GRADE 9 LEARNERS' CONCEPTIONS OF ALGEBRAIC EXPRESSION: A CASE STUDY OF ONE OF THE SCHOOLS IN THE MT AYLIFF DISTRICT

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ABSTRACT-This study involved Grades 8 and 9 in the Eastern Cape Province in the Mt Ayliff District. This paper reports on 90 learners' understanding and interpretation of the concept of an algebraic expression. The study followed a qualitative approach adopting a case study design involving one secondary school. Out of 90 learners who wrote a test on the concept of *algebraic expression*, 10 were sampled for interviews. Findings revealed that Grade 9 learners did not understand what "simplify" meant. They also did not understand that a simplification of an algebraic expression meant to construct an algebraic identity of two equivalent expressions; that is, although two algebraic expressions are different they can yield the same value for a variable, say x. Learners did not understand that a variable represented an arbitrary or any unspecified number, and that one could therefore illustrate the validity of the identity with any particular number or set of numbers. The difficulty learners had experienced in understanding an algebraic expression could be attributed to the lack of understanding the relationship between arithmetic and algebra. We recommend that the concept of a variable be introduced as early as primary school level so that learners might understand the connection between arithmetic and algebra.

Keywords: Algebraic expression, algebra, arithmetic, equivalent.

1. INTRODUCTION

There are many studies, which have looked at how learners solve or deal with algebraic problems, and serious attempts have been made to improve students' preparation for algebra (Herscovics & Kieran 1980; Jones, Inglis, Gilmore & Dowens, 2012; Linchhevski & Herscovics, 1996; Mosses & Cobb, 2001; Wu, 2001). Most of these studies looked at how learners moved from arithmetic to algebra, as well as their understanding and interpretation of an equal sign in arithmetic and an algebraic equation. Much has not been done, particularly in the South African context, in the area of equivalent algebraic expressions. Hence, the purpose of this paper is to investigate Grade 9 learners' understanding and interpretation of the concept of an algebraic expression. This study was guided by the following research questions:

How do Grade 9 learners understand the concept of simplification in an algebraic expression?

How do Grade 9 learners understand the concepts of factorisation and equivalent expression?

The study emerges from a project, which focuses on meaningful teaching and learning of mathematical concepts for Grades 8 and 9 learners in the Eastern Cape Province in the Mt Ayliff District. This paper reports on 90 Grade 9 learners' understanding and interpretation of the concept of an algebraic expression. The study has been conducted from a qualitative approach, adopting a case study approach, which focuses on one secondary school in the Mt Ayliff District. Out of 90 learners who wrote a test on the concept of *algebraic expression*, 10 were chosen for an interview.
The study revealed that Grade 9 learners in this school did not understand what "simplify" meant. They also did not understand that a simplification of an algebraic expression meant to construct an algebraic identity of two equivalent algebraic expressions; that is, although two algebraic expressions are different, they can yield the same value for the same value of a variable, say x. Learners in this grade did not understand that a variable represented an arbitrary or any unspecified number, and that one could therefore illustrate the validity of the identity with any particular number or set of numbers.

Furthermore, learners seemed to be confusing the concept of an identity in an algebraic expression with the concept of an equation. Again, learners had difficulty accepting an algebraic expression as an answer; they wanted to see an answer as a specific number. The difficulty learners had experienced in understanding an algebraic expression could be attributed to the lack of understanding the relationship between arithmetic and algebra. In other words, learners could not see algebra as generalized arithmetic.

2. HISTORY OF ALGEBRA

Many researchers have indicated that there are three stages in the development of algebra, namely rhetorical, syncopated and symbolic algebra. Rhetorical algebra is characterised by a verbal description. In rhetorical algebra, no symbols are used at all. Syncopated algebra is characterised by the use of some abbreviations for the frequently recurring quantities and operations. Symbolic algebra, which is the algebra we use today, is characterised by letters. Sfard's (1991) theory is that the historical development of algebra from rhetorical to symbolic must be reproduced in the individual to achieve the understanding of algebra. Algebra is a language through which mathematicians express relationships of which the numerals and symbols look like a shorthand version of words (Wu, 2001). Wheeler (1996) defines algebra as a symbolic and representational system because a symbol stands for an object or a situation. Wheeler (1996) further indicates that algebra is essentially about generalisation, functions, problem-solving and modelling. Some of the researchers have supported the view that the use of letters to represent numbers is a principal characteristic of algebra (for examples, see, Usiskin, 1998; Didis & Erbas, 2015).

2.1. Arithmetic to algebra

Algebraic thinking – algebraic reasoning – begins in preschool as young learners "represent addition and subtraction with objects, fingers, mental images, drawings, sound, acting out situations, verbal explanation, expressions or equations" (common Core State Standards Organization 2010:11). In Secondary school, when learners start studying algebra in more abstract and symbolic ways, focusing on the understanding of variables, expressions and equations. Algebraic thinking is present across content areas. It is central in mathematical reasoning, as can be seen from the Standards for Mathematical Practice of Common Core State Standards (2010). For example, it is stated that:

Algebra is a useful tool for generalising arithmetic and representing patterns in our world. Explaining the regularities and consistencies across many problems gives students the chance to generalise (Mathematical Practice 8);
The method we use to compute structures in our number system can and should be generalised. For example, the generalisation that $a+b=b+a$ tells us that $83+27=27+83$ without the need to compute the sum on each side of the equal sign (Mathematical Practice 7);
Symbols, especially those involving equality and variables, must be understood well conceptually for students to be successful in mathematics (Mathematical Practice 6); and,
The understanding of functions is strengthened when they are explored across representations (e.g., equations, tables and graphs). Each mathematical model provides a different view of the same relationship (Mathematical Practice 4).

Similarly, CAPS (2011, p. 6) for Grades 7–9 defines algebra as the language for investigating and communicating most of mathematics, and can be extended to the study of function and other relationships between variables. It focuses on the description of patterns and relationships through the use of symbolic expressions, graphs and tables, and the identification and analysis of regularities and change in patterns and relationships that enable learners to make predictions and solve problems. In all the definitions of algebra, the one most commonly used is that algebra should be seen as generalised arithmetic (Usiskin, 1996; Wheeler; 1996; Wu, 2001). For example, $(2+3)+5=5+(3+2)$. When we see that something similar happens in the example, we start recognising a pattern (Jones et al., 2012). After working with more examples, learners become sure that a pattern always works. When they add three numbers, they can add the first two and then add the third or add the last two numbers and then add the first number. They will be able to generalise this pattern by saying the order in which they add three numbers does not affect the answer. At a later stage, the generalisation can be represented using symbols, such as *(a+b)+c=b+(a+c)* instead of words. For example, "three plus two equals five" will be written as 3 + 2 = 5. The symbols and the words become difficult when mathematicians start using letters to represent a range of numbers as a specific number. For example, "two multiplied by itself is called two squared", which is written as *2x2=2* squared (*axa=a squared*). If learners experience difficulty in seeing the connection between arithmetic and algebra, it will be difficult for them to understand the notion of algebraic expressions (Matthews, Rittle-Johnson, Taylor & McEldoon, 2012).

2.2. Algebraic expressions

Most learners seem to experience problems dealing with algebraic expressions. An algebraic expression can contain numerals or variables or a combination of number and variable operation signs. Sfard (1991) argues that, for learners to have an understanding of algebra and algebraic expressions in particular, they should be familiarised with the historical development of mathematics, and algebraic expressions in particular from operational conception to structural conception. It is further supported by Sfard and Linchevski (1994), saying all mathematical conceptions are endowed with a "process-object duality". For example, an expression, *3(x+5)+1*, should be first understood as a computational process: add 5 to the number at hand, multiply the result by 3 and add 1. Secondly, it should be understood as a structure – an object whereby 5 has been added to a number of which the result has been multiplied by 3 and 1 has been added *[3(x+5)+1]*. This should be seen as an object, a whole, a thing, an answer. The fact that most learners could not understand an expression *[3(x+5)+1]* as an answer, is because they did not realise that an action has been done on it. Kieran (2007) indicates that one of the challenges why learners struggle with algebra, and algebraic expression in particular, is that symbolic algebra is imposed on learners without taking them through stages of rhetorical and syncopated to symbolic algebra. There has been a huge jump from rhetorical to symbolic algebra.

2.3. Transforming to equivalent expressions

Van de Walle, Karp and Bay-Williams (2014) mention that simplifying equations and solving for x have been meaningless tasks for many leaners. Learners always ask why they have to know what x is, and why and how they should solve it. Van de Walle et al. (2014) argue that algebraic expressions and equations have to be taught meaningfully. Knowing how to simplify and recognise algebraic expressions are the most important skills in algebraic thinking. Van de Walle et al. (2014) indicate that learners are often confused about what the instruction *simplify* means. Most learners have no ideas that simplify means to change an original expression into an easier one. *Simplify* also means to construct an algebraic identity; for example, *x+x=2x*, with *x+x* and *2x* being two equivalent algebraic expressions that, although different, nevertheless yield the same value for the same value of *x* (Sasman, Linchevski & Olivier, 1999) Algebraic manipulation involves equivalent transformation; for example, *3(x+2)* and *3x+3x2* represent two different methods. In a particular context, we may choose any of the two because they are identical in the sense that they yield the same value of *x*. The notion of equivalent algebraic expressions is developed to give learners an opportunity to appreciate the significance and usefulness of constructing equivalent expressions. In this study, we want to find out how Grade 9 learners understand the concept of simplification in an algebraic expression, and how they understand the concepts of factorisation and *equivalent expression*s.

3. THEORETICAL FRAMEWORK

The work done by Sfard (1991) on the theoretical framework of procedural and structural conceptions has been useful in explaining how learners understand and interpret the notion of an equal sign, and how learners transit from an arithmetic identity to an algebraic equation. Sfard (1991) indicates that any analysis of different mathematical definitions and representations brings us to the conclusion that abstract notions, such as equal signs, a number or function, can be conceived in two fundamentally different ways: structurally as an object, and operationally as processes (p. 1). She argues that the two approaches, although ostensible incomplete, are in fact complimentary. In some instances, especially in textbooks, a concept could be defined as if the mathematical notion referred to is an object (Sfard, 1991). This is a structural definition. Seeing a mathematical concept as an object means being capable of referring to it as if it were a real thing; a static structure, existing somewhere in space and time, recognising an idea at a glance and manipulating it as a whole without going into details. The same concept can also be defined in terms of its computational process, algorithms and actions rather than being an object. This reflects an operational conception of a notion. The operational conception is dynamic, sequential and detailed.

There are three stages that characterise the development of mathematical understanding in any area of mathematics (Sfard, 1991), namely interiorisation, condensation and reification. In interiorisation, some process is performed on familiar objects. At this stage, a learner becomes acquainted with processes, which will eventually give rise to a new concept like counting, which leads to natural numbers; algebraic manipulation, which turns to function; arithmetic equations, which lead to algebraic equations; verbal rhetorical expressions, which are effectively manipulated, and manipulations, which are described in prose. In condensation, the process is refined and made more manageable, as in syncopated algebra. It is a period of squeezing lengthy sequences of operations into more manageable units. At this stage, a person becomes more and more capable of thinking about a given process as a whole without feeling an urge to go into detail (Sfard, 1991, p. 19). This is the point at which a new concept is officially born. A progress in condensation will manifest itself also in growing easiness to alternate between different representations of the concept.

Reification is an act of turning computational operations into permanent object-like entities. In the reification stage, learners must move from an operational or computational orientation – for example, seeing $x+2$ as the process, whereby 2 is added to the number x to a structural orientation – for example, seeing $x+2$ as an object, a whole, a thing, an answer (Sfard, 1991). Sfard (1991) defines *reification* as an ontological shift – a sudden ability to see something familiar in a totally new light. "It is a instantaneous quantum leap – a process solidified into an object, into a static structure. Various representations of the concept become semantically unified by this abstract, purely imaginary construct, which is a new entity, which soon detaches from the process, which produced it and begins to draw its meaning from the fact of it being a member of a certain category" (Sfard 1991, p. 20). Linchevski, Olivier, Sasman and Liebenberg (1998) argue that the transition from an operational approach to a structural approach is not so much in the use of letters instead of numbers. It rather lies in the ability to perceive a process; for example, "add two to a number" as a new entity or object and the ability to operate on this object – this is referred to as *reification*. Because it is difficult to operate or manipulate these objects when expressed verbally, symbolism has been introduced to obtain structural thinking. Learners who are not able to think structurally will view $x + 2$ simply as a computational process, which results in viewing $2x$ as an answer. This is an indication that these learners have not achieved a reification stage.
In this paper, we argue that the difficulty experienced by learners in dealing with algebraic expressions is attributed to the lack of historical development of algebra from the operational to the structural conceptions. Hence, in this study, we were interested in investigating how learners interpret the notion of an equal sign. Sfard (1991) provides a summary of operational and structural conceptions of any notion in mathematics in the form of a table as indicated in Table 1.

Table 1: Summary of operational and structural conceptions

	Operational conception	**Structural conception**
General characteristics	A mathematical entity is conceived as a product of a certain process or is identified	A mathematical entity is conceived as a static structure as if it were a real object

	with the process itself	
Internal representation	Is supported by verbal representation	Is supported by visual imagery
Its place in concept development	Develops at first stages of concept formation	Evolves from the operational conception
Its role in the cognitive process	Is necessary, but not sufficient, for effective problem-solving and learning	Facilitates all cognitive processes (learning, problem-solving)

4. METHODOLOGY

The study emerges from a project focusing on mmeaningful teaching and learning of mathematical concepts for Grade 8 and 9 learners in the Eastern Cape Province in the Mt Ayliff District. (15) Schools in the Mt Ayliff District in the Eastern Province have been chosen based on their performance in the Annual National Assessment. The Project is divided into two phases. Firstly, a pre-diagnostic test is administered to determine in which mathematical concepts learners did not perform well. Secondly, after the results have been analysed, teachers are trained on the problematic mathematical concepts. This paper reports on 90 Grade 9 learners' understanding and interpretation of the concept of an algebraic expression in one school of the Mt Ayliff District in the Eastern Cape Province. The study has been conducted from a qualitative approach, adopting a case study approach, which focuses on one secondary school in the Mt Ayliff District. Out of 90 learners who wrote a test on the concept of algebraic expression, 10 learners are chosen for an interview. Learners were provided with an algebraic expression pre-test, as indicated below.

The purpose of the test was to explore learners' understanding of what an equivalent expression was. For example, in the first question, learners had to understand what *simplification* meant. If learners did not know that simplification meant to construct an algebraic identify that, although being different, nevertheless yielded the same value for the same value of *x*, they might also fail to find the solution to the problem. If a learner failed to find an equivalent expression for the first question, the probability of not solving questions 2 to 5 would be very high, because questions 2 to 5 are linked to question 1. In questions 2 to 4, learners were provided with a value of x as a decimal number to be substituted in the simplified expressions of questions 1 to 5. The reason was that learners were compelled to substitute it with a simplified expression that a complicated given expression. In question 6 learners were further compelled to simplify a given expression through factorisation before it could be substituted. This question assessed learners' knowledge of factorisation as another strategy for simplification.

TEST

Answer the following questions on the answer sheet. DO NOT USE A CALCULATOR.

1. Simplify the expression $2x(5x+6)-5x(2x+1)+3x$
2. What is the value of $2x(5x+6)-5x(2x+1)+3x$ if $x=24{,}3$?
3. What is the value of $2x(5x+6)-5x(2x+1)+3x$ if $x=7{,}6$?
4. For what value of *x* is $2x(5x+6)-5x(2x+1)+3x$ equal to 18,5?
5. For what value of *x* is $2x(5x+6)-5x(2x+1)+3x$ equal to 60?
6. How much is $\frac{x^2+5x+6}{x+3}$

6.1 if $x = 28$?

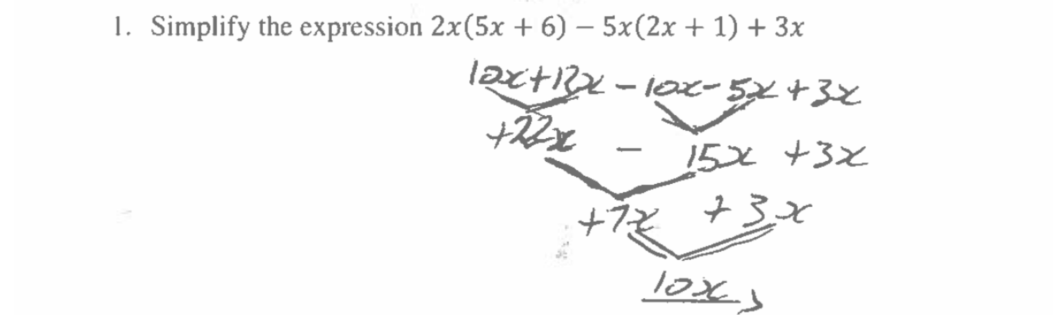
6.2 if $x = 3$

Figure 1: Test given to the leaners

5. FINDINGS AND DISCUSSION

5.1 Not knowing the concepts of *simplify* and *equivalent expression* in algebraic expressions

Most learners did not understand the concept of *simplification*. For example, L1 below found the correct answer through an incorrect procedure. He could not relate the simplified expression to the original expression. This is confirmed by what the learner said during the interview below.


1. Simplify the expression $2x(5x + 6) - 5x(2x + 1) + 3x$

$12x + 12x - 10x - 5x + 3x$

$+22x - 15x + 3x$

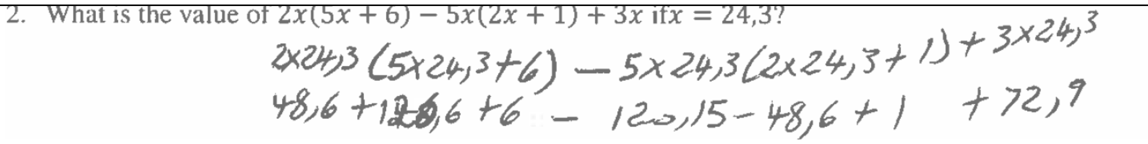
$+7x + 3x$

$10x$

Figure 2: Example of leaners' response

Researcher ®: What do you understand about the concept *simplify*?
Learner (L1): I must find the answer of this (pointing at the original expression).
R: So you got the answer as 10x. How is 10x related to the original expression (the researcher pointing at the expression $2x(5x + 6) - 5x(2x + 1) + 3x$?
L1: No sir they are not the same, 10x is the answer I got after I have worked this sum.

It is clear from the above excerpt that L1 could not see any relationship between the original expression and the answer. L1 did not understand that the two equivalent algebraic expressions, although different, nevertheless yielded the same value for x. This learner did not understand that the concept *simplify* meant constructing an equivalent expression or constructing an algebraic identity of the given algebraic expression. It was only when the learner was asked to take any value and substitute it with x that he started realising that it delivered the same answer. Furthermore, L1 failed to see that the first expression he had worked out was similar to the expression of questions 2 to 4. The purpose of these questions was for learners to realise that the first expression in the first question was similar to the second expression in the second question. So, if learners understood the equivalent expression, they could just substitute the given value of x in the answer found from the first question.

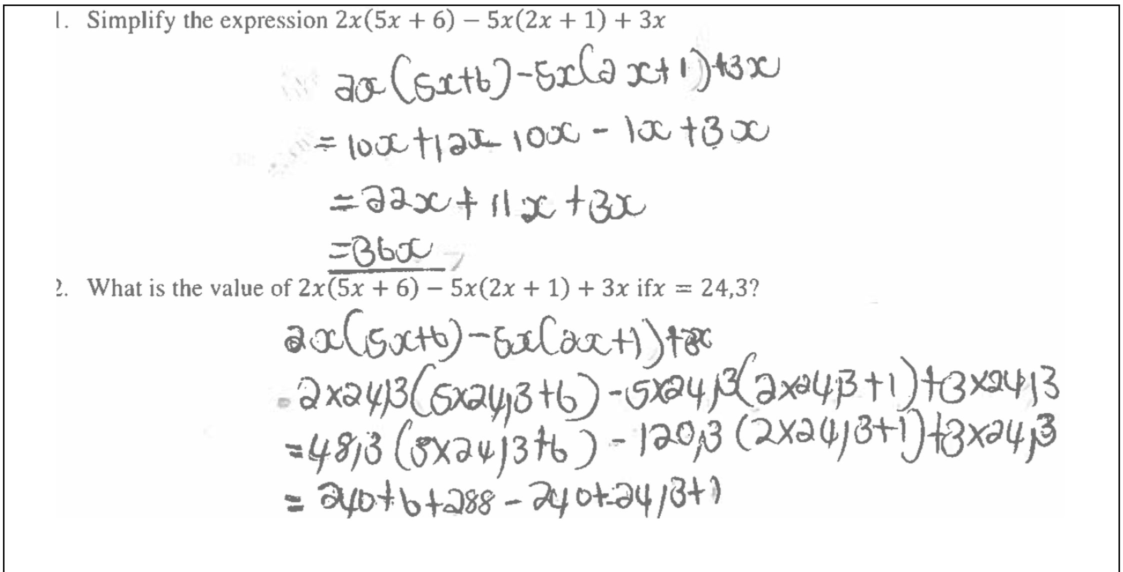

2. What is the value of $2x(5x + 6) - 5x(2x + 1) + 3x$ if $x = 24,3$?

2x24,3 (5x24,3+6) — 5x 24,3(2x24,3+ 1) + 3x24,3

48,6 + 12[illegible],6 +6 − 120,15 − 48,6 + 1 + 72,9

Figure 3: Example of leaners' response

Similarly, although L4 could not answer the first question correctly, it was clear that she did not understand the concept of *equivalent equation*. This was evident when L4 did not use her answer in question 1 to answer questions 2 to 4. This was confirmed in the interview below.



1. Simplify the expression $2x(5x+6) - 5x(2x+1) + 3x$

$2x(5x+6)-5x(2x+1)+3x$
$=10x+12x-10x-1x+3x$
$=22x+11x+3x$
$=36x$

2. What is the value of $2x(5x+6) - 5x(2x+1) + 3x$ if $x = 24,3$?

$2x(5x+6)-5x(2x+1)+3x$
$=2\times24,3(5\times24,3+6)-5\times24,3(2\times24,3+1)+3\times24,3$
$=48,3(5\times24,3+6)-120,3(2\times24,3+1)+3\times24,3$
$=240+6+288-240+24,3+1$

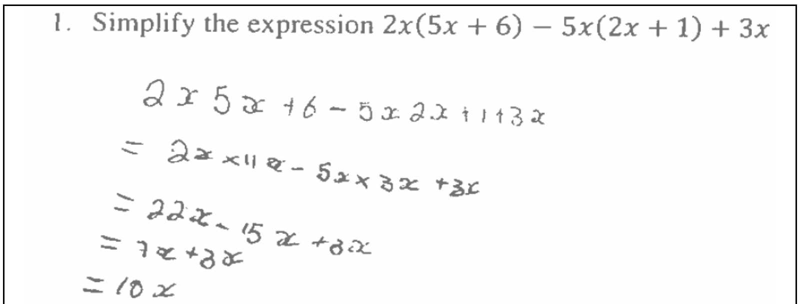
Figure 4: Example of leaners' response

R: I can see that you got your answer as 36x. Can you explain to me how you got the answer?
L4: Multiplied 2x by 5x got 10x , 2x by 6 got 12 --------------------
R: How is 36x related to the original expression $(5x+6)-5x(2x+1)+3x$?
L4: No sir 36x is the answer of $(5x+6)-5x(2x+1)+3x$

These findings are consistent with those of Steinberg, Sleeman and Ktorza (1991). In their study, they found that many students did not have a good understanding of the concept of *equivalent equation*. Most students in their study knew how to use transformation to solve simple equations, and yet, many of them did not use this knowledge to judge that the use of simple transformation gave an equivalent equation.

5.2 Inability to recognise an unlike term as an object

Most learners added and subtracted like and unlike terms without paying attention to the structure of the expression. It appeared that these learners were operating at computational level of understanding and not at structural level. For example, the following responses reflected this notion: Although they found the correct answer, L7 and L8's computational processes of working on the problem were incorrect.



1. Simplify the expression $2x(5x+6) - 5x(2x+1) + 3x$

$2x\ 5x+6-5x\ 2x+1+3x$
$=2x\times11x-5x\times3x+3x$
$=22x-15x+3x$
$=7x+3x$
$=10x$

Figure 5: Example of leaners' response

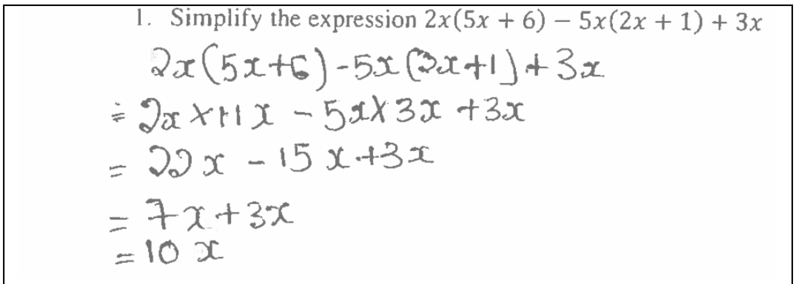


Figure 6: Example of leaners' response

When asked about how they found their answers, L7 said the following:

L7: I have added 5x and 6 and multiplied by *2* and get *22x* and this side I have added *2x* and 1 got *3x* plus 3x. So, *22x–15+3x* got *7x+3x* which *10x*.

L8 also did the same as L7. It was evident from the learners' responses above that they were not able to think structurally, because they viewed the expression *5x+6* as a computational process, which led to their answer of 10x. The inability of learners of handling this kind of problem was an indication that they had not achieved the reification stage. L5 and L6 reflected the following responses:

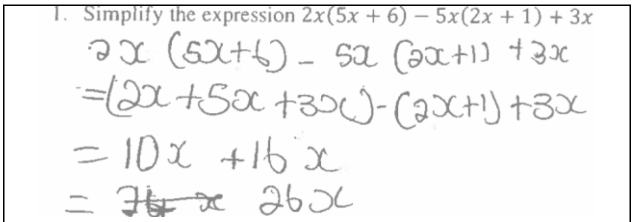


Figure 7: Example of leaners' response

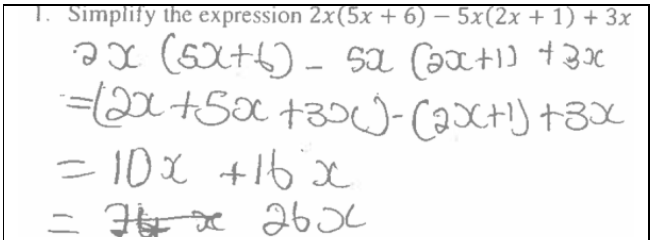


Figure 8: Example of leaners' response

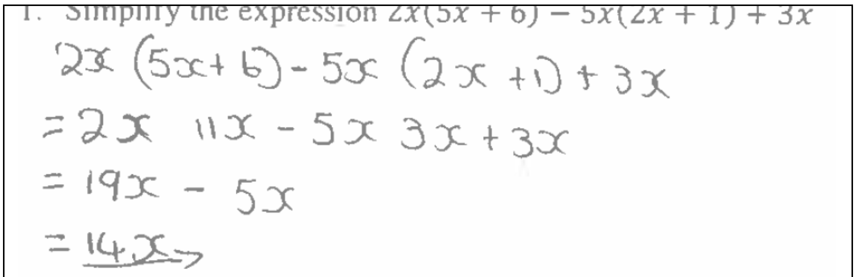


Figure 8: Example of leaners' response

When asked about their solution strategies, L5 and L6 said the following:

L5: I have just added these numbers [pointing at the *2x(5x+6)* expression] and subtracted this [pointing at $-5x(2x+1)+3x$]

L6: I have added 5x and 6 to give me 11x and 2x and 1 give me 3x, and here I got *19x–5x* and the answer is *14x*

The learners' responses above suggested that they did not see 2x(5x + 6); -5x(2x + 1) and 3x as objects on which they had done an operation (Sfard 1991).

5.3 Inability to recognise factorisation as constructing equivalent expressions

The main purpose of question 6 was for learners to find an equivalent expression for $x^2 + 5x + 6$, which is *(x+2)+(x+3)* and which would result in having $\frac{(x+3)+(x+3)}{x+3}$ and remain with (x+2) after dividing so that they could substitute 28 in the place of x to find 30 as an answer. Out of 90 learners, no one tried to factorise the expression $\frac{x^2 + 5x + 6}{x + 3}$ so that it could be easy for them to substitute the value of x, which is 28. Most learners did not solve this question because they were prohibited from using a calculator. The reason was because they were challenged to factorise first before they could substitute. Unfortunately, learners could not understand that factorisation was another way of constructing an equivalent expression. If learners had factorised first, it would have made life much easier to deal with question 6.2. For example, L2 and L3 gave the following responses:

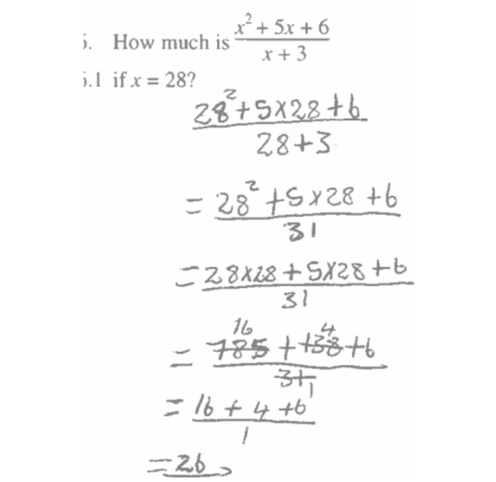


Figure 9: Example of leaners' response

R: Is there any method you can use before you substitute x with 28?
L3: No.
L2: No sir the question says how much is this if x = 28. So we have to x by 28 sir.
R: OK, what about if we factorise the numerator in this expression? Are we going to get the same expression as this one?
L2: No sir
L3: No sir I don't understand, the question said we must substitute not factorise.

It is clear from the above excerpt that learners did not view factorisation as another method of finding an equivalent expression for the original expression. They did not understand that had they factorised, it would have made life much easier to deal with question 6.2. However, this appeared not to be the case because learners were unable to find the correct answer in question 6.2 as well. For example, L3 gave the following response:

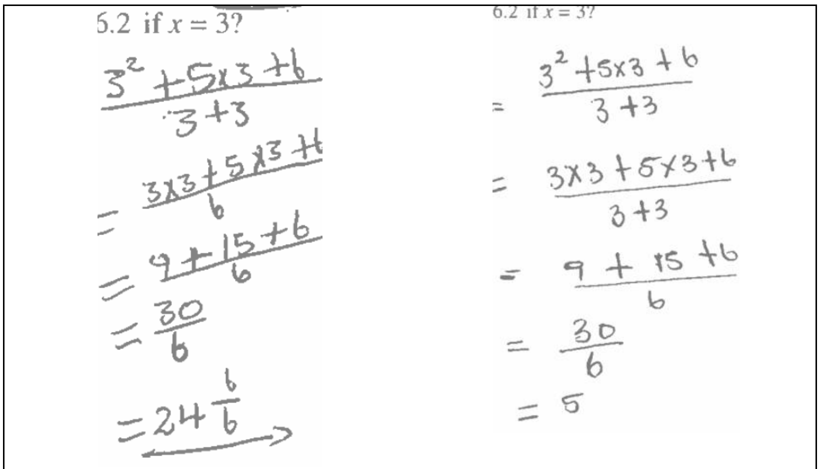


Figure 10: Example of leaners' response

6. CONCLUSION AND RECOMMENDATIONS

This paper was about 90 Grade 9 learners' understanding and interpretation of the concept of an algebraic expression. Three issues emerged very strongly from learner's understanding and interpretation of the concept of an equivalent algebraic expression. Firstly, the study revealed that Grade 9 learners in this school did not understand what *simplify* meant. They did not understand that a simplification of an algebraic expression meant to construct an algebraic identity of two equivalent algebraic expressions, namely that although two algebraic expressions may be different, they could yield the same value for the same value of a variable, say x. Learners in this class also did not understand that a variable represented an arbitrary or any unspecified number and that one could illustrate the validity of the identity with any particular number or set of numbers. Secondly, learners regarded unlike terms as the same objects. When asked to simplify an equivalent expression, they added and subtracted like and unlike terms. This suggested that these learners were unable to think structurally, because they viewed x + 2 simply as a computational process, which resulted in viewing 2x as an answer. This was an indication that these learners had not achieved the reification stage yet.
Thirdly, learners could not recognise that factorisation was another strategy of constructing an equivalent expression. This was evident when learners chose to substitute the value of x from a complex expression, which they should have factorised first before they could substitute. The difficulty learners had experienced in understanding an algebraic expression could be attributed to (1) the lack of understanding the historical development of algebra from the operational to the structural conceptions and (2) the lack of understanding the relationship between arithmetic and algebra. In other words, learners were not able to view algebra as generalised arithmetic. Therefore, we recommend that (1) the historical development of algebra be made explicit in the teaching and learning of algebra, and (2) the concept of a variable be introduced early as primary school level so that learners may understand the connection between arithmetic and algebra.

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