IMAGERY AND VISUALISATION CHARACTERISTICS OF UNDERGRADUATE STUDENTS' THINKING PROCESSES IN LEARNING SELECTED CONCEPTS OF MATHEMATICAL ANALYSIS

by

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DECLARATION

I, **Jonatan Muzangwa** declare that **Imagery and visualisation characteristics of undergraduate students' thinking processes in learning selected concepts of mathematical analysis** is my original work, and all the sources used or quoted in the study are duly acknowledged by means of complete references. I further declare that I have not previously submitted this work or part of it, at Unisa or any other higher education institution for another qualification.

Signature	Date
Jonatan Muzangwa	10 June 2018

DEDICATION

I dedicate this work to my wife, Getrude Muzangwa.

ABSTRACT

The present study investigated imagery and visualisation characteristics of undergraduate students' thinking processes in learning selected concepts of mathematical analysis. The aim was to discover the nature of images evoked by these undergraduate students and the role of imagery and visualisations when students were solving some selected problems related to mathematical analysis. The study was guided by the theory of registers of semiotic representations. Psychological notions on imagery were also fused to cater for a cognitive approach to the study.

A sample of 50 undergraduate mathematics students participated in the study. The researcher employed both quantitative and qualitative methods. Before the main study, a pilot study was conducted to account for the reliability and validity of the research instruments. The data were collected through use of a cognitive test that was composed of 12 tasks with items selected from mathematical analysis. These tasks were specially designed to capture the variables of imagery and visualisations. A structured interview was also conducted as a follow-up to the results of the cognitive test.

The study found that visual images were noticeable in the thinking processes of undergraduate students in solving problems related to mathematical analysis. The nature of the visual images evoked by the students varied from person to person. The nature of these images was also determined by the nature of the task. The most common types of imagery were diagrams, prototypes and symbols. On rare occasions the students also evoked metaphoric images. It was also observed that these images were used for illustrative purposes and to spark the idea for a proof. It was also interesting to note that some images were used to discover the limit of a converging series. The results confirmed the need to use visualisation with caution, especially when treating concepts which involve infinity.

The study recommends that instructors of mathematics should encourage visual thinking in the learning and teaching of mathematical analysis. Knowledge of the students' concept images helped the researcher to understand the nature of the learning difficulties of the students.

Further research should focus on the strengths and weaknesses of visual-mediated learning and also on the relationship between creativity and visual thinking.

Keywords: imagery; visualisation; representations; semiotic representations; mental representations; thinking processes; advanced mathematical thinking; mathematical analysis.

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APOS	Action-Process-Object-Schema
CAS	Computer Algebra System
GV	Global Visual
LV	Local Visual
MCV	Mean Content Validity
MVR	Mean Expert Ratio
MVT	Mean Value Theorem
NV	Non-Visual
OSA	Onto-Semiotic Approach
TRSR	Theory Of Registers Of Semiotic Representations
UNISA	University Of South Africa
VA	Visual/Analytic

CHAPTER ONE
INTRODUCTION

This chapter introduces the background to the study, statement of the problem, research questions, and rationale of the study, its significance and delimitation as well as a definition of key terms.

1.1 BACKGROUND TO THE STUDY

Students have learning difficulties when beginning mathematical analysis. According to Artigue et al. (1990), these difficulties stem from traditional methods which take refuge in an intensive algebraisation of analysis. Current trends in mathematics education put greater emphasis on conceptual understanding than on memorisation of mathematical concepts. Imagery and visualisation are key to an understanding of mathematics. Snow (1967) quotes G.H. Hardy (1877-1947), an English mathematician known for his work in number theory and mathematical analysis, as follows:

> The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics (Snow, 1967, p. 84).

Contrary to this amazing statement many mathematics students perceive the subject as very dull, cold and ugly. Cognitivism and constructivism relate learning to the involvement of the internal (psychological) and the external (physical) domains, hence processing of information is a way of interaction between these two domains

Mnguni (2014) considers that the process of imagery involves the formation of mental images, figures or likenesses of things. When a person thinks about a particular part of the external world, an image is evoked in the mind. The nature of the image differs from person to person (Tall, 1991). Visualizing is one process by which mental representations (imagery) can be seen. Mathematical objects can only be represented through semiotic representations such as graphs, diagrams, symbols and words (Duval, 1999). The usual instructional practice in mathematical analysis is that visual thinking is often treated with suspicion and sometimes regarded as misleading. Although this assertion is valid, we cannot disregard visualisation completely because it plays various roles in the learning of mathematical analysis. For instance, we can use visuals for illustrative purposes and to facilitate comprehension in the development of proofs (Giaquinto, 2011).

Many theories have been proposed to explain the uses and roles of visualisation in the learning and teaching of mathematics. Although the literature covers a wide variety of such theories, a majority of the studies have focused on general mathematics and geometry. This study focuses on imagery and visualisation in undergraduate students in the learning of selected concepts in mathematical analysis. Mathematical concepts, ideas and methods have a great richness of visual relationships that are intuitively representable in a variety of ways (Guzman, 2002). Visual thinking and visualisation are key elements in learning mathematics at any educational level (Boaler, Chen, Williams & Cordero, 2016). Mathematics educators pay special attention to the role of visual images in the process of learning mathematics; they also pay attention to the relevance of students' visualisation processes and abilities, and find ways to improve teaching by designing visual learning environments (Giaquinto, 2011).

It has also been noted that while mathematicians usually show their outcomes in the abstract style typical of logico-deductive formal proofs, several studies show that visualisation plays a central role in mathematicians' processes of creation, discovery or proof of new results. A typical example is the story of the great Norbert Weiner who could not resolve a complicated proof but after several minutes of thinking he was helped to complete it by means of some mysterious pictures (Guzman, 2002).

Many previous researchers have emphasised the importance of visualisation and visual reasoning in the learning of mathematics (Arcavi, 2003; Bishop, 1989; Duval, 1999; Eisenberg & Dreyfus, 1986; Giaquinto, 2011; Guzman, 2002; Presmeg, 1989; Zimmerman & Cunningham, 1991). The critical issues encountered in these researches focused primarily on finding theories which characterised the issues surrounding visualisation. Terms such as visualisation, visual thinking, spatial thinking and imagery were often used interchangeably to name mental processes that occurred during problem-solving. The limitations and difficulties concerning visualisation and even the reluctance to visualise have also been regularly discussed (Arcavi, 2003; Eisenberg & Dreyfus, 1991). Current thinking does not discount these theories, but simply builds on them to include students' learning of mathematics at higher levels.

Mathematical analysis is usually studied after the calculus course at undergraduate level mathematics. Hence, mathematical analysis is built upon the calculus of real functions on a firm logical foundation. This foundation is based on the axioms of real numbers. The focus of analysis is basically on understanding proofs and proving statements, hence attention should be given to reasons why particular steps are taken. In order for students to understand proofs and prove mathematical statements, their mathematical knowledge should be organised around definitions, lemmas, propositions, theorems, examples and counterexamples (Grouws, 1992). Real understanding in mathematics, in general, is brought about when the student has acquired procedural skill and fluency, conceptual understanding and the ability to apply mathematics (Hiebert & Carpenter, 1992). Research in mathematics education, however, shows that students generally have very weak visualisation skills in calculus, which in turn leads to a lack of meaning in the formalities of mathematical analysis, as observed by Eisenberg and Dreyfus (1991) in their paper entitled "The reluctance to visualize in mathematics", where they analysed the obstacles that are encountered by students in the visualisation processes in mathematical education.

Vinner (1983) established the interplay between concept image and concept definition. A concept image is a collection of all images, pictures, symbols, definitions and properties associated with any given mathematical concept. Mathematical thinking at undergraduate level relies heavily on definitions of mathematical terms. Theorems and propositions are derived from definitions. One of the most important components in the mental representation of concepts in the concept image of advanced mathematical thinkers is visualisation (Krussel, 1994). Visualisation is very useful in the intuition process, which is also important to mathematical thinking. The visual aspect of intuitive reasoning in mathematics is divided into three main categories, namely diagrammatic reasoning, which is predominantly graphical; analogical reasoning, which relies heavily on making connections, and prototypical reasoning that is the selection of one typical example as a representative of the concept (Fischbein, 1987). Intuition, imagination, insight, holistic thought, creativity and three-dimensional forms are categorized as right hemispheric forms of thoughts by neuroscientists (Gazzinaga, 2005).

The problems of undergraduate students when learning the concepts of mathematical analysis was explained by Engelbrecht (2010), who cited the traumatic experiences faced by these students during the transition process from calculus to analysis. In his paper "Adding structure to the transition process to advanced mathematical activity", he proposed the use of both visualisation and symbolism to enhance the conceptual understanding of mathematical analysis. Similarly, Tall (1994) mentioned the importance of visualisation and symbols on the issue of compression of mathematical knowledge in advanced mathematical thinking.

This study is designed to examine and describe the nature of imagery and visualisations used by undergraduate students engaged in studying mathematics when solving some selected concepts in mathematical analysis for the sake of assessing the quality of such mental constructions in solving these mathematical problems. According to Gauss, cited by Krussel (1994), once a mathematical argument has been constructed, none of the “scaffolding” used in the process should be evident. Only the finished product should be visible and all the scaffolding and other evidence of construction should be hidden from view. On a different note, Hadamard (1945) proposes the importance of intuition, inner experiences, visual imagery, mental images, and of playing around with ideas. Hadamard’s proposition is precisely what the current study investigates. The objective of this study is therefore to explore the types and forms of scaffolding, particularly the visual component of this scaffolding, used by students in solving problems in selected concepts of mathematical analysis. This study differs from other previous studies which have focussed more on theory. The present study brings more empirical evidence on the thinking processes of undergraduate and their use of imagery and visualisations when solving problems related to mathematical analysis. The research also adds value to previous research on mathematical analysis where the notions of imagery and visualisations are still treated with suspicion.

1.2 STATEMENT OF THE PROBLEM

Research and experience point to the fact that undergraduate students experience difficulties in understanding the concepts, proofs and proving statements of mathematical analysis (Tall, 1991). A majority of students at university level simply memorise proofs and pass the examinations (Sawyer, 1987). If the same students are asked to perform the same proof after the examination, they encounter difficulties. Skemp (1971) observes that approaches to undergraduate teaching tend to give students procedural knowledge rather than encouraging the active process of mathematical thinking. Whilst traditional methods of presenting advanced mathematical knowledge cause students to fail to understand information at a deeper level, they also result in the creation of learners who are unable to make a connection between new and previous knowledge. Dreyfus (2002), in his paper on advanced mathematical thinking processes, describes such methods as polished formalism following the sequence of definition, theorem, proof, application. Classroom teachers are aware that the process of building conceptual understanding is not a linear process. Learners sometimes use trial and error methods, are assisted by illustrative diagrams and learn through partially correct and partially wrong statements. This study explores the nature of visual thinking with which undergraduate learners engage during their encounters with mathematical tasks or concepts of an abstract nature.

Experience has also shown that no one learns mathematics in a vacuum and that the process of acquiring mathematical knowledge is cumulative. Mathematical knowledge is best understood and retained by connecting it to previously existing knowledge (Krussel, 1994). It is only when a new mathematical idea or concept is connected to an individual's existing mathematical knowledge that it begins to be understood. This process is facilitated by linking the mathematical idea to something in the experiential realm, as when mathematical induction is understood as analogous to a row of beads poised to fall, or when infinity is understood as the North Pole on a sphere or a ball (Krussel, 1994). These connections, intuitive and most often visual, are of central importance to mathematical understanding and take time to build.

The processes of analysing and making judgments of one's mathematical experience are of particular importance in the solution of non-trivial problems. Such a reflection is characteristic of advanced mathematical thinking (Schoenfeld, 1985) and enables individuals to assess the progression of their mathematical thinking and to monitor the development of mathematical understanding. It is this process that has been employed in the current study to gain access to each individual's mathematical thinking and understanding. In summary, the statement of the problem was to gain insight into the nature of imagery and visualisations evoked by undergraduate students when learning selected concepts in mathematical analysis and how these conceptual images are used in solving tasks or proving theorems.

1.3 RESEARCH QUESTIONS

The research question is the following: How do undergraduate students use imagery and visualisation in the learning of abstract mathematical analysis concepts? It was investigated through an exploration of the following four sub-questions:

- What is the nature of the imagery and visualisations that undergraduate students evoke in learning selected concepts of mathematical analysis?
- How can the visual images evoked by undergraduate students be classified?
- How do the undergraduate students deploy these images and visualisations in the process of solving problems in selected concepts of mathematical analysis?

- What role does visual thinking play in undergraduate students proving some theorems or solving problems in mathematical analysis?

1.4 RATIONALE OF THE STUDY

In this section the researcher justifies the main reasons for this study. The rationale of the study is based on learning mathematical analysis with understanding. Hiebert and Carpenter (1992) are of the view that a mathematical idea or procedure or fact is understood if it is part of an internal network. Understanding mathematics involves the ability to translate between representations of mathematical objects and to make connections between ideas, facts and procedures. Memorisation of facts or procedures without understanding often results in fragile learning while getting the right answers does not necessarily imply mathematical proficiency (Erlwanger, 1973). Fennel and Rowan (2001) argue that understanding and using mathematical concepts and procedures are enhanced when students can transfer understanding among different representations of the same idea. Students need to develop and use a wide variety of representations. Research points to the importance of intuition and multiple representations in conceptual understanding of mathematics in general.

Another fundamental reason for this study is to gain basic theoretical knowledge about what is going on in the student's mind when solving concepts related to mathematical analysis. It is an interesting experience for undergraduate students to think or recount how they went about solving a problem. There is a link between the mental and mathematical image (Dreyfus, 2002) and to represent a concept means to generate an image of it. Mathematical objects can only be accessed through their semiotic representations such as symbols, graphs and words (Duval, 1999). Visualising is one process by which mental representations can be experienced. To be successful in learning mathematics it is advantageous to have rich mental representations of concepts. Figure 1.1 is a summary of the key elements that have contributed to the study of imagery and visualisation in the learning of mathematical analysis.

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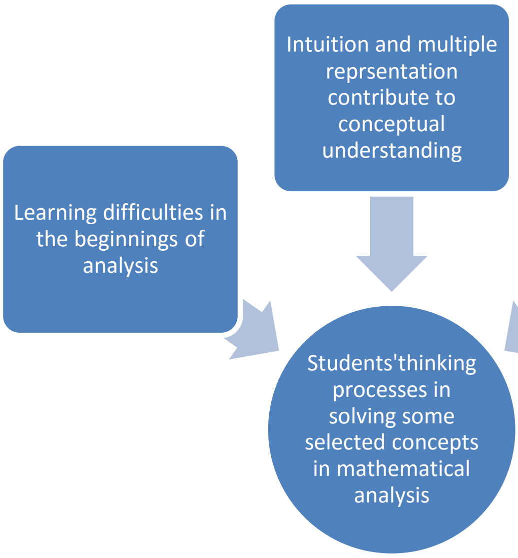


Fig.1.1: Summary of the rationale for the study

Mathematical analysis is the general theory that underpins calculus. The goal of mathematical analysis is to introduce the student to the rigorous polished formalism of laying down proofs of mathematical statements. The general thought patterns of the students are encouraged to change from a mode which relies extensively on the formation of concepts through the encapsulation of process as concept, to a mode which is structured within the realms of concept definition (Gray & Tall, 1994). However, the transition from one form of thinking to the other is a difficult one. Though mathematicians use definitions and formal language in a meaningful way to express mathematical arguments, the students' method of thinking about mathematical concepts may depend on more than the form of words used in a definition (Vinner, 1992). Visual thinking in mathematical analysis is often treated with suspicion since our visual intuitions about what happens at the limit of an infinite process lead us astray. Consequently, the important roles of intuition, imagery and visualisation are dealt with in depth in the literature review.

1.5 SIGNIFICANCE OF THE STUDY

The present study is significant because the majority of research on visualisation has been carried out at primary and secondary school levels (Clements, 1992; O'Reilly, 1995; Presmeg, 2006; Rivera, 2007; van Garderen, 2006). This research attempts to fill a gap for visualisation at undergraduate level in mathematical analysis, a field where this concept is treated with suspicion. This study has introduced the variable "imagery" which was considered irrelevant in mathematical thinking since some schools of thought regarded mathematics as an imageless subject (Hurlburt & Heavy, 1993). Current studies on imagery, however, reveal that visual imagery is very important in scientific discovery. Friedrick August von Kekule, for example, described how in 1865 the ring-like structure of benzene came to him in a dream:

> I turned my chair to the fire and dozed. Again the atoms were gambolling before my eyes. This time the smaller groups kept modestly in the background. My mental eye, rendered more acute by repeated visions of this kind ..; all twining and twisting in a snakelike motion…One of the snakes had seized hold of its own tail, and the form whirled mockingly before my eyes. As if a flash of lightning I awoke (Findlay, 1937, p. 43).

The snake biting its tail had given him the clue to discover the chemical structure of benzene. Other scientists are said to have thought in terms of images; for example, Gruber (1974) reports that Darwin's notebooks are full of images. Another testimonial is evident in Einstein's letter to Hadamard (1945, p. 82):

> Words and language, written or oral seem not to play any role in my thinking. The psychological constructs which are the elements of thought are certain signs and pictures, more or less clear, which can be reproduced and combined at liberty (Hadamard, 1945, p. 82).

From this statement we can infer that visualisation is one process by which mental representations can come into being. Visual thinking in mathematical analysis can be used for illustrative purposes but is not regarded as a means of discovery (Guiaquinto, 2011). This study revealed that some visualisations turn out to be a means of discovery in analysis.

This study is also significant since it is a study of mathematical thinking. Mathematicians and mathematics educators are keen to discover the mysterious processes of mathematical creation and thinking. Knowledge of students' thinking processes about a specific mathematical domain, such as mathematical analysis, is not well documented. It is hoped that after assessment tests and interviews with the participants in this research, the researcher will be able to gain some insight into participants' mathematical knowledge, the ways in which this knowledge is represented visually and how students deploy these images in solving problems of selected concepts in mathematical analysis. Such a discovery could be helpful in mathematics education because educators may be able to design teaching methods which encourage visualisation. Gaining insight into the reasoning processes of students also helps in discovering how mathematical knowledge is constructed. The findings of this study might contribute to the psychology of mathematics education in discovering advanced mathematical thinking processes.

1.6 SCOPE AND DELIMITATION OF THE STUDY

The study was confined to undergraduate students' thinking processes in solving mathematical tasks in the course of mathematical analysis. The tasks were selected from basic analytical concepts, graphs of functions of several variables (level curves) and double integrals. These concepts summarise most concepts in mathematical analysis and are rich in visualisation. The area of interest to the researcher is to understand students' mental imagery and visualisation processes in solving mathematical tasks. The research was delimited to a group of students at a university in Zimbabwe. The participants in the study were students majoring in mathematics in a bachelor of education in-service degree programme.

1.7 OPERATIONAL DEFINITION OF TERMS

Imagery: In this research, imagery refers to the mental representations of things that are not currently being sensed by the sense organs (Goldstein, 2011; Sternberg, 2009). The particular kind of mental imagery involved in this research is visual imagery that is, "seeing" in the absence of a visual stimulus. The ability to recreate the sensory world in the absence of physical stimuli occurs in other senses as well. People have the ability to imagine tastes, smells, tactile experiences and auditory experiences.

Visualisation: The researcher finds Arcavi's (2003) definition, which is limited to the use of figures, images and diagrams, most useful. He describes visualisation as the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images and diagrams in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

Representation: The term representation in this context refers both to process and product, to the act of capturing a mathematical concept or relationship in some form and the form itself. Moreover, the term applies to the processes and products that are observable externally, as well as those that occur internally, in the minds of people doing mathematics. Some forms of representation are diagrams, graphs and symbols (Erlwanger, 1973).

Semiotic representations: This refers to a system of signs and their meaning. All semiotic representations must be considered based on the following issues:

- The register where it was produced. What it can do and what it cannot represent.
- The properties of the object of knowledge being analyzed.
- The object itself to which it refers to (Pino-Fan, Guzmán, Duval & Font, 2015).

Mental representations: A mental representation refers to internal schemata or frames of reference, which a person uses to interact with the external world. It is what occurs in the mind when thinking of a particular part of the external world and may differ from person to person (Dreyfus, 2002).

Advanced mathematical thinking processes: Advanced mathematical thinking in the context of this research involves mathematical thinking expected at the level of university mathematics, such as the use of precise definitions and logical deductions of theorems using those definitions. The processes of advanced mathematical thinking are representation and abstraction. Representing and abstracting are complementary processes in opposite directions. A concept is often abstracted from several of its representations, and representations represent some abstract concept (Dreyfus, 2002). The use of several representations helps students to make the transition from a limited, concrete understanding of a certain topic to a more abstract and flexible understanding of the topic (Kaput et al., 1988).

Mathematical analysis: This is the branch of mathematics dealing with limits and related theories, such as differentiation, integration, measure, infinite series and analytic functions. These theories are usually studied in the context of real and complex numbers and functions (Wikepedia, 2018).

1.8 CONCLUDING REMARKS

This chapter introduced and presented the need to carry out a study on the use of imagery and visualisation in solving problems related to mathematical analysis concepts by undergraduate students. The main concern of this research was to improve the conceptual understanding of mathematical analysis. The research questions were derived from the statement of the problem and the chapter concluded by stating the rationale of the study, its significance as well as its delimitations.

CHAPTER TWO

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

This chapter introduces the theoretical perspectives used in the study and reviews the literature concerning imagery and visualisation in the learning of mathematical analysis by undergraduate students in particular.

2.1 THEORETICAL FRAMEWORK

The researcher's experience in teaching mathematics in general made him aware of the well-known epistemological problem, i.e. that it is impossible to see or know what anyone is thinking. The only way to gain access to the students' thinking processes was through their utterances, algebraic symbols, numbers, diagrams or graphs. Accordingly, the idea of using a semiotic perspective, which looks at the production of signs, became an attractive possibility as a means to an understanding of the students' mathematical activities. The theoretical framework for this study draws from the Theory of Registers of Semiotic Representations (TRSR) (Duval, 1999, 2006, 2017).

The study also incorporates other perspectives of analysis that go beyond the semiotic approach but are relevant to the role of imagery and the visualisation of students' thinking processes in mathematical analysis. Among them, the researcher highlights the visual-analytic coordination strategy (Zazkis, Dubinsky & Dautermann, 1996) as an analytical tool to explain the thinking processes of undergraduate students when solving mathematical analysis tasks. The researcher also recognizes other important frameworks such as the APOS theory (Dubinsky et al., 1991) as potential tools to analyse advanced mathematical thinking processes.

2.1.1 Theory of Registers of Semiotic Representations (TRSR)

Duval (2017) provides a useful formulation of the learning of mathematics using a semiotic perspective. The researcher agrees with Duval's (1999) views that the only possible access to mathematical objects is through their representations in their different semiotic registers. Duval (1999) argues that semiotic representations play several fundamental roles in mathematics. As an example, semiotic representations refer to mathematical objects; they allow one to communicate about mathematics and they are necessary for mathematical processing. A mathematical activity can use a variety of semiotic representation systems, each with its own possibilities. The registers of semiotic representations comprise natural language (as used in definitions and proofs), numeric, algebraic and symbolic notations, geometrical figures and Cartesian graphs. Duval (2006) further argues that mathematical activity is a transformation of one semiotic representation into another in the same or different register.

In the discipline of mathematics, a representation is a symbolic, graphic or verbal notation to express concepts and procedures of the discipline, as well as their more relevant characteristics and properties. Representations can be classified in registers of representations (Duval, 1999). In the context of cognitive psychology, the notion of representation plays an important role in the acquisition and the treatment of an individual's knowledge. Duval (1999, p. 1) points out that "representation and visualisation are at the core of understanding in mathematics". According to him,

> ...representation refers to a large range of meaning activities: steady and holistic beliefs about something, various ways to evoke and denote objects, how information is coded. On the contrary, visualisation seems to emphasize images, and empirical intuition of physical objects and actions (Duval, 1999, p. 1).

Duval (2017) further points out that no knowledge can be mobilised by an individual without a representative activity. This fact makes the study of representations very important in order to explain the understanding of the concepts and the learning of mathematics.

Comprehension of the theory on registers of semiotic representation requires consideration of three key characteristics (Pino-Fan et al.., 2017):

i. There are as many different semiotic representations of the same mathematical object as semiotic registers utilised in mathematics.
ii. Each different semiotic representation of the same mathematical object does not explicitly state the same properties of the object being represented; what is being explicitly stated is the content of the representation.
iii. The content of semiotic representations must never be confused with the mathematical objects that these represent.

It is important to point out that there are two fundamental cognitive activities within the TRSR: treatment and conversion. The activity of treatment on the one hand consists of a transformation carried out in the same register. In other words, only one register is mobilised. The activity of conversion, on the other hand, consists of the mobilisation from one register into another, where the articulation of representation becomes fundamental. According to Duval (1999), the study of the activity of conversion makes it possible to comprehend the close relation between “noesis” and “semiosis,” a relation which is essential in intellectual learning. Semiosis is the mobilisation and creation of mathematical signs while noesis is the action and effect of understanding (Duval, 1999). Semiosis is necessary for noesis. However, it must be taken into account that the operation of conversion brings some difficulties, including the fact that the representation of the source register does not have the same content as the destination register. Another difficulty lies in the treatment, which becomes complex because of the use of the register of natural language and those registers that allow ‘visualising’ (graphs, geometrical shapes, etc.). Figure 2.1 illustrates various representations in algebraic register of a function in space to its graphical representations in space and plane.

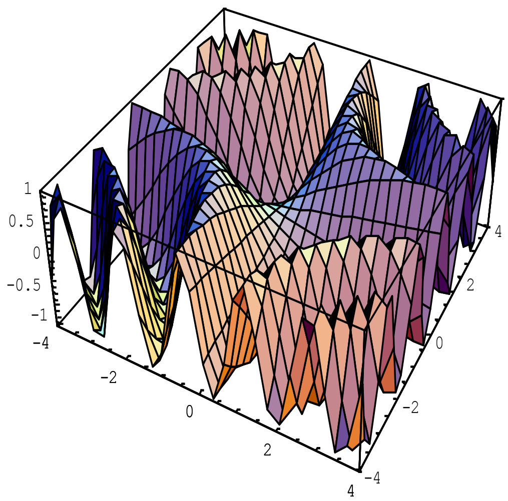
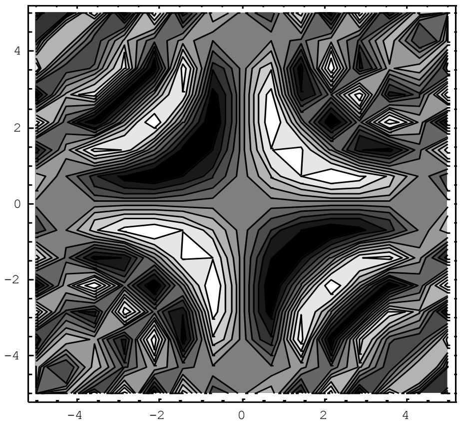
Algebraic register of z= sin xy	
Graphical representation of z = sin xy in $\mathbb{R}^3$	Graphical representation of z = sin xy in $\mathbb{R}^2$
	

Fig. 2.1: Registers of the function z = sin xy

There is a need to observe that not all semiotic systems are registers of semiotic representations. Duval (1995) defined registers of semiotic representations as all semiotic systems that allow the construction of the mathematical concepts through the following cognitive activities: (i) representation of concepts in a given register; (ii) treatment of these representations within the same register; (iii) conversion of these representations from a given register to another. For instance, with the concept of function, there are graphic, algebraic, numerical and verbal registers. There might be others, but these are the most used in teaching. It is possible to carry out processing into each register, that is, transformations of the representations in the same register in which they were created. It is also possible to realise conversions between different registers of representations that are transformations of one representation made in a register into another representation in another register. In the instance of a function, a conversion can be a translation of the function's tabular information into a graphic representation. Figure 2.2 provides a summary of how the transformations of treatment and conversion work.

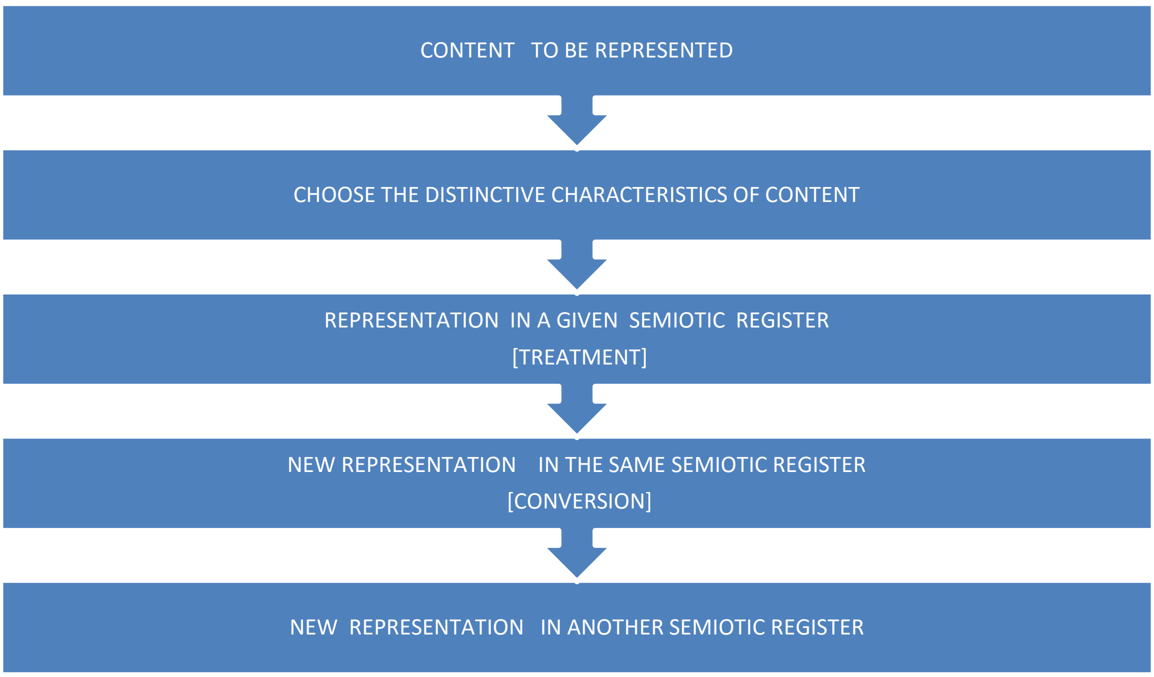


Fig. 2.2: Summary of Theory of Registers of Semiotic Representations

2.1.1.1 TRSR as a framework to study the learning of mathematical analysis

Rubio and Gomez-Chacon (2011) carried out a study to investigate students' difficulties in understanding the concept of integral. The TRSR was used as an analytical tool. Their results showed that some of the difficulties have their origins in the coordination between the analytic and graphic registers. They also noted that the use of a graphic register enhanced conceptual understanding. Guzman et al. (2015) carried out a comparative study between the TRSR and the onto-semiotic approach (OSA). The purpose of their study was to analyse the performance of a future high school teacher in a task related to the differentiability of the absolute value function. These researchers discovered that the two frameworks complement each other and result in a clear, detailed analysis of the students' performance. They also noted that the use of TRSR produced a more global analysis than the OSA framework.

Mehanovic (2011) studied the implementation of GeoGebra software to aid in treatments and conversions between registers associated with single variable integrals. In contrast to a semiotic chain (Fig. 2.3), Mehanovic (2011) succeeded in using computer software to simultaneously present an algebraic window and a geometric window in order to provide immediate access to different registers, as claimed by Duval. While Mehanovic was very successful in creating the technical tools for simultaneous access to multiple registers, the failure to successfully incorporate these tools into the classroom presentations and environment limited the success of the project.

Fig. 2.3: Example of a conversions chain associated with a semiotic chain

Berger (2010) conducted research on a semiotic view of mathematical activity with a computer algebra system (CAS). The tasks assigned to the learners were on numerical approximations of roots using MATHEMATICA. The researcher concluded that a semiotic framework enables a rich understanding of how the use of a CAS may enable, or constrain, mathematical activity. It was also observed that Duval's theory facilitated an easy analysis of the intra- and inter-registers' transformations with CAS, unlike through the use of paper and pencil or utterances.

2.1.1.2 Visual-analytic coordination strategy

Learning guided by the theory of multiple representations allows students to construct understandings that better prepare them for knowledge transfer (Goldin, 2003). A cognitive model proposed by a group of researchers has suggested that visual reasoning and analytic reasoning can and should be integrated (Zazkis et al., 1996). The researchers argue that there is mutual support between visual and analytic reasoning and this gives learners an opportunity to lay down the correct solution to a problem.

This argument was supported by Caglayan (2017) who posits that the visual-analytic (VA) model provides a beneficial teaching and learning environment which shows how visual approaches benefit from analytical thinking and how analytical approaches are enriched by visualisation. The description of the model is as follows: If a learner's thinking about a certain concept can be described in terms of a visual approach, say V_i, where V_i denotes the learner's visualisation of the concept under consideration, then the learner might be encouraged to proceed towards an analytical approach, say A_j, where A_j denotes the learner's analysis of what is being visualized. Alternatively, if the learner's thinking is inclined towards an analytical approach, then pedagogy might be designed to encourage the learner to visualize in some manner the analytical activities perceived. This model is significant in analysing mathematical activities related to mathematical analysis. Giaquinto (2011) claimed that visualisation can play an important role in analytic discovery, by providing the idea for a proof.

2.1.2 Psychological notions contributing to a cognitive approach to the study

The history of imagery can be traced back to Wilhelm Wundt (1832-1920) who proposed that images were one of the three basic elements of consciousness, along with sensations and feelings. He also proposed that because images accompany thought, studying images was a way of studying thinking (Goldstein, 2011). This idea of a link between imagery and thinking gave rise to the imageless-thought debate, with some psychologists taking up Aristotle's idea that "thought is impossible without an image", and others contending that thinking can occur without images. The researcher will outline certain areas of cognitive psychology that could help us to understand and predict the mental processes in mathematical thinking and learning.

Tomic (1993) posits that behaviourist theories, built on external observation of stimulus and response, refuse to speculate on the internal workings of the mind. Behaviourist theories claim that research should focus exclusively on observable behaviour and not on the unobserved phenomena of consciousness. Such theories are inappropriate in this study because they brand the study of imagery as being unproductive because visual images are invisible to everyone except the person experiencing them. Therefore such theories deal with the surface structure of stimulus-response behaviour and fail to explore the deep structure of mathematical thinking. Of greater value are theories of “meaningful” cognitive psychology, linking cognitive growth to the development of a knowledge domain (Tall, 1991). The need for models and images on which to hang one’s mathematical thinking is of paramount importance. Robert Sommer, Professor of Psychology and Environmental Studies at the University of California states that

> A mathematical statement leaves the hearer cold when it evokes no images or associations. It is as if the words were uttered in foreign language (Sommer, 1978, p. 73).

This statement implies that if the learner evokes no images in the learning process then there is no conceptual understanding; the learner is not able to explain. Either the learner has a local comprehension or does not understand at all what the statement means. Hiebert and Carpenter (1992) propose a framework for considering and understanding mathematics from the constructivist perspective that sheds light on analysing issues related to conceptual understanding. They make a distinction between the external and internal representations of mathematical ideas, pointing out that, in order to think and communicate ideas, people need to represent them in some way. Communication requires that the representations be external, taking the form of spoken language, written symbols, drawings or concrete objects.

2.1.2.1 Nature of visual imagery

Imagery is the ability to recreate the sensory world in the absence of physical stimuli. People have the ability to imagine tastes and smells, tactile, visual and auditory experiences. In this study, the researcher considers visual imagery as “seeing” in the absence of a visual stimulus, because of the domain of knowledge under investigation and also since understanding visual imagery provides connections to other cognitive phenomena such as perception, memory and thinking.

There is much debate among psychologists as to the nature of visual imagery. Kosslyn (1996) interprets the results of his research on imagery as supporting the idea that the mechanism responsible for imagery involves spatial representation, a representation in which different parts of an image can be described as corresponding to specific locations in space (pictorial format). Pylyshyn (1973) disagrees and proposes that the mechanism underlying imagery is not spatial but propositional. A propositional representation is one in which relationships can be represented through symbols, as when the words of language represent objects and relationship between objects. These two views provide an excellent example of how data can be interpreted in different ways.

It is a common occurrence that people can imagine melodies of familiar songs in their head. It is thus not surprising that musicians often report strong auditory imagery and that the ability to imagine melodies has played an important role in musical composition (Goldstein, 2011). Likewise, visual imagery has resulted in both scientific insights and practical applications. One example of visual imagery leading to scientific discovery is Albert Einstein's description of how he developed the theory of relativity by imagining himself travelling beside a beam of light (Intons-Peterson, 1993). Visual imagery provides a way of thinking that adds another dimension to purely verbal techniques.

Kosslyn (1983) explored whether people use a pictorial or a propositional format to remember something. He concludes that in many instances people can choose whether to use an image or a propositional format to remember something. It appears that whether we think of ourselves as 'mainly visual' or 'mainly verbal', most of us have the capacity to shift our thinking in the other direction when it is useful to do so. He also addresses the question of how and when images are actually used in thinking and memory, and suggests two main applications, namely, simulation of a real situation and symbolization which then does not actually depict an object or situation as it really appears, but represents it symbolically. Kosslyn (1983) outlined the four components of visual thinking as: generation, inspection, manipulation, and maintaining of images. Most notable is the fact that much of the power of imagery comes from the ability to modify imaged objects and to see if the changes lead to anything. In this case, emphasis is on the importance of the dynamic property of visual imagery. According to Kosslyn (1983), imagery is a complex set of processes, not a simple discrete process which an individual either possesses or does not possess.

Kossyln (1983) posits that visualisation follows an interesting pattern in cognitive development. Young children are highly proficient at visualising while adults mostly depend on images to recall facts regarding relatively unfamiliar properties. To children, many things are so relatively unfamiliar that imagery may be their only primary source of information. Another observation is that adults are highly proficient at logical deduction; a process which uses previously stored propositional information, whereas children do not possess that facility.

2.1.2.2 Concept image and concept definition

A concept is an abstract idea containing the fundamental characteristics of what it represents. Concepts arise as abstractions or generalisations from experiencing the results of transformation of the existing ideas. Concepts are stored in long-term memory. One of the important goals of research and implementation efforts in mathematics education has been to promote conceptual understanding. When students are learning a mathematical concept in the classroom, they are actively constructing their own meanings and understanding of the concept. However, in the construction process most students have difficulties if the teacher introduces the concept using a formal definition. For example, the formal definition of the limit of a sequence of real numbers is expressed in epsilon-delta form. Students in most cases want the teacher to explain the concept in a simpler way.

In real-life situations people have concept names in their minds. When a concept name is seen or heard, something is evoked by the concept name in our memory. Tall and Vinner (1981) claim that what comes to mind is not the concept's definition, even in the instance where the concept does not have a definition. Tall and Vinner (1981) referred to the evoked thing as the concept image. According to Tall and Vinner (1981), a concept image describes the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. They further stated that:

> The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure logic that neither gives us insight, nor is it chance that causes us to make mistakes. To understand how these processes occur, both successfully and erroneously, we must formulate a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived (Tall, 1991, p.7).

Reality and formality are sometimes conflicting ideas. Many mathematical concepts which we freely use are not formally defined. Students can easily find the limit of a sequence $\frac{1}{n}$ but find it difficult to define the limit of a sequence. Usually, in this process the concept is given a symbol or name which enables it to be communicated and which facilitates its mental manipulation (Tall, 1986).

Tall (1991) posits that the concept image is something nonverbal and associated in our minds with the concept name. It can be a visual representation of the concept in the case where the concept has visual representations. It can also be a collection of impressions and experiences. For example, when you hear the word “function”, the following concept images can be evoked as “$y = f(x)$’’; you might visualize a graph of a function, you might think of specific functions like $y = x^2$, or even an event. All of this points to the fact that a concept’s image varies from person to person.

In this way, Tall and Vinner (1991) distinguish the way an individual may think about a mathematical concept via the concept image and its formal mathematical definition, defined as the concept definition. The former is culturally developed and derives from the fact that mathematics is a human activity and is therefore culturally and contextually dependent. They argue that an individual will often (but not always) possess, concurrently with a concept image, a concept definition, which they describe as a verbal definition which explains the concept, a form of words used to specify the concept. The concept image is not built from the concept definition in a logical or linear fashion, but the concept definition is often not even present and yet a rich concept image, complete with examples, notation, and visual images, may be present.

In practice, there may be two distinct types of concept definition: intuitive and formal. The intuitive definition on the one hand is a mental construct, formulated informally by individuals to suit their personal needs. Note that the personal concept definition may not exist, may be incomplete, or may be inconsistent with other parts of the concept image. On the other hand, a formal concept definition is the one accepted by the mathematical community at large. It is often learned in rote fashion by the student. The formal concept definition is relatively free of social influences, and is thus an attempt at structuring mathematics as a formal system. The concept definition may in turn generate its own concept image, referred to as the concept definition image, which may or may not be related in a coherent way to the other parts of the concept image (Tall, 1986).

In a learning situation it is ideal that the student should have both the concept image and the concept definition. Concept images are refined from time to time .The role of the instructor is to use the concept image as a springboard to introduce the concept definition.

2.2 REVIEW OF LITERATURE

The review of literature was presented in the following order: historical developments of mathematical analysis, theoretical perspectives of imagery and visualisation and previous research on visual thinking in mathematical analysis.

2.2.1 Mathematical analysis

Mathematical analysis is the part of mathematics in which functions and their generalizations are studied by the method of limits. The name mathematical analysis is a short version of the old name of this part of mathematics "infinitesimal analysis" (Johnsonbaugh & Pfaffenberger, 2010). At the university where this study was conducted, mathematical analysis at undergraduate level is structured around the notions of real number, function, limits of sequences and functions, continuity, derivatives and integrals of functions of one variable. At a more advanced level the above concepts extend to analysis of several variables such as complex analysis, functional analysis, measure theory and many more. Analysis is the general theory that underpins calculus. Calculus is the foundation of mathematical analysis (Giaquinto, 2011).

2.2.2 Historical developments of mathematical analysis

Calculus was developed in the 17th and 18th centuries as a tool to describe various physical phenomena in astronomy, mechanics and electrodynamics. These physical problems could not be solved by means of geometrical and arithmetical methods alone. Many of these problems concern situations that are easy to visualise. The developments of calculus, such as the concepts of derivatives and integrals are attributed to Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1716). However, some concepts of infinitesimal calculus developed by Newton and Leibniz caused both enthusiasm and very harmful criticism.

According to Brating (2012), the 19th century is often considered as a period when mathematical analysis underwent a major change. There was an increasing concern for the lack of rigour in analysis concerning basic concepts such as functions, derivatives, and real numbers. Mathematics was often connected to the intuition of time and space. The definitions of many fundamental concepts in analysis were vague and gave rise to different views of not only the definitions but also of the theorems involving these concepts. The great mathematician, Henri Poincare, asserted:

> Weierstrass leads everything back to the consideration of series and their analytic transformations; to express it better, he reduces analysis to a sort of prolongation of arithmetic, and you may turn through all his textbooks without finding a figure. Riemann, on the contrary, at once calls geometry to his aid; each of his conceptions is an image that no one can forget, once he has caught its meaning… Among our students we notice the same differences; some prefer to treat their problems by analysis, others by geometry. The first group is incapable of seeing in space, the others are quickly tired of long calculations and become perplexed (Poincare, 1913, p. 212).

This quotation was explained by Tall (1991) who argued that such differences in the foundations of mathematics culminated in a different mathematical philosophy at the beginning of the 20th century. It is not a surprise that people have different thinking processes. The use of diagrams or pictures to explain certain concepts in mathematical analysis is not a strange thing. Mathematics educators should accommodate both visualisers and non-visualisers in teaching mathematical analysis.

Grattan-Guinness (2004) outlines four main strategies in which mathematical analysis was developed, in chronological order:

i. Newton's "fluxions" and "fluent" (1660 onwards), theory of limits deployed, though not convincingly;
ii. Leibniz's "differential" and "integral" calculus, based upon dx and $\int f(x)dx$ (1670 onwards), with infinitesimals central to, but limits absent from, all basic concepts; this was reformulated by Euler in the mid-1750s by the addition of the "differential coefficient", the forerunner of the derivative;
iii. Lagrange's algebraisation of the theory, in an attempt to avoid both limits and infinitesimals, with a new basis sought in Taylor's power-series expansion (1770 onwards), and the successive differential coefficients reconceived in terms of the coefficients of the series as the "derived functions"; and
iv. Cauchy's approach based upon a firm theory (and not just intuition) of limits (1810 onwards); from it he defined the basic notions of calculus (including the derivative as

the limiting value of the difference quotient) and also of the theories of functions and of infinite series, to create “mathematical analysis”.

Gradually Cauchy‘s tradition gained wide acceptance, with major refinements brought in by Karl Weierstrass (1815-1897) which has long been the standard way of teaching mathematical analysis.

Due to the absurdities and contradictions cited above, mathematicians felt it was essential that the foundation of their work be logically examined and rigorously established. Mathematical analysis was to be placed on a logically rigorous foundation and all intuition was thrown away. This led to the refinement of many mathematical concepts. The concept of function itself had to be clarified and such notions as limit, continuity, differentiability and integrability had to be carefully and clearly defined. Most calculus concepts underwent remarkable generalisations and abstractions.

2.2.2.1 Status of visualisation in mathematical analysis

Mancosu (2005) points out that during the 19th century visual thinking in mathematical analysis fell into disrepute. The reasons may have been that some mathematical claims that seemed obvious on account of an intuitive and immediate visualisation turned out to be incorrect due to the new emerging mathematical methods and results. Common sense would point to the fact that all continuous functions were differentiable. Weierstrass (1815-1897) constructed a continuous but nowhere differentiable function. The function is:

$f(x) = \sum b^n cos(a^n x)\pi$ where x is a real number, a is odd and 0< b< 1.

Another example which raised much debate was that of solids of infinite lengths with finite volume. For example if we rotate the curve $y = \frac{1}{x}$ about the *x*-axis and cut a plane perpendicular to the *x*-axis we obtain a solid of infinite length and finite volume. How can one visualise a solid of infinite length? How can such a shape have a finite volume? Such kind of questions created heated debates between philosophers and analysts.

According to Kadunz and Yerushalmy (2015), the history of mathematics shows visualisation to have been omitted and avoided to a certain extent. In the time of Leonhard Euler the visual was also a means for proving or establishing the existence of a mathematical object, whereas mathematicians of the 19th and 20th century reduced the use of visualisation for generating new ideas when solving problems. Heuristics was the task of visualisation. It is most likely that the gap between the two trends was one reason why the issue of visualisation became a significant topic for researchers in mathematics education.

Since the beginning of the 1980s, mathematics educators were interested in the practical challenges of teaching visualisation, in visualisation of mathematics by learners in schools, visualisation and cognitive psychology, and were looking for theoretical frameworks (Kadunz & Yerushalmy, 2015). Technology has also opened up new avenues on visualisation. As telescopes and microscopes have made the invisible to be visible, CAS has also enabled some mathematical concepts that used to be invisible to become visible.

2.2.2.2 Lessons from the history of mathematical analysis

Until the 17th century mathematical analysis was a collection of solutions to disconnected particular problems, for example, in integral calculus, the problems of the calculation of the areas of figures, the volumes of bodies with curved boundaries and the work done by a variable force, among many more. Mathematical analysis as a unified and systematic whole was put together in the works of Isaac Newton, Gottfried Leibniz, Leonhard Euler, Joseph Louis Lagrange and other scholars in the 17th and 18th century. The theory of limits was laid by Augustine Louis Cauchy at the beginning of the 19th century. A deep analysis of the original ideas of mathematical analysis was connected with the development in the 19th century and 20th century of set theory, measure theory and the theory of functions of a real variable, which has led to a variety of generalisations. A good part of the 20th century has been spent in generalising the gains already made. However, some of these developments brought about a fresh batch of paradoxical situations. The cardinality paradox was a result of those developments. In relation to the current study, it is observed that the role of visual thinking was significant in the development of Mathematical Analysis and Registers of Semiotics. Representations were central to the development of mathematical symbols. It was also observed that mathematics was used as a tool to spearhead scientific developments at every stage. It is now important to review how the 21st century mathematicians view the methods employed in mathematical analysis against a vast technological development period. Do we still regard visual thinking in mathematical analysis with suspicion?

2.2.3 Theoretical perspective of using imagery to learn mathematics

Emerging from a cognitive paradigm, the constructivist theoretical framework is representative of the contemporary notion that learning is an active, meaning-making experience. This is whereby meaning is constructed from the existing store of knowledge that learners bring to the learning task in conjunction with newly acquired knowledge, such as that gained through instruction (Douville, Pugalee, Wallace & Lock, 1998). Within the field of psychology, constructivism provides a framework for researchers concerned with investigating the mental processes involved in learning and memory and how mental representations are constructed (Dagar, 2016).

There are two principles that direct the process of learning mathematics: inductive and deductive reasoning. Inductive reasoning starts from specific cases and extends to general cases. The reverse is true for deductive reasoning. Mathematics is a cognitive process, based on thinking that requires the dual coding of imagery and language (Paivio, 1991). Imagery is fundamental to the process of thinking with numbers. Mathematics at higher levels is thinking with symbols, diagrams and language. The process requires the integration of language and imagery to understand the fundamentals and then apply them. Dual coding in mathematics requires two aspects of imagery, namely symbol/numeral (parts/details) and concept imagery (whole/gestalt) (Paivio, 1991).

2.2.3.1 Symbol/Numeral imagery

Visualizing symbols is one of the basic cognitive processes necessary for understanding mathematics, for example, " $\int_a^b f(x)dx$ " for the concept of area under a curve. When we see the numeral “5” we know it represents the concept of five of something: five people, five mangoes, and so on. Chronological relationships appear in our minds on the concept of supremum and infimum of sets, for example highest and lowest salaries at a company, highest and lowest temperatures of the day. Imagery involves our sensory systems’ way of making the abstract real. It is a means to experience mathematics.

2.2.3.2 Concept imagery

Understanding problem-solving, computing and proving statements/theorems in mathematics requires the ability to process the gestalt (the whole). Often, children or adults can visualize the symbols and the parts. However, they cannot integrate those parts into a whole, just as they can visualize individual words but cannot integrate those words into a whole to form concepts. Mathematical skill requires the ability to comprehend the gestalt, that is, see the big picture, in order to understand the process underlying mathematical logic (Dennet, 1981). Concept imagery is the ability to image the gestalt. For example: “Prove the mean value theorem for derivatives”. The basic step is the ability to state the theorem, have the diagram in mind and establish the relationships with Rolle’s theorem. Concept imagery is necessary for the process involved in critical reasoning and connects us to language and thought.

2.2.3.3 Research on imagery in mathematics

Mason (2002) argues that the purpose of assigning learning tasks to students is not only to find answers but to become aware of the generality of various possible methods: when they work, when they are efficient, when they need modifying. The sense of generality arises within the mental realm, and is expressed in words, diagrams and symbols in the visible realm. To appreciate generality one needs to move beyond the particular to the general. This is best activated by words that prompt images, supplemented and underpinned by diagrams that are seen as frames from a complex “film-like” mode of processing mathematical information that cannot be seen as single photos. For example, the expression $y = 2x + 3$ is an object. It is also the specification of a rule for calculating values of y corresponding to values of x, a function machine. In the background is a table of values which could be constructed. Associated with the expression is a graph. Most importantly, it is an expression of generality, a specification of a relationship between two quantities. Encountering the graph or the symbols will ideally trigger access to the other and to all the further associations and connections. This entire process happens in and through mental imagery.

Tall and Pinto (2001) conducted research on the transition to formal mathematical thinking involving the use of quantifiers. Their desire was to find out how students construct meaning for these quantified statements. The paper reported the case study of a student who constructed the formal definitions not from the processes of quantification, but from his own visuo-spatial imagery. The task involved the definition of a sequence. The case study revealed a student who was grappling with imagistic ideas in order to translate them into a formal definition. He constructed the concept of convergence through thought experiments that responded not only to the syntax of the definition but also attempted to give an imagined meaning for the definition. He attempted to understand the statement first as a property satisfied by his mental image of the object to be defined. He then gave meaning to the statement from his image by exploring and verifying how it worked. The crucial idea was to understand how his image characterized the mental concept of convergence that he was attempting to construct. This mental construction involved playing with the image in various ways. In order to interpret the definition, the student chose a prototype sequence $\frac{1}{n}$ and explored the concept through thought experiments by fixing epsilon and varying $n \epsilon \mathbb{N}$. He compressed information in a picture (diagram), which he evoked when writing down the definition. He was operating in a context that had both the limit processes and limit objects and he explored and refined his existing prototypical image of limit rather than encapsulating the limiting process into a limit concept.

2.2.4 Mathematical visualisation

The literature on visualisation is replete with terms such as "spatial ability", "visual thinking", "intuition" and "visual understanding", among others. Most often, these descriptions are contrasted to analytical or symbolic modes of thought as though the two representations were discrete entities. There is no consensus in the mathematics and mathematics education literature as to the precise nature of mathematical visualisation. Several authors have implied through the context of their articles, that visualisation is restricted to a mental activity and that any visualising occurs in the mind. They have essentially borrowed the visual imagery idea from cognitive psychology and, in fact, several authors have used the terms interchangeably (Burton, 2004; Glennon, 1980; Mayer, 2002; Sword, 2005; Thompson, 1996). Goldin (1987) equates visualisation to feeling, imagining and therefore confines it to mental imagery. Presmeg (1986) defines a visual image as "a mental scheme depicting visual spatial information" and notes that this "allows for the possibility that verbal, numerical or mathematical symbols may be arranged spatially to form the kind of numerical or algebraic imagery sometimes designated 'number forms'." The dangers inherent in such an approach have been clearly set out by Zimmerman and Cunningham (1991, p. 4) who wrote:

> Mathematical visualisation is not 'math appreciation through pictures'. The intuition which mathematical visualisation seeks is not a vague kind of intuition, a superficial substitute for understanding, but the kind of intuition which penetrates to the heart of an idea. It gives depth and meaning to understanding, serves as a reliable guide to problem-solving, and inspires creative discoveries. To achieve this kind of understanding, visualisation cannot be isolated from the rest of mathematics. Visual thinking and graphical representation must be linked to other modes of mathematical thinking and other forms of representation. One must learn how

> ideas can be represented symbolically, numerically, and graphically, and to move back and forth among these modes (Zimmerman & Cunningham, 1991, p. 4).

Mathematicians have always used their "mind's eye" to visualize the abstract objects and processes that arise in all branches of mathematical research. Recently, mathematics educators have directed much attention to the issue of technology in the classroom and in the curriculum, and to how technology could aid visualisation, particularly of graphs. This suggests that a description of mathematical visualisation should include not only mental imagery but also pictorial imagery in more concrete form by pencil and paper, calculator and computer. Zimmerman and Cunningham (1991, p. 4) provide an even broader and all-encompassing description of mathematical visualisation as inclined towards the ability to produce and understand how to use such a visual image correctly:

> From the perspective of mathematical visualisation, the constraint that images must be manipulated mentally, without the aid of pencil and paper, seems artificial. In fact, in mathematical visualisation what we are interested in is precisely the student's ability to draw an appropriate diagram (with pencil and paper, or in some cases, with a computer) to represent a mathematical concept or problem and use the diagram to achieve understanding, and as aid in problem-solving (Zimmerman & Cunningham, 1991, p. 4).

According to this description, visual thinking is not merely a psychological aid, facilitating the grasp of what is gathered by other means, but has epistemological functions, as means of discovery, understanding, and even proof. Zimmerman and Cunningham further argued that:

> In Mathematics, visualisation is not an end in itself but a means to an end, which understands. Notice that, typically, one does not speak about visualizing a diagram but visualizing a concept or problem. To visualize a diagram means simply to form a mental image of the diagram but to visualize a problem means to understand the problem in terms of a diagram or visual image. Mathematical visualisation is the process of forming images (mentally, or with paper and pencil, or with the aid of technology) and using such images effectively for mathematical discovery and understanding (Zimmerman & Cunningham, 1991, p. 4).

We often associate visualizing in mathematics with drawing pictures or diagrams as an aid to getting started on problems. However, visualizing has a much wider role to play in problem-solving, including supporting the development of ideas and facilitating communication of results and understanding. Some particular problems emphasise the use of visualisation to help learners understand and develop a plan to solve the problem. In producing such a visualisation, the problem-solver is identifying the key components of the problem and the relationships between them. According to Crapo, Waissel, Wallace and Willemain (2000), this process has two main elements:

i. An internal model or visualisation (described as imagery by Crapo, et al., 2000)
ii. An external representation (described as a visualisation by Crapo, et al., 2000).

An effective mathematical model is developed if it is supported by the interplay between the internal and external representations.

2.2.5 Nature of imagery and visualisations evoked by students in the learning of mathematical analysis

The main focus of the study was to investigate the nature of visualisations evoked by undergraduate students in the learning of mathematical analysis with a hope of improving the learning and teaching of this course at universities. This type of study raises some questions. Firstly, is it possible to know the nature of images evoked by learners? Secondly, is it possible to classify these images? And thirdly, is it possible to know the role of imagery in solving mathematical tasks and how these images were used? Since it is impossible to observe mental processes directly, the only easier way was to make inferences from the product of mental activities. For example, looking at students' written solutions to mathematical analysis problems and to listen to their utterances on conceptual understanding of mathematical analysis was a closer way to understand the thinking processes. Several researchers have used this strategy and yielded some positive results.

Huang (2015) conducted a study with 15 first-year engineering students taking a calculus course. The study was on students' visual thinking regarding the definite integral. The instrument used for data collection was a questionnaire containing problems related to definite integrals and interviews. These problems enabled the students' performance regarding visual thinking to be analysed. The results of the questionnaire necessitated further investigation into the visual thinking of students through clinical interviews which were video and audio-taped. During the interviews the students would think aloud while they were solving the tasks so that their responses and strategies would reveal their mental processes and images. The results of the study revealed that students with non-visualisation ability evoked visual images which were more inclined to memory images. Students with low visualisation ability's use of algebraic representations were in high percentages, and their use of graphical representations was induced from analytic thinking and this led these students to the difficulties of problem-solving. Students with high visualisation ability in the use of graphical representations were in high percentages along with algebraic representations. These students were able to link algebraic and graphical representations and this led them to succeed in problem-solving. The only problem with this study is that it is limited to the concept of integral calculus.

Krussel (1994) conducted a case study with nine advanced mathematical thinkers involving undergraduate mathematics students, mathematics graduate students and mathematics lecturers. The study was designed to examine and describe the nature of visual images used by advanced mathematical thinkers. The participants were extensively interviewed, and their responses audio-taped and transcribed. The interviews were reflective in nature, comprising a series of questions, which were asked regarding 21 different mathematical concepts. The researcher managed to classify the visual images according to how the image was used in solving advanced mathematical concepts, i.e. as prototypical, analogical or diagrammatic images.

Her study also revealed that visual images were noticeable in advanced mathematical thought. It was rare for any individual to be found as lacking a visual component to the concept image, even though at times it was no more than the associated mathematical symbol for the concept. The results of the study point to the fact that students should be provided with a variety of visual examples and images for all mathematical concepts, if possible. This could facilitate the development of their own conceptions, because it is from these that they begin to construct their own understanding, build their own scaffolding and embark on the progression from procedural to structural understanding. Whilst this study was rich in information concerning imagery and visualisations, the methodology was somehow unilateral. Participants were not given a chance to express their visualisations in written form.

Tall (1999) argues that visual intuition in mathematics has served us both well and badly. It suggests theorems that lead to great leaps of insight in research, yet it can also lead to blind alleys of error that deceive. For 2000 years Euclidean geometry was held as the archetypal theory of logical deduction until it was found, in the 19th century, that implicit visual clues had insinuated themselves without logical foundation, for instance the implicit idea that the diagonals of a rhombus meet inside the figure, when the concept of “insideness” is not formally defined in the theory. Subtleties such as these caused even more pain in the calculus. So many fondly held implicit beliefs foundered when analysis was formalised. Comfortable feelings about continuous functions and the ubiquity of differentiable functions took a sharp jolt with the realisation that many continuous functions were not differentiable anywhere. Once the real numbers had been axiomatised through the introduction of the completeness axiom, all intuition seemed to go out through the window. It is necessary to be very careful with the statement of theorems in formal analysis since any slight lack of precision is almost bound to lead to falsehood. In such an atmosphere of fear and suspicion, visual mathematics has been relegated to a minor role, with only that which can be proved by formal means being treated as real mathematics. Yet to deny visualisation is to deny the roots of many of our most profound mathematical ideas. Visualisation proves to be a fundamental source of ideas in the development of the theory of functions, limits, continuity, differentiation and integrals. To deny these ideas to students is to cut them off from the historical roots of the subject.

Much has been written about the value of visualisation and imagery in terms of the potential to enhance a global and intuitive view and understanding of various areas of mathematics (Bishop, 1989; Fischbein, 1987; Usiskin, 1987; Zimmermann & Cunningham, 1991). Fischbein (1987, p. 104), for example, comments that “a visual image not only organizes the data at hand in meaningful structures, but is also an important factor guiding the analytical development of a solution.” Bishop (1989) concludes his review by proposing that there is value in emphasizing visual representations in all aspects of the mathematics classroom. However, it is also recognised that there are difficulties concerned with visualisation and imagery (Dreyfus, 2002). If mathematical visualisation is taken to be “the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding” (Zimmermann & Cunningham, 1991, p. 3), then such difficulties can relate to the process of forming images as well as using them in solving problems. Similarly, if mental imagery is taken as involving “constructing an image from pictures, words or thoughts; representing the image as needed; and transforming that image” then difficulties can arise from the processes of constructing, representing, and transforming these visual images for the purpose of problem-solving (Wheatly, 1991).

2.2.6 Role of visualisation in proving statements in mathematical analysis

There is a common view adopted from 19th century mathematicians that visual thinking in analysis, though heuristically useful, is not a means of discovery, let alone proof. Everyone appreciates a clever mathematical picture, but the prevailing attitude is one of scepticism: diagrams, illustrations and pictures prove nothing; they are pedagogically important and heuristically useful, but only a traditional verbal/symbolic proof provides genuine evidence for a purported theorem (Mancosu, 2005). However, some recent authors (Brown, 2001; Giaquinto, 2011; Guzman, 2002) take a different view and argue using some striking examples for a positive evidential role for visual thinking in mathematical analysis.

Brown (2001) argues that visual thinking can prove things. He argues from a historical point of view using Bolzano's "purely analytic proof." Bernard Bolzano (1817) proved the intermediate value theorem. Before the formal proof by Bolzano, the theorem was only appreciated from the common sense of continuous functions. Mathematicians disregarded the geometrical diagram (Boyer, 1949). The common attitude towards Bolzano reflects the generally accepted attitude towards proofs and visual thinking. The general view was that only proofs provide mathematical knowledge and, moreover, proofs are derivations, they are verbal/symbolic entities. Images, on the other hand, are psychologically useful, often suggestive but they do not provide evidence. The following comparison between the formal proof and the geometrical representations can show which method is more convincing.

Bolzano's theorem (Intermediate zero theorem): If f is continuous on the interval $[a,b]$ and f changes sign from negative to positive (or vice versa) , then there is a c between a and b such that $f(c) = 0$.

Analytic proof: (Adapted from Apostol, 1973, p. 85)

For definiteness, assume $f(a) > 0$ and $f(b) < 0$. Let $A = \{x: x \in [a, b] \text{ and } f(x) \geq 0\}$. Then A is non-empty since $a \in A$, and A is bounded above by b. Let $c = \sup A$.

Then $a < c < b$. We will prove that $f(c) = 0$.

If $f(c) \neq 0$, there is a 1-ball , $B(c; \delta)$ in which f has the same sign as $f(c)$. If $f(c) \geq 0$, there are points $x > c$ at which $f(x) > 0$, contradicting the definition of c. If $f(c) < 0$, then $c-\frac{\delta}{2}$ is an upper bound for A, again contradicting the definition of c. Therefore we must have $f(c) = 0$ (Apostol, 1973, p. 85).

N.B. A fine and polished proof but one can see that a diagram in mind was used to complete the proof!!

“Visual proof” or “Visual representation”

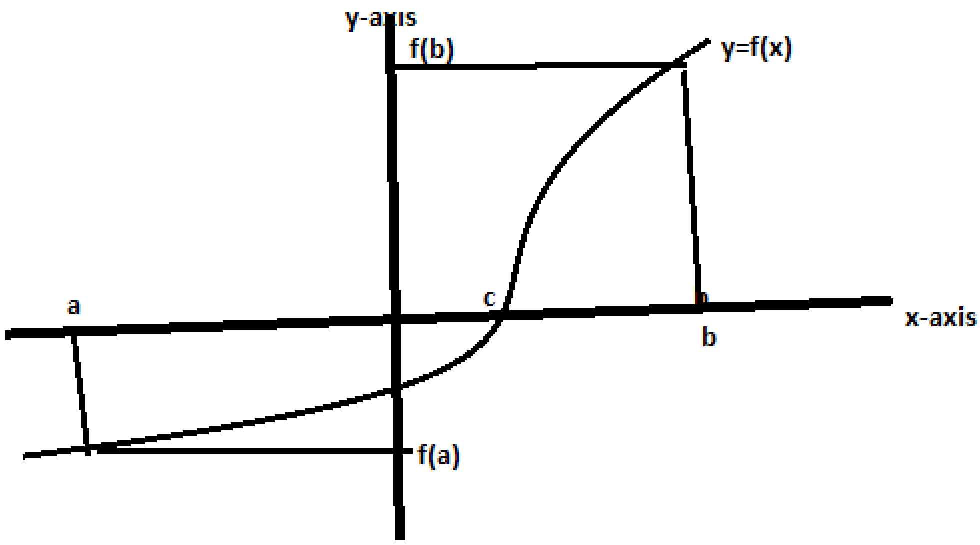


Fig. 2.4: Representation of Bolzano’s first theorem

Consider the visual evidence for the theorem. Just taking a glance at the picture, one can see a continuous curve running from below to above the x-axis, and there is no other way except for the curve to intersect the x-axis at c. If we view it that way then the proof is trivial and obvious. The diagram gives meaning to the verbal/written algorithms.

Guiaquinto (2011) challenged the assumption that the visual diagram could be proof of the theorem, although it could be a legitimate way of convincing someone. However, the examples cited by Giaquinto do not meet the class of functions which meet the conditions set by Bolzano. A simple generalisation of this theorem leads to what is now known as the Intermediate Value Theorem which states that: If f is continuous on the interval $[a,b]$ and there is a c between f(a) and f(b), then there is an x between a and b such that f(x) = c.
The theorem can be visually represented by the diagram in Figure 2.5.

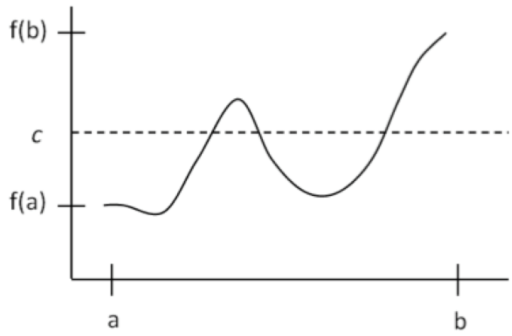


Fig. 2.5: Intermediate Value Theorem

Bolzano formally proved the theorem. A visual route of identifying continuity of functions is by drawing their graphs. If the curve of the graph is drawn without lifting the pencil tip off the paper then the function is continuous on that given interval. If we agree that the above function is continuous then it is easy to believe that the Intermediate Value Theorem is true. The important point is that the function is continuous.

Giaquinto (2011) is of the opinion that visual thinking in analysis can be a mode of discovery in a very restricted range of cases. His argument is that visual thinking is generally unreliable as a means of reaching conclusions in analysis, largely because analytic concepts, far from being faithful counterparts of visual concepts, apply to situations many of which are difficult to visualise. However, visual thinking has many distinct roles in mathematics, especially providing an idea for a proof.

Experienced mathematicians also confirm the importance of the role of visual thinking in mathematical analysis. The following testimony of Hadamard on the role of visualisation is quite representative of the image in the mathematical processes of an analyst:

> I have given a simplified proof of part (a) of Jordan's theorem [that the continuous closed curve without double points divides the plane into two different regions]. Of course, my proof is completely arithmetizable (otherwise it would be considered non-existent); but, investigating it, I never ceased thinking of the diagram (only thinking of a very twisted curve), and so do I still when remembering it (Hadamard, 1945, p. 103).

Guzman (2002) postulates that visualisation is very useful in the context of the initial process of mathematisation as well as in that of the teaching and learning of mathematics. He suggests the need to professionally develop mathematics educators so that they can introduce it to novice mathematics students. The utility of visual thinking in other areas of mathematics such as geometry is obvious but this should also be extended to mathematical analysis. Mathematical analysis arose as a need to quantitatively mathematise the spatial relationships of the objects of our ordinary life (Giaquinto, 2011).

2.3 CONCLUDING REMARKS

Chapter two started by reviewing the theory that shaped this study, Duval's (1999) Theory of Representations and Mathematical Visualisation. Some psychological notions contributing to a cognitive approach to the study were also discussed. The study also reflected back on the historical developments and status of visualisation of mathematical analysis before the 20th century and the chapter concluded with a review of current literature on the use of imagery and visualisation by undergraduate students in mathematical analysis.

CHAPTER THREE
RESEARCH METHODOLOGY

This chapter introduces the research design (mixed method case study), the method of inquiry for qualitative data, sampling procedures, research instruments, validity and reliability of instruments, the pilot study, data analysis procedures and ethical considerations.

3.1 RESEARCH PARADIGM

According to Kuhn (1962), a research paradigm is a set of common beliefs and agreements shared by researchers regarding how problems should be understood and addressed. In other words it is a philosophy of one’s understanding of the world and this governs one’s solution of a research question. A research paradigm is characterized by the following chain of assumptions: ontological paradigms (the nature of reality) which gives rise to epistemological paradigms (ways of researching and enquiring into the nature of reality) and these in turn give rise to methodological paradigms which have a bearing on the nature of instrumentation and data collection procedures (Cohen et al., 2011; Scotland, 2012).

3.1.1 Ontological considerations

This research borrows ideas from constructivist/interpretive and pragmatism paradigms where there is no single reality or truth and reality is being a constantly renegotiated, debated and interpreted in the light of its usefulness in new unpredictable situations (Creswell, 2003; Tashakkori & Teddlie, 1998; Vigotsky, 1980). These ideas were also echoed by Tall (1991) when he argued that there are not just two different kinds of mathematical thinking but many. A multiple perspectives paradigm was more ideal for this study because it encompasses multiple and possibly mixed viewpoints, representations and solutions to particular mathematical analysis problems.

3.1.2 Epistemological considerations

The study of advanced mathematical thinking has its roots in psychology. For it to be treated adequately one needs to be both a psychologist and a mathematician (Hadarmard, 1945 in Tall, 1991). A post-positivism paradigm that takes into account the subjectivity of reality and moves away from the purely objective stance was more convenient to adopt in this research.

3.1.3 Research design

The variation in mathematical thinking and learning among individuals often results in variations of activation patterns of the brain. Ideally, the results would best be interpreted through mixed methods research. The following framework is an outline of the research design.

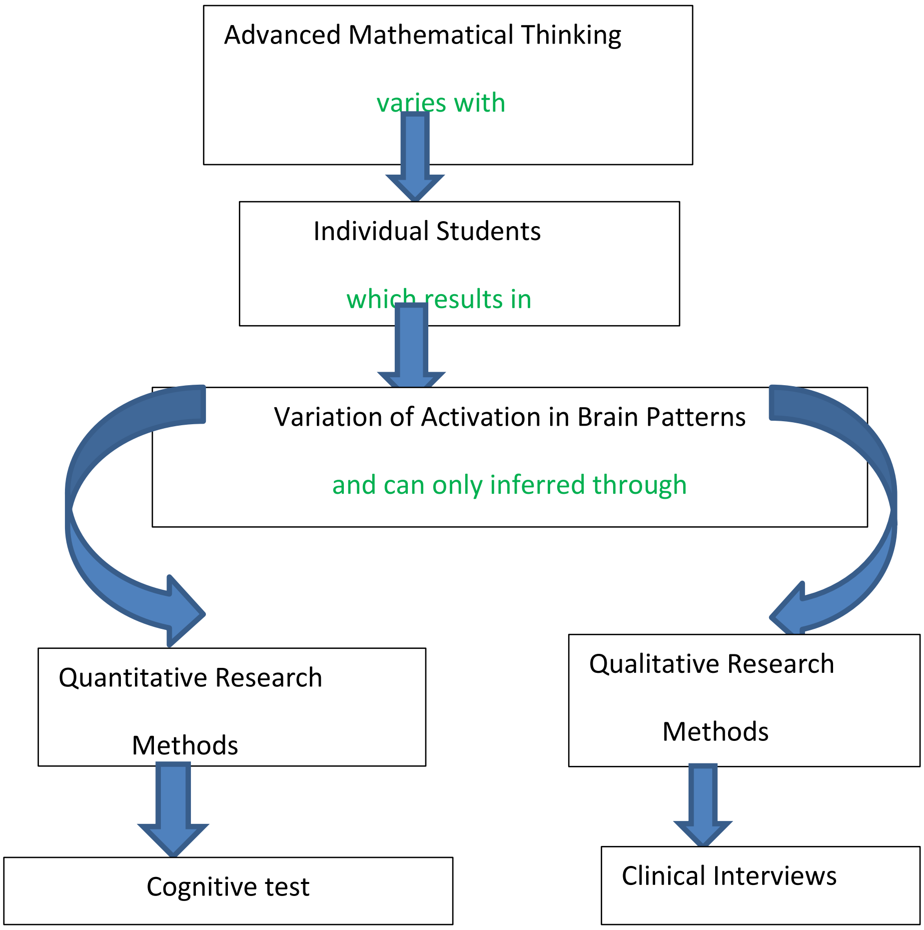


Fig 3.1: Framework of research design (selected parts from Mahmood et al., 2012)

Advanced mathematical thinking varies between different students; these differences can result in students conceptualising and processing mathematical information differently. A suitable way of investigating how students learn was to employ both quantitative methods and qualitative through the use of specially designed tasks and clinical interviews.

3.1.3.1 Mixed method case study approach

A mixed method case study approach was employed to answer the research questions. Case study research focuses intentionally on the complexity of a single case, or a bounded system, as a phenomenon of interest for disciplined investigation (Merriam, 1997; Stake, 1995). A strength of case study research is the ability to examine, in-depth, a case or a system within its real-life context to describe what happened and why (Yin, 2009). The phenomenon of interest in this investigation is complex: a study of mathematical thinking has its roots in psychology and research has indicated that there are many different kinds of mathematical thinking.

In order to have a picture of how imagery and visualisation are used to learn mathematical analysis concepts, the researcher employed several research methods to capture and describe the complexity of each case and to facilitate cross-case synthesis and explanation building (Yin, 2009). The researcher employed a mixed method approach to collect data in this case study. Mixed method research is an approach to research that is based upon the hypothesis that research questions should dictate the methodologies used (Johnson & Onwuegbuzie, 2004). The appropriate use of a mixed method approach was considered appropriate given the complexity of the research questions in this case study. The mixed method approach is a research approach in which researchers collect, analyse and integrate both quantitative and qualitative data in a single study to address the research questions (Creswell, 2013).

As mentioned earlier, it is possible to get close to knowing what one is thinking through verbal or written communication. The researcher needed both quantitative and qualitative data to gain insight into the nature of imagery and visualisations evoked by students when solving selected tasks in mathematical analysis concepts. Hadamard (1945) undertook an informal inquiry among mathematicians in America, where he queried such mathematicians as George Birkhoff, Norbert Weiner, George Polya and Albert Einstein on the mental images they used in doing mathematics. Hadamard argued for the use of subjective methods in the investigation of mathematical invention. Binet (1899) identified three principal methods of investigation: questionnaire, observation and experiment. The Newtonian mechanistic philosophy takes a positivist approach while Einstein's special theory of relativity takes a hermeneutic approach (Sokal, 1994). This study is going to employ both positivist and hermeneutic approaches.

A sample of 50 undergraduate mathematics students participated in responding to a cognitive test that had items selected from mathematical analysis concepts. From the same group 12 participated in structured interviews as a follow-up to the responses of the cognitive test. In this study the researcher undertook a case study of a group of students learning the selected concepts in a regular formal course of mathematical analysis in specially designed learning styles. The researcher had taught a course of calculus to these students and the mode of learning was through multiple representations of concepts: graphically, symbolically, numerically and through words. The researcher designed tasks that dealt with concepts in mathematical analysis where students could prefer to solve these tasks with the use of diagrams or not. The researcher also designed instruments so as to interview the participants on the nature of mental images evoked by students when dealing with the given tasks.

3.1.3.2 Method of inquiry for qualitative data: Hermeneutics and learning mathematics

According to Djauhari (2015), hermeneutics is a science of meaning as well as a theory and methodology of text interpretation. He further argued that every word that we say or write has its own meaning. Historically, the proponent of this philosophical belief was Martin Heidegger (1889-1976) who was mentored by Edmond Husserl (1859-1939). This phenomenological research methodology started as a theory of text interpretation of biblical texts, wisdom literature, and philosophical texts. Recently, however, it has gained entry into science and mathematics. The role of hermeneutics is to see the unseen (soul or mind) hidden behind a given word or text. Heidegger believes that one could not be disengaged from one's world; one's world is part of one's being and cannot be separated from one. The subject's experience was the only way to view the world. One's background is part of one's world and is handed down as a way of understanding the world (Harman, 2007). Heidegger acknowledges that each person brings his own experiences to his life and that the process varies from person to person. To clarify his argument Heidegger used the example of a hammer. One person may use the hammer as a tool to hammer a nail into a piece of wood and another person may use a hammer as a weapon. The hammer is the same with the same characteristics, but how one uses it depends on what one's life world is and how one had previously experienced the hammer (Moorman, 2013).

Djauhari (2015) also uses a simple example, from statistics, to clarify the role of hermeneutics in mathematics education. He uses the concept of "median" and notes:

> Let us ask a high school or even university student to explain the median of a data sample and all about it. According to the conversations with students, it is believed that they would give the following answer: "Median is the number separating the higher half of data from the lower half." Thus, the median of $\{3,3,5,9,11\}$ is 5 and that of $\{3,5,7,9\}$ is the average of the

two middle values $\left(\frac{5+7}{2}\right) = 6$. There is nothing wrong with these examples. However, those who were familiar with hermeneutics would immediately realize that this notion of the median is not well defined; it is not applicable for all cases of data (Djauhari, 2015, p. 2).

This method does not work where the set is composed of complex numbers or matrices. For this matter, students may have different interpretations of various mathematical concepts; mathematics is an interpretive activity. It is in a sense always embedded in some sort of social activity. The researcher was also of the view that mathematics is a straightjacket and dry type of subject, with no room for individual interpretation. Experience has, however, taught the researcher that we do not teach students only to arrive at the correct answers, but we also teach them how to explore different ways of arriving at the correct answers. Even if the answer is incorrect, the path as to how they arrived at the wrong answer may help them to find the correct answer, and that depends on the interpretation.

3.2 POPULATION AND SAMPLE

The target population of the study was all Bachelor of Education (BEd) students who specialised in mathematics at the Great Zimbabwe University. Normally, the enrolment figure of these students is low. On average, BEd mathematics students number 15 per intake. The students are enrolled twice per year according to the policy of the Great Zimbabwe University. Therefore, non-probability samples were used. Specifically for this research, the researcher selected the current group of students as the sample for the research. All BEd students who had majored in Mathematics, had taken a course of calculus and were interested in participating were purposively sampled. During the period of data collection the population of students majoring in Mathematics in the Faculty of Education at Great Zimbabwe University was 175. Cohen, Manion and Morrison (2011) argue that purposive sampling is undertaken for several kinds of research purposes including: to achieve representativeness; to enable comparisons to be made; to focus on specific, unique issues or cases; and to generate theory through the gradual accumulation of data from different sources. To achieve representativeness or comparability the researcher developed tasks that required visualisation and then investigated the thinking skills of the students as they worked through the tasks. The following Table 3.1 represents the gender distribution of the sample.

Table 3.1: Gender

Gender	**Frequency**	**Percentage**
Male	35	70
Female	15	30
Total	**50**	**100**

3.3 INSTRUMENTATION

The instrument used for data collection was a cognitive test containing items of mathematical analysis and an interview. The cognitive test was designed to collect quantitative data whilst in the clinical interviews we collected qualitative data. All students in the sample participated in responding to the cognitive test questionnaire and 12 students from the 50 students were purposively sampled to participate in the clinical interviews.

3.3.1 Cognitive test

The researcher constructed the cognitive test which was used in the study. The cognitive test was composed of typical problems in mathematical analysis. The cognitive test was comprised of problems in single variable analysis and problems in multi-variable analysis. For items on single variable analysis a majority of questions demanded the students to "make a representation" of a required concept followed by subtasks which required the student to deploy the representation. The questions were open-ended to allow multiple representations. The questions on multi-variable calculus had some tasks with closed-ended questions and also some subtasks with open-ended questions, especially the descriptive exercises. The matching exercise technique was adopted from Stewart (2008). The researcher used MATHEMATICA to construct the graphs and level curves.

First and most important, the researcher introduced the cognitive test to the students. The objectives of the test were clearly spelt out and students were invited to participate. Those who were willing to participate completed consent forms. The tests were given to the participants after the students had covered the course of analysis which is done during their second year at the university. Participants were given a whole week to familiarise themselves with the nature of the questions of the 12 tasks. After a week the students sat for the written test under the supervision of the researcher to ensure the integrity of the test. The written work was allocated three hours. At the Great Zimbabwe University the course of analysis is normally done after the students have covered a calculus course which is taken in Semester 1 of the first year.

The focus of analysis is understanding calculus from a firm logical foundation. The emphasis is on proving theorems and statements from the definition of concepts. The exercises involved concepts of basic analysis and multivariable calculus. Most of the problems were posed in the algebraic register but would allow a visual representation. The objectives for each item were conceptual understanding and application. These problems would allow the analysis of students' performance with regard to the coordination of registers, particularly the use of the graphic register (Rubio & Gomez-Chacon, 2011). A suitable area in mathematical analysis would be definite integrals, graphs of several variable functions (level curves) and double integrals since it is generally the last to be covered and also is also rich in visualisation. The purpose of this test was to explore the nature of the imagery and visualisation students invoke in solving mathematical tasks. For details of the nature of tasks see Appendices 2 and 3.

3.3.2 Clinical interview method/Task-based interview

The interview was a follow-up to the cognitive test. After participants had solved these tasks, the researcher then clinically interviewed the 12 selected participants on how the problem had been solved. Each interview lasted about an hour and was audio-taped. In order to prepare the script for the interview, the researcher analysed the written answers focused on how the students seemed to use and coordinate the different mathematical representations needed. The students were asked to think aloud while they were solving the tasks so that their responses and strategies could be described as well as allowing inferences about their mental processes and images to be made. The interviews were carried out in the assigned mathematics rooms in order to create a natural setting.

The researcher constructed an interview guide with a list of potential questions which were open-ended. The interview format was reflective in nature and designed so as to encourage respondents to reflect upon various aspects of their mathematical experience. Some of the questions were adopted from Krussel (1994). The interview questions were designed to investigate the nature of imagery and visualisations evoked by the participant at the time of solving the tasks and also to investigate how these images were deployed in solving these marked mathematical tasks. The essence of this interview was to investigate the thinking processes on imagery and visualisations evoked by the same students when they solved the given tasks. The mathematical tasks were carefully selected from popular books of mathematical analysis and multivariable calculus. The questioning tactic was, however, modified in order to assess visual thinking in mathematical analysis. It is generally agreed that the origins of the clinical method as a formal research tool coincide with Piaget's early investigations into children's thinking. Neither of the two most widely used research methods of that time, namely, naturalistic observation and standardised testing, were considered suitable for studying children's cognitive functioning (Opper, 1977).

In a clinical interview a dialogue or conversation is held between an adult interviewer and a participant. The dialogue is centred on a problem or task which has been chosen to give the respondent every opportunity to display behaviour from which mental mechanisms used in thinking about that task or solving that problem could be inferred (Opper, 1977). Several variations of the same task were presented to probe the strengths and limits of the theoretical construct thought to underlie the participant's response, and to provide additional insights into that subject's mental functioning (Posner & Gertzog, 2006).

In this research the tasks were composed of both non-routine and routine problems in mathematical analysis. Most of the problems were presented in algebraic register but would allow a visual representation. These problems would allow the analysis of students' performance with regard to the coordination of registers, particularly the use of the graphic register. The purpose of these interviews was to gain insight into the nature of imagery and visualisations that students invoke in the learning of selected concepts in mathematical analysis and how they deploy the images in solving the given tasks. Some examples of probable questions for the interviews are listed as follows:

- Do you have an image (or picture or scene) in your mind that captures this concept?
- Describe that image/participants/scene in words or in any other way you prefer.
- What first comes to mind when you think of the concept?
- What specific examples come to your mind?
- Are there specific theorems/axioms/properties that come to mind and are associated with this concept?
- When you were first introduced to the concept?
- Is your understanding different now from what it was before?
- When did your change of understanding happen?

- Are these problems new to you?
- Have you ever solved problems like this in class?
- Did the answer to this problem just come into your mind, or did you have to think about it for a while?
- Can you explain how you worked it out?
- (If no); Did you try to make a mental picture of the problem?
- (If yes); Can you describe the mental picture?

These questions were designed to enable the researcher to investigate what each student evoked in connection with the concepts involved in the given tasks. If the student evoked some images a number of follow-up questions were asked. The interview was also designed to discover the individuals' ways of connecting or linking concepts that would enable them to solve the task. In the case where the student had not attempted the question the interviewer would also probe the individual, trying to enquire how the individual might have solved the task. In the presentation of the transcripts the researcher used abbreviated subheadings to preface responses. For example, the question: "Do you have an image (or picture or scene) in your mind that captures this concept?" would simply be prefaced as "visual images".

The selected concepts in mathematical analysis were exercises presented in the twelve tasks of the cognitive test. The concepts are specifically on functions, continuity, differentiability, integrability of functions in $\mathbb{R}$, the mean value theorems of derivatives and integrals, geometry of space, functions of several variables and multiple integrals. These concepts were selected because they are rich in visualisation. Complete transcripts of responses were presented, unless the responses were insignificant.

3.4 VALIDITY OF THE COGNITIVE TESTS

Cohen et al. (2011) posit that validity is an important key to effective research, otherwise invalidity points to useless research. He argues that validity is a requirement in both quantitative and qualitative research. The instruments that were constructed provided an approximate measure of the visual thinking of undergraduate students in solving mathematical analysis problems. It is not an easy task to measure someone's thinking processes. We can only infer one's thinking through what is said by word of mouth or what is written on paper. The researcher's concern was about both face and content validity. In order to overcome this difficulty the researcher sought a judgment from the scientific community

(Neuman, 2013). Six experts volunteered to assist in judging the face and content validity of the instrument (cognitive test). Table 3.2 summarises the composition of the validators.

Table 3.2: Validation of research instruments

	Institution	Qualification	Area of research	Title
Expert 1	University of Zimbabwe	PhD	Mathematics Education	Professor
Expert 2	Midlands State University	PhD	Mathematics Education	Doctor
Expert 3	Bindura University of Science Education	PhD	Mathematics Education	Doctor
Expert 4	Zimbabwe Open University	PhD	Mathematics Education	Doctor
Expert 5	Great Zimbabwe University	PhD	Mathematics Education	Doctor
Expert 6	Great Zimbabwe University	PhD	Mathematics	Doctor

The experts rated all 12 tasks, task by task using the relevance scale in Table 3.3 adopted from Waltz et al. (2005).

Table 3.3: Rating scale for question items

Ratings of 1 and 2		Ratings of 3 and 4	
1	2	3	4
Not relevant	Somewhat relevant	Quite relevant	Highly relevant

The results of the ratings were as follows: Ratings 1 and 2 are considered to indicate items that are not relevant (-) and Ratings 3 and 4 are considered to indicate relevant items (X).

Table 3.4: Mean content validity ratio /Mean expert proportion

Item	Expert 1	Expert 2	Expert 3	Expert 4	Expert 5	Expert 6	Number of agreements	Item CVI
1	X	X	X	X	X	-	5	0.83
2	X	X	X	X	X	X	6	1.00
3	X	-	X	X	X	X	5	0.83
4	X	X	X	X	X	X	6	1.00
5	X	X	X	X	X	X	6	1.00
6	X	X	X	-	X	X	5	0.83
7	X	X	X	X	X	X	6	1.00
8	X	X	X	X	X	X	6	1.00
9	X	X	-	X	X	X	5	0.83
10	X	X	X	X	X	X	6	1.00
11	X	X	X	X	X	-	5	0.83
12	X	X	X	-	X	X	5	0.83
Proportion relevance	1.00	0.92	0.92	0.83	1.00	0.83		

The instruments were rated by six mathematics experts at a workshop of the Southern African Association of Researchers in Mathematics, Science and Technology Education (SAARMSTE) held in Zimbabwe in July 2015. Table 3.5 shows the results after calculating the mean item-content validity ratio. The mean item-content validity ratio was found by calculating the mean of all 12 test items. Likewise, the mean expert proportion was found by calculating the mean of the proportions of the six experts.

Table 3.5: Results of validation

Statistics of cognitive test	
Mean item-CVR	0.92
Mean expert proportion	0.92

The results show that although there was some disagreement on certain items, the indices are sufficiently high to warrant the researcher to proceed in using the cognitive test. Generally, it is well known that validity rarely reaches 100% (Cohen et al., 2005; Neumann, 2013).

Face validity: Five out six experts agreed that the instrument had face validity. The sixth expert was concerned about the role of imagery and visualisation in mathematical analysis.

Content validity: All lecturers agreed that the content of the instrument was valid, however, they recommended the inclusion of the infinity concept. Their recommendations were considered after the pilot study. The final instrument included the infinity concept.

3.5 PILOT STUDY

The pilot study was conducted to test the reliability and validity of the research instruments before they were deployed to the main field. The study was conducted on mathematics students at the Great Zimbabwe University during the first semester of 2015. Eight mathematics students who had completed the course of analysis were involved, four men and four women. All the participants were given a mathematical analysis cognitive test to answer some prescribed questions. The cognitive test was conducted in a session of three hours under the researcher's supervision to ensure the integrity of the test. After going through all the responses of the eight participants the researcher randomly selected one participant for an interview. The interview was audio-taped and the session lasted approximately one hour. The tape recordings were transcribed to provide a hard copy of the interview.

3.5.1 Reliability of the cognitive tests

Cohen et al. (2011) argued that reliability is a measure of consistency over time and over similar samples. A reliable instrument for a piece of research will yield similar data from similar respondents over time. The reliability being tested in this study was the ability to produce an acceptable answer from the given test items. Because visual thinking varies from person to person (Presmeg, 1989), the researcher concentrated on the solution of the problem rather than the nature of the solution.

A sample of eight students participated in the test. The test comprised five tasks, constituting a total of 10 test items. The total score was 50 marks. The Spearman Brown Prophecy coefficient was used to measure internal consistency reliability (Brown, 2001). The formula is as follows: $\rho_{xx} = \frac{2\rho_{AB}}{1+\rho_{AB}}$ where ρ_{AB} is the Pearson product moment for test A and test B. The test items were dichotomously divided into test A and test B. Test A represented even items and test B represented odd items. Table 3.6 shows the distribution of the results of the test. Participants were labelled A-H.

Table 3.6: Testing for reliability

Student	A	B	C	D	E	F	G	H
Test A	14	14	18	14	14	15	13	19
Test B	13	12	14	15	13	16	12	16

From the results ρ_{AB} = 0.61 and consequently$\rho_{xx} = \frac{2(0.61)}{1+0.61} = 0.76$, which is a satisfactory result. Hence the test can be considered as internally reliable.

3.6 VALIDITYOF THE CLINICAL INTERVIEW

An important requirement for clinical diagnosis and testing is sensitivity and specificity (Opper, 1977). The researcher ensured that the questions were sensitive enough to detect the relevant problem if it was present (and to avoid too many false negative results) but specific enough to respond to the desired variable under study (and therefore avoid too many false positive results). The researcher designed questions that diagnosed the manner of mathematical thinking experienced by the participants in solving the tasks (Krussel, 1994). Interviews were conducted in the usual assigned mathematics rooms in order to create a positive environment. The responses were first-hand information and the transcriptions were recorded without amendments. The researcher used thick descriptions to narrate the outcomes (Cohen et al., 2011). In addition, the interviews were audio-taped so that data would be presented in terms of the respondents rather than secondary data.

3.7 RELIABILITY OF THE CLINICAL INTERVIEW

Some researchers contest the use of the term “reliability” in qualitative research. Lincoln and Guba (1985) prefer to substitute “reliability” with terms such as “credibility,” “neutrality”, “dependability” etc. Bogdan and Bilken (1992) argue that in qualitative research, reliability can be regarded as a fit between what researchers’ record as data and what actually occurs in the natural setting that is being researched. Generally, interviews are considered as powerful tools of data collection. Gadd (2004) explains that if the interviewer does his job well (establishes rapport, asks questions in an acceptable manner, etc.), and if the respondent is sincere and well-motivated, accurate data may be obtained. After the students had written the cognitive test, the only reliable method to investigate the student’s mental processes and images was through clinical interviews. Reliability was also improved by using some standard questions pertaining to a specific question and also replication of the same questions to several respondents. Before the instrument was used it was pilot-tested with mathematics students in the Faculty of Science.

3.8 DATA ANALYSIS

The data was analysed using both quantitative and qualitative methods. The nature of data collected from the cognitive tests was in written form (solution of a task). The solutions were in the form of words, diagrams, graphs and algebraic symbols.

3.8.1 Analysis of results from the cognitive test

The analysis aimed to identify the students' mental processes and images used to create meanings for the problems and the justifications provided. For each task the researcher identified:

i. The visual images and representations that the students used.
ii. The deployment of the visual images and representations in solving mathematical analysis tasks.

In order to assess and interpret the visual thinking of students in solving selected concepts of mathematical analysis, the researcher constructed alternative visual thinking strategies of each task and used this guide to develop clear standards, and then classify the visual thinking of the students into various levels. The analysis of the results points to the fact that the visual thinking of the students can be categorised into a competence matrix (Huang, 2015). The competences of each task are not necessarily identical but the levels are maintained. For each task the researcher assigned three levels, namely, non-visual (NV), local visual (LV) and global visual (GV). The levels were described as follows:

i. **Non-visual**: Students at this level produce no visual image, nor do they focus on a single visual image without looking at other representations of a similar nature.
ii. **Local-visual**: Students who operate at this level have a tendency to rely more on numerical representations. They can perceive concepts in various representations but cannot make connections between these concepts. Their visual images cannot be generalised.
iii. **Global-visual**: Students who operate at this level are integrated thinkers who are able to visual concepts with analytic concepts. Under normal circumstances these students are associated with high competence skills.

The competences were taken from the following types of ability:

- Create mental images or to draw a diagram for the purpose of representing a mathematical concept;
- Extract relevant information from the diagram;
- Use diagrams as an aid in solving a problem;
- Understand mathematical transformation visually;
- Link analytic concepts with geometric concepts.

The competencies would be summarised by the following terms: Correct; Partial; Incorrect; No response in cases where the question was closed-ended. In cases where the question required the participant to prove a mathematical statement or theorem, the terms for competences were on how the images were used in proving; such as adjunct, illustrative, visual proof or not applicable. A diagram was considered to be adjunct if it helped in the proof process. It was considered as illustrative if its presence has nothing to do with discourse of the proof. A diagram was also used as a proof if it was representative of the proof "without words".

Table 3.7: Summary of method and theory of assessment of competencies

Competency	Method	Theory
Ability to create mental images or to draw a diagram for the purpose of representing a mathematical concept	Both quantitative and qualitative approaches	Duval's theory of semiotic representations
Ability to extract relevant information from the diagram	Quantitative methods	1. Duval's theory (Treatment/Conversion) 2. VA model (Zazkis et al., 1996).
Ability to use diagrams as an aid in solving a problem	Both quantitative and qualitative approaches.	1. Duval's theory (Treatment/Conversion). 2. VA model (Zazkis et al.,1996).
Ability to understand mathematical transformation visually	Both quantitative and qualitative approaches	1. Duval's theory (Treatment/Conversion). 2. VA model (Zazkis et al.,

		1996)
Ability to link analytic concepts with geometric concepts	Both quantitative and qualitative approaches	1. Duval's theory (Treatment/Conversion) 2. VA model (Zazkis et al., 1996)

Through this lens the researcher was able to analyse the imagery and visualisation evoked by students in solving mathematical analysis concepts.

The initial step was to go through the entire questionnaires (responses of the cognitive test) one by one in order to check for the accuracy of the participants. The researcher identified the themes of the research. For each task the researcher had to identify the following themes: nature of images evoked, quality of images evoked, deployment of images in solving mathematical analysis problems, and the ability to use images in proving theorems. The initial subtask of Task 1, Task 2, Task 3, Task 4, Task 5, Task 6, Task 7 and Task 8 required the students to make a representation on paper of the concept concerned. This subtask was created in order to gain an insight into the nature of the images evoked. The other subtasks were created so as to allow students to use the evoked images to solve the exercises. After the data had been collected, the responses of the students were coded into categories seeking patterns of responses and the data was analysed using SPSS (Statistical Package for Social Scientists). An analytical framework and marking guide (Appendix 7) was used to analyse the thinking processes that might have been involved in each item. The diagram in Figure 3.2 represents the analytical framework used for data analysis.

Duval (2006) argued that to analyse the complex and specific thinking processes that underlie mathematical activity, we must take into account the differences between the various semiotic representations that are used. Mathematical objects can only be accessed through a system of registers of representation and it is important to observe the transformations of treatment and conversion. There are also three issues involved in the checking of conceptual understanding namely:

i. The particular meaning/interpretation/representation a person gives to a concept, relationships between concepts, assertions, or problems;
ii. The particular solution a person provides to a problem;

iii. The particular evidence a person offers to establish or refute a mathematical assertion

The model in Figure 3.2 approximates the visual thinking processes. Normally, an analysis of the mathematical activity starts from the answer.

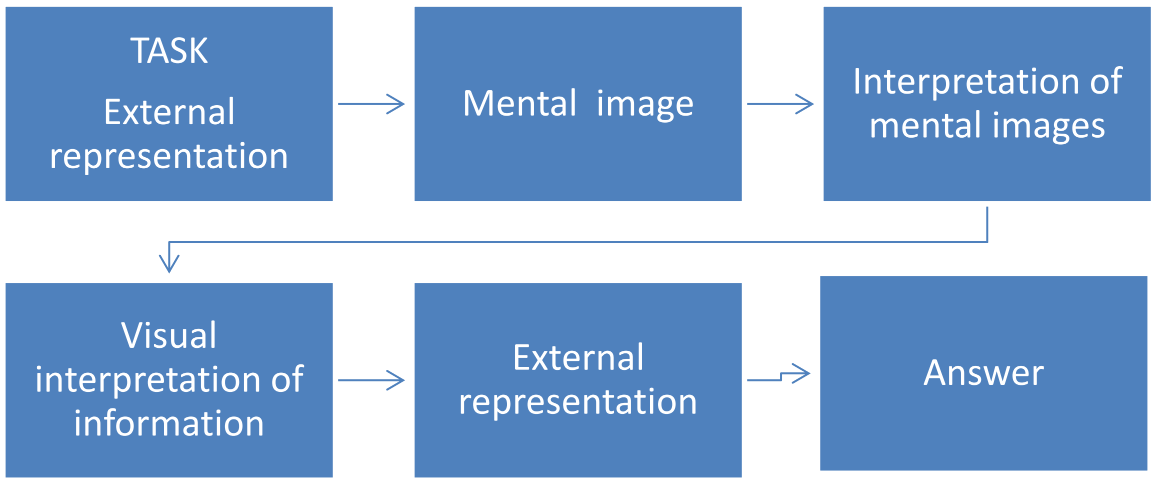


Fig. 3.2: Model used to characterize visual thinking in solving tasks

The distributions of the nature of evoked images, the quality of images and how the images were deployed were presented quantitatively using frequency tables or bar graphs.

3.8.2 Analysis of the results from the interviews

The researcher started by writing down initial perceptions for interpretation. It is known in hermeneutics that a text is open to a variety of interpretations that are used by the intent of the participant and the interpretation of the researcher. The validity of an interpretation is a function of its ability to be reproduced, and in hermeneutics, this principle is considered meaningless, as one can never reproduce or understand the life world of another (Djauhari, 2015).

The frequencies from the quantitative approach informed the qualitative analysis from a sample of 12 students. A few representative samples of images of the solutions from the whole group were presented for an interpretive qualitative analysis on the nature of the students’ thinking processes. The students’ utterances were presented in transcribed texts and the analysis was in the form of thick descriptions

3.9 ETHICAL CONSIDERATIONS

The researcher applied for ethical clearance from the Ethics Committee of the University of South Africa (Institute of Science and Technology Education) at the end of 2013. Permission was granted in 2014. Since the participants were students at the Great Zimbabwe University, the researcher applied for permission from the Registrar to carry out research with the students from the university. Permission was granted in March 2014 (Appendix 6).

The research was carried out using undergraduate mathematics students taking an in-service BEd degree. Although the students were adults, they participated through informed consent (Appendix 5). Participants voluntarily completed informed consent forms. The consent process ensured that individuals were participating voluntarily in the research, with a full knowledge of relevant risks and benefits (Cohen, Manion & Morrison, 2011).Since part of the research was qualitative in nature, data analysis frequently concerned individual cases and could involve personal and sensitive matters. It raised the question of identifiability, confidentiality, and the privacy of individuals. The issue of confidentiality was handled by using pseudonyms and making sure that the scripts were not handled by any other person except the researcher and in a few cases, of participants who were interviewed.

The research also involved six experts in mathematics education to validate the research instruments. The researcher approached the experts personally for their voluntary assistance. Their names remain anonymous for confidentiality.

3.10 CONCLUDING REMARKS

Chapter three presented the research methodology and procedures used to collect and analyse data. The study was guided by the mixed method paradigm which permitted the researcher to collect data using both quantitative and qualitative methods. The sampling procedures were outlined. The instruments for data collection were also explained and the process of validation was clarified. The chapter concluded with a description of how quantitative and qualitative data were analysed, followed by a description of ethical considerations.

CHAPTER FOUR

FINDINGS

This chapter presents the research findings as follows: results of the cognitive test, item by item, followed by the results from interviews. The chapter also presents summaries of the results of the cognitive test and those of interviews.

4.1 RESULTS OF THE COGNITIVE TEST

The cognitive test, which comprised 12 tasks was the major instrument used to investigate the nature of imagery and visualisations evoked by students in the process of learning mathematical analysis concepts. The cognitive test was divided into two categories, namely basic analysis concepts and multivariable calculus concepts. Since mathematical objects are abstract, they cannot be seen and felt in daily life (Duval, 2000). Therefore students were expected to have the skill of expressing mathematical knowledge and thoughts by using representations such as graphics, diagrams, illustrations, formulae and numerical tables. It is vital for students to recognise and use different representations of a concept and to shift from one representation to another. The TRSR was used to analyse the complex and specific thinking processes that manifested in the mathematical activity. The two types of transformation of semiotic representations that are radically different are treatments and conversions.

Tasks 1 to 9 covered basic analysis concepts and required students to make certain representations of a desired concept on paper. Tasks 10, 11 and 12 covered multivariable calculus concepts. Task 10 required students to draw a diagram and give a description of the desired concept. The objectives of Tasks 10 and 11 were to assess the following:

- How students switch representations (symbolic →words →graphical);
- Types of visual images evoked by students in solving problems in 3-dimensional space;
- How students deploy these images in solving problems of mathematical analysis in 3-dimensional space.

Task 12 was an application problem where a visual solution, if identified, could result in an easy solution.

Data was coded and captured using the SPSS analysing tool. The variables of interest to the research questions were coded. The entries were captured task by task for each student. The responses for each task were summarised using frequency distribution tables. The presentation was done task by task and assessed in response to the key questions of the research. The presentation starts with frequency distribution of the nature/types of images evoked by the students. This was followed by an assessment of the quality of images using the categories: (a) NV thinkers; (b) LV thinkers; (c) GV thinkers and then followed by a competence matrix which analysed how the images were used to solve the tasks and lastly an interpretive analysis of selected representative cases (vignettes) of the visual levels.

Basic Analysis Concepts

Results presentation format

The presentation of results followed the following pattern: description of the task, ability to create representations (external), ability to use images and procedures applied in proofs or problem solving. The presentation starts with the ability to create images, followed by application of images and procedures used by NV, LV and GV thinkers.

General observation

Overall, the students provided a diversity of visual and analytic (V/A) strategies in their exploration of the tasks assigned. Each task highlighted the results of students' conceptions which were summarised in three categories, namely: NV, LV, and GV thinkers. NV thinkers often stick to symbolic, numeric modes of representations while local visual thinkers tend to use prototypes in their thinking processes and global visual thinkers have the ability to translate correctly between multiple representations. The coordination of registers of representations as a tool to analyse students' thinking processes was also highlighted for each category.

4.1.1 Task 1

> Suppose *A* and *B* are bounded sets of real numbers
>
> a. How would you represent Sup on paper?
> b. Prove that Sup(A+B) =Sup A +Sup B
> c. Let *A* = $\{4 + \frac{1}{2^n} : n\epsilon\mathbb{N}\}$. Find the least upper bound and greatest lower bound of set *A*.

Task 1 required students first to write something which represented Sup A on paper, where A is a set of real numbers. Treatments could be defining Sup A and conversions could be drawing a diagram (illustration) of Sup A. The next exercise was to prove the statement $\sup(A + B) = supA + \sup B.$ Here the register was symbolic or algebraic, hence, thinking processes are algorithmic. The last subtask was an application problem that required students to find the supremum and the infimum of the provided sequence of numbers. Students were expected to make treatment transformations of the sequence and recognise the required supremum and infimum of the provided set. The response to the question "Make a representation of …" was recognised as the student's mental image of the concept. The quality of the type of images was assessed and also the manner in which the images were used to solve the two sub-tasks (proving the proposition and finding the supremum and the infimum).

Ability to create images

Duval's theory provides semiotic representations as the only possibility in learning mathematics; hence the expected images were diagrams (numerical table of values), symbols, words or no images. Table 4.1 represents the frequency distribution of the types of images evoked by the students in response to item (a) of Task 1.

Table 4.1: Types of images for Task 1

Image	Frequency	Percentage
Diagram/Graph	18	36
Numerical	4	8
Symbolic	15	30
Words	2	4
No images	2	4
Missing	9	18
Total	**50**	**100**

Item (a) was attempted by 41 students. Nine values were missing because these students did not attempt this task. From the 41 who attempted the item, the researcher identified four types of images, namely: diagrammatic, numeric, symbolic and words. The diagram was the common representation. For this task the specific diagram was a labelled number line. Symbols were also used to represent the definition of supremum of a set in compact form.

A further analysis of the quality of images was conducted. The quality was observed from the manner the images were presented. Table 4.2 presents the quality of the images in terms of visual levels.

Table 4.2: Visual levels for Task 1

Visual level	**Frequency**	**Percentage**
Non visual	4	8
Local visual	26	52
Global visual	11	22
Missing	9	18
Total	**50**	**100**

The NV category produced images which lacked conceptual understanding. The LV produced number lines with actual values whilst the GV category produced general line intervals with variables as limit points

Image using

Image using could be found (i) as adjunct and (ii) as illustrative. An image is used as an *adjunct* if it is part of the proof. Usually, we make reference to the image in the algorithm of proving. An image is used for *illustrative* purposes if it enhances the understanding of a proof. Image using could be seen in part (b) where the students were supposed to prove the stated proposition and in part (c) where students were supposed to deduce the supremum of the given sequence. Table 4.3 shows the distribution of the outcome.

Chi-square test for competence matrices

A Chi-square test was conducted to test the association between visual levels and competencies. The following assumptions were considered:

i. The individual observations were independent; the variables of visual levels and image role for each task were independent observations done by the researcher;
ii. Data was treated as nominal. Visual levels and image role were in nominal categories;
iii. At least 80% of the expected frequencies (for each cell) should be 5 or larger; in the contrary the results of the test would not be validated.

Table 4.3: Competence matrix for Task 1

Image role	**NV**	**LV**	**GV**	**Total**
Adjunct	0	1	2	3
Illustrative	0	17	8	25
Not applicable	4	8	1	13
Total	**4**	**26**	**11**	**41**

The majority of students in this study (on this exercise) used the image for illustrative purposes and never referred to it when they were conducting their proofs or solutions to problems. Only three in both the LV and GV levels were able to use the image to prove the proposition. The NV categories were trying to reproduce the algorithms which they had learnt in class. However, their layout was not accompanied with arguments.

Chi-square test (Task 1)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.4 : Chi-square test for Task 1

Chi-square tests

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	12.932[a]	4	.012
Likelihood ratio	13.758	4	.008
McNemar-Bowker test	.	.	.[b]
N of valid cases	41		

a. 6 cells (66.7%) have expected count less than 5. The minimum expected count is .29.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.012 was less than 0.05. A true conclusion could be obtained after further post-hoc tests.

Further tests

The method of combining categories was used in cases where the third assumption was violated. For this case the NV category was attached to LV and adjunct category to illustrative to form a 2 x 2 contingency table. The following results were obtained as shown in Table 4.4 b.

Table 4.5 : Chi-square test for task1 after post-test

Chi-square tests

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson chi-square	1.203[a]	1	.273		
Continuity correction[b]	.572	1	.450		
Likelihood ratio	1.194	1	.275		
Fisher's exact test				.322	.224
No of valid cases[b]	41				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 5.39.

b. Computed only for a 2x2 table

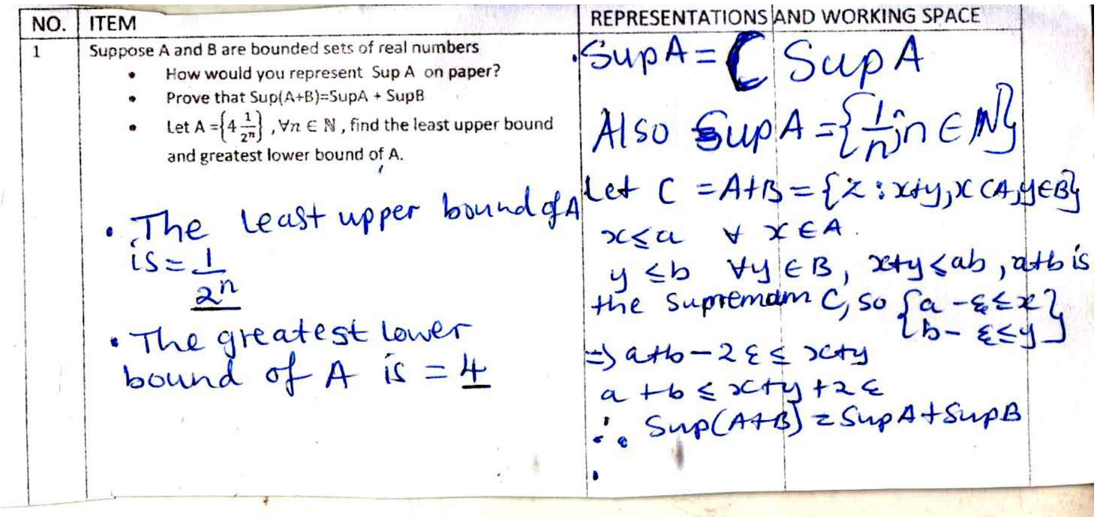
Results: There is no association between visual levels and competencies ($\alpha = 0.273 > 0.050$).

Procedures used in solving tasks

In order to conduct an analysis of the procedures used by students in solving this task, the researcher made use of the vignettes of students' work related to Task 1.

Procedures used by NV

The following picture was produced by a student in the category of non-visual thinkers.



NO.	ITEM	REPRESENTATIONS AND WORKING SPACE
1	Suppose A and B are bounded sets of real numbers • How would you represent Sup A on paper? • Prove that Sup(A+B)=SupA + SupB • Let A = $\{4\frac{1}{2^n}\}$, $\forall n \in \mathbb{N}$, find the least upper bound and greatest lower bound of A. • The least upper bound of A is = $\frac{1}{2^n}$ • The greatest lower bound of A is = $\underline{4}$	Sup A = C Sup A Also Sup A = $\{\frac{1}{n} n \in \mathbb{N}\}$ Let C = A+B = {z : x+y, x ∈ A, y ∈ B} $x \leq a \quad \forall x \in A$. $y \leq b \quad \forall y \in B$, $x+y \leq ab$, a+b is the supremum C, so $\begin{cases} a-\varepsilon \leq x \\ b-\varepsilon \leq y \end{cases}$ $\Rightarrow a+b-2\varepsilon \leq x+y$ $a+b \leq x+y+2\varepsilon$ $\therefore$ Sup(A+B) = SupA+SupB

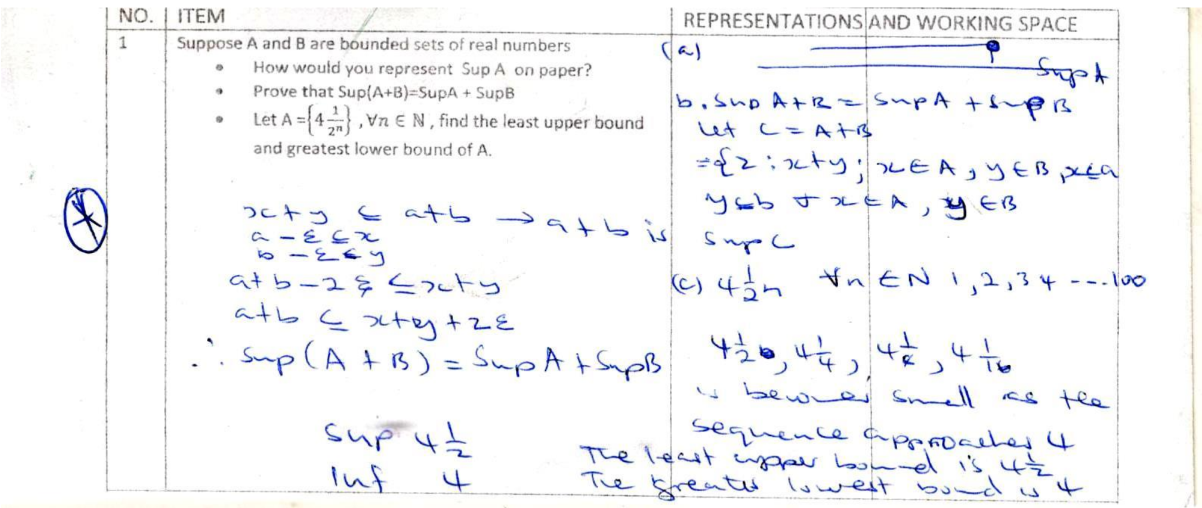
Vignette 4.1: Image of Sup A concept for NV category

Interpretive analysis

The student's concept image of $supA$ was "$supA = C\ supA$." The image does not make sense in the mathematical community. The student attempted the proof using his understanding of the concept definition of supremum. The student was not able to identify the infimum of the sequence A= $\{4 + \frac{1}{2^n}$:n∈ℕ $\}$. The student used the analytic mode of thinking. It is also noted that the student has a conversion difficulty in respect of translating from symbolic register to graphic register.

Procedures by LV

The following picture was produced by a student in the category of local visual thinkers.



NO.	ITEM	REPRESENTATIONS AND WORKING SPACE
1	Suppose A and B are bounded sets of real numbers • How would you represent Sup A on paper? • Prove that Sup(A+B)=SupA + SupB • Let A = $\{4\frac{1}{2^n}\}$, $\forall n \in \mathbb{N}$, find the least upper bound and greatest lower bound of A. $x+y \leq a+b \rightarrow a+b$ is $a-\varepsilon \leq x$ $b-\varepsilon \leq y$ $a+b-2\varepsilon \leq x+y$ $a+b \leq x+y+2\varepsilon$ $\therefore$ Sup(A+B) = SupA + SupB Sup $4\frac{1}{2}$ Inf 4	(a) Sup A b, Sup A+B = SupA + SupB Let C = A+B = {z : x+y; x ∈ A, y ∈ B, x ≤ a y ≤ b ∀ x ∈ A, y ∈ B Sup C (c) $4\frac{1}{2^n}$ ∀n ∈ N 1, 2, 3, 4 ---100 $4\frac{1}{2}$, $4\frac{1}{4}$, $4\frac{1}{8}$, $4\frac{1}{16}$ 4 becomes small as the sequence approaches 4 The least upper bound is $4\frac{1}{2}$ The greatest lower bound is 4

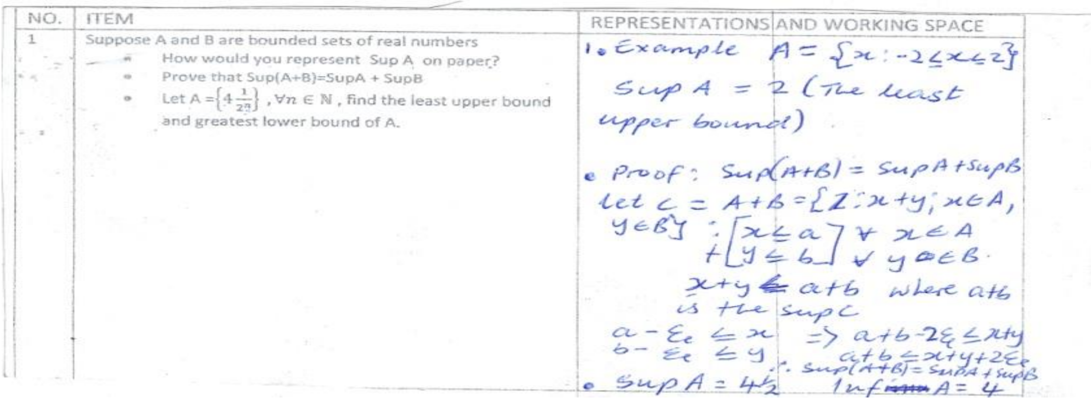
Vignette 4.2: Image of Sup A concept for LV category

Interpretive analysis

The general picture of LV images of Sup A was in the form of a diagram. However, the diagram was not well labelled. This student was not able to translate correctly in that representation. A number line makes sense if it is well labelled; in this instance the images were not clear enough to represent Sup A on paper. The proof was somehow mechanically produced because there were no arguments to validate the proof. There is a disconnection between the image and the proof. However, the student was able find the supremum and infimum of the given set. The student used a VA model to solve this subtask.

Procedures by GV

The following picture was produced by a student in the category of global visual thinkers.



NO.	ITEM	REPRESENTATIONS AND WORKING SPACE
1	Suppose A and B are bounded sets of real numbers • How would you represent Sup A on paper? • Prove that Sup(A+B)=SupA + SupB • Let A = $\{4\frac{1}{2^n}\}$, $\forall n \in \mathbb{N}$, find the least upper bound and greatest lower bound of A.	1. Example $A = \{x : -2 \le x \le 2\}$ Sup A = 2 (The least upper bound) • Proof: $Sup(A+B) = SupA + SupB$ let $c = A+B = \{z : x+y;\ x \in A, y \in B\}$ ∴ $\left[\begin{array}{l} x \le a \\ + y \le b \end{array}\right]$ $\forall x \in A$ $\forall y \in B$. $x+y \le a+b$ where $a+b$ is the sup c $a - \varepsilon_c \le x$ $b - \varepsilon_c \le y$ $\Rightarrow a+b-2\varepsilon_c \le x+y$ $a+b \le x+y+2\varepsilon_c$ ∴ $sup(A+B) = SupA + SupB$ • Sup A = 4½ Inf A = 4

Vignette 4.3: Image of Sup A concept for GV category

Interpretive analysis

For Task 1 a perfect image of set A would be in symbolic form representing a general bounded set of real numbers, for example ,$A = \{x: a < x < b\}$. The students evoked a prototypical set such as the above image, case (a) which was categorised as a correct image. Through the lens of TRSR it can be seen that the student was able to make conversions from symbolic register to graphic register and also to verbal register. Sup A is also considered as the least upper bound of a given set.

Proving that $sup(A + B) = supA + sup\,B$ lacked rigour since the student's statements lacked justification though the steps could be followed up. Proving statements normally is in algebraic register hence, statements should be backed by justification. On the application of the concept (supremum/infimum of a given sequence), the student was able to recognise the supremum of the sequence as 4½ and the infimum as 4. Possibly, the student mentally realised that the sequence was a decreasing and bounded sequence. The student used the VA model approach.

4.1.2 Task 2

Let the function f: $\mathbb{Z} \rightarrow \mathbb{N}$ be defined by

$$f(n)=\begin{cases} 2n+1, if\ n = 0,1,2,3, \ldots. \\ -2n\ ,\ if\ n = -1,-2,-3, \ldots.. \end{cases}$$

(a) Make your own representation of f(n) on paper
(b) From the representation deduce whether f(n) is one-one and/or onto.

Task 2 required students to first make a representation of the given function and then deduce from the representation whether the function was one-one and/or onto. The students were expected to translate from algebraic register to numeric register and then to graphic register. The idea was to establish that the set of integers was countable. The analysis of the responses was done by referencing to answering the research questions, i.e. the nature and types of images evoked and how these images were deployed to solve tasks related to the concept.

Ability to create images

Table 4.6 represents the distribution of the types of images evoked by the students in response to item (a) of Task 2.

Table 4.6: Types of images for Task 2

Image	Frequency	Percentage
Diagram/Graph	41	82
Numerical	1	2
Symbolic	3	6
No images	1	2
Missing	4	4
Total	**50**	**100**

A total of 46 students attempted this test item. The types of images evoked were graphs, numerals and symbols. The majority of students evoked graphical representations. A further analysis of the quality of images was conducted and Table 4.7 represents the levels of visual thinking.

Table 4.7: Visual levels for Task 2

Visual levels	Frequency	Percentage
Non visual	18	36
Local visual	20	40
Global visual	8	16
Missing	4	8
Total	**50**	**100**

Table 4.7 shows that for this particular exercise the majority of students fall into the category of LV. Very few are GV thinkers.

Image using

For this task students were required to deduce from the graphical or numerical table of values that the function was one-to-one and also to deduce the property of onto. Table 4.8 summarises the ways in which the images were used in proving the propositions.

Table 4.8: Competence matrix for Task 2

Image role	NV	LV	GV	Total
Adjunct	0	3	2	5
Illustrative	4	8	4	16
Not applicable	14	9	2	25
Total	**18**	**20**	**8**	**46**

A majority never used the diagram to deduce the property of being injective and subjective. Some resorted to the use of analytic root, which was somehow mechanically done without supporting arguments. Others used the diagram incorrectly. They used the methods of drawing a horizontal line to test for injectivity, and failed to see that the function was defined in natural numbers.

Chi-square test (Task 2)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.9: Chi-square test for Task 2

Chi-square tests

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	8.653[a]	4	.070
Likelihood ratio	10.353	4	.035
McNemar-Bowker Test	.	.	.[b]
N of valid cases	46		

a. 5 cells (55.6%) have expected count less than 5. The minimum expected count is .87.

Condition three of the assumptions of the Chi-square test was not satisfied. Even the significant level 0.07 suggests no association the test needs further post-hoc tests.

Further test

The method of combining categories was used in cases where the third assumption was violated. For this case the Adjunct category was attached to Illustrative to form a 2 x 3 contingency table. The following results were obtained:

Table 4.10 : Chi-square test for task2 after post-test

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	5.438[a]	2	.066
Likelihood ratio	5.656	2	.059
N of valid cases	46		

a. 0 cells (.t%) have expected count less than 5. The minimum expected count is 5.02.

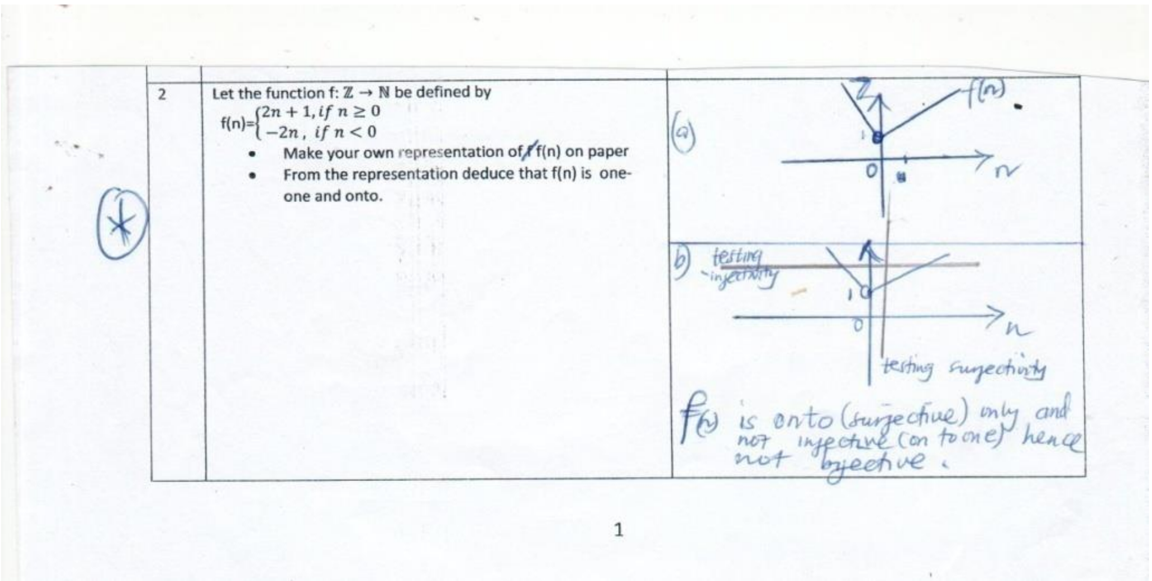
Results: There is no association between visual levels and competencies ($\alpha = 0.066 > 0.050$).

Procedures used in solving tasks

The researcher used the vignettes of students' work related to Task 2 in order to conduct an analysis of the procedures used by students in solving this task.

Procedures used by LV (a)

The following picture was produced by a student in the category of LV thinkers.



Vignette 4.4: Representation of (n) for LV category (a)

Interpretive analysis

(a) Representation of f(n)

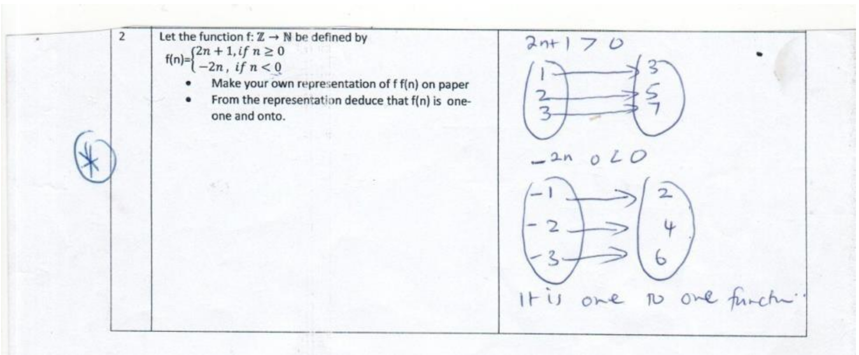
The student is a visual thinker but a local one. His image is incorrect because the function is not continuous but defined in natural numbers. The student has a conversion difficulty in the sense of Duval's theory. He has problems in translating from symbolic register to graphic register with discrete functions.

(b) One-on-one correspondence and onto

The student in this category used a visual route to establish the one-one correspondence between the two sets. The horizontal line is used for testing injectivity. However, this resulted in two different elements having the same image which was false for the function under consideration. The VA model conducted could have yielded positive results for continuous functions. The student had a problem in treating functions in real numbers the same as numerical in sequence. The vertical line was correctly used to confirm test subjectivity.

Procedures by LV (b)

The following picture was produced by a student in the category of LV thinkers.



Vignette 4.5: Representation of f(n) for LV category

Interpretive Analysis

(a) Representation of f(n)

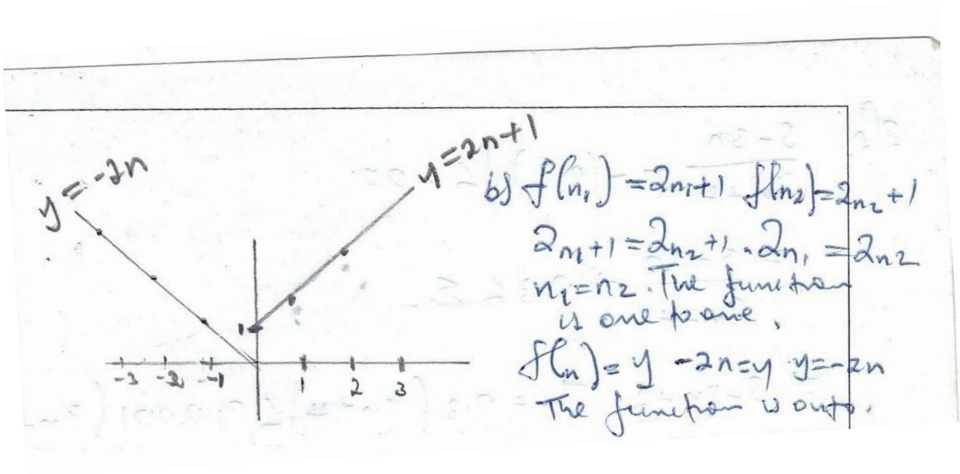
The image produced makes sense. It falls into the LV because it lacks some minor concepts which are significant in mathematical analysis. The diagram shows a corresponding relationship between two sets but the element zero is not included. Again, the diagram lacks some dots to indicate that the process continues up to infinity. This disqualifies this representation as partly correct.

(b) One-to-one correspondence/onto

The student preferred to use a VA approach. It can be recognised that the function is one-to-one deduced from the correspondence relation established. However, the student could not deduce the onto property on $f(n)$. This shows that the student has conversion difficulties in comprehending the visual meaning of onto.

Procedures by GV

The following picture was produced by a student in the category of global visual thinkers.



Vignette 4.6: Representation for f(n) for GV category

Interpretive Analysis

(a) Representation of f(n)

A correct representation for Task 2 could be a graph of dotted lines or a correspondence relationship between two sets almost similar to the above cases of partially correct images. Although Vignette 4.6 shows a graphical representation, the points are united by a continuous line and this disqualifies the image as incorrect. The student had difficulties in treatments of graphs in real numbers and graphs in natural numbers.

(b) One-to-one correspondence/onto

The student in Vignette 4.6 preferred to use the analytic approach to prove the property of injective and onto. However, the algorithms used lacked the accompanying quantifiers and also the response was disconnected from the representation.

4.1.3 Task 3

A sequence $\{u_n\}$ has its n^{th} term given by

$U_n = \frac{5-3n}{9n+4}$, $n \in \mathbb{N}$

a. Represent the following t: 1st , 5^{th} , 10^{th} ,100^{th} ,1000^{th} , 10 000^{th} , 1000 000^{th} terms of the sequence in decimal form to 6d.p. and make a good guess of the limit of the sequence.

b. Sketch the graph of $f(n) = \frac{5-3n}{9n+4}$

c. Verify that the guess (or conjecture) in (a) is correct by using the definition of the limit.

Task 3 required the students to use intuition to evaluate the limit of the sequence through completing a table of values. The sequence was given in algebraic register and students were supposed to convert them to numeric register. They were also required to make a graphical representation of the given sequence and finally justify the existence of the limit using the epsilon-delta definition of the limit. In this context, the expected images were numerical values of the sequence which were supposed to lead to a good guess of the limit. Students were also expected to make a graphical representation of the sequence. A table of values in this context was considered a diagram but if the values were not in table form then the image was categorised as numerical.

Ability to create images

Table 4.11 represents the distribution of the types of images evoked by the students in response to item (a) of Task 3.

Table 4.11: Types of images for Task 3

Image	Frequency	Percentage
Diagram/Graph	19	38
Numerical	23	46
Symbolic	1	2
No images	4	8
Missing	3	6
Total	**50**	**100**

The most common image evoked on this task was a table of values for the sequence. A diagram in this exercise meant the graphical representation of the sequence. A further analysis of the quality of images was conducted and Table 4.12 represents the levels of visual thinking.

Table 4.12: Visual levels for Task 3

Visual levels	Frequency	Percentage
NV	12	24
LV	20	40
GV	15	30
Missing	3	6
Total	**50**	**100**

LV thinkers are the majority category. This task required students to translate from numerical to graphical representation. Algebraic register was needed in the verification of the convergence of the sequence. Table 4.13 summarises the ways in which the images were used in proving the propositions.

Image using

Table 4.13: Competence matrix for Task 3

Image role	NV	LV	GV	Total
Adjunct	0	0	1	1
Illustrative	0	5	9	14
Not applicable	12	15	5	32
Total	**12**	**20**	**15**	**47**

For Task 3 students were supposed to use the definition of limit to verify that the limit was -0.333. Only one student was able to verify the existence of the limit. A majority of the students just calculated the limit using the formula of the sequence.

Chi-square test (Task 3)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.14: Chi-square test for Task 3

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	15.055[a]	4	.005
Likelihood ratio	18.123	4	.001
McNemar-Bowker test	.	.	.[b]
N of valid cases	47		

a. 5 cells (55.6%) have expected count less than 5. The minimum expected count is .26.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.005 was less than 0.05. A true conclusion could be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases where the third assumption was violated. For this case the Adjunct category was attached to Illustrative to form a 2 x 3 contingency table. The following results were obtained.

Table 4.15 : Chi-square test for task3 after post-test

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	11.153[a]	1	.001		
Continuity correction[b]	9.116	1	.003		
Likelihood ratio	11.176	1	.001		
Fisher's exact test				.001	.001
N of valid cases[b]	47				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 5.79.

b. Computed only for a 2x2 table

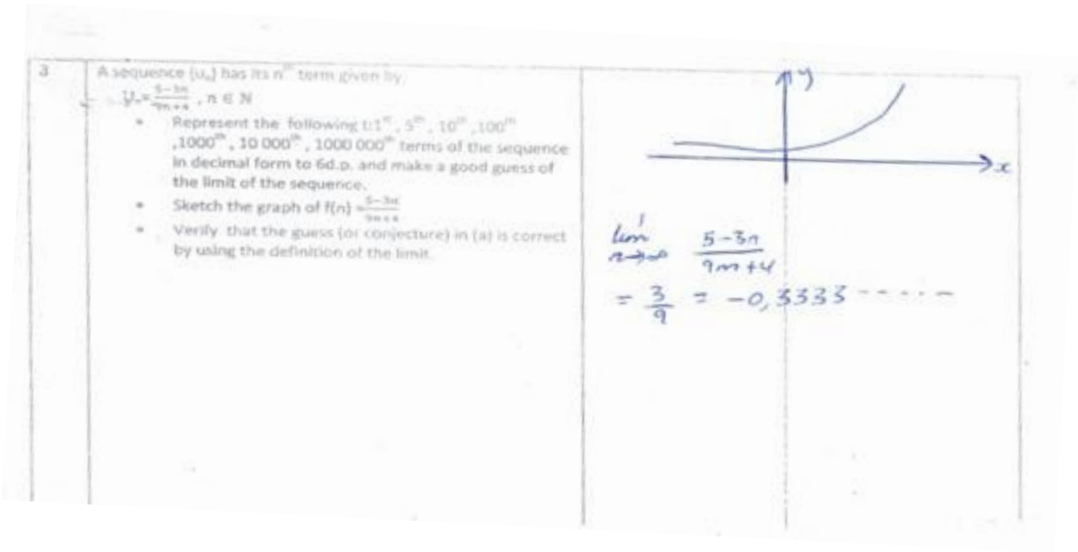
Results: There is an association between visual levels and competencies (α =0.001 < 0.050).

Procedures used in solving tasks

In order to conduct an analysis of the procedures used by students in solving this task the researcher made use of the vignettes of students' work related to Task 2.

Procedures used by NV

The following picture was produced by a student in the category of NV thinkers.



3 A sequence $\{u_n\}$ has its n^{th} term given by

$U_n = \frac{5-3n}{9n+4}$, $n \in \mathbb{N}$

- Represent the following t:1^{st}, 5^{th}, 10^{th}, 100^{th}, 1000^{th}, $10\,000^{th}$, $1000\,000^{th}$ terms of the sequence in decimal form to 6d.p. and make a good guess of the limit of the sequence.
- Sketch the graph of $f(n) = \frac{5-3n}{9n+4}$
- Verify that the guess (or conjecture) in (a) is correct by using the definition of the limit.

$$\lim_{n\to\infty} \frac{5-3n}{9n+4} = \frac{3}{9} = -0,3333\ldots$$

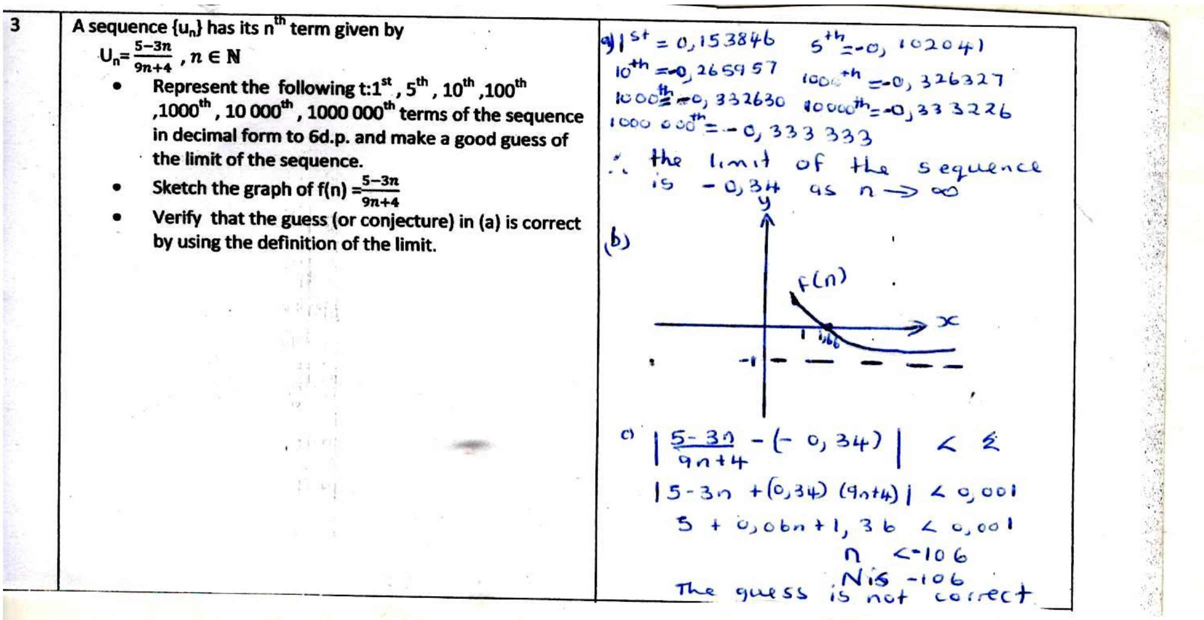
Vignette 4.7: Graphical representations of the sequence for NV category

Interpretive Analysis

Firstly the representation was incorrect. The student simply reproduced the graphical image that was in his mind. This student was able to find the limit of the sequence through the analytic method. He did not have a deeper sense of the concept of sequence.

Procedures for LV

The following picture was taken from the category for LV thinkers.



3 A sequence $\{u_n\}$ has its n^{th} term given by

$U_n = \frac{5-3n}{9n+4}$, $n \in \mathbb{N}$

- Represent the following t:1^{st}, 5^{th}, 10^{th}, 100^{th}, 1000^{th}, $10\,000^{th}$, $1000\,000^{th}$ terms of the sequence in decimal form to 6d.p. and make a good guess of the limit of the sequence.
- Sketch the graph of $f(n) = \frac{5-3n}{9n+4}$
- Verify that the guess (or conjecture) in (a) is correct by using the definition of the limit.

a) $1^{st} = 0,153846$ $5^{th} = -0,102041$
$10^{th} = -0,265957$ $100^{th} = -0,326327$
$1000^{th} = -0,332630$ $10000^{th} = -0,333226$
$1000\,000^{th} = -0,333333$
$\therefore$ the limit of the sequence is $-0,34$ as $n \to \infty$

(b) $f(n)$

c) $\left|\frac{5-3n}{9n+4} - (-0,34)\right| < \varepsilon$

$|5-3n + (0,34)(9n+4)| < 0,001$

$5 + 0,06n + 1,36 < 0,001$

$n < -106$

N is -106

The guess is not correct

Vignette 4.8: Graphical representations of the sequence for LV category

Interpretive analysis

Representation and estimate

It can be observed that the representation of the terms was correct as well as the limit of the sequence. The only error was rounding-up the limit.

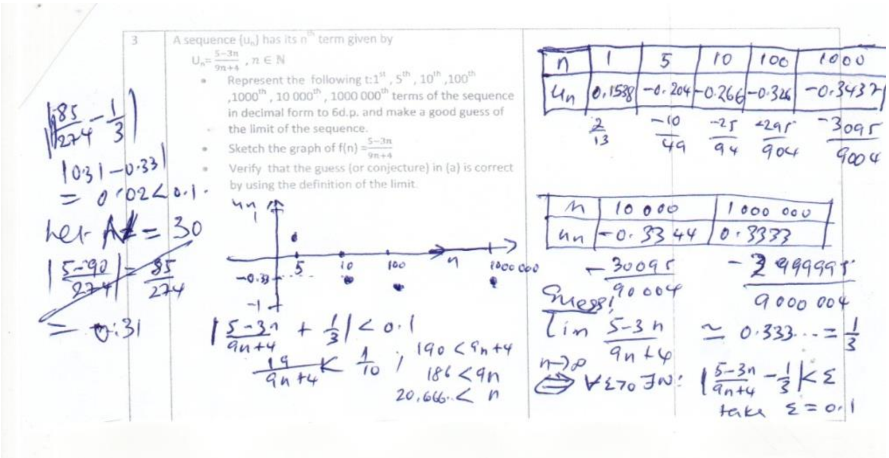
Sketch graph

The representation of the sketch graph was partially correct but the only incomprehension was uniting the points with a continuous line. This example is a treatment problem. The student applied the VA model strategy.

Verification of limit using definition

It can be observed that the student used the definition to verify the limit of the sequence. The student had difficulty when translating in algebraic register which caused him to make the wrong conclusions.

Procedures for GV



Vignette 4.9: Graphical representations for the sequence for GV category

Interpretive analysis

(a) Representation and estimate

The student in this category made a good representation of the terms of the given sequence and made a correct estimate of the limit. A common error that appeared in most cases was the sign of the limit. The correct estimate should be -0.333….

(b) Sketch graph

The students evoked correct sketch graphs. The student was able to translate from numeric to graphic register. The VA model strategy was used.

(c) Verification of limit using definition

It can be observed that students who fall in this category managed to verify using the epsilon-delta definition.

4.1.4: Task 4

For the following functions in real numbers:

(a) $f(x)=\begin{cases} 1\ for\ x\ irrational \\ -1\ for\ x\ rational \end{cases}$ (b) $f(x) = \begin{cases} \frac{x-|x|}{x}\ if\ \ x \neq 0 \\ 2\ \ if\ \ x = 0 \end{cases}$ (c) $f(x) = |x|-x$

(a) Make your own representations of the above functions on paper
(b) Discuss the kind of discontinuity, if any, of the above functions

Students were required to make representations of the above functions and then analyse the nature of continuity of these functions. Case (a) was a tricky one because analytically it is not possible to draw the graph of the function, but, visually, the graph could be two parallel horizontal lines passing through 1 and -1 respectively. This is because of the density property of rational numbers and irrational numbers. For cases (b) and (c) students were supposed to apply their concept image of the modulus function or simply construct a table of values to enable them to draw the graphs of the functions. To analyse continuity they could use the intuitive definition of "drawing the graph without lifting the pencil."

Ability to create images

Table 4.16 represents the distribution of the types of images evoked by the students in response to item (a) of Task 4.

Table 4.16: Frequency of types of images for Task 4

Image	Frequency	Percentage
Diagram/Graph	43	86
Numerical	0	0
Symbolic	0	0
No images	0	0
Missing	7	14
Total	**50**	**100**

In this exercise the type of image evoked was a diagram. The graphs of the functions were given in algebraic register. A further analysis was done to determine the quality of the images and to check the visual levels. Table 4.17 shows the distribution of visual levels.

Table 4.17: Visual levels for Task 4

Visual levels	Frequency	Percentage
Non-visuals	9	18
Local visuals	16	32
Global visuals	18	36
Missing	7	14
Total	**50**	**100**

The majority in this task were in the category of GV thinkers. Table 4.18 shows the manner in which images were used in solving mathematical analysis tasks.

Table 4.18: Competence matrix for Task 4

Image role	NV	LV	GV	Total
Adjunct	0	4	10	14
Illustrative	0	9	8	17
Not applicable	9	3	0	12
Total	**9**	**16**	**18**	**43**

The images in this task were used for illustrative purposes and also as part of the solution to the assigned tasks. Ten (10) students, which is a majority, used them as part of the solution to the given exercises.

Chi-square test (Task 4)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.19: Chi-square test for Task 4

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	33.199[a]	4	.000
Likelihood ratio	37.381	4	.000
McNemar-Bowker test	.	.	.[b]
N of valid cases	43		

a. 4 cells (44.4%) have expected count less than 5. The minimum expected count is 2.51.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases where the third assumption was violated. For this case the Adjunct category was attached to Illustrative to form a 2 x 3 contingency table. The following results were obtained:

Table 4.20 : Chi-square test for task4 after post-test

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	13.858[a]	2	.001
LikelihoodrRatio	18.206	2	.000
N of valid cases	43		

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 5.02.

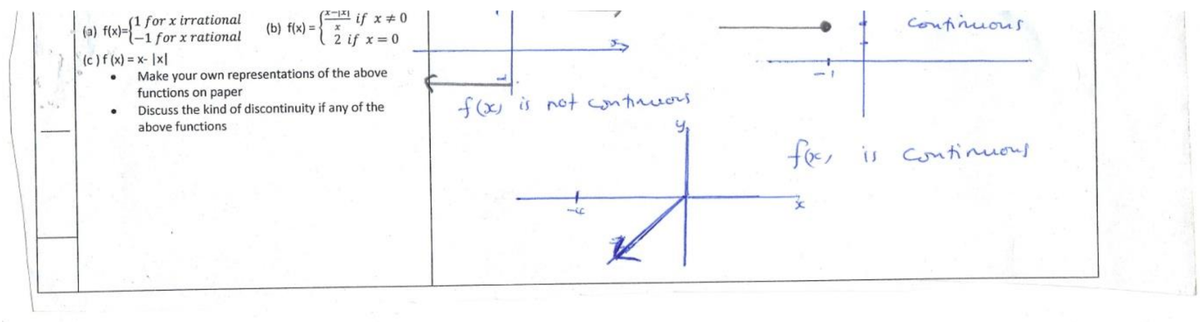
Results: There is an association between visual levels and competencies (α =0.001< 0.050).

Procedures used in solving tasks

The researcher used the vignettes of students' work and interviews related to Task 4 in order to conduct an analysis of the procedures used by the students in solving this task.

Procedures used by NV

The following picture was produced by a student in the category of non-visual thinkers.



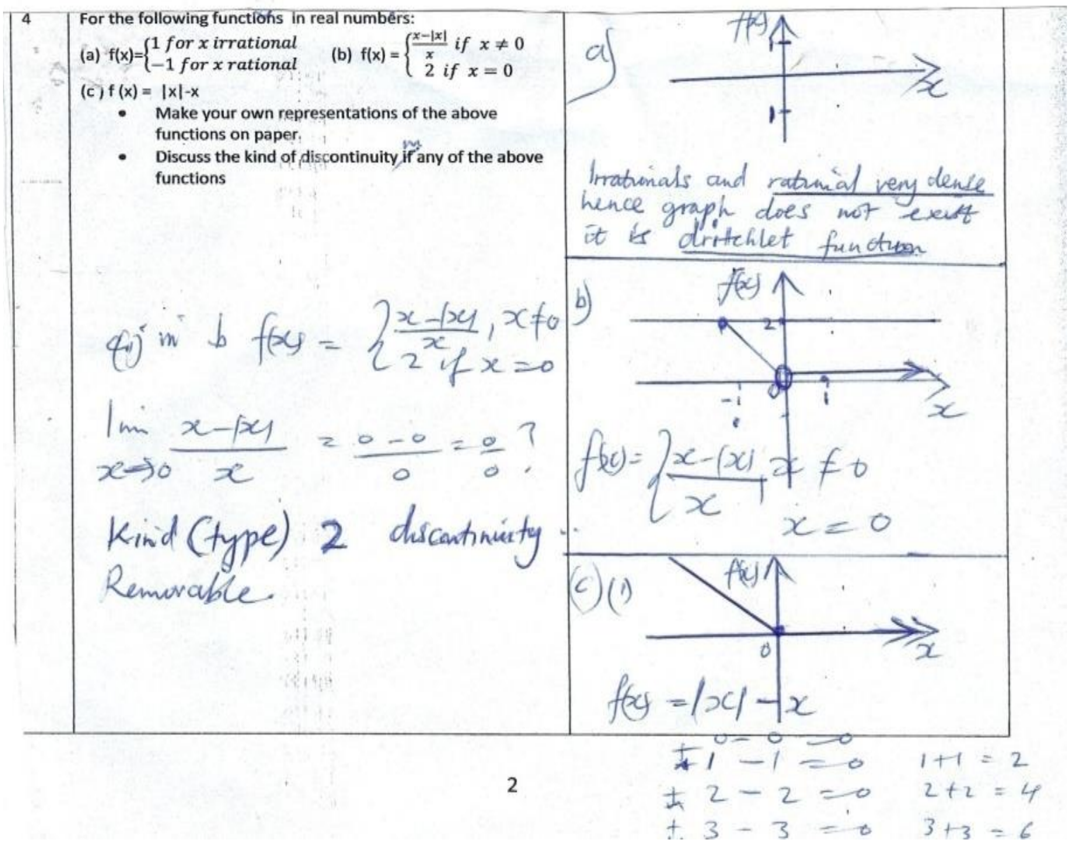
Vignette 4.10: Graphical representations of functions for category NV

Interpretive analysis

It seems that this particular student was guessing because there is no working representation. All the representations were wrong. His response to the question of continuity was correct for the first and last function but there were no supporting arguments. Possibly he used the visual approach to establish continuity.

Procedures used by LV

The following picture was produced by a student in the category of LV thinkers.



Vignette 4.11: Graphical representations of functions for category LV

Interpretive analysis

Sketching graphs

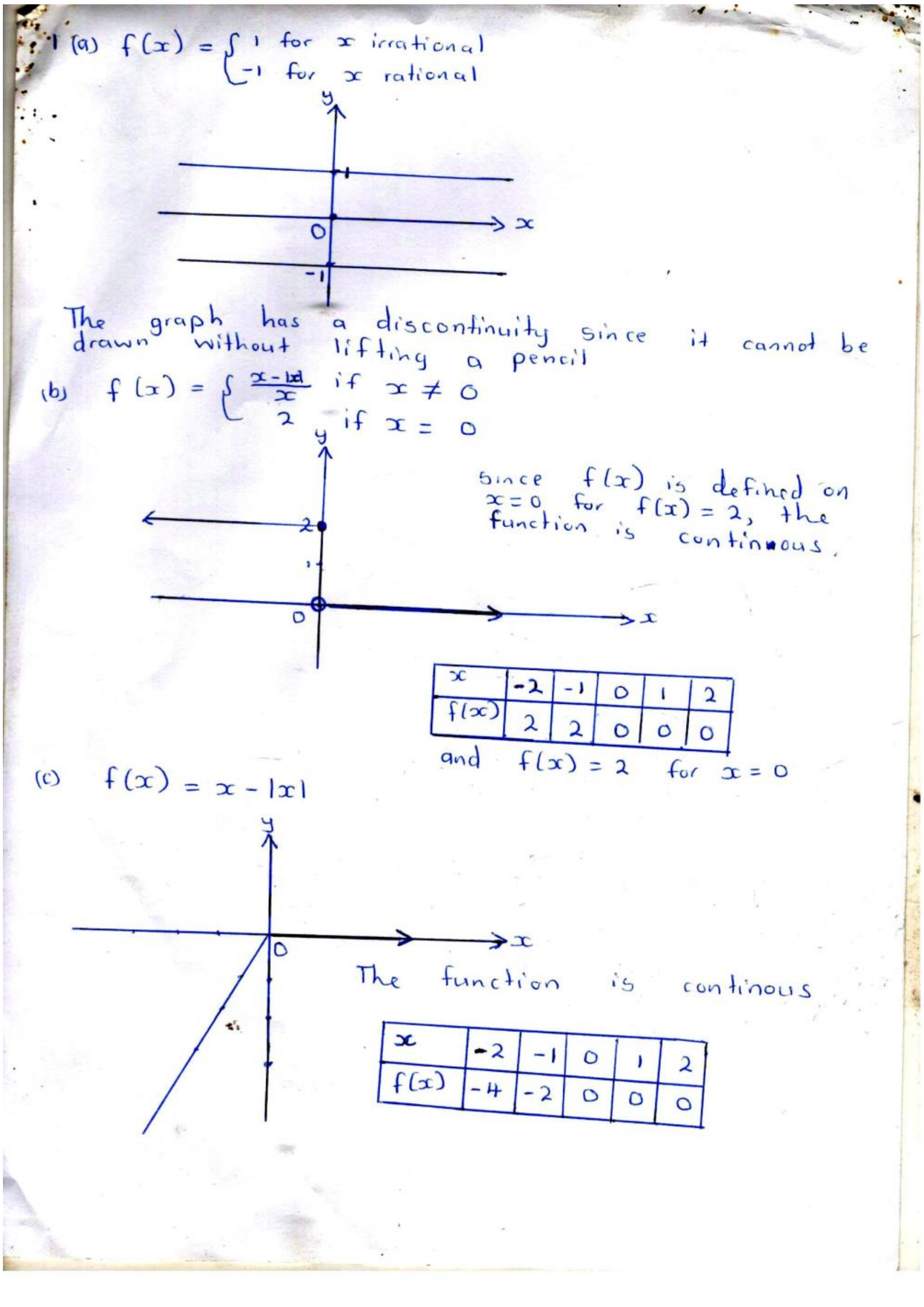
It can be observed that the argument on (a) was correct although analytically the graph cannot be drawn. Sketch (b) was partially correct for x>0 but partially wrong for x<0. The sketch of (c) was correct. The student had problems of comprehension, especially for functions presented in piecewise notation. The student used an AV approach.

Discussion of continuity

The student could not discuss continuity for (a) and (c) and failed to translate between the registers of discontinuity. For (b) there was both a removable discontinuity and a jump discontinuity.

Procedures used by GV

The following picture was produced by a student in the category of GV thinkers.



Vignette 4.12: Graphical representations of functions for category GV

For this particular question one student (Tatenda) (pseudonym) was interviewed to probe her solutions and visual thinking further. The following is part of the interview:

Researcher: "Any visual images evoked when you constructed the graph?"

Tatenda: "The picture is for values of x when f(x) is 1, x is irrational and when f(x) is -1, x is rational".

Researcher: "So, you have the picture of that graph in mind?"

Tatenda: "Yes."

Researcher: "How did you discuss the continuity/discontinuity? Any pictures in mind?"

Tatenda: "Yes, I thought that since it is one function there are two lines on the graph, so there is discontinuity".

"**Researcher** What first comes to your mind when you think of the concept of continuity and discontinuity?"

Tatenda: "When I look at a function for continuity, the first thing that comes to my mind is that a continuous function can be drawn without lifting a pencil."

Researcher: "How did you describe the behaviour of the given graph of a function? Any pictures in mind?"

Tatenda: "From the graph as you can look at it, it cannot be drawn without lifting a pencil, so I thought that it is not continuous."

Researcher: At which point is it not continuous?

Tatenda: "At x = 0, from the graph."

Interpretive Analysis

Task 4 involves solving typical examples; hence the discussion is not for general concepts but for specific examples. Tatenda first evoked a picture of the function

$$f(x) = \begin{cases} 1 \; for \;\; x \;\; irrational \\ -1 \; for \;\; x \;\; rational \end{cases}.$$

She can see it in her mind. When discussing the behaviour of this function she focused on the graph of the function and could "see" two parallel lines representing one function and from there she concluded that such a function could not be drawn without lifting a pencil, hence, it is not continuous for all values. She used this way of analysis for other graphs. The following figure was Tatenda's mental picture of the above function and the other functions. Her conclusion that the function is not continuous is correct but is based on the wrong argument, i.e. that the function cannot be drawn without lifting a pencil. The learner failed to identify that rational numbers have gaps although the density property misleads us to think that y =-1 is a continuous line. On the other functions (b) and (c) she managed to translate from symbolic representation to numerical and then to graphical register. She managed to draw a good representation of the function

$f(x) = \begin{cases} \frac{x-|x|}{x} if x \neq 0 \\ 2 \ if x = 0 \end{cases}$ but again she failed to use her intuitive definition of continuity (drawing without lifting the pencil). She tried to use the analytical definition of continuity but failed to realise that the function had a jump discontinuity at the point x = 0. For the function $f(x) = x - |x|$ she deduced from the graphical register that the function was continuous although the reason was not mentioned. The student mainly used the VA model strategy.

4.1.5 Task 5

Consider the graph of the function f below

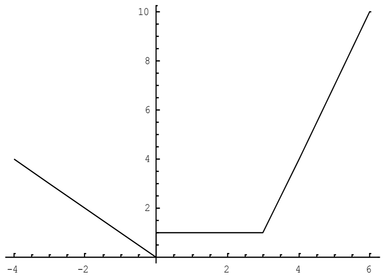


Figure 1. A conceptual task on continuity, differentiability and integration

a. Describe the behaviour of the graph of f
b. At which points is f differentiable? Differentiable to the right? To the left?
c. Describe the behaviour of the graph of the function h defined by $h(x) = \int_{-4}^{x} f(t)dt$
d. Compute $\int_{-4}^{6} f(x)dx$

The task required students to make some verbal descriptions on continuity and differentiability of a function given in graphic context. They were also supposed to compute the area under the given function. Normally, students do not have many problems in calculating derivatives of polynomial functions but the concept of lateral limits and lateral derivatives often pose some misconceptions to undergraduate students (Artigue, Menigaux & Viennot, 1990). The concept images expected in this task were the descriptions of the behaviour of the function and the computation of the area under the curve. To compute the area under the curve, it was easier to use the visual approach by dividing the region of integration into simple shapes.

Ability to create images

Task 5 was a little different from the other tasks. Students were required to demonstrate the following competencies:

- ability to extract relevant information from the diagram;
- ability to use the diagram as an aid in solving related problems; and
- ability to link geometrical with analytic concepts.

These competencies were associated with visual levels. Table 4.21 shows a distribution of the visual levels.

Table 4.21: Visual levels for Task 5

Visual levels	Frequency	Percentage
Non visuals	14	28
Local visuals	11	22
Global visuals	24	48
Missing	1	2
Total	**50**	**100**

The results show that a majority were global visual thinkers. Table 4.22 shows the distribution of competencies versus the visual levels.

Table 4.22: Competence matrix for Task 5

Image Role	NV	LV	GV	Total
Extract information	0	0	19	19
Use of diagram	0	8	4	12
Link concepts	14	3	1	18
Total	**14**	**11**	**24**	**49**

GB thinkers dominated in the ability to extract relevant information from the given diagram. NV thinkers also dominated in linking geometrical concepts to analytic concepts in the given exercises.

Chi-square test (Task 5)

H_0: There is no association between visual levels and competencies, versus

H_1: There is an association between visual levels and competencies.

Table 4 .23: Chi-square for Task5

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	56.723[a]	4	.000
Likelihood ratio	63.360	4	.000
McNemar-Bowker test	.	.	.[b]
N of valid cases	49		

a. 4 cells (44.4%) have expected count less than 5. The minimum expected count is 2.69.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion could be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases where the third assumption was violated. For this case the NV category was attached to LV category to form a 3 x 2 contingency table. The following results were obtained:

Table 4.24 : Chi-square test for task5 after post-test

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	34.550[a]	2	.000
Likelihood ratio	44.908	2	.000
N of valid cases	49		

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 5.88.

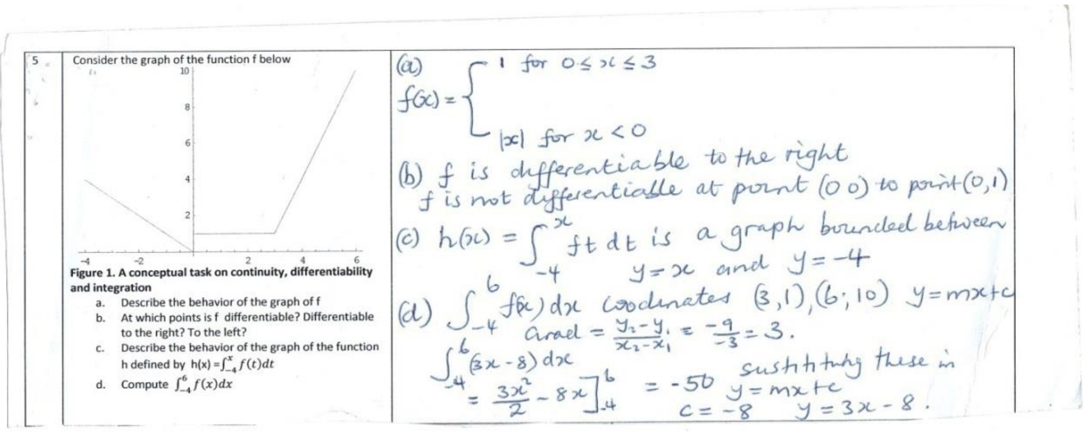
Results: There is an association between visual levels and competencies (α = 0.00 0< 0.050).

Procedures used in solving tasks

In order to conduct an analysis of the procedures used by the students in solving this task the researcher made use of the vignettes of students' work related to Task 5.

Procedures used by NV

The following picture was produced by a student in the category of NV thinkers.



Vignette: 4.13: Mathematical activity (Task 5) for NV

Interpretive analysis

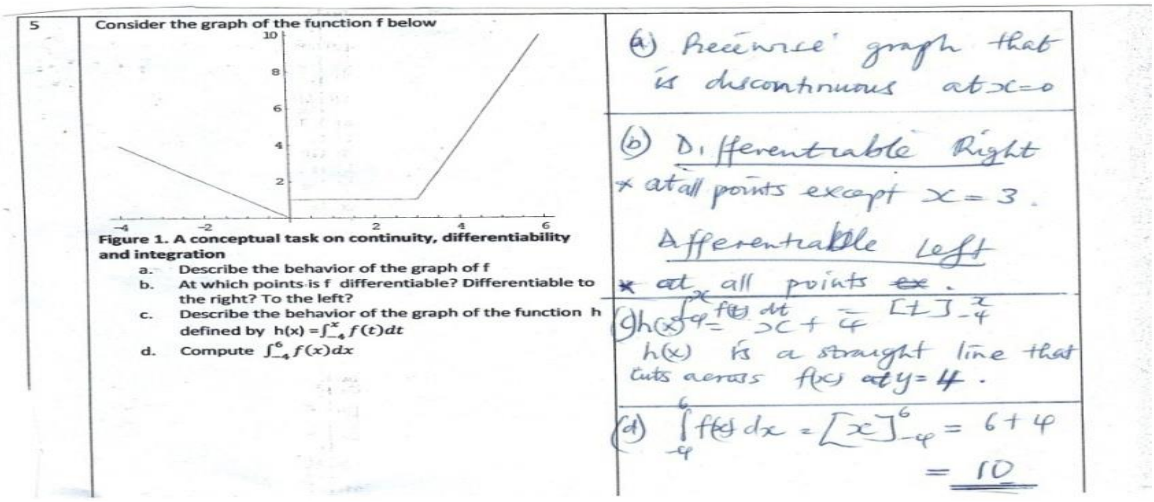
The student:

- failed to identify points of discontinuity;

- correctly identified the point where f(x) was not differentiable;
- could not describe properly the behaviour of h(x); and
- failed to use algebraic registers to calculate the required area.

Procedures used by LV

The following picture was produced by a student in the category of LV thinkers.



5 Consider the graph of the function f below

Figure 1. A conceptual task on continuity, differentiability and integration

a. Describe the behavior of the graph of f
b. At which points is f differentiable? Differentiable to the right? To the left?
c. Describe the behavior of the graph of the function h defined by $h(x) = \int_{-4}^{x} f(t)dt$
d. Compute $\int_{-4}^{6} f(x)dx$

(a) Piecewise graph that is discontinuous at $x=0$

(b) Differentiable Right at all points except $x = 3$.

Differentiable left at all points ~~ex~~.

(c) $h(x) = \int_{-4}^{x} f(t)\,dt = x + 4 = [t]_{-4}^{x}$

h(x) is a straight line that cuts across f(x) at $y = 4$.

(d) $\int_{-4}^{6} f(x)\,dx = [x]_{-4}^{6} = 6 + 4 = 10$

Vignette 4.14: Mathematical activity (Task 5) for LV

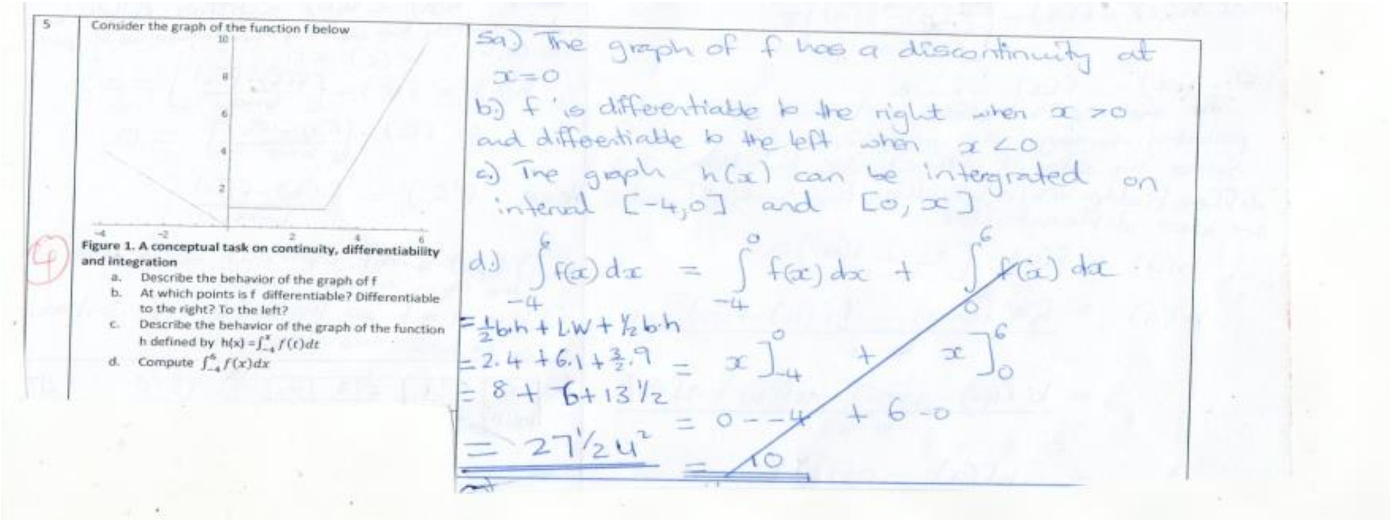
Interpretive analysis

The student:

- identified correctly the point of discontinuity of f(x);
- partially described the points where f(x) was differentiable;
- could not describe properly the behaviour of h(x);
- failed to subdivide the region to facilitate calculation of required area;
- could not link geometrical concepts with analytic concepts.

Procedures used by GV

The following picture was produced by a student in the category of GV thinkers.



Vignette 4.15: Mathematical activity (Task 5) for GV

Interpretive analysis

- The student was able to identify from the graph the point of discontinuity. He translated correctly from graphic register to language register.
- The student identified partially correctly the points where the function was differentiable.
- He could not describe the nature of the function *h(x)* convincingly.
- Student subdivided the region correctly to calculate the required area using a visual approach (VA model).

4.1.6 Task 6

Mean Value Theorem for Derivatives

Suppose that f is continuous on the interval [a,b] and differentiable on the interval (a,b) . Then there exists a number c ϵ [a,b] such that

f/(c) $=\frac{f(b)-f(a)}{b-a}$.

a. Represent on paper the meaning of the above theorem
b. Prove the theorem
c. If $f(x) = \begin{cases} 1 \; if \; x \; is \; irrational \\ -1 \; if \; x \; is \; rational \end{cases}$, investigate whether (i) f(x) and (ii) $/f(x)/$ satisfy the above theorem on [-1,1]

This task required students to make a representation of the mean value theorem for derivatives. Normally, the visual meaning of the theorem should be a diagram showing the gradient of a function at a point c which lies in the interval [a,b]. The transformation of conversion was the key to this task since students were expected to translate from multi-functional registers to mono-functional registers. Students were also supposed to use the diagram to construct the auxiliary function h(x) to prove the mean value theorem. Arriving at the desired result involved the transformation of treatment of semiotic representations. The last subtask tested how students deployed their images in solving mathematical analysis problems. The function

$$f(x) = \begin{cases} 1 \; if \; x \; is \; irrational \\ -1 \; if \; x \; is \; rational \end{cases}$$

does not satisfy the mean value theorem because it is not differentiable on (-1,1) but $|f(x)|$ satisfies the theorem because the union of irrationals and rational is the complete real number line.

Ability to create images

Table 4.25 represents the types of images evoked by students.

Table 4.25: Types of images for Task 6

Image	Frequency	Percentage
Diagram/Graph	45	90
Numerical	0	0
Symbolic	2	4
No images	3	6
Missing	0	0
Total	**50**	**100**

Students mainly used diagrams to represent the mean value theorem. A good and clear diagram also indicated an ability to prove the theorem. Table 4.26 represents the distribution of the visual levels.

Table 4.26: Visual levels for Task 6

Image	Frequency	Percentage
Non visual	13	26
Local visual	16	32
Global visual	21	42
Missing	0	0
Total	**50**	**100**

The distribution shows a gradual increase from non-visual to global visual levels.

Image using

Table 4.27 shows a competence matrix of visual level against competencies for the group on this task.

Table 4.27: Competence matrix for Task 6

Image role	NV	LV	GV	Total
Adjunct	0	4	10	14
Illustrative	0	9	8	17
Not applicable	9	3	0	12
Total	**9**	**16**	**18**	**43**

Global visualizers used the images as part of their proofs, whilst non-visual thinkers were not able to deploy their images in proving the theorem.

Chi-square test (Task 6)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.28: Chi–square test for Task 6

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	48.815[a]	4	.000
Likelihood ratio	61.772	4	.000
McNemar-Bowker test	.	.	.[b]
N of valid cases	50		

a. 3 cells (33.3%) have expected count less than 5. The minimum expected count is 1.30.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases were the third assumption was violated. For this case the NV category was attached to LV category and the Illustrative category was attached to Adjunct category to form a 2 x 2 contingency table. The following results were obtained.

Table 4.29 : Chi-square test for task 6 after post-test

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	33.422[a]	1	.000		
Continuity correction[b]	30.188	1	.000		
Likelihood ratio	42.573	1	.000		
Fisher's exact test				.000	.000
N of Valid Cases[b]	50				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 10.08.

b. Computed only for a 2x2 table

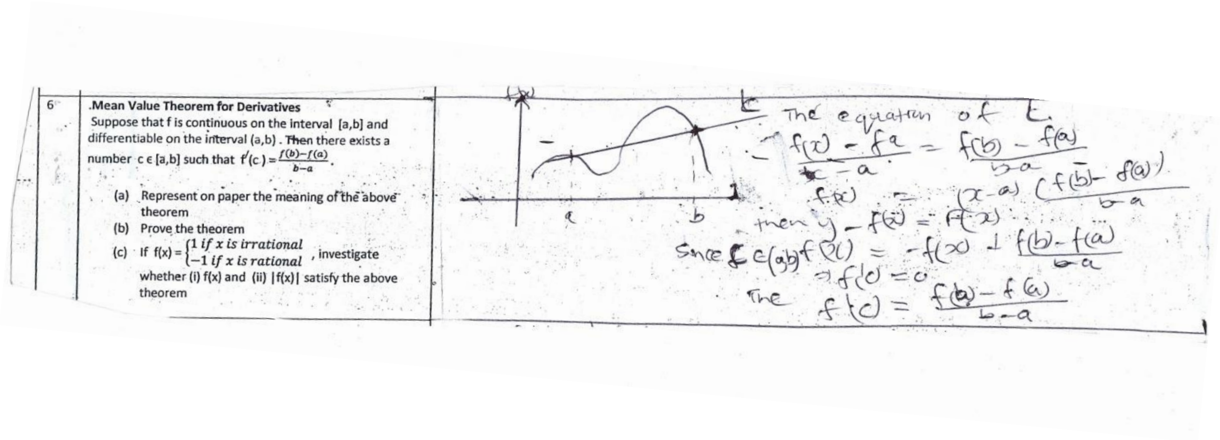
Results: There is an association between visual levels and competencies ($\alpha = 0.00\ 0 < 0.050$).

Procedures used in solving tasks

In order to conduct an analysis of the procedures used by the students in solving this task the researcher made use of the vignettes of students' work and interviews related to Task 6.

Procedures used by NV

The following picture was produced by a student in the category of non-visual thinkers.



Vignette 4.16: Representations on MVT for NV

Interpretive analysis

- *Visual representation of the theorem*

  The student's representation lacked clarity. There is a need to clearly label the diagram.

- *Proof of the theorem*

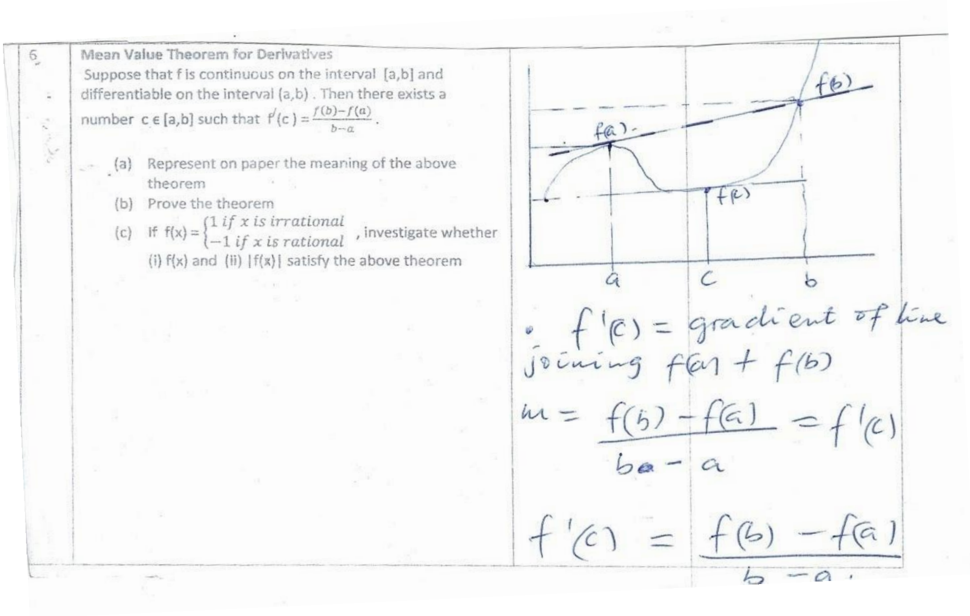
  The student attempted to deduce the auxiliary function h(x) but the arguments were not based on the diagram.

- *Application of the theorem*

  The student failed to apply the theorem to (c).

Procedures used by LV

The following picture was produced by a student in the category of local visual thinkers.



Vignette 4.17: Representations on MVT for LV

Interpretive analysis

- *Visual representation of the theorem*

  The student's diagram/representation above was correct and well labelled. It can be seen easily that $f'(c) = \frac{f(b)-f(a)}{b-a}$although the tangent lines drawn seem not to be aligned.

- *Proof of the theorem*

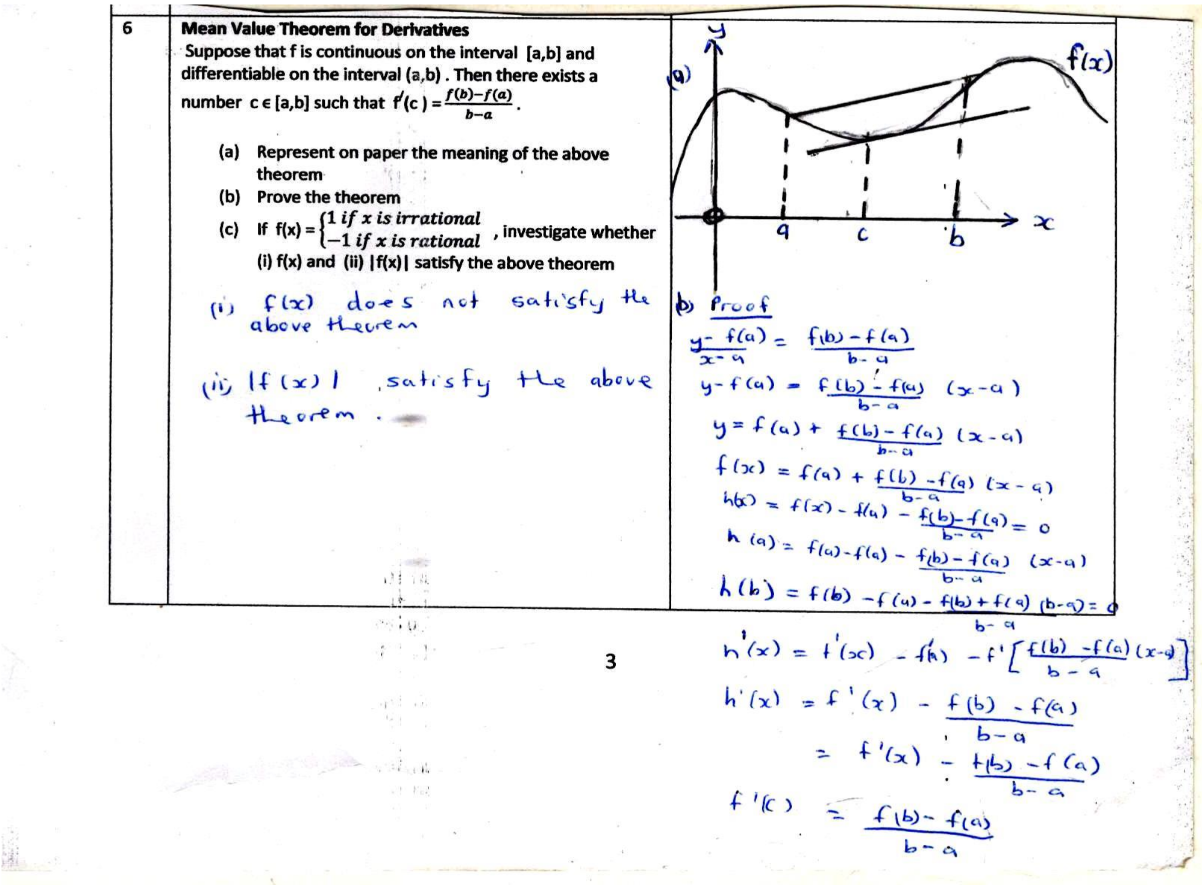
  The student failed to complete the proof and also failed to connect the visual with the analytic concepts. The analytic concepts included reference to Rolle's theorem and formulation of the auxiliary function.

- *Application of the theorem*

  The student could not apply the theorem to attempt (c).

Procedures used by GV

The following picture was produced by a student in the category of global visual thinkers.



6	**Mean Value Theorem for Derivatives** Suppose that f is continuous on the interval [a,b] and differentiable on the interval (a,b). Then there exists a number c ∈ [a,b] such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. (a) Represent on paper the meaning of the above theorem (b) Prove the theorem (c) If $f(x) = \begin{cases} 1 \text{ if } x \text{ is irrational} \\ -1 \text{ if } x \text{ is rational} \end{cases}$, investigate whether (i) f(x) and (ii) |f(x)| satisfy the above theorem	(a) y, f(x), a, c, b, x

(i) f(x) does not satisfy the above theorem

(ii) |f(x)| satisfy the above theorem.

(b) Proof

$$\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

$$y-f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$f(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a} = 0$$

$$h(a) = f(a)-f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$$h(b) = f(b) - f(a) - \frac{f(b)+f(a)}{b-a}(b-a) = 0$$

3

$$h'(x) = f'(x) - f(a) - f'\left[\frac{f(b)-f(a)}{b-a}(x-a)\right]$$

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$= f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Vignette 4.18: Representations on MVT for GV

Interpretive analysis

- *Visual representation of the theorem*

  It can be observed that the student sketched a partially correct diagram. A correct representation should have had two parallel lines, one connecting point corresponding to A and B and the other passing through point C. Once the diagram was clearly labelled the steps of the proof would also be clear and logical.

- *Proof of the theorem*

  The construction of the auxiliary function h(x) was properly done and the student was also able to apply Rolle's Theorem to reach the conclusion. Here, the diagram was used as an adjunct to the proof of the mean value theorem for derivatives. The VA model was properly used.

- *Application of the theorem*

  The student responded correctly to the questions though he did not validate the statements with supporting arguments. The student used internal visual constructs to attempt these subtasks.

A follow-up interview was conducted with the student who wrote the above vignette. His name was Peter Chako (pseudonym). The following is an excerpt of the conversation followed by an interpretive analysis.

Researcher: "Describe in words the mental images or thinking processes that came to your mind when you solved part (a)."

Peter: "I thought that f' (c) is the derivative of f at c and f(b) $-f$(a) is an increase in y and b-a is an increase in x , then the gradient of f at c is $f(b)$-$f(a)$ over b-a."

Researcher: "Were you able then to prove the theorem from the diagram?"

Peter: "No."

Researcher: "How did you investigate whether $f(x) = \begin{cases} 1 \text{ } if \text{ } x \text{ } is \text{ } irrational \\ -1 \text{ } if \text{ } x \text{ } is \text{ } rational \end{cases}$ satisfies the mean value theorem?"

Peter: "From Task 4 I see that the function is not continuous therefore it does not satisfy the theorem."

Researcher: "What about $|f(x)|$?"

Peter: "For $|f(x)|$ it gives us one and we are going to draw one graph which is continuous and differentiable."

Interpretive Analysis

Peter evoked relevant images of the geometrical meaning of derivatives. This is evidenced by his correct explanation of the concept of gradient. His challenge was to deploy the images evoked in order to prove the theorem. However, his concept of the mean value theorem is internalised since he was able to apply this theorem to deduce that the function $f(x)$

$$= \begin{cases} 1 \text{ } for \text{ } x \text{ } irrational \\ -1 \text{ } for \text{ } x \text{ } rational \end{cases}$$

does not satisfy the theorem since it was not continuous; was able also to deduce that $|f(x)|$ satisfied the conditions of the theorem since it was both continuous and differentiable everywhere.

4.1.7 Task 7

For the inequality $\ln(1+x) < x$

- Make a representation of the given inequality
- Prove the above inequality

The task required students to make a representation of the given inequality. A visual representation could be the graphs of the functions *ln(1+x)* and *x* on the same axes so as to establish the relationships. Students were also required to prove the inequality. The proof of this inequality was connected to the mean value theorem for derivatives. The students were given the information in algebraic register and needed to convert it to graphic register.

Ability to create images

Table 4.30 represents the types of images evoked by students.

Table 4.30: Types of images evoked by students on Task 7

Image	Frequency	Percentage
Diagram/Graph	39	78
Numerical	0	0
Symbolic	3	6
No images	1	2
Missing	7	14
Total	**50**	**100**

The main image evoked was a diagram. Table 4.31 represents the visual levels.

Table 4.31: Visual levels for Task7

Visual levels	Frequency	Percentage
Non visuals	28	56
Local visuals	8	16
Global visuals	7	14
Missing	7	14
Total	**50**	**100**

The non-visual route was dominant in this exercise.

Image using

Table 4.32 shows a competence matrix of visual level against competencies for the group on this task.

Table 4.32: Competence matrix for Task 7

Image role	**NV**	**LV**	**GV**	**Total**
Adjunct	0	0	1	1
Illustrative	1	2	1	4
Proof	0	1	5	6
Not Applicable	27	5	0	32
Total	**28**	**8**	**7**	**43**

A majority of non-visual thinkers were not able to apply their images in solving the task. Only one global visual thinker used the image as part of the solution.

Chi-square test (Task 7)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.33: Chi-square for Task 7

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	36.113[a]	6	.000
Likelihood ratio	34.884	6	.000
McNemar-Bowker test	.	.	.[b]
N of valid cases	43		

a. 9 cells (75.0%) have expected count less than 5. The minimum expected count is .16.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases were the third assumption was violated. For this case the NV category was attached to LV category and the Adjunct category was attached to Illustrative category to form a 2x2 contingency table. The following results were obtained:

Table 4.34 : Chi-square test for task7 after post-test

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	.157[a]	1	.692		
Continuity correction[b]	.003	1	.956		
Likelihood ratio	.158	1	.691		
Fisher's exact test				.745	.479
N of valid cases[b]	43				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 5.58.

b. Computed only for a 2 x 2 table

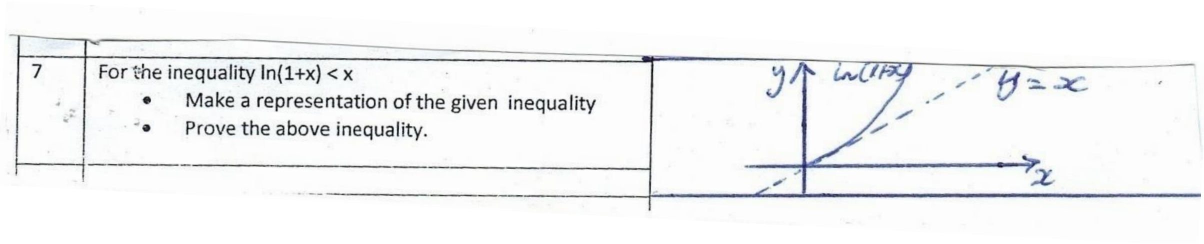
Results: There is no association between visual levels and competencies ($\alpha = 0.692 > 0.050$).

Procedures used in solving tasks

The researcher used the vignettes of students' work related to Task 7 in order to conduct an analysis of the procedures used by students in solving this task.

Procedures used by LV (a)

The following picture was produced by a student in the category of non-visual thinkers.



Vignette 4.19: Representation of inequality for LV (a)

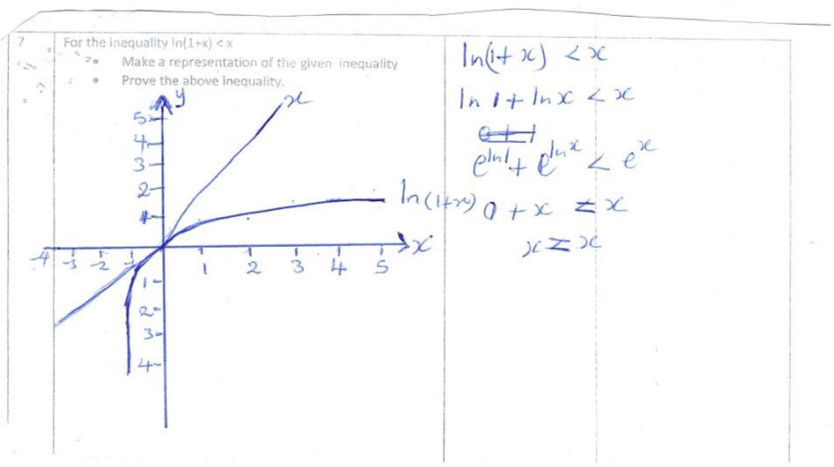
Interpretive analysis

The above case was incorrect. The student failed to note that the function $\mathrm{y} = \ln(1+\mathrm{x})$ was above the graph $y = x$ which contradicted the algebraic statement. This case shows that the source of incomprehension was the problems of conversion and treatments of semiotic representations.

- The student failed to attempt the proof. The student had difficulties in the transformation of conversion of registers (from symbolic to graphic).

Procedures used by LV (b)

The following picture was produced by a student in the category of local visual thinkers.



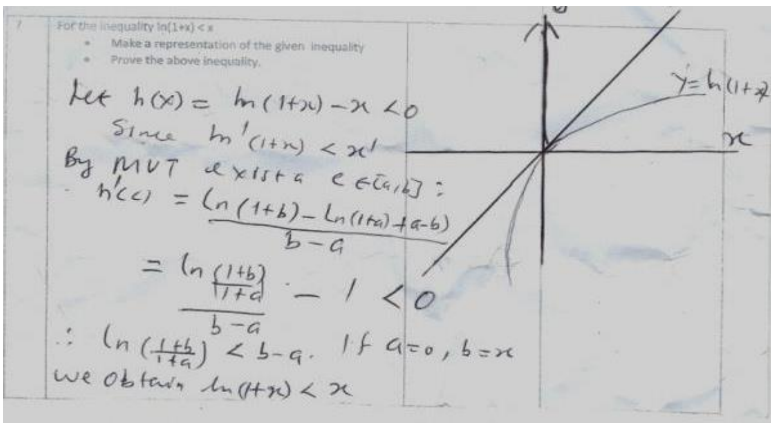
Vignette 4.20: Representation of inequality for LV (b)

Interpretive analysis

- The student managed to draw a correct representation of the inequality.
- The student did not attempt the proof. He could not link geometric concepts and algebraic concepts.

Procedures used by GV

The following picture was produced by a student in the category of global visual thinkers.



Vignette 4.21: Representation of inequality for GV

Interpretive analysis

- This student was able to produce a correct sketch as shown in the above representation. The graph of $y = x$ is above the graph of $y = \ln(1 + x)$ for every value of x to be seen in the diagram.
- The proof followed from an application of the mean value theorem by use of an auxiliary function h(x) as shown above. Here, the semiotic transformations engaged were treatments in algebraic register. The VA model was properly used.

4.1.8 Task 8

Mean Value Theorem for Integrals

Suppose that f is continuous on $[a,b]$, then there is a number $c \epsilon (a,b)$ for which

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$.

a. Make your own representation on paper of the meaning of the above theorem
b. Prove the theorem
c. Find the (i) estimated value and (ii) actual value of $\int_0^2 \sqrt{4-x^2}\, dx$

The task required students to first make a representation of the mean value theorem for integrals on paper, prove the theorem and then apply their visualisations to estimate and calculate a given definite integral. The usual way was to express the theorem as

$f(c\,)(b-a) = \int_a^b f(x)dx.$

This enabled the student to identify that the left hand side of the statement was the formula to calculate the area of a rectangle whose sides are f(c) and (b-a) in length while the right hand side represented the area under the curve f(x).

Ability to create images

Table 4.35 represents the distribution of the types of images evoked by the students in response to item *(a)* of Task 8.

Table 4.35: Types of images evoked by students on Task 8

Image	Frequency	Percentage
Diagram/Graph	35	70
Symbolic	3	6
Words	1	2
No images	6	12
Missing	5	10
Total	**50**	**100**

The type of representations produced by the students was diagrams, symbols and words. It can be observed from Table 35 that there were six cases where no images were produced and also five were missing values, which means the exercise was not attempted. The most frequent image or representation was a diagram. After observing the nature of representations produced by students, it was also necessary to determine the frequency distribution of their levels of visualisation. The majority of students evoked diagrams. Table 4.36 illustrates the distribution of visual levels.

Table 4.36: Visual levels for Task 8

Visual levels	Frequency	Percentage
Non visuals	12	24
Local visuals	10	20
Global visuals	23	46
Missing	5	10
Total	**50**	**100**

It can be observed from the table that all levels of visualisation were represented on this task. The majority fell into the category of GV thinkers. The frequency of LVs was almost the same as that of NV thinkers. The researcher was also interested to see how the students used their representations in proving the mean value theorem. The role of the image for this task emerged to be either that the representation was used as an adjunct (helper) in proving the theorem or for illustrative purposes. GV thinkers dominated in this task. Table 4.30 shows a competence matrix of visual levels against competencies for the group on this task.

Image using

Table 4.37 shows a competence matrix of visual levels against competencies for the group on this task.

Table 4.37: Competence matrix for Task 8

Image role	NV	LV	GV	Total
Adjunct	0	0	17	17
Illustrative	0	1	4	5
Not Applicable	12	9	2	23
Total	**12**	**10**	**23**	**45**

A majority of global visual thinkers used the diagram as part of proof.

Chi-square test (Task 8)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.38: Chi-square for Task 8

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	35.088[a]	4	.000
Likelihood ratio	45.401	4	.000
McNemar-Bowker rest	.	.	.[b]
N of valid cases	45		

a. 5 cells (55.6%) have expected count less than 5. The minimum expected count is 1.11.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases where the third assumption was violated. For this case the NV category was attached to LV category and the Illustrative category was attached to Adjunct category to form a 2 x 2 contingency table. The following results were obtained:

Table 4.39 : Chi-square test for task8 after post-test

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	30.683[a]	1	.000		
Continuity correction[b]	27.462	1	.000		
Likelihood ratio	36.235	1	.000		
Fisher's exact test				.000	.000
N of valid cases[b]	45				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 10.27.

b. Computed only for a 2x2 table

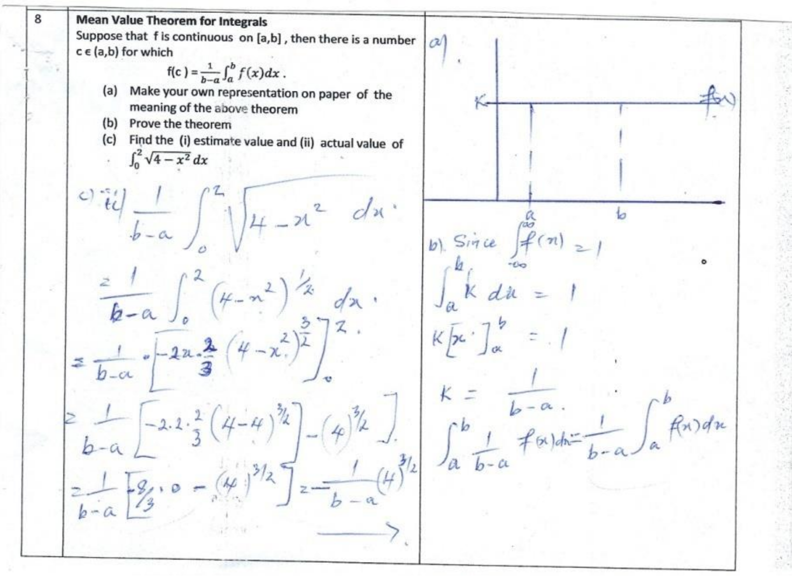
Results: There is an association between visual levels and competencies (α = 0.00 0< 0.050).

Procedures used in solving tasks

In order to conduct an analysis of the procedures used by the students in solving this task the researcher made use of the vignettes of students' work related to Task 8.

Procedures used by LV (a)

The following picture was produced by a student in the category of non-visual thinkers.



Vignette 4.22: Representation of MVT (integrals) for LV (a)

Interpretive analysis

Visual representation of the theorem

The representation used by this participant was too simple though visual (constant function). It was also not fully labelled.

Proof of the theorem

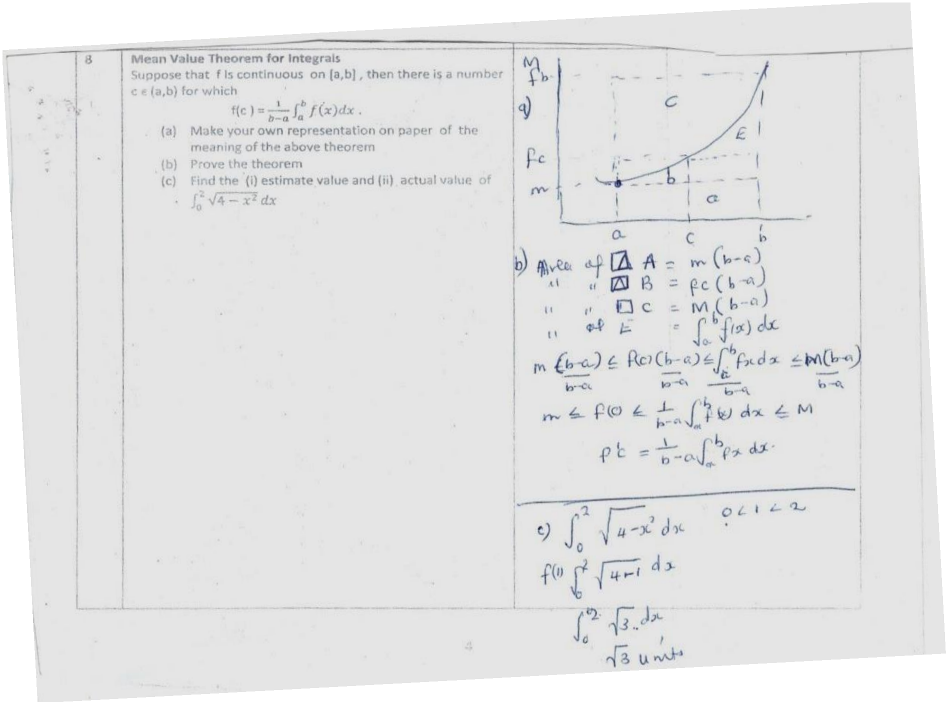
The proof was only for a constant function whereas the f(x) should be a general function which is continuous.

Application of the theorem

The student expressed difficulties with treatments within the symbolic register to calculate the actual value of the integral.

Procedures used by LV (b)

The following picture was produced by a student in the category of local visual thinkers.



8 Mean Value Theorem for Integrals

Suppose that f is continuous on [a,b], then there is a number $c \in (a,b)$ for which

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$.

(a) Make your own representation on paper of the meaning of the above theorem

(b) Prove the theorem

(c) Find the (i) estimate value and (ii) actual value of $\int_0^2 \sqrt{4-x^2}\,dx$

a)

b) Area of A = m (b-a)

" " B = fc (b-a)

" " C = M (b-a)

" of E = $\int_a^b f(x)\,dx$

$m\frac{(b-a)}{b-a} \le \frac{f(c)(b-a)}{b-a} \le \frac{\int_a^b f x\,dx}{b-a} \le \frac{M(b-a)}{b-a}$

$m \le f(c) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$

$f c = \frac{1}{b-a}\int_a^b f x\,dx$

c) $\int_0^2 \sqrt{4-x^2}\,dx$ $0 < 1 < 2$

$f(1)$ $\int_0^2 \sqrt{4-1}\,dx$

$\int_0^2 \sqrt{3}\,dx$

$\sqrt{3}$ units

Vignette 4.23: Representation of MVT (integrals) for LV

Interpretive analysis

Visual representation of the theorem

The student produced a well labelled diagram / representation of the mean value theorem.

Proof of the theorem

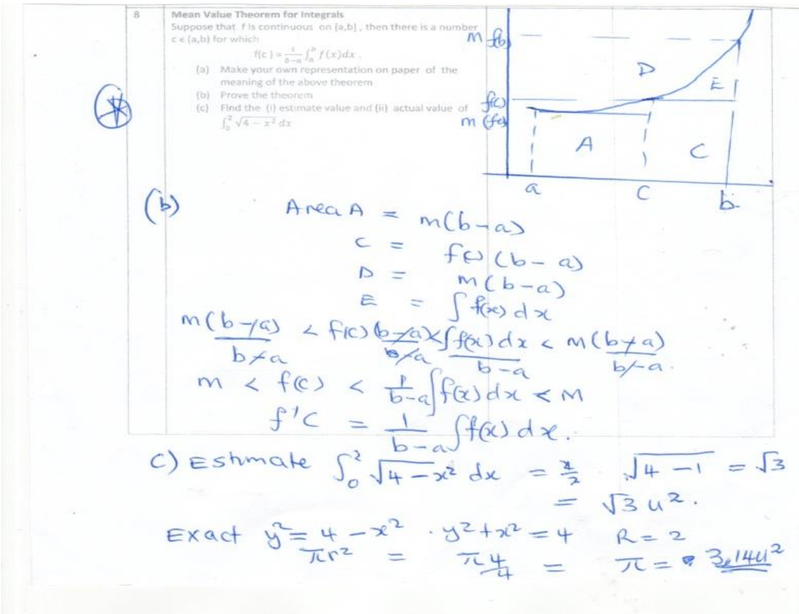
The student managed to lay out the proof in a very clear manner. There is evidence between coordination between the diagram and the algorithm of the proof. There is a connection of interpretations and treatments that is provided in the graphic and symbolic representations of the theorem. However, there is lack of clarity in establishing the equality sign of the mean value theorem. The VA model was properly used.

Application of the theorem

The student showed that he was able to translate correctly from one register to another and was also able to translate between representations of the same register. He recognised the need to use the mean value theorem of integrals correctly to estimate the integral and also realised that the function/integral represented a circle; hence the student switched to a visual solution of the problem.

Procedures used by GV

The following picture was produced by a student in the category of global visual thinkers.



Vignette 4.24: Representation of MVT (integrals) for GV

Interpretive analysis

Visual representation of the theorem

The student presented a well labelled diagram showing that the concept had a vivid geometrical meaning in his/her mind. The diagram shows that the student has good coordination between the symbolic register and graphic register.

Proof of the theorem

The student managed to lay out the proof in a very clear manner. There is evidence of coordination between the diagram and the algorithm of the proof. There is connection of interpretations and treatments that is made in the graphic and symbolic representations of the theorem. However, there is lack of clarity in establishing the equality sign of the mean value theorem. The VA model was properly used.

Application of the theorem

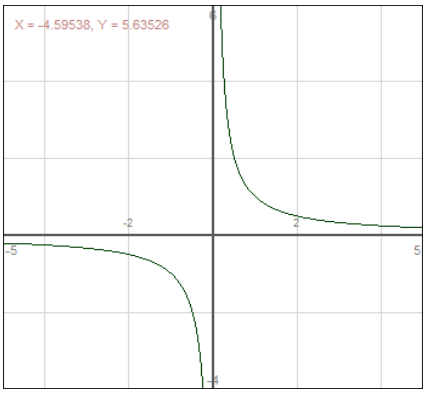
There is evidence that the student wanted to estimate the value of the integral using the mean value theorem for integrals but the work shows some deficiencies in the mathematical activity. Instead of using the expression $f'(c)$ (b-a), the student maintained the integral sign which showed some confusion. The actual value sub-task was not attempted

4.1.9 Task 9

Concepts involving infinity

1. By use of relevant diagram or otherwise show that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{32}+$........... = 1

2. Consider the graph of the function y= $\frac{1}{x}$ below



Deduce $from\ the\ graph\ \int_{-\infty}^{\infty}\frac{1}{x}dx$

The task comprised two sub-tasks. Exercise 1 required students to use a relevant diagram or otherwise to show that the infinite series was converging to one and Exercise 2 required students to deduce the value of the improper integral from the graph. The second exercise required students to observe that the function (integrand) had a point of discontinuity at x $=$ 0, therefore, the improper integral is not integrable on (-∞, ∞).

Ability to create images

Table 4.40 represents the types of images evoked by students.

Table 4.40: Types of images evoked by students on Task 9

Image	Frequency	Percentage
Diagram/Graph	39	78
Symbolic	3	6
Words	0	0
No images	1	2
Missing	7	14
Total	**50**	**100**

Since the question suggested the use of a diagram, a majority of students evoked diagrams. A further analysis was conducted to determine the visual levels. Table 4.41 highlights the visual levels.

Table 4.41: Visual levels for Task 9

Visual levels	Frequency	Percentage
Non Visuals	26	52
Local Visuals	10	20
Global Visuals	7	14
Missing	7	14
Total	**50**	**100**

Non-visual thinkers dominated in this task.

Image using

Table 4.42 illustrates the competence matrix of this task.

Table 4.42: Competence matrix for Task 9

Image role	NV	LV	GV	Total
Adjunct	0	0	1	1
Illustrative	3	2	1	6
Proof	0	1	5	6
Not applicable	23	7	0	30
Total	**26**	**10**	**7**	**43**

A majority of non-visual thinkers were not able to use images to represent the given tasks.

Chi-square test (Task 9)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.43: Chi-square for Task 9

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	32.012[a]	6	.000
Likelihood ratio	30.608	6	.000
McNemar-Bowker test	.	.	.[b]
N of valid cases	43		

a. 10 cells (83.3%) have expected count less than 5. The minimum expected count is .16.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases were the third assumption was violated. For this case the NV category was attached to LV category and the Illustrative and Adjunct category were attached to Proof category to form a 2x2 contingency table. The following results were obtained:

Table 4.44 : Chi-square test for task9 after post-test

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	25.880[a]	1	.000		
Continuity correction[b]	22.569	1	.000		
Likelihood ratio	31.432	1	.000		
Fisher's exact test				.000	.000
N of valid cases[b]	43				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 5.44.

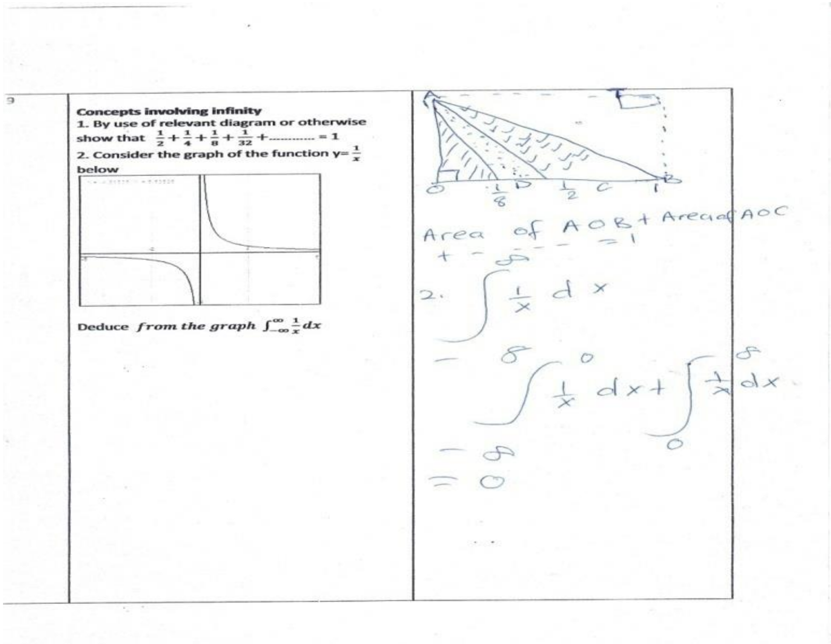
b. Computed only for a 2 x 2 table

Results: There is an association between visual levels and competencies (α =0.00 0< 0.050).

Procedures used in solving tasks

In order to conduct an analysis of the procedures used by the students in solving this task the researcher made use of the vignettes of students' work related to Task 9.

Procedures for LV



Vignette 4.25: Representation of infinite series for LV

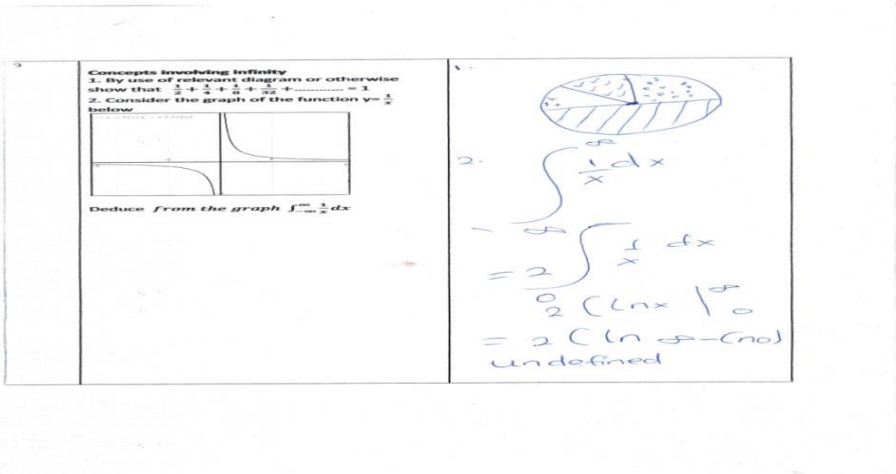
Interpretive Analysis

To attempt item (1) the student conceptualised a square with unit area and diagonally divided the square to obtain a triangle (AOB) with area ½ u^2, and made further subdivisions to obtain triangles whose areas are of the sequence: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$.Definitely, if we add the respective areas, we obtain the area of the unit square which coincides with the sum of the infinite series. This case is somehow difficult to visualize since the area of the respective triangles overlaps.

The student's response to Item 2 shows that he divided the region of integration into two parts. He went on to observe that the two regions were equal but with opposite signs. This representation misled the student to conclude that the integral was equal to zero.

Procedures for GV

Case (a)



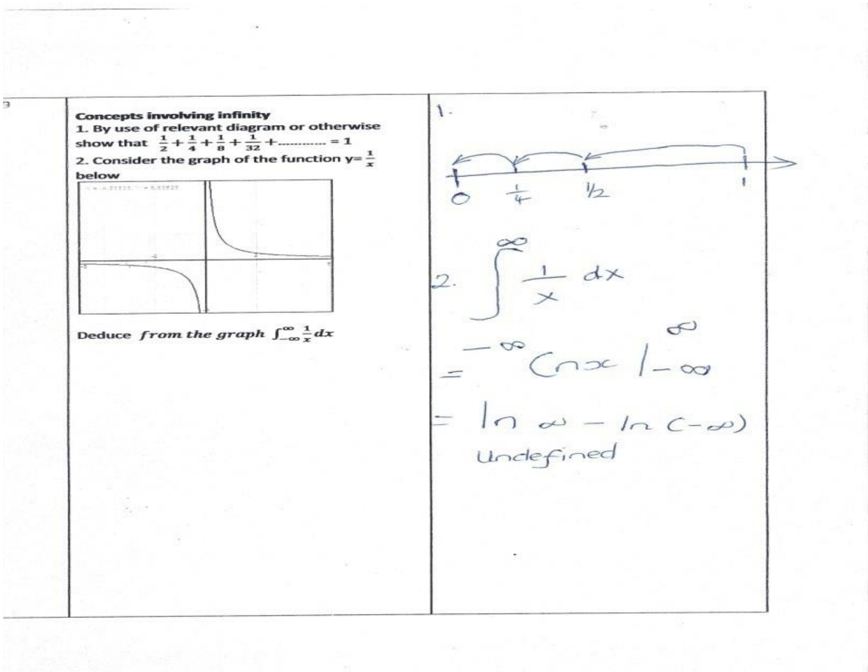
Vignette 4.26: Representation of infinite series for GV

Interpretive Analysis

The response to Exercise 1 showed that the student used a circle with some shaded regions (parts). The areas of the shaded parts represented $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots\ldots$. This coincided with the sequence of the given series. It can also be observed that the area of the circle constitutes an area which is equivalent to one whole. It can also be noted that it is difficult to visually represent very small areas and that there is no visual (diagrammatic) representation of the notion of infinity. The above representation convincingly represents the above sequence although the representation does not constitute the proof of the infinite series.

In the response to Exercise 2 the student observed from the diagram that area on the left was equal to the area on the right side of the y-axis. However, that treatment was misleading since the function was odd and the area on the left was always negative as it was below the x-axis. The student was able to translate correctly using the analytic route and found that the area of the function was unbounded.

Case (b)



Vignette 4.27: Representation of infinite series for GV

Interpretive Analysis

The student used the number line image to handle Exercise 1. The direction of movement was from one to zero and the steps generated the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \ldots$ which is also the sequence of the given infinite sequence. It can easily be observed (visually) that the sum of this sequence is equal to 1. Again, the thinking process used was convincing and showed that the sum of the infinite series was equal to 1. However, this does not constitute a proof of the given statement. The actual steps should be infinity but the representation has 4 steps, meaning that a visual representation of infinity is impossible.

The response to Item 2 shows that the student took an analytic route to solve the improper integral. However, the student observed that the integral was divergent after an evaluation of the limits. The student was not able to observe that the function to be integrated had a point of discontinuity at x = 0.

Multivariable calculus concepts

4.1.10: Task 10

Function/equation/inequality	**Describe the region corresponding to the given equation/inequality in R^3**	**Sketch the graph of the region**
(a) $X = 9$		
(b) $x^2 + y^2 + z^2 = 1$		
(c) $1 < x^2 + y^2 + z^2 < 25$		
(d) $x + y + z = 1$		

The task required students to switch representations from symbolic representations to words, then to graphical representations. The described regions functions in space ($\mathbb{R}^3$). The researcher identified three visual competencies versus three visual levels for the sake of conducting a thorough analysis of the data. The following are the visual competencies to be taken into consideration:

- To describe a concept verbally (written words)
- To represent a concept graphically
- To understand algebra and geometry as alternative languages.

The researcher scrutinised all the responses and graded the respondents in three visual levels, namely, NV, LV, and GV. The NV level was represented by those students who had one register of representation. LV level represented a group of students who could translate correctly between two representations but had difficulties in a third register of representation. GV level represented a group of students who translated correctly through all registers of representation. Table 4.45 represents the distribution of visual levels.

Table 4.45: Visual levels for Task 10

Visual levels	**Frequency**	**Percentage**
Non visuals	8	16
Local visuals	13	26
Global visuals	28	56
Missing	1	2
Total	**50**	**100**

A majority of students are GV thinkers. Table 4.46 represents the competence matrix for Task 10 with their respective Chi-square tests.

Table 4.46: Competence matrix for Task 10

Image role	NV	LV	GV	Total
Correct description	0	0	4	4
Partial correct	5	12	24	41
No description	3	1	0	4
Total	**8**	**13**	**28**	**49**

Global visual thinkers produced correct descriptions.

Chi-square test (Task 10)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.47: Chi-square for Task 10

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	22.515[a]	4	.000
Likelihood ratio	26.075	4	.000
McNemar-Bowker test	14.333	3	.002
N of valid cases	41		

a. 6 cells (66.7%) have expected count less than 5. The minimum expected count is 1.71.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases were the third assumption was violated. For this case the NV category was attached to LV category and the No Descriptions category was attached to Partial Descriptions category to form a 2x2 contingency table. The following results were obtained.

Table 4.48 : Chi-square test for task10 after post-test

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	21.339[a]	1	.000		
Continuity correction[b]	18.662	1	.000		
Likelihood ratio	27.939	1	.000		
Fisher's exact test				.000	.000
N of valid cases[b]	49				

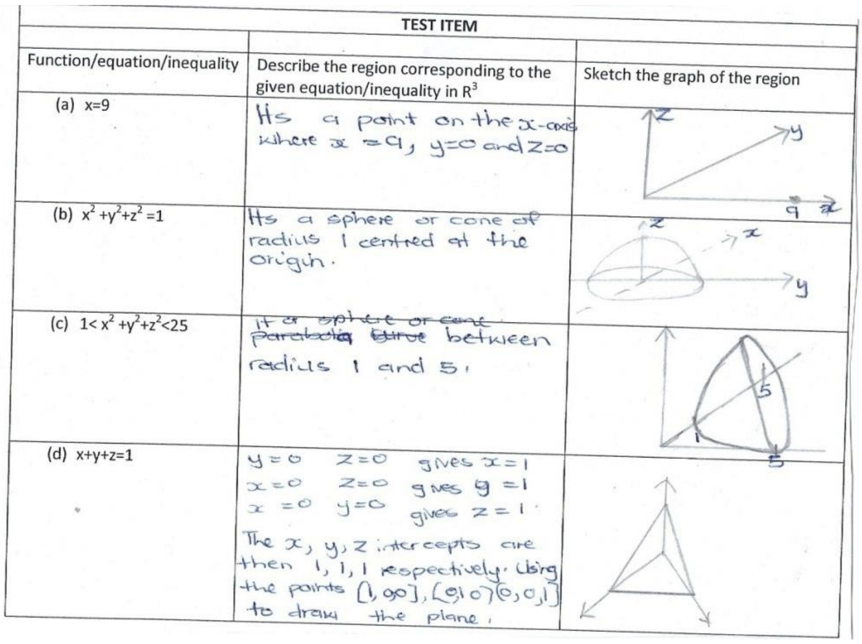
a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 7.71.

b. Computed only for a 2x2 table

Results: There is an association between visual levels and competencies (α = 0.00 0< 0.050).

Procedures for LV

The following are images evoked by students who attempted Task 10. The researcher presented two cases of partially correct responses.

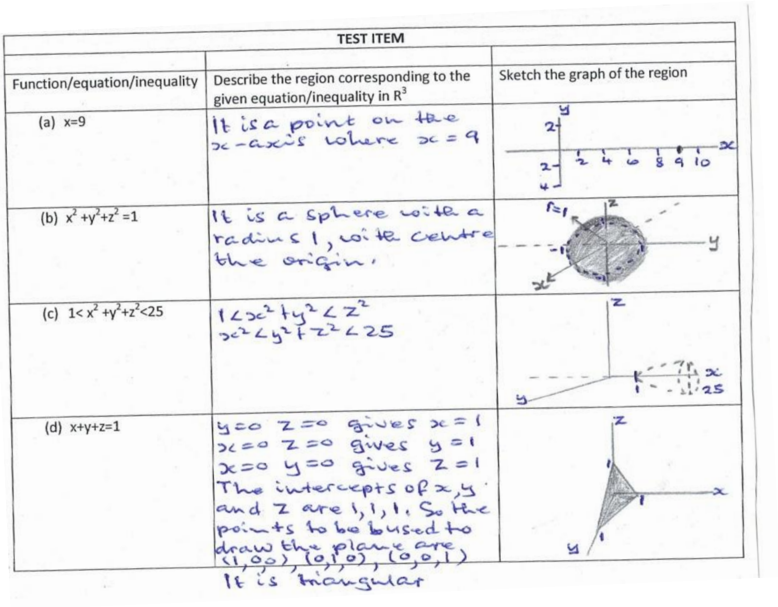


TEST ITEM

Function/equation/inequality	Describe the region corresponding to the given equation/inequality in R^3	Sketch the graph of the region
(a) x=9	Its a point on the x-axis where x = 9, y=0 and z=0	
(b) $x^2+y^2+z^2=1$	Its a sphere or cone of radius 1 centred at the origin.	
(c) $1<x^2+y^2+z^2<25$	~~it a sphere or cone~~ ~~Parabola~~ ~~circle~~ between radius 1 and 5.	
(d) x+y+z=1	y = 0 z = 0 gives x = 1 x = 0 z = 0 gives y = 1 x = 0 y = 0 gives z = 1 The x, y, z intercepts are then 1, 1, 1 respectively. Using the points (1,0,0), (0,1,0), (0,0,1) to draw the plane.	

Vignette 4.28: Case (a): Partially correct images

The above had difficulties in sketching the plane x = 9 and the region between two spheres part (c) of the task. The problems emanate from failure to translate between symbolic register to graphic register.



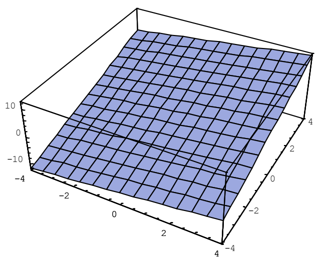
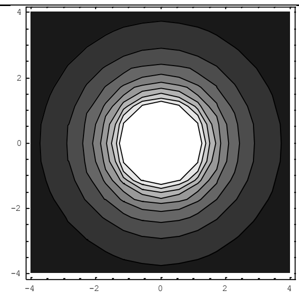
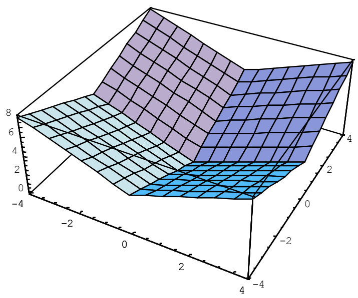
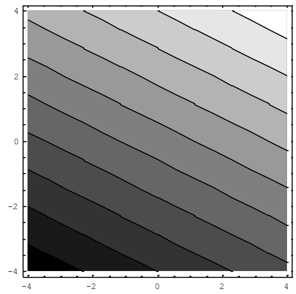
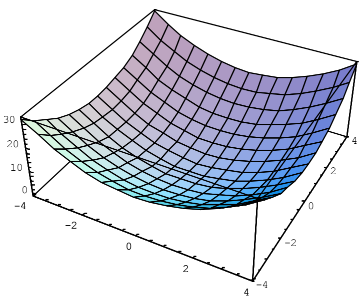
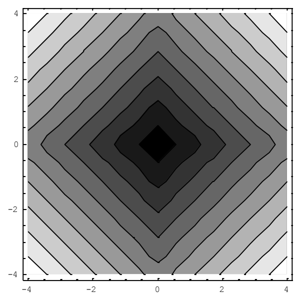
TEST ITEM		
Function/equation/inequality	Describe the region corresponding to the given equation/inequality in R^3	Sketch the graph of the region
(a) $x=9$	It is a point on the x-axis where $x = 9$	
(b) $x^2+y^2+z^2=1$	It is a sphere with a radius 1, with centre the origin.	
(c) $1<x^2+y^2+z^2<25$	$1 < x^2 + y^2 < z^2$ $x^2 < y^2 + z^2 < 25$	
(d) $x+y+z=1$	$y=0$ $z=0$ gives $x=1$ $x=0$ $z=0$ gives $y=1$ $x=0$ $y=0$ gives $z=1$ The intercepts of x, y and z are 1, 1, 1. So the points to be used to draw the plane are (1,0,0), (0,1,0), (0,0,1) It is triangular	

Vignette 4.29: Case (b) Partially correct images

The student had difficulties in dealing with multi-functional registers. The verbal descriptions and some sketchy diagrams were somehow vague, e.g. x = 9 is considered as just a point on the x-axis when it actually is a plane parallel to z-y plane. Also region (c) was not properly sketched. It seems there was a disconnection between verbal register and visual register.

4.1.11 Task 11

Match the function with its graph and level curves. Give reasons for your choice.			
FUNCTION (a-d)	GRAPH (I-IV)	LEVEL CURVE (A-D)	GIVE REASONS TO YOUR ANSWER
$z = x + 2y$			
$z = x^2 + y^2$			
$Z = \frac{1}{x^2+y^2+1}$			
$z = \lvert x \rvert + \lvert y \rvert$			

FUNCTION	GRAPH	LEVEL CURVES
$z = x^2 + y^2$		
$z = x + 2y$		
$Z = \frac{1}{x^2+y^2+1}$		

z = |x| +|y|

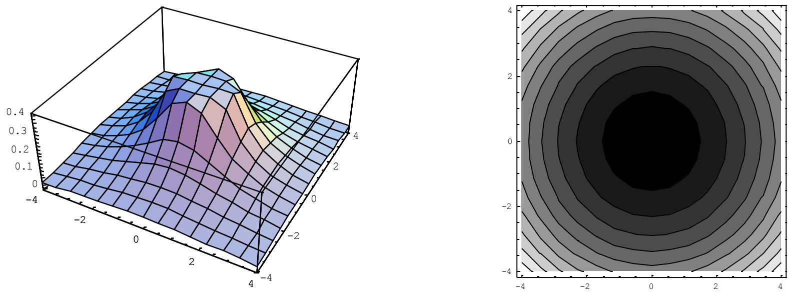


Fig. 4.1: Matching graphs with level curves

This task required students to match the function with its graph and level curves. The exercise tested how students switched representations, translated and gave reasons for their choice. Everything was provided; graphs and level curves were drawn using mathematical software. The method of analysis was similar to that of Task 10. On this task three competencies were identified, namely the ability to:

- match correctly the graph with its level curves;
- give reasons for the matching; and
- link geometrical with analytic concepts.

The visual levels were decided from the reasons given for matching the graphs. Table 4.49 illustrates the visual levels obtained after checking students' responses.

Table 4.49: Visual levels for Task 11

Visual levels	Frequency	Percentage
Non visuals	23	46
Local visuals	10	20
Global visuals	13	26
Missing	4	8
Total	**50**	**100**

Non-visual thinkers dominated in this task. Table 4.50 illustrates the competence matrices.

Table 4.50: Competence matrix for Task 11

Image role	NV	LV	GV	Total
Correct match	0	1	8	9
Partial correct	21	9	5	35
No match	2	0	0	2
Total	**23**	**10**	**13**	**46**

Global visual thinkers matched correctly.

Chi-square test (Task 11)

H_0: There is no association between visual levels and competencies; versus
H_1: There is an association between visual levels and competencies.

Table 4.51: Chi-square for Task 11

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	39.037[a]	4	.000
Likelihood ratio	16.883	4	.002
McNemar-Bowker test	24.182	2	.000
N of valid cases	31		

a. 7 cells (77.8%) have expected count less than 5. The minimum expected count is .03.

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases where the third assumption was violated. For this case the NV category was attached to LV category and the No Match category was attached to Partial Match category to form a 2 x 2 contingency table. The following results were obtained.

Table 4.52 : Chi-square test for task11 after post-test

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	46.000[a]	1	.000		
Continuity correction[b]	41.897	1	.000		
Likelihood ratio	61.578	1	.000		
Fisher's exact test				.000	.000
N of valid cases[b]	46				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 7.04.

b. Computed only for a 2x2 table

Results: There is an association between visual levels and competencies ($\alpha = 0.00\ 0< 0.050$).

Procedures used in solving tasks

In order to conduct an analysis of the procedures used by the students in solving this task the researcher made use of the vignettes of students' work related to Task 11.

Procedure for LV

The following picture was produced by a student in the category of local visual thinkers.

FUNCTION (F1-F4)	GRAPH (G1-G4)	LEVEL CURVE (L1-L4)	GIVE REASONS TO YOUR ANSWER
F1	G1	L2	The function is linear
F2	G3	L4	It is the equation of a circle
F3	G4	L3	Its an Inverse of ~~a hemisphere~~ function.
F4	G2	L1	Since f(x,y) is a modulus function.

Vignette 4.30: Matching task for LV

Interpretive analysis

The matching was correct but the descriptions were somehow vague. Translating between two different graphic registers was properly done.

Procedure for GV

Match the function with its graph and level curves. Give reasons for your choice.

TEST ITEMS			
FUNCTION (a-d)	GRAPH (I-IV)	LEVEL CURVE (A-D)	GIVE REASONS TO YOUR ANSWER
z) $=x+2y$	G1	L2	The level curves look like linear functions of the form $x+2y=c$
$z=x^2+y^2$	G3	L4	The level curves $x^2+y^2=c$ are concentric circles.
$Z=\frac{1}{x^2+y^2+1}$	G4	L3	The graph looks like a hilltop with maximum value at (0,0) and level curves are concentric circles $(0<c<1)$
$z=\lvert x\rvert+\lvert y\rvert$	G2	L1	The level curves $\lvert x\rvert+\lvert y\rvert=c$ take the form of linear function $x+y=c$ with the positive axis.

The following picture was produced by a student in the category of global visual thinkers.

Vignette 4.31: Matching task for GV

Interpretive analysis

The student managed well in matching the shapes, i.e. translating between symbolic to a three-dimensional graph, two-dimensional graph, and then to verbal descriptions. Some shapes were difficult to describe, especially G4 and G2.

4.1.12 Task 12

Double Integrals
(a)Find the volume of the solid bounded by the plane z = 0 and the paraboloid $z = \sqrt{1 - x^2 - y^2}$
(b) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and the region D in the xy-plane bounded by the line y = 2x and the parabola $y = x^2$.

The task required students to solve the two subtasks. Exercise (a) could be solved by the method of double integrals or by simply visualising that the solid under consideration was a hemisphere, hence, the volume was simply half that of the sphere. Exercise (b) could be solved using double integration only if students were able to produce good and relevant diagrams of the region of integration. The task tested students' ability to:

- Visualise regions in $\mathbb{R}^3$ from algebraic equations.
- Link geometrical concepts with analytical concepts.
- Extract relevant information from a diagram for the sake of problem solving.

Ability to create relevant images

The students' responses were scrutinised and grouped according to visual levels. Table 4.53 illustrates the distribution of visual levels.

Table 4.53: Visual level for Task 12

Visual levels	Frequency	Percentage
NV	21	42
LV	9	18
GV	1	2
Missing	19	38
TOTAL	**50**	**100**

Non-visual thinkers dominate in this exercise. Table 4.54 illustrates the competence matrix of competencies versus visual levels.

Table 4.54: Competence matrix for Task 12 (linking concepts)

Image role	NV	LV	GV	Total
Correct link	0	0	1	1
Partial correct	0	3	0	3
No link	21	6	0	27
Total	**21**	**9**	**1**	**31**

Global visual thinkers were able to link geometrical and analytical concepts.

Chi-square test (Task 12a)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.55: Chi-square test for Task 12

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	39.037[a]	4	.000
Likelihood Ratio	16.883	4	.002
N of Valid Cases	31		

Condition three of the assumptions of the Chi-square test was not satisfied . We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further -hoc tests.

Further test

The method of combining categories was used in cases were the third assumption was violated. For this case the NV category was attached to LV category and the No Link category was attached to Partial Link category to form a 2x2 contingency table. The following results were obtained.

Table 4.56 : Chi-square test for task12 after post-test

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	31.000[a]	1	.000		
Continuity correction[b]	27.125	1	.000		
Likelihood ratio	42.943	1	.000		
Fisher's exact test				.000	.000
N of valid cases[b]	31				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 7.26.

b. Computed only for a 2x2 table

Results: There is an association between visual levels and competencies (α =0.00 0< 0.050). Table 4.57 illustrates the competence for ability to extract relevant information.

Table 4.57: Competence matrix for Task 12 (extraction of information)

Image role	NV	LV	GV	Total
Correct extraction	0	0	1	1
Partial correct	0	6	0	6
Incorrect	21	3	0	24
Total	**21**	**9**	**1**	**31**

Global visual thinkers extracted relevant information from the diagrams.

Chi-square test (Task 12 b)

H_0: There is no association between visual levels and competencies; versus

H_1: There is an association between visual levels and competencies.

Table 4.58 Chi-square for Task 12

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-square	49.083[a]	4	.000
Likelihood ratio	27.402	4	.000
N of valid cases	31		

Condition three of the assumptions of the Chi-square test was not satisfied. We cannot conclude that there is an association even though the significant value 0.00 was less than 0.05. A true conclusion can be obtained after further post-hoc tests.

Further test

The method of combining categories was used in cases where the third assumption was violated. For this case the NV category was attached to LV category and the No extraction category was attached to Partial extraction category to form a 2 x 2 contingency table. The following results were obtained.

Table 4.59 : Chi-square test for task12 after post-test (extraction)

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-square	31.000[a]	1	.000		
Continuity correction[b]	27.029	1	.000		
Likelihood ratio	42.165	1	.000		
Fisher's exact test				.000	.000
N of valid cases[b]	31				

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 5.45.

b. Computed only for a 2 x 2 table

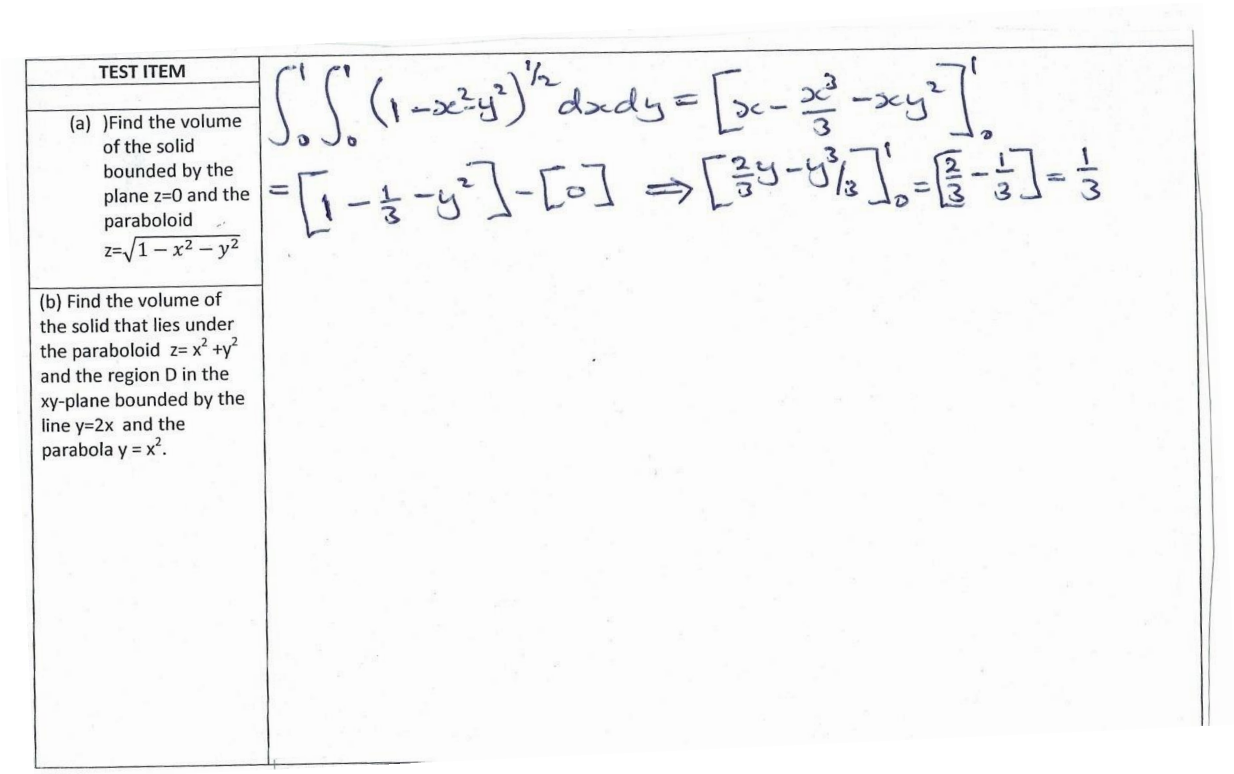
Results: There is an association between visual levels and competencies (α = 0.00 0< 0.050).

Procedures used in solving tasks

In order to conduct an analysis of the procedures used by the students in solving this task, the researcher made use of the vignettes of students' work related to Task 12.

Procedure for NV

The following picture was produced by a student in the category of non-visual thinkers.



TEST ITEM	
(a))Find the volume of the solid bounded by the plane z=0 and the paraboloid z=$\sqrt{1-x^2-y^2}$	$\int_0^1\int_0^1 (1-x^2-y^2)^{1/2}\,dx\,dy = \left[x - \frac{x^3}{3} - xy^2\right]_0^1$ $= \left[1 - \frac{1}{3} - y^2\right] - [0] \Rightarrow \left[\frac{2}{3}y - y^3/3\right]_0^1 = \left[\frac{2}{3} - \frac{1}{3}\right] = \frac{1}{3}$
(b) Find the volume of the solid that lies under the paraboloid z= x^2+y^2 and the region D in the xy-plane bounded by the line y=2x and the parabola y = x^2.	

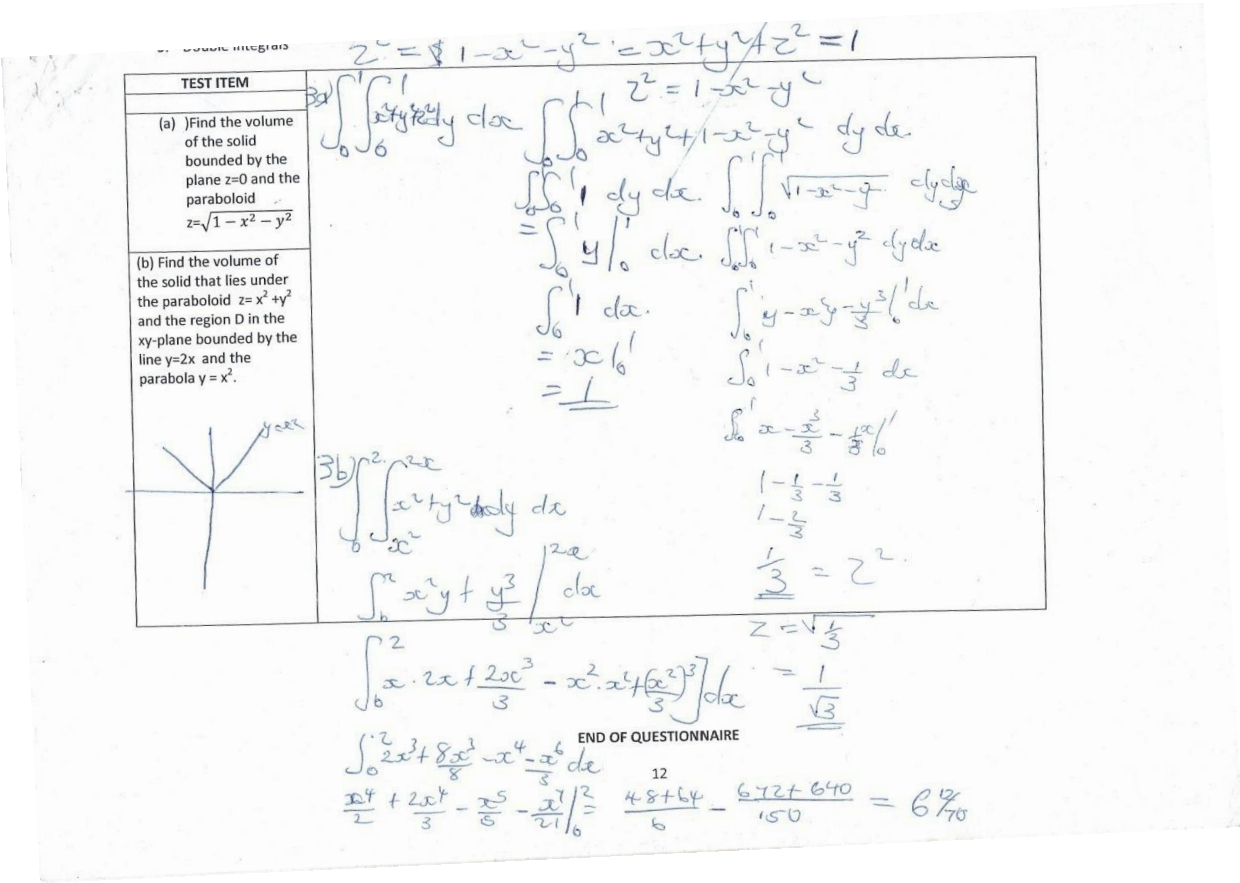
Vignette 4.32: Mathematical activities for NV category

Interpretive analysis

The student improperly translated in algebraic register. The student demonstrated some deficiencies in conceptual understanding of double integration.

Procedure for NV

The following picture was produced by a student in the category of local visual thinkers.



Vignette 4.33: Mathematical activities for LV category

Interpretive analysis

It can be observed that initially, the student's treatment of the first equation of (a) made it possible for him to visualise that the equation was that of a sphere but he could not realise this approach and resorted to an analytical route which was poorly laid out. Such problems arose from deficiencies in using the graphic register which was however paramount in identifying the region of integration.

The second part (b) was correctly done. The region of integration was poorly laid out. There is a disconnection between the graphic register and the algebraic register.

Procedure for GV

The following picture was produced by a student in the category of global visual thinkers.

TEST ITEM	
	$z=\sqrt{1-x^2-y^2}$
(a))Find the volume of the solid bounded by the plane z=0 and the paraboloid $z=\sqrt{1-x^2-y^2}$	$z^2+x^2+y^2=1$ This is now the equation of a sphere with centre (0,0,0) and radius 1 - but it is bounded by the plane $z=0$ so its now a hemisphere. ⇒ volume of hemisphere $=\frac{2}{3}\pi r^3$ $=\frac{2}{3}\cdot\frac{22}{7}\cdot 1$ $=\frac{44}{21}=2\frac{2}{21}=2,1$ units^2.
(b) Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$ and the region D in the xy-plane bounded by the line $y=2x$ and the parabola $y=x^2$.	b) $\int_0^2\int_{x^2}^{2x} x^2+y^2\,dy\,dx$ $=\int_0^2\int_{x^2}^{2x} x^2y+\frac{y^3}{3}\Big]_{x^2}^{2x}dx=\int_0^2 x^2\cdot 2x+\frac{(2x)^3}{3}-\left[x^4+\frac{(x^2)^3}{3}\right]dx$ $=\int_0^2 2x^3+\frac{8x^3}{3}-x^4-\frac{x^6}{3}\,dx=\frac{x^4}{2}+\frac{2x^4}{3}-\frac{x^5}{5}-\frac{x^7}{21}\Big]_0^2$ $=\frac{16}{2}+\frac{32}{3}-\frac{32}{5}-\frac{128}{5}-\frac{128}{21}-0=\frac{48+64}{6}-\left(\frac{672+640}{105}\right)$ $=\frac{112}{6}-\frac{1312}{105}=\frac{7840-5248}{420}=\frac{2592}{420}=6\frac{72}{420}=6,2$ units^2.

Vignette 4.34: Mathematical activity for GV category

Interpretive analysis

The student used the visual approach to solve (a). The student realised that after removing the square root sign the solid was a hemisphere of radius 1 and centre at origin (0,0,0) and he/she applied the formula to calculate the volume of the hemisphere. The analytical root was possible using polar coordinates.

It can be observed that the student was assisted by a mentally constructed diagram representing the region of integration because the limits were correctly placed. It can be noted that the student translated correctly from algebraic to the visual and then to algebraic registers.

Summary for quantitative data

The following presentation is a summary of results of quantitative data. The presentation starts with the frequency distribution table of visual levels for each task followed by frequency distribution tables pertaining to some possible answers to the research questions.

Frequency of students' visual levels for each task

Table 4.60 presents the visual levels of the sample for each task.

Table 4.60: Summary of visual levels for all tasks

Task	**Non-visual**	**Local visual**	**Global visual**
1	4	26	11
2	18	20	8
3	12	20	15
4	9	16	18
5	14	11	24
6	13	16	21
7	28	8	7
8	12	10	25
9	28	10	7
10	8	13	28
11	23	10	13
12	21	9	1
Mean	**15.83**	**14.08**	**14.83**
Standard deviation	**7.79**	**5.58**	**8.46**

General results for visual levels

Each task was analysed independently. The students were divided according to their visual thinking applied in solving the given tasks. As mentioned earlier, the Table 4.47 presents the visual levels for each task. The summary statistics show that the visual levels were almost evenly distributed, slightly biased towards non-visual level. The results show that the sample was chosen from a normally distributed population. The non-visual aspect is a common trend in the learning of mathematical analysis concepts.

Frequency of types of images evoked by students

Table 4.56 is a summary of the nature/types of images evoked by students in the learning of mathematical analysis concepts. The researcher used Tasks 1, 2, 3, 4, 6, 7 and 8 to investigate the nature of images evoked by students in the learning of mathematical analysis concepts. For these tasks, students were asked to represent on paper the meaning of a given concept.

Table 4.61: Summary of types of images evoked

Task	Diagram	Symbol	No image
1	18	21	2
2	41	4	1
3	19	24	4
4	43	0	0
6	45	2	3
7	39	3	1
8	35	4	6
Mean	**34.29**	**8.29**	**2.429**
Standard deviation	**11.24**	**9.84**	**2.070**

General results of the types/nature of images evoked

The observed outcome was that the image was either a diagram, or a symbol or no image. The summary of statistics shows that the diagram was evoked by a majority of students with a mean value of 34.29. Very few cases did not evoke images.

Table 4.62: Frequency of the nature of deployment of images in solving tasks

Task	Adjunct	Illustrative	None
Task 1	3	25	13
Task 2	5	16	25
Task 3	1	14	32
Task 4	14	17	12
Task 6	21	5	24
Task 7	1	4	32
Task 8	17	5	23
Mean	**8.86**	**12.29**	**23.00**
Standard deviation	**8.30**	**7.91**	**8.04**

General results of how the images were used to solve the given tasks

Very few students were able to use those images as adjuncts to prove theorems and mathematical statements in mathematical analysis. A majority abandoned their images and resorted to traditional approaches.

4.2 STUDENTS' THINKING PROCESSES IN LEARNING SELECTED CONCEPTS ASSESSED FROM FOLLOW-UP INTERVIEWS

A structured interview was conducted with a group of 12 students as a follow-up to the cognitive test. A pair of interviewees was individually asked the same questions for each concept. This type of interview was ideal since the researcher was interested in the variables of visual thinking and cognitive processes. The questions were therefore focused on these variables. The purpose was also to gain further insight into the deep thinking processes of students in concepts related to mathematical analysis. The interview sought to discover students' concept images of the selected concepts in mathematical analysis and how they deployed the images in solving some selected problems in the cognitive test.

Results

4.2.1 Concept 1: Supremum and infimum of sets in real numbers

Interviewee 1

Question	Response
What first comes to mind when you think of the concept of Supremum of Set A?	Supremum of a set? Well I think of upper bound of an interval of numbers.
Do you have an image in your mind that captures this concept?	The picture is that the set is bounded above.
What specific example comes to your mind?	There are so many things in my mind but $\frac{1}{n}$ comes quickly where 1 is the upper bound.
Can you explain how you worked out the supremum and infimum of A = $\{4 + \frac{1}{2^n}$:n$\epsilon\mathbb{N}$ }	I changed to improper fraction $\langle\frac{4.2^n+1}{2^n}\rangle$, and filled in the values of n and got $\frac{9}{2}$ as Supremum

	and zero as infimum.
Did you try to have a mental picture of this problem?	Yes.
If yes, can you describe the mental picture?	I see numbers in a certain pattern.

Interpretive Analysis

Interviewee 1: The student's first thought of the concept of supremum of a set is the upper bound of an interval of numbers and his visual image of the supremum of a set is a set which is bounded above. Although he has so many pictures in mind, what came first was the sequence $\frac{1}{n}$ whose upper bound is one. If I revisit his thinking processes in relation to this concept, it appears his way of thinking has some deficiencies. When he solved an exercise pertaining to finding the supremum and infimum of A = { $4 + \frac{1}{2^n}$:n$\epsilon\mathbb{N}$ }, he first converted the sequence to an improper fraction where he successfully managed the supremum but made an error in finding the infimum. The error was due to treatment transformations in algebraic register. When solving the problem he had a mental picture in the form of numbers that followed a certain pattern.

Concept 1: Supremum and infimum of sets in real numbers

Interviewee 2

Question	Response
What first comes to mind when you think of the concept of Supremum of Set A?	I think of the least upper bound …um.
Do you have an image in your mind that captures this concept?	I see a picture of a set …um.
What specific examples come to your mind?	½, 1, 2, 3, the least upper bound is 3.
Can you explain how you worked out the	I first write $4\frac{1}{2^n}$ as 4 + $\frac{1}{2^n}$ and completed some

supremum and infimum of A=$\{4+\frac{1}{2^n}$:n$\epsilon\mathbb{N}$ }	values 4 ½, …4 and 4 ½ is supremum and 4 is the infimum.
Did you try to have a mental picture of this problem?	No, I just solved.
If yes, can you describe the mental picture?	<No response>

Interpretive Analysis

Interviewee 2: The student's first thought of the concept of supremum of a set was the least upper bound and his visual image of the supremum of a set was a picture of a set. He cited some numbers ½, 1, 2, 3; the least upper bound is 3 as an example or prototype which quickly came to mind. Revisiting his thinking processes in relation to this concept, his way of thinking seems to be logical. When he solved an exercise in order to find the supremum and infimum of A = $\{4+\frac{1}{2^n}$:n$\epsilon\mathbb{N}$ }, he first converted the sequence to an algebraic sum 4 + $\frac{1}{2^n}$ for which it appeared very easy to compute the supremum and infimum. When solving the problem he had no mental picture, he just solved it.

4.2.2 Concept 2: Cardinality of Sets

Interviewee 3

Question	**Response**
What first comes to mind when you think of the concept of cardinality of a set?	Um…something to do with size.
Do you have an image in your mind that captures this concept?	I see a diagram of a number line.
What specific examples come to your mind?	If a set *A = {1, 2, 3}* cardinality of A is 3.
Can you explain how you proved that the set of	I put even numbers to positive numbers and odd

integers is countable?	numbers to negative numbers.
Are you convinced that natural numbers and integers have the same number of elements?	No!
If yes, can you describe the mental picture?	Natural numbers are numbers on the right hand of the number line.

Interpretive Analysis

Interviewee 3: The student's first thought of the concept of cardinality of a set is something to do with size and her visual image of cardinality was a diagram of a number line. She cited a set A = {1, 2, 3} cardinality *of* A *is 3* as an example or prototype which quickly came to mind. Her way of proving that the set of integers was countable also seemed very rational although the idea was intuitive. Her idea of assigning even numbers to positive integers and odd numbers to negative integers was a correct reasoning which can be summarised by the following function: $f(n)=\begin{cases} 2n+1, if n = 0,1,2,3,... \\ -2n\ ,\ if n = -1,-2,-3\ ... \end{cases}$. However, she objected to the proposition that natural numbers and integers have the same cardinality. Generally, this is called the cardinality conflict which reflects the nature of mathematical analysis.

Concept 2: Cardinality of Sets

Interviewee 4

Question	**Response**
What first comes to your mind when you think of the concept of cardinality of a set?	I think about numbers.
Do you have an image in your mind that captures this concept?	The set of natural numbers.
What specific examples come to your mind?	1, 2, 3…
Can you explain how you proved that the set of	I used the formula for calculating one-one

integers is countable?	function.
Are you convinced that natural numbers and integers have the same number of elements?	No.
If yes, can you describe the mental picture?	Integers are bigger than natural numbers.

Interpretive Analysis

Interviewee 4: The student's first thought of the concept of cardinality of a set was numbers and his visual image of cardinality was the set of natural numbers where he cited a set *A 1, 2, 3* as an example or prototype which quickly came to mind.. His way of proving that the set of integers was countable seemed logical since the usual way is to use $f(n_1) = f(n_2)$ which implies that $n_1 = n_2$. However, he also objected to the proposition that natural numbers and integers have the same cardinality. He saw the set of integers as bigger than the set of natural numbers.

4.2.3 Concept 3: Graphs of functions

Interviewee 5

Question	**Response**
What first comes to mind when you think of the concept of graphs of functions?	I think of relationship between variables, domain and range.
Do you have an image in your mind that captures this concept?	I see graphs.
What specific examples come to your mind?	y = x^2 that is shaped like a cup.
Can you explain how you represented $f(x)=\begin{cases} 1\ for x irrational \\ -1\ for x rational \end{cases}$ on paper?	I draw two lines which pass at 1 and -1.

Do you think the function is continuous?	Yes. The numbers are dense.
Can you describe the mental picture?	It's very difficult.

Interpretive Analysis

Interviewee 5: On the concept of graphs of functions, the student quickly thought of relationships between variables, graphs as objects with a domain and a range. He had images of graphs in his mind which captured this concept. A specific example that came to his mind was the graph of a parabola which, to him, is cup-like in shape. Reflecting on his solution to the function f(x)= $\begin{cases} 1\ forxirrational \\ -1\ forxrational \end{cases}$ he represented this function graphically with two parallel lines that passed through $y = 1$ and $y = -1$. The confusion lay in the density property of rational and irrational numbers. He also thought that the function was continuous though the obvious jump could be seen. It was very difficult for him to have a mental picture of the function.

Concept 3: Graphs of functions

Interviewee 6

Question	**Response**
What first comes to mind when you think of the concept of graphs of functions?	I think of a rule that associates sets A and B.
Do you have an image in your mind that captures this concept?	I see a smooth graph.
What specific examples come to your mind?	y= x.
Can you explain how you represented *f(x)* = $\begin{cases} 1\ forxirrational \\ -1\ forxrational \end{cases}$ on paper?	I could not represent it.
Do you think the function is continuous?	No.

Can you describe the mental picture?	The numbers are mixed.

Interpretive Analysis

Interviewee 6: On this concept the student thought of a rule that associates elements of set A to elements of set B. He saw graphs of functions as a process which was machine like. He visualized smooth graphs as a representation of the concept. A specific example that came to his mind was the linear graph $y = x$. Reflecting on his solution of the function *f*(x)=$\begin{cases} 1\; for\;\; x\; irrational \\ -1\; for\; x\; rational \end{cases}$ he could not represent the function graphically. He also deduced that the function and analysis were not continuous and his mental picture to the problem was a mixture of numbers.

4.2.4 Concept 4: Derivative of a function

Interviewee 7

Question	**Response**
What first comes to your mind when you think of the concept of derivative of a function?	I think of differentiation.
Do you have an image in your mind that captures this concept?	I can see signs of differentiation.
What specific example comes to your mind?	y' , dy/dx. I can see x^2 and the derivative as $2x$.
Can you explain why $f(x) = \|x\|$ is not differentiable at x = 0?	The graph has a sharp point at x = 0.
What is the derivative of |x| for x> 0?	For $x > 0$ the function is x and the derivative of x is 1.
Can you describe the mental picture?	It is a point $x = 1$.

Interpretive Analysis

Interviewee 7: The student thought of differentiation as something which came first to his mind as the concept of derivative of a function. His mental images were symbols of differentiation such as y', dy/dx . Specifically he could see x^2 and the derivative as 2x. Reflecting on why the function $|x|$ was not differentiable at $x = 0$ he positively responded that the graph had a sharp point at $x = 0$ which is the only visual image to account for local differentials of functions at a point. His translation or treatment of modulus function was correct since he managed to identify the derivative for $x > 0$. However, the mental picture of the problem was a bit confused.

Concept 4: Derivative of a function

Interviewee 8

Question	**Response**
What first comes to mind when you think of the concept of derivative of a function?	I think of maximum and minimum turning point.
Do you have an image in your mind that captures this concept?	I relate derivative to velocity.
What specific examples come to your mind?	If $f(x) = 5x^3$ then the derivative is 15 x^2.
Can you explain why $f(x) = \|x\|$ is not differentiable at x=0?	The derivatives of the sides are 1 and -1,they are not the same.
What is the derivative of |x| for $x > 0$?	The derivative is 1.
Can you describe the mental picture?	The gradient is 1…eh… because it is positive.

Interpretive Analysis

Interviewee 8: What first came to the mind of this student on the concept of derivative of a function was the idea of maximum and minimum turning points. She had a relational understanding of this concept and she related the derivative to velocity as her concept image. She specifically thought that If $f(x) = 5x^3$ then the derivative is 15 x^2. Reflecting on why the function |x| is not differentiable at x = 0, she used lateral derivatives, which is an analytical method of checking derivatives at a point. She was also able to calculate mentally the derivative at x >0. Her mental picture of the problem was a function with gradient 1.

4.2.5 Concept 5: Integral of a function

Interviewee 9

Question	**Response**
What first comes to your mind when you think of the concept of integral of a function?	I think of changing the gradient to [the] original function.
Do you have an image in your mind that captures this concept?	Area under the curve.
What specific examples come to your mind?	I see the integral sign with limits.
Can you explain the geometrical meaning of definite integral?	Area under a curve.
Explain how you would calculate $\int_{-1}^{1} \|x\|dx$.	It is zero because the other area on [under] the negative is the same with [as] the other area.
Can you describe the mental picture of the above problem?	I can see two triangles with the same area but looking at different direction [s].

Interpretive Analysis

Interviewee 9: The student's first thought on the concept of integration was changing the gradient function to the primitive function. Normally, integration is the reverse of derivatives. His mental image is the area under a curve, in other words definite integrals. The student specifically imagined the integral symbol. Geometrically, the student attributed integration to an area under a curve. His thinking processes pertaining to $\int_{-1}^{1} |x|dx$ is that the result is zero because the area on the left cancels the area of the right of the y-axis. This is a misconception of definite integration. He could see two triangles with the same area but looking in different directions, as a mental picture of the problem.

Concept 5: Integral of a function

Interviewee 10

Question	**Response**
What first comes to mind when you think of the concept of integral of a function?	I think of the integral sign.
Do you have an image in your mind that captures this concept?	I can see the sign of [an] integral.
What specific examples come to your mind?	Integral of x^2 dx.
Can you explain the geometrical meaning of definite integral?	It is a sum.
Explain how you would calculate $\int_{-1}^{1} \lvert x \rvert dx$.	It cannot be integrate[d] by formula.
Can you describe the mental picture of the above problem?	No.

Interpretive Analysis

Interviewee 10: The student's first thought on the concept of integration was the symbol of integration ($\int$). This symbol was also his mental image. The specific image or prototype was that of, $\int x^2 dx$. His geometrical meaning was that of a sum. Possibly, he referred to the Riemann sum $\sum f(x) \ \Delta x$. Pertaining to the computation of $\int_{-1}^{1} \lvert x \rvert dx$, he saw the expression as not integrable by formula. He did not have a mental picture of the problem.

4.2.6 Concept 6: Infinity

Interviewee 11

Question	Response
What first comes to mind when you think of the concept of infinity?	Nothing comes in[to] my mind.
Do you have an image in your mind that captures this concept?	Something unreachable’
What specific examples come to your mind?	I can see the symbol of infinity (∞)
Do you associate this concept with something in real life?	I associate it with sand, soil…something which is not countable.
Do you think the set of natural numbers has the same numbers as the set of even numbers?	No. Even numbers are half of natural numbers.
Do you see a picture of a number line in your mind?	Yes, I can see numbers and a straight line with some marks.

Concept 6: Infinity

Interviewee 12

Question	Response
What first comes to mind when you think of the concept of infinity?	Very big.
Do you have an image in your mind that captures this concept?	Something that increases continuously, for example, the graph of a function.
What specific examples come to your mind?	A graph of a function.
Do you associate this concept with something in real life?	Whenever I think of infinity I just think of drawing the graph of a function.

Do you think the set of natural numbers has the same numbers as the set of even numbers?	No. Impossible.
Do you see a picture of a number line in your mind?	I see a straight line with arrows on both ends.

Interpretive Analysis

Interviewee 11 and 12: It can be noted that Interviewee 11 had nothing coming to mind first on the concept of infinity but had an image of something which was "unreachable". The other student saw something "very big" and had an image of something which increased continuously. The two students had different images. The first associated the concept with "sand or soil" while the second associated it with graphs of functions. Both students agree with the idea that the set of natural did not have the same cardinality as the set of even numbers. They also had a graphical register for a real number line.

4.3 SUMMARY OF FINDINGS

This study was guided by a mixed methods paradigm. Quantitative data was collected first using a cognitive test followed by qualitative data which was collected through clinical and structured interviews. The findings were mainly summative in nature and provided evidence of the students' thinking processes in solving some tasks of selected concepts in mathematical analysis. Imagery and visualisation were the variables of interest in this study. The cognitive test had questions rich in imagery and visualisations. The findings included a wide range of students' responses. Some themes emerged in the process of quantitative data analysis. It emerged that

- Students' visual thinking was different. This difference was classified into three levels, namely; non-visual, local visual and global visual.
- Students evoked different images. There were also students who have no images in their thinking.
- Students had difficulties in making connections between visual data and algebraic concepts.
- Students had learning difficulties in visualising the concept of infinity.
- The thinking process that emerged was global-visual thinking.

- Students have difficulties in translating between different graphical registers of representations (e.g. conversion from $\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^3$).
- The Chi-square tests conducted for each test item indicated that there is an association between visual levels and competencies. This result needs a further analysis with a larger sample.

These seven themes are the highlights of the results from the cognitive test and the interviews. The visual characteristics for each student were examined on all tasks. It was observed that a student could be classified as non-visual in one exercise but on a different task he could be a global visual. All students were familiar with the selected concepts. The summary of visual thinking levels for each task is presented in Table 4.50. The common evoked images were diagrams (including graphs and table of values) and symbols (including numerals, words and algebraic terms and metaphors).

Not all questions catered for the investigation of the nature of images except those whose first question was on the representation of a concept on paper. The summary of the evoked images for the designated tasks is presented in Table 4.51. A Chi-square test for the association of visual ability competencies versus visual levels was conducted for all the tasks. The general outcome from the findings was that visual ability competencies were associated with visual levels. However, further tests should be conducted with a larger sample to confirm the association.

A follow-up interview was conducted to gain further insight into the use of imagery and visualisation characteristics of students' thinking processes in solving tasks in mathematical analysis. Clinical interviews were conducted in the pilot study. Structured interviews were conducted in the main study because the researcher wanted to focus the study around the concept of imagery and visualisations. Some follow-up questions were also included in the structured interview.

Six concepts were selected from the area of mathematical analysis. Two students were requested to respond to the same concept for comparison. The results showed that students evoked images during the learning of mathematical analysis tasks. These internal images were externally represented in the form of diagrams and symbols. It was also noted that students mainly evoked prototypical images. The issue of an imageless thought also manifested, but only on rare occasions. The findings from the interviews confirmed the finding from the written work.

CHAPTER FIVE
DISCUSSION OF FINDINGS

This chapter discusses the findings. The discussion is organised around the research questions linked to current research literature.

INTRODUCTION

The study investigated imagery and visualisation characteristics of undergraduate students' thinking processes while solving selected mathematical analysis tasks. Grouws (1992) described this process of investigation as a quest to establish what students actually consider when they are doing mathematics. Despite the cognitive paradox of access to the mathematical thinking that happens in the minds of the students, we could get close to knowing how they think through an analysis of their representations on paper, supported by word of mouth. The discussion of the findings was guided by the following research question: How do undergraduate students use imagery and visualisation in the learning of abstract mathematical analysis concepts?

This research question was investigated through an exploration of the following four sub-questions:

- What is the nature of imagery and visualisations that undergraduate students evoke in learning selected concepts of mathematical analysis?
- How can the visual images evoked by undergraduate students be classified?
- How do undergraduate students deploy these images and visualisations in the process of solving problems in selected concepts of mathematical analysis?
- What role does visual thinking play in undergraduate students proving some theorems or solving problems in mathematical analysis?

5.1 NATURE OF IMAGERY AND VISUALISATIONS THAT UNDERGRADUATE STUDENTS EVOKE IN LEARNING MATHEMATICAL ANALYSIS CONCEPTS

The cognitive development of mathematics occurs in the biological brain. The way in which the development occurs depends on the nature of the different forms of information presented to our senses (Tall, 1991). One way to access the nature of the visual thinking of students was through the representations they used in solving the given tasks and their responses in the interviews. From the cognitive test and interviews presented in the summary of the findings relating to the question being addressed in this section, it can be noted that visual thinking was at the centre of this study. In order to ascertain that visualisation was captured, the researcher provided students with a cognitive test whose items required students to make a representation of a concept on paper and that "representation" was considered to be the visual imagery of that person. By representation the researcher means diagrams, symbols, definitions and experiences that can be written down.

Vignettes 1 to 33 in the study are visual representations of the nature of images evoked by students in the learning of mathematical analysis. These images were qualitatively put into categories according to their usefulness in solving tasks of mathematical analysis. It can be observed that details of the images vary from person to person and, in some cases, some students simply left that task without attempting it, because they did not have an external representation of the concept. Further investigation of this question was conducted through clinical interviews (Concepts 1 to 6). In general, students evoked some form of representation that was related to the specific concept. The image might take the form of a diagram, symbol, numeric or a prototypical example related to the specific concept. This is in line with Berman's (2008) confession:

> I need to make my confession, which is that I am an extremely weak mental image, having little or no visual images. And the images that I am able to form are fugitive, flickering, sketchy, indistinct, weak and incomplete. So even when I try to form an image as simple as a triangle, I usually get little or nothing, or when I get something resembling a triangle, it has never, as far as I can remember, had all three sides clearly connected. For me there is always some bit of the triangle that is missing. And this deficiency in imagining a triangle applies to all my images (Berman, 2008, p. 256).

Berman's (2008) confession of having weak mental imagery is also echoed in the findings of this current research. It was noted that in some cases of non-visual thinkers, the question that demanded students to "make a representation on paper" was left blank; not even attempted. Also, the interpretive analysis of Interviewee 11 and Interviewee 12 mentioned something concerning weak mental imagery.

It was noted that students evoked both symbol imagery and concept imagery. When the researcher asked the students "What first comes to mind when you think of the concept of integral of a function?" (Concept 5) Interviewee 9 thought of integration as the reverse of the derivative. The example that he had in his mind was that of an area under a curve. This is an example of concept imagery. Interviewee 10 thought of the integral sign and what he had as an example was $\int x^2\,dx$. This is an example of symbol imagery.

Students also evoked images that made the abstract concepts of mathematical analysis real. For example, in the case of the concept of infinity, Interviewee 12 thought of something which was “very big”. His image was something which kept on growing bigger and bigger. His prototype was “graph of a function”. When asked whether the set of natural numbers had the same number of elements as the set of even numbers, he objected. His counterpart, Interviewee 11, initially had nothing on the concept of infinity in his mind. After further follow-up questions, he associated the concept of infinity with real objects such as sand and soil. He also objected to the issue of the paradox of cardinality.

This study therefore, supports the studies of Dubinsky et al. (1988) and Pinto (1998). Dubinsky et al. suggested that the process by which students construct meaning from formal mathematics presented in quantifiers occurs through reflective abstraction. The predicate with the whole variables is conceived as a mental process that is encapsulated into a statement (a mental object) by the process of quantification. This process is what Bell et al. (2003) referred to as concept imagery.

Pinto and Tall (1999) conducted a study on students’ difficulties in learning mathematics. They used a case study of a gifted student who constructed the formal definition of limit of a sequence from his own visuo-spatial imagery.

This current study also revealed that the most frequently evoked image was a diagram with an average frequency of 34. Most of these diagrams were prototypes, for example, the vignettes on Task 6. Students recalled the diagrams that were used in proving the mean value theorem for integrals. This was evidenced by their inflexibility to apply the theorem to estimate the value of the given definite integral. However, this outcome contrasts with outcomes from other studies (Bagni, 1998; Eisenberg & Dreyfus 1991; Presmeg, 1989). Research conducted by these authors revealed that the visual route in solving mathematical tasks in advanced mathematical thinking was difficult.

In summary, it is important to conduct a study on visual imagery for advanced mathematical thinking concepts because it is a way of perfecting the existing contradictions between concept images and concept definition. Even if the images are misleading, it is better that way because studies of mathematical thinking and misconceptions could enable mathematics educators to correct those errors and misconceptions.

5.2. CLASSIFICATION OF VISUAL IMAGERY OF MATHEMATICAL ANALYSIS CONCEPTS

The second sub-question explored how the visual images evoked by the students could be classified. The findings of this study on the nature of these images and summarized in Table 4.48 show that the students' visual images in solving mathematical analysis problems could be classified as diagrams, prototypes, algebraic symbols or metaphors. The first eight tasks, except for Task 5, all required students to make a representation of a given concept. That was the first item to be attempted. Those tasks were effectively used to gather data concerning the types of images of the students. Table 5.1 presents a summary of the types of images evoked:

Table 5.1: Summary of image types

Image type	Diagram	Symbol	No image
Frequency	240	58	17

For example, in Task 1, students were required to "represent Sup A". Vignette 4.1 has a symbolic image; $A = \{1/n;\ n \epsilon N\}$ is a sequence of real numbers. For this particular case, the supremum of the set was not identified. Vignette 4.2 displayed a diagram in the form of a number line. The supremum was not clearly stated. It was only inferred as a shaded spot. Vignette 4.3 again displayed a diagram in the form of a line interval. The supremum was clearly stated. This analysis was conducted for all the tasks where a representation was required. Diagrams were cited in the relevant tasks.

5.2.1 Diagrammatic image

The use of diagrams was crucial for all the tasks. Diagrams were used to represent mathematical analysis concepts, for example, the representations of Vignettes 4.2 and 4.3. Diagrams were also used in representing mathematical relations, for example, the inequality in Task 7. Graphs were also used to illustrate that $ln(1 + x) < x$.

Diagrams play an essential role in both the comprehension and communication of mathematical proofs. This role is to make the content "real" rather than formal. Diagrams are used to represent the objects and relations to which a proof refers. When successfully used, the validity of a proof can be seen in the diagram rather than justified as a step-by-step application of formal rules. For example, Task 8 on the mean value theorem for integrals, Vignette 4.22, was well-labelled so that it is self-explanatory to the mean value theorem for integrals. The diagrammatic image was clear with no superfluities. Students should therefore have the skills required in deciding which features are essential and should also be able to extract information which is important for the proof. Usually, such images require prior knowledge of the conventions used in the construction of the diagram in order that the implicit information is correctly extracted. On the other hand, the diagrams on the mean value theorems for derivatives (Vignette 4.15) were not very clear in illustrating that the gradient of the function at point c was equal to the gradient of the chord joining f(a) and f(b).Normally, in mathematical analysis, diagrams should be combined with well labelled algebraic symbols so that the proof can be seen from the image.

We also take note of the power of visual thinking in the diagrams which illustrated the theorem $1/2 + 1/2^2 + 1/2^3 + \ldots + \ldots = 1$ (Vignettes 4.25-4.27). The results showed depth and originality. Although the diagrams did not constitute proof of the theorem in the mathematical community, they were convincing. Didactical implications often point to the fact that students are given the impression that in mathematics, all is logical, accurate and provable. However, mathematical creativity is none of these things (Tall, 1991).

5.2.2 Prototypical image

Prototypical mathematical images are organised around a typical, average or modal "best example" of the given concept (Gagatsis et al., 2004). This example should cover a majority of the characteristics of the desired concept and can take the form of a diagrammatic image, symbolic image or any other form of an image. In mathematics, general examples are essential for generalisation, abstraction and analogical reasoning. However, prototypical thinking in mathematical analysis is narrow (Yerushalmy, 2005). Mathematical analysis is the general theory that underpins calculus, but normally in mathematical analysis there is frequent use of some prototypical diagrams, counter-examples and examples which facilitate comprehension of definitions and theorems.

It was noted that 57% of the diagrams used were prototypes. Most of the diagrams were those used by the teacher in the classroom. For example, representation of Sup A (Vignette 4.3) on the first item concerning the concept of supremum of a set (Sup A) seven students evoked prototypical images in symbolic form i.e. "$A=\{-2<x< 2\}$ and 2 is the supremum" while others started by drawing a directed number line and labelled the supremum on the diagrammatic image. The short numerical form to represent natural numbers n = 1,2,3…(Vignette 4.5), the vignettes that represented the mean value theorems for derivatives and integrals were prototypical in nature.

The observation therefore concurs with Krussel's (1994) research on visualisation in advanced mathematical thinking. Krussel observed that prototypical images and metaphorical images are abundant in mathematical circles, at least on an informal level. Prototypes are usually typical examples which are handed down from generation to generation in the community of mathematics educators. For example, the function $f(x) = |x|$ is used as a counterexample of a function which is continuous but not differentiable at x = 0.

5.2.3 Algebraic symbolic images

Algebraic symbols are the most common entities of mathematical analysis since logical reasoning is highly encouraged (Ottmar & Landy, 2017). For example, in this study what specifically came to students' minds for some concepts were algebraic symbols. As for the concept of derivatives, Interviewee 7 evoked symbolic images such as "$\mathrm{dy/dx}$" for the concept of derivatives and Interviewee 10 evoked "$\int_a^b f(x)dx$" for the concept of definite integral. The purpose of an algebraic symbol is to provide a convenient, brief and compact form of representation of a mathematical concept. In general, algebraic symbols are usually synonymous with the corresponding concept (Radford, 2014), but in some situations a different convention may be used. For instance, the congruence symbol "$\cong$" for congruent triangles may mean "isomorphic to" in Group Theory. Gray and Tall (1994) refer to the combination of symbols representing both the process and the output of that process as a procept (Fig. 5.1). Students in this study frequently used algebraic symbols in calculations and in proving theorems and a diagram was also combined with algebraic symbols as a label.

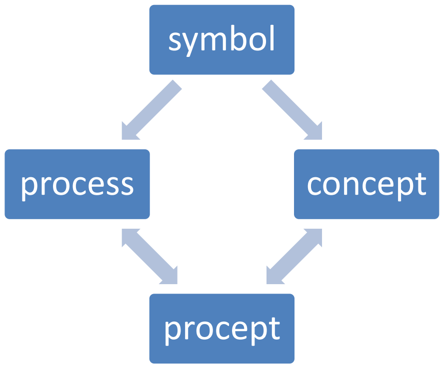


Fig.5.1: Symbol as process and concept forming a procept

5.2.4 Metaphorical images

Three students evoked metaphorical images during the interviews. Interviewee 5 associated the graph of $y = x^2$ with the shape of “a cup”. Interviewee 11 associated the infinity concept with “sand’ “or “soil”. Interviewee 12 associated infinity with something “very big”. Although such images have some traits in common with the desired concept, their use may lead to falsehoods in mathematical analysis. As for Interviewee 11 and Interviewee 12, such conceptions led them to misrepresent the proposition that natural numbers and even numbers have the same cardinality.

This observation supports Giaquinto (2011) who argues that our visual intuitions about what happens at the limit of an infinite process may sometimes lead us astray. A metaphor implicitly compares two domains of experience, giving meaning to one of these domains (target domain) by reference to structural or practical similarities in the other (the source domain) (Presmeg, 1998). Its purpose is to provide a connection with prior experiences, both mathematical and non-mathematical. In most cases it is extra-mathematical in nature and helps in the assimilation of new concepts with existing knowledge. For example, maximum and minimum turning points of a graph may be referred to as a hilltop and valley, respectively. In this research, the concept of infinity was associated with sand or soil, objects with countable particles. Complex numbers on a plane are always associated with the Riemann sphere.

5.3 STUDENTS' VISUAL THINKING IN SOLVING MATHEMATICAL ANALYSIS TASKS

The third sub-question explored how students deploy images and visualisations in the process of solving problems in selected concepts of mathematical analysis. The summary of the competencies of deployment of images and visualisations is presented in Table 4.50.

Generally the deployment of evoked images was partially correct. It was observed that some students were not able to link geometrical concepts and analytical concepts. Evidence of how students were deploying their imagery and visualisations is presented as interpretive analysis on the vignettes. Good deployment of representations means the student has learnt and understood the concept. From this point of view, understanding of a concept was built through tasks that imply the use of different representation systems and promote the flexible articulation between representations. To analyse any mathematical activity we need three important aspects, namely representation, treatment and conversion. Two themes emerged on how students deployed imagery and visualisation in solving some selected concepts of mathematical analysis, namely:

- visual representations as organizers of thought (Task1-Task10); and
- recognition of the same mathematical object through two representations whose contents are different (Task 11-Task 12).

5.3.1 Visual representations as organisers of thought

The study observed that a correct and clear representation of a concept facilitated the comprehension of solving some tasks in mathematical analysis. For example, in finding the supremum and infimum of the sequence $A = \{4 + \frac{1}{2^n} : n \epsilon \mathbb{N}\}$, those who correctly used the numerical representation were able to deduce the supremum and infimum of the sequence. Those who treated the sequence as an improper fraction were not able to recognize the solution, (see findings of Concept 1 in the previous chapter). It was also noted that students had conversion difficulties in moving from the algebraic symbolic register to the graphical register of functions involving sequences. This was evidenced by the graphical representations of the functions $f(n) = \begin{cases} 2n+1, if n = 0,1,2,3,... \\ -2n\ ,\ if n = -1,-2,-3,... \end{cases}$ and $f(n) = \frac{5-3n}{9n+4}$. A majority of students produced continuous graphs, and did not realise that the functions were defined in the set of natural numbers. For Task 2 those who used the graphical representation to deduce that the function was one-one by drawing a horizontal line observed that the function was not one-one. However, those who used the numerical table of values and those who used the analytical method produced correct results. This observation agrees with that of Duval (2006, p.121) where he noted that

> ... conversion is a representation transformation, which is more complex than treatment because any change of register requires recognition of the same represented object between two representations whose contents have very often nothing in common.

It is easier for students to use visual methods when the mathematical activity is presented in a graphical register, otherwise they should use the analytic method if the activity is presented in algebraic register.

Giaquinto (2011) also noted that the following ways in which diagrams and visual imagery of analytic concepts are represented, could help students:

- Visual illustrations of instances of an analytic concept can strengthen one's grasp of the concept.
- When a function has a visualisable curve, visualising the curve can help us grasp and think about symbolically presented operations on the function such as transformations and area under the curve.
- It can also help us to grasp the type of discontinuity and location of roots, etc.

The first two findings were also observed in the written scripts of global visual thinkers. Most students had visual learning difficulties in grasping the type of discontinuity of some classic functions.

5.3.2 Recognition of the same mathematical object through two representations whose contents are different

Mathematical activity involves the use of a variety of semiotic representation systems which comprise natural language, the registers of numeric, algebraic and symbolic notations, geometrical figures and Cartesian graphs. It was observed from Task 10 and Task 11 that students had difficulties in making verbal descriptions of sketch diagrams that they had produced and also had difficulties in justifying their matching. The tasks involved multivariable calculus concepts. This agrees with the observation of Kashefi, Ismail, Yusof (2010) that students have difficulties in representing some three-dimensional shapes. The most problematic cases were the following functions in space: $x = 9$ and the region represented by $1 < x^2 + y^2 + z^2 < 25$. The students' sketch diagrams for $x = 9$ showed a point on the real number line or a vertical straight line in two-dimensional space. This problem was noted by Duval (1999) where he proposed that the content of a representation depended more on the register of the representation than on the object represented. Only 30% of the participants were able to recognise that $1 < x^2 + y^2 + z^2 < 25$ was a region between two spheres, hence, were able to sketch the required region. It also emerged in this task that students exhibited poor discursive skills. This was evidenced by the partially correct descriptions. A total of 40% of the students produced partially correct descriptions of a sphere in Task 10 item (b); an example "The region is a sphere in R^3 with radius 1 and centre 0". The centre of the sphere in question had its centre at the origin (0, 0, 0). Generally, in the classroom we have a very specific practice of simultaneously using two registers, that is, spoken language register and symbolic expression register/graphic register on the chalkboard. Students usually take note of what is written and pay less attention to what is said.

On the matching exercise it was noted that 90% of the students were able to match a function with its graph and level curve. Most reasons given for the match were partially correct answers. Matching is typically a visual method since the process relies on mental reasoning. The results showed that, generally, students were able to make connections but had difficulties in substantiating such connections with suitable mathematical statements. The function $f(x, y) = \frac{1}{x^2+y^2+1}$ caused visual difficulties and the common errors included a confusion between inverse function and reciprocal function. Students failed to visualize that the function attained its maximum value at (0, 0, 1) and approached zero for larger values of x and y. As for exercise (d), most students were able to link the characteristics of the function with those of the modulus function in real numbers $f(x) = |x|$. This hook was the initial access point to additional information on the concept image of the function $f(x, y) = |x| + |y|$. Generally, the reasons given were based on intuition (perceptual). The roots of cognitive development in geometry were based on perception of objects in the real world. Such a cognitive development and its succession of cognitive stages have been documented in the work of van Hiele (1985). An analytical route was to use the definition of the level curve (x, y) = c; then deduce the pattern of the level curves.

In the classroom, teachers normally deliver lessons using two registers simultaneously, that is spoken language, and written language on the chalkboard. Based on this view, the study urges teachers of mathematics to be conversant in translating between all the semiotic systems used to access mathematical knowledge. Communicating mathematics either orally, in writing, using symbols or visual representations is very important to learning and using mathematics. In addition, students should use a variety of forms of communication in different settings in order to generate and share ideas.

It was also observed from this empirical study that students with a better knowledge of representation of the same mathematical concept in a variety of registers have greater prospects of successful mathematical learning. These students can translate well between the graphical and the symbolic representations.

5.4 ROLE OF IMAGERY AND VISUALISATION IN PROVING THEOREMS

In this context imagery and visualisation meant the student`s ability to draw an appropriate diagram (with pencil and paper) to represent a mathematical concept, theorem or problem and to use the diagram to achieve understanding in proving a theorem or solving a problem. It was observed that the role of visualisation varied from problem to problem and from student to student. For example, the role of visualisation in Task 1 to Task 4 was to illustrate conceptual understanding. There was no way a diagram could be used in proving the proposition that $sup(A+B) = supA + supB$ since students followed the standard proof. A majority were able to construct a diagram to illustrate the concept of supremum and a few students (6%) were helped by their diagrams to find the supremum and infimum of the given set in that task.

An important observation for the first three tasks was that a correct diagram or representation was helpful in completing proofs or solving tasks. Some representations, however, were misleading in solving tasks. This was noted especially for Task 2 where students were supposed to use the representation to deduce that f(n) was one-to-one. Students who constructed the graph of f(n) as a continuous function (Vignette 4.6 Case (c)) made a wrong deduction from their representation but those who made a partially correct representation (Vignette 4.5 Case (b)) were able to deduce that the function was one-to-one.

Quite a reasonable number of diagrams in this study were used as illustrations or adjuncts in proving some theorems. This can be observed from the correct and partially correct responses of Tasks 6, 7 and 8. For example, students used standard illustrative diagrams as representations for the mean value theorems for derivatives and integrals. Vignette 4.22 clearly illustrates the steps used to arrive at the statement of the mean value theorem for integrals. Visual representations in mathematical analysis have several roles. Giaquinto (2011) highlighted that visual illustrations may facilitate comprehension of formulas or definitions; they can be reminders of counter-examples to plausible seeming claims; they can serve as stimuli to spark an idea for a proof. These uses of visual thinking do not involve trusting it to deliver or preserve truth…but are untrustworthy when used as a means of discovery (Giaquinto, 2011, p. 163).

It was also observed that students needed to use visual diagrams carefully in solving problems involving the concept of infinity. This was noted from the following item: "Deduce from the graph " $\int_{-\infty}^{\infty} \frac{1}{x} dx$" (Task 9). The response obtained from Vignette 4.27 illustrated that the student took it from the graph that the area on the left hand side was equal to the area on the right hand side but with opposite signs therefore $\int_{-\infty}^{\infty} \frac{1}{x} dx = 0$. This exercise was also an eye-opener to the idea that visualizing could be misleading when used to discover the nature of the limit of an infinite process.

However, the study has some results where visualisation was very powerful in solving some problems in mathematical analysis and also discovering some theorems. It can be easily observed that solutions to Task 9, Item 1 of Vignette 4.25 to Vignette 4.27 visually illustrate that the limit of the sum to infinity of the series $1/2 + 1/2^2 + 1/2^3 + \ldots + \ldots = 1$. The images also produced some degree of creativity. Gardner (1973) posits that in many cases a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance. However, some mathematicians regard "proof without words" as unacceptable proofs (Eisenberg & Dreyfus, 1991).

It was also noted that the visual route to solving some problems could be very brief and powerful while the analytic route could be difficult. Evidence can be drawn from the solutions of the last task on Double Integration. The item reads as follows*: "Find the volume of the solid bounded by the plane z = 0 and the paraboloid z =*$\sqrt{1 - x^2 - y^2}$". Those who quickly identified that the required region of integration was a hemisphere could solve the problem in a sentence (Vignettes 4.32) but those who attempted to use the analytic method found the problem difficult (Vignettes 4.33 and Vignettes 4.34).

It is also important to note that students demonstrated difficulties in the application of theorems to solve problems in mathematical analysis. Task 6, Task 7 and Task 8 required the students first to make representations of the theorems, followed by proof of the theorems. Lastly they had to apply the mean value theorems to handle related exercises. Task 7 was an application of the mean value theorem for derivatives. Item (c) of Task 6 required the students to verify the conditions of the mean value theorem (derivatives) for |f(x)| where

$$f(x) = \begin{cases} 1 \; if \; x \; is \; irrational \\ -1 \; if \; x \; is \; rational \end{cases} .$$

Item (c) of Task 8 required the students to estimate the value of $\int_0^2 \sqrt{4-x^2}\, dx$ which was an application of the mean value theorem (integrals). The summary of the results is as follows in Table 5.2.

Table 5.2 Summary of application of theorem to solve item (c) of Task 8

Task	**Apply theorem**	**Not able to apply**
6	42%	58%
7	2%	98%
8	34%	66%

A similar observation was noted by Parameswaran (2009) on a study involving Rolle's theorem and the mean value theorem. The study revealed that students had difficulties in understanding Rolle's theorem, making sense of the theorem, and using it in situations when the function was not explicitly given.

5.5 CONCLUDING REMARKS

This chapter presented a discussion of the findings. The discussion focused on answers to the research questions. The main thrust was to discuss how undergraduate students used imagery and visualisation in solving some selected concepts in mathematical analysis. The nature and types of images were discussed. The manner in which these images were involved in solving tasks and proving theorems was also discussed. The findings were compared with similar studies. It can be argued that the similarities noted encourage the incorporation of using visual thinking in the teaching and learning of mathematical analysis.

CHAPTER SIX

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

INTRODUCTION

Visual thinking in mathematical analysis is often regarded as misleading in proving theorems. Often, the proofs are presented by mathematicians in a polished form. The main reason why visualisation is thought of as untrustworthy is that our visual intuitions about what happens at the limit of an infinite process sometimes lead us astray. There is no contradiction in this assertion. The concept of infinity and a limit is fundamental to mathematical analysis. However, visualisation has several roles in mathematical analysis, especially by providing the idea for proving a theorem. The present study investigated advanced mathematical thinking processes with a group of university students majoring in mathematics. The aim was to find out "how they do what they do" when they are solving some selected concepts of mathematical analysis.

A sample of 50 undergraduate mathematics students participated in the study for the purpose of collecting data. The researcher employed both quantitative and qualitative methods. All the selected students responded to a given cognitive test. The test comprised 12 tasks with items selected from mathematical analysis concepts. Each task had an item which catered for the visual processing component. The purpose of the cognitive test was to explore the nature of imagery and visualisations evoked by these students through solving tasks related to mathematical analysis. A sample of 12 students was purposefully selected from the 50 to participate in the structured interview as a follow-up to the responses of the cognitive test. The purpose of the interview was to gain insight into the mental representations of the students. This study was guided by a combination of Duval's (1999) TRSR and some psychological notions contributing to a cognitive approach (Goldstein, 2011; Kossyln, 1973; Tall & Vinner, 1991).

The main question reads: "How is visualisation used in solving abstract mathematical analysis concepts?" This question was investigated using four sub-questions. The first sub-question was on the nature of imagery and visualisations evoked by undergraduate students in solving mathematical analysis concepts. The cognitive test, combined with the interviews, was used to investigate the nature of imagery and visualisations evoked. Task 1 to Task 8 had an item each which required students to “make a representation” of a selected concept or theorem. The first three questions of the structured interviews also investigated the nature of imagery and visualisation of the selected concept.

6.1 SUMMARY OF FINDINGS

It was found in this study that undergraduate students used imagery and visualisations in solving tasks related to mathematical analysis. The mental representations (internal schemata) which were evoked by the students were inferred from what they wrote on paper and/or from their spoken words. It was observed that these psychological constructs were evoked by a majority of the selected students but a few saw nothing in their “mind’s eye”. The nature of images and visuals differed from person to person as evidenced by the categories of the responses highlighted from the cognitive test as well as the responses of the structured interviews. In the case of the concept “graph of functions” some would see a smooth graph and others would see a relationship of variables. In the case of derivatives some would see the symbols related to the derivative, for example dy/dx, whilst others would see turning points and relate these constructs to velocity and so on. Generally, these constructs helped the students to deal with other issues pertaining to the concepts once the construct had clicked in the mind.

The second sub-question was on the classification of the types of visual images evoked by the students. The cognitive test was used to investigate this question. Again, the sub-item which required students to make a representation of a given concept was used. The interview also had questions which enabled the researcher to classify the types of images evoked by the students. It was observed that the following categories emerged according to their usage rather than their nature: diagrams, prototypes, algebraic symbols and metaphors. The purpose of each category is clearly described in the discussion section. It is also possible that students could have their own ways of doing things. This category of personal symbols was not discussed in this study because such images are too subjective.

The third sub-question was on how the visual thinking was used or deployed in solving problems related to mathematical analysis. To handle this question the researcher relied more on the responses to the cognitive test. The test comprised concepts of basic analysis and multi-variable calculus concepts. It was observed that clear diagrammatic representations were associated with correct solutions. Students had difficulties in interpreting graphs. They often abandoned simple graphical solutions and opted for harder analytic solutions, for example, the responses to Task 5's last item which read "Compute $\int_{-4}^{6} f(x)dx$." Some students failed to realize that they were supposed to just calculate areas of simple geometric shapes (rectangle and triangle) and opted to find the functions and integrate them, which was a more difficult method. It also emerged that students had conversion difficulties, especially where the source register was symbolic and the target register was graphical. This problem was frequent in the concept of sequences. Students have problems in plotting the graphs of these functions. They constructed continuous curves instead of points. The source of difficulties of comprehension here were obstacles. They thought that graphs of functions were plotted by uniting the points by a curve, a concept which had been acquired as far back as secondary school mathematics.

It also emerged that students had learning difficulties in recognizing the same mathematical object through two representations whose contents could be different. This was observed in tasks dealing with multi-variable calculus where students were switching representations between symbolic, graphical and verbal registers. Some students failed to recognize the meaning of a symbolic expression in a 3-D register, for example, the meaning of $x = 9$. Again, obstacles were the source of difficulties. However, a positive outcome was also registered. The students were able to visually (perceptually) match the symbolic equation of a function with its graph in 3-D and its level curve in 2-D.

Students had epistemic problems with concepts involving the concept of infinity. Task 9 in the cognitive test had two items involving the concept of infinity. Students registered learning difficulties with Item 2 where some visualized $\int_{-\infty}^{\infty} \frac{1}{x} dx = 0$ taking it from the graphical representation of the integrand. This conception supports the idea that visualisation in analysis concepts can be misleading. An interview on this concept also revealed that students had difficulties with the cardinality paradox. They could not visualize that the set of natural numbers had the same cardinality as the set of even numbers.

However, there was also a positive result on the concept involving the limit of an infinite process. The visual diagrams representing the statement $1/2 + 1/2^2 + 1/2^3 + \ldots + \ldots = 1$ were helpful in discovering the limit of the series.

The fourth sub-question was on the role of visualisation in proving theorems and solving concepts related to mathematical analysis. The results showed that good representations were helpful in completing proofs and poor representations were misleading in the completion of proofs or in solving a problem. It also emerged that most students used standard diagrams as adjuncts and for illustrative purposes, especially in proving the mean value theorems. It was also noted that visualisation was a difficult route to use since some students preferred to stick to the analytic route even where the visual route was shorter. Visualisation could also be misleading in solving concepts involving infinity; therefore, it should be used with care. However, it can also be an eye-opener for discovering theorems, as previously mentioned.

6.2 CONCLUSIONS

The following conclusions were based on the data collected from this study. The findings were related to students' use of imagery and visualisations in the learning of abstract mathematical analysis concepts. The major findings are summarized as follows.

Conclusion 1

Students evoked mental representations (images) when they were solving concepts related to mathematical analysis and these representations were accessed through students' written work and their spoken words. There were some students who evoked no images in some selected concepts of mathematical analysis. The nature of images varied with the nature of the task and also varied from person to person. There is enough evidence from this study that visual images are noticeable in the thinking processes of undergraduate students in solving mathematical analysis concepts. In most cases, it is true that clarity of visual representations is positively associated with correct solutions. This relationship was tested and confirmed to be true in this research using the Chi-square test.

Conclusion 2

The classification was made according to the purpose of the image rather than its nature. The types of images were as follows: diagrams (including graphs), prototypes, algebraic symbols and metaphors. The use of metaphorical images was most pronounced in the study. It was also noted that clear and correct representations were associated with correct solutions.

Students experienced difficulties in using visual methods, which, in some cases, appeared easier and shorter and opted for the familiar analytic methods even though they were more difficult and longer.

Conclusion 3

Students had difficulties in plotting graphs of sequences and some could not distinguish between finding the limit and verifying the limit of a sequence. The trend originated from the conflict between concept image and concept definition. The traditional idea of presenting mathematical analysis in a polished formalism which often followed the sequence theorem-proof-application sounded a very dry approach because it was more important to discuss the students' concept images as well as the instructor's concept images.

Conclusion 4

It was also observed that in mathematical analysis it was rare that diagrams could stand alone as proofs in mathematical statements. In this research, diagrams were used as adjuncts in proving theorems and for illustrative purposes.

Conclusion 5

It was also noted that our visual intuitions about what happens at the limit of an infinite process were misleading. Students experienced/demonstrated epistemic problems with the concept of infinity. However, some diagrams which were observed in this study were an eye-opener in proving the limit of an infinite sequence.

Conclusion 6

Students had learning difficulties in recognising the same mathematical object through two or more different registers. Some students were not able to translate from graphic register to language (verbal) register. For example, descriptions of objects on a number, line, plane and space were muddled.

6.3 RECOMMENDATIONS

The following recommendations are based on the present study.

Recommendation 1

Imagery and visualisation should be encouraged in the teaching and learning of mathematical analysis. The findings of this study have shown that students evoked images in the form of diagrams, symbols, prototypes and metaphors in the learning of mathematical analysis. Mathematics educators should take the opportunity to know students' images for the purpose of correcting misconceptions that may be visible in the images. Students sometimes have their own peculiar images related to specific mathematical concepts. For instance, the concept of infinity was likened to "sand" or "soil" with countable particles in one case. It is important that students learn mathematical concepts in a multiple representation approach. In this study students were able to correctly match functions in space with their representations on a plane but had difficulties in providing reasons to support their arguments.

Recommendation 2

Visualisation should be encouraged in proving theorems and statements in mathematical analysis. This idea brings the discipline of mathematical analysis very close to reality. Well labelled diagrams are helpful for illustration and also for sparking the idea of a proof.

Recommendation 3

Visual thinking should be the starting point of learning concepts involving infinity in mathematical analysis. Although it is a fact that visualisation is unreliable whenever it is used to discover the nature of the limit of some infinite process, the mathematics instructor should convincingly supply counter-examples which reinforce the areas of weakness of visual thinking in mathematical analysis.

6.4 LIMITATIONS OF THE STUDY

The only obstacle in this study was the study design limitation. Due to financial and time factor constraints, the study was limited to only one university in Zimbabwe. The sample of the study was limited to only mathematics students in the Faculty of Education. The results could be generalized if the study was extended to other mathematics students in other universities in different countries. However, the results in this study are an eye-opener for further research.

6.5 SUGGESTIONS FOR FUTURE RESEARCH

The researcher suggests that this study could be conducted with students from other institutions and other countries and at other levels of study to see if the results will be the same. Further research that thoroughly examines the role of image schematic structures in mathematics learning is necessary for the design and implementation of mathematics curricula that contributes to concept development and success in school mathematics for students with different visualisation preferences.

Further research could also investigate methodologies which focus on visualisation. For example, when are observations significant enough to make a strong claim that true understanding and acquisition from visual-mediated learning and teaching have taken place? What theoretical framework could enable mathematics educators to scientifically investigate the relevant variables of visual thinking in mathematics? Further research may also focus on the relationship between creativity and visual thinking.

6.6 CONCLUDING REMARKS

This chapter presented the summary, conclusions and recommendations of the study. The limitations of study and suggestions for future research were also highlighted. To conclude the whole issue, the researcher proposes that imagery and visualisation are useful in the learning of mathematical analysis. Even expert mathematicians have visual images, intuitive ways of solving mathematical analysis theorems and imaginative ways of perceiving concepts.

REFERENCES

Apostol, T.M. (1973). *Mathematical analysis*. 6th edn. Reading, UK: Addison-Wesley.

Arcavi, A. (2003). The role of visual representation in the learning of mathematics, *Educational Studies in Mathematics*, 52(3), 215-24.

Artigue, M., Menigaux, J., & Viennot, L. (1990). Some aspects of students' conceptions and difficulties about differentials. *European Journal of Physics, 11*(5), 262.

Berger, M. (2010). Using CAS to solve a mathematics task: A deconstruction. *Computers & Education, 55(1), 320-332.*

Berman, D. (2008). Philosophical counselling for philosophers: A confession of images. *Philosophical Practice*, July 2008, 32: 255-266.

Binet, A. (1899). La pedagogie scientifique [Scientific pedagogy]. In *A Klein - Bulletin de Psychologie,* 2014.

Bishop, A.J., (1989). ERIC-Review of Research on Visualisation in Mathematics Education. *Focus on Learning Problems in Mathematics 11*(1), 7-16.

Boaler, J., Chen, L., Williams, C., & Cordero, M. (2016). Seeing as understanding: The importance of visual mathematics for our brain and learning. *Journal of Applied & Computational Mathematics*, *5*(5), 1-17.

Bogdan, R.G. & Bilken, S.K. (1992). *Qualitative Research for Education* (2nd Edition), Boston, MA: Allyn & Bacon.

Boyer, C.B. (1949). *The history of calculus and its conceptual development.* Dover Books on Mathematics.

Brating, K. (2012). Visualisation and intuitive reasoning in mathematics, *The Mathematics Enthusiast, 9(*1/2), 1.

Brown, J.D. (2001). Can we use the Spearman-Brown prophecy formula to defend low reliability? *Statistics,4*(3).

Caglayan, G. (2017). Real analysis students' understanding of point-wise convergence of function sequences in a DGS-assisted learning environment. *Journal of Mathematical Behavior, 49*(2018) 61-81.

Clements, D.H. (1992). Geometry and spatial reasoning. In Grouws, D.A. (1992) *Handbook of Research in Mathematical Teaching and Learning. A Project of the National Council of Teachers of Mathematics.* New York: MacMillan.

Cohen, L., Manion, L. & Morrison, K. (2011). *Research methods in education.* 7th Edition. London and New York: Routledge.

Crapo, A.W., Waissel, L.B., Wallace, W.A.,& Willemain, T.R. (2000). Visualisation and the process of modelling: A cognitive-theoretic view. In *Proceedings of the Sixth ACM SIGKDD international conference in knowledge discovery and data mining,* (pp 218-226) Boston, USA.

Creswell, J.W. (2013*). Research design: Qualitative, quantitative and mixed methods approaches.* London: Sage.

Dagar, V., & Yadav, A. (2016). Constructivism: A Paradigm for Teaching and Learning. *Arts Social Sci J 7:200. doi: 10.4172/2151-6200.1000200*

Dennet, D.C. (1981). Two approaches to mental images. In Bolck, N. (Ed). *Imagery.* Cambridge, MA: MIT Press/Bradford Books.

Douville, P., Pugalee, D.K., Wallace, J., & Lock, C.R. (2002). Investigating the effectiveness of mental imagery strategies in a constructivist approach to mathematics instruction. In paper published in the *Proceedings of The Humanistic Renaissance in Mathematics Education International Conference. Palermo, Italy: The Mathematics into the 21st Century Project.*http://math.unipa.it/~grim/SiDouville.PDF

Dreyfus, T. (2002). Advanced mathematical thinking processes. In *Advanced Mathematical Thinking,* p. 25-41, Amsterdam, Netherlands: Kluwer Academic.

Duval, R. (1999). Representation, vision and visualisation: Cognitive functions in mathematical thinking and basic issues for learning. In *Proceedings of the 21st North American Chapter of the International Group for the Psychology of Mathematics Education*, Morelos, Mexico, 3-26.

Duval, R. (2006). A cognitive analysis of problems of comprehension in learning of mathematics, *Educational Studies in Mathematics* (2006) 61: 103-131.

Duval, R. (2017). *Understanding the mathematical way of thinking: The registers of semiotic representations.* Amsterdam: Springer International.

Engelbrecht, J. (2010). Adding structure to the transition process to advanced mathematical activity. *International Journal of Mathematical Education in Science and Technology,* Volume 41, No 2, 143-154.

Erlwanger, S.H. (1973). Benny's conception of rules and answers in IPI Mathematics. *Journal of Children's Mathematical Behavior, 1*(2), 7-26.

Eisenberg, T. & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In *Visualisation in teaching and learning mathematics,* (pp. 25-37) Mathematical Association of America. Washington D.C.

Fennel, F. & Rowan, T. (2001). Representation: An important process for teaching and learning mathematics. *Teaching Children Mathematics*, 7(5), pp. 288-292.

Findlay, A. (1937). *A hundred years of chemistry*. London: Gerald Duckworth.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht: D. *Holland: Reidel.*

Galindo, M.E. (1994).*Visualisation in the calculus class: Relationship between cognitive style, gender and use of technology*. PhD thesis, Ohio State University, USA.

Gazzinaga, M.S. (2005). Forty-five years of split brain research and still going strong. *Nature Reviews – Neuroscience*, 6(8): 653-659.

Giaquinto, M.(2011). *Visual thinking in mathematics: An epistemological study*. Oxford University Press.

Goldin, G.A. (2003). Representation in school mathematics: A unifying research perspective. *A research companion to principles and standards for school mathematics*, 275-285.

Goldstein, B. (2011). *Cognitive Psychology: Connecting Mind, Research, and Everyday Experience*. Wadsworth. *Inc. ISBN*, *854969976*.

Gray, E.M. & Tall, D.O. (1994). Duality, Ambiguity and Flexibility: A proceptual view of simple arithmetic. *Journal of Research in Mathematics Education 26*(2)115-141.

Grouws, D.A. (1992). *Handbook of Research on Mathematical Teaching and Learning. A Project of the National Council of Teachers of Mathematics* (Vol. 1). IAP.

Gruber, H.E. & Barrett, P.H. (1974). *Darwin: A psychological study of scientific creativity*. (2nd). E.P. Dutton, Chicago: University of Chicago Press.

Guzman, M.D. (2002). *The role of visualisation in the teaching and learning of mathematics analysis*. Madrid, Universidad Complutense de Madrid.

Hadamard, J. (1945). Psychology of Invention in the Mathematical Field *(Princeton: Princeton University Press);* republished in 1954 by Dover Publications.

Harel, G. & Sowder, L. (2005). Advanced Mathematical-Thinking at Any Age: Its Nature and Its Development. *Mathematical Thinking and Learning.* 7(1) 27-50.

Hiebert, J, & Carpenter, T. P. (1992). Learning and teaching with understanding. In D.A. Grouws (Ed.*), Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 65-97). New York.

Huang, C.H. (2015). Calculus students' visual thinking of definite integral. *American Journal of Educational Research, 3(4), 476-482.*

Hurlburt, R.T., & Heavy, C.L. (2001). Telling what we know describing the inner experience. *Trends in Cognitive Sciences,* 5 (9) p.400-403.

Hurlburt, R.T., & Heavy, C.L. (2006*). Exploring inner experience: The descriptive experience sampling method.* Amsterdam: John Benjamins.

Intons-Peterson, M.J. (1993). *Imagery's role in creativity and discover.* Amsterdam, North Holland. In *Advances in psychology* (Vol. 98, pp. 1-37). North-Holland.

Johnsonbaugh, R. & Pfaffenberger, W.E. (2010). *Foundations of Mathematical Analysis (Dover Books on Mathematics).* New York: The Amazon Book Review.

Johnson, R.B. & Onwuegbuzie, A.J. (2004) . Mixed method research: A paradigm whose time has come. *Educational Researche , 33, 14-26.*

Kashefi, H., Ismail, Z., & Yusof, Y.M. (2010). Engineering mathematics obstacles and improvement. A comparative study of students and lecturers'perspectives through creative problem-solving. *Procedia-Social and Behavioral Sciences*, *56*, 556-564.

Kadunz, G. & Yerushalmy, M. (2015). Visualisation in the Teaching and Learning of Mathematics. In *Proceedings of the 12th International Congress in Mathematical Education.* (pp. 463-467) Springer, Cham.*DOI 10.1007/978-3-319-12688-3-41*

Kosslyn, S.M. (1996) *Image and Brain: The Resolution of the Imagery Debate*. Cambridge, MA: MIT Press.

Kosslyn, S.M. (1983). *Ghosts in the mind's machine*. Creating and using images in the brain. W. W. Norton. New York.

Kuhn, T.S. (1962). *The Structure of Scientific Revolutions*. 2nd Edition, Chicago, University of Chicago Press.

Krussel, C.E.S. (1994*). Visualisation and Reification of Concepts in Advanced Mathematical Thinking*. Unpublished Thesis. Oregon State University.

Lincoln, Y.S. & Guba, E. (1985*). Naturalist Inquiry*. Beverly Hills, CA, Sage.

Mahmood, A., Othman, M.F., Yusof, Y.M. (2012). A conceptual framework for mathematical ability analysis through the lens of cultural neuroscience. *Procedia-Social and Behavioral Sciences 56 (2012) 175-182.*

Mancosu, P., Jorgensen, K.F., Pedersen, S.A. (Eds.). (2005). *Visualisation, explanation and reasoning styles in mathematics.* (Vol 327). Springer Science and Business Media, Amsterdam.

Mason, J. (2002). Exploiting mental imagery in teaching and learning Mathematics.*Actasdo ProfMat 13* (2002):75-81.

Mehanovic, S. (2011). *The potential and challenges of use of Dynamic Software in upper secondary mathematics: Students' and Teachers' work with Integrals in GeoGebra*

Based Environments. (Doctoral Dissertation, Linkoping, Sweden, University Electronic Press.).

Meriam, S.B. (1997). *Qualitative Research and Case Study Applications in Education: Revised and Expanded from 1 Case Study Research in Education /* John Wiley & Sons.

Mnguni, L.E. (2014). The theoretical cognitive process of visualisation for science education. *Sprnger Plus 3*(1),184.

Neuman, W.L. (2013*). Social Research Methods, Qualitative and quantitative approaches.* Pearson Education.

Opper, S. (1977). Piaget's clinical method. *Journal of Children's Mathematical Behavior,* 1(4), 90-107.

Orton, A. (1980) *A cross-sectional study of the understanding of elementary calculus in adolescents and young adults,* Unpublished PhD. Thesis, Leeds University, UK.

Ottmar, E., & Landy, D. (2017). Concreteness fading of algebraic instruction: Effects on learning. Journal of the Learning Sciences, 26(1), 51-78. doi:10.1080/10508406.2016.1250212

Paivio, A. (1991). Dual coding theory: Retrospect and current status. *Canadian Journal of Psychology, 45*(3), 255-287.

Parameswaran, R. (2009). Understanding Rolle’s theorem. *The Mathematical Educator, 19*(1) *18-26.*

Pino-Fan, L., Guzmán, I., Duval, R., & Font, V. (2015). The theory of registers of semiotic representations and the onto-semiotic approach to mathematical cognition and instruction: linking looks for the study of mathematical understanding. In Beswick, K. & Wells, J. (Eds.) Proceedings of 39th Psychology of Mathematics Education conference, Vol.4, 33-40. Hobart, Australia: PME.

Pino-Fan, L., Guzmán, I., Duval, R., & Font, V. (2017). Analysis of the underlying cognitive activity in the resolution of a Task on derivability of the absolute value function: Two theoretical perspectives. PNA 11(2) 97-124

Pinto, M.M.F. & Tall, D. (2001). Following students’ development in a traditional university classroom, *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 4, pp. 57–64). Utrecht, Netherlands.

Pinto, M.M.F. & Tall, D. (1999). Students’ constructions of formal theory: Giving and extracting meaning. *Proceedings of the 23rd Conference of PME , Haifa, Israel , 3, 281-288.*

Presmeg, N.C. (1989). Visualisation in multilingual mathematics classrooms. *Focus on Learning Problems in Mathematics. 11*(1&2) 17-24.

Presmeg, N.C. (1997). *Generalization using imagery in mathematics*. In L.D. English (Ed). *Mathematical reasoning: Analogies, metaphors, and images* (pp. 299-312). Mahwah, New Jersey: Lawrence Erlbaum.

Presmeg, N.C. (1998). Metaphoric and metonymic signification in mathematics, *Journal Of Mathematical Behaviour,17*(1), 25-32.

Presmeg, N.C. (2006). Research on visualisation in learning and teaching mathematics. In A. Gutierrez & P. Boero (Eds). *Handbook of research on the psychology of mathematics education: Past, present, and future.* (pp. 205- 235). Rotterdam: Sense Publishers.

Presmeg, N.C. (2008). An overarching theory for research in visualisation in mathematics education. *Paper presented at the 11th International Congress for Mathematics education (ICME11).Monterrey, Mexico.*

Pylyshyn, Z.W. (1973). What the mind's eye tells the mind's brain: A critique of mental imagery. *Psychological Bulletin 80*(1) 1.

Sawyer, W.W. (1987). Intuitive understanding of mathematical proof. *Bulletin of the I.M.A.,23*, 61-62.

Scotland, J. (2012). Exploring the philosophical underpinnings of research: Relating ontology and epistemology to the methodology and methods of the scientific, interpretive, and critical research paradigms. *English Language Teaching, 5(9), 9-16.*

Schoenfeld, A. (1985). *Mathematical problem-solving.* New York: Academic Press.

Skemp, R.R. (2012). *The psychology of learning mathematics*. *Expanded American edition*. Routledge.

Snow, C.P. (1967). Foreword to a Mathematician's Apology by G.H. Hardy. Cambridge: Cambridge University Press.

Sommer, R. (1978). *The mind's eye: Imagery in everyday life*. New York: The Amazon Book Review.

Stake, R.E. (2005). *The art of case study research.* Thousand Oaks, CA: Sage.

Sternberg, R.J. (2009). *Applied Cognitive Psychology. Perceiving, Learning, and Remembering.* New Dehli: Centage Learning.

Stewart, J. (2008).*Calculus: Early transcendentals.* 6th edition. New York: Thomson Brooks/Cole.

Radford, L. (2014) .The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal 26 (257).*

Rubio, B.S., & Chacón, G.I. (2011). Challenges with Visualisation. The Concept of Integral with Undergraduate Students. In *Proceedings of the Seventh Congress of European Society for Research in Mathematics Education (CERME-7). 9th and 13th Feb, University of Rezeszow, Poland*.

Tall, D.O. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12* (2) 151-169.

Tall, D. (Ed.) (1991).*Advanced mathematical thinking*. (Vol 11). Springer Science and Business Media.

Tall, D. (Ed.) (1994). Understanding the Processes of Advanced Mathematical Thinking. *An invited ICMI lecture at the International Congress of Mathematicians, Zurich,* August, 1994.

Tashakkori, A. & Teddlie, C. (1998*). Mixed methodology: Combining qualitative and quantitative approaches*. London: Sage.

Tomic, W. (1993). Behaviorism and cognition in education. *Psychology Journal of Human Behavior*. 30(3/4) p.39-48.

Van Hiele, P.M. (1985). *A child's thought and geometry.* English Translation of Selected Writings of Dina Van Hiele-Geldof and Pierre M. van Hiele. 237-241. Washington DC: NSF.

Vygotsky, L.S. (1980). *Mind in society: The development of higher pedagogical processes.* Harvard University Press.

Vinner, S. (1992). The function concept as a prototype for problems in mathematics learning. In G. Harel, & E. Dubinsky, (Eds) *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA, 195-213.

Wikepedia, 2018. (https://en.wikipedia.org/wiki/Mathematical_analysis*)* (Accessed 20 December 2017).

Wheatly, G. (1998). Imagery and learning. *Focus in learning problems in Mathematics, 20* (2) 65-77.

Whiteley, W (2002). Teaching to see like a mathematician, to appear in *Proceedings of Visual Representations and Interpretation II* In *Visual Representations and Interpretations Conference*. http://www.math.yorku.ca/~whiteley/. (Accessed 13 July 2014).

Yerushalmy, M. (2005). Functions of interactive visual representations in interactive mathematics textbooks. *International Journal of Computers for Mathematical Learning*; 10 (3) 217-219.

Yin, R.K. (2009). *Case Study Research Designs and Methods* 3rdedn. New Delhi: Sage.

Zazkis, R., Dubinsky, E, & Dautermann, J. (1996). Coordinating visual and analytical strategies: A study of students' understanding of the Group D4. *Journal for Research in Mathematics Education*, 27(4), 435-457.

Zimmermann, W. & Cunningham, S. (1991). Editors' introduction: What is Mathematical visualisation? In N. Zimmermann, S. & Cunningham (Eds.*) Visualisation in Teaching and Learning Mathematics* (pp1-8). Washington, D.C: Mathematical Association of America.

APPENDICES

APPENDIX ONE: DIARY OF EVENTS

The purpose of this diary is to record the chronological order of events on the PhD research Project.

Planned activities	2013	2013	2014	2014	2014	2014	2014	2014	2015	2015	2015	2016	2016	2016	2016	2016
1.Development of research proposal.																
2.Development of chapter one																
3. Literature review																
4.Development of research design instruments																
5.Field Work in schools.																
6.Data cleaning and sorting																
7.Presentation and analysis of findings.																
8.Development and conclusions & recommendations																
9.Printing and binding of final copy.																
10.Submission of final bound copy (to be advised)																
11. Monitoring and Evaluation.																

APPENDIX TWO: COGNITIVE TEST PART 1

NAME OF STUDENT: ---REG. NO.-------------------------

Mathematical Processing Instrument 1: Cognitive test 1.

Mathematical Processing Questionnaire 1

INSTRUCTIONS

1. For each problem please show your working as much as you can.

2. Attempt all the problems given. Do not use scientific calculators, computers, textbooks or class notes. We are interested in your own mathematical thinking.

Questionnaire on Analysis Concepts

NO.	ITEM	REPRESENTATIONS AND WORKING SPACE
1	Suppose A and B are bounded sets of real numbers • How would you represent Sup A on paper? • Prove that Sup(A+B)=SupA + SupB • Let A = $=\{4+\frac{1}{2^n} : n\epsilon\mathbb{N}\}$, , find the least upper bound and greatest lower bound of A.	
2	Let the function f: $\mathbb{Z} \to \mathbb{N}$ be defined by f(n)=$\begin{cases} 2n+1, if\ n = 0,1,2,3,\ldots. \\ -2n\ ,\ if\ n = -1,-2,-3,\ldots.. \end{cases}$ • Make your own representation of f f(n) on paper • From the representation deduce thatf(n) is one-one and onto.	
3	A sequence $\{u_n\}$ has its n^{th} term given by $U_n = \frac{5-3n}{9n+4}$, $n \in \mathbb{N}$	

	a. Representthe following t:1st , 5th , 10th ,100th ,1000th , 10 000th , 1000 000th terms of the sequence in decimal form to 6d.p. and make a good guess of the limit of the sequence. b. Sketch the graph of f(n) =$\frac{5-3n}{9n+4}$ c. Verify that the guess (or conjecture) in (a) is correct by using the definition of the limit.	
4	For the following functions in real numbers: (a) f(x)=$\begin{cases} 1 \; for \; x \; irrational \\ -1 \; for \; x \; rational \end{cases}$ (b) f(x) = $\begin{cases} \frac{x-\lvert x\rvert}{x} \; if \;\; x \neq 0 \\ 2 \;\; if \;\; x = 0 \end{cases}$ (c) f (x) = |x|-x • Make your own representations of the above functions on paper • Discuss the kind of discontinuity if any of the above functions	
5	Consider the graph of the function f below	

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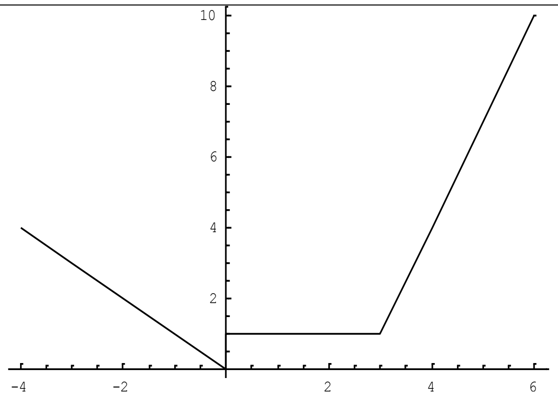

Figure 1. A conceptual task on continuity, differentiability and integration

a. Describe the behavior of the graph of f
b. At which points is f differentiable? Differentiable to the right? To the left?
c. Describe the behavior of the graph of the function h defined by h(x) = $\int_{-4}^{x} f(t)dt$
d. Compute $\int_{-4}^{6} f(x)dx$
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Mean Value Theorem for Derivatives
Suppose that f is continuous on the interval [a,b] and differentiable on the interval (a,b) . Then there exists a number c ϵ [a,b] such that $f^{/}(c) = \frac{f(b)-f(a)}{b-a}$.

• Represent on paper the meaning of the above theorem
• Prove the theorem
• If f(x) = $\begin{cases} 1 \; if \; x \; is \; irrational \\ -1 \; if \; x \; is \; rational \end{cases}$, investigate whether (i) f(x) and (ii) |f(x)| satisfy the above theorem
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<td>7</td>
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For the inequality ln(1+x) < x
• Make a representation of the given inequality
• Prove the above inequality.
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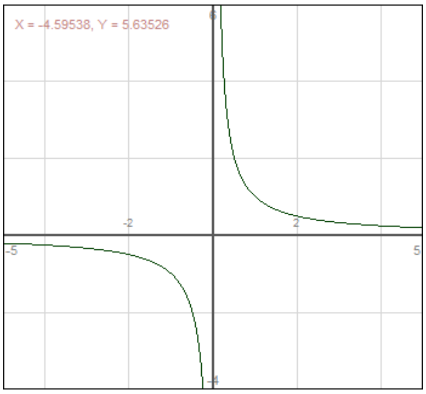
8	**Mean Value Theorem for Integrals** Suppose that f is continuous on [a,b] , then there is a number $c \in (a,b)$ for which f(c) = $\frac{1}{b-a}\int_a^b f(x)dx$. • Make your own representation on paper of the meaning of the above theorem • Prove the theorem • Find the (i) estimate value and (ii) actual value of $\int_0^2 \sqrt{4-x^2}\,dx$	
9	**Concepts involving infinity** 1.By use of relevant diagram or otherwise show that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{32}+$………… = 1 2. Consider the graph of the function y= $\frac{1}{x}$ below  Deduce $from\ the\ graph\ \int_{-\infty}^{\infty}\frac{1}{x}dx$	

APPENDIX THREE: COGNITIVE TEST PART2

NAME OF STUDENT: --REGNO-------------------------

Mathematical Processing Instrument CognitiveTest Multivariablecalculus concepts

Mathematical Processing Questionnaire Part 2

INSTRUCTIONS

2. For each problem please show your working as much as you can.
3. Attempt all the problems given. Do not use scientific calculators,computers, textbooks or class notes. We are interested in your own mathematical thinking.

1. Answer all questions on spaces provided.

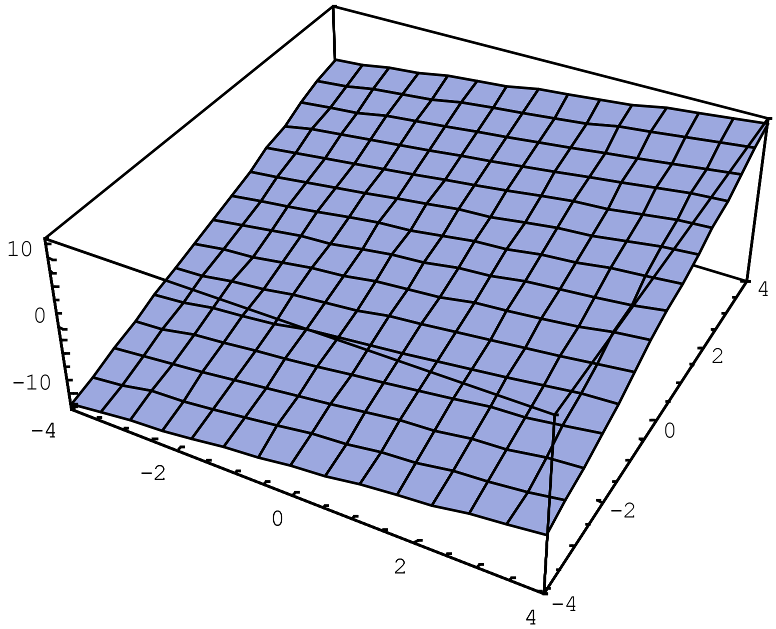
Function/equation/inequality	Describe the region corresponding to the given equation/inequality in R^3	Sketch the graph of the region
(e) X=9		
(f) $X^2 + y^2 + z^2 = 1$		
(g) $1 < X^2 + y^2 + z^2 < 25$		
(h) X+y+z=1		

2. Match the function with its graph and level curves. Give reasons for your choices.

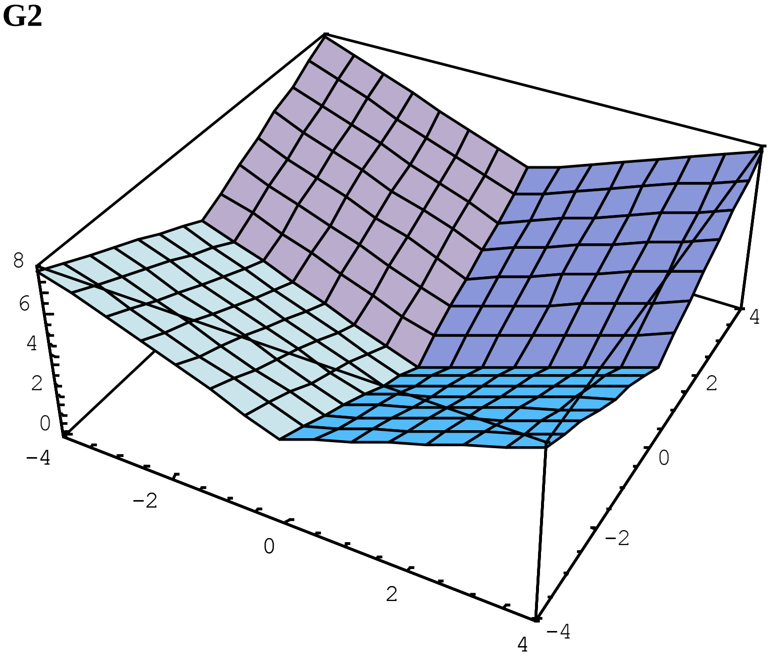
FUNCTION (F1-F4)	GRAPH (G1-G4)	LEVEL CURVE (L1-L4)	GIVE REASONS TO YOUR ANSWER
F1			
F2			
F3			
F4			

F1 .f(x,y) =x + 2y **F2:** f(x,y) = $x^2 + y^2$**F3.** f(x,y) = $\frac{1}{x^2+y^2+1}$ **F4.** f(x,y) = $|x| + |y|$

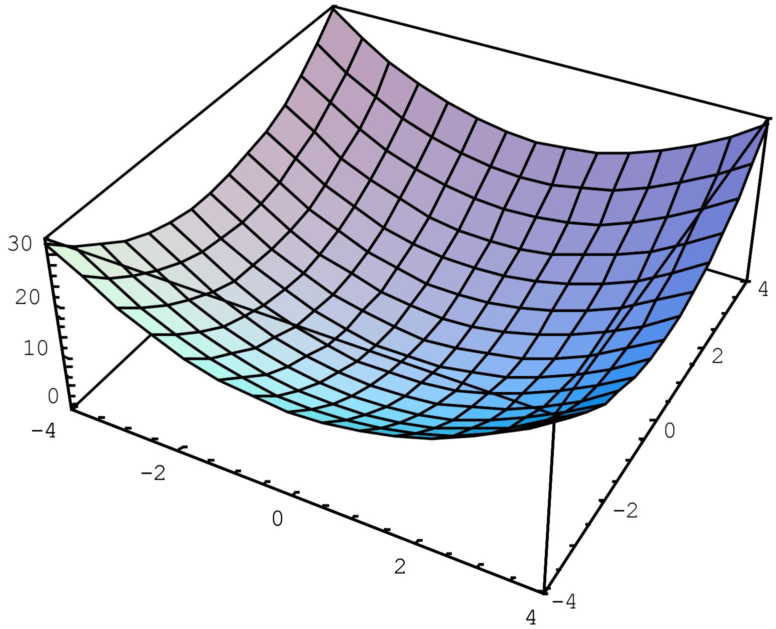
G1



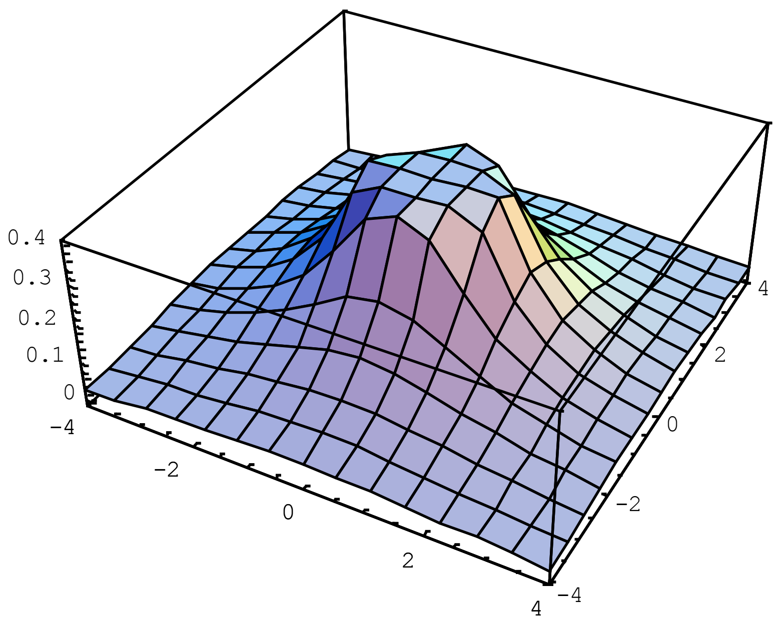
G2



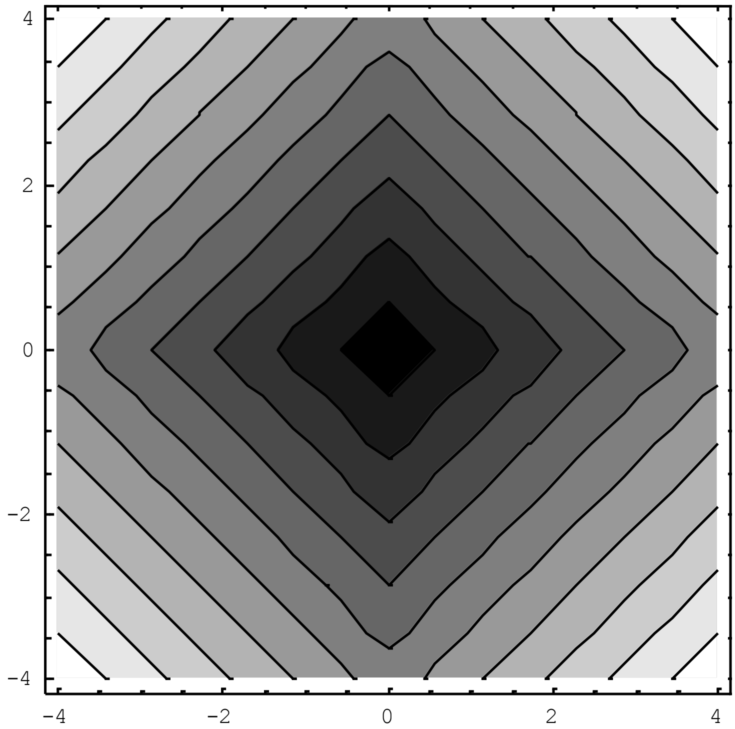
G3



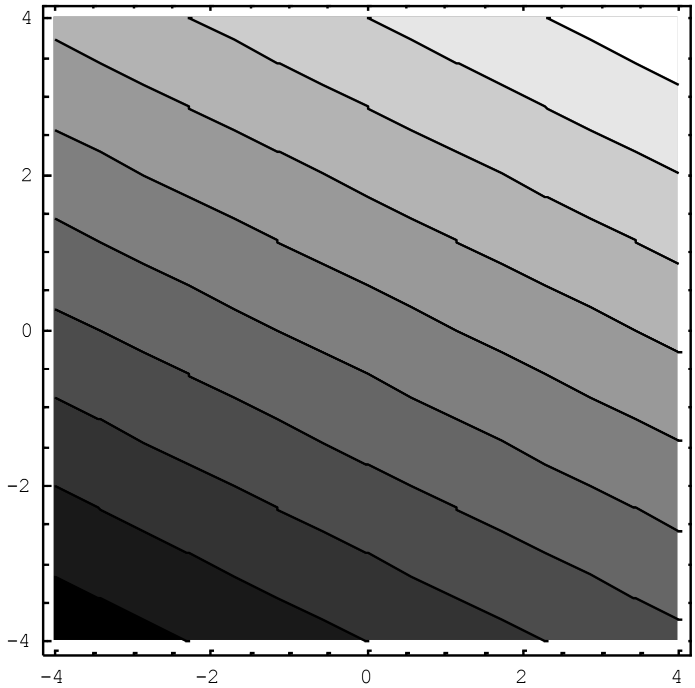
G4



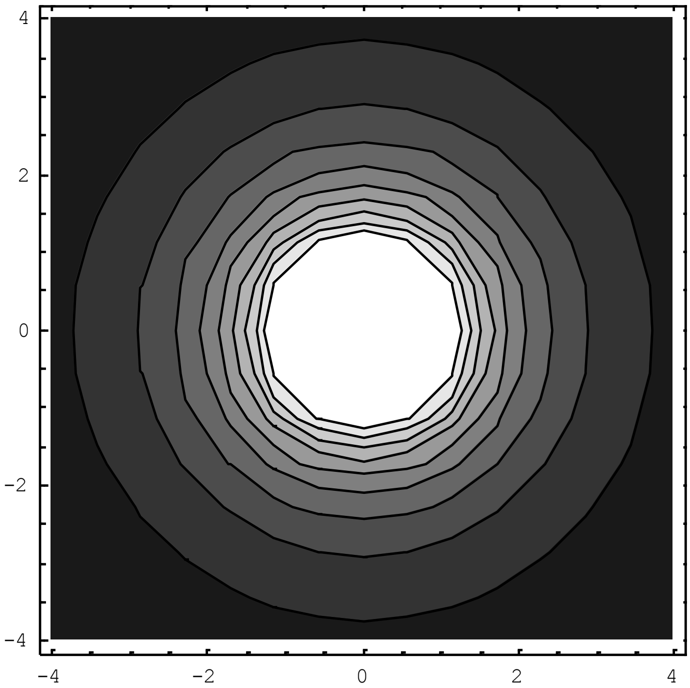
L1



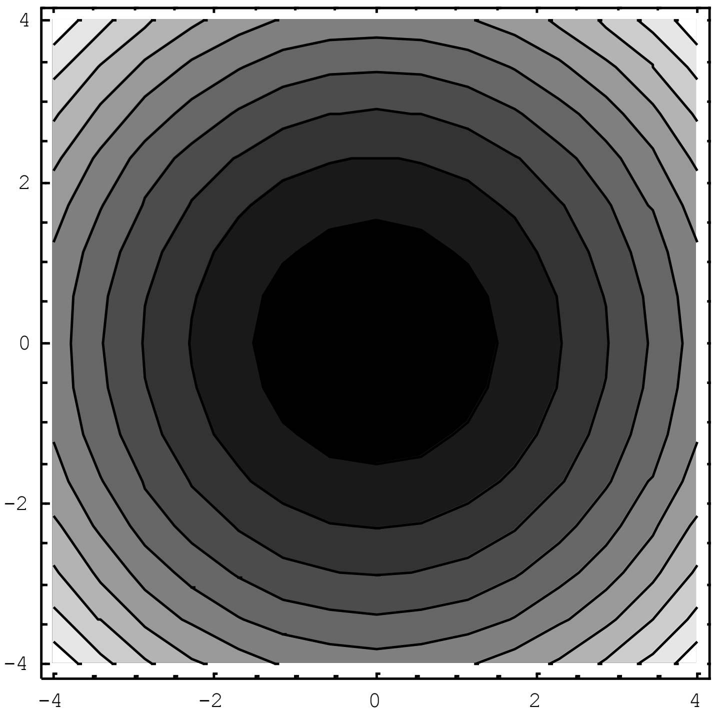
L2



L3



L4



3.Double Integrals

(a) Find the volume of the solid that lies under the paraboloid $z= x^2 +y^2$ and the region D in the xy-plane bounded by the line $y=2x$ and the parabola $y = x^2$.

(b)Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

END OF QUESTIONNAIRE

APPENDIX FOUR: CLINICAL INTERVIEW / TASK- BASED INTERVIEW INSTRUMENT

MOST PROBABLE INTERVIEW QUESTIONS TO BE ASKED BY THE RESEARCHER AFTER A PROBLEM IS SOLVED

- Are these problems new to you?
- Have you ever solved problems like this in class?
- Did the answer to this problem just come in your head, or did you have to think about it for a while?
- Can you explain how you worked it out? [If yes, there shall some follow-up questions]
- [If no] Did you try to make a mental picture of the problem?
- Can you describe the mental picture?
- Did you rely much on your diagram / mental picture when you were solving the problem?

END OF QUESTIONNAIRE

APPENDIX FIVE: CONSENT FOR PARTICIPATION IN RESEARCH (STUDENT COPY)

Dear Student,

You are a BEdMathematics student enrolled with Great Zimbabwe University. Your group is selected to participate in a study designed to determine how undergraduate students process mathematical information in a selected area of Mathematical Analysis. The information obtained will help improve mathematical analysis courses taken by future students.

If you choose to participate you will be asked to complete two questionnaires in Semester two. You may also be interviewed after completion of the two questionnaires. Completion of the questionnaire can take about three hours. Some students will be contacted to participate in one-hour interviews.

Confidentiality will be guaranteed at every stage of the study. After the data are collected all links between your name and Registration number and scores or information provided will be destroyed to guarantee anonymity.

Reliability and validity on how student's process information is found by involving as many students as possible. However, you have the right to choose not to participate in this study, and this has nothing to do with your final grade in the class. Additional information concerning this study may be obtained from Mr.JonathanMuzangwa (0775130616).

I---consent to participating in research entitled: **Imagery and visualisation characteristics of undergraduate students' thinking processes in learning selected concepts of Mathematical Analysis.** I also give my permission to the Department of Curriculum Studies to release my scores in this class to the researcher for the solely purpose of this study as the researcher is going to maintain confidentiality as mentioned above.

The researcher , Mr.JonatanMuzangwa has explained the purpose of the study , the procedures to be followed , the benefits of the study and the duration of my participation .I acknowledge that I have had the opportunity to obtain additional information regarding the study and that any questions I have raised have been answered to my full satisfaction. I also understand that I am free to withdraw consent at any time and to discontinue participation in the study without prejudice to me or to my grade in the class.

Finally, I acknowledge that I have read and fully understand the consent form. I sign it freely and voluntarily. A copy has been given to me.

Signature-------------------------------------Date----------------Signature---------------------------------- ------------

(Participant) (Researcher)

APPENDIX SIX:LETTER OF PERMISSION

Registrar
Ms S. Tirivanhu
P O Box 1235
MASVINGO
Tel: 039-252100
Fax: 039-252100

Off Old Great Zimbabwe Road
MASVINGO
E mail: stirivanhu@gzu.ac.zw

GREAT ZIMBABWE UNIVERSITY

24 March 2014

Mr. J. Muzangwa
Great Zimbabwe University
P.O. Box 1235
MASVINGO

Dear Mr. Muzangwa

REF: REQUEST FOR PERMISSION TO CONDUCT A RESEARCH WITH UNIVERSITY STUDENTS AS PARTICIPANTS

The above matter refers.

This is to confirm that your request has been approved, but please note that we would request a copy of your findings too.

Wishing you good luck in your studies.

Sincerely

24 MAR 2014

S. Tirivanhu (Ms)
Registrar

APPENDIX: SEVEN

Table 3.2 : Marking Guide

			Criteria for each task		
Task	Concepts	Subtasks	Ability to form relevant images	Ability to deploy images	Types of images
1	Supremun and Infimum	• Sketch graph • Proof • Finding Sup &Inf	• Interval of real numbers	• Sequence of numbers	• Diagram • Table of values
2&3	Sequence	• Sketch graph • Table of values	• Sketch graph • Table of values	• Use graph to make inferences	• Diagram • Table of values
4&5	Functions	Sketch graph	• Tableof values • Algebraic symbols	• Sketching graph using images	• Diagram / graph
	Continuity	Describe Continuity	• graph	• Intuitive definition • Jump points	• Prototype
	Differentiatio	Describe	• Points with	• Use of	• Diagram

	n	differentiability	jumps and sharp curves	lateral derivative	
	Integration	Evaluate Integral	• Regular shapes	• Area of regular shapes	• Diagram
6&8	Mean Value Theorem for derivatives and integrals	Geometrical meaning	• Cartesian plane • Graph of f(x) • Gradient at a point • Images of limit points • Area under a curve	• Use the images to relevant diagrams of MVT for derivatives and integrals	• Diagram • Algebraic symbols
		Proof	• Identifying a function h(x) =f(x) –y • Identifying the use of the IVT	• Use of Rolle's theorem on h(x) • Use of IVT on areas under a curve	• Diagram • Algebraic symbols

		Application	• Graphs of the given functions	• Use the premises of the MVT	• Diagrams • Algebraic symbols
3	Geometry of space	Description in own words	• Mental pictures which characterise a plane , ellipsoid , paraboloid	• Words which describe the mental pictures	• Metaphorical images
		Sketch graph	• Mental pictures which characterise a plane , ellipsoid , paraboloid	• Accurate sketch graphs of the equations	• Diagrams
4	Functions of several variables	Matching	• Mental pictures of graph in space and graph on a plane surface	• Correct matching of graph and its level curve	• Diagrams
		Give reasons	• Mental pictures of graph in space	• Correct identification of	• Diagrams • Metaphor

			and graph on a plane surface	common features	ical images
5	Multiple Integration	Sketch of region	• Sketches of regions of integration	• Correct identification of method of solution	• Diagrams
		Finding Volume	• Images of formula to apply	• Correct formula and limits for integration	• Algebraic symbols

The key below was used to code the participants' thinking processes.

Table 3.3: KEY:Codes for thinking processes

Correct relevant images	RI	2
Partially correct images	PI	1
No images	NI	0
Correct deployment of images	DI	2
Partially correct deployment of images	PD	1
No attempt	NA	0