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JULIE 1 – 3 JULY 1987
Proceedings
of the
4th South African Computer Symposium

Holiday Inn, Pretoria
1 – 3 July 1987

edited by

Pieter Kritsinger
Computer Science Department
University of Cape Town
Computer science is an emerging discipline which is having difficulty in being recognised as a worthy member of the sciences. I will paraphrase John Hopcroft, co-winner of the 1986 Turing Award, when, during a recent interview, he said that the primary reason for the lack of recognition, is the age of our researchers. Probably not one of the researchers who presented their work at this symposium is older than 45. I know of no computer scientist in South Africa who is in a position where (s)he can affect funding priorities. As far as I know we have no representation on any of the committees of the Foundation for Research Development and for our Afrikaans speaking fraternity, none who is a member of the Akademie vir Wetenskap en Kuns. It will take time and conscious effort to establish our presence. The same is true of course for our universities. Again, with one exception, I know of no dean of a science faculty, vice-principal or principal who is a computer scientist. We consequently spend an enormous amount of time trying to explain the needs of computer science and its difficulties. I believe this symposium is a further step towards accreditation by our peers and superiors from the other sciences.

The total number of papers submitted to the Programme Committee for consideration was 34. Each paper was reviewed by three persons knowledgeable in the field it represents. Of those submitted, 23 were finally selected for inclusion in the symposium. As a result the overall quality of the papers is high and as a computer science community in Africa we can be justly proud of the final programme.

This is the fourth in the series of South African computer symposia. This year the symposium is sponsored by the Computer Society of South Africa (CSSA), the South African Institute for Computer Scientists and the local IFIP Committee. The executive director of the CSSA and his staff deserve warm thanks for handling the organisation as well as they have, while the Organising Committee provided Derrick and I with very valuable advice.

Finally I would like to express my sincere appreciation to the authors, to the members of the Programme Committee and particularly the reviewers. Without the kind cooperation of everyone, this symposium would not have taken place.

Pieter Kritzinger
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AN APPROXIMATE SOLUTION METHOD FOR
MULTICLASS QUEUEING NETWORKS
WITH STATE DEPENDENT ROUTING
AND WINDOW FLOW CONTROL

Anthony E Krzesinski
Institute for Applied Computer Science
University of Stellenbosch

Abstract

A Multiclass Queueing Network (MQN) \( Q(\vec{N}, \mathcal{M}) \) consisting of \( M \) centres with index set \( \mathcal{M} \) and population vector \( \vec{N} \) is partitioned into two subnetworks \( Q(\vec{N} - \vec{V}, \mathcal{M} - \mathcal{V}) \) and \( Q(\vec{V}, \mathcal{V}) \). The centres in \( Q(\vec{V}, \mathcal{V}) \) are further partitioned into disjoint subnetworks called branches. A set of State Dependent Routing (SDR) probabilities is used to admit customers from \( Q(\vec{N} - \vec{V}, \mathcal{M} - \mathcal{V}) \) into the individual branches of \( Q(\vec{V}, \mathcal{V}) \) such that the customers are preferentially routed to the least congested branches. The SDR probabilities are such that \( Q(\vec{N}, \mathcal{M}) \) has a product form solution.

The SDR probabilities set an upper bound on the number of customers that can concurrently be present in each branch of \( Q(\vec{V}, \mathcal{V}) \). In addition, an upper bound is set on the total number of customers that can concurrently be present in \( Q(\vec{V}, \mathcal{V}) \). Once this bound is reached, the product form SDR probabilities route customers around \( Q(\vec{V}, \mathcal{V}) \).

Exact solutions for MQNs with product form SDR are computationally intractable unless each SDR branch consists of a single centre only. An approximate solution method is therefore developed for MQNs with product form SDR. The approximate solution method is next extended to admit customers in first come first served order to \( Q(\vec{V}, \mathcal{V}) \) when the total number of customers in \( Q(\vec{V}, \mathcal{V}) \) reaches its upper bound. Such a combination of SDR and blocking can be used to model adaptive routing and window flow control mechanisms in computer networks.

MQNs with SDR and blocking violate product form. Simulation solutions are used to test the accuracy of the approximate method. The accuracy of the approximation technique is found to be good. Finally, the effects of population constrained SDR on network performance measures are investigated.

\[ \text{The Institute is sponsored by Reunert Computers (pty) Ltd} \]
1 INTRODUCTION

Queueing networks are important as performance models of computer systems and computer communication networks since the performance of these systems is usually principally affected by contention for resources. A class of queueing networks called product form queueing networks has proved to be particularly suited as computer performance modeling. A queueing network is said to have a product form solution if

\[ P(n_1 \ldots n_M) = X_1(n_1) \ldots X_M(n_M)/G(N) \]

where \( P(n_1 \ldots n_M) \) is the joint queue length distribution for a network with \( M \) service centres, \( X_i(n_i) \) is a factor obtained from the analysis of queue \( i \) in isolation, and \( G(N) \) is a normalisation constant, where \( N = n_1 + \ldots + n_M \) is the total number of customers in the network, and \( n_i \) is the number of customers at centre \( i \).

Since their discovery in the early sixties [4], product form networks have been extended to networks with heterogeneous job classes, with queueing disciplines other than first come first served, with general service time distributions (for certain service disciplines) and with state dependent behaviour.

Efficient computational algorithms have been developed to compute solutions for product form queueing networks. Nonetheless, the computational and storage requirements to solve a product form network with heterogeneous jobs is prohibitively large. In addition, there are many features of real computer systems whose representation in a queueing network results in the queueing network not having a product form solution. Methods have therefore been developed to compute accurate approximate solutions for large product form networks as well as for networks containing non-product form features.

This paper addresses Multiclass Queueing Networks (MQNs) with a certain type of state dependent behaviour, namely State Dependent Routing (SDR). The form of the SDR employed is such that the MQN has a product form solution. These solutions are computationally intractible except for small MQNs and efficient approximate solutions for MQNs with SDR are therefore developed.

SDR is next combined with blocking in order to model a computer communication network with adaptive routing and Window Flow Control. The function of WFC is to limit the total number of requests that can concurrently be present in designated parts of the network thereby avoiding congestion and improving network throughput.

WFC is modelled by imposing concurrency constraints on submodels within the MQN and by queueing customers for admission to the submodels once all the submodels have reached their concurrency limits. The addition of WFC results in a MQN that does not possess a product form solution.

The paper assumes the convolution [2] and Mean Value Analysis (MVA) results [10] for MQNs of BCMP [1] centres. The exact solution method for MQNs with product form SDR [5] is also assumed. The approximate solution method is based
upon the Linearizer technique [3] appropriately extended to service centres with queue dependent service rates [6]. The WFC analysis is based upon the application of the Linearizer method to the solution of MQNs containing population constrained subnetworks [7,9].

Section 2 presents the product form SDR probabilities [5,12]. Some consequences of SDR, namely adaptive routing, concurrency constraints and blocking are discussed. Section 3 presents the joint probability distribution for a closed MQN with SDR. Section 4 summarizes the exact MVA solution of MQNs with SDR. Section 5 presents an approximate solution method for MQNs with SDR. Section 6 extends the approximate solution method to MQNs with SDR and WFC. Section 7 presents a summary of the performance impact of SDR and WFC.

2 STATE DEPENDENT ROUTING

2.1 DEFINITIONS

Let $Q(\bar{N}, M)$ denote a MQN consisting of $M$ BCMP service centres with index set $M = (1 \ldots M)$. Let the customers belong to $J$ closed chains labelled $(1 \ldots J)$. Let $\bar{N} = (N_1 \ldots N_J)$ denote the population vector where $N_j$ is the number of chain $j$ customers. Let $n_{ij}$ denote the number of chain $j$ customers at centre $i$ and let $n_i = \sum_{j=1}^{J} n_{ij}$ denote the total number of customers at centre $i$. The network state descriptor is given by $\bar{N} = (n_1 \ldots n_M)$ where the population vector at centre $i$ is given by $\bar{n}_i = (n_{i1} \ldots n_{iJ})$. Let $\mathbf{1}_j$ denote a unit vector in the $j$ direction.

Let $A - B$ denote the difference of sets $A$ and $B$. Let $A + B$ denote the union of sets $A$ and $B$. Let the set of centre indices $M$ be partitioned into two subsets $M - V$ and $V$ such that the subnetwork $Q(\bar{N}, V)$ contains all the centres which are subject to SDR. Let $Q(\bar{V}, V)$ have a single entry centre labelled $e$ and a single departure centre labelled $z$. Note that the centres $e$ and $z$ are in $M - V$.

Let $M$ also be partitioned into $B$ subsets $\mathcal{B}_1 \ldots \mathcal{B}_B$ where $M - V = \mathcal{B}_1$ and $V = \mathcal{B}_2 + \cdots + \mathcal{B}_B$. The subnetworks $Q(\bar{B}_b, \mathcal{B}_b)$ are referred to as $BRANCHES$. Each branch $Q(\bar{B}_b, \mathcal{B}_b)$ where $2 \leq b \leq B$ has a single entry centre $e_b$ directly connected to $e$ and a single departure centre $z_b$ directly connected to $z$. Let $m_b$ denote the total number of customers in branch $b$ and $V = \sum_{b=2}^{B} m_b$ denote the total number of customers in $Q(\bar{V}, V)$.

Figure 1 presents a schematic representation of a branch $Q(\bar{B}_b, \mathcal{B}_b)$. The centres in the branch (a few of the centres are represented in the figure) are arbitrarily interconnected with the restriction that all customers enter the branch via a single entry centre and depart from the branch via a single departure centre.

Figure 2 illustrates a MQN $Q(\bar{N}, M)$ consisting of four branches. The MQN $Q(\bar{N}, M)$ is partitioned into two subnetworks $Q(\bar{N} - V, M - V)$ and $Q(\bar{V}, V)$. The subnetwork $Q(\bar{V}, V)$ has a single entry centre with index $e$ and a single departure centre with index $z$. The entry centre $e$ is directly connected to the entry centres
Figure 1: The branch $Q(\bar{B}_b, B_b)$

Figure 2: A MQN $Q(\bar{N}, M)$ consisting of four branches
of the branches in \( Q(\vec{V}, V) \). The departure centres of the branches in \( Q(\vec{V}, V) \) are directly connected to the departure centre \( z \), described in section 2.1 above.

### 2.2 THE STATE DEPENDENT ROUTING PROBABILITIES

Consider a MQN whose centres are arranged into a set of branches as described in section 2.1 above. The following State Dependent Routing (SDR) and State Independent Routing (SIR) probabilities are used to describe the routing of the customers:

1. from the entry centre \( e \) of \( Q(\vec{V}, V) \) to the entry centres of the various branches in \( Q(\vec{V}, V) \)
2. among the centres within each branch and eventually to the departure centre \( z \) from \( Q(\vec{V}, V) \)
3. from the departure centre \( z \) to the centres of \( Q(\overline{N} - \vec{V}, M - V) \) and ultimately to the entry centre \( e \) of \( Q(\vec{V}, V) \).

The remainder of this section describes the routes (1), (2) and (3) in detail.

1. The SDR probability of a chain \( j \) customer, upon completing service at the entry centre \( e \) of \( Q(\vec{V}, V) \), proceeding directly to the entry centre \( e_b \) of branch \( b \) where \( 2 \leq b \leq B \) is given by

\[
P_{e_j e_b j}(\vec{N}) = \begin{cases} 
\delta_b(m_b) / \omega(V) & \omega(V) > 0 \\
0 & \omega(V) = 0 
\end{cases} \tag{1}
\]

The function \( \omega \) is defined to be non-negative and hence the function \( \delta \) is non-negative. The functional forms of the \( \delta \) and \( \omega \) are presented in section 2.3 below.

2. A set of SIR probabilities \( P_{x_j y_j} \) where \( x \) and \( y \) are in \( B_b \) and \( 2 \leq b \leq B \) is used to route the chain \( j \) customers among the centres in branch \( b \). The SIR probability \( P_{e_b l_j} = 1 \) is used to route customers from the departure centre \( e_b \) of branch \( b \) to the departure centre \( z \) of \( Q(\vec{V}, V) \). Transitions from one branch \( b \) to another branch \( f \) where \( 2 \leq b, f \leq B \) are not permitted.

3. A set of SIR probabilities \( P_{x_j y_j} \) where \( x \) and \( y \) are in \( M - V \) is used to route the chain \( j \) customers among the centres in \( Q(\overline{N} - \vec{V}, M - V) \).

It can be shown [5] that the global balance equations for a MQN with SDR and SIR probabilities as described above have a product form solution. SDR probabilities of the type presented in equation 1 above are therefore sufficient for a product form solution. It remains to be shown that equation 1 is necessary for the existence of
product form solutions. SOR probabilities other than equation 1 can exist which also yield product form solutions. However, in the context of this paper a SOR probability will refer to a probability of the type defined in equation 1.

Note that the product form SOR probabilities presented in equation 1 above are a restricted case of the SOR probabilities presented in [5].

2.3 A FUNCTIONAL FORM FOR SDR PROBABILITIES

It can be shown [5] that, in order for equation 1 to define a probability distribution it is necessary that

$$\delta_b(m_b) = C m_b + d_b \quad \text{and} \quad \omega(V) = \sum_{b=2}^{B} \delta_b(m_b) = CV + D$$

where $C$ and $d_b$ are constants, and $V = \sum_{b=2}^{B} m_b$ and $D = \sum_{b=2}^{B} d_b$. The SDR probabilities 1 can be reduced to a SIR form by setting $C = 0$ so that $P_{ej, eb} = d_b/D$.

2.4 STATE DEPENDENT ROUTING, CONCURRENCY

CONSTRAINTS, BLOCKING

Consider a closed queueing network $Q(N, M)$ consisting of a central server and two peripheral servers. Let the customers all belong to a single customer class. Let $M = \{1, 2, 3\}$ and $V = \{2, 3\}$. Let each centre define a single branch so that $B_1 = \{1\}, B_2 = \{2\}$ and $B_3 = \{3\}$. The central server centre 1 functions as both the entry centre and the exit centre for $Q(V, V)$.

Let $C = -1$ and let $(d_2, d_3) = (1, 2)$. Equation 2 yields the branch population constraints namely $\delta_2(m_2) = 1 - m_2 \geq 0$ so that $m_2 \leq 1$ and $\delta_3(m_3) = 2 - m_3 \geq 0$ so that $m_3 \leq 2$. Equation 2 yields the constraint on $Q(V, V)$ namely $\omega(V) = 3 - V \geq 0$ so that $V \leq 3$. Equation 1 yields the SDR probabilities $P_{1,2}(N) = (1 - m_2)/(3 - V)$ and $P_{1,3}(N) = (2 - m_3)/(3 - V)$.

The values of the SDR probabilities are presented in table 1 for all permissible values of $m_2$ and $m_3$. Note that the customers are preferentially routed to branch 3 until branch 3 is full ($m_3 = 2$) whereupon all customers are routed to branch 2. When branches 2 and 3 are both full ($m_2 = 1, m_3 = 2$) then a customer completing service at centre 1 will be denied entry to $Q(V, V)$ and will be returned immediately to centre 1.

SDR thus provides a product form solution for a restricted type of blocking. Note that SDR does not model blocking in its true sense where the highest priority blocked customer requesting entry into $Q(V, V)$ is admitted as soon as queueing space at the requested centre in $Q(V, V)$ becomes available. Instead, SDR blocking at centre 1 implies the busy form of waiting. After being denied access into $Q(V, V)$, the customer is returned to centre 1, is queued and is serviced before it can again attempt access into $Q(V, V)$. 


product form solutions. SDR probabilities other than equation 1 can exist which also yield product form solutions. However, in the context of this paper a SDR probability will refer to a probability of the type defined in equation 1.

Note that the product form SDR probabilities presented in equation 1 above are a restricted case of the SDR probabilities presented in [5].

2.3 A FUNCTIONAL FORM FOR SDR PROBABILITIES

It can be shown [5] that, in order for equation 1 to define a probability distribution it is necessary that

\[ \delta_b(m_b) = Cm_b + d_b \quad \text{and} \quad \omega(V) = \sum_{b=2}^{B} \delta_b(m_b) = CV + D \]  

(2)

where \( C \) and \( d_b \) are constants, and \( V = \sum_{b=2}^{B} m_b \) and \( D = \sum_{b=2}^{B} d_b \). The SDR probabilities 1 can be reduced to a SIR form by setting \( C = 0 \) so that \( P_{e_j,e_b,j} = d_b/D \).

2.4 STATE DEPENDENT ROUTING, CONCURRENCY CONSTRAINTS, BLOCKING

Consider a closed queueing network \( Q(N, M) \) consisting of a central server and two peripheral servers. Let the customers all belong to a single customer class. Let \( M = \{1, 2, 3\} \) and \( V = \{2, 3\} \). Let each centre define a single branch so that \( \beta_1 = \{1\}, \beta_2 = \{2\} \) and \( \beta_3 = \{3\} \). The central server centre 1 functions as both the entry centre and the exit centre for \( Q(V, V) \).

Let \( C = -1 \) and let \((d_2, d_3) = (1, 2)\). Equation 2 yields the branch population constraints namely \( \delta_2(m_2) = 1 - m_2 \geq 0 \) so that \( m_2 \leq 1 \) and \( \delta_3(m_3) = 2 - m_3 \geq 0 \) so that \( m_3 \leq 2 \). Equation 2 yields the constraint on \( Q(V, V) \) namely \( \omega(V) = 3 - V \geq 0 \) so that \( V \leq 3 \). Equation 1 yields the SDR probabilities \( P_{1,2}(N) = (1 - m_2)/(3 - V) \) and \( P_{1,3}(N) = (2 - m_3)/(3 - V) \).

The values of the SDR probabilities are presented in table 1 for all permissible values of \( m_2 \) and \( m_3 \). Note that the customers are preferentially routed to branch 3 until branch 3 is full \( (m_3 = 2) \) whereupon all customers are routed to branch 2. When branches 2 and 3 are both full \( (m_2 = 1, m_3 = 2) \) then a customer completing service at centre 1 will be denied entry to \( Q(V, V) \) and will be returned immediately to centre 1.

SDR thus provides a product form solution for a restricted type of blocking. Note that SDR does not model blocking in its true sense where the highest priority blocked customer requesting entry into \( Q(V, V) \) is admitted as soon as queueing space at the requested centre in \( Q(V, V) \) becomes available. Instead, SDR blocking at centre 1 implies the busy form of waiting. After being denied access into \( Q(V, V) \), the customer is returned to centre 1, is queued and is serviced before it can again attempt access into \( Q(V, V) \).
\begin{align*}
V & \quad m_2 & \quad m_3 & \quad P_{1,2} & \quad P_{1,3} & \quad P_{1,1} \\
0 & \quad 0 & \quad 0 & \quad 1/3 & \quad 2/3 & \quad 0 \\
1 & \quad 0 & \quad 1 & \quad 1/2 & \quad 1/2 & \quad 0 \\
1 & \quad 1 & \quad 0 & \quad 0 & \quad 1 & \quad 0 \\
2 & \quad 0 & \quad 2 & \quad 1 & \quad 0 & \quad 0 \\
2 & \quad 1 & \quad 1 & \quad 0 & \quad 1 & \quad 0 \\
3 & \quad 1 & \quad 2 & \quad 0 & \quad 0 & \quad 1 \\
\end{align*}

Table 1: State Dependent Routing Probabilities

The remainder of the analysis assumes that \( C \) is negative and that the \( d_b \) are positive. Without loss of generality let \( C = -1 \) so that \( \delta_b(m_b) = d_b - m_b \) and \( \omega(V) = D - V \).

3 PRODUCT FORM SOLUTION

3.1 JOINT PROBABILITY DISTRIBUTION

Consider a MQN with SDR and SIR probabilities as described in section 2 above. It can be shown [5] that the joint probability distribution \( P(\tilde{N}, M) \) is given by

\[
P(\tilde{N}, M) = G^{-1}(\tilde{N}, M) \left[ \frac{1}{\Omega(V)} \right] \prod_{b=2}^{B} \Delta_b(m_b) \prod_{i=1}^{M} f_i(\tilde{n}_i)
\]

where the normalising constant \( G(\tilde{N}, M) \) ensures that the \( P(\tilde{N}, M) \) sum to unity. The SDR terms are given by \( \Omega(V) = \omega(V - 1)\Omega(V - 1) \), \( \Omega(0) = 1 \) and \( \Delta_b(m_b) = \delta_b(m_b - 1)\Delta_b(m_b - 1), \Delta_b(0) = 1 \). The product terms \( f_i(\tilde{n}_i) \) are given by

\[
f_i(\tilde{n}_i) = \left[ \frac{n_{i1}}{\beta_i(n_i)} \right] \prod_{j=1}^{J} \frac{n_{ij}^{n_{ij}}}{n_{ij}!}
\]

where \( \alpha_i(n_i)\mu_{ij} \) is the queue dependent average service rate of a chain \( j \) customer at centre \( i \) and \( \beta_i(n_i) = \alpha_i(n_i)\beta_i(n_i - 1) \), \( \beta_i(0) = 1 \) and \( \gamma_{ij} = \xi_{ij}/\mu_{ij} \) where \( 1/\mu_{ij} \) is the average service demand of a chain \( j \) customer at centre \( i \). The \( \xi_{ij} \) are discussed in the following section.

3.2 VISIT COUNTS

In the case of a MQN with state independent routing, the \( \xi_{ir} \) can be interpreted as the number of visits (to within an arbitrary multiplicative constant) made by
class r customers to centre i. Let $\mathcal{E} = \{e_2, \ldots, e_B\}$ denote the index set of the entry centres to all the SDR branches and let $\mathcal{Z} = \{z_2, \ldots, z_B\}$ denote the index set of the departure centres from the SDR branches. It can be shown [5] that for the entry and departure centres whose indices $i$ are in $\mathcal{E} + Z$, the $\xi_{ij}$ are given by $\xi_{ij} = \xi_{e_j}$ where $e$ is the entry centre to $Q(V, V)$. These $\xi_{ij}$ cannot be interpreted as relative visit counts.

For the entry centre $e$ and the departure centre $z$ from $Q(\bar{V}, V)$, $\xi_{ej} = \xi_{zj}$. For the remaining centres whose indices $i$ are in $M - \mathcal{E} - Z$, $\xi_{ij} = \sum_k \xi_{kj} P_{kji;j} s$ so that each of these $\xi_{ij}$ can be interpreted as the relative visit counts of chain $j$ customers to centre $i$.

4 EXACT SOLUTION OF MQNs WITH SDR

The following section presents the exact analysis of MQNs with product form SDR as described in section 2 above. The analysis assumes that each branch $b$ consists of a single BCMP centre. This is done in order to anticipate the requirements of the approximate analysis where each branch $b$ will be reduced to a single approximately flow equivalent composite centre.

MQNs with product form SDR are solved using a combination of Mean Value Analysis (MVA) and convolution methods. The analysis begins by solving the two subnetworks $Q(N - \bar{V}, M - V)$ and $Q(\bar{V}, V)$ in isolation. These solutions are then combined to solve $Q(N, M)$.

4.1 MVA SOLUTION OF $Q(N - \bar{V}, M - V)$

The centres in $Q(N - \bar{V}, M - V)$ are not subject to SDR. Standard MVA and convolution methods [2,10] are applied to compute the network performance measures.

4.2 MVA SOLUTION OF $Q(\bar{V}, V)$

The centres $b$ in $Q(\bar{V}, V)$ are subject to SDR and a modified version [5] of MVA is applied.

STEP 1: The expected waiting time of a chain $j$ customer at each centre $b$ is given by

$$W_{bj}(\bar{V}, V) = \left[\mu_{bj}[d_b - Q_b(\bar{V} - \bar{y}_j, V)]]^{-1} \times \sum_{n=1}^{d_b} n[\delta_b(n - 1)/\alpha_b(n)] P_b(n-1; \bar{V} - \bar{y}_j, V)\right]$$

where $Q_b(\bar{V}, V)$ is the expected total queue length at centre $b$ and $P_b(n; \bar{V}, V)$ is the queue length distribution at centre $b$. 
**STEP 2:** Let $T_{bj}(\bar{V}, \mathcal{V})$ denote the chain $j$ throughput at centre $b$. The expected chain $j$ queue length at each centre $b$ is given by

$$Q_{bj}(\bar{V}, \mathcal{V}) = T_{bj}(\bar{V}, \mathcal{V})W_{bj}(\bar{V}, \mathcal{V})$$

$$= \xi_{bj}[d_b - Q_b(\bar{V}-\bar{I}, \mathcal{V})]T_j(\bar{V}, \mathcal{V})W_{bj}(\bar{V}, \mathcal{V})$$

Summing over all centres $b$ yields

$$T_j(\bar{V}, \mathcal{V}) = V_j/\sum_{b=2}^{B} \xi_{bj}[d_b - Q_b(\bar{V}-\bar{I}, \mathcal{V})]W_{bj}(\bar{V}, \mathcal{V})$$  \hspace{1cm} (4)

**STEP 3:** The queue length distribution at each centre $b$ is given by

$$P_b(n; \bar{V}, \mathcal{V}) = [\delta_b(n-1)/\alpha_b(n)]\sum_{j=1}^{J} \gamma_{bj}T_j(\bar{V}, \mathcal{V})P_b(n-1; \bar{V}-\bar{I}, \mathcal{V})$$  \hspace{1cm} (5)

Repeated application of equations 3, 4 and 5 yields the $Q(\bar{V}, \mathcal{V})$ performance measures for all population vectors $0 \leq \bar{V} \leq \bar{N}$.

### 4.3 CONVOLUTION SOLUTION OF $Q(\bar{N}, \mathcal{M})$

Let $G(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$ denote the normalising constant for the network $Q(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$. The $G(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$ are computed during the MVA solution of $Q(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$ as follows. For $\bar{N} \geq \bar{V} + \bar{I}_j$

$$G(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V}) = G(\bar{N}-\bar{V}-\bar{I}_j, \mathcal{M}-\mathcal{V})/T_j(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$$

Similarly, let $G(\bar{V}, \mathcal{V})$ denote the normalising constant for the network $Q(\bar{V}, \mathcal{V})$. The $G(\bar{V}, \mathcal{V})$ are computed during the MVA solution of $Q(\bar{V}, \mathcal{V})$ as follows. For $\bar{V} \geq \bar{I}_j$

$$G(\bar{V}, \mathcal{V}) = G(\bar{V}-\bar{I}_j, \mathcal{V})/[T_j(\bar{V}, \mathcal{V})\omega(V - 1)]$$

The unnormalized probability of there being $\bar{V}$ customers in $Q(\bar{V}, \mathcal{V})$ given $\bar{N}$ customers in $Q(\bar{N}, \mathcal{M})$ is given by

$$\hat{P}(\bar{V}, \mathcal{V}; \bar{N}, \mathcal{M}) = G(\bar{V}, \mathcal{V})G(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$$

The normalising constant for the network $Q(\bar{N}, \mathcal{M})$ is given by

$$G(\bar{N}, \mathcal{M}) = \sum_{\bar{V} \in \bar{N}} \hat{P}(\bar{V}, \mathcal{V}; \bar{N}, \mathcal{M})$$

The performance measures of $Q(\bar{N}, \mathcal{M})$ are now computed as follows. The average chain $j$ queue length at each centre $i$ where $i \in \mathcal{M}-\mathcal{V}$ is given by

$$Q_{ij}(\bar{N}, \mathcal{M}) = G^{-1}(\bar{N}, \mathcal{M}) \sum_{\bar{V} \in \bar{N}} Q_{ij}(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})\hat{P}(\bar{N}-\bar{V}, \mathcal{V}; \bar{N}, \mathcal{M})$$
The average chain \( j \) queue length of each centre \( i \) where \( i \in \mathcal{V} \) given by

\[
Q_{ij}(\bar{N}, \bar{M}) = G^{-1}(\bar{N}, \bar{M}) \sum_{\bar{V} \in \bar{N}} Q_{ij}(\bar{V}, \mathcal{V}) \hat{P}(\bar{V}, \mathcal{V}; \bar{N}, \bar{M})
\]

5 APPROXIMATE SOLUTION OF MQNs WITH SDR

The analysis presented above where each branch consists of a single centre is now extended to allow each branch to consist of several BCMP centres. The exact analysis, which is summarized in the Appendix, is computationally intractable except for small MQNs with few closed chains and few customers in each closed chain.

This section presents a computationally efficient approximate solution method for MQNs with SDR into multicentre branches. The analysis initially addresses product form SDR and is extended in section 6 to the non-product form case where customers are queued for admission to the branches in \( Q(\bar{V}, \mathcal{V}) \) once the population \( \bar{V} \) reaches its upper bound.

The approximation method replaces each branch with a flow equivalent composite centre. In the case of a SINGLE CLASS queueing network with population \( \bar{N} \), a total of \( \bar{N} \) values of the throughput at each composite centre must be computed. The resultant queueing network of flow equivalent centres has product form and can be solved by standard methods.

However, the above method cannot be applied to compute exact solutions for queueing networks with MULTIPLE CHAINS. If exact methods are used to reduce each branch to a flow equivalent composite centre, a total of \( \prod_{j=1}^J (N_j + 1) \) values of the throughput at each composite centre must be computed for all possible values of the population vector \( \bar{N} = (N_1 \ldots N_J) \) where \( N_j \) is the number of customers in chain \( j \). Furthermore, the exact analysis of a MQN with SDR whose centres have service rates which depend upon the value of the population vector at the centre is computationally intractable [5].

The approximate solution method for MQNs with product form SDR is based upon an approximate method for solving MQNs containing concurrency constrained subnetworks [7].

5.1 THE ALGORITHM

The approximate solution method constructs a MQN \( Q(\bar{N}, \mathcal{N}) \) consisting of \( B \) composite centres with index set \( \mathcal{N} = (1 \ldots B) \) such that centre 1 is approximately flow equivalent to \( Q(\bar{N} - \bar{V}, \mathcal{M} - \mathcal{V}) \) and centre \( b \) is approximately flow equivalent to \( Q(\bar{B}_b, \bar{B}_b) \) where \( 2 \leq b \leq B \).

When calculating the service rates of the approximately flow equivalent centres, the algorithm assumes [9] that

1. the chain \( j \) throughput of each branch \( b = 1 \ldots B \) depends upon the chain \( j \)
population in branch $b$ and upon the average populations of the other chains in branch $b$

2. the chain $j$ performance measures for $Q(\bar{N}, \mathcal{M})$ are approximately equal to the performance measures of the single chain queueing network $Q(N_j, \mathcal{M})$.

A description of the algorithm is presented below.

A.1. INITIALIZE. Employ SIR and use the Linearizer to solve the MQN $Q(\bar{N}, \mathcal{M})$.

Compute the average number $Q_{bj}$ of chain $j$ customers in branch $b$ for all $1 \leq b \leq B$ and all $1 \leq j \leq J$. Compute $Q_b = \sum_{j=1}^{J} Q_{bj}$ for all $1 \leq b \leq B$.

Compute $K_{bj} = \min(Q_{bj}, d_b Q_{bj}/Q_b)$ which is an estimate of the average number of chain $j$ customers in branch $b$ when SDR is used for all $1 \leq b \leq B$ and $1 \leq j \leq J$.

B.1. ANALYZE EACH CHAIN $j = 1 \ldots J$ IN THE AVERAGE ENVIRONMENT OF THE OTHER CHAINS.

B.2. REPLACE EACH BRANCH $b = 1 \ldots B$ WITH A FLOW EQUIVALENT CENTRE.

Compute the feasible chain $j$ populations $M = 0 \ldots g_b$ in branch $b$ where $g_b = \lceil d_b - K_b + K_{bj} \rceil$ and $K_b = \sum_{j=1}^{J} K_{bj}$. Set $g_1 = N_j$. Explanation: $K_b - K_{bj}$ is the average number of customers competing with chain $j$ for entry into branch $b$. Therefore $g_b = \lceil d_b - K_b + K_{bj} \rceil$ is an estimate of the maximum number of chain $j$ customers that can enter branch $b$ when branch $b$ is subject to SDR.

For each $1 \leq M \leq g_b$ construct a population vector $\bar{K}_b = (K_{b1} \ldots K_{bj-1} M K_{bj+1} \ldots K_bj)$. Use the Linearizer to solve the MQN $Q(\bar{K}_b, \mathcal{M})$ obtaining the chain $j$ throughput $T_j(\bar{K}_b, \mathcal{M})$. Construct a service centre with index $b$ and queue dependent service rate $\mu_b(M) = T_j(\bar{K}_b, \mathcal{M})$.

B.3. COMPUTE THE SDR PROBABILITY AND VISIT COUNT FOR EACH FLOW EQUIVALENT CENTRE $b = 2 \ldots B$.

The SDR probability from centre 1 to centre $b$ is given by $P_{1,b}(\bar{N}) = (g_b - m_b)/(G - B)$ where $G = \sum_{b=2}^{B} g_b$. The SDR visit count to centre $b$ is $\xi_b = \xi_1$.

B.4. SOLVE A SINGLE CLASS PRODUCT FORM SDR QUEUEING NETWORK CONSISTING OF $B$ FLOW EQUIVALENT CENTRES.

Use SDR MVA to solve the single chain queueing network $Q(N_j, \mathcal{M})$ consisting of $B$ centres with index set $\mathcal{M} = (1 \ldots B)$ and population $N_j$. Obtain the chain $j$ throughput $T_j(N_j, \mathcal{M})$ and the chain $j$ queue lengths $Q_{bj}(N_j, N)$ for each centre $b$ in $\mathcal{M}$.
C.1. TEST FOR CONVERGENCE.

Set $K_{bj} = Q_{bj}(N_j, \mathcal{N})$ for each $1 \leq j \leq J$ and $1 \leq b \leq B$. If the current and preceding computations of each $K_b$ have not converged then repeat steps B and C.

D.1. CALCULATE THE PERFORMANCE MEASURES of the centres in $Q(N, M)$.

Set the throughputs $T_j(N, M) = T_j(N_j, \mathcal{N})$ for each $1 \leq j \leq J$.

Use the Linearizer to solve the product form MQN $Q(K_b, B_b)$ obtaining the chain $j$ queue lengths $Q_{ij}(K_b, B_b)$ at each centre $i$ in each branch $b$ for $1 \leq b \leq B$. Set $Q_{ij}(N, M) = Q_{ij}(K_b, B_b)$.

6 AN APPROXIMATE SOLUTION METHOD FOR MQNs WITH SDR AND BLOCKING

Product form SDR probabilities can be used to model adaptive routing and to impose an upper bound $d_b$ on the number of customers that can concurrently be present in each branch $b = 2 \ldots B$. However, when all the branches in $Q(\bar{N}, \mathcal{M})$ are full the product form SDR probabilities route customers requesting entry into $Q(\bar{N}, \mathcal{M})$ directly from the entry point $e$ to the departure point $z$ of $Q(\bar{N}, \mathcal{M})$. Customers are thus denied admission to $Q(\bar{N}, \mathcal{M})$ but instead of being queued for admission to $Q(\bar{N}, \mathcal{M})$ they are immediately returned to $Q(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$. Recycling blocked customers through $Q(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$ rather than queueing them for admission to $Q(\bar{N}, \mathcal{M})$ causes an additional load on the centres in $Q(\bar{N}-\bar{V}, \mathcal{M}-\mathcal{V})$ and an underutilization of the centres in $Q(V, \mathcal{V})$.

This section presents a blocking approximation which queues blocked customers in First Come First Served (FCFS) order for admission into $Q(\bar{V}, \mathcal{V})$. The approximation works as follows.

1. Employ assumption 5.1.2 namely that for each chain $1 \leq j \leq J$ the chain $j$ throughput in $Q(\bar{V}, \mathcal{V})$ is approximately equal to the throughput of the single class network $Q(V_j, \mathcal{V})$. In STEP B.4 the chain $j$ throughputs $T_j(N_j, \mathcal{N})$ parameterize a composite centre with index $c$ that is flow equivalent to $Q(V_j, \mathcal{V})$.

2. In STEP B.2 a composite centre with index 1 was constructed that is flow equivalent to $Q(N_j, \mathcal{M}-\mathcal{V})$.

3. Construct a single class network $Q(N_j, \mathcal{A})$ consisting of two centres with index set $\mathcal{A} = \{1, c\}$. The queue dependent service rate $\mu_c(n)$ of centre $c$ is adjusted to approximate the blocking constraint by setting $\mu_c(n) = \mu_c(G)$ for $n \geq G$ where $G$ was determined in STEP B.3. Maximally $G$ customers can therefore be in service in centre $c$. Customers in excess of the $G$ in service are queued for FCFS admission to centre $c$. 
4. Assume that the chain $j$ performance measures for $Q(\bar{N}, A)$ are approximately equal to the performance measures for the single chain network $Q(N_j, A)$.

The blocking approximation is added to the algorithm after STEP B.4 as follows:

**B.5. SOLVE A SINGLE CLASS PRODUCT FORM QUEUEING NETWORK CONSISTING OF 2 FLOW EQUIVALENT CENTRES.**

Use MVA to solve the two centre single chain queueing network $Q(N_j, A)$ where the centres in $A$ have queue dependent service rates $\mu_{1j}(M) = T_j(M, V)$ for $1 \leq M \leq N_j$ and $\mu_{2j}(M) = T_j(M, V)$ for $1 \leq M \leq G$ and $\mu_{2j}(M) = \mu_{2j}(G)$ for $M > G$. Obtain the chain $j$ throughputs $T_j(N_j, A)$, the queue length distributions $P_c(n : N_j, A)$ for $1 \leq n \leq G$ and the chain $j$ queue lengths $Q_{bj}(N_j, A)$.

**B.6. CONSTRAIN THE POPULATION IN THE SDR SUBNETWORKS.** For each chain $1 \leq j \leq J$ compute a new estimate $Q_{cj}$ of the chain $j$ population at the composite centre $c$

$$Q_{cj} = \sum_{n=1}^{G} nP_c(n; N_j, A) \quad \text{and} \quad Q_{1j} = Q_{1j}(N_j, A)$$

**6.1 COMPUTATIONAL CONSIDERATIONS**

The storage and computational requirements of the SDR algorithm are of the same order as the computational requirements for the approximate solution of MQNs with concurrency constrained subnetworks [7]. These requirements are determined by the speed of convergence of the algorithm and by the computational requirements of the underlying Linearizer and the single chain MVA and convolution solution methods. It can be shown [7] that the storage requirements are $OBN$ locations where $B$ is the number of branches and $N$ is the total number of closed chain customers. The computational requirements for the MVA and Linearizer components of the algorithm are $O(IB2^BN^2)$ and $O(kIJ^4B^2N^2)$ respectively where $I$ is the number of iterations required for the SDR algorithm to converge, $J$ is the number of closed chains and $k$ is the number of iterations required for the Linearizer to converge. In practice $k = 5$ iterations are usually sufficient for the Linearizer to converge yielding tolerance errors of less than 2 percent on all network performance measures. In practice $I = 4$ iterations are usually sufficient for the SDR algorithm to converge, convergence being achieved when the difference in the value of any two successive computations of the chain $j$ population vector $K_b$ in branch $b$ is less than $1/(4000 + 16|\bar{K}_b|)$ for all $1 \leq j \leq J$ and $1 \leq b \leq B$.

The exponential term $2^B$ is present since each single class closed queueing network solved by MVA requires the solution of $2^B - 1$ networks in order to compute correct values for the empty probabilities at the queue dependent centres.
The accuracy of the approximate solution method for product form SDR is evaluated by comparing approximate and exact solutions for MQNs with SDR into single centre and multicentre branches.

The accuracy of the SDR blocking approximation is tested by comparing approximate and simulation solutions for single class networks with SDR into single centre and multi centre branches.

### 7.1 SDR INTO SINGLE CENTRE SUBNETWORKS

Consider a MQN consisting of a central server and four peripheral servers. Let each centre be a fixed rate FCFS exponential server. The peripheral centres define four branches each consisting of a single centre. The servers are assigned service rates as follows: \( \mu_1 = 100 \) and \( \mu_i = 20 \) for \( 2 \leq i \leq 5 \). Equation 1 yields the SDR probabilities \( P_{i,i}(\bar{N}) = (1-n_i)/(4-V) \) for \( 2 \leq i \leq 5 \) where \( V = n_2 + n_3 + n_4 + n_5 \).

Since the branches consist of single centres both the exact and the approximate solution methods can be applied. The results for each solution method in terms of the total queue lengths \( Q_1 \) and \( Q_i \) and throughputs \( T_1 \) and \( T_i \) at centres 1 and \( i \) (\( 2 \leq i \leq 5 \)) are presented in table 2 for for several values of the routing coefficients \( d_b \). The keys -, E and A identify whether the performance measures were computed using State Independent Routing (SIR), or for State Dependent Routing (SDR) using the exact (E) or the approximate (A) solution method.

The results in table 2 demonstrate that the approximate solution method is acceptably accurate even for small population vectors with the exception of small \( d_b \) values (\( d_b = 1 \)). The cause of this inaccuracy lies in STEP B.2 of the approximate solution method. The SDR routing coefficient \( g_b = |d_b - K_b + K_{b'}| \) (which approx-
Table 3: SDR Approximation for MQNs with Single Center Branches

<table>
<thead>
<tr>
<th>N</th>
<th>d_b</th>
<th>Q_1</th>
<th>Q_i</th>
<th>T_1</th>
<th>T_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4,4)-</td>
<td>1.0944</td>
<td>1.7264</td>
<td>55.16</td>
<td>13.80</td>
<td></td>
</tr>
<tr>
<td>(4,4)E</td>
<td>2</td>
<td>1.8538</td>
<td>1.5366</td>
<td>72.00</td>
<td>18.00</td>
</tr>
<tr>
<td>(4,4)A</td>
<td>2</td>
<td>1.7356</td>
<td>1.5661</td>
<td>70.26</td>
<td>17.57</td>
</tr>
<tr>
<td>(4,4)E</td>
<td>4</td>
<td>1.4786</td>
<td>1.6303</td>
<td>65.18</td>
<td>16.30</td>
</tr>
<tr>
<td>(4,4)A</td>
<td>4</td>
<td>1.3250</td>
<td>1.6687</td>
<td>61.76</td>
<td>15.44</td>
</tr>
<tr>
<td>(4,4)E</td>
<td>6</td>
<td>1.3494</td>
<td>1.6627</td>
<td>62.16</td>
<td>15.56</td>
</tr>
<tr>
<td>(4,4)A</td>
<td>6</td>
<td>1.2276</td>
<td>1.6931</td>
<td>59.36</td>
<td>14.84</td>
</tr>
<tr>
<td>(4,4)E</td>
<td>8</td>
<td>1.2850</td>
<td>1.6788</td>
<td>60.54</td>
<td>15.13</td>
</tr>
<tr>
<td>(4,4)A</td>
<td>8</td>
<td>1.0941</td>
<td>1.7265</td>
<td>55.16</td>
<td>13.78</td>
</tr>
</tbody>
</table>

SDR approximates the average effect of the other chains on the chain j SDR probabilities) is integral and hence \( g_b \geq 1 \). Thus for \( d_b = 1 \), \( g_b = 1 \) for both \( j = 1 \) and \( j = 2 \) thereby allowing up to 2 rather than maximally 1 customer into each peripheral centre.

Table 3 confirms that the accuracy of the SDR approximation at small \( d_b \) values improves as the population vector increases.

### 7.2 SDR INTO MULTI CENTRE SUBNETWORKS

Consider a MQN consisting of a central server and 16 peripheral servers. Let each centre be a fixed rate FCFS exponential server. The central server has service rate \( \mu_i = 500 \) and each peripheral centre has service rate \( \mu_i = 20 \) for \( 2 \leq i \leq 17 \).

The peripheral servers are partitioned into 4 branches each consisting of 4 equally utilized fixed rate centres. Each such branch \( 2 \leq b \leq 5 \) can be reduced to a flow equivalent centre with centre index \( b \) and queue dependent service rate

\[
\mu_{bj}(n_b) = a_b(n_b)\mu_{bj}
\]

where \( a_b(n_b) = n_b/(n_b + M_b - 1) \) and \( \mu_{bj} = 4\mu_{ij} \) where \( i \) is any centre in branch \( b \) and \( M_b \) is the number of centres in branch \( b \).

Equation 6 can thus be used to reduce the original queueing network to a network consisting of a central server and four peripheral servers. Each peripheral server \( 2 \leq b \leq 5 \) has a queue dependent service rate and is flow equivalent to branch \( b \).

The exact SDR solution method can be applied to the reduced 5 centre network. The approximate SDR solution method can be applied to the original 17 centre network. The results for each solution method in terms of the total queue lengths \( Q_1 \) and \( Q_b \) and throughputs \( T_1 \) and \( T_b \) at centres 1 and \( b \) \((2 \leq b \leq 5)\) are presented in table 4 for several values of the routing coefficients \( d_b \).
The results in table 4 indicate that the approximate solution method can compute acceptably accurate solutions for SDR into multi centre subnetworks.

### 7.3 SDR AND BLOCKING

**EXAMPLE A.** Consider a single class closed queueing network as in section 7.1 above consisting of a central server and four peripheral servers. Let each centre be a fixed rate FCFS exponential server. Let the central server have an average service rate $\mu_1 = 100$ and let the peripheral centres have identical average service rates $\mu_i = 20$ for $2 \leq i \leq 5$. The peripheral centres define four branches each consisting of a single centre and let $d_b = 1$ for $2 \leq b \leq 5$.

Figure 3 plots the CPU throughput $T_1$ and the peripheral SDR throughput $T = T_2 + T_3 + T_4 + T_5$ (where $T_i$ is the throughput at centre $i$) as a function of the multiprogramming level $N$ for several routing strategies, namely

- SIR
- SIR with blocking (FCFS admission into $Q(V, V)$ when $V = 4$)
- SDR
- SDR with blocking (FCFS admission into $Q(V, V)$ when $V = 4$)

The curves marked $\Box$ and $+$ in figure 3 emphasize a significant feature of product form SDR, namely that customers which are denied entry into the SDR subnetwork $Q(V, V)$ are not queued for admission but are returned to the central server where they complete another service interval before again attempting access into $Q(V, V)$. Thus for $N > 4$ there is always one or more customers at the central server and hence $T_1 = 100$ (curve $\Box$).

Note also that the SDR peripheral throughput rate $T = T_2 + T_3 + T_4 + T_5$ (curve $+$) is constant for $N \geq 4$. However, the SDR peripheral throughput rate will

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**Table 4: SDR Approximation for MQNs with Multicentre Branches**

<table>
<thead>
<tr>
<th>$N$</th>
<th>$d_b$</th>
<th>$Q_1$</th>
<th>$Q_i$</th>
<th>$T_1$</th>
<th>$T_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8,8)</td>
<td>0.4714</td>
<td>3.8822</td>
<td>162.7</td>
<td>40.66</td>
<td></td>
</tr>
<tr>
<td>(8,8)E</td>
<td>0.5437</td>
<td>3.8641</td>
<td>179.2</td>
<td>44.79</td>
<td></td>
</tr>
<tr>
<td>(8,8)A</td>
<td>0.5405</td>
<td>3.8649</td>
<td>179.4</td>
<td>44.85</td>
<td></td>
</tr>
<tr>
<td>(8,8)E</td>
<td>0.5268</td>
<td>3.8683</td>
<td>175.7</td>
<td>43.92</td>
<td></td>
</tr>
<tr>
<td>(8,8)A</td>
<td>0.5160</td>
<td>3.8710</td>
<td>173.8</td>
<td>43.46</td>
<td></td>
</tr>
<tr>
<td>(8,8)E</td>
<td>0.5153</td>
<td>3.8712</td>
<td>173.1</td>
<td>43.27</td>
<td></td>
</tr>
<tr>
<td>(8,8)A</td>
<td>0.5039</td>
<td>3.8758</td>
<td>169.3</td>
<td>42.33</td>
<td></td>
</tr>
<tr>
<td>(8,8)E</td>
<td>0.4967</td>
<td>3.8758</td>
<td>169.3</td>
<td>42.33</td>
<td></td>
</tr>
</tbody>
</table>

---
Figure 3: SDR versus SIR throughput

Figure 4: SDR into Multiclass Branches
be constant only if the central server has a fixed (load independent) average service rate. This can be demonstrated as follows. Let centre 1 have a fixed average service rate \( \mu_1 \) and let \( n_1 \) denote the number of customers at centre 1. If \( N = 4 \) then \( 0 \leq n_1 \leq 4 \) and the SDR subnetwork is subject to a Poisson arrival process \( \lambda(n_1) \) where \( \lambda(n_1) = 100 \) for \( 1 \leq n_1 \leq 4 \) and \( \lambda(0) = 0 \). Consider \( N = 5 \). Maximally 4 customers can be in \( Q(V, V) \) so that \( 1 \leq n_1 \leq 5 \). The SDR network is now subject to a Poisson arrival process with parameter \( \lambda(n_1 - 1) \) identical to the Poisson process described above with population \( N = 4 \). The SDR peripheral throughput is thus constant for \( N \geq 4 \) provided that centre 1 has a fixed average service rate.

Figure 3 also presents (curve \( \triangle \)) the SIR throughput for a product form queueing network with \( P_{i, i} = \frac{1}{4} \) as well as (curve \( \circ \)) the SIR throughput for a SIR queueing network where \( P_{i, i} = \frac{1}{4} \) and a blocking constraint is placed on the subnetwork \( Q(V, V) \) so that maximally four customers can be concurrently present in \( Q(V, V) \). Once \( V = 4 \) then customers are queued for admission in FCFS order to \( Q(V, V) \).

Figure 3 finally presents (curve \( \times \)) the SDR throughput when customers are queued for admission to \( Q(V, V) \) rather than being immediately returned to the central server as in the case of product form SDR. SDR with blocking achieves significantly greater throughput than product form SDR and SIR.

**EXAMPLE B.** Consider a single class closed queueing network consisting of a central server and three peripheral servers. Let each centre be a fixed rate FCFS exponential server. Let the central server have an average service rate \( \mu_1 = 100 \) and let the peripheral centres have identical average service rates \( \mu_i = 20 \) for \( 2 \leq i \leq 4 \). Let the centres be partitioned into branches as illustrated in Figure 5 such that \( B_1 = \{1\}, B_2 = \{2\} \) and \( B_3 = \{3, 4\} \). Two instances of SDR are investigated, namely \((d_2, d_3) = (1, 2)\) and \((d_2, d_3) = (2, 1)\). Note that in the first instance the SDR prob-
abilities preferentially route customers to branch 2 whereas in the second instance customers are preferentially routed to branch 3. In both instances the SDR subnetwork $Q(V, V)$ becomes full when $V = 3$ whereupon the blocking approximation is applied to queue blocked customers for FCFS admission into $Q(V, V)$.

Table 5 presents approximate and SLAM [13] simulation solutions for the central server throughput $T_1$ as a function of the multiprogramming level $N$ for the two sets of SDR routing coefficients. Each simulation was terminated when the relative width of the 90 per cent confidence interval for the average transit time across the SDR subnetwork was 5 per cent. Table 5 confirms that the approximate solution method can compute acceptably accurate solutions for SDR with blocking into multicentre subnetworks.

Figure 4 compares the blocking approximation for SDR and SIR networks. The SDR network is as in example B above. The SIR network is identical to the SDR network apart from the routing probabilities. Three sets of SIR routing probabilities are investigated, namely $P_{1,i} = \frac{1}{2}, \frac{1}{2}$, $P_{1,i} = \frac{1}{3}, \frac{2}{3}$ and $P_{1,i} = \frac{2}{3}, \frac{1}{3}$ where $i = 2, 3$. The SIR network also allows maximally 3 customers into $Q(V, V)$ whereupon customers are queued for FCFS admission into $Q(V, V)$.

Figure 4 reveals that SDR (curves □ and +) offers a substantial improvement in throughput over SIR (curves Δ, ◊ and ×). Note that SIR achieves greater throughput by preferentially routing customers to branch 2 (which contains a single centre with service rate $\mu_2 = 50$) irrespective of the resultant congestion at branch 2. In contrast, SDR initially ($N \leq 2$) achieves better throughput by preferentially routing customers to branch 2. However, for $N \geq 2$ SDR achieves better throughput by preferentially routing customers to branch 3 which can contain maximally 2 customers.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T_1$ (approx)</th>
<th>$T_1$ (approx)</th>
<th>$T_1$ (approx)</th>
<th>$T_1$ (approx)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.7</td>
<td>10.8</td>
<td>13.0</td>
<td>13.1</td>
</tr>
<tr>
<td>2</td>
<td>18.3</td>
<td>18.4</td>
<td>23.5</td>
<td>23.7</td>
</tr>
<tr>
<td>3</td>
<td>29.0</td>
<td>29.3</td>
<td>28.2</td>
<td>28.3</td>
</tr>
<tr>
<td>4</td>
<td>31.9</td>
<td>31.9</td>
<td>29.5</td>
<td>30.7</td>
</tr>
<tr>
<td>5</td>
<td>32.9</td>
<td>33.4</td>
<td>29.8</td>
<td>30.7</td>
</tr>
<tr>
<td>6</td>
<td>33.2</td>
<td>33.6</td>
<td>30.0</td>
<td>30.7</td>
</tr>
</tbody>
</table>

Table 5: SDR Approximation for MQNs with Multicentre Branches
8 CONCLUSION

This paper presents exact and approximate solution methods for multiclass queueing networks with product form state dependent routing. The product form solution requires that the centres in the MQN be partitioned into disjoint subnetworks called branches. The SDR probabilities set an upper bound on the number of customers that can concurrently be present in each branch as well as an upper bound on the total number of customers that can be present in all the SDR branches. Once this bound is reached, the product form SDR probabilities route customers around the SDR branches.

The approximate solution method is extended to admit customers in first come first served order to the SDR branches when the total number of customers in all the SDR branches has reached its upper bound. MQNs with SDR and blocking violate product form.

The SDR blocking approximation is tested out on several single and multichain queueing networks and is compared against simulation solutions. In general, the accuracy of the approximation method is found to be good.

References


**APPENDIX.**

**EXACT SOLUTION OF MQNs WITH SDR FOR MULTICENTRE SUBNETWORKS**

Let $Q(N,M)$ denote a MQN consisting of $B$ branches $Q(B_b, B_b)$ where $1 \leq b \leq B$. Let $M = B_1 + \cdots + B_B$ where $B_b$ is the index set of the centres in branch $b$. Let $Q(B_1, B_1)$ contain all the centres that are not subject to SDR and let $Q(V, V)$ where $V = B_2 + \cdots + B_B$ contain all the centres that are subject to SDR.

The exact analysis of $Q(N,M)$ proceeds as follows. The normalising constant $F_b(B_b, B_b)$ for each branch $2 \leq b \leq B$ is given by [2]

$$F_b(B_b, B_b) = F_b(B_b, B_b - i) + \sum_{j=1}^{J} \gamma_{B_b} F_b(B_b - I_j, B_b) \sum_{n=1}^{m_b} [1/\alpha_b(n)] P_b(n-1: B_b - I_j, B_b)$$

where $P_b(n: B_b, B_b) = [1/\alpha_b(n)] \sum_{j=1}^{J} T_j(B_b, B_b) P_b(n-1: B_b - I_j, B_b)$

and $T_j(B_b, B_b) = F_b(B_b - I_j, B_b)/F_b(B_b, B_b)$

The normalising constant $G(V, V)$ is given by [5]

$$G(V, V) = [1/\Omega(V)] \sum_{B_b \in V} \Delta_b(B_b) F_b(B_b, B_b) G(V - B_b, V - B_b)$$
The distribution of the population vector $\bar{B}_b$ in branch $b$ is given by [5]

$$P(\bar{B}_b, \bar{B}_b; \bar{V}, \bar{V}) = \frac{\Omega(\bar{V} - m_b)/\Omega(\bar{V})}{\Delta_b(m_b) F_b(\bar{B}_b, \bar{B}_b) G(\bar{V} - \bar{B}_b, \bar{V} - \bar{B}_b)/G(\bar{V}, \bar{V})}$$

The average number of chain $j$ customers in branch $b$ is given by

$$Q_{b_j}(\bar{V}, \bar{V}) = \sum_{\bar{B}_b \in \bar{V}} m_{b_j} P(\bar{B}_b, \bar{B}_b; \bar{V}, \bar{V})$$

The throughput of chain $j$ customers in branch $b$ for $\bar{V} \geq \bar{I}_j$ is given by

$$T_{b_j}(\bar{V}, \bar{V}) = \xi_{b_j} \left[ d_b - Q_{b}(\bar{V} - \bar{I}_j, \bar{V}) \right] G(\bar{V} - \bar{I}_j, \bar{V}) / \left[ G(\bar{V}, \bar{V}) \omega(\bar{V} - 1) \right]$$

The unnormalized distribution of the population vector $\bar{V}$ in $Q(\bar{V}, \bar{V})$ is given by

$$\hat{P}(\bar{V}, \bar{V}; \bar{N}, \bar{M}) = G(\bar{N} - \bar{V}, \bar{M} - \bar{V}) G(\bar{V}, \bar{V})$$

The conditioning on the population vector $\bar{V}$ can now be removed. The average number of chain $j$ customers in branch $b$ is given by

$$Q_{b_j}(\bar{N}, \bar{M}) = G^{-1}(\bar{N}, \bar{M}) \sum_{\bar{V} \in \bar{N}} Q_{b_j}(\bar{V}, \bar{V}) \hat{P}(\bar{V}, \bar{V}; \bar{N}, \bar{M})$$

The throughput of chain $j$ customers in branch $b$ is given by

$$T_{b_j}(\bar{N}, \bar{M}) = G^{-1}(\bar{N}, \bar{M}) \sum_{\bar{V} \in \bar{N}} T_{b_j}(\bar{V}, \bar{V}) \hat{P}(\bar{V}, \bar{V}; \bar{N}, \bar{M})$$

where the network normalising constant $G(\bar{N}, \bar{M})$ is given by

$$G(\bar{N}, \bar{M}) = \sum_{\bar{V} \in \bar{N}} \hat{P}(\bar{V}, \bar{V}; \bar{N}, \bar{M})$$

If each branch $b$ consists of a single BCMP centre then the above equations yield equations 3 through 5.