

Students' Difficulties in Interpreting Areas and Volumes from the Region Bounded by Graphs

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ABSTRACT

The results of an investigation where students solved problems on areas and volumes revealed that the majority of students encounter problems in interpreting the region bounded by graphs (drawn or given) especially if they give rise to 3D diagrams. The approximation of the area of the region bounded by the graphs, as well as rotation of that region were found to be the main obstacle, leading to students' failure in solving such problems. The problem existed especially if the horizontal strip was appropriate or when rotating a vertical strip about the y -axis, which led to formulation of an incorrect formula for volume. The students were seen to use the disc or washer methods even if the problem required the use of the shell method. An additional obstacle was that the majority of the students did not draw the rotated strip in 3D and the solid of revolution formulated.

Keywords: Riemann sums, areas, volumes, solids of revolution and cognitive skills.

1.1 INTRODUCTION AND BACKGROUND OF THE STUDY

This study was conducted at one college for Further Education and Training (FET) in South Africa, where students are prepared for different vocations in engineering fields. FET colleges resulted after a merger of 152 technical colleges into 50 FET colleges in 2002, in terms of the FET Act, No 98 of 1998 (Akoojee & McGrath, 2008; McGrath & Akoojee, 2009; Papier, 2008). In South Africa FET colleges mostly enrol students who did not qualify to be at Comprehensive Universities or Universities of Technologies due to poor performance in Mathematics and Science.

In this study mathematics N6 (completed in ten weeks) students calculate areas and volumes and further application thereof from the region bounded by graphs. This section constitutes 40% of the final examination paper. The researcher's previous experience in lecturing at the FET colleges for five years is that students see this section as difficult and perform poorly in it, in comparison to other parts of the curriculum. At N5 level students calculate areas and volumes using the disc and washer method whereas at N6 level they include the shell method and further applications such as calculating centroids and moments of inertia (introduced in N6). The researcher's experience as a lecturer at Universities and Universities of Technologies is the same, in that students also experience this section to be difficult.

1.2 RESEARCH QUESTION AND CONCEPTUAL FRAMEWORK

The research question in this study is:

How competent are the students in interpreting areas and volumes from the region bounded by graphs?

A conceptual framework for this study was developed as five *skill factors* of knowledge for the larger study, where six investigations were carried out. In this report the researcher focuses on one investigation (classroom observations).

All the five skill factors relating to students' competency are classified below:

Skill factor I: Graphing skills and translation between visual graphs and algebraic equations. Students are given questions where they are required to translate a given equation into a graph or to translate a given diagram into an algebraic equation, both in two and three dimensions.

Skill factor II: Translation between 2D and 3D diagrams. Students are given questions where they are required to translate from two dimensional to three dimensional diagrams or to translate from three to two dimensions. This skill factor involves three-dimensional thinking.

Skill factor III: Translation between continuous and discrete representations. Students are given questions where they are required to draw rectangles to approximate the area under a curve and to draw a representative strip that will, upon rotation, generate a *disc*, *washer* or *shell* in order to generate solid of revolution.

Skill factor IV: General manipulation skills. Students are given questions where they are required to evaluate a given definite integral, representing the volume of a solid of revolution.

Skill factor V: Consolidation and general level of cognitive development. Students are given questions where they are required to have the necessary cognitive skills to grasp the concepts involved in calculating solids of revolution and be able to fully solve problems in this regard, that is by drawing the graph, indicating the representative strip used for setting up the formula and then calculating the volume.

In this report the researcher reports on the interpretation of the region bounded by graphs mainly in relation to Skill factors II and III and partially in relation to Skill factor V and Skill factor I on translation from visual graphs to algebraic equations.

1.3 LITERATURE REVIEW

When solving problems on areas and volumes, students are expected to visualize the region bounded by the graphs, select the representative strip and draw or imagine its 3D representation. According to Rahim and Siddo (2009, p. 496), "visual justification in mathematics refers to the understanding and application of mathematical concepts using visually based representations and processes presented in diagrams ...". Visualisation is referred to by Duval (1999, p. 13) as "a cognitive activity, that is intrinsically semiotic ... neither mental nor physical". Kozhevnikov, Hegarty and Mayer (2002), refer to students who use visualisation as visualisers as they process visual-spatial information in a form of visual imagery and spatial imagery. Gutiérrez (1996) points out that the ability to translate depends on the students' visualisation ability involving the ability to imagine the rotations.

When calculating volumes, students must be in a position to imagine the rotation of the rotated strip and the solid of revolution formulated. In most cases, students lack imaginative skills as a result of their lack of preference to visualize. The results of the study conducted by

Maul and Berry (2000) to test engineering students on differentiation, integration, differential equations and their application to simple physical cases reveal that engineering students prefer verbal representations rather than visualizing when posed with questions. In one study by Rösken and Rolhka (2006), students were unable to name the limits of integration and to use visualization approaches when given a region bounded by graphs. Relating to volumes, the results from Montiel (2005) revealed that students have problems in interpreting questions on volume, including questions given in symbolic form as well as knowing which strip to use. In one study, the lack of skills in three dimensional thinking was evident in a study conducted by Gorgorió (1998) which revealed that students used 2D drawings to represent 3D objects when interpreting 2D representations of 3D objects.

1.4 DATA COLLECTION AND ANALYSIS

Data was collected through classroom observations for five days with 40 students from one FET College using a video recorder. In the presentation of the results, extracts from the classroom observations were used and analyzed mainly in terms of the two skill factors (II and III) as mentioned earlier, for one group comprising of eight students, who agreed to be video recorded. In-depth discussions are presented for lesson five only since it was the last lesson and it was assumed that the students have learnt from the first four lessons and clarified their misconceptions if any.

1.5 CLASSROOM OBSERVATIONS

1.5.1 Observations from the five lessons

Before reporting on how the students interpreted the region bounded by graph when calculating areas and volumes, a summary of how the students were taught throughout the five lessons is given below.

It was observed that translation from visual graphs to algebraic equations in 3D from Skill factor I was evident when the lecturer worked with the students to select the equations to use when calculating volumes based on the different shapes, disc, washer or shell, when interpreting the region bounded by graphs. The second skill factor, three-dimensional thinking involving the translation from 2D to 3D was addressed from the strip that was rotated to form different shapes, disc, washer or shell, which were drawn on the graphs without drawing a solid of revolution formed. The third skill factor, moving between discrete and continuous, was not well addressed since the lecturer drew the representative strip on the board without relating to the Riemann sums. The reason why a Δy or Δx strip was used was not explained or reinforced. The fifth skill factor, namely the consolidation and general level of cognitive development involves all the four skill factors, as well as the skill factor on general manipulation skills, which is not part of this report. Skill factor V was not well addressed since the interpretation of the region bounded by graphs was taught less conceptually. The reason for selecting a particular strip and drawing its 3D representation (a solid of revolution) after rotation were not addressed.

What was also observed was that students were not using textbooks in class. They used notes and questions compiled by the lecturer as hand-outs. It was also evident that there

was more emphasis on finding the formula from the formula sheet rather than deriving it visually from the diagram.

In the extracts that follow the students are referred to as STs and single students as S_1, S_2, \dots and the researcher as R. The words used by the researcher appear in normal font. The students' responses are in italic format, while the reactions and comments for all participants are in square brackets.

1.5.2 Observing the five lessons: Group work

Throughout the five lessons, students worked in different groups. The discussion that follows is of a group of eight students who worked on two questions from previous question papers, one involving volumes and the other one involving areas. During the group discussions, the researcher asked students questions to justify what they were doing as well as probing their responses. One student, S_1 was dominant and was also writing solutions down during the discussions.

Question 1 (involving calculation for volume)

There was an interesting situation when the students struggled to draw the graphs of $\frac{x^2}{9} - \frac{y^2}{4} = 1$ without showing the asymptotes until they finally succeeded as shown from Figure 1 to Figure 3 when a Δy representative strip was chosen. The question required that the students to shade the region bounded by the graphs of $\frac{x^2}{9} - \frac{y^2}{4} = 1$, the x -axis and $y = 3$ as well as showing the strip that would be used when rotating about the y -axis. In selecting the strip, a Δy strip was drawn. When the researcher asked why a Δy strip was used, one of the students argued that a Δy strip is used because rotation is about the y -axis and that if rotation was about the x -axis, a Δx strip would be used. This misconception points to the fact that the students do not have in-depth knowledge on how the strip is selected.

Figure 1: The 1st attempt

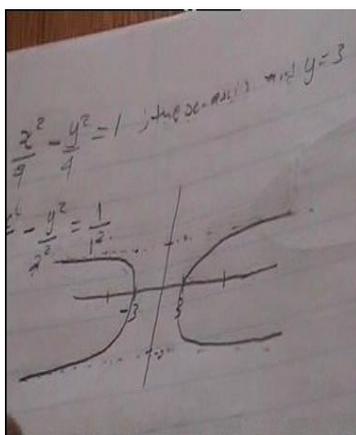


Figure 2: The 2nd attempt

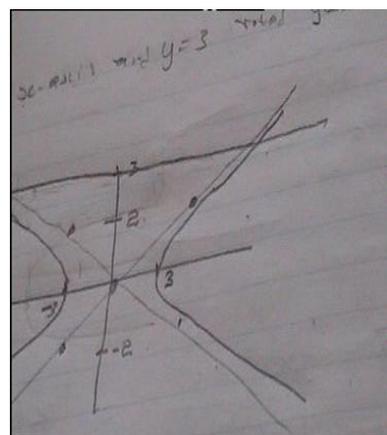
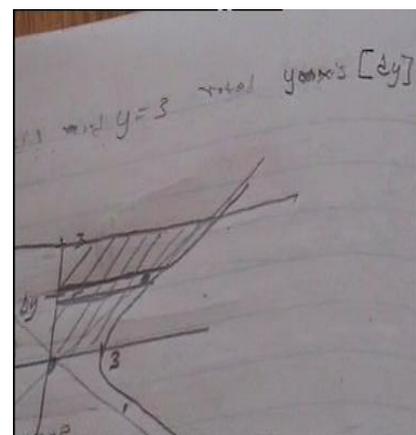


Figure 3: The last attempt



The extract below justifies what transpired during the selection of the representative strip.

R: why are you choosing that one? [referring to a Δy strip]?

S1 and S2: ... it rotates about the y -axis
 R: So when they say rotated about the y -axis. You're going to choose Δy ?
 S1: yah, the strip must be ...
 S3: with respect to y
 R: then what if we change the question and say rotates at the x -axis
 S1: we must change the strip
 R: and put it like?
 S1: like this (referring to Δx strip)
 R: Okay

The question that follows is an example of Skill factor V where all skill factors are consolidated under one question, with the main focus on skill factors II and III.

Question 2 (involving calculation for area)

2.1 Calculate the point of intersection of the graphs of $y = 2x + 2$ and $y = x^2 + 2$

Sketch the graphs and show the representative strip/element that you will use to calculate the area bounded by the graphs.

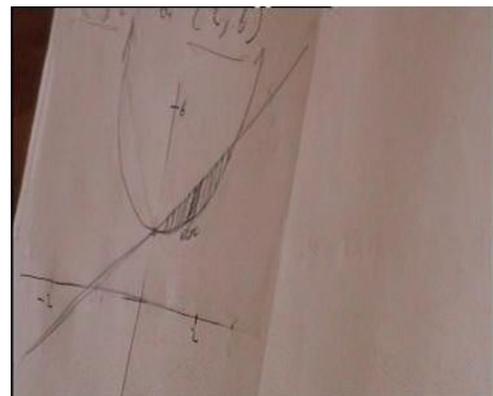
2.2 Calculate the area described in QUESTION 2.1.

2.3 Calculate the area moment of the bounded area about the x -axis as well as the distance of the centroid from the x -axis described in QUESTION 2.1.

After succeeding to draw the graphs after scaffolding as in figure 4, the region bounded by the graphs was indicated and a representative strip was drawn. The region bounded by the graphs was interpreted as in the extract below. It was evident that some of the students had problems in understanding why a Δx strip was drawn.

S2: Question says calculate the area
 S3: The next question says calculate the moment of the area ...
 S1: So this one will be rotating about which axis?
 S2: You just choose
 R: Mmh! [surprised]
 S1: It won't be the same if we chose the y axis they won't be the same.
 S2: Then lets choose ...
 R: What is your question?
 S2: Calculate the area bounded
 R: Ya! And you [referring to S1]? what were you saying
 S1: I was saying they will be equal [referring to answers] if we rotate about the y -axis and about the x -axis?
 R: Read the question ... is there anywhere where they talk about rotating?
 S1: [read the Question 2.3], they say calculate the area moment of the bounded area about the x -axis, as well as the distance of the centroid from the ... axis, so before you start you must read the whole question?
 R: I think it is important.

Figure 4: A Δx strip drawn



S1: Then, we will rotate it about the x-axis [student confused again].
 R: Do they say rotate?
 S2: We will rotate it, about the x – axis [another student also confused].
 S1: It means that we do not have to rotate
 R: They say area moment
 S1: There is no way that it won't rotate
 R: But the question says area moment, do you rotate in area moment? If they say area, do you rotate?
 Sts: [Argue about which strip to use] Lets find the formula
 S3: Lets chose the x one (as shown in Figure 4)
 S1: Let's use Δx strip

The discussion above highlights that the students did not know how to select the strip and neither when to rotate. They do not know that with area one does not rotate. They seem to prefer the Δx . They fail to explain the translation from continuous to discrete based on the strip used.

After selection of the strip, the students selected the formula to use and continued with the calculation. Through continuous scaffolding from the researcher, they managed to get the solution correct. When calculating area, the students wanted to use the centroid formula, the researcher advised them to use the area formula not that of the centroid. When the researcher asked them what the formula for area is this is what one student said.

S1: The formula for area is length times breadth that will be a change in x multiply by a change in y
 R: Why do you go back there, what did you do in the first lesson?
 S3: We will use $A = \int_c^d (x_2 - x_1) dy$
 S2: It is the formula for the washer
 S3: The washer – the washer
 S1: The question says area of this strip ok? ... this is our strip, which formula do we use? We use $A = \int_a^b (y_2 - y_1) dx$

From Question 1 and Question 2 one can conclude that the students were not competent in translating from continuous to discrete since they did not know why a particular strip was used and failed to translate from 2D to 3D as they did not draw such diagrams after rotation. They however, demonstrated some capabilities in translating from visual to algebraic both in 2D and 3D, as they were always able to select the correct formula (disc, washer or shell) and substituted correctly from the graphs.

1.5.3 Discussion from the lessons observed lessons

The students were seen to rely on the formula sheet to find the formula to calculate area and volume and not from their drawn graphs, hence that did not improve on their ability to translate from visual to algebraic in 2D and in 3D. The 3D solids generated when translating from 2D to 3D were not drawn. The students were not competent in three-dimensional thinking (translation from 2D to 3D). The different shapes, disc, washer or shell, were not

drawn. The skill factor on moving between discrete and continuous was not well addressed since the strips drawn were drawn without relating to the Riemann sums. The reason why a Δy or Δx strip was used was not clear to most students. The skill factor involving the consolidation and general level of cognitive development was not well developed, since for example in many instances during the five observation, most of the students failed to draw strips correctly hence were unable to interpret the region bounded by graphs. The students were not competent in identifying the correct representative in drawing the 3D solids formulated after rotation. As a result, the students were not competent in interpreting areas and volumes from the region bounded by graphs?

5.1.4 Summary of the classroom observations

For the lessons observed, the lecturer did not teach the students how to select the rectangular strip stemming from the Riemann sums, probably because they had been dealt with at previous levels. However, from the responses that students produced, it seems that even if the concept of Riemann sums had been done, it might have been at a more procedural level, without relating the continuous to the discrete representations. After selection of the strip, the lecturer focused mainly on the rotation of the strip and using the strip to select the formula from the formula sheet and to do calculations for area, volume, centroid and so on, without drawing the 3D representations of the rotated strip. As mentioned earlier, the students had problems in justifying how they selected the representative strip as well as rotating it. In cases where the strip was rotated correctly, a correct formula was selected and substituted correctly.

1.6 DISCUSSION AND CONCLUSIONS

According to Gutiérrez (1996) the ability to translate properly depends on the students' visualization ability, which is a kind of mathematical reasoning activity based on the use of spatial or visual elements that can assist a student to imagine the rotations. The classroom observations did not include problems where 2D diagrams were rotated to form 3D diagrams or where 3D diagrams were used to determine which 2D diagrams they originated from, as it was evident from the questions given. The rotations that were demonstrated were when representative strips were rotated about a particular axis, resulting in a shell, washer or disc, without drawing the exact solids of revolution. Failure to draw the translated diagram from 2D to 3D in this case interferes with the ability to develop visual and imaginative skills, necessary if mental images are made about rotations. Failure to draw the translated diagram from 2D to 3D in this case interferes with the ability to develop visual and imaginative skills as a cognitive activity Duval (1999).

A conclusion that could be drawn is that students' performance was not satisfactory in Skill factor II, which involves translating between two-dimensional and three-dimensional diagrams. Even if at times the students drew the strip correctly, they could not rotate it correctly as a result of failing to translate between 2D and 3D, or even interpret a 3D diagram to determine which 2D diagram it originated from. If students have difficulty in solving problems that involve three-dimensional thinking, then learning to interpret the region bounded by graphs can be problematic.

In regard to the selection of the representative strip, it was evident that the lecturer gave a clear explanation about the strip being parallel or perpendicular to a certain axis and the

shape it would generate after rotation, hence enhancing visual skills. The problem was that the reason for selecting a parallel or a perpendicular strip to the axis of rotation was not well explained, hence the student lacked competency in that regard. Similarly to Montiel's (2005) study, some students did not know when to use a Δx strip and when to use a Δy strip, mainly because they could not translate between the continuous and the discrete representation. Based on the overall poor performance, it means that the concept of Riemann sums, which is crucial when calculation areas and volumes, is lacking. In order for the students to do well with this content, the concept of Riemann sums should be dealt with at a level where conceptual understanding is reinforced starting from N4 to N6.

In conclusion, the students were not competent in interpreting areas and volumes from the region bounded by graphs. This was evident from their lack of competency in translation between 2D and 3D diagrams and in translating between continuous and discrete representations. In that regard, it seems obvious to conclude that students are likely not to cope with tasks that involve consolidation and general level of cognitive development. This might be due to their poor mathematics background as underperformers from the secondary schools.

1.7 RECOMMENDATIONS

It is recommended that the way in which the questions are asked should be more conceptual rather than being procedural. Before the students are asked to calculate the volume, they must use the Riemann sums to approximate the area of the region bounded by graphs. In that way they may be in apposition to justify whether a correct strip was chosen. The students should then be explicitly asked to represent visually what the strip represent after rotation as either as a disc, a washer or a shell. A solid of revolution generated must also be drawn. Lastly the question may now be asked to calculate the volume. In that way the formula chosen will be from the 3D representation drawn. The researcher believes that changing the ways of assessment will change the way in which teachers teach, as a result, learning may improve.

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