EXPLORING LEARNERS' APPLICATION OF PRODUCTIVE THINKING MODEL IN SOLVING OPTIMISING REAL-LIFE PROBLEMS: A CALCULUS APPROACH

Mutodi Paul
University of Limpopo
South Africa
paulmutodi2@ul.ac.za

ABSTRACT-This paper explores Grade 12 learners' productive thinking processes when solving optimisation problems using Calculus. The study was guided by the Gestalt theory of problem solving. The sample for the study consists of 50 Grade 12 learners who were randomly selected from a mathematics and science supplementary instruction centre. A test was used as a research instrument of the study. Content analysis was used as the methodology to identify learners' productive thinking processes in optimising problems by applying calculus methods. The results of the study indicate that learners struggle to decode contexts of problems and represent it symbolically, formulate equations and functions and apply calculus methods to solve the problem and interpreting the solutions. The implications for the study are that teachers should engage learners in ways that promote teaching for problem solving transfer, critical thinking and productive thinking.

Keywords: Problem-solving, productive thinking, critical thinking, optimisation, calculus.

1. INTRODUCTION

The ability to solve problems is one of the salient competencies of learning mathematics. One of the major goals of mathematics education is to help learners to acquire problem-solving skills in new situations (Taplin, 2006). Problem solving is fundamental to mathematics education because educators are interested in improving leaners' ability to solve problems. Problem solving is the process of finding a solution to a problem (Human, 2009). Problem solving as a teaching and learning approach affords learners opportunities to make sense of the mathematical concepts by using their own strategies as they decide how to proceed. Problem-solving allows learners to solve problems in multiple ways think critically beyond applying basic skills and surface knowledge. Lai (2011) posits that desire to solve a problem correctly motivates learners to acquire new ways of thinking about concepts and processes that might not have been adequately learned when first introduced.

Problem-solving is a teaching strategy that helps learners in advanced mathematics teaching (Sullivan, 2011). It involves working through details of a problem to reach a solution. The working includes series of logical and systematic mathematical operations and may be a gauge of an individual's critical thinking skills. Current studies on problem-solving revealed that it has strong connections with various cognitive abilities such as intelligence, intellect, attention and working memory (Branchini, Savardi & Bianchi, 2013). Booker and Bond (2008) observed that by learning mathematics through problem solving, learners acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. Learners experience difficulties with the applications of derivative to solve real-life problems (Schoenfeld, 2013). Learners face multiple difficulties in all the steps of solving applied derivative problems. The application of calculus to solve real-life problems involving rates of change is gaining ground in the field of mathematics. Tasks that have familiar or real-life contexts are more meaningful to learners. This is so because learners are able to draw on their everyday experiences to solve these kinds of tasks, they are more motivated to stick with the task (Taylor, 2015). Calculus is extensively applied in the solution phase in order to obtain optimum conditions for a given problem.

In South Africa calculus is introduced at Grade 12 level. The calculus concepts covered at this level include derivatives form first principles (definition), rules for differentiation, graph sketching and applications which are predominantly optimisation. Optimisation is a form of mathematical modelling that provides learners with the means to analyse and describe their world mathematically, and so allows learners to deepen their under-standing of mathematics while adding to their mathematical tools for solving real world problems. Calculus contributes about 40% of the Grade 12 South African Mathematics curriculum (Department of Basic Education, 2011). In the secondary school mathematics curriculum there are two main applications of differential calculus. One is sketching curves, and the other optimisation. Learners often struggle to solve optimisation problems since they require knowledge from other topics such as perimeter, volumes and surface areas (Brijlall & Ndlovu, 2013). This knowledge is the form of relationships which are expressed in the form of functions. Karadag (2009) urges teachers to engage learners in advanced mathematical thinking during problem-solving. He proposes three phases of problem solving. Firstly, the given "real-world" problem is translated from the context to the abstract level of calculus, the abstract problem is then solved, and the solution is translated back to the context. The formulation of these relationships requires learners to contextualise the problem context, make algebraic formulation of the situation using suitable symbols and finally formulating functional relationships. These functional relationships are then optimised. Optimization problems involve situations in which the problem-solver looks for the largest value or smallest value that a function may assume in a given interval subject to set of constraints.

Calculus is a branch of mathematics that focuses on mastery of symbolic methods for differentiation and applying these to solve a range of problems (Tall, 2006). Optimisation problems take the concept of a derivative to higher cognitive levels where the tasks may be linked to real life situations. Optimisation is one of the most difficult topics for learners taking a calculus course. A subtle change of wording may completely change the problem. Learners need to pay attention to language issues. There is also the problem of identifying the quantity that will be optimised and the constraint. There are series of steps that are followed to solve an optimisation problem. The first step is to read and re-read the problem carefully understanding the problem context. The next step is to identify the quantity to be optimized and the constraint. Learners often struggle to identify the constraint, the quantity that must be optimised and fixed value(s) or constants. Once the quantity to be optimised is identified, differentiation rules may be applied to optimise the problem. The aim of this study is to explore learners' application of productive thinking model in optimising real-life problems by using a calculus approach. The study explores how learners use the productive thinking model when solving real-life problems by optimising using the calculus approach. The study also sought to sensitise teachers for need to carefully select tasks that require learners to struggle and provide the support that learners need without diminishing the cognitive demand of the task or giving learners too much help.

2. PROBLEM STATEMENT

Many learners feel inadequate and insecure when they encounter problem-solving questions. Learners seem to have no idea of how to solve problems and are unable to use prior knowledge to new situations. Learners struggle to utilise calculus as a tool for optimising and solving real-life problems. Learners' limited understanding of calculus and derivatives is a barrier to effective conceptual understanding and problem solving. Learners lack skills of conceptualising a problem, formulating functional relationships, identifying the quantity to be optimised and the constraint(s) based on the problem context. Very little is known in terms of learners' productive thinking processes when solving optimisation problems. Research is therefore needed to explore the amount of rigour and productive thinking processes that lead to learners' correct or incorrect solutions. Research is also needed to explore how creative learners are in problem-solving, their connections of ideas in optimisation situations. Creativity and critical thinking at each stage of the problem-solving process are the main substrates of productive thinking in problem solving, but the drawback is that most learners may not operate at such a higher level of mathematical thinking.

3. RESEARCH QUESTION

This study is guided by the following research questions:

How do learners construct mathematical knowledge when solving tasks in optimisation?
How do learners apply the productive thinking model in optimising real-life problems using a calculus approach?
How much creativity and rigour do learners exhibit in their solution processes?

4. PURPOSE OF THE STUDY

The purpose of this study is to explore learners' productive thinking processes when solving optimisation problems using a calculus approach. The study also intends to assess learners' creative and critical thinking skills at each stage of the problem-solving process. The educational significance of the paper is to encourage learners to take a well-rounded look at a problem, and come up with better potential solutions. The outcomes of the research may also offer insights into other areas of algebraic reasoning where more effective pedagogic approaches need to be developed.

5. PRODUCTIVE THINKING IN PROBLEM SOLVING

There is need to define and show the connection between productive thinking and problem-solving. Problem solving refers to the processes involved in finding a solution to a problem. Andrews & Xenofontos (2015) defined a mathematical problem as a situation the problem-solver desires a goal but does not have immediate means to solve the problem. Mayer and Wittrock (2006) defined problem solving as a cognitive process directed at transforming a problem from the given state to the goal state when the problem solver is not immediately aware of a solution method. According to Taplin (2006) engaging learners in problem solving is a necessary component of learning mathematics with understanding. Teaching through problem-solving helps learners to develop real understanding of the subject matter, it teaches then to think critically, to make decisions, and to learn from their experiences both in and out of the mathematics class. Barron & Darling-Hammond (2008) confirmed that teaching through problem-solving allows learners to construct knowledge by making connections between their prior knowledge and new information.

Gestalt psychologists provided a phenomenological description of problem-solving as a problematic situation in which the problem-solvers desires a solution to a problem but has no immediate solution to the problem. Gestalts identified two types of thinking that take place during problem-solving: productive thinking and reproductive thinking. Productive thinking involves deep understanding of the phenomenal structure of a problem in order to create the solution. Recent developments in mathematics education on problem solving emphasised the connection between reasoning and various cognitive abilities such as intelligence, intellect, attention and working memory (Branchini, Savardi & Bianchi, 2015). Cunningham and MacGregor (2014) viewed productive thinking as person's ability to reconsider, reframe, rethink, or consider a problem from multiple points of view. Productive thinking involves going from a situation of confusion about the problem that is blind to the core structural features and properties of that problem, to a new state in which everything about the issue is clear, makes sense, and fit together(Wertheimer, 2005). Productive thinking involves making sense of problems and perseveres in solving them. The main focus of productive thinking is to promote creative problem solving. The thrust of the thinking process hinges on the application of a novel idea or algorithm that solves the problem.

Productive thinking is an element of problem solving which encourages learners to persevere, or continue working on a problem irrespective of struggle or difficulty. Perseverance or continuing forward irrespective of struggle or difficulty is an essential element in problem solving because the first or second approach or strategy may not result in a reasonable solution. When learners labour and struggle but continue to try to make sense of a problem, they are engaging in productive struggle. Taylor (2015) argued that learners tend to persevere on tasks that have a familiar real-life context. This is so because they are able to apply their life experiences to solve these kinds of tasks, they are more motivated to stick with the task. Other factors which influence learners' productive thinking include the socio-mathematical norms of the class (Middleton, Tallman, Hatfield & Davis, 2015); self-image and disposition to struggle with a challenging mathematical task (Star, 2015). Understanding the structure of a problem by detecting its primary and secondary elements and the relationships between them is essential because the structure not only organises the problem itself, but also contains gaps or "trouble zones" to be healed which function as cues for the directions to be followed when seeking the solution. Gestalt psychologists perceived that the structure of a problem may be a potential obstacle to initiate the solution process. On the other hand; reproductive thinking involves the mechanical application of previously learned concepts and experiences in order to solve new problem situations. Reproductive thinking leads to good performance on retention problems but poor on transfer problems.

6. CONCEPTUAL FRAMEWORK

This study is guided by the Gestalt theory of problem solving. The Gestalt theory of problem solving holds that problem solving occurs with a flash of insight Ellis(2012). The theory suggests that the best way of discovering how to find the solution to the problem is not by being taught a rule or algorithm, but by finding the underlying structure of the problem, and thereby solving the problem in a meaningful way. The drive to the solution is created by the problem-solver's perception of the structural 'givens' of the problem. It emphasises the importance of the meaning of the problem for the learner. The theory suggests that the structural quality of the problem-solver's perception assists the solution process and failure to solve problems is result of a failure to perceive the structure of the problem situation.

Insight is one of the requirements in problem-solving (Batchelder & Alexander, 2010). Topolinski and Reber (2010) defined insight as an "experience during or subsequent to problem-solving attempts, in which problem-related content comes to mind with sudden ease and provides a feeling of pleasure, the belief that the solution is true, and confidence in this belief" (p. 401-2). Insight is important for selecting productive moves at various stages and states of problem solving. Insight occurs when a problem solver moves from a state of not knowing how to solve a problem to knowing how to solve a problem (Schoenfeld, 2014). During insight, problem solvers devise a way of representing the problem that leads to the solution. Insight involves building a schema in which all the parts fit together, insight involves suddenly reorganizing the visual information so it fits together to solve the problem, insight involves restating a problem's givens or problem goal in a new way that makes the problem easier to solve, insight involves removing mental blocks, and insight involves finding a problem analog, that is, a similar problem that the problem solver already knows how to solve. Gestalt theory informs educational programs aimed at teaching learners how to represent problems.

Gestalt psychologists emphasises the process of problem solving than the solution process. The theory hypothesises that solutions come from an insight into the problem and occurred when the problem-solver restructures the problem. In particular, in the process of thinking about a problem the solver restructures the representation of the problem, leading to a flash of insight that leads to a solution. According to Metcalfe and Wiebe (1987) insight occurs when a problem solver moves from a state of not knowing how to solve a problem to knowing how to solve a problem. Insight learning occurs suddenly when a learner discovers new relationships within his/ her prior knowledge as a result of reasoning or problem solving processes that re-organize or restructure that knowledge. During insight, the problem solver gathers tools (numbers, symbols, equations) to solve the problem, devises a plan for representing the solution process. There are several ways in which conceptualizing may happen during insight. Insight involves building a schema in which all the parts fit together, insight involves suddenly reorganizing the visual information so that it fits together to solve the problem, insight involves restating a problem's givens or problem goal in a new way that makes the problem easier to solve and insight involves finding a problem analogy.

7. RESEARCH METHODOLOGY

7.1 Research design

A mixed methods research approach was utilised to explore learners' application of productive thinking model in optimising real-life problems by applying calculus methods. Creswell (2013) describes mixed methods research as a methodology for conducting research that involves collecting, analysing and integrating quantitative and qualitative research. The researcher feels that this integration provides a better understanding of the research problem than either of each alone. Quantitative data for the study was obtained from a researcher-designed test which was administered as performance instrument. The analysis of data consists of statistically analysing scores obtained in the test. Qualitative data was obtained from open-ended information that the researcher gathered through focus group interviews. The analysis of the qualitative data was in the form of words and texts which were aggregated into categories based on the diversity of ideas gathered during data collection. By mixing both quantitative and qualitative research and data, the researcher intended to gain in breadth and depth of understanding and corroboration, while offsetting the weaknesses inherent to using each approach by itself. Finally, triangulation of the results was done to identify aspects of learners' problem-solving capabilities of optimisation by using a calculus approach. This study is concerned with four processes of problem solving: understanding, formulating, solving/ executing and interpreting. The emphasis is on how learners apply the four processes when solving the problems.

A content analysis research design was used to assess learners' work. Content analysis is described as the scientific study of content of communication. It is the study of the content with reference to the meanings, contexts and intentions contained in messages (Elo & Kyngäs, 2008). Content denotes what is contained and content analysis is the analysis of what is contained in a message. Lal Das and Bhaskaran (2008) described content analysis as the scientific study of content of communication. It is the study of the content with reference to the meanings, contexts and intentions contained in messages. Furthermore, content analysis may be viewed as a form of observation and document analysis. It is a method of observation because instead of asking participants to respond to questions it analysis the documents they produced. Jensen (2013) defined content analysis as a method of studying and analysing communication in a systematic, objective, and quantitative manner for the purpose of measuring variables. Content analysis was also utilised as a research tool to determine the presence or absence of productive and reproductive thinking within learners' work. Content analysis is a procedure for the systematic, replicable analysis of text (Prasad, 2008). In this study content analysis was used to classify parts of learners' texts through the application of a structured, systematic coding scheme from which conclusions were drawn about problem-solving competencies. The units of learners' texts that were of interest during the coding process were the four phases. Descriptive statistics, such as frequency counts, were used to summarise findings from the sample and appropriate inferential statistics used to test any hypotheses that have been formulated. In order to understand how learners approach the problems and their perspectives to problem-solving, investigative methods that are capable of probing the learners' learning experiences and of eliciting data that will give more insight into the full complexity of the problem-solving process as practised by learners. Learners were engaged in a short focus –group interview questions. The interviews were conducted soon after the test had been finished so that learners were able to remember what they had written in some detail.

7.2 Participants

Participants for this study consisted of 50 Grade 12 learners selected randomly from a Mathematics and Science enrichment and supplementary instruction school. The learners were drawn from different schools in Sekhukhune District of Limpopo province. The researcher taught the topic and compiled a set of question based on the topic. A rubric or code matrix was used to assess the learners' work. Permission to engage learners was sought with the Limpopo Department of Education, the circuits, principals and supplementary instruction centre managers. Participants were issued with consent forms written in plain language statements that clearly describes the aim of the research and the nature of involvement of participants. Participants were clearly informed of their rights and any risks associated with participation. At all times the researcher observed the welfare of the participants and respect the dignity and personal privacy of the individuals. Letter codes (L1 – L50) were used to identify participants so that they remain anonymous throughout the study.

7.3 Research instruments

The research instrument for this study consists of a test administered to 50 Grade 12 learners who were randomly selected from a class of 75 selected at a supplementary instruction school. The sampling technique applied in this research was simple random sampling. Test items require learners to show comprehension of the problem, formulating equations and functional relationships, solving by optimising using derivatives and interpreting the solution and check feasibility of solutions. A scoring rubric was used to score learners' performance. The scoring key assessed learners' work based on their ability to exhibit the four attributes in their working. Comprehension of the problem was assessed by checking the formulation of relevant equations and functions using correct symbols and identifying unknown(s) variable(s) in the equations. Optimisation was assessed by checking the learners' ability to take correct derivatives and formulating correct optimisation equations. Learners' interpretation of the solutions was assessed by relating the solution to the initial problem conditions. Productive thinking was assessed by checking learners' ability to find solutions in new situations, brainstorming, and critical thinking skills. The occurrence of positive and negative instances of reproductive thinking such as fixed mind sets, functional fixedness, and confirmation bias were noted and recorded in learners' work. The test consisted of 7 questions six of which were based on real life situation and common geometrical figures. The concepts involved in these questions were mainly area and volume. The questions required learners to formulate objective and constraint functions based on common shapes such as circles, semi-circles, rectangles and squares. The first three questions were accompanied by diagrammatic representations of the real life situations. The other three questions were purely word problems in which learners were required to interpret the questions and draw diagrams and then apply their knowledge of calculus to respond to the questions. The last question requires learners to interpret and draw a graph and use the distance formula to formulate the object function and apply calculus methods to optimise the resultant function. The keywords related to the language of optimisation used in the formulation of test items were largest, minimise, maximise, most and closest.

8. RESULTS AND DISCUSSION

The scoring of learners' work was guided by a rubric which emphasises on learners' comprehension of the problem, formulating equations and functional relationships, solving by optimising using derivatives and interpreting the solution and check feasibility of solutions. The mark allocation for each question was based on the four attributes. Summary statistics of learners' performance are shown in the Table 1.

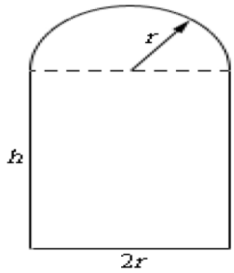
Table 1: Learners' Performance

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Total**
Comprehension of the problem	1.6	1.54	1.82	1.01	1.34	1.37	1.21	1.41
Formulating equations	1.42	1.28	1.56	0.94	0.98	1.23	1.52	1.28
Optimising using derivatives	0.85	0.74	1.45	0.88	0.86	0.99	1.25	1.00
Interpreting of solutions.	0.12	0.46	1.34	0.85	0.81	0.65	1.12	0.76
Total	3.99	4.02	6.17	3.68	3.99	4.24	5.10	4.45
Success Rate (%)	49.8	50.3	77.1	61.3	66.5	70.7	63.8	64.2

Table 1 shows learners' average performance per question and per each attribute of problem solving. The average interpretation success rate was 70.5%. Interpretation success rates were higher on questions in which diagrams were provided than word problems in which learners were asked to interpret and draw their own diagrams. Interpretation success rates for the first three questions were all above 1.50 while for non-diagram questions were below 1.40 out of 2. On the overall learners did well with interpretation followed by formulation of equations. Learners struggled to optimise or others used other methods other than optimisation by differentiation but to avail. Learners showed that they struggled with differentiation and interpretations or relating the solutions to demands of the questions. In some cases learners left their answers in terms of x and y instead of going back to original demands of the question. Question 3 recorded a high success rate which was contributed by good understanding of the question and formulation of equations which were fairly optimised and interpreted.

8.1 Question 3

A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 m of framing materials what must the dimensions of the window be to let in the most light?



8.2 Diagramming

Drawing a diagram emerged as a problem solving strategy that aids the solution process. Questions 1-3 had the problem illustrated in terms of diagrams while questions 4-6 require learners to draw their own diagrams to as to show their interpretation. The use of diagrams or tables is essential for providing a visual view of what is happening. It is a useful strategy of coming to terms with the information in a problem and provides a breakthrough towards the solution. Learners who used diagrams had better chances of solving the problem correctly. The use of a diagram helps learners to acquire a visual understanding of the problem. The diagram provides a visual representation of the problem in which learners may “see” it, understand it, and productively and critically think about it while looking for the next step.

Table 2: Diagramming effects

<table>
<tr><th rowspan="2"></th><th rowspan="2">Mean</th><th rowspan="2">N</th><th rowspan="2">Std Deviation</th><th rowspan="2">Std .Error mean</th><th colspan="2">95% confidence interval of the difference</th><th rowspan="2">t</th><th rowspan="2">df</th><th rowspan="2">Sig 2-tailed</th></tr>
<tr><th>Lower</th><th>Upper</th></tr>
<tr><td>Diagrams</td><td>1.6533</td><td>3</td><td>0.14742</td><td>0.08511</td><td rowspan="2">0.1687 3</td><td rowspan="2">0.72627</td><td rowspan="2">0.109</td><td rowspan="2">5</td><td rowspan="2">0.015</td></tr>
<tr><td>No Diagrams</td><td>1.2325</td><td>4</td><td>0.16378</td><td>0.08189</td></tr>
</table>

The effects of using diagramming during the interpretation of the problem are shown in the table above. Since the sig (2-tailed) value is less than 0.05 we conclude that there is a statistically significant difference between the two approaches. We may conclude that the difference between two mean values is not due to due to chance but due to diagramming. Drawing helps to visualise problem structures. Equations are an abstract way of modelling the problem situation and it’s a higher cognitive level of problem-solving.

8.3 Interpretation problems

In question 1 learners misinterpreted the question, they take 2400 metres as the area of the field instead of perimeter. During the interviews one the learners indicated the choice of area instead of perimeter was informed by the last part of the question which requires them to find area. Hence the most common error was.

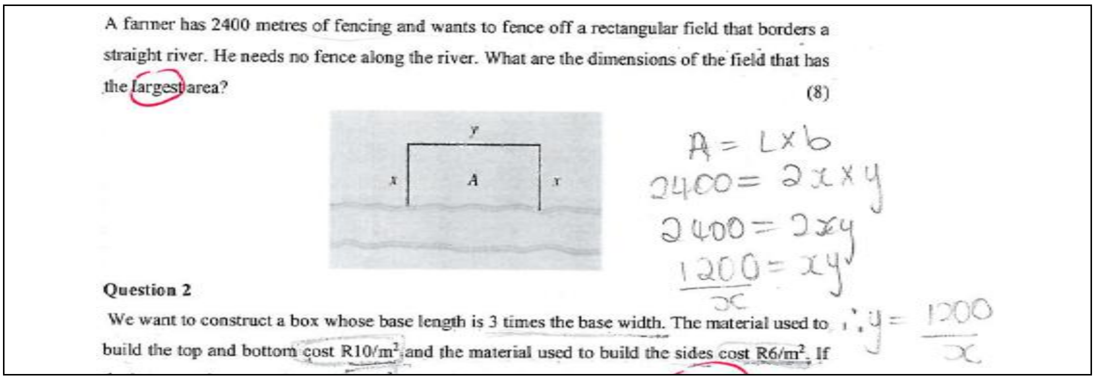


Figure 1: Sample response to Question 1

Figure 1 shows how one of participants interpreted the question 1 and equated perimeter to the area formula. The approach could not allow the problem solver to the optimisation stage.

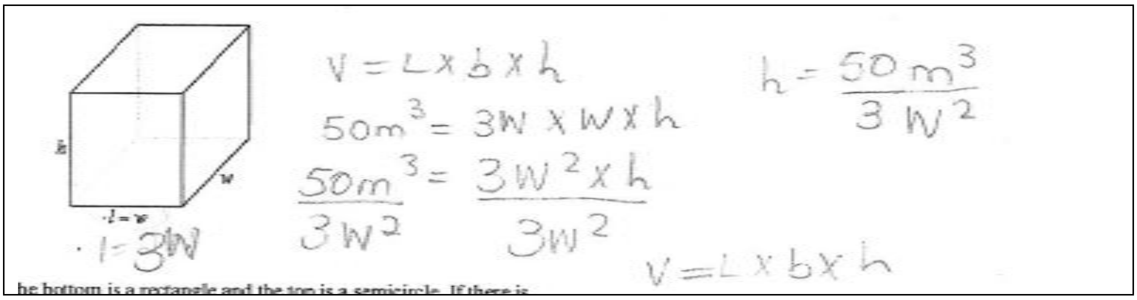


Figure 2: Sample response to Question 2

The learner understood the question partly and the concept of volume was well-handled. The relationship between length and width in the phrase "length is 3 times the base width" was correctly interpreted and the volume was correctly connected to the dimensions of the cube using correct symbols and formula. Key phrases and concepts were also underlined and some were circled .However, the learner could not connect area and cost per square metre and formulate the cost function. Thus, the learner failed to optimise the cost function since the function itself was not there.

8.4 Question 3

A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12m of framing materials what must the dimensions of the window be to let in the most light?

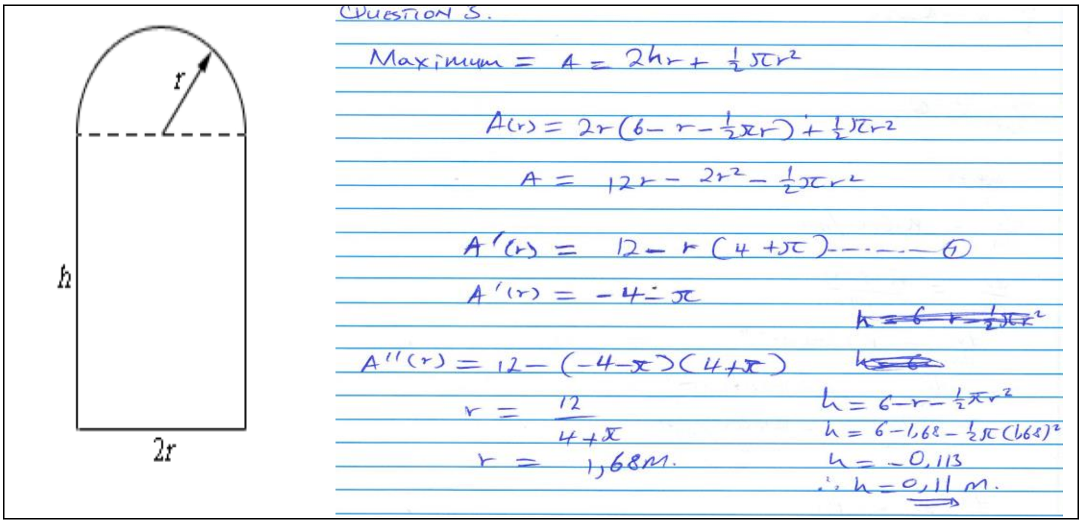


Figure 3: Sample response to Question 3

The learner begins with the word the "maximum" and equates it to the object function as a function of the height and radius. However, the constraint was worked out somewhere and substituted in the equation. Two first derivatives were found but theses have two different functions. The learner had an idea for maximising the function, however some crucial steps were omitted and some were written somewhere and then transferred to the answer sheet but some of the aspects were left out. The second derivative was wrongly expressed and does not lead to the value of r obtained. The learner seems to have the ideas but lacks the ability to organise them coherently into a meaningful mathematical argument.

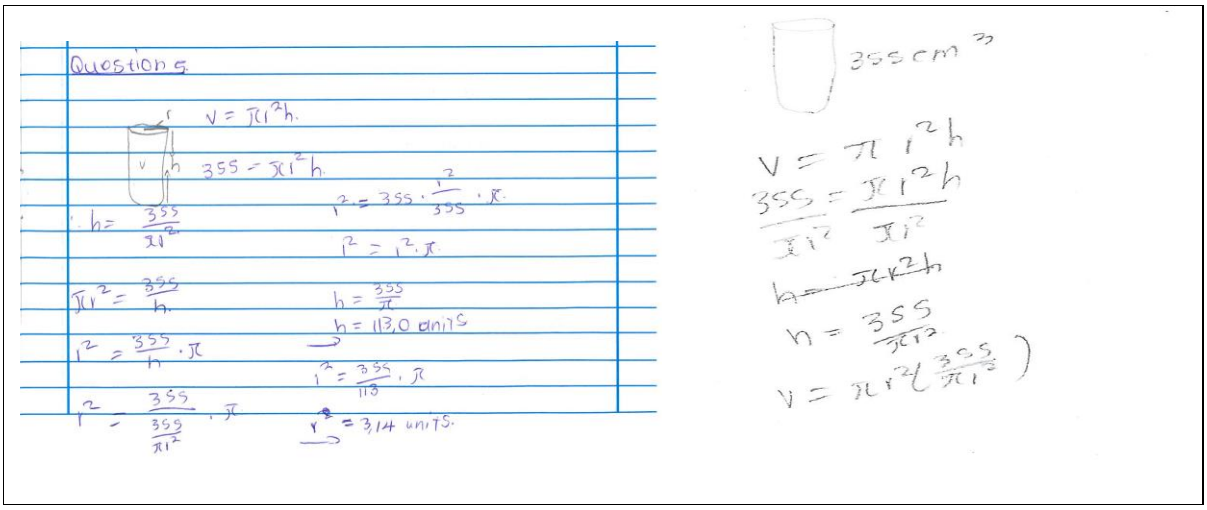


Figure 4: Sample response to Question 5

Figure 4 shows workings by two different learners. The learners started by drawing a visual representation of the problem situation. The learners quoted correct formulae for finding volume. The second part of the question was not addressed; hence the learners spent time writing the same equation with r and h as subjects of formulae. The learners failed to express the objective function, hence like in the previous; there was no function to be optimised. Thus the learners failed to answer the question correctly due inability to misinterpret the question. The second part of the question was not understood. There was no evidence of an attempt to find the surface area.

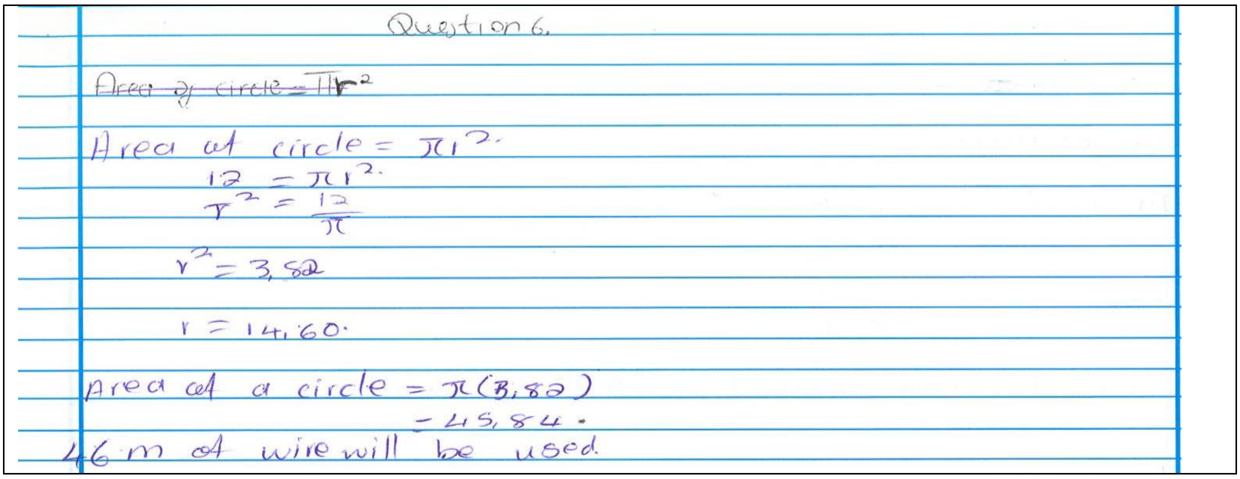


Figure 5: Sample response to Question 6

There was no attempt to represent the problem situation by a graph. The working in figure 5 above shows no evidence of understanding the question. The length of the wire was interpreted as area of the circle and the square was completely ignored. It may be guessed from this problem solving procedure that the learner read the second part of the question and interpreted is if it requires learners to find the area of the circle.

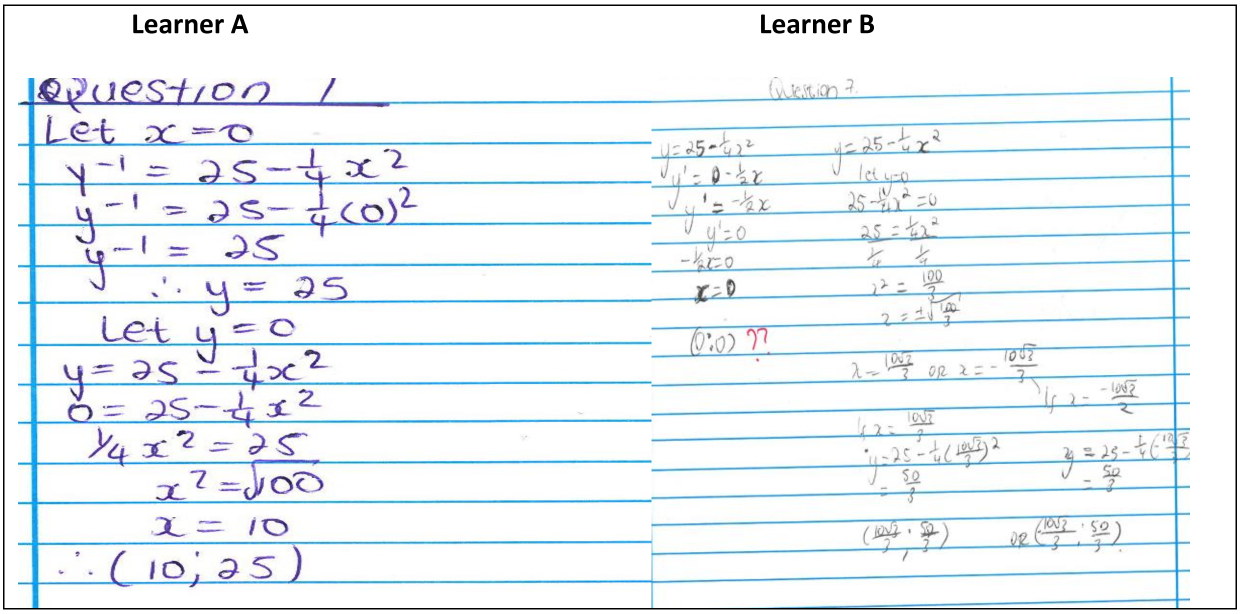


Figure 6: Sample response to Question 7

Figure 6 shows two samples of learners' responses to question 7. Most learners failed to interpret the question. Fourteen (28%) drew correct graphs and of these eight (8) managed to use the diagram to solve the problem. Twenty-two (44%) used algebraic methods by applying the distance formula but latter struggled with differentiation. Learner A's working seem to suggest the learner was trying to find x and y –intercepts but further take the two values obtained when y=0 and x=0 to represent the point (10; 25) as the point closest to the origin. Another confusing observation was the symbol (y^{-1}) in learner A's working. Learner B also misinterpreted the question and unsuccessfully proceeded to find the turning point. However, the intention was thwarted by a calculation error.

Table 3: Categorisation of learners' optimisation problems

Problem	Frequency (n)							Total
	Q1	Q2	Q3	Q4	Q5	Q6	Q7	(F)
Misunderstanding of the question	12	11	13	12	5	8	14	75
No attempt to solve the problem	8	3	4	11	8	9	7	50
Incorrect diagrams and no diagrams	0	0	0	7	13	6	5	31
No means to link the problem to optimisation	14	11	10	2	8	9	9	73
Learner used an inappropriate concept	5	8	9	7	7	7	6	49
Learner lacks knowledge of related concepts	11	7	13	6	4	11	5	57
Incorrect Interpretation of the solution	4	3	2	1	5	4	1	20

Table 3 shows the categories of problems experienced by learners during optimisation of the problems in the test. The most common problem identified was the inability to comprehend problems. Learners expressed that this was due to language barriers in some of the questions. Another common difficulty experienced was the inability to link the given problem situations to optimisation. Learners made attempts to solve the problems without associating them to optimisation. The least experienced difficulty was the inability to interpret solutions after the problem has been solved. Most learners who managed to optimise the problems were also able to relate their solutions to problem contexts.

9. CONCLUSION

The results of this investigation revealed that learners lack productive thinking processes when solving optimisation problems using Calculus. The results of the study indicate that learners struggle to decode contexts of problems and represent them symbolically, formulate equations and functions and apply calculus methods to solve the problem and interpreting the solutions. Learners' shortcomings that emerged from this study could be explained in terms of underdeveloped concepts, spatial thinking and measurement processes. Learners struggled to make meaningful connections of measurement processes such area, volume and perimeter. Learners exhibited an underdeveloped capacity to read problems for meaning and a tendency to be led astray by the language of the questions. Diagrams were not frequently utilised to understand problem situations.The study also revealed that learners lack creative and critical thinking skills at various stages of problem-solving. Learners do not follow problem-solving stages, do not read questions with understanding and do not value diagramming as a strategy. Learners revealed that they struggle to decode problem contexts and language. The implications for the study are that teachers should engage learners in ways that promote teaching for problem solving transfer, critical thinking and productive thinking during problem solving. To be successful problem-solvers, learners invest time and energy in the analysis of the problem, identifying what is being asked, what are givens, what solution is most likely? Establishing the meaning of the problem should take precedence over the drawing up the plan of action. Teachers should therefore engage learners in ways that encourage learners to view mathematics as a way of thinking rather than as a means to find answers.

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