LEARNERS' ERRORS WHEN SOLVING TRIGONOMETRIC EQUATIONS AND SUGGESTED INTERVENTIONS FROM GRADE 12 MATHEMATICS TEACHERS

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ABSTRACT-The trigonometry content area has a weighting of 40 % in Grade 12 Mathematics Paper 2. Therefore, failure to understand trigonometry concepts on the part of the learners has a ripple effect on the success rate of the learners in Mathematics. This study sought to find out the problematic concept(s) and errors that learners have in solving trigonometry equations. The article gives some valuable suggestions for possible treatment of learners' errors when solving trigonometric equations. Particular types of errors that learners made when solving trigonometric equations, the possible causes, and ways in which this information might be used to structure instructional interventions, are described from the teachers' perspectives. Data were gleaned using an open-ended questionnaire and interviews of 10 Grade 12 Mathematics teachers. The teachers' responses to the questionnaire and interviews were analyzed thematically. The most common errors that the learners made as outlined by the teachers were selected. Data revealed that learners misinterpreted sine, cosine and tangent of an angle when their values were negative; failed to identify relevant quadrant and; made invalid inference (failed to recognize the reference angle). The study found learners commit errors in solving trigonometric equations and teachers have difficulties in teaching the same content.

Keywords: errors, trigonometric equations, reference angle.

1. INTRODUCTION

The National Senior Certificate (NSC) is the gateway to Higher Education and thus all spheres of South Africa's society regard the NSC results as the main indicators of the quality of the education system (Sasman, 2011, p. 10). There is thus an understandable concern about what these results are saying about the current state of education. The low performance of Grade 12 learners in mathematics has been a perennial challenge in South Africa (Makgato & Mji, 2006; Department of Basic Education [DBE], 2012). The poor performance in mathematics prohibits learners from studying mathematics related careers such as architecture, astronomy, civil engineering, electrical engineering, or mechanical engineering. Learners drop mathematics and opt for mathematical literacy. For example, from 2008 until 2011, the proportion of learners taking mathematics, as opposed to mathematical literacy, declined from 56% to 45% because more learners opted for the easier mathematical literacy (Spaull, 2013). Mathematics in the Further Education and Training (FET) phase now covers ten main content areas. Each content area contributes towards the acquisition of specific skills. Progression in terms of concepts and skills from Grade 10 to Grade 12 for each content area are interdependent. Trigonometry is one of the content areas covered in the FET phase and is often introduced in Grade 9 with the naming of the sides of a right-angled triangle. Figure 1 shows a mind map for trigonometry content area:

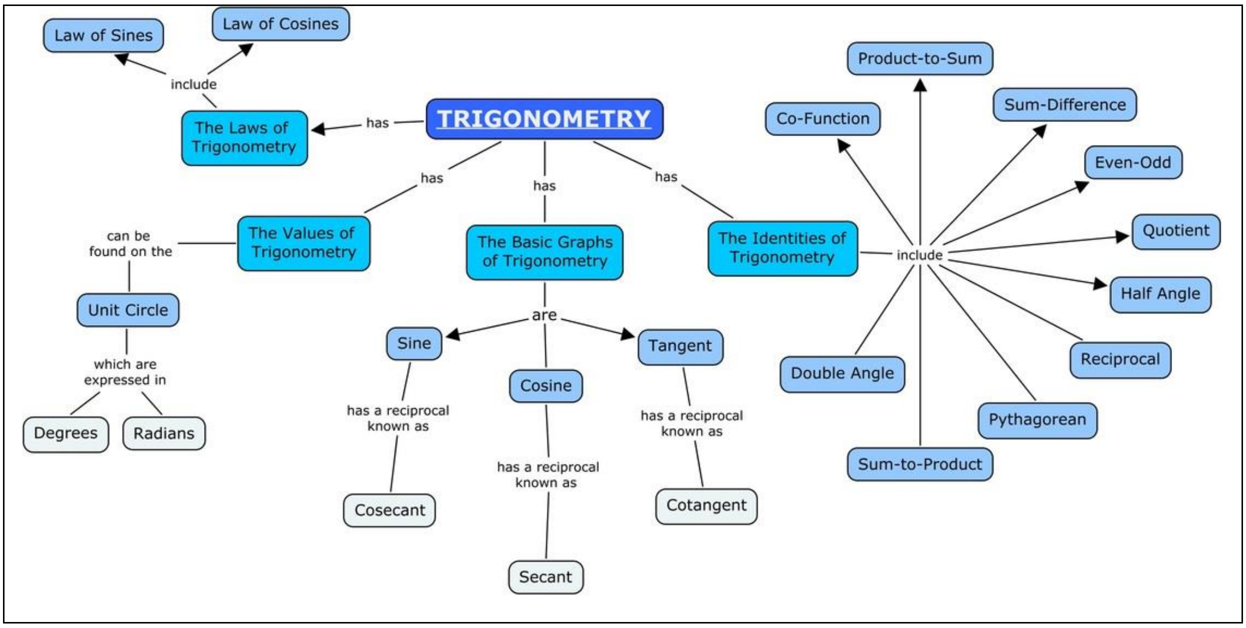


Figure 1: A mind map for trigonometry content area

The 2014 National Senior Certificate examination diagnostic report on Mathematics Paper 2 highlighted that "performance in the Trigonometry section was a cause for concern as candidates performed poorly in questions that tested basic knowledge" (DBE, 2014, p. 121). This suggests that learners have problems understanding trigonometry. The trigonometry content area has a weighting of 40 % in Grade 12 Mathematics Paper 2, hence it has a large impact on performance in Paper 2. Therefore, failure to understand trigonometry concepts on the part of the learners has a ripple effect on the success rate of the learners in Mathematics as found by Sasman (2011):

Trigonometry was the most poorly answered section in Paper 2…. Candidates struggled to identify the relevant quadrant, struggled with reduction formulae, compound and double angles, and co-functions. It was also disturbing to note that quite a few learners obtained a negative value for the radius.(p. 10).

Factors responsible for such low performance in trigonometry are often assumed and may not be well researched. In a quality teaching and learning enhancement project in Mathematics, Science and Technology, a preliminary survey in Mankweng Circuit schools proved that teachers encounter problems when teaching trigonometry (Mavhunga, Chigonga, Ndlovu & Kibirige, 2015). Apparently this suggests that learners have problems understanding trigonometry. The World Economic Forum (WEF) report (2014) rates South Africa as the worst in quality of Mathematics and Science education. "It is a matter of concern that a learner can spend 3 years in the FET band and seemingly have learnt nothing in trigonometry" (Sasman, 2011, p. 10) which suggests learners have difficulties in trigonometry. Consequently, further research is required to determine learners' problematic aspect(s) and errors in trigonometry. Thus the present study aims to identify the problematic aspect(s) of trigonometry and errors (and possible root causes) that Grade 12 Mathematics learners have when solving trigonometry problems. Grade 12 Mathematics teachers' perspectives were used as a lens to understand learners' problematic aspect(s) and errors in trigonometry. They were also asked to explain the cause and how they would address the errors by means of an intervention with learners.

2. RESEARCH QUESTIONS

The study responded to the research questions:
What aspect(s) of trigonometry teachers find learners having difficulties?

What are the errors teachers envisage learners commit in trigonometry?

What may be the causes of the identified errors as envisaged by teachers?

How do teachers deal with learners' errors?

3. LITERATURE REVIEW

The review of literature answered the question: What is the role of errors in the learning process? Attention was given to the idea of errors focusing on their value for mathematics teaching and learning. Content knowledge of Mathematics is obtained through procedural and /or conceptual learning (Kilpatrick, Swafford & Findell, 2001; Long, 2005). One of the biggest tasks facing mathematics teachers in Mathematics education is what decisions to make that will best serve the learners' needs in the reconstruction of knowledge (Ball, Hill & Bass, 2005). This is important because unless cognitive structures in the building of mathematical concepts are transformed, misconceptions may continue throughout the learners' lifetime. This means that consideration must be given to the kind of pedagogical strategies necessary for knowledge transformation to be effective. It is therefore useful to review and think about learners' possible errors or misconceptions before teaching them.

Research in mathematics education has proven that a focus on errors, as evidence of mathematical thinking on the part of learners, helps teachers to understand learner thinking, to adjust the ways they engage with learners in the classroom situation, as well as to revise their teaching approach (Adler, 2005; Brodie, 2014; Venkat & Adler, 2012). This entails paying attention to creating strategies with accompanying activities that attempt to transform erroneous knowledge in the mathematical domains, with a focus on learners rather than on content. Studies on teaching dealing with learners' errors have proven that teachers' interpretive stance is essential for the process of remediation of error (Peng & Luo, 2009; Peng, 2010; Prediger, 2010). Nevertheless, there is dearth of literature that describes teachers' interpretative stance on learners' errors in trigonometry. Research on teachers' known learners' errors in trigonometry as reflected in learners' work could provide further insight in the latter content area. Hence learners' errors from teachers' lens is the subject of this article.

Hill and Ball (2009, p. 70) see analysing learners' errors as one of the four mathematical tasks of teaching "that recur across different curriculum materials or approaches to instruction". Peng and Luo (2009) and Peng (2010) argue that the process of error analysis includes four steps: identifying, addressing, diagnosing and correcting errors. In South Africa, Adler (2005, p. 3) sees teachers' knowledge of error analysis as a component of "mathematics for teaching". What do teachers need to know and know how to do in ways that produce "mathematical proficiency", a blend of conceptual understanding, procedural fluency and mathematical reasoning and problem solving skills (Adler, 2005, p. 3) ? Teachers are required "to interpret their own learners' performance in national (and other) assessments" (DBE & Higher Education and Training, 2011, p. 2) and develop lessons on the basis of these interpretations. This requirement implies that teachers are expected to use learner data diagnostically and therefore, I argue, teachers' involvement in error analysis of classroom assessment is a professional responsibility and an integral aspect of teacher knowledge. Hence in this article I foreground concept(s) in trigonometry that learners have difficulties, errors they make and the possible remedy from the teachers' perspectives.

Research has only recently begun to engage with the question of how to use learner data beyond that of a statistical indicator of quality, that is, beyond benchmarking for external accountability (Cohen & Hill, 2001; Katz, Sutherland & Earl, 2005; Boudett, City & Murnane, 2005; Katz, Earl & Ben Jaafar, 2009). In South Africa, Reddy (2006), Dempster (2006), Long (2007) and Dempster and Zuma (2010) have each conducted small case studies on test-item profiling, arguing that this can provide useful data that can be used by teachers for formative and diagnostic purposes. Notwithstanding these important contributions, there is very little research on learners' errors in trigonometry from the teachers' perspectives. Teachers are in a position to recognize what learners do not understand in trigonometry; their reasoning behind the error; how that may affect their learning and; which instructional practices could allow or constrain them to address learner difficulties (Shepard, 2009). Teachers are perceived as having the "diagnostic competence" to distinguish reasoning about learners' errors from merely grading their answers (Prediger, 2010, p. 76). Prediger's view is consistent with Shepard's (2009, p. 34) work on formative assessment, in particular with the idea of using insights from learners' work formatively to adjust instruction. As Peng and Luo (2009) argue, if teachers identify learners' errors but interpret them with wrong mathematical knowledge, their assessment of learner performance and their plan for a teaching intervention are both meaningless.

Teachers' knowledge of what counts as the explanation of the correct answer enables them to spot the error, looking for patterns in learners' errors enables them to interpret learners' solutions and evaluate their acceptability. This article reports teachers explaining learners' errors in identified concept(s) in trigonometry from the perspective of "the mistakes or misconceptions that commonly arise during the process of learning the topic" (Hill, Ball & Schilling, 2008, p. 375). The idea is to understand how teachers go beyond the mathematical error and explain how they can remedy the error. I hope that by analysing teachers' explanations of learners' errors, this article will contribute to Black and Wiliam's (1998) well-established argument of the positive potential impact of assessment for learning (formative assessment).

4. CONCEPTUAL FRAMEWORK

The paper is conceptualized around the assumption that errors may help reveal possible ways of remediating difficulties experienced by learners (Fennema, Carpenter, Franke, Levi, Jacobs & Empson, 1996) in trigonometry. Fennema et al. (1996) are convinced that errors made by learners can become a point of departure for discussion about teaching strategies that enhance learning. Indeed errors can stimulate new ways of teaching and learning of trigonometry. Therefore, the hypothesis in this article is that it is possible to use the errors learners make as a pivot to empower teachers in coming up with effective teaching strategies. From this viewpoint, the study sought to establish the difficult aspect(s) of trigonometry and describe the learners' errors, from the teachers' perspectives in the identified problematic aspect(s) of trigonometry.

5. METHODOLOGY

5.1 Research design

The study used descriptive- interpretive qualitative research approach employing phenomenological research design. Doing phenomenological research, involves trying to understand the essence of a phenomenon by examining the views of people who have experienced that phenomenon (Giorgi, 2009). The operative word in phenomenological research is '*describe*' (Giorgi, 2009). The phenomenon that was studied was that of teachers' experiences teaching trigonometry. Therefore the aim was to describe as accurately as possible the phenomenon, refraining from any pre-given framework, but remaining true to the facts, hence the use of phenomenological research design.

5.2 Population and sample

The population was all Grade 12 Mathematics teachers in Malamulele West Circuit schools. A sample of 10 Grade 12 mathematics teachers (3 female and 7 male), 2 from each of the five purposively selected public secondary schools in the Malamulele West Circuit, was selected to participate in the study (Creswell, 1998). Boyd (2001) corroborates this sample size in stating that 2 to 10 participants are sufficient to reach saturation for a phenomenological study.

5.3 Instruments

Data were gleaned through an open-ended questionnaire (Appendix 1) before phone interviews (Hill, Thompson & Williams, 1997). The open-ended questionnaire consisted of questions directed to the teacher's experiences and views about teaching trigonometry (Welman & Kruger, 1999). Phone interviews sought to stimulate the needed level of elaboration on responses from the open-ended questionnaire (Hill et al., 1997).

5.4 Data collection

Teachers were provided with a list of open-ended questions which they responded to within five days before the phone interview. As responses often do not provide enough elaboration to understand the respondents' point, phone interviews were conducted to stimulate the needed level of elaboration sought (Hill et al., 1997). The interview was reciprocal in that both researcher and participant were engaged in a dialogue. The duration of phone interviews and the number of questions asked the respondents for elaboration, varied from one participant to the other. Questioning was flexible and carefully adapted to the problem at hand and to the individual informant's particular experiences and abilities to communicate those experiences, making each interview unique. Notes were made during the phone interview as the researcher reflected on experiences and views of the interviewee (Lofland & Lofland, 1999).

5.5 Data analysis

Grouping units of meaning into themes, data analysis involved comparing notes made during the phone interviews and the notes or sketches that teachers made in respond to the open-ended questionnaire. For credibility checks, the draft transcription and analysis of the data was presented for confirmation of correctness and/or commentary by the research participants (Elliott, Fischer, & Rennie, 1999).

6. ETHICAL ISSUES

Permission to carry out the study at the selected sites was obtained from the circuit manager and principals. During the process of data collection and processing anonymity and confidentiality were assured. Also insincerity and manipulation were guarded against.

7. RESULTS

7.1 Biographical information

The biographical profile of teachers in schools was included to determine if there are any differences in responses as a result of their different profiles. Figure 2 shows the age distribution. Of the 10 teachers, eight are in the groups 30-49 years old. That accounts for 80% of the participating teachers, of whom 62.5% are male. The average age of the group is 46 years. Of the qualifications, 60% are degrees and the rest diplomas. Figure 2 shows the variety of qualifications.

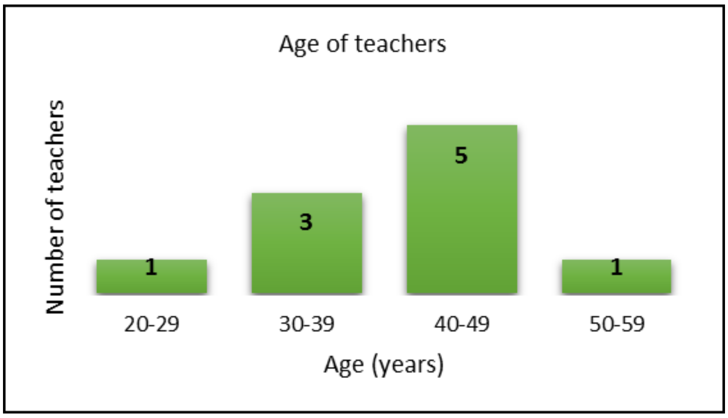


Figure 2: Age distribution of teachers

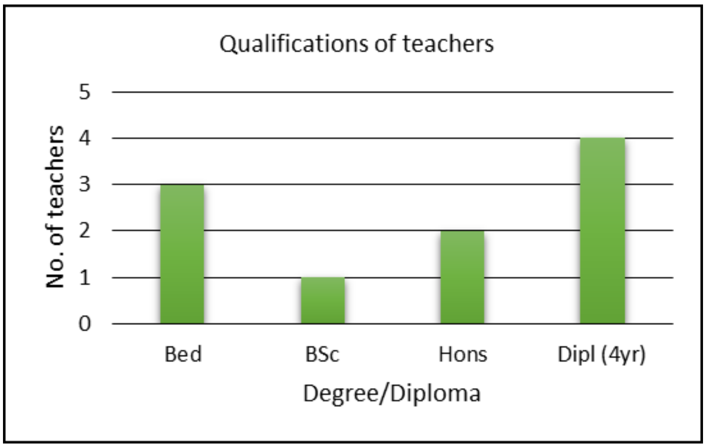


Figure 3: Variety of qualifications of teachers

The majority of teachers (90%) have 11 or more years' experience teaching Grade 12 mathematics (Figure 3). The average is 15 years. The current positions that they occupy are: six heads of department and the rest are teachers.

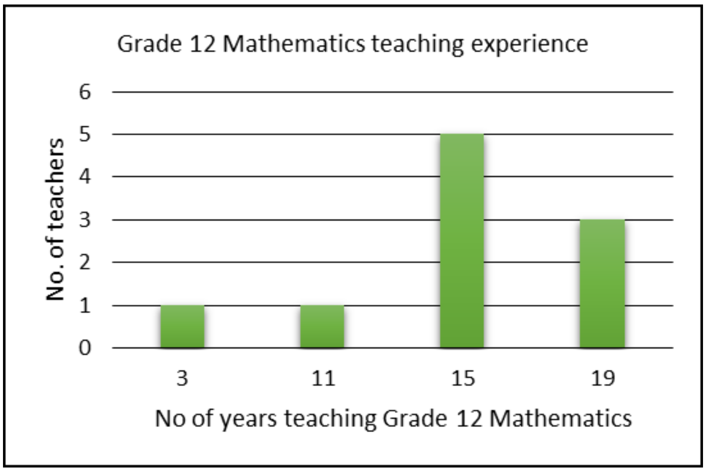


Figure 3: Teachers' experience of teaching Grade 12 Mathematics

7.2 Open-ended questionnaire and interviews

The notes and sketches that teachers made in respond to the questions put to them in the open-ended questionnaire and the notes made during the phone interviews were grouped thematically. Three themes emerged:

Solving trigonometric equations is difficult to teach and learn
Learners choose the wrong quadrants
Learners divide by a variable expression
Learners do not check the validity of the solutions
learners lack knowledge about periodicity of trigonometric functions
Learners lack enough practice in solving trigonometric equations
Learners need to justify every step when solving trigonometric equations to enhance understanding

8. DISCUSSION

8.1 The perception that 'solving trigonometric equations is difficult to teach and learn'

Teachers concurred that learners struggle to solve trigonometric equations that are *conditional* (finding solutions within given intervals) hence they encounter problems teaching the same content.

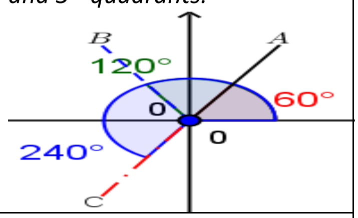
T2[3]: "…Many schools are not affording appropriate teaching and assessment time to certain content topics and solving trigonometric equations is one of them……It appears as if many teachers have problems teaching learners how to solve trigonometric equations….and, as a result, students have phobia of the trigonometric equations and it is not a favorite topic for them….".

Many of the errors gleaned from teachers' responses have their origins in poor understanding of the basics and foundational competencies taught in the earlier grades. Learners *choose the wrong quadrants* especially where the value of cosine, sine or tangent of an angle is negative; *divide by a variable expression* hence eliminating possible solutions; square both sides of a trigonometric equation and *do not check the validity of the solutions* and; *learners lack knowledge about periodicity of trigonometric functions* hence cannot link revolutions to integers. As causes of the identified errors/misconceptions/mistakes, one teacher summed up saying:

T8: "…The reason for this could be that many teachers are not confident about some content in the NCS…It also appears that many teachers are struggling to teach learners how to solve trigonometric equations, especially finding solutions within a given interval….".

Examples of some of the errors that teachers indicated are as follows.

Example 1: *Learners choosing the wrong quadrants*
Solve $2\cos\emptyset + 1 = 0$ *over the interval* $[0°; 360°)$.

Learner's working	***Comment***	***Correct working***
$\emptyset = \cos^{-1}(-0.5)$	*The quadrants where* $\cos\emptyset$ *gives a negative value are informed by the minus sign.*	$\cos\emptyset = -0.5$. *Let* $\alpha = \cos^{-1}0.5 = 60°$, *which is the reference angle.*
$\emptyset = 120°$	*Learner did not find the acute angle first (reference angle)* $\cos^{-1}0.5 = 60°$ *to then consider reflection in the y-axis (2nd quad) and point reflection (3rd quad) of the arm which makes the angle of* $60°$ *with the positive* $x-axis$ *in an anticlockwise direction.*	*Considering the quadrants where cosine is negative: 2nd and 3rd quadrants.* 
		$\emptyset = 180° \pm 60°$
		$\therefore \emptyset = 120°\ or\ 240°$

As can be seen in Example 1, the learner did not consider the issue of quadrants: *Cosine of an angle* is negative in second and fourth quadrants, something that is not evident in the learner's solution. We have $2\cos\emptyset + 1 = 0$, then the trick is to realize that we look for solutions in more than one quadrant. All the trigonometric functions are positive in two quadrants and also negative in the other two quadrants. To find values of $\emptyset$ that satisfy $\cos\emptyset = -\frac{1}{2}$, we observe that $\emptyset$ must be in either quadrant II or III since the $cosine$ function is negative only in these two quadrants. The reference angle is $60°$ since $\cos 60° = \frac{1}{2}$. Therefore, the diagram in Example 1 shows the two possible values of $\emptyset$, $120°\ and\ 240°$. Thus the solution set is $\{120°; 240°\}$.

<u>Example 2:</u> *Learners dividing by a variable expression*
Solve $\sin\theta\tan\theta = \sin\theta$ *over the interval* $[0°; 360°)$.

[3]. The ten teachers involved in the study were coded from 1-10.

Learner's working	*Comment*	*Correct working*
$\tan\theta - 1 = 0$	*Dividing by* $\sin\theta$	$\sin\theta\,(\tan\theta - 1) = 0$ (Factor)
$\tan\theta = 1$		$\sin\theta = 0$ or $\tan\theta - 1 = 0$ (Zero factor property)
$\theta = 45°$	*Which is not the only solution to the equation*	$\theta = 0$ *or* $\theta = 180°$ or $\theta = 45°$ *or* $\theta = 225°$
		The solution set is $\{0°; 45°; 180°; 225°\}$.

As can be noted in example 2, in solving quadratic trigonometric equations, trying to solve the equation by dividing each side by $\sin\theta$ would lead to just $\tan\theta = 1$, which would give the solution set as $\{45°; 225°\}$. Students will of course be tempted to divide both sides of the equation by $\sin\theta$; the only problem with this is that it will eliminate possible solutions because $\sin\theta$ can equal zero.

Example 3: *Learners not checking the validity of the solutions*

Solve $\tan\theta + \sqrt{3} = \sec\theta$ *over the interval* $[0°; 360°)$.

Learner's working	*Comment*
$(\tan\theta + \sqrt{3})^2 = \sec^2\theta$	*Square both sides*
$\tan^2\theta + 2\sqrt{3}\tan\theta + 3 = \sec^2\theta$	*Simplify*
$\tan^2\theta + 2\sqrt{3}\tan\theta + 3 = 1 + \tan^2\theta$	*Substitute:* $\sec^2\theta = 1 + \tan^2\theta$
$2\sqrt{3}\tan\theta = -2$	*Simplify*
$\tan\theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$	
$\theta = 150°$ *or* $330°$	

The possible solutions are $150°$ *and* $330°$. *Now check them. Try* $150°$ *first.*

Left side: $\tan\theta + \sqrt{3} = \tan 150° + \sqrt{3} = -\frac{\sqrt{3}}{3} + \sqrt{3} = \frac{2\sqrt{3}}{3}$

Right side: $\sec\theta = \sec 150° = -\frac{2\sqrt{3}}{3}$ *Not equal*

Therefore $150°$ *is not a solution. Now check* $330°$.

Left side: $\tan\theta + \sqrt{3} = \tan 330° + \sqrt{3} = -\frac{\sqrt{3}}{3} + \sqrt{3} = \frac{2\sqrt{3}}{3}$

Right side: $\sec\theta = \sec 330° = \frac{2\sqrt{3}}{3}$ *Equal*

So $\{330°\}$ *is the solution set.*

Recall that squaring both sides of a radical equation, such as $\sqrt{x + 14} = x + 2$, will yield all solutions but may also give extraneous values (in this radical equation, 2 is a solution, while -5 is extraneous). The same situation may occur when trigonometric equations are solved in this manner as we have already seen in Example 3.

Example 4: *Learners lacking knowledge about periodicity of trigonometric functions*

Find the general solution of $\sin 3x \cos x - \cos 3x \sin x = 0.4$

Learner's working	*Comment*	*Correct working*
$\sin(3x - x) = 0.4$	*Correctly applying compound angle identity*	
$\sin 2x = 0.4$	*Correct*	
$2x = \sin^{-1}0.4$	*The reference angle concept and periodicity of trigonometric functions are not present*	$2x = \sin^{-1}0.4 + k.360°$ ***or*** $2x = 180° - \sin^{-1}0.4 + k.360°,\ k \in \mathbb{Z}$.
$2x = 23.58°$		$2x = 23.58° + k.360°$ ***or*** $2x = 156.42° + k.360°,\ k \in \mathbb{Z}$.
$x = 11.79°$	*This is not the general solution*	$\therefore x = 11.79° + k.180°$ ***or*** $x = 78.21° + k.180°,\ k \in \mathbb{Z}$.

In Example 4, insufficient knowledge about periodicity of trigonometric functions is evident as learners struggle to find other solutions to the equation after (in some case) obtaining the reference angle. The connection of k to integers was not a common feature since learners could not link revolutions to integers. Thus, this could be a possible explanation as to why most learners struggled with finding particular (exact) or general solution of a trigonometric equation. This suggests that interventions to improve learners' performance in solving trigonometric equations should focus also on knowledge, concepts and skills learnt in earlier grades and not just on the final year of the FET phase. Furthermore, an analysis of learners' errors as depicted by the teachers indicated that most learners associated trigonometric equations with identities, quadratic and linear equations but could not go beyond that. The insufficient knowledge of periodicity of trigonometric functions hampered the finding of the general solutions. I suggest the use of graphical method of solving trigonometric equations to enable the learners to see the periodicity of trigonometric functions hence infinitely many solutions. For instance, to indicate the solution set of $\sin x = -\frac{1}{2}$ on the graph, draw the graphs of $y = \sin x$ *and* $y = -\frac{1}{2}$ on the same set of axes (Figure 4) and then find the values of x for which $\sin x = -\frac{1}{2}$ (points of intersection of the two graphs).

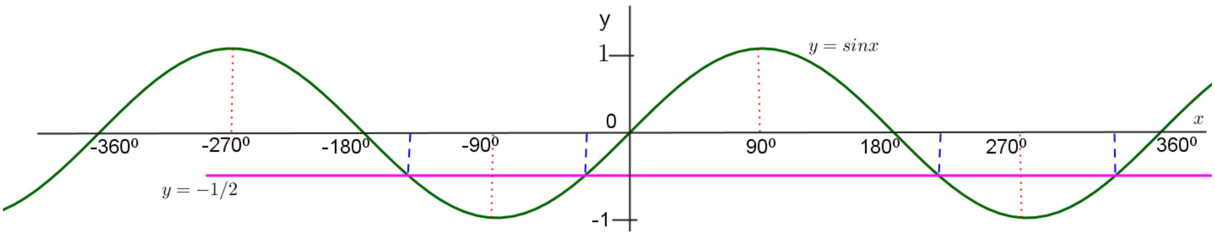


Figure 4: Trigonometric graph for solving trigonometric equstions

8.2 The perception that 'learners lack enough practice in solving trigonometric equations'
Some teachers appear to be struggling with how solving trigonometric equations must be taught as one teacher proclaimed:

T6: "…teachers often spend too much time on teaching certain sections which they know well and teaching the less familiar topics at a very superficial level…"

This view only indicates that the errors that learners exhibit can be got rid of with plenty of practice at school and at home as one teacher puts it:

T4: "…Repeat the concept of quadrant several times to make them understand…Teachers need to give learners lots of examples of the same type for learners to realise their mistakes… errors made by learners working at the board can be picked up and corrected by the other learners…"

8.4 The perception that 'learners need to justify every step when solving trigonometric equations to enhance understanding'
When asked what are the possible remedies to remove learners' errors one teacher said:

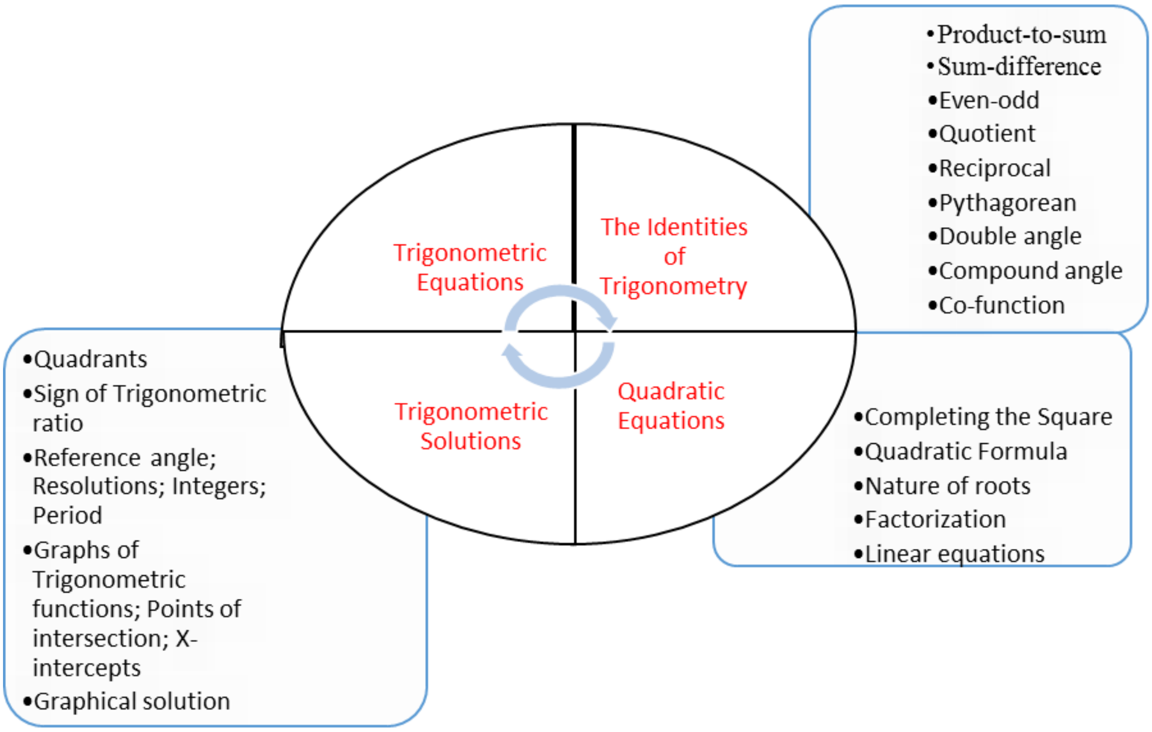
"…We need to instruct learners to write relevant ideas and concepts used at each step when solving each trigonometric equation given…"

The teacher elucidated what he meant by the following example.

Example 5: *Find the general solution of the trigonometric equation:* $\cos 2\theta + 3\sin\theta + 1 = 0$
The learners are expected to do the following:

Step	Concept identified (step justification)
$\cos 2\theta + 3\sin\theta + 1 = 0$	Quadratic trigonometric equation
$1 - 2\sin^2\theta + 3\sin\theta + 1 = 0$	Double angle identity ($cos2\theta = 1 - 2sin^2\theta$).
$-2\sin^2\theta + 3\sin\theta + 1 + 1 = 0$	Grouping like terms.
$-2\sin^2\theta + 3\sin\theta + 2 = 0$	Combining like terms
$2\sin^2\theta - 3\sin\theta - 2 = 0$	Multiplication by a negative sign. (Quadratic equation in $\sin\theta$).
$(2\sin\theta + 1)(\sin\theta - 2) = 0$	Factorizing the left side.
$2\sin\theta + 1 = 0$ or $\sin\theta - 2 = 0$	The Zero product rule.
$\sin\theta = \frac{-1}{2}$ or $\sin\theta \neq 2$	Solution of quadratic equation in $\sin\theta$. Sin θ is a bounded function ($-1 \le \sin\theta \le 1$), which means only one equation needs to be solved for solutions.
$\theta = [180 + \sin^{-1}(\frac{1}{2})] + k.360$, or $\theta = [360° - \sin^{-1}(\frac{1}{2})] + k.360°$, $k \in \mathbb{Z}$.	General solutions, quadrants (3rd or 4th), reference angle, integers, inverse functions, revolutions, period.
$\theta = 210 + k.360°$, or $\theta = 330° + k.360°$, $k \in \mathbb{Z}$.	General solutions, quadrants, reference angle, integers, revolutions, period.

At the end of the lessons concerned with the solution of trigonometric equations, the teacher should ask learners to generate a concept matrix in relation to solving trigonometric equations. The concept matrix below provides a summary of the different concepts that can be generated by the learners as they solve different types of trigonometric equations using the approach above.



The idea is to assess if learners can produce visual representations of ideas and concepts and effectively incorporate them when solving trigonometric equations. Learners will be assessed on how they: organize their knowledge of solving trigonometric equations; understand concepts related to trigonometric equations such as quadratic equations, the nature of roots, different solution methods and; manipulate algebraic processes such as factorization, completing the square and the use of different number systems. Thus the number of linkages in a network of concepts tells the teacher much about the meaning of the concept from the perspective of the learner.

9. LIMITATION OF THE STUDY

The dilemma with any study is the extent to which the results can be generalized. The problem under investigation focuses more on tapping teachers' perspectives and not leaners'. Only Grade 12 Mathematics teachers were selected from 5 low performing schools in Malamulele West Circuit in Vhembe District. Qualifications of the sample participants (60% of the teachers hold degrees) and on average 15 years' teaching experience of Grade12 Mathematics were comparable to other teachers in high performing schools. However, the learners considered here are from low performing schools implying that their mathematics was not good. Therefore, the results and findings of the study will not be generalizable to high performing schools. It is not certain that the errors discussed herein would be the same as with learners from high performing schools whose mathematical backgrounds are comparatively good. That is, it may be that the learners from high performing schools will manifest different categories of errors (if any) from those of learners from low performing schools considered in this paper. Besides, failure to conduct an in depth understanding of the problems in trigonometry from learners' perspectives and the fact that data were collected from teachers' perspectives without going into the classroom was a limiting factor. A more balanced technique would have been to analyze learners' written work and interview the latter and; observe teachers teaching to have an insight into teachers' classroom practices, their preparations, teaching strategies and the way they handle learners' errors and how feedback is given to learners. Anyway those are the limitations shaping the results of the study.

10. CONCLUSION

The article focused on reporting the types of errors that learners made when solving trigonometric equations, the possible causes, and ways in which this information might be used to structure instructional interventions, from the teachers' perspectives. The findings of this study revealed that learners misinterpreted sine, cosine and tangent of an angle when their values were negative; failed to identify relevant quadrant and; made invalid inference (failed to recognize the reference angle). Based on this, the conclusion was that, when solving trigonometric equations, it is important for students to:

decide whether the equation is linear or quadratic in form, so they can determine the solution method:

If only one trigonometric function is present, use linear method.

If more than one trigonometric function is present, either

Rearrange the equation so that one side equals 0 and avoid dividing by a variable expression. Then try to factor and set each factor equal to 0 to solve, or

Try using identities to change the form of the equation in terms of a single trigonometric function. It may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions.

If the equation is quadratic in form, factorise or use the quadratic formula.

Check that solutions are in the desired interval. All the trigonometric functions are positive in two quadrants and also negative in two quadrants.

11. RECOMMENDATIONS:

We recommend that teachers should always emphasis to learners that when solving trigonometric equations that are *conditional* by:

Linear methods

–Conditional equations with trigonometric functions can be solved using algebraic methods and trigonometric identities (Example 1). [Because $cos\ \emptyset$ is the first power of a trigonometric function, we use the same method as we would to solve the linear equation $2x + 1 = 0$].

Factorizing or the use quadratic formula

-Because we have a quadratic equation in $\sin\theta$ (Example 5), we use the same method as we would to solve the quadratic equation $2x^2 - 3x - 2 = 0$.

Squaring both sides

-Squaring both sides of a radical equation will yield all solutions but may also give extraneous values hence the need to check the validity of the solutions. The same situation may occur when trigonometric equations are solved in this manner (as we have already seen in Example 3.

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APPENDIX 1: Teachers' open-ended questionnaire questions.

The questionnaire, consisting of sections A and B. Section A collected teachers' biographic information (age, qualification and experience teaching Grade 12 Mathematics) and section B had the following open-ended questions:

Focus of question	**Question**
Experiences	What aspect of trigonometry do learners find difficult to learn? What are the errors committed by learners in trigonometry? (Illustrate learners' errors using examples).
Views	What do you think is the cause of the identified errors? What are the possible remedies to remove learners' errors?