INVESTIGATION INTO WHETHER SOME KEY PROPERTIES OF $\beta\mathbb{N}$ UNDER ADDITION ALSO APPLY IN $\beta\mathbb{N}$ UNDER MULTIPLICATION AND ELABORATION OF SOME PROPERTIES OF THE SMALLEST IDEAL OF A SEMIGROUP

by

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I declare that **INVESTIGATION INTO WHETHER SOME KEY PROPERTIES OF $\beta\mathbb{N}$ UNDER ADDITION ALSO APPLY IN $\beta\mathbb{N}$ UNDER MULTIPLICATION AND ELABORATION OF SOME PROPERTIES OF THE SMALLEST IDEAL OF A SEMIGROUP** is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

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Abstract

This dissertation will seek to explore if the properties of some of the key results on semigroups and their compactifications under the operation of addition also apply under the operation of multiplication. Considerable emphasis will be placed on the semigroup $\mathbb{N}$ of the set of natural numbers and its compactification $\beta\mathbb{N}$.

Furthermore, the dissertation will discuss the smallest ideal of a semigroup and highlight some of its fundamental properties.

Key terms

Semigroup; sub-semigroup; left ideal; right ideal; ideal; minimal left ideal; minimal right ideal; ultrafilter; semigroup compactification; commutativity; idempotents; smallest ideal.

Contents

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Glossary of symbols and Notations

The following notations will be used

$\mathbb{N}$	The set of natural numbers
ω	$\mathbb{N} \cup \{0\}$
$\in$	' Is a member of '
$\varnothing$	Empty set
$\cap$	Intersection of sets
$\cup$	Union of sets
$\subseteq$	' Is a subset of '
$\subset$	' Is a proper subset of '
$\mid S \mid$	The cardinality of a set S
c	The cardinality of the continuum
$\overline{A}$	The closure of a set A
$\beta\mathbb{N}$	The Stone-Čech compactification of $\mathbb{N}$
$(\beta\mathbb{N}, +)$	The Stone-Čech compactification of $\mathbb{N}$ under the operation of addition
$(\beta\mathbb{N}, \bullet)$	The Stone-Čech compactification of $\mathbb{N}$ under the operation of multiplication

$K(\beta\mathbb{N},+)$	The smallest ideal of $(\beta\mathbb{N},+)$ under the operation of addition
$K(\beta\mathbb{N},\bullet)$	The smallest ideal of $(\beta\mathbb{N},\cdot)$ under the operation of multiplication
S^*	$\beta S\backslash S$
$\wp_f\ (A)$	The collection of finite subsets of A
$\Lambda(S)$	The topological centre of S
$U\oplus V$	The sum of ultrafilters U and V
$U\odot V$	The product of ultrafilters U and V
$\widetilde{\phi}$	The continuous extension of the mapping ϕ
$E(S)$	The collection of idempotents of S
$C(S)$	The set of all continuous functions from the topological space S into the topological space $\mathbb{R}$.
$C^*(S)$	All bounded functions in $C(S)$

Chapter 0

Introduction

A semigroup is a pair $(S, *)$, where S is a non-empty set and $*$ is a binary associative operation on S (Hindman and Strauss, 1998). The set of natural numbers is an example of a semigroup.

Given a semigroup $(S, *)$, one can extend the operation $*$ on S to its Stone-Čech compactification $(\beta S, *)$. The resulting semigroup $(\beta S, *)$ is a right topological semigroup with S contained in its topological centre. Furthermore, the compactification $(\beta S, *)$ has a smallest two-sided ideal.

In the last thirty years, the Stone-Čech compactification of a discrete group has generated a lot of interest. Research activities in this area included that of the compactification of the set of natural numbers, $\mathbb{N}$. One of the key applications of the compactification of the set of natural numbers is that the algebra of $\beta\mathbb{N}$ led to the establishment of a simple extension of the Finite Sums Theorem which states that whenever $\mathbb{N}$ is finitely colored, there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ such that $\text{FS}(\langle x_n \rangle_{n=1}^{\infty}) \cup \text{FP}(\langle y_n \rangle_{n=1}^{\infty})$ is a monochrome, where the finite sum, FS

is given by

$$FS\left(\langle x_n\rangle_{n=1}^{\infty}\right)=\left\{\sum_{n\in F} x_n:\ \text{F is finite nonempty subset of } \mathbb{N}\right\}$$

and the finite product, FP is given by

$$FP\left(\langle y_n\rangle_{n=1}^{\infty}\right)=\left\{\prod_{n\in F} y_n:\ \text{F is finite nonempty subset of } \mathbb{N}\right\}$$

In proving this important result, $\beta\mathbb{N}$ was made into a right topological semigroup.

The set $\mathbb{N}$ is the most familiar semigroup, yet very little is known about the properties of the compactification of $\beta\mathbb{N}$, particularly under multiplication. Though considerable progress has been achieved in investigating the properties of $\beta\mathbb{N}$ under addition, a lot more research is required to investigate the properties of $\beta\mathbb{N}$ under multiplication especially on the smallest ideal $K(\beta\mathbb{N},\bullet)$. For example, it is known that

1. commutativity holds in $(\beta\mathbb{N},+)$,
2. the smallest ideal of $(\beta\mathbb{N},+)$ can be expressed as a disjoint union of the minimal right ideals as well as the the disjoint union of the minimal left ideals,
3. $(\beta\mathbb{N},+)$ has 2^c minimal left ideals and 2^c minimal right ideals,
4. there are 2^c idempotents outside the minimal ideal of $(\beta\mathbb{N},+)$, and
5. there exist copies of F_∞ in $(\beta\mathbb{N},+)$ which miss the minimal ideal of $(\beta\mathbb{N},+)$, where F_∞ is the free semigroup on countably many generators.

The question as to whether some key properties that hold in $\beta\mathbb{N}$ under addition also apply in $\beta\mathbb{N}$ under multiplication is critical and is the purpose of chapter 2. Mathematical literature has demonstrated that there are results that show that some of the properties of $\beta\mathbb{N}$ under addition also apply in $\beta\mathbb{N}$ under multiplication. For example, it is shown that $\beta\mathbb{N}$ under both addition and multiplication has the same topological centre i.e., $\mathbb{N}$ [7, pg. 130]. Furthermore, each maximal subgroup of $K(\beta\mathbb{N}, \bullet)$ contains a copy of the free group on 2^c generators [10, pg. $4-5$], a result which also holds in $K(\beta\mathbb{N}, +)$. In addition to this, broadening the understanding of the compactification of $\beta\mathbb{N}$ especially under multiplication will not only widen the application of mathematics but will also help in finding solutions to problems that have remained open for a long time. One of these problems is as follows:
Are there elements ξ, η, σ,and τ of $\mathbb{N}^*$ for which $\xi + \eta = \sigma\tau$?

In the above statement, it is clear that the knowledge of not only the sum but also the product of elements of $\beta\mathbb{N}$ is essential in order to solve the problem since $\mathbb{N}^* \subset \beta\mathbb{N}$.
In this dissertation, we will seek to show that the following key properties which hold in $(\beta\mathbb{N}, +)$ also apply in $(\beta\mathbb{N}, \bullet)$:

1. $(\beta\mathbb{N}, \bullet)$ is a semigroup

2. Commutativity does not hold in $(\beta\mathbb{N}, \bullet)$

3. $(\beta\mathbb{N}, \bullet)$ has 2^c minimal left ideals and 2^c minimal right ideals

4. $(e, \beta\mathbb{N})$ is a semigroup compactification where e is the embedding

from $\mathbb{N}$ to $\beta\mathbb{N}$.

The dissertation will then go on to bring out the idea of the smallest ideal of a semigroup and its characterisation. The fact that the smallest ideal of a semigroup can be expressed as a disjoint union of minimal left ideals and also a disjoint union of minimal right ideals will be among the key results to be highlighted in the discussion of the smallest ideal.

Chapter 0 has provided an introduction to semigroups and their compactifications. The chapter has also highlighted the key problems to be solved in the dissertation.

Chapter 1 will deal with preliminary results. The chapter will bring out some key results on semigroups and their compactifications. Greater emphasis will be given to results under the operation of addition.

Chapter 2 will consider some key properties of discussed in chapter 1 and investigate whether the properties also hold under the operation of multiplication.

Chapter 3 will discuss the smallest ideal of a semigroup and highlight some of its key properties.

Chapter 1

Preliminaries

This chapter is dedicated to literature review. Some known results about semigroups and their compactifications will be presented, beginning with a presentation of a brief historical background of semigroups and some basic details on semigroups. We will also discuss algebaic properties of semigroups. In addition, the concept of topological semigroups will be presented.

1.1 Historical Background

Semigroups have a rich and interesting historical background. This area of Mathematics, though more recent than many other branches of mathematics such as group theory and ring theory, has attracted the attention of mathematicians resulting in increased research activities in this field of study.

Historically, it is claimed that the term 'semigroup' first appeared in mathematical literature in 1904, that the first published paper on semigroups 'On semigroups and the General Isomorphism' was authored by L.E Dickson and appeared in 1905 and that the first book on semigroups appeared in 1937 . It is also documented that from 1940 to 1961, the number of papers on semigroups appearing each year grew fairly to an average of little more than 30 per year (See Theory of Generalised Groups by A.K Suschkewitsch (cited in [9, pg. 7])).

The question remains, "Why semigroups?". M. Petrich [11, pg. 1] reminds us that "of all the numerous generalisations of group or ring theory, the theory of semigroups has been undoubtedly the greatest success". He asserts that the study of semigroups started as an attempt at generalising group theory by omitting the axioms of existence of identity and inverses, and an abstraction of the multiplicative theory of rings. He further asserts that all this work provided impetus for extensive study of other properties of semigroups.

Examples of semigroups

1. The set of natural numbers under addition, that is $(\mathbb{N}, +)$, is a semigroup since addition of natural numbers is associative.

2. Consider any nonempty set S and define $*$ on S by $x * y = y$ for all $x, y \in S$. Since for any $x, y, z \in S$,
$$(x * y) * z = y * z = z = x * z = x * (y * z)\,,$$
the operation $*$ on S is associative so that $(S, *)$ is a semigroup.

3. The sets of real numbers under the operations of addition and multiplication i.e., $(\mathbb{R}, +)$ and $(\mathbb{R}, \bullet)$ are semigroups

4. Consider the set $(X_X, \circ)$, where $X_X = \{f : f : X \longrightarrow X\}$ and $\circ$ represents the composition of functions. Let $f, g, h \in X_X$. Clearly,

$$f(X) = X, \quad g(X) = X \quad \text{and } h(X) = X$$

Hence,

$$\begin{aligned} &(f \circ (g \circ h)(X)) \\ &= f((g \circ h)(X)) \\ &= f(g(h)(X)) \\ &= f(g(X)) \\ &= f(X) \\ &= X \end{aligned}$$

and

$$\begin{aligned} &((f \circ g) \circ h)(X) \\ &= (f \circ g)(h(X)) \\ &= (f \circ g)(X) \\ &= f(g(X)) \\ &= f(X) \\ &= X \end{aligned}$$

Hence associativity holds, showing that $(X_X, \circ)$ is a semigroup.

5. Let S be $(\mathbb{R}, +)$ with the topology for which the base of neighborhoods of any point $x \in \mathbb{R}$ is given by $\left\{\left[x, x + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$. We

show that this is a topological semigroup.

That $(\mathbb{R},+)$ is a semigroup is clear since associativity holds in $(\mathbb{R},+)$. Define the mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(y) = x + y$ where $x,\ y \in \mathbb{R}$. Let V be a neighborhood of $x + y$. Then there exists $V_1 \subset V$ such that $V_1 \in \left\{\left[x+y,\, x+y+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$. Clearly, $\left\{\left[y,\ y+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$ is a neighborhood base for y. Now, $f(y) = x + y$ and $f\left(y+\frac{1}{n}\right) = x + y + \frac{1}{n}$. So, if we pick a neighborhood U of y from $\left\{\left[y,\ y+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$, then $f(U) = \left\{\left[x+y,\, x+y+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$. Hence we can choose a neighborhood of y such that $f(U) = V_1 \subset V$.

Hence continuity holds and therefore we have a topological semigroup.

The definition of topological semigroup will be given in the next section.

Of interest to our study is the compactification of semigroups. In particular, we will look at the Stone-Čech compactification of the set of natural numbers.

Given a discrete semigroup $(S, *)$, one can extend the operation $*$ to βS, the Stone-Čech compactification of S, so that $(\beta S, *)$ becomes a right topological semigroup with S contained in its topological centre. That is, for each $p \in \beta S$, the function $\gamma_x : \beta S \rightarrow \beta S$ defined by $\gamma_x(p) = x * p$ is continuous. We take βS to be the set of ultrafilters on

S and identify the points of S with the principal ultrafilters.

The fact that an operation of a discrete semigroup S can be extended to βS was first implicitly established by M. Day while R. Ellis successfully did the extension of an operation from S to βS as a space of ultrafilters, where S is a group (cited in [7, pg. 107]).

To better understand the topology of βS, choose sets of the form

$\overline{A} = \{p \in \beta S : Ap\}$,where $A \subseteq S$, is a base for open sets.

It should be noted that though we are denoting the operation by $+$ because we will be concerned with the semigroup $(\beta S, +)$, the semigroup is never commutative.

1.2 Some definitions and known results on semigroups and ideals

In this section, we present some known results on semigroups and their algebraic properties. For a semigroup $(S, *)$ with $s \in S$ and $\emptyset \neq V$, $W \subseteq S$, we define the following subsets:

$$s * V = sV = \{s * v : v \in V\}$$

$$V * s = Vs = \{v * s : v \in V\} \text{ and}$$

$$V * W = VW = \{v * w : v \in V, w \in W\}.$$

Definition 1.1 Subsemigroup

Let $(S,*)$ be a semigroup. A nonempty subset T of S is a subsemigroup if $TT \subseteq T$.

Example

Consider the multiplicative semigroups $(\mathbb{N}, \bullet)$ and $(\omega, \bullet)$ with $\omega = \mathbb{N} \cup \{0\}$. Since $\mathbb{N}\mathbb{N} \subseteq \mathbb{N}$, $(\mathbb{N}, \bullet)$ is a subsemigroup of $(\omega, \bullet)$.

Definition 1.2 Left ideal, right ideal, ideal

Let $(S,*)$ be a semigroup.

1. L is a left ideal of S if and only if $\varnothing \neq L \subseteq S$ and $SL \subseteq L$
2. R is a right ideal of S if and only if $\oslash \neq R \subseteq S$ and $RS \subseteq R$
3. I is an ideal of S if and only if I is both a left ideal and a right ideal of S

Definition 1.3 Minimal left ideal, minimal right ideal, left simple semigroup, right simple semigroup, simple semigroup

Let $(S,*)$ be a semigroup.

1. L is a minimal left ideal of S if and only if L is a left ideal of S and whenever J is a left ideal of S and $J \subseteq L$, then $J = L$.

2. R is a minimal right ideal of S if and only if R is a right ideal of S and whenever J is right ideal of S and $J \subseteq R$, then $J = R$.

3. S is left simple if and only if S is a minimal left ideal of S.

4. S is right simple if and only if S is a minimal right ideal of S.

5. S is simple if and only if the only ideal of S is S.

Theorem 1.4

Let $(S, *)$ be a semigroup and $\varnothing \neq L \subseteq S$. Then a left ideal L of S is minimal if and only if $Ss = L$, where $s \in L$

Proof. See Theorem 1.31c [7, pg. 16]. ■

We now present the topological properties of semigroups and related concepts of compactness.

Definition 1.5 Right and Left topological semigroups

Let S be a semigroup endowed with a topology. S is a right topological semigroup if the map $r_t : S \longrightarrow S, s \longmapsto st$ is continuous for each $t \in S$. In other words, a right topological semigroup is a pair $(G, *)$ satisfying the following:

1. G is a topological space.

2. $*$ is a binary associative operation on G.

3. For every element x in G, the function $*_x : G \longrightarrow G$ that maps every element y of G to $y * x$ is continuous.

The definition of a left toplogical semigroup is defined similarly by using a left-right switch.

Definition 1.6 Topological semigroups

Let S be a semigroup endowed with a topology. Then S is a topological semigroup if it is both left and right topological.

Definition 1.7 Compact right topological semigroup

A semigroup is a compact right topological semigroup if it is compact (i.e., for every open cover, there exists a finite sub-cover) and is also right topological. Compact right topological semigroups have very useful properties and will often be mentioned in this study.

Definition 1.8 Homomorphisms between semigroups

Let $(S, *)$ and $(T, \bullet)$ be semigroups. A function $\varphi : S \longrightarrow T$ is a homomorphism if $\varphi(x * y) = \varphi(x) \bullet \varphi(y)$ for all x, $y \in S$.

If a homomorphism is both one-one and onto, it is known as an isomorphism.

Clearly, the composition of two homomorphisms is a homomorphism. In category theory, there is category of semigroups in which the objects are semigroups and the morphisms are homomorphisms.

1.3 Stone-Čech Compactification

This section focusses on the Stone-Čech compactification, a very important concept of our study.

It should be noted that an embedding of a topological space X into a topological space Y is a function $\varphi : X \longrightarrow Y$ which defines a homeomorphism from X onto $\varphi[X]$ i.e., φ is one-one, onto and continuous and φ^{-1} is also continuous. All the hypothesised topological spaces are Hausdorff.

As stated in our introduction, given a semigroup $(S, *)$, one can extend the operation $*$ on S to its Stone-Čech compactification $(\beta S, *)$. The resulting semigroup $(\beta S, *)$ is a right topological semigroup with S contained in its topological centre.

The knowledge of ultrafilters is crucial to gaining a clear understanding of the Stone-Čech compactification. Hence, before going into greater details, we need the following.

Definition 1.9 Ultrafilter

An ultrafilter on a set D is a filter which is not properly contained in any other filter on D.

Examples

1. Consider the set $X = \{1, 2, 3\}$. Then,
   $\mathcal{F} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$ is an ultrafilter since there is no

other filter on X properly containing $\mathcal{F}$.

2. Let $D = \{n, m, p\}$. Then the collection of subsets $\mathcal{F}= \{\{n, m\}, \{n, m, p\}\}$ is a filter but not an ultrafilter on D as it is a proper subset of $\{\{n\}, \{n, m\}, \{n, p\}, \{n, m, p\}\}$ which is another filter on D.

Definition 1.10 Principal ultrafilter

Let D be a set and $a \in D$. Then $e(a) = \{A \subseteq D : a \in A\}$ is called a principal ultrafilter.

For the set $D = \{a, b, c\}$, $e(a) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ is a principal ultrafilter with a as an element of each of $\{a\}, \{a, b\}, \{a, c\}$ and $\{a, b, c\}$.

An ultrafilter whicth is not principal is called a free ultrafilter.

Definition 1.11 βS

Given that S is a semigroup, then by βS, we mean the collection of all ultrafilters on S.

We now illustrate how to construct βS where S is a semigroup. Before this is done, we need to note that $C(S)$ denotes the set of all continuous functions from the topological space S into the topological space $\mathbb{R}$ while $C^*(S)$ consists of all bounded functions in $C(S)$.

Now, βS is a compact space in which S is dense and C^* embedded. In order to complete the construction of βS discussed in [5, pg. 154-155], it suffices to demonstrate that the definition determines βS up to homeomorphism in such a way as to leave S pointwise fixed. From Theorem 10.7 [5, pg. 143], we observe that βS is unique.

We then construct the product P_* of real lines and define the homeomorphism σ_* of S into P_* in such a way that $\sigma_*[S]$ will be C^*-embedded in P_*. Define P_*=$\mathbb{R}^{C^*(S)}$. Each $T \in P_*$ has the form $T = (T_f)_{f \in C^*}$ where the real number T_f is the value of T at f. For each $f \in C^*$, π_f will denote the projection of P onto $\mathbb{R}$ defined by $\pi_f(T) = T_f$ where T_f is the value of T at f. By Theorem 11.3 [5, pg. 155], the mapping σ_* defined by $\sigma_* x = (f(x))_{f \in C^*(S)}$, where $x \in S$ is a homeomorphism of S into P_* and $\sigma_*[S]$ is C^*-embedded in P_* by Theorem 11.3 [5, pg. 155]. To construct $\beta\mathbb{N}$, we note that we can obtain many free z-ultrafilters on the set of natural numbers $\mathbb{N}$, with each free z-ultrafilter on $\mathbb{N}$ converging to w. In a completely regular space, a $z-$ultrafilter can have at most one cluster point and distinct ultrafilters can can have the same cluster point [5, pg. 83].

On the one-point compactification $\mathbb{N}^* = \mathbb{N} \cup \{w\}$ of $\mathbb{N}$, let $\mathcal{F}$ be the filter of all sets that contain all but a finite number of even integers, and $\mathcal{L}$ those that contain all but finitely many odd integers. Any ultrafilters U and V containing $\mathcal{F}$and $\mathcal{L}$, respectively, are distinct, but both converge to w [5, pg. 47].

As an illustration of the fact that βS is a semigroup, we consider the following theorem.

Theorem 1.12

Let $(S, +)$ be a semigroup. Then $(\beta S, +)$ is a semigroup.

Proof. The proof principally borrows ideas from Theorem 4.4 [7, pg. 86-87] by essentially applying the same ideas which were used under multiplication but this time under the operation of addition and also providing a simple illustration that βS is nonempty. It should be noted that if $s \in S$ and $q \in \beta S$, then $\lim_{t \to q} s \bullet t = s \bullet q$

Since $(S, +)$ is a semigroup, it is clearly nonempty. As $(\beta S, +)$ contains at least one ultrafilter of $(S, +)$, $(\beta S, +)$ must be nonempty. Let $a, b, c \in S$ and $p, q, r \in \beta S$.

$$\text{Then } \lim_{a \longrightarrow p} \lim_{b \longrightarrow q} \lim_{c \longrightarrow r} (a + b) + c = \lim_{a \longrightarrow p} \lim_{b \longrightarrow q} (a + b) + r$$
$$= \lim_{a \longrightarrow p} (a + q) + r$$
$$= (p + q) + r$$

$$\text{Also } \lim_{a \longrightarrow p} \lim_{b \longrightarrow q} \lim_{c \longrightarrow r} a + (b + c) = \lim_{a \longrightarrow p} \lim_{b \longrightarrow q} a + (b + r)$$
$$= \lim_{a \longrightarrow p} a + (q + r)$$
$$= p + (q + r)$$

Hence, associativity holds.

Thus, $(\beta S, +)$ is nonempty and has the operation $+$ which is associative and so it is a semigroup. ■

Corollary 1.13

$(\beta \mathbb{N}, +)$ is a semigroup

Proof. After noting that $\mathbb{N}$ is a semigroup, the proof follows from Theorem 1.12 ■

In Chapter 2, we will demonstrate that if S is a semigroup under multiplication, then βS is a semigroup under the operation of multiplication. The fact that $\beta\mathbb{N}$ is a semigroup under multiplication will also be shown.

In the next theorem, we will turn to the question of commutativity in a semigroup compactification of natural numbers. Before that, we need a definition.

Definition 1.14 Addition of ultrafilters of the set of natural numbers

Let U and V be ultrafilters in $\beta\mathbb{N}$. Then the sum of U and V, denoted by $U \oplus V$ is the ultrafilter

$$U \oplus V = \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m + n \in A\} \in V\} \in U\}$$

It should be noted that the operation of addition is not commutative in $\beta\mathbb{N}$. This fact will be shown in the following theorem.

Theorem 1.15

Commutativity in $(\beta\mathbb{N}, +)$ does not hold.

Proof. Let U and V be disjoint ultrafilters. Then every element of U is not an element of V. Now,

$$U \oplus V = \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m + n \in A\} \in V\} \in U\}$$

So,

$$\{m \in \mathbb{N} \mid m+n \in A\} \in V \text{ and}$$

$$\{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m+n \in A\} \in V \in U\}$$

Clearly, the elements $n \in \mathbb{N}$ are distinct from the elements $m \in \mathbb{N}$. Assume on the contrary that commutativity holds. Then

$$V \oplus U = U \oplus V = \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m+n \in A\} \in V\} \in U\}.$$

But by definition,

$$V \oplus U = \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m+n \in A\} \in U\} \in V\}.$$

Hence,

$$\{m \in \mathbb{N} \mid m+n \in A\} \in U \text{ and}$$

$$\{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m+n \in A\} \in U\} \in V.$$

Thus,

$\{m \in \mathbb{N} \mid m+n \in A\} \in V$and $\{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m+n \in A\} \in U\} \in V$, contradicting the fact that the set of elements m of natural numbers is distinct from the set of elements n of natural numbers. Hence, commutativity does not hold. ■

Theorem 1.16

The space βX is compact and Hausdorff, where X is a discrete space.

Proof. (see [4, pg. 25]) ■

Corollary 1.17

The space $\beta\mathbb{N}$ is compact and Hausdorff

Proof. Follows immediately from Theorem 1.16 ■

Definition 1.18 Dense set

Let $(P, \leq)$ be a partially ordered set. A subset D of P is dense in P if and only if for every $a \in P$ there is some $b \in D$ such that $b \leq a$.

Definition 1.19 Semigroup Compactification

Let S be a semigroup which is also a topological space. A semigroup compactification of S is a pair (φ, T) where T is a compact right topological semigroup, $\varphi : S \longrightarrow T$ is a continuous homomorphism, $\varphi[S] \subseteq \Lambda(T)$, and $\varphi[S]$ is dense in T.

We now prove a result which shows some connections between an embedding and a semigroup compactification.

Theorem 1.20

Let $(S, +)$ be a discrete semigroup and let $e : S \longrightarrow \beta S$ be the embedding.

a. $(e, \beta S)$ is a semigroup compactification of S

b. If T is a compact right toplogical semigroup and $\varphi : S \longrightarrow T$ is a continuous homomorphism with $\varphi[S] \subseteq \Lambda(T)$, then there is a continuous homomorphism η from βS to T such that $\eta_{|S} = \varphi$.

Proof. a. It suffices to show that:

i. $(\beta S, +)$ is a compact right topological semigroup

ii. $e : S \longrightarrow \beta S$ is a continuous homomorphism

iii. $e[S] \subseteq \Lambda(\beta S)$

iv. $e[S]$ is dense in βS

That $(\beta S, +)$ is a semigroup is demonstrated in Theorem 1.12 above.

Compactness of $(\beta S, +)$ is shown in [4, pg. 25]. Theorem 4.1b [7, pg. 85] implies that $(\beta S, +)$ is right topological. Hence, $(\beta S, +)$ is a compact right topological semigroup, thereby satisfying i.

To show that $e : S \longrightarrow \beta S$ is a homomorphism, it suffices to show that $e(a+b) = e(a) + e(b)$, where $a, b \in S$

Now, $a \in e(a)$ and $b \in e(b)$

$\Longrightarrow a + b \in e(a) + e(b)$

But $a + b \in e(a+b)$

Thus $e(a) + e(b) \subseteq e(a+b)$

Conversely, $a + b \in e(a+b)$

But $a \in e(a)$ and $b \in e(b)$

$\Longrightarrow a + b \in e(a) + e(b)$

Hence, $e(a+b) \subseteq e(a) + e(b)$ and so $e(a+b) = e(a) + e(b)$

This demonstrates that e is a homomorphism.

By the definition of an embedding, e is clearly continuous.

Hence, $e : S \longrightarrow \beta S$ is a continuous homomorphism, satisfying ii.

Now, $\Lambda(\beta S) = \{x \in \beta S : \lambda_x \text{ is continuous}\}$. Clearly, $\lambda_{e(x)}$ is continuous for each $x \in \beta S$ [7, pg. 85]. Hence $e[S] \subseteq \Lambda(\beta S)$, thereby demonstrating iii.

By [2, pg. 169], any discrete space is completely regular. Thus $(S, +)$ must be completely regular. By Theorem 3.27 [7, pg. 66], $(e, \beta S)$ is a Stone-Čech compactification. Definition 3.25 [7, pg. 65] implies that $e[S]$ is dense in βS.

Hence the four requirements for a semigroup compactification have been met and this completes the proof.

b. [7, pg. 89] ■

In chapter 2, we will explore if the same applies for $(S, \bullet)$

The following section explores the concept of idempotents in semigroups.

Understanding idempotents is crucial to gaining a thorough knowledge of semigroups and their properties. Furthermore, a number of properties of the smallest ideal have several connections with the idea of an idempotent. Some of these properties will be explored in chapter 3.

1.4 Idempotents in semigroups

Definition 1.21 Idempotent

Let $(S, *)$ be a semigroup. An element $x \in S$ is an idempotent if and only if $x * x = x$.

Examples

a. In the set $\{0, 2, 4, ...\}$, the number 0 is an idempotent under the operation of addition and multiplication.

b. Consider the semigroup $(S, *)$ such that $x * y = y$ for $x, y \in S$. Then every element of S is an idempotent under the operation $*$ since for each $m \in S$, $m * m = m$

One of the key results that will be stated in this study indicates the number of idempotents in each minimal left ideal and each minimal right ideal.

Lemma 1.22

Let p be an idempotent in $\mathbb{Z}^*$. Then for every $n \in \mathbb{Z}\backslash \{0\}, n\mathbb{Z} \in p$. If $p \in \beta\mathbb{N}$ and $n \in \mathbb{N}$, then $n\mathbb{N} \in p$.

Proof. See Lemma 5.19.1 [7, pg. 115]. ■

Lemma 1.23

Let p be an idempotent in $(\beta\mathbb{N},+)$. Then for each $n \in \mathbb{N}, n\mathbb{N} \in p$.

Proof. The proof follows directly from Lemma 1.22. ■

The number of minimal left ideals and minimal right ideals in $(\beta\mathbb{N},+)$ is very important in semigroup theory. Even though the verification of the theorem stating the number of minimal left ideals and the number of minimal right ideals in $(\beta\mathbb{N},+)$ will not be shown as it has already been proved, the following section will bring out ideas that are critical for the proof of this important result.

1.5 The Semigroup H

Before we define the semigroup H, some concepts need to be elaborated.

Definition 1.24 $\wp_f$ **(A)**

The set $\wp_f(A)$ is defined as follows:

$\wp_f(A) = \{F : \varnothing \neq F \subseteq A \text{ and } F \text{ is finite}\}$

Example

Given that $A = \{1,2\}$, find $\wp_f(A)$.

Solution

It suffices to find non-empty subsets of A.

Thus $\wp_f(A) = \{\{1\},\{2\},\{1,2\}\}$

Definition 1.25 supp (x)

Given $n \in \mathbb{N}$, $supp\,(n) \in \wp_f\,(A)$ is defined by $supp\,(n) = \left\{ i : n = \Sigma\ 2^i \ \right\}$

Examples

Determine $supp\,(1)\,, supp\,(3)\,, supp\,(9)\,, supp\,(16)$

Solution

$1 = 2^0$

$\Longrightarrow supp\,(1) = \{0\}$

$3 = 2^0 + 2^1$

$\Longrightarrow supp\,(3) = \{0, 1\}$

$9 = 2^0 + 2^3$

$\Longrightarrow supp\,(9) = \{0, 3\}$

$16 = 2^4$

$\Longrightarrow supp\,(16) = \{4\}$

$max\ supp\,(n)$ and $minsupp\,(n)$ can easily be identified.

As can be observed above,

$maxsupp\,(3) = 1$ and $minsupp\,(3) = 0$

$max\ supp\,(16) = 4$ and $minsupp\,(16) = 4$

It is clear that for any natural number n,

$max\ supp\,(2^n) = minsupp\,(2^n) = n$

$\phi : \mathbb{N} \to \omega$ and $\theta : \mathbb{N} \to \omega$ may be defined by stating that $\phi\,(n) = max$ $supp\,(n)$ and $\theta\,(n) = minsupp\,(n)$.

The mapping $\overline{\phi} : \beta\mathbb{N} \longrightarrow \beta\omega$ is the continuous extension of ϕ while the mapping $\overline{\theta} : \beta\mathbb{N} \longrightarrow \beta\omega$ is the continuous extension of θ.

Definition 1.26 The semigroup *H*

The semigroup H is defined as follows:

$H = \cap_{n \in \mathbb{N}} cl_{\beta\mathbb{N}} 2^n \mathbb{N}$

Before we consider some properties of H, we need the following result.

Theorem 1.27

Let $(S, \bullet)$ be a semigroup and let $\mathcal{L} \subseteq \wp\,(S)$ have the finite intersection property. If for each $A \in \mathcal{L}$ and each $x \in A$, there exists some $B \in \mathcal{L}$ such that $xB \subseteq A$, then $\cap_{A \in \mathcal{L}} \overline{A}$ is a subsemigroup of βS.

Proof. See Theorem 4.20 [7, pg. 91]. ■

Given that D is a discrete space, let Y be a compact space, and let $f : D \longrightarrow Y$. Then $\widetilde{f}$ is the continuous function from βD to Y such that $\widetilde{f}_{|D} = f$.

To illustrate the meaning of the extension mapping $\widetilde{f}$, we can use the following setup:

$$\begin{array}{ccc} D & \xrightarrow{f} & E \\ {\scriptstyle e_D}\downarrow & & \downarrow{\scriptstyle e_E} \\ \beta D & \xrightarrow[\widetilde{f}]{} & \beta E \end{array}$$

e_D embeds D into βD and e_E embeds E into βE. The function $\widetilde{f}$ maps the principal ultrafilters corresponding to the elements of D to the principal ultrafilters corresponding to the elements of E.

The following result investigates a condition under which the extension $\widetilde{f}$ of f is injective and a condition under which $\widetilde{f}$ is surjective.

Theorem 1.28

Suppose that $f : D \rightarrow E$ is a mapping from a discrete space D to discrete space E. Then

a. $\widetilde{f} : \beta D \rightarrow \beta E$ is injective if f is injective

b. $\widetilde{f}$ is surjective if f is surjective.

Proof. a. Consider the functions as highlighted in the following diagram:

$$\begin{array}{ccc} D & \xrightarrow{f} & E \\ {\scriptstyle e_D}\downarrow & & \downarrow{\scriptstyle e_E} \\ \beta D & \xrightarrow[\widetilde{f}]{} & \beta E \end{array}$$

Clearly, the mappings e_D and e_E are injective as they are embeddings. The inverse of e_D i.e. $(e_D)^{-1}$ is also injective. The function f is injective as given in the question. Thus, $\widetilde{f}$ is inejctive as as it can be expressed as a composition of injective functions, i.e. $\widetilde{f} = e_E \circ f \circ (e_D)^{-1}$.

b. It is evident that the embaddings e_D and e_E are surjective. So if f is surjective, then $\widetilde{f}$ is must be surjective, being the composition of the functions expressed in the first part above. ■

In one of the results, we shall need to show that $(\beta\mathbb{N},+)$ contains 2^c minimal left ideals and 2^c minimal right ideal and that each of these contains 2^c idempotents. The knowledge of Theorem 1.28 above is useful in proving this result.

Lemma 1.29

a. The set H is a compact subsemigroup of $(\beta\mathbb{N},+)$, which contains all the idempotents of $(\beta\mathbb{N},+)$.

b. Furthermore, if $p \in \beta\mathbb{N}$ and $q \in H$, then $\overline{\phi}(p+q) = \overline{\phi}(q)$ and $\overline{\theta}(p+q) = \overline{\theta}(p)$.

Proof.

a. The proof is shown in Lemma 6.8 [7, pg. 129] but we show it differently using an approach which appears clearer to us as follows:

The closure of a set is closed. So, H as an intersection of closures of sets is therefore closed. Furthermore, a closed subset of a compact space is compact. Hence H, being a closed subset of a compact space $(\beta\mathbb{N},+)$ must be compact.

We now show that H is a subsemigroup. It suffices to show that the requirements of Theorem 1.26 are met. Consider the semigroup $(\mathbb{N},+)$. Let $\mathcal{L}$ be the set $2^n\mathbb{N}$, where $n \in \mathbb{N}$ ie

$$\mathcal{L} = \left\{ \left\{ 2^1, 2 \cdot 2^1, 3 \cdot 2^1, ... \right\}, \left\{ 2^2, 2 \cdot 2^2, 3 \cdot 2^2 \right\}, ..., \left\{ 2^n, 2 \cdot 2^n, 3 \cdot 2^n, ... \right\} ... \right\}$$

Clearly $\mathcal{L} \subseteq \wp(\mathbb{N})$ and it is easily observed that $\mathcal{L}$ has the finite intersection property.

Now consider an arbitrary element $A = \{2^n, 2 \cdot 2^n, 3 \cdot 2^n, ...\}$ of $\mathcal{L}$. All elements of A are of the form $r \cdot 2^n$, where $r \in \mathbb{N}$.
Let $B = \left\{2^{n+1}, 2 \cdot 2^{n+1}, 3 \cdot 2^{n+1}, ...\right\}$ which is clearly an element of $\mathcal{L}$ and choose an arbitrary element $t \cdot 2^{n+1}$ of B where $t \in \mathbb{N}$. Now, consider $r \cdot 2^n + t \cdot 2^{n+1} = r \cdot 2^n + t \cdot 2 \cdot 2^n = (r + 2t) 2^n$. Clearly, $r + 2t \in \mathbb{N}$ and so $r \cdot 2^n + t \cdot 2^{n+1} \in A$. Since $t \cdot 2^{n+1}$ is an arbitrary element of B, $r \cdot 2^n + B \subseteq A$ and so the requirements of Theorem 1.27 are met and hence $\cap_{n \in \mathbb{N}} \overline{2^n \mathbb{N}} = H$ is a subsemigroup of $(\beta\mathbb{N}, +)$.

We demonstrate that H contains all the idempotents of $(\beta\mathbb{N}, +)$. By Lemma 1.22, for each $n \in \mathbb{N}$, $n\mathbb{N} \in p$, where p is any idempotent of $(\beta\mathbb{N}, +)$. Since $2^n \in \mathbb{N}$ for every $n \in \mathbb{N}$, then it follows that $2^n\mathbb{N} \in p$ for any idempotent p of $(\beta\mathbb{N}, +)$. As $2^n\mathbb{N} \subseteq \mathbb{N}$, it follows by definition that $p \in \overline{2^n\mathbb{N}}$ for every $n \in \mathbb{N}$. Thus $p \in \cap_{n \in \mathbb{N}} \overline{2^n\mathbb{N}} = H$. Hence, H contains all the idempotents of $(\beta\mathbb{N}, +)$.

b. See Lemma 6.8 [7, pg. 129]. ■

Theorem 1.30

$(\beta\mathbb{N}, +)$ contains 2^c minimal ideals and 2^c minimal right ideals. Each of these contains 2^c idempotents.

Proof. See Theorem 6.9 [7, pg. 129]. ■

This chapter has brought out semigroups and a number of properties of semigroups with considerable emphasis on their properties under the operation of addition. The properties of semigroups that have been highlighted under addition will be discussed in chapter 2 but this time under the operation of multiplication. The idea is to investigate whether or not multiplication applies to the properties discussed in chapter 1.

Chapter 2

Some properties of semigroups under multiplication

In this chapter, we will investigate some properties of semigroups under multiplication with considerable emphasis on $\beta\mathbb{N}$ under multiplication.

A lot is known about $\beta\mathbb{N}$ under addition but far less work has been done on $\beta\mathbb{N}$ under multiplication. Hence the importance of this chapter.

The following have been verified about $\beta\mathbb{N}$ under addtion :

1. $\beta\mathbb{N}$ is a semigroup (Corollary 1.13).

2. $\beta\mathbb{N}$ is not commutative (Theorem 1.15).

3. For a discrete semigroup $(S,+)$ (including $(\mathbb{N}, +)$), $(e, \beta S)$ is a semigroup compactification as long as $e : S \to \beta S$ is the embedding (Theorem 1.20)

4. $(\beta\mathbb{N}, +)$ contains 2^c minimal left ideals and 2^c minimal right ideals (Theorem 1.30).

The question is: “Do these results hold under the operation of multiplication?”

We begin with the investigatation of whether $\beta\mathbb{N}$ is also a semigroup under multiplication and as a starting point, we deal with a general semigroup $(S, \bullet)$.

We remind the reader that as stated in chapter 0, given a semigroup $(S, *)$, one can extend the operation $*$ on S to its Stone-Čech compactification $(\beta S, *)$ which is also a semigroup.

Theorem 2.1

Let S be a semigroup under multiplication. Then $(\beta S, \bullet)$ is a semigroup.

Proof. As $(\beta S, \bullet)$ is an extension of $(S, \bullet)$, it follows that it is a semigroup. ■

Corollary 2.2

$(\beta\mathbb{N}, \bullet)$ is a semigroup

Proof. Follows directly from Theorem 2.1. ■

When we consider βS, where S is a semigroup under addition, it should be noted that addition does not necessarily imply commutativity. As stated already, we proved that commutativity does not hold in $(\beta\mathbb{N}, +)$ in Chapter 1 (Theorem 1.15). The question is: “What about commutativity in $\beta\mathbb{N}$ under multiplication?”. The next few results are aimed at exploring this matter. We begin with the following result.

Proposition 2.3

Let $\langle x_n \rangle_{n=1}^{\infty} = \{2^n : n \in \mathbb{N}\}$ and $\langle y_n \rangle_{n=1}^{\infty} = \{3^n : n \in \mathbb{N}\}$ be sequences in the compactification $\beta\mathbb{N}$. Show that

$\{y_k x_n : k, n \in \mathbb{N} \text{ and } k < n\} \cap \{x_k y_n : k, n \in \mathbb{N} \text{ and } k < n\} = \varnothing$

Proof. Assume on the contrary that the intersection is non-empty. Then there exist $a, b, c, d \in \mathbb{N}$ such that $3^a \cdot 2^b = 2^c \cdot 3^d$ where $a < b$ and $c < d$.

$\Longrightarrow \frac{3^a}{3^d} = \frac{2^c}{2^b}$

$\Longrightarrow 3^{a-d} = 2^{c-b}$

Equality between the left and right hand sides leads to the following possibilities:

Case 1

a-d = 0 and c -b = 0

$\Longrightarrow a = d$ and $c = b$. Now $a < b$ and $a = d$ and $c = b$ implies that $d < c$. But this contradicts the fact that $c < d$.

Case 2

$a - d > 0$ and $c - b > 0$

Now, in this case, 3^{a-d} gives an odd number and 2^{c-b} is an even number and so it is not possible for equality to hold.

Case 3

$a - d < 0$ and $c - b < 0$

Using a similar approach as in case 2, equality will not hold in this case as well. Hence,

$$\{y_k x_n : k, n \in \mathbb{N} \text{ and } k < n\} \cap \{x_k y_n : k, n \in \mathbb{N} \text{ and } k < n\} = \varnothing$$

when

. $$\langle x_n \rangle_{n=1}^{\infty} = \{2^n : n \in \mathbb{N}\} \, and \, \langle y_n \rangle_{n=1}^{\infty} = \{3^n : n \in \mathbb{N}\}$$ ■

Theorem 2.4

Let $(S, \bullet)$ be a semigroup. Then $(\beta S, \bullet)$ is not commutative if and only if there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ such that $\{y_k x_n : k, n \in \mathbb{N} \text{ and } k < n\} \cap \{x_k y_n : k, n \in \mathbb{N} \text{ and } k < n\} = \varnothing$

Proof. See Theorem 4.27 [7 , pg. 96]. ■

We now show that commutativity does not hold in $(\beta\mathbb{N}, \bullet)$.

Theorem 2.5

The operation $\bullet$ in $\beta\mathbb{N}$ is not commutative.

Proof. Taking the semigroup $(\mathbb{N}, \bullet)$, we see that sequences such as $\langle x_n \rangle_{n=1}^{\infty} = \{2^n : n \in \mathbb{N}\}$ and $\langle y_n \rangle_{n=1}^{\infty} = \{3^n : n \in \mathbb{N}\}$ exist such that $\{ y_k x_n : k, n \in \mathbb{N} \text{ and } k < n\} \cap \{x_k y_n : k, n \in \mathbb{N} \text{ and } k < n\} = \varnothing$. Hence it follows that $(\beta\mathbb{N}, \bullet)$ is not commutative, thereby completing the proof. ■

Alternatively, the proof is provided as follows: Let U and V be disjoint ultrafilters. Assume that commutativity holds i.e., $U \odot V = V \odot U$. Now,

$$U \odot V = \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m \bullet n \in A\} \in V\} \in U\}$$

So,

$$\{m \in \mathbb{N} \mid m \bullet n \in A\} \in V$$

and

$$\{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m \bullet n \in A\} \in V\} \in U$$

By assumption,

$$V \odot U = U \odot V = \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m \bullet n \in A\} \in V\} \in U\}$$

Now, by the defintion of multiplication of ultrafilters, we also have

$$V \odot U = \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m \bullet n \in A\} \in U\} \in V\}$$

from which we deduce that

$$\{m \in \mathbb{N} \mid m \bullet n \in A\} \in U \quad ,$$

and

$$\{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid m \bullet n \in A\} \in U\} \in V$$

contradicting the fact that the elements m are distinct from the elements n. Hence commutativity does not hold.

In chapter 1, we presented some properties of semigroup compactifications under the operation of addition. We wish to explore if those properties apply under multiplication. We remind the reader that a semigroup compactification of S is a pair (φ, T) where T is a compact right topological semigroup, $\varphi : S \longrightarrow T$ is a continuous homomorphism, $\varphi[S] \subseteq \Lambda(T)$, and $\varphi[S]$ is dense in T (Definition

1.18).

We begin with a result on semigroup compactifications under the embedding. In Chapter 1 we proved that if $(S,+)$ is a discrete semigroup and $e : S \longrightarrow \beta S$ is the embedding,

1. $(e,\ \beta S)$ is a semigroup compactification of S

2. If T is a compact right toplogical semigroup and $\varphi : S \longrightarrow T$ is a continuous homomorphism with $\varphi[S] \subseteq \Lambda(T)$, then there is a continuous homomorphism η from βS to T such that $\eta_{|S} = \varphi$.

The following theorem seeks to demonstrate that this result also holds for a semigroup S under multiplication.

Theorem 2.6

Let $(S,\bullet)$ be a discrete semigroup and let $e : S \longrightarrow \beta S$ be the embedding. Then

a. $(e,\beta S)$ is a semigroup compactification of S.

b. If T is a compact right toplogical semigroup and $\varphi : S \longrightarrow T$ is a continuous homomorphism with $\varphi[S] \subseteq \Lambda(T)$, then there is a continuous homomorphism η from βS to T such that $\eta_{|S} = \varphi$.

Proof.

a. For the first part, it suffices to show that:

i. $(\beta S,\bullet)$ is a compact right toplogical semigroup,

ii. $e : S \longrightarrow \beta S$ is a continuous homomorphism,

iii. $e[S] \subseteq \Lambda(\beta S)$,

iv. $e[S]$ is dense in βS.

That $(\beta S, \bullet)$ is a semigroup is demonstrated in Theorem 2.1 above. Compactness of $(\beta S, \bullet)$ is shown in [4, pg. 25]. Theorem 4.1b [7, pg.85] implies that $(\beta S, \bullet)$ is right topological. Hence, $(\beta S, \bullet)$ is a compact right topological semigroup, thereby satisfying i.

To show that $e : S \longrightarrow \beta S$ is a homomorphism, it suffices to show that $e(ab) = e(a)\,e(b)$, where $a, b \in S$

Now, $a \in e(a)$ and $b \in e(b)$

$\Longrightarrow ab \in e(a)\,e(b)$

But $ab \in e(ab)$

Thus $e(a)\ e(b) \subseteq e(ab)$

Conversely, $ab \in e(ab)$

But $a \in e(a)$ and $b \in e(b)$

$\Longrightarrow ab \in e(a)\,e(b)$

Hence, $e(ab) \subseteq e(a)\,e(b)$ and so $e(ab) = e(a)\,e(b)$

This demonstrates that e is a homomorphism.

By the definition of an embedding, e is clearly continuous.

Hence, $e : S \longrightarrow \beta S$ is a continuous homomorphism, satisfying ii.

Now, $\Lambda(\beta S) = \{x \in \beta S : \lambda_x \text{ is continuous}\}$. Clearly, $\lambda_{e(x)}$ is continuous for each $x \in \beta S$ [7, pg. 85]. Hence $e[S] \subseteq \Lambda(\beta S)$, thereby demonstrating iii.

By [2 , pg. 169], any discrete space is completely regular. Thus $(S, \bullet)$ must be completely regular. By Theorem 3.27 [7, pg. 66], $(e, \beta S)$ is a Stone-Čech compactification. Definition 3.25 [7, pg. 65] implies that $e[S]$ is dense in βS.

b. See [7, pg. 89]. ■

The fact that $(e, \beta\mathbb{N})$ is a semigroup compactification follows easily from the above theorem.

In chapter 1, we showed that $\beta\mathbb{N}$ under addition contains 2^c minimal left ideals and 2^c minimal right ideals, where c is the cardinality of the continuum. The next theorem seeks to show that the same holds for $\beta\mathbb{N}$ under multiplication.

We begin with the following lemma.

Lemma 2.7

The set $\beta\mathbb{N}\bullet p$ is a left ideal of $\beta\mathbb{N}$ and the set $q \bullet \beta\mathbb{N}$ is right ideal of $\beta\mathbb{N}$ where p, $q \in \beta\mathbb{N}$.

Proof. Since $(\beta\mathbb{N}, \bullet)$ is a semigroup as shown in Corollary 2.2, then associativity holds in $\beta\mathbb{N}$ under multiplication. Hence, if $x, y \in \beta\mathbb{N}$, then $x \bullet (y \bullet p) = (x \bullet y) \bullet p$. Now, $x \bullet y \in \beta\mathbb{N}$ since $\beta\mathbb{N}$ is a semigroup and so

$$x \bullet (y \bullet p) = (x \bullet y) \bullet p \in \beta\mathbb{N} \bullet p$$

$$\Longrightarrow \beta\mathbb{N} \bullet (\beta\mathbb{N} \bullet p) \subseteq \beta\mathbb{N} \bullet p$$

showing that $\beta\mathbb{N} \bullet p$ is a left ideal of $\beta\mathbb{N}$.

To prove that $q \bullet \beta\mathbb{N}$ is right ideal, we show that $(q \bullet \beta\mathbb{N}) \bullet \beta\mathbb{N} \subseteq q \bullet \beta\mathbb{N}$.

Let $a, b \in \beta\mathbb{N}$. Then $a \bullet b \in \beta\mathbb{N}$ and

$$(q \bullet a) \bullet b = q \bullet (a \bullet b) \in q \bullet \beta\mathbb{N} \Longrightarrow (q \bullet \beta\mathbb{N}) \bullet \beta\mathbb{N} \subseteq q \bullet \beta\mathbb{N}$$

which shows that $q \bullet \beta\mathbb{N}$ is a right ideal of $\beta\mathbb{N}$ ■

Theorem 2.8

$(\beta\mathbb{N}, \cdot)$ contains 2^c minimal left ideals and 2^c minimal right ideals, where c is the cardinality of the continuum.

Proof. From set theory, we know that the set of natural numbers is countable. The set is also discrete. Hence by Corollary 3.57 [7, pg.79], $|\beta\mathbb{N}| = 2^c$. By Lemma 2.7, the set $\beta\mathbb{N} \bullet p$ is a left ideal, where $p \in \beta\mathbb{N}$ and so there will be 2^c such left ideals.

Now, since $\beta\mathbb{N}$ is a compact right topological semigroup, each of the left ideals will contain a minimal left ideal. Hence there will be 2^c minimal left ideals. Similarly, there will be 2^c minimal right ideals. ■

In chapter 1, we explored the set $H = \cap_{n \in \mathbb{N}} cl_{\beta\mathbb{N}} 2^n\mathbb{N}$ and showed that the set H is a compact subsemigroup of $(\beta\mathbb{N}, +)$, which contains all the idempotents of $(\beta\mathbb{N}, +)$. Furthermore, for any $p \in \beta\mathbb{N}$ and any $q \in H$, $\overline{\phi}(p+q) = \overline{\phi}(q)$ and $\overline{\theta}(p+q) = \overline{\theta}(p)$.

In the following result, we will demonstrate that the set H is a subsemigroup of $(\beta\mathbb{N}, \bullet)$ and we shall seek to investigate the mappings $\overline{\phi}$ and $\overline{\theta}$ on $p \bullet q$

Proposition 2.9

The set H is a compact subsemigroup of $(\beta\mathbb{N}, \bullet)$. Furthermore, for any $p \in \mathbb{N}^*$ and any $q \in H$,

$$\widetilde{\phi}(p \bullet q) = \widetilde{\phi}(p) + \widetilde{\phi}(q) \text{ and } \widetilde{\theta}(p \bullet q) = \widetilde{\theta}(p) + \widetilde{\theta}(q)$$

Proof. H as a closed subset of a compact space $\beta\mathbb{N}$ is compact. Consider the semigroup $(\mathbb{N}, \bullet)$. Let $\mathcal{L}$ be the set $2^n\mathbb{N}$, where n is a natural number ie $\mathcal{L}$ is given by

$\left\{\left\{2^1, 2 \bullet 2^1, 3 \bullet 2^1, ...\right\}, \left\{2^2, 2 \bullet 2^2, 3 \bullet 2^2\right\}, ..., \left\{2^n, 2 \bullet 2^n, 3 \bullet 2^n, ...\right\} ...\right\}$.

Clearly, $\mathcal{L} \subseteq \wp(\mathbb{N})$ and it is easily observed that $\mathcal{L}$ has the finite intersection property.

Now consider an arbitrary element $A = \{2^n, 2 \bullet 2^n, 3 \bullet 2^n, ...\}$ of $\mathcal{L}$. All elements of A are of the form $r \cdot 2^n$, where $r \in \mathbb{N}$. Pick the set $B = \left\{2^{n+1}, 2 \bullet 2^{n+1}, 3 \cdot 2^{n+1}, ...\right\}$ which is clearly an element of $\mathcal{L}$. Pick

an arbitrary element $t \bullet 2^{n+1}$ of B where $t \in \mathbb{N}$.

Clearly, $(r \bullet 2^n) \bullet \left(t \cdot 2^{n+1}\right) = \left(r \bullet t \bullet 2^{n+1}\right) \bullet 2^n$. It is easily observed that $r \bullet t \bullet 2^{n+1} \in \mathbb{N}$ and thus $\left(r \bullet t \bullet 2^{n+1}\right) \bullet 2^n \in A$. So, $r \bullet 2^n \bullet B \subseteq A$. This satisfies Theorem 1.27 and so $\cap_{n\in\mathbb{N}}\overline{2^n\mathbb{N}} = H$ is a subsemigroup of $(\beta\mathbb{N}, \bullet)$.

Now, any $m, n \in \mathbb{N}$ can be expressed as follows:

$m = 2^{d_1} + ...2^{d_s}$ where d_s =max(supp (m)) and $n = 2^{c_1} + ...2^{c_t}$ where c_t =max(supp (n)) .Thus,

$\phi(mn) = \phi\left[\left(2^{d_1} + ...2^{d_s}\right)\left(2^{c_1} + ...2^{c_t}\right)\right]$. In $\left(2^{d_1} + ...2^{d_s}\right)\left(2^{c_1} + ...2^{c_t}\right)$, the highest power of 2 is $d_s + c_t$. Hence, $\phi(mn) = d_s + c_t = \phi(m) + \phi(n)$.

Following the same steps as in Lemma 6.8[7, pg. 129], we obtain the results $\widetilde{\phi}(p \bullet q) = \widetilde{\phi}(p) + \widetilde{\phi}(q)$ and $\widetilde{\theta}(p \bullet q) = \widetilde{\theta}(p) + \widetilde{\theta}(q)$ ■

Chapter 2 brought out some properties of semigroups and their compactification under the operation of multiplication. It has been observed that a large number of properties considered in chapter 1 under addition also apply in chapter 2 under multiplication.

Chapter 3 will discuss the smallest ideal and bring out some of its fundamental properties.

Chapter 3

The smallest Ideal of a semigroup

Chapters 1 and 2 presented some concepts on semigroups and investigated some properties of semigroups and their compactifications.

Among the concepts presented in chapter 1 is that of minimal left and minimal right ideals. Furthermore, we highlighted the fact that if there is a compact right topological semigroup, then this guarantees the presence of a smallest two-sided ideal. The characterisation of the smallest ideal is of utmost importance in mathematical literature.

Chapter 3 will seek to bring out fundamental properties of the smallest ideal of a semigroup. The chapter will begin with key definitions and then will proceed to the presentation of some results on the smallest ideal of a semigroup. The later part of the chapter will deal with the smallest ideal of the compactification of a semigroup, particularly the smallest ideal of the compactification of the set of natural numbers under the operations of both addition and multiplication.

We begin with the definition of the smallest ideal of a semigroup to be followed by a theorem demonstrating the existence of the smallest ideal.

Definition 3.1 Smallest ideal

Let S be a semigroup. If there is no two-sided ideal smaller than the two sided ideal I, then the ideal I is said to be the smallest ideal of S.

The smallest ideal of S is denoted by $K(S)$.

It should be noted that the smallest ideal of S is in essence the smallest minimal two-sided ideal.

Example

Consider the sets $S = \{0, 1, 2, ...\}$, $A = \{0, 2, 4, ...\}$ and $I = \{0\}$. Under the operation of multiplication, S is a semigroup, since associativity clearly holds and for any $x, y \in S$, $x \bullet y \in S$. The set $A = \{0, 2, 4, ...\}$ is an ideal of S since for any $p \in A$ and any $q \in S$, $p \bullet q = q \bullet p \in A$ (since the product of an even number and any number is an even number and the product of zero and and any number is zero). The set I is an ideal of S. In fact, I is the smallest ideal of S.

The smallest ideal has very interesting properties. A result which we will prove shortly shows that the smallest ideal of a semigroup contains all minimal right ideals and all minimal left ideals of the semigroup. Furthermore, the isomorphism between the maximal subgroups contained in a smallest ideal will be brought out. Before this result, we need the following definition.

Definition 3.2 Subgroup of a semigroup

A nonempty subset H of a semigroup S is a subgroup of S if H is a group under the restriction of the binary operation on S to H, i.e., $s \in H$ implies that $s^{-1} \in H$ and $s, t \in H$ implies that $st \in H$.

A subgroup H of G is said to be maximal if it is not properly contained in any other subgroup of the semigroup.

The following theorem gives characterisations on the two sided ideal of a compact
right topological semigroup.

Theorem 3.3

If S is any compact right topological semigroup, then S has a smallest two-sided ideal $K(S)$ which satisfies the following statements.

a. K(S) $= \cup\{L : L$ is a minimal left ideal of $S\}$.

b. K(S) $= \cup\{R : R$ is a minimal right ideal of $S\}$.

c. If L is a minimal left ideal of S and R is a minimal right ideal of S, then $L \cap R$ is a maximal subgroup of $K(S)$.

d. All maximal subgroups of S contained in $K(S)$ are isomorphic.

e. All maximal subgroups of S contained in the same minimal right ideal are isomorphic and homeomorphic via the same function.

Proof. a. S must contain a left ideal and by Corollary 2.6 [7, pg. 41], S contains a minimal left ideal. Hence, it satisifies the requirement for Theorem 1.51 [7, pg. 25] from which the result follows.

b. See Theorem 2.8 [7, pg. 42].

c. It should be noted that $E(S)$ denotes the collection of idempotents of S. Now, since S is a compact right topological semigroup, then by Theorem 2.8 [7, pg. 42], S has a smallest two sided ideal $K(S)$ such that $\{eSe : e \in E(K(S))\}$partitions $K(S)$. By Theorem 2.9c [7, pg. 42], eSe is a group. By Theorem 2.4 [10, pg. 3], we see that $R \cap L = RL = eSe$. Now, since $R \cap L =eSe$ is a group and partitions $K(S)$, then $R \cap L$ is a maximal subgroup.

d. By Corollary 2.6 [7, pg. 41], every left ideal of S contains a minimal left ideal and each minimal left ideal has an idempotent. The proof then follows from Theorem 1.66 [7 , pg. 35].

e. See Theorem 2.11 b [7 , pg. 43]. ■

One might wish to investigate the maximum number of smallest ideals that can be contained in a semigroup.

Proposition 3.4

There can be at most one smallest ideal in a semigroup.

Proof. Assume, without loss of generality that a semigroup has two smallest ideals K_1 and K_2. By Theorem 3.3a),

$K_1 = \cup\{L : L$ is a minimal left ideal of $S\}$. Since K_2 is a smallest (two-sided ideal) ideal, it is a minimal left ideal. So, $K_2 \subseteq K_1$. Similarly, K_2 being a smallest ideal can be expressed as $K_2 = \cup\{L : L$ is a minimal left ideal of $S\}$. Using the same argument above, we see that $K_1 \subseteq K_2$. Thus, $K_1 = K_2$ and so there can be at most one smallest ideal in a semigroup, completing the proof. ■

It should be noted that not every semigroup has a smallest ideal. In the set of natural numbers, there is no smallest ideal either under addition or under multiplication (See for example [7, pg. 25]).
We now consider other properties of the smallest ideal of a semigroup.

Proposition 3.5

Let S be a semigroup. If I is an ideal of S, then I is the smallest ideal if and only if $IxI = I$ for every $x \in I$.

Proof. Suppose that I is the smallest ideal of S. Then $xI \subseteq I$ and so $IxI \subseteq I$. Let $t \in S$. Since $tI \subseteq I$, $t\,(IxI) = tI\,(xI) \subseteq IxI$. Similarly, $(IxI)\,t = (Ix)\,It \subseteq IxI$. Thus IxI is an ideal of S. But I is the smallest ideal of S. Hence, $I \subseteq IxI$. So, $IxI = I$

Conversely, let $IxI = I$ where $x \in I$ and assume on the contrary that I is not the smallest ideal. Let $T \subseteq I$ be the smallest ideal. Let $y \in T$. Then $y \in I$ and so $IyI= I$. But $IyI\subseteq T$. So $I \subseteq T$. Thus $I = T$, showing that I is the smallest ideal. ■

Theorem 3.6

Let S be a semigroup. If L is a minimal left ideal of S and R is a minimal right ideal of S, then $K(S) = LR$

Proof. See Theorem 1.53 [7, pg. 25]. ■

Theorem 3.7

Let S be a semigroup and assume that $K(S)$ exists and $e \in E(S)$. Then the following statements are equivalent.

a) $e \in K(S)$

b) $K(S) = SeS$

Proof. See Theorem 1.54 [7, pg. 25-26]. ■

There are instances when the smallest ideal is commutative. We will show that if this is the case, then it is a group.

We begin with the following result.

Proposition 3.8

Suppose that a minimal left ideal L of a semigroup S is commutative. Then L is a group.

Proof. Without loss of generality, we will take an operation of multiplication on the elements of L. It suffices to verify that L is semigroup, has an identity and that each element has an inverse. Clearly, $SL \subseteq L$. Since $L \subseteq S, LL \subseteq L$. Thus L is a subsemigroup. By Lemma 1.52a [7, pg. 25], if $x \in L$,then $Lx = L$ (1)

Let L= $\{a_1, a_2, ...a_n, ...\}$. By (1), for an arbitrary element $a_n \in L$, there exists $a_i \in L$ such that $a_i a_n = a_n$ and so a_i is the left identity of a_n. We denote a_i by e. We show that e is the left identity for every element of L. Take another element a_m of L.

By (1), it is clear that there is an element, say a_j such that $a_j a_m = a_m$, i.e., a_j is the left identity of a_m.

Assume that $a_j \neq e$. Then

$$\begin{gathered} a_j (a_n a_m) \neq e (a_n a_m) \\ \Longrightarrow a_j (a_m a_n) \neq e (a_n a_m) \;\; \textit{commutativity} \\ \Longrightarrow (a_j a_m) a_n \neq (e a_n) a_m \quad \textit{associativity} \\ \Longrightarrow a_m a_n \neq a_n a_m. \end{gathered}$$

This leads to a contradition since the elements of L commute and so $a_m a_n = a_n a_m$. Hence the supposition that $a_j \neq e$ is false and so $a_j = e$. This implies that e is the left identity of every element of L. Since commutativity holds, $a_n e = a_n$. Thus e is also the right identity of a_n. Furthermore, by (1) for each element a_p of L, there exists an element a_q such that $a_p a_q = e$. Commutativity in L implies that $a_q a_p = e$. Hence a_q is the right and left inverse of a_p.

We now verify uniqueness.

Suppose that a_q and a_r are inverses of a_p . Then $a_q = a_q e = a_q (a_p a_r) = (a_q\ a_p) a_r = e a_r = a_r$ which shows that the inverse is unique.

This completes the proof. ■

Proposition 3.9

Let S be a semigroup and assume that there is a minimal left ideal of S. If $K(S)$ is commutative, then it is a group.

Proof. By Theorem 3.3a, $K(S)$ is a union of minimal left ideals of S. Hence if $K(S)$ is commutative, then each minimal ideal of S is commutative and is therefore a group by Proposition 3.8. $K(S)$, being a union of groups must be a group, thereby completing the proof. ■

We will now consider some properties of the smallest ideal of the compactification of a semigroup. Particular attention will be given to the smallest ideal of the compactification of the set of natural numbers under both addition and multiplication.

Proposition 3.10

Every maximal group in the smallest ideal of $(\beta\mathbb{N}, +)$ contains a free group on 2^c generators.

Proof. See Corollary 7.37 [7, pg. 190]. ■

One might wish to find out if Proposition 3.9 holds in the smallest ideal of $(\beta\mathbb{N}, \bullet)$

Proposition 3.11

Each maximal group in the smallest ideal of $(\beta\mathbb{N}, \bullet)$ contains a free group on 2^c generators.

Proof. Let p be an idempotent in $K(\beta\mathbb{N},\ \bullet)$. Then $p\ \beta\mathbb{N}\ p$ is a group by Theorem 1.59c [7, pg. 28]. Since p is in the smallest ideal $K(\beta\mathbb{N},\ \bullet)$, $p\beta\mathbb{N}p$ is a maximal group [7, pg. 191].

Let:

G_L be the free group on the elements of set L,

G_∞ be the free group on countably many generators

F_∞ be the free semigroup on countably many generators

$F_\infty \cup \{1\}$ be the free semigroup on the generators $\{x_n : n \in \mathbb{N}\}$ with an identity adjoined.

$T_\infty = \cap_{n=1} cl_{\beta\mathbb{N}}(\mathbb{N}\ n)$

Consider the following mappings:

$$\begin{array}{ccccc}
\beta\mathbb{N} & \xleftarrow{\ \tau\ } & T_\infty & \xleftarrow{\ \gamma_p^*\ } & G_L \\
\phi_1^\beta \downarrow & & & & \downarrow \pi_i^* \\
\beta F_\infty \cup \{1\} & \xleftarrow{\ \tau\ } & \beta F_\infty & \xleftarrow{\ \psi^*{}_{1,\,p}\ } & G_\infty
\end{array}$$

In the above diagram, τ is the inclusion map. To show that $p\ \beta\mathbb{N}\ p$ contains a free group on 2^c generators, it suffices to show that γ_p^* is injective. Let $f \in G_L$. Since

$$\gamma_p^*\ (f) = \tau^{-1}\left(\left(\phi_1^\beta\right)^{-1}\left(\tau\left(\psi^*{}_{1,\,p}\ \ (\pi_i^*\,(f))\right)\right)\right)$$

we show that each of the functions is one-one.

Define $\pi_i^*\ {:}G_L \longrightarrow G_\infty$ by $\pi_i^*\,(f) = x_{f(i)}$ where $f \in G_L$, $i \in I$ and x is a generator. Let $f,\ g \in G_L$ and suppose that $\pi_i^*\,(f) = \pi_i^*\,(g)$

$\implies x_{f(i)} = x_{g(i)}$

$\Longrightarrow f(i) = g(i)$

$\Longrightarrow f = g$

Hence, π_i^* is injective.

We now consider the mapping $\psi_{i,\,p}^* : G_\infty \longrightarrow \beta F_\infty$. This mapping can be expressed as the composite map $e \circ \alpha$ where $\alpha : G_\infty \longrightarrow F_\infty$ and $e : F_\infty \longrightarrow \beta F_\infty$. Clearly, e is one-one. If α is defined in such a way as to map elements of the free groups to their corresponding elements of free semigroups, it is also one-one. Since the composition of one-one function is one-one, $\psi_{i,p}^*$ is also one-one.

Since τ is an inclusion map, it is one-one and τ^{-1} is also one-one.

Consider the function $\phi : \mathbb{N} \cup \{0\} \longrightarrow F_\infty \cup \{1\}$ defined by

$\phi(0) = 1$ and $\phi(n) = x_{a_t} x_{a_{t-1}} ... x_{a_2} x_{a_1}$ where $n = \sum_{l=1}^{t} a_l r_l!$ each $a_l \in \{1, 2, 3, ..., r_l\}$

and $1 < r_1 < r_2 < ... < r_t$. Clearly, ϕ is injective. By Theorem 1.27a, ϕ_1^β is injective and hence its inverse is also injective.

Hence, γ_p^* is injective. ■

Theorem 3.12

In $K(\beta\mathbb{N}, \bullet)$, there are 2^c minimal left ideals and 2^c minimal right ideals.

Proof. By Theorem 2.7, $(\beta\mathbb{N}, \bullet)$ contains 2^c minimal left ideals and 2^c minimal right ideals. By Theorem 3.3 a) and b), $K(\beta\mathbb{N}, \bullet)$ can be expressed as a disjoint union of all the minimal left ideals and also as a disjoint union of all the minimal right ideals of $\beta\mathbb{N}$. Hence, the result holds. ■

Chapter 3 presented the idea of the smallest ideal of a semigroup and brought out some of its properties.

Conclusion

In summary, the disseration has provided evidence that the results discussed under the operation of addition also hold under the opeation of multiplication.

There is however need to for more research so that more results are investigated.

The idea of the smallest ideal of a semigroup was also discussed and some key results were highlighted including results on the smallest ideal of the compactification of the set of natural numbers.

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