Identifying Vertices in Graphs and Digraphs

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IDENTIFYING VERTICES IN GRAPHS AND DIGRAPHS

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I declare that *Identifying Vertices in Graphs and Digraphs* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

SIGNATURE   DATE

(MR R D SKAGGS)

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SUMMARY

The *closed neighbourhood* of a vertex in a graph is the vertex together with the set of adjacent vertices. A *differentiating-dominating* set, or *identifying code*, is a collection of vertices whose intersection with the closed neighbourhoods of each vertex is distinct and nonempty. A differentiating-dominating set in a graph serves to uniquely identify all the vertices in the graph.

Chapter 1 begins with the necessary definitions and background results and provides motivation for the following chapters. Chapter 1 includes a summary of the *lower identification parameters*, γ_L and γ_d. Chapter 2 defines co-distinguishable graphs and determines bounds on the number of edges in graphs which are distinguishable and co-distinguishable while Chapter 3 describes the maximum number of vertices needed in order to identify vertices in a graph, and includes some Nordhaus-Gaddum type results for the sum and product of the differentiating-domination number of a graph and its complement.

Chapter 4 explores criticality, in which any minor modification in the edge or vertex set of a graph causes the differentiating-domination number to change. Chapter 5 extends the identification parameters to allow for orientations of the graphs in question and considers the question of when adding orientation helps reduce the value of the identification parameter. We conclude with a survey of complexity results in Chapter 6 and a collection of interesting new research directions in Chapter 7.

Key terms: graph theory, domination, differentiating-domination, identifying code, criticality, graph complement, Nordhaus-Gaddum results, oriented graph

Chapter 1

Preliminary Results

Suppose we have a building into which we need to place fire alarms. Suppose each alarm is designed so that it can detect any fire that starts either in the room in which it is located or in any room that shares a doorway with the room. We want to (1) detect any fire that may occur or (2) use the alarms which are sounding to not only detect any fire but be able to tell exactly where the fire is located in the building. For reasons of cost, we want to use as few alarms as necessary. The first problem involves finding a minimum *domination* set of a graph. If the alarms are three-state alarms capable of distinguishing between a fire in the same room as the alarm and a fire in an adjacent room, we are trying to find a minimum *locating-domination* set. If the alarms are two-state alarms that can only sound if there is a fire somewhere nearby, we are looking for a *differentiating-domination* set of a graph. These three areas are the subject of much active research; we primarily focus on the third problem.

Unless otherwise stated, we follow the terminology of [34] with [12] as a secondary source. For any of the graph parameters π discussed here, a π-set is an optimal set with the given property.

1.1 Undirected graphs

A *simple graph* $G = (V, E)$ consists of a finite non-empty set $V(G)$ and a finite set $E(G)$ set of distinct unordered pairs of distinct elements from the set $V(G)$. The set $V(G)$ is called the *vertex* set of G while $E(G)$ is called the *edge* set of G. We say $|V(G)| = n$ is the *order* of G and $|E(G)| = m$ is the *size* of G.

A graph is typically visualised as a collection of points representing the vertices which are joined by (not necessarily straight) lines representing the edges. We typically write $vw \in E(G)$ to indicate that $e = \{v, w\} \in E(G)$; we say that v and w are *adjacent* vertices and that both v and w are *incident with* e. The *degree* $\deg(v)$ of a vertex v is the number of vertices adjacent to v; equivalently, $\deg(v)$ is the number of edges incident with v. If $\deg(v) = 1$, the vertex v is said to be a *leaf.* If for for some constant r we have $\deg(v) = r$ for all vertices $v \in V(G)$, then G is said to be r*-regular*. Every pair of vertices is adjacent in a *complete graph* of order n. A complete graph, denoted K_n, is $(n-1)$-regular and has size $n(n-1)/2$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and H contains no edges that are not in G. If a subgraph H of G is such that for every pair of vertices $u, v \in V(H)$ it is the case that $uv \in E(H)$ if and only if $uv \in E(G)$, then H is an *induced subgraph* of G. A *clique* is a complete subgraph of a given graph.

Note that since $E(G)$ is a set that consists of pairs of distinct elements, no loops or multiple edges are permitted. That is, no vertex is adjacent to itself, and if e_1 and e_2 are both incident with vertices v_1 and v_2 then $e_1 = e_2$.

A *walk* of length $k-1$ is a sequence of vertices and edges $v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k$ in which $e_i = v_i v_{i+1}$. Since the edges are determined by the vertices with which they are incident, a walk is generally given by simply listing its vertices.

A walk in which all the vertices are unique is called a *path*. If we have a path with $v_1 = v$ and $v_k = w$, we say there is a path between v and w. If there is a path between any two vertices in G, we say G is *connected*. A graph that is not connected is said to be *disconnected*; if a graph has no edges it is said to be *totally* disconnected, or trivial.

If $v_1 = v_k$, the walk is said to be *closed*. A closed walk in which all the vertices $v_2, v_3, \ldots, v_{k-1}$ are distinct is called a *cycle*. A connected acyclic graph is called a *tree*, while the disjoint union of trees is called a *forest*.

The *complement* $\overline{G}$ of a graph G has $V(\overline{G}) = V(G)$ with $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. Note that at least one of G or $\overline{G}$ is connected.

Two vertices v and w are *independent* if $vw \notin E(G)$. Two edges are independent if they do not share any incident vertices. A set of vertices or edges is independent if each pair in the set is independent.

A graph G is *bipartite* if the vertex set can be partitioned into two nonempty independent sets. Notice that all trees are bipartite. A *complete bipartite* graph $K_{r,s}$ has vertex set $V = V_1 \cup V_2$, where $|V_1| = r$, $|V_2| = s$, and two vertices v and w are adjacent if and only if $v \in V_1$ and $w \in V_2$. As a special case, $K_{1,n-1}$ is called a *star* of order n.

It is often beneficial to construct a graph from other graphs. One such construction is the disjoint union of G_1 and G_2, which is denoted $G_1 \cup G_2$ while kG denotes the disjoint union of k copies of G. The *join* of G and H, denoted $G+H$, is formed from $G \cup H$ by adding all the edges between $V(G)$ and $V(H)$.

The *open neighbourhood* $N(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v. The *closed neighbourhood* $N[v]$ of v is $N[v] = N(v) \cup \{v\}$. For $S \subseteq V$, $\bigcup_{v \in S} N(v)$ is written $N(S)$ and $\bigcup_{v \in S} N[v]$ is written $N[S]$. For $S \subseteq V$, $N_S[v] = N[v] \cap S$ and $N_S(v) = N(v) \cap S$. The *complement* of S is $\overline{S} = V(G) - S$.

A set of vertices $S \subseteq V(G)$ is a *dominating set* if $N[S] = V(G)$. That is, S is a dominating set if every vertex in G is either in S or is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the smallest cardinality of a dominating set in G. The study of domination in graphs began in the early 1960's; MathSciNet currently lists more than 1300 papers on domination-related topics.

1.2 Oriented graphs

A *directed graph* or *digraph* $D = (V, E)$ consists of a finite non-empty set $V(D)$ of vertices and a finite set $E(D)$ of ordered pairs of distinct elements from the set $V(D)$ which we call *arcs*. As with graphs, the cardinality of the vertex set is the *order* of D and the cardinality of the arc set is the *size* of D.

If $v, w \in V(D)$ and $(v, w) \in E(D)$, we say w is *adjacent from* v and that v is *adjacent to* w. If no confusion will arise we use vw to indicate the arc (v, w). We say the arc $a = vw$ is *incident from* v, *incident to* w, or *directed from* v to w.

When a digraph is represented by a figure, an arrowhead is typically used to show the order of the vertices of each arc. Directed graphs can, for example, represent a street network in which some of the streets are one-way. This information can be readily described if the arrowheads are drawn according to the permissible traffic flow.

A *walk* of length $k-1$ in a digraph is a sequence of vertices and arcs $v_1, a_1, v_2, a_2, \ldots, a_{k-1}, v_k$ in which $a_i = v_i v_{i+1}$. Since the arcs are determined by the vertices with which they are incident, a walk is generally given by simply listing its vertices. A path in a digraph is a walk $v_1, v_2, \ldots, v_n$ in which no vertex is repeated; we say the path is *from* v_1 *to* v_n. For $n \geq 3$, a cycle in a digraph is a walk $v_1, v_2, \ldots, v_n, v_1$ whose n vertices v_i are distinct.

A digraph D is said to be *symmetric* if vw is an arc of D whenever wv is. A digraph D is an *oriented graph* if whenever vw is an arc of D then wv is not an arc of D.

An oriented graph D can be obtained from a simple graph G by assigning a direction to each edge of G. We say that G is the *underlying* graph of D and that D is an *orientation* of G.

The *reversal* of an oriented graph D, denoted D^{-1}, is the oriented graph obtained by reversing the direction of each arc of D, so $V(D) = V(D^{-1})$ and $vw \in A(D^{-1})$ if and only if $wv \in A(D)$. An oriented graph D is *connected* if the underlying graph G of D is connected. A digraph D is *strongly connected* if for every pair of vertices u, v in D there is a path from u to v.

We define the *outset* of a vertex v by $O(v) = \{x \in V(D) : vx \in A(D)\}$. The *outdegree* of v, $\text{outdeg}(v)$, is $|O(v)|$, the cardinality of the outset of v. The *closed outset* of v, denoted $O[v]$, is the outset of v together with the vertex v. That is, $O[v] = O(v) \cup \{v\}$. Similarly, the *inset* of v, denoted $I(v)$, is defined as $I(v) = \{x \in V(D) : xv \in A(D)\}$, the *indegree* of v, $\text{indeg}(v)$, is $|I(v)|$ and the *closed inset* of v is $I[v] = I(v) \cup \{v\}$. The intersection of a set S and the outset of a vertex is denoted $O_S(v)$ while the intersection of S and the inset of a vertex is denoted $I_S(v)$. The intersection of S and the closed outset (resp. closed inset) of a vertex v is given by $O_S[v]$ (resp.

$I_S[v]$). For $S \subseteq V$, $\cup_{v \in S} O[v]$ is written $O[S]$ and $\cup_{v \in S} I[v]$ is written $I[S]$. The *degree* of v is given by $\deg(v) = \text{outdeg}(v) + \text{indeg}(v)$.

A set of vertices S in an oriented graph D is *dominating* if all vertices of D are elements of $O[S]$. The cardinality of a smallest dominating set in D is denoted $\gamma(D)$.

Theorem 1.1 (Ghoshal, et al. [30]) *Let D be a connected digraph with n vertices.*

(a) $\gamma(D) \leq n - 1$.

(b) $n/(1+\Delta(D)) \leq \gamma(D) \leq n - \Delta(D)$, *where $\Delta(D)$ denotes the maximum outdegree.*

Given an oriented graph D, a set of vertices S is *absorbant* if for every vertex $v \notin S$ there is a vertex $w \in S$ such that vw is an arc in D. Equivalently, S is absorbant if all vertices of D are elements of $I[S]$.

Observation 1.2 *If S is absorbant in D, then S is dominating in D^{-1}.*

Proof. Let S be an absorbant set in D. Then $I[S] = V(D)$ implies that for every vertex v in $V(D) - S$, there is a vertex w in S such that vw is an arc in D. Thus, for all vertices v in $V(D^{-1}) - S$ there is a vertex w in S such that wv is an arc in D^{-1}. Therefore all vertices of D^{-1} are elements of $O[S]$, so S is a dominating set in D^{-1}. ■

Two vertices v and w in an oriented graph D are *independent* if neither vw nor wv is an arc in D. A set of vertices is independent if each pair of vertices is independent. An independent absorbant set is called a *kernel*. Most domination-related research in directed graphs has focused on the study of kernels, which have applications in such areas as cooperative n-person games, Nim-type games, and logic [2, 30].

Theorem 1.3 (Berge [3]) *Let D be a digraph and let $S \subseteq V(D)$. If S is a kernel, then S is a maximal independent set and a minimal absorbant set.*

1.3 Identifying vertices

A set of vertices $L = \{v_1, v_2, ..., v_k\}$ was defined by Slater [49] to be *locating* if the vector $\vec{v} = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$ is unique for each vertex v in the graph. In other words, any vertex in the graph can be located if its distance from each of the vertices in the set L is known. This idea of a locating set corresponds to the use of sonar to find objects underwater. The *locating number* of a graph G is defined to be the least number of vertices in a locating set of G.

A dominating set S in a graph G is a *locating-dominating set*, or an LD-set, if for any two vertices v and w in $V(G) - S$, $N_S(v) \neq N_S(w)$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of an LD-set in G. For further details on γ_L see [34, 52].

Theorem 1.4 (Slater [51]) *Let n represent the order of a graph G.*

(a) *For a complete graph, $\gamma_L(K_n) = n - 1$.*

(b) *For paths and cycles,*

$\gamma_L(P_{5k}) = \gamma_L(C_{5k}) = 2k$, $\gamma_L(P_{5k+1}) = \gamma_L(C_{5k+1}) = \gamma_L(P_{5k+2}) = \gamma_L(C_{5k+2}) = 2k + 1$ *and*

$\gamma_L(P_{5k+3}) = \gamma_L(C_{5k+3}) = \gamma_L(P_{5k+4}) = \gamma_L(C_{5k+4}) = 2k + 2.$

Thus, an LD-set in a path requires at least 40% of the vertices.

(c) *For any tree T, $\gamma_L(T) > n/3$. That is, if $\gamma_L(T) = k$, then $n \leq 3k - 1$.*

(d) *If a graph G has $\gamma_L(G) = k$, then $n \leq k + 2^k - 1$.*

(e) *If G is an r-regular graph, then $\gamma_L(G) \geq 2n/(r+3)$.*

Rall and Slater proved the following results about planar and outerplanar graphs.

Theorem 1.5 (Rall and Slater [45]) *Let G be a graph with $\gamma_L(G) = k$. If G is planar, then $n \leq 7k - 10$ for $k \geq 4$. If G is outerplanar, then $n \leq \lfloor (7k-3)/2 \rfloor$. Furthermore, these bounds are sharp.*

Lemma 1.6 ([51]) *Suppose S is an LD-set in a graph G such that two vertices $v, w \in V(G)$ satisfy* (1) *if $vw \notin E(G)$, then $N(v) = N(w)$ or* (2) *if $vw \in E(G)$, then $N[v] = N[w]$. Then at least one of v or w is in S.*

Remark 1.7 *A set $P \subseteq V(G)$ which is locating and dominating is not necessarily locating-dominating.*

If a dominating set S in a graph G is such that for any two distinct vertices $v, w \in V(G)$, $N_S[v] \neq N_S[w]$, we say S is an *identifying code* or a *differentiating-dominating set*. The *differentiating-domination number* $\gamma_d(G)$ is the minimum cardinality of a differentiating-dominating set if G has such a set, and $\gamma_d(G) = \infty$ otherwise.

Locating-dominating sets were first studied in the context of identifying the location of fires or intruders in a building while differentiating-dominating sets, called identifying codes by coding theorists, arose in determining the location of errors in multiprocessor systems. Identifying codes were introduced in [38]; we use the notation of [31]. For more details on identifying codes in graphs of particular interest to coding theorists, see

[17, 18, 19, 20, 21, 36, 46]. An active list of references related to identifying codes is maintained at [1].

Locating-domination involves three-state devices which can distinguish between abnormalities which occur at a vertex and those which occur nearby. Differentiating-dominating sets, on the other hand, rely on more simple two-state devices. The set of processors in a network can be represented by the vertices of a graph while the connections between the processors can be represented by the edges of the graph. A selected subset of the processors can be used to monitor the system by sending periodic error reports to a central monitoring location. Each member of the selected subset of processors sends one of two reports to the central controller: (1) an error has been detected in the neighbourhood of the processor or (2) no errors have been detected. From this information, the controller must be able to determine the exact location of any faulty processor in the system.

Results from the study of identifying codes have recently been used to add robustness to location detection aspects of sensor networks [39, 47]. Since nodes in sensor networks tend to be battery-powered with relatively short lifespans, it is important that the system be able to withstand monitor failure. On the other hand, if the set of sensors can be varied over time the sensors may be able to save some battery life and maintain the network for a longer period of time. The similar problem of detecting errors in random networks is studied in [28].

Locating-domination has an advantage over differentiating-domination in that while every graph has a locating-dominating set, not every graph has a differentiating-dominating set. However, some applications seem to be modeled more accurately by the more simple two-state devices described by differentiating-domination.

An earlier study of this concept was undertaken by Sumner in [54]. He defined a graph G to be *point-determining* if and only if distinct vertices of G have distinct neighbourhoods. That is, for all v, $w \in V(G)$, $N(v) \neq N(w)$. If $\overline{G}$ is point-determining, then G is said to be *point-distinguishing*. Alternately, G is point-distinguishing if and only if $N[v] \neq N[w]$ for all v, $w \in V(G)$. If $N[v] = N[w]$ for two vertices $v, w \in V(G)$, we say v and w are *indistinguishable*. In this case, note that $\gamma_d(G) = \infty$. For example, if G is a complete graph of order at least two, then $\gamma_d(G) = \infty$. Graphs for which the differentiating-domination number is finite are said to be *distinguishable*; note that a graph G is distinguishable if and only G is point-distinguishing. The difference between the current work and the earlier work of Sumner is in the current goal to determine the minimum cardinality of sets needed to distinguish all the vertices.

Theorem 1.8 (Gimbel, et al. [31] and Karpovsky, et al. [38]) *If G is of order n, then $\gamma_d(G) \geq \lceil \log_2(n+1) \rceil$. Equivalently, if G is a graph of order n with $\gamma_d(G) = k$, then $n \leq 2^k - 1$.*

Theorem 1.9 ([31]) *Let P_n be the path on n vertices, and let C_n be the cycle on n vertices.*

(a) *For $k \geq 2$, $\gamma_d(P_{2k}) = k + 1$.*

(b) *For $k \geq 1$, $\gamma_d(P_{2k+1}) = k + 1$.*

(c) *For $n = 4$ or $n = 5$, $\gamma_d(C_n) = 3$.*

(d) *For $k \geq 3$, $\gamma_d(C_{2k}) = k$.*

(e) *For $k \geq 3$, $\gamma_d(C_{2k+1}) = k + 2$.*

Theorem 1.10 ([31]) *If G is distinguishable of order n with maximum degree $\Delta \geq 2$, then $\gamma_d(G) > \sqrt[\Delta]{n}$.*

By considering the probability of a pair of indistinguishable vertices in a random graph, it is also shown in [31] that almost every graph is distinguishable. Furthermore, if a graph is not distinguishable, it can be embedded in a distinguishable graph.

Theorem 1.11 ([31]) *If G is a graph of order n, then G embeds as an induced subgraph in a distinguishable graph of order $n + \lceil \log_2 n \rceil$.*

A lower bound on the differentiating-domination number that includes information about the order of the graph as well as the cardinality of certain vertex neighbourhoods is given as Theorem 1.3 in [38]. Let G be a graph of order n and let $V_i = |N[v_i]|$ for $1 \leq i \leq n$. Assume that the V_i are indexed such that $V_1 \geq V_2 \geq \cdots \geq V_n$.

Theorem 1.12 ([38]) *Let G be a graph of order n such that $n/2 \geq V_1$. Let K be the smallest integer such that for some l with $1 \leq l \leq \min(K, V_1)$ the two following conditions are satisfied:*

$$n \leq \sum_{j=1}^{l-1} \binom{K}{j} + \left\lfloor \frac{1}{l} \left(\sum_{i=1}^{K} V_i - \sum_{j=1}^{l-1} j \binom{K}{j} \right) \right\rfloor$$

and

$$\sum_{j=1}^{l-1} j \binom{K}{j} < \sum_{i=1}^{K} V_i \leq \sum_{j=1}^{l} j \binom{K}{j}.$$

Then $\gamma_d(G) \geq K$.

Let $K(G)$ be the set of cliques in a graph G. The *clique-graph* $C(G)$ of G is defined in [26] as having the elements of $K(G)$ as vertices with two vertices A and B adjacent if and only if the cliques A and B have a nonempty intersection in G. A graph G is *clique-critical* if its clique graph changes whenever any vertex is removed.

A set $\mathcal{L}$ satisfies the *Helly property* if for any subfamily $\mathcal{L}' \subseteq \mathcal{L}$ with $S_i \cap S_j \neq \emptyset$ for all $S_i, S_j \in \mathcal{L}'$, we have $\cap\{S : S \in \mathcal{L}'\} \neq \emptyset$.

Lim [41] defines a *supercompact graph* G to be the intersection graph of some family $\mathcal{L}$ of subsets of a set X such that $\mathcal{L}$ satisfies the Helly property and for any $x \neq y$ in X, there exists $S \in \mathcal{L}$ with $x \in S$, $y \notin S$.

Theorem 1.13 ([41]) *A graph G is supercompact if and only if G is distinguishable.*

From the work of Gimbel and Lim, we see that almost every graph is supercompact. In particular, Lim shows every clique-critical graph is supercompact and hence distinguishable

1.4 Matrix representations

By using various matrix represenations of graphs, many graph-theoretic parameters can be described as linear or integer programming problems. An underlying framework that describes relations among many known parameters is given in [53]. We describe the adjacency and neighbourhood matrices of a graph G and show how $\gamma_d(G)$ can be determined.

Let G be a simple graph of order n with vertices $v_1, v_2, \ldots, v_n$. Then G can be represented using the *adjacency matrix* $A(G)$, which is a binary $n \times n$ matrix with $a_{ij} = 1$ if and only if $v_i v_j \in E(G)$ and $a_{ij} = 0$ otherwise. If the graph G is clear from the context, we denote $A(G)$ simply as A. The *neighbourhood matrix* of G is $N = A + I$, where I is the $n \times n$ identity matrix.

Let n_{ij} denote the entry in row i and column j of the neighbourhood matrix N. Note that for each vertex v_k, the closed neighbourhood of v_k is given by $N[v_k] = \{v_i \in V(G) : n_{ik} = 1\}$.

The graph G is distinguishable if and only if each column of matrix N is unique and contains at least one 1. We call such a matrix an *identifying matrix*. Suppose N is an identifying matrix and S is a set of s rows in N. Suppose N is modified by replacing each row in S with a row consisting entirely of zeros. We denote the resulting matrix N^S_{n-s} since $n-s$ rows of N are left unchanged. If S consists of a single row i, we will use N^i_{n-1} rather than $N^{\{i\}}_{n-1}$.

If, for some i, the matrix N^i_{n-1} is an identifying matrix with $n-1$ nonzero rows then $V(G) - \{v_i\}$ is a differentiating-dominating set in G and $\gamma_d(G) \le n-1$. Thus, the problem of finding $\gamma_d(G)$ can be reformulated as the problem of finding the smallest r with a set of $n-r$ rows S such that the matrix N^S_r is an identifying matrix with r nonzero rows. A γ_d-set for G is then given by the vertices $V(G) - S$, which correspond to the nonzero rows in the matrix.

Consider the neighbourhood matrix of P_4

$$N = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Each row is unique and contains at least one 1, so N is an identifying matrix and the corresponding graph, a path with four vertices, is distinguishable. Suppose row 2 is replaced by a row consisting entirely of zeros. Then columns 3 and 4 are equivalent, so the resulting matrix is no longer identifying.

However, if either row 1 or row 4 is replaced with all zeros, then the resulting matrix is identifying. It can be easily seen that replacing any two rows with zeros results in a matrix that is not identifying, so $\gamma_d(P_4) = 3$ as given in Theorem 1.9.

Let X_S represent the column *characteristic matrix* of a set $S \subseteq V(G)$, with $x_i = 1$ if vertex $v_i \in S$ and $x_i = 0$ otherwise. Form N^* from N and X_S by letting $n^*_{ij} = n_{ij} \times x_i$ for each i, j. Let N^*_i denote the ith column of N^*.

Define $\overrightarrow{0}_n$ to be an $n \times 1$ column vector with a 0 in each entry and $\overrightarrow{1}_n$ to be an $n \times 1$ column vector with a 1 in each entry. For two $n \times 1$ column vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, define $\overrightarrow{a} \geq \overrightarrow{b}$ if $a_i \geq b_i$ for all $1 \leq i \leq n$.

Differentiating-domination can then be formulated as follows.

$$\gamma_d(G) = MIN \sum_{i=1}^{n} x_i$$

$$\begin{aligned} \text{Subject to: } & N \cdot X_S \geq \overrightarrow{1}_n \\ & N^*_i - N^*_j \neq \overrightarrow{0}_n \text{ for any } i \neq j \\ & x_i \in \{0, 1\} \end{aligned}$$

While problems of this type can be extremely difficult to solve in general, this formulation may provide an approach for computing γ_d for certain classes of graphs. It also suggests a fractional version of differentiating-domination in which x_i is permitted to take any value in the closed interval $[0, 1]$. Fractional versions of other domination-related parameters are described in [24] as well as in Chapters 3 and 10 of [34].

Chapter 2

Distinguishable and Co-distinguishable Graphs

A graph G is said to be *co-distinguishable* if the complement of G, $\overline{G}$, is distinguishable. We begin with a simple characterisation of co-distinguishable graphs in terms of open neighbourhoods of non-adjacent vertices. We then consider bounds on the size of graphs which are distinguishable and co-distinguishable.

2.1 Co-distinguishable graphs

Co-distinguishable graphs are graphs whose complements are distinguishable; these graphs can be completely characterised by considering the neighbourhoods of non-adjacent vertices.

Proposition 2.1 *A graph G is co-distinguishable if and only if there are no distinct non-adjacent vertices $v, w \in V(G)$ such that $N(v) = N(w)$.*

Proof. Suppose v and w are non-adjacent vertices in a graph G with $N_G(v) = N_G(w)$. This implies that $\overline{N_G(v)} \cup \{v\} = \overline{N_G(w)} \cup \{w\}$, which implies $N_{\overline{G}}[v] = N_{\overline{G}}[w]$. Thus, $\overline{G}$ is not distinguishable, so G is not co-distinguishable.

If G is not co-distinguishable, then there are two distinct vertices v and w such that $N_{\overline{G}}[v] = N_{\overline{G}}[w]$. This implies v and w are adjacent in $\overline{G}$ so are not adjacent in G. Since $N_{\overline{G}}[v] = V(G) - N_G(v)$ and $N_{\overline{G}}[w] = V(G) - N_G(w)$, it follows that $N_G(v) = N_G(w)$. ∎

As an immediate corollary, we have the following.

- P_1 is both distinguishable and co-distinguishable, P_2 is co-distinguishable but not distinguishable, and P_3 is distinguishable but not co-distinguishable.
- P_n is distinguishable and co-distinguishable for $n \geq 4$,
- C_n is distinguishable and co-distinguishable for $n \geq 5$,
- the complete bipartite graph $K_{m,n}$ is distinguishable but not co-distinguishable if $m > 1$ or $n > 1$, and
- the complete graph K_n is co-distinguishable but not distinguishable for $n > 1$.

Notice Proposition 2.1 implies that a co-distinguishable graph G has at most one isolated vertex, so if $|V(G)| > 1$ then G has at least one edge. The trivial example of a graph G that is both distinguishable and co-distinguishable is $G = K_1$, so in the following we assume $|V(G)| > 1$.

Observation 2.2 *Let G be distinguishable and co-distinguishable with $|V(G)| > 1$. Then:*

- *G has at most one trivial component,*

- *Every nontrivial component of G has at least 4 vertices, and*

- *G and $\overline{G}$ have at least three edges each.*

Proof. If G has two isolated vertices, say v and w, then $N(v) = N(w) = \emptyset$. Thus, by Proposition 2.1, G is not co-distinguishable.

If a component of G has exactly two vertices v and w, then $N[v] = N[w]$ so that G is not distinguishable. If a component of G has exactly three vertices u, v, and w, then either there is an induced path $u - v - w$ or the three vertices form a clique. In the former, u and w are non-adjacent vertices with $N(u) = N(w) = v$ so that G is not co-distinguishable. In the latter, $N[u] = N[v] = N[w]$ so that G is indistinguishable.

Thus, both G and $\overline{G}$ have at least one component with at least 4 vertices. Since this component is connected, each graph has at least 3 edges. ■

Note also that if G is distinguishable, co-distinguishable, and isolate-free, then $G \cup K_1$ is also distinguishable and co-distinguishable. Furthermore, if G has at most one trivial component and each non-trivial component of G is both distinguishable and co-distinguishable, then G is as well.

2.2 Bounds on size

We will determine bounds on the size of graphs which are distinguishable or co-distinguishable. We need the following lemma, which follows immediately from the observation that every forest of order n with k components has size $n - k$.

Lemma 2.3 *The minimum size of a graph of order n with k components is $n - k$.*

We now determine the maximum size of a distinguishable graph. We first define

$$H_n = \begin{cases} kP_2 & \text{if } n = 2k \\ kP_2 \cup P_1 & \text{if } n = 2k+1. \end{cases}$$

Note that H_n is not distinguishable but is a co-distinguishable graph of order n and size $\lfloor n/2 \rfloor$. Hence, the complement $\overline{H}_n$ is a distinguishable graph of order n and size $\lceil n^2/2 - n \rceil$.

Proposition 2.4 *For each $n \geq 1$, the minimum size of a co-distinguishable graph of order n is $\lfloor n/2 \rfloor$ and H_n is the only graph that attains this minimum.*

Proof. If G is co-distinguishable, then G has at most one component of order one. All the other components then have order at least two, so G has at most $1 + \lfloor (n-1)/2 \rfloor$ components.

If n is even, then G has at most $n/2$ components. Thus, by Lemma 2.3, G has at most $n/2$ edges.

If n is odd, then G has at most $\lceil n/2 \rceil$ components. Thus, G has at most $\lfloor n/2 \rfloor$ edges.

If $G \neq H_n$, then G must have at least one vertex of degree greater than 1. Furthermore, there must be at least one isolate in G for each vertex of degree greater than 1. Thus, if there are two vertices of degree greater than 1 then there must be at least two isolated vertices in G. By Proposition 2.1, G is not co-distinguishable.

If there is exactly one vertex v with $\deg(v) > 1$, then v is adjacent to at least two other vertices, w and x, of degree 1. Then $N(w) = N(x) = v$ so again Proposition 2.1 implies G is not co-distinguishable. ■

The following corollary, which recently appeared in a slightly different format as Proposition 1 in [42], now follows immediately.

Corollary 2.5 *For each $n \geq 1$, the maximum size of a distinguishable graph of order n is $\lceil n^2/2 - n \rceil$ and $\overline{H}_n$ is the only graph that attains this maximum.*

Examination of all graphs of order $n \leq 7$ as presented in [48] shows there are five distinguishable and six co-distinguishable graphs of order $n \leq 7$ and size $\lceil n^2/2 - n \rceil$.

We now consider the minimum size of a distinguishable graph. Define

$$F_n = \begin{cases} kP_3 & \text{if } n = 3k \\ kP_3 \cup P_1 & \text{if } n = 3k+1 \\ (k-1)P_3 \cup P_4 \cup P_1 & \text{if } n = 3k+2. \end{cases}$$

Note that for each n, the graph F_n is distinguishable but not co-distinguishable.

We now determine a lower bound for the size of a distinguishable graph.

Proposition 2.6 *If G is a distinguishable graph of order n with at most one isolate, then G has size at least $\lfloor 2n/3 \rfloor$.*

Proof. The graph G has at most one component of order one and all its other components have order at least three. Hence, G has at most $1 + \lfloor (n-1)/3 \rfloor$ components.

If $n = 3k$, then G has at most k components. Thus, by Lemma 2.3, G has size at least $n - k = 2k = \lfloor 2n/3 \rfloor$.

If $n = 3k+1$, then G has at most $k+1$ components, so G has size at least $n - (k+1) = 2k = \lfloor 2n/3 \rfloor$.

If $n = 3k+2$, then G has at most $k+1$ components, so G has size at least $n - (k+1) = 2k+1 = \lfloor 2n/3 \rfloor$. ■

The bound in Proposition 2.6 is attained by the graphs F_n. The only other graphs attaining this bound are $P_1 \cup P_5 \cup (k-2)P_3$ (with order $3k$ and k components) and $(k-1)P_3 \cup K_{1,3} \cup P_1$ (with order $3k+2$ and $k+1$ components).

We are now able to provide bounds on the number of edges in a graph which is both distinguishable and co-distinguishable.

Theorem 2.7 *Suppose G has order n and size m. If G is distinguishable and co-distinguishable, then $\lfloor 3n/4 \rfloor \leq m \leq \lceil (2n^2-5n)/4 \rceil$.*

Proof. From Observation 2.2, G contains at most one component of order one and all its other components have order at least four. Thus, G has at most $1 + \lfloor (n-1)/4 \rfloor$ components.

If $n = 4k$, then G has at most k components and has size at least $n-k = 3k = 3n/4$. Thus, $\overline{G}$ has size at most $n(n-1)/2 - 3n/4 = (2n^2-5n)/4$.

If $n = 4k+1$, then G has at most $k+1$ components and has size at least $n-(k+1) = 3k = \lfloor 3n/4 \rfloor$. Thus, $\overline{G}$ has size at most $\lceil n(n-1)/2 - 3n/4 \rceil = \lceil (2n^2-5n)/4 \rceil$.

If $n = 4k+2$, then G has at most $k+1$ components and has size at least $n-(k+1) = 3k+1 = \lfloor 3n/4 \rfloor$. Again, $\overline{G}$ has size at most $\lceil n(n-1)/2 - 3n/4 \rceil = \lceil (2n^2-5n)/4 \rceil$.

If $n = 4k+3$, then G has at most $k+1$ components and has size at least $n-(k+1) = 3k+2 = \lfloor 3n/4 \rfloor$. Once again, $\overline{G}$ has size at most $\lceil n(n-1)/2 - 3n/4 \rceil = \lceil (2n^2-5n)/4 \rceil$. ■

For $n \geq 4$, define

$$Q_n = \begin{cases} kP_4 & \text{if } n = 4k \\ kP_4 \cup P_1 & \text{if } n = 4k+1 \\ (k-1)P_4 \cup P_5 \cup P_1 & \text{if } n = 4k+2 \\ (k-1)P_4 \cup P_6 \cup P_1 & \text{if } n = 4k+3. \end{cases}$$

Observe that for each $n \geq 4$, the graph Q_n is both distinguishable and co-distinguishable since Q_n has at most one isolate and each non-trivial component is both distinguishable and co-distinguishable. Furthermore, the lower bound in Theorem 2.7 is achieved by Q_n and the upper bound is achieved by $\overline{Q}_n$.

Chapter 3

The Differentiating-domination Number

We determine the maximum number of vertices needed in a differentiating-dominating set in a distinguishable graph and describe some graphs which require the maximum. The final section consists of some Nordhaus-Gaddum type results on the sum and product of $\gamma_d(G)$ and $\gamma_d(\overline{G})$.

3.1 Large differentiating-dominating sets

Theorem 7 in [31] states that if G is a connected distinguishable graph of order n, then $\gamma_d(G) \leq n - 1$. This result is correct, but the proof in [31] is incorrect. It is shown there that if x is a vertex in G such that $V(G) - \{x\}$ is not a differentiating-dominating set of G, then there are two vertices, y and z, in G such that $xy \in E(G)$, $xz \notin E(G)$, and $N[z] = N[y] - \{x\}$.

This is correct, but it is then concluded that $V(G) - \{y\}$ is necessarily a differentiating-dominating set of G. This conclusion is incorrect. For example, if G is the path $wxyz$, then $V(G) - \{x\}$ is not a differentiating-dominating set of G. There are two vertices, y and z, in with $xy \in E(G)$, $xz \notin E(G)$, and $N[z] = N[y] - \{x\}$, but $V(G) - \{y\}$ is not a differentiating-dominating set either. Furthermore, if $G^* = G + K_1$ then none of $V(G^*) - \{x\}$, $V(G^*) - \{y\}$, or $V(G^*) - \{z\}$ is a differentiating-dominating set in G^*. It soon becomes clear that the result cannot be proved by this simple interchange of vertices.

We now provide a proof for the theorem based on the following lemma. A shorter proof of the bound is to appear in [32].

Lemma 3.1 *Suppose G is a distinguishable graph with $\gamma_d(G) = |V(G)|$ and v is a vertex with positive degree in G. Then there exists an induced path $pqrs$ in G such that $v = q$, $N[p] = N[q] - \{r\}$ and $N[s] = N[r] - \{q\}$.*

Proof. If $C = V(G) - \{v\}$, then C is not differentiating-dominating, so there are two vertices r and s such that $N[s] \neq N[r]$, but $N_C[s] = N_C[r]$.

Note that neither r nor s is the vertex v, since $N_C[v] = N_C[v']$ implies $N(v) = N[v'] - \{v\}$. But this implies $N[v] = N[v']$ for the vertices v and v', which contradicts the fact that G is distinguishable.

Since $N[s] \neq N[r]$ but $N_C[s] = N_C[r]$, v is adjacent to exactly one of r and s. Without loss of generality say v is adjacent to r, in which case $N_C[s] = N_C[r]$ implies

$$N[s] = N[r] - \{v\}. \tag{1}$$

Similarly, if $D = V(G) - \{r\}$, then there are two vertices p and q, neither of which is r, such that q is adjacent to r and

$$N[p] = N[q] - \{r\}. \tag{2}$$

Now $p \neq v$, since v is adjacent to r but p is not.

It now follows from (1) that p is not adjacent to s and hence (2) implies that q is not adjacent to s. Therefore $q \notin N[s]$, so $q \in N[r] - N[s]$. This implies $q = v$.

We have also shown that p is non-adjacent to both r and s, and q is non-adjacent to s. Hence $pqrs$ is an induced path in G with $q = v$. ■

Theorem 3.2 *Let G be a distinguishable graph with at least one edge. Then $\gamma_d(G) \leq n - 1$.*

Proof. Suppose, to the contrary, that $\gamma_d(G) = n$. We shall show, by repeated application of Lemma 3.1, that for every integer $k \geq 2$ there exist induced paths $P^i := p^i q^i r^i s^i$ in G for $i = 1, \ldots, k$, such that

$$q^i = p^{i-1}, s^i = r^{i-1},$$

$$N[p^i] = N[q^i] - \{r^i\},$$

and

$$N[s^i] = N[r^i] - \{q^i\}$$

for $i = 1, \ldots, k$ and

$$p^j \notin \bigcup_{i=1}^{j-1} V(P^i)$$

for $j = 2, \ldots k$.

By Lemma 3.1 there is an induced path $P^1 := p^1 q^1 r^1 s^1$ in G with

$$N[p^1] = N[q^1] - \{r^1\} \tag{1a}$$

and

$$N[s^1] = N[r^1] - \{q^1\}. \tag{1b}$$

Now, applying Lemma 3.1 to the vertex p^1, we obtain an induced path $P^2 := p^2q^2r^2s^2$ with $p^1 = q^2$ and

$$N[p^2] = N[q^2] - \{r^2\} \tag{2a}$$

and

$$N[s^2] = N[r^2] - \{q^2\}. \tag{2b}$$

Both p^2 and r^2 are adjacent to $q^2 = p^1$. But (1a) implies that $N[p^1] \subset N[q^1]$, so both p^2 and r^2 are in $N[q^1]$. Since p^2 and r^2 are not adjacent to one another, it follows that neither p^2 nor r^2 is the vertex q^1. Furthermore, since P^1 is an induced path of G, neither r^1 nor s^1 is adjacent to p^1 so $p^2, r^2 \notin \{r^1, s^1\}$. Hence

$$p^2 \notin V(P^1).$$

Since r^2 is adjacent to q^1, it follows from (2b) that s^2 is also adjacent to q^1. But s^2 is not adjacent to $q^2 = p^1$, and (1a) implies that r^1 is the only neighbour of q^1 that is not adjacent to p^1. Hence $s^2 = r^1$.

This proves the result for $k = 2$.

Now let $k \geq 3$ and suppose we have obtained $k - 1$ induced paths $P^1, \ldots, P^{k-1}$ in G with $P^i := p^iq^ir^is^i$ such that

$$q^i = p^{i-1}, s^i = r^{i-1},$$

$$N[p^i] = N[q^i] - \{r^i\}, \tag{ia}$$

and

$$N[s^i] = N[r^i] - \{q^i\} \tag{ib}$$

for $i = 1, \ldots, k-1$ and

$$p^j \notin \bigcup_{i=1}^{j-1} V(P^i)$$

for $j = 2, \ldots, k-1$.

Then it follows from (ia) that

$$N[p^{k-1}] \subset N[q^{k-1}] = N[p^{k-2}] \subset N[q^{k-2}] = \cdots = N[p^1] \subset N[q^1] \quad (4)$$

and

$$N[s^1] \subset N[r^1] = N[s^2] \subset N[r^2] = \cdots = N[s^{k-1}] \subset N[r^{k-1}]. \quad (5)$$

Now we repeat the procedure that we applied in the case $k = 2$, using the path P^{k-1} instead of P^1, to obtain a path $P^k := p^k q^k r^k s^k$, with

$$q^k = p^{k-1}, r^k = s^{k-1},$$

$$N[p^k] = N[q^k] - \{r^k\},$$

and

$$N[s^k] = N[r^k] - \{q^k\}.$$

Both p^k and r^k are adjacent to $q^k = p^{k-1}$, so it follows from (4) that both p^k and r^k are adjacent to every vertex in $\bigcup_{i=1}^{k-1}\{p^i, q^i\}$. But p^k is not adjacent to r^k since P^k is an induced path, so

$$p^k \notin \bigcup_{i=1}^{k-1} \{p^i, q^i\}.$$

Since p^k is not adjacent to $r^k = s^{k-1}$, it follows from (5) that $p^k \notin \bigcup_{i=1}^{k-1}\{r^i, s^i\}$. Hence

$$p^k \notin \bigcup_{i=1}^{k-1} V(P^i).$$

This contradicts the finiteness of G. ■

Corollary 3.3 *A graph G has $\gamma_d(G) = n$ if and only if G is totally disconnected.*

It is shown in [10] that all cardinalities between the lower bound given in Theorem 1.8 and the upper bound given in Theorem 3.2 are possible.

We now consider the union of two graphs.

Lemma 3.4 *If $G = G_1 \cup G_2$, then $\gamma_d(G) = \gamma_d(G_1) + \gamma_d(G_2)$.*

Proof. Suppose that one component of G, say G_1, is indistinguishable. Then $\gamma_d(G) = \gamma_d(G_1) = \infty$.

Now suppose G_1 and G_2 are distinguishable. It requires $\gamma_d(G_1)$ vertices to distinguish G_1 and $\gamma_d(G_2)$ vertices to distinguish G_2. Since G_1 and G_2 are disjoint, and the neighbourhood of a vertex is unaffected by non-adjacent vertices, then $\gamma_d(G_1) + \gamma_d(G_2)$ vertices are required in a γ_d-set for G. ■

Note that this result does not hold for the join of two graphs. If S_i is a differentiating-dominating set of G_i for $i = 1, 2$, then $S_1 \cup S_2$ is not necessarily a differentiating-dominating set of $G_1 + G_2$. For example, if G_i has a vertex whose open neighbourhood in S_i is the empty set for $i = 1, 2$, then $S_1 \cup S_2$ is not a differentiating-dominating set of $G_1 + G_2$. In particular, consider $G_i = P_4$ for $i = 1, 2$. By Theorem 1.9, $\gamma_d(P_4) + \gamma_d(P_4) = 6$. However, it is shown in Proposition 3.5 that $\gamma_d(P_4 + P_4) = 7$.

We now describe some families of graphs with differentiating-domination number equal to $n - 1$. One example of a graph with $\gamma_d(G) = n - 1$ is $G = P_4$. As another example, begin with P_4 then add two more vertices a and b adjacent to the four vertices in P_4 but not to each other. This graph requires five vertices in a γ_d-set. Both of these examples are special cases of an infinite family with differentiating-domination number equal to $n - 1$ which we now describe.

For $n \geq 4$, consider the family M_n given by

$$M_n = \begin{cases} kP_4 & \text{if } n = 4k \\ kP_4 \cup P_1 & \text{if } n = 4k+1 \\ kP_4 \cup P_2 & \text{if } n = 4k+2 \\ kP_4 \cup P_2 \cup P_1 & \text{if } n = 4k+3. \end{cases}$$

Now consider the complement $G = \overline{M}_n$.

Proposition 3.5 *The graph $G = \overline{M}_n$ of order n has $\gamma_d(G) = n - 1$.*

Proof. We first observe that no distinct non-adjacent vertices $v, w \in V(M_n)$ have $N(v) = N(w)$, so by Proposition 2.1 M_n is co-distinguishable. Since $G = \overline{M}_n$ is distinguishable and has at least one edge, we see by Theorem 3.2 that $\gamma_d(G) \leq n - 1$.

We now show that G has no γ_d-set of cardinality $n - 2$. Suppose to the contrary that $S = V(G) - \{v, w\}$ is a γ_d-set in G.

Case I: Suppose $vw \notin E(G)$. Then $vw \in E(M_n)$ so either v and w belong to either a P_2 or a P_4 in M_n. If v and w belong to a P_2 in M_n, then in G it must be the case that $N_S[v] = N_S[w]$ so that S cannot be a γ_d-set.

So, v and w must belong to a P_4 in M_n. Suppose the other two vertices in the P_4 are labeled a and b. Note that v, w, a, and b are adjacent to all the vertices in $G - \{v, w, a, b\}$. Since $vw \in E(M_n)$, at most one of v or w is a leaf in M_n.

If v is a leaf and w is adjacent to a, then in G, the neighbourhoods $N_S[w] = N_S[b]$ since the only difference in $N[w]$ and $N[b]$ is the fact that, in G, v is adjacent to b but not to w. Thus neither v nor w is a leaf.

If a and b are leaves in M_n, then in G we see $N_S[a] = N_S[b]$, again violating the premise that S is a γ_d-set.

Case II: Suppose $vw \in E(G)$. Then v and w are in different components of M_n.

Suppose v is in a copy of P_4 and $\deg(w) \geq 1$ in M_n. If v is a leaf, then the other leaf of the component and the vertex adjacent to v share the same neighbourhood relative to S in the graph G. Thus, v is not a leaf.

Now consider the leaf a adjacent to v and the leaf b adjacent to w in M_n. Vertex a is adjacent to all the other elements of S in G, as is vertex b. Therefore, $N_S[a] = N_S[b]$ in the graph G.

Now suppose that w is an isolate. Whether v is in P_2 or a copy of P_4, v must be adjacent to a leaf a. Observe that $N_S[a] = N_S[w]$.

Thus there is no γ_d-set of order $n-2$ in $G = \overline{M}_n$. ■

It is easy to see that $n-1$ vertices are also required to distinguish the vertices in a star.

Proposition 3.6 *If G is a star $K_{1,n-1}$, then $\gamma_d(G) = n-1$.*

Now suppose $n \geq 3$. If n is even, construct a graph G of order n by beginning with a complete graph on vertices $v_1, v_2, \ldots, v_n$. Remove the edge $v_i v_{i+1}$ for each odd value of i. If n is odd, construct a graph of order $n-1$ as described, then add vertex v_n adjacent to all other vertices to form G. Note that G is K_n minus a maximum matching. Thus, $G = \overline{H}_n$, where H_n is the family defined in Section 2.2. Furthermore, note that G is $(n-2)$-regular for even n.

Proposition 3.7 *The graph $G = \overline{H}_n$ of order n described above has $\gamma_d(G) = n-1$.*

A different construction of the same family is given in [11], which contains a proof of this result as part of Theorem 4.

Open Problem 1 *Are there r-regular graphs G of order n with $\gamma_d(G) = n-1$ for $r < n-2$?*

Open Problem 2 *Let $\Delta(G)$ represent the maximum degree in G. Are there graphs of odd order with $\gamma_d(G) = n-1$ and $\Delta(G) < n-1$?*

We will see in Proposition 3.12 that the family Q_n described in Section 2.2 also has differentiating-domination number equal to $n-1$ for $n \equiv 0, 1 \mod 4$.

3.2 Nordhaus-Gaddum type results

In a famous 1956 paper, Nordhaus and Gaddum [44] introduced the search for bounds on the sum and product of the chromatic number of a given graph and its complement. Since then, these types of inequalities have been studied for numerous other graph parameters. We will first consider $\overline{\gamma}_d(G) = \gamma_d(\overline{G})$ then provide Nordhaus-Gaddum type inequalities for the differentiating-domination number.

It is often easier to determine whether a given set S of vertices is a differentiating-dominating set in the complement of a graph G by considering certain neighbourhoods in G rather than considering the complement of the graph. The following lemma is useful for determining $\overline{\gamma}_d(G)$.

Lemma 3.8 *Let G be a co-distinguishable graph. A set S is a differentiating-dominating set of $\overline{G}$ if and only if the following hold in G:*

- $N_S(v) \neq S$ *for* $v \in V(G) - S$, *and*
- $N_S(v) \neq N_S(w)$ *for* $v, w \in V(G)$.

Proof. Suppose $N_S(v) = S$ for some $v \in V(G) - S$. Then in $\overline{G}$, the vertex v is not dominated by any vertex in S. Thus S cannot be a differentiating-dominating set of $\overline{G}$.

Suppose $N_S(v) = N_S(w)$ for two vertices $v, w \in V(G)$.

If neither v nor w is in S, then in $\overline{G}$, $N_S(v) = N_S(w)$ since v and w are adjacent to exactly the vertices in S to which they were not adjacent in G. This implies $N_S[v] = N_S[w]$ since for any vertex $a \notin S$ we have $N_S(a) = N_S[a]$.

If v and w are adjacent in G, then neither v nor w is in S else the open neighbourhoods would differ. Thus, if one or both of v and w are in S then vw is an edge in $\overline{G}$. Once again, we have $N_S[v] = N_S[w]$.

If $v \in S$, $w \notin S$, and $vw \notin E(G)$, then in $\overline{G}$ it must be the case that $N_S(v) \cup \{v\} = N_S(w)$. But this implies $N_S[v] = N_S[w]$.

In each case, S is not a differentiating-dominating set in $\overline{G}$.

To see necessity of the condition, suppose S is not a differentiating-dominating set in $\overline{G}$. Then some vertex is undominated or two vertices have the some closed neighbourhood with respect to S in $\overline{G}$.

If a vertex v is undominated, then $N_S(v) = \emptyset$ in $\overline{G}$, which implies $N_S(v) = S$ in G.

If $N_S[v] = N_S[w]$ in $\overline{G}$, then $N_S(v) = N_S(w)$ in G since v and w are each adjacent to the same elements of S in G. ■

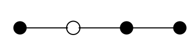

Figure 3.1: $\overline{\gamma}_d(P_4) = 3$.

As an example of how to use this lemma, consider paths and cycles. Before stating the main result, we begin with some observations.

Suppose that S is a differentiating-dominating set in the complement of a path or cycle.

1. There can never be exactly three consecutive vertices uvw in the set S since this would imply $N_S(u) = N_S(w) = v$.

2. Three consecutive vertices uvw which are not in S can occur at most once, since otherwise there would be two vertices v and v' with $N_S(v) = N_S(v') = \emptyset$.

3. If there are three consecutive vertices uvw not in S, then the four vertices to the immediate right of w and the four to the left of u are in S. Otherwise, there will be at least two vertices with the same open neighbourhood relative to S. Note, of course, that these four vertices overlap in C_7.

4. If there are exactly two consecutive vertices uv not in S, then there are four consecutive vertices in S either to the right of v or the left of u. Otherwise, either $N_S(u)$ or $N_S(v)$ will correspond to the open neighbourhood of a vertex at distance 2 from u or v.

Thus, any attempt to reduce the number of vertices required to form a $\overline{\gamma}_d$-set in a path or cycle by leaving out several consecutive vertices fails. Roughly two-thirds of the vertices are required to form a $\overline{\gamma}_d$-set.

Proposition 3.9 *For $G = P_n$ or C_n, $\overline{\gamma}_d(G) \approx 2n/3$.*

The next proposition follows immediately from the fact that $\lceil \log_2(n+1) \rceil \leq \gamma_d(G) \leq n-1$ for all graphs which are both distinguishable and co-distinguishable.

Proposition 3.10 *If G is distinguishable and co-distinguishable, then*

$$2\lceil \log_2(n+1)\rceil \leq \gamma_d(G) + \gamma_d(\overline{G}) \leq 2n-2 \text{ and}$$

$$(\lceil \log_2(n+1)\rceil)^2 \leq \gamma_d(G)\gamma_d(\overline{G}) \leq n^2-2n+1.$$

The bounds are achieved by $G = P_4$. We do not know whether there are other graphs that realise the upper bound, but we shall show that there are infinitely many graphs that realise the lower bound.

3.2.1 Lower bounds

We now provide a construction that attains the lower bounds of Proposition 3.10.

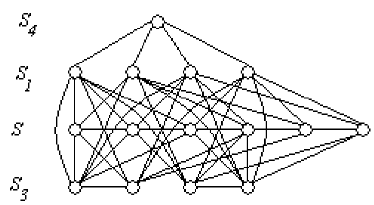

Figure 3.2: The graph G of order 15 from Proposition 3.11.

Proposition 3.11 *For each even $k \geq 4$, there is a graph G of order $2^k - 1$ which attains the lower bounds of Proposition 3.10.*

Proof. For even $k \geq 4$, construct a graph of order $2^k - 1$ as follows: Let S_1 consist of k independent vertices. For $i = 2, 3, \ldots, k$, let S_i be a set of $\binom{k}{i}$ vertices. Add edges between S_1 and each S_i so that the neighbourhoods of S_i in S_1 correspond to the $\binom{k}{i}$ distinct subsets of S_1 of cardinality i.

Let $S_1 = \{v_1, v_2, \ldots, v_k\}$ and $S_{k-1} = \{w_1, w_2, \ldots, w_k\}$. Let $S = \bigcup_{i=2}^{k-2} S_i$.

Add $k/2$ independent edges between the vertices of S_{k-1}, then add edges from S_{k-1} to $S_2, S_3, \ldots, S_{k-2}$ so that $N_S(w_i) = N_S(v_i)$ for $i = 1, 2, \ldots, k$. Call the resulting graph G. (See Figure 3.2 for the case $k = 4$.)

In $\overline{G}$, for each $i = 2, \ldots, k-2$ and $i = k$, the intersection of S_{k-1} with the neighbourhood of each vertex of S_i corresponds to a different subset of order i of S_{k-1}. In addition, note that the sets $N_{\overline{G}}(v_i) \cap S_{k-1}$ correspond to the one-element subsets of S_{k-1} while the sets $N_{\overline{G}}[w_i] \cap S_{k-1}$ correspond to the $(k-1)$-element subsets of S_{k-1}.

Since all vertices are distinguished, S_1 is a γ_d-set of G and S_{k-1} is a $\overline{\gamma}_d$-set of G. ■

It would be interesting to determine whether there any graphs of even order, other than P_4, which achieve the lower bounds of Proposition 3.10.

3.2.2 Upper Bounds

The largest values of $\gamma_d + \overline{\gamma}_d$ and $\gamma_d \overline{\gamma}_d$ we have been able to find are achieved by the family Q_n described in Section 2.2.

We now show that the graph Q_n and its complement have large differentiating-domination numbers.

Proposition 3.12 *For $G = Q_n$, $\gamma_d(G) \approx 3n/4$ and $\overline{\gamma}_d(G) \approx n - 1$ for $n \geq 4$.*

Proof. Each copy of P_4 in the construction adds 3 to the differentiating-domination number, while each K_1 adds 1, each P_5 adds 3, and each P_6 adds 4. We consider four cases.

Case I: If $n \equiv 0 \mod 4$ then there are $n/4$ disjoint copies of P_4 in G, which implies $\gamma_d(G) = 3n/4$. By Proposition 3.5, $\overline{\gamma}_d(G) = n - 1$.

Case II: If $n \equiv 1 \mod 4$ then there are $(n-1)/4$ disjoint copies of P_4 and an isolate in G. Thus, $\gamma_d(G) = 3(n-1)/4+1 = 3n/4+1/4$. By Proposition 3.5, $\overline{\gamma}_d(G) = n-1$.

Case III: If $n \equiv 2 \mod 4$, then there are $(n-6)/4$ disjoint copies of P_4, one copy of P_5, and an isolate in G. Therefore, $\gamma_d(G) = 3(n-6)/4+4 = 3n/4-1/2$.

Let S consist of $n-3$ vertices in G with vertices v, w, and x not in S. As in Proposition 3.5, if two of v, w, and x are in the same or different copies of P_4, then S is not a $\overline{\gamma}_d$-set of G. Likewise, if one of the three is in a copy of P_4 then neither of the other two can be the isolate. Similarly, if all three are in P_5 then S is not a $\overline{\gamma}_d$-set of G.

Suppose that both v and w are in P_5. Then there must be at least one vertex in P_5 which has an empty open neighbourhood with respect to S. This implies that x cannot be the isolate nor in any of the copies of P_4. Thus, G has no $\overline{\gamma}_d$-set of cardinality $n-3$.

Consider the set formed by choosing a $\overline{\gamma}_d$-set of the copy of P_5 together with all the other vertices in G. Since this set is a differentiating-dominating set of $\overline{G}$, $\overline{\gamma}_d(G) = n-2$.

Case IV: If $n \equiv 3 \mod 4$, then there are $(n-7)/4$ disjoint copies of P_4, one copy of P_6, and an isolate in G. This gives $\gamma_d(G) = 3(n-7)/4+5 = 3n/4-1/4$.

Suppose S is a set of vertices of cardinality $n-4$ with u, v, w, and x not in S. As in the previous cases, at most one of these can be the isolate or in a copy of P_4. If three are in the copy of P_6, then there must be at least two vertices with the same open neighbourhoods with respect to S. Therefore there is no $\overline{\gamma}_d$-set of G of order $n-4$.

Consider the set formed by selecting the isolate together with the leaves of the copy of P_6. Since the complement of this set is a differentiating-dominating set of $\overline{G}$, $\overline{\gamma}_d(G) = n - 3$. ■

We believe that the graphs Q_n yield the largest value of $\gamma_d + \overline{\gamma}_d$. In the construction for $n \equiv 0 \mod 4$ we have $\gamma_d + \overline{\gamma}_d = 2n - n/4 - 1 = 2n - 2 - (n/4 + 1)$. Also, $\gamma_d \overline{\gamma}_d = (n^2 - 2n + 1) - (n^2/4 - 5n/4 + 1)$. Some progress towards tightening this bound would follow from a positive solution to the following conjecture.

Conjecture 3.13 *If $\overline{\gamma}_d(G) = n - 1$, then either $G = kP_4$, $G = kP_4 \cup K_1$, or G is not distinguishable.*

One attempt that has so far proved unsuccessful involves determining the number of edges required to obtain $\gamma_d(G) = n - 1$. If G is a distinguishable star then $\overline{G}$ contains a complete component and is not distinguishable. On the other hand, if G is not a star, $\gamma_d(G) = n - 1$ seems to require that G have a large number of edges. An immediate corollary of Theorem 2.7 is that there are no graphs of order n and size $\lceil n^2/2 - n \rceil$ which are both distinguishable and co-distinguishable. We believe that if $\gamma_d(G) = n - 1$ then there would not be enough edges remaining to allow $\overline{G}$ to be distinguishable.

Chapter 4

Critical Concepts

We now consider situations in which a minor modification in the edge or vertex set of a graph causes the differentiating-domination number to change. Following [33], for each parameter π define the simple finite graph G to be

- π-*edge-critical* if $\pi(G+e) < \pi(G)$ for all $e \in E(\overline{G})$,
- π^+-*edge-critical* if $\pi(G+e) > \pi(G)$ for all $e \in E(\overline{G})$,
- π-*critical* if $\pi(G-v) < \pi(G)$ for all $v \in V(G)$,
- π^+-*critical* if $\pi(G-v) > \pi(G)$ for all $v \in V(G)$,
- π-*ER-critical* if $\pi(G-uv) > \pi(G)$ for all $uv \in E(G)$, and
- π^--*ER-critical* if $\pi(G-uv) < \pi(G)$ for all $uv \in E(G)$.

For each type of criticality with respect to $\pi = \gamma_d$, we say G is *finitely critical* if γ_d is finite both before and after the corresponding modification to G. For example, if G is a graph with $\gamma_d(G) < \gamma_d(G+e) < \infty$ for all $e \in E(\overline{G})$, then we say G is finitely γ_d^+-edge-critical.

Much of the material in this chapter appears in [27]. See [16, 33, 55] for further results and references regarding criticality with respect to other domination-related parameters. We provide basic existence results for the six types of criticality with $\pi = \gamma_d$.

4.1 γ_d-edge-critical

Adding an edge to an indistinguishable graph may result in a distinguishable graph. For example, if $G = P_2 \cup K_1$, then $\gamma_d(G) = \infty$, but $\gamma_d(G + e) = 2$ for every $e \in E\,(G)$.

There also exist finitely γ_d-edge-critical graphs.

Proposition 4.1 *Odd cycles of order $n \geq 7$ are finitely γ_d-edge-critical.*

Proof. Label the vertices of C_{2k+1} as $1, 2, \cdots, 2k+1$. By symmetry, we need consider only the addition of edges $\{1, x\}$ where $x \in \{3, 4, \cdots, k+1\}$. If x is even, the set of even vertices together with vertex 1 serves to distinguish all vertices in $C_{2k+1} + \{1, x\}$. If x is odd, then the set of odd vertices serves to distinguish all vertices. Thus, it follows from Theorem 1.9 that $\gamma_d(C_{2k+1} + e) \leq k + 1 < k + 2 = \gamma_d(C_{2k+1})$. ■

However, not all cycles are γ_d-edge-critical.

Proposition 4.2 *Even cycles C_{2k} are not γ_d-edge-critical for $k \geq 2$.*

Proof. Note that $C_4 + e$ is indistinguishable for all $e \in E(\overline{G})$ so C_4 is not γ_d-edge-critical. Furthermore, by Theorem 1.8, C_6 and C_8 are not γ_d-critical, since $\gamma_d(C_6 + e) \geq \lceil \log_2 7 \rceil = 3 = \gamma_d(C_6)$ and $\gamma_d(C_8 + e) \geq \lceil \log_2 9 \rceil = 4 = \gamma_d(C_8)$.

For $k \geq 5$, label the vertices of C_{2k} as $1, 2, \cdots, 2k$. By symmetry, we need consider only the addition of edges $\{1, x\}$ where $x \in \{3, 4, \cdots, k+1\}$. Whether x is even or odd, the set of even vertices in $C_{2k} + \{1, x\}$ serves to distinguish all vertices, so $\gamma_d(C_{2k} + e) \leq k$. We show that $\gamma_d(C_{2k} + e) \geq k$ by application of Theorem 1.12.

From Theorem 1.12, suppose $K = k-1$. For $C_{2k}+e$, note that $|N[v]| = 4$ if v is incident with e and $|N[v]| = 3$ otherwise. Thus

$$\sum_{i=1}^{K} V_i = 4 \cdot 2 + 3(K-2) = 3K + 2 = 3(k-1) + 2 = 3k - 1.$$

Since $1 \leq l \leq \min(K, 4)$, we consider each possible value of l.

Case I. If $l = 1$, then the second condition on K and l in Theorem 1.12 is not satisfied since $3k - 1 \not< \sum_{j=1}^{1} j\binom{k-1}{j} = k - 1$.

Case II. If $l = 2$, then the first condition on K and l is not satisfied since

$$2k \not\leq \sum_{j=1}^{l-1} \binom{k-1}{j} + \left\lfloor \frac{1}{l} \left(3k - 1 - \sum_{j=1}^{l-1} j \binom{k-1}{j} \right) \right\rfloor = k-1 + \left\lfloor \frac{1}{2}(2k) \right\rfloor = 2k-1.$$

Case III. If $l = 3$ then the second condition on K and l is not satisfied. The left inequality implies that $k - 1 + 2\binom{k-1}{2} < 3k - 1$. This implies that $2\binom{k-1}{2} < 2k$ which is not true for $k \geq 5$.

Case IV. Similarly, if $l = 4$ then the second condition on K and l implies $k - 1 + 2\binom{k-1}{2} + 3\binom{k-1}{3} < 3k - 1$. This implies $2\binom{k-1}{2} + 3\binom{k-1}{3} < 2k$ which is again not true for $k \geq 5$. ■

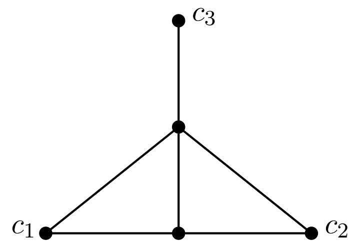

Figure 4.1: A γ_d^+-edge-critical graph.

4.2 γ_d^+-edge-critical

There are graphs for which $\gamma_d(G) < \gamma_d(G+e) = \infty$ for all $e \in E(\overline{G})$. For example, P_3, C_4 and mK_1, $m \geq 2$, have this property.

There are also γ_d^+-edge-critical graphs for which the increase is finite for some edges but infinite for others. For example, if G is the graph in Figure 4.1, then $\gamma_d(G) = 3$ and $\gamma_d(G + c_1c_3) = 4$, while $\gamma_d(G + c_1c_2) = \infty$.

The differentiation-domination number of a finitely γ_d^+-edge-critical graph is at most $|V(G)| - 3$. In order to prove this, we need the following lemma.

Lemma 4.3 *Suppose G is a finitely γ_d^+-edge-critical graph and C is a γ_d-set of G. Then the subgraph of G induced by $\overline{C}$ is a clique.*

Proof. Suppose there are two distinct non-adjacent vertices v and w in $\overline{C}$. Then C is clearly a dd-set of $G + vw$, so $\gamma_d(G + vw) \leq |C| = \gamma_d(G)$. But then G is not γ_d^+-edge-critical. ■

Theorem 4.4 *Suppose G is a connected finitely γ_d^+-edge-critical graph. Then $\gamma_d(G) \leq |V(G)| - 3$.*

Proof. It follows from Theorem 3.2 that $\gamma_d(G) \neq |V(G)| - 1$.

Suppose $\gamma_d(G) = |V(G)| - 2$ and that $C = V(G) - \{x_1, x_2\}$ is a γ_d-set of G. By Lemma 4.3, x_1 and x_2 are adjacent. Let $B = N(x_1) \cap N(x_2)$ and for $i = 1, 2$, $A_i = (N(x_i) \cap C) - B$. At least one of A_1 and A_2 is nonempty, otherwise C would not distinguish between x_1 and x_2. Without loss of generality let $y_1 \in A_1 = (N(x_1) \cap C) - N(x_2)$.

Since $\gamma_d(G) < \gamma_d(G+x_2y_1)$, the set C is not a differentiating-dominating set of $G + x_2y_1$. Hence there is a vertex y_2 such that

$$N_C[x_2] \cup \{y_1\} = N_C[y_2].$$

If y_2 is adjacent to x_1, then

$$N_{G+x_2y_1}[x_2] = N_C[x_2] \cup \{x_1, y_1\} = N_C[y_2] \cup \{x_1\} = N_{G+x_2y_1}[y_2].$$

But then $G + x_2y_1$ is indistinguishable, contradicting that G is finitely γ_d^+-edge-critical. Hence y_2 is not adjacent to x_1, so $y_2 \in A_2$. In particular, $y_2 \neq y_1$.

Since $N_C[x_2] = N_G(x_2) - \{x_1\}$,

$$(N_G[x_2] - \{x_1\}) \cup \{y_1\} = N_C[y_2] \cup \{x_2\},$$

which implies

$$N_G[x_2] - \{x_1\} = N_G[y_2] - \{y_1\}. \tag{1}$$

Similarly, since C is not a differentiating-dominating set of $G + x_1y_2$, there is a vertex w such that

$$N_C[x_1] \cup \{y_2\} = N_C[w]$$

and w is not adjacent to x_2. Now, w is adjacent to y_2, but y_1 is the only neighbour of y_2 that is nonadjacent to x_2. Hence $w = y_1$, which implies that

$$N_G[x_1] - \{x_2\} = N_G[y_1] - \{y_2\}. \tag{2}$$

Let $C^* = V(G) - \{x_1, y_2\}$. Since x_1 is not adjacent to y_2, it follows from Lemma 4.3 that C^* is not a γ_d-set in G. Hence there are two vertices u and v such that

$$N_{C^*}[u] = N_{C^*}[v]. \tag{3}$$

Note that C^* is obtained from C by interchanging y_2 and x_2. We consider three cases.

Case I. Suppose $u, v \notin \{x_1, y_1\}$. Let $z \in N_C[u]$.

If $z \neq y_2$, then $z \in C^*$ and hence $z \in N_{C^*}[u] = N_{C^*}[v]$ by (3). Since $z \in C$ by our assumption, it follows that $z \in N_C[v]$.

If $z = y_2$ then it follows from (1) that x_2 is a neighbour of u. Since $x_2 \in C^*$, it follows from (3) that x_2 is a neighbour of v. But then it follows from (1) that y_2 is a neighbour of v. Hence $y_2 \in N_C[v]$. This proves that $N_C[u] \subseteq N_C[v]$. The proof that $N_C[v] \subseteq N_C[u]$ is similar. Hence $N_C[u] = N_C[v]$, contradicting our assumption that C is a γ_d-set.

Case II. If $u = x_1$, then $N_{C^*}[x_1] = N_{C^*}[v]$ by (3). Note that $N_{C^*}[x_1] = N_G[x_1] - \{x_1\}$, so (2) and (3) imply that

$$\begin{aligned} N_{C^*}[x_1] &= ((N_G[y_1] - \{y_2\}) \cup \{x_2\}) - \{x_1\} = (N_G[y_1] - \{y_2, x_1\}) \cup \{x_2\} \\ &= N_{C^*}[y_1] \cup \{x_2\}. \end{aligned}$$

Application of (3) thus implies

$$N_{C^*}[v] = N_{C^*}[y_1] \cup \{x_2\}. \tag{4}$$

If $v \in C^*$, then $N_{C^*}[v] = N_G[v] - x_1$, and (4) implies

$$N_G[v] - \{x_1\} = N_{C^*}[y_1] \cup \{x_2\} = (N_G[y_1] - \{y_2, x_1\}) \cup \{x_2\},$$

which implies $N_G[v] = N_G[x_1]$ by (2). However, this contradicts our assumption that G is distinguishable.

Therefore $v \notin C^*$, so $v = y_2$. Recall that $y_2 \in A_2$, so y_2 is not adjacent to x_1. Since $N_{C^*}[x_1] = N_G(x_1)$ and $N_{C^*}[y_2] = N_G(y_2)$, (3) implies $N_G(x_1) = N_G(y_2)$. Therefore $N_{G+x_1y_2}[x_1] = N_{G+x_1y_2}[y_2]$, hence $\gamma_d(G + x_1y_2) = \infty$, contradicting our assumption that G is finitely γ_d^+-edge-critical.

Case III. If $u = y_1$, then (3) implies $N_{C^*}[y_1] = N_{C^*}[v]$ for some vertex v. Since $N_{C^*}[y_1] = N_G[y_1] - \{x_1, y_2\} = N_G(x_1) - \{x_2\}$ by (2), $N_{C^*}[v] = N_G(x_1) - \{x_2\}$. This implies that v is not adjacent to x_2. Furthermore, by (1), v is not adjacent to y_2.

If v is not adjacent to x_1, then by (2) v also is not adjacent to y_1. This contradicts the fact that $N_{C^*}[y_1] = N_{C^*}[v]$, so v is adjacent to x_1. Therefore $N_G[v] = N_G[x_1] - \{x_2\} = N_G[y_1] - \{y_2\}$, which implies $N_{G+vy_2}[v] = N_{G+vy_2}[y_1]$. Thus $\gamma_d(G + vy_2) = \infty$, again contradicting our assumption that G is finitely γ_d^+-edge-critical. ■

Proposition 4.5 *If G is a connected finitely γ_d^+-edge-critical graph, then $\gamma_d(G) \geq 4$.*

Proof. Suppose to the contrary that G is a connected finitely γ_d^+-edge-critical graphs of order n with $\gamma_d(G) < 4$. Let C be a γ_d-set in G.

By Theorem 1.8, we see that if $\gamma_d(G) = 1$, then $n = 1$. Furthermore, if $\gamma_d(G) = 2$ then either $n = 2$ or $n = 3$, and if $\gamma_d(G) = 3$ then $3 \leq n \leq 7$. By Theorem 4.4, $\gamma_d(G) + 3 \leq n$. Thus, it must be the case that $\gamma_d(G) = 3$ and either $n = 6$ or $n = 7$.

Note the subgraph induced by C is either a path of length two or consists of three isolates and, by Lemma 4.3, the subgraph induced by $\overline{C}$ is complete.

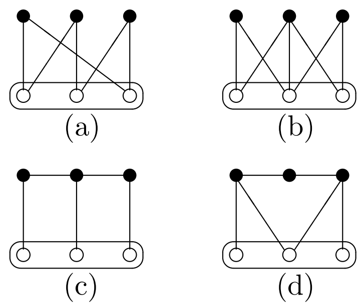

Figure 4.2: Four graphs with $n = 6$ and $\gamma_d(G) = 3$.

Figure 4.3: Two graphs with $n = 7$ and $\gamma_d(G) = 3$.

Figure 4.2 shows the four possible graphs for $n = 6$ and Figure 4.3 shows the two possible graphs for $n = 7$.

None of these six graphs is finitely γ_d^+-edge-critical. In Figure 4.2(a) and 4.2(c), an edge can be added that leaves the differentiating-domination number unchanged. In all the other cases, an edge can be added that results in an indistinguishable graph. ■

We now describe a method for the construction of a finitely γ_d^+-edge-critical graph G with $\gamma_d(G) = k$ for each even k greater than or equal to 4.

Construction 4.1 *Let $k \in 2\mathbb{Z}$, $k \geq 4$. To construct a finitely γ_d^+-edge-critical graph G with $\gamma_d(G) = k$:*

1. *Begin with G_1, a K_k minus a maximal matching. Note that G is a*

$(k-2)$-regular graph on k vertices.

2. *Create K, a copy of K_s for $s = \binom{k}{k-2} + 1$.*

3. *Join one vertex, a, of K to each vertex in G_1.*

4. *Join each remaining vertex in K to a unique set of $k-2$ vertices in G_1.*

Lemma 4.6 *Each graph produced in Construction 4.1 has $\gamma_d(G) = k$.*

Proof. We claim the vertices $V(G_1)$ form a differentiating-dominating set of order k in the graph G. Each vertex $v \in V(G_1)$ has $k-1$ distinct elements in the closed neighbourhood with respect to G_1. The vertex $a \in V(K)$ has k neighbours in G_1, and each of the other vertices in K is adjacent to a unique set of $k-2$ vertices in G_1. Thus, for any $v, w \in V(G)$, $N_{G_1}[v] \neq N_{G_1}[w]$.

Suppose S is a differentiating-dominating set of order $k-1$ in G. If $S \subset V(G_1)$, then there is a vertex $v \in V(G_1)$ with $v \notin S$. Then v is adjacent to exactly $k-2$ elements in S. Thus, there is a vertex $w \in V(K)$ which is adjacent to the same $k-2$ elements in S, so $N_S[v] = N_S[w]$.

If S contains a vertex from K, then there are two vertices from G_1 not in S. But then there is a vertex $v \in V(K)$ such that $N_S[v] = N_S[a]$.

Thus, $V(G_1)$ is a γ_d-set of order k in the graph G. ■

Theorem 4.7 *Each graph produced in Construction 4.1 is finitely γ_d^+-edge-critical.*

Proof. We claim that the addition of any edge $e \in E(\overline{G})$ yields a distinguishable graph $G^* = G + e$ with $\gamma_d(G) < \gamma_d(G^*)$.

To see that G^* is distinguishable, we consider two cases: (1) the added edge is between vertices v and w in G_1 or (2) the added edge is between a vertex v in K and a vertex x in G_1. In the first case, both v and w are adjacent to all the vertices in G_1 so G_1 is no longer a γ_d-set. In the second case, v and some $w \in G_1$ are adjacent to the same $k-1$ vertices in G_1. However, in both cases $N[v] \neq N[w]$ since there is at least one k_1 in K that is adjacent to v but not w. Since all the other closed neighbourhoods are unaffected by the edge addition, G^* is distinguishable.

By the construction, each vertex $v \in K$ with $v \neq a$ is non-adjacent to exactly two of the vertices in G_1. Thus, if S is any set of order k in G that contains more than one vertex from K, then there is at least one vertex $x \in K$ with $x \neq a$ such that $N_S[x] = N_S[a]$. Hence, such a set is not a γ_d-set in G or G^*.

Suppose then that S contains one vertex x in K and $k-1$ vertices in G_1. Let b be the vertex in $G_1 - S$. We show that S is not a γ_d-set in G^*.

Suppose the added edge is between two vertices v and w in G_1. Then v and w are adjacent to all the vertices in G_1. If x is adjacent to either v or w, then S does not distinguish that vertex from a. If x is adjacent to neither v nor w, then S does not distinguish v and w.

Now suppose the added edge is between u in K and v in G_1. Then there is exactly one vertex w in G_1 that is not adjacent to u. If w is not in S, then S does not distinguish a and u since every vertex in S is in the closed neighbourhood of each. If w is in S, then S does not distinguish between u and the vertex in K that is not adjacent to either w or b.

Since G^* is distinguishable but has no γ_d-set of order k, we have $\gamma_d(G) < \gamma_d(G^*) < \infty$. ■

The smallest finitely γ_d^+-edge-critical graph produced by Construction 4.1 is of order 11. This graph is finitely γ_d^+-critical with $\gamma_d(G) = 4$, which shows the bound established in Proposition 4.5 is sharp. By Theorem 4.4, the difference between n and $\gamma_d(G)$ can be no less than 3; this example has a difference of 7. It is not known whether there exists a graph for which the difference is less than 7.

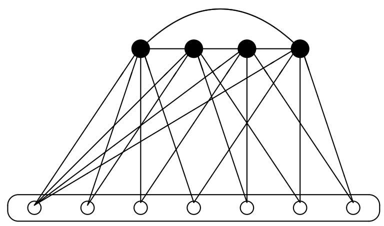

Figure 4.4: A finitely γ_d^+-critical graph with $\gamma_d(G) = 4$.

Open Problem 3 *Is there a finitely γ_d^+-edge critical graph with a γ_d-set of odd order?*

4.3 γ_d-critical

Any edgeless graph with at least one vertex is γ_d-critical. The following proposition shows some other easy examples of connected graphs that are γ_d-critical.

Proposition 4.8 *If $G = P_2$, C_4, or $K_{n,m}$ with $n > m \geq 3$, then G is γ_d-critical.*

Proof. Deletion of a vertex from P_2 leaves a singleton. Since $\gamma_d(P_2) = \infty$ and $\gamma_d(K_1) = 1$, P_2 is γ_d-critical.

Deletion of a vertex from C_4 yields P_3. Since $\gamma_d(C_4) = 3$ and $\gamma_d(P_3) = 2$, C_4 is γ_d-critical.

For $n \geq m \geq 2$, $\gamma_d(K_{n,m}) = n + m - 2$. So for $n \geq m \geq 3$, deletion of a vertex reduces the differentiating-domination number by one. ■

Proposition 4.9 *Odd cycles C_{2k+1} are finitely γ_d-critical for $k \geq 3$.*

Proof. Deletion of any single vertex from C_{2k+1} yields P_{2k}. By Theorem 1.9, if $k \geq 3$ then $\gamma_d(C_{2k+1}) = k + 2 > k + 1 = \gamma_d(P_{2k})$. ■

Proposition 4.10 *If $\delta(G) \geq 1$, $\gamma_d(G) = n - 1$, and $G - v$ is distinguishable for each $v \in V(G)$, then G is γ_d-critical.*

Proof. This follows immediately from Theorem 3.2. ■

4.4 γ_d^+-critical

Proposition 4.11 *There are no γ_d^+-critical graphs.*

Proof. Suppose G is a γ_d^+-critical graph of order n. Let $C \subseteq V(G)$ be a γ_d-set in G. Since G is γ_d^+-critical, such a set C exists.

Suppose there exists a vertex $v \notin C$. If v is deleted, then C is differentiating-dominating in $G - v$, although C may no longer be a γ_d-set. This implies that deletion of v either reduces the differentiating-domination number or leaves it unchanged. Therefore, if G is γ_d^+-critical, then $\gamma_d(G) = n$.

By Corollary 3.3, G must be edgeless so deletion of a vertex reduces the differentiating-domination number. Thus there are no γ_d^+-critical graphs. ■

Corollary 4.12 *If G is supercompact, then $G - v$ is supercompact for at least one $v \in V(G)$.*

Proof. This follows from Theorem 1.13 and the proof of Proposition 4.11. ■

4.5 γ_d-ER-critical

Consider a graph G with $\gamma_d(G) < \infty$ but for which $G - e$ is not distinguishable for any $e \in E(G)$. Entringer and Gassman [25] called graphs with this property *line-critical point distinguishing.* They showed that the only connected nontrivial line-critical point distinguishing graph is the path on three points.

Theorem 4.13 (Entringer and Gassman [25]) *A graph has $\gamma_d(G) < \infty$ and $\gamma_d(G-e) = \infty$ for all $e \in E(G)$ if and only if it is the union of isolated vertices and disjoint paths of length two.*

Proposition 4.14 *Even cycles C_{2k} are finitely γ_d-ER-critical for $k \geq 3$.*

Proof. Deletion of any single edge from C_{2k} yields P_{2k}. By Theorem 1.9, if $k \geq 3$ then $\gamma_d(C_{2k}) = k < k+1 = \gamma_d(P_{2k})$. ■

4.6 γ_d^--ER-critical

Removing an edge can cause the differentiating-domination number to decrease from the infinite to a finite value.

Proposition 4.15 *Odd cycles C_{2k+1} are finitely γ_d^--ER-critical for $k \geq 3$.*

Proof. Deletion of any single edge from C_{2k+1} yields P_{2k+1}. By Theorem 1.9, if $k \geq 3$ then $\gamma_d(C_{2k+1}) = k+2 > k+1 = \gamma_d(P_{2k+1})$. ■

4.7 Related topics

The maximum cardinality of a minimal locating-dominating set in a graph G is called the *upper locating-domination number* and is denoted $\Gamma_L(G)$.

The *upper differentiating-domination number* of G, $\Gamma_d(G)$, is the maximum cardinality of a minimal differentiating-dominating set in G. While very little is known about these parameters, it would be interesting to consider criticality for γ_L, Γ_L, and Γ_d.

We now briefly discuss various concepts related to criticality which have apparently not been previously studied in the literature. First, we begin with an open problem whose solution is likely necessary if progress on these topics is to occur.

Open Problem 4 *Provide a characterisation for each type of criticality.*

Vertex identification parameters often arise in the context of detecting and locating errors in computer or sensor networks. A 1-edge-robust differentiating-dominating set in a graph G is defined in [35] as a set which distinguishes all the vertices in any graph G' obtained by modifying G through the addition or deletion of a single edge. This leads to the more general concept of *noncritical* graphs in which, say, any vertex could be deleted without changing the value of the parameter in question. While it seems noncriticality with respect to edge or vertex deletion is related to fault-tolerant networks, a framework of noncriticality in terms of other graph alterations would be interesting.

Alternately, we define *pseudocritical* graphs to be graphs in which every deletion/addition causes the parameter to change, but not necessarily in the same direction. For example, the path of length 3, with $\gamma_d(P_4) = 3$, is *vertex pseudocritical.* Deletion of a leaf causes γ_d to decrease to 2 while deletion of an interior vertex causes γ_d to increase to ∞.

Open Problem 5 *Are there pseudocritical graphs for which the differentiation-domination number remains finite after each alteration?*

We could also consider "chains" of critical graphs. Define a graph to be *extracritical* if it remains critical after modification. Suppose, for example, that G is γ_d-critical and G' is the result of a vertex deletion. If G' is also γ_d-critical, then we say G is *Type I extracritical.* If G' is not γ_d-critical but is, say, γ_d-ER-critical, then we say G is *Type II extracritical.* Very little is currently known about extracriticality in the context of differentiating-domination. Any hope for progress in this direction seems to rely heavily on Open Problem 4.

Chapter 5

Identifying Vertices in Oriented Graphs

Suppose a computer network is designed so that all error signals are sent only one way. That is, all edges in the corresponding graph receive an orientation which describes the direction in which an error message may be sent. Once again, we are interested in finding a minimum set of processors capable of determining the exact location of any errors in the system. We shall see that in many cases this "one-way" communication reduces the number of processors required for error detection and location, while in other instances orientation causes an increase in the number of processors needed.

Recall that if v is a vertex in a digraph D, then $O(v) = \{x \in V(D) : vx \in A(D)\}$ is the outset of v, while the closed outset of v is $O[v] = O(v) \cup \{v\}$. Similarly, the inset of v is $I(v) = \{x \in V(D) : xv \in A(D)\}$ and the closed inset of v is $I[v] = I(v) \cup \{v\}$. The intersection of a set S and the outset of a vertex is denoted $O_S(v)$ while the intersection of S and the inset of a vertex is denoted $I_S(v)$. The intersection of S and the closed outset (resp. closed inset) of a vertex v is given by $O_S[v]$ (resp. $I_S[v]$). For $S \subseteq V$, $\bigcup_{v \in S} O[v]$ is written $O[S]$ and $\bigcup_{v \in S} I[v]$ is written $I[S]$.

Given a digraph $D = (V, A)$, a set of vertices S is a *differentiating-dominating set*, or a dd-set, of D if:

1. $\bigcup_{v \in S} I[v] = I[S] = V(D)$ and
2. $O_S[v]$ is unique for all vertices $v \in V(D)$.

Just as in the undirected case, we may also speak of identifying codes in a digraph. The minimum cardinality of a dd-set in a digraph D is denoted $\gamma_d(D)$.

Definition 5.1 *For an undirected graph G, the oriented differentiating-domination number $\vec{\gamma}_d(G)$ is given by*

$$\vec{\gamma}_d(G) = min\{\gamma_d(D) : D \text{ is an orientation of } G\}$$

If $I[S] = V(D)$ and $O_S(v)$ is unique for all vertices v not in S, we say S is a *locating-dominating set*, or an LD-set. The *locating-dominating number*, denoted $\gamma_L(D)$, and the *oriented locating-domination number*, denoted $\vec{\gamma}_L(G)$, are defined in a fashion similar to Definition 5.1.

Observation 5.1 *Let D be a digraph.*

(a) *D has at least one locating-dominating set. Vacuously, all the vertices form an LD-set.*

(b) *If w is a vertex with outdeg$(w) \geq 1$ then $V(D) - \{w\}$ is an LD-set.*

(c) *Both dd-sets and LD-sets are absorbant.*

Proposition 5.2 *If K is a kernel in a digraph D such that every vertex $v \in D$ has a unique outset in K, then K is a minimal dd-set.*

Proof. If K is a kernel, then K is minimal absorbant by Theorem 1.3. Since K is a dd-set that is minimal absorbant, K is therefore a minimal dd-set. ■

A similar problem recently has been studied in [7, 8], in which detection is described in a sense of out-domination rather than absorption. That is, v dominates w, and hence gathers information about w, if the edge is directed from v to w. While this indicates a certain hierarchical structure relevant in many systems, it seems that absorbant sets describe the idea of detection based on incoming messages. In any case, simply reversing all edges yields equivalent results between the two methods.

5.1 Locating-domination

Proposition 5.3 *For all graphs G, $\vec{\gamma}_L(G) \leq \gamma_L(G)$.*

Proof. We show that every LD-set in a graph G is an LD-set in some orientation of the edges of G.

Let S be an LD-set in G. For each vertex v not in S, orient the edges in such a way that vw is an arc for all $w \in N_S(v)$. Arbitrarily orient all remaining edges.

Since S is dominating, each vertex in $V_G - S$ is adjacent to at least one vertex in S. With the described orientation, S is absorbant. Since $N_S(v)$ is unique for each vertex $v \notin S$, each relevant outset is unique under the orientation. ■

We can now determine the number of vertices required in a locating-dominating set in a complete graph.

Proposition 5.4 *For a complete graph K_n with $n > 2$, $\vec{\gamma}_L(K_n)$ is the smallest k such that $n - k \leq 2^k - 1$.*

Proof. Let D be an orientation of K_n and suppose S is an LD-set of D with $|S| = k$. Since the vertices in $D - S$ have distinct outsets in S and S has only $2^k - 1$ nonempty subsets, it follows that $n - k \leq 2^k - 1$.

Conversely, if k is any integer such that $n - k \leq 2^k - 1$ then we can choose any subset S of cardinality k in $V(K_n)$ and orient the edges such that the outsets of the $n - k$ vertices not in S correspond to $n - k$ distinct subsets of S. Then S is an LD-set of the resulting orientation D of K_n. ■

A similar counting argument allows us to determine the number of vertices needed in a complete bipartite graph.

Proposition 5.5 *If $K_{r,b}$ is a complete bipartite graph with $2^{r-1} \leq b \leq 2^r - 1$, then $\vec{\gamma}_L(K_{r,b}) = r$.*

Proof. Given a complete bipartite graph $K_{r,b}$ with $2^{r-1} \leq b \leq 2^r - 1$, let R be the partition of order r and let B be the partition of order b.

We claim R is a $\vec{\gamma}_L$-set. There are $\binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r} = 2^r - 1$ possible outsets, so orient the edges from B toward R accordingly. That is, $\binom{r}{1}$ vertices in B can have one vertex in $O_R(v)$, $\binom{r}{2}$ vertices can have two vertices in $O_R(v)$, and so forth. Orient any remaining edges from R to B.

Since all the vertices in B have unique outsets in the kernel R, R is a $\vec{\gamma}_L$-set. ■

5.2 Differentiating-domination

While some graphs are indistinguishable, the following result shows that this problem can be avoided if orientation is permitted.

Lemma 5.6 *For an undirected graph G of order n, the vertex set $V(G)$ is a dd-set under any orientation.*

Proof. Clearly, the entire vertex set is absorbant. Let $v, w \in V(G)$. If $vw \notin E(G)$ then $O[v] \neq O[w]$, so without loss of generality suppose $v \in O[w]$ under some orientation of the edges of G. Then $w \notin O[v]$, so $O[v] \neq O[w]$. ∎

The following proposition now follows immediately.

Proposition 5.7 *For any undirected graph G of order n, $\vec{\gamma}_d(G) \leq n$.*

We now characterise the graphs that realise the bound in Proposition 5.7.

Theorem 5.8 *For an undirected graph G of order n, $\vec{\gamma}_d(G) = n$ if and only if $G = kP_2 \cup mK_1$ for $k, m \geq 0$.*

Proof. The sufficiency of the condition is clear. We prove the necessity by contradiction.

Suppose to the contrary that $\vec{\gamma}_d(G) = n$ and there is a vertex $v \in V(G)$ with $\deg v > 1$. Then v is adjacent to at least two vertices in G, say w and x.

Arbitrarily orient the edges in the subgraph of G induced by $S = V(G) - v$. Note that S serves to distinguish the vertices in S. Orient the edges between v and S so that $\text{outdeg}(v) = 1$ with vw as an arc.

Case I. If $O_S[v] \neq O_S[w]$, then S is a differentiating-dominating set of order $n - 1$.

Case II. If $O_S[v] = O_S[w]$, then it must be the case that $\text{outdeg}(w) = 0$ under the given orientation. We consider two subcases.

1. If w is not adjacent to x, then reverse the arc between x and v so that vx is an arc. Then $O_S[v] \neq O_S[w]$ since $x \in O_S[v]$ but $x \notin O_S[w]$. Similarly, $O_S[v] \neq O_S[x]$. Therefore, S is a dd-set of order $n-1$.

2. If w is adjacent to x, then xw is an arc. Reverse the arc between v and x so that vx is an arc and reverse the arc between v and w so that wx is an arc. Then $O_S[v] \neq O_S[x]$ since $w \in O_S[x]$ but $w \notin O_S[v]$. Furthermore, x and w are distinguishable, as are v and w. Thus, S is a dd-set of order $n-1$.

In all cases, the set S serves as a differentiating-domination set of order $n-1$. Therefore, $\vec{\gamma}_d(G) \leq n-1$, which contradicts the assumption that $\vec{\gamma}_d(G) = n$. Thus, there can be no vertex $v \in V(G)$ with $\deg v > 1$. ∎

Proposition 5.9 *For a complete graph K_n, $\vec{\gamma}_d(K_n) = \lceil \log_2(n+1) \rceil$.*

Proof. Let S be a differentiating-domination set in K_n of cardinality k. There are only $2^k - 1$ nonempty subsets of S that could serve as closed outsets of vertices in K_n, so $n \leq 2^k - 1$.

Conversely, if $n \leq 2^{k-1} - 1$ then a set S' of order $k-1$ would suffice since the $2^{k-1} - 1$ distinct nonempty subsets of the $k-1$ elements can serve as outsets for the purpose of distinguishing vertices. Thus $2^{k-1} \leq n \leq 2^k - 1$, which implies $k = \lceil \log_2(n+1) \rceil$. ∎

Proposition 5.10 *If $K_{r,b}$ is a complete bipartite graph with $2^{r-1} - r \leq b \leq 2^r - r - 1$, then $\vec{\gamma}_d(K_{r,b}) = r$.*

Proof. Given a complete bipartite graph $K_{r,b}$ with $2^{r-1} - r \leq b \leq 2^r - r - 1$, let R be the partite set of order r and let B be the partite set of order b.

We claim R is a $\vec{\gamma}_d$-set. Since the vertices in R must be used to distinguish themselves, there are $\binom{r}{2} + \cdots + \binom{r}{r} = 2^r - r - 1$ possible outsets. Orient the edges from B toward R so that each of the possible outsets is the outset of a corresponding vertex in B. Since there are no more than $2^r - r - 1$ vertices in B, the outsets of the vertices in B can all be unique. Orient any remaining edges from R to B.

Since all the vertices in $K_{r,b}$ have unique outsets in the kernel R, R is a differentiating-domination set with r elements.

A set with $r-1$ elements can serve to distinguish at most $2^{r-1} - 1$ vertices, but between 2^{r-1} and $2^r - 1$ nonempty and unique outsets are needed to distinguish the vertices in $K_{r,b}$. Therefore, R is a $\vec{\gamma}_d$-set of $K_{r,b}$. ■

It is easy to see that $\vec{\gamma}_d(K_{1,1}) = 2$ and that $\vec{\gamma}_d(K_{1,b}) = b$ for $b > 1$. It can also be seen that $\vec{\gamma}_d(K_{2,b}) = 3$ for $b = 2, 3$. We now consider *balanced* complete bipartite graphs $K_{r,r}$.

Proposition 5.11 *Let $r \geq 5$. If k is an integer such that $r \leq 2^k + k - 1$, then $\vec{\gamma}_d(K_{r,r}) \leq 2k$.*

Proof. Let $A \cup B$ be the partition of the vertex set of $K_{r,r}$. Select k vertices from A and k vertices from B to form a set S. We claim S is a dd-set under an appropriate orientation of $K_{r,r}$.

Label the vertices in $S \cap A$ with consecutive odd integers beginning with 1 and label the vertices in $S \cap B$ with consecutive even integers beginning with 2. Orient the edges in the subgraph induced by S to form a directed cycle $\{1, 2, 3, \ldots, 2k, 1\}$. Arbitrarily orient the remaining edges in the subgraph induced by S. Note that each vertex in S has a unique closed outset determined by this orientation, so the vertices in S are distinguished by S.

Consider the vertices in $A - S$. There are $2^k - 1$ nonempty and unique outsets in $S \cap B$ for distinguishing the $r - k$ vertices in $A - S$. Orient the edges from $A - S$ toward $S \cap B$ so that each of these possible outsets is the outset of a unique vertex in $A - S$. Orient the edges between $S \cap A$ and $B - S$ in a similar fashion, then orient all remaining edges into either $A - S$ or $B - S$.

The described orientation is possible since $r - k \leq 2^k - 1$. Since all vertices in $K_{r,r}$ are distinguished, S is a dd-set under this orientation. ■

We shall see in Chapter 6 that in general it is quite difficult to determine $\vec{\gamma}_d(G)$ even for restricted classes of graphs.

5.3 Effects of orientation

While orientation drastically reduces the number of vertices needed to distinguish vertices in complete graphs and complete bipartite graphs, there are examples in which orientation results in an increase in the number of vertices required. Consider the *corona* of G, denoted $G \circ K_1$, which is formed by joining a new vertex v' to each $v \in V(G)$. Consider $P_3 \circ K_1$. Three vertices are required in a γ_d-set, but at least four vertices are required in an identification set. This gives rise to the following question.

Open Problem 6 *Under what conditions is* $\vec{\gamma}_d(G) < \gamma_d(G)$*?*

This question is partially answered by the following proposition.

Proposition 5.12 *If G is a tree, then* $\gamma_d(G \circ K_1) < \vec{\gamma}_d(G \circ K_1)$.

Proof. Since G is distinguishable, all the vertices of G form a differentiating-dominating set. Since G is connected and each pendant vertex in the corona is adjacent to exactly one vertex in G, the set of all the vertices in G is also differentiating-dominating in $G \circ K_1$.

Let T be an orientation of $G \circ K_1$. For each vertex v in G, either v or its private neighbour in the corona must be in any $\vec{\gamma}_d$-set. Since G is introverted, there is a v_T in $V(G)$ with $\text{outdeg}_G(v_T) = 0$. Consider the arc joining v_T and its private neighbour in the corona. Regardless of the direction of the orientation, both of these vertices must be in any dd-set. Therefore, $\vec{\gamma}_d(G \circ K_1) > \gamma_d(G \circ K_1)$. ■

There are other graphs which have $\gamma_d(G) < \vec{\gamma}_d(G)$. In fact, it is possible to satisfy this inequality with the two values arbitrarily far apart. Let G be formed by joining two copies of $P_3 \circ K_1$ at an additional vertex v as shown in Figure 5.1. Clearly, $\gamma_d(G) = 6$. However, $\vec{\gamma}_d(G) = 7$. In the best case, orient the vertical edges in Figure 5.1 upward and orient the horizontal edges toward v. Then the six dark vertices together with v serve to distinguish the vertices.

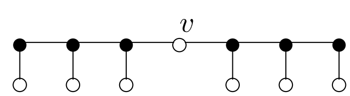

Figure 5.1: A graph in which a minimum dd-set under any orientation has more than $\gamma_d(G)$ vertices.

Suppose w is a leaf of G. Attach G', a copy of G, to w by adding edge $v'w$. A copy of G can then be added to any leaf of the resulting graph as well. This can be repeated as many times as desired to yield a family of graphs G^* with $\gamma_d(G^*) = 6n/13$ and $\vec{\gamma}_d(G^*) = 7n/13$.

We now consider some situations in which orientation does not cause an increase in the number of vertices needed for identification purposes.

Proposition 5.13 *If G has an independent γ_d-set, then $\vec{\gamma}_d(G) \leq \gamma_d(G)$.*

Proof. Let C be an independent γ_d-set in G. Orient the edges from vertices in $V(G) - C$ into C and arbitrarily orient all remaining edges. Clearly C is a dd-set, so $\vec{\gamma}_d(G) \leq \gamma_d(G)$. ■

Corollary 5.14 *For odd paths and even cycles G, $\vec{\gamma}_d(G) \leq \gamma_d(G)$.*

While we do not yet have a proof, examples of short paths and cycles indicate that orientation neither reduces nor increases the number of vertices needed to distinguish vertices.

Conjecture 5.15 *For all paths and cycles G, $\vec{\gamma}_d(G) = \gamma_d(G)$.*

Chapter 6

A Survey of Complexity Results

We briefly discuss computational complexity in two specific classes of graphs for certain parameters related to vertex identification and domination. We then provide a short overview of NP-completeness and a summary of known NP-completeness results related to the the parameters in question.

6.1 Permutation graphs and trees

Many graph problems which are difficult in general can be solved easily when the problem is restricted to the class of trees. Section 12.4.1 of [34] describes linear algorithms for computing many domination-related parameters in trees, while a linear algorithm for finding a minimum cardinality locating-dominating set in an acyclic graph is given in [50]. A linear algorithm for finding a minimum cardinality differentiating-dominating set in oriented trees is given in [7].

We now consider another class of graphs which is briefly discussed in Section 12.4.3 of [34]. Suppose $\mathcal{P} = [p_1, p_2, \ldots, p_n]$ is a permutation of the numbers $1, 2, \ldots, n$, with the elements simply listed in the order in which they occur. Let $\mathcal{P}(i)$ be the element in position i and $\mathcal{P}^{-1}(i)$ be the position of i in $\mathcal{P}$. For example, if $\mathcal{P} = [4, 3, 5, 1, 2]$, then $\mathcal{P}(3) = 5$ and $\mathcal{P}^{-1}(5) = 3$. As described in [5], a *permutation graph* (or *inversion* graph) $G[\mathcal{P}]$ has vertex set $1, 2, \ldots, n$ with $ij \in E(G)$ if and only if $(i - j)(\mathcal{P}^{-1}(i) - \mathcal{P}^{-1}(j)) < 0$. The permutation graph for $\mathcal{P} = [4, 3, 5, 1, 2]$ is shown.

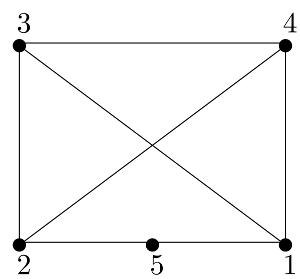

Figure 6.1: A permutation graph.

It is easy to see whether a permutation graph is distinguishable. For $a > b$, an (a, b)-block is a subsequence in $\mathcal{P}$ of $a - (b + 1)$ values which consists of a permutation of the numbers $b+1, b+2, \ldots, a-1$. For example, in the permutation $\mathcal{P} = [4, 2, 3, 1, 5, 6]$, the subsequence $[4, 2, 3, 1]$ is a $(4, 1)$-block.

Lemma 6.1 *If there is a* $(j+1, j)$*-block in a permutation* $\mathcal{P}$*, then* $G = G[\mathcal{P}]$ *is indistinguishable.*

Proof. Note that j is adjacent to $j + 1$ in G. Suppose $x > j + 1$ precedes $j + 1$ somewhere in $\mathcal{P}$, so that x is adjacent to $j + 1$ in G. Then x also precedes j and x is adjacent to j. Similarly, if $y < j$ follows j somewhere in $\mathcal{P}$, then y is adjacent to both j and $j+1$. In any case, $N[j] = N[j+1]$. ■

Proposition 6.2 *A permutation graph $G = G[\mathcal{P}]$ is indistinguishable if and only if $\mathcal{P}$ contains an (a,b)-block.*

Proof. If $\mathcal{P}$ contains an (a,b)-block for some a and b, then clearly $N[a] = N[b]$ so G is indistinguishable.

Now suppose $\mathcal{P}$ does not contain an (a,b)-block. Suppose $X = \{x_1, x_2, \ldots x_k\}$ is a subset of $\mathcal{P}$ and that vXw is a subsequence of values in $\mathcal{P}$. We need only consider the case in which $v > w$, since otherwise $N[v] \neq N[w]$. If X is not empty, then X contains either a value larger than v or a value smaller than w. Either way, $N[v] \neq N[w]$ in $G[\mathcal{P}]$.

If X is empty, then by Lemma 6.1 we see that $v > w + 1 = y$. If y precedes v in $\mathcal{P}$, then $y \notin N[v]$ but $y \in N[w]$. Alternatively, if v precedes y, then $y \in N[v]$ but $y \notin N[w]$. Again, $N[v] \neq N[w]$.

Therefore, $G[\mathcal{P}]$ is distinguishable. ■

An algorithm for finding a minimum dominating set in permutation graphs is given in [5]. It is shown through aggregate analysis that the algorithm runs in O(n) time in amortized sense; see Chapter 17 of [23] for more information on amortized analysis. The simple description provided by Lemma 6.1 and Proposition 6.2 of whether a permutation graph is distinguishable, together with methods developed in [5], seem to indicate that a polynomial-time algorithm may be found for finding a γ_d-set for a permutation graph.

6.2 NP-completeness results

It is well known that some problems can be solved in polynomial time in terms of the size of the input, while other problems seem to be much more difficult to solve but can be easily checked. The problems which can be solved in polynomial time are said to be in the class P. Problems which can "checked" in poynomial time are said to be in the class NP. That is, for a problem in NP it can be determined in polynomial time whether a potential answer is actually correct.

There is an important subset of problems in the class NP known as the class of NP-complete problems. If A is a member of this class and B is any problem in NP, then each instance of problem B can be transformed in polynomial time into an instance of problem A in such a way that a "Yes" answer to the instance of B corresponds to a "Yes" answer to the corresponding instance of A. "No" answers correspond as well. Thus, any procedure for solving the problem A could be used to solve problem B. However, there are no known polynomial time algorithms for solving any of the NP-complete problems in general. See [34] or a text such as [23] for much more information on this topic.

The typical decision questions have been considered for nearly all the domination and vertex identification parameters in question. Nearly all these problems are NP-complete in general, but may have polynomial, even linear, algorithms when restricted to appropriate classes of graphs. We follow the standard form for stating decision problems as given in [29].

DOMINATING SET

INSTANCE: A graph G and a positive integer k

QUESTION: Is $\gamma(G) \leq k$?

LOCATING-DOMINATION

INSTANCE: A graph G and a positive integer k

QUESTION: Is $\gamma_L(G) \leq k$?

DIFFERENTIATING-DOMINATION

INSTANCE: A graph G and a positive integer k

QUESTION: Is $\gamma_d(G) \leq k$?

DIRECTED LOCATING-DOMINATION

INSTANCE: A digraph D and a positive integer k

QUESTION: Is $\gamma_L(D) \leq k$?

DIRECTED DIFFERENTIATING-DOMINATION

INSTANCE: A digraph D and a positive integer k

QUESTION: Is $\gamma_d(D) \leq k$?

Theorem 6.3 *The following problems are NP-complete.*

1. (D. Johnson, cited in [34]) *DOMINATING SET is NP-complete.*

2. ([22]) *LOCATING-DOMINATION is NP-complete.*

3. ([9]) *LOCATING-DOMINATION and DIFFERENTIATING-DOMINATION are NP-complete even when restricted to bipartite graphs.*

4. ([43]) *DIFFERENTIATING-DOMINATION is NP-complete even when restricted to bipartite planar unit disk graphs.*

5. ([8]) *DIRECTED LOCATING-DOMINATION and DIRECTED DIFFERENTIATING-DOMINATION are NP-complete and remain so even when restricted to strongly connected, directed, asymmetric, bipartite graphs or to directed, asymmetric, bipartite graphs without directed cycles.*

The fact that these problems are all NP-complete makes it unlikely that polynomial-time algorithms will be found. It then becomes a topic of interest to develop heuristics or provably good approximation algorithms to find sets which are reasonably close to optimal cardinality while maintaining the desired properties. However, all the proofs cited in Theorem 6.3 utilise the NP-completeness of the well-known 3-satisfiability problem, and as such do not directly yield approximation algorithms. Fortunately, other progress has been made on finding approximations for differentiating-domination.

A greedy algorithm for finding a minimal differentiating-dominating set is given in [47]. The algorithm begins by selecting the entire vertex set V of the input graph G as a potential differentiating-dominating set; if G is indistinguishable the algorithm notes this and stops. Otherwise, G is distinguishable so V is differentiating-dominating. Each vertex is then considered for deletion from the differentiating-dominating set according to some predetermined order. If deletion of a vertex v yields a differentiating-dominating set, then v is deleted. Otherwise, v is retained in the set and the next vertex is considered. The algorithm stops when all vertices have been considered for deletion and outputs a minimal differentiating-dominating set.

Unfortunately, the cardinality of a minimal differentiating-dominating set may be much larger than that of a γ_d-set. Consider P_7 labeled $v_1, \ldots, v_7$ in the usual manner. Suppose the vertices are considered for deletion in the order $v_1, v_6, v_2, v_3, v_4, v_5, v_7$. Then v_1 and v_6 can be deleted from the differentiating-dominating set, but deletion of any the other vertices leaves either two indistinguishable vertices or at least one vertex undominated. Thus, the algorithm outputs a differentiating-dominating set of cardinality 5 when in fact $\gamma_d(P_7) = 4$.

Proposition 3 in [42] shows that the difference between the cardinality of a minimum and minimal differentiating-domination set can be arbitrarily large, so the algorithm just described has no performance guarantees in general. However, consider the *set cover problem* described in [40], in which we have a set T of n elements. Let $\mathcal{S}$ be a set of subsets of T. A set $C \subseteq \mathcal{S}$ is a *cover* of T if $\bigcup C = T$. The problem SET COVER asks for a cover of minimum cardinality. While SET COVER is NP-complete, a well-known greedy algorithm is to begin with an empty cover and at each step to add the set from $\mathcal{S}$ which covers the largest number of uncovered elements in T. Consider the ratio of the cardinality of the greedy cover to the cardinality of the minimum cover. A result of Slavik cited in [40] is that the difference of the upper and lower bounds for this ratio is less than 1.1. A greedy algorithm for finding a minimal differentiating-dominating set C in a distinguishable graph G is given in [40]; it is based on a greedy solution to a set cover problem on a set of $n(n-1)/2$ elements. Reductions between DIFFERENTIATING-DOMINATION and SET COVER are given and it is shown that the bounds on the approximation of SET COVER hold for DIFFERENTIATING-DOMINATION.

Chapter 7

Future Research

We conclude with a sample of topics for future research. Many potential avenues of research have been discovered during this study; the following seem to be among the most interesting.

1. A version of locating-domination in which vertices can be identified from a distance is described in [4], and [38] introduced a distance version of identifying codes. It would be interesting to consider the effects of distance on both criticality and orientation.

2. *Total* or *open domination* is defined in Section 6.3 of [34] to mean that a vertex dominates its open neighbourhood but not itself. That is, S is a total dominating set if $V = N(S)$. For differentiating-domination, this would mean a vertex may dominate and identify errors in its open neighbourhood but not at itself. Thus, S is a total differentiating-dominating set if $N(S) = V$ and $N_S(v) \neq N_S(w)$ for any v, w. The minimum cardinality of such a set is the total differentiating-domination number.

3. A set S of vertices is defined in Section 7.6 of [34] to be a *global dominating set* of a graph G if S is dominating in both G and $\overline{G}$. Global differentiating-domination would involve finding a smallest set that is differentiating-dominating in both G and $\overline{G}$.

4. For an undirected graph G and a set of colours C, let $\psi : E(G) \rightarrow C$ be a proper edge colouring. The *colour set* of a vertex with respect to ψ is the set of colours of edges incident with the vertex. A colouring is defined in [37] to be *neighbour-distinguishing* if each set of adjacent vertices has a unique colour set.

5. A *discriminating code* in a bipartite undirected graph $G = (A \cup B, E)$ is defined in [6] to be a subset $S \subseteq B$ such that $N_B(v) \neq \emptyset$ and $N_B(v) \neq N_B(w)$ for $v, w \in A$. This is very similar to locating-domination and differentiating-domination except that the only elements which must be distinguished are those in A, and only elements from B are permitted to be used for this purpose.

6. Consider the famous chessboard result that it is possible to place a maximum of n independent Queens on an $n \times n$ board. Two recent papers deal with increasing the number of Queens by "placing Pawns" which block Queens from attacking one another [13, 14]. This idea of *separation* on the chessboard has also been applied to changing the number of pieces required for domination and other parameters, as well as to other pieces such as Rooks and Bishops. A general framework for separation is currently being developed [15]. Let $\mathcal{F}$ be a family of graphs on a set of vertices V. The *transit graph* $T(\mathcal{F})$ is the graph on V with $ab \in E(T)$ if and only if there is a path between a and b in at

least one member of $\mathcal{F}$. Then separation on $T(\mathcal{F})$ is defined in terms of removing a given vertex from each member of $\mathcal{F}$ and generating the transit graph of the modified family.

Since every graph G can be described as the transit graph of a family of graphs on $V(G)$, the family with the fewest members, as well as the number of members necessary to generate G, are both of interest. Vertex identification, in which a vertex is determined based on the paths on which it lies, then becomes possible as well.

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