logical arrangement of this information it into new schemas.

However, if these two cognitive processes are not properly executed by the learners, they can negatively impact on their geometric understanding. For example, faulty assimilation, which might be due to poor integration or faulty accommodation due to poor reconfiguration of their schemas, may hinder the learners' understanding of geometry concepts, resulting them experiencing learning difficulties. Furthermore, in extreme situations where a learner's schema cannot assimilate or accommodate incoming information, they will resort memorisation.

On the other hand, geometry errors and misconceptions that learners might make or have can be explored through the lens of cognitive constructivism by tracing the root causes back to the way learners mentally manipulate incoming geometric information. In this case, manipulation of geometric information depends on the learner's prior knowledge. It could be argued that the reason why learners experience learning difficulties, as suggested in literature, is that they fail to link their prior knowledge to incoming geometric information. Thus, assimilation and/or accommodation is not realised in the learning environment.

Similarly, these two principles of cognitive constructivism are useful to this study, as they can be used to explain the challenges that learners have in developing topological relations like connectedness, enclosure and rectilinearity, as well as Euclidean concepts of angularity, parallelism and distance as major relationships that require formations of schemas by learners (Brau, 2018; Sapire & Mays, 2008). Thus, adopting cognitive constructivism in this paper helps to explain and authenticate any challenges that learners might have that relate to geometric reasoning and logical argumentation, since these two are related to how learners mentally manipulate incoming geometry information. Thus, cognitive constructivism, which was used as the theoretical framework for this study, aims to explain the identified geometry challenges that related to the way in which learners process geometric information received during teaching and learning.

2.5.3 Social constructivism: Vygotsky's Theory

Research by Van de Walle (2007) and Watson (2001) indicates that geometry instruction can be more effective if it involves active learner engagement with authentic or "real-life" examples, where learners practice geometry activities in groups, as the teacher guides the way to more formal geometry riders. Thus, Vygotsky (1979) proposes social constructivism where learning occurs within each learner's Zone of Proximal Development (ZPD). According to Vygotsky (1979, p. 16), the ZDP is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined by problem-solving under adult guidance or in collaboration with more capable peers". The theory proposes that this interaction should occur in symbolic space created through the interaction of learners with more capable peers or adults in their culture. In this vein, social interactions in the ZDP play a major role in geometric understanding in that there are certain geometry concepts that learners can only master with the help of a more knowledgeable peer or educator, through scaffolding. However, if there is no collaborative learning through peer interactions or mediation by the educator within their

ZDP, learners are likely to encounter learning difficulties. In terms of the South African curriculum, learners need to work effectively as individuals and with others as a team (DBE, 2011).

Furthermore, constructivism promotes a social learning process where learners learn from each other or by educator facilitation. This means that learners should be allowed to express themselves through discussion, communication, asking questions, sharing ideas and reflecting on their own thinking as well as that of others to negotiate a shared meaning to geometry concepts. In view of this, Usiskin (1987) argues that geometry learning is socio-cultural and interpretive, where learners bring different but converging interpretations to geometry problem solving within their culture and learning environment. Thus, it can be argued that, if such socio-cultural interactions are not realised in a geometry classroom environment and learners are not being assisted within their ZDP to reach their full potential, learning difficulties are more likely to arise.

As a result, the theory by Vygotsky (1986) informed this study because understanding of Euclidean geometry requires learning to proceed through social interactions that result in conceptual development by learners. In this view, the researcher is left questioning whether geometry learning and teaching in South African schools result in effective mediation within the learners' ZPD to promote higher levels comprehension. Therefore, poor social interactions between the educator and the learners or amongst the learners in the geometry classroom environment could be the reason why learners are experiencing learning difficulties in Euclidean geometry. This study sought to explore learning difficulties in Euclidean geometry through the lens of social constructivism, because of its emphasis on learning as a social process.

In view of the above, previous studies by Bleeker et al. (2011); Hock et al. (2015); Howse and Howse (2015); Vojkuvkova (2012); Watan and Sugiman (2018) have been reviewed and valuable insights on theoretical frameworks related to constructivist geometry learning and teaching have been gained from them. Two more anchor theories underpinning this research that emanated from constructivism originated from the two seminal works produced in the latter part of the previous century, during the 1980s. These theories have seen recent applications, adaptations and modifications; however, their value for this study cannot be denied. These theories are the Van Hiele (1986) levels of geometrical thinking and Hoffer's (1981) geometric thinking skills. These theories can be explained in terms of the broader constructivist theory (Piaget, 1969), as it embraces major aspects geometry instruction.

These theories, together with their implications for this study, are explained below:

2.5.4 Van Hiele Theory of the development of geometrical thinking

Amongst the above-mentioned constructivist theories to geometry thinking, Van Hiele theory was the most significant theoretical framework adopted by this study, with other models stated above playing a supportive role in exploring the learning difficulties experienced by Grade 12 learners. This study used the Van Hiele theory of levels of geometric thinking as its theoretical framework, mainly because it has been the most influential theory in terms of geometry education globally (De Villiers, 1996). In addition, the Van Hiele theory is useful in determining how learners develop their understanding of geometry and their spatial sense; a shortfall in any of these two results in learners experiencing learning difficulties (Alex, 2012). In this respect, the Van Hiele model of geometric thinking relates to this study in that it is a constructivist framework to understand the geometry thinking process by learners so that learning difficulties are understood as deviations from its prescriptions.

Furthermore, the Van Hiele theory was adopted as the theoretical framework for this study because it was deemed to have strong implications for learning difficulties that learners experience in Grade 12 Euclidean geometry in that:

- it assists the researcher with sources of possible challenges learners experience in geometry learning in terms of the reasoning level at which learners are operating in geometry and the associated challenges envisaged at each level
- it provides the researcher with a broader knowledge base of how learners learn Euclidean geometry and how to address the difficulties related to geometry instruction and learning
- Van Hiele description of geometry learning at basic level puts more emphasis on visualisation, which, if combined with shape manipulation and conjecturing abilities, learners will be better prepared for higher levels of geometrical thinking.

Moreover, learners in a geometry class may be operating at different levels of reasoning ability; therefore, before beginning instruction, it is important for the educator to assess the learners’ reasoning levels. This enables the instructor to differentiate instruction on the basis of learners’ readiness. Failure to implement such measures may result in learners experiencing learning difficulties, which is the focus of this study. Similarly, for this study, the Van Hiele theory helps the researcher to identify and interpret the learning difficulties, as the model gives a framework for classifying different learner competences in geometry through reasoning levels.

To that effect, the Van Hiele model has five levels of geometrical reasoning (Musser et al., 2011). Van Hiele (1999) emphasises that a learner progresses and develops his/her geometrical thinking skills through these levels, as shown in the below table:

Table 2.2

Van Hiele's levels of geometric thinking

Level	Name	Description
Level 0	**Visualisation**	• learner recognises figures by appearance alone • compares figures to known prototypes (original thinking), for example, it is a rectangle because it looks like a door • properties of shape not known at this stage • perception about shapes is made on intuition, not reasoning • learners can only focus on a single feature, not properties
Level 1	**Descriptive/ Analysis**	• properties of shapes are stated but no logical order • learners identify and examine each element of a figure to understand it in detail • can recognise and name properties of figures without relationships between properties • properties of figures are seen as independent of each other • learners cannot make short deductions
Level 2	**Informal deduction**	• combining shapes and properties and relate shapes • learners at this level see relations between properties • can create meaningful definitions and informal arguments to justify their reasoning • understand logical implications and class inclusions, for example, for an equilateral triangle, because all sides are equal, it implies that all angles are equal • formal deduction is not known at this stage. For example, solving problems that require application of a chain of deductions
Level 3	**Deduction**	• application of formal deductions like proofs • learners can reason from known facts, circumstances, or observations • learner can construct proof, understand role of axioms, definitions, and theorems • learner can give formal proofs of theorems and apply them to solve riders
Level 4	**Rigor**	• formal reasoning about establishing and comparing mathematical systems

	(Outside the scope of expectations for secondary school curriculum)	• understand the use of indirect proofs and proof by contra-positive and non-Euclidean systems

Table 4.2 above shows that in Van Hiele's view, the geometric reasoning levels are hierarchical, where the amount of reasoning expected increases with the levels. Thus, the hierarchical nature suggests that a learner is not expected to master higher levels of geometric reasoning like abstraction, deduction and rigor before lower geometric reasoning levels like recognition and analysis. Thus, at base level, the hierarchical nature of Van Hiele's geometric reasoning merely demonstrates that if a learner has an underdeveloped visualisation, it means the learner has not created in his/her mind the types of figures and relationships that are the focus of thought for the next level – the descriptive level, and the subsequent levels. As a result, the failure by any learner to master lower-level geometric reasoning in the hierarchy acts as barrier to functioning at a higher level and that learner will encounter difficulties in solving geometrical problems.

In addition, learners should master each preceding reasoning level to progress to the next level in their geometric understanding. Van de Walle (2007) emphasises this view by arguing that learners can only be adequately prepared for deductive reasoning if their thinking has grown to level 2 by the end of the eighth grade. This simply points to the idea that the difficulties that learners experience with geometry might stem from their failure to progress in their reasoning through the hierarchy of reasoning levels, as proposed by the Van Hiele model.

Furthermore, since visualisation skills are needed at every level in the hierarchy, a learner with faulty visualisation cannot progress to levels higher up in the hierarchy, because Van Hiele (1999) proposes that learners can only arrive at levels above level 0 after achieving all prior levels. Van Hiele (1999) further argues that the ability to recognise and name shapes has been viewed as important for geometric conceptualisation. Thus, the hierarchical nature of geometric reasoning means a learner cannot describe properties of a shape if he cannot recognise it. The same applies to dealing with deductive systems of properties without the knowledge of relationships among properties. Thus, learners' failure to successfully master and navigate through these levels could be the reasons for their learning difficulties in Euclidean geometry.

Moreover, the Van Hiele hierarchical levels to geometric reasoning are relevant in establishing why learners make certain errors by pointing out the level at which individual learners are operating together with their geometric understanding. In this vein, the Van Hiele hierarchy of levels is relevant to this study in that it provides the basis on which researchers can explore the learning difficulties by tracing the identified challenges to the level at with the learner is operating at in the hierarchy according to the model. Put differently, the significance of these levels to this study was that they assist the researcher to link geometry challenges to the level of growth of learners' thinking. In addition, the hierarchy of reasoning levels acts as a guideline for pedagogy; it enriches learners' geometrical knowledge, together with learners' proof construction of theorems and riders (Mogari, 2002; Watan & Sugiman, 2018).

To that end, educators can effectively intervene in addressing learners' geometric errors and misconceptions through planning instruction, which is meant to address identified challenges at given levels with reference to Van Hiele's model.

The figure below summarises the hierarchical nature of Van Hiele's geometric thinking levels as alluded to above:

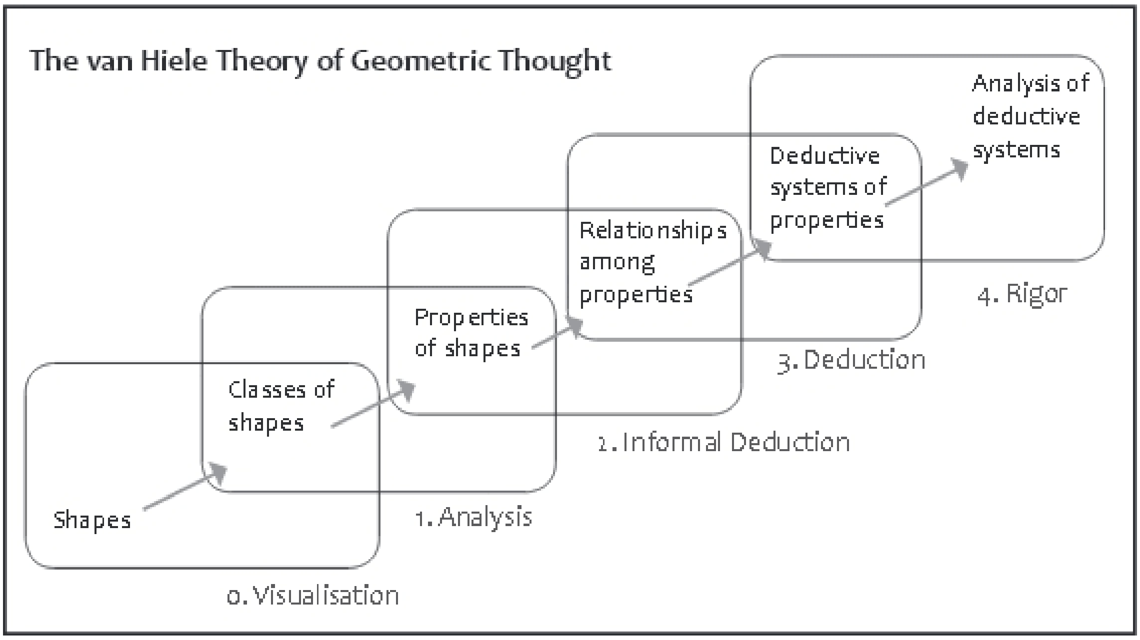


Figure 2.3

The Van Hiele theory of geometric thought (Van de Walle 2004, p. 347)

The diagram above demeonstrates that the understanding of geometric concepts is hierarchical and cummulative (builds on), with the level of mastery and geometric information acquired by the leraner in prior levels dertemining his/her preparedness to functioning at the next level.

The characteristictics of these van Hiele levels are explained below:

2.5.5 The properties of Van Hiele model

The five properties that characterise the Van Hiele model are: sequential, advancement, intrinsic and extrinsic, linguistic and mismatch.

(1) Sequential: learners should pass through the levels in a sequence. Thus, in line the Van Hiele theory, learners cannot master level (n) if they have not mastered level (n - 1). Most probably, the disregard of a sequential instruction is the reason why educators fail to reach the learners in Euclidean geometry while, on the other hand, learners struggle to grasp geometry concepts. If learners have not passed the levels as indicated above, they cannot adequately operate at any given higher reasoning level despite their ability to perform algorithmically at any level without understanding (rote learning). Thus, the strategies of previous levels should be acquired first. Hence, Jones (2002) suggests that if learners are to succeed at a certain level, they cannot by-pass Van Hiele's levels.

Furthermore, studies by Baiduri et al. (2020); Yew and Saleh (2019) on the efficacy of geometry teaching methods were in favour of Van Hiele-based instruction as an effective method of instruction in improving learners' achievement in Euclidean geometry, as compared to traditional methods. These researchers' arguments were in favour of the Van Hiele model because it emphasises the sequential understanding of geometric concepts from inductive to deductive reasoning.

(2) Advancement: according to the Van Hiele model, progression from one level to the other depends on teaching method (geometrical experience) rather than the learner's age.

(3) Intrinsic and extrinsic: a geometric concept studied in the previous level becomes a topic of study in the next level; for example, if learners identify shapes in level 1, their properties will be the topic of discussion at level 2.

(4) Linguistic: each Van Hiele level has its own characteristic linguistic symbols, with simple symbols in the first level and more complicated ones at higher levels. Research reveals that the educator and some learners who progress to higher levels of geometrical thinking seem to speak the same language, which other learners who have not yet reached that level cannot communicate. It means the educator should not use the language that is above the learners' level of comprehension. Hence, Pierre Van Hiele (1986) observed that two people reasoning at different linguistic levels would not understand each other.

(5) Mismatch: learners operating at a certain Van Hiele level find it difficult to understand the vocabulary and concepts typical of higher levels. Learners should first understand the content and linguistic symbol of a typical level for them to comprehend its contents and processes. Furthermore, it is the disregard of the hierarchical nature of the levels by the educator and the learners operating at different levels that account for the challenges that learners experience in the process of learning Euclidean geometry.

Studies by Alex and Mammen (2012), Alex (2016); Jones (2002); Ural (2016) and Walle (1994) all invariably suggested that lower secondary learners perform at visual and descriptive levels, with almost 40% of these learners finishing secondary school below the descriptive level. The explanations to this challenge, as alluded to by Van Hiele (1986), are that educators are required to teach a curriculum that is at a higher level than the learners' level of understanding. Hence, Van Hiele (1986) further asserts that higher levels of geometrical thinking are rarely achieved by learners due to improper sequencing of geometrical materials. That means, if the teacher uses a language, a textbook or a teaching method at a higher level than that of the learner, there will be a serious communication breakdown, which could lead to frustration and a lack of understanding by the learners. Thus, for learners to understand the skills, they need to think and work through these levels.

More recently, studies have used the Van Hiele theory to investigate problems experienced by learners (and pre-service teachers) in geometry, including those by Mostafa et al. (2017) and Naidoo and Kapofu (2020). Findings from research that used the Van Hiele model to explore learners' geometry misconceptions were that theory was useful in extracting, measuring, understanding and addressing learners' difficulties in Euclidean geometry (Alex & Mammen, 2014; Luneta, 2015). In this investigation, the Van Hiele theory is relevant as it assisted in diagnosing the misconceptions that learners displayed in answering questions, as set out in the test to check the learner's development of geometrical reasoning.

2.5.6 Hoffer's geometric thinking skills

The above-mentioned assumptions to geometric thinking put forward by Van Hiele induced Hoffer (1981) to identify five basic skills that have the greatest bearing in determining the extent of success in enhancing the learner's levels of geometrical thinking. Thus, Vorster (2012) describes a reasoning level as the depth of understanding by a learner to be reached, together with how well he/she can express their thinking using words, drawings or symbols. In this vein, these geometric thinking skills, in Hoffer's view, are meant to address limitations to the Van Hiele reasoning levels by describing the skills a learner needs to master to be able to solve geometry problems.

Thus, for a learner to think at a certain Van Hiele level, attention should be paid to the development of visual skills, verbal skills, logical skills, drawing skills and applied skills, depending on the operational level demanded by the geometrical problem. However, if a learner fails to master these geometric thinking skills required at a given reasoning level, the learner will encounter difficulties in solving geometry problems. Thus, this study adopted the geometric thinking skills identified by Hoffer in 1981 to interpret the learning difficulties learners experience in Euclidean geometry. In this vein, a skill is viewed as something that a person has learnt to do (Vorster, 2012).

Hoffer (1981) identifies five basic skills underpinning the understanding in Euclidean geometry as:

(a) The skill to use your sight (eye)

Success in Euclidean geometry depends on the learner's ability to interpret diagrams and oblique projections, and to imagine or visualise a situation from written or oral descriptions. Visual skills entail recognition and observation; hence, learners are required to develop and implement visual competences. Thus, learners most probably face challenges in geometry because of insufficiently developed recognition and observation skills.

(b) Verbal skills

The place of language in learning mathematics, in particular Euclidean geometry, is essential for describing spatial relations in words, describing shapes and the relationships between them in words, and formulating assumptions, definitions and theorems. As a result, learners with reading and comprehension challenges, communication or verbal formulation are likely to encounter difficulties in learning Euclidean geometry

(c) Drawing skills

Learners need to develop drawing skills in order for them to understand the properties of shapes, together with relationships between geometric shapes for doing proofs (Patkin & Sarfaty, 2012). They should be able to draw simple perspective sketches.

(d) Logical skills

Learners need to have logical and reasoning skills when learning geometry. Thus, learners seem to struggle in geometry most probably because they cannot build arguments in a hierarchical manner, ground claims in arguments, identify valid and non-valid argument and differentiate between a reason and a conclusion. Hoffer (1981) argues that learners struggle to find similarities and differences among shapes and understand that shapes are independent of their orientation, position and size.

(e) Application skills

Research revealed that geometry is a theoretical model of the real world around us. It is imperative for learners to realise that the surroundings and comprehending geometry are mutually related aspects. Learners need to apply geometric thinking in different situations and their inability to make this application could disrupt their learning of geometry (Vorster, 2012).

A link between Van Hiele's model of geometric reasoning levels and the associated thinking skills by Hoffer is useful for this study because it enables the researcher to effectively use research instruments to explore the learning difficulties based on these established theories, which, in a way, assists in explaining the learning challenges that learners experience in Euclidean geometry. Thus, Hoffer's model is relevant to this study as it assists the researcher in identifying and categorising learners' challenges in Euclidean geometry based on their responses to questions from the research instruments. Based on that, the researcher's view is that if learners lack these basic skills as postulated by Hoffer (1981), their achievement in Euclidean geometry is highly likely to be compromised and hence they encounter learning difficulties.

The researcher is of the view that the Van Hiele theory should be considered in explaining the learners' development of geometric concepts through levels; however, the theory has gaps regarding reflecting learners' mental representation of geometric concepts which contribute to learning difficulties in geometry. These gaps can be overcome by using Hoffer's geometric thinking skills to identify learner challenges in terms of concepts lacking in learners' geometry thinking.

Furthermore, adopting Hoffer's model enabled the researcher to explore learner challenges in terms of the errors and misconceptions learners have in Euclidean geometry in relation to learners' faulty schema, prior knowledge and incorrect reasoning, and whether learners have the relevant procedural and conceptual knowledge by the time they are in Grade 12.

2.5.7 Integration of theoretical frameworks in geometrical knowledge generation and this study

The Van Hiele theory as explained in literature needs to be understood in the context of the broader theory of constructivism, which puts more emphasis on the active participation of learners in self-generation of knowledge during teaching and learning. The CAPS document (2011a) is based on the constructivist framework, where the focus of instruction is shifted from being teacher centred to instruction that puts learners at the centre of teaching and learning. Hence, by transitivity principle, the Van Hiele model can be viewed as a constructivist theory to geometry instruction. Constructivism marks the departure from the traditional approaches to mathematics instruction where learners were expected to unquestioningly absorb mathematical structure, as they were invented by an expert (Fujita et al., 2014). Application of geometric concepts in disciplines like architecture, art, interior design and science demands that educators cite geometrical examples from real-world contexts during teaching and learning to enhance learners' understanding of geometry.

Similarly, and Brau (2018); McInemey and McInemey (2002) argue that in traditional approaches, educators were holders of geometrical knowledge to be learnt and they had to pass this knowledge on to passive learners through drills and practice; hence, persistent geometry challenges. However, through the constructivist elements in the Van Hiele levels of geometric thinking, learners themselves are the ones to develop through the reasoning levels and they should apply the reasoning skills to solve geometry problems. In view of those learning difficulties, Mcyntire (2018) argues that Van Hiele's theory of geometrical thinking is the theory behind the teaching of Euclidean geometry in schools, but many educators still prefer using the traditional teaching approaches, hence performance remains below expectation. In this vein, failure by learners to develop and master the requisite geometry thinking results in them experiencing learning difficulties.

Moreover, Aina (2017) argues that learning only takes place when learners have ownership of the body of knowledge being learnt in their social environments (social constructivism). In the case of Euclidean geometry, educators should know the learners' prior knowledge and then, through negotiation and scaffolding, help them navigate the Van Hiele's hierarchical levels of geometrical thinking within their ZPD. If this is not done, learning difficulties will arise. This notion further affirmed Van Hiele's (1986) views that teaching with indoctrination inhibits learning, and educators should not be all knowing and believe that learners are there merely to be instructed; but educators should treat learners as dignified opponents who can introduce new arguments. This view shows that the Van Hiele theory is consistent with the constructivist framework to geometry teaching and learning.

Moreover, constructivists view language not only as a means of transferring information, but also that language should be meaningful, and not a source for that meaning (Yager, 1991). Thus, within the geometric thinking levels as postulated by Van Hiele (1986), he emphasised the importance of understanding the associated geometry language at every level in the hierarchy, as learners make direct encounters with reality. Thus, if learners do not master the relevant geometric language, they experience learning difficulties. In that case, Van Hiele explains the process as explication, which enables learners to move from one level to the next. This is so because learners, through mediation in their ZPD, are expected to have cognitive constructions of geometrical language relevant to each level, and failure by learners to develop geometrical thinking skills as demonstrated by Hoffer leads to misconceptions, resulting in them making geometry errors. Therefore, there was a need to explore learning difficulties in geometry with due diligence to the above theoretical frameworks.

2.6 Conclusion

This chapter explored literature on other studies that is relevant to the current investigation, with a comprehensive discussion of the major themes of Euclidean geometry to include the importance of geometry as a section of the secondary school mathematics curriculum and the factors that contribute to the learning difficulties encountered in Euclidean geometry problem solving. In addition, theoretical frameworks underpinning this study were also explored through the lens of the widely accepted theory of constructivism. Constructivism as the anchor theory was also discussed through its affiliate theories to geometry learning such as cognitive constructivism and social constructivism, in addition to the Van Hiele theory of geometric thinking and Hoffer's model of geometrical skills. These theoretical frames were also explored in terms of how they are integrated, together with the ways in which they are used to help in answering the research question(s) for this study.

The next chapter explores the research methodology to the study.

CHAPTER 3: RESEARCH METHODOLOGY

3.1 Introduction

The previous chapter focused on literature and the theoretical frameworks that guided this study. In this chapter, attention was on the research paradigm(s) that guided the research, methodology and research design for the study. Thus, in this chapter, the researcher firstly gave an account of the research paradigms as guiding principle to the choice of methodology, the methods adopted and the research design for this investigation, before proceeding to describe the targeted population and the procedures through which the research data were collected. Secondly, the data collection instruments, how they were developed and their subsequent administration to research participants as presented in section 3.7 were discussed. Furthermore, an overview of the polity study, together with its relevance in the refinement of the questionnaires, was also given. The procedures for analysing the test, questionnaires and interview responses were also provided in section 3.10. The chapter concluded by discussing validity, reliability and trustworthiness, followed by an account of ethical considerations, together with limitations to the study.

3.2 Research paradigm

A paradigm in an educational study refers to a framework that governs the way of exploring and interpreting knowledge as well as motivations and goals of the study (Mackenzie & Knipe, 2006). Moreover, according to Huitt (2011), a paradigm is viewed as the perspective of reality held by the researcher. Apart from the above, Shanon-Baker (2016) describes a paradigm as an assumption about how things work, sometimes expressed as a “worldview” involving a shared understanding of reality. Based these researchers’ views, a paradigm can be described as a set of beliefs and agreements held by researchers on how the problem under study can be understood and addressed.

More elaborately, adopting a specific paradigm(s) in educational research assists in guiding how the research problem(s) are solved, as well as influencing the author’s choice of the research method(s) (Teherani et al., 2015). The same idea was elaborated by Blaikie and Priest (2019) who argue that, for a researcher to properly position himself within the research, there should be proper alignment between the belief system underpinning the research approach, the research question(s) and the choice of the research methodology and methods.

Therefore, it is imperative for every educational research to be framed around any of the four basic research paradigms, namely positivism, post-positivism, interpretivism/constructivism and critical theory. These paradigms represent different belief systems around which a study can be framed. For example, positivism is a paradigm that focuses on observable behaviour that is aimed at building scientific knowledge (Grix, 2004; Taylor, 2013). In addition, research further revealed that proponents of positivism emphasise observation and give reasons for human behaviour; hence, positivists hold the belief that true knowledge emanates from the experience of human senses through observation and experimenting (Trochim, 2006). Moreover, many researchers concur that positivism has its roots in Natural Sciences, and it should be understood within the principles and assumptions of science, where scientific methods are used in knowledge generation rather than through speculation (Creswell & Poth, 2016). Thus, positivism as an educational research paradigm is anchored on the belief that there is a single, objective reality that can be observed through science.

However, over the years, positivism has been criticised for its flaws in studies dealing with human behaviour, where scientific methods were found to be less effective in addressing human challenges (Trochim, 2006). Similarly, McMillan (2014) argues that a study with humans could not be but done in the same way as the study of nature, which confines the research to an accepted set of rules to conducting and reporting results. As a result, due to limitations associated with positivism, as suggested above, this investigation could not adopt positivism as a research paradigm because the reported study explores humans and their actions.

On the other hand, post-positivism focuses on the imperfect and probabilistic nature of knowledge, where knowledge is viewed as never static but dynamic (Trochim, 2006). In view of post-positivism, Chilisa (2012) argues in favour of an imperfect reality, which gives room for error in scientific observations. Arguing from a similar perspective, these post-positivist researchers believe that knowledge is never fixed, but is conjectural. On the bases of this belief, one can safely conclude that through the post-positivist paradigm, knowledge is can only be estimated to some degree of certainty.

Furthermore, the critical theory paradigm focuses more on the notion of social justice and advocates for equitable resources distribution, and an environment that is more inclusive of human diversity (Lan, 2018; Taylor, 2008). The major goal of critical theory is to address issues of power and social politics and the inequalities in the social world (Cohen et al., 2007). Proponents of the critical paradigm believe that the world is characterised by power struggles and their duty is to expose them and then build an environment that is fairer, more equitable and more inclusive of all humans. In view of this, critical theorists aim to eradicate all forms of societal injustices and social inequalities that institutions like legislature, media, educational research and schools seem to replicate and perpetuate. Hence, Brown and Duenas (2019) allude to the existence of multiple subjective realities that are influenced by societal power relations, thus, influencing the social, political, economic ethnic and gender values. The paradigm could not be applied to this study because the aim of the research was not to correct social injustice.

Altson and Bowles (2018) argue that a researcher(s) needs to have a set of assumptions (paradigm) that guides their approach to an investigation in terms of factors like, personal skills, the background of the study, their view of research problem, accessibility of participants and their perspective to the study before deciding on the methodology to follow. Against this backdrop, this study was framed around the interpretive/constructivist paradigm that Cohen et al. (2007) view as helpful in the study of individuals with their opinions, attitudes and different human behaviours. Constructivists are of the view that reality can be fully understood only through subjective interpretation and real-life intervention (Bryman, 2008). The constructivist/interpretivist paradigm affiliates to the notion of multiple subjective realities that one must actively interpret to fully understand and enhance one's ability to construct knowledge (Brown & Duenas, 2019).

Furthermore, the researcher believes that the multiple realities alluded to above ought to be socially constructed by and among individuals, in terms of how learners develop geometrical knowledge at an individual level. As a result, the researcher's choice of constructivism over the other worldviews mentioned above was informed by his strong belief in the role of interactions as a quest for the development of geometrical knowledge. In addition, the adoption of constructivism in the reported study was also based on the main belief that research can never be objectively observed from outside, rather it should be observed from inside through direct experiences of the people (Mack, 2010).

Through this study, constructivism fulfilled the researcher's intention to explore challenges experienced by learners in Euclidean geometry by interpreting their responses from face-to-face interactions during activities like written tests, questionnaires and interviews. Thus, the choice of the paradigm for this study is well supported by Van de Walle (2016) who argues that learners are expected to actively participate in their own geometrical knowledge development, as proposed by the widely accepted theory of constructivism. Against this backdrop, by exploring the way in which learners construct geometrical knowledge, the researcher hoped to generate answers to the research question(s). Hence, the philosophical perspective of this study was that there are multiple local and specific constructed realities associated with the learning and teaching of Euclidean geometry, without which learners are bound to experience difficulties in solving geometry problems.

3.3 Research methodology

The main purpose of the research methodology is to help understand the process of research by spelling out approaches (methods) used to collect data and needed by the researcher for inferences and interpretations (Cohen et al., 2009; Ravitch & Carl, 2016). Moreover, Creswell (2014) views a research methodology as the different methods, techniques and processes that are used in the implementation of a research design. Furthermore, Creswell (2014) argues that selecting the correct method(s) to address a problem or a challenge in any study plays a pivotal role in achieving the purpose of the research.

3.3.1 Research approach

Educational researchers often choose between qualitative, quantitative and mixed methods research approaches when conducting research studies. In this study, a mixed methods approach (both quantitative and qualitative research methodology) was chosen to elicit learners' levels of geometry competence, as well as their views, feeling and attitudes towards the challenges they experience with geometry through the administration of tests, questionnaires and semi-structured interviews. In the same breath, Creswell (2014) views a mixed methods methodology as an approach to research where the researcher brings together qualitative and quantitative elements, thereby increasing the depth and breadth of comprehension and corroboration. In line with the reported study, the research adopted an explanatory mixed methods approach where quantitative and qualitative methods were applied sequentially.

3.3.2 Rational for choosing the mixed methods approach

The reason for using a mixed methods approach in this study was that it provided the researcher with more advanced and detailed methods to handle both quantitative and qualitative data (Creswell, 2014). Furthermore, Shannon-Baker (2016) argues that the mixed methods approach ensures that evaluation will be able to capture both statistical information and the views and opinions of learners, educators and the rational of their perspective. In this study, a mixed methods approach was used to provide multiple perspectives to geometry challenges. Thus, the justification for choosing the mixed methods approach is that it provided the researcher with advanced and comprehensive methods to handle both quantitative and qualitative data (Creswell, 2014).

Other benefits for using the mixed methods approach for this study are summarised by Creswell (2008) as follows:

- The insufficient argument – either the quantitative or qualitative approach may be insufficient by itself.
- Multiple angles argument – quantitative and qualitative approaches provide different “pictures”.
- The more evidence argument – the better argument: combined quantitative and qualitative approaches provide more evidence.
- Community of practice argument – mixed methods may be the preferred approach in a scholarly community.
- Eager to learn argument – it is the latest methodology.
- “It is intuitive” argument – it mirrors “real life”

Based on the above-mentioned, the researcher chose the mixed methods approach over other appraoches because it is the only and the latest methodology that effectively integrates quantitative and qualitative methods, thus providing multiple angles to the research proplem. The researcher felt that using either quantitative or quantitative method alone would not provide data that sufficiently answer the research question(s). In this respect, quantitative and qualitative methods adopted by the researcher are elaborated below:

3.3.3 Quantitative research methods

Quantitative methods involve measurement, establishing relationship, generalising and replication (Bryman, 2016). To supplement this assertion, Mungai (2019) argues that the major strength of quantitative approaches is the use of statistical methods to make generalisations from small representative samples to large populations. As result, it can be argued that quantitative methods were relevant for this study because they enabled the researcher to use statistics to generalise the learning difficulties encountered by small representative samples of participants to large populations. To validate the feasibility of using statistical methods in this study, Creswell (2016) argues that quantitative methods involve numerical data that show trends, behaviour or perspectives of a specific population by investigating a sample of that population. In this case, the researcher was interested in identifying learning difficulties in Euclidean geometry as encountered by Grade 12 learners. That was achieved by developing quantitative descriptions of data obtained from learners’ test scripts, together with questionnaire responses from different groups of respondents to the investigation.

3.3.4 Qualitative methods

Unlike quantitative methods, qualitative methods are not concerned with numbers and statistics but seek to understand how respondents of the research experience life and what a particular social phenomenon means to them, and from that, they develop or deepen their understanding (Alston & Bowles, 2018). Therefore, qualitative methods were considered relevant to this study because they enabled the researcher to explore and interpret the learning difficulties in Euclidean geometry from the perspectives of the participants and they helped to complement quantitative data gathered through tests and questionnaires.
Furthermore, the justification for using qualitative methods in this study was that they emphasised viewpoints of participants, transparency of the process and flexibility in exploring learning difficulties in Euclidean geometry. Furthermore, the researcher found this method suitable because Creswell (2014) emphasises that qualitative research aims to gather full information in a real setting to enable a clear understanding of everything being answered by respondents, together with observations by the researcher.

To that effect, the qualitative approach was also suitable for this study because it was the most suitable method for exploring and interpreting existing problems (Cohen et al., 2007; Henning, 2004; Leedy & Ormrod, 2001; Mertens, 2014). In the same vein, the approach assisted the researcher by giving the participants the opportunity to give comprehensive verbal explanations of the types and reasons for their challenges in Euclidean geometry to complement their written responses in the test and questionnaires.

3.4 Research design

A research design is a group of instruments and guidelines to be adhered to in addressing the research problem (Creswell, 2014). Thus, a research design is viewed as a comprehensive roadmap/package of plans, strategies and procedures for the study ranging from underlying worldviews to detailed methods of data collection and analysis (Luneta, 2013). Thus, some of the constituencies of a research design include methods to be used, data to be gathered, how, where, data collection sources, together with the associated circumstances for data collection. Similarly, McMillan and Schumacher (2014) view a research design as a general plan for conducting a study, constituting how the research is set up, the procedures involved, subjects involved and the methods of data collection.

A mixed methods approach consists of numerous research designs, such as convergent parallel designs, sequential exploratory designs and sequential explanatory designs (Creswell, 2014). In a convergent parallel design, the researchers collect the data concurrently. In exploratory sequential design, the researcher collects qualitative data to generate information that is used to conduct the quantitative phase of the study, whereas in sequential explanatory design, which was relevant to this study, the researcher uses qualitative questions to explain findings from the quantitative data. To achieve this, the researcher engaged in a two-phase data collection process where qualitative data were collected in phase two to provide explanations for findings from quantitative data collected in phase one. Since the logic of the study is more explanatory, quantitative data were collected first by administering a test and questionnaire to learner participants, followed by qualitative questions through interviews. Based on learner responses to the quantitative phase, the researcher purposefully selected participants to provide qualitative data to the challenges experienced with Euclidean geometry. The data collection phases are explained in detail in section 3.8.

Furthermore, the type of sequential explanatory design adopted is sequential explanatory triangulation design.

3.4.1 Sequential explanatory triangulation mixed-methods design

A triangulation mixed methods design was adopted for this study, and it is viewed as an approach that correlates different data sets collected on the same topic (Creswell, 2014). Furthermore, Cohen et al. (2007) refer to a methodological triangulation design as the researcher's use of different methods to collect data on the same object of study to determine consistency of results, hence ensuring validity. Similarly, triangulation enabled collecting, analysing and integrating data from quantitative and qualitative methods, respectively, in a single study (Creswell, 2014).

The adoption of the triangulation design for this study was on the assumption that collecting a variety of data enables the researcher to have a better understanding of the research problem. In support of that assumption, Mnguni (2019) argues that the mixing of approaches sequentially in a single study provides the research with a total strength that is greater than either quantitative or qualitative research. Thus, this study achieved methodological triangulation (evidence from several sources) by using quantitative (questionnaires and tests) and qualitative (interviews) methods, respectively, to enable the exploration of learning difficulties in Euclidean geometry in participants of selected secondary schools.

3.4.2 Aims and advantages of the explanatory sequential design

The major aim of this design was to facilitate the use of qualitative data to explain quantitative results or to further elaborate quantitative findings. This aim is supported by Creswell (2014) who argues that an explanatory design is mainly used when quantitative data collection is clearly warranted by follow-up analysis and there is a need to elucidate quantitative findings. Similarly, in this study, the design enabled the researcher not only to give descriptions of the learning difficulties in Euclidean geometry as evidenced by quantitative data collected, but also to provide qualitative explanations of those challenges from the perspectives of the participants. This means that through sequential explanatory triangulation design, the researcher used qualitative questions to provide explanations for findings from quantitative questions.

In addition, the choice of explanatory design was informed by the following inherent advantages: Firstly, according to Creswell (2014), sequential explanatory designs give the research a sequential integrative character, where data collection phases are logically linked. That means that quantitative results are clarified through follow-up participant explanations. Put differently, this design has the benefit of enabling one phase of data collection to build from another phase in sequence; for example, collecting qualitative data after the quantitative phase explains quantitative data in more depth. It can also be argued that a sequential explanatory design helps to combine quantitative and qualitative paradigms in meaningful ways through a philosophical convergence of viewpoints, methods and conclusions.

Specifically, a quantitative descriptive design, which used a survey design, was chosen for the quantitative aspects of the research. Quantitative aspects of the study used a written test and questionnaire while qualitative aspects of the study used interviews to explore the learning difficulties in geometry. Moreover, a case-study design was applied for the qualitative part of the study.

As a means of fulfilling this purpose, both quantitative descriptive design (survey design) and qualitative case study design are outlined below:

3.4.3 Descriptive quantitative designs

According to research, descriptive designs are used to collect information about variables without changing them or manipulating the environment (McMillan & Schumacher, 2014). In addition, Grove et al. (2013) suggest that descriptive designs may be used to identify problems with current practice or to justify the current practice. The choice of this design was informed by the research question(s), where the intention of the researcher was to identify the learning difficulties in Euclidean geometry as they are experienced by Grade 12 learners. McMillan and Schumacher (2014) view descriptive designs as simply using quantitative summaries to characterise an existing phenomenon using numbers, statistics, structure and control. This study used a survey design as a type of descriptive design for the quantitative aspects of the study of learning difficulties in geometry. The descriptive designs used followed an explanatory sequential design, where qualitative data were used to explain what was observed in quantitative data (Creswell, 2014).

MacMillan and Schumacher (2014) describe a survey design as a situation in which the researcher administer a questionnaire or interviews to a selected sample of subjects to explore participants' attitudes, beliefs, opinions and other types of information. In the reported study, the research employed the survey design to explore participants' perspectives on the learning difficulties they experience in Euclidean geometry.

A survey design was chosen for this study because this approach allows information to be generalised to the larger population by making inferences from a smaller group of respondents (the sample). The design was suitable for this study because the researcher used relatively smaller samples of learners and educators to explore the learning difficulties that Grade 12 learners experience in Euclidean geometry. In the same vein, Mnguni (2019) argues that through the survey, the investigator can select a sample of subjects and then administer a questionnaire or conduct interviews to collect data. As a result, the survey design informed the researcher's choice of research instruments. Overall, a survey design was deemed suitable for this study because previous researchers have successfully used it to describe attitudes, beliefs and other types of information, which the researcher in this study found useful to explore learning difficulties in Euclidean geometry.

3.4.4 Case study design

This study adopted a case study design that catered for descriptions, analysis and naturalistic summaries of the geometry challenges gathered through interviews and documents sources. Cohen et al. (2007) emphasise that the case study design provides a detailed description and in-depth perceptions and meanings held by participants in their unique situations. Thus, the case study design for this study used a report format to explain learning difficulties in geometry as gathered from the perspectives of the research participants.

Moreover, the case study design was specifically used for the qualitative aspects of the study, where learner challenges were explored on a case-by-case basis in terms of interview responses. The design was suitable for this study because data were extracted directly from individuals in their natural environment to determine participants' different perspectives to learning difficulties in geometry. Therefore, the case study design was chosen for its ability to offer descriptions and summaries of geometry challenges based on the identified patterns and themes.

3.6 Location, population and sampling

3.6.1 Location and population samples

The reported study was undertaken in the Ngaka Modiri Molema District in the North-West province. The investigation comprised a total study population sample of 60 randomly selected learners from a total of six purposefully sampled secondary schools in the district (10 learners per school), four mathematics educators and two mathematics HoDs for senior and FET phases. In addition, sub-samples of 12 learners and four educators were selected from the study population to take part in the interviews section of the investigation. The respondents came from different secondary schools showing a mix of urban and rural, private and government, fully resourced and under-resourced schools.

3.6.2 Sampling procedures

Tavakoli (2012) describes a sample as the number of subjects selected from the population for the purpose of collecting the required data. For this study, cluster sampling, simple random sampling and purposive sampling methods were used to choose the participants. In case of cluster sampling, Salawu (2017) argues that the entire population of the study should be divided into externally homogeneous but internally heterogeneous groups described as clusters. In support of that, McMillan (2014) emphasises that cluster sampling is more applicable in studies in which the researchers can only identify groups or clusters because the entire population cannot used. For this study, the researcher could not access all the schools in the district; hence, cluster sampling was used. In this study, the researcher started by profiling schools into three categories/clusters of low-, average- and high-performing schools. Two schools were randomly selected from each of the three clusters. Based on the above, cluster sampling was relevant for this study because it enabled the study population to be more representative of all Grade 12 learners' competence levels, since participants were drawn from school clusters representing all performance levels. That probably ensured that the research findings were not compromised by favouring a certain level of learner competences. As a result, the researcher hoped that using the cluster sampling method would render more authentic findings because it enabled the study to cater for learners from different socio-economic environments and performance groups.

3.6.3 Simple random sampling

McMillan and Schumacher (2014) describe simple random sampling as a procedure for choosing small samples of research participants from a population in such a way that every member of the population has the same chance of being chosen, thereby enhancing the validity of the research findings. In view of the above, simple random sampling was used in this study, both for selecting schools from the three clusters of low, average and high performing schools as mentioned above and for selecting participants to take part in the study (learners, educator and school mathematics HoDs) because the samples used for the study were relatively smaller but representative of the entire population. For this study, a sample of 10 learners per school was randomly selected from the two schools in each of the three clusters. A sample of 12 mathematics educators and four HoDs also came from the selected schools. Participants for a pilot study were randomly selected from any school that was not in the study population. In this view, random sampling was found relevant for the study in that it gave every member of the accessible population an equal chance of taking part in the investigation.

3.6.4 Purposive sampling

McMillan and Schumacher (2014) argue that purposive sampling is sampling that allows selection of small groups or individuals that are likely to have knowledge and information about the research problem, in this case, learning difficulties in Euclidean geometry, particularly in the domain of circle geometry. Furthermore, Cohen et al. (2007) describe purposive sampling as choosing participants that possess a particular characteristic that adds value to the research. Mnguni (2019) views purposive sampling as a situation where the researcher selects cases with a specific purpose in mind. Based on that view of purposive sampling, one can conclude that through this method, only those participants that are likely to provide useful information are selected. Thus, the selection of learners for interviews depended on their responses in the written test.

As a result, sampling of learners for interviews was purposefully done to those that showed poor performance in the written test, as they were likely to provide valuable verbal information on their challenges with Euclidean geometry problems. Creswell (2014) argues that purposive sampling is suitable for smaller groups of participants and has the potential to provide valuable data. To that end, the choice of purposive sampling for interviews for this study is well supported by the above assertion since small samples were used from where the researcher managed to gather valuable data. Purposive sampling of learner interviewees provided qualitative data which complemented the data collected from quantitative tests and questionnaires.

3.7 Researcher's role

The researcher's role in this study was mainly informed by his worldview, interpretive/constructivist, as mentioned in the research paradigm section where emphasis was placed on multiple socially constructed realities. By the nature of the research problem (learning difficulties in Euclidean geometry), the researcher could not take an outsider role but had to be actively involved in the process of generating empirical evidence. Since the research was evidence based, the researcher's role was also guided by the principles applicable to all educational studies in line with the National Research Council (2001). Guided by these principles, the researcher had to pose significant questions that could be investigated empirically. Thus, the researcher was involved in the development of research instruments (test, questionnaires and interview schedules) that were used to gather research data. Furthermore, the researcher had to link the research to the relevant conceptual framework and use methods that allowed direct investigation of the research questions.

To fulfil these roles, the researcher was physically involved with the participants in their natural settings, from selection of subjects, to getting involved in the direct administration of instruments (tests and questionnaires), data collection through face-to-face interviews to data analysis. Thus, because of the nature of the adopted methodology (mixed methods approach), the researcher had to strike a balance between being detached from the study (ideal for quantitative) to avoid biases and being immersed in the situation (ideal for qualitative) or phenomenon being studied (McMillan, 2014). Thus, the researcher had to assume an interactive social role, through involvement in face-to-face interviews with participants by exercising disciplined subjectivity and reflexivity to gather data that were used to argument that which was objectively collected through quantitative research instruments (tests and questionnaires).

3.8 Research instruments

The researcher used both primary and secondary data collection instruments for this study. Primary data were collected through a test, questionnaires and semi-structured interviews, while secondary data were gathered through documentary reviews in the form of Department of Basic Education diagnostic reports (2016-2020).

The table below gives a summary of the data collection plan adopted by the researcher. The table presents an overview of the instrumentation that was used in relation to the important research question(s) that the researcher intended to answer.

Table 3.1

A plan for data collection

Research Question(s) (main research)' What are the learning difficulties that the Grade 12 learners experience in the learning of Euclidean geometry?	**Respondents**	**Sample size**	**Method used to collect data**
1. What are the Grade 12 learners' learning difficulties related to visualisation in Euclidean geometry?	Learners Educators HODs	60 4 2	Written test, questionnaire, interviews Questionnaire, interviews Questionnaire
2. What are the Grade 12 learners' learning difficulties related to the analysis level in Euclidean geometry?	Learners Educators HODs	60 4 2	Written test, questionnaires, interviews Questionnaire, interviews Questionnaire
3. What are the Grade 12 learners' learning difficulties related to the informal deduction level in Euclidean Geometry?	Learners Educators HODs	60 4 2	Written test, questionnaires, inteviews Questionnaire, interviews Questionnaire
4. What are the Grade 12 learners' learning difficulties related to the formal deduction level in Euclidean geometry?	Learners Educators HODs	60 4 2	Written test, questionnaires, interviews Questionnaire, interviews Questionnaire

Based on the table above, a detailed explanation is given of how each of the research instruments was administered below:

3.8.1 Test

A written geometry test was administered to learner participants, with questions designed to collect quantitative data on learner challenges, considering the themes highlighted in the literature, to include learners' visualisation, geometric reasoning and proof, geometric misconceptions, and errors and the requisite geometrical skills as argued by (Hoffer, 1981) (see appendix C). That test worksheet was conceptualised in terms of constructivism, together with the adoption of Vygotsky's educational theory that promotes scaffolding, as explained in the literature above. The Van Hiele levels were assigned based on the following criteria: if a respondent scored 50% and higher for questions at a given level, he/she was considered to have "passed" that level, and thus was viewed as competent at that level.

The above-mentioned competence levels allocation was successfully used by Van Putten et al. (2010) in a study of attitudes and understanding of Euclidean geometry in pre-service mathematics education (PME) students at the University of Pretoria. Similarly, Usiskin (1987) uses the same classification in the study of the Cognitive Development and Achievement in Secondary school Geometry (CDASSG) project to describe learner behaviour at each Van Hiele level; hence, the relevance for this study.

In addition, the researcher used a performance and error recording sheet to record the challenges in terms of errors and misconceptions that learners had in Euclidean geometry, as evidenced by their responses to the written test. Learners' errors and misconceptions were recorded per question to give a picture of the challenges that learners demonstrated at each Van Hiele level of geometrical thinking.

In this regard, a paper-and-pencil test was found to be a suitable technique in that it allowed learners to demonstrate their geometric skills in writing and it assisted the researcher with information on learners' thought processes regarding Euclidean geometry. The final items of the test were designed and developed by the first author of this document, with reference to the descriptions of learner behaviour at each of the levels according to the Van Hiele model of geometric thinking. In addition, the test items were designed based on the South African Grades 10 to 12 circle geometry curriculum and were deemed suitable to identifying the challenges experienced by learners in geometry.

Furthermore, the design of the test content was such that specific questions fall within a given Van Hiele level. In this case, the test contained six questions arranged hierarchically in increasing levels of difficulty (hierarchical geometry thinking), as stated by the Van Hiele levels, which are visualisation, descriptive/analysis, abstraction and deduction, together with the requisite geometrical thinking skills according to Hoffer (1981). Thus, a detailed breakdown of both the concept analysis distribution of the test question items in terms of the Van Hiele levels is given below:

Table 3.2

A concept analysis matrix used to develop test items based on the Van Hiele levels

Van Hiele levels and Descriptions	**Euclidean Geometry concepts**	**Question Number**
Level 0 (visualisation)	Explore learners' learning difficulties with visualisation, in terms of: • identifying and naming angles equal to x and y as corresponding, alternate, and co-interior angles.	1
Level 1 (analysis)	Explore learners' learning difficulties with analysis of geometric diagrams, in terms of: • describing components of a circle (definition of circle) • comprehension and conceptualisation of geometry language/terminology related to properties of shapes.	2
Level 2 (Informal deduction)	Explore learners' learning difficulties with informal deduction, in terms of: • choosing correct properties or relationships for triangles and quadrilaterals • deducing interrelationships between the circle, sides of triangles, quadrilaterals to angles • making short deductions about triangles, quadrilaterals and parallel lines (class inclusions) • writing correct geometrical statements and reasons.	3 & 4
Level 3 (Formal deduction)	Explore learners' learning difficulties with formal deduction in terms of: • their knowledge of geometric proof knowledge of proving a theorem • mental imaging to proving geometric shape like cyclic quadrilaterals • logical arrangement geometrical statements	5 & 6

	to reach conclusions • writing relevant reasons in a multi-steps rider.	
Total: **55 Marks**		**6 questions**

From table 3.2 above, levels 2 and 3 were given two questions each to ensure a thorough exploration of learning difficulties, since Grade 12 questions mainly test learners on these levels of geometrical thinking. Although levels 0 and 1 had two questions each, the researcher increased the number of sub-questions to ensure that concepts to be mastered at these levels are thoroughly explored. As a way of exploring learners' thinking processes and finding out their challenges, questions 3 to 6 had many sub-questions covering each of the Van Hiele levels, from visualisation to formal deduction (0-3). For the contents of the test questions, see appendix C.

The alignment of test items to specific Van Hiele levels was meant to enable a thorough investigation of learners' challenges by checking learners' competences at specific levels of geometrical thinking in the hierarchy. Thus, every question was matched to specific Van Hiele levels of geometric understanding so that the quality of information relevant to the research problem is maintained.

Moreover, the National Curriculum Statement of South Africa (2011) recommends that Grade 12 learners should master geometry concepts up to level 3 (deduction), according to the Van Hiele model. Comparison of mathematical systems using axioms at Van Hiele level 4 "Rigor" is not included in the high school curriculum, but only reserved for studies after Grade 12 geometry. Moreover, the researcher sought expert judgement on the test items from the senior member of the mathematics department, especially at the design stage of the test to ensure its suitability for answering the research question(s).

3.8.2 Questionnaire

Bhandari (2021) views a questionnaire as a list of carefully selected items or questions used to gather data on attitudes, experiences or opinions of research participants about the issues being investigated. Therefore, the participants to whom the questionnaire was administered were able to supply their own answers or make an informed decision in the process of selecting the appropriate responses from the options given. With reference to this study, the latter was applicable where the researcher designed self-administered questionnaires that comprised Likert-type closed questions relating to teaching and learning difficulties in Euclidean geometry, which were given to learners, educators and school heads of the mathematics department, each with a set of questions directed at a given group of participants.

For this reason, the researcher adopted only the format of the questionnaire (Likert-type format) from that used by Adolphus (2011), but the questions constituting the questionnaire were self-formulated. Thus, in formulating the questions, together with maintaining the validity of the questionnaire, the researcher was guided by the literature review, the geometry CAPS curriculum and the worldview upon which the study is anchored (constructivism). In addition, the questionnaire was also peer reviewed by colleagues in similar geometry research fields, in addition to guidance from the research supervisor.

A Likert scale was included as a type of a response scale in which the respondents expressed a degree of agreement or disagreement with a statement (McMillan & Schumacher, 2010). A modified four-point Likert-type rating scale that used questionnaire responses of the type, *Strongly Agreed, Agreed, Disagreed and Strongly Disagreed* was adopted. To this end, the major reason for choosing this instrument was that it has been successfully used by previous researchers in similar studies; for example, Adolphus (2011) who used a Likert-type rating scale in a similar study in Niger River State, although that study focused on factors and not learning difficulties. A questionnaire was convenient in this investigation because it is self-administered and can be completed by participants at different locations (Bhandari, 2021).

3.8.3 Interviews

The researcher conducted semi-structured focus individual interviews with smaller, selected sub-samples of learners, taking into consideration various performance levels in the test. In this vein, special focus was placed on learners whose test scripts showed most errors, and that helped the researcher to select learners for interviews since sources for those errors had to be identified and explained by the learners themselves during the interview process. Hence, interviews with learners were conducted after the test had been administered. Tengan (2022) argues that semi-structured interviews use a predetermined thematic framework of asking questions together with soliciting insights and divergent thinking from the interviewee. Therefore, semi-structured interviews are flexible and useful as explanatory tools in survey methodology.

In this study, interview data were collected from a group of 12 learners and six educators from the sample of selected schools that took part in the investigation. To facilitate that, codes like L 1, L2 and E1, E2 were used to represent learner and educator interviewees, respectively. The interviews lasted about 15 to 20 minutes with each participant. In emphasising the importance of adopting interviews, Dowling and Brown (2010) argue that interviews enable the researcher to probe challenging issues in more detail, provide clarification and prompt responses from participants. Furthermore, Johnson and Christensen (2004) state that if the process of interviewing is done by a researcher with quality interviewing skills, there is a high possibility of gathering the most valuable information for the research. Interviews were suitable for this investigation because the researcher wanted to gain insight and divergent thinking from the interviewees on learning difficulties that learners have in Euclidean geometry as well as minimising the limitations of questionnaires. Interviews enabled the researcher to clarify questions, ask further probing questions and observe non-verbal communications.

In addition, interview protocol enabled the researcher to investigate both the experiences and feelings of the participants with Euclidean geometry. The interviews were tape-recorded and later transcribed for further analysis (Creswell & Poth, 2016). Interviews were relevant in this study because of their descriptive nature and could give qualitative data in terms of participants' feeling, attitudes, motivations, perspectives and even suggestions that complement their responses to the questionnaires and tests. Hence, the interviews especially with learners acted as a follow-up to the questions the researcher asked in the written test (see appendices D, E & F).

The interview sessions were slotted last in the list of research instrument administration to complement and address different sets of questions that were not fully addressed through the sole use of quantitative methods (test and the questionnaires) (Cutis et al., 2000). The data that were gathered from interviews were meant to answer questions relating to the extent to which learners experience difficulties that relate to visualisation, description, analysis and deduction in Euclidean geometry.

3.8.4 Document reviews

Documentary research is viewed as the use of documents, outside sources and any available written information to support an argument or viewpoint of an academic work (Omari, 2011). In addition to collect quantitative and qualitative primary data, as mentioned above, document review also formed part of the secondary data collection process. The researcher consulted the National Senior Certificate Report and the Mathematics Paper 2 Diagnostic Reports (2016-2020) with a particular focus on the trends in learner performance in Euclidean geometry questions, common errors, possible reasons for poor performance and possible ways to alleviate those challenges. Data on challenges experienced in

Euclidean geometry questions, particularly in the domain of circle geometry from the perspective of the examiners in the reports were used to complement that which was gathered from the said data collection instruments. Data gathered from document reviews of those reports were used to explain how learners were progressing in terms of the Van Hiele levels and to provide information to support the existence of learning difficulties in Euclidean geometry.

3.9 Pilot study

A pilot study was used to troubleshoot any problems the research may experience, mainly with a specific focus on things like the timeframes and checking the clarity of directions and items of the questionnaire. Thus, the pilot study assisted in detecting possible problems in the proposed research, which enabled the researcher to make possible adjustments before the commencement of the actual study. Thus, Hallway and Jefferson (2007) describe a pilot study as small version of the proposed research that is meant to refine the methodology. The researcher used a pilot study to test the reliability and to refine the research instruments. McMillan and Schumacher (2014) argue that researchers should conduct a pilot study of their questionnaires before using them in the main study. A pilot study was done with nine participants (six learners, three educators) from schools that were not meant for the actual investigation but that had characteristics that were similar with those that took part in the main research. The purpose of the pilot study was to give the researcher an idea on whether intended questions for participants could yield data to sufficiently answer the research question(s).

After administering the test, in the pilot study, the researcher realised that the one-hour duration was not enough to complete the test since some learners were really struggling and needed more time, because such learners were a possible source for the learning difficulties under investigation. The researcher then realised there was a need to adjust the test duration to 1 hour and 15 minutes. During the administration of the questionnaire, some learners indicated that they could not understand the questions, probably because of the language barrier; therefore, the researcher had to rephrase some of the questions by using simple English and writing short sentences. After piloting the interviews with learners, the researcher realised that the structure of the interview questions was appropriate for different groups of participants, but there had to be more flexibility in terms of the time. Therefore, 15 to 20 minutes were added, as some learners were not comfortable at first, hence they needed more time to acclimatise to interview atmosphere.

3.10 Data collection procedure

The researcher asked for permission to conduct the investigation from the Education Department at the Ngaka Modiri Molema district office and from the principals of the schools that took part in the study. Participants were first given a clear explanation of the purpose of the study before data collection started. The data collection process started with the gathering of data for the pilot study, with small samples of participants that were used to check the feasibility of research instruments to allow modification, where possible. McMillan and Schumacher (2014) argue that before using questionnaires in the main study, researchers should conduct a pilot study of their questionnaires. The pilot study was done with participants from schools that were not in the main study but that fell in the same categories as the researched schools. Data gathered from the pilot study showed that the three research instruments were feasible for the study after correcting the identified shortcomings.

The data collection consisted of two phases: Phase 1 and Phase 2. Phase 1 involved quantitative data collection, where data were collected through administering a written test to learners. As part of Phase 1, learners were first given a pencil-and-paper Euclidean geometry diagnostic test from which the researcher gathered data on learning difficulties in Euclidean geometry as indicated by their responses. In addition, from the test responses, the researcher gathered data on how learners fared according to the Van Hiele levels. Moreover, as part of Phase 1, the researcher also gathered quantitative data through administrating questionnaires to selected learners, educators and HoDs. More specifically, the researcher then used the test results to purposefully sample the learners to take part in Phase 2 of the data collection process. In this way, the selection of learners to take part in the second phase of data collection was informed by their responses in the test. Those who performed badly in the test were considered for the phase 2, since they were likely to provide valuable information on their challenges in Euclidean geometry.

Phase 2 focused on the qualitative data collection process where the researcher used interviews and documentation reviews to gather data to complement quantitative information that was collected in Phase 1. In this phase, the researcher collected data from those learners who did not perform well in the written test as a follow-up process aimed at providing detailed verbal explanations to the reasons behind their geometry challenges, as explained above. In this case, researcher had face-to-face interactions with the interviewees to collect qualitative data on geometry challenges from the perspectives of the participants with the aim of answering the research question(s). That means, during the interviews, the researcher probed interviewees to provide further details on the challenges they experienced with Euclidean geometry based on the responses they provided in the written test. Thus, in this phase of data collection, the researcher was more interested in gathering qualitative data from learners' experiences with geometry content. During the process, the researcher recorded learners' responses for further transcription and analysis to determine geometry challenges from the perspective of the participants.

Furthermore, the researcher also consulted previous diagnostic reports on the geometry section of Paper 2, as part of the secondary data collection. The researcher gathered data from document reviews by identifying the most frequent geometry challenges, as identified by the examiners as data that would supplement and help to authenticate learners' interview responses.

In view of the above, a design plan in figure 3 below shows a sequential flow summary of both the quantitative and qualitative phases of data collection and analysis, where data from the two phases were then triangulated for the purpose of answering the research question(s).

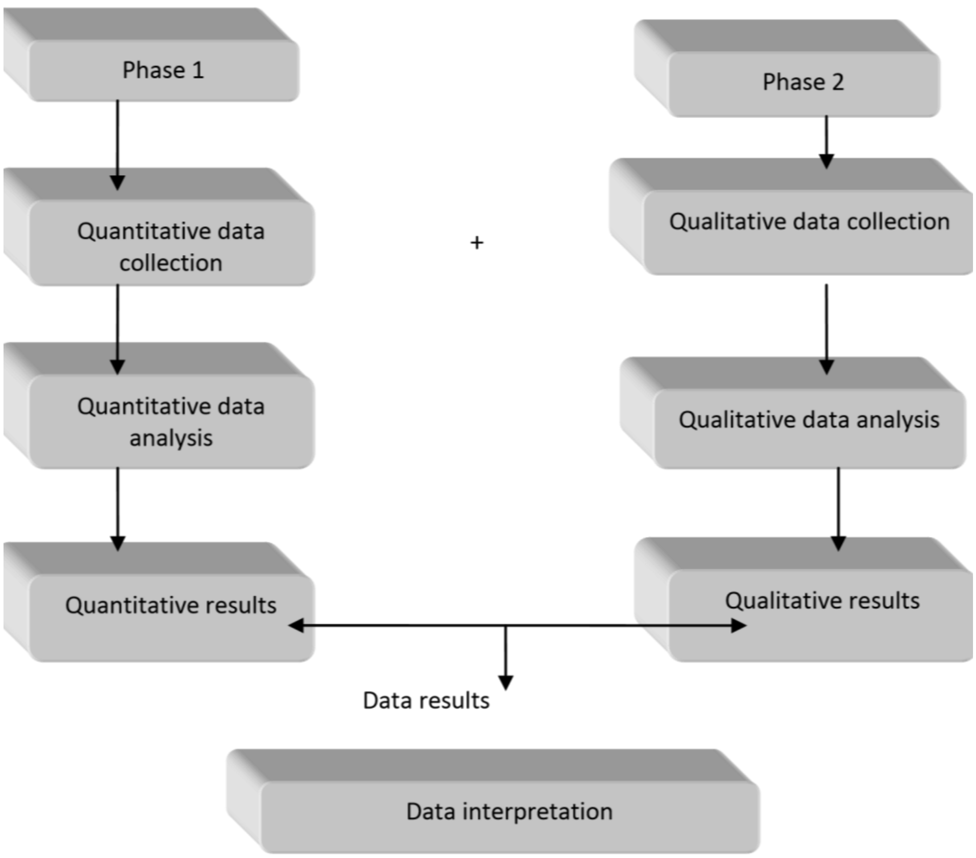


Figure 3.2

Data collection and analysis procedure (adapted from Creswell, 2008:12)

3.11 Data analysis procedure

The procedure for data analysis was divided into broad categories for quantitative and qualitative data analysis. Quantitative data analysis was based on the results from a written test and the questionnaires. To illustrate the procedures for quantitative data analysis for the test, the Van Hiele theory was used as a point of reference to categorise learners by totals and percentages as having achieved or not achieved at a given level. Furthermore, quantitative data analysis of the questionnaires was done in terms the percentages of respondents that agreed or disagreed with the proposed geometry learning difficulties at each Van Hiele level as part of the analysis that data was presented in tables, graphs and figures from where descriptive summaries were done with reference to the research questions.

Subsequently, qualitative data were analysed from the interviews and document reviews. As a procedure for analysis, data from qualitative aspects of the investigation were examined using interpretive analysis for patterns and themes on learner difficulties. The procedure involved scrutinising audiotape recordings and transcriptions of the interviews and document reviews of examiners' reports for patterns, trends and themes in geometry learning difficulties experienced by learners at different Van Hiele levels. On completion of the whole process and with the help of an educational professional with extensive knowledge of Euclidean geometry, data from both quantitative and qualitative analysis were merged, compared and confirmed through triangulation to further maintain validity of the study.

Details of the analysis procedures for individual research instruments are provided below:

3.11.1 Quantitative data analysis

To answer the research questions, the analysis of the written test (see appendix C) was done using a table to determine learners' achievements and difficulties at different Van Hiele levels of geometric thinking. According to that analysis, learners were considered as having difficulties at specific Van Hiele level if they scored below 50% per question at each level. Firstly, using these criteria, test data were analysed graphically using a double bar graph across all Van Hiele levels per question in terms of achieved and not achieved 50% to give an overview of average learner performance (see figure 4.1, chapter 4). As a follow-up on that, a more detailed analysis of the test responses per question was presented based on the identified abilities and difficulties using a matrix table for each question at a given Van Hiele level, as shown below:

Table 3.2.1

Frequency distribution assigning learners' achievement levels to each Van Hiele level/question

Learner's achievement levels	**Numbers achieved**	**(%)** **Achieved**	**Numbers not achieved**	**(%)** **Not achieved**
Visualisation level (0)				
Analysis Level (1)				
Informal deductive Level (2)				
Formal deductive Level (3)				

Using this frequency distribution table, the researcher analysed the actual learner competences against the expected level of achievement at each Van Hiele level to determine whether the learners possess the required conceptual knowledge at each level. This enabled the identification and explanation thereof to the learning difficulties per question, in terms of learners' geometrical errors, misconceptions and underlying learning difficulties. Those identified misconceptions and the resultant errors were analysed in terms of Watson's (1980) error classification model (see table 2.1, section 2.4.6.2). That analysis was used for different test questions to analyse difficulties related to motivation and comprehension as identified from learner responses at different Van Hiele levels. In addition, a comparative test analysis was done in terms of their abilities and difficulties at visual and analysis levels and the data were presented using a bar graph (see figure 4.6).

Overall, for the purpose of drawing conclusions from the pen-and-paper test responses, thematic analysis was used to locate, analyse and report patterns of learners' difficulties across questions and across Van Hiele's levels.

The questionnaire data were analysed to determine participants' level of agreement and disagreement (measured in percentage) to the possible learning difficulties at different levels of Van Hiele's model. In addition, analysis was done through determining the trends of both learners' and educators' perspectives per question or a cluster of questions, and then quantified into the number of learners that strongly, agreed, disagreed and strongly disagreed (indicated as percentages). As a result, the identified learning difficulties related to different Van Hiele's levels were then corroborated with the learning difficulties identified from the test data from questions set at those levels. Furthermore, the analysis of questionnaire responses was done by determining the level of agreement and disagreement by participants on difficulties related to themes like visualisation, geometry terminology, geometric proofs, logical thinking, where respondents' perspectives had to be shown by ticking the appropriate box (see appendices D, E & F). From the analysis of mean responses for sampled question(s), that data were used to supplement and authenticate the learning difficulties identified in the test. For analysing the questionnaires as mentioned above, a matrix table used at each Van Hiele level is shown below:

Table 3.2.2

A questionnaire matrix for learners' levels of agreement and disagreement proposed difficulties

<table>
<tr><th colspan="2">Van Hiele's level-based questions</th><th>Strongly Agree (%)</th><th>Agree (%)</th><th>Disagree (%)</th><th>Strongly Disagree (%)</th></tr>
<tr><td></td><td>Visualisation</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>Analysis (terminology)</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>Deductive reasoning (proofs)</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>Average response/question</td><td></td><td></td><td></td><td></td></tr>
</table>

As part of the data triangulation process, mean percentage responses identified for each question or a group of related questions was used to explain the types of difficulties learners encountered at Van Hiele's levels, as identified in the test. In addition, those levels of agreement and disagreement by percentage responses were used to validate learning difficulties identified in terms of geometry errors and misconceptions held by learners from the test.

Furthermore, a similar matrix as that given above for learners was used for the educators' questionnaire where they were required to indicate their views to the problems experienced by learners in their understanding of Euclidean geometry. Educator responses to the questionnaire were analysed in terms of how their level of agreement and disagreement to proposed difficulties at each Van Hiele level correlated with those given by the learners. Thus, patterns/trends were also established from educators' questionnaire responses by looking at the questions that had the highest average reposes. From those averages, the researcher was able triangulate that data with test and interview responses to answer the research question(s). On the other hand, the responses of the school departmental heads were analysed in terms of their general perspectives to poor performance at different Van Hiele levels. The questionnaire for educators, especial section B were analysed in terms of what they consider to be the contributing factors to geometry difficulties that the learners showed in the written test.

3.11.2 Qualitative data analysis

Interview schedules for learners were done based on purposive sampling as stated in section 3.5.4 where only learners with the requisite knowledge of the challenges were interviewed. For an analysis of interview data, audio tape recordings and transcriptions of verbatim quotes from learners and educators were considered as data sources. The focus of the interview questions and the analysis thereof were done on participants' perspectives to their challenges at each Van Hiele level as presented in the test for learners and from their interactions with learners for educators. The analysis of interview responses was done by considering the general trend in their learning difficulties at each level: visualisation, analysis, informal deduction and formal deduction level; together with the possible reasons they gave for those learning difficulties. Thus, verbatim transcriptions extracts were presented at each Van Hiele level of the test question, after which a brief explanation of the identified trends in learner difficulties was given as part of data analysis. Furthermore, the analysis of interviews responses was done to find the extent to which learners could visualise and analyse geometry diagrams, and possessed the requisite informal and formal deductive knowledge as presented in appendices G and H. This was the basis for data triangulation in line with the mixed methods design chosen by the researcher.

Furthermore, at each Van Hiele level of data analysis, learners were considered to have failed that level upon failing to demonstrate a certain domain-specific level of competences, for example, the expected level of achievement, according to Van Hiele, at visualisation, analysis, and so on. Therefore, interview responses were analysed with reference to those expectations, enabling the researcher to determine whether learners were competent at a given Van Hiele level from their interview responses.

Just like the analysis of learners' interview responses, educators' interview responses were also analysed in terms of how the responses agreed with those given by learners. The researcher managed to establish the difficulties that grade 12 learners experience in geometry from the analysis of interview responses from both the learners and educators.

3.11.3 Integrating quantitative and qualitative analysis

The researcher used an integrated approach to analyse quantitative and qualitative data concurrently. Thus, the researcher used multiple data analysis methods to analyse the data including a pen-and-paper test, questionnaires and verbatim interview transcriptions. The implementation of that approach included the use of numerical methods where data were analysed using frequency distribution tables, bar graphs and pie charts for quantitative data. On the other hand, thematic analysis was used for interview schedules to cater for the qualitative aspects of the research because it emphasises, pinpoints, examines and records patterns (themes) in data sets. Those qualitative patterns were analysed in terms of participants' perspectives to the learning difficulties exhibited by most learners at a given Van Hiele level, together with the factors responsible for those learning difficulties. In this way, the researcher used the integrative approach by corroborating quantitative and qualitative data. The research achieved integrative analysis by using qualitative data to confirm and authenticated quantitative data. That was done by analysing the perspectives to geometry learning difficulties, based on their attitudes, geometric experience, levels of motivation and confidence levels of both learners and educators in Euclidean geometry to explain the learning difficulties identified from the test.

3.12 Issues of reliability and validity and trustworthiness

Validity is regarded as the degree to which inferences made on numerical scores are suitable, meaningful and useful to the sample (McMillan & Schumacher, 2014). In addition, validity seeks to explain whether a particular research instrument measures what it is intended to measure. Thus, the researcher considered not only the instruments used, but also the participants and the degree of triangulation, as they all form the basis for validity (Winter, 2000).

In this investigation, both construct and content validity was taken into consideration to determine if the test, questionnaire and interview questions really measured the concepts they purported to measure. Even though validity cannot be absolutely achieved in real terms, the researcher tried to maximise it through careful sampling, choosing the rightful instruments and employing appropriate data analysis procedures. This investigation attempted to maintain construct validity, as alluded to by McMillan (2014), by ensuring the optimum level at which interventions and measured variables represent intended, theoretical and underlying psychological constructs and elements. That was upheld in this study by ensuring that there is a balance between the subjects, the instruments used and the procedure for collecting data.

In addition, to ensure content validity of the research instrument(s) items, the researcher consulted the curriculum, Grade 12 textbooks and past examination papers relating to the geometry content Grade 12 level learners were expected to know, and then requested senior members in the mathematic department comprising school mathematic HoDs to assist with instrument validation prior to administration to participants. Trustworthiness was promoted by triangulation, member checking and consultation with experts. These mathematics experts examined the questionnaires and the test items to ensure content validity.

On the other hand, Cohen et al. (2007) and Sürücü and Maslakci (2020) emphasise that reliability can be viewed as the relationship between the researcher's recorded data and what really happened in the natural setting where the research was done. In this study, reliability was ensured by triangulating data captured from written responses and correlating it with interview responses. Thus, in this study, the researcher ensured credibility through triangulation, where similar questions were asked to different participants (learners and teachers) to obtain data from a variety of sources using different methods to answer the same researcher question(s). The same notion was also conceptualised by Cohen et al. (2007) who point out that validity and reliability are enhanced if participants provide honest and well-thought-out answers, hence the collection of rich data.

To further enhance credibility, researcher also used member checking, where interview responses were taken back to participants for confirmation and validation of the data supplied. Thus, these credibility checks were applied to the qualitative aspects of the study by allowing interviewees to double-check their statements and close gaps from earlier interviews. In the same way, to ensure validity and reliability of data collected from the geometry test, the researcher drafted questions from externally set Grade 12 papers to ensure good quality items.

Ethical considerations are discussed in the next section.

3.13 Ethical considerations

Academic research is viewed as a "scientific human endeavour that is organized according to a range of protocols, methods, guidelines and legislation" (Gerrish & Lacey, 2010, p. 15). In this respect, Creswell (2014) argues that researchers are obliged to respect the needs, values, rights and desires of the participants. Thus, to ensure that the study was conducted ethically, the above-mentioned was be taken into consideration. Similarly, ethical considerations are relevant in situations where one conducts face-to-face interviews with a vulnerable group of respondents like learners. In this study, the position of the researcher was that of inquiry and confidentiality, thus the researcher signed a code of ethical conduct before commencing the data collection process. Neuman (2003) argues that the researcher bears both moral and professional imperatives to be ethical, even under circumstances where the participants are not well informed or do not care about the importance of ethical considerations.

Before conducting the study, the researcher received ethical approval from various authorities, which included ethical clearance from Unisa, certificate number 2021/02/10/49266764/10/AM approval from the Ngaka Modiri Molema district offices and approval from principals of participating schools (see appendices A & B). Parents whose children were considered to be minors (below 18 years) were requested to voluntarily sign consent letters for their children. Details of the study, together with the data collection procedures, were explained in those consent letters (see appendices J & K) and none of the research participants were forced to take part in this research.

Furthermore, the researcher explained to participants the probable associated risks and discomforts and gave them complete assurance of anonymity and confidentiality of all responses to tests, questionnaires and interviews**.** Participants were briefed of the importance of the study and were kindly requested to cooperate throughout the research process. Polit and Beck (2014) argue that informed consent and voluntary participation are important, as they enable participants to understand the information and allow them the freedom of choice of whether to participate or decline. As result, participants were informed that their participation was voluntarily and that they were free to withdraw their participation at any stage of the study if they felt it necessary and not reasons for quitting would be required from them. The researcher promised the respondents that their names and identities would not be revealed during data collection, data analysis and reporting of results; hence, anonymity and confidentiality of participants' responses were guaranteed.

To facilitate of the above-mentioned protection of the research subjects, the researcher had to ensure that data collected were analysed in terms of the research objectives and were treated in a highly confidential manner. This was with specific reference to the respect for anonymity of participants (learners and educators), together with their respective schools. Similarly, the researcher had to ensure that his own biases and personal prejudices were limited. To ensure adherence to the ethical protocols mentioned above, various codes such as L1, L2, L3 to identify learners; T1, T2, T3 to identify educators; H1, H2 to identify heads of mathematics departments; and S1, S2, S3 to identify the selected schools were used for easy identification and for ethical reasons. These codes were used to identify participants in the analysis of their responses in the tests, questionnaires and interviews. In this respect, ethical considerations were very important for the researcher during data analysis.

Moreover, to ensure that there was no conflict of interest associated with working with participants known to the researcher or at his workplace, the researcher followed and fully implemented the code of ethical integrity, which is binding on all educational researchers, regardless of how close or familiar one is to the participants. The researcher had to stick to universal research protocols of impartiality, fairness, honesty, anonymity and confidentiality, regardless of the nature of participants. The researcher had to remain objective by sticking to the research objectives and not allowing subjective feelings to interfere with the research process. Impartiality was minimised by only working within the confines of questions set in the data collection instruments and the researcher attempted to be objective in the process of administration and interactions with participants.

Hard copies and soft copies of the participants' responses would be kept by the researcher in a safe place for a period of four years for future academic reference, while all information captured electronically would be stored on the researcher's computer with password protection. Thus, any future used of that data in cases where it was applicable would be subject to approval and further assessment by the Ethics Review Committee. After the period mentioned above, hard copies would be burned, and electronic copies deleted permanently from the computer by using relevant software.

3.14 Limitations

Despite the possibility of the research results being compromised in terms of generalisability to the larger population because of relatively smaller samples that were used, the researcher ensured diversity in the selection of participants for the results to be more authentic. Another limitation was inherent in self-reporting instruments like the interviews, where the researcher relied on the truthfulness of participants in their responses. To cater for that, the research had to ensure that the instruments met all the validity and reliability standards by seeking expert knowledge in the design stage. The data collection process was limited because of the Covid-19 pandemic because both the researcher and the respondents had to interact within the confines of the 'new normal' characterised by the need to follow strict health protocols to minimise chances of contracting the Coronavirus. Free interactions that are characteristic of educational investigations were limited because of the fear of contracting the disease during the process. Even in the process of requesting permission from school principals to undertake the study, some of them were very sceptical to allow administration of research instruments at their schools because of Covid-19 issues. As a result, the research had to provide details of how learners' safety was going to be assured before permission was granted.

3.15 Chapter conclusion

The focus of this chapter was on the research design and methodology, together with a brief description of interpretivism and constructivism as paradigms within which the study was orientated. The design of the study was explained as a mixed methods design comprising both quantitative descriptive design and qualitative case study design on a total of 60 mathematics learners, 12 educators and four mathematics school HoDs. The overall instrumentation (test, questionnaires and semi-structured interviews), sampling procedures, pilot study, data collection and analysis procedures were explained alongside the research ethics and validity measures. Overall, the next chapter discusses the results of this investigation, their analysis and the presentation of evidence of data collection in the form of samples of participants' written responses.

CHAPTER 4: DATA ANALYSIS AND INTERPRETATION

4.1 Introduction

The previous chapter discussed how data were collected. In this chapter, the analysis of the learning difficulties experienced by Grade 12 learners in Euclidean geometry is presented through participants' responses to the following research instruments: a pencil-and-paper test, questionnaires, semi-structured interviews and anecdotal/examination diagnostic reports (documentary reviews). Thus, predetermined data categorisation was used as the basis for analysis and interpretation of the research data. In view of that, Renner and Taylor-Powell (2003) suggest that data analysis based on pre-set categories involves starting with a list of themes or categories before embarking on the actual categorisation of data, from where data are then searched for text that matches those categories. In the case of this study, the predetermined categories included the Van Hiele's levels from which data were collected to provide insight into the following research questions that guided the study:

1. What are the Grade 12 learners' learning difficulties related to visualisation in Euclidean geometry?
2. What are the Grade 12 learners' learning difficulties related to the analysis level in Euclidean geometry?
3. What are the Grade 12 learners' learning difficulties related to the informal deduction level in Euclidean Geometry?
4. What are the Grade 12 learners' learning difficulties related to the formal deduction level in Euclidean geometry?

As a result, the research questions as set at each Van Hiele level provided direction regarding what to look for in the research data. This enabled the researcher to explore the learning difficulties by extracting from the data gathered geometric statements and phrases that relate to the research questions.

Furthermore, data analysis was informed by the theoretical framework of constructivism, as illustrated in the methodology chapter. Descriptions of participants' responses in each of the above-mentioned research instruments are also presented. Furthermore, the findings as demonstrated through participant responses are presented using tables, bar graphs, pie charts and narrations to answer the research questions. In addition, the findings are discussed with reference to what the literature has exposed.

4.2 Demographic characteristics of the participants

This section provides information of the teacher participants in terms of (gender, age, qualifications and experience) and learners' participants (age, gender) as indicated below:

Table 4.1

Distribution of educator respondents by gender, age, qualification, experience and designation

Educator code	Gender	Age	Qualifications	Experience	Designation
E1	Female	36	ACE, Bed	13	PL 1
E2	Male	34	BSc, PGCE	12	PL 1
E3	Male	47	UDE, Bed.	21	PL 2 (HOD)
E4	Female	45	UDE, Bed	18	PL 1
E5	Male	38	BSc, PGCE	15	PL 1
E6	Male	52	UDE, ACE, Bed.	28	PL 2 (HOD)

Table 4.1 above; shows the demographics of educator participants in terms of gender, ages, qualifications, teaching experience and their designations. The data above as presented in table 4.1 indicate that the researcher was both gender and age sensitive in the choice of educators who took part in this study. Thus, two females and four males constituted the educator sample. That was done to limit the impact of gender biases and was associated with the belief the male educators are more conversant with geometry than their female counterparts. Similarly, there was a consideration of maturity in terms of the ages of the selected educators to only those above 30 years based on the assumption that maturity of educators would positively influence the quality of the content knowledge through experience and of their interpretation of the geometry learning difficulties encountered by the learners they teach.

More importantly, data on educator qualifications and their experience in geometry teaching was considered relevant, as the two directly impact on the amount of geometrical knowledge the researcher assumed that educators acquired over time through teaching geometry. Furthermore, data the presented in table 4.1 also revealed that all educators in the sample had at least an education degree, thereby bolstering the researcher's assumption that these educators had the requisite geometry content knowledge and were more likely to present an authentic and academic standpoint in their interview responses to learners' geometry learning difficulties. Lastly, data on educator demographics in terms of their designations represented the need for educators to interpret the learning difficulties from both their experiences with learners as classroom practitioners and from a management perspective as HoDs for a more balanced view to answering the research question.

Furthermore, the distribution of learner respondents to the study by gender and ages is presented in table 4.2 below:

Table 4.2

Distribution of learner respondents by gender and age

Ages	**Males**	**Females**	**Total**
17	0	2	2
18	13	19	32
19	10	11	21
20	3	1	4
Total	**26**	**34**	**60**

The results from table 4.2 above show the distribution of learner participants by age and gender even though literature reveals that the Van Hiele levels are not age dependent (Van Hiele, 1986). In this instance, it was important that the sampled learners were in the same age ranges for the purpose of uniformity. Similarly, the same age distribution was important to ensure that the findings would not be compromised by maturity and exposure of learners in terms of their geometry conceptualisation. As a result, the data presented on the table showed that most of the learners were between 18 and 19 years, which is the right age for grade 12 learners.

The implication was that those learners were at the right age to be taught Grade 12 geometry. Therefore, with all other factors kept constant, those learners were expected to be operating at a similar level of competences in geometry, making it much easier to explore the learning difficulties. In terms of teacher interactions with these learners, they are less likely to experiences discipline issues as most of the learners were not over or under-aged to be in grade 12.

Furthermore, table 4.2 shows the distribution by gender where the number of males to females sampled was almost the same. That distribution was intended to eliminate biases to results that are associated with gender stereotypes, even though the sample had a number of females slightly higher than males. That was assumed not to have a noticeable impact on the findings since such a phenomenon is characteristic of South African classroom gender distribution where the number of girls is slightly higher than that of boys. The researcher purposefully selected this gender distribution for it to be representative of a normal grade 12 class.

4.3 Presentation and interpretation of the findings

In this chapter, the researcher used research questions as the reference points to present and interpret the data. Data were collected with different research instruments (pencil-and-paper test, questionnaire, interviews and documentation) to explore learning difficulties in geometry as experienced by grade 12 learners. From those research instruments, the researcher considered the test as the major source of data to determine learner abilities and learning difficulties in Euclidean geometry for several reasons. Firstly, the test was considered as the major source of data for this study by enabling the presentation and interpretation of geometry questions in terms of the different Van Hiele levels. In addition, the test was the most convenient research instrument through which data on learner abilities and difficulties in geometry could be gathered directly from learner responses. Thus, choosing the test as the major source of data enabled the researcher to set the question at different levels of difficulty according to the Van Hiele model, making it easier to categorise learning difficulties into different reasoning levels for easy analysis. Similarly, the test provided primary data for the researcher to analyse learning difficulties, as they are experienced by learners through writing.

Furthermore, the test was considered the most relevant source of data because it enabled the researcher to compare expected learner achievements at each Van Hiele level to the actual learner achievement per question. Thus, using the test as the major source of data enabled the researcher to directly gather quantitative information to be interpreted through triangulation with data from other sources like interviews, questions and document analysis. It was relevant to use the test as the major data source because the researcher believed that having collected test data, it would be easier to supplement and authenticate the identified geometry learning difficulties with data from other sources.

Moreover, the details of the process followed are provided below:

Firstly, the overall learner performance in the test for the six schools (S1 – S6) and for learners (L1 – L60) is presented. Furthermore, averages for learner performance for the test as percentage were also analysed in relation to national performance graphs, as illustrated in figures 1.1 and 1.2 to establish any correlation between research test averages to the national geometry performance averages. In addition, also presented as part of data analysis were the learners’ sampled responses to the above-mentioned test items for each of the six test questions as evidence of the challenges that the learners encountered in solving Euclidean geometry problems. Therefore, the test results were analysed based on the Van Hiele levels of geometrical thinking to establish Grade 12 learning difficulties, in accordance with the levels of geometry thinking in the hierarchy.

Table 4.3

Overall learner performance as a percentage for the geometry test

S1 %		**S2 %**		**S3 %**		**S4 %**		**S5 %**		**S6 %**	
L1	29	**L1**	42	**L1**	33	**L1**	40	**L1**	29	**L1**	35
L2	18	**L2**	36	**L2**	15	**L2**	32	**L2**	55	**L2**	58
L3	18	**L3**	36	**L3**	24	**L3**	76	**L3**	27	**L3**	43
L4	36	**L4**	29	**L4**	11	**L4**	74	**L4**	16	**L4**	61
L5	38	**L5**	07	**L5**	11	**L5**	20	**L5**	20	**L5**	31
L6	29	**L6**	31	**L6**	20	**L6**	43	**L6**	18	**L6**	42
L7	46	**L7**	24	**L7**	18	**L7**	12	**L7**	49	**L7**	18
L8	05	**L8**	20	**L8**	13	**L8**	34	**L8**	20	**L8**	72
L9	24	**L9**	22	**L9**	25	**L9**	12	**L9**	09	**L9**	61
L10	05	**L10**	42	**L10**	20	**L10**	54	**L10**	15	**L10**	25
AVERAGE/ SCHOOL	**24.8**		**31.3**		**19.0**		**39.6**		**25.8**		**50.8**

KEY:
S1 -S6 (sampled schools)
L1 – L10 (Learner code/per school)

Table 4.3 above gives an overview of the marks as percentage for the 10 learners from each of the six selected schools. These results indicated that 58.3% of the learners who wrote the test scored less than 30%, with only 41.2% managing to pass according to the CAPS grading system, the pass mark of which is 30%. This data seemed to indicate that learners were not performing well in Euclidean geometry.

The results further indicated that, on average, most learners performed below 50% in geometry. In the same vein, these averages seemed to tally with national statistics, which revealed that the average performance for geometry questions was below 50% in the National Senior Certificate examination diagnostic reports (2016-2020). In view of that, one can conclude that learners do have difficulties in Euclidean geometry.

An overview of the average performance of the sampled schools showed that only one school achieved, and the other five schools did not achieve 50% overall. This is worrisome, because the other five schools performed below 40%, which was an indication that learners were experiencing learning difficulties in geometry. Such geometry performance, as indicated by these figures, seemed to tally with national averages as revealed through document analysis of previous annual reports (figure 1.2 chapter 1) (DBE, 2016-2020). Therefore, both document reviews and results from the test administered to learner respondents indicated that Grade 12 learners' performance in geometry has been consistently poor. Thus, the level of learner performance as indicated by the results in table 4.3 above was a confirmation that they were experiencing learning difficulties in geometry.

Furthermore, learner performance was analysed per question for the purpose of determining the Van Hiele levels at which the learners are operating, together with the learner challenges associated with each level. The table below presents data on the number of learners that managed to reach the 50% mark for each question, as stated in the researcher's criterion for determine concept mastery of each Van Hiele level in chapter 3, section 3.7.

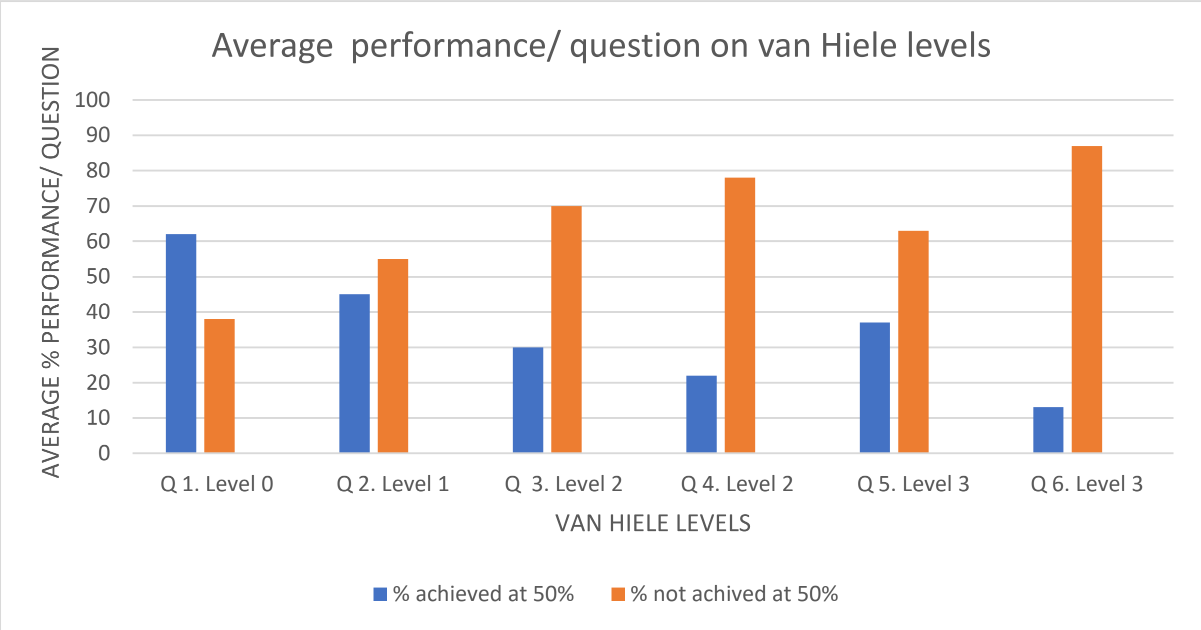


Figure 4.1

Analysis of average learner performance per question according to the Van Hiele levels

Figure 4.1 above indicates a general trend in the number of learners who achieved and those who did not achieve 50% to test questions administered to learners that related to different Van Hiele levels. Thus, figure 4.1 above indicates a general decline in learner performance as the cognitive levels of the questions increase along the Van Hiele hierarchy of geometrical thinking. In this respect, data gathered indicate that 37 learners (62%) mastered question 1, which was set at visualisation level 0, with 23 learners (38%) not achieving at the same level. Moreover, question 2 was set to explore learner challenges at the analysis level 1, of which data gathered indicated that 22 learners (47%) demonstrated a good mastery, as compared to 28 learners (53%) who failed to score the set target achievement of (50%) and above.

In the same vein, questions 3 and 4 were set to explore learner abilities and the associated learning difficulties at informal deduction level 2. Therefore, data gathered from learner responses to question 3 showed that 20 learners (30%) achieved at this level and 40 learners (70%) failed to score a pass mark of 50%. Just like question 3, question 4 was set at analysis level, but at a higher cognitive level within level 2, as it had more lines and properties to be explored than question 3. As result, only 13 learners (22%) achieved at this level against 47 learners (78%) who did not achieve at analysis level.

Furthermore, questions 5 and 6 were set at the formal deduction level 3. In the case of question 5 (proof of a theorem), 22 learners (37%) showed a good understanding of the concepts involved in the proof, as compared to the other 38 learners (63%) who did not achieve at 50%, as set in the test scoring criterion. Lastly, question 6 was set to explore learner competences at the application level of deductive reasoning where evidence from data gathered. It was found that about 8 learners (13%) of the respondents achieved at this level, as compared to the other 48 learners (87%) who did not achieve a minimum set (50%) pass mark per question.

Overall, from the data gathered as shown in the figure 4.1 above, there is a general decline in learner achievement as the cognitive levels of the questions increase along the Van Hiele's hierarchy of geometrical thinking (blue bars) as compared to a general increase in the heights the orange bars (not achieved). Therefore, such an inverse relationship between the blue and orange bars may be an indication of a possible incremental existence of learning challenges as learning progresses from concrete to abstract geometric thinking.
Therefore, the next section presents a detailed question-by-question analysis of learner responses to the test items at different Van Hiele levels with the aim of answering the research questions as presented in section 4.1 above.

4.4 Grade 12 learners' current level of abilities and difficulties in Euclidean geometry problem solving, with specific reference to visualisation, analysis and deductive reasoning (proofs) in the domain of lines, angles and circle theorems

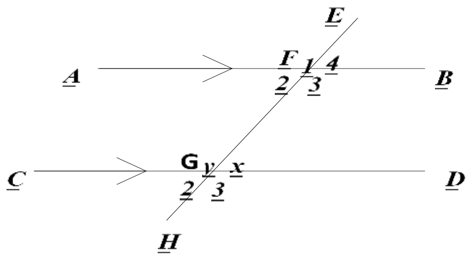
To determine learners' current level of abilities, the test administered consisted of questions in accordance with Van Hiele's levels. As a result, the researcher presents learners' abilities and difficulties related to each level as per the test questions.

4.4.1 Learning difficulties related to Van Hiele's visualisation

Learning difficulties related to Van Hiele's visualisation level consisted of one question (question 1) in the test administered, as shown below:

QUESTION 1

Consider the diagram drawn below.



1.1 Name two angles which are equal to x and explain?

(i)______________________Reason ________________________________

(ii) _____________________ Reason ________________________________ (4)

1.2 Name two angles which are equal to y

(i)_______________________ Reason ______________________________

(ii) _______________________Reason _______________________________ (4)

Expected solutions

(i) $x = F_4$ √ Reason $corresponding\ \angle s = AB \parallel CB$ √

(ii) $x = F_2$ √ Reason $alternating\ \angle s =\ AB \parallel CD$ √ (4)

1.2 Name two angles which are equal to y

(i) $y =\ F_1$ √ Reason $corresponding\ \angle s = AB \parallel CD$ √

(ii) $y = F_3$ √ Reason $alterneting\ \angle s = AB \parallel CD$ √ (4)

[8]

The purpose of the above question was to explore learners' basic geometrical knowledge related to angles, parallel lines and transversals. In this vein, the reason for setting question 1 at Van Hiele's visualisation level 0 was to enable the researcher to establish whether learners had the requisite visual competences at the basic level, since any insufficiencies in visual and observational skills at this level result in learners experiencing geometry challenges at all subsequent levels. Similarly, learners were asked basic knowledge about angles, parallel lines and transversals that gave the researcher a chance to determine their readiness to progress to the next level in the hierarchy. That is because learners needed to have well developed concrete visual competences before they were introduced to abstract thinking, hence the importance of question1.

Furthermore, question 1 was deemed suitable for exploring learners' current abilities, together with their learning difficulties at visual level 0. It required the learners to have what

Vorster (2012) refers to as 'the skill of the eye', which, in this case, was exploring learners' ability to recognise and name angles that are equal those given (x and y) from the diagram. It also required learners to state the position of the identified angles as corresponding, alternate or vertically opposite by visualising their position with respect to parallel lines and a transversal. Furthermore, at Van Hiele's visualisation level 0; properties of shape are not yet known and aspects of question 1 offered the researcher a platform to explore learners' basic geometrical knowledge of lines and angles, as they are the basis for understanding properties of shape at the analysis and the subsequent levels in the hierarchy. The data on the overall learner performance (achieved/not achieved) at Van Hiele's visualisation level 0 at 50% achievement level per question as set in the test scoring criterion (section 3.7) are presented in the table below:

Table 4.4

Frequency distribution of learners' abilities and difficulties for question 1 related to Van Hiele's visualisation level (0)

Learners' levels of difficulties in relation to Van Hiele's basic level	**Numbers of learners who achieved basic level**	**(%) achieved**	**Numbers of learners who did not achieve basic level**	**(%) not achieved**
Basic level (0) Visualisation	37	62%	23	38%

Table 4.4 illustrates learners' abilities and difficulties resulting from question 1 associated with Van Hiele's visualisation level 0. In this study, learners were regarded to have difficulties at Van Hiele's visualisation level 0 if they failed to:

- use intuition to build perception about angles between parallel lines and the transversal and not reasoning
- recognises angles equal to $x \text{ and } y$ by their position with respect to the parallel lines and the transversal, and not properties
- use terms like corresponding, alternate or vertically opposite or co-interior to describe the identified angles
- state the parallel lines for corresponding and alternating angles between parallel lines

- build intuition about parallel, co-interior and alternating angles from the visual appearance of angles and not properties.

The results as presented in table 4.4 above revealed that 37 learners (62%) managed to achieve 50% and above in question 1 at Van Hiele's basic level 0 (visualisation). This achievement rate of 62% showed that these learners had sufficient basic geometrical knowledge and were ready the progress to level 1 (analysis level). The responses from the scripts of learners who achieved indicated that most of them had a good mastery of visual and spatial relations. For example, those who achieved were able to recognise angles equal to (x and y) by referring to their respective positions with respect to parallel lines and the transversal without using their properties. Despite those learners achieving at 50% and above, some of them still had difficulties with integration of visualisation and visual thinking. Thus, some of the learners who achieved correctly identified the required angles but struggled to explain why those angles were equal to those given.

On the other hand, 23 learners (38%) did not achieve 50% or higher at Van Hiele's basic level 0. It was seen that those learners who did not achieve experienced difficulties with all the above-mentioned expectations at Van Hiele's visualisation level 0. Most of those learners could not use spatial skills to identify angles equal to x and y. Similarly, those learners failed to use intuition to build perception about the angles between parallel lines.
However, details of individual learner performance to question 1 are provided in the extracts below:

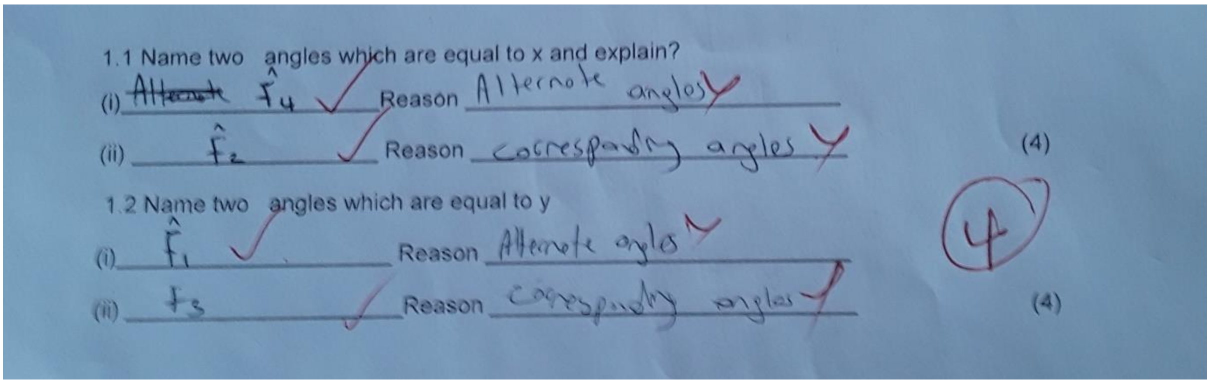


Figure 4.2

Extract from L1 (not specifying parallel lines)

The results from figure 4.2 above showed that even though L1 achieved at 50% and above, the learner still displayed some misconceptions in some aspects of that question. For example, the results as presented in figure 4.2 above revealed that L1 had the misconception that corresponding and alternate angles are equal, regardless of the orientation of the lines. Thus, the learner identified angles correctly but gave incomplete explanations. In addition, L1 made an error of not stating the parallel lines as part of the reason to her statement. Thus, despite having achieved in question 1, L1 held the view that corresponding and alternate angles are always equal if there is a transversal to any two lines. That view was incorrect since such angles are only equal if the lines being intersected by the transversal are parallel to each other. As a result, learners like L1 ended up making an error by not stating the parallel lines as part of their reasons to statements.

Furthermore, L23 managed to achieve in question 1 but showed a misconception about co-interior angle as presented in the extract below:

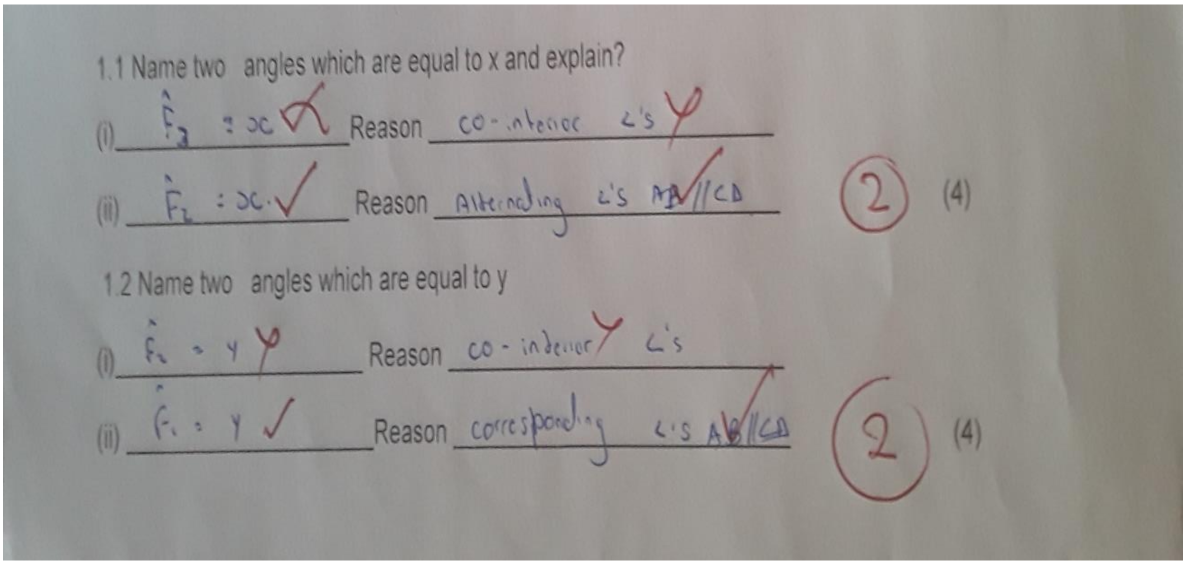


Figure 4.3

Extract from L23 (misconception on co-interior angles)

Figure 4.3 above shows that despite having achieved at 50% in question 1, L23's responses showed that the learner had a wrong conception of co-interior angles. For example, L23's responses to questions 1.1 and 1.2 revealed the learner had a misconception that since alternate and corresponding angles between parallel lines and the transversal are equal, the same applies for co-interior angle. Thus, the learner had a wrong conception of co-interior angles between parallel lines by viewing them as being equal instead being supplementary (add up 180°). Such responses were an indication that there were some of the learners who lacked basic visual skills and basic knowledge related to line geometry.

On the other hand, table 4.4 above shows that 23 learners (38%) of the sample did not achieve at the basic level 0 (visualisation). In this regard, the learners who did not achieve a level mastery of 50% and above showed that they lacked basic geometrical knowledge; hence, they encountered more serious geometry learning difficulties.

An example of such learners who did not achieve at basic level 0 is L37, whose extract for question 1 is shown below:

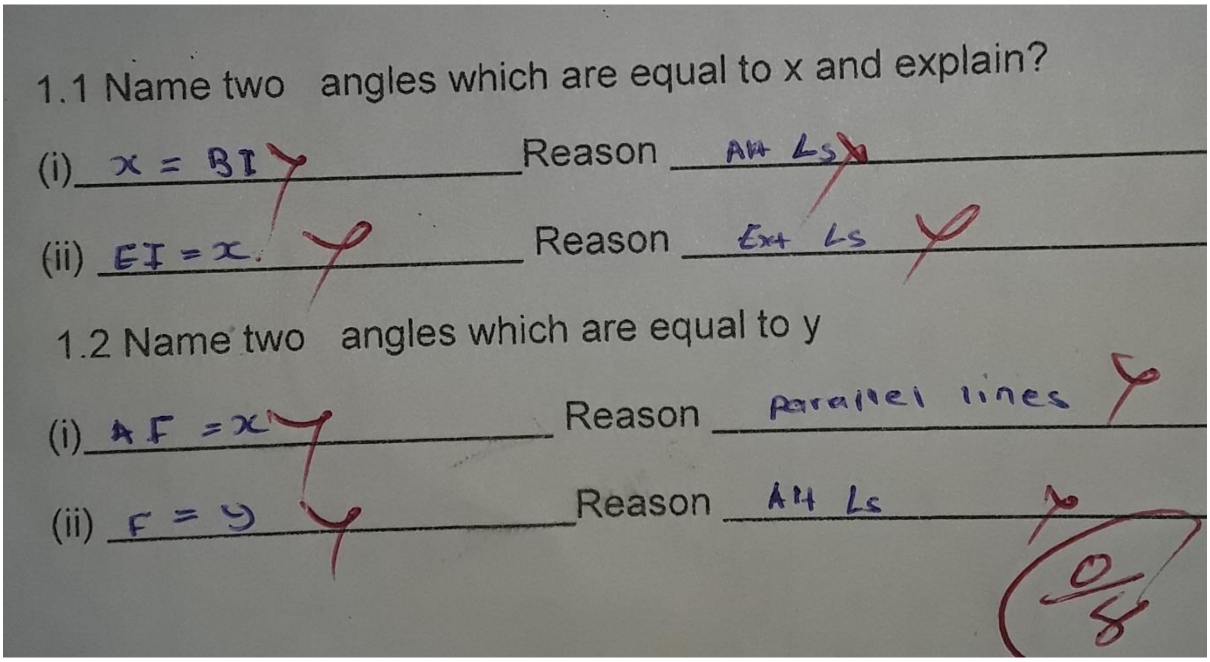


Figure 4.4

Extract from L37 (failure to differentiate a line from an angle)

Figure 4.4 above shows that L37 could not even recognise and distinguish an angle from a line. For example, the learner identified $EF\ and\ AF$ as being equal to angle$x$. Thus, the responses as indicated in figure 4.4 showed that the learner had challenges at visual level because he/she could not differentiate a line from an angle. The learner in this case lacked basic knowledge at visualisation level, which is 'naming' of shapes, lines and angles. From such responses, it was beyond any reasonable doubt that this learner had insufficient basis geometrical knowledge, possibly because both his/her concept image (mental image of the angles) and the concept definition were not connected. As a result, the research could safely conclude based on the above-mentioned challenges that L37 was not yet ready to progress to the next level (analysis).

Furthermore, data gathered from learners who did not achieve at basic level indicated that those learners had difficulties in recognising angles from the diagram that were equal to $x\ and\ y$ and could not state whether the angles were alternate, corresponding or vertically opposite. In addition, most of them did not state parallel lines for alternate and corresponding angles. Similarly, failure by learners to state parallel lines in their reasons to statements for corresponding angles, alternate angles or co-interior angles was an indication of their poor conception of the condition(s) to be satisfied for those statements to be true. Their answers showed that learners often made assumptions in geometric information, resulting in them making errors. As such, one can conclude that some of the errors that learners commit in geometry are a result of the assumptions in geometric information.

Therefore, reference to that is from figure 4.5 in the extract from L28 as shown below:

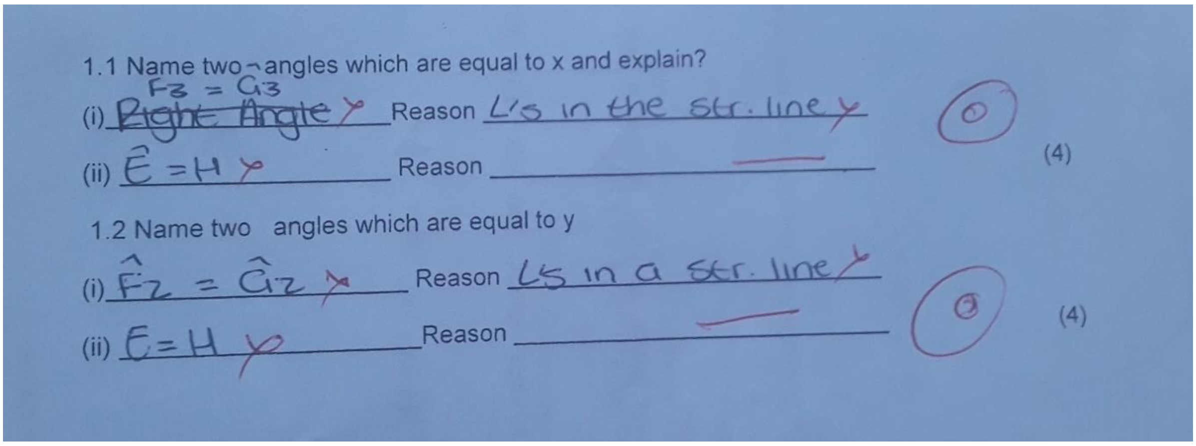


Figure 4.5

Extract from L28 (incorrect angles and reasons)

Figure 4.5 above is an example of a learner who did not achieve due to a lack of basic visualisation knowledge. Thus, L28 failed to identify angles equal to x and y and, as a result, the learner could not give correct reasons for the statements. Thus, the information as revealed in the learner's responses suggested that he lacked basic visualisation and spatial skills by not being able to recognise equal angles or identify angles occupying similar positions from the diagram. Such responses at basic visualisation level indicated that learners had serious gaps in conceptualisation of basic geometrical knowledge and could not progress to the next Van Hiele level of geometric thinking.

Moreover, through triangulation, the above-mentioned learner challenges from test responses at basic level 0 (visualisation) were then confirmed by their responses to the questionnaire. In this respect, an analysis was done of the key questionnaire question at visualisation, whose main purpose was to capture learner perspectives of challenges associated with naming different lines, angles, triangles and quadrilaterals. The results were as indicated in table 4.4 below:

Table 4.5

Frequency distribution of learners' responses to the questionnaire showing their level agreement/disagreement to difficulties associated with question 1 at Van Hiele's visualisation level

VISUALISATION		**Strongly Agree**	**Agree**	**Disagree**	**Strongly Disagree**
2.1	I have challenges with naming different lines, angles, tringles and quadrilaterals	4 (7%)	16 (27%)	31 (52%)	9 (15%)

From table 4.5 above; the questionnaire data indicate that 20 out of 60 learners experienced difficulties at the visualisation level 0. Of those 20 learners, 16 (27%) agreed and 4 (7%) strongly agreed that they experienced challenges related to naming different lines, angles, tringles and quadrilaterals. In addition, the data indicated that there were those learners who seemed not to have challenges with visualisation (naming geometric shapes). In this regard, 52% of the learner respondents disagreed and 15% strongly disagreed to having any challenges with naming geometric shapes. Therefore, an overview of the above percentages suggested that despite some learners (34%) having challenges with aspects of visualisation, a large number (66%) of the respondents to questionnaire item 2.1 regarding visualisation had a good mastery of basic geometrical knowledge. Thus, the difficulties those few learners experienced at visualisation level showed that they lacked basic geometrical knowledge from previous grades. This could be the reason why more or less 38% of the learners did not achieve in question 1 of the test at basic level, as indicated in table 4.5.

In view of the above, interview data gathered from a group of learners (for L28, L1, L42 and L44) who were purposefully sampled from those who did not achieve 50% and above in the written test corroborated the difficulties related to visualisation level. Some of those learners' perspectives to the challenges they had at Van Hiele's basic visualisation level 0 are resented in the quotations below:

> **L28:** *I know triangles I can see a cyclic quadrilateral if it is drawn inside a circle. I can even name other angles equal to x but, I get confused when there are many lines in the diagram, especially writing those things of corresponding, alternate and co-interior. We did them in grade 10 and I know them, but sometimes I mix them.*

L1*: Sometimes diagrams are complicated, especially those with circles. I know vertically opposite angles are equal and I know that opposite angles are formed when lines cross each other. My problem is those co-interior and corresponding angles; if there are many shapes inside a circle.*

L42*: I don't have a problem with angles; we did them from grade 9, especially those that form* $Z \text{ and } F$ *and the transversal line. I know they are called alternate and co-interior. Sometimes I forget to write the parallel lines in my answers. I only have a problem with identifying different shapes like triangles and quadrilaterals inside those geometry diagrams … We did shapes even from primary, but my problem started in Grade 11 when the shapes were drawn inside a circle with those many lines. It is difficult for me see different shapes from those lines. I get confused.*

L37: *Those things of parallel lines only did in grade 8. Our teacher only gave us a few examples and said you do with them in higher grades. At grade 11, the teacher just started with the theorems. So, for that question in the test about parallel line, I was using what they told us in grade 8 and I was not sure about those angles and reasons; that's why I got those things wrong. Geometry is really a problem to me, but I also think our teachers don't teach us well, and they are always in a hurry, and we don't understand.*

L20: *I get confused because during the lesson, we understand, but when I get home and try the same things, I remember nothing! At least with parallel lines, I can see those angles like vertically opposite, and corresponding, but sometimes I confuse corresponding and co-interior angles and I lose marks. My problem is those shapes like triangles and quadrilaterals when they are … and they ask me to calculate angles; it's like, we don't understand anything at all. Ooh! Worse when those shapes are inside a circle, they confuse me more. I think those theorems are … that's why I am a little bit confused.*

From the sampled learner interview transcriptions above, it is evident that despite most learners having a good mastery of the basic visualisation level, there were some learners who still had gaps in their understanding of concepts in theorem group lines and angles, especially when they are drawn inside the circle. In this vein, L28's interview response as presented in the extract above probably suggested that despite having basic visualisation skills, especially on stand-alone shapes, his ability to correctly identify and name angles diminished with an increase in the number of lines within a geometric diagram. The same sentiment was echoed by L1, who also lamented experiencing visual challenges with the identification of co-interior and corresponding angles, especially if they are integrated in a circle and when many other lines are involved.

Furthermore, these interview responses as presented by the learner quotations above suggested that some of the challenges learners experienced with basic geometrical knowledge were possibly a result of poor concept formation in lower grades. This assertion was confirmed by the general trend in learner responses where they concurred to having basic geometrical knowledge, but they all seemed to be lacking fluency in their conception. In addition, these interview responses confirmed the challenges identified from the test whereby most of the learners who did not achieve in question 1, struggled to correctly identify angles equal to$x\ and\ y$. From their interview responses, those learners also seemed to concur that visual challenges become more pronounced if a circle is integrated into the framework of the ordinary shapes they are used to, resulting in them failing to correctly identify angles and recognise geometric shapes.

Moreover, a common trend in the above quotations was that there were some learners who still lacked the requisite visual and spatial skills for geometry problem solving. That researcher's view was affirmed by Ferrara and Mammana's (2014) findings, which indicated that learners who have limited spatial skills and visual thinking skills experience difficulties in their approach to geometry problems. Similarly, Barut and Retnawati (2020) found that insufficiency in providing proper visual reasoning capabilities in the interpretation of geometric diagrams often results in learners experiencing geometry challenges.

On the other hand, educators presented their own perspectives of the possible challenges that learners at grade 12 encountered at visual level, together with the possible factors related to those challenges. To that effect, transcriptions from sampled educator interviewees are presented below:

> ***E2:*** *From my observation, most learners seem to understand the parallel-transversal lines, triangle geometry when diagrams are separately structured but if you integrate and infuse them, they get confused, especially recognising the angle-lines relationships, for triangles and quadrilaterals, especially if they are inscribed by a circle … My understanding and experience is that learners enjoy and understand algebra from infancy, and I can confirm that through instruction learners don't develop enough visualisation and visual thinking, possible because educator put more emphasis in analytical skills rather than visual, spatial and logical skills that are essential in geometry.*
>
> **E4:** *Most learners mix up concepts used in parallel lines and triangle geometry, and that has something to do with their visualisation skills; for example, where they are*

supposed to say co-interior angles, they identify them as corresponding angles. As for triangles, most of them struggle to distinguish similar triangle from congruent triangle due to poorly developed visualisation skills … These learner challenges are possibly because learners are not exposed to concrete objects, for example the use of paper cuts, geoboards, use of technology to manipulate geometric shapes, in addition to involving learners in the actual construction of geometric shape as these improve their visual and spatial skills. And, during instruction, as educators we tend theories too much hence learners fail to develop basic geometry skills.

The above quotations from educators affirmed the participants' assertions that there were learners that still had difficulties with basic geometry concepts at Van Hiele's visualisation level. The above-mentioned interview responses from educators suggested a lack of development among learners' visual and spatial skills. That is evidenced by responses from E2 and E4 where both attributed learner challenges at visualisation level to poor development of visualisation skills. In addition, educator E2 suggested a lack of active learner participation and manipulation of with real geometric objects as one of the reasons for poor development of learners' visualisation skills. Furthermore, the interviewed educators attributed the lack of basic geometry skills by learners to a lack of geometry fluency at visualisation level, where learners fail to link visualisation to visual thinking. For example, E4 attributed such difficulties to educators not giving learners the opportunity to work with concrete objects to reinforce conceptualisation of knowledge at basic level (visualisation).
Furthermore, through their interview responses, educators argued that learners experience visualisation difficulties when many lines are integrated in one geometric diagram. In view of that, educators emphasised the need for learners to acquire basic geometrical knowledge at visualisation level because poor development of such knowledge results in them experiencing difficulties in the successive Van Hiele levels.

Furthermore, data gathered from learners' replies to question 1 as presented in table 4.5 above indicated that 38% of the sampled learners did not achieve at the basic Van Hiele level 0 and the percentage was reasonably lower when compared to the 62% achievement at the same level. Furthermore, interview responses from both learners and educators confirmed that visualisation challenges arise with an increase in the complexity of geometric diagrams in terms of the identification of lines, angles and the related geometric shapes.
On the other hand, school departmental heads for mathematics expressed their learners' agreement and disagreement with the possible challenges related to visualisation. Their perspectives were based on their assessment/moderation of learners' scripts for formal tasks. Some of their perspectives are presented in table 4.6 below:

Table 4.6

Frequency distribution of HoDs' responses to the questionnaire showing their level agreement/disagreement with difficulties associated with question 1 at Van Hiele's visualisation level

Visualisation		**Strongly Agree**	**Agree**	**Disagree**	**Strongly Disagree**
1.3	Learners struggle with intuition to perceive the different parts of geometrical figures	1	2	1	0
2.1	Educators give too little attention to the development of spatial reasoning	1	3	0	0
3.5	Learners struggle to link their visual to their verbal skills when presented with a geometrical diagram.	0	3	1	0

Table 4.6 above shows the frequency distribution of departmental heads' questionnaire response to selected questions at Van Hiele's visualisation level. The results as indicated above suggested that some learners can use intuition to recognise simple geometric diagrams, because 75% of the responses strongly agreed or just agreed to the question. However, there was 100% agreement to the fact that educators do not emphasise the development of spatial skills, which is supported by educators' responses to question 3.3 of their questionnaire where approximately 70% of the educators agreed that learners lacked flexibility in their spatial orientation as they encoded geometric information in terms of fixed attributes. Lastly, the results presented in table 4.6 above showed that 75% of the departmental heads' responses agreed to question 3.5 of the questionnaires, and they also agreed that learners had difficulties in translating from visual to verbal representation. Similarly, HoDs' perspectives concurred with educators' responses to question 3.2 of their questionnaire where they either agreed or strongly agreed that approximately 68% of learners had challenges with the way in which they acquire and process visual information (visual cognition).

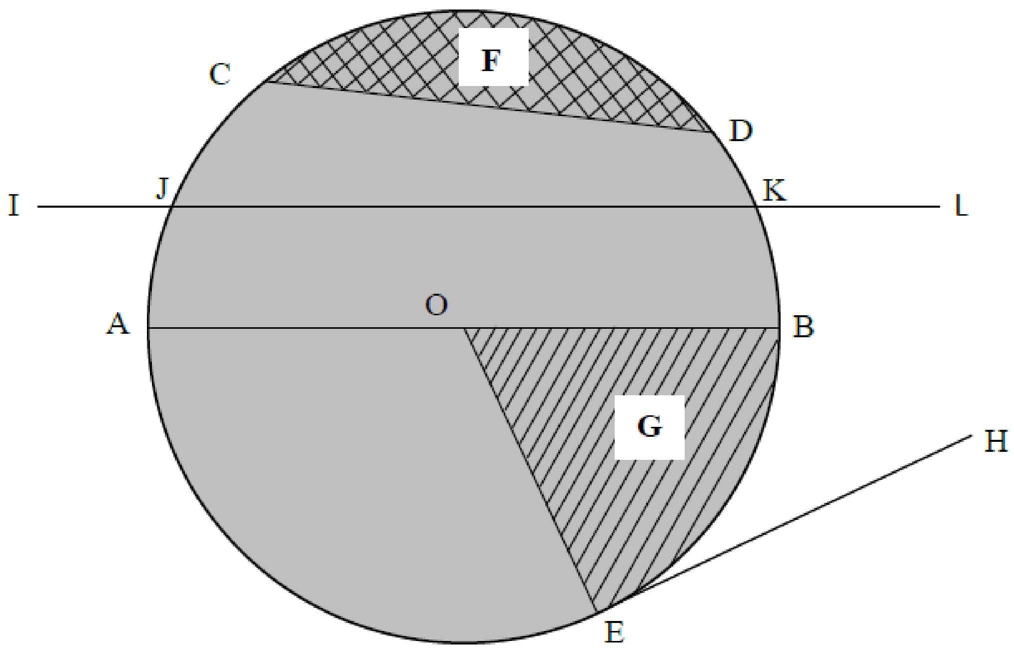
Therefore, both the educators and departmental heads were of the view that difficulties experienced by learners at visual level were a result of learners' and educators' approach to geometry and curriculum configuration, which does give special attention to development of visual abilities as the basis for understanding more complex geometry at higher levels of Van Hiele's hierarchy of geometry thinking.

Overall, corroboration of data from the test, interview responses and questionnaire perspectives pointed to the fact that despite most learners having a good mastery of basic visualisation skills, as indicated by the percentage of those who achieved (table 4.4), some learners still had challenges with visual and spatial relations. This means that one cannot safely conclude that even those who achieved at visual level would function efficiently at the next Van Hiele level (analysis).

4.4.2 Learning difficulties related to Van Hiele's analysis/ descriptive level 1

Learning difficulties related to Van Hiele's analysis level consisted of one question (question 2) in the test administered, as shown below:

QUESTION 2 (Source): Northwest province Grade 11 investigation 2019



2.1 Describe the following terms and use the figure above to give an example of each.

(a) Circle

(b) Diameter

(c) Chord

(d) Radius

(e) Segment

(f) Sector

(g) Arc

(h) Secant

(i) Tangent

(j) Angle **[10]**

Expected solutions to Question 2

(a) **Circle:** *A round figure whose boundary (circumference) consists of points equidistant from the centre.* (1)

(b) **Diameter**: *A line joining two points of circle passing through the centre of the circle. It is double the radius.* (1)

(c) **Radius:** *A line joining any two points on the circumference of the circle, OB or OE or OA* (1)

(d) **Chord:** *A line with end points on the circumference of the circle, CD* (1)

(e) **Segment:** *The shape enclosed between a chord and one of the arcs joining the ends of that chord, F* (1)

(f) **Sector:** *The enclose area between an arc and two radii at either end of that arc, G* (1)

(g) **Arc*:*** *An arc of a circle is any piece of the curve which makes the circle* (1)

(h) **Secant:** *A line passing through two points on the circle, IL* (1)

(i) **Tangent:** *A line touching the circle at only one point. EH* (1)

(j) **Angle:** *The space between two intersecting lines close to the point where they meet E* (1)

[10]

The purpose of question 2 was to explore learner abilities and difficulties with Van Hiele's analysis/descriptive level thinking skills. The main goal was to explore the extent to which Grade 12 learners could describe the different components of a circle in terms of their properties, as this is the basis for understanding circle geometry. Therefore, question 2 was deemed relevant in exploring the learning difficulties experienced by learners at analysis level. In addition, question 2 was also meant to reinforce the concept of visualisation that was introduced in question 1 (visual level).

Furthermore, question 2 was considered suitable for Van Hiele's analysis level because it enabled the exploration of the challenges that learners encountered when giving their own definitions of the different components of a circle in terms of their properties. Thus, question 2 gave the researcher the opportunity to investigate the extent of learners' conceptual understanding of circle components, as they are the foundation of the mastery of circle theorems that are introduced in the succeeding deduction levels of Van Hiele's hierarchy of geometric thinking. Therefore, an investigation into the challenges related to learners' understanding of concepts like tangent, angle, segment and diameter gave the researcher an opportunity to determine their readiness to progress to higher levels of geometry reasoning within the hierarchy because learners' mastery of the properties of circle components was deemed necessary for their successful progression to higher levels of geometric thinking.

In addition, one other outstanding characteristic of question 2 which made it relevant for exploring geometry challenges at analysis level is that it enabled the researcher to investigate learners' competences in transitioning from visual to descriptive level since the learners had to first recognise the circle components in terms of the letters of the alphabet as given on the diagram, and then they had to give their respective definitions using their properties. Thus, through question 2, the researcher was able to identify difficulties that learners encountered when transitioning from iconic (figure) to enactive (descriptions) geometric representations.

To that effect, data gathered on how learners performed in terms of their competences and difficulties encountered with transitioning between iconic and enactive geometrical representation are presented below:

Performance levels for visual and descriptive/verbal skills in question 2

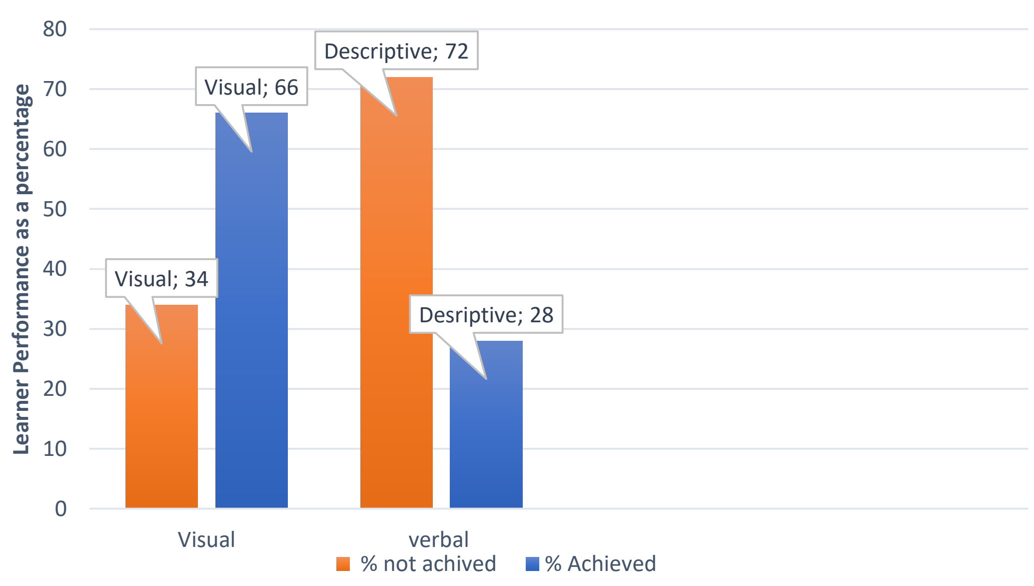


Figure 4.6

Performance levels for visual and descriptive/verbal skills

Figure 4.6 above indicates that only 20 learners (34%) did not master the visual concepts embedded in question 2, as compared to 40 (66%) who achieved at visual level, even though the question was set at analysis level according to the Van Hiele model of geometry thinking. This confirms Van Hiele's hierarchical model as mentioned in literature that learners can only function at a given level if they have mastered the preceding levels.

On the other hand, 72% of the learners did not achieve the descriptive/verbal skills required at analysis level in question 2, as compared to only 28% who achieved the verbal skills. The disparities in the number of learners who achieved in the semiotic representation to question 2 (66% visual vs 28% analysis) possibly point to the fact that learners struggle with transitioning between different geometrical representations when presented with a geometry problem, especially if their geometry conceptualisation is weak.

Furthermore, the graph above shows that most learners could identify a circle's components but could not give verbal descriptions of the identified parts of a circle, which is part of analysis.

However, besides using the above-mentioned semiotic representations (iconic and enactive) to explore learning difficulties at analysis level, data were also gathered in terms of the overall learner performance to question 2, as summarised in the table below. Thus, the table below gives the frequency distribution of those who achieved and those who did not achieve a 50% pass rate to question 2 at analysis level, as set in the scoring criterion in section 3.7.

Table 4.7

Frequency distribution of learners' abilities and difficulties for question 2 related to Van Hiele's analysis/descriptive level

Learners' levels of difficulties in relation to Van Hiele's analysis level	**Numbers of learners who achieved at analysis level**	**(%) achieved**	**Numbers of learners who did not achieve at analysis level**	**(%) not achieved**
Analysis level 1	27	45%	33	55%

Table 4.7 illustrates learners' abilities and difficulties resulting from question 2 associated with Van Hiele's analysis level 1. In this study, learners are regarded to have difficulties at Van Hiele's visualisation level if they failed to:

- recognise, state and describe in detail the properties of the components of the circle

- identify and examine each element of a circle to understand it in detail as independent of each other
- translate the visual representations of the circle components (A-L) into verbal descriptions of the circle components without using properties
- give detailed verbal descriptions of the less familiar components like tangent, segment chord as a build up to theorems related to those components
- recognise and describe the components of the circle without having to know the relationships between their properties.

The results presented in table 4.7 above revealed that 27 learners (45%) managed to achieve 50% and above in this question at Van Hiele's analysis level 1. In this vein, responses from learner scripts for those who achieved indicated that some of them managed to achieve most of the above-mentioned expectations at Van Hiele's analysis level. For example, most of them managed to recognise; state and describe in detail the properties of the components of the circle, which probably means that those who achieved had a good mastery of the analysis level and the implication was that those learners were ready to be taught circle theorems. Despite those learners achieving at 50% and above, some of them still had difficulties to give detailed descriptions of a circle's components that were less familiar to them (like the tangent and secant) than more well-known components like circle, radius, diameter and circumference. That was probably because those they managed to describe were the ones the learners were introduced to as early as at primary school level.
The same table also depicts that 33 (55%) of the sampled learners failed to achieve 50% in question 2. As identified from the test responses for those who did not achieve, most of them managed to identify different line segments in terms of the given letters of the alphabet but failed to describe circle components in terms of their properties. The implication is that those learners probably had difficulties to make short deductions; one of the competences expected at Van Hiele's analysis level. Thus, their lack of knowledge of the components of a circle resulted in them failing to even describe its properties. The implications of that were that those learners were likely to experience some challenges applying to those properties to make short deductions.

Therefore, failure by those learners to achieve in question 2 pointed to the fact that they either had challenges with geometry conceptualisation or with language, or both. More so, such learning difficulties at analysis level are probably because educators do not devote time to the teaching of geometric technical concepts. Thus, based on the data from table 4.7, one can safely say that the learners who did not achieve a 50% and above lacked the requisite geometrical knowledge at analysis level; hence, they could not proceed to be taught circle theorems.

In addition, data gathered indicated that most the learners who did not achieve made some errors at analysis level that the researcher explained using Watson's classification of errors, as discussed in section 2.4.6 (Theoretical framework). In this respect, some of the identified errors in view of Watson's classification include the following:

***Comprehension errors**,* where learners did not understand what was expected of them in question 2. In this case, 22% of the learners who did not achieve in question 2 showed that they did not understand whether they were supposed to name the circle components, define the components, or both. Similarly, data from learners' responses also indicated that learners who struggled with comprehension of question 2 also made coding errors, since these two are interrelated.

Encoding errors, where some the learners (28%) failed to write the answers in an acceptable form, as presented in the examination guideline. That means, instead of giving definitions of circle components, some learners, especially those who did not achieve at visual level, just identified the components of the circle in terms of the given letters of the alphabet, whereas the acceptable form was to define the circle components.

The extract below is from one of the learners who did not achieve due to coding difficulties. Learners such as L20 whose responses are shown in figure 4.7 below identified circle components using letters at the end of line segments and arcs, but the acceptable format was to give definitions of the components.

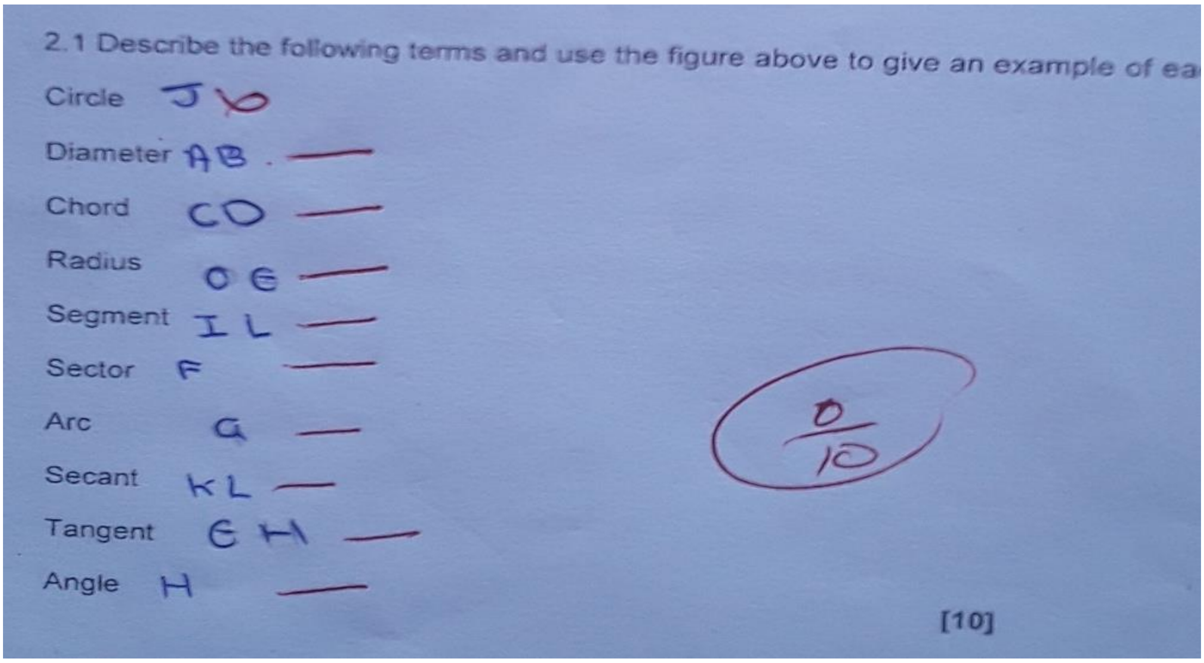


Figure 4.7

Extract from L20 (comprehension and encoding errors)

The data gathered indicated that about 25% of the learners who did not achieve (like L20) identified circle components, instead of giving definitions. Such learners showed that they lacked basic knowledge at analysis level. In view of the above, Duval (2006) cautions educators about the importance of exposing geometry learners to tasks with different registers of representation; for example, non-discursive (geometric figures) and discursive representations (properties and definitions in a natural language). Thus, the common trend across learner scripts was that learners fared well at visual level (identifying components of circle using labels (A-L)) but had challenges in formulating appropriate definitions (descriptive level of Van Hiele's hierarchy of geometrical thinking). As a result, the identified challenges on that question were:

- Translating visual components of a circle into verbal descriptions
- Using geometry terminology to describe straight lines, curved lines and shapes within a circle.

Another identified error according to Watson's classification was poor motivation.
Poor motivation, the respondents who showed a lack of motivation according to Watson's classification were those who did not even attempt question 2. Thus, some of the learners who did not achieve 50% in question 2 did not even try to answer the question, which might be an indication that those learners lacked motivation to reason at analysis level. To substantiate this, data gathered showed that 6 (10%) of the learners who did not achieve 50% in question 2 showed that they lacked motivation at Van Hiele's analysis level. As a result, the figure below is an extract of the test of one of those learners who displayed motivation challenges to geometry problem solving, as stated above.

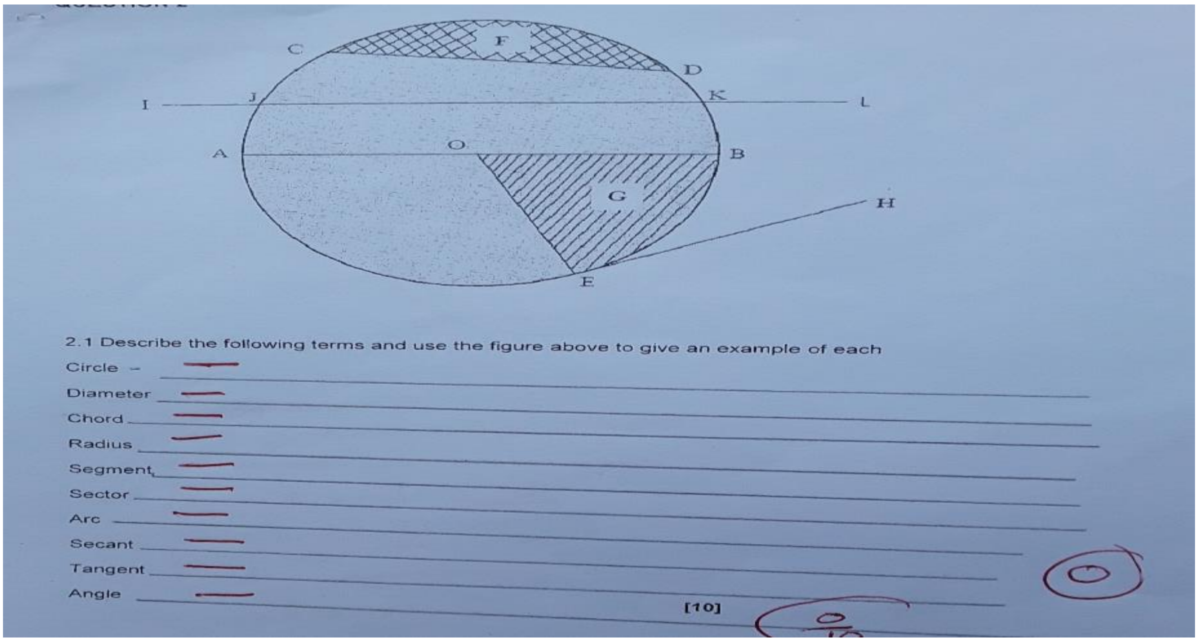


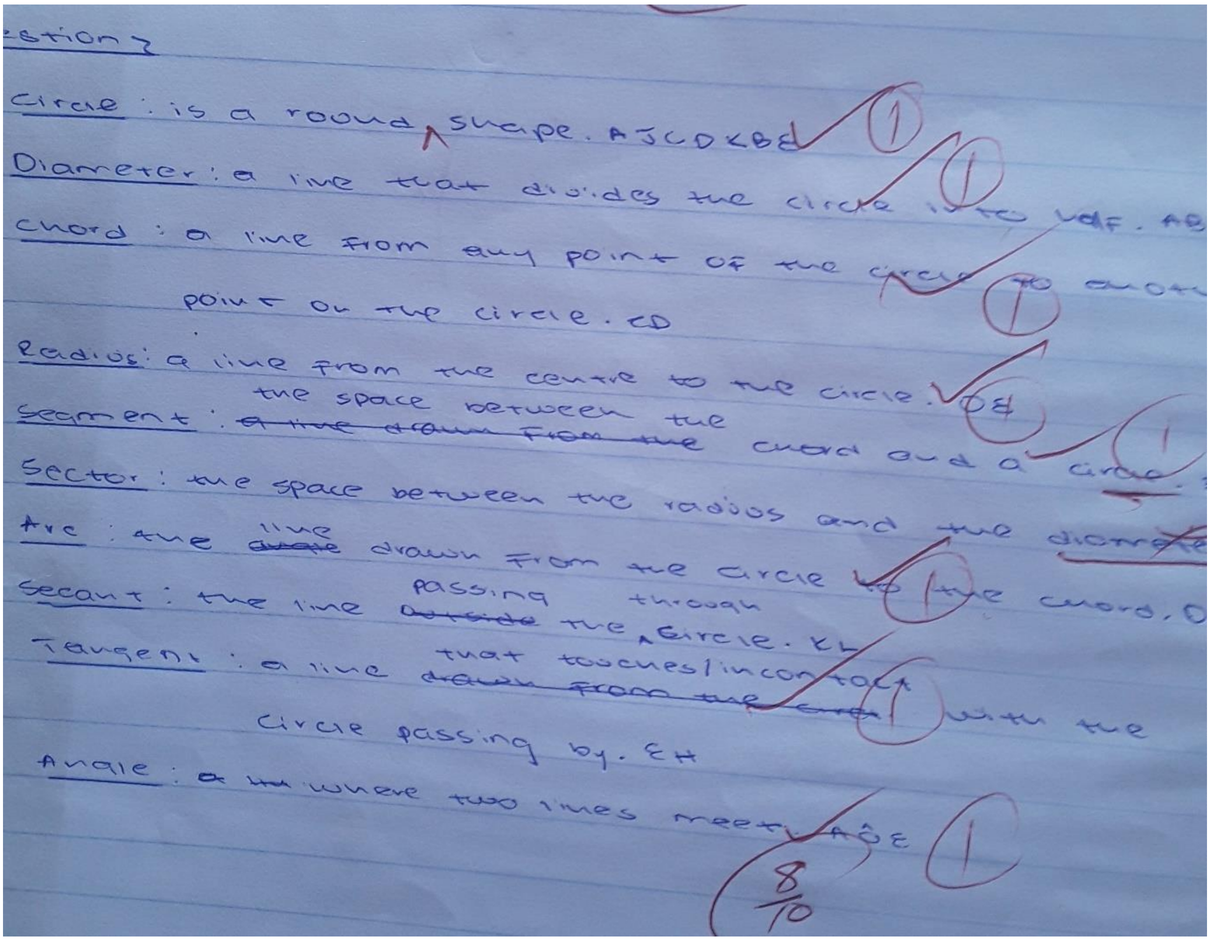
Figure 4.8

Extract from L18 (motivation errors)

Figure 4.8 as presented above clearly shows that such learners lacked the motivation to try when confronted with a geometric problem. Data gathered from the abovementioned 10% of learners who did not achieve indicated that some learners did not attempt the question at all, or some only attempted the first three items and then did not attempt all the subsequent items. Therefore, the identified challenges related to poor motivation were also confirmed by document analysis of the examiner's reports (2016, 2017 and 2019) whose findings were that learners did not know where to start when confronted with a geometric problem, hence they do not even bother to attempt.

Similarly, the challenge of poor learner motivation in geometry, as indicated in learner responses to question 2 confirms literature findings in section 2.4.1, where research indicated that not listening to learners' voices and their contributions during geometry lessons, resulted in them developing low self-efficacy and decreased levels of motivation (DBE, 2018). Furthermore, the above errors and the associated difficulties to question 2 as explained above using Watson's classification of errors confirms finding by Siyepu (2014) that learners in South Africa have problems with the fundamentals of geometric understanding, which are features and properties of shapes.

Of the 27 learners who achieved 50% and above in question 2, 13 learners showed an exceptional geometry concept mastery at analysis level, and those learners showed that they were ready to progress to the next level in Van Hiele's hierarchy of geometric thinking (informal deduction level). An example of such learners is L23 whose extract to question 2 is shown below:



Circle: is a round shape. AJCDKBE 1
Diameter: a line that divides the circle into half. AB 1
Chord: a line from any point of the circle to another point on the circle. CD 1
Radius: a line from the centre to the circle. OE
Segment: ~~a line drawn from the~~ the space between the chord and a circle. 1
Sector: the space between the radius and the diameter
Arc: the line drawn from the circle to the chord. 1
Secant: the line ~~outside~~ passing through the circle. KL
Tangent: a line ~~drawn from the~~ that touches/in contact with the circle passing by. EH 1
Angle: where two lines meet. AÔE 1
8/10

Figure 4.9

Extract from L23 (demonstration of mastery of analysis level)

The responses of L23 as indicated in figure 4.9 above demonstrate that the learner had basic knowledge at analysis level. In this respect, when comparing the extract for L23 in figure 4.9 to that of L18 and L20 (figures 4.7 and 4.8) above, there was a huge discrepancy in their mastery of Van Hiele's analysis level, which is a clear testimony that learners can be at the same grade level but operating at completely different Van Hiele levels and having different geometry challenges.

Furthermore, transcriptions from learner interviews for those who did not achieve 50% in question 2 indicated various perspectives they hold of the challenges they had at analysis level. When asked whether they experienced difficulties related to identifying and describing properties of plane shapes (polygons and circle) in terms of their angles, sides and components, some learners responded as follows:

> ***Learner L20:*** *I am better with properties of different triangles, but my challenge is circles like the one given in the test. I know those parts of the circle, but my problem was explaining their meanings, so I end up naming them using the letters that were given. Also, that question was not clear to me; that's why I decided to use letters of the alphabet given. I did not know if I was supposed to explain those parts of a circle or just name them.*

> ***Learner L38:*** *I can identify different triangles like scalene, isosceles, right-angled, and equilateral but my problem is describing those things of similarity and congruency; I confuse them. I also know how to describe the sides and angles of a square, rectangle, but my problem is on parallelogram, especially properties of diagonals, bisectors, perpendicular and parallel. We did them in grade 10 but I still don't know how to explain them. So, for me, I am not good at explaining when it comes to geometry.*

> ***Learner L55:*** *As for the circle that was given in question 2, I was able to explain only radius, diameter, but the other parts like sector, segment, secant, tangent, arc and chord, our teacher showed us in grade 11 but did not describe them. So, for that question, my only problem was explaining them, but I can see them from the diagram.*

> ***L28:*** *Eh Sir! To be honest with you, Sir, I think our teacher did not introduce us well to this geometry, he didn't even explain what geometry is? He didn't even explain to us what kind of geometry we were doing. Worse still, he did tell us how some lines are called such as chord and secant … a diameter. He just showed us those parts of the circle without explaining what they mean, and he just went straight to those theorems. It was my first time to be asked to explain those words in that test.*

> ***L39:*** *I think our educators thought that since we started doing geometry … at those [lower] grades, maybe we know, what is segment, what is chord, what is diameter, for grade 8 and grade 9 I think that's why he didn't think of explain those things … only to find that even in the past grades, our teachers did not teach us those things. They just showed us those parts of the circle from the diagram … Even now, I can see those parts from the diagram, but I cannot explain like giving its definition.*

An overview of learners' interview responses as presented in the quotations above revealed that most of the learners could recognise the various parts of a circle, but the challenges were in giving detailed descriptions of those components. This is an indication that most learners still have gaps in their understanding of properties of shapes. Thus, 56% of the interview respondents confirmed to having difficulties with explaining components of geometric diagrams in terms of their properties, especially those that relate to circles, quadrilaterals and triangles. Such responses by learners suggested that some of the challenges that learners had with analysis in geometry possibly emanated from the fact that educators do not devote enough time to concept explanation. That was evidenced in L20's interview responses where the learner confessed that she had challenges in describing the components of a circle, despite being able to identify them using the given letters.

Thus, those interview responses indicated that learners had knowledge of circle components but could not give detailed descriptions of those components from their properties. Overall, learners' interview responses possibly suggested that some of the challenges they had at analysis level related to a lack of the requisite geometry terminology and fluency in their conception of properties of shapes.

Furthermore, questionnaire data were also gathered from a sample of educators for their views on the possible challenges that learners encounter at Van Hiele's analysis level, as presented below.

Table 4.8

Frequency distribution of educator responses to the questionnaire showing their level of agreement/disagreement to difficulties associated with question 2 at Van Hiele's analysis level

Descriptive/Analysis level 1		**Strongly Agree**	**Agree**	**Disagree**	**Strongly Disagree**
2.1	Geometry language and terminology is far higher than the learners' level of comprehension.	8 (67%)	3 (25%)	1 (8%)	0 (8%)
2.2	Learners interpret/analyse geometry vocabulary incorrectly	6 (50%)	3 (25%)	2 (16)	1 (8%)
2.4	Learners cannot derive geometric definition of shape properties from their visual representations	8 (67%)	2 (16%)	1 (8%)	1 (8%)
	Average % responses	(61%)	(22%)	(11%)	(8%)

From table 4.8 above, the questionnaire data from the sampled educators indicate that out of 12 educators strongly agreed, while the other 3 just agreed that learners were limited in their description of geometric properties of shapes due terminology comprehension difficulties. In addition, 6 out of 12 learners (50%) strongly agreed and 25% agreed that poor comprehension results in learners interpreting/analysing geometry vocabulary incorrectly. On the other hand, a total of 3 out of 12 educators showed some level of disagreement to the learning challenge, as indicated in 2.1.

More so, 67% and 16% of educators strongly agreed and just agreed, respectively, to the view in item 2.4 that even though most learners could identify parts of geometric diagrams, they failed to derive the definitions of those components form what they were seeing on the diagram. Only 1% disagreed and 1% strongly disagreed with the existence of the challenge highlighted in 2.4. Based on that, it could be concluded that learners do not capitalise on their visual capabilities from the preceding Van Hiele level to analysis properties of shape at the descriptive level. Thus, an overview of the above with an average of 83% in agreement and 17% average in disagreement shows that a bigger proportion of the learners still experienced geometry language/terminology barriers in the analysis of properties of geometric shapes.

Through triangulation, the data as presented above indicated that most learners struggled to give definitions of the components of a circle, even though they could identify them using letters of the alphabet, as presented in the diagram. Data gathered from the scripts of those 33 learners (55%) who did not achieve in question 2 probably suggested that geometry language and terminology use was far higher than their level of comprehension, resulting in them interpreting incorrectly. In this vein, failure by learners to analyse the components of a circle through verbal descriptions meant that they were not ready to be taught circle theorems, a point of focus in the next Van Hiele level (informal deductive/formal deduction levels).

On the other hand, interview responses from educators also confirmed learner challenges with geometry terminology, especially learner fluency in the use of geometry language, and the conceptualisation of properties because they were used to analyse geometric diagrams. When reviewing the responses given by educators during interviews, it was found that 75% of them agreed that learners' challenges at Van Hiele's analysis level were a result of poor concept formation in the preceding level, especially visual level. Thus, based on the data collected, it could be concluded that learners are limited in terms of appropriate vocabulary to express distinguishing properties of a geometric figure or compare those properties of a shape.

4.4.3 Learning difficulties related to Van Hiele's informal deduction level 2

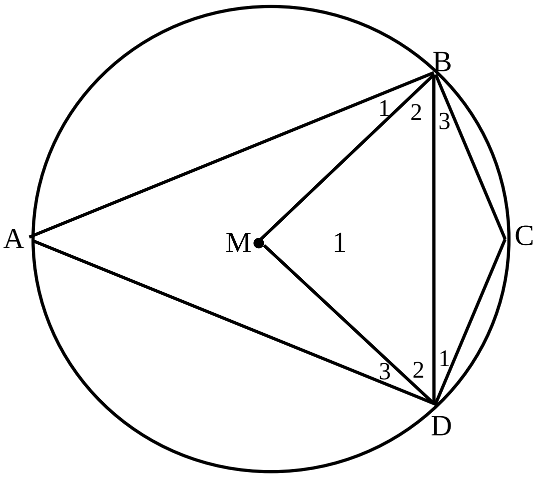
Learning difficulties related to Van Hiele's informal deduction level consisted of two questions (question 3 and question 4) in the test administered to learners, as shown below: At the informal deduction level, learners are expected to manipulate shapes and relate them in terms their properties. In addition, at this level learners are expected to establish relationships among properties of shape in terms of sides and angles, using informal arguments to justify their reasoning. Similarly, learners had to demonstrate abilities and difficulties in using logical implications and understanding those implications in terms of short deductions like solving one-step riders. However, the main reason for setting two questions at Van Hiele level 2 was to capture learners' competences and challenges along the continuum of the informal deduction level. In the same vein, question 3 was set at a lower cognitive level of informal deduction (cognitive levels 1 and 2) to explore learning difficulties as learners transition from Van Hiele's analysis level 1. Therefore, asking learners question 3 at a transitional level was meant to be a diagnosis to the anticipated learner challenges as they move to more challenging informal deduction problems.

On the other hand, question 4 was set at a higher informal deduction level (cognitive level 3) where learners are expected to have mastered basic knowledge for informal deduction and are ready to transition to formal deductive reasoning at Van Hiele's level 3. The level of complexity of the diagram for question 4 was higher than that for question 3 in that it involved more lines and angles; hence, demanded more in terms of learners' conceptual understanding of the relationships among properties of shapes and application of circle theorems. Thus, even though the two questions were at Van Hiele's level 2, they were set at different cognitive levels to ensure that learners' challenges were explored across the breadth of Van Hiele's informal deduction level.

4.4.3.1 Question 3 in the test administered to learners explored learning difficulties related to Van Hiele's informal deduction level 3 (cognitive levels 1 and 2) as shown below:

QUESTION 3 (Source: Northwest, Ngaka Modiri Molema district Grade 12 P2 Spring Camp material 2020)

3.1 In the diagram, M is the centre of the circle with A, B, C and D on the circumference of the circle. If $\hat{B}_2 = 25°$, find, with reasons, the size of the following angles:



Angles	Reason
$\hat{D}_2 =$	
$\hat{M}_1 =$	
$\hat{A} =$	
$\hat{C} =$	

Expected Solutions to question 3:

Angles	**Reason**
$\hat{D}_2 =$ MB=MD $\hat{B}_2 = 25°$ $\hat{D}_2 = \hat{B}_2 = 25°$ √ S	radii given $\angle$s opp $=$ sides √ R (2)
$\hat{M}_1 =$ 180°--($\hat{B}_2 + \hat{D}_2$) $180° - 50° = 130°$ √ S	sum of angles of a Δ √ (2)
$\hat{A} = \frac{130°}{2} = 65°$ √ S	$\angle$ at circum $= \frac{1}{2} \times \angle$ at centre √. R (2) OR angle at centre $= 2 \times$ angle at circum
$\hat{C} =$ $180° - A$ $= 180° - 65°$ $= 115°$ √	sum opp $\angle$s of cyclic quad $= 180°$ √ R (2) [8]

As indicated above, question 3 was set at basic level of informal deduction (cognitive levels 1 and 2). Therefore, the main characteristics of question 3 that made it suitable for the above-mentioned level of informal deduction are that it infused sum-angle property of inscribed triangles and quadrilaterals (analysis) with the knowledge of transitivity, and relationships between properties of shapes in terms of sides and angles (informal deduction) using short deductions in the form of one-step riders. More precisely, this question was set to explore whether learners could perceive relationships between shapes and their properties using simple analytical means (calculations) to find the angles and provide sufficient conditions for their solutions in terms of the following theorems and their converses:

- The angle subtended by an arc at the centre of the circle is twice the angle it subtends at the circumference.
- Opposite angles of cyclic quadrilateral are supplementary.
- Angles opposite equal sides of triangle are equal.

As result, data gathered on the overall learner performance (achieved/not achieved) at Van Hiele's basic informal deduction level 2 at 50% achievement level to question 3 as set in the test scoring criterion (section 3.7) are presented in the table below:

Table 4.9

Frequency distribution of learners' abilities and difficulties for question 3 related to Van Hiele's informal deduction level

Learners' levels of difficulties in relation to Van Hiele's informal deduction level	**Numbers of learners who achieved basic level (cognitive level 1& 2)**	**(%) achieved**	**Numbers of learners who did not achieve basic level (cognitive level 1 & 2)**	**(%) not achieved**
Cognitive level (1 & 2) of informal deduction	18	30%	42	70%

Table 4.9 illustrates learners' abilities and difficulties resulting from question 3 associated with Van Hiele's informal deduction level 1 (cognitive levels 1 and 2). In this study, learners were regarded as having difficulties at Van Hiele's informal deduction level if they failed to:

- use properties of isosceles triangles in terms of the relationships between its sides and angles to calculate missing angles
- use appropriate theorem and relationships between properties of a shape to find angles subtended by an arc at the centre and at the circumference
- apply theorems about cyclic quadrilaterals to calculate missing angles
- write correct statements and justify their statements with correct reasons
- understand logical implications and class inclusions by relating properties triangles and cyclic quadrilaterals to relevant theorems.

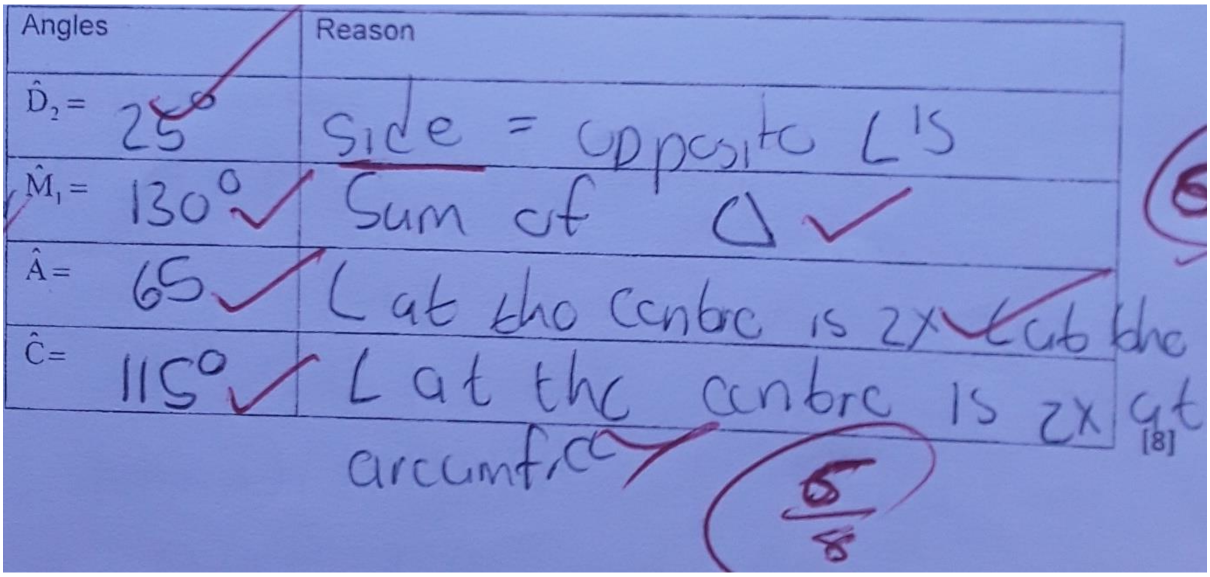
Table 4.9 above shows the frequency distribution of learners' responses to question 3 where 18 learners (30%) achieved 50% and above in this question at Van Hiele's informal deduction level 2, with 8 of them scoring full marks for the question. Data gathered from those learners' scripts indicated they were able to relate the properties of triangles and quadrilateral in terms of their sides and angles. Special mention was made of those learners who scored full marks to question 3 because they showed a good mastery of class inclusion. In addition, those learners were able to relate properties of triangles, quadrilaterals and their angles, and then link them to relevant theorems. Thus, they were able to combine shapes and properties, and establish relationships among the properties.

However, most of the learners who achieved, especially those who did not score full marks for question 3, seemed to experience difficulties with their conceptualisation of logical implication. For example, on analysing the diagram for question 3 above, learners were expected to show that, for an isosceles triangle, if two sides are equal, it implies that the angles opposite those equal sides are also equal. Thus, data gathered also indicated that those learners had difficulties with the integration of triangles and quadrilaterals to the circle, hence making informal deductions a challenge. Despite having gaps at cognitive level 3 of informal deduction, the same learners had previously demonstrated a good mastery of informal deduction at cognitive levels 1 and 2 as set in question.

On the other hand, table 4.9 above also shows that 42 (70%) of the sampled learners did not achieve 50% to question 3. Even though some of the learners were able to identify the properties of triangles and quadrilaterals in terms of sides and angles; most failed to give meaningful definitions and informal arguments to justify their reasoning. As a result, those who did not achieve in question 3, demonstrated poor conceptualisation of logical implications and class inclusions. Therefore, those learners were considered not ready to be taught geometry at higher Van Hiele levels.

An overview of the data from the sampled learner scripts indicated that 10 of the learners who achieved had challenges with some aspects of basic informal deductive knowledge, especially the poor understanding of relationships among properties of shapes (isosceles triangles, cyclic quadrilaterals) in terms of sides-to-angle, and angle-to-angle relations.

An elaboration of specific competences and difficulties at the informal deduction level for those who achieved and those who did not achieve in question 3 is presented in the following extracts. Therefore, the evidence to support the competences for the learners who achieved in question 3 is provided in the extracts bellow. For example, L 36 was one such learner whose response is shown in the extract below:



Angles	Reason
$\hat{D}_2 =$ 25°	Side = opposito ∠'s
$\hat{M}_1 =$ 130°	Sum of △
$\hat{A} =$ 65	∠ at tho centre is 2x ∠ at the
$\hat{C} =$ 115°	∠ at the centre is 2x at circumfrce

[8]

5/8

Figure 4.10

Extract from L36 (correct statements with some reasons wrong)

Figure 4.10 is representative of one of the common trends of challenges as identified from learners' responses to question 3 for the sample of 18 who achieved 50% but did not score full marks for the question. A common challenge among those learners' responses to question 3 was the lack of fluency in the way they presented reasons to statements. Thus, despite these learners having knowledge of the relationships between properties of shapes, as presented in the diagram, they failed to present the reasons for the way they were presented in the examination guideline. In this case, L36 correctly stated that $\widehat{D}_2 = 25°$ and had an idea of the relationships between properties of isosceles triangles in terms of its sides and angles but lacked the fluency to write the reason. The learner wrote =

$opposite\ angles$, instead of $\angle\, s\ opposite = sides.$ This response showed that the learner had conceptual gaps in his/her understanding of circle theorems, hence could not present relationships among properties in the acceptable way.

Similarly, L36 stated that $\hat{C} = 115°$, which possibly suggest L36 was good at using analytical means (calculations) at informal deduction level, but had challenges finding the appropriate relationships in terms of statement/reason, as the response given appeared to suggest that the learner struggled when it comes to scrutinising the given diagram and any relevant information for clues about which theorem to use to answer the question. As a result, they ended up writing correct statements with incorrect reasons. Thus, the statements as presented by L36 show that the learners made two types of errors: conceptual errors that are caused by inadequate mastery of basic concepts, facts and geometric skills; and procedural errors where a learner has the knowledge of concepts and properties of shapes but fails to present the reasons correctly. As a result, such responses showed that some of the learners who achieved stills had gaps in the application of postulates, axioms and theorems using short and informal deductions.

Furthermore, from the sample of learners who achieved, data also revealed that there were 8 learners who achieved full marks for question 3. Thus, L57 was one such learner whose response is presented in the extract below:

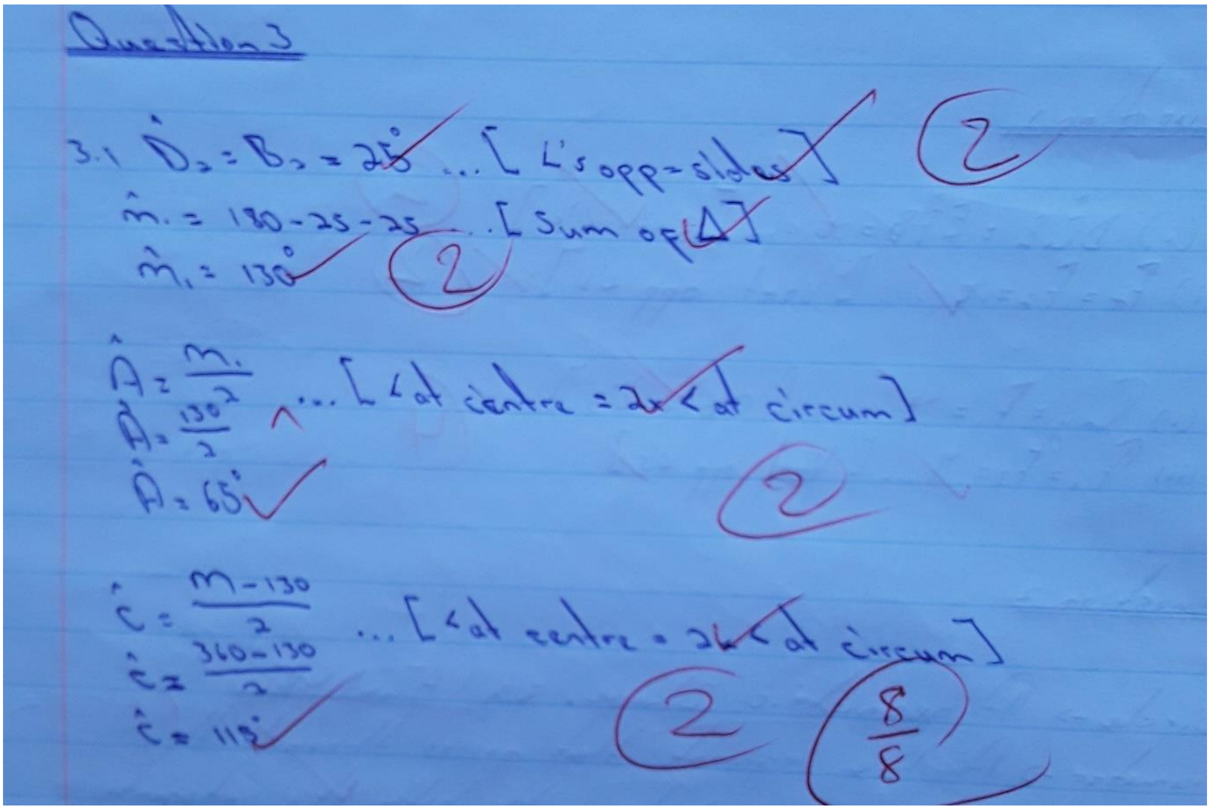

Question 3
3.1 $\hat{D}_2 = \hat{B}_2 = 25°$ ... [∠'s opp = sides]
2
$\hat{m}_1 = 180 - 25 - 25$ ... [Sum of Δ]
$\hat{m}_1 = 130°$
2
$\hat{A} = \frac{m_1}{2}$ ... [∠ at centre = 2x ∠ at circum]
$\hat{A} = \frac{130°}{2}$
$\hat{A} = 65°$
2
$\hat{C} = \frac{m - 130}{2}$ ... [∠ at centre = 2x ∠ at circum]
$\hat{C} = \frac{360 - 130}{2}$
$\hat{C} = 115°$
2
$\frac{8}{8}$

Figure 4.11: Extract from L57 (correct statements and reasons)

The representation of responses to question 3 as shown in Figure 4.11 above is a clear indication that those learners who obtained full marks had a good mastery of basic knowledge at the informal deduction level. This conclusion is based on the idea that those learners were able to integrate concepts gained from the previous Van Hiele level (analysis level 1) into informal deduction.

On the other hand, the data from table 4.8 also indicate that 70% of the learners did not achieve in question 3, which probably means that those learners lacked knowledge of basic concepts at the informal deduction level and were likely to encounter serious challenges with question 4, which was set at a higher cognitive level of informal deduction. Therefore, such a huge discrepancy of only 30% who achieved compared to only 70% who did not achieve suggests a big gap in the level of learners' geometry conceptualisation between Van Hiele's levels 1 and level 2. That probably supports Van Hiele's findings that, if learners are to perform at any advanced level, they should have mastered concepts in prior levels along the hierarchy of geometrical thinking. This means that those learners who did not achieve were not ready to be taught higher levels of informal deduction and even formal deduction level reasoning.

A further analysis of the data gathered from learner responses to question 3 showed that 17 learners (about a quarter of the 70% who did not achieve) obtained 0 out of 8 marks. An example of such learners is L39, whose responses to question 3 are shown below:

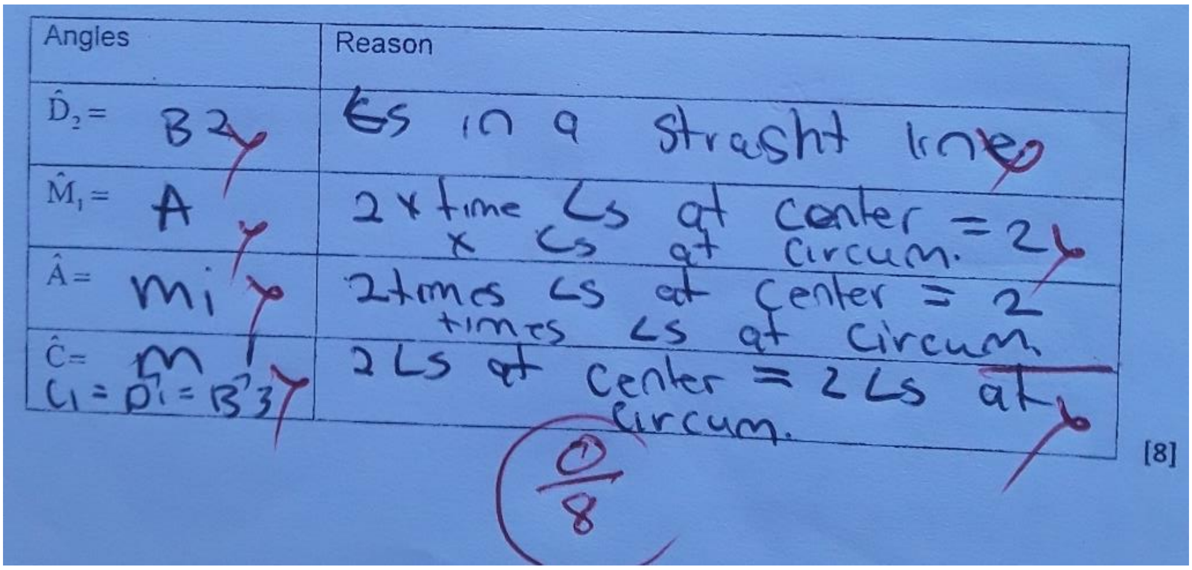


Figure 4.12

Extract from L39 (incorrect statements and reasons)

L39's responses, as presented in figure 4.12 above, showed a complete breakdown in the learner's basic conceptual knowledge at the informal deduction level because, the learner did not even understand what was expected of him/her. Thus, in view of Watson's model of classification of errors, L39 displayed comprehension errors at informal deduction level. That was probably a challenge because the learner was expected to find the missing angles but instead, named other angles that are equal to the ones whose sizes were supposed to be calculated. As a result of those interpretation challenges, the learners also gave incorrect reasons for all the statements, which is an indication that he/she lacked basic knowledge of relationships among properties of shapes, which is an indication of poor conceptualisation. Responses such as those presented by L39 above showed that the learner was operating at a level far below Van Hiele's informal deduction level.

Similarly, L22's response to question 3 is presented in figure 4.13 below:

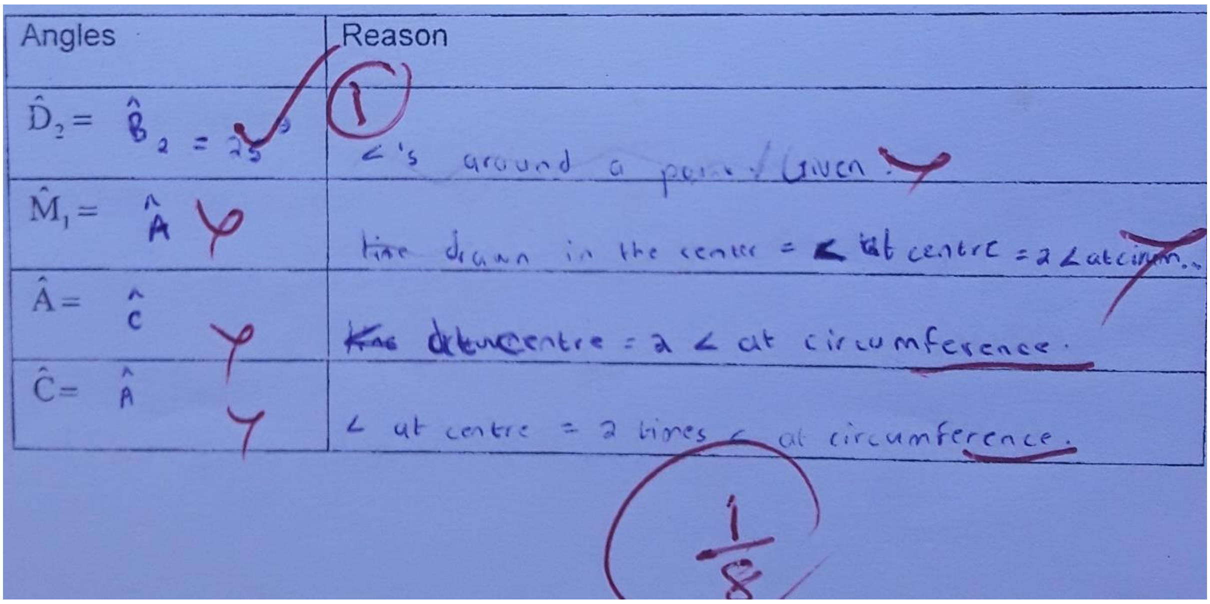


Figure 4.13

Extract from L22 (incorrect statements and reasons)

Figure 4.13 show that L22 had serious challenges with informal deductive reasoning. Even though L22 managed to calculate $\hat{A} = 25°$ with an incorrect reason for the statement, the learner wrote incorrect statements and reasons for all the other items of question 3. Therefore, an analysis of figures 4.12 and 4.13 indicated that these learners had similar challenges with basic knowledge at informal deduction level. In view of that, a trend identified by the research for most of the learners who did not achieve in question 3 was that they knew the theorem that related the angle subtended by an arc or chord at the centre and the circumference, but they did not know when the theorem was applicable. In addition, most of them did not realise that they were expected to write the sizes of angles instead of naming other angles they thought were equal to those given. Therefore, one can conclude that these learners had language and comprehension challenges, in addition to poor knowledge of relationships among properties of shapes despite the question clearly stating that they should write sizes of angles.

Furthermore, those challenges at the basic level of informal deduction, as identified from the scripts of the learners who did not achieve, were also confirmed by their interview responses, as indicated in the transcriptions below:

> **L22:** *I think the problem is our Grade 11 teacher who told us to memorise those theorems as they are in the textbook, and I am not good at that at cramming. As for that question 3, the diagram was easy to understand because it did not have many lines, but I got stuck on calculating those angles. At least I saw that the triangle inside that circle was an isosceles triangle, and I used the property that base angles are equal to write* $\widehat{D}_2 = 25°$*, but with the other angles I did not know what to do and I thought they wanted us to write angles that were equal* $\hat{A}$*,* $\widehat{M}_1$*. So, I just named other angles … My major problem is with those reasons, I only remembered that which says,* $\angle\, at\ center = 2\angle\, at\ the\ circum$ *and I wrote it for all the other angles it was the only theorem I remembered that time.*

> ***L39:*** *I may know the relationships between properties, but sometimes I forget how to write the reasons or theorem using the correct geometry language, as it is in the exam guideline … To be honest, that question was really confusion. I did not even see that* $25°$ *was given on top. Since there was no angle given on the diagram; I ended up writing other angles equal to* $\widehat{D}_{2,}\,\hat{A},\ \widehat{M}_1$*. As for the reasons I used to remember them in Grade 11, now they confuse me. My problem is to choose the correct reason to use, I was not sure about the theorems for those for triangles and cyclic quadrilaterals and I used* $\angle\, at\ center = 2\angle\, at\ the\ circum$.

> ***L20:*** *Sir, I don't feel good because I know that theorems are needed when proving riders, but they are difficulty for to remembers, even if I try to cram them. That diagram in the test was simple, but I realised that my problem is I don't know those theorems, riders. Eish! I know that riders have a lot of marks and if I can only remember correct theorems, I know I can get those marks.*

> ***L57:*** *I do not have any problem with theorems using properties of triangles and quadrilaterals. I know the theorems to use for those two. Our teacher told us to study the diagram first and write all the angles I can find before I answer the questions. We were told to write those reasons the way they are written in the exam*

guideline, so I know them. As for question 3, it was easy to see the relationships between angles of the triangle and the cyclic quadrilateral because the lines were not many in that diagram and $25°$ *was given, so only needed to know theorems used when those two shapes are drawn inside a circle.*

L38: *I struggle with these questions because our educator did not teach us riders. The worst part is that the question papers have lots of riders. I always loose marks in this geometry section because I can't write something that I don't know. I think the problem is our teacher don't teach us properties of shapes because they are in a hurry to finish the ATP, when we tell them that we don't understand those riders they say we can't be stuck on one topic for the whole term. I think that's why we are a little bit confused with … and theorems.*

The above extracts from learners' interview responses revealed that learners had a good mastery of the basics of informal deduction at lower cognitive levels (1 and 2) like the identification of properties of shapes, and they were aware that they were supposed to write a statement and reason. However, the extracts from the interview responses above suggest that most of the learners had difficulties with the interpretation of question 3 in terms of writing correct statements and reasons. Thus, data from the extracts above show that learners had a poor understanding of circle theorems and the correct application thereof. For example, L22 and L39 confirmed that they had challenges in writing the correct reasons to statements.

In this vein, these interview responses seemed to concur with findings from test responses above in that in both situations, most learners were found to experience challenges with relating statements to reasons, especially where the circle was integrated into the diagram.
On the other hand, the learners' perspectives to challenges they encountered at informal deduction were also confirmed by educators' responses to the questionnaire. Some of the sampled educators' responses to the questionnaire are presented below:

Table 4.10

Frequency distribution of educators' questionnaire responses to learner challenges related to Van Hiele's informal deduction level 2 using question 3

Informal deduction level 2		**Strongly Agree**	**Agree**	**Disagree**	**Strongly Disagree**
4.1	Some learners struggle to relate geometric statements to correct	1(16%)	4 (67%)	1 (16%)	0%

	reasons even for one-step-riders				
4.2	Learners misuse theorems by making unjustified assumptions	3 (50%)	3 (50%)	0%	0%
4.3	Learners rush through a geometric problem and fail to read important instructions and given information carefully when using diagrams in geometry	1 (16%)	3 (50%)	2 (33%)	0%
Average response/ question		27%	56%	17%	0%

Table 4.10 above shows the distribution of educators' responses to question 4.1 of the questionnaire administered to them. The data in the table indicated that 16% of the educator sample strongly agreed and 67% agreed that learners cannot present correct statement and reasons in problems involving one-step riders and only 16% disagreed with that standpoint. Similarly, on the issue of learners overgeneralising and giving unjustified reasons to geometric statements, 3 (50%) of the educators strongly agreed and the other 3 (50%) agreed, while none of the sampled educators disagreed with the view. In addition, the responses to question 4.3 showed that 16% strongly agreed to that view, 50% agreed and the other 33% disagreed with that view. Overall, 27% and 56%, respectively, either strongly agreed or agreed that learners had challenges with informal deduction. Only 17% and 0% disagreed and strongly disagreed, respectively, to learners having those challenges at informal deduction level.

Overall, educators' responses to the questionnaire confirmed that learners lacked basic knowledge at informal deduction level, especially relating properties' shapes in terms of correct statements and reasons.

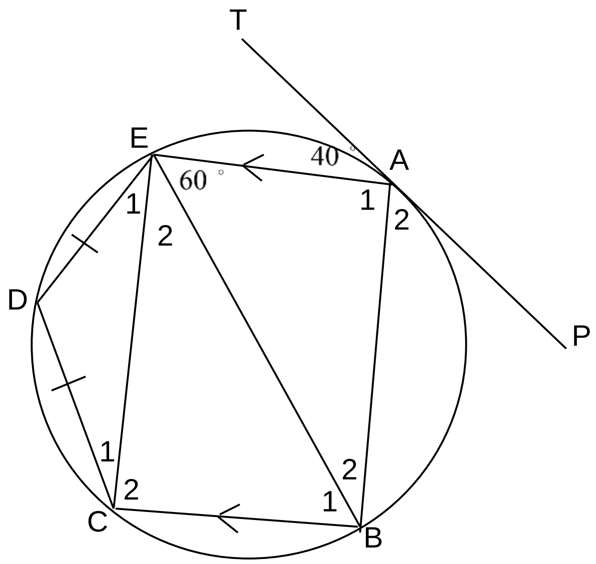
4.4.4 Learning difficulties related to Van Hiele's informal deduction level 2

4.4.4.1 Question 4 in the test administered to learners explored learning difficulties related to Van Hiele's informal deduction level 3 (cognitive level, as shown below

QUESTION 4 (Source: Gauteng province Grade 12 Paper 2 Preparatory Exam 2018)

In the diagram below, TAP is the tangent to circle ABCDE at A. AE || BC and DC =DE.

$T\hat{A}E = 40°$ and $A\hat{E}B = 60°$



4.1 Name the following shapes

4.1.1 CDE (1)

4.1.2 BCDE (1)

4.2 Determine with reasons, the sizes of the following angles:

4.2.1 $\hat{B}_2$ (2)

4.2.2 $\hat{B}_1$ (2)

4.2.3 $\hat{D}$ (2)

4.2.4 $\hat{E}_1$ (3)

[11]

Expected solutions to question 4:

4.1.1. $CDE\ is\ an\ isosceles\ triangle$ √ (1)

4.1.2. $BCDE\ is\ a\ cyclic\ quadrileteral$ √ (1)

4.2

4.2.1. $\hat{B}_2 = 40°$ √ $(\tan/\ chord)$ √ S/R (2)

4.2.2. $\hat{B}_1 = 60°$ √ $(alt\ \angle s = AE \parallel BC$ √ S/R (2)

4.2.3. $\hat{D}\ = 180° - 60°$ √ $(opp\ \angle s\ of\ a\ cyclic\ quad$ √ SR (2)

$= 120°$

4.2.4. $\hat{E}_1 = \frac{180°-120}{2}$ √ $(sum\ of\ \angle\ of\ \Delta)$ √ S/R

$= 30°$ √ (3)

The purpose of question 4 was to explore learners' abilities and difficulties with Van Hiele's informal deduction level 2 at cognitive level 3. Thus, the reason for setting question 4 at a higher cognitive level of informal deduction was to explore learner challenges associated with the following theorems:

- Tan chord theorem
- Theorems related to cyclic quadrilaterals (exterior and interior opposite angle, and sum of opposite angles) and the parallel-transverse line relationships
- Equilateral triangle properties

In addition, question 4 was intended to explore learner competences and challenges in geometric problems where many concepts and relationships are integrated in a single diagram. Therefore, learning difficulties related to Van Hiele's informal deduction level 2 (cognitive level 3) consisted of question 4 in the test administered to learners.

Moreover, question 4 above was set at informal deduction level, just like question 3, but at a higher cognitive level to enable the exploration of learners' competences and challenges as they transition into Van Hiele's formal deduction level 3. Therefore, the major characteristics of question 4 that made it suitable for that level of informal deduction were that learners had to uncover a variety of relationships imbedded in one geometrical diagram, thereby setting the basis for logic, which was the area of focus for the next Van Hiele level (formal deduction). Thus, at informal deduction level, learners should understand the connection between properties within geometric shapes and from one group of shapes to the other. Therefore, in case of question 4, learners had to demonstrate their competences on a more complex geometrical diagram with more lines and angles, triangles, cyclic quadrilaterals and application of related circle theorems. To have achieved at the expected level of informal deduction at cognitive level 3, learners had to uncover and apply the following relationships among properties of shape: tan chord theorem, opposite angles of cyclic quad, sum of angle of a triangle and alternate angles between parallel lines.

Furthermore, question 4 was relevant in that infused concepts learnt in the previous Van Hiele's levels, like name geometric shapes (visualisation level 0) and the concept of parallel lines from question 1. As a result, learners had to show their competences in concept integration. Thus, question 4 was more comprehensive than question 3 in that it integrated concepts from previous levels into the hierarchy up to informal deduction level as confirmation of Van Hiele's assertion that for learners to master the next level in the hierarchy, they should have mastered concepts at the preceding levels.

Therefore, data on the overall learner performance (achieved/not achieved) in question 4 at Van Hiele's informal deduction level 2 at 50% achievement level per question, as set in the test scoring criterion (section 3.7), are presented in the table below:

Table 1

Frequency distribution of learners' abilities and difficulties for question 4 related to Van Hiele's Informal deduction level

Learners' levels of difficulties in relation to Van Hiele's informal deduction level	**Numbers of learners who achieved basic level**	**(%) achieved**	**Numbers of learners who did not achieve basic level**	**(%) not achieved**
Informal deduction (Cognitive level 3)	13	22%	47	78%

Table 4.11 illustrates learners' abilities and difficulties resulting from question 4 associated with Van Hiele's informal deduction level at cognitive level 3. In this study, learners were regarded to have difficulties at Van Hiele's informal deduction level if they failed to:

- understand and present relationships between properties of parallel lines, triangles, cyclic quadrilaterals and how they relate to each other and the circle
- create own definitions and present informal arguments to justify their geometrical statements
- understand logical implications and class inclusions by making short deductions leading to the calculation of the missing angles
- use the relationships between properties of shapes to write correct statements and reasons
- do short deductions from relationships between properties of shape to find missing angles.

An overview of the data collected from learner responses to question 4 in relation to the above-mentioned expectations showed that some learners were able to create meaningful definitions and informal arguments. Some also managed to relate properties of isosceles triangles, cyclic quadrilaterals and parallel lines to solve one-step riders. Moreover, some of the learners were able to make short deductions from relationships between properties of triangles and cyclic quadrilaterals integrated in a circle. However, most learners failed to use logical implications and class inclusion to determine the missing angles. Similarly, some of the learners failed to write correct statements and reasons.

In this vein, the data indicated in table 4.11 above showed that 13 learners (22%) achieved 50% and above in question 4, with 6 (10%) of those learners who achieved at Van Hiele's informal deduction level 3 (cognitive level 3) scoring full marks. The achievement rate of 22% showed that the learners had sufficient geometrical knowledge at higher cognitive levels of informal deduction therefore, they were ready the progress to level 3 (formal deduction).

Furthermore, a common trend among the learners who achieved 50%, as shown in table 4.11 was that they were able to recognise and name shapes ADE and BCDE as an isosceles triangle and a cyclic quadrilateral, respectively. That knowledge of basic shapes (visualisation level 0) was necessary at informal deduction level 2 for learners to select relevant theorems for their statements in question 4.2 based on their understanding of the relationships among the properties of those two shapes in circle geometry.

For example, L30 was one such learner who achieved in question 4, as shown below:

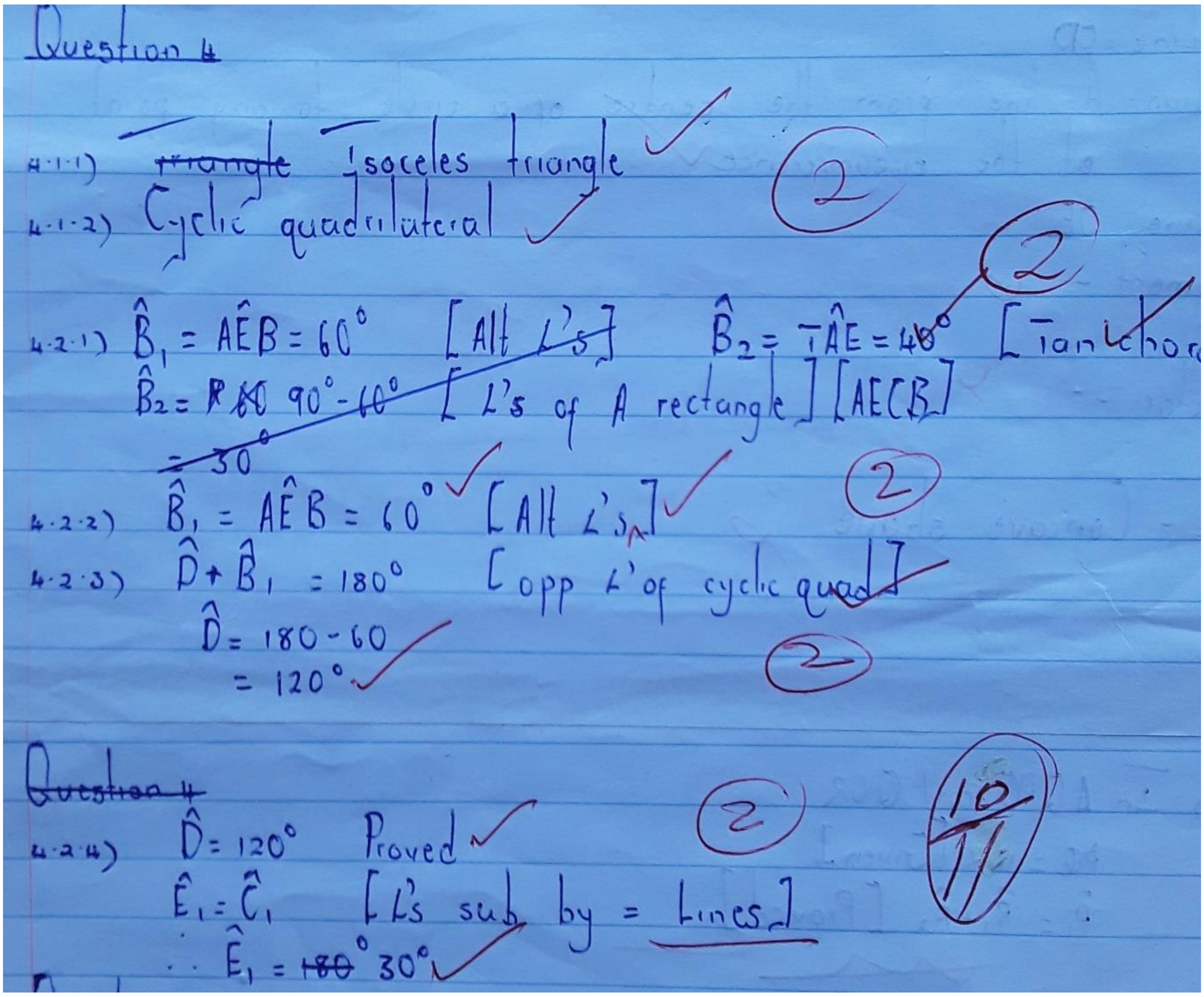


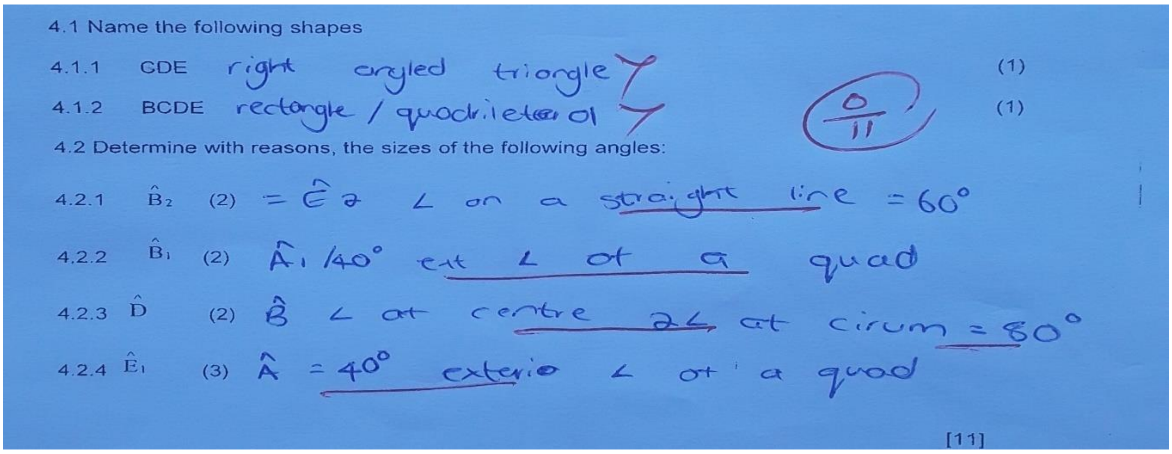
Figure 4.14

Extract from L30 (Good mastery of informal deduction)

Learner responses from figure 4.14 above indicated that L30 had a good mastery of higher levels of informal deduction and was ready to be taught formal deduction. However, the data also showed that despite those learners achieving in question 4, some had gaps in their conception of theorems. For example, as shown in figure 4.14 above, item 4.2.2, L30 did not specify the parallel lines for alternate angles to be equal. Furthermore, on item 4.2.4, L30 had an idea of the relevant theorem $\angle s\ opposite = sides$, but the learner wrote $\angle s\ subtended\ by = sides$. In this case, despite the learners having knowledge of the applicable theorem, the learner lacked geometrical fluency (writing the reason according to the format in the examination guideline).

On the other hand, the data in table 4.11 above also indicate that 47 learners (78%) of the learner sample did not achieve 50% in question 4. Therefore, an analysis of learner responses to question 4 also showed that failure by those learners to achieve the pass mark of 50% in question 4 was attributed to poor conceptual understanding of relationships among properties of shapes in more complex geometric diagrams. Thus, a common trend among learner responses to question 4 for those learners who did not achieve was that most of them could not name the shape CDE and $BCDE$. Therefore, they could not relate properties of shapes they did not know, as indicated by the fact they struggled to answer the items that required related properties of those shapes in terms of statement and reason.

In view of those learners who did not achieve, as shown in table 4.11 above, the data also indicated that 14 learners (23%) obtained 0 out of 11 in question 4. An example of such learners was L25 whose reply to question 4 is shown below.


4.1 Name the following shapes

4.1.1 GDE right angled triangle (1)

4.1.2 BCDE rectangle / quadrilateral (1)

$\frac{0}{11}$

4.2 Determine with reasons, the sizes of the following angles:

4.2.1 $\hat{B}_2$ (2) $= \hat{E}_2$ ∠ on a straight line $= 60°$

4.2.2 $\hat{B}_1$ (2) $\hat{A}_1$ 140° ext ∠ of a quad

4.2.3 $\hat{D}$ (2) $\hat{B}$ ∠ at centre 2∠ at cirum $= 80°$

4.2.4 $\hat{E}_1$ (3) $\hat{A} = 40°$ exterio ∠ of a quad

[11]

Figure 4.15

Extract from L25 (failure to relate shape properties to theorems)

Therefore, figure 4.15 above shows that L25 was challenged even at the Van Hiele levels prior to informal deduction level 2, levels 0 and 1, since the learners could not recognise geometric figures from the diagram. This could be the reason why those learners failed to score any mark for item 4.2 of question 4. In this regard, the fact that L25 failed to write correct statements and reasons was probably because he had difficulties with distinguishing between necessary and sufficient conditions to correctly relate properties of those two shapes to the circle using theorems. That is so because, according to Van Hiele’s theory, such reasoning is evident in learners working at informal deduction level 2 of the model. Therefore, the statements written by L25 were not in any way related to the reasons given, which was an indication that reasons were used indiscriminately. When interviewed, the learner pointed out that he had a challenge with identifying different shapes from a geometric diagram; worse still, establishing the relationship among their properties together with the relevant theorems. The implication is that such learners were not ready to be taught informal deductive reasoning.

Similarly, data gathered from learners who did not achieve in question 4, especially item 4.2 indicated that the understanding of theorems as applied in solving one-step riders was largely dependent on learners' ability to translate a diagram into a verbal statement or vice versa. In this respect, making such translations was a challenge for most learners in their responses to question 4.2 items, hence learners had either the statement, the reason or both wrong. For example, L37's extract is shown below:

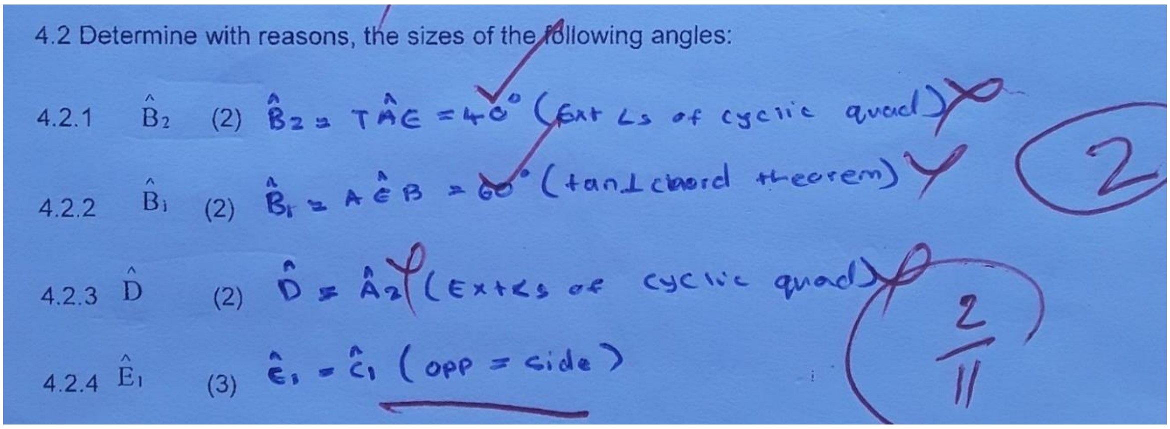


Figure 4.16

Extract from L37 (misconceptions about exterior angle)

Figure 4.16, like figure 4.15, shows common trends in the types of misconceptions held by learners at informal deduction level. One such misconception, displayed by one- third of the learners who did not achieve was involving the 'exterior angle of a cyclic quad' as a reason to a geometric statement. That was evident from L37's and L25's responses to item 4.2 above where they considered the 40° given in the diagram as exterior angle of cyclic quadrilateral. They understand an exterior angle as any angle on the outside of vertex, not knowing that an exterior angle is formed between one side of the shape and an extension of the other at the vertex. Thus, these learners showed that they experienced challenges with their conceptual understanding of lines and angle properties in more complex geometric diagrams.

Consequently, from the data gathered, it was seen that about two-thirds of the learners who did achieve 50% in question 4 made unjustified deductions based on wrong reasons, which was a sign that some of these learners were guessing the solutions.

Furthermore, 5 of the learners who did not achieve in question 4 also took part in the interview session and some of those learners' perspectives are presented in the quotations below:

L25: *On the issue of solving geometry problems using known properties of geometrical shapes, sometimes I can calculate the missing angles, but my problem is choosing the correct reason to use. Those theorems are too many and I end up guessing the one to use. And most of the time my reasons are wrong. I think the other thing causing these problems is our teacher, he did not take time on teaching riders at all! They just teach us how the theorems [are] proved … but riders they just touched them a bit. No, I am not able to memorise theorems and the arrangement used in the textbook. During examinations, I am unable to remember them because I get confused on which theorem to use and how it is written in the textbook, especially if the diagram has many lines and shapes inside the circle.*

A similar view was also given by L37 regarding informal deduction, as indicated in the interview extracts below:

L37. *Yes, I know some of those properties of shapes, especially triangles if the diagram is simple, with only a few lines and angles, but when it is complex like that one in question 4 of that test, then they confuse me. I get more confused, especially when those shapes are inside a circle. As with those problems where I must use known properties of geometrical shapes to solve, I struggle to choose the right reasons for my statement. I always get such questions wrong. I think the problem is our teacher normally teach those theorems without explaining them or helps us to identify and discover the relationships on our own.*

L60: *I can calculate some of the angles, but my problem is giving the correct reason. I know different types of triangles, and I can use the properties to calculate angles; my problem when they mix the triangles with quadrilaterals in one diagram, I get stuck. When the shapes are mixed, I cannot see the relationships between the properties of those shapes.*

L22: *My only problem is writing those statements and reasons. Those theorems are too many, even though in the test I remember some of them, my problem is choosing the correct one to my statement. When I write those reasons in the test, I will be thinking that I am writing the correct ones but when I gets back my answer script most of them will be wrong. It tried to memorise them but still I get them wrong. As for me, question 4 in that test was better because the diagram did not have many lines, so I got some few answers correct. My problem was in question 5; those many lines confused me.*

> ***L58:*** *I think they taught us in a good way. If I was going to rate, I would rate them at 95% because I understood most of the things about Euclidean geometry and geometry proofs, especially those riders, where we must calculate angles giving reasons. I have problems in some questions, but most of the questions I get them correct. Our teachers at grade11and 12 gave us a chance to do those riders on the chalkboard, correcting us where we wrote wrong statements and reasons, now I am confident with those questions. I always get good marks in those questions.*

The above extracts from learners' interview responses revealed that learners had a good mastery of the analytical (calculation) aspects as their responses point to the fact that they can calculate missing angles on a geometric diagram. However, information from the above interview responses above suggested that most of the learners had difficulties with the application of relevant theorems to support their statements. Similarly, the general trend as can be deduced from the extracts above was that learners lacked the understanding and active classroom involvement in the derivation of theorems from relationships among properties of shapes.

Furthermore, some of the learners interviewed confirmed their reliance on rote memorisation, hence they end up confusing the relationships, which has a negative impact on performance, as in their interview responses, most learners confirmed that they end up guessing the reasons to their statements. In this regard, these learners attributed the challenges they had with informal deduction to poor teaching methods which discourage active learner participation in conceptual development. To this end, the interview extracts above suggest that learners had a poor understanding of the relationship between properties of shapes and the correct application thereof.

On the other hand, the existence of challenges at higher levels of informal deduction was confirmed by sampled educators in both their interview and questionnaire responses. For example**, E3** participant stated that:

> *In most instances, at informal deduction, learners lack the knowledge to correctly apply theorems. Most learners struggle to find where to start when answering question that involve relationships among properties of shape. Mostly, they fail to link the statement and the correct reason for that specific question especial if the geometric diagram has many concept/theorems imbedded it. In my view, most of them don't have the conceptual understanding of the theorems as they relate to the geometrical diagram but rely on memorising them and they end up mixing them.*

Moreover, another educator **E5** elaborated that:

At informal deduction, learners struggle to link the given angle(s) and what they are asked to find to the relevant theorems. That means most of them end up making unjustified reasons to geometric statements. That is mainly due to poor concept formation by learners of the relationships among geometric shapes and the relationship among properties of a shape … While memorisation helps to commit the memory, understanding the concepts helps learners to gain knowledge and appreciate geometry content. Concepts that are understood stay longer in the child's memory than those that are memorised, that means our learners prefer memorisation to understanding hence the struggle with deductive reasoning (informal and formal).

Educator's interview responses as presented in the extracts above confirmed learners' sentiments to their geometry difficulties associated with higher levels of informal deduction. The general trend in the educators' interview responses as presented above was the view that most learners have difficulties with their understanding and application of theorems. As a result, when faced with problems that require theorem application, they ended up guessing the reasons to statements. This is what E5 referred to as learners making unjustified reasons to geometric statements.

Similarly, the educators attributed those poor deductive skills to poor concept formation by learners during instruction. In the same vein, in their interview responses, the learners above viewed poor conceptualisation as the reason why they end up resorting to rote memorisation of theorems. From a similar point of view, the educator extracts above suggest that the poor performance at informal deduction level was due to failure by learners to use logical argumentation to establish relationships between properties of shapes. Therefore, educators were of the view that learners had challenges with deductive reasoning because they tended to rely more on memorisation of geometry facts than knowledge development.

On the other hand, questionnaire responses by the 2 departmental heads who took part in the study revealed that educators were not using discovery methods to enhance learners' deductive thinking skills. For example, the response from H2 is presented below:

***H2:** For learners to be organised in solving multistep riders, they must be taught to discover the relations by singling out from a geometric diagram the shapes to be studied to better understand the applicable theorems … is the most appropriate approach because it discourages rote memorisation of theorems and the wrong application thereof … and curriculum-wise, the geometry content these learners are exposed to is much higher than their level of understanding, worse still most of our educators not well trained or workshopped to teach geometry.*

> ***H1**: Educators are not fully conversant with the topic, especially at the level of teaching learners deductive reasoning. Most of the educators underestimate the importance of letting learners explore geometric diagrams to discover the relationships between properties of shape and use it as the basis to develop deductive reasoning skills … and they emphasise drill and memorisation of theorems without teaching them applications skills.*

The questionnaire response as presented above by HODs' suggest that learners were not prepared well enough by their educators at lower Van Hiele levels to be able to function at the deduction level. As a result, they lacked the creativity that is needed to equip learners with deductive thinking skills. Furthermore, the above responses suggested that most educators lacked both teaching approach and geometry content knowledge to solve geometry riders.

Moreover, the existence of learning difficulties at the informal deduction level was confirmed through document reviews of the examiners' reports (2016 to 2020). Therefore, a review of these documents seems to confirm that some of the challenges identified from learners' scripts to question 4 have been persistently experienced by grade 12 learners.

Furthermore, the 2016 document reported that, at informal deduction level, learners do not scrutinise the given information together with the diagram for clues about theorems to use in answering the question(s). In addition, most of them end up making assumptions of geometric diagrams even though it is clearly stated from question paper instructions that not all diagrams are drawn to scale. Therefore, these observations by the examiners concur very well with data gathered from learner responses to question 4 at informal deduction level, especially for those learners who did not achieve as already suggested in table 4.10, as most of them guessed or made unjustified assumptions. Moreover, the Examiners' Report DBE (2017) cautions that learners are not exposed to the hands-on activities to establish the relationships among properties of shapes, and that could be the reason why most of them had gaps in their understanding short deductions. This suggestion was made in view of the challenges learners faced with establishing relationships between sides, angles and the related theorems as was the case with learners' responses to question 4.

More recently, the Examiner's Report DBE (2020) also argues that learners should avoid making conclusions that are unjustified and not related to their geometrical statements. Thus, at informal deduction level learners were found lacking in their conceptual understanding of relationships among properties of shapes, as they struggled to prove correct reasons for their statements. Therefore, to guard against such challenges, educators should ensure that when giving classwork or any class-based assessment task, they should not condone learners giving incorrect reasons. It was further argued that from the informal deduction level to the higher Van Hiele levels, learners should be taught logical and implication skills because most of them were found to be persistently making irrelevant reasons and/or statements.

Triangulation of challenges in both questions 3 and 4 at informal deduction level 2

In respect of learners' performance in questions 3 and 4, the table below gives an overview of those who achieved and those who did not achieve 50% per question.

Table 02

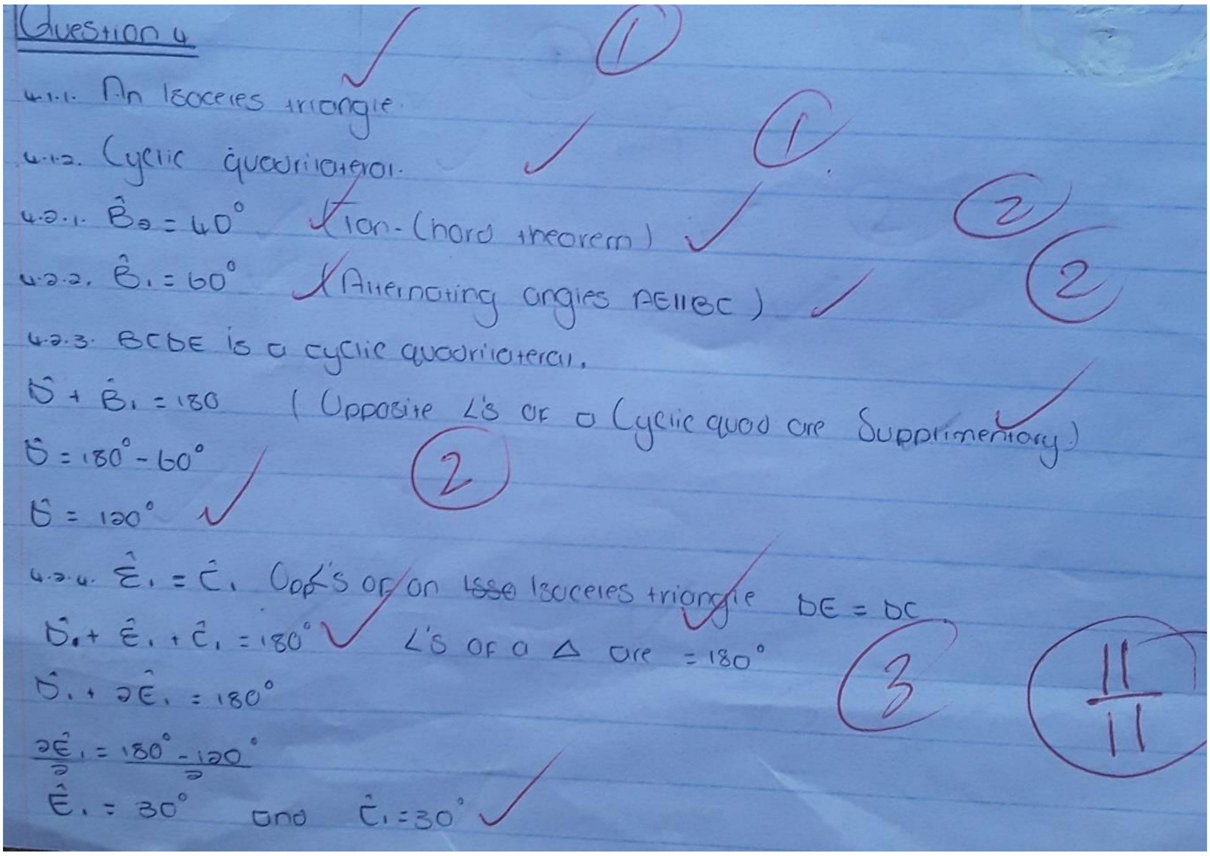
Frequency distribution of learners' abilities and difficulties for questions 3 and 4 related to Van Hiele's informal deduction level

Informal deduction level	**Numbers of learners who achieved basic level**	**(%) achieved**	**Numbers of learners who did not achieve basic level**	**(%) not achieved**	**Got 0 marks/ Question**	**Got full marks/ Question**
Question 3 Cognitive level (1 & 2)	18	30%	42	70%	8 (13%)	9 (15%)
Question 4 (Cognitive level 3)	13	22%	47	78%	14 (23%)	6 (10%)

Table 4.12 above was meant to demonstrate that when learners' achievement, or lack thereof, is presented side by side, the difference in performance can be seen clearly. As a result, data gathered from learner responses to both questions 3 and 4 showed that the sampled learners experienced difficulties at both the entry and exit levels of informal deduction. Therefore, comparatively, learners performed much better in question 3 than in question 4. There was a definite descending trend in learner achievement with an increase in the cognitive level of the questions within the informal deduction level, as indicated in table

4.12 where 18 learners achieved in question 3 as compared to 13 who achieved in question 4.

Similarly, there was an increase of 8% in the number of learners who did not achieve in question 4 from question 3. These figures probably suggest that as the cognitive level of the questions increase, together with an increase in the complexity of geometrical diagrams within a Van Hiele level, learner achievement decreases, with those that obtained 0 per question increasing from question 3 to question 4 while those who obtained full marks decreasing from question 3 to 4. Despite the challenges encountered by learners at cognitive level 3 of informal deduction as presented in question 4, one of the 6 learners who performed exceptionally well was L42 who managed to get a total for that question. In support of that, L42's extract is presented below:


Question 4

4.1.1. An Isoceles triangle (1)

4.1.2. Cyclic quadrilateral. (1)

4.2.1. $\hat{B}_2 = 40^\circ$ (tan-chord theorem) (2)

4.2.2. $\hat{B}_1 = 60^\circ$ (Alternating angles AE∥BC) (2)

4.2.3. BCDE is a cyclic quadrilateral.

$\hat{D} + \hat{B}_1 = 180$ (Opposite ∠'s of a cyclic quad are supplimentary)

$\hat{D} = 180^\circ - 60^\circ$

$\hat{D} = 120^\circ$ (2)

4.2.4. $\hat{E}_1 = \hat{C}_1$ Opp ∠'s of an ~~isse~~ Isoceles triangle DE = DC

$\hat{D}_1 + \hat{E}_1 + \hat{C}_1 = 180^\circ$ ∠'s of a △ are $= 180^\circ$

$\hat{D}_1 + 2\hat{E}_1 = 180^\circ$

$\frac{2\hat{E}_1}{2} = \frac{180^\circ - 120^\circ}{2}$

$\hat{E}_1 = 30^\circ$ and $\hat{C}_1 = 30^\circ$ (3)

$\frac{11}{11}$

Figure 4.17

Extract from L42 (correct statements and correct reasons)

In figure 4.17 it can be seen that L42 he/she had an exceptionally good mastery at high levels of informal deduction and was ready to be taught.

Furthermore, a general trend was identified in terms of the answers that learners gave to questions 3 and 4 from their test scripts, where those answers were categorised into four groups, as shown in the table below:

Table 03

Classification of learners' answers from their scripts

Category of response	**Question 3 and 4, average number of learners at: (50% of the items)**		**Explanation**
	Number	**%**	
Group 1	16	(27%)	Correct statement and correct reason
Group 2	11	(18%)	Incorrect statement and correct reason
Group 3	13	(22%)	Correct statement and incorrect reason
Group 4	20	(33%)	Incorrect statement and incorrect reason

The results from learners' scripts based on the classifications presented in table 4.13 showed that learners who achieved in questions 3 and 4 as suggested in table 4.12 had most of their answers in group 1. In this case, there was an average of 16 learners in group 1 for both questions 3 and 4. Those learners had a good mastery of informal deduction in both questions 3 and 4; therefore, they were able to give correct statements and reason. Furthermore, those who performed average in both questions were placed in group 3, where their statements to both questions 3 and 4 were correct for 50% or more of the items, even though their reasons were incorrect. However, looking closer at the answers to questions 3 and 4 for group 2 and 4 responses, it was found that there were more learners who wrote either incorrect statements, correct reasons, or both. In both instances, learners were considered as having failed that item and their responses were incorrect for more than 50% of the items. Therefore, that group constituted of most of the learners who did not achieve in both questions. A further analysis of group 2 and 4 responses showed that there were more learners who did not achieve in question 4 than there were who did not achieve in question

3. Overall, group 4 contained most of those learners who performed dismally in both questions, especially those who obtained 0 in both questions, as indicated in table 4.12.

Furthermore, the data presented in table 4.13 above are summarised graphically in terms of percentages in the form of a pie chart as shown below:

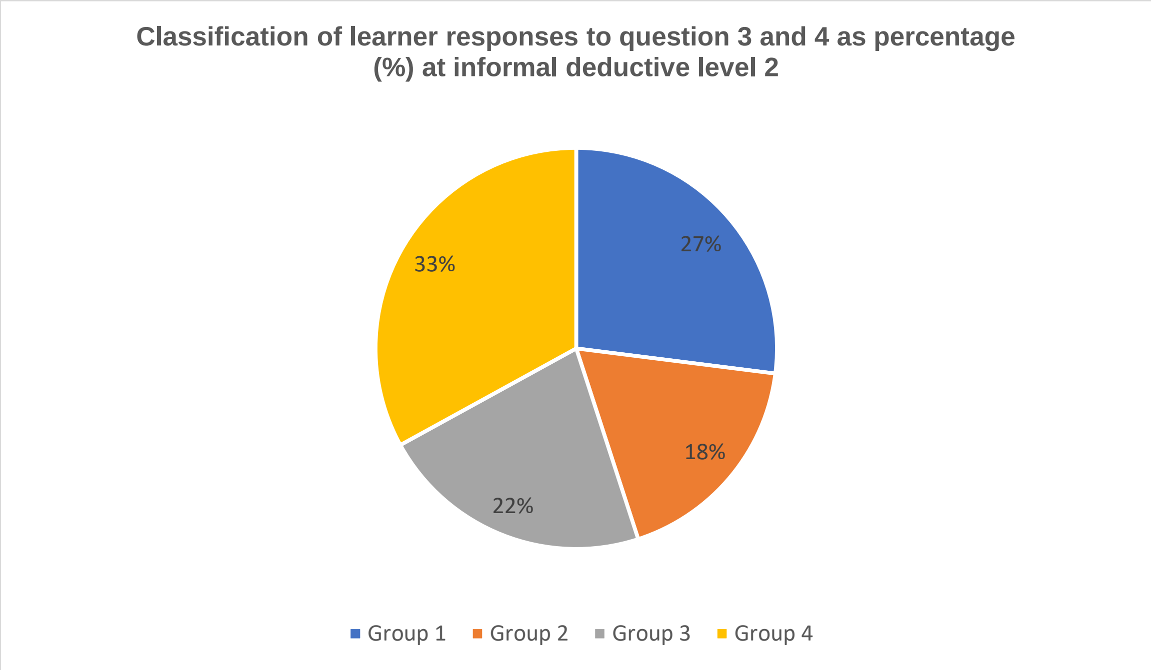


Figure 4.18

Classification of learner responses (%) to question 3 and 4

Figure 4.17 gives a visual impression of learner responses in terms of statement and reason at Van Hiele's informal deduction level 2, where group 4 had the biggest part (33%). The implication was that most learners wrote incorrect statements and reasons to both questions; therefore, they did not achieve. On the other hand, the group 2 had the smallest part, meaning that 18% of the groups' responses had both statement and reason incorrect for 50% and above of the items, with the other two groups falling in the middle.

Moreover, a common challenge as identified by the researcher from questions 3 and 4 was that most learners could not establish correct relationships between lines, angles and geometric shapes (triangles and cyclic quadrilaterals) in terms of their properties. Furthermore, learner responses to these two questions indicated that most learners had difficulties with theorem mastery and correct application in one-step riders. In addition, these responses also indicated that as the number of lines, angles and shapes increase within a circle (geometric diagram), learners struggle to make logical geometrical connections, resulting in them overgeneralising or making unjustified assumptions. In this vein, this research found challenges existed due to a lack of or poor conceptualisation of theorem at informal deduction level, as confirmed by learner responses to the questionnaire on the impact of memorisation in Euclidean geometry.

As result, that challenge was probably a result of the fact that learners tended to learn theorems through memorisation, hence ended up performing poorly.

That assertion was supported by learners' questionnaire responses to whether they understand or just memorise circle theorems as presented in the extract below:

Table 04

Memorisation of geometry theorems

	Frequency	**Percentage**
Strongly Agree	22	37
Agree	26	43
Disagree	8	13
Strongly Disagree	4	7
Total	60	100

Table 4.14 shows that 22 learners (37%) strongly agreed that they relied on memorisation to learn circle theorems, while 26 (43%) agreed with the statement, 8 (13%) disagreed and only 4 (7%) strongly disagreed. The implication was that, at informal deduction level, most learners relied on memorisation of theorems instead of understanding the relationships among properties of shapes from which the theorems were derived. Due to the fact that learners memorise theorems, they do not achieve because they forget what they were taught. Thus, it can safely be concluded that learners experienced challenges with conceptual understanding of theorems, resulting in them performing poorly at informal deduction level. It was also noted mostly from test scripts of those who did not achieve at informal deduction level that due to poor conceptualisation, learners could not distinguish between different shapes in the diagram and their properties.

Overall, the two questions enabled the learners to analyse the properties of geometry and determine their understanding and challenges related to the conceptualisation of properties of shapes and the relation between those figures. In addition, questions 3 and 4 enabled the researcher to determine the difficulties learners have with logical argumentation using properties of figures, even then though were not yet ready to create a new proof from scratch.

4.4.5 Learning difficulties related to Van Hiele's formal deduction level 3

Learning difficulties related to Van Hiele's formal deduction level consisted of two questions (5 and 6), where question 5 was set at cognitive level 2 and question 6 at cognitive level 3 and 4; in the test administered to learner respondents.

4.4.5.1 Question 5 in the test administered to learners explored learning difficulties related to Van Hiele's formal deduction level 3, cognitive levels 1 and 2, as shown below:

QUESTI0N 5 (Source: Northwest Ngaka Modiri Molema District Grade 11 Investigation 2019)

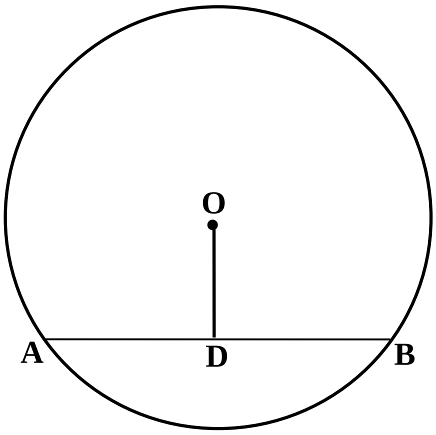
Complete the following statement

5.1 A line drawn from the centre of the circle, perpendicular to a chord………………… (1)

5.2 Given: Circle below with center O and Chord $\mathrm{AB}, \mathrm{OD} \perp \mathrm{AB}$

Required To Prove : AD = DB (5)

[6]



Expected solution to question 5:

5.1. *A line draw from the centre of the circle perpendicular*

to a chord bisect the chord √ (1)

5.2. *Consruct OA and OB.*√

In $\Delta\, ADO$ *and* $\Delta\, BDO$

$\widehat{D}_1$ and $\widehat{D}_2 = 90°$ $(OD \perp AD)$ √

$OA = 0B$ $(radii)$ √ S

$DO = DO$ $(common)$ √ S/R

$\Delta ADO \equiv \Delta BDO$ (RHS) √ S/R

$Therefore$ $\boldsymbol{AD = BD}$ $(congruency)$ (5)

[6]

Question 5 as presented above was set at Van Hiele's formal deduction level 3, cognitive levels 1 and 2. The reason for setting question 5 at lower cognitive levels of formal deduction was to explore their ability and difficulties in proving theorems. In this regard, question 5, as set in the test, was meant to explore abilities and difficulties in proving a theorem which states that a line drawn from the centre of the circle perpendicular to the chord, bisects the chord. The purpose of that question was to determine whether learners could use formal deduction to logically present a series of short statements and reasons, to prove a theorem as a form of reasoning that learners were expected to master before applying theorems to solve multistep geometric riders (cognitive levels 3 and 4). As a result, question 5 was meant to determine the challenges that learners experienced with logical reasoning as required in formal proofs of theorems. Similarly, the purpose of question 5 was to explore the learning difficulties that learners encountered when proving a theorem using an explicit sequence of established rules of deduction. Therefore, question 5 was relevant in that it prepared learners for the application of theorems at higher levels of formal deduction when required to solve multistep geometric riders.

Furthermore, those learner competences on the application of theorems to solve multistep riders were explored in question 6, which was also set at formal deduction level like question 5, but at higher cognitive levels (levels 3 and 4). Thus, at those cognitive levels, learners were required to demonstrate abilities and difficulties in using logical reasoning required in the application of circle theorems to solve multistep geometric riders. In this regard, a detailed analysis of question 6 and its purpose is provided in section 4.3.1.6 below.

In view of the above, data on the overall learner performance (achieved/not achieved) to question 5 at Van Hiele's formal deduction level 3 (theorem proof) at 50% achievement level per question as set in the test scoring criterion (section 3.7) are presented in the table below:

Table 4.15

Frequency distribution of learners' abilities and difficulties for question 5 related to Van Hiele's formal deduction level 3 (theorem proof)

Learners' levels of difficulties in relation to Van Hiele's formal deduction level	**Numbers of learners who achieved basic level**	**(%) achieved**	**Numbers of learners who did not achieve basic level**	**(%) not achieved**
Formal deduction (theorem proof)	22	37%	38	63%

The results presented in table 4.15 above illustrate learners' abilities and difficulties resulting from question 5 associated with Van Hiele's formal deduction level 3 (theorem proof). In this study, learners are regarded to have difficulties at Van Hiele's formal deduction level if they failed to:

- complete the theorem which states that a line drawn from the centre of the circle perpendicular to the chord at the point of contact, bisects the chord
- do the relevant constructions to the given diagram before presenting the proof
- reason from known facts, circumstances or observations about the relationships between sides and angles of isosceles triangles from construction
- understand the role of axioms, definitions and theorems to prove that the two triangles from construction are congruent
- apply formal deductions from congruency to present a series of statements and reasons leading to the conclusion that $AD = DB$.

The results presented in table 4.15 above revealed that 22 learners (37%) achieved 50% and above in question 5 at Van Hiele's formal deduction level 3, with 5 of them scoring full marks for the question. Data gathered from their scripts in relation to the expected competence levels at formal deduction level 3 indicated that those who achieved were able to apply formal deductions like proofs. Furthermore, those learners were able to reason from known facts, circumstances or observations and were able to arrive at the expected conclusion. In addition, data showed that most of the learners who achieved could construct proof, although little was known of their abilities to understand the role of axioms, definitions and theorems in geometry conceptualisation. In the same vein, most of those who achieved showed that they were able to use knowledge from previous Van Hiele levels as the basis to formally prove the theorem as presented in those solutions to question 5 above.

However, despite some of the learners managing to achieve 50% and above in the question, many were found to be experiencing difficulties with doing the necessary constructions to the given diagram before logically presenting the necessary steps in terms of statements and reasons. Thus, data gathered also suggested that those learners had difficulties with reasoning from known geometric facts to logical deductions. As a result, learners who achieved demonstrated a good mastery of basic levels of formal deduction and were considered as being ready to solve problems involving the application of formal deduction, like geometric riders. Therefore, examples of such learners' responses to question 5 are presented in the extracts below:

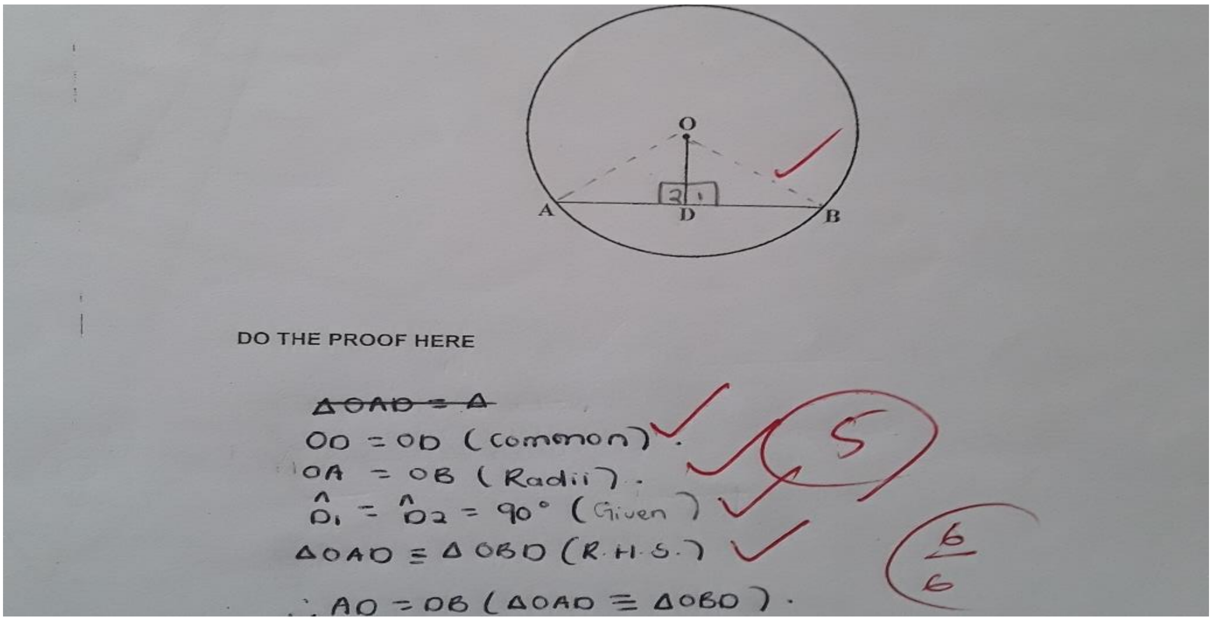


Figure 4.19

Extract from L18 (Good mastery of theorem proof)

Figure 4.20 above is an extract from L18's response, an example of the learners who achieved full marks for question 5. The logical presentation of steps in L18's response above shows that the learner had a good mastery of theorem proof. Everything seemed to be well organised, from correct constructions, logical presentation of geometric facts in the form of correct statements and reasons to making relevant conclusions. Such a presentation of a geometric proof was a clear indication that even those learners seemed to experience difficulties with formal deductive reasoning and remain stuck at lower levels of Van Hiele's hierarchy of geometry thinking. There were only a few learners who could reach a high level of geometrical reasoning.

On the other hand, data from table 4.15 above also indicate that 38 learners (63%) of the learner sample did not achieve 50% for question 5. Therefore, analysis of learner responses to question 5 showed that failure by those learners to achieve the pass mark of 50% in that question was attributed to their inability to reason from known geometric facts. Thus, a common trend among learner responses to question 5 for those learners who did not achieve was that such learners showed that a lack of basic knowledge from previous Van Hiele levels. Such basic knowledge those learners lacked included establishing relationships among properties of shape, which was the basis for logical reasoning.

In addition, a common trend among most of the responses from those who did not achieve was that most of them knew that they had to first prove that $\Delta ODA \equiv \Delta ODB$ but instead they merely concluded that the triangles are congruent without providing the necessary and sufficient conditions for those triangles to be congruent. This meant that most of those learners lacked conceptual understanding of the necessary and sufficient conditions for triangles to be congruent. That was evidenced by their failure to relate properties of $\Delta\, AOD\ and\ \Delta\, BOD$ through a series of logically related statements and reasons.

As a result, some learning difficulties as experienced by those who did not achieve are presented in the extracts below:

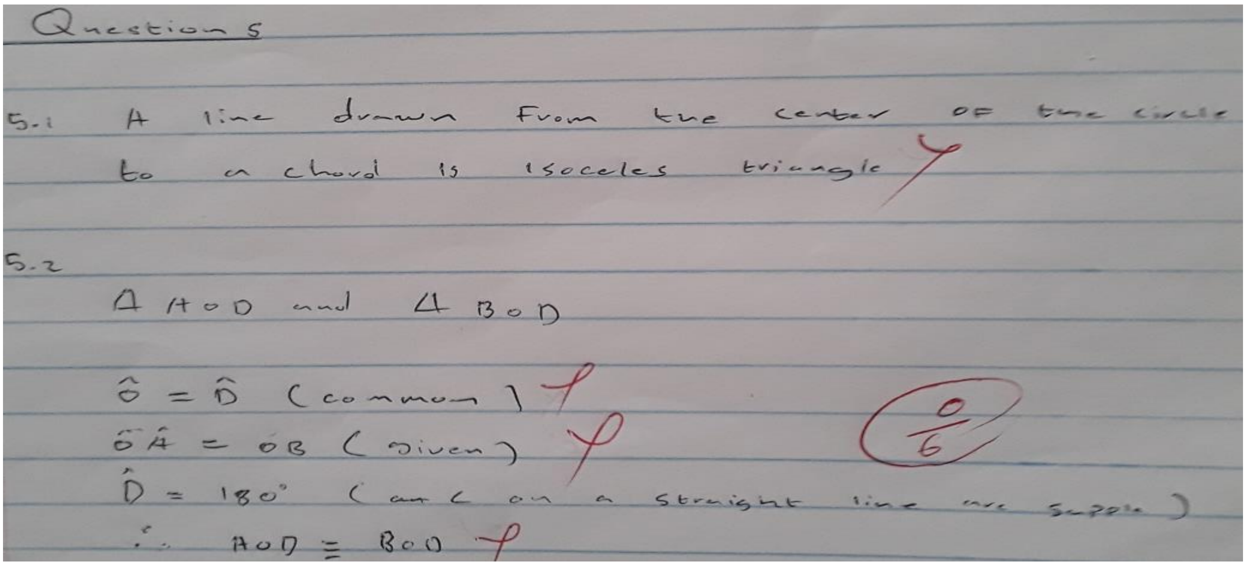


Figure 4.20

Extract from L10 (No knowledge of applicable theorem)

From figure 4.21 above, the learner's response to item 5.1 indicated that the learner the did not have any idea of the theorem he/she was required to prove. Thus, instead of completing the theorem by saying “bisects the chord”, the learner wrote is an isosceles triangle. That on its own hindered the learner from giving a correct response to item 5.2, as the learners lacked knowledge of the relevant theorem to that proof. In addition, L10 lacked the basics of theorem proof, which is construction. In this regard, the learner went on to write the proof without doing the necessary constructions; hence, both the statements and the reasons were incorrect.

Furthermore, a common trend identified by the researcher from learners’ responses was that all the learners who gave the wrong answers for item 5.1, also failed to correctly complete the proof of the theorem.

Similarly, some of those who did not achieve in question 5 showed the relevant constructions but they had challenges with the logical presentation of ideas leading to their conclusions. For example, L34’s responses as indicated below:

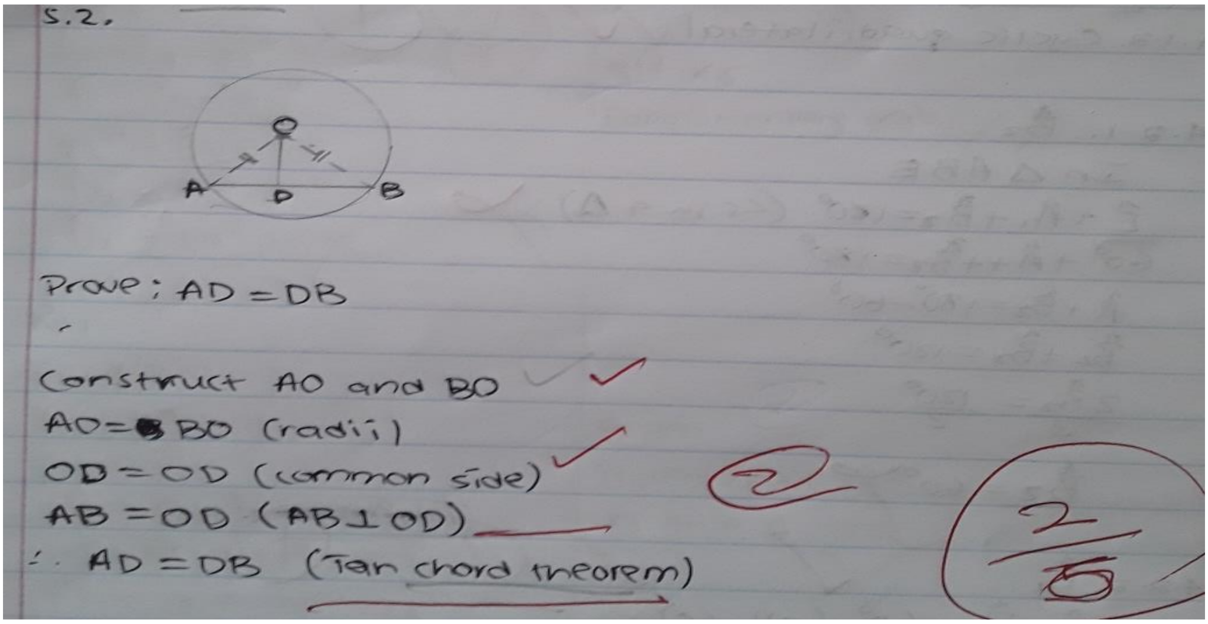


Figure 4.21

Extract from L34 (lack of fluency and logic)

The response, as shown in figure 4.21 above, indicated that the learner made unjustified assumptions, leading to wrong statements and reasons. The learner had knowledge of construction, although adjacent angles at D were not indicated as equal to 90°. In addition, the learner managed to show that $\mathrm{OD} = \mathrm{OD}\ (common\ side)$ and identified that $\mathrm{OD} = \mathrm{OD}$ $(common\ side)$. The learner could not deduce the subsequent steps leading to the conclusion that $\mathrm{AD} = \mathrm{DB}$. The solution presented in figure 4.21 shows that the learner had an idea of formal deduction (writing a statement and a reason), but he/she lacked fluency and logical presentation ideas.

Moreover, the general trend in learners' responses to question 5 was that those who failed to complete the theorem in item 5.1 in words also had difficulties proving the theorem in item 5.2. Such challenges also suggested that geometry is not given the necessary attention, especially the process of involving learners in deriving the theorem rather than using the telling method. As a result, there are higher chances that learners will resort to memorisation of proofs of theorems as they are presented in the textbook without understanding the underlying concepts.

However, information gathered from learner responses suggested that there were weaknesses associated with the use proofs of theorems to determine learner proficiency at formal deduction level. Even though there was a 37% achievement rate as revealed in table 4.15, those results were not a guarantee that these learners had a good mastery of formal deductive reasoning. A threat to that assumption is that those learners who achieved in question 5 could have memorised the proof as it is presented in the textbooks; hence, giving a false impression that a learner has mastered geometric proofs of theorems. However, that fact that the learners managed to present those geometric facts logically can be viewed as evidence that they understand some form of formal deductive reasoning. As a result, with such abilities, the learners can apply formal deductive reasoning to solving multi-step geometry riders.

Furthermore, the existence of learning difficulties associated with proofs of theorems was also highlighted in learners' interview responses, as presented in the quotations below:

> ***L34:*** *Our teacher told us that, when proving a theorem, we must start with construction. That is where my problem is; because those lines sometimes confuse me, I get stuck; especially on that one for the tan-chord theorem because it has many lines. Sometimes I join the diameter to the wrong angle, then when I try to the proof steps that I know from the textbook, my statements will be different from the reasons. I am not good at memorising all the proofs, especially those statements and reasons. The other thing is I can write the steps, but the reasons are too many and I end up mixing the theorems… As for that question 5 in the test, I knew the construction but my problem is writing those steps for congruent triangles. I knew that I was supposed to prove that the two triangles from construction were congruent; I think I only wrote the first two steps of the proof then I got stuck and I ended up guessing the other steps.*

L10: *Those theorems are too many and I cannot remember all of them. Especially that in question 5; I really got stuck start from the theorem they wanted me to complete. I completely forgot that theorem, I just wrote something. Worse was the proof itself; I did not know where to start. The only thing I remembered about that theorem is I should prove that,* $\Delta\, AOD \equiv \Delta BOD$*. That is what I wrote at the end.*

L25: *As for me, I don't have any problems with proving the theorems. We have all of them in our classwork books. Our teacher told us those that come in the examination, and I have practiced them. I make sure that my construction is correct, and as for the reasons; our teacher told us to write them as they are in the exam guideline… I always get good marks in proof questions, even though at times I make minor mistakes when writing some of the statements, especially the tan-chord theorem. Once I get that one correct, I am done with proofs.*

L37: *I can blame myself failing proofs because I have memorised the theorems as they are in the textbook, but when I am in the test, they confuse me. I don't do well in proofs because our teacher just told us "copy them as they are in the book", and I keep on missing some of the steps. Sometimes I mix them, and my teacher ends up marking it wrong. My problem is arranging the statements and reasons in the correct order… Proving that theorem was very confusing, and it was difficult to make sense of the … theorem. These proofs of Euclidean geometry are so difficult … with the circles, the lines, and angles, so it's like all mixed up, that it is why it is so hard … we can't understand it … other things like finding angles equal to* x *is better but, it becomes a nightmare to me if we are asked to provide reasons.*

An overview of learners' interview responses as presented in the extracts above revealed that some of them could remember that constructions were needed before proving. However, some of the learners confirmed having difficulties with construction. Therefore, most of those who did not achieve in question 5 did not show or state the relevant constructions. In this regard, the general trend among learner responses was that they had difficulties with presenting the actual proof. In this regard, some learners indicated they were stuck, not knowing where to start or how to arrange their ideas logically. As can be deduced from their responses, most of the learners blamed those difficulties on their reliance on memorisation of reasons to statements, which they end up forgetting or mixing up. In addition, their responses suggested that those learners had difficulties with logical reasoning. This was evidenced by the fact that most of them confirmed not being able to solve the geometric proof.

Furthermore, these extracts suggest that learners lacked fluency in their understanding geometric proofs, in that they tended to have an idea of the steps involved in a geometric proof, but they failed to present them in a coherent manner. For example, the issue raised by L10, that he knew he had to prove congruence of the two triangles, did not know where to start. That points to the idea that most learners at that level encountered challenges with logical reasoning because they did not realise that ideas are derived from the previous ones through deductive reasoning. For example, L37 lamented missing some steps in a geometric proof and at times mixing up the ideas. In this regard, learners blamed those difficulties to their learning style to proofs, where they prioritised memorisation and not conceptual understanding.

In support of the existence of these challenges with proofs of theorems, the 2017-2019 DBE examiners' report repeatedly confirmed that many candidates did not show the relevant constructions on the given proof diagrams. Some did not even state it or make incorrect constructions but proceeded to prove the theorem; and that constituted a breakdown in the proof.

Despite having those challenges, as alluded to by most of the interviewed learners, there were some that had a good mastery of proof of theorems, such as L25 who argued that he could do the necessary constructions and present the proof in a logical and coherent manner.

On the other hand, these learners' perspectives to challenges they encountered at formal deduction level 3 (proofs of theorems) were also confirmed by educators' interview responses. Some of the sampled educators' interview responses are presented in the quotations below:

> ***E1:*** *When proving geometry theorems, learners have shown that they lack knowledge of relating concepts in a logical way which leads to the proof, despite that the fact that those proofs are there in their textbooks. They have challenges with correct matching of statements and reasons. Some try to proceed with the proofs of the theorems without doing the relevant constructions.*
>
> ***E5:*** *In most instances, learners resort memorisation of theorems as they are presented in the textbook with very little understanding of the relationships among properties of shape to the relevant theorems. Thus, some signs of rote memorisation at the expense of understanding are revealed by the fact that any change of orientation of the diagram from the way it is presented in the textbooks results in learners getting stuck. In addition, their fluency or lack of it thereof in their presentation of geometric proof show that they are merely reciting memorised steps*

with little understanding. It is also seen by the way they present their work, where, in most instances, they just don't complete those proofs.

E3: *Most of the learners make wrong constructions. Some fail to arrange their steps, accordingly, taking into consideration the relationships among properties of the shapes involved. Some are the ones that start by writing what they are supposed to prove as the first statement of the proof; instead of using other relationships that lead to the required proof … Some even get completely stuck, especially with the constructions because most of them cannot mentally manipulate the geometrical diagram from the orientation that they developed the first time they were taught about that theorem or the diagram they have memorised.*

E2: *Learners find it difficult to structure logically a proof of a theorem, which means they have difficulties connecting steps from the relevant first statements to the supposed last concluding statement. When it comes to proof of individual theorems, learners tend to memorise and find it difficult to do the same proof if only the orientation of the diagram changes, rather than the concept.*

E6: *Drawing the relevant constructions is a challenge for most learners. Thus, most of the learners miss the proof by not doing the constructions at all or doing wrong constructions. Besides, the other challenge is most of them cannot do step-by-step logical statements and reasons. That is probably because most of them lack a logical thinking skill which is informed by their understanding of relationships among properties of shapes. Most of the learners have difficulties recalling or remembering the steps leading to the proof of a theorem.*

The above extracts from educators' interview responses revealed that most of the learners had challenges with formal deductive reasoning at the level of proving geometry theorems. Most of the educators' interview responses suggested that those learners lacked basic formal deductive reasoning skills needed to logically present geometric ideas in a way that leads to the desired proof. In this regard, information from the above interview responses suggested that most of the learners resorted rote memorisation of proofs with little understanding of the underlying concepts. Those educator perspectives were based on their assessment of learner presentations of proofs in tests and examinations where they were found to be mixing the steps, doing wrong constructions and making disconnected statements and reasons. That was attributed to the fact that proving riders is an abstract process that many learners find difficult to understand. Therefore, the general trend across educator responses as presented above indicated that despite some learners scoring good marks in the question on proofs, most of them lacked both conceptual and procedural fluency in their presentations; a sign that most these learners had challenges with formal deduction.

Triangulation of data from participants' responses to question 5

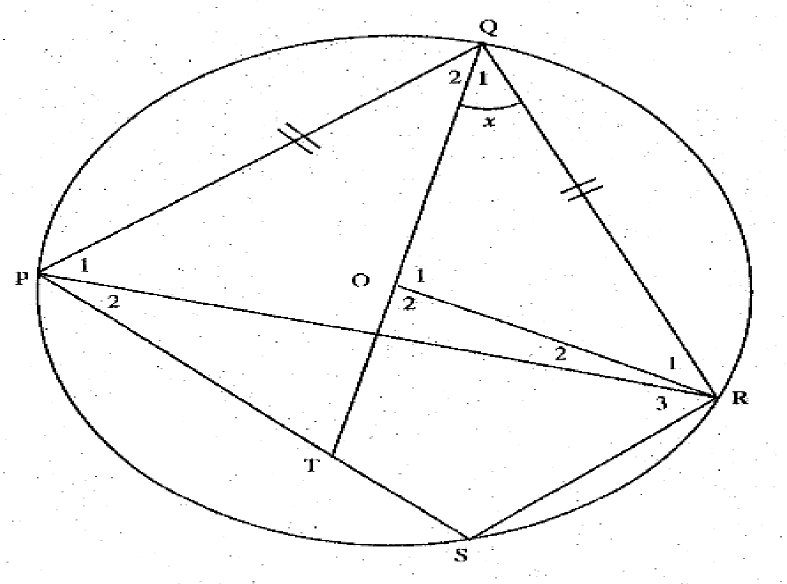
Triangulation of data gathered from learners' and educators' responses on learning difficulties associated with proofs of theorems can be summarised as follows:

- The responses showed that most of the learners relied on memorising proofs of theorems through rote learning, as correct statements were supported by incorrect reasons.
- Learners struggled with logical arrangement of geometrical ideas.
- Some learners write geometric statements without giving reasons.
- In some instances, learners write the required information to prove as a given statement.
- Some learners concluded without stating the necessary and sufficient conditions for a triangle to be congruent.

4.4.5.2 Question 6 in the test administered to learners explored learning difficulties related to Van Hiele's formal deduction level 3, cognitive levels 3 and 4, as shown below

QUESTION 6 (Source: Mpumalanga province Grade 12 P2 Preparatory Exam 2018)

The diagram below shows that O is the centre of circle PQRS. QOT is a straight line such that T lies on PS. PQ = QR and $\hat{Q}_1 = x$.



6.1 Calculate, with reasons $\hat{P}_1$ in terms of x. (4)

6.2 Prove that TQ bisects $P\hat{Q}R$. (3)

6.3 Prove that STOR is a cyclic quadrilateral. (5)

[12]

Expected solutions to question 6

6.1. $OQ = OR\ (radii)$ √ S/R

$\hat{Q}_1 = \hat{R}_1 = x\ (\angle s\ opp = sides)$ √ S

$\hat{O}_1 = 180° - 2x\ (sum\ of\ \angle s\ of\ a\ \Delta)$ √ S/R

$\hat{P}_1 = \frac{180°-2x}{2}\ (\angle\ at\ the\ circum = \frac{1}{2}\angle\ at\ centre)$

$= 90° - x$ √ S (4)

6.2. $PQ = RQ\ (given)$

$\hat{P}_1 = \hat{R}_1 + \hat{R}_2 = 90° - x\ (\angle s\ opp = sides)$ √ S

$\hat{Q}_1 + \hat{Q}_2 + \hat{P}_1 + \hat{R}_1 + \hat{R}_2 = 180°\ (sum\ of\ \angle s\ of\ \Delta)$ √ S/R

$x + \hat{Q}_2 + 90° - x + 90 - x = 180°$

$\hat{Q}_2 = 180° - 180° + 2x - x$

$\hat{Q}_2 = x$

$\hat{Q}_1 = \hat{Q}_2 = x$ √ S (3)

$\boldsymbol{TQ\ bisects\ P\hat{Q}R}$

6.3. $\hat{Q} + \hat{S} = 180°\ (\ opp\ \angle s\ of\ cyclic\ quad\ are\ supplementary)$ √ S/R

$but\ \hat{Q} = 2x\ (above)$

$\hat{S} = 180° - 2x\ (\hat{Q} = 2x\ from\ above)$ √ S/R

$OR = OS\ (radii)$

$\hat{Q}_1 = \hat{R}_1 = x\ (\angle s\ opp = sides)$

$\hat{O}_1 + \hat{R}_1 + \hat{Q}_1 = 180°\ (\angle s\ of\ \Delta)$

$\hat{O}_1 = 180° - 2x$ √ S

$\hat{O}_1 = \hat{S} = 180 - 2x$

$Therefore, STOR\ is\ a\ cyclic\ quad\ (ext\ \angle\ of\ cyclic\ quad = int\ opp\ \angle)$ √ S (5)

[12]

As alluded to in the previous section, both question 5 and 6 were set at formal deduction level, although the two questions were at different cognitive levels. Question 6 was set at the Van Hiele's formal deduction level 3 (cognitive levels 3 and 4), the highest level of geometrical reasoning in South Africa's secondary school mathematics. The question 6 was different from question 5 in that it is set at a higher cognitive level of formal deductive thinking where the learners had to demonstrate their competences in applying geometry theorem to solve multistep riders. In addition, question 6 was meant to explore earner abilities and challenges on the application of logic in proving the necessary and sufficient conditions to make certain conclusions about lines, angles or given shapes within a geometric diagram. Thus, in addition to using facts, axioms and relationships among properties to make conclusions about a given theorem as in question 5, question 6 focused more on the application of those theorems to formulate a series of deductions for certain conditions to be true. For example, learners had to present a series of deductions using theorems to show that a given line is bisector of an angle, and a certain quadrilateral whose vertices are not on the circumference of a given circle, is cyclic.

Furthermore, such questions items demanded high levels of formal deduction far above an ordinary proof of theorem as set in question 5. Therefore, the purpose of setting questing 6 at this level of formal deduction was to ascertain the difficulties learners experienced at that level of deductive reasoning, as it is the highest level of geometry thinking they are expected to have acquired by the time they are in Grade 12. More importantly, the purpose of setting question 6 at that level of formal deduction was to explore learners' learning difficulties as they present geometric statements orally or in writing, perform geometric manipulations, check validity of geometric statements and draw conclusions from geometric statements. Similarly, question 6 was set to determine challenges that learners had in applying theorems to solve multistep geometric riders.

Therefore, data on the overall learner performance (achieved/not achieved) in question 6 at Van Hiele's formal deduction level 3 at 50% achievement level per question, as set in the test scoring criterion (section 3.7), are presented in the table below:

Table 4.16

Frequency distribution of learners' abilities and difficulties for question 6 related to Van Hiele's formal deduction level 3 (cognitive levels 3 and 4)

Learners' levels of difficulties in relation to Van Hiele's formal deduction level	**Numbers of learners who achieved basic level**	**(%) achieved**	**Numbers of learners who did not achieve basic level**	**(%) not achieved**

Formal deduction (application of theorems)	8	13%	52	87%

Table 4.16 illustrates learners' abilities and difficulties resulting from question 6 associated with Van Hiele's formal deduction level 3. In this study, learners were regarded to have difficulties at Van Hiele's formal deduction level if they failed to:

- apply formal deductions to calculate the missing angles in terms of x
- reason from known facts, circumstances, theorems or observations to prove that line TQ bisects $\hat{Q}$
- apply axioms, definitions and theorems and knowledge of relationships between properties of shapes to write a series of statements and reason leading to the proof that $STOR$ is a cyclic quadrilateral.
- demonstrate their knowledge relevant theorems to solving multistep riders, angle-side relationships for tringles and apply logical thinking to prove that a quadrilateral is cyclic without physically seeing the circle through its vertices.
- give correct statements and reasons in the solution process.

The results presented in table 4.16 revealed that only 8 learners achieved 50% and above in question 6. Those learners who achieved showed a good mastery of the above-mentioned expected competences at Van Hiele's formal deduction level Thus, some of the competences shown in learners' responses were their ability to reason from facts, circumstances and observations, and applying formal deduction to find another angle in terms of the specified variable. In the case of question 6, those learners were able to express $\hat{P}_1$ in terms of x. Furthermore, some of those learners were able to use the sum of the angles of a triangle to express the missing angle in terms of x. Despite those learners demonstrating a good mastery of the concepts involved, some failed to correctly apply theorems to prove that $STOR$ is a cyclic quadrilateral.

On the other hand, table 4.16 reveals that 52 learners (87%) did not achieve 50% in question 6. Thus, considering the number of learners who did not achieve against those who achieved, one can safely conclude that learner performance in question 6 was very poor, which is an indication that learners had serious difficulties with high levels of formal deductive reasoning.

To better understand the learning difficulties associated with a high level of formal deductive reasoning, a further analysis of question 6 was done per item of this question in terms of the three categories: those who did not attempt the question, those who attempted but had it wrong and those who had each item correct. Therefore, a summary of the distribution of learner performance according to these categories is presented in table 4.17 below:

Table 07

Frequency distribution of Learners' questionnaire responses to learning difficulties related to Van Hiele's formal deduction level 3 using question 6

Question 6 items	**(%) of learners who did not attempt at all**	**(%) of learners who attempted but got wrong**	**(%) of learners who attempted and got it correct**
6.1	9 (16%)	41 (73%)	10 (17%)
6.2	16 (27%)	37 (62%)	7 (12%)
6.3	20 (33%)	34 (57%)	6 (10%)

The overall impression given by the data presented in table 4.17 above was that learners performed badly at the Van Hiele level 3 (formal deduction level). This conclusion was reached because most of the respondents did not even attempt the question, and most of those who attempted it had it wrong. On the other hand, from table 4.17 it could be seen that only a few learners attempted question 6 and had it right. In addition, the data as presented in table 4.17 show that as the cognitive levels of the questions increased with the question's items, the number of learners who achieved per item decreased.

The following are some of the extracts from learner scripts for question 6, analysed per item according to the three categories as indicated in table 4.17 above. Firstly, there were 9 learners did not achieve in 6.1 because they did not even attempt the question and they were considered as having failed that question item. Interpreted according to Watson's classification of errors (1980), those learners lacked motivation, probably due to insufficient formal deductive knowledge.

Secondly, table 4.17 shows that there were 41 learners who attempted item 6.1 but did not get it right. In support of that, the sampled extracts below for L34, L21 and L40 were used to explain some of the learning difficulties experienced by such learners and the analysis thereof:

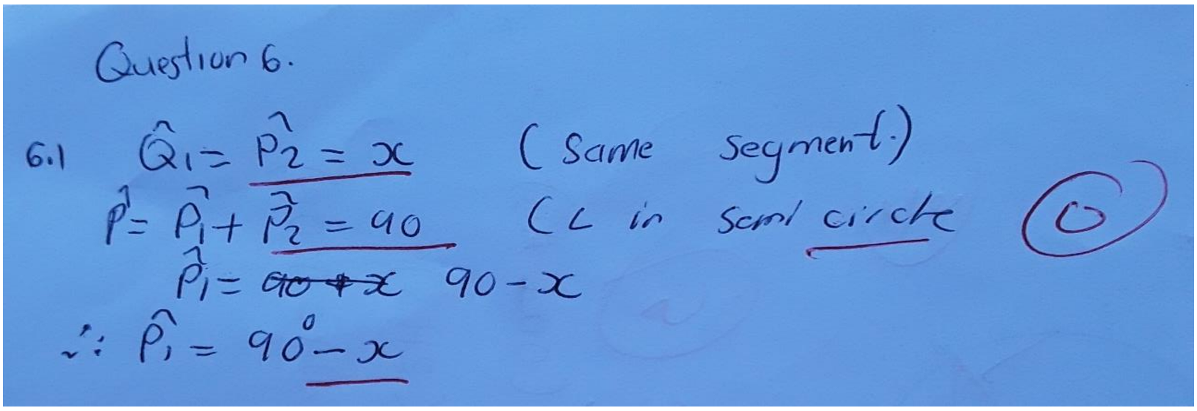


Figure 4.22

Extract from L40 (wrong conception of angles in the same segment)

From figure 4.22 above, it can be seen that L40 had an idea that statements had to be written with reasons using properties of shapes; however, there were many content gaps in this learner's conceptualisation of the relationships, as presented in the diagram. Some of the identified misconceptions included the learner's misunderstanding of angles in the same segment. Because L40 failed to define a segment in question 2, this challenge was carried over to the application of the concept (segment) at Van Hiele's formal deduction level. In that case, the learner seemed not to know that for angles in the same segment to be equal, they should be subtended by the same arc/chord at the circumference. However, the learner failed to realise that $\hat{P}_2\, and\, \hat{Q}_1$ were not subtended by the same chord and, worse still, point T was not on the circumference, hence the reason given was incorrect.

Furthermore, the second reason given ($\angle\, in\, a\, semi - circle$) shows another misconception held by the learners about that theorem. The learner did not realise that TQ was not a diameter because point T was not on the circumference, even though the line passed through the centre and subtending $\hat{P}$ at the circumference. Therefore, even though the learner finally showed that $\hat{P}_1 = 90° - \theta,$ this conclusion was from incorrect deductions, hence it was wrong.

On the other hand, L21 held a similar misconception, as shown in the extract below:

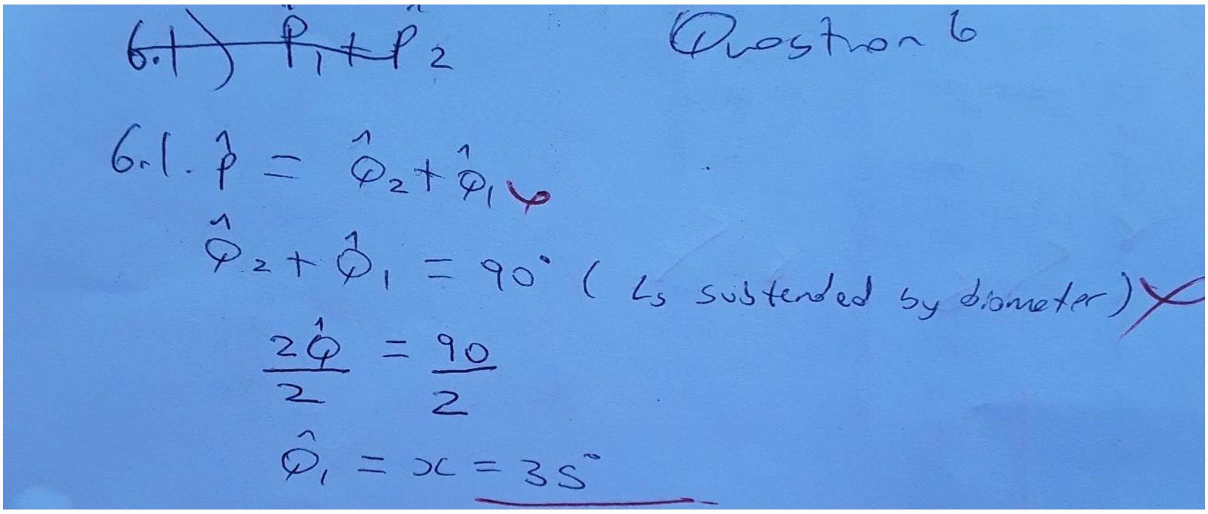


Figure 4.23

Extract from L21 (Wrong conception of angles in semi-circle)

From figure 4.23 above, the learner response to 6.1 also suggested that the learner made unjustified assumptions by writing some statements without reasons. This showed that the learner lacked fluency in his understanding of geometry theorems. Besides, just like L40, L21 above had a misconception about angle in a semi-circle. Similarly, L21 assumed that PR was a diameter, even though it did not pass through the centre. In view of these challenges, learners like L21 and L40 above were regarded as not yet ready for formal deductive reasoning.

Despite those learners showing insufficiencies in their deductive reasoning capabilities; table 4.17 shows that there were 10 learners who achieved 50% and above for item 6.1. One of those learners was L58 who scored full marks, as shown in the extract below:

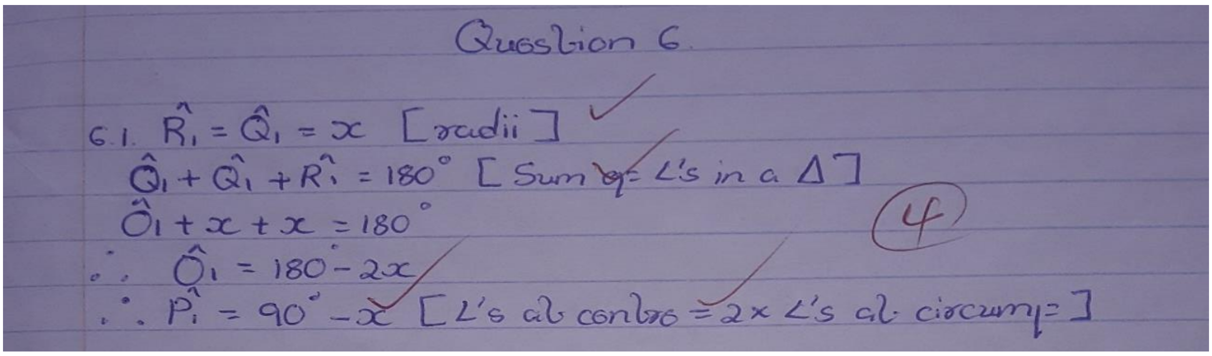


Figure 4.24

Extract from L58 (Good mastery of multistep geometric riders)

In contrast to L21 and L40 above, L58 was one of the learners who achieved full marks in question 6.1. The learners showed a good mastery of formal deductive reasoning, where both statements and reasons were presented correctly.

Thus, the presentation in figure 4.24 above suggests that L58 had a good mastery of both conceptual and procedural knowledge at Van Hiele's formal deduction level. In addition, according to Van Hiele's expected level of competence at formal induction level, the learner was able to apply formal deductions to prove a given statement, reason from known facts, circumstances or observations, and apply relevant theorems to solve multistep geometric riders.

Unfortunately, learners who were found to be operating at that level of reasoning, as shown in table 4.17 above, constituted only 10% of the sample of 60 learners. This is an indication that very few learners in Grade 12 can operate at Van Hiele's formal deduction level.

Furthermore, a similar analysis was also done of item 6.2, and the data gathered showed that 16 learners did not even attempt to answer that question item. Thus, data as presented in table 4.17 showed an increase in the number of learners who did not attempt 6.2 when compared to those who did not attempt 6.1. This increase suggested that more learners experienced learning difficulties as the cognitive levels of the questions increased, which was an indication that most learners remained stuck at the lower levels of Van Hiele's hierarchy of geometric thinking.

In view of question item 6.2, learners were required to prove that $TQ\ bisects\ P\hat{Q}R$. Those learners who did not attempt the question showed that they lacked basic knowledge to establish the specified relationships in terms of the relevant theorems.

Secondly, there were 37 learners who attempted 6.2 but had it wrong. Some of the sampled learner responses to question 6.2 for those who did not achieve included L60 and L55, as illustrated in the extracts below:

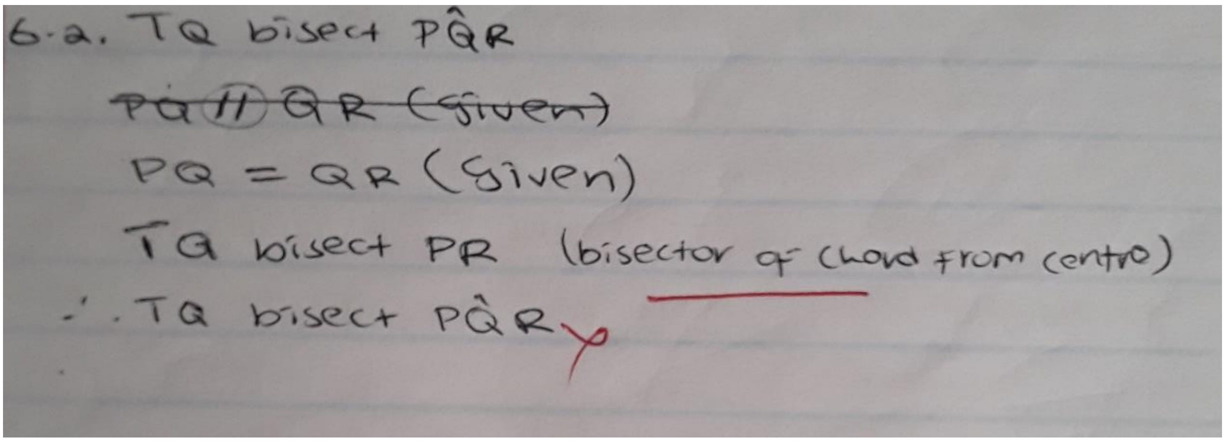

6.2. TQ bisect $P\hat{Q}R$
~~PQ // QR (given)~~
PQ = QR (given)
TQ bisect PR (bisector of chord from centro)
∴ TQ bisect $P\hat{Q}R$

Figure 4.25

Extract from L60 (Poor distinction of given from required)

Figure 4.25 above suggests that L60 had difficulties with distinguishing between the given and the required in a geometric proof. For example, the learner started by writing TQ bisects $P\hat{Q}R$. In this case, the learner wrote the required as the given and therefore had nothing to prove. Similarly, other learners among those who did not achieve were found to have made circular arguments by assuming what they were supposed to prove and then made conclusions like that which they had already assumed.

Furthermore, L60 wrote TQ bisects PR, which was an incorrect assumption based on poor understanding of the theorem about bisector of a chord from the centre. Thus, the learner had a challenge in identifying the necessary and sufficient conditions for a geometrical statement to be true. Based on this data, the researcher concluded that learners struggled to make formal deductions based on the relationships between shapes, their properties and the relevant theorems. Thus, despite having knowledge of the theorems applicable within a given diagram, learners lacked the specifics in terms of the underlying conditions for those theorems to suit the given relationships.

Moreover, among those who did not achieve and showed similar challenges to L60, was L55 whose response to 6.2 is shown in the extract below:

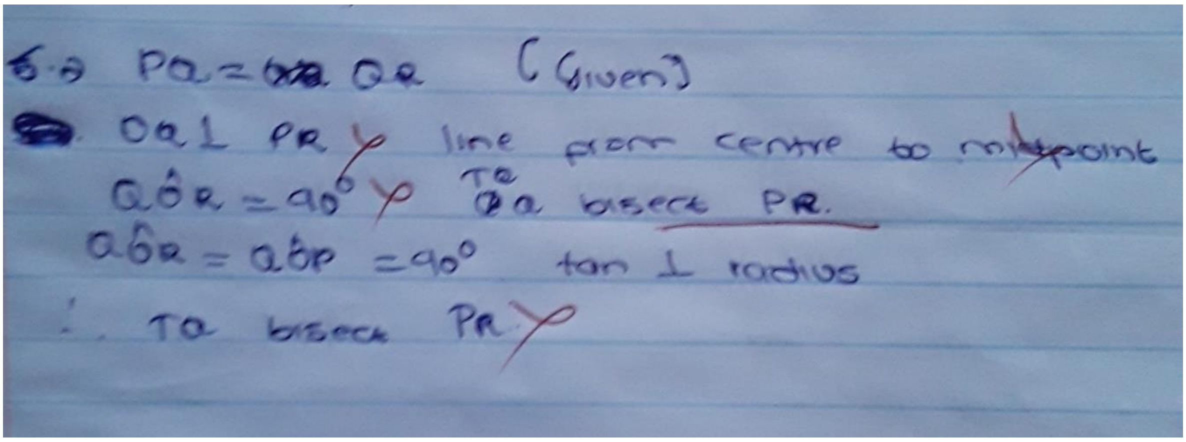


Figure 4.26

Extract from L55 (Making unjustified assumptions)

From figure 4.26 above, it can be deduced that L55's response suggests that the learner had difficulties with choosing correct statements and reasons. In other words, the learners made unjustified assumptions based on a misconception about a line from the centre of a circle. The researcher's conclusion was that learners held the view that any line from the centre of the circle to the chord always bisects the chord, and then concluded that $OQ \perp PR$ disregards the necessary and sufficient conditions for that theorem to be true. Thus, the learner assumed that $QT\ bisceted\ PR$. In this regard, the learner showed a lack of fluency in his understanding of circle theorems.

Despite learners having difficulties with item 6.2 as indicated in the presentations above, table 4.17 shows that there were those that attempted and had the question correct. One such learner was L58 whose response to 6.2 is presented in figure 4.27 below:

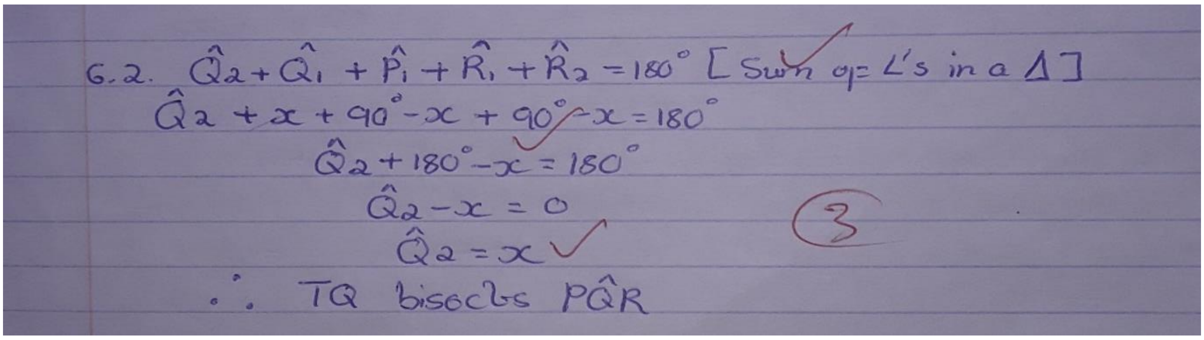


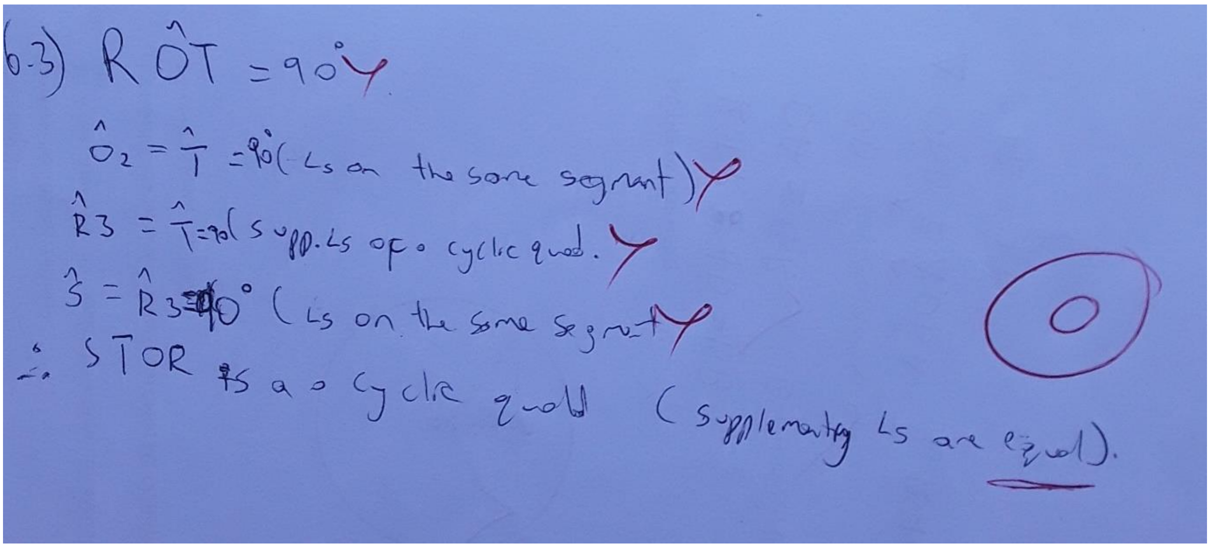
Figure 4.27

Extract from L58 (Good mastery of logical reasoning)

From figure 4.27, the data as presented showed that the learner had a good understanding of formal deduction. Thus, according to table 4.17, only 7 learners (12%) attempted item 6.2 of this category and managed to achieve 50%. In this category, L58 was one of the learners who obtained full marks for 6.2. Such a performance is an indication that the learner had a good understanding of formal deductive reasoning. The learner, according to the presentation of the proof, showed that he had a clear understanding of the relationships between properties of shapes and the theorems applicable to those relationships. In addition, the data show that that the learners could logically arrange ideas, which one of the competences for a learner to operate at formal deductive reasoning.

Furthermore, item 6.3 was set at a higher level of formal deductive reasoning, where learners were supposed to demonstrate the highest level of logical arrangement of geometrical ideas. Moreover, the question required learners to demonstrate their abilities to use visual imaging together with their understanding of properties of cyclic quadrilaterals to prove a theorem without a circle drawn passing through the vertices of the quadrilateral. In the same vein, as indicated in table 4.16 above, only 10% of the learners attempted question 6 and only 6 learners (10%) scored 50% and above for question 6.3. It showed that most of the Grade 12 learners were not ready for high levels formal deductive thinking.

The extract below is an example of a learner who attempted 6.3 but did not get it correct.



6.3) $R\hat{O}T = 90°$

$\hat{O}_2 = \hat{T} = 90°$ (∠s on the same segment)

$\hat{R}3 = \hat{T} = 90$ (supp. ∠s of a cyclic quad.

$\hat{3} = \hat{R}3 = 90°$ (∠s on the same segment

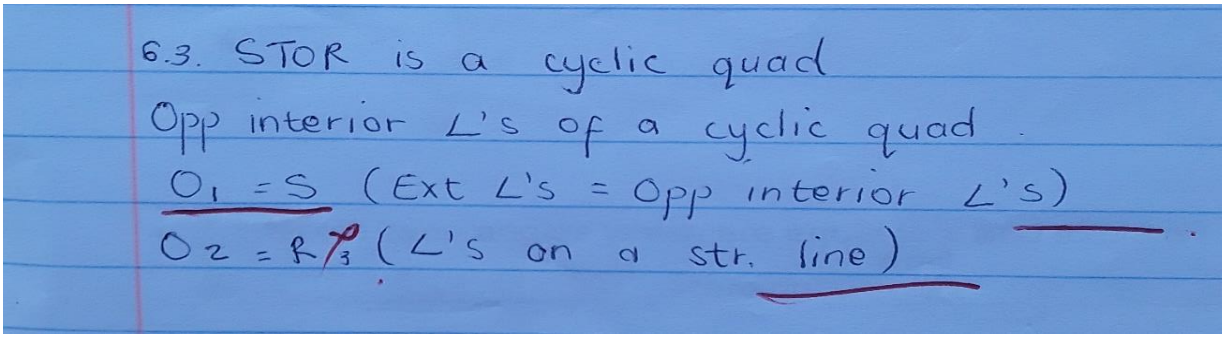
∴ STOR is a cyclic quad (supplementary ∠s are equal).

0

Figure 4.28

Extract from L21: (unjustified statements and reasons)

From the responses given by L21 above, there is a clear indication that the learner was not ready for formal deductive thinking. The learner's presentation in figure 4.28 shows that the learner did not start by analysing the diagram for clues and relationships that could help him to solve the problem. In this case, learners were required to use the relationship between the exterior and the opposite interior angle to prove that STOR is a cyclic quadrilateral. Thus, in the case above, the learner just started with an assumption that $R\hat{O}T = 90°$, without justifying that statement with a reason. Furthermore, L21 just wrote the statements and reasons that did not correspond and were not in any way related to the problem solution, which was an indication that he/she had content gaps emanating from the lower levels of van Hiele's hierarchy of geometric thinking.



6.3. STOR is a cyclic quad

Opp interior ∠'s of a cyclic quad

$O_1 = S$ (Ext ∠'s = Opp interior ∠'s)

$O_2 = R_3$ (∠'s on a str. line)

Figure 4.29

Extract from L2: (No clue of the solution strategy)

Similarly, L2 also started with the assumption that STOR is a cyclic quadrilateral, which means this learner had difficulties in differentiating between the required and the given information in a geometric proof. Furthermore, the challenge of overgeneralisation was common among learner responses at formal deduction level. Thus, data from L2's response above suggested that the learner had an idea that STOR had to be a cyclic quadrilateral $\hat{O}_1 = \hat{S}$, but the learner had no clue of the sequence of deductive steps that could lead to that conclusion. As a result, the learner went straight to the conclusion that $\hat{O}_1 = \hat{S}\ (ext\ \angle = opp\ int\ \angle)$. One can conclude that L2 lacked procedural knowledge of that proof because the learner knew what was to be proven but did not have the solution strategy (a chain of steps leading to the solution). This presentation also confirms Barut and Retnawati's (2020) assertion as alluded to in literature that learners lacked the visualisation ability, and most importantly, some demonstrated insufficiency in providing proper visual reasoning capabilities in the interpretation of geometric diagrams based on their formal definition.

Similarly, in item 6.3, about 90% of the sampled learners had difficulties transitioning from natural language representations to diagrammatic representations through mental imaging since the circle was not drawn, but they were required to prove that STOR is a cyclic quadrilateral. Therefore, the learners failed to mentally visualise the necessary and sufficient conditions for STOR to be a cyclic quadrilateral in the absence of the circle.

However, a few of the learners showed a good mastery of the concepts involved in question 6.3. A good example is L58 who scored full marks in item 6.3, as shown in the extract below:

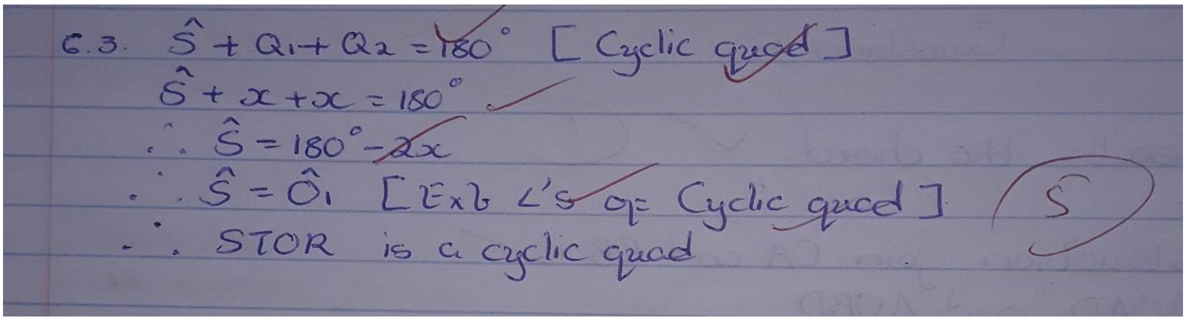


Figure 4.30

Extract from L58 (Good mastery of multistep geometric riders)

The data from figure 4.30 above revealed that L58 had a good mastery of formal deductive thinking. Thus, the learners demonstrated high levels of logical reasoning by linking statements and reasons in a coherent manner. The learner had a clear idea of the necessary and sufficient conditions to be satisfied for a quadrilateral STOR to be cyclic. The learners managed to link the solution to 6.1 (figure 4.24) and 6.3 (figure 4.30) and concluded that if the exterior angle of a quadrilateral is equal to the interior opposite angle, it is cyclic. However, the major challenge was that only 10% of the learners were able to reason at such a level of deductive reasoning; with the other 90% operating far below this level.

In view of all the above, poor performance in question 6 confirmed the importance of visualisation in geometry, where visualisation of figures is important for successful identification of relationships in a geometric diagram. Thus, from the data collected at the formal deduction level (question 6), the researcher observed that learners often made several correct statements or reasons but could not prove that STOR is a cyclic quadrilateral. This view is supported by Presmeg (2006) who argues that visualisation involves visual imagery with or without a diagram, as an important component of the solution. That explains why learners experience spatial errors such as the ability to visualise geometry, and studies done by Saputra et al. (2018) suggested that difficulties in visualisation geometry affect their overall mathematical abilities. As a result, only 6 learners achieved in question 6, and the rest did not achieve, which shows that those learners had challenges with visual imagery, as they failed to apply the properties of a cyclic quadrilateral without physically seeing the circle.

Furthermore, the existence of learning difficulties associated with question 6 at Van Hiele's formal deduction level 3 at cognitive levels 3 and 4 was also highlighted in learners' interview responses, as presented in the quotations below:

> ***L40:*** *I can't link the steps correctly but sometimes I write the correct statements. My major problem with geometrical proofs is writing the correct reasons. Sometimes I don't understand the question and just write what think. I think it is because some proofs are too long, and if I can't finish then it is difficult to arrange the steps in the correct order… I can at least try those questions saying "find another angle equal to* x*; those ones I enjoy them. My problem is where I must prove that* $STOR$ *is cyclic quadrilateral like the one in question 6 of that test I really didn't know where to start, but I wrote something even though I got wrong.*

> ***L55:*** *The challenge is I am not able to understand which theorem to use when proving something. The problem comes when I have show many steps to get to the answer, I don't know how to arrange the statements to get the correct answer. Especially, that question 6 which was saying "prove that line was a bisector of an angle", I was completely lost, I only knew that bisect means cutting into two equal parts, but my problem was I did not know where to start.*

> ***L60:*** *Yes, sometimes I provide the statement that is needed in the question, but sometimes I provide wrong reasons for my statements, or I mix things. I have a*

problem with question that say "prove that …" I sometimes I just start with that statement and write (given). After that, I struggle to write other steps. At times, I only write the theorem that I know. When I get my answer sheet, most of my answers are wrong. To be honest, I am not good in geometry, and I always get low marks… Yooo! Those questions are confusing; especially that question 6.3 where they wanted us to prove that shape is cyclic quadrilateral. I wanted to ask you if they did not forget to draw the circle because our teacher told us that for a shape to be a cyclic quadrilateral there must be a circle passing through its four corners. As for that diagram, there was no circle passing through $STOR$ so I could not prove it, and I did not write it.

L58: *I have a problem here and there with those proofs, but as for that question 6, I really enjoyed it. Even that question which requires us to prove that $STOR$ is a cyclic quadrilateral. I got it right because our teacher told us that for such questions, just prove one property of a cyclic quad and then conclude, even if the circle is not there.*

L37: *The most confusing thing about those questions for proving is that they don't give us straightforward questions. They give us those questions that need many steps, so I have a challenge of arranging all those statements and reasons to get to the answer. Sometimes, I try but most of the time, I get stuck after writing the first few steps. If there are many lines inside a circle, I confuse the theorems and most of the time my answers are wrong. Like that diagram in question 6 of the test, did not know where to start; so, I just wrote what I thought was correct.*

An overview of the learner responses as presented in the extracts above seemed to concur with the learning difficulties identified from their test responses. Thus, a general trend established from the interview extracts above is that most learners had challenges solving multistep riders where a series of statements and reasons was needed to reach certain conclusions. Similarly, the learner responses as presented above seemed to suggest that most learners had some knowledge of the theorems applicable to solving question 6, but a common challenge among them was selecting those theorems that were relevant to specific statements.

Furthermore, the analysis of the above interview response suggested that the level of formal deductive reasoning expected in question 6 was far higher than the actual level of learner comprehension. That is supported by the fact that in their interview responses, most of the learners confirmed that they became stuck when they tried to answer the question and they ended up guessing. Such learner responses were supported by the fact that in their answers to the test, most of them were found to be making unjustified assumptions with the purpose of reaching a critical step in the proof. In addition, those who attempted to justify their statements lacked fluency and logic in their presentation of solutions. That is supported by learners like L55, L60 and L37 who in their interview responses said that they were completely lost, they mixed the steps or they ended up guessing the theorems. That was an indication the learners have difficulties with distinguishing different forms of geometrical reasoning, including explanation, argumentation, verification and proof.

On the other hand, the major difficulty identified from these interview responses pertained to the use of visual imagery in conjunction with logical reasoning. In this regard, learner responses suggested that most learners were still stagnant at the basic visualisation level 0, where they rely on seeing the diagram to establish the relevant theorems. In addition, the same learner responses above suggested that learners had difficulties in thinking through multistep problems and giving reasons for the problems. These conclusions were based on the learners' interview responses where most learners confirmed becoming stuck on question 6.3 because there was no circle passing through STOR. As a result, they could not prove that it is a cyclic quadrilateral. Therefore, the researcher concluded that diagrams play a significant role in expressing geometric information in a way that promotes understanding by highlighting the relevant properties to be analysed.

However, despite most of the learners confirming to having difficulties with formal deductive reasoning as evidenced by their poor performance in question 6 of the test, there were some learners whose responses suggested that they had acquired the requisite deductive reasoning competences. Such responses were from learners like L8 as indicated in the extracts above. From their responses in both the test and the interviews, learners like L8 who achieved showed that they could operate at Van Hiele's formal deduction level. This was proved through responses that they were comfortable with formal deductive reasoning although those learners were a very small proportion of the learners' sample.

On the other hand, learners expressed their views on a questionnaire to the learning difficulties at Van Hiele's formal deduction level, as indicated below:

Table 08

Frequency distribution of learners' questionnaire responses to learning difficulties related to Van Hiele's formal deduction level 3 using question 6

Formal deduction level 3		**Strongly Agree**	**Agree**	**Disagree**	**Strongly Disagree**
4.1	It is difficult to choose the correct theorem to use for a	15 (25%)	25 (42%)	15 (25%)	5 (8%)

	given proof problem.				
4.2	I find it difficult to relate correct reasons to a given geometric statements.	20 (33%)	26(43%)	9 (15%)	5 (8%)
4.3	When solving multistep geometric riders, I struggle with logical arrangement of the steps.	44 (73%)	10 (17%)	6 (10%)	0 (0%)
4.4	I just need to memorise the steps and theorems to solve a proof problem.	14 (23)	5 (8%)	21 (35%)	10 (17%)
Average response/ question		39%	16%	13%	5%

Learners' responses to the questionnaire in relation to learning difficulties at formal deduction level 3 as presented in table 4.18 above indicated that 15% of the sample strongly agreed and 25% agreed that they struggled to choose correct theorems to apply to given multistep geometric problems. On the other hand, only 15% and 5%, respectively, disagreed and strongly disagreed. That data suggest that most of the learners lacked the necessary deductive reasoning skills in theorem selection. In addition, 44% and 10% strongly agreed and agreed, respectively, to having challenges with logical arrangement ideas in the presentation of multistep geometric riders. Only 6% disagreed with that statement.

Moreover, that data suggested that most learners lacked logical reasoning skills when solving riders. That confirmed that the learner responses to the interviews also led to that challenge. Overall, an average of 39% and 16%, respectively, either strongly agreed or agreed that learners had challenges with formal deduction. On the other hand, only 13% and 5%, respectively, disagreed and strongly disagreed that learners had challenges at formal deduction level. Therefore, by proportion, the general view held by most learners was that they experienced difficulties with the level of geometrical thinking expected at formal deduction level, hence the poor performance.

Furthermore, to triangulate the learning difficulties as encountered by learners at Van Hiele's formal deduction level, data were also gathered from educator interviews as indicated in the extracts below:

E1: *One of the difficulties that learners encounter with geometric proofs is that they tend to mix up steps in the process of solving multistep geometric riders. That is probably because they lack the skills to approach proof questions, for example, most of them struggle to link the relationships among properties leading to the proof. Most of the questions at this level of formal deduction are high-order questions with a series of interrelated steps that require logical reasoning, which these learners are lacking. As a result, most of the learners struggle to go through the steps correctly.*

E3: *As with questions that require learners to prove that certain geometrical conditions are true, learners experience difficulties with distinguishing between the given and the required to prove. Most of the learners start with what they are asked to prove instead of using other steps, which lead to the proof or concept to be proven. One other thing is most of them are not able to link the relevant theorems to the situation given on the geometric diagram. As a result, they end up guessing or making unjustified assumptions leading to wrong solutions steps. For example, learners tend to assume that all quadrilaterals drawn in conjunction with a circle are cyclic quadrilaterals regardless of the position of the vertices.*

E5: *Learners lack the creativity needed in proofs, they fail to structure a proof, deciding on where to start and where to end. In addition, most of them do not have the skills to interrelate theorems to reach a logical conclusion. Most of them fail to unpack the properties of a shape to be proven especially when it is embedded in a geometric diagram with many lines due to poor mental imagery. Thus, learners fail to develop mental images of a complete diagram in cases where the circle may not be drawn but is implied. For example, in that case where learners were required to prove STOR is a cyclic quadrilateral when its vertices were not on the circumference of circle … Such difficulties are probably a result of the fact that most of our Grade 12 learners lack being observational, and the ability to break down the supposed geometry problem into smaller pieces and apply deductive reasoning. I case of solving multistep riders, they, these learners need to single out the steps leading to the problem solution and the accompanying theorems.*

E6: *From my experience with these learners, geometric proofs are a challenge. In most instances, they take what they must prove as given. In such cases, they end up getting stuck on where to start and how to do the proof. That is possibly the reason why most learners end making unjustified geometric statements … These difficulties with deductive reasoning may be because our learners are not exposed* *to enough geometric riders. In view of that, I will recommend geometric workbooks for our learners from Grade 10-12.*

The educator quotations above seemed to concur with the data gathered from the test and the learners' interview responses in terms of the identified learning difficulties. In this regard, a general view held by the educator respondents was that learners experience difficulties with choosing correct theorems for their statements. In addition, the educators' interview extracts as presented above suggested that despite some learners who had basic knowledge of circle theorems, most learners struggled with logical arrangement of ideas in a geometric proof. For example, E5 suggested that learners had to combine components of a multistep rider into a series of interrelated theorems to arrive at the problem solution.

Furthermore, educators lamented the lack of logical thinking skills among learners. The extracts showed that most learners had challenges distinguishing the given from the required, which resulted in them failing to continue with the proofs to reach the desired conclusions. In addition, educators' interview responses suggested that learners lacked the skills to reason from general to specific situations (deductive reasoning). For example, E3 highlighted that most learners lacked the solution strategies to solve multistep riders, resulting in them not knowing where to start.

On the other hand, the different perspectives held by the respondents regarding the learning difficulties in geometry at Van Hiele's formal deduction level 3 as alluded to above seemed to concur with the data gathered through document reviews of the examiners' reports by the DBE (2016 – 2020). In those reports, examiners repeatedly highlighted that in the questions where they were asked to prove, many candidates had no idea where to start. In the same report, the examiners cautioned that learners had to refrain from making assumptions and had to be reminded that all statement must be accompanied by reasons. More precisely, in 2018, 2019 and 2020, examiners continuously reported that, generally, candidates often loose marks for naming angles incorrectly, or for giving incorrect or incomplete reasons. They further cautioned that the statements must be logical and should lead to solving the problem. From those common challenges they encountered from learners' responses over the years, they concluded that the fact that at Grade 12 level learners are naming angles incorrectly showed that this issue was not dealt with effectively in lower grades.

Thus, across most of those reports, examiners argued that most learners lacked conceptual understanding and procedural fluency when solving proof problems. In this regard, examiners emphasise that most learners at formal deduction level lacked the knowledge to construct relationships between pieces of information leading to the required proof. Similarly, the reports highlighted a lack of procedural fluency among learners by indicating that they were failing to demonstrate abilities to recall and accurately execute the relevant steps that lead to the desired conclusions. To this end, the examiners attributed those challenges to learners' lack of understanding and underestimation of the role played by terminology, definitions axioms and theorems at formal deductive reasoning.

4.5 Conclusion

This chapter out outlined the data analysis and gave details of different perspectives held by the research participants on the difficulties that grade 12 learners encounter in Euclidean geometry. The analysis was done using the test as the main research instrument, where learning difficulties were explored in terms of different Van Hiele levels. In addition, supplementary data were analysed from questionnaires, interview schedules and document analysis. The theoretical framework of constructivism was used to analyse the data. The following chapter discusses the findings in relation to each research question guiding the study.

CHAPTER 5: DISCUSSION OF FINDINGS, RECOMMENDATIONS AND CONCLUSION

5.1 Introduction

The previous chapter presented the analysis of results from the test, questionnaires, interviews and document reviews on the learning difficulties Grade 12 learners experience in Euclidean geometry. In the analysis of results, the test was the major source of research data in which learners' learning difficulties were explored at different Van Hiele levels. Moreover, through triangulation, those test results were supplemented by the analysis follow-up of participants' views to questionnaires and interview responses, together with the results from document reviews to ensure validity of the study. In this chapter, section 5.2 presents a discussion of the findings that emerged from the analysis of results by summarising the identified learning difficulties experienced by Grade 12 learners at different Van Hiele's levels. This discussion is followed by the recommendations from the study in section 5.3. Furthermore, the limitations of the study are presented in section 5.4, which gives the concluding remarks.

5.2 Summary discussion of research findings

This study aimed at exploring and describing the learners' learning difficulties in Euclidean geometry. Therefore, the findings of this investigation aimed to provide answers to the main research question and its sub-questions. To achieve this purpose, the summary of findings was presented according to each research question as outlined in below:

5.2.1 Research question 1: What are the Grade 12 learners' learning difficulties related to visualisation in Euclidean geometry?

The analysis of results from learners' test scripts showed that 38% of the learners did not achieve at Van Hiele's visualisation level 0. Research findings at this level showed that those learners who did not achieve experienced difficulties with recognising and naming from the diagram other angles equal to x and y. Hence, these findings concurred with literature where Bronkhorst et al. (2021) emphasised the development of visualisation skills in geometry at enactive, iconic and symbolic levels as a prerequisite for basic geometry understanding. On the other hand, 68% of the learner respondents achieved at this level, which is an indication that they were able to use their spatial and visualisation skills correctly. According to the demands of question 1, the object of thought at Van Hiele's level 0 was identifying and naming angles using their relative positions with respect to parallel lines and the transversal. In this respect, the findings suggested that most of the learners had a good mastery of Van Hiele's visualisation level 0. Similarly, the findings showed that those who achieved had well-developed visualisation and spatial skills because they were able to use intuition to recognise and name angles equal to x and y from their positions on the diagram and not from their properties.

However, of concern was the 38% who did not achieve at visualisation level 0. In this regard, the findings revealed that these learners had difficulties to recognise other angles equal to x and y using their standard positions from the diagram. In view of that, the findings suggested that Grade 12 learners had a poor conception of angles and lines because even the learners who managed to identify angles equal the x and y from the diagram failed to state whether those angles were corresponding, alternating or vertically opposite angles. Therefore, those difficulties with basic visualisation skills were an indication that there are some Grade 12 learners who still have not yet mastered level 0 concepts.

Furthermore, even though on the diagram parallel lines were indicated by arrows, learners did not state the parallel lines in their solutions. Therefore, that fact that most learners did not state parallel in their solutions to the problem indicated poor conception of parallel lines. Similarly, the challenge of not stating parallel lines was also referred to in the examiners' reports (2016-2021) as a common challenge, and it was reported as a recurring omission by learners in their examination scripts. For example, the NSC Diagnostic report (2017) states that some candidates still do not state the parallel lines when working with alternate angles, corresponding angles or co-interior angles.

Furthermore, the test results and follow-up interview responses from learners who did not achieve at visualisation level suggested that learners have difficulties in integrating their visualisation and visual thinking skills in geometry problem solving. That is so because the general trend from learners' responses to interviews questions at this level revealed that learners could not translate what they saw using their previous encounters with the given diagram into meaningful visual images in their minds. As a result, those learners had difficulties recognising angles that were equal to x and y, and then use mental imaging from the identified standard positions of the angles on the diagram to name them as corresponding, alternating or co-interior.

Thus, the major learning difficulty identified at visualisation level 0 regarding question 1 was learners' inability to integrate their visualisation with visual thinking skills. That was so because some learners could not remember, reason and understand or visualise other angles that were spatially related angles x and y from the diagram. Therefore, educators should encourage learners to name the angles correctly, because the fact that learners in Grade 12 cannot name angles correctly is an indication that geometry of lines was not thoroughly dealt with in lower grades. In support of this finding, some learners (like L37) reported becoming stuck and sometimes they confused the names of angles. In this regard, the learners blamed it on the teachers who they felt did not teach them well, because they were always in a hurry to complete the syllabus. As a result, they do not equip learners with sufficient visualisation skills, especially for diagrams involving the integration of many lines and angles.

Overall, the identified learning difficulties at Van Hiele's visualisation levels as revealed by the research results above were consistent with Mammana's (2014) findings, which indicated that some learners have limited spatial skills and visual thinking skills; hence, they experience difficulties in geometry problem solving. Similarly, these findings support the view held by Kirby (1991) that poorly developed visualisation skills limit the chances for the less gifted child to succeed in geometry problem solving.

5.2.2 Research question 2: What are the Grade 12 learners' learning difficulties related to the analysis level in Euclidean geometry?

Results from question 2 revealed that 33 (55%) of the learners were regarded as having failed to achieve at Van Hiele's analysis level 1 of geometric thinking. Findings from these results suggested that these learners had difficulties in giving standard or non-standard definitions of the components of the circle. On the other hand, only 27 (45%) managed to achieve 50% and above at analysis level, which was an indication that those learners were able to use their analytical skills to give either a standard or non-standard definition of the circle components as shown on the geometric diagram.

However, the findings from the test suggested that 55% of the learners failed to demonstrate their knowledge of the components of a circle. The same results revealed that, besides most learners being able recognise the components of a circle in terms of the given letters of the alphabet from the diagram, most of them failed to describe the components from their properties. Thus, the findings revealed that learners' poor analytical skills resulted in them having difficulties to transition from visual to verbal descriptions of the components of a circle. This was confirmed by a comparison of learners' demonstration of visual to verbal skills, as indicated in figure 4.2 where 72% of those who did not achieve failed to describe the circle components despite having identified them using letters from the diagram.

Furthermore, learners' thought processes from the follow-up interview and questionnaire responses suggested that learners could visually identify components of a circle but have difficulties to describe the characteristic features like chord, segments, tangents and even the more common ones like angle, radius and diameter. Therefore, in their interview responses, learners attributed these difficulties at analysis level to the failure by educators to involve them practically in the discovery of those circle properties to enhance their descriptive abilities. Instead, learners argued that those concepts are just mentioned in passing and educators rush to introduce them circle theorems. Those learners' interview perspectives were consistent with those of De Villiers (1997) who argues from a theoretical point of view that instead of teaching learners' definitions of geometric concepts (for example, radius, diameter tangent, etc.), we should rather strive to develop learners' ability to develop their own definition through practical engagements. Therefore, such remarks suggested that learners experienced difficulties in giving detailed descriptions of those components. Thus, educators are urged to avoiding robbing learners of the most intellectually enriching activities.

Moreover, learners' use of precise geometry language was found to be limited and those who attempted to describe the components of a circle used their own words, which distorted what they wanted to explain. In this regard, those who did not achieve at analysis level showed little or no understanding of geometry terms, since most of them failed to apply correct geometric terminology to describe the circle components. On the other hand, the findings revealed that learners had difficulties with comprehension and encoding of geometry questions as was evidenced by the results from learner's scripts, as some did not attempt question 2, probably because they did not understand the question. Thus, these findings suggested that educators underestimate the importance of developing learners' analytical skills, especially their ability to describe the properties of a circle, as it is the basis for understanding circle theorems and deductive thinking in the next Van Hiele's levels in the hierarchy.

In view of the above findings, the researcher summarised learning difficulties identified at analysis levels as follows:

- A lack of analytical skills of the components of a circle
- A lack of appropriate geometry language to describe the properties of shapes (circle)
- Poor transitioning skills from visual to verbal descriptions of the properties of a circle.

As a result, learners found to be lacking in these skills were those who did not achieve at analysis level and were deemed not ready to progress to the next Van Hiele level.

5.2.3 Research question 3: What are the Grade 12 learners' learning difficulties related to informal deduction level in Euclidean geometry?

The findings at informal deduction levels were presented over a continuum of cognitive levels from 1 and 2 in question 3 and 3 in question 4. This was done to ensure that learners' learning difficulties are investigated at entry and exit levels of informal deductive reasoning, as it is the level at which Grade 12 learners are expected to develop ground knowledge of circle theorems and their applications in doing short deductions.

As a result, learning difficulties at the informal deduction level were investigated and explained as learners demonstrated their competences in solving problems involving relationships between properties of shapes, leading to the application of relevant theorems. In this regard, the results from question 3 which was set at cognitive levels 1 and 2 of informal deduction revealed that 42 (70%) learners had difficulties inter-relating the properties of triangles and cyclic quadrilaterals to the components of the circles that were introduced at the analysis level. Thus, the learners who did not achieve in question 3, showed poor conception of circle theorems, especially those related to a cyclic quadrilateral and angles subtended by a chord at the centre and circumference of circle. In addition, one of the major challenges related to the poor conception was the learners' failure to work with abstract statements about properties of shapes (triangles and cyclic quadrilaterals) that were presented in the geometrical diagram. Similarly, for those who did not achieve in question 3, the findings revealed that learners had difficulties to make conclusions based more on logic than on intuition, resulting in them writing incorrect statements, reasons or both. In other words, those learners had difficulties in solving simple riders in terms of correct statements and reasons. Those difficulties were probably a result of their poor conception of relationships between properties of shapes and their understanding of circle theorems.

In respect of the above, those difficulties were found to be emanating from learners' poor conceptualisation properties of shape from the previous Van Hiele level (analysis). The evidence to that effect was that most of the learners who did not achieve at analysis level were the ones who performed poorly at cognitive level (1 and 2) of informal deduction. This confirmed Van Hiele's (1999) argument as stated in section 2.5.5 of literature that learners cannot operate at level (n) if they did not master level (n - 1).

However, findings from the analysis of learners' scripts as presented in table 4.9 showed that only 18 (30%) of the learners achieved in question 3, indicating that only a few learners had mastered the basis knowledge of informal deduction where they had to use relationships between the properties of a shape to make short deductions in the form of statements and reasons.

Of concern were challenges encountered by the other 48 (70%) learners who did not achieve in question 3, as already alluded to above. In view of that, the findings further revealed that those learners had difficulties in using relationships between properties of shapes to calculate missing angles and then gave correct reasons for their statements.

Thus, findings from the analysis of learner scripts to question 3 showed that learners had difficulties in using logical implications and class inclusion to relate properties of shapes and to then link them to relevant theorems. In addition, follow-up responses from interviews and questionnaires showed the learners' difficulties at Van Hiele's informal deduction level were compounded by the integration of a circle to ordinary plane shapes that most learners confirmed to have knowledge of from lower grades. Thus, research results showed that learners had difficulties with the interpretation of geometric diagrams, especially where several shapes are integrated with a circle.

Thus, a general trend in the learners' interview response to question 3 was that they attributed their poor performance at informal deduction level to the integration of a circle in the matrix of the properties of the ordinary plane shapes. Such responses were a confirmation of the findings already made at Van Hiele's analysis level to test question 2 where it was stated that educators underestimated the importance of teaching circle components in detail before introducing learners to circle theorems.

Furthermore, question 4 set at a higher cognitive level of informal deduction (cognitive level 3) as a way of determining learning difficulties related to more complex geometric diagrams that prepare learners for the demands of the next level, the Van Hiele's formal deduction level. Thus, question 4 had more shapes, more lines and more angles integrated into the circle in single diagram, making it suitable for investigating learners learning difficulties with relationships among properties within and across a variety of shapes.

The findings at this level of informal deduction reasoning were that 47 (78%) of the learners did not achieve 50% and above in question 4 and only 13 (22%) achieved at cognitive level 3 of informal deduction. Therefore, the findings indicated an increase in the number of learners who did not achieve in question 4 as compared to those who did not achieve in question 3, which suggested that learners' abilities to interpret geometric diagrams tended to diminish with an increase in the complexity of the diagram. It was also found that learners had difficulties integrating many concepts in one geometric diagram, where in the case of question 4, learners had difficulties in using abstraction to establish the relationships between properties of parallel lines, isosceles tringles, cyclic quadrilaterals and the circle in terms of relevant theorem. These findings were consistent with those of De Villiers (1999), Siyepu (2005) and Ahmed and Bora (2018) who all found that in South African high school geometry, learners' performance is mostly poor in geometry questions involving the understanding of features and relationships among properties of shapes.

To this end, follow-up interview responses from both learners and educators confirmed that one of the major difficulties learners had with informal deductive reasoning was their inability to conceptualise angle-side properties of shape to relevant theorems. In the same vein, the findings also revealed that learners lacked the knowledge of theorems and they ended up giving unjustified reasons for their statements.

Overall, triangulation of the findings from questions 3 and 4 and follow-up interview responses from learners and educators revealed the following learning difficulties at informal deduction level:

- Inability to use relationships between properties of a shape to choose the correct solution strategy.
- Inability to use properties of a shape and relevant theorems to write correct statements and reasons.
- Inadequate knowledge of applicable theorems to suite the given problem solution.
- Poor logical skills and understanding of theorems, resulting in them making unjustified conclusions.
- Inability to correctly present short deductive statements and reasons (one-step riders).

5.2.4 Research question 4: What are the Grade 12 learners' learning difficulties related to formal deduction level in Euclidean geometry?

As previously alluded to in the data analysis section, both questions 5 and 6 were at set formal deduction level 3, which is the highest level of deductive reasoning examinable at high school level in South Africa. To that effect, question 5 investigated learners' learning difficulties with proofs of theorems, while question 6 explored challenges related to application of theorems to solve multistep geometric riders.

5.2.4.1 Findings related to learners deductive reasoning in proving theorems

The results from question 5 revealed that 38 (68%) of the learners had learning difficulties with completing and proving the theorem which states that 'a line drawn from the centre perpendicular to the chord, bisects the chord'. Thus, proving that AD = DB from the diagram. The findings from the analysis of question 5.1 revealed that most of the learners who did not achieve 50% and above in question 5 failed to write the theorems as stated above. Such performance is an indication that most learners had a poor mastery of circle theorems; as a result, they were limited in their proving abilities. For example, most of those who did not achieve in this question were confused about when to use midpoint and when to use perpendicular in the statement. Similarly, in their interview responses, learners blamed their poor conception of circle theorems on the way educators teach them in class. Most of the learners lamented a lack of practical activities involving construction and measurement to enable them to discover the theorems rather than being given the theorem as a finished product and then told to memorise them as they appear in the textbook. In that regard, they accused their educators of using the ‘telling method’, which means they must know the theorem as it is and apply it in problem solving. They argued that such an approach affected their performance because they tend to confuse or omit important steps when presenting the proof.

Furthermore, research findings showed that when presented with a proof of a theorem, most of the learners had difficulties to do the relevant constructions. As evidence to that, learners’ responses as presented on their scripts showed that most of the learners who did not achieve proceeded to present the proof without doing the construction to the diagram, and most of those who attempted, did the wrong constructions. That was an indication that most learners lacked the requisite knowledge of the relevant steps in proving circle theorems.

Moreover, it was also found that learners had difficulties with proving congruency of triangles, together with the logical arrangement of statements and reasons leading to the conclusions that AD = DB. As a follow-up to this, the results suggested that learners had difficulties with class inclusion in terms of proving congruency. Thus, their difficulties with class inclusion were that they could not reason about relationships between the constructed triangles and classify them as congruent based on their sides-angles properties relationships.

Furthermore, findings from learners’ test scores and their interview responses on proofs of theorems suggested that learners were not given sufficient experiences to develop geometry deductive and logical thinking skills, as proofs of theorems are presented as ‘finished products’ in their textbooks. Therefore, results from follow-up interview responses indicated that educators do not use the Van Hiele model when teaching proofs of theorems, which promotes concept discovery, probably because these theorems are presented in the learners’ textbooks. However, Siyepu (2014) cautions that by using the ‘traditional teaching methods’, educators run the risk of depriving their learners of requisite deductive reasoning skills to prove geometry theorems. As a result, it was noted that most of the learners resorted to rote memorisation of the proofs due poor conceptualisation and hence developed misconceptions, leading to errors in their solution strategy.

However, the findings showed that only 22 (37%) of the learners demonstrated a good mastery of proofs of theorems. Findings from these learners’ test scripts and interview responses confirmed that they did not have any challenges with proofs of theorems. Findings from their interview responses attributed their understanding of theorems to that fact that they were involved in actual discovery of the theorems through construction and measurement. Some of the learners from well-resourced schools confirmed that they were exposed to manipulative devices for teaching geometry like geoboards and animated geometry video lessons, which enhanced their understanding of the underlying concepts. As a result, they did not rely on memorisation of proofs but the understanding of the underlying logical reasoning and deductive skills.

5.2.4.2 Findings related to learners' deductive reasoning in solving multistep riders

Learner abilities, in the application of theorems to solve multistep riders were very limited, as revealed by their answers to question 6. As a result, as many as 52 (87%) of the learners had learning difficulties in applying deductive reasoning to calculate the missing angles in terms of a variable x. Similarly, learners were required to prove that line TQ bisects $\hat{Q}$. In response to this question, learners wrote all sorts of incorrect answers such as those presented in figures 4.25 and 4.26. Therefore, that question proved to be difficult for most of the learners, judging from their test scores and follow-up learner's interview responses. Moreover, the poor learner performance was consistent with Siyepu's (2014) research findings which revealed that learning to write a geometry proof 'is the most difficult topic' for many high school learners. The explanation given for that was that if the Van Hiele model is not used during instruction, learners' conception of deductive reasoning will be lacking the skills to break the problem into its constituent parts and higher order understanding of the processes involved. It was also noted that since that question was at cognitive level 3, there was no obvious route to the solution (non-routine). Therefore, the findings suggested that learners had difficulties with high levels of formal deduction mainly because most of them do not know the solution strategy.

Furthermore, the worst learner performance was item 6.3 of question 6, where findings revealed that most of the learners experienced difficulties to prove that STOR is a cyclic quadrilateral. What made the question even more difficult for most of the learners was the fact that there was no circle passing through STOR. This question proved to be beyond the level of comprehension of almost 90% of the learners in the sample judging by their test scores for the question item and from their interview responses. On the other hand, only 8 (13%) learners achieved at this level of formal deduction. Thus, the findings suggested that most learners held a general misconception that a quadrilateral is only cyclic if there is a visible circle drawn passing its four vertices. As a result, in their interview responses, most of the learners reported becoming stuck and not knowing what to write.

Thus, judging from such responses, the major difficulty demonstrated by these learners seemed to have been the understanding of the nature of a proof. In view of that, learners argued that they were not exposed to proof skills where they had to make abstractions of an incomplete diagram to prove a theorem. That could be the reason why most of the learners did not even attempt the question. These findings were consistent with those of Mateya (2008), which revealed that out of 50 learners who participated in the study, 19 (38%) were at Van Hiele level 0, 11 (22%) at level 1, 13 (26%) at level 2 and only 4 (8%) at level 3. Those results were an indication that in geometry, very few learners reach the formal deduction level by the time they are in Grade 12. Furthermore, the NSC diagnostic report (2019) confirmed that learners had difficulties related to proving a cyclic quadrilateral where there is no visible circle, while examiners reported that candidates could not prove a cyclic quadrilateral where there was a visible circle, despite having made several correct statements in their solutions. It was further reported that either the reason for the quadrilateral to be cyclic was missing or the theorem was given instead of the converse, and most of the candidates had no idea where to start. Those findings were consistent with the findings of this study which indicated that only 8 (13%) achieved with the other 52 (87%) operating levels below the demands of their curriculum. Learners also have difficulties with logical reasoning.

In respect of the above, the findings of this study were validated because learners were found to be experiencing difficulties with the conceptualisation of the proof of theorems, especially the understanding of the necessary sufficient conditions for certain theorems to hold up in terms relationships between properties of shapes. Thus, the findings from both the test responses and the diagnostic reports showed that learners had difficulties with high levels of formal deductive reasoning; their logical reasoning was especially limited. Therefore, judging from learners' dismal performance in this question, the findings suggested that educators did not teach them to reason deductively in terms of abstract relationships. For example, developing in them the knowledge that even if there is no visible circle, a quadrilateral still qualifies to be cyclic if it can be proven that opposite angles are supplementary or an exterior angle is equal to the interior opposite. Therefore, poor performance in such proof questions revealed that learners lacked skills to integrate properties of shape to relevant theorems, resulting in them being limited in their choice of the appropriate deductive strategy and procedural knowledge to solve the problem.

Overall, triangulation of findings from test questions 5 and 6 and follow-up interview responses from learners and educators revealed the following learning difficulties at formal deduction level:

- The findings suggested that learners' difficulties with geometric proofs are due their lack of basic geometry concepts and geometry problem-solving techniques, leading to most of them being under-prepared to work with sophisticated concepts requiring abstraction. That was evidenced by the test results where only eight learners from a sample of 60 learners achieved in question 6.
- Learners were found to be lacking the knowledge of definitions, conceptual understanding of theorems and concept imaging skills; hence, they failed to answer proof type questions.
- It was also found that learners did not know how to begin a proof, resulting in most of them not even attempting those questions and most of those that did attempt, omitted important steps. That was probably because writing a proof in each domain requires learners to have knowledge of the concepts in that domain, failing which they are bound to experience learning difficulties. For example, learners showed a lack of knowledge in the domain of congruency in question 5 and cyclic quadrilaterals in question 6 of the test. As a result, they could not do the proofs.
- It was also found that, because proofs are presented as finished products in learners' textbooks and therefore they are not challenged to think deductively learners find no reason to do proofs of theorems. These findings were consistent with De Villiers's (2004) finding that if geometry proofs are done properly, they help learners with explanations, intellectual challenge and systematisation.

5.3 Implications of findings and recommendations

The following general implications and recommendation were proposed to assist in dealing with the learning difficulties that Grade 12 learners experiencing in Euclidean geometry. That was done with the hope that the recommendations can improve geometry teaching practices and possibly improve learners' performance in Euclidean geometry. Therefore, from the research findings, the following implications and the resultant recommendations emanated from the types and extent of learning difficulties that Grade 12 learners experience at each Van Hiele level.

5.3.1 Learners should be exposed to a variety of geometric shapes and manipulatives

Many learners demonstrated a good mastery of basic knowledge at the Van Hiele's visualisation level by being able to recognise and name angles equal to x and y from the diagram. The implication was that most of the Grade 12 learners can operate at Van Hiele's visualisation level 0. However, it is recommended that educators should cater for those who still have visualisation challenges by exposing them to different shapes to develop the visualisation and visual thinking skills since it is the basis for more advance geometry thinking. Therefore, the educator should focus on developing in learners what Vorster (2012) referred to as ‘the skill of the eye’, where they are taught ways to recognise and name geometric shapes, lines and angles from various vintage points. The researcher also recommends that educators can enhance learners’ visualisation and visual thinking skills by involving them in exploration and hands-on activities with a variety of geometry diagrams to be able to translate their intuitive spatial knowledge into more formal visualisation skills.

Furthermore, manipulatives (physical or concrete material like puzzles, paper cuts, pattern blocks and geoboards that are used as teaching aids to engage learners in hands-on geometry learning) should be used mainly to teach concepts at visualisation level. Therefore, recommending the used manipulatives at visualisation level is consistent with John (2012) who suggests that educational tools like iPads can give learners access to geometry sketch programmes, geometry vocabulary programmes and self-passed geometry lessons, and online lectures that can enhance learners’ performance. Thus, exposure of learners to such technologies can probably help them integrate their natural intuition of shapes into more organised visual thinking skills. In addition, it is believed that the use of manipulatives in teaching geometry reduces the prevalence of learner misconceptions and errors by enhancing conceptual understanding. Therefore, a more encompassing recommendation at Van Hiele’s visualisation level is that learners must be involved in the discovery of basic geometry concepts, and educators should use the Van Hiele model as a framework for teaching geometry instead of resorting to the use of traditional geometry teaching methods.

5.3.2 Learners should be exposed to conceptualisation of shapes and their properties

It was found that many learners experienced difficulties at Van Hiele’s analysis level, especially with the description of the components of a circle. The implication is that learners who have limited analytic abilities relating to shapes are more likely to face difficulties with the understanding of circle theorems in the next Van Hiele level, the informal deduction level. In respect of this, regarding the challenges at Van Hiele’s analysis level, Clements and Sarama (2000) caution that educators should reconsider their teaching of shapes only through examples without exposing the knowledge of shapes through their properties. Similarly, Lowrie et al. (2018) argue that if educators choose not to introduce the correct geometry language, they eliminate any opportunity for learners to choose to learn that language. Thus, the implications derived from the studies above, supported by those emanating from the findings of this study, suggest that a sizeable number of learners do not have grounded analytical skills. This conclusion was also reached because learners indicated in their interview responses that they had limited direct encounters with geometric shapes in terms of their properties and limited geometry terminology to describe those properties of shapes. As a result, learners were found to be limited in their understanding of circle theorems as that understanding is largely dependent on their active involvement in discovery of the properties of a shape.

Furthermore, the implication of these findings to educators was that educators should encourage learners to develop geometry expressive language by talking about geometric concepts and discovering properties by themselves. This suggestion is based on the researcher's view that the educator's role in the learners' understanding of key concepts related to relationships of the properties of shapes is critical for enhancing the development of deeper analytical skills. As a result, learners can use that as building blocks upon which they can develop their understanding of circle theorems; an area of focus for the next Van Hiele level. Therefore, it is recommended that educators should ground learners' knowledge of properties of shapes in conjunction with the use of appropriate geometry vocabulary to describe shapes in terms of their properties.

5.3.3 Unconventional instructional methods should be used to develop learners' conceptual understanding

The following recommendations were made based on the identified of learning difficulties related to the incorrect application of theorems to justify statements in geometry problem solving. In this vein, an overview of the findings showed that Van Hiele's levels 2 and 3 concepts were the most difficult for many learners. That conclusion was based on the findings that learners' test scores for the questions set at those levels were very low. In support of the need to use unconventional teaching methods, Acquah (2011) suggests that there is an urgent need to change the traditional mode of geometry teaching to one that is more rewarding for both educators and learners. In view of this, it is recommended that educators should implement the Van Hiele model as a framework for teaching geometry, as this would enable learners to graduate from one level of geometry thinking to the next in a hierarchical order, allowing them to reach the requisite levels of geometry thinking in Grade 12.

Furthermore, regarding difficulties related to learners' informal deductive thinking skills, the researcher recommended that educators should place more emphasis on the development of logical thinking skills when introducing short deductions. This recommendation is based on the research findings that learners had difficulties in proving geometry problems involving short deductions, especially the writing of a correct statement and reason. Similarly, learners were found to be lacking in their logical thinking skills where they had to present a series of correct statements that had to be accompanied by relevant reasons. That prevalence of those difficulties in learners' responses to the test and the questionnaires concurred with Egodawatte's (2011) argument that learners lacked logic in mastering the requisite skills, facts and concepts, resulting in them using irrelevant rules or strategies in geometry problem solving. To help achieve that goal, learners should be taught how to integrate their understanding of relationships between properties of shapes to relevant theorems. Therefore, equipping learners with such knowledge decreases their chances of relying on rote memorisation of geometric facts, which they reported as their main method used to remember circle theorems. Therefore, it is recommended that educators should teach theorems with relevant understanding instead of promoting memorisation. In view of that and as alluded to in the literature section, using the Van Hiele model of geometry teaching enhances learners' geometry conceptualisation.

Furthermore, through this study, it is recommended that learners should not make any geometrical assumptions. In view of this, it is recommended that learners should first prove that any statement they make about the relationship between sides or angles is true before using it. In addition, learners are encouraged give geometric reasons as they are presented in the examination guidelines. These recommendations were prompted by the findings from both the test and interviews where learners reported that they become stuck, become confused with statements and reasons, make unjustified conclusions or just guess the reasons to statements. All those difficulties are believed to be a result of educators' poor pedagogical skills that favour traditional methods of teaching geometry at the expense of implementing the Van Hiele's model of development of deductive reasoning.

5.4 Summary of recommendations

In view of the above, this study is believed to add the following to the teaching and learning of Euclidean geometry:

- If taken into consideration by learners, educators and curriculum planners, the suggested recommendations to classroom practices can probably assist in improving learner performance in Euclidean geometry.
- This study successfully used the Van Hiele's theory of geometry learning to explore learners' learning difficulties in geometry, and it is hoped that this framework, if employed by educators during instruction, can help them to easily structure the way in which geometry concepts are intruded to learners to ground their conceptual development.

- Through the findings and recommendations of this study, educators should consider the need to discuss with learners the importance of using the Van Hiele model of geometric thinking to help overcome the related learning difficulties. That can be achieved by ensuring that learners master lower-level geometry concepts before being taught higher levels of geometric thinking.
- The results of the study also suggested that educators should not underestimate the importance of grounding learners' understanding of the relevant geometry vocabulary/terminology as the basis on which they can build deductive reasoning.

5.5 Limitations

Just like any other research study, this investigation had its own limitations. Firstly, the research was conducted in only six schools in the same district. Therefore, chances are, if a similar study is conducted in many schools in many districts and different provinces of the country, the results might be different and more generalisable to the learning difficulties experienced by Grade 12 learners in South Africa as a whole. Therefore, the results of this study are more relevant to the participants of the six schools involved in the study. However, the findings of this study can give insight into the learning difficulties that Grade 12 learners experience in Euclidean geometry. Secondly, the phenomenological approach employed by the researcher of using interviews and a questionnaire to gather research data was limited in that it relied more on the participants' willingness to give honest and objectives perspectives to their lived experiences in geometry teaching and learning. The researcher tried to counter those elements of subjectivity by using test results in the package of mixed methods. Therefore, further studies need to be conducted at a much larger scale to find out more about the prevalence and extent of the learners' geometry learning difficulties in the South African context, to have a positive influence at national level.

5.6 Conclusion

The purpose of this research was to investigate learners' learning difficulties in Euclidean geometry using the Van Hiele levels of geometry thinking. In this vein, the findings of this study revealed that most of the learners could recognise and name different types of angles, evidence being that about 87% of the learners attained basic visualisation level 0, according to the Van Hiele model of geometric thinking. Moreover, it was found that at analysis level, learners had limited geometry vocabulary/terminology and that negatively affected their understanding of properties of shapes, as most of them encountered difficulties in describing circle components using appropriate geometry language.

In this regard, to answer sub-question 2, the researcher concluded that these learners have difficulties in using correct geometry terminology to describe the properties of geometrical shapes (circle, triangles and quadrilaterals). This was shown by their lack of analytical skills to describe the components of a circle as presented in the geometry test. As a result, those deficiencies were responsible for learners' limited abilities in using geometry terminology to describe the properties and the relationships between properties of the shapes within geometric diagrams.

Furthermore, to answer sub-questions 3 and 4, research findings suggested that most of the Grade 12 learners were experiencing learning difficulties in their understanding of concepts related to deductive reasoning. In this vein, these findings revealed that most of the learners lacked both conceptual knowledge and procedural fluency to solve geometry problems at Van Hiele's deduction level 2 and 3 successfully. Therefore, the lack of conceptual understanding being referred to in this instance included, for example, their inability to relate properties of geometric shapes, apply correct and relevant theorems as reasons for geometric statements and logical presentation of their ideas when solving multistep geometric riders.

In support of challenges stated above, the learners' lack of procedural fluency to answer questions that required formal deductive skills has been persistently reported in NSC diagnostic reports (2016-2021). This is an indication that such challenges call for urgent attention by educators to ensure improved geometry performance. In addition, it was noted that the failure by learners grasp the requisite basic geometrical knowledge at each of Van Hiele's levels during geometry instruction was responsible for the misconceptions and resultant errors that were identified from their test responses, as explained in the data analysis section in the previous chapter. Similarly, it was noted through this study that educators' knowledge of learners' learning difficulties in terms of their geometry misconceptions and errors provides them with insight into learners' procedural and conceptual misunderstandings and can be used teaching points.

Therefore, the findings of this study support the claim by Vorster (2012) that the Van Hiele theory is the best framework for exploring learners geometric reasoning, despite educators' preference for traditional teaching methods that supress learners' discovery of geometry concepts through active participation during instruction. In addition, the study recommended that educators should receive training in how to implement the Van Hiele theory in teaching, more especially at the stages and levels on which new concepts and figures are introduced. Similarly, educators should be empowered on ways to identify learning difficulties in terms of misconceptions and errors that are experienced by learners at each Van Hiele level and then use them to improve their teaching strategies. Therefore, this study recommended that to help curb the identified learning difficulties, educators should understand the Van Hiele theory and implement it in their teaching of Euclidean geometry.

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APPENDIX A: APPROVAL FROM DEPARTMENT OF EDUCATION

education

Lefapha la Thuto la Bokone Bophirima
Noord-Wes Departement van Onderwys
North West Department of Education
NORTH WEST PROVINCE

Private Bag X10, Mmabatho 2735
Tel.: (018) 388-1964 / 3383
Fax: (018) 086 513 9881
email: dntlabathi@nwpg.gov.za
email: omolete@nwpg.gov.za

OFFICE OF THE DISTRICT MANAGER:
NGAKA MODIRI MOLEMA DISTRICT

Enquiries S.O. Molete
Telephone 018 - 388 – 3383

To : Mr F. Mudhefi

From : Mr S.D. Ntlabathi
District Director

Date : 05 May 2021

PERMISSION TO CONDUCT RESEARCH SECONDARY SCHOOLS IN NGAKA MODIRI MOLEMA DISTRICT

Permission is hereby granted to **Mr F. Mudhefi,** who is a Masters Student at UNISA to conduct a research under the topic: ***An exploration into the learning difficulties experienced by Grade 12 learners in the learning of Euclidean Geometry: A case of Ngaka Modiri Molema District***.

Permission is granted for him on the basis that prior arrangement is made with targeted School Managers to avoid disruption of formal learning, teaching and examinations and further that participation will be voluntary. The School Managers and Management Teams are therefore requested to provide him with necessary support and ensure compliance with COVID-19 regulations.

Your cooperation and support in this regard is highly appreciated

Yours in education,

Mr S.D. Ntlabathi
District Director

APPENDIX B: UNISA ETHICS CLEARANCE LETTER

UNISA university of south africa

UNISA COLLEGE OF EDUCATION ETHICS REVIEW COMMITTEE

Date: 2021/02/10

Ref: **2021/02/10/49266764/10/AM**
Name: Mr F Mudhefi
Student No.:49266764

Dear Mr F Mudhefi

Decision: Ethics Approval from 2021/02/10 to 2024/02/10

Researcher(s): Name: Mr F Mudhefi
E-mail address: 49266764@mylife.unisa.ac.za
Telephone: 0784365819

Supervisor(s): Name: Mr KS Mabotja
E-mail address: mabotks@unisa.ac.za
Telephone: 0124292282

Title of research:

An exploration into the learning difficulties experienced by grade 12 learners in the learning of Euclidean Geometry: A case of Ngaka Modiri Molema district.

Qualification: MEd Mathematics Education

Thank you for the application for research ethics clearance by the UNISA College of Education Ethics Review Committee for the above mentioned research. Ethics approval is granted for the period 2021/02/10 to 2024/02/10.

*The **medium risk** application was reviewed by the Ethics Review Committee on 2021/02/10 in compliance with the UNISA Policy on Research Ethics and the Standard Operating Procedure on Research Ethics Risk Assessment.*

The proposed research may now commence with the provisions that:

1. The researcher will ensure that the research project adheres to the relevant guidelines set out in the Unisa Covid-19 position statement on research ethics attached.
2. The researcher(s) will ensure that the research project adheres to the values and principles expressed in the UNISA Policy on Research Ethics.

University of South Africa
Preller Street, Muckleneuk Ridge, City of Tshwane
PO Box 392 UNISA 0003 South Africa
Telephone: +27 12 429 3111 Facsimile: +27 12 429 4150

3. Any adverse circumstance arising in the undertaking of the research project that is relevant to the ethicality of the study should be communicated in writing to the UNISA College of Education Ethics Review Committee.
4. The researcher(s) will conduct the study according to the methods and procedures set out in the approved application.
5. Any changes that can affect the study-related risks for the research participants, particularly in terms of assurances made with regards to the protection of participants' privacy and the confidentiality of the data, should be reported to the Committee in writing.
6. The researcher will ensure that the research project adheres to any applicable national legislation, professional codes of conduct, institutional guidelines and scientific standards relevant to the specific field of study. Adherence to the following South African legislation is important, if applicable: Protection of Personal Information Act, no 4 of 2013; Children's act no 38 of 2005 and the National Health Act, no 61 of 2003.
7. Only de-identified research data may be used for secondary research purposes in future on condition that the research objectives are similar to those of the original research. Secondary use of identifiable human research data requires additional ethics clearance.
8. No field work activities may continue after the expiry date **2024/02/10**. Submission of a completed research ethics progress report will constitute an application for renewal of Ethics Research Committee approval.

Note:

The reference number **2021/02/10/49266764/10/AM** *should be clearly indicated on all forms of communication with the intended research participants, as well as with the Committee.*

Kind regards,

Prof AT Motlhabane
CHAIRPERSON: CEDU RERC
motlhat@unisa.ac.za

Prof PM Sebate
EXECUTIVE DEAN
Sebatpm@unisa.ac.za

APPENDIX C: GEOMETRY TEST

GEOMETRY TEST (WORKSHEET) GRADE 12 TOTAL MARKS: 55

TIME: $1\frac{1}{2}$ HOURS

Van Hiele's levels of Geometric Thinking: Investigating Grade 12 learners' difficulties in different progression levels.

INSTRUCTIONS TO THE LEARNER

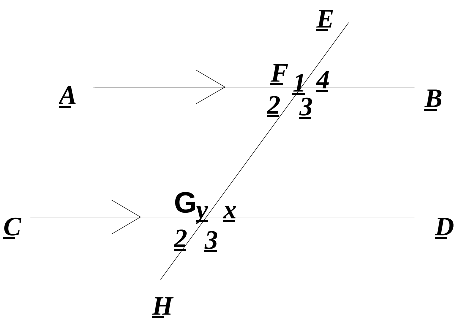
Answer all questions

All your work should be neat and legible

Give reasons for your statements where necessary

QUESTION 1

Consider the diagram drawn below.



1.1 Name two angles which are equal to x and explain?

(i)____________________Reason ______________________________

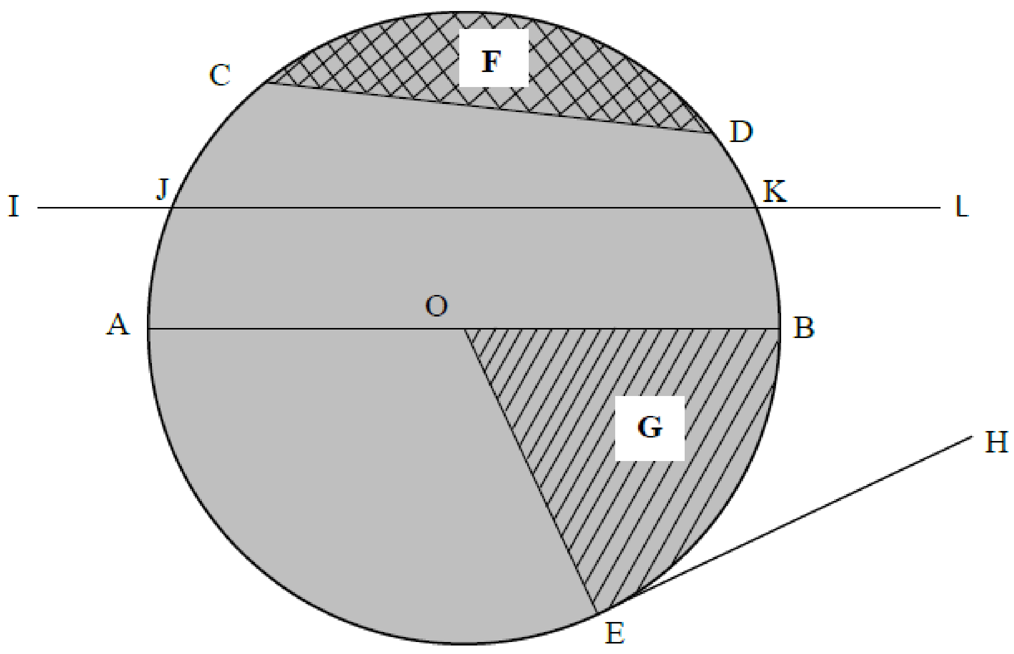
(ii) ____________________ Reason ______________________________ (4)

1.2 Name two angles which are equal to y

(i)_____________________ Reason ____________________________

(ii) _____________________Reason ____________________________ (4)

QUESTION 2



2.1 Describe the following terms and use the figure above to give an example of each

Circle

Diameter

Chord

Radius

Segment

Sector

Arc

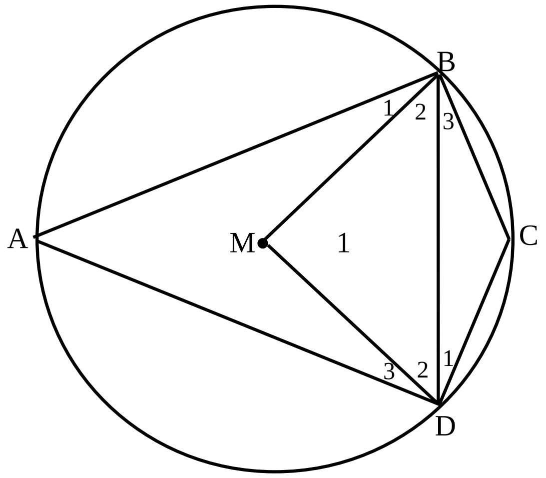
Secant

Tangent

Angle [10]

QUESTION 3

3.1 I n the diagram M is the centre of the circle with A, B, C, and D on the circumference of the circle. If $\hat{B}_2 = 25°$,find with reasons the size of the following angles



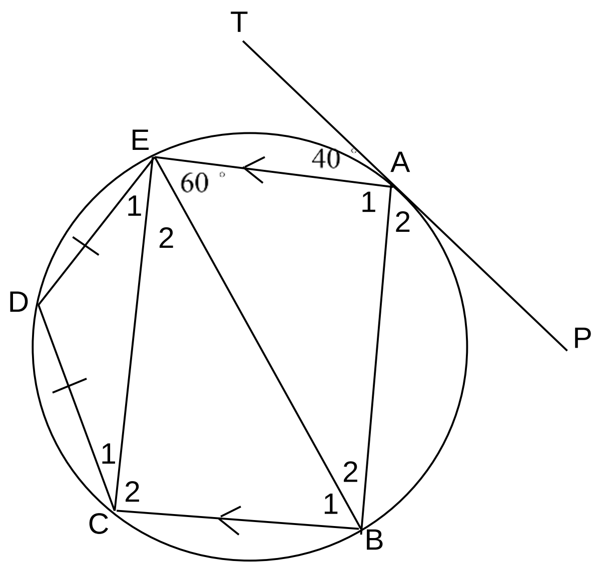
Angles	Reason
$\hat{D}_2 =$	
$\hat{M}_1 =$	
$\hat{A} =$	
$\hat{C} =$	

[8]

QUESTION 4

In the diagram below, TAP is the tangent to circle ABCDE at A. AE || BC and DC =DE.

$T\hat{A}E = 40°$ and $A\hat{E}B = 60°$



4.1 Name the following shapes

4.1.1 CDE (1)

4.1.2 BCDE (1)

4.2 Determine with reasons, the sizes of the following angles:

4.2.1 $\hat{B}_2$ (2)

4.2.2 $\hat{B}_1$ (2)

4.2.3 $\hat{D}$ (2)

4.2.4 $\hat{E}_1$ (3)

[11]

QUESTI0N 5

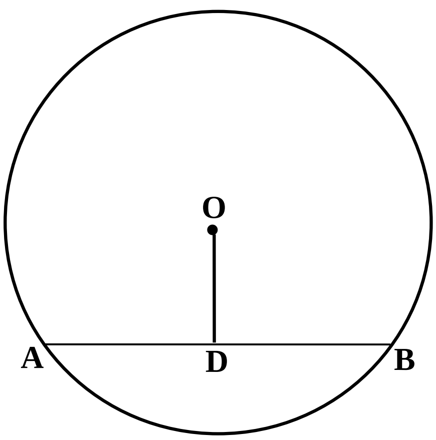
Complete the following statement

5.1 A line drawn from the center of the circle, perpendicular to a chord………………… (1)

5.2 Given: Circle below with center O and Chord AB, OD perp to AB

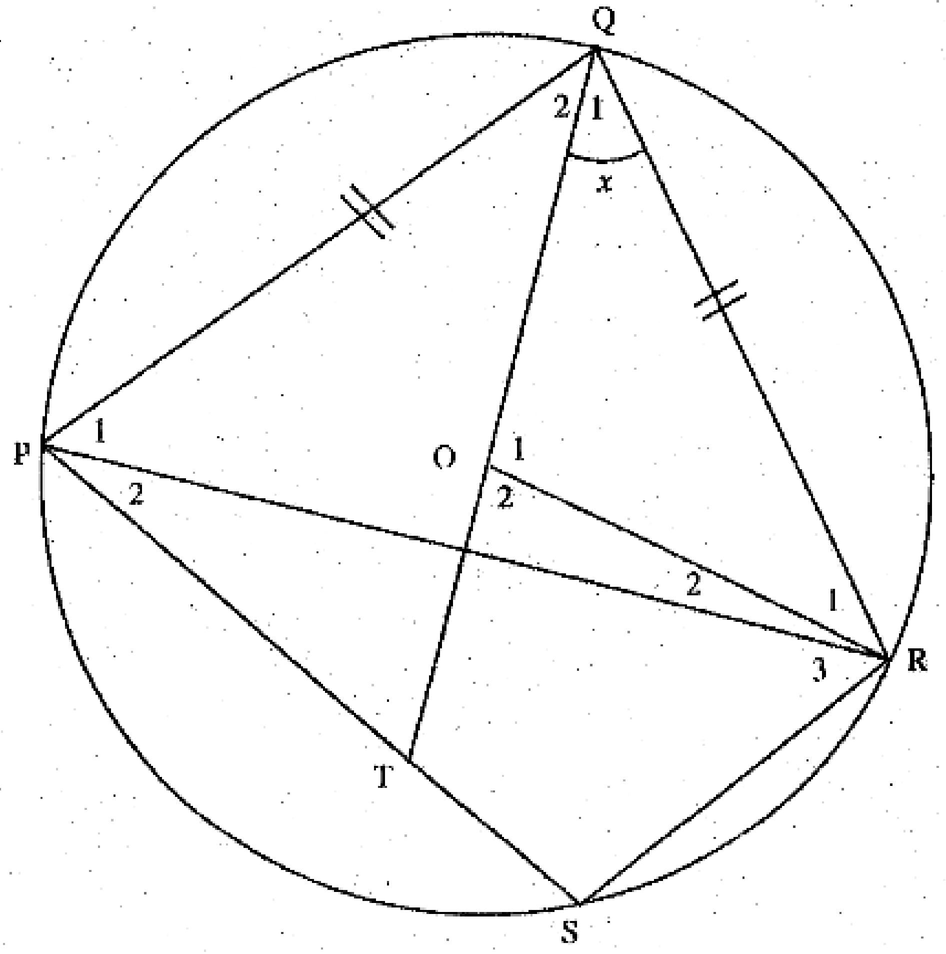
Required To Prove : AD = DB (5)

[6]



QUESTION 6

The diagram below, shows O is the centre of circle PQRS. QOT is a straight line such that T lies on PS. PQ = QR and $\hat{Q}_1 = x$.



6.1	Calculate, with reasons $\hat{P}_1$ in terms of *x*.	(4)
6.2	Prove that TQ bisects $P\hat{Q}R$.	(3)
6.3	Prove that STOR is a cyclic quadrilateral.	(5)

[12]

TOTAL: 55 MARKS

APPENDIX D: QUESTIONNAIRE SCHEDULE FOR LEARNERS

QUESTIONNAIRE SCHEDULE FOR GRADE 12 LEARNERS

INSTRUCTIONS TO THE LEARNER

Please answer all questions in this questionnaire.

Your name is not required, you will be provided with a code.

Give your honest response to all questions.

Indicate your response by a tick in the correct column.

Do not tick the column written Mean Response

TIME: 20 MINUTES

GENERAL INFORMATION

1.1 **LEARNER CODE…………………………………………**

1.2 HOME LANGUAGE

Setswana	English	Afrikaans	Others

LEARNING DIFFICULTIES IN EUCLIDEAN GEOMETRY

1	**GEOMETRY RELATED VARIABLES**	**Strongly Agree**	**Agree**	**Disagree**	**Strongly Disagree**	**Mean Response**

1.1	Euclidean Geometry is a difficult topic					
1.2	Geometry seems to be too technical for me to understand.					
1.3	Geometry learning does not link content to our individual differences.					
1.4	Geometry is my worst topic in mathematics.					
1.5	Only brilliant learners can understand geometry.					
2	**VISUALISATION**					
2.1	I have a challenge in naming different lines, angles, triangles, and quadrilaterals					
2.2	I cannot identify different components of circle					
2.3	Geometric diagrams have no link with real-life objects.					
2.4	It is difficult to interpret geometrical diagrams, especially doing diagram analysis.					
2.5	It is difficult to relate geometry diagrams to relevant theorems.					
3	**PROPERTIES OF GEOMETRIC FIGURES**					
3.1	Geometry concepts are difficult to understand.					
3.2	I have challenges with					

	identifying properties of geometric shapes.					
3.3	I have challenges with calculations involving geometric properties of plane shapes, unknown angles, intersecting and parallel lines.					
3.4	Geometric diagrams with many intersecting lines and angles are really confusing					
3.5	I find it difficult to solve geometry problems using known properties geometrical shapes.					
4	**GEOMETRIC PROOFS**					
4.1	It is difficult to choose the right theorem to use for a give proof problem.					
4.2	I find it difficult to choose the correct reason for a given geometric statement.					
4.3	When solving multi-step geometric riders, I struggle with the correct arrangement of the steps.					
4.4	I have a challenge proving cyclic quadrilateral if the circle is not drawn.					
4.5	I just need memorize the steps and theorems					

	when doing a geometric proof					
5	**GEOMETRY TERMINOLOGY**					
5.1	Terminology used in Euclidean geometry is difficult to understand.					
5.2	Euclidean geometry language is difficult to understand especially when the teacher is using English.					
5.3	Symbols for parallel, perpendicular, similarity and congruency are very confusing.					
5.4	I struggle to answer geometry questions because the language used is too technical.					
5.5	I just need to memorize geometry terms to be able to answer questions.					
6	**GEOMETRY QUESTIONS**					
6.1	It is difficult to recognize/see geometrical diagrams in theorem groups, lines triangle, quadrilateral, and circles.					
6.2	Identification of properties of shape is a real challenge.					

6.3	I struggle with identification of lines and angles in geometrical diagram.					
6.4	I have a challenge in selecting the correct theorems when solving geometric riders.					
7	**CONTRIBUTING FACTORS**					
7.1	I have phobia for Euclidean geometry					
7.2	I did not understand geometry from lower grades.					
7.3	We are only taught to memorize geometry facts without understanding.					
7.4	I cannot relate geometry to everyday life.					
7.5						

APPENDIX E: QUESTIONNAIRE SCHEDULE FOR EDUCATORS

College of Education

UNISA

QUESTIONNAIRE SCHEDULE FOR EDUCATORS

AN EXPLORATION OF LEARNING DIFFICULTIES IN EUCLIDEAN GEOMETRY AT SECONDARY SCHOOLS IN NGAKA MODIRI MELAMA DISTRICT.

Aim: This questionnaire intended to gather information about the difficulties grade12 learners have in the learning and teaching of Euclidean geometry and suggest possible ways to alleviate those challenges.

TIME: 20 MINUTES

GENERAL INSTRUCTIONS

Please answer all questions.

Your name is not required, only code can be used.

Give your true response to all questions.

INSTRUCTIONS ON HOW TO USE THE KEY

1. Indicate by ticking in the appropriate column next to each item to indicate your level of agreement or disagreement to the given statement
2. Use the key below to choose the correct response according to your opinion.
3. Only one tick per questions is acceptable.
4. Leave do not tick the column for mean response.

Key: 1 =strongly agree. 2 =Agree. 3=Disagree. 4=Strongly Disagree. M/R=Mean response

	ITEMS	**1**	**2**	**3**	**4**	**M/R**
1	**GENERAL VIEWS ON EUCLIDEAN GEOMETRY CHALLENGES**					
1.1	I think Euclidean geometry is a challenging topic to teach					
1.2	Geometry is only for talented learners					
1.3	Learners are generally unmotivated to do geometry.					
1.4	Geometry is not given special attention in lower grades					
1.5	Most learners fail to acquire the requisite					

	geometrical reasoning that prepare them for high school geometry, which is targeted at van Hiele level 3; hence, they are operating at lower levels at Grade 12.					
2	**CONTENT RELATED VIEWS ON EUCLIDEAN GEOMETRY**					
2.1	Geometry languages and terminology is far higher than the learners' level of comprehension.					
2.2	Learners interpret geometry vocabulary incorrectly					
2.3	Learners have a clear understanding of lines, angles, triangles, quadrilaterals, and circles.					
2.4	Learners struggle to use definitions of quadrilaterals when given a task using forms of visual representations of shapes.					
2.5	Learners cannot link geometric definitions to visual representations					
3	**VISUALISATION**					
3.1	Learners have a challenge in visualising and explaining their thinking.					
3.2	Learners have challenges in the way they acquire and process visual information (visual cognition).					
3.3	Learners are not flexible in their spatial orientation as they encode geometrical information in terms fixed attributes (symbols and marks).					
3.4	Geometry languages and terminology is far higher than the learners' level of comprehension.					
3.5	Learners struggle to use visual reception and visualisation concurrently when doing geometry proofs.					
4	**GEOMETRICAL PROOF**					
4.1	Learners struggle to understand the nature of a proof if they do not have problem-solving strategy at hand.					
4.2	Learners misuse theorems by making unjustified assumptions					

4.3	Learners struggle to do relevant constructions on geometrical diagrams to highlight important properties.					
4.4	Visualisation is a pre-requisite for proving and problem solving.					
4.5	Learners rush through a geometric problem and fail to read instructions and given information carefully when using diagrams in geometry					
5	**VIEWS ON CONTRIBUTING FACTORS**					
5.1	As educators we are not teaching inductive reasoning					
5.2	Geometry lessons are detached from real-life situations					
5.3	The role of diagrams and gestures and digital technologies is underestimated.					
5.4	Lack of background knowledge, poor reasoning skills and visualising skills impact on classroom instruction.					
5.5	Teachers to use instructional material to make teaching real					

APPENDIX F: QUESTIONNAIRE FOR MATHEMATICS HEADS OF DEPARTMENT

College of Education
UNISA

SELF-ADMINISTERED, CLOSED QUESTIONS QUESTIONNAIRE FOR SCHOOL MATHEMATICS HEADS OF DEPARTMENT'S VIEWS TO LEARNING DIFFICULTIES IN EUCLIDEAN GEOMETRY.

TIME: 20 MINUTES

SECTION A

GENERAL INSTRUCTIONS

Please answer all questions.

Your name is not required, only code can be used.

Give your true response to all questions.

INSTRUCTIONS ON HOW TO USE THE KEY

1. Indicate by ticking in the appropriate column next to each item to indicate your level of agreement or disagreement to the given statement
2. Use the key below to choose the correct response according to your opinion.
3. Only one tick per questions is acceptable.
4. Leave do not tick the column for mean response**.**

Key: 1 =strongly agree. 2 =Agree. 3=Disagree. 4=Strongly Disagree. M/R=Mean Response

	CRITERIA	**1**	**2**	**3**	**4**	**M/R**
	GENERAL ISSUES RELATED TO EUCLIDEAN GEOMETRY					
1.1	Curriculum design does not give special attention to concept development					
1.2	The current crop of teachers lacks the necessary skills to teach Euclidean geometry.					
1.3	There is not enough monitoring of the standard of assessment tasks and quality thereof.					

1.4	Teachers are not making use of improvement plans as well as recommendations from geometry question reports.					
1.5	Secondary school learners' attitudes towards Euclidean Geometry are responsible for their poor performance.					
1.	**INSTRUCTION RELATED ISSUES**					
2.1	General integration of digital technologies for geometry teaching by educators still lacking.					
2.2	Neither what the learners learn in geometry nor the methods by which they learn is satisfactory.					
2.3	Teachers give too little attention to the development of spatial reasoning.					
2.4	Teachers need to empower learners with methods by which they can establish for themselves geometry truth, intellectual autonomy.					
2.5	Teachers should make learners do more practical work than theoretical					
2.6	Teacher's geometry content knowledge negatively impacts on learner performance					
3	**CONTENT RELATED ISSUES**					
3.1	Most grade 12 learners have mastered recognition of geometrical figures					
3.2	Learners struggle to perceive the different parts of geometrical figures					
3.3	Only exceptionally talented learners have the skills to do geometrical proofs					
3.4	Learners cannot easily modify geometrical diagrams or their elements when solving geometry problems					
3.5	Learners struggle to link their visual to their verbal skills when solving geometry problems.					

SECTION B

COMPLETE THE FOLLOWING QUESTIONS BY GIVING A BRIEF EXPLANATION

What can be done to motivate educators to ensure improved learner performance in geometry?

………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
…………………………………………………………………………………………………

What are some of the learning difficulties grade 12 learners encounter in the learning and teaching of Euclidean geometry?

………………………………………………………………………………………………………
………………………………………………………………………………………………………
……………………………………………………………………………

State some of the possible factors that Educators to consider contribute to grade 12 learners' challenges you stated in question 2?

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………………………………………………………………………………………………………
……………………………………………………………………………

Research indicates that most learners' competence levels rarely go beyond visualisation (Van Hiele level 0). what could be challenges the challenges that hinder learners from operating deduction level, (level 3)?

………………………………………………………………………………………………………
………………………………………………………………………………………………………
……………………………………………………………………………

Give any tips you can give grade 12 learners to help them grasp Euclidean geometry concepts much easier?

………………………………………………………………………………………………………
………………………………………………………………………………………………………

APPENDIX G: INTERVIEW SCHEDULE FOR EDUCATORS

College of Education

UNISA

INTERVIEWS QUESTIONS PROPOSED FOR MATHEMATICS EDUCATORS

ESTIMATED TIME: 15 - 20 MINUTES

How are you?

My name is Fungirai Mudhefi. I am conducting research on the learning difficulties experienced by Grade 12 learners in Euclidean geometry.

It is my pleasure to meet you, and I really appreciate your willingness to be part of this interview, hence I am going to ask you a few questions pertaining to learner performance in Euclidean geometry.

Learners seem to lack competency in the domain Euclidean geometry, especially the understanding of concepts related to lines (parallel and transversal), angles, triangles, quadrilaterals, and the related theorems etc. Please explain (specifying which concepts) as to whether you encounter such challenges with your Grade 12 learners.

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

Do learners through instruction develop enough geometrical visualisation and visual thinking skills by the time they reach grade 12, if not; what could be the reasons behind the poor development of these skills? Explain briefly.

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

In your view, as a classroom practitioner, do learners have challenges with the issue of geometry language and terminology when answering geometry questions? Give a brief explanation.

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

In most instances Euclidean geometry riders combines different shapes like tringles, quadrilaterals, circle(s) in one diagram and learners struggle in relating these figures to their properties and relevant theorems. How can you describe the challenges learners have in solving geometric riders?

………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
……………………………………………

Proofs of theorems usually combine several steps that involve logical as well as correct matching of statements and reasons. What are the challenges learners encounter in presenting geometric proofs, mention any three?

………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………

Do learners at Grade 12 have challenges with logical arrangement of ideas in terms of proof questions? Please briefly explain any challenges they might have from your own assessment.

………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………

Generally, most learners at Grade 12 rarely reach level 3 (deductive reasoning) as proposed by the van Hiele model of geometrical thinking. What challenges do learners have with deductive reasoning? Please give your view.

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………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………

What are some of the misconceptions and errors that learners encounter when solving geometrical problems that involve lines, angles, and geometric figures, state any three?

………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………

In your opinion, what are the possible factors that contribute to Grade 12 learners experiencing difficulties in Euclidean geometry. Briefly explain each of the factors you identified.

………

Previous research has indicated that educators do not apply constructivist model of geometrical thinking like the van Hiele model of geometric thinking during teaching and learning. In your view what challenges are likely be encountered by learners if educators resort traditional approaches to geometry instruction?

……

THANK YOU FOR YOUR PARTICIPATION

APPENDIX H: INTERVIEW SCHEDULE FOR LEARNERS

INTERVIEWS QUESTIONS PROPOSED FOR MATHEMATICS LEARNERS

ESTIMATED TIME: 15 - 20 MINUTES

How are you?

My name is Mudhefi, F. I am conducting research on learning difficulties encountered by Grade 12 learners in Euclidean geometry.

It is my pleasure to meet you, and I really appreciate your willingness to be part of this interview, hence I am going to ask you a few questions pertaining to the challenges you experience in Euclidean geometry.

1. Do you have any challenges with recognition of geometrical diagrams in theorem groups lines, triangle quadrilaterals and circles? Please explain.

………………………………………………………………………………………………………
………………………………………………………………………………………………………
………………………………………

2. Identifying and describing the properties of plane shapes (polygons) in terms of angles and sides is often a challenge in Euclidean geometry. Do you also experience difficulties with these? Please explain.

………………………………………………………………………………………………………
………………………………………………………………………………………………………
……………………………

3. What challenges do you have when solving geometry problems using known properties of geometrical shapes?

………………………………………………………………………………………………………
………………………………………………………………………………………………………
……………………………………

4. Do you have any problems with geometrical proofs; to include the logical(correct) arrangement of statement and reasons? Meaning to say, do you have any challenges with writing correct reasons to geometric statements. Please explain the challenges if you have any.

……
……
…………………………………
5. Do you experience any difficulties with the language and terminology that is used in Euclidean geometry? Comment on this briefly.

……
……
…………………………………
6. You are normally taught to memorize theorems and steps for proving those theorems as they appear in the textbook. Are you able remember those theorems during examinations?

……
……
………………………………………
7. Do examples given by your educators during geometry lessons relate in any way to your daily lives? Please explain.

……
……
………………………………………………
8. Are you given opportunities to work with a variety of geometrical objects to improve your conceptual understanding of geometric terms? Give more details.

……
……
…………………………………………………
9. What are some of the factors you consider to be responsible for the difficulties, errors, and misconceptions that you have in your learning of Euclidean geometry? Explain.

……
……
……
10. What do you think your teacher should do to enhance your geometry understanding during teaching and learning? Explain.

……
……
……
………………………………………………

THANK YOU FOR YOUR PARTICIPATION

APPENDIX I: REQUESTING PERMISSION TO CONDUCT RESEARCH

REQUEST FOR PERMISSION TO CONDUCT RESEARCH IN NGAKA MODIRI MOLEMA DISTRICT

Title of the research: An exploration into the learning difficulties experienced by grade 12 learners in the learning of Euclidean geometry: A case of Ngaka Modiri Molema district.

Date______________________

Northwest department of Education (District director)

Contact details: ______________________

email**:** ______________________

Dear Sir

I, FUNGIRAI MUDHEFI am doing research under the supervision of MR K. S. MABOTJA towards a Master of Education in Mathematics Education at the University of South Africa. The research is not funded by any organization or body. If there are any costs involving research materials; they will be met by to researcher.

We request for permission to invite your members (learners and educators) to participate in a study entitled:

An exploration into the learning difficulties experienced by grade 12 learners in the learning of Euclidean geometry: A case of Ngaka Modiri Molema district.

The aim of the study is to:

Identify learning difficulties grade 12 learners encounter in the learning of Euclidean geometry and suggest possible strategies to alleviate those challenges.

The study will entail using a mixed-methods research design, where both quantitative and qualitative data are collected from randomly selected samples of learners, educators, and school mathematics departmental heads from your area. Your district forms part of the schools that are earmarked for my research since it has schools representing different social-economic status that are likely to provide my study with authentic and representative findings. Data will be collected through administration of tests, questionnaire, and semi-structured interviews from which the researcher hopes to use the data to answer the research question. Participants are requested to provide honest answers to the activities given by the researcher. If permission is granted, research instruments will be administered to participants at selected schools while ensuring that normal learning is not disrupted.

The benefits of this study are:

* To provide a general learner awareness of the difficulties encountered in Euclidean geometry.
* To provide Grade 12 learners with the knowledge and skills that will help them improve their performance in Euclidean geometry.
* To equip both educators and Grade 12 learners with strategies to alleviate challenges encountered in Euclidean Geometry.
* To help build learner confidence when solving geometry problems and hence improve their performance in geometry.

Potential risks:

By risk categories standards, the study can be considered a low risk since the study does not solicit for personal and sensitive information. The foreseeable risk may just be that of inconvenience since activities will be done during working hours but with prior arrangements that risk is negligible. There is no reimbursement or any incentives for participation in the research.

Feedback procedure will entail a written report of the research findings which will be forwarded **upon request** to all schools that took part in the research, the other copy will probably be forwarded to your district mathematics department to make subject specialists aware of some of the challenges learners are having with Euclidean geometry for their consideration during Professional Support Forums (PSF)s.

Yours sincerely

____

signature of researcher

____

name of the above signatory

____

above signatory’s position

APPENDIX J: REQUEST FOR PERMISSION TO CONDUCT RESEARCH AT SCHOOLS

REQUEST FOR PERMISSION TO CONDUCT RESEARCH AT SCHOOLS

Title of the research: An exploration into the learning difficulties experienced by grade 12 learners in the learning of Euclidean geometry: A case of Ngaka Modiri Molema district.

Date_______________________

The principal

Northwest department of Education (Ngaka Modiri Molema District)

Contact details______________________

email address)_______________________

Dear Sir/ Madan

I, FUNGIRAI MUDHEFI am doing research under supervision of MR K. S MABOTJA towards a Master of Education in Mathematics Education at the University of South Africa. The research is not funded by any organization or body. If there are any costs involving research materials; they will be met by to researcher.

We request for permission to invite your members (learners and educators) to participate in a study entitled:

An exploration into the learning difficulties experienced by grade 12 learners in the learning of Euclidean geometry: A case of Ngaka Modiri Molema district.

The aim of the study is to:

Identify learning difficulties grade 12 learners encounter in the learning of Euclidean geometry and suggest possible strategies to alleviate those challenges.

The study will entail using a mixed-methods research approach where both quantitative and qualitative data are collected from randomly selected samples of learners, educators, and a mathematics departmental head from your school. You are part of the schools that are earmarked for my research since your school is the category (rural/urban/ private) in terms of its social-economic status and categories low, medium, and high in terms of learner performance in mathematics. Hence, granting me permission to undertake the study at your school is likely to provide this study with authentic and representative findings. Data will be collected through administration of a test, questionnaire, and semi-structured interviews from which the researcher will use the data to answer the research question(s). Participants are requested to provide honest answers to the activities given by the researcher. If permission is granted, research instruments will be administered to participants at your school while ensuring that normal learning time is not disrupted.

The benefits of this study are:

* To provide a general learner awareness of the difficulties they encountered in Euclidean geometry.
* To provide learners with the knowledge and skills that will help them improve their performance in Euclidean geometry.
* To help build learner confidence when solving geometry problems.

Potential risks:

By risk categories standards, the study is considered as a medium to low risk since it does not solicit for sensitive personal information. The foreseeable risk may just be that of inconvenience since activities will be done during working hours. The study involves participants answering questionnaire, a test, and interviews where confidentiality can be compromised but researcher will ensure that participants do not work on these activities in groups to ensure confidentiality.

There will be no reimbursement or any incentives for participation in the research.

Feedback procedure will entail a written report of only the research findings which will be forwarded to your school upon request, the other copy will probably be forwarded to the district mathematics department to make subject specialists aware of some of the challenges learners are having with Euclidean geometry for their consideration during Professional Support Forums (PSF)s.

Yours sincerely

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Signature of researcher

____

Name of the Principal

____

Signature of Principal

APPENDIX K: CONSENT/ASSENT TO PARTICIPATE IN THIS STUDY (Return slip)

CONSENT/ASSENT TO PARTICIPATE IN THIS STUDY (Return slip)

I, __________________ (participant name), confirm that the person asking my consent to take part in this research has told me about the nature, procedure, potential benefits, and anticipated inconvenience of participation.

I have read (or had explained to me) and understood the study as explained in the information sheet.

I have had sufficient opportunity to ask questions and am prepared to participate in the study.

I understand that my participation is voluntary and that I am free to withdraw at any time without penalty (if applicable).

I am aware that the findings of this study will be processed into a research report, journal publications and/or conference proceedings, but that my participation will be kept confidential unless otherwise specified.

I agree to the recording of the ________________ (tests/ questionnaire/ interview).

I have received a signed copy of the informed consent agreement.

Participant Name & Surname (please print) ____________________________________

___________________________ __________________________________
Participant Signature Date

Researcher's Name & Surname (please print) ____________________________________

____________________________ _________________________________
Researcher's signature Date

APPENDIX L: LANGUAGE EDITING PROOF

LETITIA GREENBERG

LANGUAGE SPECIALIST

31 Aland Road
Valhalla
0185

Cellphone: 084 077 1580
Email: lgletitia@gmail.com

30 August 2022

To whom it may concern

With this letter, I confirm that I have language edited the proposal titled *An exploration of learning difficulties experienced by grade 12 learners in Euclidean geometry: a case of Ngaka Modiri Molema district* by Fungirai Mudhefi with student number 49266764.

With a relevant degree and honours degree, I am fully qualified to undertake such editing.

Yours faithfully

Letitia Greenberg

APPENDIX M: ORIGINALITY REPORT

AN EXPLORATION OF LEARNING DIFFICULTIES EXPERIENCED BY GRADE 12 LEARNERS IN EULIDEAN GEOMETRY: A CASE OF NGAKA MODIRI MOLEMA

ORIGINALITY REPORT

14%	13%	3%	3%
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