STRUCTURE AND REPRESENTATION OF REAL LOCALLY C*- AND LOCALLY JB-ALGEBRAS

by

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CHAPTER 1.

INTRODUCTION

The abstract Banach associative symmetrical *-algebras over $\mathbb{C}$, so called C*-algebras, were introduced first in 1943 by Gelfand and Naimark[24]. In the present time the theory of C*-algebras has become a vast portion of functional analysis having connections and applications in almost all branches of modern mathematics and theoretical physics[51,55].

From the 1940's and the beginning of 1950's there were numerous attempts made to extend the theory of C*-algebras to a category wider than Banach algebras. For example, in 1952, while working on the theory of locally-multiplicatively-convex algebras as projective limits of projective families of Banach algebras, Arens in the paper[8] and Michael in the monograph[48] independently for the first time studied projective limits of projective families of functional algebras in the commutative case and projective limits of projective families of operator algebras in the non-commutative case. In 1971 Inoue in the paper[33] explicitly studied topological *-algebras which are topologically -isomorphic to projective limits of projective families of C*-algebras and obtained their basic properties. He as well suggested a name of *locally* C*-*algebras* for that category. For the present state of the theory of locally C*-algebras see the monograph of Fragoulopoulou[20].

Also there were many attempts to extend the theory of C*-algebras to non-associative algebras which are close in properties to associative algebras (in particular, to Jordan algebras). In fact, the real Jordan analogues of C*-algebras, so called JB-algebras, were first introduced in 1978 by Alfsen, Shultz and Størmer in[1]. One of the main results of the aforementioned paper stated that modulo factorization over a unique Jordan ideal each JB-algebra is isometrically isomorphic to a JC-algebra, i.e. an operator norm closed Jordan subalgebra of the Jordan algebra of all bounded self-adjoint operators with symmetric multiplication acting on a complex Hilbert space.

Projective limits of Banach algebras have been studied sporadically by many authors since 1952, when they were first introduced by Arens[8] and Michael[48]. Projective limits of complex C*-algebras were first mentioned by Arens. They have since been studied under various names by Wenjen[75], Sya Do-Shin[69], Brooks[12], Inoue[33], Schmüdgen[62], Fritzsche[22,23], Fragoulopoulou[21], Phillips[58], etc.

We will follow Inoue[33] in the usage of the name "locally C*-algebras" for these objects.

At the same time, in parallel with the theory of complex C*-algebras, a theory of their real and Jordan analogues, namely real C*-algebras and JB-algebras, has been actively developed by various authors[9,30,45].

In chapter 2 we present definitions and basic theorems on complex and real C*-algebras, JB-algebras and complex locally C*-algebras to be used further.

In chapter 3 we define a real locally Hilbert space H^R and an algebra of operators $L(H^R)$ (not bounded anymore) acting on H^R.

In chapter 4 we give new definitions and study several properties of locally C*- and locally JB-algebras. Then we show that a real locally C*-algebra (locally JB-algebra) is locally isometric to some closed subalgebra of $L(H^R)$.

In chapter 5 we study complex and real Abelian locally C*-algebras.

In chapter 6 we study universal enveloping algebras for locally JB-algebras.

In chapter 7 we define and study dual space characterizations of real locally C* and locally JB-algebras.

In chapter 8 we define barreled real locally C* and locally JB-algebras and study their representations as unbounded operators acting on dense subspaces of some Hilbert spaces.

It is beneficial to extend the existing theory to the case of real and Jordan analogues of complex locally C*-algebras. The present thesis is devoted to study such analogues, which we call real locally C*- and locally JB-algebras.

CHAPTER 2.

PRELIMINARIES

2.1 C*-algebras and Locally C*-algebras

In this chapter we give some preliminaries on complex locally C*-algebras.

Definition 1 *Let B be an algebra. A subset U of B is called **idempotent**, if*

$$UU \subseteq U \tag{2.1}$$

in the sense that $\forall x, y \in U$, the product $xy \in U$.

Definition 2 *Let B be a locally convex algebra over $\mathbb{C}$. B is **locally m-convex** iff there exists a basis of neighborhoods of zero entirely composed of convex idempotent sets U_i .*

In every locally convex topological space the topology can be defined by a basis of continuous seminorms[61]. If the algebra over $\mathbb{C}$ is a locally m-convex one, the basis can be chosen in such a way that each seminorm is a submultiplicative one[48]. In every locally m-convex algebra over $\mathbb{C}$, the multiplication law is jointly continuous, and if the algebra has a unit, inversion is continuous on the group of invertible elements.[8]

Definition 3 *An **involution** on an algebra B over $\mathbb{C}$ is defined as a conjugate anti-isomorphism of period two, $^* : B \longrightarrow B$, which satisfies the following conditions:*

$$(x + y)^* = x^* + y^* \tag{2.2}$$

$$(\lambda x)^* = \overline{\lambda} x^* \tag{2.3}$$

$$(xy)^* = y^* x^* \tag{2.4}$$

$$(x^*)^* = x \tag{2.5}$$

for all $x, y \in B$, and each $\lambda \in \mathbb{C}$. An algebra over $\mathbb{C}$ on which there is an involution defined is called a ***complex involutive algebra*** *or *-algebra over $\mathbb{C}$.*

Definition 4 *A locally convex *-algebra B over $\mathbb{C}$ with unit is called* ***symmetric*** *if for every $x \in B$, $(\mathbf{1} + x^* x)$ is invertible and the inverse element $(\mathbf{1} + x^* x)^{-1}$ is bounded (in the sense of Allan[4]).*

Definition 5 *A symmetric element, i.e.*

$$x = x^*,$$

*of a complex topological *-algebra with unit is called* ***Hermitian****, iff its spectrum is contained in $\mathbb{R}$. If every symmetric element is Hermitian, then involution is called Hermitian.*

Definition 6 *Let B be a vector space. A real function $p : B \to \mathbb{R}$ on B is called a* ***seminorm****, if:*

$$1) \quad p(x) \geq 0, \forall x \in B \tag{2.6}$$

$$2) \quad p(\lambda x) = |\lambda|\, p(x) \qquad \forall \lambda \in \mathbb{R} \text{ or } \mathbb{C}, \text{ and } x \in B \tag{2.7}$$

$$3) \quad p(x + y) \leq p(x) + p(y) \qquad \forall x, y \in B \tag{2.8}$$

One can see that $p(\mathbf{0}) = 0$.

If $p(x) = 0$ implies $x = \mathbf{0}$, the seminorm is called a ***norm*** *and is usually denoted by $\|.\|$.*

Definition 7 *a) A seminorm p defined on an algebra B over $\mathbb{C}$ (or $\mathbb{R}$) is called **submultiplicative** or **m-seminorm** if it satisfies the following condition:*

$$p(xy) \le p(x)p(y), \quad \forall x, y \in B. \tag{2.9}$$

*b) A seminorm p defined on an algebra B over $\mathbb{C}$ (or $\mathbb{R}$) is called ***-invariant** if it satisfies:*

$$p(x) = p(x^*), \quad \forall x \in B. \tag{2.10}$$

Definition 8 *If a space with a norm is complete, then it is called a **Banach space**. If an algebra is a Banach space with a submultiplicative norm, then it is called **Banach algebra.***

Definition 9 *A submultiplicative *-invariant seminorm p (norm ||.||) defined on a *-algebra B over $\mathbb{C}$ (or $\mathbb{R}$) is called **regular (C*-regular)** if the following condition is true:*

$$p(a)^2 = p(a^*a), \ (||a||^2 = ||a^*a||), \ \forall a \in B. \tag{2.11}$$

*A submultiplicative *-invariant seminorm p (norm ||.||) is **strongly regular (strongly C*-regular)**[56] if*

$$p(a)^2 \le p(a^*a + b^*b), \ (||a||^2 \le ||a^*a + b^*b||), \ \forall a, b \in B. \tag{2.12}$$

Definition 10 *A is a **complex C*-algebra (C*-algebra)** if it is a complex Banach *-algebra with a C*-regular norm.*

Theorem 1 (Gelfand-Naimark) *Let A be a complex C*-algebra. Then there exists a complex Hilbert space H such that A is *-isomorphic to a norm-closed self-adjoint complex subalgebra of $\mathcal{B}(H)$.*

Definition 11 *Let A be a* ***real algebra*** *(an algebra over the field of real numbers) and let B be the Abelian group of the Cartesian product* $A \times A$*; we define multiplication and scalar multiplication:*

$$(i) \quad (x,y)(s,t) = (xs - yt, xt + ys), \tag{2.13}$$

$$(ii) \quad (\lambda + i\mu)(x,y) = (\lambda x - \mu y, \lambda y + \mu x), \tag{2.14}$$

for any $x,y,t,s \in A,\ (x,y),(t,s) \in B,\ \lambda,\mu \in \mathbb{R}$*. Then B is called the* ***complexification of*** *A. We will use the following notation:* $B = A \dot{+} iA$*.*

Definition 12 *A* ***natural embedding*** *of A in B (as a real subalgebra or subspace) is the map:*

$$e_n : x \mapsto (x,0) \qquad x \in A, \quad (x,0) \in B. \tag{2.15}$$

There is a ***natural imaginary embedding*** *of A in B:*

$$e_i : x \mapsto (0,x) \qquad x \in A, \quad (0,x) \in B, \tag{2.16}$$

We will be dealing with real algebras or real spaces such that

$$A \cap iA = \{0\}, \tag{2.17}$$

*and we will call this property "****essential****".*

Remark 1 *Everywhere below we will consider only essential Hilbert spaces, unital essential real C*- and unital Jordan algebras.*

Definition 13 *If A is a real *-algebra, then the* ***involution on the complexification*** *B can be defined as:*

$$(x,y)^* = (x^*, -y^*), \ \ x,y \in A, (x,y) \in B. \tag{2.18}$$

Definition 14 *A* ***real C*-algebra*** *is a real Banach *-algebra with a C*-regular norm whose complexification can be equipped with a C*-regular norm which makes it a complex C*-algebra.*

Definition 15 *Let B be a complex algebra and $b \in B$. The* ***spectrum*** *of b is the set*

$$\sigma(b) = \{\lambda \in \mathbb{C} : \lambda - b \ \ \textit{not invertible in } B\}.$$

Let A be a real algebra and $a \in A$. The ***spectrum*** *of a is*

$$\sigma_R(a) = \sigma(a + i\mathbf{0}),\ a + i0 \in B,\ B = A \dot{+} iA.$$

Definition 16 *A complex Banach *-algebra B is said to be* ***Hermitian*** *if*

$$\sigma(b) \subset \mathbb{R}, \forall b \in B_H = \{b \in B : \ b^* = b\}. \tag{2.19}$$

*For a real Banach *-algebra A to be* ***Hermitian*** *the spectrum $\sigma_R(a)$ of the Hermitian element a should also be real.*

Definition 17 *Let a real Banach *-algebra A. $a \in A$ is said to be* ***positive*** *$(a \geq 0)$ if $a = a^*$ and $\sigma_R(a) \subset [0, \infty)$.*

Definition 18 *(i) A real Banach *-algebra A is said to be* ***skew Hermitian*** *if*

$$\sigma(k) \subset i\mathbb{R}, \forall k \in A_K = \{a \in A : \ a^* = -a\}.$$

*(ii) A real Banach *-algebra A is said to be* ***symmetric*** *if*

$$a^*a \geq 0, \quad \forall a \in A.$$

Theorem 2 *Let A be a real Banach *-algebra. Then the following statements are equivalent*[45]*:*

(1) A is a real C^*-algebra.

*(2) there exists a real Hilbert space H such that A is *-isomorphic to a norm-closed subalgebra of real $\mathcal{B}(H)$.*

(3) A is Hermitian and norm is regular.

(4) A is symmetric and norm is regular.

(5) $1 + \tilde{a}^*\tilde{a}$ is invertible in $\widetilde{A}$ and norm is regular, where $\widetilde{A}$ is the algebra with associated unity.

(6) Norm is strongly regular: $||a||^2 \leq ||a^*a + b^*b||, \ \forall a, b \in A.$[56]

If p is a submultiplicative seminorm on an algebra B, the unit semiball $U_p(1)$ corresponding to p, that is

$$U_p(1) = \{x \in B : p(x) \leq 1\} \tag{2.20}$$

is idempotent. Moreover, $U_p(1)$ is an absolutely convex (balanced and convex) absorbing subset of B.

Definition 19 *Given an absorbing absolutely-convex subset $U \subset B$, the function*

$$p_U : B \to \mathbb{R}_+ :$$

$$x \to p_U(x) = \inf\{\lambda > 0 : x \in \lambda U\} \tag{2.21}$$

called the gauge or ***Minkowski functional*** *of U, is a seminorm. One can see that a real-valued function p on the algebra B is an m-seminorm iff*

$$p = p_U$$

for some absorbing, absolutely-convex and idempotent subset $U \subset B$

In fact, one can take $U = U_p(1)$.

Definition 20 [33]*By a **topological algebra** we mean a topological vector space which is also an algebra, such that the ring multiplication is separately continuous. A topological algebra B is often denoted by (B, τ), where τ is the topology of the underlying topological vector space of B. The topology τ is determined by a **fundamental 0-neighbourhood system**, say $\mathcal{B}$, consisting of absorbing, balanced sets with the property*

$$\forall V \in \mathcal{B} \quad \exists U \in \mathcal{B}$$

satisfying the condition $U + U \subseteq V$.

Since translations by y in (B, τ), i.e. the maps

$$x \to x + y :$$

$$(B, \tau) \to (B, \tau)$$

$y \in B$, are homeomorphisms, an x-neighbourhood in (B, τ) is of the form

$$x + V$$

with $V \in \mathcal{B}$.

Definition 21 *A closed, absorbing and absolutely convex subset of a topological algebra (B, τ) is called a **barrel**.*

A locally convex algebra is a topological algebra in which the underlying topological vector space is a locally convex space. The topology τ of a locally convex algebra (B, τ) is defined by a fundamental 0-neighbourhood system consisting of closed absolutely convex sets. Equivalently, the same topology τ is determined by a family of nonzero seminorms. Such a family, is always assumed without a loss of generality to be saturated (definition 25) .

Definition 22 *A family $\{B_\alpha, g_\alpha^\beta; \alpha, \beta \in \Lambda, \alpha \preceq \beta\}$ consisting of topological algebras B_α, and continuous morphisms g_α^β with dense images $g_\alpha^\beta(B_\beta)$ in B_α, is called a **projective family** if for $\alpha \preceq \beta \in \Lambda$ (Λ is a directed set of indices), the maps*

$$g_\alpha^\beta : B_\beta \longrightarrow B_\alpha \tag{2.22}$$

fulfill the conditions

$$g_\alpha^\beta([x]_\beta) = [x]_\alpha, \quad [x]_\alpha \in B_\alpha, \ [x]_\beta \in B_\beta, \tag{2.23}$$

$$g_\alpha^\alpha([x]_\alpha) = [x]_\alpha,$$

and for all γ such that $\gamma \succeq \alpha$ and $\gamma \succeq \beta, \ \alpha, \beta, \gamma \in \Lambda$

$$g_\alpha^\beta \circ g_\beta^\gamma = g_\alpha^\gamma. \tag{2.24}$$

Definition 23 *The smallest subalgebra B of a direct product*

$$\prod_{\alpha \in \Lambda} B_\alpha \tag{2.25}$$

*is called a **projective limit** of the projective family $\{B_\alpha, g_\alpha^\beta; \alpha, \beta \in \Lambda, \alpha \preceq \beta\}$ if the **natural projections**[71]) π_α*

$$\pi_\alpha(x) = [x]_\alpha, \quad \forall x \in B, \ [x]_\alpha \in B_\alpha, \ \forall \alpha \in \Lambda \tag{2.26}$$

are continuous morphisms with dense images $\pi_\alpha(B)$ *in* B_α*, and*

$$\pi_\alpha = g_\alpha^\beta \circ \pi_\beta, \quad \forall \alpha \preceq \beta \in \Lambda. \tag{2.27}$$

The projective limit algebra B *above is denoted by*

$$B = \varprojlim B_\alpha, \ \alpha \in \Lambda \quad or \quad B = \varprojlim g_\alpha^\beta B_\beta, \quad \forall \alpha \preceq \beta \in \Lambda. \tag{2.28}$$

If $B_\alpha, \ \alpha \in \Lambda$ *are *-algebras, then* g_α^β *and* π_α *are *-morphisms*

$$g_\alpha^\beta([x]_\beta^*) = [x]_\alpha^*, \quad [x]_\alpha \in B_\alpha, \ [x]_\beta \in B_\beta, \tag{2.29}$$

and

$$\pi_\alpha(x^*) = [x]_\alpha^*, \quad x \in B, \ [x]_\alpha \in B_\alpha, \ \forall \alpha \in \Lambda. \tag{2.30}$$

The ***projective topology*** τ_B *is formed by all finite intersections*

$$\bigcap_{i=1}^{k} \pi_{\alpha_i}^{-1}(O_{\tau_{\alpha_i}}([x]_{\alpha_i})), \tag{2.31}$$

where $O_{\tau_{\alpha_i}}([x]_{\alpha_i})$ *are the open balls of radius* ε *in* B_{α_i} *with the center in* $[x]_{\alpha_i}$*,* $\alpha_i \in \Lambda$*, and* $i = \overline{1,k}$*.*

One can notice that this topology is the coarsest under which all π_α are continuous.

Definition 24 *A family of seminorms* $\{p_\alpha(.)\}$ *on an algebra* B *is a* ***separating family****, if for any* $x \neq 0$ *there exists* α'*, such that* $p_{\alpha'}(x) \neq 0$*.*

Definition 25 *A family of seminorms* $\{p_\alpha(.)\}$ *on an algebra* B *is a* ***saturated family****, if for any finite subset* F *of* Λ *there exists* $p_F \in S(B)$ *:*

$$p_F(x) = \max_{\alpha \in F}\{p_\alpha(x)\} \tag{2.32}$$

$\forall x \in B$*, where* $S(B)$ *is the set of all seminorms on* B*.*

Definition 26 *A topological *-algebra B over $\mathbb{C}$ is called a* ***complex locally C*-algebra*** *if there exists a separating saturated family of C*-regular submultiplicative seminorms $\{p_\alpha\}_{\alpha\in\Lambda}$ such that $\forall\alpha \preceq \beta$*

$$p_\alpha(v) \leq p_\beta(v), \quad \forall v \in B. \tag{2.33}$$

If seminorms are not C-regular and only submultiplicative, then B is called a* ***complex lmc *-algebra****.*

Theorem 3 [33]*A complex locally C*-algebra B is isomorphic to a projective limit*

$$B' = \varprojlim g_\alpha^\beta B_\beta,$$

of a projective family of complex C-algebras $\{B_\alpha, g_\alpha^\beta; \alpha, \beta \in \Lambda\}$.*

Real C*-algebras and Their Representations

Definition 27 *An algebra A is* ***symmetric*** *if for any element $x \in A$ the spectrum of the element x^*x is a nonnegative real number: $\sigma(x^*x) \in \mathbb{R}_+$.*

Theorem 4 [45]*Let A be a real Banach *-algebra Then, A is symmetric with a C*-regular norm iff it is real C*-algebra.*

Let $H^{\mathbb{R}}$ be a real Hilbert space. then $H^{\mathbb{C}} = H^{\mathbb{R}} \dot{+} iH^{\mathbb{R}}$ becomes a complex Hilbert space if we define a scalar product $< \xi + i\eta, \xi' + i\eta' >_{H^{\mathbb{C}}} = < \xi,\ \xi' > + < \eta,\ \eta' > + i < \eta,\ \xi' > - i < \eta',\ \xi >,\ \forall \xi, \eta, \xi', \eta' \in H^{\mathbb{R}},$ where $< .,. >$ is the scalar product on $H^{\mathbb{R}}$.

Then $||\xi + i\eta||^2 = ||\xi - i\eta||^2 = ||\xi||^2 + ||\eta||^2$ and complex Hilbert space $H^{\mathbb{C}}$ is a complexification of $H^{\mathbb{R}}$, and the previous equality does not mean that $H^{\mathbb{R}} \perp iH^{\mathbb{R}}$ in $H^{\mathbb{C}}$.

Theorem 5 *Let $H^{\mathbb{R}}$ be a real Hilbert space and $H^{\mathbb{C}}$ be as above. Then $\mathcal{B}(H^{\mathbb{C}}) = \mathcal{B}(H^{\mathbb{R}}) \dot{+} \mathcal{B}(H^{\mathbb{R}})$ is a complexification of $\mathcal{B}(H^{\mathbb{R}})$, and $< \overline{\xi_c}, \overline{\eta_c} >_{H^{\mathbb{C}}} = \ < \overline{\xi_c, \eta_c} >_{H^{\mathbb{C}}} = \ < \eta_c, \xi_c >_{H^{\mathbb{C}}},\ \overline{R}^* = \overline{R^*},\ T_c^* = (T^*)_c T^*,\ T^* = T^*,\ (TS)_c = T_c S_c$,*

where $\xi_c = \xi + i\xi,\ T_c(\xi + i\eta) = T(\xi) + iT(\eta),\ \forall \xi, \eta \in H^{\mathbb{C}},\ R \in \mathcal{B}(H^{\mathbb{C}}),\ T, S \in \mathcal{B}(H^{\mathbb{R}})$.

Theorem 6 [32,27]*Let A be a symmetric real Banach *-algebra with C*-regular norm. Then there exists a real Hilbert space H_R such that A is real *-isomorphic to a norm closed subalgebra of $\mathcal{B}(H_R)$.*

Representations of Complex Locally C*-algebras

In[33] Inoue introduced a complex locally Hilbert space.

Definition 28 *Let Λ be a directed set and $H_\alpha^{\mathbb{C}}; \alpha \in \Lambda$ be a family of complex Hilbert spaces with the inner product $< x, y >_\alpha, x, y \in H_\alpha^{\mathbb{C}}$, such that if $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$, then $H_\alpha^{\mathbb{C}} \subset H_\beta^{\mathbb{C}}$ and $< x, y >_\alpha = < x, y >_\beta$. We consider $H^{\mathbb{C}} = \cup H_\alpha^{\mathbb{C}}$ and endow it with the structure of vector space and inductive topology (defining the family of closed sets as a collection of $H^{\mathbb{C}}$ and all closed subsets in each $H_\delta^{\mathbb{C}}$). This space $H^{\mathbb{C}}$ with the inner product and inductive topology will be called a* ***locally Hilbert space****.*

Inoue as well[33] introduced the algebra $L(H^{\mathbb{C}})$ of all continuous linear operators on $H^{\mathbb{C}}$, it consists of all complex continuous linear operators

$$T : H^{\mathbb{C}} \to H^{\mathbb{C}} \tag{2.34}$$

whose restrictions $T|_{H_\alpha^{\mathbb{C}}}$ are invariant on $H_\alpha^{\mathbb{C}}$ and belong to the C*-algebras $\mathcal{B}(H_\alpha),\ \alpha \in \Lambda$, of bounded linear operators.

Theorem 7 (Inoue) [33] *An arbitrary complex locally C*-algebra B is *-isomorphic to a locally C*-subalgebra in $L(H^{\mathbb{C}})$ complete in projective topology of $L(H^{\mathbb{C}})$, where $H^{\mathbb{C}}$ is a locally Hilbert space.*

2.2 Jordan Algebras and JB-algebras

Definition 29 *A* ***real Jordan algebra*** *J is a real linear space with a binary operation "$\bullet$" such that: $\forall a, b, c \in J, \ \ \forall \gamma \in \mathbb{R}$*

i) Commutativity:

$$a \bullet b = b \bullet a \tag{2.35}$$

ii) Distributivity:

$$(a + b) \bullet c = a \bullet c + b \bullet c \tag{2.36}$$

iii) Module property:

$$\gamma(a \bullet b) = (\gamma a) \bullet b = a \bullet (\gamma b) \tag{2.37}$$

iv) Weak associativity:

$$(a^2 \bullet b) \bullet a = a^2 \bullet (b \bullet a) \tag{2.38}$$

Throughout the dissertation we consider only real Jordan algebras, so we will omit the word "real".

The abstract Jordan analogues of complex C*-algebras, so called *JB-algebras*, were first defined by Alfsen, Schultz and Størmer in[1] as the real Banach–Jordan algebras satisfying for all pairs of elements a and b the inequalities of submultiplicativity and fineness and the regularity identity (definition 30 below). The basic theory of JB-algebras is fully treated in the monograph of Hanche-Olsen and Størmer.[30]

Definition 30 *A **Banach-Jordan algebra** is a Jordan algebra which is as well a Banach algebra with submultiplicative norm:*

$$\|a \bullet b\| \leq \|a\|\|b\| \tag{2.39}$$

*A **JB-algebra** is a Banach-Jordan algebra with the norm satisfying:*

a) JB-regularity:

$$\|a^2\| = \|a\|^2 \tag{2.40}$$

b) Fineness:

$$\|a^2\| \leq \|a^2 + b^2\| \tag{2.41}$$

*A submultiplicative norm $\|.\|$ with properties a) and b) is called **JB-regular norm**.*

Definition 31 *Let B be a real associative algebra. Then $B^J = (B, \bullet)$, where "$\bullet$" is the **symmetric multiplication***

$$a \bullet b = \frac{1}{2}(ab + ba), \tag{2.42}$$

$a, b \in B$, is a Jordan algebra.

*A Jordan algebra which can be obtained in such a way is called **special**. A Jordan algebra which cannot be obtained in such a way is called **exceptional**.*

An example of an exceptional Jordan algebra is the algebra M_3^8 of all 3×3 symmetric matrices over Cayley numbers (or octonions).[30]

Example 1 *Let B be a complex or real C*-algebra. Then its self-adjoint part B_{SA} with the symmetric multiplication is a JB-algebra. Each norm closed Jordan subalgebra of B_{SA} is as well a JB-algebra.*

Example 2 *Let H be a complex Hilbert space, $\mathcal{B}(H)$ be the C^*-algebra of all linear bounded operators on H. Then its self-adjoint part $\mathcal{B}(H)_{SA}$ with the symmetric multiplication is also a JB-algebra.*

Definition 32 *A JB-algebra which is isometrically Jordan isomorphic to an operator norm closed Jordan subalgebra of $\mathcal{B}(H)_{SA}$, is called a **JC-algebra**.*

Theorem 8 *Let A be a JC-algebra and J be a norm closed ideal in A. Then A/J is a JC-algebra, and in particular each homomorphic image of A is a JC-algebra.*

Each special JB-algebra is isometrically isomorphic to a JC-algebra[1]).

A homomorphic image of a special Jordan algebra does not have to be special. However, if you have a surjection from a special JB-algebra A onto a JB-algebra B, then B has to be special.

By a factor representation of a JB-algebra we mean a Jordan homomorphism from our JB-algebra onto a dense subalgebra of a JBW-factor. Recall that a JBW-algebra is a JB-algebra with a Banach predual space, and a JBW-factor is a JBW-algebra with its center being trivial, thus being composed of real scalar multiples of the identity element in the algebra.

Each JB-algebra has a separating family of factor representations[1]).

Theorem 9 [1] *Let A be a JB-algebra. Then*

(i) there is a unique (up to isomorphic Jordan isomorphism) Jordan ideal K in A such that A/K has a faithful isometric Jordan representation as a JC-algebra, and

(ii) every factor representation of A not annihilating K is onto the algebra M_3^8.

With such properties we call K an "exceptional ideal".

Another important result[1] states that all JBW-factors except M_3^8 are special (and thus isometrically isomorphic to JW-algebras - weakly operator closed Jordan subalgebras of $\mathcal{B}(H)_{SA}$ for some complex Hilbert space H), i.e. M_3^8 is the only exceptional JBW-factor.

CHAPTER 3.

PROPERTIES OF LOCALLY ADMISSIBLE OPERATOR ALGEBRAS

3.1 Real Locally Hilbert Spaces

We discuss here real analogues $H^{\mathbb{R}}$ of complex locally Hilbert spaces $H^{\mathbb{C}}$ and real analogies $L(H^{\mathbb{R}})$ of complex admissible continuous linear operator algebras $L(H^{\mathbb{C}})$.[33]

Definition 33 *(i) Let $\{H^{\mathbb{R}}_{\alpha};\ \alpha \in \Lambda\}$ be a family of real Hilbert spaces,indexed by a directed set Λ, such that*

$$\forall \alpha \preceq \beta \in \Lambda, \quad H^{\mathbb{R}}_{\alpha} \subset H^{\mathbb{R}}_{\beta}. \tag{3.1}$$

and let $H^{\mathbb{R}} = \cup H^{\mathbb{R}}_{\delta}$ be the union of this family.

(ii) Let us define a linear vector space structure on $H^{\mathbb{R}}$: $\forall \nu, \xi \in H^{\mathbb{R}}\ \exists \alpha, \beta, \gamma \in \Lambda:\ \nu \in H^{\mathbb{R}}_{\alpha},\ \xi \in H^{\mathbb{R}}_{\beta},\ \ \alpha \preceq \gamma,\ \beta \preceq \gamma,\ a, b \in \mathbb{R}$ and, correspondingly $\nu, \xi \in H^{\mathbb{R}}_{\gamma}$

$$a\nu \overset{H^{\mathbb{R}}}{+} b\xi = a \cdot \nu \overset{\gamma}{+} b \cdot \xi\ . \tag{3.2}$$

(iii) Let the vector space $H^{\mathbb{R}}$ be equipped with the inner product:

$$< \nu, \xi >_{H^{\mathbb{R}}} = \ < \nu, \xi >_{\gamma}, \tag{3.3}$$

where "$\cdot$" is the multiplication of a real number by a vector in $H^{\mathbb{R}}_{\gamma}$, $\overset{\gamma}{+}$ is an operation of addition of vectors ν and ξ in $H^{\mathbb{R}}_{\gamma}$, $< \nu, \xi >_{\gamma}$ is the inner product in $H^{\mathbb{R}}_{\gamma}$, $\nu, \xi \in H^{\mathbb{R}}_{\gamma}$ and $a, b \in \mathbb{R}$ are real numbers.

(iv) Let us determine the topology $\tau_{H^{\mathbb{R}}}$ on $H^{\mathbb{R}}$ by defining U as a ***closed subset*** *in $H^{\mathbb{R}}$ iff it is the whole $H^{\mathbb{R}}$, or there exists an $\alpha \in \Lambda$, such that U is a closed subset in*

$H^{\mathbb{R}}_{\alpha}$. To show that this is a topology one may argue along the same lines as,[33] lemma 5.1.

Iff conditions (i)-(iv) are satisfied, then we call $H^{\mathbb{R}}$ a ***real locally Hilbert space.***

Here and after we will consider only essential real Hilbert spaces (definition 12, $H^{\mathbb{R}} \cap iH^{\mathbb{R}} = \{\mathbf{0}\} \in H^{\mathbb{R}}$).

Remark 2 *A real locally Hilbert space $H^{\mathbb{R}}$ is a topological space T_1-space (proof is analogous to Inoue[33]).*

Let $\nu \in H^{\mathbb{R}}_{\alpha},\ \xi \in H^{\mathbb{R}}_{\beta}$. If $\alpha \preceq \beta$, then the inner product is $< \nu, \xi >_{H^{\mathbb{R}}} = < \nu, \xi >_{\beta}$. If α and β are not compatible, then there exists γ, greater than α and β separately (because Λ is a directed set). Then, $< \nu, \xi >_{H^{\mathbb{R}}} = < \nu, \xi >_{\gamma}$.

Remark 3 *Recall[41] that if F is a closed subspace in a Hilbert space H, then $H = F \oplus F^{\perp}$ where $F^{\perp} = \{\eta \in H : \ < \xi, \eta >= 0,\ \forall \xi \in F\}$ is orthogonal complement of F in H. It implies that any vector $\mu \in H$ can be uniquely presented as $\mu = \xi + \eta$, where $\xi \in F,\ \eta \in F^{\perp}$.*

Lemma 1 *Let $H^{\mathbb{R}}$ be a locally Hilbert space and arbitrary $\xi \in H^{\mathbb{R}}$. For each $\alpha \in \Lambda$, there exist unique $\xi_{\alpha} \in H^{\mathbb{R}}_{\alpha}$ and $\xi' \in (H^{\mathbb{R}}_{\alpha})^{\perp}_{H^{\mathbb{R}}}$ such that $\xi = \xi_{\alpha} + \xi'$.*

Proof. For any $\xi \in H^{\mathbb{R}}$ there exists $\beta \in \Lambda$ such that $\xi \in H^{\mathbb{R}}_{\beta}$. Let $\alpha \in \Lambda$ be arbitrary. There exists $\gamma \in \Lambda$, such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. The Hilbert space $H^{\mathbb{R}}_{\alpha}$ is a closed subspace of $H^{\mathbb{R}}_{\gamma}$ and $\xi \in H^{\mathbb{R}}_{\gamma}$. Thus, due to remark 3 there exist unique $\xi' \in (H^{\mathbb{R}}_{\alpha})^{\perp}_{H^{\mathbb{R}}_{\gamma}} \subset (H^{\mathbb{R}}_{\alpha})^{\perp}_{H^{\mathbb{R}}}$ and $\xi_{\alpha} \in H^{\mathbb{R}}_{\alpha}$, such that $\xi = \xi_{\alpha} + \xi'$. ■

Theorem 10 *Let $H^{\mathbb{R}}$ be a real locally Hilbert space. Then a complexification*

$$H^{\mathbb{C}} = H^{\mathbb{R}} \dot{+} iH^{\mathbb{R}} \tag{3.4}$$

of $H^{\mathbb{R}}$ can be equipped with a structure of a complex locally Hilbert space $H^{\mathbb{C}}$, in such a way that

(i) the family of complex Hilbert spaces $\{H^{\mathbb{C}}_\alpha; \alpha \in \Lambda\}$, $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$, $H^{\mathbb{C}}_\alpha \subset H^{\mathbb{C}}_\beta$, each $H^{\mathbb{C}}_\alpha$ is given as

$$H^{\mathbb{C}}_\alpha = H^{\mathbb{R}}_\alpha \dot{+} iH^{\mathbb{R}}_\alpha, \tag{3.5}$$

(ii) $H^{\mathbb{C}}$ can be equipped with the inner product

$$\ll v, w \gg \; = \ll \xi + i\eta, \xi' + i\eta' \gg = <\xi, \xi'> + <\eta, \eta'> + i(<\eta, \xi'> - <\xi, \eta'>), \tag{3.6}$$

where $\xi, \xi', \eta, \eta' \in H^{\mathbb{R}}$, $v = \xi + i\eta$, $w = \xi' + i\eta'$, $v, w \in H^{\mathbb{C}}$ and $< ., .>$ the inner product on $H^{\mathbb{R}}$.

(iii) $H^{\mathbb{C}}$ can be equipped with some topology $\tau_{H^{\mathbb{C}}}$, such that $\tau_{H^{\mathbb{C}}} \supset \tau_{H^{\mathbb{R}}}$, where $\tau_{H^{\mathbb{R}}}$ is a topology on $H^{\mathbb{R}}$.

Proof. (i) First, note that for any $\alpha \in \Lambda$ a complexification $H^{\mathbb{C}}_\alpha = H^{\mathbb{R}}_\alpha \dot{+} iH^{\mathbb{R}}_\alpha$ is complex Hilbert space[45] ($H^{\mathbb{R}}_\alpha \cap iH^{\mathbb{R}}_\alpha = \{0\}$). Let $\xi_\alpha + i\eta_\alpha \in H^{\mathbb{C}}_\alpha$, $\xi_\alpha, \eta_\alpha \in H^{\mathbb{R}}_\alpha$, then for any $\beta \succeq \alpha$ $\xi_\alpha, \eta_\alpha \in H^{\mathbb{R}}_\beta$ and $\xi_\alpha + i\eta_\alpha \in H^{\mathbb{C}}_\beta$. Thus there exists a family of complex Hilbert spaces $H^{\mathbb{C}}_\alpha$, such that $H^{\mathbb{C}}_\alpha \subset H^{\mathbb{C}}_\beta$ as long as $\alpha \preceq \beta \in \Lambda$.

Second, we define a linear vector space structure on $H^{\mathbb{C}}$: $\forall \nu, w \in H^{\mathbb{C}}$ $\exists \alpha, \beta, \gamma \in \Lambda :$ $\nu \in H^{\mathbb{C}}_\alpha$, $w \in H^{\mathbb{C}}_\beta$, $\alpha \preceq \gamma$, $\beta \preceq \gamma$, $\mathbf{a}, \mathbf{b} \in \mathbb{C}$ and, correspondingly $\nu \in H^{\mathbb{C}}_\gamma$, $w \in H^{\mathbb{C}}_\gamma$

$$\mathbf{a}\nu \overset{H^{\mathbb{C}}}{+} \mathbf{b}w = \mathbf{a} \cdot \nu \overset{H^{\mathbb{C}}_\gamma}{+} \mathbf{b} \cdot w \; . \tag{3.7}$$

(ii) The bilinear form $\ll .,. \gg$ is the inner product on $H^{\mathbb{C}}$. Indeed, it holds conjugate symmetry

$$\overline{\ll w, v \gg} = <\xi', \xi> + <\eta', \eta> - i(<\xi', \eta> - <\eta', \xi>) \tag{3.8}$$

$$= <\xi, \xi'> + <\eta, \eta'> + i(<\xi, \eta'> - <\eta, \xi'>) = \ll v, w \gg,$$

additiveness in the first slot

$$\ll v + z, w \gg = <\xi + \xi'', \xi'> + <\eta + \eta'', \eta'> + i(<\eta + \eta'', \xi'> - <\xi + \xi'', \eta'>) \tag{3.9}$$

$$= <\xi, \xi'> + <\eta, \eta'> + i(<\eta, \xi'> - <\xi, \eta'>) + <\xi'', \xi'>$$

$$+ <\eta'', \eta'> + i(<\eta'', \xi'> - <\xi'', \eta'>) = \ll v, w \gg + \ll z, w \gg,$$

homogeneousness in the first slot

$$\ll (a + ib)v, w \gg = \ll av, w \gg + \ll ibv, w \gg = a[<\xi, \xi'> + <\eta, \eta'> \tag{3.10}$$

$$+ i(<\eta, \xi'> - <\xi, \eta'>)] + b[-<\eta, \xi'> + <\xi, \eta'> + i(<\xi', \xi> + <\eta', \eta>)]$$

$$= (a + ib)[<\xi, \xi'> + <\eta, \eta'>] + (b - ia)[-<\eta, \xi'> + <\xi, \eta'>)]$$

$$= (a + ib)[<\xi, \xi'> + <\eta, \eta'>] + (a + ib)i[<\eta, \xi'> - <\xi, \eta'>)]$$

$$= (a + ib) \ll v, w \gg,$$

and positive definiteness

$$\ll v, v \gg = <\xi, \xi> + <\eta, \eta> + i(<\eta, \xi> - <\xi, \eta>) = <\xi, \xi> + <\eta, \eta> \; \geq 0, \tag{3.11}$$

$$\ll v, v \gg = 0 \;\; \Leftrightarrow \xi = 0 \text{ and } \eta = 0,$$

where $z = \xi'' + i\eta'' \in H^{\mathbb{C}}, \; a, b \in \mathbb{R}$.

(iii) Closed subsets on $H^{\mathbb{C}}$ will be formed by $H^{\mathbb{C}}$ itself and all closed subsets in each $H^{\mathbb{C}}_{\alpha}, \; \alpha \in \Lambda$ closed in the norm, generated by the corresponding scalar product

$\ll .,. \gg_\alpha$. If we embed $H^{\mathbb{R}} \hookrightarrow H^{\mathbb{C}}$, then any closed subset in $H^{\mathbb{R}}$ will be a closed subset in $H^{\mathbb{C}}$. ■

Corollary 1 *Let $H^{\mathbb{C}} = H^{\mathbb{R}} \dot{+} iH^{\mathbb{R}}$ be a complex locally Hilbert space and let*

$$\tilde{H}^{\mathbb{C}} = \underset{\alpha\in\Lambda}{\cup} H_\alpha^{\mathbb{C}}, \tag{3.12}$$

be a union of complex Hilbert spaces, where $H_\alpha^{\mathbb{C}} = H_\alpha^{\mathbb{R}} \dot{+} iH_\alpha^{\mathbb{R}}$. Then $\tilde{H}^{\mathbb{C}}$ can be equipped with linear structure and inner product to become a complex locally Hilbert space which coincides with $H^{\mathbb{C}}$.

Proof. Let the subject family for the $\tilde{H}^{\mathbb{C}}$ be $\{H_\alpha^{\mathbb{C}};\ \alpha \in \Lambda\}$. Then the linear structure and inner product will be imposed for $\tilde{H}^{\mathbb{C}}$.

Two locally Hilbert spaces are equivalent if subject families of Hilbert spaces coincide. Indeed, for $H^{\mathbb{C}}$ and $\tilde{H}^{\mathbb{C}}$ their subject families $\{H_\alpha^{\mathbb{C}};\ \alpha \in \Lambda\}$ are the same; this is why $H^{\mathbb{C}}$ and $\tilde{H}^{\mathbb{C}}$ coincide. ■

Corollary 2 *For any vector $\omega \in H^{\mathbb{C}} = H_\alpha^{\mathbb{R}} \dot{+} iH_\alpha^{\mathbb{R}}$ there exist a unique pair of vectors $\xi, \eta \in H^{\mathbb{R}}$ such that $\omega = \xi + i\eta$.*

Proof. Suppose on the contrary that $\omega = \xi_1 + i\eta_1$, $\omega = \xi_2 + i\eta_2$, $\xi_1 \neq \xi_2$, $\eta_1 \neq \eta_2$. Then, $0 + i0 = \xi_1 - \xi_2 + i(\eta_1 - \eta_2)$ and $\xi_1 = \xi_2$, $\eta_1 = \eta_2$ due to $H_\alpha^{\mathbb{R}} \cap iH_\alpha^{\mathbb{R}} = 0 \in H^{\mathbb{R}}$. ■

3.2 Properties of Admissible Operators

Definition 34 *Let $H^{\mathbb{R}}$ be a locally Hilbert space with a directed family $\{H_\alpha^{\mathbb{R}}; \alpha \in \Lambda\}$. We define a* ***locally Hilbert space projection***

$$P_\alpha :\ H^{\mathbb{R}} \to H_\alpha^{\mathbb{R}}, \tag{3.13}$$

from $H^{\mathbb{R}}$ *onto* $H^{\mathbb{R}}_{\alpha}$ *in the following manner: Let* ξ *be an arbitrary vector from* $H^{\mathbb{R}}$. *According to lemma 1 there exists* $\xi_\alpha \in H^{\mathbb{R}}_{\alpha},\ \xi' \in (H^{\mathbb{R}}_{\alpha})^{\perp}_{H^{\mathbb{R}}}$ *such that* $\xi = \xi_\alpha + \xi'$ *and* $P_\alpha(\xi) = P_\alpha(\xi_\alpha + \xi') = \xi_\alpha$.

Lemma 2 *There exists one-to-one correspondence between a projective family of vectors* $\{\xi_\alpha\}_{\alpha\in\Lambda},\ \xi_\alpha \in H^{\mathbb{R}}_{\alpha}$ *and a vector* $\xi \in H^{\mathbb{R}}$, *such that* $P_\alpha(\xi) = \xi_\alpha,\ \forall\alpha \in \Lambda$.

Proof. Let on the contrary $\xi' \in H^{\mathbb{R}}$ be one more vector corresponding to the projective family $\{\xi_\alpha\}_{\alpha\in\Lambda}$. Then $P_\alpha(\xi') = \xi_\alpha,\ \forall\alpha \in \Lambda$; it contradicts to $\xi_\beta = P_\beta(\xi) \neq P_\beta(\xi')$ for some $\beta \in \Lambda$.

Conversely, let some $\beta \in \Lambda$ be such that $\xi'_\beta \neq \xi_\beta$ and at least two projective families $\{\xi_\alpha\}_{\alpha\in\Lambda}$ and $\{\xi'_\alpha\}_{\alpha\in\Lambda}$ correspond to $\xi \in H^{\mathbb{R}}$. It is impossible because $P_\beta(\xi) = \xi_\beta \neq \xi'_\beta = P_\beta(\xi)$. ■

Definition 35 *We define a* ***Hilbert space projection***

$$P_{\alpha\beta} :\ H^{\mathbb{R}}_{\beta} \to H^{\mathbb{R}}_{\alpha}, \tag{3.14}$$

from $H^{\mathbb{R}}_{\beta}$ *onto* $H^{\mathbb{R}}_{\alpha}$ *in the following manner:* $P_{\alpha\beta}(\xi_\beta) = P_{\alpha\beta}(\xi_\alpha + \xi') = \xi_\alpha$, *where* $\xi_\beta = \xi_\alpha + \xi'$ *with* $\xi_\alpha \in H^{\mathbb{R}}_{\alpha},\ \xi' \in (H^{\mathbb{R}}_{\alpha})^{\perp}_{H^{\mathbb{R}}_{\beta}}$.

Definition 36 *A family of vectors* $\{\xi_\alpha\}_{\alpha\in\Lambda},\ \xi_\alpha \in H^{\mathbb{R}}_{\alpha}$ *is called* ***projective family of vectors*** *iff it satisfies the following formula*

$$P_{\alpha\beta}(\xi_\beta) = \xi_\alpha,\quad \forall\alpha \preceq \beta,\ \alpha,\beta \in \Lambda. \tag{3.15}$$

Definition 37 *Let*

$$T :\ H^{\mathbb{R}} \to H^{\mathbb{R}}, \tag{3.16}$$

be a real linear operator on a real locally Hilbert space $H^{\mathbb{R}}$.

*We define a **restriction operator** T_α as follows:*

$$T_\alpha = T|_{H_\alpha^{\mathbb{R}}}, \quad (T_\alpha(\xi) = T(\xi), \ \ \textit{for} \ \ \xi \in H_\alpha^{\mathbb{R}}). \tag{3.17}$$

Definition 38 *Let T be invariant for each $H_\alpha^{\mathbb{R}}$*

$$T(H_\alpha^{\mathbb{R}}) \subset H_\alpha^{\mathbb{R}}, \quad \forall \alpha \in \Lambda. \tag{3.18}$$

and

$$T_\alpha \circ P_{\alpha\beta} = P_{\alpha\beta} \circ T_\beta, \ \ \forall \alpha, \beta \in \Lambda, \ \alpha \preceq \beta. \tag{3.19}$$

*In this case we call T an **admissible operator** on $H^{\mathbb{R}}$. Now $L(H^{\mathbb{R}})$ will denote the set of all continuous linear admissible operators on $H^{\mathbb{R}}$.*

Lemma 3 *An admissible real linear operator T on $H^{\mathbb{R}}$ is continuous iff T_α is a bounded real linear operator on $H_\alpha^{\mathbb{R}}$ for each $\alpha \in \Lambda$.*

Proof. With minor modifications, repeat the proof of,[33] lemma 5.2. ■

Lemma 4 *Let $H^{\mathbb{R}}$ be a real locally Hilbert space and arbitrary $\alpha \preceq \beta, \ \alpha, \beta \in \Lambda$. Then*

$$P_{\alpha\beta} \circ P_\beta = P_\alpha. \tag{3.20}$$

Proof. We choose $\xi \in H^{\mathbb{R}}$ as follows: $\xi = \xi_\beta + \xi'$ with $\xi_\beta \in H_\beta^{\mathbb{R}}, \ \xi' \in (H_\beta^{\mathbb{R}})^{\perp}_{H^{\mathbb{R}}}$. Also $\xi_\beta = \xi_\alpha + \xi''$ with $\xi_\alpha \in H_\alpha^{\mathbb{R}}, \ \xi'' \in (H_\alpha^{\mathbb{R}})^{\perp}_{H_\beta^{\mathbb{R}}}$ and $\xi = \xi_\alpha + \xi'''$ with $\xi_\alpha \in H_\alpha^{\mathbb{R}}, \ \xi''' \in (H_\alpha^{\mathbb{R}})^{\perp}_{H^{\mathbb{R}}}$. Then

$$P_{\alpha\beta} P_\beta(\xi) = P_{\alpha\beta} P_\beta(\xi_\beta + \xi') = P_{\alpha\beta}(\xi_\beta) = P_{\alpha\beta}(\xi_\alpha + \xi'') \tag{3.21}$$

$$= \xi_\alpha = P_\alpha(\xi_\alpha) = P_\alpha(\xi_\alpha + \xi''') = P_\alpha(\xi),$$

which proves the lemma. ■

3.3 Projective Family of Operators

Definition 39 *A family of operators* $\{T_\alpha\}_{\alpha\in\Lambda}$ *is called* ***projective family of operators*** *if it satisfies the following condition*

$$P_{\alpha\beta}\circ T_\beta \circ P_\beta = T_\alpha \circ P_\alpha, \quad \alpha \preceq \beta,\ \alpha,\beta\in\Lambda. \tag{3.22}$$

Lemma 5 *Any admissible operator* $T\in L(H^{\mathbb{R}})$ *defines a unique projective family of operators* $\{T_\alpha\}_{\alpha\in\Lambda}$*, such that* $T|_{H^{\mathbb{R}}_\alpha} = T_\alpha \in \mathcal{B}(H^{\mathbb{R}}_\alpha)$.

Conversely, every projective family of operators $\{T_\alpha\}_{\alpha\in\Lambda}$, $T_\alpha\in\mathcal{B}(H^{\mathbb{R}}_\alpha)$ *uniquely defines an admissible operator* $T\in L(H^{\mathbb{R}})$ *such that* $T|_{H^{\mathbb{R}}_\alpha} = T_\alpha\in\mathcal{B}(H^{\mathbb{R}}_\alpha)$.

Proof. Let T be an admissible operator and $H^{\mathbb{R}} = \bigcup_{\alpha\in\Lambda} H^{\mathbb{R}}_\alpha$ be a locally Hilbert space. An operator T_α is linear because the linear operator T is restricted on some linear subspace. T_α is continuous because $H^{\mathbb{R}}_\alpha$ is closed in $H^{\mathbb{R}}$. Since T_α is linear and continuous, then it is bounded and $T_\alpha\in\mathcal{B}(H^{\mathbb{R}}_\alpha)$.

For a given $\alpha\in\Lambda$ we have $\xi=\xi_\beta+\xi'$, $\xi\in H^{\mathbb{R}}$, $\xi_\alpha,\eta_\alpha\in H^{\mathbb{R}}_\alpha$, $\xi_\beta,\eta_\beta\in H^{\mathbb{R}}_\beta$, $\xi'\in(H^{\mathbb{R}}_\alpha)^\perp_{H^{\mathbb{R}}}$, $\eta^\perp\in(H^{\mathbb{R}}_\alpha)^\perp_{H^{\mathbb{R}}_\beta}$, $T_\alpha(\xi_\alpha)=\eta_\alpha$, $T_\beta(\xi_\beta)=\eta_\beta$. Then

$$P_{\alpha\beta}(T_\beta(P_\beta(\xi)) = P_{\alpha\beta}(T_\beta(P_\beta(\xi_\beta+\xi')) = P_{\alpha\beta}(T_\beta(\xi_\beta)) = P_{\alpha\beta}(\eta_\beta) \tag{3.23}$$

$$= P_{\alpha\beta}(\eta_\alpha+\eta^\perp) = \eta_\alpha = T_\alpha(\xi_\alpha) = T_\alpha P_\alpha(\xi_\alpha+\xi') = T_\alpha(P_\alpha(\xi)).$$

Let $\{T_\alpha\}_{\alpha\in\Lambda}$ be a projective family of operators with

$$T_\alpha P_{\alpha\beta}(\xi_\beta) = T_\alpha P_{\alpha\beta}(\xi_\alpha+\xi^\perp) = T_\alpha(\xi_\alpha) = \eta_\alpha, \tag{3.24}$$

and

$$P_{\alpha\beta}T_\beta(\xi_\beta) = P_{\alpha\beta}(\eta_\beta) = P_{\alpha\beta}(\eta_\alpha+\eta^\perp) = \eta_\alpha, \quad \xi^\perp\in(H^{\mathbb{R}}_\alpha)^\perp_{H^{\mathbb{R}}_\beta}. \tag{3.25}$$

From the previous two equations we conclude that formula 3.19 is valid, which proves that T is admissible. ■

Corollary 3 *A continuous linear operator T from $H^{\mathbb{R}}$ to $H^{\mathbb{R}}$ is admissible iff there exists a projective family of operators $\{T_\alpha\}_{\alpha\in\Lambda}$, $T_\alpha \in \mathcal{B}(H_\alpha^{\mathbb{R}})$ such that*

$$P_\alpha \circ T = T_\alpha \circ P_\alpha. \tag{3.26}$$

Proof. From the previous lemma 5 an admissible operator determines a unique projective family of operators and vice versa. It remains to prove the formula 5. Let all notations be as in the above lemma and $\eta' \in (H_\alpha^{\mathbb{R}})^{\perp}_{H^{\mathbb{R}}},\ T(\xi) = \eta$

$$P_\alpha T(\xi) = P_\alpha(\eta) = P_\alpha(\eta_\alpha + \eta') = \eta_\alpha,$$

and

$$T_\alpha P_\alpha(\xi_\alpha + \xi') = T_\alpha(\xi_\alpha) = \eta_\alpha.$$

Last two equations prove the corollary. ■

Lemma 6 *The set of all admissible real linear operators $L(H^{\mathbb{R}})$ can be equipped with operations turning it to a real *-algebra.*

Proof. Let us introduce addition, scalar multiplication, multiplication and involution for admissible operators and show that the result is again an admissible operator. Let T and R be admissible operators.

a) Addition: We define $T + R$

$$(T + R)(\xi) = T(\xi) + R(\xi),\ \forall \xi \in H^{\mathbb{R}},\ \forall T, R \in L(H^{\mathbb{R}}). \tag{3.27}$$

We show that thus defined operator is admissible. $T+R$ is obviously continuous. Let ξ_α be the vector in Hilbert space $H_\alpha^{\mathbb{R}}$, then $T(\xi_\alpha) \in H_\alpha^{\mathbb{R}}$, $R(\xi_\alpha) \in H_\alpha^{\mathbb{R}}$ (because T and R are admissible), the sum $(T+R)(\xi_\alpha) = T(\xi_\alpha) + R(\xi_\alpha) \in H_\alpha^{\mathbb{R}}$ due to linearity of the Hilbert space $H_\alpha^{\mathbb{R}}$.

If $T_\alpha \circ P_{\alpha\beta} = P_{\alpha\beta} \circ T_\beta$ and $R_\alpha \circ P_{\alpha\beta} = P_{\alpha\beta} \circ R_\beta$, then $(T_\alpha + R_\alpha) \circ P_{\alpha\beta} = P_{\alpha\beta} \circ (T_\beta + R_\beta)$. Therefore $(T+R)$ is admissible.

b) Multiplication by a scalar: We define λR as

$$\lambda R(\xi) = \lambda(R(\xi)),\ \forall \xi \in H^{\mathbb{R}},\ \forall R \in L(H^{\mathbb{R}}),\ \forall \lambda \in \mathbb{R}, \tag{3.28}$$

λR is obviously continuous. Let $\xi_\alpha \in H_\alpha^{\mathbb{R}}$, then $R(\xi_\alpha) \in H_\alpha^{\mathbb{R}}$. Thus $\lambda(R(\xi_\alpha)) \in H_\alpha^{\mathbb{R}}$, because $H_\alpha^{\mathbb{R}}$ is a linear space.

If $R_\alpha \circ P_{\alpha\beta} = P_{\alpha\beta} \circ R_\beta$, then $(\lambda R_\alpha) \circ P_{\alpha\beta} = P_{\alpha\beta} \circ (\lambda R_\beta)$. So, λR is admissible.

c) Multiplication: define TR as

$$TR(\xi) = T(R(\xi)),\quad \forall \xi \in H^{\mathbb{R}},\ \forall T, R \in L(H^{\mathbb{R}}). \tag{3.29}$$

TR is continuous as a composition of continuous mappings. Let now $\xi_\alpha \in H_\alpha^{\mathbb{R}}$, then $TR(\xi_\alpha) = T(R(\xi_\alpha)) \in H_\alpha^{\mathbb{R}}$, because $R(\xi_\alpha) \in H_\alpha^{\mathbb{R}}$, and $T(H_\alpha^{\mathbb{R}}) \subset H_\alpha^{\mathbb{R}}$.

If $\xi^\perp \in (H_\alpha^{\mathbb{R}})^\perp_{H_\beta^{\mathbb{R}}}$, $\theta^\perp \in (H_\alpha^{\mathbb{R}})^\perp_{H_\beta^{\mathbb{R}}}$, $R_\alpha(\xi_\alpha) = \eta_\alpha$, $T_\alpha(\eta_\alpha) = \theta_\alpha$, $R_\beta(\xi_\beta) = \eta_\beta$, $T_\beta(\eta_\beta) = \theta_\beta$, then

$$(T_\alpha \circ R_\alpha) \circ P_{\alpha\beta}(\xi_\beta) = (T_\alpha \circ R_\alpha) \circ P_{\alpha\beta}(\xi_\alpha + \xi^\perp) = T_\alpha \circ R_\alpha(\xi_\alpha) = T_\alpha(\eta_\alpha) = \theta_\alpha,$$

and

$$P_{\alpha\beta} \circ (T_\beta \circ R_\beta)(\xi_\beta) = P_{\alpha\beta} \circ T_\beta(\eta_\beta) = P_{\alpha\beta}(\theta_\beta) = P_{\alpha\beta}(\theta_\alpha + \theta^\perp) = \theta_\alpha.$$

Thus, the product of admissible operators is admissible.

d) Involution: Let $T \in L(H^{\mathbb{R}})$ and $T_\alpha \in \mathcal{B}(H^{\mathbb{R}}_\alpha)$ is as above and T^*_α is the involution of T_α for all $\alpha \in \Lambda$.

Observe that

$$< T^*_\alpha(\xi_\alpha), \eta_\alpha >_{H^{\mathbb{R}}_\alpha} = \ < \xi_\alpha, T_\alpha(\eta_\alpha) >_{H^{\mathbb{R}}_\alpha} = \ < \xi_\alpha, T_\beta(\eta_\alpha) >_{H^{\mathbb{R}}_\alpha},$$

and

$$< T^*_\alpha(\xi_\alpha), \eta_\alpha >_{H^{\mathbb{R}}_\alpha} = \ < T^*_\beta(\xi_\alpha), \eta_\alpha >_{H^{\mathbb{R}}_\beta} = \ < \xi_\alpha, T_\beta(\eta_\alpha) >_{H^{\mathbb{R}}_\beta} .$$

From the last two equations we deduce $< T^*_\alpha(\xi_\alpha), \eta_\alpha >_{H^{\mathbb{R}}_\alpha} = \ < T^*_\beta(\xi_\alpha), \eta_\alpha >_{H^{\mathbb{R}}_\beta}$. Recall[41] that there is one to one correspondence between operators acting on a Hilbert space and continuous sesquilinear forms. As a corollary we get that two linear bounded operators A and B on a Hilbert space H are equal iff $< A(\eta), \xi > = < B(\eta), \xi >, \ \forall \xi, \eta \in H$. Then

$$T^*_\beta|_{H^{\mathbb{R}}_\alpha} = T^*_\alpha, \ \forall \alpha \preceq \beta, \ \alpha, \beta \in \Lambda. \tag{3.30}$$

Let us now show that $\{T^*_\alpha\}_{\alpha \in \Lambda}$ is a projective family of operators.

For a given $\alpha \in \Lambda$ we have $\xi_\alpha, \eta^*_\alpha \in H^{\mathbb{R}}_\alpha$, $T_\alpha \circ P_{\alpha\beta} = P_{\alpha\beta} \circ T_\beta$ and $R_\alpha \circ P_{\alpha\beta} = P_{\alpha\beta} \circ R_\beta$, then $(T_\alpha + R_\alpha) \circ P_{\alpha\beta} = P_{\alpha\beta} \circ (T_\beta + R_\beta).$, $\eta^*_\beta \in H^{\mathbb{R}}_\beta$, $\xi^\perp \in (H^{\mathbb{R}}_\alpha)^\perp_{H^{\mathbb{R}}_\beta}$, $\eta^\perp \in (H^{\mathbb{R}}_\alpha)^\perp_{H^{\mathbb{R}}_\beta}$, $T^*_\alpha(\xi_\alpha) = \eta^*_\alpha$, $T^*_\beta(\xi_\beta) = \eta^*_\beta$.

Then

$$T^*_\alpha P_{\alpha\beta}(\xi_\beta) = T^*_\alpha P_{\alpha\beta}(\xi_\alpha + \xi^\perp) = T^*_\alpha(\xi_\alpha) = \eta^*_\alpha, \tag{3.31}$$

and

$$P_{\alpha\beta} T^*_\beta(\xi_\beta) = P_{\alpha\beta}(\eta^*_\beta) = P_{\alpha\beta}(\eta^*_\alpha + \eta^\perp) = \eta^*_\alpha. \tag{3.32}$$

So, we conclude that $\{T^*_\alpha\}_{\alpha\in\Lambda}$ is a projective family of operators ($T^*_\alpha \circ P_{\alpha\beta} = P_{\alpha\beta}\circ T^*_\beta$) and by lemma 5 there is a unique admissible operator $T^* \in L(H^{\mathbb{R}})$ determined by a projective family of operators $\{T^*_\alpha\}_{\alpha\in\Lambda}$. ■

The following theorem is important to develop Gelfand-Naimark theory for algebras that are projective limits of operator algebras:

Theorem 11 *A real *-algebra of all admissible continuous real linear operators $L(H^{\mathbb{R}})$ on a real locally Hilbert space $H^{\mathbb{R}}$ can be equipped with a topology so that it is real *-isomorphic and homeomorphic to the projective limit with projective topology of a projective family $\{\mathcal{B}(H^{\mathbb{R}}_\alpha), g^\beta_\alpha,\ \alpha,\beta\in\Lambda\}$ of algebras $\mathcal{B}(H^{\mathbb{R}}_\alpha)$ of all bounded linear operators on real Hilbert spaces $H^{\mathbb{R}}_\alpha$.*

Proof. Let T be an arbitrary operator from $L(H^{\mathbb{R}})$ and let us define $T_\alpha : H^{\mathbb{R}}_\alpha \to H^{\mathbb{R}}_\alpha$ as follows: for any $\xi_\alpha \in H^{\mathbb{R}}_\alpha$, $T_\alpha(\xi_\alpha) = T(\xi_\alpha)$.

Due to the fact that $H^{\mathbb{R}}_\alpha$ is a linear subspace of $H^{\mathbb{R}}$ and $T(H^{\mathbb{R}}_\alpha) \subset H^{\mathbb{R}}_\alpha$, it follows that T_α is a linear operator from $H^{\mathbb{R}}_\alpha$ to $H^{\mathbb{R}}_\alpha$.

From continuity (by lemma 3 T_α is continuous) and linearity of T_α it follows that T_α is bounded and thus $T_\alpha \in \mathcal{B}(H^{\mathbb{R}}_\alpha)$.

Let us define $\pi_\alpha : L(H^{\mathbb{R}}) \to \mathcal{B}(H^{\mathbb{R}}_\alpha)$ as follows

$$\forall\alpha\in\Lambda,\ \forall T\in L(H^{\mathbb{R}}),\ \pi_\alpha(T) = T_\alpha, \tag{3.33}$$

We show that π_α is a *-homomorphism.

In fact, $\forall\xi_\alpha \in H^{\mathbb{R}}_\alpha$, $T, S \in \mathcal{B}(H^{\mathbb{R}}_\alpha)$, $b \in \mathbb{R}$, $S_\alpha(\xi_\alpha) = \eta_\alpha$, $T_\alpha(\eta_\alpha) = \zeta_\alpha$, $Q_\alpha(\xi_\alpha) = T_\alpha S_\alpha(\xi_\alpha) = \zeta_\alpha$, $T^*(\xi) = \eta^*$, $(\eta_\alpha)^* = \eta^*_\alpha$. Thus

i) π_α commutes with multiplication

$$\pi_\alpha(TS)(\xi_\alpha) = \pi_\alpha(Q)(\xi_\alpha) = Q_\alpha(\xi_\alpha) = \zeta_\alpha,$$

and

$$\pi_\alpha(T)(\pi_\alpha(S)(\xi_\alpha)) = \pi_\alpha(S_\alpha(\xi_\alpha)) = T_\alpha(\eta_\alpha) = \zeta_\alpha.$$

Then $\pi_\alpha(TS) = \pi_\alpha(T)(\pi_\alpha(S)$.

ii) π_α commutes with addition

$$\begin{aligned} \pi_\alpha(T+S)(\xi_\alpha) &= P_\alpha((T+S)(\xi)) = \\ P_\alpha(T(\xi)+S(\xi)) &= P_\alpha(\zeta+\eta) = \zeta_\alpha + \eta_\alpha, \end{aligned}$$

and

$$\pi_\alpha(T)(\xi_\alpha) + \pi_\alpha(S)(\xi_\alpha) = T_\alpha(\xi_\alpha) + S_\alpha(\xi_\alpha) = \zeta_\alpha + \eta_\alpha.$$

Then $\pi_\alpha(T+S) = \pi_\alpha(T) + \pi_\alpha(S)$.

iii) π_α commutes with multiplication by real scalars

$$\pi_\alpha(bT)(\xi_\alpha) = P_\alpha(bT(\xi)) = P_\alpha(b\eta) = b\eta_\alpha,$$

and

$$\pi_\alpha(T)(\xi_\alpha) = bP_\alpha(T(\xi)) = bP_\alpha(\eta) = b\eta_\alpha.$$

Then $\pi_\alpha(bT) = b\pi_\alpha(T)$.

iv) π_α commutes with involution

$$\begin{aligned} < \ & \pi_\alpha(T^*)(\xi_\alpha), \eta_\alpha >=< P_\alpha(T^*(\xi)), \eta_\alpha >=< T^*_\alpha(\xi_\alpha), \eta_\alpha >= \\ < \ & \xi_\alpha, T_\alpha(\eta_\alpha) >=< \xi_\alpha, \pi_\alpha(T)(\eta_\alpha) >=< \pi_\alpha(T)^*(\xi_\alpha), \eta_\alpha > . \end{aligned}$$

Then $\pi_\alpha(T^*) = \pi_\alpha(T)^*$.

In view of i)-iv) π_α is a *-homomorphism.

We prove that π_α is surjective.

Let $T_\alpha \in \mathcal{B}(H_\alpha^{\mathbb{R}})$ be an arbitrary fixed operator.

If $\forall \delta \preceq \alpha$, let ξ_δ be an arbitrary vector in $H_\delta^{\mathbb{R}}$. We define

$$T_\delta(\xi_\delta) = P_{\delta\alpha}(T_\alpha(\xi_\delta)) \tag{3.34}$$

Such defined T_δ is a linear operator:

i) additivity

$$T_\delta(\xi_\delta + \eta_\delta) = P_{\delta\alpha}(T_\alpha(\xi_\delta + \eta_\delta)) = P_{\delta\alpha}(T_\alpha(\xi_\alpha + \eta_\alpha))$$

$$= P_{\delta\alpha}(T_\alpha(\xi_\alpha) + T_\alpha(\eta_\alpha)) = T_\delta(\xi_\delta) + T_\delta(\eta_\delta).$$

ii) homogeneity

$$T_\delta(b\xi_\delta) = P_{\delta\alpha}(T_\alpha(b\xi_\delta)) = P_{\delta\alpha}(T_\alpha(b(\xi_\delta)))$$

$$= bP_{\delta\alpha}(T_\alpha(\xi_\delta)) = bT_\delta(\xi_\delta).$$

Thus $\forall \delta \succeq \alpha$, we define $T_\delta(\xi_\delta) = T_\alpha(P_{\alpha\gamma}(\xi_\delta))$, when $\xi_\delta \in H_\delta^{\mathbb{R}}$.

If δ is not comparable with α, then $\exists \gamma : \alpha \preceq \gamma$, $\delta \preceq \gamma$, and

$$T_\delta(\xi_\delta) = P_{\delta\alpha}(T_\gamma(\xi_\delta)).$$

The family $\{T_\delta\}_{\delta \in \Lambda}$ is a projective family of operators and by lemma 5 there exists a unique $T \in L(H_\alpha^{\mathbb{R}})$ such that $\pi_\alpha(T) = T_\alpha$.

So, π_α is surjective.

Let the morphisms $g_\alpha^\beta : \mathcal{B}(H_\beta^\mathbb{R}) \to \mathcal{B}(H_\alpha^\mathbb{R})$ be defined in the following manner

$$\forall T_\beta \neq 0,\ T_\beta \in \mathcal{B}(H_\beta^\mathbb{R}),\ \alpha \preceq \beta,\ \alpha, \beta \in \Lambda,\ g_\alpha^\beta(T_\beta) = \pi_\alpha(\pi_\beta^{-1}(T_\beta)).$$

Show that the morphism g_α^β is well defined. Let $T \in A,\ \pi_\beta(T) = T_\beta,\ \pi_\alpha(T) = T_\alpha,\ T' \neq T,\ T' \in \pi_\beta^{-1}(T_\beta)$.

Note that if surjective morphisms $\pi_\alpha,\ \pi_\beta$ and g_α^β are such that $\pi_\beta = g_\alpha^\beta \circ \pi_\alpha$, then $\ker \pi_\beta$ includes $\ker \pi_\alpha$ as a subset: $\ker \pi_\alpha \subseteq \ker \pi_\beta$.

Then, as long as $\pi_\beta(T') = T_\beta,\ T' = T + R,$ where $R \in \ker \pi_\beta$,or $R \in \ker \pi_\alpha$. So $\pi_\alpha(T') = \pi_\alpha(T + R) = T_\alpha$.

If we denote $\pi_\alpha(\pi_\beta^{-1}(T_\beta)) = T_\alpha \in \mathcal{B}(H_\alpha^\mathbb{R})$, one can see that

$$T_\alpha = \pi_\alpha(T) = (g_\alpha^\beta \circ \pi_\beta)(T),\ \forall T \in L(H^\mathbb{R}),\ \alpha \preceq \beta \in \Lambda.$$

We show that g_α^β are surjective *-homomorphisms.

Note first that $g_\alpha^\beta(\mathcal{B}(H_\beta^\mathbb{R})) \subset \mathcal{B}(H_\alpha^\mathbb{R})$. For any $T_\alpha \in \mathcal{B}(H_\alpha^\mathbb{R})$ there exists T_β, such that

$$T_\beta = (\pi_\beta(\pi_\alpha^{-1}(T_\alpha))), \tag{3.35}$$

which means that $g_\alpha^\beta(\mathcal{B}(H_\beta^\mathbb{R})) \supset \mathcal{B}(H_\alpha^\mathbb{R})$ - the surjection is shown. To prove that g_α^β's are *-homomorphisms we note that

$$g_\alpha^\beta(T_\beta S_\beta) = g_\alpha^\beta(\pi_\beta(T)\pi_\beta(S)) = g_\alpha^\beta(\pi_\beta(TS)) = \pi_\alpha(TS) \tag{3.36}$$

$$= \pi_\alpha(T)\pi_\alpha(S) = g_\alpha^\beta(\pi_\beta(T))g_\alpha^\beta(\pi_\beta(S)) = g_\alpha^\beta(T_\beta)g_\alpha^\beta(S_\beta).$$

and

$$g_\alpha^\beta(T_\beta^*) = g_\alpha^\beta(\pi_\beta(T^*)) = \pi_\alpha(T^*) = (\pi_\alpha(T))^* \tag{3.37}$$

$$= (g_\alpha^\beta(\pi_\beta(T)))^* = (g_\alpha^\beta(T_\beta))^*.$$

Due to the fact that for any $\alpha \preceq \beta$ g_α^β is surjective *-homomorphism between two real C*-algebras $\mathcal{B}(H_\beta^\mathbb{R})$ and $\mathcal{B}(H_\alpha^\mathbb{R})$, it is a contraction and thus is automatically continuous.[45]

We introduce seminorms as follows

$$p_\alpha(T) = ||T_\alpha||_{\mathcal{B}(H_\alpha^\mathbb{R})},\ T \in L(H^\mathbb{R}),\ \alpha \in \Lambda. \tag{3.38}$$

The set $\{p_\alpha\}_{\alpha\in\Lambda}$ is a family of C*-regular seminorms because each norm $||.||_{\mathcal{B}(H_\alpha^\mathbb{R})}$ on $\mathcal{B}(H_\alpha^\mathbb{R})$ is C*-regular[50] $\forall \alpha \in \Lambda$.

Now we show that the family of seminorms $\{p_\alpha\}_{\alpha\in\Lambda}$ is separating and saturated. Let T be any nonzero operator in $L(H^\mathbb{R})$, on the contrary assume that for all $\alpha \in \Lambda$ $p_\alpha(T) = 0$. It means that $||T_\alpha||_{\mathcal{B}(H_\alpha^\mathbb{R})} = 0$ which is equivalent $T_\alpha = 0_\alpha$ for any α i.e. $T = 0$. It contradicts to the original assumption.

Due to the fact that for any finite set of indices $\{\alpha_1, ..., \alpha_n\}$ $\exists \gamma \in \Lambda : \gamma \succeq \alpha_i, \forall i \in \overline{1,n}$. It means that for each pair γ, α_i there exists $g_{\alpha_i}^\gamma$ which is surjective contraction. This implies that

$$p_{\alpha_i}(T) = ||T_{\alpha_i}||_{\mathcal{B}(H_{\alpha_i}^\mathbb{R})} = ||g_{\alpha_i}^\gamma(T_\gamma)||_{\mathcal{B}(H_{\alpha_i}^\mathbb{R})} \le ||T_\gamma||_{\mathcal{B}(H_\gamma^\mathbb{R})} = p_\gamma(T) \tag{3.39}$$

$\forall T \in L(H^\mathbb{R})$ and $\forall i \in \overline{1,n}$.

We prove now that π_α is continuous for all $\alpha \in \Lambda$.

Let

$$U_\alpha = \{T_\alpha : ||T_\alpha||_{\mathcal{B}(H_\alpha^\mathbb{R})} < \varepsilon\} \tag{3.40}$$

be ε-neighbourhood of zero in $\mathcal{B}(H_\alpha^\mathbb{R})$. Then

$$\pi_\alpha^{-1}(U_\alpha) = \{T : p_\alpha(T) < \varepsilon,\ p_{\alpha_i}(T) < \varepsilon,\ \forall i \in \overline{1,n},\ \forall n\} \tag{3.41}$$

it is obviously an open set in A, where A is a projective limit of the projective family $\{\mathcal{B}(H^{\mathbb{R}}_\alpha), g^\beta_\alpha,\ \alpha, \beta \in \Lambda\}$ of C* operator algebras $\mathcal{B}(H^{\mathbb{R}}_\alpha)$:

$$A = \varprojlim(g^\beta_\alpha \mathcal{B}(H^{\mathbb{R}}_\beta)). \tag{3.42}$$

We equip A with the projective topology which consists of the following neighbourhoods of zero

$$\check{O}(\alpha_1, .., \alpha_n; \varepsilon) = \bigcap_{i=1}^{n} \{\check{\pi}^{-1}_{\alpha_i}(T_{\alpha_i} : ||T_{\alpha_i}||_{\mathcal{B}(H^{\mathbb{R}}_{\alpha_i})} < \varepsilon)\}. \tag{3.43}$$

We define $\varphi : L(H^{\mathbb{R}}) \to A$ as follows, $\varphi(T) = x_T$ such that $\pi_\alpha(T) = \check{\pi}_\alpha(x_T)$ for all $\alpha \in \Lambda$, arbitrary $T \in L(H^{\mathbb{R}})$, and $x_T \in A$.

Then φ is a *-homomorphism:

i) $\varphi(T + S) = x_T + x_S$

ii) $\varphi(bT) = bx_T$

iii) $\varphi(TS) = x_T\ x_S$

iv) $\varphi(T)^* = x^*_T$

i) - iv) follow from the *-homomorphism properties of π_α and $\check{\pi}_\alpha$.

Bijection follows from the fact that $\{T_\alpha\}_{\alpha\in\Lambda}$ uniquely determines $T \in L(H^{\mathbb{R}})$ and $x_T \in A$, the element of direct product.

The topology on $L(H^{\mathbb{R}})$ is defined with the ε-neighbourhood of zero

$$O(\alpha_1, .., \alpha_n; \varepsilon) = \{T : \forall \alpha_1, .., \alpha_n,\ p_{\alpha_i}(T) < \varepsilon)\}. \tag{3.44}$$

There is an obvious one to one correspondence between $O(\alpha_1, .., \alpha_n; \varepsilon)$, ε-neighbourhood of zero of $L(H^{\mathbb{R}})$ and $\check{O}(\alpha_1, .., \alpha_n; \varepsilon)$, ε-neighbourhood of zero of A; it proves that φ is a homeomorphism. ■

Lemma 7 *Let $L(H^{\mathbb{C}})$ be the algebra of all continuous linear admissible operators on $H^{\mathbb{C}}$. Let for any $\alpha \in \Lambda$, $\hat{\pi}_\alpha : L(H^{\mathbb{C}}) \to \mathcal{B}(H_\alpha^{\mathbb{C}}) : \hat{\pi}_\alpha(\hat{T}) = \hat{T}_\alpha$, such that for any $\zeta_\alpha \in H_\alpha^{\mathbb{C}}$, $\hat{T}_\alpha(\zeta_\alpha) = \hat{T}(\zeta_\alpha)$.*

*Then $\hat{\pi}_\alpha$ is a continuous surjective *-homomorphism.*

Proof. The proof word by word repeats the proof in theorem 11 that $\pi_\alpha : L(H^{\mathbb{R}}) \to \mathcal{B}(H_\alpha^{\mathbb{R}})$ is a continuous surjective *-homomorphism. ■

Lemma 8 *Let the morphisms $\hat{g}_\alpha^\beta : \mathcal{B}(H_\beta^{\mathbb{C}}) \to \mathcal{B}(H_\alpha^{\mathbb{C}})$ be defined in the following manner*

$$\forall T_\beta^c \in \mathcal{B}(H_\beta^{\mathbb{C}}),\ \alpha \preceq \beta \in \Lambda,\ \hat{g}_\alpha^\beta(T_\beta^c) = \hat{\pi}_\alpha(\hat{\pi}_\beta^{-1}(T_\beta)). \tag{3.45}$$

*Thus defined $\hat{g}_\alpha^\beta$ is a *-homomorphism and a surjection, hence a contraction and therefore continuous.*

Proof. The proof word by word repeats the proof that $g_\alpha^\beta : \mathcal{B}(H_\beta^{\mathbb{R}}) \to \mathcal{B}(H_\alpha^{\mathbb{R}})$ is continuous *-homomorphic surjective contraction in theorem 11. ■

Theorem 12 *An algebra of all continuous linear admissible operators $L(H^{\mathbb{C}})$ is *-isomorphic and homeomorphic to a projective limit of a projective family $\{\mathcal{B}(H_\alpha^{\mathbb{C}}), \hat{g}_\alpha^\beta,\ \alpha \preceq \beta \in \Lambda\}$ of all bounded linear operators on Hilbert spaces $H_\alpha^{\mathbb{C}}$.*

Proof. Let $H^{\mathbb{C}}$ be given as $H^{\mathbb{C}} = \underset{\delta\in\Lambda}{\cup} H_\delta^{\mathbb{C}}$, where $H_\alpha^{\mathbb{C}} \subset H_\beta^{\mathbb{C}}$, whenever $\alpha \preceq \beta$.

We need to establish that $\forall T_\alpha \in \mathcal{B}(H_\alpha^{\mathbb{R}}),\ \exists T \in L(H^{\mathbb{R}}),\ \pi_\alpha(T) = T_\alpha$. So, $\forall T_\alpha \in \mathcal{B}(H_\alpha^{\mathbb{R}})\ \xi' \in (H_\alpha^{\mathbb{R}})^{\perp}_{H^{\mathbb{R}}}$:

$$T(\xi) = T_\alpha(\xi_\alpha).$$

By lemma 5, for a family $\{T_\alpha\}_{\alpha\in\Lambda}$ there exists $T \in L(H^{\mathbb{R}}) : T|_{H^{\mathbb{R}}_\gamma} = T_\gamma$. Thus, by the previous lemmas 7 and 8, $\{\mathcal{B}(H^{\mathbb{C}}_\alpha), \hat{g}^\beta_\alpha,\ \alpha \preceq \beta \in \Lambda\}$ is a projective family whose projective limit is *-isomorphic and homeomorphic to $L(H^{\mathbb{C}})$. ■

Theorem 13 *Let $H^{\mathbb{R}}$ be a real locally Hilbert space. Then a complex linear algebra $L(H^{\mathbb{R}}) \dot{+} iL(H^{\mathbb{R}})$ can be identified with $L(H^{\mathbb{C}})$ the algebra of all continuous complex linear admissible operators on $H^{\mathbb{C}} = H^{\mathbb{R}} \dot{+} iH^{\mathbb{R}}$, a complex locally Hilbert space.*

Proof. Let $T \in L(H^{\mathbb{R}})$ be an arbitrary admissible operator acting on a locally real Hilbert space $H^{\mathbb{R}}$ and let us define

$$T^c : H^{\mathbb{C}} \to H^{\mathbb{C}}, \quad T^c(\xi + i\eta) = T(\xi) + iT(\eta),\ \forall \xi, \eta \in H^{\mathbb{R}},\ T^c \in L(H^{\mathbb{C}}). \tag{3.46}$$

We prove that $T^c \in L(H^{\mathbb{C}})$.

Let T be an admissible operator, $\xi \in H^{\mathbb{R}}_{\alpha'}$ and $\eta \in H^{\mathbb{R}}_{\alpha''}$. Then $\xi + i\eta \in H^{\mathbb{C}}_\alpha$, where $\alpha \succeq \alpha'$ and $\alpha \succeq \alpha''$. So, $T(\xi) + iT(\eta) = \zeta + i\theta \in H^{\mathbb{C}}_\alpha$. The uniqueness of the last element follows from essentiality of $H^{\mathbb{C}}_\alpha$.

To complete the proof of the admissibility of T^c it remains to show that T^c is homogeneous

$$T^c\left((a+ib)(\xi+i\eta)\right) = T^c\left(a\xi - b\eta + i(a\eta + b\xi)\right) \tag{3.47}$$

$$= T\left(a\xi - b\eta\right) + iT\left(a\eta + b\xi\right) = a[T\left(\xi\right) + iT(\eta)] + ib[T\left(\xi\right) + iT(\eta)]$$

$$= (a+ib)[T\left(\xi\right) + iT(\eta)] = (a+ib)T^c\left(\xi + i\eta\right),$$

and additive

$$T^c\left((\xi+i\eta) + (\xi' + i\eta')\right) = T^c\left(\xi + \xi' + i(\eta + \eta')\right) = T\left((\xi + \xi') + iT(\eta + \eta')\right) \tag{3.48}$$

$$= T(\xi) + iT(\eta) + T\left(\xi'\right) + iT(\eta')) = T^c(\xi + i\eta) + T^c(\xi' + i\eta'),$$

where $\xi + i\eta \in H^{\mathbb{C}}_{\alpha}$, $a, b \in \mathbb{R}$.

Then we identify $T \in L(H^{\mathbb{R}})$ with $T^c \in L(H^{\mathbb{C}})$ as follows

$j(T)(\xi + i\eta) = T(\xi) + iT(\eta) \in H^{\mathbb{C}}$, which provides an embedding

$$j: \ L(H^{\mathbb{R}}) \hookrightarrow L(H^{\mathbb{C}}). \tag{3.49}$$

Let us show that j is a *-isomorphism. Observe that

$$\ll \xi + i\eta, \ (T^c)^*(\xi' + i\eta') \gg \, = \, \ll T^c(\xi + i\eta), \xi' + i\eta' \gg$$

$$= \, \ll T(\xi) + iT(\eta), \ \xi' + i\eta' \gg \, = \, <T(\xi), \xi'> + <T(\eta), \eta'> + i(<T(\eta), \xi'> - <T(\xi), \eta'>),$$

and

$$\begin{aligned} \ll \ & \xi + i\eta, \ (T^*)^c(\xi' + i\eta') \gg \, = \, \ll \xi + i\eta, \ T^*(\xi') + iT^*(\eta') \gg \\ = \ & <\xi, T^*(\xi')> + <\eta, T^*(\eta')> + i(<\eta, T^*(\xi')> - <\xi, T^*(\eta')>) \\ = \ & <T(\xi), \xi'> + <T(\eta), \eta'> + i(<T(\eta), \xi'> - <T(\xi), \eta'>), \end{aligned}$$

thus $j(T^*) = j(T)^*$.

We show that $j(TS) = j(T)j(S)$:

$$j(TS)(\xi + i\eta) = TS(\xi) + iTS(\eta) = j(T)(S(\xi) + iS(\eta)) = j(T)j(S)(\xi + i\eta).$$

Now, if $T \neq S$, then

$$\exists \xi \in H^{\mathbb{R}} : T(\xi) \neq S(\xi), \ \Rightarrow \ j(T)(\xi + i\eta) = T(\xi) + iT(\eta), \tag{3.50}$$

$$j(S) = S(\xi) + iS(\eta), \ \Rightarrow \ j(T) \neq \ j(S), \ \ T, S \in L(H^{\mathbb{R}}).$$

Formula 3.49 means that $T_\alpha \mapsto T^c_\alpha$ for any $T_\alpha \in L(H^{\mathbb{R}})$ with the following embeddings

$$j_\alpha : \ \mathcal{B}(H^{\mathbb{R}}_{\alpha}) \hookrightarrow \mathcal{B}(H^{\mathbb{C}}_{\alpha}), \ \forall \alpha \in \Lambda. \tag{3.51}$$

Let now $\hat{T}$ be an arbitrary operator from $L(H^{\mathbb{C}})$. Then for arbitrary $\xi \in H^{\mathbb{R}}$ there exist $\zeta, \theta \in H^{\mathbb{R}}$

$$\hat{T}(\xi) = \zeta + i\theta.$$

We define $R : H^{\mathbb{R}} \to H^{\mathbb{R}} : R(\xi) = \zeta,\ S : H^{\mathbb{R}} \to H^{\mathbb{R}} : S(\xi) = \theta.$

We prove that R and S are real linear admissible operators on $H^{\mathbb{R}}$.

First, show that R and S are real linear operators

$$\begin{aligned} R(\xi + a\eta) + iS(\xi + a\eta) &= \hat{T}(\xi + a\eta) = \hat{T}(\xi) + a\hat{T}(\eta) = \\ R(\xi) + iS(\xi) + a[R(\eta) + iS(\eta)] &= R(\xi) + aR(\eta) + i[S(\xi) + aS(\eta)], \quad a \in \mathbb{R}. \end{aligned} \tag{3.52}$$

From the above and uniqueness of representation of the element from $H^{\mathbb{C}}$ (corollary 2) it follows that R and S are linear.

Second, we prove that

$$\hat{T}(\xi + i\eta) = R^c(\xi + i\eta) + iS^c(\xi + i\eta). \tag{3.53}$$

Indeed,

$$\hat{T}(\xi + i\eta) = \hat{T}(\xi) + i\hat{T}(\eta) = \tag{3.54}$$

$$R(\xi) + iS(\xi) + i[R(\eta) + iS(\eta)] = R(\xi) - S(\eta) + i[R(\eta) + S(\xi)],$$

on the other hand

$$R^c(\xi + i\eta) + iS^c(\xi + i\eta) = \tag{3.55}$$

$$R(\xi) + iR(\eta) + i[S(\xi) + iS(\eta)] = R(\xi) - S(\eta) + i[R(\eta) + S(\xi)].$$

Third, we prove that R (and S) is admissible. Let for any $\alpha \in \Lambda$ and any arbitrary vector $\xi_\alpha \in H^{\mathbb{R}}_\alpha$ set $\xi_\alpha + i\xi_\alpha \in H^{\mathbb{C}}_\alpha$. Then $\hat{T}(\xi_\alpha + i\xi_\alpha) = \zeta_\alpha + i\zeta_\alpha \in H^{\mathbb{C}}_\alpha$ or $R(\xi_\alpha) = \zeta_\alpha \in H^{\mathbb{R}}_\alpha$. ■

CHAPTER 4.

FUNDAMENTAL DEFINITIONS AND PROPERTIES

4.1 Projective Limits of Real C*-algebras

The Hausdorff projective limits of projective families of Banach algebras as natural locally-convex generalizations of Banach algebras have been studied sporadically by many authors since the early 1950's, when they were first introduced by Arens[8] and Michael[48]. The Hausdorff projective limits of projective families of C*-algebras were first mentioned by Arens[8]. They have since been studied under various names (i.e. locally C*-algebras, pro-C*-algebras, b*-algebras, LMC*-algebras) by many authors. Development of the subject is reflected in the monograph of Fragoulopoulou[20]. We will follow Inoue[33] in the usage of the name **locally** C***-algebras** for these algebras. The Hausdorff projective limits of projective families of real C*-algebras and JB-algebras were first introduced under the name of **real locally** C***-algebras** and resp. **locally** JB**-algebras**[36].

Factor Algebras and Arens-Michael Decompositions

Let us introduce a real lmc *-algebra as a real topological *-algebra with the special family of C*-regular seminorms $\{p_\alpha\}$ and a relation $\sim_\alpha$ on the real lmc *-algebra.

Definition 40 *If A is a real topological *-algebra with topology generated by a separating saturated family of submultiplicative seminorms $\{p_\alpha\}$, then it is called a* ***real lmc *-algebra****.*

If in addition $\{p_\alpha\}$ are regular (strongly regular) by definition 9, then A is called a ***regular real lmc *-algebra (strongly regular real lmc *-algebra).***

Definition 41 *We say that two elements x and y from a real lmc *-algebra A are α-**equivalent,** $x \sim_\alpha y$ if $x - y \in N_\alpha$, where*

$$N_\alpha = \{z : p_\alpha(z^* z) = 0, \ z \in A\}. \tag{4.1}$$

The proofs of lemmata 9, 10, 11 and 12 are elementary and can be omitted.

Lemma 9 *Let A be a regular real lmc *-algebra. Then the set N_α (defined above in the formula 4.1) is a *-ideal and *-subalgebra in A.*

Lemma 10 *Let A be a real lmc *-algebra and N_α be as above (formula 4.1). Then the relation $\sim_\alpha$ is an equivalence relation.*

Remark 4 *We have just obtained that $\sim_\alpha$ is an equivalence relation. Then, based on the Partition Theorem, $\sim_\alpha$ induces a partition of A into equivalence classes. Such classes $[x]_\alpha$ and $[y]_\alpha$ either coincide or $[x]_\alpha \cap [y]_\alpha = \varnothing$. We denote by A_α the factor set*

$$A_\alpha = A/\sim_\alpha = A/N_\alpha. \tag{4.2}$$

The elements of A_α are the classes

$$[x]_\alpha = \{z \in A : z \in x + N_\alpha, \ x \in A\}, \tag{4.3}$$

which are subsets in A.

Lemma 11 *Let A be a real lmc *-algebra. Then for any $\alpha \in \Lambda$ the factor algebra $A_\alpha = A/N_\alpha$, is a real *-algebra.*

Lemma 12 *Let A be a strongly regular real lmc *-algebra. Then for any $\alpha \in \Lambda$ the factoralgebra $A_\alpha = A/N_\alpha$, can be equipped with a strongly regular submultiplicative norm.*

Lemma 13 *Let A be a strongly regular real lmc *-algebra. Then for any $\alpha \in \Lambda$ the norm completion $\bar{A}_\alpha$ of the factor algebra $A_\alpha = A/N_\alpha$ is a real C*-algebra.*

Proof. By lemmas 11 and 12 A_α is a normed real strongly regular algebra for any $\alpha \in \Lambda$. Define a relation $\sim$ between Cauchy sequences of elements of A_α as follows

$$\{_k[x]_\alpha\}_{k=1}^\infty \sim \{_k[y]_\alpha\}_{k=1}^\infty \quad \text{iff} \quad \lim\ ||_k[x]_\alpha\ -\ _k[y]_\alpha||_\alpha\ = 0, \quad _k[x]_\alpha, _k[y]_\alpha \in A_\alpha \tag{4.4}$$

It is a routine to show that is $\sim$ an equivalence relation (reflexive, symmetric and transitive). Denote the classes of equivalence $[\hat{x}]_\alpha$ of all Cauchy sequences as $\bar{A}_\alpha$. It is one more routine to show that $\bar{A}_\alpha$ is a real Banach *-algebra.

Because of norm continuity on $\bar{A}_\alpha$ the extension of a strongly regular norm is also strongly regular. Then from lemma 12 and theorem 2 it follows that $\bar{A}_\alpha$ is a real C*-algebra. ■

Proposition 1 *Let A be a complete strongly regular real lmc *-algebra with a family of seminorms $\{p_\alpha\}_{\alpha\in\Lambda}$, generating a topology on A. Then A is *-isomorphic and homeomorphic to a projective limit with a projective topology of a projective family of real C*-algebras $\bar{A}_\alpha = \overline{A/N_\alpha}$, $\alpha \in \Lambda$.*

Proof. As is shown in lemma 13 $\bar{A}_\alpha$ is a real C*-algebra whose elements are classes of equivalence $[\hat{x}]_\alpha$ of all Cauchy sequences $\{_k[x]_\alpha\}_{k=1}^\infty$ of elements $_k[x]_\alpha \in A_\alpha$.

Denote a stationary sequence as $\{[x]_\alpha, ..., [x]_\alpha, ...\} = \{[x]_\alpha\}_{k=1}^\infty \in [\hat{x}_s]_\alpha$. Some classes of equivalence $[\hat{x}_s]_\alpha \in \bar{A}_\alpha$ contain such an element. We define an injection φ_α as

$$\varphi_\alpha : A_\alpha \hookrightarrow \bar{A}_\alpha, \quad \varphi_\alpha([x]_\alpha) = [\hat{x}_s]_\alpha, \quad \forall \alpha \in \Lambda, \tag{4.5}$$

By construction $\varphi_\alpha(A_\alpha)$ is dense in $\bar{A}_\alpha,\ \alpha \in \Lambda$.

Define π_α, a projection A to $A_\alpha = A/N_\alpha$

$$\pi_\alpha(x) = [x]_\alpha, \quad \forall \alpha \in \Lambda, \tag{4.6}$$

Then we define $g_\alpha^\beta : \varphi_\beta(A_\beta) \longrightarrow \varphi_\alpha(A_\alpha)$, a surjective homomorphism

$$g_\alpha^\beta([x]_\beta) = [x]_\alpha, \quad (g_\alpha^\beta \circ g_\beta^\gamma)([x]_\gamma) = g_\alpha^\gamma([x]_\gamma), \quad \forall \alpha \preceq \beta,\ \beta \preceq \gamma,\ \alpha \preceq \gamma,\ \alpha, \beta, \gamma \in \Lambda \tag{4.7}$$

so that

$$(g_\alpha^\beta \circ \pi_\beta)(x) = \pi_\alpha(x), \quad \forall \alpha \preceq \beta,\ x \in A,\ \alpha, \beta \in \Lambda. \tag{4.8}$$

Our next step is to extend morphisms $g_\alpha^\beta,\ \alpha \preceq \beta \in \Lambda$, mapping $\varphi_\beta(A_\beta)$ onto $\varphi_\alpha(A_\alpha)$, to morphisms $\hat{g}_\alpha^\beta$ mapping $\bar{A}_\beta$ on $\bar{A}_\alpha$.

Consider a Cauchy sequence $\{{}_k[x]_\beta :\ {}_k[x]_\beta \in A_\beta\}_{k=1}^\infty \in [\hat{x}]_\beta$. Then $\{g_\beta^\alpha({}_k[x]_\alpha) :\ g_\beta^\alpha({}_k[x]_\alpha) \in A_\alpha\}_{k=1}^\infty$ will also be a Cauchy sequence

$$||g_\alpha^\beta({}_k[x]_\alpha) - g_\alpha^\beta({}_m[x]_\alpha)||_\beta \le ||({}_k[x]_\alpha) - ({}_m[x]_\alpha)||_\alpha \ < \varepsilon, \quad \alpha \preceq \beta, \quad k, m \ge N. \tag{4.9}$$

We define

$$\hat{g}_\alpha^\beta([x]_\beta) = \lim_{k \to \infty} g_\alpha^\beta({}_k[x]_\beta), \quad \hat{g}_\alpha^\beta([x]_\beta) \in \bar{A}_\alpha. \tag{4.10}$$

If $[x]_\beta \in A_\beta$ then $\bar{g}_\alpha^\beta(\varphi_\alpha[x]_\beta) = g_\alpha^\beta([x]_\beta)$, where φ_α is defined by formula 4.5. Thus, $\bar{g}_\alpha^\beta(\bar{A}_\beta)$ is dense in $\bar{A}_\alpha$ and $(\hat{g}_\alpha^\beta \circ \hat{g}_\alpha^\beta)([\hat{x}]_\gamma) = \hat{g}_\alpha^\gamma([\hat{x}]_\gamma),\ [\hat{x}]_\gamma \in \bar{A}_\gamma$.

Let A' be a projective limit of a projective family $\{\bar{A}_\beta,\ \hat{g}^\beta_\alpha\}$ with $(\hat{g}^\beta_\alpha \circ \hat{\pi}_\beta)(x') = \hat{\pi}_\alpha(x),\ \ \forall\alpha \preceq \beta,\ \alpha, \beta \in \Lambda,\ x' \in A'$.

Let φ be a map from A to A' such that

$$\varphi(x) = x' \quad \text{iff} \quad \varphi_\alpha(\pi_\alpha(x)) = \hat{\pi}_\alpha(x'), \quad \forall\alpha \in \Lambda. \tag{4.11}$$

We show that the map φ is a *-isomorphism and homeomorphism.

Note that $\varphi_\alpha(A_\alpha)$ is dense in $\bar{A}_\alpha$ for any $\alpha \in \Lambda$, so $\varphi(A)$ (by definition) is dense in A'.

There are two topologies on $\varphi(A)$: a topology τ_φ translated from A by φ, and a projective topology τ_p on A'. There is a one-to-one correspondence between open sets in the topology τ_φ and open sets in the topology τ_p. Indeed, for any $\varepsilon > 0$ and any finite set of indices $\alpha_1, ..., \alpha_k,\ i = \overline{1,k}$ an open set $\{\varphi(x) : p_{\alpha_i}(\varphi(x)) < \varepsilon, \forall i = \overline{1,k}\} \in \tau_\varphi$ can be put in one-to-one correspondence with an open set $\{\varphi(x) : \hat{p}_{\alpha_i}(\varphi(x)) < \varepsilon, \forall i = \overline{1,k}\} \in \tau_p$, where $\{p_\alpha\}_{\alpha\in\Lambda}$ and $\{\hat{p}_\alpha\}_{\alpha\in\Lambda}$are seminorms on A and A' respectively. We have

$$p_{\alpha_i}(\varphi(x)) = ||\pi_{\alpha_i}(\varphi(x))||_{\alpha_i} = ||\hat{\pi}_{\alpha_i}(\varphi(x))||_{\alpha_i} = \hat{p}_{\alpha_i}(\varphi(x)).$$

So, $\tau_\varphi \equiv \tau_p$. From completeness of A it follows that $\varphi(A)$ is complete. Also, $\varphi(A)$ is dense in A', then $\varphi(A)$ coincides with A'.

Also $\varphi(x^*) = \varphi(x)^*$ and $\varphi(xy) = \varphi(x)\varphi(y)$. It is an exercise to see that these properties hold. ■

Norm Extension to Complexification Algebra

Lemma 14 *Let W be a norm closed real essential *-subalgebra of $\mathcal{B}(H^{\mathbb{R}})$. Then, $Q = W + iW$ is normed closed in the norm of $\mathcal{B}(H^{\mathbb{C}})$.*

Proof. Let $a_n + ib_n$ be a Cauchy sequence in $Q = W \dot{+} iW$, $a_n, b_n \in W$. Then $\forall \varepsilon > 0\ \exists N(\varepsilon) : \forall n, k \geq N$, $||(a_n + ib_n) - (a_k + ib_k)|| < \varepsilon$. Note that (page 13)

$$||(a_n - a_k)(\xi)||^2 + ||(b_n - b_k)(\xi)||^2 = ||(a_n - a_k)(\xi) + i(b_n - b_k)(\xi)||^2 =$$
$$||(a_n + ib_n)(\xi) - (a_k + ib_k)(\xi)||^2 < \varepsilon^2,\ \xi \in H^{\mathbb{R}},\ ||\xi|| = 1.$$

Then $||(a_n - a_k)(\xi)|| < \varepsilon$ and $||(b_n - b_k)(\xi)|| < \varepsilon$. So $||a_n - a_k|| < \varepsilon$ and $||b_n - b_k|| < \varepsilon$.

Thus, a_n and b_n are Cauchy sequences in the subalgebra W. Due to closedness of W there exist a_0 and b_0, such that $\lim a_n = a_0$ and $\lim a_n = a_0$ respectively. Therefore $\forall \varepsilon > 0\ \exists M(\varepsilon) : \forall n > M$, $||a_n - a_0|| < \varepsilon/2$, and similarly for b_0 : $||b_n - b_0|| < \varepsilon/2$. We show that $a_n + ib_n$ converges to $a_0 + ib_0$. Indeed

$$||(a_n + ib_n) - (a_0 + ib_0)|| = ||(a_n - a_0) + (ib_n - ib_0)|| \leq ||a_n - a_0|| + ||b_n - b_0|| < \varepsilon,$$

which completes the proof. ■

Proposition 2 (lmc Regular Extension) *Let A be a strongly regular real lmc*-algebra. Then for $B = A \dot{+} iA$, a complexification of A, there exists a separating saturated family of complex regular seminorms $\{\hat{p}_\alpha\}$, generating topology on B, such that for any p_α, $\hat{p}_\alpha(x + i0) = p_\alpha(x)$, where $0, x \in A$.*

Proof. Let $\{p_\alpha\}$ be a family of seminorms generating the topology on A.

By proposition 1 A is isomorphic and homeomorphic to a projective limit $\varprojlim \bar{A}_\alpha$ of the projective family of real C*-algebras $\bar{A}_\alpha$. A Gelfand-Naimark type theorem for real C*-algebras states that each $\bar{A}_\alpha$ can be represented by $\psi_\alpha(\bar{A}_\alpha)$, a norm closed *-subalgebra of $\mathcal{B}(H_\alpha^{\mathbb{R}})$[32,45] ($H_\alpha^{\mathbb{R}}$ is a real Hilbert space).

We denote on $\mathcal{B}(H^{\mathbb{R}}_\alpha)$ the operator norm $||T_\alpha||_\alpha$, hence

$$||T_\alpha||_\alpha = \tilde{p}_\alpha(T) = p_\alpha(x), \quad \psi_\alpha([x]_\alpha) = T_\alpha,\ \pi_\alpha(x) = [x]_\alpha, \tag{4.12}$$

$$\psi : A \to L(H^R),\ \psi_\alpha : \bar{A}_\alpha \to \mathcal{B}(H^{\mathbb{R}}_\alpha),\ x \in A,\ [x]_\alpha \in \bar{A}_\alpha,$$

where $\pi_\alpha : A \to \bar{A}_\alpha$ are projections and $\{\tilde{p}_\alpha\}$ is a separating saturated family of strongly regular seminorms on $L(H^R)$.

Then, we complexify $H^{\mathbb{R}}_\alpha$: $H^{\mathbb{C}}_\alpha = H^{\mathbb{R}}_\alpha \dot{+} iH^{\mathbb{R}}_\alpha$, define a scalar product $\ll .,. \gg |_{H^{\mathbb{C}}_\alpha}$ and an algebra of all admissible operators $L(H^C)$ (as in theorem 10) on a locally Hilbert space $H^{\mathbb{C}}_\alpha$ with regular separating family of seminorms $\{\widehat{p}_\alpha\}$ acting on $L(H^C)$.

Define, as in the proof of theorem 13 $j_\alpha : \mathcal{B}(H^{\mathbb{R}}_\alpha) \hookrightarrow \mathcal{B}(H^{\mathbb{C}}_\alpha),\ \forall \alpha \in \Lambda$ as isometrically isomorphic embeddings, where real C*-algebras $\bar{A}_\alpha$ are embedded in $\mathcal{B}(H^{\mathbb{C}}_\alpha)$ with the closed images $j_\alpha(\psi_\alpha(\bar{A}_\alpha))$.

The conservation of the order for seminorms follows from the fact that locally Hilbert space is presented as a union of embedded Hilbert spaces $H^C = \cup H^C_\alpha$, and, if $\alpha \preceq \beta$ then $H^{\mathbb{C}}_\alpha \subset H^{\mathbb{C}}_\beta$ (theorem 10), and

$$\widehat{\tilde{p}}_\alpha(\widehat{T}) = ||\pi_\alpha(\widehat{T})||_{\mathcal{B}(H^{\mathbb{C}}_\alpha)} = ||g^\beta_\alpha(\pi_\beta(\widehat{T}))||_{\mathcal{B}(H^{\mathbb{C}}_\alpha)} \le ||\pi_\beta(\widehat{T})||_{\mathcal{B}(H^{\mathbb{C}}_\beta)} = \widehat{\tilde{p}}_\beta(\widehat{T}), \tag{4.13}$$

where $g^\beta_\alpha : \bar{A}_\beta \to \bar{A}_\alpha,\ \alpha \preceq \beta$ are surjective morphisms.

The last step is to assign the values of seminorms on $L(H^{\mathbb{C}})$ to B, the complexification of A

$$\widehat{p}_\alpha(v) = \widehat{\tilde{p}}_\alpha(\widehat{T}),\ \ \widehat{p}_\alpha(v) \le \widehat{p}_\alpha(v),\ \alpha \preceq \beta, \tag{4.14}$$

$$\widehat{\psi} : B \to L(H^{\mathbb{C}}),\ \widehat{\psi}(v) = \widehat{T},\ \widehat{\pi}_\alpha(v) = [v]_\alpha,\ v \in B,$$

where $\widehat{\pi}_\alpha : B \to B_\alpha$ are projections; note that the family of seminorms $\{\widehat{\widetilde{p}}_\alpha\}$ on $L(H^{\mathbb{C}})$ is separating and saturated ($L(H^{\mathbb{C}})$ is a locally C*-algebra).

We prove that $\{\widehat{p}_\alpha\}_{\alpha\in\Lambda}$ is separating family. Let $v \in B$, $v \neq 0$, $\hat{\psi}(v) = \widehat{T}$, $\widehat{T} \neq 0$, then $\exists\gamma \in \Lambda : \widehat{\widetilde{p}}_\gamma(\widehat{T}) \neq 0$, hence $\widehat{p}_\gamma(v) = \widehat{\widetilde{p}}_\gamma(\hat{\psi}(v)) = \widehat{\widetilde{p}}_\gamma(\widehat{T}), \neq 0$.

We prove that $\{\widehat{\widetilde{p}}_\alpha\}$ is saturated: for any finite subset of indices $F = \{\alpha_1, ..., \alpha_n\}$, $\alpha_k \in \Lambda$ $(k = \overline{1,n})$, $\exists\delta \in \Lambda : \widehat{\widetilde{p}}_\delta(\widehat{T}) = \max_{\alpha\in F}\{\widehat{\widetilde{p}}_\alpha(\widehat{T})\}$, $\widehat{\widetilde{p}}_\delta \in S(L(H^{\mathbb{C}}))$, $\widehat{T} \in L(H^{\mathbb{C}})$. Then $\widehat{p}_\delta(v) = \widehat{\widetilde{p}}_\delta(\hat{\psi}(v))$. Suppose on the contrary that $\exists w \in B : \widehat{p}_\delta(w) \neq \max_{\alpha\in F}\{\widehat{p}_\alpha(w)\}$. Thus $\widehat{\widetilde{p}}_\delta(\hat{\psi}(w)) \neq \max_{\alpha\in F}\{\widehat{\widetilde{p}}_\alpha(\hat{\psi}(w))\}$, which contradicts the fact that $\{\widehat{\widetilde{p}}_\alpha\}$ is saturated.

So the family of seminorms $\{\widehat{p}_\alpha\}_{\alpha\in\Lambda}$ is separating and saturated. ■

Gelfand-Naimark Type Theorem for Real Locally C*-algebras

The following Theorem here is a Gelfand-Naimark type theorem, which is a real analogue of Inoue's theorem[33]:

Proposition 3 *Let A be a projective limit of a projective family of real C*-algebras A_α, $\alpha \in \Lambda$. Then there exists a real locally Hilbert space $H^{\mathbb{R}}$ such that A is real *-isomorphic and homeomorphic to a closed *-subalgebra of $L(H^{\mathbb{R}})$.*

Proof. Let an Arens-Michael decomposition of A be $A \cong \varprojlim g_\alpha^\beta A_\beta$ and any A_β, $\beta \in \Lambda$ be a real C*-algebra. Then A_β is isometrically *-isomorphic to a closed subalgebra $\psi_\beta(A_\beta)$ of $\mathcal{B}(H^{\mathbb{R}}_\beta)$, $\psi_\beta : A_\beta \hookrightarrow \mathcal{B}(H^{\mathbb{R}}_\beta)$, where $H^{\mathbb{R}}_\beta$, $\beta \in \Lambda$ is a real Hilbert space. We define an orthogonal direct sum of $H^{\mathbb{R}}_\delta$ with $\delta \preceq \beta$:

$$\mathbf{H}^{\mathbb{R}}_{\boldsymbol{\beta}} = \oplus_{\delta\preceq\beta} H^{\mathbb{R}}_\delta. \tag{4.15}$$

Thus, $\mathbf{H}^{\mathbf{R}}_{\boldsymbol{\alpha}} \subset \mathbf{H}^{\mathbf{R}}_{\boldsymbol{\beta}}$ with $\alpha \preceq \beta$ and $H^R = \varinjlim \mathbf{H}^{\mathbf{R}}_{\boldsymbol{\beta}}$ is a locally Hilbert space.[20]

Define the operator

$$R^x_\beta : \ \mathbf{H}^R_\beta \!\rightarrow \mathbf{H}^R_\beta; \ \ \widehat{\xi}_\beta = (\xi_\delta)_{\delta\preceq\beta} \longmapsto R^x_\beta(\widehat{\xi}_\beta) = (\psi_\delta(x_\delta)(\xi_\delta))_{\delta\preceq\beta}. \tag{4.16}$$

where $\xi_\delta \in H^{\mathbb{R}}_\delta, \ \widehat{\xi}_\beta = (\xi_\delta)_{\delta\preceq\beta} = (..., \xi_\delta, ..., \xi_\beta)_{all \ \ \delta\preceq\beta} \in \mathbf{H}^{\mathbb{R}}_\beta$.

It is additive

$$R^x_\beta(\widehat{\xi}_\beta + \widehat{\eta}_\beta) = (\psi_\delta(x_\delta)(\xi_\delta + \eta_\beta))_{\delta\preceq\beta} = (\psi_\delta(x_\delta)(\xi_\delta) + \psi_\delta(x_\delta)(\eta_\delta))_{\delta\preceq\beta} \tag{4.17}$$

$$= (\psi_\delta(x_\delta)(\xi_\delta))_{\delta\preceq\beta} + (\psi_\delta(x_\delta)(\eta_\delta))_{\delta\preceq\beta} = R^x_\beta(\widehat{\xi}_\beta) + R^x_\beta(\widehat{\eta}_\beta),$$

and homogeneous

$$R^x_\beta(a\widehat{\xi}_\beta) = (\psi_\delta(x_\delta)(a\xi_\delta))_{\delta\preceq\beta} = (a\psi_\delta(x_\delta)(\xi_\delta))_{\delta\preceq\beta} = aR^x_\beta(\widehat{\xi}_\beta). \tag{4.18}$$

It is also bounded

$$||\psi_\delta(x_\delta)(\xi_\delta)||_\delta \leq ||\psi_\delta(x_\delta)||_\delta \ ||\xi_\delta||_\delta = p_\delta(x)||\xi_\delta||_\delta \leq p_\beta(x)||\xi_\delta||_\delta,$$

$$\forall\delta \preceq \beta \in \Lambda, x \in A, \ p_\delta(x) \leq p_\beta(x).$$

So, $R^x_\beta \in \mathcal{B}(\mathbf{H}^{\mathbf{R}}_{\boldsymbol{\beta}}), \ \forall\beta \in \Lambda$.

Then, let ψ be defined as follows

$$\psi : A \hookrightarrow L(\mathbf{H}^{\mathbf{R}}) : x \longmapsto \psi(x) = \varinjlim R^x_\beta(\widehat{\xi}_\beta), \tag{4.19}$$

$$\psi(x)|_{\mathbf{H}^{\mathbf{R}}_{\boldsymbol{\beta}}} = \psi_\beta(x_\beta) = R^x_\beta(\widehat{\xi}_\beta), \quad x \in A, \ \beta \in \Lambda, \ \widehat{\xi}_\beta \in \mathbf{H}^{\mathbb{R}}_\beta.$$

We have

$$p_\beta(\psi(x)) = ||R^x_\beta(\widehat{\xi}_\beta)||_\beta = \sup\{||\psi(x)||_\delta : \forall\delta \preceq \beta \in \Lambda\} \tag{4.20}$$

$$= \sup\{||x_\delta||_\delta : \forall\delta \preceq \beta \in \Lambda\} = \sup\{p_\delta(x) : \forall\delta \preceq \beta \in \Lambda\} = p_\beta(x).$$

We show that ψ is a *-isomorphism:

$$\psi(x^*)|_{\mathbf{H}^{\mathbf{R}}_{\beta}} = \psi_\beta(x^*_\beta) = R^{x^*}_\beta(\widehat{\xi}_\beta) = (\psi_\delta(x^*_\delta)(\xi_\delta))_{\delta\preceq\beta} \tag{4.21}$$

$$= (\psi_\delta(x_\delta)^*(\xi_\delta))_{\delta\preceq\beta} = (R^x_\beta(\widehat{\xi}_\beta))^* = (\psi(x)|_{\mathbf{H}^{\mathbf{R}}_{\beta}})^*.$$

and

$$\psi(xy)|_{\mathbf{H}^{\mathbf{R}}_{\beta}} = \psi_\beta((xy)_\beta) = R^{xy}_\beta(\widehat{\xi}_\beta) = (\psi_\delta((xy)_\delta)(\xi_\delta))_{\delta\preceq\beta} \tag{4.22}$$

$$= (\psi_\delta(x_\delta)(\xi_\delta)\ \psi_\delta(x_\delta)(\xi_\delta))_{\delta\preceq\beta} = (\psi_\delta(x_\delta)(\xi_\delta))_{\delta\preceq\beta}(\psi_\delta(x_\delta)(\xi_\delta))_{\delta\preceq\beta}$$

$$= (R^x_\beta(\widehat{\xi}_\beta))(R^y_\beta(\widehat{\xi}_\beta)) = (\psi(x)|_{\mathbf{H}^{\mathbf{R}}_{\beta}})(\psi(x)|_{\mathbf{H}^{\mathbf{R}}_{\beta}}),$$

Also, if $x, y \in A,\ x \neq y$, then $\psi(x)|_{\mathbf{H}^{\mathbf{R}}_{\beta}} = \psi_\beta(x_\beta) = R^x_\beta(\widehat{\xi}_\beta) = (\psi_\delta(x_\delta)(\xi_\delta))_{\delta\preceq\beta}$, $\psi(y)|_{\mathbf{H}^{\mathbf{R}}_{\beta}} = \psi_\beta(y_\beta) = R^y_\beta(\widehat{\xi}_\beta) = (\psi_\delta(y_\delta)(\xi_\delta))_{\delta\preceq\beta}$. It means that $\exists\delta' : \psi_{\delta'}(x_{\delta'})(\xi_{\delta'}) \neq \psi_{\delta'}(y_{\delta'})(\xi_{\delta'}) \in \mathcal{B}(H^{\mathbb{R}}_{\delta'})$. Hence $\psi(x)|_{\mathbf{H}^{\mathbf{R}}_{\beta}} \neq \psi(y)|_{\mathbf{H}^{\mathbf{R}}_{\beta}}$ and ψ is injective.

For any finite subset $\alpha_1, ..., \alpha_k \subset \Lambda$ consider now the open sets $U_{\varepsilon,\ \alpha_1,\ldots,\alpha_k}$ in A

$$U_{\varepsilon,\ \alpha_1,\ldots,\alpha_k} = \{x :\ x \in A,\ p_{\alpha_i}(x) < \varepsilon,\ \alpha_1, ..., \alpha_k \subset \Lambda\} \tag{4.23}$$

and open sets $V_{\varepsilon,\ \alpha_1,\ldots,\alpha_k}$ in $\psi(A)$

$$V_{\varepsilon,\ \alpha_1,\ldots,\alpha_k} = \{R^x :\ R^x \in \psi(x),\ \hat{p}_{\alpha_i}(R^x) < \varepsilon,\ \alpha_1, ..., \alpha_k \subset \Lambda\} \tag{4.24}$$

We have $\psi(U_{\varepsilon,\ \alpha_1,\ldots,\alpha_k}) = V_{\varepsilon,\ \alpha_1,\ldots,\alpha_k}$. Due to the fact that all open sets in the topologies of the respective algebras are of these forms, that $p_{\alpha_i}(x) = \hat{p}_{\alpha_i}(R^x) < \varepsilon$, and open sets are in one to one correspondence, the proof that ψ is a homeomorphism is complete. ■

Proposition 4 *Let A be a projective limit of a projective family of real C^*-algebras. Then there exists a complex locally Hilbert space $\mathbf{H}^C$ such that A is real *- isomorphic and homeomorphic to a closed real essential *-subalgebra of $L(\mathbf{H}^C)$.*

Proof. As it is shown in proposition 3 $\psi : A \hookrightarrow L(H^{\mathbb{R}})$, and as it is shown in the proof of theorem 13 (formula 3.49), $j : L(\mathbf{H}^{\mathbf{R}}) \hookrightarrow L(\mathbf{H}^{\mathbf{C}})$ are *-isomorphisms. Therefore $j \circ \psi : A \hookrightarrow L(H^{\mathbb{C}})$ is also a *-isomorphism. In addition, from the fact that $L(\mathbf{H}^{\mathbf{R}}) \cap iL(\mathbf{H}^{\mathbf{R}}) = 0_{L(\mathbf{H}^{\mathbf{C}})}$ and $L(\mathbf{H}^{\mathbf{R}}) \dotplus iL(\mathbf{H}^{\mathbf{R}}) = L(\mathbf{H}^{\mathbf{C}})$, it follows that $(j \circ \psi)(A) \cap i(j \circ \psi)(A) = 0_B$, where $B = A \dotplus iA$ is a complexification of A. Due to the fact that $\psi(A)$ is closed in $L(\mathbf{H}^{\mathbf{R}})$, and $j(L(\mathbf{H}^{\mathbf{R}}))$ is closed in $L(\mathbf{H}^{\mathbb{C}})$, then $\breve{A} = (j \circ \psi)(A)$ is closed in $L(\mathbf{H}^{\mathbf{C}})$. ■

Connections with complex locally C*-algebras

It is well known that real C*-algebras are related to the complex C*-algebras through the actions of a *-antiautomorphism of period 2 on it[30]. Analogous results below extend the known results to the case of real locally C*-algebras and the locally JB-algebras respectively.

Proposition 5 *Let A be a strongly regular real lmc*-algebra and B be the complex locally C*-algebra, $B = A \dotplus iA$. Define φ as*

$$\varphi(x + iy) = x^* + iy^*, \ \forall (x + iy) \in B. \tag{4.25}$$

*Then φ is a *-antiautomorphism of period 2 of B and A is *-isomorphic and homeomorphic to $A'-$ a subalgebra of B*

$$A' = \{v \in B : \varphi(v) = v^*\}. \tag{4.26}$$

Proof. Let $B = A \dotplus iA$ be a locally C*-algebra. For involution

$$[\varphi(x + iy)]^* = [x^* + iy^*]^* = x^{**} - iy^{**} = x - iy = \varphi(x^* - iy^*) = \varphi((x + iy)^*).$$

The map ψ is an antiautomorphism since

$$\varphi((x_1+iy_1)(x_2+iy_2)) = \varphi((x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2)) =$$

$$(x_2^*x_1^*-y_2^*y_1^*)+i(x_2^*y_1^*+y_2^*x_1^*) = (x_2^*+iy_2^*)(x_1^*+iy_1^*) = \varphi(x_2+iy_2)\varphi(x_1+iy_1), \tag{4.27}$$

and of period 2 since

$$\varphi^2(x+iy) = x^{**}+iy^{**} = x+iy. \tag{4.28}$$

Then, for the element x^*+iy^* $(x, y \in A)$ we have:

$$\varphi(x+iy) = x^*+iy^*$$

So $\varphi(x+iy) = x^*-iy^*$ is true iff $y = 0 \in A$.

Let $\psi : A \to A' \subset B,\ \psi(x) = x+i0$ be a map from A to B.

The fact that $\psi(A)$ is *-isomorphic and homeomorphic to A follows from $a)-d)$:

$a)$ $\psi(xy) = xy+i0 = (x+i0)(y+i0) = \psi(x)\psi(y)$.

$b)$ $x \neq y,\ \psi(x) = x+i0,\ \psi(y) = y+i0$ then $\psi(x) \neq \psi(y)$.

$c)$ $\psi(x^*) = x^*+i0 = (x+i0)^* = \psi(x)^*$.

$d)$ $p_\alpha(x) = \hat{p}_\alpha(x+i0),\ \forall\alpha \in \Lambda$.where $\{p_\alpha(x)\}$ and $\{\hat{p}_\alpha(x)\}$ are families of seminorms in A and B respectively with one-to-one correspondence of open sets in topologies τ_A and $\tau_{A'}$. ■

Definitions of Real Locally C*-algebras

Now we will combine equivalent definitions of real locally C*-algebras.

Theorem 14 (Real Locally C*-algebras' Main Theorem) *For a complete real lmc *-algebra A the following conditions are equivalent:*

*1) A is (isomorphic and homeomorphic to) a strongly regular real lmc *-algebra.*

2) A is (isomorphic and homeomorphic to) a projective limit of a projective family of real C-algebras equipped with projective topology.*

*3) A is topologically real *- isomorphic and homeomorphic to a closed *-subalgebra of real admissible operators $L(\mathbf{H}^R)$, where $\mathbf{H}^R$ is a real locally Hilbert space.*

*4) A is topologically real *- isomorphic and homeomorphic to a closed real *-subalgebra of admissible operators $L(\mathbf{H}^C)$, where $\mathbf{H}^C$ is a locally Hilbert space.*

5) Let $B = A + iA$ be a complexification of A. There exists a topology τ_B on B, such that

a) $\tau_B|_A = \tau_A$ (A naturally embedded in B).

b) (B, τ_B) is complex locally C-algebra.*

Proof. It follows from Propositions 1, 3, 4 and 5. ■

Definition 42 *A complete real lmc *-algebra A is called a* ***real locally C*-algebra*** *if it satisfies any of five and thus all conditions of the theorem 14.*

Examples

We will present here a couple examples of real locally C*-algebras.

Example 3 *The product $\prod_{\alpha\in\mathbb{I}} A_\alpha$ of real C*-algebras A_α, with the product topology, is a projective limit of real C*-algebras.*

Example 4 *Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$,). Then the algebra $C(X)$ of all continuous, not necessarily bounded real-valued functions on X, with the* *topology of uniform convergence on compact subsets, is a projective limit of real C*-algebras. It is known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated*[76].

4.2 Complexification of Real Locally C*-algebras

Theorem 15 (Projective Limit Complexification) *Let A be a real locally C*-algebra with Arens-Michael decomposition $A \cong \varprojlim g_\alpha^\beta A_\beta$. A complexification $B = A \dot{+} iA$ of A is *-isomorphic and homeomorphic to a projective limit $\tilde{B} \cong \varprojlim \tilde{g}_\alpha^\beta B_\beta$ of the complexifications $B_\beta = A_\beta \dot{+} iA_\beta$ of real C*-algebras $A_\beta,\ \beta \in \Lambda$.*

Proof. Let $\{A_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$ be a projective family of real C*-algebras. Define projections $\tilde{\pi}_\alpha$ on $\tilde{B}$ and morphisms $\tilde{g}_\alpha^\beta$ as

$$\begin{aligned} \tilde{\pi}_\alpha &: \tilde{B} \to B_\alpha,\ \tilde{\pi}_\alpha(\tilde{a}) = \pi_\alpha(x) + i\pi_\alpha(y), \\ \tilde{g}_\alpha^\beta &: B_\beta \to B_\alpha,\ \tilde{g}_\alpha^\beta(\tilde{a}_\beta) = g_\alpha^\beta(x_\beta) + i g_\alpha^\beta(y_\beta), \end{aligned} \tag{4.29}$$

such that

$$\tilde{g}_\beta^\alpha \circ \tilde{\pi}_\beta = \tilde{\pi}_\alpha \text{ and } \tilde{g}_\alpha^\beta \circ \tilde{g}_\beta^\gamma = \tilde{g}_\alpha^\gamma,\quad \forall \gamma \succeq \alpha,\ \gamma \succeq \beta,\ \alpha, \beta, \gamma \in \Lambda. \tag{4.30}$$

Indeed,

$$\tilde{\pi}_\alpha(\tilde{a}) = \pi_\alpha(x) + i\pi_\alpha(y) = g_\alpha^\beta \pi_\beta(x) + i g_\alpha^\beta \pi_\beta(y) = \tilde{g}_\beta^\alpha \circ \tilde{\pi}_\beta(\tilde{a}) \tag{4.31}$$

and

$$\begin{aligned} \tilde{g}_\alpha^\beta \circ \tilde{g}_\beta^\gamma(\tilde{a}_\gamma) &= \tilde{g}_\alpha^\beta(g_\beta^\gamma(x_\gamma) + i g_\beta^\gamma(y_\gamma)) = \tilde{g}_\alpha^\beta(\tilde{a}_\beta) = g_\alpha^\beta(x_\beta) + i g_\alpha^\beta(y_\beta) \\ &= x_\alpha + i y_\alpha = \tilde{a}_\alpha = \tilde{g}_\alpha^\gamma(\tilde{a}_\gamma),\ \forall \gamma \succeq \alpha,\ \gamma \succeq \beta,\ \alpha, \beta, \gamma \in \Lambda. \end{aligned} \tag{4.32}$$

By proposition 2, B_α are complex C*-algebras with norms $||.||_{B_\alpha} \leq ||.||_{B_\beta}$ for $\alpha, \preceq \beta,\ \alpha, \beta \in \Lambda$.

Thus, taking into account 4.31 and 4.32 $\{B_\alpha; \tilde{g}^\beta_\alpha; \alpha, \beta \in \Lambda\}$ is a projective family of complex C*-algebras.

Define the following embeddings

$$\begin{aligned} j_\alpha &: \mathcal{B}(H^{\mathbb{R}}_\alpha) \hookrightarrow \mathcal{B}(H^{\mathbb{C}}_\alpha),\ \forall \alpha \in \Lambda \\ j &: L(H^{\mathbb{R}}) \hookrightarrow L(H^{\mathbb{C}}), \\ \psi_\alpha &: A_\alpha \hookrightarrow \mathcal{B}(H^{\mathbb{R}}_\alpha),\ \forall \alpha \in \Lambda, \\ \psi &: A \cong \varprojlim g^\beta_\alpha A_\beta \hookrightarrow \varprojlim \hat{g}^\beta_\alpha \mathcal{B}(H^{\mathbb{R}}_\beta) \cong L(H^{\mathbb{R}}), \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \tilde{\psi}_\alpha &: B_\alpha \hookrightarrow \mathcal{B}(H^{\mathbb{C}}_\alpha),\ H^{\mathbb{C}}_\beta = H^{\mathbb{R}}_\beta \dot{+} H^{\mathbb{R}}_\beta,\ \forall \alpha \in \Lambda, \\ \tilde{\psi} &: \tilde{B} \cong \varprojlim \tilde{g}^\beta_\alpha B_\beta \hookrightarrow L(H^{\mathbb{C}}),\ H^{\mathbb{C}} = H^{\mathbb{R}} \dot{+} H^{\mathbb{R}}, \\ \bar{\psi} &: B = A \dot{+} iA \hookrightarrow L(H^{\mathbb{C}}) = L(H^{\mathbb{R}}) \dot{+} iL(H^{\mathbb{R}}). \end{aligned} \tag{4.34}$$

Let $T^a \in \bar{\psi}(B) :\ \bar{\psi}(a) = T^a,\ a \in B$. We prove that $T^a \in \tilde{\psi}(\tilde{B}),\ \exists \tilde{a} \in \tilde{B} :\ \tilde{\psi}(\tilde{a}) = T^a$.

If $T^a \in L(H^{\mathbb{C}})$, then there exists a pair $(R^a)^c$ and $(R^a)^c$ from $L(H^{\mathbb{C}})$ of the form $j(R^a) = (R^a)^c,\ j(S^a) = (S^a)^c$ for some pair R^a and S^a, such that $T^a = (R^a)^c + i(S^a)^c$ as in the proof of theorem 13 (formula 3.53).

Let $a \in B$, then there exist $x, y \in A$, such that $a = x + iy$ and $\forall \alpha \in \Lambda, \exists x_\alpha, y_\alpha \in A_\alpha : \pi_\alpha(x) = x_\alpha,\ \pi_\alpha(y) = y_\alpha,\ x_\alpha + iy_\alpha \in B_\alpha$. Hence $\tilde{g}^\alpha_\beta(x_\beta + iy_\beta) = x_\alpha + iy_\alpha$, and there exists a unique $\tilde{a} \in \varprojlim \tilde{g}^\beta_\alpha B_\beta :\ \tilde{\pi}_\alpha(\tilde{a}) = x_\alpha + iy_\alpha \in B_\alpha$.

Prove that $\tilde{\psi}(\tilde{a}) = T^a$: $\psi(x) = R^a,\ \psi(y) = S^a$, so $\forall\alpha \in \Lambda,\ \psi_\alpha(x_\alpha) = R^a_\alpha,\ \psi_\alpha(y_\alpha) = S^a_\alpha$, and $\tilde{\psi}_\alpha(x_\alpha + iy_\alpha) = (R^a_\alpha)^c + i(S^a_\alpha)^c = T^a_\alpha$.

Hence we conclude that $\tilde{\psi}(\tilde{a}) = T^a$.

Now, on the other hand $T^{\tilde{a}} \in \tilde{\psi}(\tilde{B}) : \tilde{\psi}(\tilde{a}) = T^{\tilde{a}},\ \tilde{a} \in \tilde{B}$. We will prove that $\exists a \in B : \bar{\psi}(a) = T^{\tilde{a}}$.

Then $\forall\tilde{a}_\alpha \in \tilde{B}_\alpha,\ \exists T^{\tilde{a}}_\alpha : \psi_\alpha(\tilde{a}_\alpha) = T^{\tilde{a}}_\alpha,\ T^{\tilde{a}}_\alpha = (R^{\tilde{a}}_\alpha)^c + i(S^{\tilde{a}}_\alpha)^c$ with a pair of operators $(R^{\tilde{a}}_\alpha)^c, (S^{\tilde{a}}_\alpha)^c \in L(H^{\mathbb{C}})$.

We have $\psi_\alpha(x_\alpha) = R^{\tilde{a}}_\alpha,\ \psi_\alpha(y_\alpha) = S^{\tilde{a}}_\alpha,\ \exists R^{\tilde{a}}, S^{\tilde{a}} \in L(H^{\mathbb{R}}),\ x \in A,\ \psi(x) = R^{\tilde{a}},\ \psi(y) = S^{\tilde{a}}$.

Therefore, for $x + iy \in B$: $\bar{\psi}(x + iy) = (R^{\tilde{a}}_\alpha)^c + i(S^{\tilde{a}}_\alpha)^c = T^{\tilde{a}}$.

It completes the proof of the theorem. ■

4.3 Locally Isometry of Locally C*-algebras

A *-isomorphism between C*-algebras is automatically an isometry,[51] so the following definition makes sense.

Definition 43 *Let $\{B_\alpha; g^\beta_\alpha; \alpha, \beta \in \Lambda\}$ and $\{C_\alpha; f^\beta_\alpha; \alpha, \beta \in \Lambda\}$ be projective families of C*-algebras, $(B, \tau_B) = \varprojlim B_\alpha$ and $(C, \tau_C) = \varprojlim C_\alpha$ projective limits with respective projective topologies and $\psi : B \to C$, be a *-isomorphism. The morphism ψ is called a* ***locally *-isometry*** *if for each $\alpha \in \Lambda$ there exists $\psi_\alpha : B_\alpha \to C_\alpha$, such that $\psi_\alpha(B_\alpha)$ is isometrically *-isomorphic to C_α with $\psi_\alpha \circ {}_B\pi_\alpha = {}_C\pi_\alpha \circ \psi$.*

Let us formulate the following result due to Nassopoulos (proposition 2.1[54]):

Proposition 6 *Let* $\{B_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda_1\}$ *and* $\{C_\alpha; f_\alpha^\beta; \alpha, \beta \in \Lambda_2\}$ *be projective families of C*-algebras,* $(B, \tau_B) = \varprojlim B_\alpha$ *and* $(C, \tau_C) = \varprojlim C_\alpha$ *projective limits with respective projective topologies and* ψ *be a *-homomorphism* $\psi : B \to C$ *from* B *onto* C.

Then the following two statements are equivalent:

(i) ψ *is continuous;*

(ii) ψ *is decomposable, in the sense that for each* $\beta \in \Lambda_2$ *there exists* $\alpha \in \Lambda_1$, *and a unique homomorphism*

$$\psi_\beta^\alpha : B_\alpha \to C_\beta, \tag{4.35}$$

so that

$$\psi_\beta^\alpha \circ \ {}_B\pi_\alpha = \ {}_C\pi_\beta \circ \psi, \tag{4.36}$$

where

$${}_B\pi_\alpha : B \to B_\alpha, \text{ and } {}_C\pi_\beta : C \to C_\beta, \tag{4.37}$$

are natural projections.

The next theorem explains the true meaning of a locally *-isometric mapping.

Proposition 7 *Let* $\{B_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$ *and* $\{C_\alpha; f_\alpha^\beta; \alpha, \beta \in \Lambda\}$ *be projective families of C*-algebras,* $(B, \tau_B) = \varprojlim B_\alpha$ *and* $(C, \tau_C) = \varprojlim C_\alpha$ *projective limits with respective projective topologies and* $\psi : B \to C$ *be an algebraic *-isomorphism from* B *to* C. *Then* ψ *is a locally *-isometric mapping iff* ψ *is a homeomorphism.*

Proof. Let ψ be a locally *-isometric mapping. Thus, due to proposition 6 ψ and ψ^{-1} are both decomposable, and therefore both are continuous.

Conversely, if ψ and ψ^{-1} are both continuous, then due to proposition 6 they are both decomposable, and as a result ψ is locally *-isometric. ■

Corollary 4 *Let* $\{B_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda_1\}$ *and* $\{C_\alpha; f_\alpha^\beta; \alpha, \beta \in \Lambda_2\}$ *be projective families of C*-algebras,* $(B, \tau_B) = \varprojlim B_\alpha$ *and* $(C, \tau_C) = \varprojlim C_\alpha$ *projective limits with respective projective topologies and* $\psi : B \to C$ *be an algebraic *-isomorphism from* B *to* C*. If* ψ *is a homeomorphism, then* Λ_1 *can be identified with* Λ_2 *and* ψ *is a locally *-isometric mapping.*

Proof. Follows by applying Propositions 6 and 7 to ψ and ψ^{-1}. ■

Let now B and C be locally C*-algebras of type Λ, and the Arens-Michael decomposition of B be

$$B \cong \varprojlim g_\alpha^\beta B_\beta,$$

and the Arens-Michael decomposition of C be

$$C \cong \varprojlim f_\alpha^\beta C_\beta, \ \alpha \in \Lambda.$$

Definition 44 *We call* B *being surjective *-homomorphic to* C *iff there exists a surjective *-homomorphism (which we call a surjective locally *-homomorphism)*

$$\psi : B \to C,$$

such that for each $\alpha \in \Lambda$ *there exists a surjective *-homomorphism*

$$\psi_\alpha : B_\alpha \to C_\alpha,$$

such that

$${}_C\pi_\alpha \circ \psi = \psi_\alpha \circ {}_B\pi_\alpha.$$

Theorem 16 *Let* $\{B_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$ *and* $\{C_\alpha; f_\alpha^\beta; \alpha, \beta \in \Lambda\}$ *be projective families of C*-algebras,* $(B, \tau_B) = \varprojlim g_\alpha^\beta B_\beta$ *and* $(C, \tau_C) = \varprojlim f_\alpha^\beta C_\beta$ *be projective limits with* *respective projective topologies and $\psi : B \to C$ be a surjective locally *-homomorphism. Then $B/\ker\psi$ is locally *-isometric to C.*

Proof. The statement of the theorem follows from the fact that for each $\alpha \in \Lambda$, $B_\alpha/\ker\psi_\alpha$is a C*-algebra isometrically *-isomorphic to C_α, and the family $B_\alpha/\ker\psi_\alpha, \alpha \in \Lambda$, forms a projective family of C*-algebras such that its projective limit is locally *-isomorphic to the projective limit of the projective family of C*-algebras C_α, which is locally *-isomorphic to C. ■

4.4 Locally Isometry of Projective Limits of JB-algebras

In the present section we obtain the main result that modulo a certain closed Jordan ideal each locally JB-algebra is locally Jordan isomorphic to a locally JC-algebra of continuous linear self-adjoint operators acting on a certain locally Hilbert space.

Let now $\{B_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$ and $\{C_\alpha; f_\alpha^\beta; \alpha, \beta \in \Lambda\}$ be projective families of JB-algebras, with $(B, \tau_B) = \varprojlim B_\alpha$ and $(C, \tau_C) = \varprojlim C_\alpha$ the projective limits with respective projective topologies.

Definition 45 *We call B as surjective Jordan homomorphic to C if there exists a surjective Jordan homomorphism (which we call a surjective locally Jordan homomorphism)*

$$\psi : B \to C,$$

such that for each $\alpha \in \Lambda$ there exists a surjective Jordan homomorphism

$$\psi_\alpha : B_\alpha \to C_\alpha,$$

such that

$$_C\pi_\alpha \circ \psi = \psi_\alpha \circ_B \pi_\alpha.$$

Similarly to the C*-algebra case we establish the following:

Theorem 17 *Let $\{B_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$ and $\{C_\alpha; f_\alpha^\beta; \alpha, \beta \in \Lambda\}$ be projective families of JB-algebras, $(B, \tau_B) = \varprojlim B_\alpha$ and $(C, \tau_C) = \varprojlim C_\alpha$ projective limits with respective projective topologies Let B and C be surjective Jordan homomorphic locally JB-algebras of type Λ, and $\psi : B \to C$,be a surjective locally Jordan homomorphism. Then $B/\ker\psi$ is locally Jordan isometric to C.*

Proof. The statement of the theorem follows from the fact that for each $\alpha \in \Lambda$, $B_\alpha/\ker\psi_\alpha$is a JB-algebra isometrically Jordan isomorphic to C_α, and the family $B_\alpha/\ker\psi_\alpha, \alpha \in \Lambda$, forms a projective family of JB-algebras such that its projective limit is locally Jordan isomorphic to the projective limit of the projective family of JB-algebras C_α, which is locally Jordan isomorphic to C. ■

Proposition 8 *Let*

$$g_\alpha^\beta : A_\beta \longrightarrow A_\alpha,$$

be a surjection from the JB-algebra A_β onto the JB-algebra A_α. Let K_β be the exceptional ideal of A_β. Then

$$K_\alpha = g_\alpha^\beta(K_\beta), \tag{4.38}$$

is the exceptional ideal of A_α.

Proof. Let us assume on the contrary that

$$g_\alpha^\beta(K_\beta) = K_\alpha^{'} \neq K_\alpha. \tag{4.39}$$

Let us define a mapping

$$\tilde{g}_{\alpha}^{\beta} : A_{\beta}/K_{\beta} \longrightarrow A_{\alpha}/K_{\alpha}^{'}, \tag{4.40}$$

as follows:

$$\tilde{g}_{\alpha}^{\beta}(x + K_{\beta}) = (y + K_{\alpha}^{'}), \quad \text{iff} \quad g_{\alpha}^{\beta}(x) = y,$$

where $x \in A_{\beta}$ and $y \in A_{\alpha}$. One can see that $\tilde{g}_{\alpha}^{\beta}$ is a surjective Jordan homomorphism from A_{β}/K_{β} onto $A_{\alpha}/K_{\alpha}^{'}$. Thus, because A_{β}/K_{β} is isometrically Jordan isomorphic to a JC-algebra, then $A_{\alpha}/K_{\alpha}^{'}$ is, as its Jordan homomorphic image, is also isometrically Jordan isomorphic to a JC-algebra (theorem 8). On the other hand, let φ be a factor representation not annihilating $K_{\alpha}^{'}$ which is a JBW-factor $M \neq M_3^8$ (existence of such a factor representation follows from corollary 5.7[1] and theorem 8). Then $\varphi \circ g_{\alpha}^{\beta}$, is a factor representation of A_{β} on M not annihilating K_{β}, which contradicts the fact that $M \neq M_3^8$ (theorem 9 (ii)). Thus, due to the uniqueness of the exceptional ideal in JB-algebra $K_{\alpha}^{'} = K_{\alpha}$. ■

Proposition 9 *Let $A_{\alpha}, K_{\alpha}, A_{\beta}, K_{\beta}, g_{\alpha}^{\beta}$ be the same as in proposition 8. Then there exists a natural surjection*

$$\tilde{g}_{\alpha}^{\beta} : A_{\beta}/K_{\beta} \longrightarrow A_{\alpha}/K_{\alpha}, \tag{4.41}$$

such that

$$\tilde{g}_{\alpha}^{\beta}(x + K_{\beta}) = (y + K_{\alpha}) \quad \textit{iff} \quad g_{\alpha}^{\beta}(x) = y, \tag{4.42}$$

where $x \in A_{\beta}$ and $y \in A_{\alpha}$.

Proof. Let us set

$$\tilde{g}_{\alpha}^{\beta}(x + K_{\beta}) = g_{\alpha}^{\beta}(x) + g_{\alpha}^{\beta}(K_{\beta}).$$

The result now follows from proposition 8. ■

So, without a loss of generality we can (due to theorem 9 (i)) assume now that each

$$M_\alpha = A_\alpha / K_\alpha,$$

$\alpha \in \Lambda$, is a special JB-algebra. Thus, M_α is isometrically isomorphic to a JC-algebra. Let us show that these isometric Jordan isomorphisms

$$M_\alpha \hookrightarrow B(H_\alpha)_{SA} \;\text{ and }\; M_\beta \hookrightarrow B(H_\beta)_{SA} \tag{4.43}$$

can be chosen in such a way that $H_\alpha \subset H_\beta$,if $\alpha \preceq \beta; \quad \alpha, \beta \in \Lambda$.

Proposition 10 *For the special JB-algebras*

$$M_\alpha = A_\alpha / K_\alpha \;\; \textit{and} \;\; M_\beta = A_\beta / K_\beta \tag{4.44}$$

there exist Hilbert spaces H_α and H_β, such that $H_\alpha \subset H_\beta$, M_α is isometrically Jordan isomorphic to a norm closed Jordan subalgebra of $B(H_\alpha)_{SA}$ and M_β is isometrically Jordan isomorphic to a norm closed Jordan subalgebra of $B(H_\beta)_{SA}$, $\alpha \preceq \beta; \quad \alpha, \beta \in \Lambda$.

Proof. Proposition 10 follows from theorem 9 except for the fact that $H_\alpha \subset H_\beta$. From the fact that M_α is a special JB-algebra it follows that all its factor representations are JW-factors. Let H_α be the complex Hilbert which is a direct sum of all Hilbert spaces of the factor representations of M_α. Let φ be a factor representation of M_α on a JW-factor N, then

$$\tilde{\varphi} = \varphi \circ \tilde{g}_\alpha^\beta, \tag{4.45}$$

where

$$\tilde{g}_\alpha^\beta : M_\beta \longrightarrow M_\alpha,$$

is the natural surjection from M_β onto M_α. One can see that $\tilde{\varphi}$ is the factor representation of M_β on N, thus $H_\alpha \subset H_\beta$. ■

Theorem 18 *Let $\{A_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$ be a projective family of JB-algebras, $(A, \tau_A) = \varprojlim A_\alpha$ projective limit with respective projective topology. Then there exists a unique, up to a locally Jordan isomorphism, closed Jordan ideal K in A, such that:*

1) $K \cong \varprojlim K_\alpha$,where K_α is the exceptional ideal of A_α for each $\alpha \in \Lambda$;

2) A/K is locally Jordan isomorphic to a projective limit of a projective family of JC-algebras, namely A_α/K_α.

Proof. To prove the first part of the statement notice that from proposition 8 it follows that the family K_α, $\alpha \in \Lambda$, is a projective family of JB-algebras, where each K_α is a closed ideal in the JB-algebra A_α. Thus

$$K = \varprojlim K_\alpha \tag{4.46}$$

is the unique, up to a locally Jordan isomorphism, closed Jordan ideal K of type Λ in

$$A = \varprojlim A_\alpha,$$

$\alpha \in \Lambda$.

To prove the second part of the statement notice that for each $\alpha \in \Lambda$, A_α/K_α is isometrically isomorphic to a JC-algebra. The family A_α/K_α is a projective family of special JB-algebras, and, according to Proposition 5 the Hilbert spaces H_α and H_β of the representations of the algebras A_α/K_α and A_β/K_β can be chosen so that $H_\alpha \subset H_\beta$ iff $\alpha \preceq \beta$; $\alpha, \beta \in \Lambda$.Thus, the family of Hilbert spaces $H_\alpha, \alpha \in \Lambda$ is inductive, and the locally Hilbert space

$$H = \varinjlim H_\alpha,$$

is its inductive limit.[33] We as well get that the family $B(H_\alpha), \alpha \in \Lambda$ is projective. Let

$$L(H) = \varprojlim B(H_\alpha),$$

be the locally C*-algebra of continuous linear operators on the locally Hilbert space H which is the projective limit of C*-algebras $B(H_\alpha),\ \alpha \in \Lambda$.[33] One can now see that

$$A/K = \varprojlim A_\alpha/K_\alpha, \tag{4.47}$$

is locally Jordan isomorphic to a locally JC-subalgebra of the locally JC-algebra $L(H)_{SA}$.

■

4.5 Properties of Projective Limits of JB-algebras

In this subsection we introduce a class of Jordan algebras that are Jordan analogues of complex locally C*-algebras.

Definition 46 *A Jordan algebra J with topology generated by a separating saturated family $\{p_\alpha\}_{\alpha\in\Lambda}$ of JB-regular seminorms is called a* ***Jordan lmc algebra****.*

A Jordan lmc algebra with the family of JB-regular seminorms is called ***JB-regular Jordan lmc algebra****. Let us first introduce an equivalence relation on J.*

Definition 47 *Let J be a JB-regular Jordan lmc algebra and*

$$M_\alpha = \{x : p_\alpha(x^2) = 0, \forall x \in J\}. \tag{4.48}$$

Two elements x and y from J are equivalent $x \sim_\alpha y$ if $x - y \in M_\alpha$

Lemma 15 *Let J be a JB-regular Jordan lmc algebra. Then the set M_α is a Jordan ideal in J and the relation $\sim_\alpha$is an equivalence relation for any $\alpha \in \Lambda$.*

Proof. We show that M_α is a Jordan ideal. One can see that $p_\alpha(x) = 0$ if $x \in M_\alpha$. In fact, JB-regularity implies $0 = p_\alpha(x^2) = p_\alpha^2(x)$, so $p_\alpha(x) = 0$.

a) If $y \in J$ and $x \in M_\alpha$, then

$$p_\alpha\left((x \bullet y)^2\right) \leq p_\alpha^2(x \bullet y) \leq p_\alpha^2(x) p_\alpha^2(y) = 0, \tag{4.49}$$

which means that $x \bullet y \in M_\alpha$.

b) If $x \in M_\alpha$ and $y \in M_\alpha$, then linear combination $\mu x + \eta y \in M_\alpha$, $\mu, \eta \in \mathbb{R}$. This is correct because

$$p_\alpha(\mu x + \eta y) \leq p_\alpha(\mu x) + p_\alpha(\eta y) = |\mu| p_\alpha(x) + |\eta| p_\alpha(y) = 0.$$

The relation $\sim_\alpha$is an equivalence one because it is

(i) reflexive: $x \sim_\alpha x$ because $x - x = 0 \in M_\alpha$,

(ii) symmetric: if $x - y \in M_\alpha$, then equivalently $y - x \in M_\alpha$ (additive inverse belongs to the ideal) or $y \sim_\alpha x$, and

(iii) transitive: $x - y \in M_\alpha$ and $y - z \in M_\alpha$, then $(x - y) + (y - z) = x - z \in M_\alpha$ (sum of two elements of the ideal belongs to the ideal) or $x \sim_\alpha z$. ■

Remark 5 *The relation $\sim_\alpha$ induces a partition of J into classes of equivalency with respect to $\sim_\alpha$. These classes we identify with elements of a factor set $J_\alpha = J/M_\alpha$. The elements of J_α are at the same time the subsets $[.]_\alpha$ in J. If $x \in J$ is a representative of some class of equivalency then we denote such a class by $[x]_\alpha$. Note that if $x \sim_\alpha y$,* *then classes* $[x]_\alpha$ *and* $[y]_\alpha$ *are identical. If the elements* x *and* y *are not equivalent, then* $[x]_\alpha \cap [y]_\alpha = \varnothing$.

Lemma 16 *Let* J *be a Jordan lmc algebra. Then* $J_\alpha = J/M_\alpha$ *is a Jordan algebra with a JB-regular norm.*

Proof. We define addition, multiplication by real scalars and multiplication and show that J_α is complete under these operations. Let $\forall [x]_\alpha, [y]_\alpha \in J_\alpha$, $x, y \in J$, and $\forall \mu, \xi \in R$.

(i) sum

$$\mu[x]_\alpha + \xi[y]_\alpha = \{(\mu x + \xi y) + r : r \in M_\alpha\} \tag{4.50}$$

$$= \{(\mu x + r) + (\xi y + r) : r \in M_\alpha\}$$

$$= \{(\mu x' + r) + (\xi y' + r) : r \in M_\alpha\} = \mu[x']_\alpha + \xi[y']_\alpha,$$

(ii) product $[x]_\alpha \bullet [y]_\alpha = [x \bullet y]_\alpha \in J_\alpha$.

$$\begin{aligned} [x]_\alpha \bullet [y]_\alpha &= \{(x + r_1) \bullet (y + r_2) : r_1, r_2 \in M_\alpha\} \\ &= \{x \bullet y + x \bullet r_2 + r_1 \bullet y + r_1 r_2 : r_1, r_2 \in M_\alpha\} \\ &= \{x \bullet y + r : r \in M_\alpha,\ x \bullet r_2 + r_1 \bullet y + r_1 r_2 = r\} = [x \bullet y]_\alpha \end{aligned} \tag{4.51}$$

We show that the result of Jordan multiplication does not depend on the choice of the representative element. Let $x \sim_\alpha x'$ and $y \sim_\alpha y'$.Then

$$\begin{aligned} [x']_\alpha \bullet [y']_\alpha &= \{(x' + r') \bullet (y' + r'') : r', r'' \in M_\alpha\} \\ &= \{((x + r_1) + r') \bullet ((y + r_2) + r'') : r_1, r_2, r', r'' \in M_\alpha\} \\ &= \{(x + (r_1 + r')) \bullet (y + (r_2 + r'')) : (r_1 + r'), (r_2 + r'') \in M_\alpha\}. \end{aligned} \tag{4.52}$$

(iii) Jordan associativity takes place:

$$([x]_\alpha^2 \bullet [y]_\alpha) \bullet [x]_\alpha = [x^2 \bullet y]_\alpha \bullet [x]_\alpha = [(x^2 \bullet y) \bullet x]_\alpha = [x^2 \bullet (y \bullet x)]_\alpha = \tag{4.53}$$

$$[x^2]_\alpha \bullet [y \bullet x]_\alpha = [x]_\alpha^2 \bullet ([y]_\alpha \bullet [x]_\alpha)$$

Let us show that $\|[x]_\alpha\|_\alpha = p_\alpha(x)$ is a JB-norm.

(a) Norm nonnegativity

$$\|[x]_\alpha\|_\alpha = p_\alpha(x) \geq 0 \tag{4.54}$$

and equal to zero iff $[x]_\alpha = [0]_\alpha$, indeed, $\|[0]_\alpha\|_\alpha = \|[0]_\alpha\|_\alpha + M_\alpha$;

(b) Triangle inequality:

$$\|[x]_\alpha + [y]_\alpha\|_\alpha = p_\alpha(x+y) \leq p_\alpha(x) + p_\alpha(y) = \|[x]_\alpha\|_\alpha + \|[y]_\alpha\|_\alpha; \tag{4.55}$$

(c) Homogeneity:

$$\|\lambda[x]_\alpha\|_\alpha = p_\alpha(\lambda x) = |\lambda| p_\alpha(x) = |\lambda| \|[x]_\alpha\|_\alpha. \tag{4.56}$$

(d) Regularity:

$$\|[x]_\alpha^2\|_\alpha = p_\alpha(x^2) = p_\alpha^2(x) = (\|[x]_\alpha\|_\alpha)^2, \tag{4.57}$$

(e) Fineness:

$$\|[x]_\alpha^2\|_\alpha = p_\alpha(x^2) \leq p_\alpha(x^2 + y^2) = \|[x]_\alpha^2 + [y]_\alpha^2\|_\alpha. \tag{4.58}$$

(f) Submultiplicative:

$$\|[x]_\alpha \bullet [y]_\alpha\|_\alpha = p_\alpha(x \bullet y) \leq p_\alpha(x) p_\alpha(y) = \|[x]_\alpha\|_\alpha \|[y]_\alpha\|_\alpha. \tag{4.59}$$

Then, combining (i)-(iii) and (a)-(f) we conclude that $J_\alpha, \forall \alpha \in \Lambda$ is a Jordan algebra with the JB-regular norm. ■

Lemma 17 *Let J be a JB-regular Jordan lmc algebra. Then a completion of J_α in the norm $||.||_\alpha$, $\overline{J_\alpha} = \overline{J/M_\alpha}$ $\forall \alpha \in \Lambda$, is a JB-algebra.*

Proof. To prove this lemma we repeat the same steps of the proof of lemma 13. ■

Proposition 11 *Let J be a complete JB-regular Jordan lmc algebra. Then J is isomorphic and homeomorphic to a projective limit with a projective topology of projective family of JB-algebras $\bar{J}_\alpha = \overline{J/M_\alpha}$, $\alpha \in \Lambda$.*

Proof. To prove this proposition we repeat the same steps of the proof of proposition 1. ■

Theorem 19 (Locally JB-algebras' Main Theorem) *For a complete Jordan lmc algebra J the following conditions are equivalent:*

1) J is (isomorphic and homeomorphic to) a JB-regular Jordan lmc algebra.

2) J is (isomorphic and homeomorphic to) a projective limit of a projective family of JB-algebras, equipped with the projective topology.

If J in addition is a locally JC-algebra, then

3) J is Jordan isomorphic and homeomorphic to a closed subalgebra of a Jordan algebra with symmetric multiplication of admissible operators $L(\mathbf{H}^C)_{SA}$, where $\mathbf{H}^C$ is a complex locally Hilbert space.

Proof. 1) and 2) follow from propositions 11, 10 and theorem 18. 3) follows from example 2 and theorem 18 (i). ■

Definition 48 *A complete Jordan lmc algebra J is called a* ***locally JB-algebra*** *if it satisfies any one of two conditions of theorem 19 and thus both of them.*

<u>Examples</u>

Example 5 *The self-adjoint part of any complex locally C*-algebra is a locally JB-algebra.*

Example 6 *The self-adjoint part of any real locally C*-algebra is a locally JB-algebra.*

Example 7 *The product $\prod_{\alpha\in\mathbb{I}} J_\alpha$ of JB-algebras J_α, with the product topology, is a locally JB-algebra.*

Example 8 *Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$, see[76]). Then the algebra $C(X)$ of all continuous, not necessarily bounded real-valued functions on X, with the topology of uniform convergence on compact subsets, is a locally JB-algebra.*

CHAPTER 5.

ABELIAN COMPLEX, REAL LOCALLY C*- AND LOCALLY JB-ALGEBRAS

5.1 Introduction

One of the most fundamental results in the theory of C*-algebras was discovered by Gelfand and Naimark in the seminal paper[24]. It says that any unital abelian C*-algebra B is isometrically *-isomorphic to the algebra $C(X)$ of all continuous complex-valued functions on a compact Hausdorff topological space X (the spectrum of the algebra). Moreover, any *-homomorphism of unital abelian C*-algebras is related to a continuous map of the underlying Hausdorff compacts. Briefly speaking, the category of unital abelian C*-algebras with their *-homomorphisms is dual to the category of compact Hausdorff spaces and their continuous mappings.

Following their complex brethren a real counterpart of the theory of complex C*-algebras was born a few years later, see monographs[27] and.[45] The complexification $B = A \dotplus iA$ of each real C*-algebra A can be endowed with a structure of a C*-algebra, and the process of complexification naturally generates an order 2 involutory linear antiautomorphism

$$\Phi : B \to B, \quad \Phi(x + iy) = x^* + iy^*, \quad x, y \in A, \tag{5.1}$$

where

$$A \cong \{a \in B : \Phi(a) = a^*\}, \quad \forall a \in B. \tag{5.2}$$

Similarly, the aforementioned process of complexification naturally generates an order 2 involutory conjugate-linear automorphism

$$\Psi : B \to B, \Psi(x + iy) = x - iy, \quad x, y \in A, \tag{5.3}$$

where

$$A \cong \{a \in B : \Psi(a) = a\}, \tag{5.4}$$

and

$$\Phi(a) = \Psi(a^*) = (\Psi(a))^*, \quad \forall a \in B \tag{5.5}$$

Arens and Kaplansky were able in late 1940's to extend Gelfand-Naimark duality to a duality between pairs (B, Ψ) and (X, h), where B is an abelian complex C*-algebra, Ψ and X are as above, and

$$h : X \to X, \tag{5.6}$$

is called a topological involution on X, and is an order 2 homeomorphism on X. [6]

Real Jordan analogues of C*-algebras, so called JB-algebras were introduced in 1970's by Alfsen, Shultz and Størmer.[1] In the same paper they produced a Gelfand-Naimark type theorem on the functional representation of abelian unital JB-algebras. The development of the subject is reflected in the monograph of Hanche-Olsen and Størmer[30], which we will use for further references on the general theory of JB-algebras.

In the study of functional topological algebras $C(X)$, where X is a topological space, the Shilov program devoted to the interaction of Functional analysis and General topology, following the ideas of the aforementioned Gelfand-Naimark duality, naturally asks: "To what extent does the spectrum determine the algebra and conversely, is the topological space recoverable from the structure of the algebra of functions associated with it?" So, it is very natural to generalize Gelfand-Naimark and Arens-Kaplansky type results to respectively locally C*-algebras, real locally C*-algebras and locally JB-algebras.

There were numerous attempts to extend the aforementioned Gelfand-Naimark duality to locally C*-algebras. The main difficulty was to find a proper category to which the spectrum normally belongs such that:

1) within this category the spectrum admits a dual decomposition to Arens-Michael decomposition of the algebra into a projective limit of a projective family of unital abelian C*-algebras;

2) the algebra of functions associated with each of its objects should be complete;

3) the intrinsic structure of the spectrum is convenient in the sense of Steenrod[67].

We introduce a type Λ for the unital locally C*-algebra B, and the notion of locally *-homomorphisms between two algebras of the same type Λ. When a locally *-homomorphism is an isomorphism, we come to the notion of a locally isometric *-isomorphism between two unital locally C*-algebras. Then we describe the structure of the spectrum $\mathfrak{M}(B)$ and show how to topologize $\mathfrak{M}(B)$ to turn it into a certain compactly generated topological space with an extra filtration property, so that B is locally isometrically *-isomorphic to the functional algebra $C(\mathfrak{M}(B))$. After that we use the structure of $\mathfrak{M}(B)$ to obtain a version of an Arens-Kaplansky type theorem for real unital abelian locally C*-algebras and a version of a Gelfand-Naimark type theorem for unital abelian locally JB-algebras.

Definition 49 *A topological *-algebra* (B, τ) *over* $\mathbb{C}$ *is called a* ***locally C*-algebra*** (***of type*** Λ)*, where* Λ *is a directed set, if there exists a projective family of C*-algebras*

$$\{B_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\},$$

so that

$$B \cong \varprojlim B_\alpha, \ \alpha \in \Lambda,$$

i.e. B *is topologically *-iseomorphic (*-isomorphic and homeomorphic) to a projective limit of a projective family* B_α, $\alpha \in \Lambda$, *of C*-algebras, i.e. there exists its Arens-Michael decomposition of* B *of weight* Λ, *composed entirely of C*-algebras.*

Definition 50 *A space* X *is* T_4 *if any pair of disjoint closed subsets of* X *have disjoint neighbourhoods.* X *is* ***normal*** *if it satisfies both* T_1 *and* T_4 *axioms.*

A space X *is* $T_{3\frac{1}{2}}$ *if for any closed set* $Y \subset X$ *and any points* $x \in X \backslash Y$ *and* $y \in Y$ *there exists a continuous function such that* $f(y) = 0$ *and* $f(x) = 1$.

A space which satisfies both $T_{3\frac{1}{2}}$ *and* T_1 *axioms is called* ***completely regular*** *or* ***Tychonoff space***.

Let now X *be a Tychonoff space. A* ***filtration*** *of* X *is a directed family*

$$F = \{X_\alpha, \alpha \in \Lambda\}, \tag{5.7}$$

of compact subsets of X, *such that:*

(i) Ordered by inclusion, the family F *is filtered, meaning:*

a) Every one-point subset of X *is in* F,

b) A compact subset of an element of F *is again in* F, *c) Every finite union of X_α belongs to some X_β in F (the saturation property);*

(ii) The inductive limit of the family F being taken in the subcategory $Tych$ is homeomorphic to the space X, i.e. we can think that

$$X = \bigcup_{\alpha \in \Lambda} X_\alpha, \tag{5.8}$$

and the topology on X is defined as follows: a subset $Y \subset X$, is closed iff $Y \cap X_\alpha$ is closed for every compact element $X_\alpha \subset F$, $\alpha \in \Lambda$.

Definition 51 *By a **filtered space (of type** Λ) we understand a pair*

$$(X, F) = (X, X_\alpha, \alpha \in \Lambda), \tag{5.9}$$

consisting of a Tychonoff space X and a fixed filtration F on it.

By a morphism from a filtered space $(X, X_\alpha, \alpha \in \Lambda_1)$ of type Λ_1 to a filtered space $(Y, Y_\delta, \delta \in \Lambda_2)$ of type Λ_2 we mean a continuous map.

$$u : X \to Y,$$

subject to the following extra condition: for every $\alpha \in \Lambda_1$ there exists $\delta \in \Lambda_2$, such that

$$u(X_\alpha) \subset Y_\delta.$$

Definition 52 *With these morphisms the filtered spaces form a category denoted by $Filt$.*

*Two filtered spaces $(X, X_\alpha, \alpha \in \Lambda)$ and $(Y, Y_\alpha, \alpha \in \Lambda)$ of type Λ are called **locally homeomorphic**, iff there exists a homeomorphism $u : X \to Y$, such that for each $\alpha \in \Lambda$, $u(X_\alpha)$ is homeomorphic to Y_α.*

Let now

$$(X, X_\alpha, \alpha \in \Lambda),$$

be a filtered space of type Λ. *Let us consider the algebra* $C(X)$ *of all continuous complex-valued functions on* X *equipped with the supremum seminorms* $\|.\|_\alpha$, $\alpha \in \Lambda$, *corresponding to the* X_α. *This family of seminorms*

$$\Gamma = \{\|.\|_\alpha, \alpha \in \Lambda\}, \tag{5.10}$$

generates on $C(X)$ *a* ***locally convex topology*** τ_Γ, *such that* $(C(X), \tau_\Gamma)$ *becomes a unital* ***abelian functional locally C*-algebra of type*** Λ, *which has an Arens-Michael decomposition*

$$(C(X), \tau_\Gamma) = \varprojlim (C(X_\alpha), \|.\|_\alpha),\ \ \alpha \in \Lambda, \tag{5.11}$$

into a projective limit of a projective family of functional abelian unital C-algebras*[54]

$$(C(X_\alpha), \|.\|_\alpha),\ \ \alpha \in \Lambda. \tag{5.12}$$

5.2 Gelfand-Naimark type Theorem for Abelian Complex Locally C*-algebras

Let (B, τ_Γ) be a locally C*-algebra, where the Hausdorff locally convex topology τ_Γ is generated by

$$\Gamma = \{p_\alpha\}_{\alpha \in \Lambda},$$

(where Λ is a directed set) a saturated separating family of C*-seminorms on B. One has an Arens-Michael decomposition

$$B \cong \varprojlim B_\alpha,$$

where

$$B_\alpha = B/\ker(p_\alpha),$$

is a projective family of C*-algebras.

Because a *-isomorphism between C*-algebras is automatically an isometry[51] , the following definition makes sense.

Definition 53 *Let*

$$\psi : B \to C,$$

*be a *-isomorphism from a locally C*-algebra B of type Λ to a locally C*-algebra C of type Λ. A homomorphism ψ is called* ***locally *-isometry*** *iff for each Arens-Michael decomposition of B,*

$$B \cong \varprojlim B_\alpha,$$

there exists an Arens-Michael decomposition of C,

$$C \cong \varprojlim C_\alpha,$$

*$\alpha \in \Lambda$, such that $\psi(B_\alpha)$ is *-isomorphic to C_α, for each $\alpha \in \Lambda$.*

At first, let us recall the following result due to Nassopoulos[54]:

Proposition 12 *For a *-homomorphism*

$$\psi : B \to C,$$

from a locally C-algebra B of type Λ_1 onto a locally C*-algebra C of type Λ_2, the following two statements are equivalent:*

(i) ψ is continuous;

(ii) ψ is decomposable, in the sense that for each $\beta \in \Lambda_2$ there exists $\alpha \in \Lambda_1$, and a unique morphism

$$\psi_\beta^\alpha : B_\alpha \to C_\beta,$$

so that

$$\psi_\beta^\alpha \circ \ {}_B\pi_\alpha = \ {}_C\pi_\beta \circ \psi, \tag{5.13}$$

where ${}_B\pi_\alpha : B \to B_\alpha$, and ${}_C\pi_\beta : C \to C_\beta$, are natural projections.

The following statement explains the true meaning of the notion of the locally *-isometric mapping: if $\psi : B \to C$, is an algebraic *-isomorphism from a locally C*-algebra B of type Λ to a locally C*-algebra C of type Λ, then ψ is a locally *-isometric mapping iff ψ is a homeomorphism (see proposition 7).

Also, if $\psi : B \to C$, is an algebraic *-isomorphism and homeomorphism from a locally C*-algebra B of type Λ_1 to a locally C*-algebra C of type Λ_2, then Λ_1 can be identified with Λ_2 (means B and C have the same type), and ψ is a locally *-isometric mapping (see corollary 4).

Now, let us proceed to the notion of a global spectrum of a locally C*-algebra. By a character on a topological algebra we understand a non-zero complex-valued morphism on it. Let (B, τ_Γ) be a unital locally C*-algebra, and $\mathfrak{M}(B)$ be the set of all continuous characters on B, that is

$$\mathfrak{M}(B) = \{\varphi \in B' : \varphi \neq 0 \text{ and } \varphi(xy) = \varphi(x)\varphi(y), \ \ \forall x, y \in B\}. \tag{5.14}$$

Let us endow $\mathfrak{M}(B)$ with the relative topology $s|_{\mathfrak{M}(B)}$ from B'_s, where by s we mean the weak *-topology $\sigma(B', B)$ on B'. That topology is the topology of simple or point-wise convergence on B'. The resulting Tychonoff topological space $(\mathfrak{M}(B), s)$ is called the **global** or **topological spectrum** of (B, τ_Γ). In what follows we will refer to $(\mathfrak{M}(B), s)$ by writing simply $\mathfrak{M}(B)$.

Let (B, τ_Γ) be a unital abelian complex locally C*-algebra, where the Hausdorff locally-convex topology τ_Γ is generated by $\Gamma = \{\|.\|_\alpha\}_{\alpha\in\Lambda}$, (where Λ is a directed set) - a saturated separating family of C*-seminorms on B. One has an Arens-Michael decomposition

$$B \cong \varprojlim B_\alpha,$$

where

$$B_\alpha = B/\ker(\|.\|_\alpha),\ \alpha \in \Lambda$$

is a projective family of C*-algebras[33]. Let $\mathfrak{M}(B_\alpha)$ denote the topological spectrum of C*-algebra $B_\alpha, \alpha \in \Lambda$. It is well known that each $\mathfrak{M}(B_\alpha)$ is a Hausdorff compact[51].

From[47] it follows that $\mathfrak{M}(B)$ can be identified with the inductive limit of the family $\mathfrak{M}(B_\alpha)$ equipped with direct limit topology, i.e.

$$\mathfrak{M}(B) = \bigcup_{\alpha\in\Lambda} \mathfrak{M}(B_\alpha),\ \text{ and } \mathfrak{M}(B) \cong \varinjlim \mathfrak{M}(B_\alpha). \tag{5.15}$$

Recall the following:

Definition 54 *A topological space X is called* ***functionally Hausdorff*** *if for any two different points $x, y \in X$ there exists a continuous function*

$$f : X \to [0,1],$$

such that

$$f(x) = 0 \text{ and } f(y) = 1.$$

Theorem 20 *Let (X, F_X) be a functionally Hausdorff filtered space of type Λ, and $C(X)$ be the algebra of all continuous complex-valued functions on the space X, such that their restrictions on each compact $X_\alpha \in F_X, \alpha \in \Lambda$, are continuous, with the locally convex topology generated by supremum seminorms on $C(X_\alpha)$, $\alpha \in \Lambda$. Then there exists a family of surjective *-morphisms*

$$g_\alpha^\beta : C(X_\beta) \to C(X_\alpha),\ \forall \alpha \preceq \beta,\ \alpha, \beta \in \Lambda, \tag{5.16}$$

and with these morphisms the family of functional unital abelian C-algebras $C(X_\alpha), \alpha \in \Lambda$, forms a projective family, and its projective limit $\varprojlim C(X_\alpha)$ with projective topology is a unital abelian locally C*-algebra of type Λ, which is locally *-isometric to $C(X)$.*

Proof. Let $f \in C(X)$ be arbitrary. Denote by

$$f_\alpha = f|_{X_\alpha},\ f_\alpha \in C(X_\alpha), \quad \alpha \in \Lambda.$$

We define

$$\pi_\alpha : C(X) \to C(X_\alpha), \quad \alpha \in \Lambda,$$

$$\pi_\alpha(f) = f|_{X_\alpha}. \tag{5.17}$$

We call a family of functions $\{f_\alpha\}_{\alpha\in\Lambda},\ f_\alpha \in C(X_\alpha)$ inductive if $\forall \alpha \preceq \beta$, $f_\beta|_{X_\alpha} = f_\alpha$. Note that each inductive family of functions generates a unique function $f \in C(X)$, such that $f|_{X_\alpha} = f_\alpha$, $\forall \alpha \in \Lambda$. On the contrary assume that there exists another $f' \in C(X),\ f' \neq f$. It means that there exists $x \in X : f'(x) \neq f(x)$. However we know that $\exists \gamma \in \Lambda : x \in X_\gamma$. Observe that the restriction of $f'|_{X_\gamma} = f|_{X_\gamma}$, or $f'(x) = f(x)$, which proves that any inductive family establishes a unique function f.

From the Tietze Extension theorem[65] it follows that f_α is extendable to f_β, where $\beta \succeq \alpha$[1]. Then we can build the inductive family of functions $\{f_\alpha\}$, which will determine some $f \in C(X)$. So, $\pi_\alpha(f) = f_\alpha$ is a surjection.

Now, for every $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$ we define a mapping

$$g_\alpha^\beta : C(X_\beta) \to C(X_\alpha),$$

as

$$g_\alpha^\beta(\pi_\beta(f)) = \pi_\alpha(f), \ \forall f \in C(X).$$

Each g_α^β is a morphism from the C*-algebra $C(X_\beta)$ with supremum norm onto $C(X_\alpha)$ with supremum norm. Note that g_α^β is a surjective *-homomorphism because the composition of g_α^β with a surjective *-homomorphism π_β is a surjective *-homomorphism π_α. Thus, the family $C(X_\alpha)$ with morphisms g_α^β, $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$, is a projective family. Its projective limit $\varprojlim C(X_\alpha)$, $\alpha \in \Lambda$, with projective topology is a unital functional locally C*-algebra of type Λ which is locally *-isometric to $C(X)$. ■

Remark 6 *Now we can formulate the main result for abelian complex locally C*-algebras. The same result was formulated in the monograph of Fragoulopoulou,*[20] *with the Michael topology (whose character space consists of equicontinuous subsets). Our proof is different and close to that of Nassopoulos*[54] *who presented it in a more categorical manner than the one given in.*[20]

[1] Tietze's Extension theorem requires the space to be normal (satisfies T_1 and T_4 axioms) to extend a continuous function from a closed subset continuously to the whole space. Theorem 6.1.9[65] states that every compact Hausdorff space is normal, which allows us to apply Tietze's Extension theorem for our purposes.

Theorem 21 (Gelfand-Naimark type Theorem) *Each unital abelian locally C*-algebra* (B, τ_Γ) *of type* Λ *is locally *-isometric to the algebra* $C(\mathfrak{M}(B))$ *of all continuous complex-valued functions on the functionally Hausdorff space* $\mathfrak{M}(B)$, *such that their restrictions on each compact* $\mathfrak{M}(B_\alpha), \alpha \in \Lambda$, *are continuous, equipped with the corresponding to* $C(\mathfrak{M}(B_\alpha))$ *supremum seminorms* $\widehat{\|.\|}_\alpha$, $\alpha \in \Lambda$. *This family of seminorms generates on* $C(\mathfrak{M}(B))$ *a locally convex topology* $\tau_{\widehat{\Gamma}}$,

$$\widehat{\Gamma} = \{\widehat{\|.\|}_\alpha, \alpha \in \Lambda\},$$

such that $(C(\mathfrak{M}(B)), \tau_{\widehat{\Gamma}})$ *becomes a unital abelian functional locally C*-algebra of type* Λ, *for which there exists an Arens-Michael decomposition*

$$(C(\mathfrak{M}(B)), \tau_{\widehat{\Gamma}}) \cong \varprojlim (C(\mathfrak{M}(B_\alpha), \widehat{\|.\|}_\alpha), \tag{5.18}$$

$\alpha \in \Lambda$, *into a projective limit of a projective family of functional abelian unital C*-algebras*

$$(C(\mathfrak{M}(B_\alpha), \widehat{\|.\|}_\alpha),$$

$\alpha \in \Lambda$.

Proof. We will start with showing that if (B, τ_Γ) is unital abelian locally C*-algebra of type Λ, and $\mathfrak{M}(B)$ is its global spectrum, then $\mathfrak{M}(B)$ with the inductive limit topology is a compactly generated, functionally Hausdorff space, and $\mathfrak{M}(B_\alpha), \alpha \in \Lambda$ is a distinguished family of generating compacts in it, and B is locally *-isometric to the algebra $C(\mathfrak{M}(B))$ of all continuous complex-valued functions on the space $\mathfrak{M}(B)$, such that their restrictions on each compact $\mathfrak{M}(B_\alpha), \alpha \in \Lambda$, are continuous.

Indeed, let (B, τ_Γ) be an unital abelian locally C*-algebra of type Λ, and

$$B \cong \varprojlim B_\alpha,$$

be its Arens-Michael decomposition as a projective limit of the projective family of unital abelian C*-algebras B_α, $\alpha \in \Lambda$.

From the fact that each B_α, $\alpha \in \Lambda$ is unital and abelian, it follows[51] that each space $\mathfrak{M}(B_\alpha)$, $\alpha \in \Lambda$ is a Hausdorff compact. Thus, $\mathfrak{M}(B)$ is compactly generated. Due to the fact that

$$\mathfrak{M}(B) \cong \varinjlim \mathfrak{M}(B_\alpha), \ \alpha \in \Lambda, \tag{5.19}$$

when $\mathfrak{M}(B)$ is equipped with the inductive limit topology, one can easily see that $\mathfrak{M}(B_\alpha), \alpha \in \Lambda$ is a distinguished family of generating compacts in $\mathfrak{M}(B)$.

Again, as each B_α, $\alpha \in \Lambda$, is unital and abelian, it follows[51] that each algebra B_α is isometrically *-isomorphic to $C(\mathfrak{M}(B_\alpha))$. Let

$$\varphi_\alpha : B_\alpha \to C(\mathfrak{M}(B_\alpha)), \tag{5.20}$$

$$a_\alpha \mapsto f_{a_\alpha},$$

where $a_\alpha \in B_\alpha$, $f_{a_\alpha} \in C(\mathfrak{M}(B_\alpha))$ be that isomorphism, where for each $h_\alpha \in \mathfrak{M}(B_\alpha)$,

$$h_\alpha(a_\alpha) = f_{a_\alpha}(h_\alpha). \tag{5.21}$$

As B_α, $\alpha \in \Lambda$, is a projective family of C*-algebras, for each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$, there exists a surjective *-homomorphism

$$g_\alpha^\beta : B_\beta \to B_\alpha,$$

such that

$$g_\alpha^\gamma = g_\alpha^\beta \circ g_\beta^\gamma, \quad g_\alpha^\alpha = id, \quad \forall \alpha, \beta, \gamma \in \Lambda: \ \alpha \preceq \beta \preceq \gamma,$$

and a surjective *-homomorphism

$$\pi_\alpha : B \to B_\alpha,$$

such that

$$\pi_\alpha = g_\alpha^\beta \circ \pi_\beta, \quad \forall \alpha, \beta \in \Lambda, \quad \alpha \preceq \beta.$$

Let f_{a_β} be an arbitrary function from $C(\mathfrak{M}(B_\beta))$, such that

$$\varphi_\beta^{-1}(f_{a_\beta}) = a_\beta, \quad \beta \in \Lambda.$$

Let us define for $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$,

$$\widetilde{g}_\alpha^\beta : C(\mathfrak{M}(B_\beta)) \to C(\mathfrak{M}(B_\alpha)),$$

$$\widetilde{g}_\alpha^\beta(\varphi_\beta(a_\beta)) \mapsto \varphi_\alpha(g_\alpha^\beta(a_\beta)),$$

where $a_\beta \in B_\beta$. One can easily see that with these morphisms the family $C(\mathfrak{M}(B_\alpha))$ is a projective family, and let $\varprojlim C(\mathfrak{M}(B_\alpha))$, $\alpha \in \Lambda$, be its projective limit algebra equipped with its projective topology generated by supremum seminorms $\widehat{\|.\|}_\alpha$, $\alpha \in \Lambda$ built from the C*-supremum norms on functional C*-algebras $C(\mathfrak{M}(B_\alpha))$. Let

$$\widetilde{\pi}_\alpha : \varprojlim C(\mathfrak{M}(B_\alpha)) \to C(\mathfrak{M}(B_\alpha)), \tag{5.22}$$

be the natural projection which is a surjective *-homomorphism from $\varprojlim C(\mathfrak{M}(B_\alpha))$ onto $C(\mathfrak{M}(B_\alpha))$, such that

$$\widetilde{\pi}_\alpha = \widetilde{g}_\alpha^\beta \circ \widetilde{\pi}_\beta,$$

for each $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$.

Let now

$$\varphi : B \to \varprojlim C(\mathfrak{M}(B_\alpha)),$$

be such that

$$\widetilde{\pi}_\alpha \circ \varphi = \varphi_\alpha \circ \pi_\alpha, \quad \forall \alpha \in \Lambda.$$

It is routine to check that φ is a locally *-isometry from B onto $\varprojlim C(\mathfrak{M}(B_\alpha))$. On the other hand, let $C(\mathfrak{M}(B))$ be the set of all continuous complex-valued functions f on $\mathfrak{M}(B)$ such that $f|_{\mathfrak{M}(B_\alpha)}$ is continuous for all $\alpha \in \Lambda$. It is shown in[18] that $C(\mathfrak{M}(B))$ is naturally endowed with operations and topology turning it to a locally C*-algebra, and that $\varprojlim C(\mathfrak{M}(B_\alpha))$ is *-isomorphic to $C(\mathfrak{M}(B))$, and from the proof one can see that the aforementioned isomorphism is a locally *-isometry. Thus, B is locally *-isometric to $C(\mathfrak{M}(B))$.

So, without a loss of generality we now can say that $C(\mathfrak{M}(B))$ is locally *-isometric to $\varprojlim C(\mathfrak{M}(B_\alpha))$, $\alpha \in \Lambda$, and

$$\varphi : B \to C(\mathfrak{M}(B)) \cong \varprojlim C(\mathfrak{M}(B_\alpha)), \tag{5.23}$$

is a locally *-isometry, and

$$\widetilde{\pi}_\alpha : C(\mathfrak{M}(B)) \cong \varprojlim C(\mathfrak{M}(B_\alpha)) \to C(\mathfrak{M}(B_\alpha)), \tag{5.24}$$

is a natural projection for all $\alpha \in \Lambda$.

Assume that non-zero $h, t \in \mathfrak{M}(B)$ cannot be separated by continuous functions. Thus, for each $a \in B$,

$$(\varphi(a))(h) = (\varphi(a))(t).$$

On the other hand, h and t are continuous on B. Thus, there exist indices α_h and α_t in Λ, such that $h_{\alpha_h} \in \mathfrak{M}(B_{\alpha_h})$ and $t_{\alpha_t} \in \mathfrak{M}(B_{\alpha_t})$, where

$$h(a) = h_{\alpha_h}(\pi_{\alpha_h}(a)) \text{ and } t(a) = t_{\alpha_t}(\pi_{\alpha_t}(a)).$$

Let $\beta \in \Lambda$ be such that $\beta \succeq \alpha_h$ and $\beta \succeq \alpha_h$. We can now define $h_\beta, t_\beta \in \mathfrak{M}(B_\beta)$ such that

$$h_\beta(\pi_\beta(a)) = h_{\alpha_h}(g^\beta_{\alpha_h}(\pi_\beta(a)) = h(a) \text{ and } t_\beta(\pi_\beta(a)) = t_{\alpha_t}(g^\beta_{\alpha_t}(\pi_\beta(a)) = t(a). \tag{5.25}$$

So, we get that

$$h(a) = h_\beta(\pi_\beta(a)) = (\varphi_\beta(\pi_\beta(a)))(h_\beta) = (\varphi(a))(h)$$

$$= (\varphi(a))(t) = (\varphi_\beta(\pi_\beta(a)))(t_\beta) = t_\beta(\pi_\beta(a)) = t(a),$$

for all $a \in B$. Thus, $h = t$. ■

The following corollary is a version of the Spectral theorem for locally C*-algebras.

Corollary 5 (Spectral Theorem) *Let B be a unital locally C*-algebra of type Λ, a be its self-adjoint element, and $LC^*(a)$ be the unital abelian locally C*-subalgebra of type Λ of B generated by a and 1_B. Then $LC^*(a)$ is locally *-isometric to the functional locally C*-algebra $C(\mathfrak{M}(LC^*(a)))$ of type Λ with the Arens-Michael decomposition $\varprojlim C(Sp(\pi_\alpha(a))), \alpha \in \Lambda$.*

Proof. Follows from theorems 21 and 20, and from the fact that $Sp(a) = \cup_{\alpha\in\Lambda} Sp(\pi_\alpha(a))$[20,47]. ■

From a categorical prospective we get the following generalization of Gelfand duality for the category of unital locally C*-algebras of type Λ with local *-isometries vs functionally Hausdorff filtered spaces of type Λ with local homeomorphisms.

Theorem 22 *Let (X, F_X) and $C(X)$ be as in theorem 20. Then the functor*

$$X \mapsto C(X),$$

is a contravariant category equivalence from the category of functionally Hausdorff filtered spaces of type Λ with locally continuous morphisms to the category of unital abelian locally C-algebras of type Λ with local homomorphisms.*

Proof. Direct functor is established as follows:

Let (X, F_X) and (Y, F_Y) be two functionally Hausdorff filtered spaces of type Λ, and

$$\phi : (X, F_X) \to (Y, F_Y) \tag{5.26}$$

locally continuous, i.e. ϕ is continuous, and

$$\phi_\alpha : X_\alpha \to Y_\alpha$$

is continuous for each $\alpha \in \Lambda$, where

$$\phi_\alpha = \phi|_{X_\alpha},$$

and $\phi_\alpha(X_\alpha) \subset Y_\alpha$. One can observe that

$$C(\phi) : C(Y) \to C(X) \tag{5.27}$$

will be a homomorphism given by

$$C(\phi)(f_Y) = f_Y \circ \phi,$$

where $f_Y \in C(Y)$, and for each $\alpha \in \Lambda$, $C(\phi_\alpha) : C(Y_\alpha) \to C(X_\alpha)$ will be a homomorphism given by

$$C(\phi_\alpha)(f_{Y_\alpha}) = f_{Y_\alpha} \circ \phi_\alpha,$$

where $f_{Y_\alpha} = f|_{Y_\alpha}$, and $f_{Y_\alpha} \in C(Y_\alpha)$.

Now, we need an inverse functor. Such a functor is supposed to assign to each unital abelian locally C*-algebra B of type Λ the spectrum $\mathfrak{M}(B)$ of B as in theorem 20 above.

Let the mapping

$$\nabla : B_1 \to B_2 \tag{5.28}$$

be a locally continuous unital homomorphism from a unital abelian locally C*-algebra B_1 of type Λ to a unital abelian locally C*-algebra B_2 of type Λ. It means that for any Arens-Michael locally *-isometric decomposition of

$$B_1 \cong \varprojlim {}_1B_\alpha,$$

as a projective limit of the projective family of unital abelian C*-algebras ${}_1B_\alpha$, $\alpha \in \Lambda$, there exists an Arens-Michael locally *-isometric decomposition of

$$B_2 \cong \varprojlim {}_2B_\alpha,$$

as a projective limit of the projective family of unital abelian C*-algebras ${}_2B_\alpha$, $\alpha \in \Lambda$, where for each $\alpha \in \Lambda$,

$$\nabla_\alpha : {}_1B_\alpha \to {}_2B_\alpha, \tag{5.29}$$

is defined as

$$\nabla_\alpha({}_1\pi_\alpha(a)) = {}_2\pi_\alpha(\nabla(a)), \tag{5.30}$$

for each $a \in B_1$, and is a continuous (as a composition of continuous mappings) unital homomorphism of C*-algebras. Let us notice that for every $b \in B_2$, the function

$$x \mapsto x(b),$$

defines a continuous function from $\mathfrak{M}(B_2)$ to $\mathbb{C}$, such that for each $\alpha \in \Lambda$,

$$x_\alpha \mapsto x_\alpha({}_2\pi_\alpha(b)),$$

is continuous, where

$$x_\alpha = x|_{{}_2B_\alpha}.$$

Consider now the function

$$x \mapsto x \circ \nabla.$$

It is a continuous mapping from $\mathfrak{M}(B_2)$ to $\mathfrak{M}(B_1)$, such that for each $\alpha \in \Lambda$,

$$x_\alpha \mapsto x_\alpha \circ \nabla_\alpha,$$

is a continuous mapping from $\mathfrak{M}({}_2B_\alpha)$ to $\mathfrak{M}({}_1B_\alpha)$, i.e.

$$x \mapsto x \circ \nabla,$$

is locally continuous.

Finally, from theorems 21 and 20 it now follows that these two functors are inverses of each other. ∎

Remark 7 *Some elements of theorems 21, 20 and 22 first appeared in the paper of Phillips*[59] *in the realm of a quasi-topological structure on $\mathfrak{M}(B)$.*

Remark 8 *Examples of Weidner*[74] *(see also*[59]*) show that you cannot make the restriction on (X, F_X) any weaker.*

5.3 Arens-Kaplansky type Theorem for Abelian Real Locally C*-algebras

Let X be a topological space. By a **topological involution** on X we understand a homeomorphism

$$h : X \to X, \tag{5.31}$$

such that

$$h(h(x)) = x, \quad \forall x \in X.$$

Let now

$$(X, F) = (X,\ X_\alpha,\ \alpha \in \Lambda),$$

be a filtered space of type Λ. A topological involution h on X is called a **locally topological involution** on (X, F_X) iff for each $X_\alpha \in F$, $\alpha \in \Lambda$,

$$h_\alpha = h|_{X_\alpha} : X_\alpha \to X_\alpha, \tag{5.32}$$

is a topological involution on X_α.

The following example is a motivation for what follows.

Example 9 *Let $C(X)$ be a locally C*-algebra of a type Λ of all continuous complex-valued functions on a functionally Hausdorff filtered space (X, F_X) of type Λ, such that their restrictions on each Hausdorff compact $X_\alpha, \alpha \in \Lambda$, are continuous, equipped with the supremum seminorms $\|.\|_\alpha$, $\alpha \in \Lambda$, corresponding to $C(X_\alpha)$'s. Let now h be a locally topological involution on (X, F). We define*

$$C(X, h) = \{f \in C(X) : f(h(x)) = \overline{f}(x), \forall x \in X\}. \tag{5.33}$$

*The algebra $C(X,h)$ is a *-subalgebra of $C(X)$ over the field of real numbers. One can see that $C(X,h)$ is a real locally C^*-algebra of type Λ with the topology inherited from $C(X)$.*

The following theorem is valid:

Theorem 23 *Let*

$$(X, F_X) = (X, X_\alpha, \alpha \in \Lambda), \tag{5.34}$$

and h be a locally topological involution on (X, F_X). Let $C(X)$ be a locally C^*-algebra of a type Λ of all continuous complex-valued functions on a functionally Hausdorff filtered space (X, F_X) of type Λ, such that their restrictions on each Hausdorff compact $X_\alpha, \alpha \in \Lambda$, are continuous, equipped with the supremum seminorms $\|.\|_\alpha$, $\alpha \in \Lambda$, corresponding to the $C(X_\alpha)$'s. We define a mapping

$$\Psi : C(X) \to C(X), \tag{5.35}$$

as

$$\Psi(f)(x) = \overline{f}(h(x)),$$

for any $f \in C(X)$ and all $x \in X$. Then:

*(i) Ψ is a conjugate-linear *-automorphism of $C(X)$ of order 2, and*

$$C(X,h) = \{f \in C(X) : \Psi(f) = f\}, \tag{5.36}$$

*is a real *-subalgebra in $C(X)$;*

(ii)

$$C(X) = C(X,h) \dot{+} iC(X,h), \tag{5.37}$$

or, alternatively, each $f \in C(X)$ *has a unique decomposition as* $u + iv$, *with* $u, v \in C(X, h)$;

(iii) let

$$\widehat{\pi}_\alpha : C(X, h) \to C(X_\alpha, h_\alpha), \tag{5.38}$$

$\alpha \in \Lambda$, *be a natural surjective projection from* $C(X, h)$ *onto* $C(X_\alpha, h_\alpha)$. *There exists*

$$\Psi_\alpha : C(X_\alpha) \to C(X_\alpha)$$

such that

$$\Psi_\alpha(\widehat{\pi}_\alpha(f(x)) = \widehat{\pi}_\alpha(\Psi(f(x))), \tag{5.39}$$

for any $f \in C(X)$ *and all* $x \in X, \alpha \in \Lambda$. *In addition,* Ψ *is locally isometric, i.e. for each* $\alpha \in \Lambda$, Ψ_α *is a conjugate-linear *-automorphism of* $C(X_\alpha)$ *of order* 2, *and an isometry;*

(iv) there exists an Arens-Michael decomposition of $C(X, h)$ *into a projective limit of a projective family of real unital functional* C^**-algebras*

$$C(X, h) \cong \varprojlim C(X_\alpha, h_\alpha), \tag{5.40}$$

$\alpha \in \Lambda$, *where each* h_α *is a topological involution on* X_α, *i.e.* $C(X, h)$ *is a real locally* C^**-algebra of type* Λ;

(v) let $(C(X))_{\mathbb{R}}$ *(resp.* $(C(X_\alpha))_{\mathbb{R}}$, $\alpha \in \Lambda$*) denote the algebra* $C(X)$ *regarded as a real algebra (resp.* $C(X_\alpha)$, $\alpha \in \Lambda$ *regarded as a real algebra) (over the field of real scalars).*[40] *For* $f \in C(X)$, *we define*

$$P(f) = \frac{1}{2}[f + \Psi(f)]. \tag{5.41}$$

Then P is a continuous linear surjective mapping

$$P : (C(X))_{\mathbb{R}} \to C(X, h),$$

satisfying

$$P^2 = P.$$

*(vi) every continuous conjugate-linear *-automorphism of $C(X)$ of order 2 arises from a locally topological involution on X (a functionally Hausdorff filtered space (X, F_X) of type Λ) in a manner described above.*

Proof. One can see (i) by direct verification.

To establish (ii), notice that since Ψ is a conjugate-linear *-automorphism of order two,

$$\Psi(f + \Psi(f)) = \Psi(f) + f \text{ and } \Psi(\frac{1}{i}(f - \Psi(f)) = \frac{1}{i}(f - \Psi(f)), \quad f \in C(X).$$

From this we get that

$$f = \frac{1}{2}(f + \Psi(f)) + i(\frac{1}{2i}(f - \Psi(f))),$$

where $\frac{1}{2}(f + \Psi(f)), \frac{1}{2i}(f - \Psi(f)) \in C(X, h)$.

Now we prove uniqueness of decomposition.

Note first that $\Psi(f + ig) = f - ig$, where $f, g \in C(X, h)$.

We get

$$f = \frac{1}{2}((f + ig) + \Psi(f + ig)) \text{ and } g = \frac{1}{2i}((f + ig) - \Psi(f + ig)). \tag{5.42}$$

Suppose on the contrary that there exist $f', g' \in C(X, h) : f' + ig' = f + ig$, and

$f' = \frac{1}{2}((f'+ig')+\Psi(f'+ig'))$ and $g' = \frac{1}{2i}((f'+ig')-\Psi(f'+ig'))$. It means that

$f' = \frac{1}{2}((f+ig)+\Psi(f+ig))$ and $g' = \frac{1}{2i}((f+ig)-\Psi(f+ig))$,

or $f' = f$, and $g' = g$; which proves the uniqueness of decomposition.

One can see (iii) by direct verification: $\widehat{\pi}_\alpha(C(X,h)) = C(X,h)|_{X_\alpha, h_\alpha}$, $h_\alpha : X_\alpha \to X_\alpha$, is surjective by construction.

To show (iv), let

$$C(X) \cong \varprojlim C(X_\alpha),$$

$\alpha \in \Lambda$ be a locally *-isometric Arens-Michael decomposition of $C(X)$ as a projective limit of the projective family of unital abelian functional C*-algebras $C(X_\alpha)$ with supremum norms, $\alpha \in \Lambda$. Let

$$\pi_\alpha : C(X) \to C(X_\alpha),$$

be the natural projection from $C(X)$ onto $C(X_\alpha)$, and

$$g_\alpha^\beta : C(X_\beta) \to C(X_\alpha),$$

be the connecting surjections for all $\alpha, \beta \in \Lambda$, $\alpha \preceq \beta$. We define

$$\Psi_\alpha : C(X_\alpha) \to C(X_\alpha),$$

as

$$\Psi_\alpha \circ \pi_\alpha = \pi_\alpha \circ \Psi,$$

for each $\alpha \in \Lambda$.

If the topological involution

$$h_\alpha : X_\alpha \to X_\alpha,$$

is defined as

$$h_\alpha = h|_{X_\alpha},$$

let

$$C(X_\alpha, h_\alpha) = \{f_\alpha \in C(X_\alpha) : \Psi_\alpha(f_\alpha) = f_\alpha\}, \tag{5.43}$$

for any $\alpha \in \Lambda$ and arbitrary $f_\alpha \in C(X_\alpha)$. From the fact that Ψ_α is a conjugate-linear *-automorphism of order two on a C*-algebra $C(X_\alpha)$ it follows that $C(X_\alpha, h_\alpha)$ is a unital real C*-algebra with topology inherited from $C(X_\alpha)$, and

$$C(X_\alpha) = C(X_\alpha, h_\alpha) \dotplus iC(X_\alpha, h_\alpha),$$

and

$$C(X_\alpha, h_\alpha) \cap iC(X_\alpha, h_\alpha) = \mathbf{0}_{C(X_\alpha)}.$$

Let

$$\tilde{g}_\alpha^\beta : C(X_\beta, h_\beta) \to C(X_\alpha, h_\alpha),$$

be defined as

$$\tilde{g}_\alpha^\beta = g_\alpha^\beta|_{C(X_\beta, h_\beta)}.$$

From the fact that g_α^β was a surjective *-homomorphism of complex C*-algebras it follows that $\tilde{g}_\alpha^\beta$ is a surjective *-homomorphism of real C*-algebras for all $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$. The family of real C*-algebras $C(X_\alpha, h_\alpha)$ with the morphisms $\tilde{g}_\alpha^\beta$ forms a projective family, and its projective limit $\varprojlim C(X_\alpha, h_\alpha)$ is locally *-isomorphic to $C(X, h)$, thus, $C(X, h)$ is a real unital locally C*-algebra of type Λ.

Statement in (v) immediately follows from (i) and (iii).

To show (vi), let Ψ be an arbitrary continuous conjugate-linear involutory *-automorphism of $C(X)$ of order 2, where X is a functionally Hausdorff filtered space (X, F_X) of type Λ. Then from theorem 20 there exists a locally *-isometric Arens-Michael decomposition

$$C(X) \cong \varprojlim C(X_\alpha),$$

$\alpha \in \Lambda$, of $C(X)$ as a projective limit of the projective family of unital abelian functional C*-algebras $C(X_\alpha)$ with supremum norms, $\alpha \in \Lambda$, and each $X_\alpha \subset F_X$. Let

$$\pi_\alpha : C(X) \to C(X_\alpha),$$

be the natural projection from $C(X)$ onto $C(X_\alpha)$, and

$$g_\alpha^\beta : C(X_\beta) \to C(X_\alpha),$$

be the connecting surjections for all $\alpha, \beta \in \Lambda$, $\alpha \preceq \beta$. We define

$$\Psi_\alpha : C(X_\alpha) \to C(X_\alpha),$$

as

$$\pi_\alpha \circ \Psi = \Psi_\alpha \circ \pi_\alpha,$$

for each $\alpha \in \Lambda$. From[40] it follows that each $\Psi_\alpha, \alpha \in \Lambda$, arises from a locally topological involution h_α on X_α. Direct verification shows that there exists a locally topological involution h on (X, F_X), such that for each $\alpha \in \Lambda$, $h_\alpha = h|_{X_\alpha}$. Thus

$$\Psi(f(x)) = \overline{f}(h(x)),$$

for any $f \in C(X)$ and all $x \in X$ because for each $\alpha \in \Lambda$,

$$\Psi_\alpha(f_\alpha(x_\alpha)) = \overline{f_\alpha}(h_\alpha(x_\alpha)),$$

for any $f_\alpha \in C(X_\alpha)$ and all $x_\alpha \in X_\alpha$. ■

Now we can formulate and prove an Arens-Kaplansky type theorem for real abelian locally C*-algebras:

Theorem 24 (Arens-Kaplansky type Theorem) *Let A be a real unital abelian locally C*-algebra of type Λ, and $B = A \dotplus iA$, be its complexification Then A is a real locally *-isometric to the real locally C*-subalgebra*

$$D = \{d \in C(\mathfrak{M}(B)) : d(h(x)) = \overline{d(x)},\ \textit{for any } x \in \mathfrak{M}(B)\}, \tag{5.44}$$

where h is a locally topological involution on the filtered space $\mathfrak{M}(B)$.

Proof. Let $A \cong \varprojlim A_\alpha, \alpha \in \Lambda$, be the Arens-Michael decomposition of A as a projective limit of real unital abelian C*-algebras $A_\alpha,\ \alpha \in \Lambda$. Then $B_\alpha = A_\alpha \dotplus iA_\alpha$ is a unital abelian C*-algebra for each $\alpha \in \Lambda$, and $B \cong \varprojlim B_\alpha,\ \alpha \in \Lambda$, is the Arens-Michael decomposition of B as a projective limit of unital abelian C*-algebras $B_\alpha,\ \alpha \in \Lambda$. Each B_α is isometrically *-isomorphic to $C(\mathfrak{M}(B_\alpha))$, where $\mathfrak{M}(B_\alpha)$ is a Hausdorff compact. From theorem 20 above it follows that B is locally *-isometric to $C(\mathfrak{M}(B))$, where $\mathfrak{M}(B)$ is a functionally Hausdorff filtered space of type Λ.

From theorem 23 it follows that there exists a continuous conjugate-linear involutory *-antiautomorphism Ψ of $C(\mathfrak{M}(B))$ of order 2, which generates a continuous conjugate-linear involutory *-antiautomorphism Ψ' of B of order 2. We define $\Psi'_\alpha : C(\mathfrak{M}(B_\alpha)) \to C(\mathfrak{M}(B_\alpha))$, so that

$$\Psi'_\alpha \circ \pi_\alpha = \pi_\alpha \circ \Psi',$$

where π_α is the natural projection from B onto $B_\alpha, \alpha \in \Lambda$, which is a conjugate-linear involutory antiautomorphism of B_α of order 2. Therefore there exists a family of topological involutions h_α on each $\mathfrak{M}(B_\alpha)$ which generate Ψ'_α for each $\alpha \in \Lambda$, and they in turn generate a locally topological involution h on the space $\mathfrak{M}(B)$, which, as one can directly verify, satisfies the condition of the theorem. ■

From the categorical prospective we get the following generalization of Arens-Kaplansky duality for the category of pairs $(C(X), \Psi)$ of unital abelian locally C*-algebras of type Λ with conjugate-linear involutory antiautomorphisms, with morphisms being local *-isometries, vs pairs (X, h) of functionally Hausdorff filtered spaces of type Λ and local topological involutions with morphisms being local homeomorphisms.

Theorem 25 *Let (X, h) and $C(X, h)$ be as in theorem 24. Then the functor*

$$(X, h) \mapsto C(X, h), \tag{5.45}$$

is a contravariant category equivalence from the category of pairs of functionally Hausdorff filtered spaces of type Λ with local topological involutions and morphisms being local homeomorphisms to the category of pairs of unital abelian locally C-algebras of type Λ with continuous conjugate-linear *-antiautomorphisms of order two with morphisms being local *-isometries.*

Proof. Direct functor is established in the following way:

Let $((X, F_X),_X h)$ and $((Y, F_Y),_Y h)$ be two pairs of functionally Hausdorff filtered spaces of type Λ local topological involutions, with

$$\phi : ((X, F_X),_X h) \to ((Y, F_Y),_Y h), \tag{5.46}$$

satisfying (i) and (ii) below

(i) ϕ is locally continuous, i.e.

$$\phi : X \to Y,$$

is continuous, and

$$\phi_\alpha : X_\alpha \to Y_\alpha, \quad \forall \alpha \in \Lambda,$$

is continuous, where

$$\phi_\alpha = \phi|_{X_\alpha},$$

and $\phi_\alpha(X_\alpha) \subset Y_\alpha$;

(ii) also

$$\phi({}_X h(x)) = {}_Y h(\phi(x)), \quad \forall x \in X$$

and thus

$$\phi_\alpha({}_X h_\alpha(x_\alpha)) = {}_Y h_\alpha(\phi_\alpha(x_\alpha)), \quad \forall x_\alpha \in X_\alpha.$$

One can observe that

$$C(\varphi) : C(Y) \to C(X),$$

given by

$$C(\varphi)(f_Y) = f_Y \circ \phi,$$

will be a homomorphism, where $f_Y \in C(Y)$, and for each $\alpha \in \Lambda$,

$$C(\phi_\alpha) : C(Y_\alpha) \to C(X_\alpha),$$

given by

$$C(\phi_\alpha)(f_{Y_\alpha}) = f_{Y_\alpha} \circ \phi_\alpha,$$

will be a homomorphism, where $f_{Y_\alpha} = f|_{Y_\alpha}$, and $f_{Y_\alpha} \in C(Y_\alpha)$. The restriction of $C(\phi)$ on $C(Y,_Y h)$ is the required homomorphism

$$C(\phi)|_{C(Y,_Y h)} : C(Y,_Y h) \to C(X,_X h),$$

such that

$$C(\phi)(f_Y({}_Y h(\phi(x)))) = f_Y \circ \phi({}_X h(x)),$$

where $f_Y \in C(Y)$, and for each $\alpha \in \Lambda$ and $x \in X$, and for each $\alpha \in \Lambda$,

$$C(\phi_\alpha)|_{C(Y_\alpha,_Y h_\alpha)} : C(Y_\alpha,_Y h_\alpha) \to C(X_\alpha,_X h_\alpha)$$

given by

$$C(\phi_\alpha)(f_{Y_\alpha}({}_Y h_\alpha(\phi_\alpha(x_\alpha)))) = f_{Y_\alpha} \circ \phi_\alpha({}_X h_\alpha(x_\alpha)), \tag{5.47}$$

will be a homomorphism, where $f_{Y_\alpha} = f|_{Y_\alpha}$, and $f_{Y_\alpha} \in C(Y_\alpha)$ and $x_\alpha \in X_\alpha$.

Now, we need an inverse functor. Such a functor is supposed to assign to each pair (B, Ψ) composed of a unital abelian locally C*-algebra B of type Λ with its continuous conjugate-linear *-automorphism Ψ, the pair $(\mathfrak{M}(B), h)$ composed of the spectrum $\mathfrak{M}(B)$ of B with h being a locally topological involution of $\mathfrak{M}(B)$ as in theorem 24 above.

Let

$$\nabla : (B_1, \Psi_1) \to (B_2, \Psi_2),$$

be a locally continuous unital homomorphism from a unital abelian locally C*-algebra B_1 of type Λ to a unital abelian locally C*-algebra B_2 of type Λ, such that

$$\nabla \circ \Psi_1 = \Psi_2 \circ \nabla.$$

It means that for any Arens-Michael locally *-isometric decomposition of

$$B_1 \cong \varprojlim {}_1B_\alpha,$$

as a projective limit of the projective family of unital abelian C*-algebras ${}_1B_\alpha$, $\alpha \in \Lambda$, there exists an Arens-Michael locally *-isometric decomposition of

$$B_2 \cong \varprojlim {}_2B_\alpha,$$

as a projective limit of the projective family of unital abelian C*-algebras ${}_2B_\alpha$, $\alpha \in \Lambda$, such that for each $\alpha \in \Lambda$,

$$\nabla_\alpha : ({}_1B_\alpha, {}_1\Psi_\alpha) \to ({}_2B_\alpha, {}_2\Psi_\alpha),$$

is defined as

$$\nabla_\alpha({}_1\pi_\alpha(a)) = {}_2\pi_\alpha(\nabla(a)),$$

for each $a \in B_1$, and is a continuous (as a composition of continuous mappings) unital homomorphism of C*-algebras, and satisfies the condition

$$\nabla_\alpha \circ {}_1\Psi_\alpha = {}_2\Psi_\alpha \circ \nabla_\alpha.$$

Let us notice that for every $b \in B_2$, the function

$$x \mapsto x(b),$$

defines a continuous function from $\mathfrak{M}(B_2)$ to $\mathbb{C}$, such that for each $\alpha \in \Lambda$,

$$x_\alpha \mapsto x_\alpha({}_2\pi_\alpha(b)),$$

is continuous, where

$$x_\alpha = x|_{{}_2B_\alpha}.$$

If now ${}_{\mathfrak{M}(B_2)}h$ is the locally topological involution on $\mathfrak{M}(B_2)$ which generates Ψ_2, then ${}_{\mathfrak{M}(B_2)}h(x) \mapsto x(\Psi_2(b))$ defines a continuous function from $\mathfrak{M}(B_2)$ to $\mathbb{C}$, such that

$$ {}_{\mathfrak{M}({}_2B_\alpha)}h_\alpha(x_\alpha) \mapsto x_\alpha({}_2\Psi_\alpha({}_2\pi_\alpha(b))),\ \forall \alpha \in \Lambda. $$

Now, consider the function

$$ x \mapsto x \circ \nabla \circ \Psi_1. $$

It is a locally continuous mapping from $(\mathfrak{M}(B_2), {}_{\mathfrak{M}(B_2)}h)$ to $(\mathfrak{M}(B_1), {}_{\mathfrak{M}(B_1)}h)$, such that for each $\alpha \in \Lambda$,

$$ x_\alpha \mapsto x_\alpha \circ \nabla_\alpha \circ\ {}_1\Psi_\alpha, $$

is a continuous mapping from $(\mathfrak{M}({}_2B_\alpha), {}_{\mathfrak{M}(B_2)}h_\alpha)$ to $(\mathfrak{M}({}_1B_\alpha), {}_{\mathfrak{M}(B_1)}h_\alpha)$ i.e.

$$ x \mapsto x \circ \nabla \circ \Psi_1, $$

is locally continuous.

Finally, from theorems 23 and 24 it follows that these two functors are inverses of each other. ■

5.4 Gelfand-Naimark type Theorem for Abelian Locally JB-algebras

Let B be a C*-algebra. Then

$$ B_{sa} = \{a \in B : a = a^*\}, $$

with symmetric multiplication

$$ a \bullet b = \frac{1}{2}(ab + ba), \tag{5.48} $$

is a Jordan algebra which is a JB-algebra (more precisely, a JC-algebra[30]). Turumaru showed[72] that a C*-algebra B is abelian iff $(B_{sa}, \bullet)$ is an abelian Jordan algebra under symmetric multiplication.

We start by proving a version of Turumaru's theorem for locally C*-algebras.

Theorem 26 *A unital locally C*-algebra (B, τ_B) of type Λ is abelian if the locally JB-algebra $(B_{sa}, \bullet), \tau_{(B_{sa},\bullet)}$ is an abelian locally JC-algebra of type Λ. Conversely, if (A, τ_A) is an abelian locally JB-algebra of type Λ, then there exist operations and topology τ_B on its complexification $B = A \dotplus iA$, turning (B, τ_B) into an abelian unital locally C*-algebra (B, τ_B) of type Λ.*

Proof. Let (B, τ_B) be a unital abelian locally C*-algebra (B, τ_B) of type Λ, and $B \cong \varprojlim B_\alpha, \alpha \in \Lambda$ be the Arens-Michael decomposition as a projective limit of the projective family of unital abelian C*-algebras $B_\alpha, \alpha \in \Lambda$. It is easy to see that each connecting surjective C*-morphism

$$g_\alpha^\beta : B_\beta \to B_\alpha,$$

has a property that

$$g_\alpha^\beta((B_\beta)_{sa}, \bullet)) = ((B_\alpha)_{sa}, \bullet), \tag{5.49}$$

for each pair $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$. Thus, the family of JC-algebras $((B_\alpha)_{sa}, \bullet)$ is projective, and its projective limit with projective topology is Jordan isomorphic and homeomorphic to the abelian locally JC-algebra $((B_{sa}, \bullet), \tau_{(B_{sa},\bullet)})$, where

$$\tau_{(B_{sa},\bullet)} = \tau_B|_{(B_{sa},\bullet)}. \tag{5.50}$$

Conversely, let $((A.\bullet), \tau_A)$ be an abelian unital locally JB-algebra of type Λ and let $A \cong \varprojlim A_\alpha, \;\; \alpha \in \Lambda$, be its Arens-Michael decomposition as a projective limit of the projective family of abelian unital JB-algebras $(A_\alpha, \bullet), \alpha \in \Lambda$, where the topology τ_A is generated by a saturated separating family of JB-seminorms p_α, such that for each

$$p_\alpha(x) = \|\pi_\alpha(x)\|_{A_\alpha}, \;\; \alpha \in \Lambda, \; x \in A,$$

where

$$\pi_\alpha : A \longrightarrow A_\alpha,$$

is the natural projection from A onto $A_\alpha, \; \alpha \in \Lambda$. Let B be the complexification of A. One can easily see that with the product

$$(x+iy)(z+iw) = (x \bullet z - y \bullet w) + i(x \bullet w + y \bullet z) = (z+iw)(x+iy),$$

and involution

$$(x+iy)^* = x - iy,$$

$x, y, z, w \in A$ we turn B into an abelian unital associative *-algebra with a unit

$$\mathbf{1}_B = \mathbf{1}_A = \mathbf{1}_A + i\mathbf{0}_A.$$

Let us for each $\alpha \in \Lambda$ extend seminorms on B

$$\widehat{p}_\alpha : B \longrightarrow \mathbb{R}$$

as

$$\widehat{p}_\alpha(x+iy) = \sqrt{p_\alpha(x^2+y^2)}, \quad x, y \in A. \tag{5.51}$$

Let $x + iy = a \in B$, where $x, y \in A$. Then

$$\widehat{p}_\alpha(a)^2 = p_\alpha(x^2+y^2) = p_\alpha((x-iy)(x+y)) = p_\alpha((x+iy)^*(x+iy)) = \widehat{p}_\alpha(a^*a), \tag{5.52}$$

and

$$\widehat{p}_\alpha(\lambda a) = \sqrt{p_\alpha(\lambda^2 x^2 + \lambda^2 y^2)} = |\lambda| \sqrt{p_\alpha(x^2 + y^2)} = |\lambda| \widehat{p}_\alpha(a) \tag{5.53}$$

for each $\alpha \in \Lambda$. Let a and b are elements of B. Then a^*a and b^*b are elements of $A = B_{SA}$. Therefore

$$\widehat{p}^2_\alpha(ab) = \widehat{p}_\alpha((ab)^*ab) = \widehat{p}_\alpha(b^*a^*ab) = \widehat{p}_\alpha((a^*a)(b^*b)) = \widehat{p}_\alpha(\frac{1}{2}((a^*a)(b^*b) + (b^*b)(a^*a))) \tag{5.54}$$

$$= p_\alpha((a^*a) \bullet (b^*b)) \leq p_\alpha((a^*a)p_\alpha((b^*b) = \widehat{p}_\alpha((a^*a)\widehat{p}_\alpha((b^*b) = \widehat{p}_\alpha(a)^2\widehat{p}_\alpha(b)^2,$$

which implies that

$$\widehat{p}_\alpha(ab) \leq \widehat{p}_\alpha(a)\widehat{p}_\alpha(b).$$

Now, if $a, b \in B$, and $a = x + iy$, $b = z + iw$, where $x, y, z, w \in A$, then we have

$$a^*b + b^*a = 2x \bullet z + 2y \bullet w \in A. \tag{5.55}$$

Due to the fact that p_α is a JB-seminorm for each $\alpha \in \Lambda$,

$$p^2_\alpha(x \bullet z + y \bullet w) = p^2_\alpha(x \bullet z + y \bullet w) \leq p_\alpha((x \bullet z + y \bullet w)^2 + (x \bullet w - y \bullet z)^2) \tag{5.56}$$

$$= p_\alpha((x^2 + y^2) \bullet (z^2 + w^2)) \leq p_\alpha(x^2 + y^2)p_\alpha(z^2 + w^2) = \widehat{p}^2_\alpha(a)\widehat{p}^2_\alpha(b).$$

Thus, we get that

$$\widehat{p}^2_\alpha(a + b) = \widehat{p}_\alpha((a + b)^*(a + b)) = p_\alpha((a + b)^*(a + b)) = \widehat{p}_\alpha(a^*a + (a^*b + b^*a) + b^*b)$$

$$= p_\alpha(a^*a + (2x \bullet z + 2y \bullet w) + b^*b) \leq p_\alpha(a^*a) + p_\alpha(2x \bullet z + 2y \bullet w) + p_\alpha(b^*b)$$

$$= \widehat{p}^2_\alpha(a) + 2p_\alpha(x \bullet z + y \bullet w) + \widehat{p}^2_\alpha(b) = \widehat{p}^2_\alpha(a) + 2\widehat{p}_\alpha(a)\widehat{p}_\alpha(b) + \widehat{p}^2_\alpha(b) = (\widehat{p}_\alpha(a) + \widehat{p}_\alpha(b))^2,$$

which implies

$$\widehat{p}_\alpha(a + b) \leq \widehat{p}_\alpha(a) + \widehat{p}_\alpha(b).$$

We have proved that for each $\alpha \in \Lambda$, $\widehat{p}_\alpha$ is a C*-seminorm on B. Also note, that the family of C*-seminorms $\widehat{p}_\alpha, \alpha \in \Lambda$ is saturated and separating as a consequence of saturation and separability of the family of seminorms $p_\alpha, \alpha \in \Lambda$. Thus (B, τ_B) is a locally C*-algebra of type Λ, where the topology τ_B is generated by family $\widehat{p}_\alpha,\ \alpha \in \Lambda$.

■

From theorem 26 it follows that the representation theory of unital abelian locally JB-algebras A is the representation theory of pairs of the form $(B, *)$, where $B = A + iA$ is the unital abelian locally C*-algebras of a complexification of A and " * " is its involution (which is a continuous order 2 involutory linear antiautomorphisms on B). So, our given unital abelian locally JB-algebra A is Jordan isomorphic and homeomorphic to the Jordan algebra $(B_{sa}, \bullet)$ of self-adjoint elements of the locally C*-algebra B of its complexification with topology on B extended from the topology on A, with symmetric multiplication, which is the set of fixed points under the actions of " * " on B.

Theorem 27 *Each unital abelian locally JB-algebra (A, τ_A) of type Λ is locally *-isometric to the algebra $C^{\mathbb{R}}(\mathfrak{M}(B))$ of all continuous real-valued functions on the functionally Hausdorff space $\mathfrak{M}(B, \tau_B)$ (where*

$$(B, \tau_B) = (A, \tau_A) \dot{+} i(A, \tau_A),$$

is a complexification of (A, τ_A)), such that their restrictions on each compact $\mathfrak{M}(B_\alpha), \alpha \in \Lambda$ (where $B_\alpha = A_\alpha \dot{+} iA_\alpha$), are continuous, equipped with the corresponding to $C^{\mathbb{R}}(\mathfrak{M}(B_\alpha))$ supremum seminorms $\|.\|_\alpha$, $\alpha \in \Lambda$. This family of seminorms generates on $C^{\mathbb{R}}(\mathfrak{M}(B))$ a locally convex topology $\tau_{C^{\mathbb{R}}(\mathfrak{M}(B))}$, such that $(C^{\mathbb{R}}(\mathfrak{M}(B)), \tau_{C^{\mathbb{R}}(\mathfrak{M}(B))})$ becomes an abelian *unital functional locally JB-algebra of type* Λ, *for which there exists an Arens-Michael decomposition*

$$(C^{\mathbb{R}}(\mathfrak{M}(B)), \tau_{C^{\mathbb{R}}(\mathfrak{M}(B))}) \cong \varprojlim C^{\mathbb{R}}(\mathfrak{M}(B_\alpha)), \tag{5.57}$$

$\alpha \in \Lambda$, *into a projective limit of a projective family of functional abelian unital JB-algebras* $C^{\mathbb{R}}(\mathfrak{M}(B_\alpha))$, $\alpha \in \Lambda$.

Proof. Let $(B, \tau_B) = (A, \tau_A) \dot{+} i(A, \tau_A)$ be the abelian unital locally C*-algebra (see theorem 26 above). If

$$(A, \tau_A) \cong \varprojlim A_\alpha, \quad \alpha \in \Lambda,$$

is the Arens-Michael decomposition of (A, τ_A), then one can easily see that there exists an Arens-Michael decomposition of (B, τ_B)

$$(B, \tau_B) \cong \varprojlim B_\alpha, \quad \alpha \in \Lambda,$$

such that each

$$B_\alpha = A_\alpha \dot{+} iA_\alpha, \quad \alpha \in \Lambda.$$

Indeed, if

$$g_\alpha^\beta : A_\beta \to A_\alpha,$$

be the contractive JB-surjection for each $\alpha \preceq \beta$, $\alpha, \beta \in \Lambda$, we can define

$$\tilde{g}_\alpha^\beta : B_\beta \to B_\alpha,$$

as

$$\tilde{g}_\alpha^\beta(x_\beta + iy_\beta) = g_\alpha^\beta(x_\beta) + ig_\alpha^\beta(y_\beta),$$

where

$$x_\beta = \pi_\beta(x) \in A_\beta \text{ and } \ y_\beta = \pi_\beta(y) \in A_\beta,$$

while

$$g_\alpha^\beta(x_\beta) = \pi_\alpha(x) = x_\alpha \in A_\alpha \text{ and } g_\alpha^\beta(y_\beta) = \pi_\alpha(y) = y_\alpha \in A_\alpha, \quad \forall x, y \in A. \tag{5.58}$$

One can see that $\tilde{g}_\alpha^\beta$ is a surjective *-homomorphism, and from the fact that each B_α is a C*-algebra it follows that they are contractions and thus are continuous.

Now, we apply to (B, τ_B) the theorem 20 above to get that (B, τ_B) is locally *-isometric to the abelian unital locally C*-algebra

$$(C(\mathfrak{M}(B)), \tau_{C(\mathfrak{M}(B))}) \cong \varprojlim C(\mathfrak{M}(B_\alpha)), \tag{5.59}$$

and $(C^{\mathbb{R}}(\mathfrak{M}(B)), \tau_{C^{\mathbb{R}}(\mathfrak{M}(B)})$ is obviously its self-adjoint part. Thus, $(C^{\mathbb{R}}(\mathfrak{M}(B)), \tau_{C(\mathfrak{M}(B))})$ is locally Jordan isometric to $\varprojlim C^{\mathbb{R}}(\mathfrak{M}(B_\alpha))$, where each $C^{\mathbb{R}}(\mathfrak{M}(B_\alpha))$ is Jordan isometrically isomorphic to the self adjoint part of $C(\mathfrak{M}(B_\alpha))$ for each $\alpha \in \Lambda$. ■

The following corollary is a version of the Spectral theorem for locally JB-algebras.

Corollary 6 (Spectral Theorem) *Let A be a unital locally JB-algebra of type Λ, x be its element, and $LJB(x)$ be the unital locally JB-subalgebra of type Λ of A generated by x and $\mathbf{1}_A$. Then $LJB(x)$ is locally Jordan isometric to the functional locally JB-algebra $C^{\mathbb{R}}(\mathfrak{M}(LC^*(x)))$ of type Λ with the Arens-Michael decomposition $\varprojlim C^{\mathbb{R}}(Sp(\pi_\alpha(a)))$, $\alpha \in \Lambda$.*

Proof. Follows from theorems 25, 26, and 27 above. ■

CHAPTER 6.

UNIVERSAL ENVELOPING ALGEBRAS FOR LOCALLY JB-ALGEBRAS

Universal Specialization and Universal Enveloping

Let us first define universal specialization:

Definition 55 [30] *Let J be a Jordan algebra. A* ***universal specialization*** *of J is an associative real algebra U with a Jordan homomorphism u of J into $U^J = (U, \bullet)$ (definition 31) such that:*

(i) $u(J)$ generates U as an algebra.

(ii) If A is a real associative algebra and $\phi : J \longrightarrow A^J$ is a homomorphism, then there exists a homomorphism $\hat{\phi} : U \longrightarrow A^J$ such that $\phi = \hat{\phi} \circ u$.

In view of their relevance to the current investigation, we remind the reader of the two theorems noted in the preliminaries. The first theorem states that if G is a JC-algebra and K is a norm closed ideal in G, then the factor algebra G/K is also a JC-algebra (theorem 8).

According to the second theorem if J is a JB-algebra, then there is a unique Jordan ideal K such that J/K has a faithful Jordan representation as a JC-algebra, and every factor representation of J not annihilating J is onto the exceptional algebra of M_8^3 (theorem 9).

Theorem 28 [30] *Let G be a JB-algebra. Then there exists up to a *-isomorphism, a unique C^*-algebra $C_u^*(A)$ such that:*

i) ψ - a Jordan homomorphism $\psi : G \to C_u^*(G)_{SA}$ (where $C_u^*(G)_{SA}$ denotes the self-adjoint part of $C_u^*(G)$) is defined in such a way that $\psi(G)$ generates $C_u^*(G)$ as a C^*-algebra.

*ii) If Q is a C^*-algebra and ϕ is a Jordan homomorphism from G to Q_{SA}, then there exists a *-homomorphism $\widehat{\phi}$ from $C_u^*(G)$ to Q such that $\phi = \widehat{\phi} \circ \psi$.*

*iii) There exists a unique *-antiautomorphism α of $C_u^*(G)$ of order 2, such that $\alpha(\psi(x)) = \psi(x)$ for any $x \in G$.*

Such an algebra $C_u^*(A)$ with properties (i), (ii) and (iii) is called a "universal enveloping algebra" or a "universal envelope" of a JB-algebra G.

6.1 Representations of Locally JB-algebras as Locally JC-algebras

The main goal of this chapter is to generalize the theory of enveloping of JB-algebras to locally JB ones. There were numerous attempts to extend the theory of C*-algebras to non-associative algebras which are close to associative, in particular to Jordan algebras. In 1978 Alfsen, Schultz and Størmer published their celebrated paper,[1] in which they introduced and studied JB-algebras, which are real non-associative analogues of C*-algebras. They obtained for these algebras representation theorems analogous to the Gelfand-Naimark ones.

Definition 56 *An algebra R is called a **locally JC-algebra** if there exists a projective family $\{R_\alpha, \widehat{g}_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$ of JC-algebras R_α with morphisms $\widehat{g}_\alpha^\beta$, such that its projective limit is locally isomorphic to R.*

We need the following lemma:

Lemma 18 (Alfsen and Shultz) [2]*Let A be a JB-algebra. Then there exists a Hilbert space H of dimension large enough such that for every Jordan homomorphism*

$$\phi : A \longrightarrow B_{sa},$$

where B is any C-algebra, the C*-subalgebra B_ϕ of B generated by $\phi(A)$, can be *-isomorphically imbedded in $\mathcal{B}(H)$.*

6.2 Universal Representations of Locally JB-algebras and their Universal Locally C*-algebra Envelopes

Now we are able to formulate and prove the main result of the current chapter:

Theorem 29 *Let A be a locally JB-algebra of type Λ. Then there exists up to a locally *-isomorphism a unique locally C*-algebra $LC^*_u(A)$ of type Λ such that:*

(i) ψ_A - a locally Jordan homomorphism

$$\psi_A : A \longrightarrow LC^*_u(A)_{sa} \tag{6.1}$$

and $\psi_A(A)$ generates $LC^*_u(A)$ as a locally C-algebra.*

(ii) If B is a locally C-algebra of type Λ, and*

$$\phi : A \longrightarrow B_{sa} \tag{6.2}$$

is a locally Jordan homomorphism, then there exists a unique locally *-homomorphism

$$\widehat{\phi} : LC^*_u(A) \longrightarrow B, \tag{6.3}$$

such that

$$\phi = \widehat{\phi} \circ \psi_A. \tag{6.4}$$

(*iii*) *There exists a unique locally *-antiautomorphism* Φ *of* $LC_u^*(A)$ *of order two, such that*

$$\Phi(\psi_A(a)) = \psi_A(a), \tag{6.5}$$

for all $a \in A$.

Proof. Let $A \cong \varprojlim A_\alpha,\ \alpha \in \Lambda$, be the Arens-Michael decomposition of the locally JB-algebra A of type Λ into a projective limit of the projective family of JB-algebras A_α, $\alpha \in \Lambda$.

For a given $\alpha \in \Lambda$, let H_α be the Hilbert space from lemma 18. Let $\{{}_\alpha\psi_\xi\}_{\xi \in I_\alpha}$ be the set of all Jordan homomorphisms from A_α into $B(H_\alpha)$, organized into a family with the index set I_α, and consider the direct sum

$$\psi_{A_\alpha} = \bigoplus_{\xi \in I_\alpha} {}_\alpha\psi_\xi. \tag{6.6}$$

One can see that ψ_{A_α} is Jordan homomorphism from A_α into $\mathcal{B}(H_\alpha^u)$, where

$$H_\alpha^u = \bigoplus_{\xi \in I_\alpha} {}_\xi H_\alpha, \tag{6.7}$$

and

$${}_\xi H_\alpha = H_\alpha,\ \forall \xi \in I_\alpha. \tag{6.8}$$

From the fact that each factor representation of a special JB-algebra is into a JBW-factor which is not M_3^8 (see[1]), and thus, is a JW-factor, it follows that the dimension of the Hilbert space H_α^u is large enough so that the JC-subalgebra $\psi_{A_\alpha}(A_\alpha)$ of $\mathcal{B}(H_\alpha^u)_{sa}$ is isometrically isomorphic to A_α/K_α, where K_α is the unique exceptional closed Jordan ideal of A_α (see theorem 18 above).

Let

$$g_\alpha^\beta : A_\beta \longrightarrow A_\alpha,\ \alpha \preceq \beta,\ \alpha, \beta \in \Lambda.$$

be the canonical surjection from A_β onto A_α. Due to the fact

$${}_\alpha\psi_\xi \circ g_\alpha^\beta : A_\beta \longrightarrow \mathcal{B}(H_\alpha), \tag{6.9}$$

is a Jordan homomorphism from A_β into $\mathcal{B}(H_\alpha)$, it follows that

$$H_\alpha^u \subset H_\beta^u,\ \alpha \preceq \beta,\ \alpha, \beta \in \Lambda. \tag{6.10}$$

Thus, the family of Hilbert spaces $H_\alpha^u, \alpha \in \Lambda$ is inductive. Let the locally Hilbert space

$$H^u = \varinjlim H_\alpha^u$$

be its inductive limit. We also get that the family $\mathcal{B}(H_\alpha)$, $\alpha \in \Lambda$ is projective. Let

$$L(H^u) = \varprojlim \mathcal{B}(H_\alpha^u),$$

be the locally C*-algebra of type Λ of continuous linear operators on the locally Hilbert space H^u which is the projective limit of C*-algebras $\mathcal{B}(H_\alpha^u), \alpha \in \Lambda$ (proposition 3). Let

$$\psi_A : A \longrightarrow L(H^u)_{sa},$$

be the locally Jordan homomorphism such that

$${}_{L(H^u)}\pi_\alpha \circ \psi_A = \psi_{A_\alpha} \circ\ {}_A\pi_\alpha, \tag{6.11}$$

where

$${}_{L(H^u)}\pi_\alpha : L(H^u) \longrightarrow \mathcal{B}(H_\alpha^u),$$

is the canonical projection from $L(H^u)$ onto $\mathcal{B}(H^u_\alpha)$, and

$$_A\pi_\alpha : A \longrightarrow A_\alpha,\ \forall \alpha \in \Lambda.$$

is the canonical projection from A onto A_α. From theorem 18

$$K \cong \varprojlim K_\alpha,\ \ \alpha \in \Lambda.$$

Let now $LC^*_u(A)$ be a locally C*-subalgebra of type Λ of $L(H^u)$ generated by $\psi_A(A)$, and for each $\alpha \in \Lambda$, $C^*_u(A_\alpha)$ be the C*-subalgebra of $\mathcal{B}(H^u_\alpha)$ generated by $\psi_{A_\alpha}(A_\alpha)$.

Then condition (i) is satisfied by construction.

We now show that (ii) is satisfied.

Let $B \cong \varprojlim B_\alpha$ and ϕ be as it is mentioned in (ii). Then

$$_B\pi_\alpha \circ \phi = \phi_\alpha \circ \ _A\pi_\alpha, \tag{6.12}$$

where

$$\phi_\alpha : A_\alpha \longrightarrow (B_\alpha)_{sa}$$

is a Jordan homomorphism, and

$$_B\pi_\alpha : B \longrightarrow B_\alpha$$

is the canonical projection from B onto B_α, $\alpha \in \Lambda$. By replacing B by the locally C*-subalgebra of B generated in B by $\phi(A)$, we may assume without a loss of generality that B is generated by $\phi(A)$, and therefore B_α is generated by $\phi_\alpha(A_\alpha)$ for each $\alpha \in \Lambda$.

Thus, for each $\alpha \in \Lambda$, by lemma 18, without loss of generality we may assume that $(B_\alpha \hookrightarrow \mathcal{B}(H_\alpha))$

$$B_\alpha \subset \mathcal{B}(H_\alpha), \tag{6.13}$$

so that ϕ_α is a Jordan homomorphism from A_α into $\mathcal{B}(H_\alpha)_{sa}$. Thus

$$\phi_\alpha = \ _\alpha\psi_{\xi_\alpha}$$

for some $\xi_\alpha \in I_\alpha$. Let

$$\bigoplus_{\xi\in I_\alpha} {}_{\mathcal{B}(\ _\xi H_\alpha)}\pi_{\xi_\alpha} \tag{6.14}$$

be projection onto ξ_α-th coordinate in the direct sum

$$\bigoplus_{\xi\in I_\alpha} \mathcal{B}(\ _\xi H_\alpha).$$

Then, for each $a_\alpha \in A_\alpha$, the identity

$$\bigoplus_{\xi\in I_\alpha} {}_{\mathcal{B}(\ _\xi H_\alpha)}\pi_{\xi_\alpha}(\psi_{A_\alpha}(a_\alpha)) = \ _\alpha\psi_{\xi_\alpha}(a_\alpha) = \phi_\alpha(a_\alpha),$$

is valid, where $\alpha \in \Lambda$.

Now, let for each $\alpha \in \Lambda$,

$$\widehat{\phi}_\alpha : C^*_u(A_\alpha) \longrightarrow \mathcal{B}_\alpha,$$

be the *-homomorphism obtained by restricting

$$\bigoplus_{\xi\in I_\alpha} {}_{\mathcal{B}(\ _\xi H_\alpha)}\pi_{\xi_\alpha}$$

to the C*-subalgebra $C^*_u(A_\alpha)$ of $\mathcal{B}(H^u_\alpha)$. Then, for each $\alpha \in \Lambda$,

$$\phi_\alpha = \widehat{\phi}_\alpha \circ \psi_{A_\alpha}.$$

Let now

$$\widehat{\phi} : LC^*_u(A) \longrightarrow B, \tag{6.15}$$

be a locally *-homomophism such that

$$ {}_{B}\pi_{\alpha} \circ \widehat{\phi} = \widehat{\phi}_{\alpha} \circ \ {}_{LC_u^*(A)}\pi_{\alpha}, \tag{6.16} $$

where

$$ {}_{LC_u^*(A_\alpha)}\pi_{\alpha} : LC_u^*(A) \longrightarrow C_u^*(A_\alpha), $$

is the canonical projection from $LC_u^*(A)$ onto $C_u^*(A_\alpha)$, for each $\alpha \in \Lambda$. Thus

$$ \phi = \widehat{\phi} \circ \psi_A, $$

which proves (ii).

Now we show that (iii) is satisfied.

Let $B \cong \varprojlim B_\alpha, \alpha \in \Lambda$, be a decomposition of an arbitrary locally C*-algebra B of type Λ into a projective limit of the projective family of C*-algebras B_α. By the *opposite locally* C*-*algebra* B^{op} for the algebra B we understand the same set with the same involution, but the multiplication in it satisfies the following identity

$$ a^{op} b^{op} = (ba)^{op}, $$

where

$$ a \mapsto a^{op}, $$

is the identity map from B onto B^{op}, $a, b \in B$. One can easily see that

$$ B^{op} \cong \varprojlim B_\alpha^{op}, \alpha \in \Lambda, \tag{6.17} $$

and

$$ a \mapsto a^{op}, $$

$a \in B$, is a *-antiautomorphism from B onto B^{op}, such that for each $\alpha \in \Lambda$,

$$a_\alpha \mapsto a_\alpha^{op},$$

$a_\alpha \in B$, is a *-antiautomorphism from B_α onto B_α^{op}, where B_α^{op} is the opposite C*-algebra for the C*-algebra B_α.

Let now

$$\iota : LC_u^*(A) \longrightarrow LC_u^*(A)^{op}, \tag{6.18}$$

be the identity mapping. One can then see that for each $\alpha \in \Lambda$,

$$_{LC_u^*(A)^{op}}\pi_\alpha \circ \iota = \iota_\alpha \circ \ _{LC_u^*(A)}\pi_\alpha, \tag{6.19}$$

where

$$\iota_\alpha : B_\alpha \longrightarrow B_\alpha^{op},$$

is the identity mapping.

Let

$$\psi_A : A \longrightarrow LC_u^*(A),$$

be as above. By property (ii), the locally Jordan homomorphism

$$a \mapsto \iota(\psi_A(a))$$

can be lifted to a locally *-homomorphism

$$\widehat{\phi} : LC_u^*(A) \longrightarrow LC_u^*(A)^{op},$$

such that

$$\widehat{\phi} \circ \psi_A(a) = \iota \circ \psi_A(a), \ \forall a \in A,$$

and

$$\widehat{\phi}_\alpha : C_u^*(A_\alpha) \longrightarrow C_u^*(A_\alpha)^{op},\ \forall \alpha \in \Lambda,$$

is such that

$$\widehat{\phi}_\alpha \circ \psi_{A_\alpha}(a_\alpha) = \iota_\alpha \circ \psi_{A_\alpha}(a_\alpha),\ \forall a_\alpha \in A_\alpha.$$

Now we define

$$\Phi = \iota^{-1} \circ \widehat{\phi}. \tag{6.20}$$

One can see that

$$\Phi_\alpha = \iota_\alpha^{-1} \circ \widehat{\phi}_\alpha,\ \forall \alpha \in \Lambda, \tag{6.21}$$

is a *-antiautomorphism of $C_u^*(A_\alpha)$ into $C_u^*(A_\alpha)$ that fixes $\psi_{A_\alpha}(A_\alpha)$. Thus

$$\Phi_\alpha \circ \Phi_\alpha,$$

is a *-homomorphism of $C_u^*(A_\alpha)$ into itself that also fixes $\psi_{A_\alpha}(A_\alpha)$. Since $\psi_{A_\alpha}(A_\alpha)$ generates $C_u^*(A_\alpha)$ as a C*-algebra, $\Phi_\alpha \circ \Phi_\alpha$,is the identity mapping on $C_u^*(A_\alpha)$.

Therefore, Φ is a locally *-antiautomorphism of $LC_u^*(A)$ of order two that fixes $\psi_A(A)$. If now Ψ is another locally *-antiautomorphism of $LC_u^*(A)$ with the desired properties, then

$$\Phi_\alpha^{-1} \circ \Psi_\alpha,\ \forall \alpha \in \Lambda \tag{6.22}$$

should be the identity mapping on $C_u^*(A_\alpha)$, where

$$\Psi_\alpha : C_u^*(A_\alpha) \longrightarrow C_u^*(A_\alpha), \tag{6.23}$$

is the *-antiautomorphism of $C_u^*(A_\alpha)$, such that

$$\cdot_{LC_u^*(A)}\pi_\alpha \circ \Psi(a) = \Psi_\alpha \circ \cdot_{LC_u^*(A)}\pi_\alpha(a)$$

for all $a \in LC_u^*(A)$. Thus

$$\Phi^{-1} \circ \Psi,$$

should be the identity mapping on $LC_u^*(A)$, and

$$\Phi = \Psi,$$

which proves (iii).

We now show the uniqueness of the pair $(LC_u^*(A), \psi_A)$. On the contrary, let $(LC_u^*(A)^{'}, \psi_A^{'})$ be another pair with the same properties (i),(ii) and (iii). If we apply the property (ii), then there exists a locally *-homomorphism

$$\widehat{\phi} : LC_u^*(A) \longrightarrow LC_u^*(A)^{'}$$

from $LC_u^*(A)$ onto $LC_u^*(A)^{'}$ carrying ψ_A to $\psi_A^{'}$. On the other hand there exists a locally *-homomorphism

$$\widehat{\phi}^{'} : LC_u^*(A)^{'} \longrightarrow LC_u^*(A),$$

carrying $\psi_A^{'}$ to ψ_A. The composition mapping

$$\widehat{\phi} \circ \widehat{\phi}^{'}$$

agrees with the identity mapping on $\psi_A(A)$ in $LC_u^*(A)$, and thus, due to property (i) is equal to the identity mapping on $LC_u^*(A)$. On the other hand

$$\widehat{\phi}^{'} \circ \widehat{\phi}$$

agrees with the identity mapping on $\psi_A^{'}(A)$ in $LC_u^*(A)^{'}$, and thus, due to property (i), is equal to the identity mapping on $LC_u^*(A)^{'}$. Thus $\widehat{\phi}$ is a locally *-isomorphism, which proves the uniqueness. ■

Corollary 7 *Let A be a locally JB-algebra of type Λ, and*

$$A \cong \varprojlim A_\alpha$$

be its Arens-Michael decomposition into a projective limit of the projective family of JB-algebras A_α. Then the family of universal enveloping C^*-algebras $C_u^*(A_\alpha)$ forms a projective family, and

$$LC_u^*(A) \cong \varprojlim C_u^*(A_\alpha).$$

Proof. *Follows from the proof of the preceding theorem 29.* ■

CHAPTER 7.

DUAL SPACE CHARACTERIZATIONS OF REAL LOCALLY C*- AND LOCALLY JB-ALGEBRAS

7.1 Preliminary Theorems

Dual Space properties of Real C*-, Complex C*-, JB- and Complex Locally C*- algebras

Definition 57 *Let A be a real Banach *-algebra and $B = A \dot{+} iA$ be its complexification. We define the following two operations:*

∇-***operation****: $B \to B$:*

$$\nabla(\xi + i\eta) = \xi - i\eta; \tag{7.1}$$

and γ-***operation****: $B^* \to B^*$:*

$$\gamma(u(\xi_c)) = \overline{u(\nabla(\xi_c))}, \quad \xi_c = \xi + i\eta \in B, \quad \xi, \eta \in A, \quad u \in B^*. \tag{7.2}$$

According to Li[45] *the operation ∇ is conjugate linear, isometric, of 2^{nd} degree and*

$$A = \{\xi_c \in B : \nabla(\xi_c) = \xi_c\}.$$

Proposition 13 [45]*Let A be a real Banach algebra, A^* be its dual space, B be a complexification of A, and B^* be a dual space of B.*

(i) If $u \in B^*$ and $\gamma(u) = u$, then $u|_A \in A^*$ and $\hat{\rho}(u|_A) = \rho(u)$, where $\hat{\rho}$ and ρ are norms on B^* and A^* respectively.

(ii) For any $w \in A^*$ denote

$$w_c(\xi + i\eta) = w(\xi) + iw(\eta), \quad \forall \xi, \eta \in A. \tag{7.3}$$

Then

$$w_c \in B^*,\ \gamma(w_c) = w_c \ \ \textit{and}\ \ \rho(w) = \hat{\rho}(w_c). \tag{7.4}$$

In particular, if $u \in B^*$ *and* $\gamma(u) = u$ *then*

$$(u|_A)_c = u.$$

(iii) A^* *can be isometrically embedded into* B^*,

$$A^* = \{u \in B^* : \gamma(u) = u\} \tag{7.5}$$

and $B^* = A^* \dot{+} iA^*$ *is a complexification of* A^*.[45]

Definition 58 [45] *A spectrum of* $a \in A$*, a unital real Banach algebra is defined as*

$$\sigma(a) = \sigma_{\tilde{B}}(a) = \{\lambda \in \mathbb{C} : a - \lambda\!\!\!/ \textit{ is not invertible in } \tilde{B} = \tilde{A} \dot{+} i\tilde{A}\},$$

where $\tilde{A}$ *is a unitization of* A.

*A real Banach *-algebra* A *is called Hermitian if*

$$\sigma(h) \subset \mathbb{R},\ \forall h \in A_H = \{a \in A : a^* = a\},$$

A *is called skew-Hermitian if*

$$\sigma(k) \subset i\mathbb{R},\ \forall k \in A_K = \{a \in A : a^* = -a\}.$$

A real linear functional u *on a real *-algebra* A *is called* ***Hermitian*** *if*

$$u(a^*) = u(a) \ \ \textit{for any } a \in A, \tag{7.6}$$

or $u(a)|_{A_K} = 0$.

Definition 59 *A complex linear functional u on a complex *-algebra B is called* ***Hermitian*** *if*

$$u(b^*) = \overline{u(b)}, \text{ for any } b \in B. \tag{7.7}$$

Definition 60 *A real linear functional u on a real *-algebra A is* ***positive,*** *denoted by* $u(a) \geq 0$, *if*

$$u(a^*a) \geq 0 \quad \text{for any } a \in A. \tag{7.8}$$

In the complex case with identity, $u \geq 0$ *implies that u is Hermitian. This is no longer valid in the real case.*

Theorem 30 ***(Jordan–Grothendieck)***[27] *Let B be a complex C*-algebra and let* $\hat{u} \in B^*-$ *dual space of B, be a continuous complex linear Hermitian functional. Then u can be decomposed into a unique difference of two positive continuous linear functionals* $\hat{v}$ *and* $\hat{w}$ *with the property:* $\hat{\rho}(u) = \hat{\rho}(v) + \hat{\rho}(w)$, *where* $\hat{\rho}(.)$ *is the norm on* B^*.

Notation 1 *Let B be a complex locally C*-algebra and* $\{U_\alpha,\ p_\alpha < 1\}_{\alpha\in\Lambda}$ *be directed by set theoretic inclusion with the associated family* $\{p_\alpha\}_{\alpha\in\Lambda}$ *of C*-seminorms. All linear functionals on B bounded on* U_α *we denote as* $B^*(\alpha)$.

It is well known fact that linear functionals are continuous iff they are bounded on some U_α.[35]

Theorem 31 ***(Inoue)***[33] *Let B be a complex locally C*-algebra. Then every Hermitian continuous functional* $\hat{u}$ *from* $B^*(\alpha)$ *can be decomposed into a unique difference of two positive continuous linear functionals* $\hat{v}$ *and* $\hat{w}$ *satisfying the property:* $\hat{\rho}_\alpha(\hat{u}) = \hat{\rho}_\alpha(\hat{v}) + \hat{\rho}_\alpha(\hat{w})$, *where* $\hat{v},\hat{w} \in B^*(\alpha)$ *and* $\hat{\rho}_\alpha(.)$ *is the norm on* $B^*(\alpha)$, *for any* $\alpha \in \Lambda$.

Theorem 32 ***(Li)***[45] *Let A be a real C^* algebra and let $u \in A_H^*$ be a continuous real linear Hermitian functional on A. Then u can be uniquely represented as a difference of two real linear positive functionals v and w which satisfy the formula: $\rho(u) = \rho(v) + \rho(w)$, where $\rho(.)$ is the norm on A^*.*

Remark 9 *Please note that complex linear combination of positive linear functionals spans the entire dual space of complex C^* algebra, but the real linear combination of positive linear functionals spans just the Hermitian portion of the dual space of real C^* algebra.*

Definition 61 *A directed net $\{a_\alpha\}$, $\alpha \in \Lambda$ in some Banach space E is called* ***monotone increasing (nondecreasing)*** *or* ***monotone decreasing (nonincreasing)*** *if any $\gamma, \delta \in \Lambda$, $\gamma \preceq \delta$ stipulate that $a_\gamma < a_\delta$ $(a_\gamma \leq a_\delta)$ or $a_\gamma > a_\delta$ $(a_\gamma \geq a_\delta)$ respectively, where $a_\gamma, a_\delta \in E$.*

Definition 62 *Let u be a continuous linear functional on a (real or complex) von Neumann or JBW algebra A. Then u is called* ***normal*** *if for any monotone nondecreasing (nonincreasing) net $\{a_\alpha\}$, $\alpha \in \Lambda$ with supremum (infimum)*

$$\sup(\{a_\alpha\}) = a \quad (\inf(\{a_\alpha\}) = b), \quad a_\alpha, a, b \in A, \tag{7.9}$$

the following equation is true

$$\sup(u\{a_\alpha\}) = u(a) \quad (\inf(u\{a_\alpha\}) = u(b)). \tag{7.10}$$

Theorem 33 ***(Li)***[70, 45] *Let M be a real von Neumann algebra and let $u \in M_{*H}$ be a (real) linear normal Hermitian functional from the predual space. Then u can be* *decomposed into a unique difference of two real positive normal functionals v and w which satisfy the formula: $\rho(u) = \rho(v) + \rho(w)$, where $\rho(.)$ is the norm on M_*.*

Theorem 34 [30]*Let M be a JB-algebra. Then if M is a JBW-algebra (that is M is a Banach dual space), the predual is unique and consists of the normal linear functionals on M.*

Theorem 35 [3,30] *Let W be a JBW-algebra and let $u \in W_*$ be a normal bounded linear functional from predual space. Then u can be decomposed into a unique difference of two positive normal functionals v and w with the property: $\rho(u) = \rho(v) + \rho(w)$,where $\rho(.)$ is the norm on W_*.*

<u>Dual Characterization Theorems for C* and Locally C* algebras</u>

Definition 63 *Two norms $||.||_1$ and $||.||_2$ on Banach space B are **equivalent** if there exist two real numbers λ_1 and λ_2 such that*

$$\lambda_1||.||_1 \leq ||.||_2 \leq \lambda_2||.||_1.$$

Theorem 36 ***(Grothendieck–Murphy)***[28, 50] *Let B be a complex Banach *-algebra and any continuous complex linear functional on B be a complex linear combination of complex positive linear functionals. Then there exists a C*-regular norm on B, equivalent to the original norm.*

Theorem 37 ***(Bhatt–Karia)***[10] *Let (B, τ) be a complex lmc *-algebra, where τ is the topology generated by separating saturated family of Banach seminorms. Then there exists an equivalent family of C*-regular seminorms on B, making B a complex locally*

C^*-algebra iff B is Hermitian (definition 16) and every Hermitian continuous functional u is a difference of two positive continuous functionals.

7.2 Theorems of Decomposition of Functionals

Inductive Limit Functionals and Complexification of Dual Space

Definition 64 *Let $A \cong \varprojlim(g_\alpha^\beta A_\beta)$ be a projective limit of the projective family of (real or complex) Banach algebras $\{A_\alpha, g_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$. A family of functionals $\{u_\alpha : u_\alpha \in A_\alpha^*\}_{\alpha\in\Lambda}$, is called **inductive** if there exists $\alpha' \in \Lambda$ such that*

$$u_\beta(a_\beta) = u_{\alpha'}(g_{\alpha'}^\beta(a_\beta)), \quad \forall \beta \in \Lambda: \ \beta \succeq \alpha', \tag{7.11}$$

where $a_\alpha \in A_\alpha$, $a_\beta \in A_\beta$, and g_α^β are morphisms from A_β onto A_α.

Definition 65 *A functional u on a projective limit $A \cong \varprojlim(g_\alpha^\beta A_\beta)$ of the projective family of (real or complex) Banach algebras $\{A_\alpha, g_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$ is called an **inductive limit functional** of the inductive family of functionals if there exists $\alpha' \in \Lambda_1$, $\Lambda_1 \subset \Lambda$, such that*

$$u(a) = u_\alpha(\pi_\alpha(a)), \quad \forall \alpha \in \Lambda_1: \ \alpha \succeq \alpha', \tag{7.12}$$

for any $a \in A$, $a_{\alpha'} \in A_{\alpha'}$ and $\pi_{\alpha'}$ is a projection from A onto $A_{\alpha'}$, $\pi_{\alpha'}(a) = a_{\alpha'}$. We will write

$$u = \varinjlim(u_\beta g_\alpha^\beta)_{\alpha\preceq\beta\in\Lambda}. \tag{7.13}$$

Lemma 19 *Let $A = \varprojlim(g_\alpha^\beta A_\beta)$ be an lmc *-algebra (Jordan lmc algebra) with a given Arens-Michael decomposition into a projective limit of the projective family of Banach *-algebras (Banach Jordan algebras) $\{A_\alpha, g_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$. Then* *(i) for any functional $u_{\alpha'} \in A^*_{\alpha'}$ there exists an inductive family of functionals $\{u_\alpha\}_{u_\alpha \in A^*_\alpha,\ \alpha\in\Lambda}$, where $\alpha \succeq \alpha'$.*

(ii) for any functional $u \in A^*$ there exists an inductive family of functionals $\{u_\alpha\}_{u_\alpha \in A^*_\alpha,\ \alpha\in\Lambda}$, such that u is the inductive limit functional of this family.

(iii) two functionals $u, v \in A^*$ are equal iff there exists $\gamma \in \Lambda_1 \cap \Lambda_2$ (Λ_1 and Λ_2 are subsets of Λ) such that $u_\gamma = v_\gamma$, where $\{u_\alpha\}_{u_\alpha \in A^*_\alpha,\ \alpha\in\Lambda_1}$ and $\{v_\beta\}_{u_\beta \in A^*_\beta,\ \beta\in\Lambda_2}$ are inductive families of functionals u and v respectively.

Proof. (i) We define for any $\alpha \succeq \alpha'$ $u_\alpha(a_\alpha) = u_{\alpha'}(g^\alpha_{\alpha'}(a_\alpha))$, then the set of functionals $\{u_\alpha\}_{u_\alpha \in A^*_\alpha,\ \alpha\succeq\alpha'\in\Lambda}$ is the required inductive family (definition 64).

(ii) Let us choose $\alpha' \in \Lambda$ and define $u_{\alpha'}(\pi_{\alpha'}(a)) = u(a)$. Then, for any $\alpha \succeq \alpha'$ we will determine the inductive family as in (i). Indeed u is an inductive limit functional for $\{u_\alpha\}_{u_\alpha \in A^*_\alpha,\ \alpha\succeq\alpha'\in\Lambda}$.

(iii) If two inductive limit functionals are equal $u = v$, then for large enough α the corresponding inductive family is equivalent:

$$u_\alpha(\pi_\alpha(a)) = u(a) = v(a) = v_\alpha(\pi_\alpha(a)).$$

Conversely, let $\{u_\alpha\}_{u_\alpha \in A^*_\alpha,\ \alpha\succeq\alpha'\in\Lambda_1}$ and $\{v_\alpha\}_{v_\alpha \in A^*_\alpha,\ \alpha\succeq\alpha''\in\Lambda_2}$ be inductive families of functionals and $u_\gamma = v_\gamma$ for some $\gamma \in \Lambda_1 \cap \Lambda_2$. Then for any $\beta \succeq \gamma$

$$u_\beta(a_\beta) = u_\gamma(g^\beta_\gamma(a_\beta)) = v_\gamma(g^\beta_\gamma(a_\beta)) = v_\beta(a_\beta),$$

which means that the families are identical and corresponding projective limits are equal: $u = v$. ■

Notice that the inductive limit functional is completely determined by some member of the inductive family u_γ with fixed value of γ.

Lemma 20 *Let* $A \cong \varprojlim(g_\alpha^\beta A_\beta)$ *be an lmc *-algebra (Jordan lmc algebra) with a given Arens-Michael decomposition of a projective limit of the projective family of Banach *-algebras (Banach Jordan algebras)* $\{A_\alpha, g_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$*. Then an inductive limit functional* u

i) $u \in A^*$ *is positive iff* $\{u_\alpha\}_{u_\alpha\in A_\alpha^*,\ \alpha\succeq\alpha'\in\Lambda_1}$ *are positive.*

*ii) This property relates to lmc *-algebras only:* $u \in A^*$ *is Hermitian iff* $\{u_\alpha\}_{u_\alpha\in A_\alpha^*,\ \alpha\succeq\alpha'\in\Lambda_1}$ *are Hermitian.*

iii) Three functionals u, v *and* w *satisfy the following*

$$u = v - w, \quad u, v, w \in A^*,$$

iff

$$u_\alpha = v_\alpha - w_\alpha$$

for any $\alpha \succeq \gamma \in \Lambda_1 \cap \Lambda_2 \cap \Lambda_3, \quad u_\alpha, v_\alpha, w_\alpha \in A_\alpha^*$*, where* $\{u_\alpha\}_{\alpha\succeq\alpha'\in\Lambda_1}$*,* $\{v_\alpha\}_{\alpha\succeq\alpha''\in\Lambda_2}$ *and* $\{w_\alpha\}_{\alpha\succeq\alpha'''\in\Lambda_3}$ *are inductive families of functionals* u*,* v *and* w *respectively*

Proof. i) Let us assume that $u \geq 0$ and suppose by way of contradiction that $\exists\delta \in \Lambda_1 : \ \forall\alpha \succeq \delta,\ a_\alpha \in A_{\alpha+} : u_\alpha(a_\alpha) < 0$. Then

$$0 > u_\alpha(a_\alpha)_{\alpha\succeq\delta} = u_\alpha(\pi_\alpha(a))_{\alpha\succeq\delta} = u(a) \geq 0.$$

The last inequality is a contradiction.

Conversely, assume that $\exists\alpha' \in \Lambda_1 : \forall\alpha \in \Lambda$ with $\alpha \succeq \alpha',\ u_\alpha(a_\alpha) \geq 0$ and suppose by way of contradiction that $u(a) < 0$. Then

$$0 > u(a) = u_\alpha(\pi_\alpha(a))_{\alpha\succeq\alpha'} = u_\alpha(a_\alpha)_{\alpha\succeq\alpha'} \geq 0,$$

The last inequality demonstrates a contradiction.

ii) Let $u(a^*) = \overline{u(a)}$. Then

$$u(a^*) = u_\alpha(\pi_\alpha(a^*)) = u_\alpha(a^*_\alpha) \quad \forall \alpha \in \Lambda_1,\ \alpha \succeq \alpha', \tag{7.14}$$

and

$$\overline{u(a)} = \overline{u_\alpha(\pi_\alpha(a))} = \overline{u_\alpha(a_\alpha)} \quad \forall \alpha \in \Lambda_1,\ \alpha \succeq \alpha'. \tag{7.15}$$

So we conclude that $u_\alpha(a^*_\alpha) = \overline{u_\alpha(a_\alpha)},\ \forall \alpha \succeq \alpha'$.

Conversely, assume that $\forall \alpha \in \Lambda_1,\ \alpha \succeq \alpha',\ u_\alpha(a^*_\alpha) = \overline{u_\alpha(a_\alpha)}$ and suppose by way of contrary that $\exists a \in A:\ u(a^*) \neq \overline{u(a)}$. Then from equations 7.14 and 7.15 we conclude that $u_\alpha(a^*_\alpha) \neq \overline{u_\alpha(a_\alpha)}$.

iii) $u = v - w$, iff by the formula (7.12) $\forall \alpha \in \Lambda_1 \cap \Lambda_2 \cap \Lambda_3,\ \alpha \succeq \alpha',\ \alpha \succeq \alpha'',\ \alpha \succeq \alpha''',\ u_\alpha(\pi_\alpha(a)) = v_\alpha(\pi_\alpha(a)) - w_\alpha(\pi_\alpha(a))$ or $u_\alpha(a_\alpha) = v_\alpha(a_\alpha) - w_\alpha(a_\alpha)$. ■

Proposition 14 *Let $A \cong \varprojlim(g^\beta_\alpha A_\beta)$ be a real lmc algebra, an Arens-Michael decomposition into a projective limit of a projective family $\{A_\alpha, g^\beta_\alpha\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$ of real Banach algebras, B be a complexification of A, B^* be the dual space of B and A^* correspondingly be the dual space of A.*

(i) If $u \in B^*$ and $\gamma(u_\alpha) = u$, (γ is given by definition 57) then $u|_A \in A^*$ and $\hat{\rho}(u_\alpha|_{A_\alpha}) = \rho(u_\alpha)$ for any $\alpha \in \Lambda$.

(ii) For any inductive limit functional $w \in A^*$ define the inductive limit functional $u \in B^*$ such that

$$u_\alpha(\xi_\alpha + i\eta_\alpha) = w_\alpha(\xi_\alpha) + iw_\alpha(\eta_\alpha), \quad \forall \xi_\alpha, \eta_\alpha \in A_\alpha,\ \forall \alpha \succeq \alpha' \in \Lambda_1. \tag{7.16}$$

where $\{w_\alpha\}$ and $\{u_\alpha\}$ are inductive families of inductive limit functionals w and u correspondingly.$w_c \in B^*,\ \gamma(w_c) = w_c$ and $\rho(w) = \hat{\rho}(w_c)$.

Then

$$u \in B^*,\ \gamma(u_\alpha) = u_\alpha \quad \text{and} \quad \rho_\alpha(w_\alpha) = \hat{\rho}_\alpha(u_\alpha), \quad \forall \alpha \succeq \alpha' \in \Lambda_1. \tag{7.17}$$

In particular, if $u \in B^*$ *and* $\gamma(u) = u$ *then* $(u|_A)_c = w$.

(iii) A^* *can be locally isometrically embedded into* A_c^*,

$$A^* = \{u \in B^* : \gamma(u) = u,\ \forall \alpha \succeq \alpha' \in \Lambda_1\} \tag{7.18}$$

and $B^* = A^* \dot{+} iA^*$ *is a complexification of* A^*.

Proof. (i) By proposition 13 (A_α is real Banach algebra) if $u_\alpha \in B_\alpha^*$, $B_\alpha^* = A_\alpha^* \dot{+} iA_\alpha^*$and $\gamma(u_\alpha) = u_\alpha$, then $u_\alpha|_{A_\alpha} \in A_\alpha^*$ and $||u_\alpha|_{A_\alpha}|| = ||u_\alpha||$.

(ii) Using proposition 13 we conclude that $\gamma(u_\alpha) = u_\alpha$ and $\rho_\alpha(w_\alpha) = \hat{\rho}_\alpha(u_\alpha)$ for any $\alpha \succeq \alpha' \in \Lambda_1$.

(iii) According to proposition 13 for any $\alpha \in \Lambda_1$, $A_\alpha^* = \{u_\alpha \in B_\alpha^* : \gamma(u_\alpha) = u_\alpha,\ \forall \alpha \succeq \alpha' \in \Lambda_1\}$, which proves (iii). ■

Jordan-Grothendieck Type Theorems

Notation 2 *Let* A *be a (real or complex) Banach *-algebra with dual space* A^**. In the sequel* A_+^**, and* A_H^* *will respectively be the set of positive continuous linear functionals and the set of Hermitian continuous linear functionals.*

Theorem 38 *Let* $A \cong \varprojlim(g_\alpha^\beta A_\beta)$ *be a real locally C*-algebra, an Arens-Michael decomposition into a projective limit of projective family* $\{A_\alpha, g_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$ *of real C*-algebras. Then every Hermitian continuous functional* u *can be decomposed into a unique difference of two positive continuous functionals* $u = v - w$ *and*

$$\rho_\alpha(u_\alpha) = \rho_\alpha(v_\alpha) + \rho_\alpha(w_\alpha) \quad \text{for any } \alpha \succeq \gamma \in \Lambda_1 \cap \Lambda_2 \cap \Lambda_3. \tag{7.19}$$

where $\rho_\alpha(.)$ are norms on A^*_α, dual spaces of A_α.[1]

Proof. A functional $u \in A^*_H$ is the inductive limit of the inductive family $\{u_\alpha\}$ acting on A_α by lemma 19.

By lemma 20 (ii) A_α are Hermitian real C*-algebras and by theorem 32 $u_\alpha = v_\alpha - w_\alpha$, for α large enough. Then, by lemma 20 (iii) $u = v - w$ and by theorem 32 $\rho_\alpha(u_\alpha) = \rho_\alpha(v_\alpha) + \rho_\alpha(w_\alpha) \quad$ for any $\alpha \succeq \gamma \in \Lambda_1 \cap \Lambda_2 \cap \Lambda_3$. ∎

Theorem 39 *Let J be a JB-algebra and let $u \in J^*$ be a continuous linear functional of the dual space J^*. Then u can be decomposed into a unique difference of two positive continuous functionals $u = v - w$ and the norm of u is equal to the sum of norms of these positive functionals*

$$\rho(u) = \rho(v) + \rho(w). \tag{7.20}$$

Proof. Let M be Banach dual space to J^*. Then by the theorem 33 M_* can be identified with J^*, it consists of normal functionals and J^{**}, completion of J, is a JBW-algebra. Moreover, by theorem 33 $u = v - w$, where v and w are positive normal functionals and $\rho(u) = \rho(v) + \rho(w)$. ∎

Theorem 40 *Let $\mathbf{J}$ be a locally JB-algebra, an Arens-Michael projective limit decomposition of a projective family $\{J_\alpha, g^\beta_\alpha\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$ of JB-algebras and let $\mathbf{u} \in \mathbf{J}^*$ be a continuous real linear functional of the dual space $\mathbf{J}^*$. Then $\mathbf{u}$ can be decomposed into*

[1] *It is fair to mention that Konrad Schmüdgen in the article "The order structure of topological *-algebras of unbounded operators.I" (published in Rep. Mathematical Phys. 7 (1975), no.2, 215–227) proved the analogue of Grothendieck's characterization of C*-algebras: a metric barrelled topological *-algebra A with unit is an $A\widehat{O}$*-algebra if and only if every continuous Hermitian linear functional on A is the difference of two continuous positive linear functionals.*

a unique difference of two continuous positive functionals $u = v - w$ *and each norm* $\rho_\alpha(u_\alpha)$ *is equal to the sum of norms of functionals*

$$\rho_\alpha(u_\alpha) = \rho_\alpha(v_\alpha) + \rho_\alpha(w_\alpha), \quad \forall \alpha \succeq \gamma \in \Lambda_1 \cap \Lambda_2 \cap \Lambda_3,$$

where $u_\alpha, v_\alpha, w_\alpha \in J^*_\alpha$, $\mathbf{J} = \varprojlim(g^\beta_\alpha J_\beta)$ *for any* $\alpha \in \Lambda$.

Proof. Our proof is similar to the proof of theorem 38. Note that J_α are JB-algebras. By lemma 20 (iii) $u_\alpha = v_\alpha - w_\alpha$, $\alpha \succeq \gamma \in \Lambda_1 \cap \Lambda_2 \cap \Lambda_3$, and so $u = v - w$. By theorem 39 $\rho_\alpha(u_\alpha) = \rho_\alpha(v_\alpha) + \rho_\alpha(w_\alpha)$. ∎

7.3 Dual Space Characterization for Real C*- and Real Locally C*-algebras

Theorem 41 *Let* $B \cong \varprojlim(g^\beta_\alpha B_\beta)$ *be a complex lmc *-algebra, and any complex linear functional be complex linear combinations of positive linear functionals. Then there exists an equivalent system of saturated separating regular seminorms such that* B *becomes a complex locally C*-algebra.*

Proof. A functional $u \in B^*$ is the inductive limit of the inductive family $\{u_\alpha\}_{\alpha \succeq \alpha' \in \Lambda_1}$ acting on B_α and by lemma 20 u_α are linear combinations of positive linear functionals. Then by the Grothendieck-Murphy theorem 36 there exist regular norms $||.||^R_\alpha$ on B_α making these algebras as complex C* ones. Observe that any morphism $g^\beta_\alpha : B_\beta \to B_\alpha$ for any $\alpha, \beta \in \Lambda$ does not increase the norm[50]:

$$||g^\beta_\alpha(b_\beta)||^R_\alpha \leq ||b_\beta||^R_\beta, \quad \forall b_\beta \in B_\beta, b_\alpha \in B_\alpha$$

i.e. the order of the norms preserves. Finally we conclude that the projective limit of projective family of complex C*-algebras $\{B_\alpha, g^\beta_\alpha\}$ is complex locally C*-algebra B. ∎

Definition 66 *A real linear positive functional* u *on a real *-algebra* A *is called **strongly positive** if*

$$u_c(\xi + i\eta) = u(\xi) + iu(\eta), \quad \xi, \eta \in A$$

is a positive functional on B*.*

Definition 67 *A real linear functional* u *on a real *-algebra* A *is called **skew Hermitian** if* $u(a) = 0$ *for any* $a \in A_H$

Lemma 21 *Let* A *be a real Banach *-algebra and* B *its complexification. Then any continuous linear functional* $f \in B^*$

a) can be presented in the form

$$f(z) = g(z) + ih(z), \quad z = a + ib \in B,$$

such that $g(a)$ *and* $h(a)$*, the restrictions on the algebra* A*, are real functionals.*

b) is a complex linear combination of real linear functionals.

Proof. a) We define $g(z)$ and $h(z)$ in the following manner:

$$g(z) = \frac{f(z) + \gamma(f(z))}{2} \text{ and } h(z) = \frac{f(z) - \gamma(f(z))}{2i} \tag{7.21}$$

where γ and ∇ are conjugate linear transformations (definition 57):

$$\gamma(f(z)) = \overline{f(\nabla(z))}, \tag{7.22}$$

$$\nabla(a + ib) = a - ib, \tag{7.23}$$

for any $z = a + ib \in B,\ a, b \in A$.

Then

$$g(a) = \frac{f(a) + \gamma(f(a))}{2} = \frac{f(a) + \overline{f(a)}}{2} = \frac{\text{Re}(f(a)) + i\,\text{Im}(f(a)) + \overline{\text{Re}(f(a)) + i\,\text{Im}(f(a))}}{2} = \tag{7.24}$$

$$\frac{\text{Re}(f(a)) + i\,\text{Im}(f(a)) + \text{Re}(f(a)) - i\,\text{Im}(f(a))}{2} = \text{Re}(f(a))$$

Correspondingly,

$$h(a) = \text{Im}(f(a)).$$

It proves that $g(a)$ and $h(a)$ are real functionals on A.

b) The functional g is linear:

$$g(z) = g(a) + ig(b),$$

the same is true for h:

$$h(z) = h(a) + ih(b)$$

Then,

$$f(z) = [g(a) - h(b)] + i[g(b) + h(a)]$$

which is a complex linear combination of the real functionals $g(a)$ and $h(a)$. ■

Remark 10 *This statement partly appears without proof in Li's book*[45] *for real Banach algebras.*

Theorem 42 *Let A be a real Banach *-algebra such that any real Hermitian and real skew Hermitian linear functionals are real linear combinations of strongly positive linear functionals. Then there exists a regular norm on A equivalent to the original norm.*

Proof. Notice that any element $a \in A$ can be presented as the sum of Hermitian and skew Hermitian elements

$$a = a_H + a_K, \text{ where } a_H = \frac{a + a^*}{2}, \ a_K = \frac{a - a^*}{2}$$

Let w be a real functional on A. Then w can be represented as the sum of Hermitian and skew Hermitian functionals

$$w(a) = w_H(a) + w_K(a) \text{ where } w_H(a) = w(a_H) \text{ and } w_K(a) = w(a_K)$$

From the premise of our theorem we know that real skew Hermitian linear functionals u and v can be represented as

$$u = \Sigma\alpha_i p_i \text{ and } v = \Sigma\beta_j q_j$$

where $\alpha_i, \beta_j \in \mathbb{R}$, p_i and q_j strongly positive linear functionals. Correspondingly a complexification w_c of $w(a)$ will be the complex linear combination of positive functionals. Note that any complex functional on B is a complexification of some real functional (proposition 14 iii). By the Grothendieck-Murphy theorem 36 the algebra B acquires a regular norm equivalent to the original norm. The restriction of this norm is a regular norm on A, equivalent to the original one. ■

Theorem 43 *Let A be a real lmc *-algebra, $A \cong \varprojlim(g_\alpha^\beta A_\beta)$ an Arens-Michael decomposition of a projective limit of projective family $\{A_\alpha, g_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$ of real Banach *-algebras, such that any real Hermitian and real skew Hermitian linear functionals are real linear combinations of strongly positive linear functionals. Then there exists an equivalent system of saturated separating regular seminorms such that A becomes a real locally C*-algebra.*

Proof. A Hermitian (skew Hermitian) functional $u \in A_H^*$ is the inductive limit of the inductive family $\{u_\alpha\}$ acting on A_α and by Lemma (20) u_α.are also Hermitian (skew Hermitian) and real linear combinations of strongly positive linear functionals. Then by the previous theorem 42 there exist regular norms $||.||_\alpha^R$ on A_α making these algebras into real C* ones. Note that any morphism $g_\alpha^\beta : A_\alpha \to A_\alpha$ for any $\alpha, \beta \in \Lambda$ does not increase the norm[50]:

$$||g_\alpha^\beta(x_\beta)||_\alpha^R \leq ||x_\beta||_\beta^R, \quad \forall x_\beta \in A_\beta, x_\alpha \in A_\alpha$$

Finally we conclude that projective limit projective family of real C*-algebras $\{A_\alpha, g_\alpha^\beta\}$ is a real locally C*-algebra A. ■

7.4 Dual Space Characterization for JB- and Locally JB-algebras

Definition 68 *A real Banach Jordan algebra J is* ***of complex type*** *if it is isometrically isomorphic to the self adjoint part of some complex Banach *-algebra*

Definition 69 *A real Banach Jordan algebra (Jordan lmc algebra) J is called* ***envelopable*** *by a complex Banach *-algebra (complex lmc *-algebra) B (the envelope) if :*

*(1) The algebra B exists, unique up to isometric isomorphism and is generated as a Banach *-algebra (complex lmc *-algebra) by $\psi(A)$, where ψ is a Jordan contractive homomorphism from J to the self-adjoint part of B.*

*(2) if C is another complex Banach *-algebra (complex lmc *-algebra) and ϕ is a Jordan homomorphism from J to C_{SA} a self-adjoint part of C, then there exists a*

**-homomorphism $\widehat{\phi}$ from B to C such that*

$$\phi = \widehat{\phi} \circ \psi.$$

*(3) there is a *-antiautomorphism φ of $\hat{B}$ of the order 2, such that*

$$\varphi(\psi(x)) = \psi(x) \tag{7.25}$$

for any $x \in J$.

Theorem 44 *Let J be an envelopable real Banach Jordan algebra of complex type, such that any continuous linear functional u from the dual space J^* is a linear combination of strongly positive linear functionals. Then J is a JB-algebra.*

Proof. Any element $a \in B$ is a linear combination of the elements of the algebra J (because J generates B) and correspondingly, any continuous linear functional $v \in B^*$ on B is the linear combination of functionals on J.

As a result of the last statement v is the linear combination of strongly positive linear functionals, and by the theorem 42 the envelope B is the C*-algebra with the regular norm equivalent to the original Banach norm.

Then, the real Banach Jordan algebra J acquires a regular norm and therefore becomes a JB-algebra. ■

Theorem 45 *Let J be an envelopable Jordan lmc algebra, an Arens-Michael decomposition into a projective limit of projective family $\{J_\alpha, g_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$ of real Banach Jordan algebras, such that any continuous linear functional u is a linear combination of positive linear functionals. Then J is a locally JB-algebra and the envelope $\hat{B}$ is a locally C*-algebra (up to locally isomorphism).*

Proof. Let J_α be Jordan Banach algebras forming with morphisms g_α^β and a projective family $\{J_\alpha, g_\alpha^\beta\}_{\alpha,\beta\in\Lambda;\alpha\preceq\beta}$. By the previous theorem (44) all universal envelopes $\hat{B}_\alpha$ of the algebras J_α are C*-algebras and J_α turn out to be JB-algebras. We know that any morphism between two JB-algebras $g_\alpha^\beta : J_\alpha \to J_\alpha$, $\alpha, \beta \in \Lambda$ does not increase the norm[30]:

$$||g_\alpha^\beta(a_\beta)||_\alpha^R \leq ||a_\beta||_\beta^R, \quad \forall a_\alpha \in J_\alpha, a_\beta \in J_\beta.$$

Now we see that the projective limit of projective family of JB-algebras $\{J_\alpha, g_\alpha^\beta\}$ is a locally JB-algebra J and the envelopes $\hat{B}$ is the projective limit of projective family of complex C*-algebras $\{B_\alpha, \hat{g}_\alpha^\beta\}$, where B_α are envelopes of J_α and

$$\hat{g}_\alpha^\beta(b_\beta) = \hat{g}_\alpha^\beta(\Sigma c_i a_{\beta,i}) = \Sigma c_i g_\alpha^\beta(a_{\beta,i}), \quad b_\beta \in B_\beta,\ a_{\beta,i} \in J_\beta\ c_i \in \mathbb{C}. \tag{7.26}$$

■

CHAPTER 8.

REPRESENTATIONS OF BARRELED REAL LOCALLY C*- AND BARRELED LOCALLY JB-ALGEBRAS ON REAL AND COMPLEX HILBERT SPACES

Representations of Barreled Locally C*-algebras   Lassner introduced and started the study in[42] of so called **Lassner algebras**, or topological algebras which are topologically *-isomorphic to a topological Op*-algebras of unbounded operators defined on a common domain- a dense subspace $\mathcal{D}^{\mathbb{C}}$ of a complex Hilbert space $H^{\mathbb{C}}$. Details on the development of the theory of Op*-algebras one can find in the monograph Schmüdgen[63]**.** Op*-algebras are in a way an unbounded analogue of operator complex C*-algebras. In a view of the celebrated GNS construction (see for example[51]) which establishes that each abstract complex C*-algebra is isometrically *-isomorphic to a C*-algebra on bounded operators acting on a certain complex Hilbert space $H^{\mathbb{C}}$, it was thus interesting to learn whether or not each complex locally C*-algebra can be represented as an operator Op*-algebra. This was first done by Sya in[69] and then by Brooks in[12] who established that a metrizable complex locally C*-algebra is topologically *-isomorphic to an Op*-algebra on a certain complex Hilbert space $H^{\mathbb{C}}$. Lassner refined this result in[43] by showing that each barrelled complex locally C*-algebra is topologically *-isomorphic to a certain Op*-algebra on a certain complex Hilbert space $H^{\mathbb{C}}$.

We introduce real and Jordan analogues of complex Op*-algebras, study real and complex Hilbert space representations of barrelled real locally C*-algebras and locally JB-algebras. Generalizing the result of Lassner[43] it is shown that in the case of real barrelled locally C*-algebras a realization by *-algebras of operators defined on a dense subspace $\mathcal{D}^{\mathbb{R}}$ of a real Hilbert space $H^{\mathbb{R}}$ always exists. In the case of the barrelled locally JB-algebras it is shown that modulo a Jordan ideal a factor algebra can also be realized as a Jordan subalgebra of a symmetric part of a real Lassner operator algebra.

8.1 Real and Jordan Lassner Algebras

Let $\mathcal{D}^{\mathbb{R}}$ be a real unitary (pre-Hilbert) space with the scalar product

$$< \xi, \eta >: \mathcal{D}^{\mathbb{R}} \times \mathcal{D}^{\mathbb{R}} \to \mathbb{R}, \tag{8.1}$$

$$< t\xi, \eta >=< \xi, t\eta >= t < \xi, \eta >, \quad \xi, \eta \in \mathcal{D}^{\mathbb{R}}, \ t \in \mathbb{R}. \tag{8.2}$$

Let

$$\|\xi\| =< \xi, \xi >^{\frac{1}{2}}, \tag{8.3}$$

be the norm in $\mathcal{D}^{\mathbb{R}}$, and let $H^{\mathbb{R}}$ denote the real Hilbert space which is the completion of $\mathcal{D}^{\mathbb{R}}$ in this norm.

Let $\mathcal{L}_{+}^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ be the algebra of all unbounded closable operators a on $H^{\mathbb{R}}$ with domain

$$Dom(a) = \mathcal{D}^{\mathbb{R}}, \tag{8.4}$$

which satisfy the following conditions:

1. $\mathcal{D}^{\mathbb{R}}$ is invariant under the action of a:

$$a(\mathcal{D}^{\mathbb{R}}) \subset \mathcal{D}^{\mathbb{R}}; \tag{8.5}$$

2. the adjoint operator a^* exists;

3. the domain $Dom(a^*)$ of the adjoint to a operator a^* contains $\mathcal{D}^{\mathbb{R}}$:

$$Dom(a^*) \supset \mathcal{D}^{\mathbb{R}},$$

and $\mathcal{D}^{\mathbb{R}}$ is invariant under the action of a^* :

$$a^*(\mathcal{D}^{\mathbb{R}}) \subset \mathcal{D}^{\mathbb{R}}.$$

Let a^+ be the restriction of a^* to $\mathcal{D}^{\mathbb{R}}$:

$$a^+ = a^* \restriction \mathcal{D}^{\mathbb{R}}. \tag{8.6}$$

Generally speaking the product ab of two closable operators a and b on $\mathcal{D}^{\mathbb{R}}$ does not have to be closable, however the following lemma is valid:

Lemma 22 *$\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ is a real algebra, and when equipped with the involution*

$$a \to a^+, \tag{8.7}$$

*$\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ becomes a real *-algebra.*

Proof. Same as the proof of the analogous statement for complex algebras[63]. ■

As in the case of complex algebras[42] , the following two lemmata are valid.

Lemma 23 *If*

$$\mathcal{D}^{\mathbb{R}} = H^{\mathbb{R}}, \tag{8.8}$$

then

$$\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}) = B(H^{\mathbb{R}}), \tag{8.9}$$

where $B(H^{\mathbb{R}})$ is the algebra of all bounded linear operators on $H^{\mathbb{R}}$.

Proof. Follows from the Closed Graph Theorem[61]. ■

The following lemma gives us one more scenario when the algebra $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ is composed of all bounded operators on $H^{\mathbb{R}}$.

Lemma 24 *If the only one operator*

$$a \in \mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}), \tag{8.10}$$

is closed, then

$$\mathcal{D}^{\mathbb{R}} = H^{\mathbb{R}}$$

and, due to the previous lemma 23

$$\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}) = B(H^{\mathbb{R}}).$$

Proof. Same as the proof of the analogous statement for complex algebras[63]. ■

As in the complex case ([42]), the following proposition is valid:

Proposition 15 *If there exists a norm $\|.\|_1$ on $\mathcal{D}^{\mathbb{R}}$ stronger than the real Hilbert space norm $\|.\|$ defined above, with respect to which a symmetric operator*

$$a = a^+ \in \mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}), \tag{8.11}$$

is continuous,

$$\|a(\xi)\|_1 \leq C \|\xi\|_1 , \tag{8.12}$$

($\xi \in \mathcal{D}^{\mathbb{R}}$, and a positive constant $C \in \mathbb{R}$), then a is bounded

$$a \in B(H^{\mathbb{R}}). \tag{8.13}$$

Proof. Same as the proof of the analogous statement for complex algebras[63]. ■

Let us now consider a complexification of $\mathcal{D}^{\mathbb{R}}$:

$$\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{\mathbb{R}} \dotplus i\mathcal{D}^{\mathbb{R}}. \tag{8.14}$$

Using the Polarization Identity we define a scalar product in $\mathcal{D}^{\mathbb{C}}$ by

$$< \xi + i\eta,\ \xi' + i\eta' > \ = \ < \xi,\ \xi' > + < \eta,\ \eta' > +i < \eta,\ \xi' > -i < \xi,\ \eta' >, \tag{8.15}$$

where $\xi, \xi', \eta, \eta' \in \mathcal{D}^{\mathbb{R}}$. Let the norm in $\mathcal{D}^{\mathbb{C}}$ be defined as

$$\|\xi + i\eta\| = \ < \xi + i\eta,\ \xi + i\eta >^{\frac{1}{2}} . \tag{8.16}$$

Clearly,

$$\|\xi + i\eta\|^2 = \|\xi - i\eta\|^2 = \|\xi\|^2 + \|\eta\|^2 , \tag{8.17}$$

for any $\xi, \eta \in \mathcal{D}^{\mathbb{R}}$.

Let $H^{\mathbb{C}}$ be completion of $\mathcal{D}^{\mathbb{C}}$ in this norm.

Lemma 25 *The completion $H^{\mathbb{C}}$ of the complexification $\mathcal{D}^{\mathbb{C}}$ of $\mathcal{D}^{\mathbb{R}}$ is equal to the complexification of the completion $H^{\mathbb{R}}$ of $\mathcal{D}^{\mathbb{R}}$:*

$$H^{\mathbb{C}} = H^{\mathbb{R}} \dotplus iH^{\mathbb{R}}. \tag{8.18}$$

Proof. Note that any Cauchy sequence $\xi_n + i\eta_n \in \mathcal{D}^{\mathbb{C}}$ converges iff Cauchy sequences $\xi_n, \eta_n \in \mathcal{D}^{\mathbb{R}}$ converge. Then the limits of Cauchy sequences are such that $\xi_0 + i\eta_0 \in H^{\mathbb{C}}$, $\xi_0, \eta_0 \in H^{\mathbb{R}}$. ■

Let now $(S^{\mathbb{C}}, \langle , \rangle)$ be an arbitrary complex unitary space, and the complex Hilbert space $(K^{\mathbb{C}}, \langle , \rangle)$ be its completion in the norm

$$\|.\| = \langle ., . \rangle^{\frac{1}{2}} .$$

Lemma 26 *There exists a real unitary space*

$$(S^{\mathbb{R}}, \langle , \rangle_r), \tag{8.19}$$

and its completion

$$(K^{\mathbb{R}}, \langle , \rangle_r), \tag{8.20}$$

in the norm

$$\|.\|_r = \langle ., . \rangle_r^{\frac{1}{2}} , \tag{8.21}$$

is a real Hilbert space, such that

$$S^{\mathbb{C}} = S^{\mathbb{R}} \dot{+} iS^{\mathbb{R}}, \tag{8.22}$$

$$K^{\mathbb{C}} = K^{\mathbb{R}} \dot{+} iK^{\mathbb{R}}, \tag{8.23}$$

and

$$\|\xi + i\eta\|^2 = \|\xi - i\eta\|^2 = \|\xi\|_r^2 + \|\eta\|_r^2 ,$$

where $\xi, \eta \in K^{\mathbb{R}}$.

Proof. Follows from the previous lemma and a discussion after theorem 4. ■

Let us turn to the complexification of $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$. Let, as before, $\mathcal{D}^{\mathbb{R}}$ be a real unitary space, and

$$\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{\mathbb{R}} \dot{+} i\mathcal{D}^{\mathbb{R}},$$

be a complex unitary space which is a complexification of $\mathcal{D}^{\mathbb{R}}$.

For every $a \in \mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ we define

$$a_c : \mathcal{D}^{\mathbb{C}} \to \mathcal{D}^{\mathbb{C}}, \tag{8.24}$$

$$a_c(\xi + i\eta) = a(\xi) + ia(\eta), \tag{8.25}$$

and

$$a_c^+ : \mathcal{D}^{\mathbb{C}} \to \mathcal{D}^{\mathbb{C}}, \tag{8.26}$$

$$a_c^+(\xi + i\eta) = a^+(\xi) + ia^+(\eta), \tag{8.27}$$

for every $\xi, \eta \in \mathcal{D}^{\mathbb{R}}$.

Let us denote the set of all a_c by $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$. A complex unital *-subalgebra of $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ is called a **complex** Op*-**algebra**[42]. The following proposition is valid:

Proposition 16 *(i) The complex *-algebra*

$$\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}) = \mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}) \dot{+} i\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}), \tag{8.28}$$

is a maximal complex Op-algebra of unbounded operators on a complex unitary space* $\mathcal{D}^{\mathbb{C}}$*, and if we identify* a *with* a_c *for every* $a \in \mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$*, then* $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ *can be embedded into* $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$*;*

(ii) Let "−" be a mapping on $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ *:*

$$- : \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}) \to \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}), \tag{8.29}$$

$$\overline{(a + ib)} = a - ib, \tag{8.30}$$

$\forall a, b \in \mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$*. Then*

$$\overline{a_c}(\xi_c) = \overline{a_c(\overline{\xi_c})}, \tag{8.31}$$

$\xi_c \in \mathcal{D}^{\mathbb{C}}$*, and "−" is conjugate-linear;*

(iii)

$$\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}) = \left\{ a_c \in \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}) : \overline{a_c} = a_c \right\}, \tag{8.32}$$

and if $a_c \in \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}),$ *then*

$$a_c = \overline{a_c}, \tag{8.33}$$

is equivalent to

$$\overline{a_c(\xi_c)} = a_c(\overline{\xi_c}), \tag{8.34}$$

for every $\xi_c \in \mathcal{D}^{\mathbb{C}},$ *and is equivalent to*

$$a_c(\mathcal{D}^{\mathbb{R}}) \subset \mathcal{D}^{\mathbb{R}}; \tag{8.35}$$

(iv)

$$\overline{a_c b_c} = \overline{a_c}\,\overline{b_c}, \tag{8.36}$$

for every $a_c, b_c \in \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}});$

(v)

$$< \overline{\xi_c},\ \overline{\eta_c} >= \overline{< \xi_c,\ \eta_c >} = \quad < \eta_c,\ \xi_c >,$$

for every $\xi_c, \eta_c \in \mathcal{D}^{\mathbb{C}};$

(vi)

$$\overline{a_c}^{+} = \overline{a_c^{+}}, \tag{8.37}$$

$$a_c^{+} = (a^{+})_c, \tag{8.38}$$

$$(ab)_c = a_c b_c, \tag{8.39}$$

where $a, a^+, b, b^+ \in \mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}),$ *and* $a_c, a_c^+, b_c, b_c^+ \in \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}});$

(vii) Let φ *be a mapping on* $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}):$

$$\varphi : \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}) \to \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}), \tag{8.40}$$

$$\varphi(a + ib) = a^+ + ib^+, \tag{8.41}$$

$\forall a, b \in \mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$. *Then* φ *is an order 2 *-antiautomorphism of* $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ *and*

$$\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}) = \left\{a_c \in \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}) : \varphi(a_c) = a_c^+\right\}. \tag{8.42}$$

Proof. The proofs can be obtained with some modifications of the analogous results from[45] for the algebras of bounded linear operators on real and complex Hilbert spaces, using the ideas from.[63] ■

Real Op*-algebras Now we can define a real analogue of complex Op*-algebras.

Definition 70 *A real unital *-subalgebra of* $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ *is called a* ***real*** *Op*-**algebra**.*

Since we have got an embedding of $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ into its complexification $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$, it is now possible to describe real Op*-algebras within $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ as well.

Proposition 17 *(i) A real *-subalgebra* $\mathcal{A}$ *of* $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ *is a real Op*-algebra iff*

$$\mathcal{A} \cap i\mathcal{A} = \{\mathbf{0}\}; \tag{8.43}$$

(ii) Let $\mathcal{A}$ *be a real Op*-algebra. Then*

$$\mathfrak{B} = \mathcal{A} \dotplus i\mathcal{A}, \tag{8.44}$$

is a complex Op-algebra;*

(iii) Every real Op-algebra* $\mathcal{A}$ *is a fixed point algebra of* $(\mathfrak{B}, -)$, *i.e.*

$$\mathcal{A} = (b_c \in \mathfrak{B} : \overline{b_c} = b_c), \quad \mathfrak{B} = \mathcal{A} \dotplus i\mathcal{A}, \tag{8.45}$$

*where "$-$" is an order 2 conjugate linear *-algebraic isomorphism of* $\mathfrak{B}$.

(iv) For every real Op-algebra* $\mathcal{A}$ *there is an order 2 *-automorphism* φ *on* $\mathfrak{B}$, *such that*

$$\mathcal{A} = (b_c \in \mathfrak{B} : \varphi(b_c) = b_c^+), \quad \mathfrak{B} = \mathcal{A} \dotplus i\mathcal{A},. \tag{8.46}$$

Proof. The proofs are obtained with the use of Proposition 16 and modifications of the analogous results from[45] for the algebras of bounded linear operators on real and complex Hilbert spaces, using the ideas from.[63] ■

Jordan Op-algebras Now we are ready to define a Jordan analogue of a complex Op*-algebra. Let the symmetric part of $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ be

$$\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s = \{a_c \in \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}) : a_c = a_c^+\}.$$

Generally speaking $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s$ is not algebraically closed under operator multiplication from $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$, however, if we define

$$a_c \bullet b_c = \frac{1}{2}(a_c b_c + b_c a_c), \tag{8.47}$$

$a_c, b_c \in \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s$, then $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s$ is closed under "•", and

$$(\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s, \bullet), \tag{8.48}$$

is a Jordan algebra.

Definition 71 *A real unital Jordan subalgebra of $(\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s, \bullet)$ is called a **Jordan** Op-**algebra**.*

Examples of Jordan Op-algebras include symmetric parts of complex and real Op*-algebras, however, not all Jordan Op-algebras need be symmetric parts of complex or real Op*-algebras even in the case of bounded linear operators (see[30] for details).

Now we discuss an issue of a topologization of $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ and $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$.

Let $\mathcal{A}$ be a real Op*-subalgebra of $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$. The algebra $\mathcal{A}$ defines a topological structure on $\mathcal{D}^{\mathbb{R}}$.

Lemma 27 *The family of seminorms*

$$\{p_a = \|a(.)\| : a \in \mathcal{A}\}\,, \tag{8.49}$$

separates points, and the corresponding topology $\tau_{\mathcal{A}}$ is the weakest locally convex topology in which the operator $a \in \mathcal{A}$ is continuous as a map

$$(\mathcal{D}^{\mathbb{R}}, \tau_{\mathcal{A}}) \to H^{\mathbb{R}}.$$

Since $\mathbf{1} \in \mathcal{A}$ by assumption, $\tau_{\mathcal{A}}$ is stronger than the topology induced by the real Hilbert space $H^{\mathbb{R}}$ norm on $\mathcal{D}^{\mathbb{R}}$.

Definition 72 *Let $\mathcal{M}$ be the collection of all bounded sets in $(\mathcal{D}^{\mathbb{R}}, \tau_{\mathcal{A}})$. We associate with each $M \in \mathcal{M}$ the seminorm s_M :*

$$s_M(a) = \sup_{\xi,\eta \in M} |< \xi, a(\eta) >|\,. \tag{8.50}$$

The system of seminorms

$$\{s_M : M \in \mathcal{M}\}\,,$$

separates points, therefore it defines on $\mathcal{A}$ a locally convex topology $\tau_{\mathcal{D}^{\mathbb{R}}}$, which is called the ***uniform topology****.*

Proposition 18 *A real Op*-subalgebra $\mathcal{A}$ of $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ with the uniform topology $\mathcal{A}[\tau_{\mathcal{D}^{\mathbb{R}}}]$, is a real topological *-algebra.*

Now we define real operator Lassner algebras.

Definition 73 *A real Op*-subalgebra $\mathcal{A}$ of $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ with the uniform topology $\mathcal{A}[\tau_{\mathcal{D}^{\mathbb{R}}}]$, is called a* ***real operator Lassner algebra****.*

Let now $\mathfrak{B}$ be a complex Op*-subalgebra of $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$. Similarly to real real case just discussed, the algebra $\mathfrak{B}$ defines a topological structure on $\mathcal{D}^{\mathbb{C}}$.

Lemma 28 *The family of seminorms*

$$\{p_{a_c} = \|a_c(.)\| : a_c \in \mathfrak{B}\}\,, \tag{8.51}$$

separates points, and the corresponding topology $\tau_{\mathfrak{B}}$ is the weakest locally convex topology in which the operator $a_c \in \mathfrak{B}$ is continuous as a map

$$(\mathcal{D}^{\mathbb{C}}, \tau_{\mathfrak{B}}) \to H^{\mathbb{C}}.$$

Proof. Follows the same strategy as the proof of the analogous statement for complex algebras from[42]. ■

Since $\mathfrak{B}$ is a complex Op*-algebra, $\mathbf{1} \in \mathfrak{B}$ by assumption, $\tau_{\mathfrak{B}}$ is stronger then the topology induced by the real Hilbert space $H^{\mathbb{C}}$ norm on $\mathcal{D}^{\mathbb{C}}$.

Similarly, if $\mathcal{N}_c$ be a set of all bounded sets in $(\mathcal{D}^{\mathbb{C}}, \tau_{\mathfrak{B}})$. We associate with each $N_c \in \mathcal{N}_c$ the seminorm s_{N_c} :

$$s_{N_c}(a_c) = \sup_{\xi_c, \eta_c \in N_c} |< \xi_c, a_c(\eta_c) >|\,. \tag{8.52}$$

The system of seminorms

$$\{s_{N_c} : N_c \in \mathcal{N}_c\}\,,$$

separates points, therefore it defines on $\mathfrak{B}$ a locally convex topology $\tau_{\mathcal{D}^{\mathbb{C}}}$, which is also called the **uniform topology**.

Proposition 19 [42] *A complex Op*-subalgebra $\mathfrak{B}$ of $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ with the uniform topology $\mathfrak{B}[\tau_{\mathcal{D}^{\mathbb{C}}}]$ is a complex topological *-algebra.*

Definition 74 *A complex Op*-subalgebra* $\mathfrak{B}$ *of* $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ *with the uniform topology* $\mathfrak{B}[\tau_{\mathcal{D}^{\mathbb{C}}}]$ *is called a* ***complex operator Lassner algebra****.*

One can see that real and complex Lassner algebras can be thought of as unbounded generalizations of operator real and complex C*-algebras.

Similarly to the case of a relation between real and complex Op*-algebras, we have the following theorem:

Theorem 46 *For every real Lassner operator algebra* $\mathcal{A}[\tau_{\mathcal{D}^{\mathbb{R}}}]$ *there exists a complex Lassner operator algebra* $\mathfrak{B}[\tau_{\mathcal{D}^{\mathbb{C}}}]$*, and its *-antiautomorphism* φ *with a period* 2*, such that*

$$\mathcal{A}[\tau_{\mathcal{D}^{\mathbb{R}}}] = \left\{a_c \in \mathfrak{B}[\tau_{\mathcal{D}^{\mathbb{C}}}] : \varphi(a_c) = a_c^+\right\},$$

for every $a_c, a_c^+ \in \mathfrak{B}[\tau_{\mathcal{D}^{\mathbb{C}}}]$*,*

$$\mathfrak{B}[\tau_{\mathcal{D}^{\mathbb{C}}}] = \mathcal{A}[\tau_{\mathcal{D}^{\mathbb{R}}}] \dot{+} i\mathcal{A}[\tau_{\mathcal{D}^{\mathbb{R}}}], \tag{8.53}$$

and

$$\tau_{\mathcal{D}^{\mathbb{C}}} \restriction \mathcal{A} = \tau_{\mathcal{D}^{\mathbb{R}}}. \tag{8.54}$$

Proof. The result follows from Propositions 15, 16, 17, 18 and 19. ■

Now we define Jordan Lassner operator algebras which will serve as unbounded analogues of JC-algebras of bounded linear self-adjoint operators on a complex Hilbert spaces.

Proposition 20 *A real Op-subalgebra* A *of* $(\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s, \bullet)$ *with the uniform topology* $A[\tau_{\mathcal{D}^{\mathbb{C}}}]$*,is a Jordan topological algebra.*

Proof. The result follows from Proposition 19. ■

Definition 75 *A Jordan Op-subalgebra A of* $(\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s, \bullet)$ *with the uniform topology* $A[\tau_{\mathcal{D}^{\mathbb{C}}}]$*,is called a* ***Jordan operator Lassner algebra.***

Examples of Jordan operator Lassner algebras include symmetric parts of complex and real Lassner operator algebras, however, again, we note that not all Jordan Lassner operator algebras need be symmetric parts of complex or real Lassner operator algebras even in the case of bounded linear operators (see[30] for details).

8.2 Representations of Barreled Real Locally C*- and Locally JB-algebras

In the present section we discuss Gelfand-Naimark type theory for real locally C*- and locally JB-algebras, where representations take place in certain $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$ or $\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$.

Recall that according to the celebrated GNS construction (see for example[51]), for each abstract complex C*-algebra $\mathfrak{A}$ there exists a complex Hilbert space $H^{\mathbb{C}}$ so that $\mathfrak{A}$ is isometrically *-isomorphic to a closed in operator norm topology *-subalgebra of the algebra $\mathcal{B}(H^{\mathbb{C}})$ of all bounded linear operators on $H^{\mathbb{C}}$. Analogous constructions exist in the case of real C*- and JB-algebras (for GNS construction for real C*-algebras see,[45] and for JB-algebras see[30]). If we start with complex locally C*-algebras, and set ourselves the task of finding a sort of unbounded analogue of Gelfand-Naimark theory, i.e. to find their representations as complex operator Lassner algebras, it is not possible to do it to the same extent as the theory for C*-algebras because not every complex Lassner algebra is a locally C*-algebra. The problem here is not only just that it may not be complete as in the commutative case (the functional algebra $C(\Delta)$, see[29] for details): but for a large group of cases of pre-Hilbert spaces $\mathcal{D}^{\mathbb{C}}$, the multiplication in $\mathcal{L}_{+}^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$ is not even jointly continuous (see[42] for details). So, the most we can get here is to find some sufficient conditions under which a locally C*-algebra will be topologically *-isomorphic to a certain complex Lassner operator algebra. It was done under conditions of metrizability by Sya in[69] and Brooks in.[12] In[43] Lassner showed that barrelledness is the sufficient condition under which a locally C*-algebra will be topologically *-isomorphic to a certain complex Lassner operator algebra. Our aim in the present section is to obtain real and Jordan analogues of Lassner's result from[43] for real locally C*-algebras and locally JB-algebras.

Now, let us define the representations we are going to deal with.

Definition 76 *A *-representation $a_c \to b_c(a_c)$ of a complex *-algebra $\mathfrak{A}$ is a *-homomorphism of $\mathfrak{A}$ onto an complex Op*-subalgebra $\mathfrak{B}$ of $\mathcal{L}_{+}^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})$, $a_c \in \mathfrak{A}$, $b_c \in \mathfrak{B}$.*

*A *-representation of a complex locally convex *-algebra $\mathfrak{A}$ is said to be **weakly continuous**, if $< \xi_c, b_c(a_c)\eta_c >$ depends continuously on a_c for all $\xi_c, \eta_c \in \mathcal{D}^{\mathbb{C}}$.*

*A *-representation is said to be **uniformly continuous**, if $a_c \to b_c(a_c)$ is a continuous mapping of $\mathfrak{A}$ onto the complex Lassner operator algebra $\mathfrak{B}[\tau_{\mathcal{D}^{\mathbb{C}}}]$.*

The following theorem was established by Lassner.

Theorem 47 (Lassner) [42]*Let $a_c \to b_c(a_c)$ be a *-representation of a complex locally convex *-algebra $\mathfrak{A}$. If for any $\xi_c \in \mathcal{D}^{\mathbb{C}}$,*

$$f_{\xi_c}(a_c) =< \xi_c, b_c(a_c)\xi_c) >, \tag{8.55}$$

is continuous in a_c, *then* $a_c \to b_c(a_c)$ *is weakly continuous. If furthermore* $\mathfrak{A}$ *is barrelled, then* $a_c \to b_c(a_c)$ *is also uniformly continuous.*

Now we define the *-representations of real *-algebras.

Definition 77 *A *-representation* $a \to b(a)$, *of a real *-algebra* $\mathcal{A}$ *is a *-homomorphism of* $\mathcal{A}$ *onto a real Op*-subalgebra* $\mathcal{B}$ *of* $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$, $a \in \mathcal{A}$, $b \in \mathcal{B}$.

*A *-representation of a real locally convex *-algebra* $\mathcal{A}$ *is said to be* ***weakly continuous****, if* $< \xi, b(a)\eta >$ *depends continuously on* a *for all* $\xi, \eta \in \mathcal{D}^{\mathbb{R}}$.

*A *-representation is said to be* ***uniformly continuous****, if* $a \to b(a)$, *is a continuous mapping of* $\mathcal{A}$ *onto the real Lassner operator algebra* $\mathcal{B}[\tau_{\mathcal{D}^{\mathbb{R}}}]$.

We are now ready to state the real analogue of theorem 47.

Theorem 48 *Let* $a \to b(a)$, *be a *-representation of a real locally convex *-algebra* $\mathcal{A}$, *such that*

$$\mathcal{A} \cap i\mathcal{A} = \{\mathbf{0}\}.$$

If for any $\xi \in \mathcal{D}^{\mathbb{R}}$,

$$f(a) = \ < \xi, b(a)\xi >, \tag{8.56}$$

is continuous in a, *then* $a \to b(a)$, *is weakly continuous. If furthermore* $\mathcal{A}$ *is barrelled, then* $a \to b(a)$,*is also uniformly continuous.*

Proof. Let us set

$$\mathfrak{A} = \mathcal{A} \dot{+} i\mathcal{A}.$$

Because a complexification of a barrelled space is again a barrelled space,[61] $\mathfrak{A}$ satisfies all the conditions of Theorem 47. After we apply Theorem 47 to $\mathfrak{A}$, and then reduce the representation to $\mathcal{A}$, we will get $\mathcal{B}$ as a range of this reduced map in $\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}})$, and, because the topology $\tau_{\mathcal{D}^{\mathbb{R}}}$ coincides with the topology $\tau_{\mathcal{D}^{\mathbb{C}}}$ on

$$\mathcal{L}_+^{\mathbb{R}}(\mathcal{D}^{\mathbb{R}}) \subset \mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}}),$$

the theorem is fully proven. ■

Now we are ready to state the definition of Jordan representations we need.

Definition 78 *A Jordan representation $a \rightarrow b_c(a)$ of a Jordan algebra A is a Jordan homomorphism of A onto an Jordan Op-subalgebra B of $(\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s, \bullet)$, $a \in A$, $b_c \in B$.*

A Jordan representation of a real locally convex Jordan algebra A is said to be ***weakly continuous****, if $< \xi_c, b_c(a)\eta_c >$ depends continuously on a for all $\xi_c, \eta_c \in \mathcal{D}^{\mathbb{C}}$.*

A Jordan representation is said to be ***uniformly continuous****, if $a \rightarrow b_c(a)$ is a continuous mapping of A onto the Jordan Lassner operator algebra $B[\tau_{\mathcal{D}^{\mathbb{C}}}]$.*

Let us now state a Jordan algebra analogue of theorem 47.

Theorem 49 *Let $a \rightarrow b_c(a)$ be a Jordan representation of a locally JB-algebra A onto a Jordan Op-sub-algebra B of $(\mathcal{L}_+^{\mathbb{C}}(\mathcal{D}^{\mathbb{C}})_s, \bullet)$. If for any $\xi_c \in \mathcal{D}^{\mathbb{C}}$,*

$$f(a) = \quad < \xi_c, b_c(a)\xi_c >,$$

is continuous in a, then $a \rightarrow b_c(a)$ is weakly continuous. If furthermore A is barrelled, then $a \rightarrow b_c(a)$ is also uniformly continuous.

Proof. In accordance with theorem 29, for each locally JB-algebra A there exists a universal enveloping locally C*-algebra $\mathfrak{A}$, and its involutory *-antiautomorphism ψ with period 2, such that there exists a unique Jordan ideal A_{ex} of A, so that a factor algebra A/A_{ex} is topologically Jordan isomorphic to a Jordan subalgebra of $(\mathfrak{A}_{SA}, \bullet)$, which is composed of those $a \in \mathfrak{A}_{SA}$, satisfying the identity

$$\psi(a) = a.$$

One can easily see that if A is barrelled, so will $\mathfrak{A}$ be. Now, by applying theorem 47 to $\mathfrak{A}$, and by reducing representation to A/A_{ex} which is topologically Jordan isomorphic to a Jordan subalgebra of $(\mathfrak{A}_{SA}, \bullet)$ we obtain the required result. ■

Now, let us recall Lassner's sufficient Gelfand-Naimark type theorem for complex locally C*-algebras.

Theorem 50 (Lassner) [43]*Each complex barrelled locally C*-algebra is topologically *-isomorphic to a complex Lassner operator algebra.*

The real version of this theorem will be as follows:

Theorem 51 *Each real barrelled locally C*-algebra is topologically *-isomorphic to a real Lassner operator algebra.*

Proof. Let $\mathcal{A}$ be a real barrelled locally C*-algebra, and

$$\mathfrak{A} = \mathcal{A} \dot{+} i\mathcal{A},$$

be its complexification, which is complex barrelled C*-algebra, and

$$\mathcal{A} \cap i\mathcal{A} = \{\mathbf{0}\}.$$

Now we apply theorem 50 to $\mathfrak{A}$, and restrict the representation to

$$\mathcal{A} \subset \mathfrak{A}.$$

■

Now we are ready to formulate a sufficient Gelfand-Naimark type theorem for locally JB-algebras.

Theorem 52 *For each barrelled locally JB-algebra there exists a unique Jordan ideal so that a factor algebra modulo this ideal is topologically Jordan isomorphic to a Jordan Lassner operator algebra.*

Proof. Let A be a barrelled locally JB-algebra. By theorem 49 and theorem 29, $\mathfrak{A}$ is the barrelled universal enveloping locally C*-algebra for A. Now, apply theorem 50 to the algebra $\mathfrak{A}$ and reduce the representation to subalgebra of $(\mathfrak{A}_{SA}, \bullet)$, which is topologically Jordan isomorphic to the factor algebra A/A_{ex} (where A/A_{ex} is as in the proof of theorem 49). ■

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