Proof of the F+V= E+2 formula using the heuristic method and the significance of the proof for school mathematics

Chrysoula Demetriou-Hadjichristou
Saint Nicholas School – Limassol, Cyprus
Email: chrysoh@cytanet.com.cy

Abstract
In their attempt to express their disagreement regarding the fact that mathematical knowledge is given as a granted in advance and is unmistakable, Polya and his student Lakatos try to bridge the gap between the philosophy of mathematics and physical sciences, not accepting that mathematics is an empirical science. Lakatos classifies mathematics as a quasi-empirical science. His motto is that mathematics, just like the physical sciences, is not absolute truths and its development can be derived through criticism and the correction of its theories. The view of the philosophical theories of "school" mathematics leads, according to Ernest, to two new, different conflicting theories, those of *absolutism* and *fallibility.* As a result, the second theory makes the curriculum of "school" mathematics humane with values, social context and meaning, so that the application of the curriculum to "school" mathematics does not resemble a misunderstanding game.

1. The aim of study
The aim of this article is to prove the F+V=E+2 formula of the polyhedrons, which has troubled generations of scholars of mathematics for the truth it carries but also for the methods of approach. According to Putman, "mathematical knowledge resembles empirical knowledge, which holds the success of our practical ideas as a mathematical criterion, just like in Science. Mathematical knowledge can be corrected and is not absolute, (Kotsier, 1991, p. 293). Kotsier (1991) interprets the phrase "the success of our practical ideas" and says that it is success that brings results of value and credibility that can be useful in the physical sciences. According to Kotsier, then, Euclidian theories are replaced with some other, more successful theories that can be corrected. He examines the history of mathematics from the time of the Ancient Greeks to today and notes that one may come to the conclusion that the improvements in mathematics, when compared to those of the physical sciences, are very few. He also adds that, during the '60s, Lakatos, as well as Putman in the '70s, used the new Heuristic Method, which was introduced by Polya, under the term quasi-empiricism. In 1967, Lakatos distinguished between the quasi-Empiricist and Euclidian theories, and the differences have already been mentioned.

Finally, according to Kotsier (1991), and given the proof of the polyhedrons formula, Jardan and Becker mention that when Euler assumed that the formula was applicable to all polyhedrons, he made the following mistake: though the formula was correct as a theorem, its truth was dependent on one limiting factor – it was applicable only to convex polyhedrons. With the help of the Proportional Reasoning we can say that this is valid in the 19th century with the Euclidian theories and the appearance of hyperbolic geometry. Just like the polyhedrons' formula was not refuted, the truth of the Euclidian geometry was not refuted either, but it was limited to its own "field" of truth, as Kotsier (1991) says. Therefore, let us welcome the new Heuristic Methods theories in the new century, which are becoming acceptable the moment when "the physical sciences are distinguished from the geometrical ones" (Kotsier, 1991, p. 280) and let us take the first step in inducting the philosophy of refutation into "school" mathematics, to reveal the humane educational practices.

2. Polya's scientific, philosophical context

In 1934 there was a revolution in the philosophy of science when Karl Popper stated that it is not possible, nor necessary to justify the laws of Science, thus justifying inductive thinking. Popper claimed that scientific theories are not derived from facts inductively; instead a theory can be scientific only if it can be judged, challenged and examined even if it can be refuted. Through this process, a theory can become credible and establish itself temporarily until new facts come to surface to refute it again. The idea of refutation and its possibility as scientific knowledge, are obvious with Popper (Davis & Hersh, 1981, p. 330).

Within the philosophical set of the somewhat acceptance of Popper's ideas regarding science and the prevalence of fundamentalists in mathematics, Lakatos, a student of Polya, appeared as a philosopher with a mathematics background and a supporter of Popper's theory of scientific knowledge. Polya suggested that Lakatos uses the history of the Euler-Descartes formula of the polyhedrons in his doctoral thesis. Lakatos had already translated one of Polya's books, "How to solve it", to Hungarian to earn money, and considered Popper and Polya as his mentors. His work "Proof and Refutations" is his doctoral study on polyhedrons, which was published in 1963-64 in four parts in the "British Journal for Philosophy of Science". His work was published two years after his death on the 2nd of February 1974 because he was downright negative about it being published as a book. Locates' ideas about semi-empiricism became widely known through his work.

Through his role as a teacher in his book Proofs and Refutations, Lakatos, as an expressionist of the theory of semi-empiricism recommends to his students to think of "technical terms" for the proof of the conjecture. He proposes terms such as "Experimental Thinking" or "Semi-empirical Thinking" which suggest the disassociation from the initial conjecture to sub-conjectures or lemmas in which a big part of knowledge is possibly incorporated (Lakatos, 1976, p. 9). In the book Proofs and Refutations (Lakatos, 1976) the dual role of Lakatos is depicted as that of the "teacher" as well as that of his "students", as presented in an imaginary class.

It is possible to observe that the students come up with numerous counter-examples to prove the initial conjecture. The presentation of these counter-examples by Polya and

Lakatos in Proofs and Refutations leads to the modification of the phrasing of the theory, to the improvement of the proof and to its further process. The pattern of this process is characterised by "new counter-examples", "new applications" and every step of the proof is subject to criticism, as mentioned by Lakatos' analysts, Worrall and Zahar (Lakatos, 1976).

Many sub-conjectures or lemmas have been stated with different counter-examples (like Polya did) checking each conjecture with a different counter-example. Each counter-example judges the proof (initial conjecture) and not the hypothesis, as Lakatos mentions thus reminding us that Colombus did not reach India, but he did discover something important. Even if the proofs do not prove something, they definitely improve the hypothesis. Finding the "exceptions" improves it as well, but Lakatos believes that **improvement is independent of the proof** (Lakatos, 1976, p. 14). Lakatos considers that the internal unification of the "logic of discovery" and the "logic of justification" is the most important aspect of the method of lemmas. Most mathematicians, he adds, will either prove or refute a hypothesis. They want to improve their hypothesis without refutation, without ever reducing the mistakes, but with the monotonous increase of the truth.

The students, lead by the term "counter-examples" as "monster creations" or as "medical cases" of polyhedrons, which spread panic and for which there is no conjecture, have called these "exceptions". Poincare says that: "some times, logic creates monsters" (Lakatos, 1976, p.22). According to Worrall and Zahar then, counter-examples are those for which the conjecture is valid and exceptions are the correct naming of those polyhedrons that do not fit the conjecture (initial hypothesis). Students distinguish three kinds of "Mathematical sentences":

1. Those that are always true and for which there is generally no limitation or exception;
2. Those that are the result of some initially wrong sentences and can, in no way, become acceptable; and,
3. Those that, despite the fact that they hinge on the initially true sentences, can take in some cases limitations or exceptions (Lakatos, 1976, p.24).

Exceptions prove the rules, says one student, and we should not confuse the wrong theory with those subject to some limitations. With reference to Euler's conjecture and whether it is right or wrong, students admit that it belongs to the **third category** of sentences for which the conjecture is initially right but takes some exceptions in some cases. They also admit that if the theories coexist with the exceptions, this leads to confusion and chaos in mathematics (Lakatos, 1976, p.25).

Finally, Polya (1954) proceeds to an important limitation of his conjecture and with the use of analogy, he proves the formula of polyhedrons.

3. Proof of the formula using analogy

Polya admits that the damage done to the polyhedrons formula by the "doughnut-shaped" polyhedron, was as if it "killed" the initial conjecture, which is subject to one important "limitation". Polya (1954) says that polyhedrons are **convex** as well as those polyhedrons that are derived from them just like the "roof-shaped" polyhedron and the "cut cube". He believes that there is no danger of someone being led from a convex shape to a non-convex polyhedron as with the doughnut-shaped. He concludes that the relationship exists **for any convex polyhedron**, however he is a bit hesitant to say whether the conjecture can prove to be true or not.

According to Lakatos (1976), the case of the "picture-frame" polyhedron that was the recipient of the damage, belongs to the exceptions, which do not refute the conjecture, but instead they improve it without proving it. Polya continues to support the conjecture regarding the polyhedrons despite his difficulty in trusting its validity. He does not hope to have more "help" from the verification of other polyhedrons. He believes he has exhausted all the obvious sources of such polyhedrons.

Polya (1954) wonders, like he does with each of his inductive procedures, if there is another kind of analogy, which can help prove the truth of the formula and he realises: **"Polygons are proportional to polyhedrons"** (Polya, 1954, p. 43). He says that a polygon's vertices are equal to its edges, V=E. This relationship is valid for convex polygons and seems quite simple. This sheds a little light to the much more complex relationship of F+V=E+2, which seems to be valid for every convex polyhedron. In order to bring the two relationships closer together, he claims that: "polyhedrons are three-dimensional and their faces are polygons, which are two-dimensional with one-dimensional edges and zero-dimensional vertices. By rewriting the equation and adjusting the amounts depending on the dimensions, the relationship of the polygons is written as such: V-E+1=1 (I) (V=E ➔ V-E=0)

Polya compares the above to the relationship of the polyhedrons, which is written as such:
i. V – E + F – 1 = 1 & ii. V + F = E + 2
where 0, 1, 2 indicate the corresponding dimensions to V, E, F.
Polya (1954) ends up with the conclusion that: **"the analogy seems to be complete"** (p. 43). This means that the dimensions in the polygons formula (V=E) satisfy the relationship (0=1). By using the analogy, he also proved that the dimensions satisfy the relationship (0=1) of the polyhedrons formula (V+F=E+2). Given that the polygons formula (V=E) is correct, the analogy enhances the correctness of the formula (V+F=E+2) of the polyhedrons, which was the initial conjecture for which the F+V=E+2 is true (Polya 1954, p. 43). He has therefore proved that by using the **analogy** and the term **dimension**, polygons are proportional to polyhedrons, establishing thus the truth of the Euler-Descartes formula.

In the subsequent section a discussion is provided to refer to the significance of the aforementioned proof and the effect of the Heuristic Method on mathematicians who are "doing" mathematics and on philosophers regarding the Euclidean theories.

4. Lakatos' Ideas and his Scholars

We will look at some thoughts about the scientific-philosophical context of the "Proof" using the Heuristic method, as suggested by Polya, in regards to the Euclidean proofs. As Davis and Hersh (1981) mention, according to some scholars the name of the mathematical game is proof, without it there is no mathematics. However, others have refuted that this is nonsense. There are many games in mathematics (Davis and Hersh, 1981, p. 154) and one of them is the Heuristic method with which Polya, according to Lakatos' (1976) scholars Worral and Zahra, managed to describe the proof of the Euler-Descartes formula by "trial and error" nicely. It is, therefore, because of Polya that the Heuristic method was reintroduced in the 19th century (Lakatos, 1976, p.74).

When Lakatos undertook the role of the teacher later on, he applied the "trial and error" method to an imaginary class where he observed that the F+V=E+2 relationship was valid for all polyhedrons. In his book Proofs and Refutations, Lakatos (1976) says that many of his "students" had tried to refute this conjecture and to check it with many different ways successfully. The results confirm the conjecture and suggest that it may be "proved" (p. 6). Epsilon, a student, was considered to be the first Euclidean to accept Lakatos' proof. He takes on the teacher's challenge to prove the F+V=E+2 formula telling him that "**I will start to prove where you have stopped with the improvement"** (Lakatos, p. 106). Being a Euclidean, Epsilon does not accept the existence of counter-examples as the absolutely correct proof. He uses axioms as absolute truths. He relies on the fact that proofs develop *through logic*, and since there are no doubts about logic, there is certainty about the proof. Any possible doubts may concern only the lemmas. In this closed Euclidean system the use of counter-examples is not allowed. This approach is definitely in contradiction to the one introduced by Lakatos.

Epsilon finally proved the formula with the help of the Vector Algebra (p. 116) saying "I followed the great school of mathematics of the rationalists, who have kept mathematics safe from doubt in the name of the initial definitions, and have shown the strict proof of their propositions" (Lakatos, 1976, p. 121).

For Lakatos, the proof of the non formalist mathematics in this general context does not mean a mechanical procedure that imposes the truth through continuous conjectures and conclusions. This rather means explanations, verifications, processes that make the conjecture more valid, more convincing while it becomes more analytical and precise under the pressure of the counter-examples. Every step of the proof is, too, subject to **criticism,** which is the main pattern in a semi-empirical system, even a form of reservation or the result of a counter-example in a very special attempt. A "local counter-example" according to Lakatos is a counter-example that questions a given step in an attempt. A "general

counter-example" is that which questions not the attempt, but the conclusion itself (Lakatos, 1986).
In the semi-empirical system, the typical flow of things is to bring Lies back from the false basic sentences **"down-up"** to the hypothesis says Lakatos (1986). In this case, logic is an *instrument of criticism.* The main pattern of semi-empirical criticism is *refutation.* On the contrary, in the Euclidean system, the typical flow is the transfer of the Truth from the total of the axioms **"up-down"** to the rest of the system. In this case, logic is an *instrument of Proof*. The main pattern of the Euclidean criticism is *suspicion:* **Do proofs really prove?** (Lakatos, 1986).

At this point, Lakatos' (Proof & Refutations) analysts Worrall & Zahar claim that those who, because of the deductive guessing, believe that the path of discovery leads from the axioms and/or definitions to proofs and theories may completely forget the possibility of existence and value of naïve guessing. Worrall & Zahar add that it is a fact that in the Heuristic method proof is dangerous in deductivism (from general to specific), whereas in the physical sciences proof is more dangerous in inductivism (from specific to general), (Lakatos, 1976, p. 73).

In his role as a teacher, Lakatos finds it mandatory to separate the two heuristic models and therefore makes the following distinction: the one model is the guess from deductive thinking, which he considers to be the best, whereas the other model is naïve guessing, which is considered to be better than non-guessing. Naïve guessing is not inductive, says the teacher, and his students react. The students claim that naïve guessing can be done inductively and distinguish two kinds of guessing: those that come before the standardization of the conjecture and those that come after. The first suggest the conjectures, whereas the second "support" them. They believe that both kinds provide some kind of conflict between the conjecture and the "facts". This double kind of conflict is the heart of inductivism: the first kind creates the **"Heuristic Inductivism"** and the second kind creates the "**Inductive Verification**" or "Inductive Logic" (Lakatos, 1976, p. 73).
The teacher reacts violently claiming that facts do not suggest conjectures, nor do they "support" them. He believes that the "dead", "forgotten" conjectures that have already been studied, suggest the facts. Naïve conjectures, the teacher adds, are not inductive, but we are lead there through trial and error, through conjecture and refutation. The teacher also says that what is taught is that the path of discovery leads from the facts to the conjectures and from the conjectures to proof (Inductive Method). This way, it is possible that the alternative heuristic method of **deductivism (from general to specific)** may be forgotten, says Lakatos (1976).

Many mathematicians and supporters of the Euclidean method were terrified by Lakatos' thoughts and wondered about the proofs they would use and whether the conjectures could be verified (Worrall & Zahar, 1976). On the other hand, they knew from experience that proofs are indisputable. They also knew from dogmatic teaching that authentic proofs should be true. Mathematicians who teach applied mathematics believe that the proofs used by mathematicians who teach mathematics are adequate, therefore truly proven. Those mathematicians teaching mathematics value only the “adequate” proofs of rationalists (Lakatos, 1976, p. 29). When G. H. Hardy, a rationalist mathematician, supporter of formal proof according to Worrall & Zahar, was asked to characterise the mathematical proof, being familiar with the subject, he replied: “Is such strictness applied elsewhere as it is applied to mathematical proof?” (p. 29), and he adds,

> “At the end of our analysis we may prove nothing, but a few points… proof is what Littlewood and I call gas, designed rhetorically to affect psychology, pictures on the lecture board, stimulation strategies for the students’ imagination (Lakatos, 1976, p. 29).

According to Worrall & Zahar, R. L. Wilder thinks that proof is just a procedure that allows control and that it is an application of the suggestions of our intuition. “Intuition is not an indisputable guide, but a non-indisputable guide is better than no guide at all”, says Putnam (1979). He continues to claim that the subjective meaning of self-evident as well as the meaning of intuition-clairvoyance are involved in the semi-empirical method. If our intuition was unreliable we would never think of a correct or an almost correct theory to examine. Putman’s objection to someone’s attempt to argue against the a-priori nature of mathematical knowledge is the method of mathematical proof which seems to be, at first, the only method used by mathematicians. Mathematical proof is simply there, drawing conclusions from the set axioms, and is governed by rules of inference, once and for all. Of course, he continues, the meaning of proof is not degraded. On the contrary, proof and the semi-empirical method are considered to be **complementary**. Proof has the great advantage of not increasing the risk of contradictions, whereas the introduction of new axioms or new items increases the risk of contradiction. «**For this reason, proof will continue to be the main method of mathematical verification»** (Putman, 1979, p.63).
As Davis and Hersh (1981) mention, in his attempt to show the difference between the Euclidean theories and the pseudo-empirical theories, according to which mathematics is basically about conjectures and not undeniable truths, Lakatos emphasizes that his theory is «pseudo-empirical» (not clearly empirical as physical sciences are). He even says that «the potential creators of lies» or «the basic statements» of mathematics are not – in contrast to the propositions of the physical sciences – mere spatiotemporal statements. At this point, Lakatos was questioned by Davis and Hersh (1981) regarding the fact that he was unable to explain in his theory the «items» of the non-standard mathematical theories. They, then, wonder «to what kind of beings was he referring to when he was talking about numbers, triangles or about probabilities beyond the system of axioms and definitions?» (p. 335).

Davis and Hersh (1981) defend Lakatos when his analysts, Worrall and Zahar, claim that «he undermines the achievements of mathematical strictness a little» and that «under no circumstances are these proofs indisputable»(p.339). They add that modern and contemporary typical proof is indisputable and the only source of doubt about the truth of the conclusions is the doubt about the truth of the conjectures. According to Worrall and Zahar, Davis and Hersh (1981) say that if we treat the theorem not merely as phrasing the conclusions drawn from it, but as a hypothetical sentence of the kind: "if the hypothesis is true, then the conclusion is true" then based on this hypothetical form, the achievements of the logic first class make the truth of the theory undeniable/indisputable. Therefore, according to Worrall and Zahar, Lakatos was wrong about the possibility of error, as mentioned by Davis and Hersh (1981). Davis and Hersh (1981) strongly disagree with Lakatos' analysts and say that Lakatos is right and knows the possibility of the presence or not of error in a given proof. This is obvious from the fact that in the introduction of his book Proofs and Refutations where he attacked the Formalists by saying: "the mistake is identifying with mathematics, that is, what Mathematicians really do in their everyday life with the model or their demonstration in meta- mathematics or with the first class logic" (Davis & Hersh,1981, p. 337).

According to Davis and Hersh (1981), this is the real situation: on the one hand, we have the real mathematics with proofs backed by the consensus of the "experts", but on the other hand a standard proof cannot be checked by a machine, nor by a mathematician who does not fully possess the way of thinking regarding that particular area of mathematics where the proof is focused. Whether a proof is complete or correct, is the result of discussion and clarification of any possible doubts that may arise, according to Davis and Hersh (1981). Once a proof is acknowledged/accepted, its results are considered to be true (with high probability) and it may be generations later that an error is spotted in it. If a theory is widely known and used, its proof is studied frequently, alternative proofs are invented, it had known applications and generalisations and it is pertinent to known results in relevant fields, then it ends up being considered "a stable foundation". Of course, Davis and Hersh (1981) say that this was both Arithmetic and Euclidean Geometry are considered to be a stable foundation (p. 338).

Worrall and Zahar are once again judged by Davis and Hersh (1981) about the meaning of the real proof. Worrall and Zahar say that a real proof is nothing but an **incomplete, formal proof**. Davis and Hersh (1981) claim that in real mathematical practice, we distinguish between a complete, informal proof in problem mathematical solving (that looks like persuasive) and an incomplete, formal proof, mentioned that both *"as typical proofs are incomplete".*

Polya points out that even "incomplete" proofs establish the connection between mathematical facts and this helps us memorise them.

The success of the proof of the polyhedrons formula using the Heuristic method, and the discussions of the philosophers and other Mathematicians, were and continue to be quite interesting and do not cease to fascinate scholars regarding the topics of mathematics that should be taught at schools. In one of is credible and very important articles about the nature and *teaching of mathematics*, Ernest (2008) refers to the two concepts about the character of mathematical knowledge and about the views on teaching. *He distinguishes between the absolutism and the fallibility.*

Referring to the philosophy of fallibility, he makes extensive reference to Lakatos' theory emphasizing that mathematical knowledge is considered to be fallible, open to revision regarding the proofs of its theories and its fundamental meanings (Lakatos, 1976). This concept makes the philosophy of fallibility more human and is interdependent with the values and nature of school mathematics because it rejects the absolutist image of mathematics. He claims that mathematics is a human, personal, intuitive, active, collaborative, creative, investigative, cultural, historical experience, alive with human situation, it is a pleasant experience filled with joy, curiosity and beauty. Educators have supported/encouraged the educational reformation of "school" mathematics and the adoption of humane perspective on it.

Ernest (2008) concisely and clearly develops his theory on the relationship between his personal views and thinking on mathematics, the values and images of mathematics as projected in class. He then records his findings in a table and compares the two philosophies of mathematics, absolutism and fallibility. He states that many educators with humane values and perspective of school mathematics are forced to adopt a "strategy of compromise" thus excluding school mathematics from social values and leading educators to experience stress and psychological pressure. Ernest concludes by saying that if we want to change the *negative* image of mathematics, *we need to change the image projected in classrooms* regarding the curriculum and content, as well as *social* context. He considers the sex-related performance in mathematics an example. He also says that the relationship between educators and students needs to be improved, the competitiveness imposed among students should be lessened and so should the extent of the negative impact derived from the students' mistakes, as these lead to their public humiliation, their failure and effects on their self-esteem and self-image. For all the above reasons it is mandatory to find new approaches in teaching and learning school mathematics!

References

Αναπολιτάνος, Δ. Α. (1985). *Εισαγωγή στη Φιλοσοφία των Μαθηματικών*. Αθήνα : Νεφέλη.

Artmann, B. (1999). A short history of the regular polyedra. *In MAVI-8 Proceedings –8th European Workshop Research on Mathematics Beliefs* (pp. 1-7). Edited by G. Philippou. Nicosia: University of Cyprus.

Boeyére, C. B. (1991). *A history of mathematics*. New York: J. Willy & Sons.

Davis, P. J. & Hersh, R. (1981). *Η Μαθηματική Εμπειρία.*(Μετ. Γ. Αναστασιάδης). Αθήνα: Τροχαλία.

Heath, T. (1956). *Euclid the thirteen books of the elements*. *Vol. 3. (Books X-XIII).* New York: Dover Puplications.

Heath, T. (1981). *A history of Greek mathematics. Vol. I. From Thales to Euclid.* New York: Oxford University.

Kotsier, T. (1991). *Lakatos' Philosophy of mathematics*: *A historical approach*. Amstertam: NH.

Lakatos, I. (1976). *Proofs and refutations. The logic of mathematical discovery.* Edited by John Worrall & Elie Zahar. Cambridge University Press.

Lakatos, I. (1986). A Renaissance of Empiricism in the recent Philosophy of Mathematics. *New directions in the Philosophy of Mathematics* (pp. 29-49). Edited by Tomas Tymoczko. Boston: Birkhauser.

Παπαδόπουλος, Σ. (1985). *Θεμέλεια και Βασική Δομή της Ευκλείδιας Γεωμετρίας.* Αθήνα: Gutenberg.

Πλάτων «Τίμαιος» (Εισαγωγή, Μετάφραση, Σχόλια Θ. Βλυζιώτη & Χ. Παπαναστασίου). Αθήνα: Ε. & Μ. Ι. Ζαχαρόπουλος Ε. Π. Ε.

Pólya, G. (1954). *Induction and Analogy in Mathematics. Vol. I of Mathematics and plausible reasoning.* New Jersey: Princeton University Press.

Pólya, G. (1968). *Mathematics and plausible reasoning. Vol. II Patterns of plausible inference.* New Jersey: Princeton Universitty Press.

Putnam, H. (1979). What is Mathematical Truth. *New directions in the philosophy of mathematics* (pp. 49-67). (Edited by Thomas Tymoczko). Boston: Birkhauser.