EXPLORING WORK ON EQUIVALENCE TO COMPARE AND SEQUENCING VARIED FORMS OF FRACTIONS

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ABSTRACT–This paper reports on the notion of retained knowledge of utilizing equivalence when comparing fractions-decimals-percentage (FDP). It is acknowledged that learning and teaching of fractions is an area of mathematics that both learners and teachers find particularly challenging and researchers proclaim that difficulties learners have with fractions are conceptual in nature. The aim of this study was to quantify how a cohort of students-participants understood and utilized these relationships in comparison exercises of decimal, percentage and fraction forms of quantities. All the student-participants (117) admitted into the Foundation Phase teacher training programme took the conceptual and procedural knowledge assessment in a 67 questions test with 93 items. The test items comprised of multiple choice, short answer and open calculations to show their own understanding and procedural steps towards a solution. Only the correct final answer was rewarded. The study reports on how this cohort of students performed when comparisons involved a) two, three, same, or different forms of FDP items; b) between two, three items, as well as c) when it was multiple choice question items. The identified misconceptions and associated errors need scaffolding by both the class cohort community and tutor teams with the aim of enabling the student teachers connect the equivalence and comparison of FDPs before the student teachers complete the programme.

Keywords: Fractions; errors; misconceptions, mathematics education.

1. INTRODUCTION

Researchers have established that learning and teaching of fractions is an area of mathematics that both learners and teachers find particularly challenging and complex (Ahmad, Salim, & Zainuddin, 2004; Reinup, 2010; Moss & Case, 1999; Pearn & Stephens, 2004, 2007; Clarke & Roche, 2009; Harvey, 2011, Pantziara & Philippou, 2012). This difficulty seems to generate misconceptions and associated errors in the fractions-decimals-percentages (FDP) relationship. Zhou (2011) asserts that difficulties learners have with fractions are conceptual in nature. Many learners appear to be depending on what and how they have been taught and both teacher and learners appear to have instrumental understanding (operational or procedural) of fractions without knowing why the procedures are used (Post, Harel, Behr, & Lesh, 1991). Reinup (2010) and Zhou (2011) call the combined and balanced ability to follow a procedure with understanding relative thinking and relational understanding which would have been more ideal.

This paper reports on how much relational understanding of both conceptual and procedural knowledge of fractions-decimals-percentages (FDP) the first year foundation phase teacher students have retained after they have completed their schooling career. The misconceptions and associated errors in the relational understanding between the FDP concepts is the focus of this report. The student teachers (as participants) are assumed to have accumulated knowledge concepts fractions-decimals-percentages (FDP) during their school time of about 12 years. The recollection of the participants' uninfluenced and raw knowledge of the concepts fractions, decimals and percentages was captured as early as possible in the year. Comparing, ordering, or sequencing of the FDP was captured as part of the 93 question items.

Under the content area Number, Operation and Relationships of Mathematics, the extension of the whole number to fractions is a concept taught and learnt in schools universally as shown in many international comparative assessments like the Trends in International Mathematics and Science Study (TIMSS). Dubinsky (1991), Gray and Tall (2007), Sfard (1991) as well as Baroody, Feil, & Johnson, (2007) asserted that students acquire new mathematical ideas in irregular ways, some mainly procedurally/operationally or for others mainly conceptually/structurally as two separate lanes of developments. A concept is something personally and individually conceived in the mind as an image (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981, Tall, 1995). Tall and Vinner (1981) further assert that the concepts on FDP develop schemes and relate them (figure 1). From these concepts of fractions, decimals and percentages the students invariably formed conceptual schemes or conceptual entities of what they managed to pull and put together from their learning opportunity (Greeno, 1983). The "saved" scheme could be the same as provided in the learning opportunity or very different as well as depending on how the previous related conceptual schemes have accommodated the new situation or concept. Kieren (1980) suggests that learning mathematics must be understood to be developmental in nature that involves formation of FDP schemes, mental structures and information processing capacities necessary to work with them individually and together at various levels of relationships.

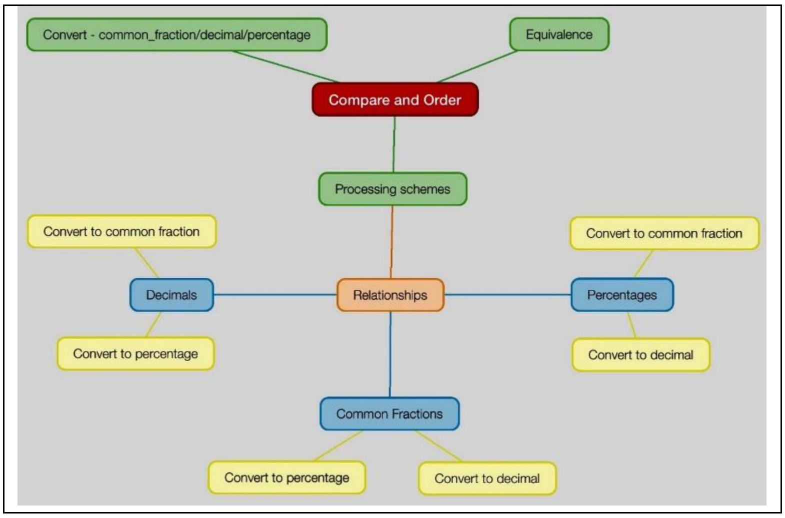

Figure 1: Conceptual schemes to process adapted from Charalambous, & Pitta-Pantazi, 2005, 2007

These schemes formed by students of what FDP could be sets of isolated conceptual entities or concept images and the opportunities to develop explicit links amongst them were not facilitated properly or enough (Maseko, 2015). Each of the three concept forms can be converted to the other two forms though with unequal difficulty as students struggle to work with fractions with understanding termed fluency (Brown & Quinn, 2006). Around 50% of the responses in converting one fraction form to the other to demonstrate the relational link between the concepts fractions, decimals and percentages were incorrect with 15% chose not to respond at all (Maseko, 2015). Some of the procedures and operations in common fractions do facilitate the transition to decimals and or percentages concepts and vice versa. Fractions, Decimals and Percentages are different ways of showing the same number value. A half is written as ½ (a fraction); 0.5 (a decimal) and 50% (a percentage). The complex schemes (Figure 1) that are related like fractions-decimals-percentages need an equally expansive tool to explain the student's capacity to work with progressively complex levels of reasoning process necessary to answer complicated question (Kieren, 1980).

An expansive tool like Structural Observation of Learning Outcomes (SOLO) offers levels of growth in understanding that could help facilitate how I can explain the process towards comparing and sequencing FDP (Figure 2). SOLO describes the processes involved in answering a question on a scale of increasing difficulty or complexity (Biggs & Collis, 1982; Collis & Romberg, 1992). The structures and levels of complexity are depicted clearest in this framework and explicit reference of how connections need to be identified, processed and then execute in many of the questions combining the three concepts and schemes.

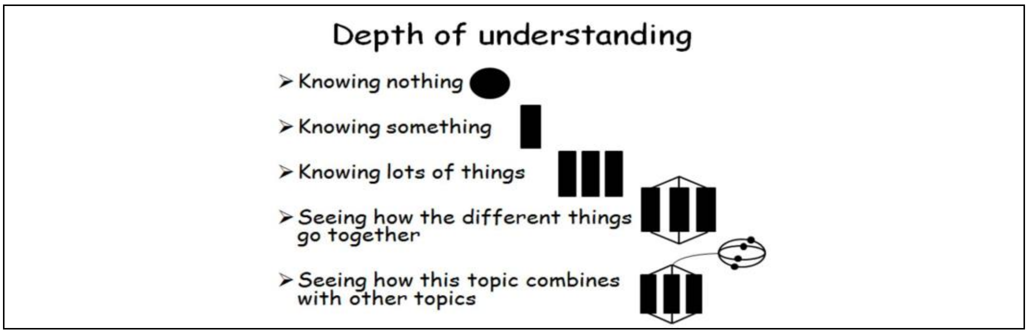

Figure 2: Structural Observation of Learning Outcomes (SOLO) - Biggs & Collis, 1982

The process of executing the compare or order or sort of a set of quantities that involve fractions, decimals and percentages needs equivalence as well as form of likeness. Equivalence of, and comparison between the decimal, percentage and fraction forms of quantities needs to be understood and mastered. Comparing and ordering the various forms of quantities demands higher skill of differentiation and linking between the FDP and level of seeing how the FDP connect and combine to create equivalent forms (Pantziara & Philippou, 2012). The SOLO framework provides an explicit visual tool to differentiate the levels. The aim of this study was to quantify how a cohort of students understood and utilised these relationships in comparison exercises of decimal, percentage and fraction forms of quantities. As Mathematics teachers in training, they need to be conversant and well-grounded with these conceptual connections to facilitate smoother transitions from being learners themselves to being teachers to others. This well-grounded transition will enable the student-teacher to seek appropriate teaching approaches to minimize unintended misconceptions that might arise in young foundation phase learners.

2. THE RESEARCH DESIGN AND METHODS

It was an exploratory investigation that utilized a quantitative approach in data collection method through a 67 questions questionnaire with 93 test items to establish a level of Foundation Phase Programme student's conceptual understanding of fractions-decimals-percentages (FDP). All these students have completed and passed their school exit (Grade 12) examination and qualified to study at a university. Participants in this investigation were the whole group of 2015 first year students in the Foundation Phase Programme. One hundred and seventeen (117) students-participants with 80% females and 20% males constituted the group who wrote the test. All first year (freshman) students do a mathematics course on fractions-decimals-percentages in the first year of the programme and were part of the study out of convenience. The aim was to collect their uninfluenced and raw knowledge of the concepts as early as possible in the year. These students were unaware that there would be a test and therefore did not attempt to prepare for it which would potentially affect the results thereof. The questionnaire worked as a formative assessment that allowed me to identify misconceptions and student weaknesses in FDP. When McMillan, Cohen, Abrams, Cauley, Pannozzo and Hearn (2010) investigated the effect of formative assessment on students, they defined formative assessment as assessment that embraces appropriate identification of student weaknesses, helps monitor progress, enables the provision of specific feedback, as well as it includes instructional correctives different from previous instruction. The instructional correctives apply to the selection of approach, activities and tasks I provide to deal with the identified misconceptions and associated errors in the FDP section of the Foundation Phase teacher development programme.

2.1 Design of questionnaire test items

The design of the questionnaire involved test items types that were multiple-choice items, short-answer as well as extended-response (free-response) items on fractions-decimals-percentage as utilized in Trends in International Mathematics and Science Study (TIMSS) assessments (TIMSS, 2012). These question items demanded students to demonstrate cognitive skills from knowing facts and procedures, using concepts appropriately, solving routine problems as well as to reasoning in the process of solving a problem as covered in the TIMSS assessments (Mullis & Martin, 2014, Martin, Mullis, & Hooper, 2016). For this report the students particularly worked on skills like transfer or convert between common fraction to either decimal or percentage form and vice versa to establish equivalence and then decide to respond to order or compare. Question items were gathered from other research reports like Ginther, Ng and Begle (1976), the experimental work of the Rational Number Project in Behr, Wachsmuth, Post, and Lesh (1984), Post, Wachsmuth, Lesh, and Behr (1985), Baturo (2004), Brown and Quinn (2006), to name a few and adapted the questions to fit the challenge to the first year university students. The challenge was to cover many competences in the mathematics' skills including describe, recognise, convert, compare and order fractions, decimals and percentages (Department of Education, 2011). After the students wrote the test, every script was assessed and marks recorded in a spreadsheet. Every response or lack of it (no response) was recorded and the initial analyses of the scripts are discussed in the next sections.

2.2 Overview analysis of selected collected data

The students' responses to the questionnaire tasks revealed a set of common misconceptions and associated errors on working with comparing fractions-decimals-percentages. Students used a portion of the 80 minutes to complete this part of the test. The students' responses were collected and coded in two ways: as dichotomous items (correct [**Cor**] or incorrect [**Incor**]) as used by Gould (2005). Every test item was examined and a record was kept in the form of:

how many students-participants attempted each question item,

how many students-participants answered correctly (**Cor**),

how many students-participants had chosen a wrong answer (**Incor**) (where applicable), and,

what was the performance in the particular categories/group

The selections of items to be included in the category of compare or order the quantities were informed by instructions to be followed in each case like arrange the given terms in ascending order, descending order, or compare two quantities that may be same or different forms.

3. DISCUSSION OF RESPONSES

A brief discussion of the responses to these test items shows a summary of the breakdown of the responses or lack thereof with correct (Cor), incorrect (Incor) as well as no response (NR). This report covers test items dealing with comparing and sequencing fractions of same forms as well as mixed forms. Responses to each question is briefly analysed and examples of common errors, unique errors as well as underlying misconceptions are discussed where possible. The purpose of the discussion is to provide quick detection of specific error types, which served as a guideline for remedial-teaching. To facilitate reading, the subsection numbering will correspond and signify the category under discussion.

3.1 Comparing and sequencing fractions: same forms of fractions (39%:58%:3%)

The category 3.1 dealt with comparing values of the same common fractions form. The category had three question items and compared only two quantities each time.

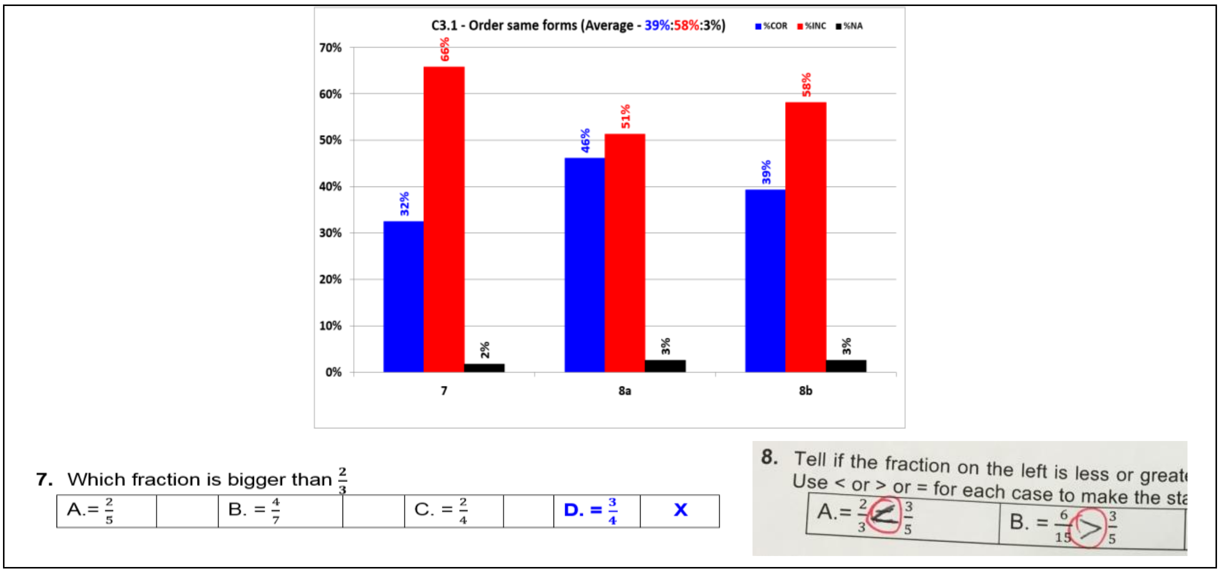

Figure 3: Comparing and sequencing fractions: same forms of fractions

This group had three items 7, 8A and 8B that the students-participants had to respond to. Almost all the students-participants (3%) responded to these questions. Thirty nine percent (39%) of the 117 students-participants provided the correct answers. The rest (58%) made incorrect choices of the sign to make the each statement true in the comparisons. Individually the question items in this category have similar results in responses. The correct entries were 32%, 46% and 39% for items 7, 8A and 8B respectively. All three question items demanded them to recall equivalence of fractions to facilitate the decision-making process. All the options in question item 7 had their denominator bigger than that of the fraction to compare against. Alternatively, the question demanded that the participant knew the real position of the fraction on a number line.

The incorrect options chosen and resulting in a distribution of 13% for A, 28% for B and C had 26% of the participants. No clear pattern could be deduced from these choices until a face-to-face interview is done where this particular student would share some light. In the case of items 8A and 8B the spread of the incorrect options was 49% for A and 3% for C in 8A. Similarly 8B had 49% for B implying $\frac{6}{15} > \frac{3}{5}$ and 9% for C implying $\frac{6}{15} = \frac{3}{5}$ which were both incorrect. It is possible, though, to suppose that “the whole number bias” was at play in all three question items because in both cases the numerators and denominators were “bigger” to make their responses correct for whole numbers (DeWolf & Vosniadou, 2015).

3.2 Comparing and sequencing fractions: mixed forms (28%:61%:11%)

The category 3.2 dealt with mixed forms where comparing common fractions, decimals as well as percentage. This category demanded a higher level of cognitive processing of these packages (schemes) of knowledge and appeal to relationships and bridges between and among the three forms. We will discuss each question item in the following section.

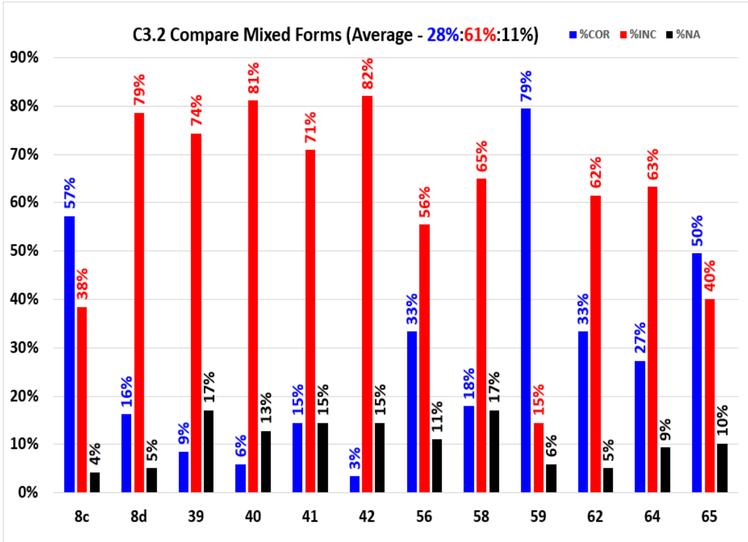

Figure 4: Comparing and sequencing fractions: mixed forms of fractions

This category had twelve items including 8C and 8D that the students-participants had to respond to. On average only 28% of the 117 participants got the correct answer whilst eleven percent (11%) of these participants did not respond to these questions. The rest (61%) provided incorrect responses in these mixed forms fractions comparisons. This category demanded that the student retrieved a conceptual scheme of converting to one form to another (fraction, decimal or percentage) before making a correct comparison. When one of the conceptual schemes is faulty or insufficiently developed or totally not retrieved, an error is due to happen.

8C	8D
A<, B>, C=	

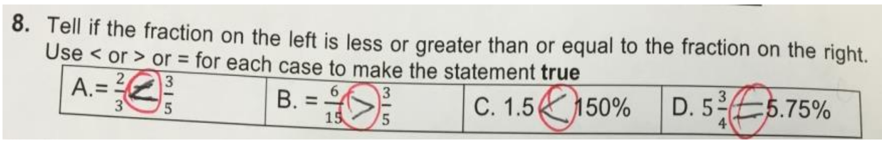

Figure 5: Question 8 which was about the comparison between two quantities

Item 8C mixed decimal and percentage forms to construct the question. This particular participant apparently used the values 1.5 versus 150 to decide on the sign to make the statement “true”. The conceptual error seemingly made was not observing the need to convert one to the other form before deciding on the sign to relate them. The face appearance of the quantities seems to have been the influence staying the whole number bias of 150 is bigger than 1.5 (Ni & Zhou, 2005).

Again for item 8D the numbers utilised in forming the question item wanted the participant to go to basics and avoid guessing a response. 8D is a typical example of two values so similar until the participant observes the percentage sign. This particular student in this example provided the exact solicited response. The resulting distribution on these two items was 57% correct for item 8C versus a low 16% correct for 8D. 38% created an incorrect statement for item 8C versus more than double (79%) for item 8D. Both question items 8C and 8D received a good size of responses as only 4% and 5% were left blank respectively.

The next group of question items had four quantities to work with in the comparison exercises. The student needed to make decisions like converting to which form, or why that would be easier, which schemes to draw from, even make the choice of the most appropriate sequence or order to respond to the question at hand (see Figure 6). Each question demanded this multiple processing to facilitate finding the solution within the SOLO framework levels as well.

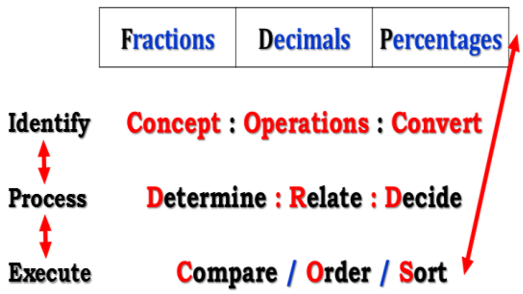

Figure 6: Conceptual schemes to process

Each question item will be discussed individually. Question item 39 will be the first to discuss.

39. Order from **least to greatest**: **A:** 13.6%, **B:** 20.25, **C:** 316%, **D:** $30\frac{1}{5}$	13.6%; 316%; 20.25; $30\frac{1}{5}$ **(ACBD)**

The combination of the sequence in Question 39 is too large (24) but the correct permutation is supplied. The route used in the process was to convert the decimal fractions to percentage as two quantities were in that form already to save time. Only 9% of the participants offered the correct order with almost three-quarters (74%) providing incorrect sequences. It is equally interesting to note that 13.6% was the most chosen as the smallest (31%) whilst others took $30\frac{1}{5}$ (27%), 20.25 (14%) and 316% (11%) as their smallest.

40. Order from **greatest to least**: **A:** 0.9, **B:** $\frac{4}{5}$, **C:** 81%, **D:** $1\frac{1}{5}$	$1\frac{1}{5}$; 0.9, 81%, $\frac{4}{5}$ **(DACB)**

For this question only 6% of the participants offered the correct order with 81% who provided incorrect permutations. Again the possible combination of the sequence is large (24) but every participant must adopt a particular strategy within the limited time. A practically quicker route was to convert all quantities to percentages because of the 81% that could delay the response if converted to common or decimal fractions. This decision-making process is possible for well-developed schemes of knowledge bases like converting to percent form, simplifying fractions, meaning of percentage notation beyond one hundred, etc. For this question only 15% chose $\frac{4}{5}$ as the biggest whilst a majority (50%) took 81% as their greatest. Apparently the whole number bias misconception of 81 overpowering all the other quantities as if there is no percentage sign that reduces to a quantity close to all the others in the group (Ni & Zhou, 2005).

41. Put in **descending** order: **A:** 33.3%, **B:** 0.33, **C:** 0.30, **D:** $33\frac{1}{3}$	$33\frac{1}{3}$; 33.3%; 0.33; 0.30 **(DABC)**

Question 41 plays around the place value of 3 in the two decimals, percentage and fraction forms. The teaser was between options A and D for the participant who is leaning towards the whole number bias behaviour. Secondly B and C were testing the participant's alertness around place value after the decimal point and place on a number line. For this question only 15% of the participants offered the correct order with 71% who provided incorrect permutations. Again a practically quicker route was to convert all quantities to percentages because of the 33.3% that could delay the response if converted to common or decimal fractions.

42. Put in **ascending** order: A: 1.5, B: $\frac{1}{5}$, C: 21%, D: $20\frac{1}{5}$	$\frac{1}{5}$; 21%; 1.5; $20\frac{1}{5}$ (BCAD)

Question 42 plays around the positions of 1 and 5 mainly in the two fractions, percentage and decimal forms. The teaser was between options A and B for the participant who has problems with the whole number bias behaviour. Secondly C and D were testing the participant's alertness around meaning of the percentage and fraction forms. For this question a very, very 3% of the participants offered the correct order with more than four fifths (82%) proving incorrect permutations. In my opinion a practically quicker route was to convert all quantities to percentages because of the 21% that could delay the response if converted to common or decimal fractions.

56. Which statement is true?			
A: $0.37 > 59\%$	B: $75\% > 0.8$	C: $59\% < 0.53$	D: $0.095 > 5.9\%$ X

This question needed the participant to test each case to make choice. Of the 117 participants 33% offered the correct response when 11% offered none. Most of them chose B(37%) while others chose C(13%) and A(6%) which were incorrect options in this question. The forms involved in the cognitive demand are all three with two taken at a time. The participant needed to decide between each pair and determine which of the forms is easier to convert to (relate) and then decide on the response to the question. In each case the student would have chosen the route of converting every decimal fraction to percentage form by multiplying by 100. The process of elimination would help finding the eventual true statement because 37% is not > 59%, 75% is not > 80%, 59% is not < 53% but 9.5% > 5.9%.

58. Which makes this a true statement $5\frac{4}{5} <$ _____			
A: $5\frac{3}{4}$	B: $\frac{50}{10}$	C: 5.8	D: 590% X

Working on this question demanded the participant to test each option across the three options in the consideration. The question demanded a cognitive level involving all three forms with two taken at a time. The participant needed to decide between each pair and decide which of the forms is easier to convert to (relate) and then decide on the response to the question. In this case the student would have battled to choose a common form though a route to convert every case to decimal form and would facilitate a quicker decision-making in finding the correct response against 5.8. The process of elimination would highlight that 5.75 is not > 5.8, 5.00 is not > 5.8, 5.8 is not > 5.8 but 5.9 > 5.8. The breakdown of responses yielded a very flat distribution in the form of 18% got the correct answer D when 17% of the participants offered no response. The spread of the incorrect choices were A(23%), B(20%) and D(22%).

59. Which statement is **true**?	
A: $12\% = 0.12 = \frac{12}{100}$ X	B: $12\% = 0.12 = \frac{1}{12}$
C: $12\% = 1.20 = \frac{12}{100}$	D: $12\% = 1.20 = \frac{1}{12}$

This question had supplied options but needed working out to facilitate eliminating the false statements from the list. All the options were laced with traps from common errors and misconception when conversions are executed between all three forms in questions. With 12 as the main number and had the influence of the place value in many disguised forms of the fractions-decimals-percentage.

Converting 12% to decimal yields 0.12 and not 1.20 but $\frac{1}{12} = 0.083$ whilst converting percentage to fraction form yielded $12\% = \frac{12}{100}$. Of the 117 participants 79% offered the correct answer. The rest, 15% and 6%, were the incorrect and no response respectively.

62. Which statement is **false**?			
A: $0.7\% = 0.7$ X	B: $0.07\% = 0.0007$	C: $7\% = 0.07$	D: $700\% = 7$

This question focused on the ability of the participant's working with converting decimals to percentages as well as comparing the resultant quantities. All the options are laced with traps from common errors to soliciting misconceptions when conversions are executed between all three forms in questions around the number 7 as the decimal point is shifted in position. The influence of the place value of 7 disguises the value possibilities between the three forms. The false statement (A) was where the incorrect conversion effects no change in the decimal point offered by 33% of the participants. The rest (62%) chose B(13%), C(11%) as well as D(38%) when 5% offered no answer.

64. Which of the following is in **ascending** order?	
A: 0.8; $\frac{2}{3}$; 0.67; 70%	B: $\frac{2}{3}$; 0.67; 70%; 0.8 X
C: 70%; $\frac{2}{3}$; 0.67; 0.8	D: 0.67; 0.8; $\frac{2}{3}$; 70%;

In this question decimals, fraction and percentage were compared and then arranged in ascending order. The item tests the misconception resulting from number of digits after the decimal point (0.67 versus 0.8). When that is clarified the meaning of 0.67 and two-thirds would be the next challenge. The magnitude 70% would be in the mix as well in the decision to sort the four values in an ascending order. Only 27% managed to choose the correct option B when 9% did not offer any response to the problem. Almost two-thirds (63%) chose the incorrect options in this spread A(21%), C(17%) and D(25%). It could be as a result of a combination of the pitfalls as explained by a maze of possibilities a participant needs to consider before arriving at a decision.

65. Which of the following is in **descending** order?	
A: $\frac{3}{4}$; 85%; 0.71; $\frac{4}{5}$	B: 85%; $\frac{3}{4}$; 0.71; $\frac{4}{5}$
C: 85%; $\frac{4}{5}$; $\frac{3}{4}$; 0.71 X	D: 0.71; 85%; $\frac{4}{5}$; $\frac{3}{4}$

In this question 50% managed to choose the correct option C when 10% did not offer any response to the problem. Around two-fifths (40%) chose the incorrect options in this spread A(5%), B(11%) and D(24%). Working with decimals, fraction and percentage in this question were compared and then arranged in a descending order. For a participant who converted everything to decimal fractions, the item tested the misconception resulting from the number of digits after the decimal point (0.85 versus 0.8 as well as 0.71 against 0.75) as mentioned by . When the values were converted to percentage form, the results would yield “clearer and simpler” numbers to compare (85%, 80%, 75% and 71%). It is safe to conclude that the correct answer was easier to get and possibly from participants who worked with percentages to avoid the pitfalls of decimals common error. It could be as a result of a combination of the pitfalls as explained by a maze of possibilities a participant needs to consider before arriving at a decision.

This set of question items demanded a higher level of problem processing involving identifying the form of fraction, decided which way to take in search of the solution towards executing the instruction. When two terms are compared the errors depend on which forms are involved. When all three forms are compared these group of questions demanded converting to either form to facilitate the decision-making to respond to an item. Every student-participant worked out the solutions to arrive at the responses in this group of questions.

4. CONCLUSION

The task in the questionnaire revealed a set of common misconceptions and associated errors displayed by the students-participants that seems to be mainly whole number bias in many forms (Ni & Zhou, 2005). The face value of number symbols seemed to have drawn the attention of the students taking the test. Secondly the decimals quantity value misconceptions played a role in the comparisons of two or quantities. The shorter-is-larger and the longer-is-larger misconceptions were observed as in item 64 to arrange the quantities (Stacey & Steinle, 1998, DeWolf, Bassok, & Holyoak, 2015). This reports on how the students-participants performed in question items where they were comparing fractions in same, different or mixed forms. The fractions in mixed form refer to a combination of more than one in the fractions-decimals-percentage. There were between two and four quantities to be compared. There were three levels of decision making in this process of comparison. Firstly decide which form to convert to, secondly, the conversion process of the concerned quantities to the same form and lastly, decide on equivalence. The student-participants needed to know how to convert between all three fraction forms. The equivalent fraction form allowed the comparison when the format is the same. They still needed to observe that this section wanted them to connect the three forms of fractions as well know which form works well in a particular question to avoid falling into the many traps called misconceptions and errors. Evidently most student-participants showed misconceptions and errors around this connection from a lot "whole number bias" connected errors. The foundation phase programme will have to make a plan to close these gaps. The role of the teacher is to guide the learner towards understanding the concepts and mastering the associated procedural skills (Pantziara & Philippou, 2012).

REFERENCES

Ahmad, A., Salim, S. S., & Zainuddin, R. (2004). A Study On Students' Performance In Solving Word Problems Involving Fractions: Towards The Development Of a Cognitive Tool. In Proceedings of International Conference of Computers in Education (ICCE), Melbourne, Australia.

Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. Journal for Research in Mathematics Education, 115–131.

Baturo, A. R. (2004). Empowering Andrea to help year-5 students construct fraction understanding. In *Proceedings of the 28th Conference of the International* (Vol. 2, pp. 95–102).

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, 323–341.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, 323–341.

Biggs, J. B., & Collis, K. F. (1982). Evaluating the quality of learning. *New York*.

Brown, G., & Quinn, R. J. (2006). Algebra students' difficulty with fractions: An error analysis. Australian Mathematics Teacher, 62(4), 28–40.

Charalambous, C. Y., & Pitta-Pantazi, D. (2005). Revisiting a theoretical model on fractions: Implications for teaching and research. International Group for the Psychology of Mathematics Education, 233.

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. Educational Studies in Mathematics, 64(3), 293–316.

Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. Educational Studies in Mathematics, 72(1), 127–138.

Collis, K. F., & Romberg, T. A. (1992). Collis-Romberg Mathematical Problem Solving Profiles.

DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction, 37*, 39-49

DeWolf, M., Bassok, M., & Holyoak, K. J. (2015). From rational numbers to algebra: Separable contributions of decimal magnitude and relational understanding of fractions. *Journal of Experimental Child Psychology*, *133*, 72–84.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In Advanced mathematical thinking (pp. 95–126). Springer.

Ginther, J., Ng, K., & Begle, E. G. (1976). A survey of student achievement with fractions. SMESG Working Paper No. 20

Gould, P. (2005). Year 6 students' methods of comparing the size of fractions. In *Proceedings of the Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 393–400). Citeseer.

Gray, E., & Tall, D. (2007). Abstraction as a natural process of mental compression. Mathematics Education Research Journal, 19(2), 23–40.

Greeno, G. J. (1983). Conceptual entities. In Genter & Stevens (Eds.), Mental Models, 227–252.

Harvey, R. (2011). Challenging and extending a student teacher's concepts of fractions using an elastic strip. Mathematics: Traditions and [new] Practices, 333–339.

Kieren, T. E. (1980). *Five faces of mathematical knowledge building*. Department of Secondary Education, University of Alberta.

Martin, M.O., Mullis, I.V.S., and Hooper, M. (Eds.). (2016). *Methods and Procedures in TIMSS 2015*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Maseko, J. (2015). First year students in the foundation phase programme's capacity to work with fractions – what do they display? In *Towards Effective Teaching and Meaningful Learning in Mathematics, Science and Technology* (pp. 225–235). Mopani Camp in Kruger National Park, Limpopo, South Africa: UNISA.

McMillan, J. H., Cohen, J., Abrams, L., Cauley, K., Pannozzo, G., & Hearn, J. (2010). Understanding Secondary Teachers' Formative Assessment Practices and Their Relationship to Student Motivation. *Online Submission*.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. Journal for Research in Mathematics Education, 122–147.

Mullis, I. V., & Martin, M. O. (2014). TIMMS Advanced 2015 Assessment Frameworks. International Association for the Evaluation of Educational Achievement.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. Educational Psychologist, 40(1), 27–52.

Pantziara, M., & Philippou, G. (2012). Levels of students'"conception" of fractions. Educational Studies in Mathematics, 79(1), 61–83.

Pearn, C., & Stephens, M. (2004). Why do you have to probe to discover what Year 8 students really think about fractions. In Mathematics education for the third millennium: Towards 2010. Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia. Townsville (pp. 27–30).

Pearn, C., & Stephens, M. (2007). Whole number knowledge and number lines help develop fraction concepts. Mathematics: Essential Research, Essential Practice, 2, 601–610.

Post, T. R., Harel, G., Behr, M., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. Integrating Research on Teaching and Learning Mathematics, 177–198.

Reinup, R. (2010). Teaching Number Line, Fractions, Decimals and Percentages as an Integrated System. Didactics as Design Science, 71.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22(1), 1–36.

Stacey, K., & Steinle, V. (1999). Understanding decimals: The path to expertise. In *Making the difference: Proceedings of the 22 nd annual conference of the Mathematics Education Research Group of Australasia Incorporated* (pp. 446-453).

Star, J. R. (2005). Reconceptualizing procedural knowledge. Journal for Research in Mathematics Education, 36(5), 404–411.

Tall, D. (1995). Cognitive growth in elementary and advanced mathematical thinking. In *PME conference* (Vol. 1, pp. 1–61). THE PROGRAM COMMITTEE OF THE 18TH PME CONFERENCE.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational Studies in Mathematics, 12(2), 151–169.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In *Proceedings of the fourth international conference for the psychology of mathematics education* (pp. 177–184).

Zhou, Z. (2011). Assessment and Intervention: The Case of Fractions. The Oxford Handbook of School Psychology, 351.