

Implication in three-valued logics of partial information

Katarina Britz

Department of Computer Science and Information Systems, University of South Africa, Pretoria
britzk@unisa.ac.za

Abstract

In formal logic, both semantic entailment and the conditional connective are used to formalize the intuitive notion of implication. The former is defined in the meta-language of the logic, and the latter in the language of the logic. Their interaction determines to what extent the conditional connective relates to entailment as an implication should. This paper addresses this question for a number of related three-valued logics based on Kleene's strong truth tables, and defines a suitable implication for Partial Logic.

Keywords: paraconsistency; partial information; semantic consequence; three-valued logic.

Computing Review Categories: F.4.1 F.3.1. F.3.2

1 Introduction

The strong three-valued truth tables of Kleene [11] were motivated by the undeterminedness of certain propositions. This has led to the definition of a number of three-valued logics based on these truth tables, some of which include additional connectives, and employing different notions of semantic consequence. The advent of the computer and information sciences gave rise to a number of new applications of three-valued logics, for example in program verification [4], formal specification systems [9], operational semantics of process algebra [5] and logic program semantics [12].

Avron [3] characterizes the consequence relations and conditional connectives of a number of three-valued logics. The three-valued conditional connectives of Monteiro [15], Lukasiewicz [13], da Costa [7] and Sobociński [17] are shown to be *internal implications* for a family of closely related three-valued logics. Informally, an internal implication is a conditional connective which corresponds closely to semantic entailment; this correspondence is made precise in the next section. The availability of an internal implication makes it possible to interleave proofs at the meta- and object-level, since statements about entailment and sentences phrased in terms of internal implication are inter-translatable.

In this paper, I consider the problem of defining an internal implication for Partial Logic [6], a three-valued logic which is used in a number of computing applications. The paper also presents a unified introduction to propositional three-valued logics based on Kleene's strong truth tables, as matrix consequence relations.

2 Preliminaries

A standard way to define a semantic consequence relation employs the notion of a logical matrix [18]. In its simplest form, a logic is determined by a single matrix which consists of an abstract algebra of the same similarity type as the language of the logic (viewed as free algebra generated by the sentential symbols in the language), and a set of designated elements. For example, the determining matrix of classical propositional logic is the tuple $\langle \mathcal{B}_2, \{t\} \rangle$, where \mathcal{B}_2 is the two-element Boolean algebra with base set $\{t, f\}$ and operations \wedge (meet) \vee (join) and \neg (complement). The similarity type of \mathcal{B}_2 is $(2, 2, 1)$, indicating that meet and join are binary operations, and complement is a unary operation. On the other hand, the connectives \wedge , \vee and \neg can be chosen as primitive connectives of classical propositional logic. The remaining connectives, such as material implication, can be defined in terms of these primitive connectives: $\phi \supset \psi \equiv_{\text{DEF}} \neg\phi \vee \psi$. The similarity type of the language of classical propositional logic, viewed as an abstract algebra \mathcal{L} , is therefore also $(2, 2, 1)$. Valuations are defined as mappings from \mathcal{L} to \mathcal{B}_2 that preserve the operations \wedge , \vee and \neg . Let v be a valuation. Then, for any propositions ϕ and ψ ,

$$\begin{aligned}v(\phi \wedge \psi) &= v(\phi) \wedge v(\psi) \\v(\phi \vee \psi) &= v(\phi) \vee v(\psi) \\v(\neg\phi) &= \neg v(\phi)\end{aligned}$$

The operations occurring here on the left-hand of the equality symbols are from \mathcal{L} , while those occurring on the right-hand of the equality symbols are from \mathcal{B}_2 . Formally, matrix consequence relations are defined as follows:

Definition 2.1 Let $M = \langle \mathcal{A}, D \rangle$ be a logical matrix, and L a propositional language with associated free algebra \mathcal{L} of same similarity type as \mathcal{A} . The matrix

consequence relation \models_M determined by L and M is defined as follows: For any $\Gamma \subseteq L$, $\phi \in L$ and homomorphism $v : \mathcal{L} \rightarrow A$,

$$\Gamma \models_M \phi \text{ iff } v(\Gamma) \subseteq D \text{ implies } v(\phi) \in D.$$

More generally, a logic is determined by a class of matrices \mathcal{M} . In this paper, the elements of \mathcal{M} will differ only in their sets of designated elements, while sharing the same abstract algebra. The matrix consequence relation determined by \mathcal{M} is defined as the intersection of the matrix consequence relations determined by each matrix.

Definition 2.2 Let $\mathcal{M} = \{\langle \mathcal{A}, D_i \rangle : i \in I\}$ be a class of matrices, with \mathcal{A} of same similarity type as \mathcal{L} . Then

$$\Gamma \models_{\mathcal{M}} \phi \text{ iff for every } M \in \mathcal{M}, \Gamma \models_M \phi.$$

In a logic determined by a single matrix, a valuation v which assigns a designated value to each element of a formula set Γ , is called a *model* of Γ . A proposition ϕ is called a *logical theorem* if it always takes on a designated value. Γ is called a *theory* if $\Gamma = Cn(\Gamma) = \{\phi \mid \Gamma \models \phi\}$. In a logic determined by a class of matrices $\mathcal{M} = \{\langle \mathcal{A}, D_i \rangle : i \in I\}$, v is a model of Γ if it assigns a designated value to each element of Γ in each determining matrix $\langle \mathcal{A}, D_i \rangle$. A theorem is similarly defined as a proposition which takes on a designated value in each matrix $\langle \mathcal{A}, D_i \rangle$.

The semantic consequence relation of a logic determines the consequences of a given theory on a meta-level. On the other hand, the conditional connective of a logic usually determines the consequences of a premiss on the object-level. It is therefore natural to ask to what extent the behaviour of an implication connective mirrors the behaviour of the semantic consequence relation. The following two rules test the extent of this correspondence:

$$\begin{aligned} \text{If } \Gamma, \alpha \models \beta \text{ then } \Gamma \models \alpha \rightarrow \beta & \quad (\text{Deduction}) \\ \text{If } \Gamma \models \alpha \rightarrow \beta \text{ then } \Gamma, \alpha \models \beta & \quad (\text{Modus Ponens - MP}) \end{aligned}$$

Following [3], we call an implication connective in a logic an *internal implication* if the rules of Deduction and MP hold.

3 Kleene Logic

The language L of propositional Kleene Logic is built in the standard way from a denumerable set of sentential symbols, and connectives \wedge , \vee and \neg . In this paper, lower case Greek symbols denote propositions in L , and upper case Greek symbols denote sets of propositions. The characteristic abstract algebra for (strong) Kleene Logic is the Kleene lattice $\mathcal{K}_3 = \{\{t, u, f\}, \wedge, \vee, \neg\}$, with base set $\{t, u, f\}$ and operations \wedge , \vee and \neg defined by the following tables:

\wedge	t	u	f	\vee	t	u	f	\neg	t	f
t	t	u	f	t	t	t	t	t	t	f
u	u	u	f	u	t	u	u	u	u	u
f	f	f	f	f	t	u	f	f	f	t

Let x and y denote arbitrary elements of \mathcal{K}_3 . The partial order associated with \mathcal{K}_3 is the *truth order* defined by: $x \leq y$ iff $x = x \wedge y$.

The algebra \mathcal{K}_3 is of the same similarity type as \mathcal{L} . Any assignment of elements of \mathcal{K}_3 to sentential symbols can therefore be extended to a homomorphism $v : \mathcal{L} \rightarrow \mathcal{K}_3$. Let Val_3 denote the set of all such valuations.

The strong truth tables of Kleene do not fix a unique logic. In order to do that, we also need a syntactic notion of derivability, or a semantic consequence relation. The abstract algebra \mathcal{K}_3 forms the basis of any definition of matrix consequence based on Kleene's strong truth tables. Designating only t yields the logic usually referred to as Kleene Logic, abbreviated KL, with determining matrix $\langle \mathcal{K}_3, \{t\} \rangle$ and the following semantic consequence relation:

Definition 3.1 Given any $\Gamma \subseteq L$ and $\phi \in L$, $\Gamma \models_{KL} \phi$ iff $\forall v \in Val_3, v(\Gamma) \subseteq \{t\}$ implies $v(\phi) \in \{t\}$.

MP holds in Kleene Logic, but the Deduction rule does not. Kleene Logic also has the peculiar property that it does not have any logical theorems. In particular, because the law of the excluded middle, $\models_{KL} \phi \vee \neg\phi$, does not hold, material implication is not an internal implication. More generally, it is not possible to define any internal implication in Kleene Logic. For if \rightarrow is intended as such a conditional, then the Deduction rule should hold. Since $\phi \models_{KL} \phi$, this would imply that $\phi \rightarrow \phi$ is logical theorem.

A logic in which all truth functions are obtainable, is called *expressively complete*. Kleene Logic can be made expressively complete by the addition of a unary nonmonotonic *definedness* connective Δ , to obtain the Logic of Partial Functions LPF [4]:

Δ	t	f
t	t	f
u	f	f
f	t	f

The following connective, here called Monteiro implication, forms the unique internal implication for LPF [15, 3]:

$$x \rightarrow_M y = \begin{cases} y & \text{if } x = t; \\ t & \text{otherwise.} \end{cases}$$

\rightarrow_M and Δ are interdefinable, for example \rightarrow_M can be defined in LPF by:

$$\phi \rightarrow_M \psi \equiv_{DEF} \neg \Delta \phi \vee \neg \phi \vee \psi.$$

4 Logic of Paradox

Contemporary paraconsistent logic dates back to the work of Lukasiewicz [14], Vasil'ev and Jaškowski [10]. The proof-theoretic aim of paraconsistent logics is to provide a framework for reasoning about systems that may be inconsistent. Formalisms such as theory change deal with inconsistencies in knowledge bases by avoiding them, and by removing them once they are located. Paraconsistent logics, on the other hand, reason non-explosively in the presence of inconsistencies. In classical logic, a theory is consistent if and only if it has a model. The trademark of paraconsistent logics is that inconsistent theories can have models.

The paraconsistent Logic of Paradox was introduced to accommodate the logical paradoxes [16]. A sentence in the Logic of Paradox can be either true (and not false), or false (and not true), or paradoxical (both true and false). This yields a three-valued logic based on Kleene's truth tables, but in which the third truth value indicates that a sentence is paradoxical, as opposed to being undefined or undetermined. Since a paradoxical sentence is true (as well as false), the third truth value in the matrix for the Logic of Paradox is also designated.

That is, designating both t and u in the abstract algebra \mathcal{K}_3 yields the Logic of Paradox LP, with determining matrix $\langle \mathcal{K}_3, \{t, u\} \rangle$ and semantic consequence relation defined as follows:

Definition 4.1 *Given any $\Gamma \subseteq L$ and $\phi \in L$, $\Gamma \models_{LP} \phi$ iff $\forall v \in Val_3$, $v(\Gamma) \subseteq \{t, u\}$ implies $v(\phi) \in \{t, u\}$.*

The Logic of Paradox has the same logical theorems as classical logic. The Deduction rule holds, but MP does not [16].

The Logic of Paradox has been studied extensively, and under different names. Gibbins [9] gives a three-valued semantics to the sequent calculus LPP2 in which the third truth value indicates undefinedness and with consequence relation defined as follows:

Given any $\Gamma \subseteq L$ and $\phi \in L$, $\Gamma \models_{LPP2} \phi$ iff $\forall v \in Val_3$, $v(\phi) \in \{f\}$ implies $v(\Gamma) \cap \{f\} \neq \emptyset$.

This relation clearly coincides with that of Definition 4.1. On this reading, $\Gamma \models_{LP} \phi$ iff it is either impossible that all elements of Γ become true through an increase in information, or it is possible that ϕ may become true through an increase in information.

As in the case of Kleene Logic, the Logic of Paradox can be uniquely enriched with an internal implication, here called da Costa implication [7, 3]:

$$x \rightarrow_c y = \left\{ \begin{array}{l} t \text{ if } x = f; \\ y \text{ otherwise.} \end{array} \right\}$$

5 Partial Logic

Definitions 3.1 and 4.1 both ignore to some extent the truth order on the elements of \mathcal{K}_3 . The former regards u and f as equally untrue, while the latter regards u

and t as equally true. The truth order gives an indication of how close a sentence is to being true or false. This information is partly lost if a simple partition between designated and non-designated elements of the algebra is used. The truth order of a logic can be used in one of two ways: either define a corresponding conditional connective in the logic, or use it at the meta-level in the definition of semantic consequence. The former is used in the definition of the Lukasiewicz conditional [13], which can also be defined in LPPF:

$$\phi \rightarrow_L \psi \equiv_{DEF} (\phi \rightarrow_M \psi) \wedge (\neg\psi \rightarrow_M \neg\phi)$$

Or, equivalently,

$$\phi \rightarrow_L \psi \equiv_{DEF} \neg\phi \vee (\neg\Delta\phi \wedge \neg\Delta\psi) \vee \psi$$

The truth order is used in that $v(\phi \rightarrow_L \psi) = t$ iff $v(\phi) \leq v(\psi)$.

The second option mentioned above is to use the truth order at the meta-level in the definition of a semantic consequence relation. This defines a three-valued logic based on Kleene's strong truth tables, Partial Logic, abbreviated PL [6]. The determining matrices of KL and LP are $\langle \mathcal{K}_3, \{t\} \rangle$ and $\langle \mathcal{K}_3, \{t, u\} \rangle$ respectively. Unlike these logics, Partial Logic is not determined by a single matrix, but by both these matrices. Further, the sets $\{t\}$ and $\{t, u\}$ are precisely the proper filters of \mathcal{K}_3 (i.e. the proper subsets of \mathcal{K}_3 that are upwardly closed under \leq). Partial Logic is therefore defined as the matrix consequence relation obtained from the following set of matrices:

$$\mathcal{M}_{PL} = \{ \langle \mathcal{K}_3, f \rangle : f \text{ is a proper filter of } \mathcal{K}_3 \}$$

Definition 5.1 *Given any $\Gamma \subseteq L$ and $\phi \in L$, $\Gamma \models_{PL} \phi$ iff $\Gamma \models_{KL} \phi$ and $\Gamma \models_{LP} \phi$.*

This consequence relation is discussed in [6], and is used in [9] to give semantics to the sequent calculus LPP3. Neither MP nor the Deduction rule hold in Partial Logic. Like Kleene Logic, Partial Logic has no logical theorems, in particular, the law of the excluded middle does not hold, and like the Logic of Paradox, Partial Logic is not explosive.

There are several paraconsistent logics based on Kleene's strong truth tables. Examples are Sobociński logic [17], the Logic of Paradox [16], the relevance logic RM3 [1, 8], and Partial Logic [6]. With the exception of Partial Logic, these logics are all determined by a single three-valued matrix in which the third truth value is designated. In fact, no logic determined by a single three-valued matrix in which the non-classical value is not designated, is paraconsistent [2]. For a three-valued logic determined by a class of matrices to be paraconsistent, it is necessary that the non-classical value be designated in at least one determining matrix. (Of course, their direct product is not three-valued.)

We now address the problem of defining an internal implication for Partial Logic. It is not difficult to check that the only binary operation in \mathcal{K}_3 which

does not violate either MP or the Deduction rule with respect to \models_{PL} , is defined by:

\rightarrow	t	u	f
t	t	u	f
u	t	t	f
f	t	t	t

For example, $t \rightarrow u \neq f$ by Deduction, and $t \rightarrow u \neq t$ by MP. Similarly, $u \rightarrow f = f$ by MP, and $u \rightarrow u = t$ by Deduction. (Remember that \models_{PL} corresponds to \leq in \mathcal{K}_3 .) Equivalently,

$$x \rightarrow y = \left\{ \begin{array}{l} t \text{ if } x \leq y \\ y \text{ otherwise} \end{array} \right\}$$

Call the logic obtained by adding this internal implication to Partial Logic PL_{\rightarrow} . As in the case of Monteiro implication, \rightarrow can be defined in terms of the definedness connective Δ :

$$\phi \rightarrow \psi \equiv_{\text{DEF}} (\neg\phi \vee \neg\Delta\phi \vee \psi) \wedge (\neg\Delta\psi \vee \Delta\phi \vee \psi)$$

Conversely, taking \rightarrow as primitive connective, we have:

$$\Delta\phi \equiv_{\text{DEF}} \neg((\phi \rightarrow \neg\phi) \wedge (\neg\phi \rightarrow \phi))$$

Since both Monteiro and da Costa implications make Kleene logic expressively complete, \rightarrow can be defined in terms of either of these connectives, but these definitions are not simple or intuitively obvious.

We can also define an equivalence connective \leftrightarrow in terms of \rightarrow :

$$\phi \leftrightarrow \psi \equiv_{\text{DEF}} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi).$$

This equivalence connective has the desirable property that $\phi \leftrightarrow \psi$ is a logical theorem if and only if ϕ and ψ always takes on the same truth value: $v(\phi \leftrightarrow \psi)$ always takes on a designated value (in each matrix) iff $(\forall v \in \text{Val}_3, v((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)) = t \text{ iff } v(\phi \rightarrow \psi) = t \text{ and } v(\psi \rightarrow \phi) = t \text{ iff } v(\phi) \leq v(\psi) \text{ and } v(\psi) \leq v(\phi) \text{ iff } v(\phi) = v(\psi))$. This property is lacking in the internal implications of Kleene Logic and the Logic of Paradox. It is shared by the equivalence connective obtained from Łukasiewicz implication in Kleene Logic, but the latter is not an internal implication.

Unlike Partial Logic, PL_{\rightarrow} has logical theorems. Since the truth tables of all the connectives in the logic agree on the classical truth values $\{t, f\}$, these are also classical logical theorems. Of course, not all classical logical theorems are logical theorems of PL_{\rightarrow} , since each logical theorem must contain at least one implication.

The addition of the internal implication \rightarrow to Partial Logic yields a logic with the same expressive power as is yielded by the addition of the definedness connective Δ . Whether \rightarrow is a primitive or derived connective is not the issue, but rather that, being an internal

implication, it relates closely to the semantic consequence operation $\models_{PL_{\rightarrow}}$, thus providing an implication corresponding to semantic entailment on the object-level. This naturally raises the question of what benefit an internal implication is to applications of a given logic. For example, does it significantly simplify proofs about program correctness or specifications? What effect would the addition of an internal implication to Partial Logic have on, say, the semantics of the specification language VDM? Other open questions involve the axiomatization of the internal implication of Partial Logic as a sequent calculus, and its properties. Some of these questions are currently under investigation, and will be addressed in a separate paper.

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