EXPLORING GRADE 12 LEARNERS' APPROACHES FOR SOLVING TRIGONOMETRIC EQUATIONS

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Abstract: This study analysed 12 learners' preferred problem solving approach for solving trigonometrical equations. The study addressed the question of whether algebraic or graphical representation format provides different affordances with respect to learners' successes in solving trigonometric equations. Data were obtained from 55 randomly selected grade 12 learners in Sekhukhune District, South Africa. Ten (10) learners were interviewed to see how they solved equations, how they represented solutions, and connected representations together. Learners' responses were analysed qualitatively by looking at the representations, perspective, and the connections between representations that were used for answering questions. A test and four focus group interviews were the main instruments used to gather data. Content and Implicative statistical analyses were performed to evaluate the relationship between learners' approach and their ability to solve trigonometric equations. Results indicated that learners prefer to use the algebraic approach to solve trigonometric tasks. Learners considered algebraic manipulation to be a more important mathematical skill than graphing, and perceived teachers' bias toward using algebraic representations over graphical ones. Implications of findings for teaching trigonometry are discussed.

Keywords: Representations, mode, graphical, algebraic, Trigonometric equations, general solutions.

Introduction

Algebraic and graphical approaches are two major representations commonly used in mathematics problem-solving. Mathematically proficient learners can explain correspondences between algebraic and graphical solutions of trigonometrical equations, check their answers to problems using a different method, they continually ask themselves whether their answers make sense, and they consider alternative approaches and identify correspondences between different approaches. (Kersaint et al, 2014).

A study by Lee et al, (2013), shows that learners generally prefer algebraic representations. Algebraic and graphical representations provide different affordances with respect to computation and interpretation problems. Graphical representation allow easier visualisation of overall patterns than algebraic representation, while algebraic representation allow ease in computation of exact points (Gettinger et al, 2013). Fyfe et al (2014) argued that graphical representations are grounded in familiar experiences, connect with learners' prior knowledge, and have an identifiable perceptual correspondence with their referents. However, they may contain extraneous perceptual details that distract learners from relevant information or inhibit transfer of knowledge to novel situations (Kaminsky, Sloutsky & Heckler, 2009). In contrast, algebraic representations eliminate extraneous surface details, are more arbitrarily related to their referents, and represent the underlying structure of the referent more efficiently. Thus, they allow greater flexibility and generalizability to multiple contexts, but may appear as meaningless symbols to learners who lack conceptual understanding (Nathan, 2012).

Learners should be to be able to use a variety of representations and move from one form to another. Each representation has strengths and in facilitating meaningful learning. Algebraic representations are essential to enhance learners' manipulative skills while graphical representations provide a visual view of the concept (Bansilal & Naidooo, 2012). Because of the complexity of algebraic problem solving, it is important for learners to understand and appreciate the affordances of different representations. The graphical approach give a more elaborate and complete view of mathematical concepts (Bayazit

& Aksoy, 2010); nevertheless, learners need a sort of visual ability and geometric skills such as reading a graph, predicting the development of a graph and interpreting the meanings embedded in the graphical representations. Atiyah (2001) argued that geometric representations are superior over algebraic expressions in illuminating mathematical ideas. However, from a pedagogical point of view neither of these representations can be given priority over the other in solving trigonometric equations (Sullivan, Mousley and Jorgensen, 2009). Integrating symbolisms gives a better understanding of the solutions.

Learners lack the ability to formulate mathematical problems, represent them, and solve them using appropriate representation (Schoenfeld, 2014). Knowing multiple ways of solving a problem leads to efficient and effective problem solving (Polya, 2004). Learners with well-developed knowledge of representations, procedures, and a robust conceptual understanding know how to solve problems using more than one representation. According to English (2013), an image of a mathematical idea cannot be separated from the concept itself; and it should be regarded as an essential part of thinking. Learners' ability to see interrelations between representations and between ideas is a crucial stage in developing conceptual understanding of mathematics. The underlying philosophy is simply that a well-chosen representation could convey part of the meaning of a mathematical concept, yet establishing connections among the representations could provide a more coherent and unified message.

There are on-going debates about which representation maximizes problem–solving opportunities for different learners, and why particular practices work while others do not work. These debates focused on the value, purpose and scope of student-generated representations and effective ways for learners to learn the representational conventions of mathematics discourse. Mielicki (2007) posted that learners prefer graphs to equations since graphs involve "less math" because they are visual and obviate the need to choose and execute any kind of algebraic procedure. Herman (2007) stated that learners prefer algebraic solution methods to be valued higher in their math courses than graphical solution methods. Furthermore, algebraic representations lead to more precise solutions than graphs, and equations are easier than graphs because the correct procedure is apparent. Giere and Moffatt (2003) recommended that learners should learn how to use representations in mathematics as thinking tools for predicting, understanding and making claims, rather than memorizing "correct" representations for knowledge display. This implies that learners are likely to learn more effectively when they see the aptness of representational conventions used in this subject.

The study asks the questions, what mode of approach or representation learners prefer to use when solving trigonometric equations, why do learners prefer an algebraic or graphical approach and how is this preference affect their performance during problem solving. The aim of this study is seek answers to these questions. This study seeks to check whether algebraic or graphical representation format provides different affordances with respect to learners' successes in solving trigonometric equations. The study further seeks to identify and evaluate the stability of these approaches in solving trigonometric equations.

Problem Statement

Selecting an appropriate approach for a given mathematical task is a critical part of successful problem solving. This selection entails understanding the affordances of different approaches in different contexts (Mielicki & Wiley, 2016). Previous researches indicate that learners do not select appropriate representations during problem solving (Ainsworth, 2006; Yee and Bostic, 2014). Learners prefer to solve problems by using algebraic representations (equations) rather than graphical ones (Herman, 2007). Lynch and Star (2014) found that algebraic strategies are most preferred by learners during problem-solving even after learners had received instruction on graphical approach. Herman (2007) also concluded that learners consider algebraic manipulation to be a more important mathematical skill than graphing, and perceived an instructor bias towards using algebraic representations over graphical ones. While Knuth (2000) and Herman (2007) emphasised that graphical representation should facilitate algebraic problem solving, Mielicki (2007) suggest that the affordances of graphical representation are most apparent for interpretation problems. Knuth (2000) claimed that learners' preference for algebraic representation over graphical representations is driven by more exposure to equations than graphs during instruction.

Research Questions

The following research questions guide the study:

1) What mode of representation do learners use when solving trigonometric equations?

2) Why do learners prefer an algebraic or graphical approach and how is this preference affect their performance during problem solving?

Conceptual Framework

The framework developed by Moschkovich, Schoenfeld, and Arcavi, (1993) guides the study. The framework distinguishes between a process and an object view of a mathematical concept. In the process view an equation is treated as a link between two quantities on either side of an equal sign. In this view, equations are perceived as rules for computation. In the algebraic representation it has been proposed that the object view enables one to take the object and perform an action on it, using it for example in a further process. In the object view, an equation is considered to be an entity which can be manipulated. According to Sfard (1991), learners typically progress from a procedural (process) to a structural (object) view of a concept through the process of reification, and exposure to multiple representations of a concept may support this process.

A trigonometric equation can be solved algebraically or graphically. Different representations of equations are thought to provoke problem-solving approaches more consistent with either the process or object view (Sfard, 1991). Algebraic (equation-based) representations are associated with approaches grounded in the process view because these representations facilitate computation of exact values. Graphical representations are associated with object view approaches because graphs facilitate perception of the equation as a whole. An important characteristic of competence in algebra is understanding the connection between multiple representations of algebraic concepts(Ainsworth, 1999). Multiple representations can serve as tools to aid learners in the development of algebraic understanding. Schliemann, Goodrow, and Lara-Roth (2001) found that instruction emphasizing multiple representations (tables and graphs) facilitated the shift from computation of unknown values (process view) to generalization of patterns between variables (object view).

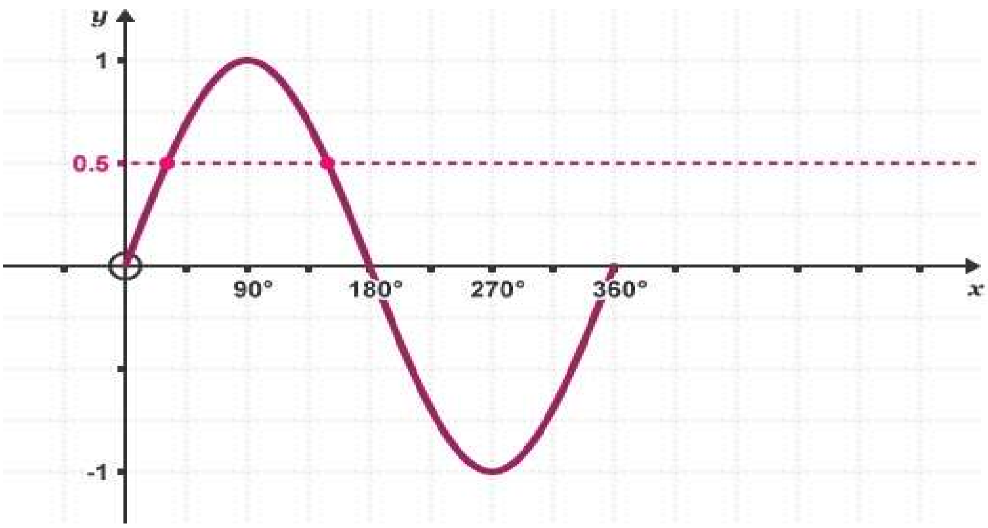
Literature Review

Solving trigonometric equations requires learners to find angles that satisfy the equation. If a specific interval for the solution is given, then learners need only find the value of the angles within the given interval that satisfy the equation. If no interval is given, then they need to find the general solution. The periodic nature of trigonometric functions means that there are many values that satisfy a given equation. The unit circle is one of the representations that are vital to solving trigonometric equations. Weber (2005) stated that understanding the process that creates the unit circle is crucial to understanding trigonometric functions and solutions to trigonometric equations. Weber (2005) found that if learners cannot connect a rotation on the unit circle with a point on the graph of a trigonometric function then using it to solve problems or connect to other representations will not be possible.

The problems that learners experience when working with representations are caused by the emphasis placed on memorizing the information, like the unit circle, without establishing a conceptual understanding for what they mean. Gur (2009) found that most instruction and textbooks emphasize this, concluding that many learners' errors are simply a mechanical error in the application of a rule. Weber (2005) also found that much of trigonometry instruction focused on procedures and paper-and-pencil computations without an emphasis on applying the process. Such experiences fail to assist learners to form connections between representations or see trigonometry as unified construct.

Gur (2009) argued that most trigonometry instruction is teacher-centred so learners have difficulty constructing new knowledge about concepts, processes, and procepts dealing with trigonometry. According to Gur (2009), a concept produces a mathematical object, a process involves the ability to use operations, and a procept is the ability to think of operations and objects by using both concepts and processes. Simply memorizing information without grounding them conceptually also hinders learners learning trigonometry. Gur (2009) found that learners memorized definitions and formulae for trigonometry without understanding of what they mean.

Some teachers choose to introduce trigonometry through the unit circle centred on the origin, O; as a point, P, moves around the circle the coordinates of P are $(\cos\theta, \sin\theta)$ where θ is the anglebetween OP and the positive x -axis, measured in an anticlockwise direction. Viewed in this way, values such as sin120^0 have a clear meaning. Spending time investigating the graphs of trigonometric functions, through sketching and plotting, builds an understanding of the symmetries and periodicities in the graphs that are a direct consequence of the symmetry of the unit circle. With this understanding, learners should be able to find all solutions, in a given range, of equations such as sin θ = 0.5.



Trigonometric equations can be solved in degrees or radians using CAST and its period to find other solutions within the range, including multiple or compound angles and the wave function. Therefore since the trig equation we are solving is sin and it is positive (0.5), then we are in the 1st and 2nd quadrants.

We have already found the first solution which is the acute angle from the 1st quadrant, so to find the second solution; we need to use the rule in the 2nd quadrant.

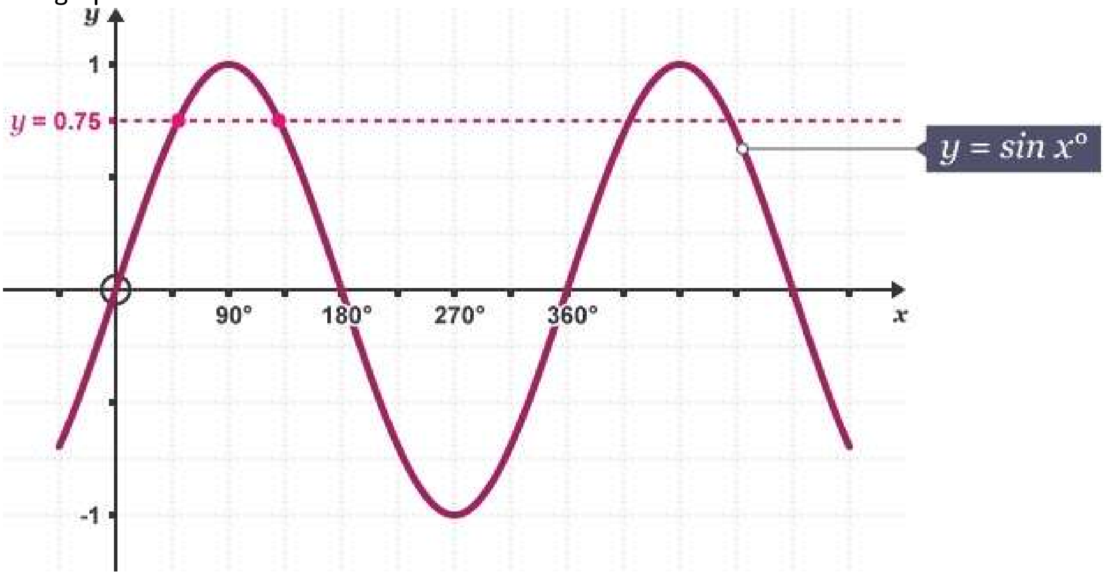
$x = 180° - 30°$
$x = 150°$
$x = 30°, 150°$

General in the first quadrant: $x = 360k + 30°$ General in the second quadrant: $x = 360k + 150°$Use the graphical approach to solve the equation $4 \sin x° - 3 = 0$, where $0° \leq x° \leq 360°$.

$4 \sin x° = 3$

$\sin x° = \frac{3}{4}$

The graph of this function looks like this:



From the graph of the function, we can see that we should be expecting 2 solutions: 1 solution between $0°$ and $90°$ and the other between $90°$ and$180°$.

Since sin is positive this means that we will be in the two quadrants where the sine function is positive - the first and second quadrants.

First quadrant

$sin\ x° = \frac{3}{4}$

$x° = sin^{-1}\left(\frac{3}{4}\right)$

$x° = 48.59037...$

$x° = 48.6°$(to 1 d.p.)

Second quadrant

$x° = 180° - 48.6°$

$x° = 131.4°$

$x° = 48.6°, 131.4°$

Research Design

A test and four focus group interviews were the main instruments used to gather data. Content and Implicative statistical analyses were performed to evaluate the relation between learners' approach and their ability to solve trigonometric equations. Content analysis is the study of the content with reference to the meanings, contexts and intentions contained in messages (Prasad, 2008). Content analysis falls in the interface of observation and document analysis. It is defined as a method of observation in the sense that instead of asking people to respond to questions, it takes the communications that people have produced and asks questions of theses communications. Therefore, it is also considered as an unobtrusive or non-reactive method of social research. On the other hand, Statistical Implicative Analysis (SIA) is a statistical method of analysing data that seeks to understand the links between variables. This method was designed to answer the question: “If an object has a property, does it also have another one?”

Population and Sampling

The study population consists of learners enrolled in Grades 12 in Sekhukhune District in Limpopo Province. The accessible population for the study consists of 250 grade 12 learners attending mathematics enrichment classes in mathematics and science. Data were obtained from 55 randomly selected grade 12 learners in Sekhukhune District, South Africa. Ten (10) learners of varying mathematical ability were interviewed to see how they solved equations, how they represented solutions, and connected representations together. Learners’ responses were analysed qualitatively by looking at the representations, perspective, and the connections between representations that were used for answering questions.

Research Instruments

A test consisting of trigonometric equations whose solution processes require learners to use mixed approaches: algebraic and graphical approaches was used. Focus group interviews were utilised to obtain in-depth explanations as to why learners prefer certain approaches. Participants for focus group interviews were purposefully selected based on their responses to test questions. Content and Implicative statistical analyses were performed to evaluate the relationship between learners’ approach and their ability to solve trigonometric equations. The analysis also attempted to discoverwhere student knowledge is lacking and how this relates to the ability of learners to use multiple representation and perspectives.

Results

Results indicated that learners prefer to use the algebraic approach to solve trigonometric tasks. Follow-up interviews with the learners suggested that learners considered algebraic manipulation to be a more important mathematical skill than graphing, and perceived teachers’ bias toward using algebraic representations over graphical ones. The table below shows the nature of f the question and the approaches selected by the learners:

Table 1: Representation Preferences

Item	Representational Type	Process/ Object	Approach			Total
			Graphical	Algebraic	Mixed	
1	Algebraic	Solving, general solution	3	46	6	55
2	Graphical	Drawing, interpretation	29	23	3	55
3	Graphical	Interpretation	19	32	4	55
4	Algebraic, Graphical	Drawing & solving	21	28	6	55

5	Algebraic, Graphical	Interpretation & solving	19	32	4	55
	Success Rate (%)		26	66	100	

Table 1 shows learners' preferred approach for each of the question representation types. The table indicates that learners mostly prefer an algebraic approach even when the question is presented graphically. Very few learners were able to link graphical and algebraic approaches; hence the frequencies for mixed approaches were low throughout. Learners struggled either to draw or interpret graphs that lead to trigonometric equations. Learners lack graphing skills and hence it was too difficult for them to obtain correct answers from incorrect graphs. The success rates for the three approaches were 26% (graphical), 66% (algebraic) and 100% (mixed approaches).

Interview Responses

When asked which representation they preferred, most participants reported a preference for graphical representation. When asked whether they felt that problems were easier when presented with a particular representation, most participants responded that the problems presented with graphs drawn were easier than those presented with equations. The number of participants whoreported more experience with algebraic equations was more than the number of participants who reported more experience with graphs equations. Learners who reported a preference for graphical representation also reported that the problems presented with graphs were easier than problems presented with equations. Participants who reported a preference for algebraic representation reported that algebraic problems were easier than graphical problems.

Table 2: Learners' preferred solution approaches

Preferences for Approaches	Graphical Approach	Algebraic Approach	Mixed
Which would you choose?	6	44	5
Which problems were easier?	3	50	2
Which do you have more exposure to?	5	47	3

Learners' responses in table 2 shows that majority of the participants prefer an algebraic approach to solve trigonometric equations. Participants also indicated that trigonometric equations or problems presented in algebraic form were easy to solve compared to those presented graphically. Learners confirmed that they have more exposure to the algebraic approach than the graphical approach.

Discussion

The aim of this study was to establish learners' preferred approach to solving trigonometric equations. The findings of the study point to the fact that learners prefer an algebraic approach to solve trigonometric equations. Learners find the algebraic approach to be more efficient than graphical representation. They indicated that the affordances of graphical representation are limited to particular problem types. The study found that very few learners are flexible in switching between algebraic and graphical approaches in representing solutions to trigonometric equations. Participants had difficulties in implementing the graphical approach. Many learners have not mastered even the fundamentals of the graphical approach in the domain of trigonometric equations. Learners' understanding is limited to the use of algebraic representations.

Another important finding emerging from findings is that learners lack the skills to relate and switch relation between the graphical approach and algebraic approach when solving trigonometric equations. This finding is inconsistent with Mwakapenda (2008), who indicated that graphical approach enables learners to manipulate an equation as an entity, and thus learners are capable of finding the connections and relations between the different representations involved in problems.The findings of this study suggest that learners lack a coherent understanding of the concept of an infinity number of solutions of trigonometric equations and cannot relate it to the relationships between algebraic and graphic representations to provide successful solutions.

The study suggests that the difficulties learners have when solving trigonometric equations by implementing the graphical approach may be due to the fact that learners cannot interpret the information a from a graph. Thus, learners may be unaware that the graphical representation offers a means for determining a solution. Furthermore, the graphical representations create cognitive difficulties that limit learners' ability to make connections between the algebraic and the graphical representations. To many learners, the graphical approach seems to have no connection to the algebraic representations of the same concept. Consequently, learners fail to establish the necessary connections between them and furthermore to switch from one form to another them during problem solving.

Conclusion

In conclusion, the data for this study presents two major findings. Firstly, there are challenges connected to the use of graphical approach in solving trigonometric equations. Learners indicated that they lack skills of graphing graphs of equations. Further the graphical approach offer estimate solutions compared to exact solutions from the algebraic approach. Secondly, most teachers frequently use the algebraic approach in most instructional sessions; hence learners have more exposure to this representation. The implication drawn from this study is that teachers should encourage learners to utilise the opportunities afforded by the use of the graphical representation for problem solving and building connections. Mathematics teachers should promote learners' flexibility at shifting between algebraic and graphical representations so that they could develop a much better understanding of the mathematical concepts

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