The effect of using Lakatos’ Heuristic Method to Teach Surface Area of Cone on Students’ Learning: The case of Secondary School Mathematics Students in Cyprus

By

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ABSTRACT

The purpose of this study was to examine the effect of using Lakatos’ heuristic method to teach the surface area of the cone (SAC) on students’ learning. The Lakatos (1976) heuristic framework and the Oh (2010) model of “the enhanced-conflict map” were employed as framework for the study. The first research question examined the impact of the Lakatosian heuristic method on students’ learning of the SAC, which was addressed in three sub-questions: the impact of the method on the students’ achievement, the impact of the method on their conceptual learning and the impact of the method on their higher order thinking skills. The second question examined whether the heuristic method of teaching the SAC helped students to sustain their learning better than the traditional method (Euclidean method). The third question examined whether the heuristic method of teaching SAC could change students’ readiness level, according to Bloom’s taxonomy.

A pre-test and post-test quasi-experimental research design was used in the study that involved a total of 198 Grade 11 students (98 in the experimental group and 100 in the control group) from two schools in Cyprus.

The instruments used for data collection were cognitive tests, lesson observations (video-recorded), interviews and questionnaire. Data was analysed using inferential statistics and the Oh (2010) model of the enhanced conflict map. Student achievement within time was the dependent variable and the method of training the independent variable. Therefore, time was the “within” factor and each group was measured three times (pre-test, post-test and delayed). The differences in students’ achievement within each group over time were examined.

Results indicated that the average mean score achievement of the students in the experimental group was double that of the students in the control group. The Jun-Young Oh’s model of the enhanced conflict map showed that students in both groups changed from alternative conceptions to scientific conceptions with the experimental group showing greater improvement. It was also observed that from the post-test to delayed test, the Lakatosian method of teaching the SAC has a significant positive effect on students’ achievement at all levels of Bloom’s taxonomy, especially at the higher order thinking (HOT) levels (application and analysis-synthesis levels) as
compared to the Euclidean method of teaching. In addition, the Lakatosian method helped the students to sustain their learning over time better than the Euclidean method did and also helped them to change their readiness level, especially at the HOT levels. The Lakatosian method helped students to foster skills that promote active learning. Of great importance was the use of mathematical language, as well as, the enhanced perception in the experimental group in comparison with the control group, through the use of the Lakatosian method.

The results of this study are promising. It is recommended that pre-service teachers should be trained on how to effectively implement the Lakatosian heuristic method in their teaching.

**Key terms:** Conceptual learning, Cyprus secondary schools, Euclidean method, higher order thinking, Lakatosian heuristic method, surface area of a cone.
Declaration

I declare that The effect of using Lakatos’ Heuristic Method to Teach Surface Area of Cone on Students’ Learning: The case of Secondary School Mathematics Students in Cyprus is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

15.2.15

(Chrysoula Dimitriou-Hadjichristou)
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CHAPTER ONE

INTRODUCTION AND CONCEPTUALIZATION

1.1 BACKGROUND

The rapid advancement of technology over the last decade and the manner in which students of the 21st century consider and handle problems, create a radically different picture from the past when mathematical problem solving was a matter of using paper and pencil. As a teacher trainer (2001–2010) in the pre-service programme of Mathematics of Secondary Education at the Cyprus Pedagogical Institute, I dealt with the problem of teaching and learning geometry with the application of information technology (IT) (e.g. via the use of mathematical applets) as well as the application of techniques (e.g. the use of heuristic methods) that help develop a student’s social skills. Today, there is a need for teachers to give learners a “chance to become involved with the activities of non-deductive methods at young age” (De Villiers, 2010, p. 205). Given the ever-changing educational environment, it is expected that teachers adjust their methods of teaching in a way that allows students to acquire the aptitude that is needed for them to develop social skills. According to the European Commission Strategy of Europe 2020, this is the key ingredient for the new knowledge (European Commission, April, 2013).

Pólya (1954, p. vi) refers to the two kinds of mathematical reasoning, the demonstrative and the plausible, explaining that “we secure our mathematical knowledge by demonstrative reasoning, but we support our conjectures by plausible reasoning”. Because the two approaches complement each other, he suggests that all students of mathematics should try to learn both kinds of reasoning. He further maintains that while “a proof is a result of demonstrative reasoning”, it is discovered by plausible reasoning, i.e. by guessing.

Pólya (1954) strongly emphasizes the importance of experimentation in the discovery or invention of new mathematics. In this regard, one of the most productive mathematicians, Euler Leonard, proposed that “the properties of the numbers known
today have been mostly discovered by observation and discovered long before their truth was verified by rigid demonstration” (Pólya, 1954, p. 3).

Experimentation, according to De Villiers (2010, p. 205), means all non-deductive methods including “intuitive, inductive or analogical reasoning employed in the following instances:

(a) Mathematical conjecturing and/or statements are numerically or visually evaluated, by means of special cases, accurate geometric construction and measurement;

(b) Conjectures, generalizations or conclusions are made on the basis of intuition, analogy or experience obtained through any of the preceding experimental methods”.

Pupils, in collaboration with the rest of the class during group work, and a teacher who will guide them with solving a problem from the experiment to the proof (down-up), should have the privilege of conjecturing and verifying, or refuting, it on the basis of intuition, analogy or experience obtained through any of the preceding experimental methods. Freudenthal (1973)—strongly criticized the traditional practice of the direct provision of geometry definitions claiming that most definitions are not preconceived, but the finishing touch of the organizing activity, and that the child should not be denied such a privilege. However, according to Feyerabend in his report on Lakatos’ letter concerning secondary school student dropouts, there is a need “to wrestle the terror of the excessive number of drop outs by concentrating on the below average students”. Both researchers refer to the promotion of discovery methods of teaching (as cited in Motterlini, 1999, p. 376).

Sriraman and English (2010) observed that proof and refutations may not be directly employed in the teaching of mathematics but “may very well serve as a basis for the philosophy of mathematics, such as a social constructivist philosophy of mathematics, which in turn can be used as a basis to develop a theory of learning such a constructivism” (p.10). Radical constructivism, based on Lakatos’ theory (1976), can then be seen as the “hard core” of Lakatos’ epistemology, whereas, social constructivism can be seen as the “protective belt”, which is fallible and amenable to refutations. Moreover, as Steffe (1992, p.184) notes, “[it] is a lot easier to integrate
models in the ‘protective belt’ of a research programme that has been established to serve certain purposes than it is to integrate epistemological hard cores”.

Steffe (1992) also supports that “constructivism (radical), as an epistemology, forms the hard core of social constructivism, which is a model in what Lakatos (1970) calls its protective belt” (p.184). Lakatos (1970) considers that proofs and refutations may well form the basis for the development of a theory of teaching and learning mathematics, in as far as maths is a socially constructed science amenable to refutations just as constructivists see it. The social constructivist thesis is that mathematics is a social construction, a cultural product, fallible, like any other branch of knowledge, “in contrast to the two activities of guessing and proving, which are rigidly separated in the Euclidean tradition” (Lakatos, 1976, p. 138). Lakatos (1976) developed a prototype of this theory in his book, Proof and refutations, which he applied within the context of a utopian class.

According to Ernest (1997), Lakatos introduced three central themes into the philosophy of mathematics: history, methodology and fallible epistemology. This is in contrast with the Platonic viewpoint, which views mathematics as a unified body of knowledge with an ontological certainty and an infallible underlying structure.

According to Feyerabend (1970), Lakatos considered that the unreasonable features of science occur only in the material world and in the world of (psychological) thought; they are absent from the world of ideas [from] Plato’s and Popper’s “third world”, authors who claimed that the growth of knowledge takes place in the “third world”, where we can achieve progress.

In addition to emphasizing the cultural nature of mathematics, Ernest (1991) presents social constructivism as a philosophy of mathematics education. According to Ernest (2002), the view of mathematics as a social construction entails that: (i) the origins of mathematics are social or cultural. This is not only controversial but it is also convincingly supported by many authors, e.g. Bishop (1988) and Wilder (1981); and (ii) the justification of mathematical knowledge rests on its quasi-empirical basis, the controversial view put forward by a growing number of philosophers representing the new wave in the philosophy of mathematics (Davis & Hersh, 1980; Kitcher, 1983; Lakatos 1976, 1978; Tymoczko, 1986; Wittgenstein, 1956). The Lakatosian heuristic
method also serves human constructivism, which allows knowledge in general, and mathematical knowledge more specifically, to be open to criticism in order to become a sound method.

The education system in Cyprus is based on the principles of encyclopaedism which promoted teacher-centric methods (Persianis, 1998). This is corroborated by Karagiorgi & Symeou (2006; 2007) who state that “the Greek educational system, was influenced by the French system with its underlying epistemological tradition of encyclopaedism and its extensive centralization and uniformity” (2006, p.14). These systems endorse a teacher-centred model of education, which “has for many years been repetitively used strictly and monotonously, leading to the failure of introducing innovative knowledge and cultivating social dynamics, creativity and intellectual beauty that would lead to a conflict and to scientific criticism” (Chazan, 1990, p. 14).

The Euclidean method of teaching geometry, which is strictly based on a teacher-centred model, encourages the accumulation of basic knowledge by the student, without simultaneously promoting the acquisition of any essential social objectives. Specifically, the up-down teaching process in the traditional Euclidean method is mostly used by Cypriot teachers, possibly because teachers were not taught another teaching and learning methodology when studying at university. Cypriot teachers of mathematics—after being on a waiting list for the pre-service programme (based on the regulations of the government’s centralized system) for approximately ten years—finally started to teach in a classroom, and naturally applied the teacher-centred approach (lecturing/exposing), thereby promoting teaching situations of “direct transmissions” (Tikva, 2010), since this was the most common and popular teaching method at the time (Toumasis, 2000).

1.2 CYPRUS’ EDUCATION SYSTEM

The study of various geometrical shapes is conducted empirically from the primary to the lower level of Gymnasium in the Cyprus curriculum. The method adopted is the finding or verification of the properties and relationship of the geometrical shapes based on measurement and using geometrical instruments. However, measurement cannot be precise and its results cannot be generalized. Theoretical or Euclidean Geometry studied at the Secondary level uses logic to put our knowledge about space in order. This knowledge already exists, but it is scattered. Geometry puts it in logical
order and adds new knowledge to the existing. Every new result emerges from the previous ones by using a process called *proof* and which is based on the *laws of Logic* (Argyropoulos, Vlamos, Katsoulis, Markatis, & Sideris, 2010, p.3).

1.3 A BRIEF DISCISSION OF LAKATOS’ HEURISTIC THEORY
Hersh (1978) in introducing Lakatos’ (1971) article ‘Cauchy and the Continuum’ referred to Lakatos as “one of the most original contributors in recent times to the Philosophy of Mathematics, and among mathematicians, one of the least known”. According to Hersh (2014), Lakatos explores the contrast between Euclidean theories such as the traditional foundationalist philosophies of mathematics and quasi-empiricist theories that regard mathematics as conjectural and fallible. Lakatos’ main quest is summed up in the question: “what are the ‘objects’ of informal mathematical theories?” (Hersh, 1978, p.150). Lakatos’ theory is that mathematics like the natural sciences, is fallible, not indubitable; grows by the criticism and correction of theories, which are never entirely free of ambiguity or of the possibility of error or oversight. Starting from a problem or a conjecture, there is a simultaneous search for proofs and counterexamples. New proofs explain old counterexamples and new counterexamples undermine old proofs (Lakatos, 1976). “Lakatos goes on to draw the contrast between the ‘Euclidean” such as the traditional foundationist philosophies of mathematics and the “quasi-empiricist” theories which regard mathematics as intrinsically conjectural and infallible)” (Hersh, 1978, p.150).

1.3.1 CONCEPTUAL BACKGROUND OF LAKATOSIAN HEURISTIC
To Lakatos (1976), ‘proof’, in the context of informal mathematics, is not a mechanical procedure which carries “the Truth” in an unbreakable chain from assumptions to conclusions. Rather, it means explanations, justifications and elaborations which make the conjecture more plausible and more convincing, while it is being made more detailed and accurate under the microscope of counterexamples. Each step of the proof is itself subject to criticism, which may be a mere skepticism or may be the generation of a counterexample to a particular argument.

Lakatos (1976) introduces the notion of a “local counterexample”, which is a counterexample that challenges a step in an argument. A counterexample that challenges a conclusion rather than an argument is called a “global counterexample”. Thus, according to Hersh (1978), Lakatos applies his epistemological analysis, not to
formalized mathematics, but “to informal mathematics or mathematics in a process of growth and discovery. This is how mathematics is perceived by mathematicians and students of mathematics” (p.150).

Figure 1.0 is a schematic representation of the process of the Lakatosian (1976) heuristic method of Proof and Refutations.

![Diagram of the Lakatosian method](image)

**Figure 1.0: Schematic representation of Lakatosian method (Davis & Hersh, 1980, p. 292)**

1.4 SURFACE AREA OF A CONE (SAC)

A cone is a three-dimensional (3-dim or 3D) geometric shape that tapers smoothly from a flat base (usually circular) to a point called the vertex. More precisely, it is a solid figure bounded by a base in a plane and by a curved surface area (called the lateral surface) formed by the locus of all straight line segments joining the vertex to the perimeter of the base, such that there is a circular cross section. The term “cone” sometimes refers only to the surface of this solid figure, i.e. the lateral surface. In common usage in elementary geometry, cones are assumed to be right circular, where “right” means that the axis passes through the centre of the base (suitably defined)
perpendicular to its plane, and “circular” means that the base is a circle. In contrast to the right cone is the oblique cone, in which the axis does not pass perpendicularly through the centre of the base. However, this study examines only the surface area of a right circular cone (SAC) that is taught in the Cypriot curriculum.

According to Christou (1999), there are three types of curricula: the official educational policy (intended curriculum), the curriculum taught by the teacher in the classroom (analytical curriculum) and what students learn in a classroom (hidden curriculum). The distinction among the aforementioned three types of curricula is a result of common restrictions in the educational practice, e.g., either in a lack of sufficient time for the coverage of the official curriculum or due to the structure of the analytical curriculum, which is brief, inclusive and self-contained (Philippou & Christou, 1996). Therefore, teachers usually take initiatives and make on-the-fly decisions about its application in the reality of a classroom. According to the intended curriculum, it is expected that students in Mathematics of Form B (Appendix A) be taught the subject of Geometry in forty-five teaching hours (t. hs. of 45 minutes each). Specifically, the topics taught are as follows:

1. inscribed quadrilaterals in a circle (4 t. hs.);
2. regular polygons (4 t. hs.);
3. measurement of a circle (4 t. hs.);
4. locus, the analysis-synthesis method, the constructions’ use of a compass (8 t. hs.);
5. space geometry (position of two-lines, the theorem of three vertical lines, skew lines, angle between two planes) (5 t. hs.);
6. polyhedron (measure and construction) (10 t. hs.); and
7. solids of revolution (10 t. hs.) (Cyprus Ministry of Education, 2010).

Ten out of the total forty-five periods are used for the solids (cylinder, cone, elliptical cone) where students must know how: (i) to define the solids and their elements; ii) to measure and to apply the corresponding formulas of the volumes and their surface area of revolution around an axis in the same plane. The curriculum does not clearly state whether the SAC or any other solid has to be proved. The analytical curriculum differs from the intended curriculum. Students’ learning depends on how the teacher uses the
textbook in the classroom and the teaching method followed. All textbooks (old and new) which are used in the Cyprus curriculum include the Euclidean proof, which approaches the SAC according to the method of the “limits of the pyramids” as a strict mathematical proof (the proof is presented in section 1.2.2 as proof 1). The same proofs were included in the older mathematical textbooks by Kanellou (1977) and Papanikolaou (1975), which were used in the analytical curriculum of Cyprus before 2000.

From 2000 onward, in the new mathematics textbooks, the mathematical proof of the SAC was abandoned and simpler proofs were introduced, e.g. Proof by using the length of arc (this proof is presented in section 1.2.2 as proof 2). Although these proofs did not require prior mathematical knowledge about the limits, they did encourage an understanding of the SAC concepts in a constructive manner. Under these circumstances, teachers have no clear target instructions for the curriculum about how to teach the SAC and the solids in general. Sometimes they merely apply the formula without proving it. Most of them teach the proof of the SAC in the way it was presented in the old textbooks of Kanellou (1977) and Papanikolaou (1975).

As a result, it is interesting to focus on the SAC. In addition, the following reasons corroborate the choice of the specified subject: First, given the researcher’s experience of the pre-service programme, it was obvious that many students’ difficulties were due to misconceptions regarding the following:

(1) how to construct/deconstruct the cone from 3-dimensional to 2-dimensional and vice versa;
(2) the kind of shape that could form/create a cone when it is rotated about an axis;
(3) the relationship between the solid cone and its shape in a plane, that is a sector; and
(4) how to prove the SAC.

Second, the notion of the SAC covers many geometrical objectives in solid and plane geometry. Subtopics that could be examined are as follows:

(1) elements of a circle;
(2) rotation of a plane, a line or a point about the axis;
According to the report of the professors Boekaerts, Leuven and Sinkinson of the Evaluation Committee, Cypriot student-teachers during their school practice in the pre-service programme experience tension between what university teachers have taught them and what the seconded teachers (teacher-trainers) want them to do, in contrast to what is possible in the school. This occurs because many mentors in the host schools do not encourage student-teachers to put into practice what they have been taught in the course (European Evaluation Committee, 2009).

1.4.1 Definition of the SAC

Up until 1980, the definition of the SAC that was used in the Cyprus syllabus of secondary education was based on this principle: “A right conic surface (area) is the curved surface (area) (Figure 1.1) that is generated by line (ε) that intersects the axis of the rotation xx’ at point K” (Papanikolaou, 1975, p. 364, definition 545, Figures 538–9).

Figure 1.1: Generation of the surface area of a cone

Source: Papanikolaou, 1975, p. 365, Figures 538–539
Papanikolaou (1975) was the first who defined the conic surface area, as mentioned above, and then he defined the cone as Apollonius of Perga did as cited in Densmore (2010, p. xxv).

According to Papanikolaou (1975, p. 366), the cone definition is: right circular cone. If the conical surface is intersected by surface level Π perpendicular to its axis KK' (542 in Fig. 1.2), the solid outlined by the upper conical point K of the surface of the cone and the intersection plane is called a cone, while according to Kanellou (1977), the cone is defined as a solid, which is formed by rotating a right-angled triangle, about one of its vertical sides (Fig. 1.3).

![Figure 1.2: Cone definition](source: Papanikolaou, 1975, p.365, Figures 542-543)

![Figure 1.3: Cone definition](source: Kanellou, 1977, p.176, Figure175)

After Papanikolaou (1975) had defined the conic surface area, as mentioned above, he defined the SAC. He refers to the SAC as a “sum of infinite circles” (Fig. 1.4) in order
to prove the following theorem: “The intersections of a surface area of a cone by perpendicular planes to its axis of symmetry are circles and the ratios of their radius are equal to the ratios of their distances from the vertex of the cone” (p. 365).

![Figure 1.4: SAC theorem](image)

**Source:** Papanikolaou, 1975, p.365, Figure 541, theorem 547

The definitions of the SAC in Kanellou’s (1977) school textbook, and in Papanikolaou’s (1975) school textbook are similar in that they both define the curved surface area of a cone which is formed by rotation as: “the limit to which the area of the lateral surface of the well-formed pyramid extends when the sum \( v \rightarrow \infty \) of the sides of the base of this pyramid grows indefinitely” (Kanellou, 1977, p. 177; Papanikolaou, 1975, p. 367).

The more recent school books of Thomaides, Xenos and Poullos (2000) and Argyropoulos, Vlamos, Katsoulis, Markatis, & Sideris (2010), used in the Cypriot curriculum for secondary education, have adopted the following SAC definition, which seems simpler than the one used in the old books.

Thomaides, Xenos and Poullos (2000, p. 335) define SAC as:

The surface area [which is] formed by the rotation of the hypotenuse of a right-angled triangle KAB (A=90°) around one of its vertical sides (i.e. KA) which is the lateral (parapleyri ‘παράπλευρη’) area of a right cone.

Argyropoulos et al. (2010, p. 311) give a similar definition of the SAC which is developed in section 1.2.2 (proof 2).
In actual fact, all definitions are derived from the same principle, i.e. the principle used in Euclid’s *Elements*. According to Flaumenhaft, as cited in Densmore (2010, p. xxv), “cones had been defined by Euclid prior to the work of Apollonious. Euclid’s definitions are, however, different from those of Apollonious”.

When Euclid gives definitions of solid figures at the beginning of the Eleventh Book of the *Elements*, he says that a cone is the figure comprehended when, taking one of the sides about the right angle in a right-angled triangle, you keep the side fixed and carry the triangle all the way around to the same position from which you began to move it. The cone axis is the straight line which remains fixed, about which the triangle is turned; and the base is the circle swept out by the straight line which is carried round. Euclid like Apollonious, defines a cone by generating it (Densmore, 2010, p. xxv).

Papanikolaou (1975), in *Euclidean geometry*, was inspired by Apollonious to define the conic surface first. Definitions used in the textbooks of Argyropoulos et al. (2010), which replaced Thomaides et al. (2000), are based primarily on the proofs that endorse the heuristic methods. The problem in teaching and learning the SAC lies in the method of teaching and learning the surface areas that are created from the sides of the right-angled triangle when it is rotated about its axis (one of its vertical sides), where each side creates a locus of a different surface area. According to Flaumenhaft, as cited in Densmore (2010, p.xxv), Apollonius states the following about the cone’s different areas: “A cone’s surface is heterogeneous: a cone has two kinds of surface – one of them being conic, while the other one (the base) is planar”. He also observes that “a conic surface is not the same thing as a cone’s surface”.

A conic surface has two parts, on opposite sides of the vertex, each one of which is itself a conic surface. One of the two surfaces is generated by the part of the moving straight line that extends above the fixed point; and the other one of the two surfaces, by the part of the moving straight line that extends below the generative circle. But although the movement of the straight line about the circumference of the circle generates a conic surface in which there are two surfaces, these surfaces are both of the same kind – conic (Densmore, 2010, p.xxv).
Vagueness on the SAC definition is also observed in the recent mathematics textbook which follows the new analytical curriculum that was developed according to the Educational Reformation (Cyprus Ministry of Education, 2010).

1.4.2 Proofs of the SAC
Despite the fact that proofs (which are analyzed below) are not mentioned in the intended curriculum, they appear in textbooks and sometimes are taught by teachers of the analytical curriculum. The third proof, discussed below, has possibly been suggested by supporters of heuristic methods, as an experimental approach to teaching the SAC and is not mentioned in any textbook.

1.4.2.1 Proof 1: Proof by using limits
The first proof is based on the use of limits. Since, in the curriculum in Cyprus, the chapter on limits is taught before the chapter on solids, it is difficult for students to recall the notion of the limits on their own as this proof requires a high level of mathematical knowledge. This proof has always been used in the curriculum as a “traditional” proof that is taught by teachers and is mentioned in the old curriculum as well as in the textbooks, e.g. those of Kanellou (1977) and Papanikolaou (1975). Papanikolaou (1975) defines the measurement of a SAC less rigidly than Kanellou (1977) does. Papanikolaou’s (1975) proof is as follows:

The lateral surface of a cone or the convex of a cone generated by rotation is defined as the threshold that is approximated by the lateral surface of a regular pyramid with base radius R and lateral contract edge \( \lambda \), when the sum of the sides of its base extends to infinity. About the lateral surface of a regular pyramid we know that \( Sarea = \frac{Pv.h}{2} \), where \( Pv \) is the perimeter of the base of \( v \)-sided polygon and \( h \) is the lateral height. Then the convex of the cone equals to \( Sarea = \lim_{v \to \infty} \frac{Pv.h}{2} = \frac{2\pi Rh}{2} = \pi R\lambda, h = \lambda \)

(Papanikolaou, 1975, p. 367).

Kanellou’s (1977) cone proof for the SAC is more rigid, by giving restrictions to the base side of a pyramid: He has taken into consideration that the surface of a regular pyramid is equal to \( \frac{1}{2}P_v.OI \) (Figure 1.5), where \( P_v \) is a perimeter of a base of a cone
and $\Gamma \Delta$ the base side of a regular polygon, which is defined as $|O \Delta - O I| < 1 \Delta$ or $|\lambda - O I| < \Gamma \Delta / 2$. While $v$ increases, $|\Gamma \Delta| / 2$ decreases, is less than any positive number $e$. So it is true that $|\lambda - O I| < e \Rightarrow \lim_{v \to \infty} O I = \lambda$, for all $e > 0$. So, by definition

$$\lim_{v \to \infty} \frac{1}{2} P_v O I = \frac{1}{2} \lim_{v \to \infty} P_v \lim_{v \to \infty} O I = \frac{1}{2} \cdot 2\pi \rho \lambda = \pi \rho \lambda$$

![Figure 1.5: Measurement of the SAC](source: Kanellou, 1977, p.176)

1.4.2.2 Proof 2: Proof by using the length of arc

Since 2001, the second proof has been presented in the textbook by Thomaides et al. (2000, p. 337), which was later replaced by Argyropoulos et al. (2010, p. 311).

According to Thomaides et al. (2000, p. 337):

The lateral surface of a cone ($E_C$) is approximated through its rotation. That is, it can be considered as a circular sector with arc length equal to the length of the cycle of the base of the cone $2\pi r$ and with radius of the circular area equal to the lateral height $l$. The ratio of the area of the circular sector ($E_S$) to the area of the circle of the base of the cone is equal to the ratio of their radii. That is $\frac{E_C}{E_S} = \frac{r}{R}$

Therefore $E_S = \frac{E_C R}{r} = \frac{\pi r^2}{r} R = \pi r R = \pi l R = l$.

(Thomaides et al., 2000, p. 337).
The idea of the first proof is also mentioned heuristically in the current textbook (Argyropoulos et al., 2010, p. 311), as shown below (Fig. 1.6):

Consider a right triangle KOB with a right angle at O (Fig.1.6) that is rotated around its vertical side KO. The hypotenuse KB of the right triangle upon rotation intersects a fixed point K and subscribes a convex, whereas the vertical side OB subscribes a circular disk with center O and radius OB, which is in a plane perpendicular to KO at point O. The convex produced by the hypotenuse KB is called the lateral or curved surface of the cone, the random position of KB is called the original place or side of the cone. The vertical side KO remains constant during the rotation and is called the axis or height, point K [is called] the peak and the circle subscribed by the vertical side OB is called the base, and the base radius is called the radius of the cone.

The convex can be generated on the horizontal plane. For this purpose, we subscribe into the cone a cone-shaped n-angular pyramid that we subsequently expand on the horizontal plane. The expansion of the lateral surface of a regular pyramid consists of equal equilateral triangles, which we generate the one alongside the other as subscribed triangles within the circle.

Figure 1.6: Surface area of a cone

Source: Argyropoulos et al., 2010, Figures 38–40, p. 311
For K random point on the horizontal plane and $\lambda$ the length of the original cone, and for constantly doubling the number $n$ of the vertices of the subscribed pyramid, the lengths of the equal strings $AB$, $B\Gamma$ ... become constantly smaller and the polygonal line $A'B'$ ... $A'$ in the expanded surface gradually approximates the arc of the circle. The circle has length $AA' = 2\pi r$. If we call $\phi$ the angle $A'K'A'$ of the sector in degrees, we have the relationship 
$$\frac{360}{2\pi\lambda} = \frac{\phi}{2\pi\rho} \iff \phi = \frac{\rho}{\lambda}360^\circ$$

Therefore, the developed curved surface of a cone with lateral side ($\lambda$) and base radius $\rho$ is a circular section of radius $\lambda$ and arc length $2\pi\rho$ or, in degrees, $\phi = \frac{\rho}{\lambda}360^\circ$. From the above we consider that the SAC equals $E = \pi\rho\lambda$ (Argyropoulos et al., 2010, p. 311).

1.4.2.3 Proof 3: Proof by using the parallelogram

According to the third proof, if one cuts the cone by the lateral height ($\lambda$) so a sector is formed, radius ($\lambda$) and the length of its arc = perimeter of a based circle of a cone $= 2\pi r$. If then, one cuts this arc into small sectors and forms a parallelogram with base equal to $(\pi r)$ and lateral height ($\lambda$) (Fig. 1.7) the area of the parallelogram is the SAC which is equal to $\pi r\lambda$.

![Figure 1.7: Proof by using the parallelogram](image)

In this study it was expected that the students in the experimental group would prove the SAC by using one of the preceding three proofs. It was specifically expected that they would recall the second proof that is based on the statement: the perimeter of the base of a cone equals the length of the arc of the sector formed by the SAC in 2-dim.
Knowledge of this statement is derived from the curriculum and is accessible via the heuristic method.

1.5 STATEMENT OF THE PROBLEM

In their interviews as representatives of the European Evaluation Committee (2009) of the pre-service programme in Cyprus, Boekaerts, Leuven and Sinkinson revealed that pre-service teachers of mathematics offered lessons that were still lecture-based, which encourages automated knowledge and memorising. This approach contradicts the goals of the community of knowledge, which are ideally required by the citizens of tomorrow and which emphasize the development of basic skills in mathematics and science. According to the Strategy of Europe 2020, “There is a strong relationship between a lack of basic skills, including skills in mathematics and science, and early school leaving” (European Commission, 2013).

Despite changing textbooks, there is still confusion and teachers are mainly teaching the Euclidean geometry proof using traditional methods. This may well be the primary reason why Cypriot students are having difficulty proving the SAC, using limits in an up-down approach. As a consequence, this kind of traditional teaching may be responsible for the students’ lack of comprehension of the shape of a cone in 2-dim and, in addition, it may make it hard for them to discover a different way of proving the SAC.

Consequently, a reform of the Cyprus Mathematics Education System (Cyprus Ministry of Education, 2010) has suggested not only an upgrade of the various curricula, but also corresponding improvements in the teaching methods. I believe one way of improving the teaching methods would be by introducing students to the problem-solving method of learning. According to De Villiers (2003; 2009; 2010), introducing students early to the art of problem-solving will give them opportunity for exploration, conjecturing, refuting, reformulating, explaining and understanding. De Villiers (2012) claims that:

instead of defining a proof in terms of its verification function (or any other function for that matter), it is suggested that proof should rather be defined simply as a deductive or logical argument that shows how a
particular result can be derived from other proven or assumed results, nothing more nothing less (p.7).

This may help students achieve higher-order thinking by stimulating their abstract thinking.

Such heuristic methodology is Lakatosian, in line with *Proof and refutations* (Lakatos, 1976) which has been adopted in this study in order to examine whether students can acquire higher-order thinking skills. Therefore, the focal point of the study is to explore the effect of using Lakatos’ heuristic method of teaching the SAC on students’ learning.

1.6 OBJECTIVES OF THE STUDY

The fundamental objective of this study is to examine whether the application of the Lakatosian heuristic method would enable secondary school students (aged between 16 and 17) to achieve higher-order thinking (higher objectives) in the area of geometry, especially relating to the SAC. Essentially, the study intends to examine and evaluate the Lakatosian method as a quasi-empirical method by comparing the traditional Euclidean teaching method with the Lakatosian method. Chazan (1990) has acknowledged the problems associated with the traditional method and suggested an alternative approach (p. 18) to teaching high school geometry, stating that mathematics, like science, is quasi-empirical knowledge. According to Chazan (1990, p. 15), Scheffler (1965), explaining the difference between mathematics and science, points out that:

> Mathematics doesn’t need laboratories or experiments; they (mathematicians) conduct no surveys and collect no statistics. They work with pencil and paper only, and yet they arrive at the firmest of all truths, incapable of being overthrown by experience.

By modelling classroom teaching on Lakatos’ well-known historical dialogue, the study examines whether this method could help students change their conceptualizations from “alternative” to “scientific” and help them acquire more robust higher-order thinking skills than students taught using the traditional method (the Euclidian method).
1.7 SIGNIFICANCE OF THE STUDY

Until now, there has been no study in the field of mathematics education on the Lakatosian method applied to school mathematics, especially in geometry, that has examined the concept of the SAC. This study contributes to the development of an alternative approach to teaching this particular topic in the mathematics curriculum of secondary schools in Cyprus.

The study provides an innovative and alternative approach to the traditional way of teaching mathematics, which aims to be more practical for teachers of geometry in secondary schools and more beneficial to students of mathematics. This is substantiated by demonstrating that the study may have had better-than-expected outcomes in students’ performance in mathematics by using the Lakatosian method compared to the traditional method of teaching, where the rates in European assessments such as TIMSS (Trends in International Mathematics and Science Study) for Cyprus were 2 degrees lower than the difference in average scores between 1995 and 2007 of the analogous U.S. difference ($p<0.05$). The automated knowledge and memorising provided by the traditional method may be one of the reasons why Cypriot students’ performance in mathematics and science is standing at one of the lowest rates in the European assessment.

1.8 RATIONALE OF THE STUDY

This study was motivated by the expressed desire for the principles of the educational reform of the Cypriot Ministry of Education (2010) that wider learning experiences should be provided to Cypriot students through a qualitative curriculum programme. Given the importance of mathematics in secondary schools, and the targeted reform of the educational system in Cyprus, the current study determines the potential impact of the introduction of the Lakatosian method to the mathematics curriculum.

The study is also motivated by the need to examine how weaknesses in the current methods of teaching can be overcome and how to develop a new method that would be implemented effectively so that students and teachers could develop more interest in the exploration of mathematical principles. Thus, the rationale for this research study was to develop, through Lakatosian heuristics, a new trend in teaching of Geometry, by examining the assumptions thoroughly, then by eliminating the “guilty lemmas” (Lakatos, 1976, p. 127) and, finally, by reconstructing the original hypothesis, while
taking any prior criticism as indications for proof, in contrast to the Euclidean method, where a priori logic is considered proof. In Euclidean geometry, students cannot have the satisfaction of using exploration and discovery methods because the up-down scenario prevents them from becoming involved in the process of “constructive definitions” (De Villiers, 1998), or the discovery of concepts, while, according to Vinner (1991) (cited in De Villiers, 1998, p. 2), “knowing the definition of a concept, does not at all guarantee understanding of the concept”. The use of a “human” method in teaching mathematics, as social activity, promotes pedagogical objectives by encouraging students “to doubt the current orthodoxies” (Lakatos, 1976). Such a method can stimulate curiosity and develop it into interest.

1.9 RESEARCH QUESTIONS
The following questions formed the basis of this research study:

(1) What is the impact of the Lakatos (1976) heuristic method on students’ learning of the SAC?
(2) Can the heuristic method of teaching the SAC help students to sustain their learning better than the traditional method?
(3) Can the heuristic method of teaching and learning the SAC change students’ readiness level according to the Bloom taxonomy?

In order to answer question 1, the following sub-questions were formulated:

i. What is the impact of using the heuristic method to teach the SAC on students’ achievement?
ii. What is the impact of using the heuristic method to teach the SAC on students’ conceptual change?
iii. What is the impact of using the heuristic method to teach the SAC on students’ attainment of higher order thinking?

1.10 DEFINITION OF TERMS

1.10.1 Lakatosian heuristic method
This is a methodology of teaching and learning applied by Lakatos (1976) in a utopian class.
1.10.2 Thought-experiment

The term thought-experiment or quasi-experiment “suggests a decomposition of the original conjecture into sub-conjectures or lemmas, thus embedding it in a possibly quite distant body of knowledge” (Lakatos, 1976, p. 9). According to Lakatos (1976, p.9, cf. 1), Szabó (1958) said that “Thought-experiment (deiknimi, ‘δείκνυμι’) was the most ancient pattern of mathematical proof. It prevailed in pre-Euclidean Greek mathematics”.

1.10.3 Down-up/Up-down

1.10.3.1 Traditional method: Teacher-centred approach (lecturing/exposing)

In this teaching method, the teacher does not give the student a chance to create a hypothesis or to criticize a conjecture. Therefore, the student is not encouraged to refute the conjecture, to come up with counter-examples, or use strategies of problem solving. As a result of this, the inductive method, which requires carefully chosen examples to introduce a concept or to prove a theorem or a mathematical statement, does not exceed the deductive method. This method encourages the up-down scenario of the Euclidean axiomatic system - “A deductive system with injections of infallible truth that inundate the whole system from the top” (Koetsier, 2002, p.193).

1.10.3.2 Quasi-empirical method

In this method the teacher encourages the student “to discover the solution to problems”, such as a certain proof or a certain formula, in contrast to a traditional method where a “suitably programmed Turing machine could solve [the problem] in a finite time” (Lakatos, 1976, p. 4). The teacher’s aim is to encourage students who are used to working in small groups to come up with counter-examples or use strategies of problem solving in their attempt to discover the path(s) toward the solution(s) of their problem or a conjecture. Both the teacher’s and students’ aim is

\[
to \text{elaborate the point of informal, quasi-empirical mathematics that does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvements of guesses by speculation and criticism, by the logic of proofs and refutations (Lakatos, 1976, p. 5).}
\]
Thus, this method is based on the quasi-empirical system where the typical flow of things is to bring “lies” back from the false “basic sentences” (or according to Popper (1959, p.78) a ‘basic statements’) in a down-up direction of the original hypothesis (Chazan, 1990).

1.11 OUTLINE OF CHAPTERS

The structure of this study is as follows:

Chapter 1 gives a brief introduction of the study describing its background, statement of the problem, research questions, its significance, a brief definition of terms and structure of the thesis.

Chapter 2 provides the theoretical framework that guided the study and review of the literature that are related to the study.

Chapter 3 describes the methodology of the study, which includes research design, sample selection method, data collection instrument and the ethical issues considered in the study.

Chapter 4 discusses the pilot study.

Chapter 5 focuses on the data analysis. The results of data analyses are presented and the results that were used to draw together the findings of the study and answer the research questions.

Chapter 6 highlights and discusses the major findings.

Chapter 7 summarizes the main findings and concludes the study.

1.12 CONCLUSION

The objective of this study was to explore the effect of using Lakatos’ heuristic method to teach the SAC on students’ learning. In this chapter the study is contextualized. The research questions and the significance of the study are presented. The definition of terms used in the study and the structure of the thesis are also presented.
CHAPTER TWO

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

2.1 THEORETICAL FRAMEWORK
This study focuses on the Lakatosian heuristic theory and its application to the Euclidean Geometry topic of the surface area of a cone (SAC). Here the objective is to explore the effect of using the Lakatosian heuristic method to teach SAC on students’ learning. This chapter discusses the Lakatos (1976) theoretical framework and the Jun–Young Oh (2010) model of “the enhanced conflict maps” that are used to explain the students’ conceptual changes/development.

2.1.1 Lakatosian heuristic theory
According to Lakatos (1970), falsificationism is the process of replacing an existing theory with a series of theories that constitute namely ‘problem shift’ in the logic of scientific discovery. Problem shift is a “succession of theories and not one given theory which is appraised as a scientific or pseudo-scientific” (p. 132). Such theories are distinguished by a remarkable continuity, which welds them into research programs. Such research programs consist of methodological rules: some tell us what paths of research to avoid (negative heuristic) and others tell us what paths to pursue (positive heuristic). The negative heuristic of a research program isolates a “hard core” of propositions that are not exposed to falsification. The positive heuristic is a strategy for constructing a series of theories in such a manner that short comings at any particular stage can be overcome (Lakatos, 1970).

Niaz (1998) observes that “given the parallel between the scientific process of theory development and an individual’s acquisition of knowledge, it is not surprising that students resist changes in their major theoretical frameworks” (p.111). According to Lakatos (1970), scientists do not abandon a theory on the basis of contradictory evidence alone and “there is no falsification before the emergence of a better theory” (p.119). As an illustration of this point, “Niaz (1991, 1993c, 1995a) has drawn a parallel between the methodology of idealization (simplifying assumptions) used by scientists and the construction of strategies (models) by students to facilitate conceptual understanding” (Niaz, 1998, p.111).
Niaz (1995a), as cited in Niaz (1998), has shown that student performance on algorithm and on chemistry problems can be interpreted as a process of progressive transitions (models) that facilitate different degrees of explanatory/heuristic power to student conceptual understanding. Lakatos (1970) shared a similar view as he has referred to the “rational reconstruction of scientific research programs” (p.111). As a prerequisite for conceptual change, it is essential that students are provided with alternative views that apparently contradict their previous thinking. This is based on the Lakatosian thesis that “[t]he history of science can be conceived as that of competing rival research programs” (Lakatos, 1970, p. 155).

Learning is an active process of knowledge construction, where cognitive conflicts must have been engendered by the students themselves in trying to cope with different problem solving strategies. Students’ alternative conceptions are not considered wrong, but rather regarded as models; perhaps in the same sense as used by scientists to simplify the complexity of a problem (Tsai, 2000). According to Strike & Posner (1992, p.153), students’ alternative conceptions may be misconceptions or false beliefs thus, “may be a candidate for change”, and the change is from alternative to scientific conceptions.

According to Chinn, & Brewer (1993), students resist changes in their core beliefs (cf. “hard core” Lakatos, 1970), more strongly than they resist change in other more peripheral aspects of a topic (Laburú & Niaz, 2002, p. 213). For this reason students look for an auxiliary hypothesis to defend their core beliefs. The new or alternative view must appear initially plausible to the students. Auxiliary hypotheses used by students to defend their core beliefs may provide clues and guidance for the construction of novel teaching strategies. This is based on the Lakatosian thesis that scientists do not abandon a theory on the basis of contradictory evidence alone, and “there is no falsification before the emergence of a better theory” (Lakatos, 1970, p.119).

Niaz (1998, pp. 111-112) first suggested the following criteria and then Oh (2010) used them in order to validate students’ alternative conceptions, which are the students’ beliefs or what Lakatos (1976) has called “core belief”: 
1. **Deletion criterion**: Faced with a similar problem in Piagetian theory, Beilin (1985) has proposed a ‘deletion criterion’: “If a construct in the theory can be deleted without apparent damage to the identification of the theory as Piagetian’s, then it is not part of the hard core. If on the other hand, deletion detracts materially from the theory or alters it in irreparable ways, then it is a part of the hard core” (p.109-110).

2. **Hard core and protective belt propositions**: According to Chinn and Brewer (1993), Lakatos (1970) has distinguished between two types of propositions within a theory: hard core propositions and protective belt [soft core] propositions. Hard core propositions cannot be altered without “scraping” the entire theory, but protective belt propositions can be altered while preserving the key central hypothesis” (p.10, original italics).

3. **Auxiliary hypothesis**: Given the opportunity for conceptual changes, students invariably tend to accept changes in their frameworks (soft core) but resist changes to the hard core by offering an ‘auxiliary hypotheses’. In the history of science Lakatos (1970, p.153), for example, considers Pauli’s “exclusion principle” as an ‘auxiliary hypothesis’, that protected the hard core of Bohr’s theory.

The theoretical framework of students’ beliefs (core belief) that concerns their alternative conceptions is a typical conceptualization reaction when the students are called to conceive an idea or understand a concept. This reaction is more resilient than instruction in a traditional classroom (Niaz, 1998).

On the other hand, students who responded correctly to an initial item A, that can be considered as a core belief of students’ understanding, performed extremely well in the following items B, C, D…, considered as the dispensable part (soft core/positive heuristic) compared to the initial item A. It is plausible, according to Niaz (1998) that the rate of the students' formation of alternative conceptions as a cognitive reaction to the pressure of conceptualization indicates that these beliefs constitute the hard core (negative heuristic) of their framework in the Lakatosian sense (Lakatos, 1970). Following the Lakatosian framework, students' understanding of items B, C, D, …,
would represent the soft core (positive heuristic) of their frameworks, which yields comparatively lesser resistance to conceptual change.

Chinn & Brewer (1993), as cited in Niaz (1998), taking their cue from Lakatos, have emphasized that students resist changes in their major theoretical frameworks (i.e. item A), by offering an ‘auxiliary hypothesis’. So students’ understanding of item A, which was characterized as a negative heuristic, needs a “strong restructuring” in order to change. Respectively, “weak restructuring” would correspond to Lakatos’ idea of changes in the soft core of a research program. According to Niaz (1998), students understanding of item A would require ‘strong restructuring’ but items B, C, D...would require ‘weak restructuring’.

The following section explains how students’ conceptual makeup changes from alternative conceptual to scientific conceptual with the help of the enhanced conflict map, which is based on the Lakatosian method.

2.1.2 The Jun-Young Oh’s model based on Lakatosian theory

The Tsai (2000) “conflict map” was enhanced by Oh (2010) based on the Lakatosian methodology. The enhanced conflict maps (Fig. 2.1) shows that students’ alternative conceptions were generated before the intervention and the second conflict map expresses the change in students’ conceptions after the learning experience. The first conflict map, which was suggested before the learning experience, includes discrepant events and allows students to relinquish the core concepts, overcome cognitive conflict with scientific concepts, and eventually learn new concepts. Therefore, conflict maps are effective tools for learning new concepts (Oh, 2010).

According to Oh (2010), as shown in Figure 2.1, Losee (2001) said that “the dotted circle line indicates a ‘protective belt’ of auxiliary hypotheses that are created around the hard core of non-falsifiable propositions” (p. 1141).
The conflict map is a useful tool for teachers in order to understand the student progress/development between their alternative concepts and target scientific concepts. The following section provides in detail the students’ conceptual change according to the Oh (2010) conflict map.

### 2.1.2.1 Reflecting on the Results of Scientific Conceptual Change

According to Oh (2010), it is plausible to suggest that the results obtained in his study reflect a change in the soft core of students’ beliefs. In Figure 2.1 the “naïve scientific concept” element suggests that in a research program, the core does not change; instead, auxiliary hypotheses verified through-experiments are continuously added. Based on the Lakatosian methodology, the structure of the enhanced conflict map (Fig. 2.1) and the teaching sequences assisted this study to assess whether this method could be successfully applied in the case of the concept of the surface area of a cone (SAC). The general theory of this method as outlined in Figure 2.1, which is the basis of this study, assisted the researcher to interpret the mathematical consequences that model conceptual changes.
According to Oh (2010) the information presented in Figure 2.1 suggests that the teaching sequence is the discrepant event (or P1, P2) first, followed by the core of naïve scientific programs, the critical event, other scientific concepts (C2, C3 …), other supporting perceptions (P3, P4 …) and, finally P1’, P2’…. Thus, the steps, arising from the Lakatosian method are:

**Step 1: Students’ alternative conception.**

It concerns students’ understanding of the structure of existing conceptions via discrepant events (P1, P2).

**Step 2: Naïve scientific concept**

It concerns students’ processing that bridges alternative to scientific concept. According to Oh (2010), the enhanced conflict map as shown in Figure 2.1 is explained as follows:

1. The core of naïve scientific conception (centre thick line circle).
2. The critical events between alternative conceptions and naïve scientific concept.
3. Protective belt of naïve scientific concept (small dotted line circle).
4. The refined protective belt of naïve scientific concept (large dotted line circle).
5. Reflecting the results of scientific conceptual change/shift (P1’, P2’……).
6. Can new scientific concepts explain the perceptions (P1’, P2’……) that symbolized students’ alternative conceptions completely, partly, or not at all?

**Step 3: Target scientific conception**

The stage of the target scientific conception depends on students’ reconstruction of their naïve scientific concepts. In the naïve scientific concept (dotted line circle) students tried to support their alternative conceptions/perceptions by using critical events in their attempt to move to a final stage that is the scientific. Thus, through
successive “progressive shifts” they can possibly achieve their target scientific concept (large thin line circle).

Oh (2010) in the same manner as Niaz (1998) suggested the criteria, referred to section 2.1.1, based on Lakatos (1976), core belief for confirming the structure of students’ alternative conceptions.

According to Oh (2010), questions about alternative conceptions “refer to the ideas that the students had before learning, emphasized the core belief” while “questions emphasized the core belief and one of the students’ alternative concepts, showed the relationship with the soft core, and used graphs and questions to show discrepant events” (p. 1146). Thus, the meaning of the conceptions that followed the discrepant events and critical events was examined and the answers were important in examining the structure of alternative conceptions. Therefore, the structure of students’ alternative conceptions toward discrepant events should be explored, and students should have minimal understanding about the main core of the scientific concepts that will be studied to resolve suggested discrepant events. This is the Condition of Resolving Conflict 1 (The Designed Discrepant Events), as shown in Figure 2.1.

2.1.3 Application of the Lakatosian theory and Oh’s (2010) model

This study takes into consideration the above theoretical framework and the fact that mathematics is based on constructivism. The goal is to compare and contrast the Lakatosian method with the Euclidean method. Emphasis is placed on the process of changing alternative conceptions to scientific concepts, given that such alternative conceptions are the “core beliefs” of the students. The latter come to contrast with the traditional problem-solving method in the axiomatic system. The teaching strategies of the Lakatosian heuristic in this study are based on those identified by Oh (2010), and are the following:

1. Understanding the student’s alternative conceptions: negative and positive heuristic operation in existing research program.
2. Looking for student’s core belief (hard core), on a topic that can be an appropriate starting point for teaching strategies.
3. Suggesting discrepant events: degenerative research program in an existing research program.
Discrepant events engender cognitive conflict when coping with core beliefs’ problem solving strategies (Oh, 2010, pp.1141-1142).

In this study it was important to carefully follow the students’ reaction to discrepant events as an important clue in helping the researcher understand not only the structure of alternative conceptions but also the nature of scientific concepts and how students’ scientific concepts changed after the intervention. The Lakatosian heuristic methodology is used to evaluate how the students’ conceptual makeup changed from alternative to scientific with the help of Oh’s (2010) model of enhanced conflict map. The conceptual change teaching strategy used in this study is based on an interactive approach within an intact classroom where the researcher worked in “small groups” as Laburú, & Niaz (2002, p.213) suggested. Teaching students in small groups by using the Lakatosian method may help students develop “experience in solving problems and experience in watching other people solving problems (and that) must be the basis on which heuristic is build” (Pólya, 1973, p.130).

2.2 LITERATURE REVIEW

2.2.1 The Origins of Geometry
Mathematics was created and developed in ancient Greece as a purely theoretical science. Along with theory, the ancient Greeks invented the terminology of the science, defined its ‘main notions’, practiced critical thinking, introduced the mathematical proof method and constructed the deductive interference (Exarchakos, 2001, p.7).

For the first time in the history of science, the philosopher is not interested in solving a particular practical problem, but rather he is interested in finding a general and abstract rule that will be valid for all constructions and all of imaginary areas and volumes. Thus, a process of proof begins, with particular rules and thoughts. In the works of the ancient Greeks, one could find almost all mathematical proof methods that are still used today, such as the deduction, the reduction, the analysis, the synthesis and the induction methods (Exarchakos, 1999).

The “axiomatic systematization” is one other field that emerged from ancient Greece. There are many elements of the axiomatic systematization in ancient Greek texts, for
example, the Pythagoreans, who introduced a form of proof much more improved than that of Thales (Exarchakos, 1999) who said that “mathematics remained for a long period of time (with the Egyptians mainly) at a very infant stage and that the change that followed was because of the revolution that brought about happy idea someone had grasped during a trial…” (in Kant, 1976, p.40). According to Kant (1976), Thales was an ‘inventor’ since he discovered some small but substantial geometrical proof (the sum of the angles of a triangle), which “as everyone agrees, do not need to be proved” (p.41). This points out two important facts: i) the importance of letting students create on their own while “in the mind of the person who first proved it there was a bright light and ii) the importance of the a priori proof that established mathematics as a science, which are based on empirical principles” (Kant, 1976, p.41).

Pythagoreans proposed that the proving process should not merely be deductive inference, but rather it should be linked to facts (hypotheses, as they called them) and some initial inferences. They defined a set of rules for proofs that could be used in the proving process and introduced definitions for specific mathematical objects. Aristotle also gave us all the elements of axiomatic systematization. He makes a point about the initial inferences, the definitions, the axioms, the proving process and the proof (Exarchakos, 2001, p.8).

According to Exarchakos (2000, p. 38) a mathematical theory (A) is axiomatically systematized if the following conditions are met:

1. All the initial notions or the initial terms of theory A have been put down.
2. With the help of the initial notions, other wider meanings of theory have been defined, called definitions. The initial terms and the definitions constitute the language of theory.
3. An infinite number of statements in theory A have been defined, the axioms, which we accept as valid in theory A without proof.
4. The proving rules have been defined that are valid in theory A.
5. Any other statement of the theory emerges (proved) with the help of the above elements and/or other statements of the theory, of
which the validity we have previously proved.

Axiomatic systematization was introduced by Euclid in the 3rd century BC (330-270 BC). Euclid recorded the initial notions or initial terms of the geometrical objects, like the point, the line etc. He recorded their basic properties and established the definitions, the postulates (axioms) and the common notions (Exarchakos, 2001, p. 8).

Hence, Geometry (Ancient Geometry γεωμετρία; geo-‘earth’, metron-‘measurement’) was born. Geometry was then called Euclidean Geometry and set a standard for the following centuries. It is a branch of mathematics concerned with questions of shape, size, relative position of figures, and the properties of space. A mathematician who works in the field of Geometry is called a geometer. Geometry arose independently in a number of early cultures as a body of practical knowledge concerning lengths, areas and volumes with elements of a formal mathematical science emerging in the West as early as Thales and Pythagoras (6th Century BC). This is the point where differentiation of practical and theoretical Geometry, commonly known as Euclidean Geometry, began (Argyropoulos, et. al., 2010, p.3).

Archimedes developed ingenious techniques for calculating areas and volumes, in many ways anticipating modern integral calculus and contributing to the theory of Riemann. According to Exarchakos (2000), Archimedes may be considered as one of the most important scientists of all times. He set the foundations of Theoretical Mechanics and defined laws and principles that are still valid today. He made a connection between Mechanics and Geometry and he invented Archimedes’ Principle.

The introduction of coordinates by René Descartes and concurrent developments of Algebra marked a new stage for Geometry, since geometry figures, such as planes and curves, could now be represented analytically with functions and equations. “This played a key role in the emergence of “infinitesimal calculus” in the 17th century” (Exarchakos, 2000, p.52). The subject of Geometry was further enriched by the study of intrinsic coordinates and structure of geometric objects that originated with Euler and Gauss and subsequently led to the creation of topology and differential Geometry. In Euclid’s time there was no clear distinction between physical space and geometrical space. Since the 19th century discovery of non-Euclidean Geometry, the concept of
space has undergone a radical transformation. With the rise of formal mathematics in
the 20th century, “space” (point, line and plane) lost its intuitive contents, and now
there is a distinction between physical space, geometrical space and abstract space.
Modern Geometry has multiple strong bonds with physics, as seen in pseudo-
Riemannian Geometry and general relativity.

According to Lakatos (1978), “Euclidean heuristic separates the process of finding the
truth and of proving it” (p.72). He claims that the discovery of the truth has an
element of guessing the necessary axioms or the appropriate statements (that have
already been proved) through which we can start the process of deductive proof. The
idea of “improving by proving” that is of proving a conjecture via a series of gradual
improvements/revisions and proofs and refutations, never occurred in the Euclidean
System, and to support his opinion, Lakatos (1978) claims that:

The Greeks did not find a process of decision for their Geometry though
they dreamt of one. However, they found a compromising solution: a
heuristic procedure, which does not always produce the desired result, but
which is still a heuristic rule, a standard pattern of the logic of discovery.
This heuristic method was the method of analysis-synthesis (Lakatos, 1978,
p.72).

2.2.2 Euclidean Geometry as an Axiomatic System

Euclidean Geometry is a mathematical system attributed to the Alexandrian-Greek
Mathematician Euclid, which he described in his textbook, *Elements*. It consists of a
small set of intuitively appealing axioms, and many other propositions (theorems)
generated from them. Although many of Euclid’s results had been stated by earlier
mathematicians, Euclid was the first who showed how these propositions could fit into
a comprehensive deductive and logical system. He postulated the following (the 5
axioms):

(1) There is only one straight line passing between two points.
(2) Every finite straight line extends continuously and rectilinearly
from both its ends.
(3) Given a center and a radius, a circle can be drawn.
(4) All right angles are equal.
(5) If a straight line intersects two other lines and forms adjacent angles whose sum is smaller than two right angles, then the straight lines when extended indefinitely will meet on that side of the plane where the smaller (than the 2 right-angles) angles are formed (Exarchakos, 2001, pp.20-21, vol. I).

The *Elements* also included the following five “common notions”, the number of which is different in the various editions and translations of the original textbook of *Elements*.

(1) Things that are equal to the same thing are also equal to one another (Transitive property of equality).
(2) If equals are added to equals, then their sums are equal (Addition property of equality).
(3) If equals are subtracted from equals, then their remainders are equal (Subtraction property of equality).
(4) Things that are identical to one another are equal to one another (Reflexive Property).
(5) The whole is greater than its part (Exarchakos, 2001, p.23).

### 2.2.3 Euclid’s Elements

The *Elements* consists of Euclid’s thirteen books called “*Euclid’s Elements*”. They are mainly a systematization of earlier knowledge of Geometry. Its superiority over earlier treatments was rapidly recognized, with the result that there was little interest in preserving the earlier ones, and they are now nearly all lost. According to Exarchakos (2001, p. 9), we may consider that the most recent complete edition of the *Elements* was written in ancient Greek and Latin by Danish philhellene historian of the sciences J. L. Heiberg and is included in “Euclid’s Opera Omnia”. Heiberg’s edition was translated to English by T. L. Heath in 3 volumes with interesting and useful comments, and was also translated in Greek by Evagelo Stamati between 1952-1957 in four volumes.
The *Elements* begins with plane geometry still taught in Secondary School as the first Axiomatic System including the first examples of formal proof. *Elements* also tackles Solid Geometry, that is the geometry of solids in 3-dim. Much of the *Elements* states results of what is now called Algebra and Number Theory, explained in geometrical language. The cone that is researched in this study refers to Euclid’s Books XI–XIII. These books are referred to the *geometry of solids*, and they concern the comparison of the cone with the cylinder. It is stated that “each cone is equal to 1/3 of a cylinder with the same base and equal height” (Exarchakos, 2001, p.122).

According to Pappus (“Synagogi”, VII, p. 672), as cited in Exarchakos (2001, p.49-50), Euclid had written “Conics” (“*Konica-Kovika*”), which consisted of four books and was lost after the era of Pappus. Pappus in “Sinagogi” (Book VII, p.672) states that Euclid's “Conics” consisted of four books that included the conical intersections (conics), thus testifying that he knew about that work before Apollonius of Perga. Later, Apollonius revised these books and wrote another 4 books of his own thus completing his work of “conics”. Pappus also attributes to Euclid the two-volume work entitled “Locus-to-surfaces” (“Synagogi”, VII, p.636), which he put together in the mathematical collection “Treasure of Analysis”.

According to Exarchakos (2001), Pappus says that Euclid knew about the conical intersections and used the properties of the directrices of the conics. Based on these properties Euclid defined the “locus of the surfaces” and proved that:

the locus of the points, of which the ratio of the distance from a given straight line and the distance from a given plane is a constant, is a cone (p.50).

Euclid through the ‘*Elements*’ has led humanity’s way of thinking for 2300 years and that the axiomatic systematization which Euclid introduced as well as the proving processes remain powerful and unchanged until today. For more than two thousand years, the adjective “Euclidean” was unnecessary because no other sort of Geometry had been conceived. Euclid’s axioms seemed so intuitively obvious (with the possible exception of the parallel postulate) that any theorem proved from them was deemed true in an absolute, often metaphysical, sense. Today, however, many other self-contained non-Euclidean Geometry systems are known. An implication of Einstein’s
theory of general relativity is that the physical space itself is not Euclidean and that Euclidean space is a good approximation of it only where the gravitational field is weak (Exarchakos, 2000, p.55).

2.2.4 Lakatos heuristic method

This study presents Lakatos’ (1976) pattern of mathematical discovery and Lakatos’ positions about “Analysis–Synthesis” as they are developed in his book *Proof and Refutations*. In a couple of sentences, he summarized the methodological framework according to which this study is developed. His method consists of the following stages:

1. [Statement of the] **Primitive conjecture.**
2. **Proof** (a rough thought-experiment or arguments, decomposing the primitive conjecture into subconjectures or lemmas).
3. ‘Global’ counterexamples (counterexamples to the primitive conjecture) emerge.

Proof is re-examined: the ‘guilty lemmas’, to which the global counter-example is a ‘local’ counterexample is spotted. This guilty lemma may have previously remained “hidden” or it may have been misidentified. Now it is made explicit, and built into the primitive conjecture as a condition. The theorem—the improved conjecture—supersedes the primitive conjecture, with the new proof-generated concept as its paramount new feature (Lakatos, 1976, p.127).

The heuristic method according to Lakatos (1978) was a method of analysis and synthesis and he stated it as “a rule” (p.72). Thus the analysis-synthesis rule is the following:

a) Draw conclusions from your conjecture, one after the other, assuming that it is true. If you reach a false conclusion, then your conjecture was false.

b) If you reach an indubitably true conclusion, your conjecture may have
been true.

c) In case of (b), reverse the process, work backwards and try to deduce your original conjecture via the reverse route from the indubitable truth to the dubitable conjecture. If you succeed you have proved your conjecture (Lakatos, 1978, p.72-73).

Lakatos (1978) calls “analysis” the parts (a) and (b) in the above, whereas he calls “synthesis” the part (c) (p.73). In Lakatos (1978, p.70-71), he uses an introduction to the subject in the form of a dialogue, like what he did in his *Proof and Refutations* book (Lakatos, 1976) and then concludes with the Analysis-Synthesis rule. The following figure illustrates Analysis-Synthesis:

\[
\text{Analysis: } P \rightarrow P_1 \rightarrow P_2 \rightarrow P_3
\]

\[\uparrow \quad \uparrow \]

\[Q_1 \quad Q_2\]

\[
\text{Synthesis: } P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P
\]

\[\uparrow \quad \uparrow \]

\[Q'_3 \quad Q'_2\]

**Figure 2.2: Lakatos’s Analysis-Synthesis**

**Source:** In Lakatos, 1978, p.71

Where (P): \(V - E + F = 2\) for all polyhedrons (Lakatos, 1976, p.6, §2).

(Q\(_1\)) (lemma): all polyhedrons are ‘simple’ (Lakatos, 1976, p.34).

(P\(_1\)): \(V - E + F = 1\) for all flat polygon network.

(Q\(_2\)) (lemma): all its faces are simply-connected (Lakatos, 1976, p.36).

(P\(_2\)): \(V - E + F = 1\) for all triangulated networks (Lakatos, 1976, p.7).

(P\(_3\)): \(V - E + F = 1\) holds true for the triangle (Lakatos, 1976, p.8)
Q_3, Q_2 (the refuted lemmas): “if we have a conjecture (P) that has already been refuted by a counterexample, we should put the refutation aside and try to test the conjecture by a thought-experiment” (Lakatos, 1976, p.75).

This study is an attempt to explain the above experiment according to Lakatos’ last project (Lakatos, 1978, pp.93-103). Lakatos (1976) in the role of a teacher presented A proof of the conjecture by following a thought-experiment that is presented below:

**Main Step (1):** We assume a polyhedron (for instance a cube) is empty and made of soft rubber. In this polyhedron (P): V–E+F=2 is true (primitive conjecture).

**1st In-between Step:** We assume the polyhedron is simple i.e. we agree that lemma Q_1 is true, we cut off one of the faces of the rubber cube, and lay the rest on a flat surface without destroying it (1 in Figure 2.3).

**Main Step (2):** To this flat network will be true: (P_1): V–E+F=1 if and only if (P) is true.

Explanation: according to Lakatos (1976) “for this flat network we have removed one face” (p.7). So the conjecture V–E+F′=2−1=1, (F′=F−1) is true.

**2nd In-between Step:** We accept the validity of Q_2.

**Main Step (3):** We draw triangular network, by drawing diagonals in those polygons which are not already (possibly curvilinear) triangles. Therefore (P_2): V–E+F=1 holds true (2 in Figure 2.3).

Explanation: according to Lakatos (1976), “by drawing each diagonal we increase both edges (E) and faces (F) by one, so V–E+F will not alter (Figure 1.1)” (p.7).

**3rd In-between Step**: Then, from the triangular network we remove the triangles, one by one. This is done in two ways:

i) We remove one edge, therefore one face and one edge disappear (3(a) in Figure 2.3) or

ii) We remove two edges and one vertex, therefore one face, two edges and one vertex disappear (3(b) in Figure 2.3).
**Main Step (4):** One triangle remains at the end of this procedure (of removing triangles), for which \((P_3)\): \(V - E + F = 1\) is true.

Explanation: according to Lakatos (1976), “if \(V - E + F = 1\) before a triangle is removed, it remains so after the triangle is removed. At the end of this procedure we get a simple triangle. For this \(V - E + F = 1\) holds. Thus, we have proved our conjecture” (p. 8). We can see the above procedure illustrated with the figures below.

![Fig. 1](image1.png) ![Fig. 2](image2.png)  
![Fig. 3](image3.png)  

**Figure 2.3: Lakatos’s Proof**

Source: In Lakatos, 1976, p.8; In Lakatos, 1978, pp.94-95, Figures 1-3

**Conclusion:** Lakatos (1978, pp.94-95) puts together into the following figure the last two descriptions of the proof that is called “Euler’s conjecture” that “for every polyhedron \(V - E + F = 2\) is true”:

\[
Q_1 \quad Q_2 \quad Q_3
\]

\[
E (P) \rightarrow E' (T_P)
\]

**Figure 2.4: Lakatos’s Analysis**

where: \(E(P)\) stands for “*All polyhedra are Eulerian*” (i.e. \(V - E + F = 2\) is valid, \(P\) is a free variable taking values over a set of all polyhedra).

---

*the third lemma \(Q_3\) is not listed either in Lakatos (1976, pp.7-75) or in Lakatos (1997, pp.70-77) whereas the other lemmas are listed in either of the two Lakatos works. In Lakatos (1997, p.93) Lakatos discusses analysis-synthesis (in his reply to Hintikka & Remes, 1974).
Where \( Q_1, Q_2, Q_3 \) are all in-between steps (lemmas) and

\[ E'(Tp) \text{ stands for “All triangles (Tp) are quasi-Eulerian” (i.e. the statement } V-E+F=1 \text{ is true).} \]

The reversal of the steps of the analysis is now possible because of the ability to use the refuted lemmas. These lemmas are strong assumptions that allow us to work backwards (regression analysis). In (Lakatos, 1978, p.95) he examines all shapes from the triangle to the polyhedron and derives Euler’s theorem from the fact that a triangle has three vertices, three edges and one face. The diagram we could have is this:

\[
\begin{array}{c}
Q'_3 \quad Q'_2 \quad Q'_1 \\
E'(Tp) \quad \rightarrow \quad E(P)
\end{array}
\]

Where \( Q'_3, Q'_2, Q'_1 \) are the refuted lemmas.

**Figure 2.5: Lakatos’s Synthesis**

According to Lakatos (1978), the heuristic characteristics of the analysis could be briefly explained below:

1. The analysis provides the hidden assumptions (lemmas) needed for the synthesis. In this particular example, \( Q_1, Q_2, Q_3 \) and their refutations \( Q'_1, Q'_2, Q'_3 \).

2. In the analysis process, there is fertile ground for the introduction of creative innovations. In the particular case the creative innovation was the idea that “polyhedra are ‘really’ closed triangulated rubber surfaces” (Lakatos, 1978, p.95).

3. The analysis was performed on one specific polyhedron and therefore “the universal lemmas were only suggested but not made explicit” (Lakatos, 1978, p.95). However, finally, through the analysis-synthesis a theorem was stated, which emerged through a process of proof (proof generated theorem). The assumptions brought about by the lemmas have been incorporated into this theorem. We have an improvement of the initial conjecture. In this particular
example, “we did not prove that $V-E+F=2$ is valid for all polyhedral (though this was our initial aim) (p.95). “After a process of imaginative-critical analysis-synthesis we arrive at the proposition ‘All Cauchy polyhedra are Eulerian’ and $V-E+F=2$ is valid” (Lakatos, 1978, p. 95).

2.2.5 Related research studies applying Lakatosian method

According to Sriraman (2008), Lakatos presented in his book the so-called “generic” case, to be one of the few special instances in the history of mathematics, “which reveals the abounding world of actually doing mathematics, the world of a working mathematician, of informal mathematics, characterized by conjectures, failed proofs, thought experiment, examples, counterexamples and so forth” (p.484).

Literature is very sparse on studies that used the Lakatosian heuristic method in mathematics education. However, some researchers have implemented the Lakatosian technique as an interdisciplinary method in their fields and have made important assumptions as a result. For example, in physics, Laburú & Niaz (2002) used the Lakatosian method to analyze Secondary School students’ (grade 9) interaction (arguments, controversy, conflict) as they participated in an intact classroom activity designed to facilitate their understanding of heat energy and temperature. Laburú & Niaz (2002) stated that this method helped them to analyze situations of cognition and to categorize them into 3 models (alternative, transitory and scientific models). Actually, they said that the “methodology used in this study also provided a glimpse of how particular students grappled with the conflicts in order to facilitate progressive transition in understanding” (p.217).

According to Sriraman (2006, p.173), it is very important that teachers use “students’ insights of a typical problem to lead them to discover mathematical structures”. He also believes that the Lakatosian method helps students to create “mathematical experiences that necessitate the creation of new tools that are useful pedagogical techniques” (p.173) and help students to foster independent thinking in the classroom (Sriraman & English, 2004).

In a study that used the Lakatosian method to facilitate open classical analogy (OCA) used for conjecturing in discourse–rich mathematics classroom, Lee & Sriraman (2010) observed that 14-year-old students were ready to use symbols and conventional
expressions for proving task problems. Also, students developed “similar vocabulary showing the occurrence of spontaneous interaction between conjecturing and justifying” (p.134). This shows that OCA tasks solving, via the Lakatosian method, promotes students’ development of meta-skills for abstraction through purposeful attention, shifting to relational similarity itself (Lee & Sriraman, 2010, p.136). Another finding is that the Lakatosian method, applied through the OCA task solving process, can be regarded as a heuristic used to view a familiar object from a new perspective or deal with it in a new way, which is not offered with the traditional method. Another finding from the study was that the students’ desire for innovation was one of the main driving forces of knowledge construction through conjecturing.

Nunokawa (1996), referring to Lakatos (1976), emphasizes that thought-experiments cannot be discussed at the same level as usual experiments and observations. He distinguishes between mathematical education and natural sciences pointing out that Lakatos’ theory may not directly suggest the introduction of experiments in the mathematical classroom but that the Lakatosian method can be applied in mathematical education. This is confirmed by Furinghetti and Paola (2002) mentioning that “Nunokawa (1996) has discussed the application of Lakatos’ ideas to mathematical problem solving” (p. 398). They applied Lakatos (1976) theory to mathematical proof in their class working in small groups (2 or 3 persons) with students (17 years old with a scientific orientation) with one computer per group. By creating in their classrooms environments suitable to exploration, production of conjectures, validation of these conjectures they present conditions of students’ learning. To this purpose they propose to their students open problems. As in this study the problem proposed about the SAC was characterized by the following:

The statement of the problem is short, so that it can be easily understood, it fosters discovery and all students are able to start the solution process.

The statement of the problem does not suggest the method of solution, or the solution itself, but it creates a situation stimulating the production of conjectures.

The problem is set in a conceptual domain which students are familiar with. Thus, students are able to master the situation rather quickly and to get involved in attempt
of conjecturing, planning solution paths and counter-examples in a reasonable time (according to Arsac et al., 1988 as cited in Furinghetti & Paola 2002, p.398).

Yim, Song & Kim (2008), explored how the constructions of mathematically gifted fifth and sixth grade students using Euler’s polyhedron theorem (F+V=E+2) compare to those of mathematicians as discussed by Lakatos (1976). In the study, the students explored two types of justifications of the theorem. The solid figures suggested that counterexamples were categorized as: i) solids with curved surfaces (such as a cone), ii) solids made of multiple polyhedra sharing points, lines, or faces, iii) polyhedra with holes, and iv) polyhedra containing polyhedra. In addition to using the monster-barring method, the students suggested two new types of conjectures to resolve the conflicts between counterexamples and the theorem, the exception-baring method and the monster-adjustment method. The students’ constructions resembled those presented by mathematicians as discussed by Lakatos (1976). Students in this study due to the Lakatosian approach referred to a special case of solid to which the formula of (F+V=E+2) can applied that is a cone having a curved surface area (F=1), only one vertex (V=1) and no edges (E=0).

**2.3. SUMMARY AND APPRAISAL OF THE REVIEW**

Based on the literature reviewed here, it can be stated that the Lakatosian heuristic method is:

A) An analysis-synthesis method, according to Andrianos (2003): The historical evolution of ideas is very prominent in Lakatos views of the analysis-synthesis method.

B) A teaching and learning method, according to Oh (2010):

Tsai (2000) recommends the use of the conflict maps, which, based on the Lakatosian method, interpret the students’ alternative conceptions. These alternative conceptions and their formation are an application of the theory of constructivism, which has inspired Lakatos’ methodology as well. Tsai’s (2000) model was developed by Oh (2010) by applying the Lakatosian method in teaching and learning science subjects.
2.4 CONCLUSION

The literature review reveals some studies that applied the promising Lakatosian ‘Research Program’ theory. The efforts of the researchers to provide students with alternative concepts, especially in the science field, by adopting the Lakatosian methodology in science education were successful. The gap identified by the researcher through the readings, indicated that there is not enough research undertaken in the area of mathematics with regard to the Lakatosian methodology as a valuable teaching and learning method that opposes the traditional method. This study, therefore, aimed to close the gap in existing research and show that the Lakatosian methodology could be a useful framework that explains students’ conceptual change in the particular subject of the surface area of a cone (SAC).

In the next chapter, the paradigm that guided the researcher in formulating the study will be discussed. The research design, the sample and sampling technique as well as the instrumentation and their validity and reliability will be presented. Finally, the data analysis and the ethical factors regarding the design, collection, and reporting of the data that were taken into consideration will be discussed.
CHAPTER THREE

RESEARCH METHODOLOGY

3.1 INTRODUCTION
The study sought to interrogate the effect of the Lakatosian method in Cyprus Secondary School 11-grade students’ learning of the surface area of a cone (SAC). This chapter outlines the research methodology used to conduct the study. It focuses on techniques and procedures by presenting the research paradigm which Nieuwenhuis (2007, p.47) defines as “a set of assumptions or beliefs about fundamental aspects of reality which gives rise to a particular worldview; it addresses fundamental assumptions taken on faith such as belief about the nature of reality (ontology) the relationship between knower and known (epistemology) and assumptions about methodologies”. In order to accomplish its aim, this chapter discusses the research approach, research design, data-collection techniques and data analysis forms employed in the study.

3.2 RESEARCH PARADIGM
A positivist paradigm guided the researcher in formulating the research. According to the positivist epistemology, science is seen as the way to get at the truth to understand the world well enough so that it might be predicted and controlled. “The world and the universe are deterministic; they operate by laws of cause and effect that are discernible if we apply the unique approach of the scientific method” (Krauss, 2005, p.760). A positivism paradigm which involved a quantitative approach was employed for the measurements of data which was used to discover and confirm causes and effects.

Healy and Perry (in Krauss, 2005) point out that positivism predominates in science and assumes that science quantitatively measures independent facts about a single apprehensive reality. Therefore, the data and analysis are value-free and data do not change because they are observed. The paradigm is based on the notion that all knowledge should be based on practical experience or observations. Positivism may be characterised by its claim that science provides us with the clearest possible ideal of knowledge.
Positivism implies a particular stance concerning the social scientist as an observer of social reality. The end-product of investigations by social scientists can be formulated in terms parallel to those of natural science (Cohen, Manion, & Morrison, 2007). It was imperative for the researcher to adopt the positivism paradigm as she sought to find the effect of the Lakatosian method in Cyprus Secondary school 11-grade students learning of the SAC.

3.2.1 The research approach
The researcher sought to compare the Lakatosian heuristic method of teaching the SAC with the Euclidean deductive method (traditional method).

The researcher maximized objectivity and minimized her involvement with the respondents during the progression of the study. This is influenced by the principles of the positivism paradigm. -The researcher was aware of the fact that she was part of the world and that posed a challenge in detaching herself from the research. Hence, to eliminate bias the research study used the quantitative approach. Quantitative research methods are deductive in nature, in the sense that inferences from tests of statistical hypothesis lead to general inferences about characteristics of a population (Harwell, 2012). Cognitive test was the main instrument of data collection and statistical analysis (which is independent of the researcher) was used to answer the research questions.

3.3 RESEARCH DESIGN
This study explored the ramifications of using Lakatos’ heuristic method to teach students the Geometry topic of the SAC. To address this problem and provide answers to the research questions, a pre-test and a post-test quasi-experimental research design (Creswell, 2012) was employed. This research design entailed the use of two intact groups namely control and experimental. The particular research design was preferred because it did not require random assignment to groups. Such a requirement would not be suitable within the settings of Secondary Education, where a researcher would not be allowed the freedom to create artificial groups for the study. The design also allowed the researcher to compare the control and experimental groups in order to examine the effect of Lakatos’ heuristic method of teaching the SAC on students’ learning.
The research design is symbolically presented below:

\[ N_1: O_1 \times_1 O_2 \quad \text{Experimental Group I (Lakatosian heuristic method)} \]
\[ N_2: O_3 \times_2 O_4 \quad \text{Control Group II (Traditional method)} \]

**Figure 3.1: Pre-test post-test quasi-experimental design**

The first row represents the experimental group while the second row is the control group. \( O_1 \ O_3 \) represent pre-test; \( O_2 \ O_4 \) represent post-tests; \( X_1 \) is the Lakatosian heuristic method used to teach the experimental group \( X_2 \) is the traditional method used to teach the control group.

Both experimental and control groups’ pre-tests were examined to see if they were significantly different. It was also important to examine the pre/post tests before and after the intervention not only within the groups, but also between them.

This design was adopted for two reasons: a) according to Morrell, & Carroll (2010, p.176), “using control groups would be more appropriate in case the researcher was trying an intervention that was not part of the normal curriculum” such as in this study, where the Lakatosian method is not normally used and differs from the traditional Euclidean method used in the current curriculum; and b) this design was based on similar studies applied in science topics by Niaz (1998) and Oh (2010), who studied students’ conceptual changes (from alternative conceptions to scientific concepts) due to the use of the Lakatosian method.

### 3.4 POPULATION

The population of the study was the total number of students in the city of Limassol, Cyprus that attended the mathematics subject of Form B (Appendix A). There were ten Lyceums in the city of Limassol, according to the Cyprus Ministry of Education (2010) with an average of three classes, of optional mathematics, in Form B in each school, and an average range between 15-22 students in each class. Therefore, the population of students comprised around 450-600 students.

### 3.5 SAMPLE AND SAMPLING TECHNIQUES

The sample consisted of the experimental and the control group of students aged between 16-17 years old (in Form B) studying optional Mathematics of seven teaching
periods per week (Appendix A). Convenience sampling (Morrell, & Carroll, 2010, p.100) was used to conduct the intervention study. More specifically, the students of Form B of Secondary School (SN-L) formed the experimental group whereas students from another school (SL-P) in the same district (Limassol city), formed the control group. The sample of the study consisted of 98 students in the experimental group from four intact classes and 100 students in the control group from four intact classes.

3.6 INSTRUMENTATION

This study used main and auxiliary instruments. The main instruments were the tests (Appendix B), the Lesson observations (video-recording) and the Lesson plan (in section 3.6.1). The auxiliary instruments were the questionnaire (Appendix C), the Interview (Appendix D) and the Class lists (Appendix F). During the Lesson observation, the entire intervention was video-recorded in both groups (experimental and control). The Lesson plan played an important role in the process of the application of the Lakatosian method in the experimental group. The tests (pre-to post and delayed) were the same and they were given to both groups. The questionnaire that followed open-ended survey questions (Morrell, & Carroll, 2010, p. 108) around the SAC was given to both groups. The interview was conducted with the experimental groups. The interview was semi-structured (Papanastasiou, & Papanastasiou, 2005), was based on the students’ answers to the test questions that was conducted during the week after the intervention in the experimental group only and before the post-test. The Class lists helped in the recording of dialogues in the experimental group. An independent observer was recording the dialogues of each team, of the experimental group, for the duration of the experiment and the problem solving session.

3.6.1 Description of Instruments

Pre-test: According to Niaz (1998), the pre-test focuses on discovering the structure of students’ alternative conceptions following the criteria for classification of students’ responses as part of the Lakatosian “core belief”. The pre-test paper was administered one week before the intervention. The focus of this test was not on the general number of correct answers but on the students’ conceptual understanding of the hard core beliefs.
**Post-test:** The post-test was used to measure the students’ conceptual changes. It consisted of the same set of questions as the pre-test. It was administered two weeks after the intervention and students were asked to state the reasoning behind their selections. “Their concepts that are not consistent with the consensus of the scientific community are called *alternative conceptions*” (Mulford, & Robinson, 2002, p.739) and referred to the ideas that students had before learning. The conceptions examined after two weeks were those that followed the discrepant events and critical events of the learning process. The answers were important in examining the structure of alternative conceptions. These questions partly represented the discrepant events themselves.

In order to examine whether the students were capable of achieving higher-order skills (Higher-Order Thinking [HOT]) the students’ scores on the questions were analysed using the various levels of Bloom’s taxonomy. This taxonomy was first described as a hierarchical model for representing the cognitive domain (Bloom, 1956). The model was revisited in 2001 by Anderson and a team of cognitive psychologists. As a result, a number of significant changes were made to the terminology and structure of the taxonomy (Anderson et al., 2001). Bloom’s taxonomy has been applied to the education domain of computer science for course design and evaluation (Scott, 2003), structuring assessment (Lister et al., 2003) and comparing the cognitive difficulty level of computer science courses (Oliver et al., 2004) (In Thompson, Luxton-Reilly, Whalley, Hu, & Robbins, 2008).

**Questionnaire:** The intention of the questionnaire (Appendix C), and the interview that followed, was to put the *phenomenon of the intervention* under scrutiny from different angles, until the phenomenon has been “saturated”. According to Akerlind (2005) it is approached from different angles until the interviewer is reasonably sure that the interviewee’s conception of phenomenon has been comprehended. She suggests extended use of the open question “explain why” that follows the closed Yes/No question, is another helpful follow-up question as the interviewer tries to define the “horizons” of the intervention. In this study the *internal horizon* of the questionnaire (Part A) was the structure of the phenomenon/intervention, i.e. the parts that structure the intervention. The researcher targeted the *internal horizon* by asking the “how interesting was…” and the “how do you understand it…” questions. The
external horizon of the questionnaire (Part B) in this study indicated what the students liked or disliked and what they did not want to be repeated in such intervention.

**Interviews**: The interview (Appendix D) was conducted in the experimental group only. In order to change their alternative conceptions the students needed to be exposed to discrepant events—that is, situations where their incorrect knowledge does not work (Mulford, & Robinson, 2002, p.743). The questions of the interview had a twofold goal: First, the students were asked to clarify if they changed or not any remaining misconceptions/alternative conceptions which were presented in the pre-test and then, to explain how they responded in their questionnaire. If the answers on the questionnaires concerning the pre-test as well as the questionnaire were not identical with the correct responses in the interviews, they were considered wrong in the analysis of the pre-test. The role of the researcher was not to give the correct answer to differentiate the experimental and the control group in the post-test results. The use of interviews was auxiliary. However, the role of interviews was to clarify and to support findings from other instruments such as the questionnaire and the pre-test results.

**Lesson plan**: The Lesson Plan which was developed in the experimental group was based on the following four sections:

**Pre-existing knowledge-Aims**: To examine if the student knew the basic knowledge of Pythagoras theorem and the elements of a circle (radius, area and perimeter of a circle, area and perimeter of a sector, the relationship between radius and degrees).

**Notion of the Surface Area of a Cone-Aims**: To examine if the student knew the geometrical meaning of the Surface Area and the Volume of a Cone, especially to explore if students know that Volume is formed by rotating the Surface Area and Area is formed by rotating the Line, about the axis of symmetry.

**Perceptions of the students-Aims**: To examine if the student knew how to: i) construct a cone from 2-dim to 3-dim, ii) deconstruct/developed a cone from 3-dim to 2-dim.

**Questions-problem solving-Aims**: To examine if the student: i) knew how to solve problems related to a cone. ii) could apply their knowledge in problem solving.
Lesson Plan process: The four sections that the lesson plan was divided will be developed as follows: First of all, the teacher had to examine section A: The pre-existing knowledge about the following objectives:

(1) Pythagoras theorem
(2) Elements of a circle (radius, diameter, length of arc)
(3) Area and perimeter of a circle and the sector of a circle
(4) Area of a triangle \( A = \frac{1}{2} ab \sin C \)
(5) Transformation of radians to degrees and vice versa.

The teacher showed on the whiteboard the following table of two columns to check the pre-existing knowledge by matching the results in (A) column to that of the (B) column giving the chance to all of the students in the classroom to react as a whole group and to give the correct answer. The teacher’s role was to manipulate the students’ answers and to give reflecting thinking on all concepts about the notion regarding the area of a cone as mentioned before to cover the above objectives.
In **section B** the teacher had to check students’ knowledge about the *Notions of the surface area of a cone* constructively by checking the following objectives:

1. Rotations 360° and/or 180° of a shape (rectangle, square, triangle), about a line.
2. Generalization “if an area/line/point, turns about the line, then it will form the volume/surface area/curves of revolution.

In order to cover the first objective, the teacher used an exercise (Fig. 3.2) by asking the students to rotate the shapes (rectangle, square and triangle) about the vertical axes and then to provide their results, which was a solid shape. The digital educational
programme of the Cyprus Ministry of Education (2010) was used to show the second objective. Students should be able to explain all the elements of the solid shapes by giving their names (i.e. cone, cylinder).

![Figure 3.2: The rotation of the shapes about vertical axis](image)

In order to cover the objectives, the researcher also used the mathematical applet to show the second objective developed in the GeoGebra software. This applet indicates how the SAC is formed by the rotation of a line segment about the vertical axis according to Papanikolaou’s (1975) definition. It is considered by the researcher very important for students’ cognitive development about the topic of this study.

![Figure 3.3: A math applet in GeoGebra software about the definition of a SAC](image)

The role of the teacher in this section B was to declare the misconceptions/alternative conceptions between the two and the three dimensions of the constructive and deconstructive way students used regarding the SAC. By using the mathematical applet, as it is shown in (Figure 3.3), the lateral height (DE’) turns about the vertical axis (DZ) and students must realize the area formed—that is the SAC and they have to tell the locus of the point E—that is a circle centre Z and radius (ZE). With this activity they had to generalize the basic principle that is: if an area/line/point, turns about the
axis of the rotation, then the volume/surface area/curve of revolution was formed. Also, it is important, for the students, to observe that this activity helps clarify the misconception that the SAC is formed when the hypotenuse (DE’) of a right angle triangle (DZE’) turns about the vertical axis (DZ) forms the curved SAC while the other vertical side (ZE) forms the base circle of a cone.

In section C the teacher had to “check” the Perceptions of the students about the construction/deconstruction of a cone from 3-dim to 2-dim and vice versa by posing the problem: A cone hat is given. Find the material needed to make it if its lateral height is l. The students had to imagine the shape of the sector of the cone-hat when it is developed in 2-dim and then to measure its lateral height (l) and the in-centre angle of the sector in order to find/calculate the material needed to make it. According to Herron as cited in Mulford & Robinson (2002) “their level of understanding should be extended beyond the simple ability to use words to describe the concept” (p.734). Thus, the teacher was able to realize students’ understanding as well as their perceptions about the SAC from their explanations in their team work about how to construct/deconstruct a cone-hat.

Then, in section D the thought-experiment starts by asking the students to prove the formula of the SAC. The students had about 20-25 minutes in each team of the experimental group to prove that the SAC is \( S=\pi rl \), (\( r \) is the base radius and \( l \) is the lateral height) by giving the in-centre angle \( \theta \) (in degrees or in radians) of its sector. Students also had 5-10 minutes to show their presentations to the whole classroom. The difference in teaching the control group was that the teacher showed the proof of the formula \( S=\pi rl \) on the whiteboard as used in the traditional (up-down) teaching method, in contrast to the down-up heuristic method where the students had to discover the way to solve the problem posed and hence to prove the formula. Both methods are explained in Chapter 5.

**Lesson observation:** As explained in section 3.6.3.1 video recording was done in both groups. These students were advised by the researcher not to interrupt any process of the students’ discussion during the intervention. Also they were advised to read and to follow very carefully the instructions given in their class list. The study of the videos and the class lists helped the researcher in the analysis of the students’ way of thinking, their level of acquisition and their reactions. Also these instruments helped
the researcher to use the actual students’ dialogues that showed a presence of authentic counter-examples.

3.6.2 Development of the instruments

Lesson plan: According to Chazan (1990, p.19), there are four steps to appropriately develop the lesson plan in an intervention to apply the Lakatosian heuristic in high school mathematics. Note that the quasi-empirical Lakatosian method differs from the traditional Geometry teaching method in high schools. Lakatos’ (1976) method differs in the following:

1. Inclusion of exploration and conjecturing.
2. Presentation of demonstrative reasoning as explanatory process.
3. Treatment of proving as a social activity.
4. Emphasis on deductive proof as part of an explanatory process, rather than its end result.

The lesson plan as explained in section 3.6.1 used in this study was designed around the central idea that these four steps were the basic rules. The knowledge of the students about the notions of the surface area of a cone was checked constructively using the math applets in GeoGebra Software, prepared by the researcher, and aimed at exploring the ideas and conjectures of the students while simultaneously presenting a demonstrative way of thinking in order to comprehend and explain the construction of a cone.

Subsequently, students’ perceptions of the construction and deconstruction of a cone, from 3-dim to 2-dim and vice versa, were examined with the help of video presentations of ways to construct and deconstruct a piece of paper to make a cone.

As a result of these two activities, the emphasis was placed on the third and the fourth steps of the alternative approach in the development of the lesson plan that is, the Treatment of proving as a social activity and the Emphasis on deductive proof as part of an explanatory process. Students were given a cone-hat (hat made from paper in the shape of a cone) and asked to solve the following problem: Find the material
needed to make it, if its lateral height is l. This was a social activity, a real problem that needed an explanatory process for its resolution, with special emphasis on the deductive proof by first using the heuristic method (videos, cut and paste method), which acted as an “auxiliary hypothesis” (Lakatos, 1970), and used by teacher/students to help students to defend their core beliefs. The explanatory process led to the deductive proof not in the end, but during the teaching of the lesson, as seen in the lesson plan used in the experimental group during the intervention.

The Tests: As explained in section 3.6.1, there were several research goals that were served by the tests. The pre-test, the post-test and the delayed test (Appendix B) were the same and consisted of the following: Each one of the tests consisted of twelve questions aligned to the curriculum, following Bloom’s Taxonomy (Appendix J).

The first section: Pre-existing knowledge about the cone consisted of three questions belonging to the knowledge (K) level of Bloom’s taxonomy. The first question was open-ended and aimed to identify whether the student knew the basic knowledge of Pythagoras theorem. The second question was about matching column A to column B and its aim was to identify whether the student had the basic knowledge of the elements of a circle (radius, area and perimeter of a circle, area and perimeter of a sector, the relationship between radians and degrees). The third question was to complete sentences concerning the relationship between radius and degrees.

The second section: Notion of the surface area of a cone consisted of five questions (Appendix B, tasks: 4-8); three of them (two were multiple-choice tasks: 4 and 6) belonged to the (K) level; one (task 7 consisted of 3 parts) was an-open ended question; Part (a) and (b) belonged to the comprehensive/understanding (U) level of Bloom’s taxonomy, whereas part (c) belonged to the application level (A). Its aim was to ascertain that the student had knowledge of the geometrical meaning of the surface area and the volume of a cone, (i.e. that the student was aware that volumes are formed by rotating areas and those areas are formed by rotating lines). The last two belonged to the application level (A) of Bloom’s taxonomy. Both were open-ended questions, one concerned the shape of the previous question and the other examined the cross section of a cone.
The third section: *Perceptions of the students about the construction /deconstruction of a cone* consisted of two multiple-choice questions; one at the application level (A) of the Bloom’s taxonomy and its aim was to explore whether the student knew how to construct a cone from 2-dim to 3-dim, and the other at the (U) level and aimed to identify whether the student knew how to deconstruct a cone from 3-dim to 2-dim.

The fourth section: *Problem solving* consisted of two open-ended questions/problems (Papanikolaou, 1975, p.368) at the level of analysis-synthesis (A-S) regarding the SAC.

**Questionnaire:** The questionnaire B (Appendix C) as an auxiliary instrument, as explained in section 3.6.1, was used in both groups (experimental and control), immediately after the intervention and differed from the pilot study questionnaire A (Appendix C). This questionnaire B consisted of closed questions (Part A) and open-ended questions (Part B). Part A consisted of ten closed questions similar to those in the test aiming to double check students’ responses immediately after the intervention in both groups (experimental and control). Part B assisted in clarifying whether students found the lesson interesting and aimed to clarify the extent to which the students understood the concept of the SAC developed in the intervention by applying the Lakatosian method compared to those students in the control group.

**Interviews:** Interviews were a follow-up to the questions asked in the questionnaire B (Appendix C). It was divided into two parts (A & B). The interview followed the week after the intervention, with a focus on all the participants in the experimental group only. The completion of the questionnaire started with constructive feedback. **Part A** consisted of ten closed questions (yes/no) based on some specific tasks of the pre-test which were important for the researcher to evaluate students’ learning. Also it consisted of questions from basic teaching points such as *The arc of a circle radius l equals to lθ=2πρ, ρ= base radius of a cone and θ rad is the incentre angle of a sector of the circle radius l,* enabled the researcher to clarify students’ misunderstandings between the two spaces. **Part B** was similar to the pilot study’s questionnaire. In this part the researcher expected the students to clarify their responses when asked to compare the two methods, i.e. what they meant by “interesting lesson” and “not monotonous”, and what made the lesson interesting for them; if they would want to repeat such a lesson and ‘why’ (they
were asked to give reasons about what they liked; what made them understand this lesson better than previous lessons and what they had learned from the lesson. Then, the researcher asked the students to explain what they had written down, what they had learned and to show all the steps in constructing/deconstructing as well as proving the SAC. The students’ responses to the interview were used to corroborate their responses to the questionnaire. This way, the researcher also clarified any misunderstandings or unclear points in their answers to the questionnaire while it was also possible for the researcher to find out what made the lesson interesting for them, and whether they would like to repeat such a lesson by giving reasons for the ‘whys’ (Sriraman, 2006). Moreover, the researcher had a chance to find out what had helped the students understand this lesson better than previous lessons, and what they had learned from the lesson. The interviews were semi-structured and free flowing, to the comfort of the researcher and the students, and allowing for more give and take (Morrell & Carroll, 2010). The only difference between the interviews and the tests was that the interviewer had extra information from the questionnaire regarding the thoughts of the students about the new method applied.

**Lesson observation for both groups:** Lesson observation took place for both groups in parallel to the process of the Intervention in the experimental group and control group. Both methods’ applications will be explained in detail in sections 5.4 and 5.5.

The purpose of the Lesson observation was twofold. *Firstly,* it was used to ground the discussion in the interview. *Secondly,* it was used to explore the breadth of variation in the activities which teachers used in both groups and was used to explore students’ reactions. By examining the classroom dialogues the researcher observed the use of authentic counter-examples and positive/negative heuristic, in students’ arguments. Additionally, the degree of difficulty in following an informal, non-traditional style of mathematical conversation in the traditional Euclidean method became apparent (Bemboni, Kesari & Patronis, 2003). Because the researcher was interested in the breadth of variation, each team of the experimental group was observed by using video recording (that recorded the activities and students’ dialogues) and the main video recorder that recorded the researcher continuously in each step of teaching. In the control group a student was video-recorded during the whole process. Also, Class lists for each team in the experimental group were used by the observers who were
students of higher grade, to write down the dialogues in each team of the experimental group.

3.6.3 Validity and Reliability of the Instruments

The validity of an instrument is the extent to which the instrument measures what it proposes to measure while an instrument’s reliability is its ability to obtain the same response each time it is administered.

3.6.3.1 Validity

Table 3.1 provides a detailed strategy of how the validity of each instrument used in the study was established.

<table>
<thead>
<tr>
<th>Instruments</th>
<th>Types of validity</th>
<th>Strategy to establish validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tests</td>
<td>Content validity</td>
<td>Asked expert teachers to check if the test questions were in line with the curriculum and assessment criteria used in Cyprus (a validation rubric was used for this). Used Bloom’s taxonomy to check the distribution of the questions in terms of depth of knowledge of the questions (content complexity analysis). A Table of Specification was used for the content validity.</td>
</tr>
<tr>
<td>Questionnaire</td>
<td>Content validity</td>
<td>Used experts in psychometric testing to validate the instrument.</td>
</tr>
<tr>
<td>Interview</td>
<td>Construct validity</td>
<td>The interview was based on the questionnaire, as an auxiliary instrument, to show that the questions of the questionnaire were really measuring those constructs and not something</td>
</tr>
</tbody>
</table>

Table 3.1: Validity Framework of the study
The lesson plan was based on Lakatos’ method, as a main instrument, to show that the process of the lesson plan really applied the constructs referred to the method and not something else.

<table>
<thead>
<tr>
<th>Lesson plan</th>
<th>Construct validity</th>
<th>The lesson plan was based on Lakatos’ method, as a main instrument, to show that the process of the lesson plan really applied the constructs referred to the method and not something else.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lists</td>
<td>Communicative validity</td>
<td>Used participants’ own words by comparing the interviews based on the questionnaire of both groups (experimental and control). Used the actual students’ dialogue that showed a presence of authentic counter-examples.</td>
</tr>
<tr>
<td>Interpreive validity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lesson observation rubric</td>
<td>Construct validity</td>
<td>Used expert teachers and researchers. The study of the videos and the Class lists helped the researcher in the students’ analysis of their way of thinking, their level of acquisition and their reactions, by using: Video recording in both groups and Class lists in the experimental group only.</td>
</tr>
</tbody>
</table>

### 3.6.4 Reliability

**Reliability of the test:** The Kuder-Richardson (KR-21) formula was used to measure the reliability of the test. For measuring internal consistency Kuder-Richardson was preferred because the questions were dichotomously scored (Morrell, & Carroll, 2010). Using the data from the pre-test and post-test of the pilot study reliability values of 0.61 and 0.74 (Table 3.2) were obtained meaning that the test was reliable.
<table>
<thead>
<tr>
<th>Instrument</th>
<th>Reliability KR-21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td>0,61</td>
</tr>
<tr>
<td>Post-test</td>
<td>0,74</td>
</tr>
</tbody>
</table>

**Reliability of the questionnaire:** The Cronbach alpha (Cronbach $\alpha$) formula was used to measure the internal consistency reliability of the questionnaire. The $\alpha$ value was found to be 0,80 meaning that the questionnaire was reliable (Morrell, & Carroll, 2010).

**Reliability of the interview:** In this study the interview was about the questionnaires administered a week after the intervention, before the post-tests. The interview was based on the questionnaire, as an auxiliary instrument. This was done to determine whether the questions of the questionnaire were really measuring those constructs and not something else. According to Cohen & Manion (1997) the reliability checks the convergence of information between different methods independently measuring the same object, that is, the examined concept. The multiple methods (interview, questionnaire and tests) approach used in this study provided the convergence of information from independent measurements of the same subject.

In this study interviews were conducted by the researcher right after they completed the questionnaire. In the interview, they had to explain their feelings about the lesson and to declare what they learned in their teams, during the intervention. According to Papanastasiou et al. (2005, p.168) “if the answers of the subjects agree then the reliability of the interview may be considered as high”. First, the researcher examined whether what they said confirmed what they wrote in their questionnaires. Then, she put together all the teams’ questionnaires (cluster analysis) to clarify students’ common ideas (e.g. about the meaning of the word ‘interesting’ or the way of proving the SAC). One role of the interview was to verify all meanings of what they wrote. For example, they explained that by ‘interesting’ they meant they preferred to work in groups, to share ideas, to use math applets and to find the solution by themselves.
Reliability of the lesson observation rubric: The reports of higher grade students’ observation of the lesson using the rubric and the researcher observation were found to corroborate hence were used to establish inter-rater reliability (Creswell, 2012) of the lesson observation rubric.

3.7 DATA ANALYSIS

Descriptive statistics including means, standard deviations, frequencies and percentages were calculated to describe the characteristics of the sample. In order to examine potential differences between the experimental group and the control group, pre and post intervention, a two-way Mixed designs (combining independent and repeated measures factors) ANOVA ($2 \times 3$) was used (Baguley, 2004). The between subjects factor was the method of training in both groups (experimental and control) and the within subjects factor was the time in the three periods within time (pre-to post and delayed). The assumptions of normality, equal variances and sphericity were examined with Shapiro Wilk test, Levene’s test and Mauchly’s test. Further analysis of the interaction between the method of training and time was conducted with the Bonferroni method. The analysis was done with SPSS (version 20) and the level of significance was set at $p<0.05$.

According to Morrell and Carroll (2010, p.183) “two-way repeated measures ANOVA (factorial ANOVA) was used as there were two independent variables (factors) influencing one dependent variable” that is the conceptual learning. One factor was the method of training and the second factor was time while the Training method was the “between” factor. This is because the ANOVA measured differences between groups using different training. Time was the within factor, because each group was measured three times (pre-test to post-test and delayed test). Therefore, the analysis involved establishing the difference within each group over time.

The logic of a two-way analysis of variance is a direct extension of the rationale underling one-way ANOVA. In the one-way ANOVA the total variability of observed scores is partitioned into two parts, a between–and a within-groups calculations. In the case of the two-factor design, scores might differ from one
another because of group differences, but the group differences are more complicated because there are two classification schemes. For example, in the two-way analysis scores might differ from one another because of group differences due to factor A, factor B, and/or the interaction of the two factors (Marcoulides, & Hershberger, 1997, p.58).

3.8 ETHICAL ISSUES
Ethical factors regarding the design, collection, and reporting of the data were taken into consideration. These areas were carefully considered for all stages of the research. Students participating were told about the purpose of the research and how their input would be used. Privacy was guaranteed, and it was impressed that participants could decline to answer questions if they so wished. Students were assured that there would be no detrimental effects on their studies because of participating or non-participation in the research. Consent was obtained from the students, parents and the Education Department before the commencement of the research study.

The researcher added a note to all distributed questionnaires, stating that all replies and completed questionnaires would be treated in strict confidence and that respondents would not be named in any way in the thesis. The researcher explained that the information in the questionnaires would be classified as survey data and no part of the information would be shared with their teachers or third parties. Also, the respondents were informed that they had the right to withdraw their participation in the research at any time.

The intention of this study was to benefit education and not to cause any concerns in any way. The researcher focused on all the ethical issues relating to this study to ensure that this study was in line with the standards set by the University of South Africa Ethics Committee. The researcher had in fact obtained ethics clearance from this committee. Also, the researcher obtained written permission from the Cyprus Pedagogical Institute for working in two of the particular state schools involved. Furthermore she obtained permission from the principals of the involved schools.

In reporting the findings fictitious names for students, for schools and for teachers are used. However, for the video recording and for the scope of the study, parents and students were informed by letter (students’ consent form) (Appendix E).
3.9 SUMMARY
In this chapter the methodology concerning the data and the tools used in this study to examine the effect of Lakatosian method on the SAC were discussed. In particular, the process of the application in this study was developed under the process (Fig. 3.4), which signified that it was first necessary to identify students’ prior conceptions through the pre-test. The questionnaire, which followed immediately after the intervention, and the interviews that ensued the week after the intervention, helped the researcher to clarify students’ conceptions from alternative into scientific and they were explained according to Oh (2010) model. Therefore, the questionnaire survey and the interviews had to be refined as an auxiliary instrument.

![Figure 3.4: Research process](image)

3.10 PROJECTION
In the next chapter, the pilot study conducted in 2012 will be presented as a way to “provide hindsight before the fact” (Morrell, & Carroll, 2010, p.112).
CHAPTER FOUR

PILOT STUDY

4.1 INTRODUCTION

A pilot study was conducted, in 2012-13, in a central school (SN-L) in Limassol-Cyprus, where 56 of 448 students of this school, participated. Thirty five per cent (35%) of the students were from the urban areas, and most of the students of this school were from medium social economic status. Thirty two per cent (32%) of the 11-grade students were studying in Form B (Appendix A), and had taken Mathematics lessons (Form B) of common core of three teaching periods (45min) per week or optional Mathematics of seven teaching periods per week. The sample of this study represented forty five per cent (45%) of those students who had optional mathematics in Form B (7 teaching periods per week). This study, examines the students’ conceptual learning using cognitive test questions which measured students’ achievements according to Bloom’s taxonomy cognitive levels.

4.2 THE STUDY

The instrument for data collection in this study was a cognitive test. The test items were dichotomously scored and they measured a common factor (surface area of a cone). The students were allowed to answer the questions at their own pace; their responses were not influenced by speed.

To ensure that the test measures what it was supposed to measure we asked expert teachers to check if the test questions were in line with the curriculum and assessment criteria used in Cyprus. The teachers strongly agreed that the items conformed to the mathematics curriculum and assessment criteria of Cyprus.

Also, Bloom’s taxonomy (Bloom, 1956; Abbott, 2012) was used to check the distribution of the questions in terms of depth of knowledge of the questions and their content complexity (Webb, 2002). The first 10 of the 12 test tasks (Appendix, B) were closed questions (i.e. either multiple choice or corresponding/matching questions), while the last two problem solving tasks were open questions. So all of the tasks were marked as right or wrong; when a student got the concept correct it was
recorded as one (1). If it was wrong it was recorded as zero (0). This means that there was no partially correct answer. The only exceptions were the open-ended questions in the problem solving tasks (11 and 12). Both of these questions had two parts (a) and (b). For example, in task 11, those students who gave a correct answer only in one part of the question (a) or (b) it was considered wrong and recorded as zero. This meant that they had to understand the whole concept in order to obtain the correct answer.

Task 12: A cone-hat having surface area of a sector of a circle with in-centre angle of $60^0$ and radius $r=12$cm is to be made using a material. Find the height of this hat.

The above task was considered correct if student correctly found the radius ($R=2$cm) of the cone and correctly substituted the lateral height ($\lambda$) that is equal to the radius ($r$) and the base radius ($R$) of the cone either by using the Pythagoras theorem formula, to find the height ($h$) of the cone, or just by marking ($\lambda$ and $R$) on their figure of the right angle triangle which they drew in the cone shape. The students that left their answers in surd form were marked correct.

4.3 METHODOLOGY

The pre-test and post-test quasi experimental research design (Morrell & Carroll, 2010; Creswell, 2012) was adopted in this study. The aim was to explore the main research question: what is the impact/effect of the Lakatosian heuristic method on students’ learning of the SAC? Two intact groups (the experimental and the control groups) were used in this pilot study. Student achievement within time was the dependent variable and the method of training the independent variable. Therefore, time was the within factor and each group was measured three times (pre-to post and delayed). The differences in students’ achievement within each group over time were examined.

4.3.1 Participants of the Pilot study

Two groups of Secondary school Cyprus students (N=56) 36 students in the control group and 20 students in the experimental group participated in the Pilot study. The experimental group was taught using the Lakatosian heuristic method while the control group was taught using the traditional Euclidean Geometry method.
4.3.2 Data collection instrument

Data was collected using a cognitive test and interview. The same test was administered to both groups as a pre-test, post-test and delayed test. The pre-test was administered a week before the intervention, the post-test was administered a week after the intervention and the delayed test was administered two weeks after the intervention.

The test consisted of twelve tasks (Appendix B) aligned to the curriculum. The tasks were based on Bloom’s taxonomy cognitive levels. It examined the students’ knowledge of the cone, notion of the SAC, perceptions of the students about the cone and problem solving skills on the SAC. The test was divided into four sections:

The first section consisted of the first three questions at the knowledge (K) level of Bloom’s taxonomy. The first question was open ended and aimed to identify whether the students had the basic knowledge of the Pythagorean theorem. The second question was about matching words and its aim was to identify whether the students had the basic knowledge of the elements of a circle (radius, area and perimeter of a circle, area and perimeter of a sector, and the relationship between radians and degrees). The third question was completing sentences referring to the relationship between radians and degrees.

The second section consisted of five questions (Appendix B, tasks: 4-8). Three of them belonged to the knowledge (K) level of the Bloom’s taxonomy. Task 7 consists of 3 parts; part (a) and (b) belonged to the comprehension/understanding (U) level of Bloom’s taxonomy, whereas part (c) belonged to the application level (A). Its aim was to ascertain that the students had knowledge of the geometrical meaning of the SAC and the volume of a cone, (i.e. that the student was aware that volumes are formed by rotating areas and that areas are formed by rotating lines). Task 8 belonged to the application level (A) of Bloom’s taxonomy. Tasks 7c and 8 were open-ended questions, whereas 7c referred to the shape that was formed in tasks 7a and 7b, and task 8 examined students’ knowledge of the cross section of a cone.
The third section examined students’ perceptions of the construction of a cone. It consisted of two multiple-choice questions. One question (task 9) focused on the application level (A) of Bloom’s taxonomy. It examined students’ knowledge of the construction of a cone from 2-dim to 3-dim and task 10 was at the understanding level of Bloom’s taxonomy. Its aim was to identify whether the students knew how to deconstruct (develop) a cone from 3-dim to 2-dim.

The fourth section (tasks 11 and 12), was made up of two open-ended questions at the problem solving and Analysis-Synthesis levels of Bloom’s taxonomy. These sections of the test were based on different modes of thinking. According to Tall, & Mejia-Ramos (2010), these modes of thinking are “the three mental worlds of mathematics” such as: i) the perceptual–symbolic world; ii) conceptual embodied world; and iii) axiomatic–formal world (p.138).

4.3.3. Interviews
Interview were only conducted with the experimental group out of all the participants the only was interviewed. All of the students were asked to answer the questions of the questionnaire A (Appendix C) where their answers were incomplete or where the examiner needed to get clarity.

The researcher asked the students to clarify the meaning of their responses when asked to compare the two methods. For example, what they meant by “interesting lesson” and “not monotonous”, and what made the lesson interesting to them; if they would want to repeat such a lesson and ‘why’ (they were asked to give reasons about what they liked); what made them understand this lesson better than previous lessons and what they had learned from the lesson. Then, the researcher asked students to explain what they had written down, what they had learned and to show all the steps in constructing as well as proving the SAC.

4.4 PILOT STUDY’S FINDINGS
Descriptive statistics including means, standard deviations, and frequencies and percentages were calculated to describe the characteristics of the sample. In order to examine potential differences between the experimental and the control group, a two-
way mixed ANOVA (2×3) with repeated measurements was used within time (pre-to post and delayed). Essentially, data analysis was the same at that of the main study.

The data were analysed using two-way analysis of variance.

According to Baguley (2004, p.7) the Mixed measures ANOVA requires that multisample sphericity holds. The sphericity assumption in the pilot study was not violated since the p value of the sphericity test was p = 0.483 and the test statistic epsilon (ε) was ε = 0.973, as shown in Table 4.1. Thus, the homogeneity of variance assumption follows the null hypothesis being tested using ANOVA.

| Table 4.1: Mauchly’s Test of Sphericity (Pilot study) |
|---------------------------------|------|-------|-------|----------|---------|
| Within Subjects                | W    | Chi-Square | Approx. df | Sig. | Epsilon³ |
| TIME                           | 0.973 | 1.456    | 2          | 0.483 | 0.974    |
| Greenhouse-Geisser            |      |          |            |      | Lower-bound |
| Huynh-Feldt                   |      |          |            |      | 1         |
| Lower-bound                   |      |          |            |      | 0.5       |

4.4.1 Within time pre-to post and delayed tests

The experimental group showed a significant improvement F(1,54) = 4.116 (p<0.05) when compared to the control group in within time and group (time*group) interaction as also shown in Table 4.2.

| Table 4.2: Pilot study’s tests of within times (pre-to post and delayed test) |
|-------------------------------|------|-------|--------|--------|--------|
| Source                        | Time | Type III | Df | Mean Square | F  | Sig. |
| Time                          | Linear | 547.41 | 1    | 547,405 | 69.867 | 0.001 |
| Time                          | Quadratic | 44,362 | 1    | 44,362 | 6.761  | 0.012 |
| Time*group                    | Linear | 140,334 | 1    | 140,334 | 17.911 | 0.001 |
| Time*group                    | Quadratic | 27,005 | 1    | 27,005 | 4.116  | 0.047 |
| Error(Time)                   | Linear | 423,09 | 54   | 7,835  |        |       |
| Error(Time)                   | Quadratic | 354,33 | 54   | 6,562  |        |       |
Table 4.3 compares each trial with the adjacent trials. This table provides redundant information, since there are only two measurement times to compare. The results are identical to the within subject effects as shown in Table 4.2. The two way ANOVA revealed a significant main effect for time (pre-to post and delayed test) F(1,54)=69.86, p<0.001 (Table 4.2) and a significantly main effect between groups (experimental and control) F(1,54)=11.58, p<0.001 (Table 4.3).

The interaction between time and groups (time and group) as shown in Table 4.2 was also statistically significant F(1,54)=17.91, p<0.001 (Table 4.2). While all groups increased their scores within time (pre-to post and delayed test) the increase of means scores in the experimental group was much higher than the increase of means in the control group within all times as shown in Table 4.4 (pilot study’s descriptive statistics). Both groups had significant higher means in the post-test and delayed test than in the pre-test as shown in Table 4.4.

**Table 4.3: Pilot’s study tests of Between-Subjects Effects**

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III</th>
<th>Df</th>
<th>Mean</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sum of Square</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>39951</td>
<td>1</td>
<td>39951</td>
<td>970.186</td>
<td>0.001</td>
</tr>
<tr>
<td>Group</td>
<td>477.01</td>
<td>1</td>
<td>477.011</td>
<td>11.584</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>2223.7</td>
<td>54</td>
<td>41.179</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 4.4: Pilot’s study Descriptive Statistics**

<table>
<thead>
<tr>
<th>Group</th>
<th>Mean</th>
<th>S.D</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental group</td>
<td>13.70</td>
<td>3.97492</td>
<td>20</td>
</tr>
<tr>
<td>Control group</td>
<td>13.1111</td>
<td>3.78552</td>
<td>36</td>
</tr>
<tr>
<td>Post-test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental group</td>
<td>19.20</td>
<td>4.1877</td>
<td>20</td>
</tr>
<tr>
<td>Control group</td>
<td>14.50</td>
<td>4.75995</td>
<td>36</td>
</tr>
<tr>
<td>Delayed test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental group</td>
<td>20.65</td>
<td>3.24889</td>
<td>20</td>
</tr>
<tr>
<td>Control group</td>
<td>15.3889</td>
<td>4.99301</td>
<td>36</td>
</tr>
</tbody>
</table>
Furthermore, as shown in Table 4.4, the experimental group had higher means than the control group in the pre-intervention test, 13,70 (±3,97) versus 13,11 (±3,79), in the post-intervention test, 19,2 (±4,19) versus 14,50 (±4,76), and in the delayed test 20,65 (±3,25) versus 15,38 (±4,99).

Figure 4.1 shows the plot of the mean scores for each combination of factor level. Also, it shows how the groups performed on pre, post and delayed tests. The graph shows that the performance of the experimental group was better than the control group. Even though both groups had low performance in the pre-test the experimental group achieved higher than the control group better in the post-test and the delayed tests.

![Figure 4.1: Plot of the Pilot study’s mean](image)

**4.5 DISCUSSION OF THE FINDINGS**

Judging by the interview (as explained in 4.3.3) as well as by examining the questionnaire A (Appendix C), the researcher noticed that Lakatosian heuristic method made students to acquire more “explanatory power” (Lakatos, 1970, p.137) than did the Euclidean method. The Lakatosian heuristic method led the students to a series of “evolving models” (Niaz, 1998; Laburú, & Niaz, 2002) or “progressive transitions” (Lakatos, 1976) which lead to a greater conceptual understanding. For example, in the test’s task 7a the students were asked: When a right-angle triangle is turned 360° about the vertical line, then a 3-dimensional solid is formed. Draw this solid and name it. In the pre-test many students gave answers like it is an isosceles triangle by drawing a symmetrical triangle about its axis (vertical side) of symmetry with or without a base circle. Few students sketched a cone as a cross section of a cone (an isosceles or an
equilateral triangle) in 2-dim arguing *it is a cone* and only one student tried to draw
the locus of its hypotenuse, as a cone’s surface area (Fig. 4.2), but she couldn’t name
it. However, in the post-test, as well as the students’ interviews, the results of the
experimental group showed a *conceptual understanding* by recognising a shape of a
cone in a space as well as *sequential apprehension* (Duval, 2002) that was about how
to construct or deconstruct a shape, such as a cone. They also gave reasons to
distinguish the solid cone from the SAC.

![Figure 4.2: Student’s art of the SAC](image)

It is likely that the misunderstanding students do have concerning the definition of the
notion of a cone and/or a SAC is due to the Euclidean teaching method which leads to
a series of misconceptions. For example, the belief that a cone (a solid cone) is
constructed by a (rotated) right angle triangle. This is exactly what many students see
when a right angle triangle is rotated on one of its vertical sides. This way of
imagining a cone gives rise to misunderstandings about the construction of a cone in
3-dim, as well as about the development of a cone on surface level (from 3-dim to 2-
dim). Also, the responses of students in the experimental group included reasons such
as *the angle of the sector in task 9a (Appendix B) was greater than that in the task 9b
or the task 9a is the correct answer because it depends on its base radius*. Students in
the experimental group improved better than the control group on task 7c: *if the
hypotenuse of the above shape is turned 360° over the vertical line, what is the
difference between the new shape and the previous solid shape?* The experimental
group increased from 14% in the pre-test to 57% in the post-test while the control
group increased from 24% in the pre-test to 50% in the post-test (Appendix H).
Students' responses in task 7c in the post-test contradicted their answers in the pre-test
especially in the experimental group. This contradiction may have occurred in relation
to the students’ learning of the definition of the SAC. The results, as shown in
Appendix H, improved in similar ways in the post-test of both groups but the
The experimental group showed greater improvement. It is likely that the Lakatosian method led the experimental group students not only to better perceptions of physical objects but also better visuo–spatial reasoning using internal conceptions built from external conceptions (Tall & Mejia-Ramos, 2010).

When comparing the results of the experimental group (24% in the pre-test and 91% in the post-test) and the control group (18% in the pre-test and 32% in the post-test) on task 9a (Appendix H), a significant difference exists.

Tasks 9 and 10 are related to the perceptual thinking, which “refers to the recognition of a shape in space” (Panaoura, 2012, p.4), of the students as a development of the “sequential apprehension, which is required whenever one wants to construct a Figure or describe its construction” (Panaoura, 2012, p.4). Task 9 concerns the construction of the cone from 2D (2-dim) to 3D (3-dim) while task 10 relates to deconstruction from 3-dim to 2-dim. According to Duval (2002), in mathematical tasks there is a relationship between the way an idea is “conceived” and the “perceptual, discursive and operative apprehension” (p.13). For many students, there is an “inhibition of operative apprehension and the lack of an interrelation between the perceptual and the discursive apprehension” (p.13), resulting in the non-understanding of the concept studied.

The lack of interrelation between the two apprehensions (discursive and perceptual) may be the main reason why most of the students in the control group gave contradictory answers to task 9 and 10. For example, many of them on task 9: A cone-shaped tall hat is asked to be made for the junior school carnival show. Circle only one of the following shapes that is the proper one to be used for the model of the hat, answered ‘true’ to Figure 4.3 which is the correct answer while they answered that when a cone is deconstructed in 2-dim is a triangle or a circle (task 10, Appendix B). The interplay between perceptual and discursive apprehension, due to the Lakatosian heuristic method of teaching, may have led the experimental group students to the operative apprehensions, “that we can get an insight to a problem solution when looking at a figure” (Panaoura, 2012, p.4), that made them double their achievements compared to the control group, in the post-test. Tasks 9 and 10 are related to the perceptual thinking of the students as a development of the sequential apprehension,
which is required whenever one wants to construct a figure or describe its construction.

![Figure 4.3: Test task 9](image)

The different answers of the pre-test by the same students to tasks 9 and 10 may be also due to their strong belief about the SAC which may have spawned from their confusion in manipulating between the two dimensions (from 2-dim to 3-dim and vice versa). The statistically significant difference between the post-test of both groups may be due to the positive influence of the Lakatosian heuristic method which may have encouraged the students to use mathematical discourse to led to discovery of important mathematical concepts.

According to Truxaw (2005) the “inductive teaching/learning cycle model” emphasise the cyclic nature of the discourse (i.e. recursively establishing common understanding, exploring, conjecturing, testing and revising hypothesis) and it is used to progressively build new meaning. According to Truxaw, & De Franco (2007), participating in a mathematical community through discourse is an important step for learning mathematics and for conceptual understanding. Students in the experimental group in agreement with Wachira and Pourdavood (2013) noted that mathematical communication is necessary for ideas to become objects of reflection, refinement and amendment. This teaching and learning process is not promoted within the context of the traditional teaching of geometry in Cyprus.

4.6 CONCLUSION OF PILOT STUDY’S FINDINGS

Results obtained from this study show that the performance of the experimental group was generally better than those of the control group. Based on these results, it is safe
to conclude that the Lakatosian heuristic method could help students’ learning of the SAC.

In addition, from the post-test to delayed-test the results test showed that even short periods of appropriate experiences could facilitate students’ learning of the SAC. So, the Lakatosian method may help students sustain learning over a longer period than the Euclidian method,

The following points were improved in the main study:

1. Both classes (experimental and control) were tested in the same topic, in two different schools, to avoid the risk of content validity interacting with other classes. The threat was limited by using two different schools that were far from each other and their students had no interaction or common activities.

2. Class lists of recorded dialogue, as a new element in the main study, played an important role in interpreting the data used in each group of the experimental group during the intervention.

3. The time allocated to complete the questionnaire in both groups (experimental and control) was rightly calculated to 25-30 minutes, compared to the pilot study that was spending a whole period of 45 minutes.

4. Also, due to time constraints on the part of the participating schools, the post-test and the delayed test, in the pilot study, were administered one week and two weeks respectively after the interventions, instead of at least 2 weeks and 4 weeks apart after the intervention, as recommended by Niaz (1998). This recommendation was followed in the main study.

In the next chapter, the findings of the main study will be presented. The research questions’ analysis will highlight the impact of the Lakatosian method in teaching the SAC compared to the Euclidean method. Throughout the analysis of data (pre-, to post and delayed test), students’ authentic dialogues in their attempt to prove the SAC will aid us to realize better the difference between the Lakatosian heuristic method and the Euclidian method.
CHAPTER FIVE

FINDINGS

5.1 INTRODUCTION
This chapter presents the results based on the data analysis. The data was analysed using two-way mixed analysis of variance. In this study, student achievement within time was the dependent variable and method of training the independent variable. The study examined the effects of the methods of training (the Lakatosian as a heuristic method and the Euclidean Geometry as a traditional method) on students’ learning of SAC. Therefore, time is the within factor and each group was measured three times (pre-, to post and delayed test). The differences within each group over time were examined. The three central research questions of this study referred to in chapter 1 were examined.

The chapter is organized into three sections. Section 1 reports the findings within times pre-to post-tests which examined research question 1, especially the sub-questions 1(i) and 1(ii) as well as the findings of students’ achievements (higher order thinking skills) which examined sub-question 1(iii). Section 2 reports the findings within times post-and delayed-tests which examined research question 2 and Section 3 reports the findings of students’ achievement (from their point of readiness) according to Bloom’s taxonomy levels (cognitive learning) which examined research question 3.

5.1 RESULTS
Descriptive statistics including means, standard deviations, frequencies and percentages were calculated to describe the characteristics of this sample. In order to examine potential differences between the experimental and the control group, a two-way mixed ANOVA (2×3) was used within time pre-to post and delayed. Also, the schools’ demographics, how both methods applied in the experimental and the control groups, as well as the thought-experiment analysis in order to show authentic students’ dialogues during the intervention based on the three stages of Lakatosian method are presented. Then the findings of the three research questions will be presented in order
to answer the main question of this study: *what is the impact of Lakatos (1976) heuristic method on students’ learning of the surface area of a cone (SAC)?*

### 5.2 PARTICIPANT’ DEMOGRAPHICS

The schools’ and students’ demographics for the two groups were similar. For example, 55-60% were students from low class families with a low level of educational attainment, 10-20% of whom received a school and mess allowance. Out of these students, 20-25% were foreign language speakers who were either immigrants or refugees from Russia, Romania, Bulgaria, Pakistan, Syria, Iran or Moldavia. Around 30-35% of the students were Cypriots from families who came as refugees from the “north” part of Cyprus as a result of the Turkish invasion in 1974. Their parents were of an average level of educational attainment. Only about 3-5% of the students were of high socioeconomic backgrounds having parents with higher education qualifications (tertiary education). The difference between the two schools was that in school SN-L (experimental group) there was a higher percentage (25%) of immigrant students or political refugees as compared to the SL-P school (control group). This is because the SN-L is close to the town where immigrants and political refugees live.

### 5.3 THE EUCLIDEAN METHOD

The Euclidean method was applied traditionally. This means the classroom was arranged in a manner, where the tables were arranged in a line. This suggests that students were unable to collaborate with each other. After drawing a right angle triangle on the blackboard, which was rotated on its vertical side, the teacher showed the students the elements of the cone on the blackboard (lateral side, height, vertex, radius of a base circle) and asked students to name them. She verbally defined the solid cone and the curved surface area of a cone, by writing them on the blackboard. Then, she took a cone hat, she opened it and by asking specific questions such as *what do you think the curved area of the cone is in the two dimensional space?* She showed students that the curved surface area of the cone was equal to the sector of a circle, after showing them its expansion. Afterwards, she took the marker and started writing the following proof on the whiteboard according to the textbook by Argyropoulos et al., (2010):
Teacher: Consider a vertical segment KO to a plane circle and the side KA to be the hypotenuse (Figure 5.1) of a right angle triangle KOA with a right angle at O. (NB: the teacher drew first the side KO to be vertical on a plane circle and then she drew the sides KA and OA of the right angle triangle KOA). The side KA is rotated around its vertical side KO to form the SAC. The hypotenuse KA of the right triangle upon rotation intersects a fixed point K and subtends a convex, whereas the horizontal side OA (of the triangle) subscribes a circular disk with center O and radius OA that is the locus of the side OA (Figure 5.1), which belongs in a perpendicular plane to KO at point O. The convex produced by the hypotenuse KA is called the lateral side of a cone that formed the curved surface of the cone (SAC). The vertical side KO remains constant during the rotation and is called the axis or height. Point K (NB: is called) the peak, the circle is the locus of the horizontal side OA of a right angle triangle KOA, it is called the base of a cone, having base radius OA, that is called the radius of the cone.

![Figure 5.1: Teacher’s drawing in the control group](image)

(NB: she continues to explain the process to prove the SAC by drawing the following sketch (Figure 5.2a), that it was similar to what the textbook used (Figure 5.2b), on the whiteboard).
(NB: when the teacher explained how to measure the arc of the circle she was showing the drawing similar to that of the textbook (Figure 5.2b) to explain that it was the same with the circumference of a base circle of a cone. Then she continued to explain to the students how to find the in centre angle $\phi$ of the sector).

Teacher: *If we call $\phi$ the in-centre angle of the sector in degrees, we have the relationship* \[
\frac{360}{2\pi \lambda} = \frac{\phi}{2\pi \rho} \iff \phi = \frac{\rho}{\lambda} 360^\circ
\]

*Therefore, the developed curved surface of a cone with side of length $\lambda$ and radius $\rho$ is a circular section of radius $\lambda$ and length of arc $2\pi \rho$ or, in degrees, $\phi = \frac{\rho}{\lambda} 360^\circ$. From the above we consider that the SAC equals to $S=\pi \rho \lambda$.***
(NB: she also referred to the 2\textsuperscript{nd} approach by using the limits of the area of a pyramid to prove the SAC).

Teacher: \textit{we can also prove the SAC by using the limits of a pyramid such as:}
\[
S = \lim_{n \to \infty} \left( P_{\text{base}} \frac{h_{\text{slant}}}{2} \right) = \lim_{n \to \infty} \left( 2\pi \rho \frac{\lambda}{2} \right) = \pi \rho \lambda \sim \text{what Argyropoulos et al., (2010) suggested in their textbook.} \text{ (NB: Then she gave some exercises to the students from the textbook of the proper subject).}
\]

\textbf{5.4 THE LAKATOSIAN METHOD}

Lakatos heuristic method was applied in the experimental group by using a lesson plan that consisted of 4 sections. In section A, the teacher checked the students’ \textit{Pre-existing knowledge of the concept of a cone} through questioning.

In section B the teacher checked students’ knowledge about the \textit{Notions of the surface area of a cone constructively} by checking: i) if students knew how to rotate a shape (rectangle, square, triangle) about a line and if they could name the resulting solids; ii) if students knew the generalization of the premises that if an area or a line or a point, turns about the line, then it will form the volume or a curved surface area or the curve/line, respectively. In order to cover the above objectives, the teacher used an exercise (in section 3.6.1: Figure 3.2).

Also in order to cover the objectives, the teacher also used the mathematical applets to show the second objective developed in the GeoGebra software as well as the digital educational programme of the Cyprus Ministry of Education (2010).

In section C: the teacher had to check the \textit{Perceptions of the students} about the construction/deconstruction of a cone from 3-dim to 2-dim and vice versa. Finally, in section D the teacher examined the \textit{Problem Solving} skills by posing a problem: \textit{A cone hat is given. Find the material needed to make it if its lateral height is }l. \textit{The thought-experiment} started by asking the students who were working in small groups, first to solve the problem and then to prove the formula (\(S = \pi \rho l\), \(\rho = \text{base radius}, l = \text{lateral height}\)) of the cone. Students had 25-30 minutes to prove the formula of the SAC and to present their solutions to the whole classroom.

In the next section, the thought-experiment analysis applied in the SAC is discussed.
5.5 THOUGHT-EXPERIMENT

For the development of the thought-experiment, the researcher posed the problem: *A cone hat model is given. Find the material needed to make it and then if its lateral height is \( \lambda \), prove the formula for the SAC that is \( S = \pi \rho \lambda \), \( \rho \) = base radius of a cone hat.*

The tendency was to encourage students in the experimental group during the thought-experiment, to come up with different plausible arguments about the problem posed. At the beginning, students worked individually and then collectively in their groups to solve the problem. To do the necessary measurements, each team had its own cone hat model, so as to identify the data (e.g. the measurement of the in-centre angle of the sector and its radius). Then, they had to measure the area of the given sector that is the SAC. The students’ conceptualization was *first stage* of the Lakatosian method. That is the stage 1: *Primitive conjecture* (Lakatos, 1976, p.127).

This stage required the use of student pre-existing knowledge (e.g. the area of a sector and the elements of a circle) as well as by making “naïve” conjectures (Lakatos, 1976) about the SAC (e.g. that when a cone developed in 2-dim its surface area is the area of a sector of a circle radius \( \lambda \)). Students observed that the radius \( r \) of the sector was the required cone’s lateral height \( \lambda \), established their conjecture “as a reasonable hypothesis about a general mathematics relation based on an incomplete evidence” (Stylianides, 2010, p.41). In contrast, the control group students were not able to see this hypothesis, so as to move to the next stage (see Chapter 6: Discussion of Finding).

By developing the cone hat in 2-dim, students were able to establish their conjecture \( \lambda = r \), so as to make a new conjecture about the relation between the length of the arc of a sector and the perimeter of the base of a cone. When making a conjecture, students formulated a hypothesis about a generalization whose domain extends beyond the domain of cases that the solver checked, whereas it is possible for the generalization to cover only the *examined case* (Stylianides, 2010). Such a case in this study was the process to prove the SAC and to make generalization, to enable students to handle high level problem solving tasks about the cone.

The students had about 20-25 minutes in each team of the experimental group to prove the SAC’s formula, while they also had 5-10 minutes to show their presentations to the whole classroom.
Stage 2: Proof (a rough thought-experiment or arguments, decomposing the ‘primitive conjecture’ into sub conjectures or lemmas) (Lakatos, 1976, p.127). In this section students decomposed the concept of the primitive conjecture that was to prove the surface area of the cone hat into sub-conjectures. This process helped them to find the relations between the two dimensions (a plane and a space) in order to realize how these relations are used in constructing/deconstructing the cone (solid or ‘funnel’ shape). They dealt with counter examples to give answers to the question: what is the lateral height of the cone in relation to the radius of its sector when the cone is deconstructed in 2-dim? They tried to justify their answers by using the proof and refutation (Lakatos, 1976) method in accordance with what Lakatos did in his utopian class. For example, in their attempt to prove the SAC, the students’ argumentation in the experimental group, was as follows, when they were asked:

How was the cone constructed?

By using the List as well as the video recording tapes, the following dialogue developed in the experimental group.

S1: It’s a circle!

S2: No! It’s a sector because the circle cannot make a cone hat.

Researcher: Bravo this is correct answer. How do you find its area?

S2: We have to divide the sector in triangles [primitive conjecture], however, what will be the base of the triangles?

After some hard thinking.

S1: The smallest the triangles the closest the height will reach the lateral height.

S2: Do you mean that the height of the triangle will be the lateral height?

(NB: she wrote down that the height of a triangle equals the lateral height of a cone (h=λ)).

S1: Yes! So the area of the sector will be the area of the SAC.

(NB: she found a counter example to alter S2’s process of thinking).
So, the area of the sector will be 
\[ E_{\text{sector}} = \frac{\pi r^2 \theta}{360} = \frac{\pi \lambda^2 \theta}{360} = \frac{\pi \lambda^2 \frac{180}{\pi} \mu^\epsilon}{360} = \frac{\lambda^2}{2} \mu^\epsilon \]

After some very hard thinking.

S2: *But, the radius* (r) *of the sector equals the lateral side* (λ) *of a cone* (r=λ). *What about the in-centre angle* μ *of a sector?*

[S3 was working silently by herself in S2’s primitive idea, in their team]

S3: *Look! If we are adding the bases of the triangles, they are equal to the length of the arc of the sector. So,* \( r \mu^\epsilon = \lambda \mu. \)

(NB: she realized after some hard conceptualization that the radius of a sector (r) equals to the lateral height (λ) of a cone).

S2: *So my idea becomes easy* \( \frac{b_1 + b_2 + \ldots + b_N}{2} \lambda = \frac{\lambda \mu^\epsilon}{2} \lambda. \)

S3: *Yes! This is exactly the same as S2’s idea. However, it’s obvious that the length of the arc of a sector equals to the base circle circumference having radius* ρ.

(NB: and she wrote down the statement \( \lambda \mu^\epsilon = 2\pi \rho \)).

S2: *HMm! What is the radius* ρ?

S1: *It is the radius of the base of a cone, while the circumference of the circle is* 2πρ.

S3: *Yes, by connecting the two edges [meant radius] of the sector a cone in 3-dim a cone hat is formed.*

Researcher: *Very well! Excellent! You have proved the formula.*

(NB: they were feeling satisfied enough since the following day they told me that this process made them feel proud of themselves and at the same time they wanted to convince themselves that they were capable of proving it as their grades in school were less than 15/20 and this made them feel low profile in the class).

In this way, students were led to the proof ‘in the idea of improving by proving’ (Lakatos, 1976, p.138) through a different conceptualization. It helped them to
“disembark from the monotonous traditional teaching method which was more concerned with convincing the mind than with enlightenment” (Barbin, 2010, p.237).

**Stage 3:** *Global counter examples (counter examples of the primitive conjecture)* emerge (Lakatos, 1976, p.127). In this section by using the Lakatosian method students were able to observe ‘guilty lemma’ (Lakatos, 1976, p.145). This referred to about the relation connecting/ between the two spaces. That is how students could “see” that the circumference ($2\pi\rho$) of a circle of a cone base in 3-dim equals to the length of the arc ($\lambda\mu\xi$) of the sector (having radius $\lambda$ the lateral height of a cone and an in-centre angle $\mu$) of a cone when it is developed in 2-dim. So that the relation ($\lambda\mu\xi=2\pi\rho$) did not remain hidden and they successfully proved the formula of the SAC. Moreover, the experimental group students proved what was requested (the formula of the SAC) with more than one approach which they fully understood, as also shown by their answers (see the following section findings) in the questionnaires which was given immediately after the interventions in both groups (experimental and control). When comparing the two groups, the students of the control group gave one word answers. For example, they said that: *we learnt the formula $\pi\rho\lambda$ or we learnt the formula of the SAC*, they were unable to give a full and clear answer of the procedure to prove the formula. Thus, by applying the Lakatosian method in the classroom, a teacher might enable students’ interaction, to generate discussion, demonstration and arguments that can lead to definition, axiomatization and proof (Confrey, & Costa, 1996, p.163).

Students in the experimental group took into consideration all of the parameters referred to in a *product as a problem* (Mousoulides et al., 2007) which is characterized by all of the ‘components’ (argumentation, experimentations, justification) (Umland, & Sriraman, 2014) of the heuristic methods. In contrast to the traditional methods, experimental group students were engaged by the researcher, who created and managed opportunities for them to “do proofs” (Herbst & Chazan, 2003, p.4), being able to prove the SAC by using more than one heuristic. During the thought-experiment students in the experimental group discovered five different heuristic approaches to prove the SAC. The findings will be discussed in detail in the next sections.
5.6 ANALYSIS AND DISCUSSION OF STUDENTS’ APPROACH OF SAC’s PROOF

The experimental group students’ five approaches to prove the SAC are by: i) development of the pyramid in 2-dim, ii) using the limits of the pyramid, in 3-dim, having a regular polygon base inscribed in a circle, iii) using cut and paste method making a parallelogram iv) considering a sector as a development of a curved surface area of a cone (SAC) in 2-dim, and v) considering a sector as a part of a circle.

We think that many of the students who chose the first two approaches were inspired by the math applets (Figure 5.3) shown in the intervention during the thought-experiment. One of the applets showed initially the base of a triangular pyramid (Figure 5.3a), as a polygon (regular) inscribed in a circle having a constant height. It was divided into several n-sided polygons and as n increased \( n \to \infty \), then they (polygons) tended to the circumference of a circle. So the limit of these pyramids was a cone having the same base and the same vertex (D). The other math applet (Figure 5.3b) showed the locus of a hypotenuse (DE’) of a right angle triangle (DZE’) turned about one of its vertical sides (DZ) so as to form the SAC.

![Figure 5.3a](image1.png) ![Figure 5.3b](image2.png)

**Figure 5.3: Math applets created the SAC**

Both of these math applets helped students to realize how the SAC is created enabling them to ‘build’ (De Villiers, 2010) the definition of the SAC. The visualization of the math applets also helped them to solve their misconceptions about the definition used in their textbooks. Such as when a right angle triangle (a plane triangle) turned about one of its vertical sides it forms a cone (a solid cone). Both applets had been shown before the students started to work in their teams about the problem posed in the thought-experiment. So, they had time to distinguish well enough, the difference
between the solid cone’s as well as the SAC’s definition during their implementation in the whole classroom.

Those students who were inspired by the first math applets used in the intervention were led to two different ways to prove the required SAC. These were by using: i) the pyramid as a solid (3-dim) and, ii) the development of the pyramid in (2-dim).

The first students, having visualizing perception, were inspired by visualization process to prove SAC. They perceived a solid in three dimension (space) and they proved the SAC by the use of limits of the n-sided pyramids when its sides (n) tends to the infinite \( n \to \infty \), as it is shown in Figure 5.4a. On the other hand, students who had a visual perception, imagined the development of the solid pyramid in 2-dim having been inspired by an icon representation of a cone. The visual process of the development of a cone in a plane helped them to prove the SAC heuristically, as a sum of the isosceles triangles of the n-sided pyramid by adding them (Figure 5.4b).

According to Duval (2002, p. 317) “visualization perception differs from a visual perception” so these students were inspired from the same math applets, used to teach them the creation of the SAC, by proving the same formula in two different ways.
The visualization process of the math applets helped students to realize how the SAC is created by i) the locus of the edges of the pyramid (Figure 5.3a) as \( n \to \infty \), or ii) the locus of the hypotenuse of a right angle triangle when it is rotated about one of its vertical sides (Figure 5.3b).

In the third case, only one student from the experimental group discovered that by using the cut and paste method, he could make a parallelogram. She was cutting the sector of a cone, when it is developed in 2-dim, in several equal smaller sectors. By putting them one next to the other upside down (Figure 5.5), she transformed it (sector) in a new shape of a rectangle (or a parallelogram). Thus, she considered the base of a parallelogram as a half perimeter of the circle (considering a perimeter of a circle equals to the arc of a sector) and the height \((v)\) of a parallelogram to be the radius of a sector which equals the lateral height \((\lambda)\) of a cone, that is \(v=\lambda\).

![Figure 5.5: Proving the SAC by making a parallelogram](image)

In the fourth case, many students inspired from the deconstruction of a cone-hat in 2-dim found out that a sector is a development of a curved surface area of a cone in 2-dim. Therefore, students in each group managed to prove by themselves and by the guidance of their teacher, *what the relation was between the elements of a cone hat in 3-dim and its development in 2-dim.* By using a cone hat model, the students discovered the mathematical relations between the lateral height of a cone and its radius when it is developed in 2-dim proving the formula of the SAC (Fig. 5.6). Moving from the 2-dim space to the 3-dim and vice versa, requires a great ability which students should have acquired at an earlier stage.
Figure 5.6: Proving the SAC by using a sector

The fifth case depends on students’ deductive reasoning. This approach was used by two students in the experimental group. They considered a sector as a part of a circle highlighting the historical fact that the initial driving of forces of mathematical knowledge are plausible conjectures and heuristic thinking and that logical arguments and deductive reasoning come into play later (Liu, 2009). This approach is considered to be difficult. While the students that used the previous four approaches used inductive reasoning to prove and justify their steps these two students like ‘Epsilon’ student of Lakatos (1976) used deductive reasoning. According to Lakatos student (Epsilon) was probably “the first-even Euclidean to appreciate the heuristic value of the proof-procedure” (Lakatos, 1976, p.106). Both students purely had a mathematical reasoning just like Epsilon providing absolute certainty (p.120) recalling that the mathematical logic proved the formula of the SAC (Figure 5.7). They proved that the SAC is a sector, which was equal to (the area of a circle (O, ρ=λ))*(part of a circle (having an in center angle θ)) 

\[ S_{\text{sector}} = \pi\lambda^2 \cdot \frac{\theta}{360} = \pi\lambda^2 \cdot \frac{2\pi\rho}{2\pi\lambda} = \pi\rho\lambda. \]
5.7 ANSWERING OF RESEARCH QUESTIONS

The analysis of the data to address the research questions is divided into three sections representing each of the three questions. The first research question consisted of three parts answering to the main question of section 1, which was: what is the impact of Lakatos (1976) heuristic method on students’ learning of the SAC?

The sub-questions were:

i. What is the impact of using the heuristic method to teach the SAC on students’ achievement?

ii. What is the impact of using heuristic method to teach the SAC on students’ conceptual change?

iii. What is the impact using heuristic method to teach the SAC on students’ attainment of higher order thinking?

To address the research questions, both quantitative and qualitative data analysis techniques were employed to analyse the data.
5.7.1 RESEARCH QUESTION ONE

**Section 1: The findings within times pre-to post tests**

**Analysis within times pre-to post tests:** Two-way repeated measures ANOVA must meet the assumption of sphericity which requires that the repeated measures demonstrate homogeneity of variance (i.e. each group of data has similar variance) and homogeneity of covariance (i.e. the correlation of each repeated measure with the dependent variable is similar) (Harwell, 2012). In this case, there was only a pre-to a post or/and delayed-test, so sphericity is irrelevant as shown in Table 5.1.

<table>
<thead>
<tr>
<th>Table 5.1: Mauchly’s Test of Sphericity (pre-to post test)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within Subjects Effect</td>
</tr>
<tr>
<td>-------------------------</td>
</tr>
<tr>
<td>Time</td>
</tr>
</tbody>
</table>

In the case of pre-to post and delayed-test, the test analysis principle is fairly simple. The null hypothesis is that sphericity holds with a test value of epsilon (e=1). The results show that the p value of the Mauchly’s test of sphericity was (p=0.042) and the actual test statistic was 0.968 (Table 5.2). Thus, it is likely that sphericity is intact or that any violation is very minor (Baguley, 2004).

<table>
<thead>
<tr>
<th>Table 5.2: Mauchly's Test of Sphericity (post-to delayed test)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within Subjects Effect</td>
</tr>
<tr>
<td>------------------------</td>
</tr>
<tr>
<td>factor1</td>
</tr>
</tbody>
</table>

**Analysis within times pre-to post-test and post-to delayed test:** Table 5.3 compares each trial with the adjacent trials. With a pre-to post-test design, the table provides
redundant information. All students in both classes took the pre-to post-test. The two-way repeated measures ANOVA results indicate a significant difference (p<0.01).

Table 5.3: Tests of Within-Subjects contrasts (pre-to post-tests)

<table>
<thead>
<tr>
<th>Source</th>
<th>TYPE</th>
<th>III Df</th>
<th>Mean Sum of Squares</th>
<th>Mean F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>Linear</td>
<td>1</td>
<td>1619,757</td>
<td>119,421</td>
<td>0,001</td>
</tr>
<tr>
<td>Time*Group</td>
<td>Linear</td>
<td>1</td>
<td>212,484</td>
<td>15,666</td>
<td>0,001</td>
</tr>
<tr>
<td>Error(Time)</td>
<td>Linear</td>
<td>196</td>
<td>2658,425</td>
<td>13,563</td>
<td></td>
</tr>
</tbody>
</table>

Since there are only two measurement times to compare, the results are identical to the within-subject effects shown in Table 5.3. The two-way repeated measures ANOVA revealed a significant main effect within time (pre-to post-test) F(1,196)=119,42, p<0,001 (Table 5.3) and a non-significant main effect between groups (experimental and control) F(1,196)= 0,367 as shown in Table 5.4.

Table 5.4: Tests of Between-Subjects effects (pre-to post-tests)

<table>
<thead>
<tr>
<th>Source</th>
<th>Type</th>
<th>III Df</th>
<th>Mean Sum of Squares</th>
<th>Mean F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td></td>
<td>1</td>
<td>77237,45</td>
<td>2057,222</td>
<td>0,001</td>
</tr>
<tr>
<td>Group</td>
<td></td>
<td>1</td>
<td>13,777</td>
<td>0,367</td>
<td>0,545</td>
</tr>
<tr>
<td>Error</td>
<td></td>
<td>196</td>
<td>7358,728</td>
<td>37,545</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.5 shows the means, standard deviation and group size for the two groups. Both groups had significantly higher means in the post-tests than in the pre-test as shown in Table 5.5. Post-test mean score of the experimental group (mean group
The experimental group had lower mean score than the control group in the pre-test 11.39 (±5,12) versus 12.49 (±3,91) but higher mean score in the post-test than the control group 16.91 (±5,93) versus 15.07 (±5,076).

The interaction within group and time (pre-and post-test) had significant effect F(1,196)=15.66, p<0.01 (Table 5.3). Both groups (experimental and control) scored higher in the post-test than the pre-test. The increase in the mean score (Table 5.5) was much higher in the experimental group than in the control group (mean difference=16.90-11.40=5.50 versus 15.07-12.49=2.58).

Figures 5.8a & 5.8b show the plots of the mean scores for each combination of factors level and how those groups performed on pre-and post-test and on post-test and delayed-test. In both graphs the performance of the experimental group was better than the control group. From Figure 5.8a it can be seen that even though both groups had low achievement on the pre-test (Figure 5.8a) they showed increases in scores in the post-test. However, the gradient of the experimental group (Figure 5.8a) increased at a higher rate than the gradient of the control group. In the post-test to delayed test the experimental group gradient increases rapidly while the control group showed a decrease with a very slow rate of change (Figure 5.8b). The post-test to delayed-test results are discussed in the research question 2 (section 5.7.2) of this chapter.

Table 5.5: Descriptive Statistics (pre-to post-tests)

<table>
<thead>
<tr>
<th>Group</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental group</td>
<td>11.398</td>
<td>5.12069</td>
<td>98</td>
</tr>
<tr>
<td>Control group</td>
<td>12.49</td>
<td>3.90673</td>
<td>100</td>
</tr>
<tr>
<td>Post-test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental group</td>
<td>16.9082</td>
<td>5.93102</td>
<td>98</td>
</tr>
<tr>
<td>Control group</td>
<td>15.07</td>
<td>5.0757</td>
<td>100</td>
</tr>
</tbody>
</table>

The interaction within group and time (pre-and post-test) had significant effect F(1,196)=15.66, p<0.01 (Table 5.3). Both groups (experimental and control) scored higher in the post-test than the pre-test. The increase in the mean score (Table 5.5) was much higher in the experimental group than in the control group (mean difference=16.90-11.40=5.50 versus 15.07-12.49=2.58).

Figures 5.8a & 5.8b show the plots of the mean scores for each combination of factors level and how those groups performed on pre-and post-test and on post-test and delayed-test. In both graphs the performance of the experimental group was better than the control group. From Figure 5.8a it can be seen that even though both groups had low achievement on the pre-test (Figure 5.8a) they showed increases in scores in the post-test. However, the gradient of the experimental group (Figure 5.8a) increased at a higher rate than the gradient of the control group. In the post-test to delayed test the experimental group gradient increases rapidly while the control group showed a decrease with a very slow rate of change (Figure 5.8b). The post-test to delayed-test results are discussed in the research question 2 (section 5.7.2) of this chapter.
Therefore, using the heuristic method to teach the SAC led to improved students’ achievement. These results encouraged the researcher to conclude that the Lakatosian method is open to commensurability. Thus, the students’ alternative conceptions have commensurability with scientific concepts. Despite “the low commensurability between the core concepts of the scientific concept and students’ alternative conceptions” (Oh, 2010, p. 1157) the Lakatosian method works positively compared to the Euclidean method. Therefore, it helps experimental students’ protective belt which supports their concept of the SAC to extent their core change alternative concepts into scientific. How this method helps students to improve their achievement as well as their conceptual changes will be discussed in chapter 6 (section 6.3).

**5.7.1a The impact of using a heuristic method to teach the SAC on students’ conceptual change**

To address the research question 1(ii), a detailed analysis of the students’ answers to the tests questions as well as in the interviews and the questionnaire that followed the intervention in both groups was performed.

In order to explain the structure of students’ alternative concepts about the constructing/deconstructing of the cone as well as the creation of the SAC, the enhanced conflict map was used. The Jun-Young Oh’s model of the enhanced conflict maps was used to explain the changes that were noted as regards to the students’ conceptual development.
Here, we shall present the Webb’s (2002) cognitive analysis (Appendix K) of the task’s test as well as the enhanced conflict map’s analysis about the SAC according to Jun-Young Oh’s model, based on the Lakatosian theory.

A. Enhanced conflict map’s analysis about the SAC according to Jun-Young Oh’s (2010) model based on the Lakatosian theory

This study focused on the Lakatosian heuristic theory and its application to the Euclidean Geometry topic of the SAC. The theoretical framework developed by Lakatos (1976) and the Oh (2010) model of the enhanced conflict maps were used to explain the cognitive changes that were noted as regards the students’ conceptual development.

Figure 5.9: Cognitive conflict map about the SAC based on Jun-Young Oh’s (2010) enhanced conflict map

On the basis of the students’ answers to the test questions, their “main viewpoints” (Lakatos, 1970, hard core) about the SAC were examined, and the Lakatos (1976) “criteria” which were also applied by Niaz (1998) were applied in order to ascertain the students’ alternative conceptions. Moreover, Oh’s (2010) model (section 2.1.2) was used to interpret the students’ conceptions (alternative or scientific). It was used
to explain the observed changes in the students’ conceptual development. The cognitive conflict map (Fig. 5.9) based on Oh’s (2010) enhanced conflict map (Fig. 2.1 in section 2.1.2) showed/explained the structure of students’ alternative concepts about the SAC. It first showed the structure of students’ alternative conceptions about the concept of the SAC suggested by students’ before learning (intervention) as well as the discrepant events “allowing students to relinquish the core concepts, overcome cognitive conflicts with scientific concepts, and learn new concepts” (Oh, 2010, p.1148).

B. Analysis of the students’ answers

The analyses of the students’ answers to the test questions together with the results from the questionnaires (of both groups) and the experimental group interviews were used to compare the two groups. How the Jun-Young Oh’s model was used to analyze the students’ answers according to the Lakatosian method will be explained.

From the analysis of students’ answers to the first section of the pre-test questions: *Pre-existing knowledge about the cone* (Appendix J: Bloom’s taxonomy analysis; Appendix K: Norman Webb’s cognitive analysis) it was evident that the students in both groups knew the basics of the Pythagorean theorem (task 1: 75.5% and 65%) and the elements of a circle (radius, area and perimeter of a circle, area and perimeter of a sector, the relationship between radians and degrees) (task 2: 68.125% and 71.625%). More than a half of students in both groups (experimental and control) (52.04% and 59% respectively) in the pre-test were aware of the SAC’s formula (i.e. $\pi \rho \lambda$, $\rho$ is the base radius of a cone and $\lambda$ is the lateral height). These results were increased in the post-test in both groups up to the (84.69% and 84%) and remained high in the delayed test (85.71% and 83% respectively). Task 3 referred to the relations between the radians and the degrees. Half of the students remembered how to translate the degrees in radians and vice versa. As shown in the pre-test results both groups translated it easier from radians to degrees (task 3b: 56.12% and 54% respectively). From the pre-test it was obvious that students had a difficulty in how to use the general transformation from the degrees into the radians and vice versa in both groups (task 3c: 18% and 11% respectively). This difficulty remained in the post-test while about one in three students could give correct answer in both groups (38% and 26%). The researcher’s contribution/explanation during the lesson plan which reminded them
how to use the proportionality \( \frac{\theta^0}{180} = \frac{c^\circ}{\pi} \) might have helped to increase the delayed test results (57%) in the experimental group compared to the control (24%).

In the second section: *Notion about the construction/deconstruction of the SAC tasks* 4 (cylinder) and 5 (cone) referred to the solid’s definition. Students’ results in the pre-test on tasks 4 and 5 were 53.06% and 58.16% versus 70% and 69% in the experimental group and control group respectively (Appendix I). Most of the students knew how to draw the solid cone in the related task 7a as shown in the pre-test results which were 65.30% and 63% respectively in experimental and control group. However, the students showed weakness in a related task (7c Appendix I): *If the hypotenuse on the above shape is turned 360\(^\circ\) over the vertical line, what is the difference between the new shape and the previous solid shape?* The experimental and control groups scores 10.2% and 26%. The weakness was also shown in their responses in the interview meaning that they were working mechanically in the pre-test. However, they were able to give the general definition of the *Surface area* but they couldn’t draw the graphical representation of the shape formed. The pre-test results (44.89% and 65% respectively for the experimental and control groups) of task 6: *If a line segment AB turns 360\(^\circ\) over a line (e)//AB, then the shape formed, will be……*, (Appendix A) which was reversed in the post-test (84.69% and 53% for the experimental and control groups respectively) were surprising. The reason the experimental group supported their weakness, in the pre-test, during the interviews was that they didn’t realize that the line segment (AB) was *rotated* about the line (e)//AB but they translated it, so they considered the “infinite lines” as the answer (Appendix B: task 6a).

In task 2 (questionnaire B - Appendix C) that is similar to task 6, the experimental group and the control group scored 99.87% 87.30% respectively. The control group’s high score compared to their post-test score (53%) might be due to the fact that as a closed question (having only ‘yes or no’ options) it had only one true answer (the cylinder) which seemed very logical compared to the multiple choice of the post-test (task 6). This task presupposed them to think critically about the correct answer to task 7. It might be that the low percentages in the post-test results in the control group
were due to the traditional method, compared to those of the experimental group, who were not able to realize the connection between the tasks 6 and 7.

Students’ interviews showed that they were unable to “see” the aim of task 6 which was a pre-required knowledge to help them on how to *create graphically* the SAC (in task 7c). The inability, to “see” the main role of task 6, was established in the pre-test task 7c. Despite both groups being aware of the SAC’s formula as mentioned in the first section, their pre-test results of task 7c (Appendix I, 10,2% and 26%) showed that it did not contribute/affect positively to the graphical representation of the SAC. Owing to the Lakatosian method the post-test results of task 7c increased in the experimental group compared to the control group (Appendix I, task 7c: 73,47% and 49%) respectively and remained at a high level in the delayed test results of the experimental group (task 7c: 82,65%) versus (task 7c: 23%) in the control group.

Therefore, the increased percentage results in task 7c (Appendix I, i.e. SAC creation) in the experimental group within time (pre-post and delayed test) (task 7c: 10,2%, 73,46% and 82,65%) compared to the control group results (task 7c: 26%, 49% and 23%) signified the effect of the Lakatosian method in students’ learning. During the interview the experimental group students were excited with a new method. They supported their decision reasonably in test task 7c such as *it is a surface area of a cone or it is an open cone curved area or the lateral height generates the surface area of a cone* referring to the definition of the SAC in contrast to the control group meaningless answers in task 7c such as *the new shape is a cone without base or it is a cone*. However, their post-test conceptual learning about the definition of the SAC as they clarified in the interviews as well as their answers in the multiple choice task 1: A(3) results (experimental group (53%) and control group (25%)) of questionnaire B (Appendix C: Part B) was due to the use of the experimentation and the argumentation in the experimental group. In the interviews they supported that working in small teams (4 or 5 students) as well as the use of the videos contributed positively to understand the lesson. Some of the experimental group ideas (students’ answers in questionnaire B: Part B, task B2) were based on their conceptualization of the positive effect of the math applets on the correct use of the graphical representation of the SAC (task 7c) such as:
the visualization in 3-dim space by the use of the computers’ software helped me to realize better that the surface area of a cone created from its lateral height or the experimentation with the cone-hat helped to realize that the surface area of a cone differs from the solid cone or the discussion about the hypotenuse of a right angle triangle which formed a cone was fruitful, especially the use of the computer.

Even though the experimental group students understood the graphical representation of the SAC as well as its difference from the solid cone compared to the control group (as shown in the post and delayed test results in task 7c), almost half (100-53=47%) of the experimental group had a difficulty in recognizing the definition of the SAC compared to those (100-25=75%) of the control group. This was confirmed from the questionnaire B results (Appendix C: questionnaire B (Part B: task 1 A(3)) given immediately after the intervention. However, the experimental group “built” the definition by using their own words as compared to the control group but it seemed more difficult for them (the experimental group) to recognize the definition of the SAC compared to its formula. It was easier for both groups to remember the formula as was mentioned in the pre-test results in the first section instead of defining it.

The pre-test results in the last task 8 of this section in both groups (experimental and control) was (Appendix I, task 8: 21.428% and 21%) respectively. In task 8 students had to recognize the cross section of the cone and then to describe it. Both tasks 7c and 8 were characterized as a complex reasoning of Norman Webb’s taxonomy. Both test tasks 8 and 7b referred on the cross section were the predetermined knowledge for the test task 11 in the fourth section: The problem solving. For many students’ misconception in the pre-test task 8 was that the cross section was a line through the vertex of a cone vertical to its base instead of a plane. In the interviews they supported/clarified that they realized the notion of the cross section by the use of the math applet (Figure 5.3a) where the isosceles triangle was rotated 180° about its height so as to also form the SAC. These task 8 as well as task 7c and 9a were characterized as the hard core of students’ beliefs as explained in section 2.1.1 according to Lakatos (1976) criteria.

The third section: Students’ Perceptions about the construction/deconstruction of a cone consisted of task 9 (construction) and task 10 (deconstruction). The pre-test results of
the true answer (Appendix B: task 9a) in both groups (experimental and control) were (Appendix I, task 9a) 22.45% and 24% respectively. This task was characterized as a students’ core belief (negative heuristic). However, in questionnaire B (Part B: Task B(3)) a percentage of 85% of the experimental group compared to 65.85% of the control group replied (yes) which meant that they understood the lesson according to how the cone was constructed from 2-dim to the 3-dim. However, the post-test results (Appendix I, task 9a) 55.1% and 40% respectively increased slightly in both experimental and control groups. This means that the perception of how the cone was constructed was a hard belief. A strong belief was that the cone was constructed in 3-dim from a right angle triangle in 2-dim. The similar low pre-test results of this belief in task 9a with task 7c might be supported by the control group confusion between how to construct a cone rather than how to create it.

An experimental group student in the questionnaire B (Appendix C: Part B: task B(3)) wrote the following answer; it is interesting that the cone in 2-dim is a shape of a sector having only one radius while this sector in 3-dim transformed a cone which has two radius, one is the base radius and the other is the radius of a sector which becomes the lateral height of a cone! This explanation might be a key for many students to realize the connection between the two spaces. However, in the interviews they explained by using the piece of paper of what they did in their experimentation (cone hat) in the experimental group, in their attempt to show to the researcher that they understood well the hidden relation (The arc of a circle (O, λ) centre O and radius λ equals to λθ=2πρ, ρ= base radius of a cone formed by this sector) which was connecting the two spaces. This also was confirmed in the questionnaire B (Appendix C: Part A: task 7). The questionnaire results (questionnaire B-Part A: (task 7)) in both groups achieved the high rate of the (96.87% and 88%) correct answers (λθ=2πρ) in both groups respectively. The superiority of the experimental group in the delayed-test was obvious, as shown in Appendix I, on how to construct a cone (task 9a: 65.3%) compared to the control group (task 9a: 43%). According to Lakatos (1976) the conceptual understanding needs time to change students’ hard core beliefs.

Task 10 (deconstruction) pre-test students’ analysis in both groups (experimental and control) showed higher percentage results (task 10d: 32.65% and 33%) respectively, than in the task 9a (construction) (task 9a: 22.45% and 24%) respectively. The results
were almost doubled in the post-test analysis of the task 10d (61.22% and 56%) respectively in both groups and remained high in the delayed test (task 10d: 72% and 64.29%) respectively with a superiority in the experimental group (Appendix I). Students in the interview emphasized that the cone hat model contributed positively on students’ perception of how the cone was developed in 2-dim as well as the argumentation and the experimentation in their team in their attempt to prove the SAC.

Task 10 predetermined the problem solving task 12 in order to find the height of the cone. It is important to say that the tasks (11 and 12) in the problem solving section were not referred to in their textbooks.

Section D: Problem solving refers to tasks 11 (cross section side) and 12 (height of the cone) of the analysis-synthesis (A-S) level of Bloom’s taxonomy regarding the SAC. This is level 4 (extended reasoning) of Webb (2002) which requires complex reasoning. Students had to know all the previous levels of cognitive thinking about the concepts of the SAC. In order to acquire the problem solving skills of the target task, in this level 4, they had to have an “extended period of time to apply significant conceptual understanding and higher order thinking” (Webb, 2002, p.4). In order to solve task 12, students had to realize how the SAC is related (construct/deconstruct) in both dimensions and know how to prove the SAC, by observing first, the relations that are connected to the SAC in 3-dim when it is developed in 2-dim and vice versa. The post-test results in Appendix I, of task 12 (30.6%) of the experimental group compared to the control group (16%) increased with in the experimental group showing greater increase in score while only one student in each group (who are distinguished in the Cyprus mathematics society) were able to solve it in the pre-test. The positive effect of the Lakatosian method in the post-test results compared to the traditional method signified the difficulty of students’ conceptual learning in a small period of 2-4 weeks. However, students in the experimental group were able to sustain their knowledge as shown in the delayed test of the problem solving tasks. In task 12 the experimental group was able to achieve the complex reasoning up to (40.81%) compared to the control group (19%), as shown in Appendix I.

5.9 CONCLUSION

However, the students in both groups had the basic knowledge as shown in the pre-test (first section) to solve the test questions and they knew the general definition of how
the solid was formed as well as the surface area (second section). Their perceptions about how to construct/deconstruct a cone as well as the creation of the SAC (in section 3) were very strong which resulted in having difficulty solving tasks in section four. Due to the Lakatosian method the results in the post-test as well as in the delayed test were increased by changing students’ alternative conceptions between the difference of the construction/deconstruction of a cone and the creation of the SAC which was the key to help them solve the problem more easily.

It is very important to mention what a student of the experimental group wrote in the questionnaire B (Appendix C: Part B: task 3) when asked if the lesson was interesting:

> it is much easier to remember a formula, if you know where the maths formula is originated, and also you make yourself more able to invent formulas and solve problems in different ways. For example, when a problem cannot be solved using the traditional ways you become more able to discover new formulas to solve it.

This student was the one who proved the SAC by making the parallelogram method (in section 1.2.2.3).

C. An explanation of how the Oh’s (2010) model changed students’ alternative to scientific concept about the SAC

Discrepant perceptions (P₁, P₂): The responses which represented the discrepant perceptions (P₁, P₂) were determined by the students’ responses in the pre-tests. For example, their responses to tasks 9 and 10 were their discrepant perceptions which became discrepant events in step 2: (the naïve scientific concept). This is because at this level, following the influence of the intervention, the students in the experimental group reflected on their modifications about their initial perceptions.

Before, as well as during the intervention, students were had conflict (due to the lack of experience) between their alternative conceptions and their naïve scientific concepts. In this study, the perceptions including discrepant events arose from the different answers that the students gave to two questions testing the same concept. For example, tasks 9 and 10 that were on constructing/deconstructing the cone respectively. In this study, there were two types of discrepant events about the construction/deconstruction of the cone.
Type 1: Those that contradicted each other i.e. those that had two different answers to related tasks where at least one was wrong (or both were wrong).

Type 2: Those that had the same but wrong answers to two related tasks.

For example, the responses of students to tasks 9 and 10 (related tasks) of the pre-test were the discrepant events ($P_1$ and $P_2$) of type 1. This was because answers were both wrong and contradicted each other. This can be explained as follows:

Firstly, students gave the wrong answer in task 9c (Appendix B) that is:

$P_1$: the cone hat was constructed (from 2-dim to 3-dim) of the right angle triangle and the wrong answer in task 10 (Appendix B) that is:

$P_2$: the cone when it is developed (from 3-dim to 2-dim) will be an isosceles triangle (task 10b) or a circle (Appendix B, task 10c). The students did not realize that both tasks 9 and 10 examined the same concept about the construction/deconstruction of the cone. They ought not to have different answers to task 9 which was on the construction of the cone from 2-dim to 3-dim and to task 10 which was on the deconstruction of the cone from the 3-dim to the 2-dim. Thus, the different responses were the students’ discrepant perception in the pre-test that contradicted each other. Many students did not resolve their contradiction, which continued to exist as discrepant events even after the intervention, especially in the control group.

Hard core propositions in students’ alternative conceptions: According to Lakatos (1976) (as mentioned in section 2.1.1) the students’ hard core (core belief) propositions concerned the wrong students’ belief. These were perceptions which scraped the theory. For example, the hard core belief (task 9) that a right angle triangle constructs a cone instead of creating a solid cone or the surface area of a cone when it is rotated about one of its vertical sides, altered the definition of the cone and its surface area. In this first stage students were in the alternative conception level of the Jun-Young Oh’s model.

According to Oh (2010, p.1152), the ratios of the incorrect to the correct responses in the pre-test tasks are considered to be the hard core. The students’ explanation characterizing the hard core responses were definitely discrepant. It concerned the definition of the SAC, the cross section of the cone as well as the construction of the
cone from 2-dim to 3-dim. For example, students answered that the cross section of a cone was *a line vertical to the base* or *a right angle triangle*” or just *a line*. These explanations were considered to be hard core for obstinate, in other words they obstructed the students’ conceptual change. As shown in the analysis of students’ answers the hard core tasks were tasks 7c, 8 and 9a. The answers in these pre-test tasks also influenced the success ratio in tasks 11 and 12 which were directly connected (task 8 to task 11 and task 9 to task 12). According to Lakatos’s (1970) theory, the hard core is the *negative heuristic*, as the results from the pre-tests showed tasks 7c, 8 and 9 were so. Thus, the *hard core* tasks were more difficult to change.

The students’ core belief seemed to be task 9a, of the construction of the cone, confirming that students were obstinate to change their core beliefs easily which was due to the teaching method concerning how they were taught the definition of the SAC even though they had learnt the definition by heart.

According to the Lakatosian sense, students resist changes in their major theoretical framework by accepting auxiliary hypotheses. For example, in this study the auxiliary hypothesis, about the SAC in task 7c was that *the cone has no base*, used by students especially in the control group. During the intervention the “responses to discrepant events become critical clues that enable us to approach scientific concepts” (Oh, 2010, p.1157). Critical events in this study concerned the construction /deconstruction of the SAC due to the teaching method; of how the definition of the cone as well as the SAC contributed positively or negatively to students’ understanding of the new concept. The use of the math applets contributed positively in the experimental group to distinguish the difference between the definition of the solid cone and the definition of the SAC. So, the students were able to move to the *second stage* of the model (naïve/relative scientific concept). They tried to find *relevant scientific concepts* supporting alternative conceptions. In this study such *relevant scientific concepts* might be the students’ answers that *the construction of a SAC from 2-dim to 3-dim is a circle* (Figure 4.3, task 9d). This might be considered as “a soft core of the alternative conceptions for less obstinate conceptual change” (Oh, 2010, p.1153). According to Niaz (1998) the *relevant scientific concepts* considered to be a response that was relatively less obstinate of correct answers which were characterized as a soft core (positive heuristic). In this study, however, “the circle” is not the correct answer. The students who considered this task as true were those who were able to change their
core beliefs (negative heuristic). Furthermore, these students were less obstinate in the post-test. What is more, these students had a higher success rate in the relevant scientific concept supporting naïve scientific concepts. During the intervention, students in the experimental groups were constructing their relevant scientific concepts observing that \( R=\lambda \) or \( 2\pi R_0 \) by supporting their naïve scientific concept \( A \) sector forms a SAC. Students tried to find other supporting perceptions (sometimes wrong) derived from their environment (e.g. the use of their hands to form a cone and then they visualized its development by opening them) or relative scientific concepts such as “a sector forms a cone” instead of the correct answer that a sector constructed the SAC. Those who finally supported their target scientific concepts achieved the third stage (scientific concepts) of the concept mapping were able to conclude that the sector with the smaller in-centre angle/length of arc or the smaller area formed/constructed the tallest cone hat.

The difference between the alternative perceptions and the relevant scientific concepts is that the first perceptions are influenced by the environment and intuition as an “advanced organizer” (Umland, & Sriraman, 2014, p.17). This often affects the formulation of the students’ opinions. In the case of relevant scientific concepts, they already started to think about supporting naïve scientific concepts. During this process a personal belief about the truth of an idea is formed and acts as “a guide for more formal analytical methods of establishing the truth” (Umland, & Sriraman, 2014, p.17). The scientific concept level indicated students’ understanding, whereas the role of the protective belt propositions (all of the tasks in section A and B) represented the “dispensable part (soft core/positive heuristic)” (Oh, 2010, p.1153). According to the Lakatosian framework, “Students’ understanding of some questions would represent the soft core (positive heuristic) of their framework which offers relatively less resistance to conceptual change” (Oh, 2010, p.1153). The experimental group students were able to “construct the definition” of the SAC and distinguish the difference between the terms construct/deconstruct and create a cone/SAC.

5.7.1b The impact of using the heuristic method to teach the SAC on students’ attainment of higher order thinking

This section answers the question “what is the impact of teaching the SAC using the heuristic method on students’ higher order thinking?” The results of the tasks were
analysed on the basis of the category from which they were chosen in accordance with Bloom’s taxonomy within times pre-to post and delayed tests, the results of which are presented (Appendix I).

The results of a detailed analysis of the four cognitive levels (knowledge, understanding, application, analysis-synthesis) of Bloom’s taxonomy (Appendix J) within time (pre-to post and delayed test) as well as between groups are presented. The purpose of this analysis is to show whether students attained higher order thinking as a result of being taught using the Lakatosian method, as compared to the Euclidean method.

A. Analyzing Bloom’s taxonomy levels within time (pre-to post and delayed test)

We are interested in finding out whether the students were able to achieve higher order thinking by examining the pre-to post and delayed test of Bloom’s taxonomy levels.

Given that the analysis of the results in the pilot study was conducted in two phases (within pre-to post-test or post-to delayed test) it was deemed expedient in the main study, to check the three phases at the same time (pre-post and delayed test at all Bloom’s taxonomy levels mentioned above). This was because it was considered that there would be a better/direct comparison of the three times (pre-to post and delayed test) at all levels.

Initially, on the basis of Table 5.6 of the knowledge level and Table 5.9 of the understanding level as shown below, the interaction between groups and time was not statistically significant as well as between groups. However, the interaction within time (pre-to post and delayed test) in both levels (the knowledge and the understanding) was statistically significant.
Table 5.6: Tests of within-subjects contrasts (pre-to post and delayed test) of knowledge level

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III</th>
<th>Df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sum of Squares</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time Linear</td>
<td>379,132</td>
<td>1</td>
<td>379,132</td>
<td>44,377</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Time and Group Linear</td>
<td>55,243</td>
<td>1</td>
<td>55,243</td>
<td>6,466</td>
<td>0.012</td>
<td>0.032</td>
</tr>
<tr>
<td>Error (factor1)</td>
<td>1674,5</td>
<td>196</td>
<td>8,543</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As shown in the Table 5.6, the interaction within groups and time was not statistically significant, $F(1,196)=6.466$, $p=0.012$ (Table 5.6) as well as between groups $F(1,196)=0.537$, $p=0.465$ (Table 5.7).

Table 5.7: Tests of Between-Subjects Effects of Knowledge Level

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sum of Squares</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>71121,1</td>
<td>1</td>
<td>71121,1</td>
<td>2393,82</td>
<td>0.001</td>
<td>0.924</td>
</tr>
<tr>
<td>Group</td>
<td>15,951</td>
<td>1</td>
<td>15,951</td>
<td>0.537</td>
<td>0.465</td>
<td>0.003</td>
</tr>
<tr>
<td>Error</td>
<td>5823,22</td>
<td>196</td>
<td>29,71</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

However as shown in Table 5.6, the effect on the knowledge level was significant within time $F(1,196)=44.38$ ($p<0.001$). While all groups increased their scores within time (pre-to post and delayed test) (Table 5.6), the increase of means scores in the experimental group was higher than the increase of means in the control group within all times as shown in Table 5.8 (Descriptive Statistics).
Table 5.8: Descriptive Statistics of knowledge level (Experimental group:1, Control group:2)

<table>
<thead>
<tr>
<th>GROUP</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td>1</td>
<td>9,1224</td>
<td>4,18703</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>10,30</td>
<td>3,15108</td>
</tr>
<tr>
<td>Post-test</td>
<td>1</td>
<td>11,3878</td>
<td>4,00935</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>11,51</td>
<td>4,05392</td>
</tr>
<tr>
<td>Delayed test</td>
<td>1</td>
<td>11,8265</td>
<td>4,4906</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>11,51</td>
<td>3,87036</td>
</tr>
</tbody>
</table>

Furthermore, as shown in Table 5.8, the experimental group had lower means than the control group in the pre-test at the knowledge level, 9,12 (±4,19) versus 10,30 (±3,15), and almost the same means in the post-test 11,38 (±4,01) versus 11,51 (±4,054), only in the delayed test it was a bit higher 11,82 (±4,49) versus 11,51 (±3,87).

Table 5.9: Tests of Within-Subjects Contrasts (pre-to post and delayed test) of Understanding level

<table>
<thead>
<tr>
<th>Source</th>
<th>factor1</th>
<th>Type III</th>
<th>Df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>Linear</td>
<td>29,546</td>
<td>1</td>
<td>29,546</td>
<td>45,667</td>
<td>0,001</td>
<td>0,189</td>
</tr>
<tr>
<td>Time and group</td>
<td>Linear</td>
<td>0,738</td>
<td>1</td>
<td>0,738</td>
<td>1,14</td>
<td>0,287</td>
<td>0,006</td>
</tr>
<tr>
<td>Error (factor1)</td>
<td>Linear</td>
<td>126,808</td>
<td>196</td>
<td>0,647</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Subsequently, as shown in the Table 5.9 of the understanding level, the interaction between groups and time was not statistically significant F(1,196)=1,14, p=0,287 (Table 5.9) as well as between groups F(1,196)=0,001, p=0,976 (Table 5.10). While
all groups increased their scores within time (pre-to post and delayed test) (Table 5.9) the increase of means scores in the experimental group was higher than the increase of means in the control group within all times as shown in Table 5.11 (Descriptive Statistics) the effect in the understanding level is significant F(1,196)=44,38 (p<0,001) (Table 5.6).

Table 5.10: Tests of Between-Subjects effects of understanding level

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III</th>
<th>Df</th>
<th>Sum of Squares</th>
<th>Mean</th>
<th>F</th>
<th>Sig.</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td></td>
<td>1</td>
<td>1970,75</td>
<td>1970,75</td>
<td>1268,88</td>
<td>0,000</td>
<td>0,866</td>
</tr>
<tr>
<td>Group</td>
<td></td>
<td>1</td>
<td>0,001</td>
<td>0,001</td>
<td>0,001</td>
<td>0,976</td>
<td>0,001</td>
</tr>
<tr>
<td>Error</td>
<td></td>
<td>196</td>
<td>304,416</td>
<td></td>
<td>1,553</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Furthermore, as shown in Table 5.11, the same is observed, as with the results of the knowledge level. The experimental group had also lower means than the control group in the pre-test of the understanding level, 1,44(±0,85) versus 1,54 (±0,87), and almost the same means in the post-test, 1,96 (±1,23) versus 1,92 (±0,97), only in the delayed test it was a bit higher 2,07 (±1,067) versus 2.00(±0,829).

Table 5.11: Descriptive Statistics of understanding level

(Experimental group:1, Control group:2)

<table>
<thead>
<tr>
<th>Group</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td>1</td>
<td>1,4388</td>
<td>0,8503</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1,54</td>
<td>0,86946</td>
</tr>
<tr>
<td>Post-test</td>
<td>1</td>
<td>1,9592</td>
<td>1,23454</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1,92</td>
<td>0,9711</td>
</tr>
<tr>
<td>Delayed test</td>
<td>1</td>
<td>2,0714</td>
<td>1,06732</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2,00</td>
<td>0,82878</td>
</tr>
</tbody>
</table>
While all groups increased their scores within time (pre-to post and delayed test) (Table 5.9) the increase of means scores of the experimental group was higher than the increase of means of the control group within all times as shown in Table 5.11 (Descriptive Statistics) the effect in the Understanding level is statistically significant within times F(1,196)=45,667 (p<0,001) (Table 5.9).

Table 5.12: Tests of Within-Subjects contrasts (pre-to post and delayed test) of application level

<table>
<thead>
<tr>
<th>Source</th>
<th>factor1</th>
<th>Type III</th>
<th>Df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>Linear</td>
<td>Linear</td>
<td>1</td>
<td>43,12</td>
<td>77,622</td>
<td>0,001</td>
<td>0,284</td>
</tr>
<tr>
<td></td>
<td>Time and Group</td>
<td>Linear</td>
<td>1</td>
<td>11,443</td>
<td>20,599</td>
<td>0,001</td>
<td>0,095</td>
</tr>
<tr>
<td></td>
<td>Error</td>
<td>Linear</td>
<td>196</td>
<td>0,556</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Given the analysis of the results of the pre-to post and delayed tests in the remaining two levels-application and the analysis-synthesis levels, a significant main effect is observed in both levels within times, between groups as well as between groups and time. As shown in Table 5.12 above the significant effect in the application level within time (pre-post and delayed test) F(1,196)=77,62 (p<0,001) was observed whereas a significant effect in between groups (for either experimental or control) F(1,196)=23,403 (p<0,001) (Table 5.13) was also observed.
Table 5.13: Tests of Between-Subjects effects of application level

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>Type III</td>
<td>df</td>
<td>Mean Square</td>
<td>F</td>
<td>Sig.</td>
<td>Partial Eta Squared</td>
</tr>
<tr>
<td>Intercept</td>
<td>Type III</td>
<td>df</td>
<td>Mean Square</td>
<td>F</td>
<td>Sig.</td>
<td>Partial Eta Squared</td>
</tr>
</tbody>
</table>

Also, a significant effect in the analysis-synthesis level within times (pre-to post and delayed test) $F(1, 196) = 22.66$ (p<0.001) (Table 5.14) was observed.

Table 5.14: Tests of Within-Subjects contrasts (pre-to post and delayed test) of analysis-synthesis level

<table>
<thead>
<tr>
<th>Source</th>
<th>factor1</th>
<th>Type III</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>Linear</td>
<td>99,8</td>
<td>1</td>
<td>99,8</td>
<td>139,883</td>
<td>0.001</td>
<td>0.416</td>
</tr>
<tr>
<td>Time and Group</td>
<td>Linear</td>
<td>16,163</td>
<td>1</td>
<td>16,163</td>
<td>22,655</td>
<td>0.001</td>
<td>0.104</td>
</tr>
<tr>
<td>Error (factor1)</td>
<td>Linear</td>
<td>139,837</td>
<td>196</td>
<td>0.713</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A significant effect in between groups (for either experimental or control) $F(1, 196)=40.083$ (p<0.001) (Table 5.15) was also observed. In addition, there is a statistically significant main effect within Time and Group $F(1, 196) = 20.599$ (p<0.001) (Table 5.12) in the application level and $F(1, 196)=22.66$ (p<0.001) (Table 5.14) in the analysis-synthesis level respectively.
Table 5.15: Tests of Between-Subjects effects of analysis-synthesis level

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III</th>
<th>df</th>
<th>Mean of Sum of Squares</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>339,171</td>
<td>1</td>
<td>339,171</td>
<td>1</td>
<td>234,618</td>
<td>0,001</td>
<td>0,545</td>
<td></td>
</tr>
<tr>
<td>Group</td>
<td>57,945</td>
<td>1</td>
<td>57,945</td>
<td>1</td>
<td>40,083</td>
<td>0,001</td>
<td>0,17</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>283,343</td>
<td>196</td>
<td>1,446</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Furthermore, as shown in Table 5.16, the experimental group and the control group had the same average means in the pre-test, 0.56 (±0.77) versus 0.50 (±0.72), and in the post-intervention test, 1.47 (±1.19) versus 0.86 (±0.84), and in the delayed test 1.56 (±1.075) versus 0.82 (±0.8). The predominance of the experimental group as compared to the control group in the application level was apparent within all times.

Table 5.16: Descriptive Statistics within time (pre-to post and delayed test) of application level

(Experimental group: 1, Control group: 2)

<table>
<thead>
<tr>
<th>GROUP</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre test</td>
<td>1</td>
<td>0.5612</td>
<td>0.77415</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.50</td>
<td>0.71774</td>
</tr>
<tr>
<td>Post test</td>
<td>1</td>
<td>1.4796</td>
<td>1.19474</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.86</td>
<td>0.84112</td>
</tr>
<tr>
<td>Delayed test</td>
<td>1</td>
<td>1.5612</td>
<td>1.07518</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.82</td>
<td>0.7962</td>
</tr>
</tbody>
</table>

Furthermore, as shown in Table 5.17, the experimental group had higher means than the control group in the pre-intervention test, 0.2 (±0.61) versus 0.10 (±0.33), in the post-intervention test, 1.38 (±1.21) versus 0.53 (±0.93), and in the delayed test 1.61 (±1.27) versus 0.70(±1.06).
Table 5.17: Descriptive Statistics (pre-to post and delayed test) of analysis-synthesis level

(Experimental group: 1, Control group: 2)

<table>
<thead>
<tr>
<th>Group</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.2041</td>
<td>0.60852</td>
<td>98</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>0.33333</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.3878</td>
<td>1.20679</td>
<td>98</td>
</tr>
<tr>
<td>2</td>
<td>0.53</td>
<td>0.92611</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Delayed test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.6122</td>
<td>1.2733</td>
<td>98</td>
</tr>
<tr>
<td>2</td>
<td>0.70</td>
<td>1.05887</td>
<td>100</td>
</tr>
</tbody>
</table>

The predominance of the experimental group as compared to the control group in the analysis-synthesis level was also apparent within all times.

While both groups increased their scores within time (pre-to post and delayed test) at all of Bloom’s taxonomy levels in both the application and the analysis-synthesis levels, the increase in means scores in the experimental group was higher than the increase in means in the control group within all times as shown in Table 5.16 as well as in the analysis-synthesis level as shown in Table 5.17. This means that the Lakatosian method as compared to the Euclidean method is more effective within pre-to post and delayed test periods in both higher levels of Bloom’s taxonomy. This means that the experimental method affected the students’ higher order thinking more positively.

By the hierarchical nature of the Bloom’s taxonomy levels, when one level is achieved one could move easily to the next level. So, students are able to move to the understanding level and then to the application level once they have effectively familiarized themselves with the knowledge level (i.e. remembering level or retrieval level 1 of the new version of Bloom’s Taxonomy (Marzano, 2001)), which is the most fundamental level of the taxonomy.
From the results (Appendix J) we can see that the Lakatosian method had no significant effect on students’ achievement at both the knowledge level and the understanding/understanding level of Bloom’s taxonomy. However, a significant main effect was observed between the experimental group and the control group at the application and analysis–synthesis levels (both are at higher levels of the Bloom’s taxonomy).

Figure 5.10, shows the plot of the mean scores for each combination of factor level. Also, it shows how the groups performed in the pre, post and delayed tests. The graphs show that the students’ scores increased regardless of the method used. However, the post-test and the delayed tests mean scores of the experimental group show better improvement than the control group within time (pre-to post and delayed test) (as shown in graphs (a) and (b) of Figure 5.10).

The graphs (c) and (d) in Figure 5.10 show higher increases within all times in scores of the experimental group compared to the control group in both the application and analysis–synthesis levels of Bloom’s taxonomy. It can be concluded that the method may work positively over a longer period of time during which students can sustain their knowledge.
Figure 5.10: Bloom’s taxonomy cognitive levels within time (pre-to post and delayed test)

B. Analysis within times pre-to post and delayed tests

Table 5.18 compares each trial with the adjacent trials. With a pre-to post and delayed test design, Table 5.18 provides redundant information, since there are only two measurement times to compare. The two-way repeated measures ANOVA revealed a significant main effect for time (pre-to post and delayed test) $F(1,196) = 186.826$, $p<0.001$ (Table 5.18) and a significant main effect between groups (experimental and control) $F(1,196) = 5.83$ $p<0.005$ as shown in Table 5.19.
Table 5.18: Tests of Within-Subjects contrast (pre-to post and delayed tests)

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>Quadratic</td>
<td>1</td>
<td>358,001</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>1</td>
<td>540,329</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Time and group</td>
<td>Quadratic</td>
<td>1</td>
<td>11,638</td>
<td>0.318</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>196</td>
<td>2389,058</td>
<td>0.318</td>
<td></td>
</tr>
<tr>
<td>Error(Time)</td>
<td>Quadratic</td>
<td>196</td>
<td>2271,794</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The interaction within *groups and time* was also statistically significant F(1,196)=44,329, p<0.001 (Table 5.18). While all groups increased their scores *within time* (pre-to post and delayed test) the increase of means scores in the experimental group was much higher than the increase of means in the control group *within all times*. As shown in Table 5.19 (Descriptive Statistics) both groups had higher means in the post-test and delayed test than in the pre-test.

Table 5.19: Tests of Between-Subjects effects (pre-to post-and delayed tests)

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III</th>
<th>Df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>131703,4</td>
<td>1</td>
<td>131703,4</td>
<td>2485,578</td>
<td>0.001</td>
</tr>
<tr>
<td>Group</td>
<td>308,859</td>
<td>1</td>
<td>308,859</td>
<td>5.829</td>
<td>0.017</td>
</tr>
<tr>
<td>Error</td>
<td>10385,46</td>
<td>196</td>
<td>52,987</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Furthermore, as shown in Table 5.20, the experimental group had lower means than the control group in the pre-test, 11,39 (±5,12) versus 12,49 (±3,91), and higher means in the post-intervention test, 16,91 (±5,93) versus 15,07 (±5,076), and in the delayed test 18,53 (±4,96) versus 14,95 (±5,17).

Table 5.20: Descriptive Statistics

(Experimental group: 1, Control group: 2)

<table>
<thead>
<tr>
<th>Group</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td>1</td>
<td>11,398</td>
<td>5,12069</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>12,49</td>
<td>3,90673</td>
</tr>
<tr>
<td>Post-test</td>
<td>1</td>
<td>16,9082</td>
<td>5,93102</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>15,07</td>
<td>5,0757</td>
</tr>
<tr>
<td>Delayed</td>
<td>1</td>
<td>18,5306</td>
<td>4,9559</td>
</tr>
<tr>
<td>test</td>
<td>2</td>
<td>14,95</td>
<td>5,17058</td>
</tr>
</tbody>
</table>

Figure 5.11, shows the plot of the mean scores for each combination of factor level. Also, it shows how the groups performed on pre-to post and delayed tests. The graph shows that the performance of the experimental group (group 1) was better than the control group (group 2). Even though both groups had low performance (while the experimental group had lower than the control group) in the pre-test, they demonstrated increases in scores regardless of the method used. However, the post-test and the delayed tests of the experimental group suggested an increasingly higher improvement when compared to the control group within time (pre-to post and delayed test).
Figure 5.11: Plot of Mean scores of the experimental and control groups within time (pre-to post and delayed tests)
5.7.2 RESEARCH QUESTION TWO

Section 2: within times post–to delayed tests

This section examined Research question 2 which answered the question whether a heuristic method of teaching the SAC could help students sustain their learning better than the traditional method. This section, as well as question 1(i), includes the results of the quantitative data collection of this study about the groups and the indicators of significant findings, when the descriptive data are compared. As mentioned above, the test analysis was conducted using a two-way repeated measures ANOVA (2×3), between the experimental and the control group, within time (post-to delayed test). The results are shown in Table 5.21.

The two-way repeated measures ANOVA revealed a significant main effect within time (post-to delayed-test) $F(1,196)=2297.037$, $p<0.001$ (Table 5.21) and a significant main effect between groups (experimental and control) $F(1,196)=15.74$, $p<0.001$ as shown in Table 5.21.

<table>
<thead>
<tr>
<th>Source</th>
<th>Type III Sum of Squares</th>
<th>Df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>106039.2</td>
<td>1</td>
<td>106039.2</td>
<td>2297.037</td>
<td>0.001</td>
</tr>
<tr>
<td>Group</td>
<td>726.663</td>
<td>1</td>
<td>726.663</td>
<td>15.74</td>
<td>0.001</td>
</tr>
<tr>
<td>Error</td>
<td>9048.046</td>
<td>196</td>
<td>46.164</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Even though a significant main effect is observed within time (post-to delayed test) $F(1,196)=5.633$, $p<0.005$ (Table 5.22), the interaction within group and time (post-to delayed test) $F(1,196)=7.576$, $p<0.005$ (Table 5.22), has a trend significant main effect. The way in which the methods affected students’ conceptual change/learning will be explained in the results of Bloom’s taxonomy levels (see following section 3 in this chapter).
Table 5.22: Tests of Within-Subjects contrasts (post-to delayed test)

<table>
<thead>
<tr>
<th>Source</th>
<th>TEST Type</th>
<th>III Sum of Squares</th>
<th>Df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>Linear</td>
<td>55,864</td>
<td>1</td>
<td>55,864</td>
<td>5,633</td>
<td>0,019</td>
</tr>
<tr>
<td>Time and Group</td>
<td>Linear</td>
<td>75,137</td>
<td>1</td>
<td>75,137</td>
<td>7,576</td>
<td>0,006</td>
</tr>
<tr>
<td>Error(time)</td>
<td>Linear</td>
<td>1943,795</td>
<td>196</td>
<td>9,917</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5.7.3 RESEARCH QUESTION THREE

Section 3 examined research question 3; whether the heuristic method of teaching the SAC changed students’ readiness level according to Bloom’s taxonomy. This question was examined in section 1 by using the Oh’s (2010) model in which it was ascertained if students’ alternative conceptions might change to scientific concepts about the SAC.

Section 3: Students’ change of level of readiness according to Bloom’s Taxonomy levels

As shown in Table 5.23 the students in the experimental group showed increased level of readiness from the pre-test to the post test at all Bloom’s taxonomy levels. The percentage of knowledge level (K) of the test (1-6), where the pre-existing knowledge task (1-2) included, was 62.5%. The remaining three levels had the same percentage in the test, 12.5% each. Tasks 7a, 7b and 10a were at the understanding level (U), while 7c, 8 and 9a were at the application level (A) and 11 and 12 were at the analysis-synthesis (A-S) level. The difference from pre-to post-test out of 62.5% at the knowledge level for the experimental group as compared to the control group was 14.03% (52.04%-38.01%) versus 3.92% (43.88%-39.96%), for the understanding level out of 12.5% it was 3.99% (9.86%-5.87%) versus 1.08% (7.25%-6.17%), for the application level out of 12.5% the difference was 4.94% (7.19%-2.23%) versus 2.37% (5.33%-2.96%) and the analysis–synthesis level out of 12.5% the difference was 5.14% (5.87%-0.73%) versus 2.63% (3.17%-0.54%).
Table 5.23: Analysis of the Bloom’s taxonomy levels (inclusive pre-existing knowledge)

<table>
<thead>
<tr>
<th>LEVELS</th>
<th>K(1-6)</th>
<th>U(7a,7b,10d)</th>
<th>A(7c,8,9a)</th>
<th>A-S(11-12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental-group (n=98)</td>
<td>15marks</td>
<td>3marks</td>
<td>3marks</td>
<td>3marks</td>
</tr>
<tr>
<td></td>
<td>38,01</td>
<td>5,87</td>
<td>2,25</td>
<td>0,73</td>
</tr>
<tr>
<td>Control-group (n=100)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>39,96</td>
<td>6,17</td>
<td>2,96</td>
<td>0,54</td>
</tr>
<tr>
<td>Post-test</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental-group (n=98)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>52,04</td>
<td>9,86</td>
<td>7,19</td>
<td>5,78</td>
</tr>
<tr>
<td>Control-group (n=100)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>43,88</td>
<td>7,25</td>
<td>5,33</td>
<td>3,17</td>
</tr>
<tr>
<td>Delayed test</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental-group (n=98)</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>50,68</td>
<td>10,54</td>
<td>7,36</td>
<td>6,72</td>
</tr>
<tr>
<td>Control-group (n=100)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>48,58</td>
<td>8,46</td>
<td>3,54</td>
<td>3,08</td>
</tr>
<tr>
<td>Test (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>62,5</td>
<td>12,5</td>
<td>12,5</td>
<td>12,5</td>
</tr>
</tbody>
</table>

Also, as shown in Table 5.24, the knowledge level (Tasks 3-6), not including the pre-existing tasks (1-2), was 40% of the test and the remaining three levels were 20% each. The difference from pre-to post-test of the experimental group as compared to the control group at the knowledge level was 10,47% (29,18%-18,71%) versus 0,4% (21,8%-21,4%), 6,39% (15,78%-9,39%) versus 1,73% (11,6%-9,87%) at the
understanding level, 7.15% (10.75%-3.60%) versus 4.6% (9.6%-5%) at the application and 8.8% (9.25%-1.17%) versus 4.2% (5.07%-0.87%) at the analysis–synthesis.

Table 5.24: Analysis of the Bloom’s taxonomy levels (exclusive of pre-existing knowledge)

<table>
<thead>
<tr>
<th>Levels</th>
<th>K(3-6)</th>
<th>U(7a,7b,10d)</th>
<th>A(7c,8,9a)</th>
<th>A-S(11,12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td>6 marks</td>
<td>3 marks</td>
<td>3 marks</td>
<td>3 marks</td>
</tr>
<tr>
<td>Experimental-group(n=98)</td>
<td>18.71</td>
<td>9.39</td>
<td>3.6</td>
<td>1.17</td>
</tr>
<tr>
<td>Control-group(n=100)</td>
<td>21.4</td>
<td>9.87</td>
<td>5</td>
<td>0.87</td>
</tr>
<tr>
<td>Post-test</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental-group(n=98)</td>
<td>29.18</td>
<td>15.78</td>
<td>10.75</td>
<td>9.25</td>
</tr>
<tr>
<td>Control-group(n=100)</td>
<td>21.8</td>
<td>11.6</td>
<td>9.6</td>
<td>5.07</td>
</tr>
<tr>
<td>Delayed test</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental-group(n=98)</td>
<td>29.73</td>
<td>16.87</td>
<td>11.77</td>
<td>10.75</td>
</tr>
<tr>
<td>Control-group(n=100)</td>
<td>28.27</td>
<td>13.53</td>
<td>5.67</td>
<td>4.93</td>
</tr>
<tr>
<td>Test (%)</td>
<td>40</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

As shown in Figure 5.12 increases were observed in both groups from the pre-test to the post-test and the students of the experimental group improved better than the control group irrespective of whether all the tasks of the knowledge level (1-6) including the pre-existing knowledge (1-3) (Figure 5.12a and 5.12b) were included or not, in other words it included tasks (3-6) where knowledge level (Figure 5.12c and 5.12d) was 40% of the test and each of the remaining levels were 20%.
From the post to the delayed test, it was observed that the experimental group maintained the percentages it achieved in the post test, as compared to the control group, with the greatest difference being at the analysis-synthesis level (highest level of the first 4 levels of Bloom’s taxonomy) which was 0,94% (6,72% - 5,78%) as against -0,09% (3,08% - 3,17%), a decrease, recorded by the control group (Figure 5.12a and 5.12b). The control group recorded the highest percentages increase as compared to the experimental group within times post and delayed test at the knowledge level (48,58% - 43,88% =5,4%) versus (50,68% - 52,04% =1,36%), whereas as regards the remaining levels, a small variation of the percentages is observed, as compared to the experimental group, which not only maintains, but increases its percentages in all levels.

<table>
<thead>
<tr>
<th>Levels</th>
<th>K(1-6)</th>
<th>U</th>
<th>A</th>
<th>A/S</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>62,5</td>
<td>12,5</td>
<td>12,5</td>
<td>12,5</td>
</tr>
</tbody>
</table>
Extra analysis between pre-to post and post–to delayed test was considered necessary aiming to observe the main effect of Bloom’s taxonomy levels within those specific times.

**Pre-to post-test analysis:** Given the analysis of the results of the pre-to post-tests a significant effect within time (pre-to post-test) in all of the taxonomy levels was observed, whereas a significant effect in between groups (Appendix L) was observed only in the higher order thinking (HOT) levels of Bloom’s taxonomy. However, there is a statistically significant main effect within time and group mainly in the application level $F(1,196)=14,108, p=0.001$ independent of the within time effect. This demonstrates that the Lakatosian method when compared to the Euclidean method is more effective within pre-to post-test periods, signifying that the Lakatosian method positively affected the students’ achievement at the HOT levels of Bloom’s taxonomy levels.
Post-to delayed test analysis: Post-to delayed test Table (Appendix M) indicates that a significant main effect was observed at HOT levels of Bloom’s taxonomy levels in between groups (experimental and control), independently of the within time effect. However, no significant main effect was observed within time and group at all of Bloom’s taxonomy levels (at the knowledge level (F(1, 196)=0.556, p=0.457) as well as at the understanding level (F(1, 196)=0.043, p=0.835), the application level (F(1, 196)=0.693, p=0.406) and the analysis–synthesis level (F(1, 196)=0.117, p=0.733). Despite the fact that a statistically significant main effect within time and group as well as within time was not indicated, a significant effect was shown at the Analysis–Synthesis level (F(1, 196)=6.108, p=0.014) in within time (post-to delayed test).

A comparison of both tables (Appendices L and M) between groups demonstrates that, from the pre- to post and from the post-to delayed test respectively, the main effect shifts from non-significant to significant, respectively, in all of Bloom’s taxonomy levels, as it is also shown in Tables 5.7, 5.10, 5.13 and 5.15 between groups within time (pre-to post and delayed). From the analysis of the results we can assume that the Lakatosian method had no significant effect on Bloom’s taxonomy lower order thinking (LOT) levels between the groups (experimental and control) within pre-to post-test (Appendix L), as well as within post-to delayed test (Appendix M). However, the Lakatosian method had a significant effect between the groups in the HOT levels (Table 5.13 and 5.15), as well as at all levels within time (pre-to post and delayed test) (Tables 5.6, 5.9, 5.12 and 5.14). This was strongly supported and examined within both times (pre-to post and post-to delayed tests) that the main effect was observed in the HOT levels (application and analysis-synthesis) level of Bloom’s taxonomy. We can conclude that the method may function positively at the higher levels of Bloom’s taxonomy over a longer period of time, during which students can sustain their knowledge (Table 5.20), and that more time may be needed to cause a change in their alternative conceptions.

A. Analysis of the results of the Students’ achievement

Table 5.23 shows the percentages of the correct tasks solutions of the test (pre-to post and delayed) after grouping the questions according to Bloom’s taxonomy levels. For example, all the knowledge tasks (tasks 1-6) were added together.
As shown in Table 5.23 almost all of the students achieved high scores at the knowledge level of Bloom’s taxonomy. This means that students understood well the definition of the SAC.

The achievement of the students on the notion of the SAC from graphical to verbal translation (tasks 7a, 7b and 10a) (Table 5.23) was also high in both groups especially in the experimental group, where the students achieved 10.54% out of 12.5% in the delayed test as compared to 8.46% by the control group. The main effect was at the application level of Bloom’s taxonomy (tasks 7c, 8 and 9a) where students had to apply the definition of the SAC (task 7c) and also find the cross section of the cone that was an isosceles triangle (task 8). The answers of the students who said that the cross section of a cone was an equilateral triangle were also considered as a true answer, whereas the answer of students, who said just “triangle”, was considered wrong, while some of them (the students) actually meant right angle triangle as verified from the interviews. The application level results in the experimental group achieved 7.19% out of 12.5% in the post test (3 times higher than the pre-test), and 7.36% in the delayed test, as compared to the control group who doubled (5.33% out of 12.5%) their results in the post test and went down in the delayed test to 3.54%.

In test’s section C: Perceptions of the students about the construction of a cone, as well as in section D: Problem solving, students in the experimental group showed an extremely high achievement at both levels (Application, Understanding) of Bloom’s taxonomy. At the Application level as shown in the Appendix I (section C: task 9a) and at the Understanding level (section D: task 10d) students in the experimental group achieved their highest level of achievement from the pre to the post test up to (55.1% in task 9a and 61.22% in task 10d) as compared to the control group (40% in task 9a and 56% in task 10d). However, students in the experimental group sustained their knowledge. (65.3% in task 9a and 72% in task 10d in the delayed test). As shown in the Appendix I, in the delayed test students in the experimental group reached 65.3%, in task 9a, that it was considered as the students’ core beliefs, while students in the control group reached only up to 43%. This shows the effectiveness of the Lakatosian method over the traditional method at the application level of Bloom’s taxonomy.
The results in section D: *Problem solving* - the analysis–synthesis level (Table 5.23: tasks 11 & 12) show a main effect in the experimental group as compared to the control group. Half of the students in the experimental group achieved higher order thinking and they sustained it in the delayed test up to 6.72% out of 12.5% from the 0.73% in the pre-test, as compared to the control group that achieved 3.08% out of 12.5%. Their post test achievement in the experimental group was 5.78% out of 12.5% compared to the control group (3.17% out of 12.5%). If we consider that they both started from the same point of readiness in the pre-test nearly 0.73% in the experimental compared to the 0.54% in the control group, the superiority of the experimental group was obvious, as they achieved half of their total achievement compared to the control group who only reached 25%.

### 5.8 CONCLUSION

The present study set out to accomplish three aims. Each aim was developed in the following three sections. First, section 1 of this chapter examined the first research question which referred to the impact of Lakatos (1976) heuristic method on students’ learning of the SAC, which was developed in 3 sub-questions in three different parts. Second, section 2, examined the second research question, whether a heuristic method of teaching the SAC helped students to sustain their learning better than the traditional method. Third, section 3 of this chapter examined the third research question whether the heuristic method of teaching can change students’ readiness level according to Bloom’s taxonomy.

The three parts in section 1 which were developed had to do with the impact of using the heuristic method to teach the SAC. This was achieved by using the heuristic method to teach 11-grade Cypriot secondary school students’ SAC and examining the impact on their achievement, conceptual learning, as well as their higher order thinking skills.

**Part 1** examined the impact of using the heuristic method to teach the SAC on the students’ achievement. Both the experimental group and the control group showed significant increases in scores. The analysis within times pre–to post tests showed a significant main effect within times as well as within group and time effect. This indicates that the performance of students taught using the Lakatosian method, within particular time (pre-to post-test) was better than the performance of the students
taught using the Euclidian (traditional) method. It is important to mention that within the particular time (pre-to post-test), the increase in the average mean scores of the students in the experimental group was double that of the students in the control group as shown in Table 5.5 (mean difference 5.50 [16.90-11.40] versus 2.58 [15.07-12.49]).

They also resolved their misconceptions (alternative conceptions) about the definition of the solid cone and the SAC. According to De Villiers (2010) the ‘constructive defining’ helped them to construct the proper definition about the SAC. This took place when “a given definition of a concept was changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process” (p. 17). This process helped them to realize the constructive concept such as the concept of the SAC in order for students to be able to achieve their higher order thinking by solving tasks 11 and 12 which was one of the targets of this study.

Part 2 examined the impact of teaching the SAC using the Lakatosian heuristic method on students’ conceptual learning. According to Webb’s criteria (Webb, 1997), cognitive complexity analysis (Webb, 2002) was performed on the test questions (Appendix K) to ascertain their cognitive demands and the items were grouped according to Bloom’s taxonomy. The Jun–Young Oh’s model of the enhanced conflict map was used to explain the students’ conceptual development. The cognitive conflict map (Fig. 5.9) showed the structure of students’ alternative concepts about constructing/deconstructing of the cone as well as the creation of the SAC.

The analysis of the pre-test and the post-test results showed students’ changes from alternative conceptions to scientific conceptions (Table 5.17). Task 7c was on the definition of the SAC, it was considered to be the hard core (Lakatos, 1976). According to Lakatos (1970) theory, the hard core is the negative heuristic. Thus, the hard core tasks are more difficult to change. However, in this study, the rate of change observed in the students in the experimental group signifies that the Lakatosian method was better than the traditional method in making the students to change from the alternative conceptions to the scientific conceptions.
A student’s misconceptions about the definition of the SAC including discrepant events (Oh, 2010) were noticed from the different answers that the student gave to two related questions about the constructing/deconstructing of the cone in the related test’s tasks 9 and 10. In this study, the following two types of the discrepant events about the construction/deconstruction of the cone and the creation of the SAC came up:

**Type 1:** Those that contradicted each other i.e. those that had two different answers to related tasks where at least one is wrong (or both are wrong).

**Type 2:** Those that had same but wrong answers to two related tasks.

The responses to discrepant events become critical clues that enabled students to approach scientific concepts (Oh, 2010). The, critical events concerned: 1) the constructing/deconstructing the cone, the creation of the SAC and, 2) the definition of the cone/SAC. Also the alternative perceptions such as “a right angle triangle constructs a cone” (task 9c) are supporting alternative perceptions (misconceptions) in students’ hard core (negative heuristic). The Lakatosian method as shown in the post-test results (Table 5.23) helped students by accepting “auxiliary hypothesis” to overcome their obstacles by finding mathematical relations (Hersh, 2014) between the two spaces (2-dim and 3-dim). These relations according to Oh (2010) are called relevant scientific concepts such as \( r=\lambda, 2\pi\rho=r\theta \). These relations are supporting the naive scientific concept such as ‘a sector forms a SAC’. Thus, it seems that the Lakatosian helped the students to overcome their misconceptions about the constructing/deconstructing of the cone. After they had learned the definition of the SAC due to the effect of the Lakatosian method by using “constructing definitions” (De Villiers, 2010) students achieved their higher order thinking skills. As shown in the results (Table 5.23) students in the experimental group scored almost twice (70.8%) as much as the students in the control group (38.77%) on problem solving tasks at the highest level of Bloom’s taxonomy (section D, problem solving tasks 11 and 12).

**Part 3** examined the impact of using the heuristic method to teach the SAC based on students’ attainment of higher order thinking skills. It was observed that within post-to delayed test the Lakatosian method of teaching the SAC had a significant positive effect on students’ achievement at all levels of Bloom’s taxonomy especially, at the
application and analysis-synthesis level as compared to the Euclidean method of teaching. The experimental group students’ achievements in the understanding level demonstrated that they were capable of achieving the highest level of Bloom’s taxonomy. That is the analysis–synthesis level. They also did so in an easier way as shown in Tables 5.23 and 5.24 (tasks 11-12).

Section 2, examined the second research question, which was whether a Lakatosian heuristic method of teaching the SAC helped students to sustain their learning better than the Euclidean (traditional) method. According to the results of the post–to delayed test it was observed that the experimental group’s mean increased from post-test to delayed test as shown in Table 5.20 (18.53%-16.91%=1.62%). This means that the Lakatosian method helped the students to sustain their learning over time than the Euclidean method. However, the Lakatosian method showed higher improvement when compared to the Euclidean method within the same time (Table 5.21 and 5.22).

Section 3 examined the research question 3 which was whether the heuristic method of teaching the SAC changed students’ readiness level according to Bloom’s taxonomy. As shown in Figure 5.11 as well as from the results of Tables 5.23 and 5.24 both groups archived better in the post-test than the pre-test. However, the experimental group students had better improvement than the students in the control group. The difference in the improvement of the students in the two groups was even more evident at the higher levels of Bloom’s taxonomy. From the post-test to delayed test, it was observed that the experimental group maintained the percentages it achieved in the post test, compared to the control group, with the greatest difference being at the analysis-synthesis level (6.72% -5.78%=0.94%) versus (3.08% -3.17%=-0.09%) where they decreased (Table 5.23).

In chapter 6 the findings of the study and the implications are discussed. In the process attempts are made to relate the findings to relevant literature.
CHAPTER SIX

DISCUSSION OF FINDINGS

6.1 INTRODUCTION
In this chapter the findings of the study are discussed. The study examined the effects of the methods of teaching (Lakatosian heuristic method and the Euclidean geometry method) on students’ learning of the SAC. The differences within the groups over time (pre-, post- and delayed tests) were examined in order to address the three research questions of this study listed in Chapter 1. First, the summary of the findings is presented followed by a discussion of the three research questions organized into three sections similar to the analysis in Chapter 5. The first research question is divided into three parts to highlight the impact of using the Lakatosian heuristic method in teaching the SAC of the following: (i) students’ achievement, (ii) students’ conceptual learning, and (iii) students higher-order thinking skills. In part 2, Jun-Young Oh’s model is used to explain in detail how students’ conceptual learning about the SAC changed. Furthermore, part 2 explains how a model that elicits thinking and skills emerged because of the Lakatosian method, and discusses its application in the classroom. It was considered necessary to discuss not only how the model eliciting skills emerged, but also the reasons why the Lakatosian method led to a model eliciting thinking and skills. Finally, students’ misconceptions are discussed. In part 3, students’ higher order thinking skills are discussed, according to Bloom’s taxonomy levels.

In section 2, the second research question is discussed in terms of post- to delayed test, in order to highlight whether using the Lakatosian method of teaching the SAC helped students to sustain their learning better than the Euclidean, as a traditional method, did. In section 3, the third research question discusses whether the Lakatosian heuristic method of teaching the SAC can change students’ readiness level according to Bloom’s taxonomy. To do this, the concept of the SAC in different registers is used to illuminate how students’ conceptual thinking affects their cognitive changes.
6.2 SUMMARY OF FINDINGS: FIRST RESEARCH QUESTION

The central question of this study is: What is the impact of Lakatos’ (1976) heuristic method on students’ learning of the SAC? In order to address this question the study will first discuss the findings as related to each of the three sub-questions to illustrate the impact of teaching the SAC using the Lakatosian method of the following: (i) students’ achievement, (ii) students’ conceptual change, and (iii) students’ higher order thinking skills. It will then draw upon the insights revealed by reflecting on the first research question, as well as its sub-questions, to answer the second and the third research questions of this study, i.e. can the heuristic method of teaching the SAC help students to sustain their learning better than the traditional method?, and, can the heuristic method of teaching the SAC change students’ readiness level according to Bloom’s taxonomy?

In addressing the first research question of this study analysis focused on three aspects of the impact of teaching the SAC using the Lakatosian method compared to the Euclidean method. While summarizing the findings related to each of these aspects, particular attention was paid to how they influenced and developed students’ understanding and the most prominent role of the Lakatosian heuristic compared to the Euclidean method in facilitating learning. From a socio-cultural perspective understanding is defined as “participating in a community of people who are becoming adept at doing and making sense of mathematics as well as coming to value such activity” (Hiebert, & Grouws, 2007, p. 382). This definition is useful in viewing students’ learning as expressed through their contributions to class discussions and their ability to participate in the discovery approach to learning as an “active process of knowledge construction” (Tsai, 2000), in which students in the process of knowledge construction make their own conjectures and either they, or the teacher, then try to refute the conjecture by offering counter-examples (Mikropoulos, & Bellou, 2013). This ideology seems to establish new roles for the following: (i) classroom learning, (ii) the teacher as a facilitator of learning, and, (iii) the learner as an autonomous thinker and explorer “who expresses his/her own point of view, asks questions for understanding, builds arguments, exchanges ideas and cooperates with others in problem solving, rather than a passive recipient of information that reproduces listened/written ideas and works in isolation” (Singer, & Moscovici, 2008).
6.2.1 Impact of teaching the SAC on students’ achievement

Sub-question one of the first research question focuses on the impact of teaching the SAC using the Lakatosian heuristic method as compared to the Euclidean method on students’ achievement. The pre- and post-test results show the effectiveness of the Lakatosian method over the Euclidean method. The superiority of the Lakatosian method could be due to its ability to engage students in visualization by using the mathematical applets and the cone hat model, observing and discovering the mathematical relations of the SAC in the two spaces (2-dim and 3-dim), and building by themselves the definition of a concept of the SAC as distinguished by the concept of “create” and the process of “construct/deconstruct” the SAC.

6.2.2 Impact of teaching the SAC on students’ conceptual change

Findings presented in part 2 of sub-question (iii) of the first research question focus on the impact of teaching the SAC using the Lakatosian method, as compared to the Euclidean method on students’ conceptual change. The cognitive conflict maps (Figure 5.9) explain how Jun-Young Oh’s model of the enhanced conflict map (Figure 2.1) based on the Lakatosian method illustrates, the students’ conceptual learning, as well as how they succeeded in changing their alternative conceptions to scientific conceptions about the SAC. The analysis of the findings in Chapter 5, as well as the discussion in this chapter, gave the researcher the opportunity to investigate the topic, through the use of participants’ own words. The interviews, based on the questionnaire of both groups (experimental and control), could be compared and “sculptured”, i.e. the researcher could engage with the deepest meaning of the discussions. The researcher orchestrated the actual dialogues of the students, thereby lending authenticity to the class discussion, in an attempt to avoid the interpretative evaluation and effect a communicative validity. During all class discussions the researcher (i) encouraged students’ participation by asking them to justify, explain, clarify and elaborate their answers; (ii) supported students in developing ideas and guided their arguments through facilitation, redirection, refutation of the primitive conjecture (i.e. the right-angled triangle constructs a cone) into a new conjecture that the right-angled triangle creates a cone when it is rotated about one of its vertical sides, aiming to “to proof by improve[ment]” (Lakatos,1976). This was achieved by introducing students to the process of “conjecture-proof-critique-accept or reject’
(Sriraman, 2006). The use of counter-examples during the test analysis, as well as class discussions, resulted in the two types of discrepant events of the construction/deconstruction of the SAC: type 1 and type 2 (as shown in section 1: part 2 (iii) of this chapter).

The main misconception of students in the pre-test caused their inability to construct the cone from 2-dim-to 3-dim and vice versa. As a result of their inability to access a correct “schematic production” (Duval, 2006, p. 104), they could not become fully involved in problem-solving tasks on the SAC. Their core belief (Lakatos, 1976) that the cone in three dimensions was the “icon” (Duval, 2006) of an isosceles triangle above a circle (Densmore, 2010, p. 7), led them to the most extreme case (Figure 6.3(I)) where they “saw” the wrong “mental model”.

By the Lakatosian method students fostered mathematical modelling by demonstrating model eliciting activities. A model of a cone constructed by a sector of the smallest arc is the tallest was demonstrated by a low-achieving student during the post-test in the experimental group.

6.2.3 Impact of teaching the SAC on students’ higher-order thinking skills

Findings presented in part 3 of sub-question (iii) of the first research question focused on the impact of teaching the SAC by using the Lakatosian method to examine whether students can achieve higher-order thinking skills. Students in the experimental group achieved better than the control group students at all levels of Bloom’s taxonomy, especially in the HOT levels of application and analysis-synthesis. The achievement of the experimental group at higher-order levels made them more successful in problem solving concerning the SAC than the control group.

6.3 SUMMARY OF FINDINGS: SECOND RESEARCH QUESTION

Findings have proved that the Lakatosian method may help students sustain knowledge about the SAC over a longer period than the traditional method does. Given the analysis of the post- to delayed tests, a significant main effect (Table 5.21) was observed in all levels of Bloom’s taxonomy in both groups (experimental and control) regardless of the within time effect (pre-, post- and delayed). Time effect on the groups (time*group) is not significant for the knowledge and understanding levels.
(Table 5.6 and 5.9), while the application level and analysis-synthesis level indicated a statistically significant main effect not only within time effect on group (time*group) but also within time (post- to delayed) (Table 5.22, see section 2 Chapter 5). Hence, students were able to move to the understanding level and then to the HOT levels (application and analysis–synthesis) once they had effectively familiarized themselves with the knowledge level, which was the most fundamental level of the taxonomy.

6.4 SUMMARY OF FINDINGS: THIRD RESEARCH QUESTION
Findings presented in the third research question proved that the Lakatosian heuristic method of teaching the SAC had a positive impact on students’ readiness point at all levels of Bloom’s taxonomy, especially on the HOT levels. Students’ performance, according to Bloom’s taxonomy readiness in both methods, within all time (pre-, post- and delayed test) increased from the LOT to the HOT levels. The Lakatosian had proved better than the traditional method at all levels, especially in HOT levels (application and analysis-synthesis) within pre-to post-test times.

6.5 DISCUSSION OF RESEARCH QUESTION ONE
This chapter is organized, similar to chapter 5, into the three following sections. These sections correspond to each research question.

Research question one is divided into three parts as follows:

6.5.1 Impact of teaching the SAC on students’ achievement
Results obtained from this study showed that the performance of the experimental group were generally better than those of the control group. In general, the results of the students taught using the Lakatosian heuristic performance were better than those using the traditional method, within time effect on the group (time*group) and within time (pre-, post- and delayed tests), as shown in Table 5.18. Based on these results, it is safe to conclude that the Lakatosian heuristic method could help students change their alternative concepts of the SAC into scientific concepts. Results of the pre- and post-test (Table 5.3 and 5.4) showed that the Lakatosian method produced a lower performance in the pre-test Figure 5.11 than it did in the post-test. Despite the fact that both groups (experimental and control) showed increases in scores, the superiority of the heuristic method is obvious. Thus, it signified the positive impact on students’ achievement of teaching the SAC using the heuristic method. The Lakatosian heuristic
method encouraged students to argue (Sriraman, & Umland, 2014). By observing the
great gap between valid deductive reasoning using theorems, and the common use of
arguments, Duval points out the importance of language in geometry (Duval, 2006).
Many researchers emphasize that the role of teachers is to encourage students to
participate in mathematical discussions and conversations in the classroom and to use
mathematical language themselves in order to become proficient in mathematics as
well as to better grasp the underlining mathematical meaning of concepts
(Kotsopoulos, 2007).

In contrast to the Lakatosian heuristic method, the misunderstandings that students
experienced arising from the traditional teaching method, point out the fact that
“reasoning cannot be explained in purely logical means; it is deeply dependent upon
the systems and representations that are used” (Hanna, Jahnke & Pulte, 2010, p. 3).
For example, when learning mathematics, semiotic and sign representations in any
mathematics activities must be taken explicitly into account in teaching. The
opposition between mental and semiotic representation is no longer relevant, because
it rests on the confusion in getting access to knowledge objects (phenomenology) and
the need to consider the semiotic representation at the level of the mind’s structure
(Duval, 2006). For this reason, in the traditional teaching method, it is very difficult
for students to understand the construction/deconstruction from 2-dim to 3-dim and
vice versa, because in order to understand a task or reasoning, one must grasp the
whole structure of such task or reasoning (Duval, 2002). For example, students’
visual perception of a sector of a circle in 2-dim, does not give a clear and full picture
of the object: “It needs exploration through physical movements, because it never
gives a complete apprehension of the object” (Duval, 2002, p. 315).

On the contrary, the use of math applets or the paper cone hat model helped students
to imagine and fully understand the concept of the SAC. In addition, the use of maths
applets enabled visualization that could immediately assist a student to fully
comprehend any organization of relations (Duval, 1999). As a result of this procedure,
students distinguished between the conceptual learning of the concept of “create” a
SAC and the process of “constructing/deconstructing” the cone. Moreover, this can be
clarified by using visualization that in a series of transformations has intense power
over the mind of a student. As a result, students were able to develop cognitive
activities on their own (Duval, 2002, p. 315). Furthermore, they were now able to observe and discover the relations (mathematical or verbal) concerning the SAC by connecting the two spaces (2-dim and 3-dim). According to Duval (2006, p. 108), this can be done by: (i) using a semiotic representation even when there is a choice of the kind of semiotic representation, (ii) not confusing mathematical objects with the semiotic representations, and (iii) in geometry, by combining the use of at least two representation systems, one for verbal expressions of properties or numerical expression of magnitude, and the other for visualization.

Therefore, teachers should seek teaching methods that will help students in their conceptual learning, by analyzing both mathematical and cognitive thinking when they introduce a new mathematical concept. They should place students in problem-solving situations, in order for them to construct concepts (Duval, 1996 as cited in Duval, 2002, p. 313). In this study, Jun-Young Oh’s model (Figure 2.1), to be developed in the next section (6.5.2, part 2), there emerged such relations (verbal and mathematical) about the SAC. These relations are deemed to be the procedure that the students developed regarding the relative scientific concepts, between naïve-scientific and target-scientific concepts. Through the relative-scientific concepts students were supporting their alternative scientific concepts so as to be led to the final stage of the model which was to prove the SAC. Thus, achieving their higher-order thinking skills, such relative scientific concepts were for students to realize that the length of the arc of a sector in 2-dim, and the circumference of a base circle of a cone, were equal. Another key relation for students’ conceptual learning helped them connect the two dimensions, i.e. \((\lambda=\rho)\), where the lateral height \((\lambda)\) of a cone equals to the radius \((\rho)\) of a sector.

Therefore, the Lakatosian heuristic method as compared to the Euclidean method might help students learn mathematics with understanding. They could actively build knowledge from experience and prior knowledge such as conceptual learning, which allows them to apply and possibly adapt some acquired mathematical ideas to new situations (NCTM, 2000).

6.5.2 Impact of teaching the SAC on students’ conceptual change

The positive effect of teaching the SAC on students’ conceptual change as well as their learning using the Lakatosian method was examined. According to the results in
Chapter 5, the positive impact of teaching the SAC using the Lakatosian method as compared to the Euclidean method on students’ achievement allowed students to change their core belief by using auxiliary hypotheses (Lakatos, 1976). It is evident from the results of this study that through the Lakatosian method, students, in their attempt to invent/discover auxiliary hypotheses that verified thought-experiments, could foster their mathematical modelling skills. This method helped all the students not only to explain but also to learn. What is more, many of them were able to be involved in problem solving in an easier way than those taught using the traditional method. It was observed that the percentage students of the experimental group who solved the tasks 11 and 12 (section 5, Table 5.23 and 5.24), was double that of those of the control group. Not only high-ability students but also low achievers were given the potential to pose problems by fostering mathematical modelling skills that will be discussed in section 6.8.

6.5.2.1 Analysis and discussion of the steps of Jun-Young Oh’s model based on the Lakatosian method

In this section, the three steps of Jun-Young Oh’s model will be discussed and analysed with the aim of explaining the way in which students’ conceptions changed from alternative to scientific conceptions. Therefore, in this study, an analysis of the steps of Jun-Young Oh’s model (as explained in Chapter 2) based on the Lakatosian method leads us to make the assumptions discussed below.

6.5.2.1a Students’ alternative conceptions using Jun-Young Oh’s model

Figure 2.1 reflects students’ learning of the structure of existing alternative conceptions. For example, in pre-test tasks 9 and 10 the students gave the wrong answer in task 9 (Appendix B: task 9d) that is P1: “A circle constructs a cone” and in task 10 they gave the correct answer (Appendix B: task 10d) that is P2: “A cone is deconstructed in a sector” and vice versa. According to Jun-Young Oh’s model these are called discrepant events (P1, P2) regarding the same concept. This may be due to the students’ confusion or misconception of the definition of the SAC. Students’ ignorance, or non-understanding, of the definitions of the concepts of the solid cone and the SAC due to traditional teaching methods played a part in the creation of conflicts between students’ alternative conceptions and their naïve scientific concept (Figure 5.9 of Jun-Young Oh’s model) about the SAC. This confusion was resolved
During the thought-experiment in the experimental group, first, by using students’ pre-existing knowledge as a “requisite condition for constructing meaning” (Pines & West, 1986 as cited in Oh, 2010, p. 10). Yet, despite the fact that the same students in the pre-test had a conflict between their alternative conceptions and their scientific conceptions, in the post-test they constructed the definition of the SAC. The Lakatosian method contributed effectively to the construction of the mathematical concept. It depends strictly on a person’s capacity to use several registers of semiotic representations of the same concept represent: to them in a given register, to treat these representations within the same register and to convert these representations from a given register into another (D’ Amore, 1999, p. 5). According to D’ Amore (1999), these three elements (represent, treat, convert) as well as the above considerations, draw attention to the deep connection existing between noetic (i.e. conceptual acquisition of the object) and constructivism. Construction of knowledge in mathematics may be seen as the unification of those three actions of the concepts, i.e. to represent the concept of the SAC, to treat the obtained representations (in 2-dim and 3-dim) within a given register and to convert the representations from one register into another (D’ Amore, 1999).

In this study, as shown from the results in chapter 5, despite the fact that students in both groups (experimental and control) answered with a relatively high success rate in the test on pre-existing knowledge, on the pre-test tasks (3–6) (Table 5.24) and tasks (7a–7b) (Appendix I) the Lakatosian method apparently had a positive impact on the experimental group of students. They were able to produce “new knowledge” (Arntzenius, 1995, p. 367), such as the definition of SAC, as compared to the control group students who tried to learn definitions by heart (Kotsopoulos, 2007) according to the traditional method of teaching.

Both groups (experimental and control) had an average mean score of 64% on the pre-test results of task 7a, regarding the definition of a solid cone. In the post-test almost all students of the experimental group answered correctly (92,85%) compared to the control group (68%) as shown in Appendix I. In the pre-test, the experimental group students had a lower percentage of correct answers on task 7c regarding the definition of the SAC (10,2%) compared to the control group students (26%). However, in the post-test the experimental group achieved better results (Appendix I, task 7c:73,47%)
compared to the control group students, of which only half (49%) answered correctly. It seems that the method which was used may have helped the experimental group students to understand the definition of the SAC better than those in the control group. Moreover, the control group students could not interpret the result of task 7c (i.e. of how a ‘funnel’ cone is created). Judging from these results, as well as the interview conducted in the following week, the intervention indicated that students in the pre-test were confused about the definition of the SAC. It seems that for many of the control group students this confusion continued to exist after the intervention. As the pre- to post-test results (Appendix I) of task 7c concern the definition of the SAC, the experimental group performed far better than the control group did. The lack of understanding of the definitions also affected problem solving of the relevant tasks 11 and 12 (Table 5.23 or 5.24). The students who were taught using the Lakatosian method achieved a higher score in problem-solving tasks 11 and 12. The post-test results were almost double (46,24%) in the experimental group compared to those taught using the traditional method (25,35%). It is important to note that the students of the experimental group maintained their percentages of success (53,75%) in the delayed test as well, compared to the control group students who remained at the same percentage (24,65%).

6.5.2.1b Students’ relative/naive conceptions using Jun-Young Oh’s model

Figure 2.1 reflects the processing strategies of students that bridge the alternative to scientific concepts. The role of the Lakatosian method positively influenced the students’ conceptual changes according to the test results of the post-test that was given two weeks after the intervention. The purpose of the tests (pre-, post- and delayed) judging by interviews was to determine the core of alternative concepts and to show the protective belt (Lakatos, 1976) role in influencing students’ conceptual changes. The protective belt of naïve scientific concept (Figure 2.1: small dotted line circle) characterizes the difference in their correct answers, between the pre-test and post-test tasks 3-6, and tasks 7a, 7b and 10d in their attempt to protect their core beliefs. In this study the high percentages in the difference of the protective belt in the experimental group compared to the control group were tasks (3-6): 28,18-18,71=10,47 (26,175%) and tasks (7a, 7b and 10d): 15,78-9,39=6,48 (32,4%) versus tasks (3-6): 21,8-21,4=10,4 (2,5%) and tasks (7a, 7b and 10d): 11,6-9,87=1,73(8,65%) as shown in chapter 5 (Table 5.24), respectively. The same phenomenon of high
percentages of the experimental group students’ from pre-test to post-test was observed in the results of similar studies in different subjects (i.e. in chemistry) where the Lakatosian method was applied by Oh (2010, p. 1154). This means that the Lakatosian method provides students with the potential, through discussions between students and students-teacher, to defend their beliefs in order to transform them from negative to positive heuristic (Lakatos, 1976) thereby improving their performance as is evident from pre-test to post-test results (section 1: part 1 (ii) in this study).

The comparatively large difference between the experimental group and the control group in performance is due to the potential of the Lakatosian method “to present the possibilities for mathematizing during classroom discourse in the spirit of Lakatos” (Sriraman, 2006). In addition, the ability of high school students to improve by prove (Lakatos, 1976) after introducing them to the process of “conjecture-proof-critique-accept or reject in geometry classes” (Oh, 2010) gives the opportunity to students of the experimental group to move from the protective belt to the enhanced protective belt, and change their alternative conceptions supported by the relevant scientific concept into target-scientific concepts.

The aim of this study was not only to determine the correct answers but also to investigate how the students’ core beliefs were changed from alternative to scientific concepts.

To investigate these aims, as explained in Chapter 3 of the methodology, a lesson observation was conducted during the intervention while using the Lakatosian heuristic method in the experimental group and the traditional Euclidean method in the control group in two different schools. The purpose of the lesson observation was two-fold. First, it was used to ground the discussion in the interview. Second, it was used to explore the breadth of variation in the activities which teachers used in both groups (experimental and control) and to explore students’ reactions; hence addressing the first research question, i.e. analysing the structures given by students to a real problem situation using both methods (traditional and heuristic), and exploring the impact of the Lakatosian method on students’ conceptual learning. By examining the actual classroom dialogues the researcher showed the use of authentic counter-examples and positive/negative heuristics in students’ arguments. Additionally, the difference in the degree of difficulty in following an informal, non-traditional style of mathematical
conversation became apparent in contrast to the traditional Euclidean method (Bemboni et al., 2003).

As explained, the students’ core beliefs (task 7c) were resistant to change (pre-test and post-test) in the control group compared to the experimental groups. Although the protective belt was changed in both groups, it was more easily changed in the experimental group. The semantic role of the heuristic tools (cone hat and maths applets) in constructing the model of the cone hat helped students to change their alternative conceptions into scientific concepts. Therefore, they fostered model eliciting activities (Mousoulides et al., 2007, p. 33) even in students with a low profile in mathematics. Student S(A) was one of them who tried to give an answer to task 9 (Appendix B). In his attempt to explain why the particular sector formed the tallest cone hat (task 9a), he justified his explanation by fostering a mathematical modelling that will be explained in section 6.8.

6.5.2.1c Students’ scientific conceptions using Jun-Young Oh’s model

Figure 2.1 reflects the students’ reconstruction and bridges a naïve scientific concept from an alternative conception to the target scientific concept that was to prove the SAC. In this study the target scientific concept was achieved when students were able to solve the test tasks 11–12. These tasks were directly related to proving the formula of the SAC. The results (Table 5.24) in the post-test tasks (46,25%) versus (25,35%) for the experimental group and control group respectively, confirm that the Lakatosian heuristic method is more effective than the Euclidean method in teaching SAC. The superiority of the Lakatosian method was also shown in the groups’ achievements in the delayed test, which was 53,75% in the experimental group as compared to 24,65% in the control group.

The role of teachers who use the Lakatosian framework educationally is very important; they should help “students’ alternative conceptions to be used as direct material to achieve their scientific concepts, as they are commensurable and coexist in part” (Oh, 2010, p. 1157). Owing to the low commensurability between the core concepts in the scientific concept and students’ alternative conceptions, the role of the relevant scientific concept (C2, C3 in Figure 2.1) such as “the lateral height equals to the radius of a sector (ρ=λ)”, which supports naïve scientific concepts, “the perimeter of a base circle equals to the length of the arc of a sector (2πR=ρθc)” is very important.
The predominance of the Lakatosian method as compared to the Euclidean method is precisely because “the Lakatosian methodology is open to commensurability” (Oh, 2010, p. 1157).

Next, the third part of the first research question will be discussed.

6.5.3 Discussion of the findings of students’ higher order thinking skills by analyzing Bloom’s taxonomy levels of the within time (pre-, post- and delayed)

A grain analysis of the results of Bloom’s taxonomy levels in chapter 5 (part 3) has explained in detail how the students achieved higher order thinking skills on the SAC, between groups (experimental and control), within time (pre-, post- and delayed test).

Conceptual changes within time (pre-, post- and delayed) as a dependent variable were measured for each group three times (pre-, post- and delayed test). The difference from the post-test to the delayed tests was also examined in order to find out whether the students were able to sustain their achievements. This will be examined in the second research question in section 6.6 of this chapter.

6.5.3.1a Discussion of the effect within time at all levels of Bloom’s taxonomy

Given the analysis of the results of the pre-, post- and delayed tests a significant main effect within time in all of Bloom’s taxonomy levels was observed in both methods in both groups (experimental or control) (in section 5). However, no significant main effect was observed in the lower order thinking (LOT) levels (knowledge and understanding) (Table 5.6 and 5.9). Despite this, there was a statistically significant main effect within time (pre- to post) which was affected by the method applied on the groups (time*group), at the HOT (application and analysis-synthesis) levels (Table 5.12 and 5.14). This means that the Lakatosian method, when compared to the Euclidean, was more effective within time pre- to post, especially at the HOT levels. This signifies that the experimental method affected positively the students’ higher order thinking skills at all of Bloom’s taxonomy levels (Figure 5.10) with greater positive effect at HOT levels (Table 5.13 and 5.15). Students’ achievements in the acquisition level demonstrated that they were capable of achieving the highest level of the taxonomy, which is the analysis–synthesis level that was involved in problem solving.
It was also observed that within time (post-to delayed test) the Lakatosian method had a significant difference compared to the traditional one, at all of Bloom’s taxonomy levels. This will be discussed in the second research question in the next section.

6.6 DISCUSSION OF RESEARCH QUESTION TWO

Section 2: Discussion on the findings within times post-to delayed-tests

In this section, the second research question: Can heuristic method of teaching the SAC help students to sustain their learning better than the traditional method? will be discussed. In the analysis of the post-to delayed tests, a significant main effect (Table 5.21) was observed in all of Bloom’s taxonomy levels in both groups (experimental and control). In the pre-to post-test, we found no significant effect at the LOT levels (knowledge and understanding) (Table 5.6 and 5.9). At the application level and analysis-synthesis level a statistically significant main effect not only within group and time but also within time post-to delayed (Table 5.22 see section 5.7.2 in chapter 5) was found. So, the students were able to move to the understanding level and then to the higher levels such as application and analysis-synthesis levels once they effectively familiarized themselves with the knowledge level, which is the most fundamental level of the taxonomy. Students’ achievements in the acquisition level demonstrated that they were capable of achieving the highest level of the taxonomy, which is the analysis-synthesis level. They also did so in an easier way, when taught using the Lakatosian method compared to those of the control group when taught using the Euclidean method. The superiority of the Lakatosian method may further be attributed to the group approach in terms of gaining greater understanding of mathematical tasks and that consequently made them able to retain this knowledge and understanding longer (Dhlamini, & Mogari, 2012).

In addition, the results of post-to delayed test showed that even short periods of appropriate experience could facilitate students’ learning about the SAC. So, we conclude that the Lakatosian method may help students sustain knowledge over a longer period than the traditional method, while additional time is needed for someone to change their alternative to scientific conceptions (Lakatos, 1970). This leads to a plausible conclusion that the hard core of students’ belief is “constructed slowly and that any change will perhaps also follow a similar process” (Niaz, 1998, p. 123).
The Lakatosian heuristic method had an even more positive impact in making students sustain their learning, than the Euclidean method did at all the levels of Bloom’s taxonomy especially at HOT levels. It was considered useful to observe whether this impact was related to the students’ readiness level, according to Bloom’s taxonomy. This will be discussed in the next section.

6.7 DISCUSSION OF RESEARCH QUESTION THREE

Section 3: Discussion of the findings of change in students’ readiness level according to Bloom’s Taxonomy

This question was examined in section 1 (part 2) of this chapter with the help of Jun-Young Oh’s model. The way in which students changed their alternative concepts into scientific concepts about the SAC was explained. In addition, in section 1 (part 3) the impact of the Lakatosian heuristic method on students’ achievement, as well as how they achieved conceptual learning due to the thought-experiment, was explained.

In this section, the positive effect of the Lakatosian method will be discussed within time (pre-, post- and delayed test) (Appendix I). Students’ change in readiness level according to Bloom’s taxonomy will be discussed as well. Also the results of the analysis in section 3 of Chapter 5 will be discussed after a general reference to the conceptual thinking as well as the different registers which influence students’ readiness level.

6.7.1a Why the Lakatosian heuristic led to the change of readiness level

The Lakatosian method, which complies with the prescriptions of the National Research Council (2001) and with the researchers who have been engaged in the history and science of mathematical concepts, particularly emphasise conceptual learning. The Lakatosian method is in line with the methods intended to promote not only mathematical but also cognitive learning. Many researchers agree that this can be developed by the use of visualization rather than visual perception; visual perception requires exploration through physical movements because it never gives a complete apprehension of the object. Through the operative apprehension when looking at a figure/object we may gain insight into a problem solution (Panaoura, 2012, p. 4). Therefore, a long training on visualization is required (Duval, 1999, p. 9).
The Lakatosian method, owing to its heuristic tools, can be used for the development of teaching processes. These are the three domains of visualization, reasoning and communication (VRC) which occur when people (students) are engaged in reasoning, sharing their ideas and using visual images to assist the three processes (Moore-Russo, Viglietti, Chiu & Bateman, 2013, p. 99) which provide students with a real problem. In their engagement with real problem situations students initially pass from the “non-proof arguments” to the “proofs” (Stylianides, 2009). This study proved the formula of the SAC \( S=\pi R\lambda \), where \( \lambda \) is the lateral height of a cone and \( R \) the base radius. This process is characterized as the “hierarchy of arguments” (Stylianides, 2009, p.280) which was taken into consideration when students of the Lakatosian method first had to “build the definitions” (De Villiers, 2010, p. 17) of the SAC with the use of math applets as a “computer tutor” (Anderson et al., 1987 as cited in Duval, 1999) by using the correct mathematical register far from the “prehistoric learning-by-definition model of mathematics education” (Kotsopoulos, 2007, p.304), and then to prove it.

According to Adler (1999) and Zazkis (2000), students must use everyday language to build the mathematical register as everyday language has a positive impact on the understanding of the definitions. Thus, students are allowed to learn the definition by using mathematical language themselves and are able to see through the outwardly familiar language to the underlining mathematical meaning (Adler, 1999). The understanding of definitions helped the experimental group in the post-test to achieve better results and even successfully solve the intended problem in tasks 11-12 better than those in the control group did.

Moreover, according to Peressini & Knuth (1998), students by situating themselves in the physiological perspective of the development of understanding, take part through their participation in the social interaction of the classroom. “This helps them to convey meanings adequately, and to generate new meanings” (Peressini & Knuth, 1998, p.108) in their attempt to give reasons for their interaction during the thought-experiment (Lakatos, 1976). Reasoning refers to a set of processes and abilities that act as a visible tool of problem solving and enable us to go beyond problem solving (Pittalis & Christou, 2010). Moreover, the lack of abstract thinking and perception of the object/concept under study may prevent the student from having the spatial ability...
to solve the problem. This may be an important obstacle in learning which is further hampered by traditional teaching methods in terms of understanding a concept (Tall, & Mejia-Ramos, 2010).

The Lakatosian method as a quasi-empirical method promotes two components, taking into account the cognitive analysis of the test tasks working in small groups during the thought-experiment: (i) the mathematical component (conjecture/proof/ non-proof arguments) concerns the empirical discussion on the task, which derives from the application of the mathematical component, so as to lead the student from the conjecture to the proof or the non-proof, through argumentation on the concept studied, and (ii) the learner component concerns the students’ perception (Stylianides, 2010, p. 43). In contrast to the above, problem solving in the traditional method is posed by the axiomatic-formal world (Tall, & Mejia-Ramos, 2010) and is proved through set-theoretic definitions. Thus, the Lakatosian method, in contrast to the traditional method, led students to change their readiness level.

6.7.1b Change in students’ level of readiness according to Bloom’s taxonomy

The philosophy of Bloom’s taxonomy is based on the fact that students must know the prior levels in order to successfully achieve a higher level. Any failure in the prior levels creates cognitive gaps which, if not closed, may have serious consequences on the development of higher intellectual faculties, as regards the concept under study (i.e. the SAC). Therefore, students who fail to fill in the gaps of the level required to understand a concept will have gaps in tasks which require higher-order thinking skills. When students fail, it is difficult to provide opportunities for higher-level thinking (Sousa, 2009, p. 55).

Findings in section 3, chapter 5 (Figure 5.12), as well as results of Tables 5.23 and 5.24, show that students’ performance, according to Bloom’s taxonomy in both methods within all time pre-, post- and delayed test, improved. The level of readiness of the experimental group students was better than that of the group in all areas, especially in higher order thinking (HOT) levels (application and analysis-synthesis) within all times. This superiority increased from the lower order thinking (LOT) to the HOT levels from pre- to post-test. As shown in Table 5.23 almost all students also changed their level of readiness in all of Bloom’s taxonomy levels on the SAC within post- to delayed tests. The analysis of the results showed that all students realized well
enough the definition of the SAC (see the results of task 7c Appendix I). However, students in the experimental group achieved better results changing their readiness levels in the pre- to post-test in the HOT levels. This means that the Lakatosian method may help students avoid learning by heart (Kotsopoulou, 2007).

The pre- to post-test results in Appendix I, for both groups in tasks 5 and 6 (knowledge level) concerning the definitions of the solid cone and the SAC, and their related tasks 7a and 7c (understanding level) respectively, show a superiority in the experimental group. Thus, the average difference in tasks 5 and 7a, referring to the definition of a solid cone, from pre- to post-test is: task 5: 91,83%-58,16%=33,67%; task 7a:92,85%-65,3%=27,55%, in the experimental group, versus task 5: 65%-69%=−5% and task 7a:68%-63%=−5% in the control group. Although students had been taught the definition of the cone according to their textbooks (see section 2) using the traditional method, the results of the control group remained almost at the same level. Also a great confusion was observed in the related tasks 6 and 7c, which referred to the definition of the SAC’s results from pre- to post-test are: task 6: 84,69%-44,89%=39,8% and task 7c: 73,47%-10,2%=63,27% in the experimental group versus task 6: 53%-65%−12% and task 7c: 49%-26%=23% in the control group respectively. This difficulty concerns the definition of the curved surface area. Students could not comprehend how the segment AB was rotated about the line (e)\parallel AB (task 6) so as to form a cylinder. By confusing the axis of the rotation (3-dim) with the axis of symmetry (2-dim) they said that the correct answer was a line AB symmetrical to AB (task 6c) or infinite lines parallel to AB (task 6a). However, the better results in pre-test task 6 were not implied in task 7c. Both groups had difficulty in seeing the hidden relation between task 6 and task 7c that was the general definition of a curved surface area (in task 6), the knowledge of which could lead them to the graphical representation of the SAC’s definition in task 7c. However, the experimental group in the post-test results could change 7 times their level of readiness compared to the control group who doubled their results from pre-to post-test in task 7c. The maths applets as a heuristic tool in the Lakatosian method, played an important role in the students’ understanding of the cone, as well as the SAC’s definition, which enabled them the visualization “in a holistic manner” (Samson, 2012, p. 8).
The difficulty in the control group from pre- to post-test results in both definitions, especially in that of the SAC, might be due to pseudo-conceptual thought processes because “these thought processes are often formed in a spontaneous way” (Vinner, 1997, p. 101). As Vinner explains, students when presented with a task (e.g. the graphical definition of the SAC in task 7(c) they start looking for ways which would enable them to perform that task. These ways are not necessarily taught by their teachers in the traditional classroom but “natural cognitive reactions to certain cognitive stimuli”. For example, the control group students answered in the questionnaire “without going through any reflective procedure”, in task 7c said that it was a cone (meaning a solid cone) without base instead of a surface area of a cone.

The control group was also confused about the graphical notion of the creation of the SAC (task 7c) compared to the creation of a cone (task 7a) while the traditional teaching method was unable to give a global but only a local visualization of the object which “can make somebody get at once a complete apprehension of any organization of relations” (Duval, 1999, p.7). However, the experimental group, because of the heuristic tools, helped students to change their level of readiness from the knowledge to the understanding level and to achieve the application level of the definition of the SAC (task 7c) better than the control group did. This was supported by the experimental group interviews as well as by their questionnaires, which were conducted immediately after the intervention in both groups.

In the application level, the experimental group changed their level of readiness which resulted in achieving within pre- to post-test three times higher results compared to the control group who doubled their pre- to post-test results (table 5.23). The related task 8 (application level) and 7b (understanding level) referred to the cross section of a cone. However, half of both groups gave correct answers in the pre-test task 7b (Appendix I, 42.85% and 52%) in the experimental and the control group respectively, while they had difficulty in pre-test task 8 (Appendix I, 21.43% and 21%) respectively (Appendix I). They were unable to visualize the cone in 3-dim when the isosceles triangle was rotated about its height (180°) to form the cone and could not see that the isosceles triangle was the cone’s cross section as well. Hence, they were confused in task 8 when they read that “the cross section of a cone is a plane through the vertex to the centre of the base of a cone…” They translated it as a
line segment connecting the two points, that of the vertex and the centre of a circle, by giving as correct answer the line instead of a plane to be a cross section of a cone. This misconception or alternative framework (Vinner, 1997) is due to the following two reasons: (i) their lack of exploration through physical movements, because/as it never gives a complete apprehension of the object (Duval, 2002, p.315), and (ii) their inability to develop cognitive activities on their own connecting the two spaces (2-dim and 3-dim) (Duval, 2006). For example, the control group showed their inability to translate the verbal expression of task 8 as well as their lack of visualization. This made them consider the cross section as a line instead of a plane. Duval (2006, p. 108) mentions that in geometry a concept is comprehended “by combining the use of at least two representation systems, one for verbal expressions of properties or numerical expression of magnitude, and the other for visualization”.

In section C: Perceptions of the students about the construction of a cone, as well as in section D: Problem solving, students in the experimental group showed an extremely high achievement in both readiness levels of Bloom’s taxonomy. Students in the experimental group in the HOT levels not only achieved their highest level from the pre- to post-test (Appendix I) compared to the control group but they also sustained their knowledge according to their Bloom’s taxonomy readiness level.

Tasks 9 (application level) and 10 (understanding level) belonged to complex reasoning (Appendix K) of section C. Recognition skills belong to Webb’s complex reasoning level which requires “reasoning, planning, using evidence and a higher level of thinking skills than the previous two levels” (Webb, 2002, p. 4). The students on this level of reasoning must be able to explain their thinking as implemented in visualization skills and activities such as observation.

Many students in pre-test results in both groups (experimental and control) considered a circle as the true answer in task 9d (Appendix I, 23.47% and 11%) respectively. As a counter-example of the true task 9a, students in the experimental group during the thought-experiment explained that if they had rubber material they could construct a cone by holding it up. This thought is similar to what Lakatos (1976, p. 7) explains in his utopian class, i.e. the development of a thin rubber cube in a flat network to prove the formula F+V=E+2. In task 9c, students considered that a right-angled triangle constructed a cone by giving a counter-example that it could be rotated about one of
its vertical sides. This task confused them by the processes of the creation of a cone as they had been taught in the traditional teaching method. The use of vocabulary in mathematics language is of great importance (Duval, 2006, p. 120). The paper-hat worked as a heuristic tool helping the experimental group to change their alternative conceptions about the correct shape (sector) which was used to construct a cone hat. However, their difficulty in seeing the relation between the two spaces in the traditional method prevented them from proving the SAC easily, as well as from solving the related task 12. The control group changed their readiness level according to the post-test results to 16% in task 12 achieving the analysis-synthesis level, compared to those in the experimental group who had doubled this achievement (30.6%) (Appendix I).

Students’ misunderstanding about the SAC due to the traditional teaching method, which promotes the “mindless rigidity of traditional mathematics” (Sriraman, & English, 2010, p. 21) does not promote students’ cognitive conflicts, in contrast to the Lakatosian method which emphasizes the “whys and the deeper structures of Mathematics rather than the hows”. Thus, they have not engendered themselves in trying to cope with different problem solving strategies (Tsai, 2000). As a result, they cannot contribute to learning the concept of the SAC because they are prevented from developing inductive as well as deductive reasoning. For example, the negative impact due to the traditional method in understanding the definition of the SAC led to the control group students’ misconception of the process of the construction/deconstruction of the cone as a proceptual thinking that is different from the concept of the creation of the SAC as a conceptual thinking.

Table 5.23 illustrates that the average achievements of the students at the application and the analysis-synthesis levels in both groups and within all times were the lowest in the taxonomy levels achievements. However, the students’ results on perception of the construction (task 9a and 9b in section C of Appendix I) of a cone were better than those of the notion of the creation of the SAC (task 7c in section B of Appendix I) within all times, in both groups, with the experimental group achieving higher than the control group. A great misunderstanding was observed in the notion of the creation of the cone graphically, when a right-angled triangle was rotated about one of its vertical sides (task 7a) compared to the perception of the construction of a cone
from 2-dim to 3-dim (task 9a). However, the high test results about the notion of a cone (task 7a), compared to their perception of how a cone was constructed (task 9a) might be due to pseudo-learning (Vinner, 1997, p. 98). Their perception of how the solid cone was constructed (task 9a and 9b) or deconstructed (task 10d) was also a major misunderstanding.

Furthermore, results show (Section B: task 7c-8, application level; Section C: task 9-10, application level) that the average achievements of the students at this level (Appendix I) in both groups and within time (pre-, post- and delayed) were the lowest in the taxonomy levels achievements. However, the results on perception (Section C Application level) were better than those of the notion of the SAC (Section B Application level) within all times in the experimental group than the control group. The implication is that teachers, as well as students, should not view sense perception (Hersh, 2014, p. 24) to be less important than mathematical intuition. However, “the misalignments between perceptions of students and the mathematical expressions dealing with objects, relations, and reasoning cause learning difficulties” (Van Hiele, 1983, p. 226). It is the teachers’ role to use appropriate teaching methods, like the Lakatosian heuristic method (Sriraman, & Mousoulides, 2014), to give students the opportunity to express their sense perception.

Finally, it is very important for teachers to encourage students’ argumentation (Sriraman, & Umland, 2014) and to allow this by implementing it through discourse (Umland, & Sriraman, 2014, lines 89–90). An important step for learning mathematics and for conceptual understanding in mathematical communication is necessary for ideas to become objects of reflection, refinement, discussion and amendment (Truxaw, & De Franco, 2007 as cited in Wachira, & Pourdavood, 2013, p. 5). Therefore, teachers must seek teaching methods that will help students in their conceptual learning, by analyzing both mathematical and cognitive thinking when they introduce a new mathematical concept (Duval, 2002, p. 313). In this way “learning is considered to be easier due to changes in the brain at the level of neuronal connections, and the ease with which particular synapses are activated” (Goswami, 2008, p. 264 as cited in Taber, 2009) so that students can easily find the related concepts among related tasks which have already been discussed in this section and to
change their level of readiness in achieving higher order thinking according to Bloom’s taxonomy levels.

6.8 EMERGENCE OF MODEL ELICITING ACTIVITY SKILLS DUE TO THE LAKATOSIAN METHOD

6.8.1 How student S(A)’s model emerged

As was discussed earlier, owing to the Lakatosian method students were fostering mathematical modelling. Two weeks after the intervention, when students were solving the post-test, instead of doing his test, student S(A) tried to answer task 9:

A cone-shaped tall hat is requested to be made for the junior school carnival show. Circle only one of the following shapes that is the proper one to be used for the model of the hat.

S(A) had been a low-achieving student. He was inspired to develop a real model in his attempt to solve test task 9 (Appendix B). He first thought of how to create the SAC. He did so by visualizing the cone and reasoning about what he “can see” mentally as a “mental model” (Hersh, 2014, p. 20), inspired from the thought-experiment during the intervention. With this model he explained why the cone with the smaller base circle was the tallest and he described his model.

S(A) student showed the following model in order to explain why the sector formed the tallest cone, giving a clever model justifying his explanation. He cut two pieces of congruent right-angled triangles and by putting them vertically on his desk, as shown in Figure 6.1a, he first transformed them and then turned them around their vertical sides (Figure 6.1b). He was sceptical and then said:
When we transform these two same (he meant congruent) triangles we will have different heights as well as different base radii (Figure 6.1b). Then he continued saying: By rotating them around their vertical sides we will have a cone...so the tallest cone has a smaller circumference as well as smaller area (Figure 6.1b).

Note that by smaller area S(A) meant the smaller sector (task 9a). The researcher encouraged him to continue, pleased with the whole precise justification. Then, he continued enthusiastically to say:

When the base radius decreases, the height increases, while the hypotenuse is the same.

So he justified that the correct answer has the sector with the smaller length of arc or with the smaller in centre angle, because a smaller base radius can be formed, while the base circle circumference is smaller.

Figure 6.1: Student S(A)’s emerging model eliciting activity skills on the SAC

In task 9, S(A) conceptualized why the taller hat is depended on the cone’s base radius and not on the radius of its sector when it is developed in 2-dim. Moreover, many experimental group students discovered the “guilty lemma” (Lakatos, 1976, p. 145), concluding that all of the sectors were cut from the same circle, so they have the same radius. Hence, they deduced that the construction of the tallest cone-hat does not depend on the radius of the sector. By observing the base of the cone-hat it was made clear that the smaller the arc the taller the hat. From that observation, in the post-test, they realized that the tallest hat depended also on the in-centre angle of the sector of a cone. Therefore, they could experience deductive learning (as shown in Figure 6.2) by justifying student S(A)’s model.
The above model is a true example of the proposal by Higgins (1971), i.e. that “heuristic teaching maximizes student activity; it approaches content through problems; it employs problem-solving techniques in instructional properties; it allows for uncertainty in, and alternate approaches towards, solutions” (as cited in Hughes, 1974, p. 293). Furthermore, an application of the above proposed model is presented.

**6.8.2 Attempt to apply student S(A)’s model**

In an attempt to apply the model with two students with low self-confidence in mathematics the following dialogue took place. The two female students (A and K) who answered that the tallest hat gave answer 9b (a sector of in-centre angle \(\theta_2\)) of task 9 instead of the correct answer 9a (a sector of an in-centre angle \(\theta_1 < \theta_2\)). When they were asked to explain, the following dialogue took place:

S(A): *The shape with the greatest arc creates a greater base cone therefore, it must be the tallest hat.*

S(K): *When I connect the radii after I close shape 9b, the highest cone will be created because it depends on the length of the arc* (she meant that the greater the length of the arc of the sector, the taller would be the cone).

S(A): *But the radius? Does the radius not matter?*

S(K): *No, the radii are equal.*

(Student A thinks hard insisting that the radius plays some role, but which radius? She cannot correlate the two areas so as to understand that the radius of the sector does not affect the final answer but the radius of the base of the cone. She tries to imagine the
closed 3-dim cone, by making various gestures. Student K also fails to change her initial conception (“hard core” as a negative heuristic) (Lakatos, 1976), believing that the tallest cone depended on the greater length of the arc, therefore on the greater surface it occupies).

Researcher: *I heard you very attentively, and you cannot decide which one is finally the tallest hat that can be created out of the two sectors?*

(The researcher shows them the model of the S(A) student after cutting two equal triangles and placing them on their desk as S(A) did (Figure 6.1a) and asks the following question).

Researcher: *What is the relation between the shapes of task 9a and 9b with this model? Does it help you to think of a relation if I rotate these triangles like this (Figure 6.1b)?*

(Their faces glowed!)

S(K): *But of course! The tallest hat has the smallest surface*, (the researcher asks her to explain) *in other words the area is created by the perimeter of the base of the cone which is equal to the arc of the sector!*

Researcher: *What do you mean the area is created by the perimeter?*

S(K): *I mean the less material…. so the smaller area of the sector depends on the less arc of a sector.*

S(A): *I got it. It is obvious now that the model having the smallest height forms a greater area, contrary to the model with the greatest height.*

Researcher: *Correct. Good observation.*

(Student K insists in her effort to consolidate what she invented. Both students are using S(A)’s model by rotating both triangles around the imaginary axis of rotation.)

S(K): *Therefore, the surface covering a smaller material, I mean area, has a smaller arc.*

S(A): *From this, we conclude that shape 9a has an in centre smaller angle.*
Researcher: *Write this down symbolically.*

S(A): $\mu_1^0 < \mu_2^0$

Researcher: *How do you link this with the formula of the sector? Remember the formula when the angle is given in degrees.*

(Without too much thought, she multiplies the two parts of the inequality with $\pi r/180$).

Researcher: *What is r?*

S(A): *The radius of the sector.*

S(K): *But these [she meant the formula $\pi r \mu^0/180$] are equal to the length of the circumference of the circle $2\pi \rho$, as shown by the model!*

(She showed us the locus of the circle on the paper by rotating one of the triangles of the model).

Researcher: *What is $\rho$?*

S(K): *The radius of the circle as shown in the paper* (meant the circles created by the rotation of the model).

S(A): *So, should I replace it?* (she meant the formula).

Researcher: *Sure!*

S(K): *Yes, so we proved, thanks to the model, that for the radii, $\rho_1 < \rho_2$ applies.*

S(A): *But what we proved is also shown by the model.*

S(K): *Yes, therefore, if $\nu_1 > \nu_2$ then $\rho_1 < \rho_2$!*

In the following section the model activity thinking and skills, due to the Lakatosian method, as well as the misconceptions about the construction and deconstruction of a cone, emerging from the pre-test in both groups in an attempt to prove the SAC will be presented.
6.8.3 Why the Lakatosian method fostered model eliciting activity skills

According to Mousoulides et al., (2007), modelling activities include three types of product: *product as tools, product as constructions and product as problem*. Each product includes several other functions each of them fulfilling a special purpose in instruction. In this study, the *product as a tool* was a model of a paper cone hat. This *tool* must fulfil a functional or operational role of “descriptions and explanations illustrating and verifying the results of an experiment or investigation or it may describe why something that appears superficially correct is mathematically incorrect” (Mousoulides et al., 2007, p. 34). It was obvious from the pilot study results (conducted in 2012), as well as the main study (conducted in 2014), that students had difficulty in imagining how to construct or deconstruct a cone. Because of the Lakatosian method, the real model of a paper cone hat because of its “functional and operational role” (Duval, 2006, p. 127) helped them to solve the problem that many students have with “mathematical specificity and the cognitive complexity of conversion and changing representations” (Duval, 2006, p. 127) between two-dimensional spaces. This method may embody students’ conceptual thinking first (Tall, & Mejia-Ramos, 2010) by the “heuristic/explanatory power” (c.f. Lakatos, 1970, p. 137) and then (or simultaneously) by the *proceptual* thinking (Tall, & Mejia-Ramos, 2010). Thus, students are led from explanatory to *deductive thinking* by fostering students’ mathematical modelling activity skills.

*Product as a construction* normally requires students to use given criteria to develop a mathematical item. They do not define the nature of the product; rather they set parameters for the design of the product” (Mousoulides et al., 2007, p. 34). In this study the *product of the construction* was the relations (mathematical or verbal). Furthermore, there were important points that students would not have observed without using the model cone hat: (i) the radius of a sector equals to the lateral side of a cone, (ii) the length of the arc of a sector (cone hat in 2-dim) is equal to the circumference of a circle (cone hat in 3-dim), (iii) the base radius of a circle (R) of a cone differs from the radius (r) of a sector. The parameter to design the cone hat as a *product* was also the main difference between how to *construct/deconstruct* and how to *create* a cone. The *product as a construction* can be the verbal or mathematical relations resulting from the model of the cone hat in the form of *spatial construction* (Mousoulides et al., 2007). This model led to the proof of the SAC.
According to Lesh & Doerr (2003a, 2003b), and English, & Lesh (2003) the *product as a problem* is the ability to pose problems. During modelling cycles involved in *model eliciting activities* skills, students are engaged in problem posing, i.e. they are repeatedly revising or refining their conception of the given problem. During the *model eliciting activities* skills, students find ways to judge strengths and weaknesses of alternative ways of thinking and whether a given response is appropriate and adequate. In this study, a *model eliciting activity* skill was used to develop a model of the SAC that described why the cone with the smaller base circle was the tallest one.

### 6.8.4 Misconceptions of students on the construction/deconstruction of a cone

The main misconception of the students in the pre-test was their inability to construct the cone from 2-dim to 3-dim (see task 9 and 10 pre-post test results in Appendix I). As a result of their inability to gain access to a correct “schematic production” (Duval, 2006, p. 104), they could not solve the problem. Their *core belief* (Lakatos, 1976) that the 3-dim cone was the “icon” (Duval, 2006) of an isosceles triangle above a circle (Densmore, 2010, p. 7), led them to the most extreme case where they “saw” the wrong “mental model” (Hersh, 2014, p. 30). Thus, they wrongly interpreted the data of the test task 12: *A cone hat having a surface area the sector of a circle with in-centre angle of 60° and radius r=12cm is to be made using a material. Find the height of this hat* (Appendix B).

Initially, they thought of the cone in 3-dim as a flat “icon” (Duval, 2006) of an isosceles triangle which was rotated 180° about its vertical axis, that is the altitude of a triangle, to *create* a cone. Judging from the interviews, they were unable to imagine the cone in 2-dim as a sector of a circle. For example, student S(D) wrongly interpreted the data of task 12 because she thought that the in-centre angle (60°) of a given sector was a part of a line angle of the base of the triangle, as shown in the following Figure 6.3(I).

This visual perception hides two major misconceptions: i) the confusion between constructive/deconstructive a cone and creating a SAC, ii) the confusion about visualizing the SAC between the two spaces. Both led students to the inability to find relations (verbal or mathematical) between the 2-dim spaces.
Influenced by their misconception that the cone in a 3-dim space is a triangle above a circle, the students could not comprehend how a cone is constructed in 3-dim from 2-dim and vice versa. For example, student S(D) by solving the pre-test, imagined it as an isosceles triangle that rotates 180° about its axis (height of a triangle) to create the cone (misconception 1). When she tried to justify the answer of task 12, S(D) thought that the in-centre angle of the sector was the same as an in-centre angle of any shape trying to find a centre and a radius of a shape (i.e. triangle) as shown in Figure 6.3(I). Therefore, S(D) was wrongly led to believe that the foot of the perpendicular from the vertex of a triangle to its base was a centre of a ‘circle’ (in such a case triangle) with radius ρ=12. As a result, S(D) “saw” the in-centre angle as shown in the extreme case Figure 6.3(I) below. The great confusion that can be seen in this conceptualization shows her inability to visualize the SAC between the two spaces (misconception 2). As a result of this, students were not able to find relations between the two dimensions.

Figure 6.3: Misconceptions of students S(D) about the construction of a cone in task 12
However, experimental group student S(D), due to the Lakatosian method, overcame her misconceptions as the excerpt from the post test shows in Figure 6.3(II). She realised how to construct/deconstruct a cone as well as how to find relations between the two spaces ($2\pi R = \lambda \theta c$). She could transform the angle of $60^0$ into the radians of $\pi/3$. However, she wrongly substituted it at the beginning in the formula $(r\theta c)$ of the arc of a circle, but afterwards she corrected it by $\pi/3$. Therefore, she realized that the radius of the sector ($r$) was equal to the lateral height ($\lambda$) of a cone by substituting it on the true formula of the arc of a sector as $(\lambda \theta c)$. Despite the fact that she could not solve the problem posed in the correct way, she achieved higher objectives because of the Lakatosian method.

Consequently, the control group students, even in the post-test (instead of constructing a cone from a given sector) wrongly continued to draw it as a process creation by rotating $360^0$ a right-angled triangle (Figure 6.4) about one of its vertical sides to create a cone. This misconception resulted from the method of teaching used in which they were “learning by heart” (Kotsopoulos, 2007). Therefore, they applied the theorem that the opposite side ($p$) to $30^0$ is half of the hypotenuse (Figure 6.4). They wrongly calculated the hypotenuse as well as the cone’s height by applying the Pythagorean theorem in a right-angled triangle.

![Figure 6.4: Misconceptions of control group on the construction of a cone](image)

As a result of the two misconceptions already referred to in this section, students made a series of mistakes when they were engaged in problem-solving tasks. Duval (2006) explains that these misconceptions arise from the cognitive conflict between two opposite requirements, the object and the semiotic representation. By asking how they [students] could distinguish the represented object apart from the semiotic representation, Duval believes that “the ability to change from one representation
system to another is very often the critical threshold for progress in learning and for problem solving” (p. 107).

Furthermore, some common examples of a post-test result of these mistakes about the SAC come especially from the control group students. Owing to the first misconception, although they realised that the lateral height (λ) was a radius of the sector (r=12), they were unable to see the cone as a construction from 2-dim to 3-dim. So, they wrongly used the sine rule (Figure 6.5) to solve the problem, thereby showing their ‘hidden assumption’ (coming from their confusion about the SAC definition) that the cone is constructed instead of being created by a rotation of a right-angled triangle about one of its vertical sides. Another common mistake of the control group students evident in the post-test results, which was due to the second misconception referred to above, was their inability to relate the concepts between the two spaces (i.e. the radius (r) of a sector is the same with a lateral height (λ) of a cone).

Figure 6.5: Misconceptions of the control group

The results of the post-test showed that even the weaker experimental group students could reach the true solution (Figure 6.6) with the help of the visual representation, given that the negative heuristic becomes positive. On the contrary, others reached the solution below (Figure 6.6) easily, having a “reliable knowledge about the mathematical ideas or thoughts or concepts” (Hersh, 2014, p. 27) and advanced “mental models” in their mind having “true facts about imaginary objects” (p. 27). It seems that students in the experimental group either solved the problem by representing drawings or by not drawing them (Figure 6.7); they “didn’t give up on devolutions because of previous lack of didactic actions” (p. 6) as observed in traditional teaching. However, they took direct and personal responsibility for “the
knowledge construction” of the context of the task to represent or to treat or to convert (D’Amore, 1999, p. 6).

Thus, visualization played an important role in teaching the SAC using the Lakatosian heuristic method. Therefore, students were able to develop cognitive activities on their own as well, observed and discovered the mathematical relations about a concept (Duval, 1999, p.7). Such relations are deemed to be the procedures that the students developed about their relative scientific concepts, between naïve scientific and the target scientific concept. Through the relative scientific concepts, students supported their alternative scientific concepts so as to be led to the final stage of the model which was to prove the SAC. Thus, they acquired higher order thinking skills.

Figure 6.6: Visual way of solving task 12 by experimental group
Relative scientific concepts became scientific. For example, students realized that the length of the arc of a sector in 2-dim and the circumference of a base circle in 3-dim were equal ($r\theta=2\pi R$, $R=$radius of a base circle). Also, the relation ($\lambda=r$) connecting the two dimensions that is the lateral height ($\lambda$) of a cone in (3-dim) equals the radius ($r$) of a sector in (2-dim).

Also, teaching the SAC using Lakatosian heuristic method positively affected students’ conceptual change: “Lakatos provided the context for valuable mathematical thinking and for activities that encouraged participants to make use of their ‘mathematical’ intuition and ability” (Yim et al., 2008, p. 126). In addition, due to the Lakatosian heuristic method, students were able to demonstrate model eliciting activities’ skills, even students with a low profile in mathematics. They needed to “(a) develop a model(s) that describes a real-life situation, (b) use their models to describe, revise, and refine their ideas; and (c) use a number of representational media to explain (and document) their conceptual systems” (Mousoulides et al., 2007, p. 33) in such a way as to discover the hidden theory, even giving some inspiration to their teachers.

6.9 CONCLUSION

In the Cypriot mathematics curriculum, the definition of the solid cone is identical both in the new textbooks and the old, which defined and proved rigidly/deductively that “the cone is defined as a solid, which is formed by rotating a right-angled triangle, about one of its vertical sides” (Kanellou, 1977, p. 176); in the latest books the definition of the SAC is “The surface area [which is] formed by the rotation of the hypotenuse of a right-angled triangle KAB ($A=90^0$) around one of its vertical sides (i.e. KA) which is the lateral ($\pi\varphi\text{παράπλευρη}$) area of a right cone”
(Thomaides et al., 2000, p. 335). Hence, the problem students have in learning the SAC is not due to its definition but to the teaching and learning method.

This study has revealed that using the Lakatosian heuristic method to teach the SAC has the potential to help students learn the concept and achieve better than the Euclidean method currently used in teaching the topic in Cypriot secondary schools. The study highlights the possibilities of making students move from alternative conceptions to the scientific conception more easily thereby enabling them to achieve higher order thinking using the heuristic method as opposed to the Euclidean method.

The study has established that the Lakatosian heuristic method in teaching the SAC has a positive effect on students’ achievements at all levels of Bloom’s taxonomy, especially the HOT levels of application and analysis-synthesis, by changing their readiness level in a better way than the traditional method would. The Lakatosian heuristic method, as a constructive method, enables students to develop deep understanding as they build new knowledge based on their previous knowledge, in light of the view that “knowledge cannot be transferred but must be discovered and constructed by the child” (Bowers, 2007, p. 70 as cited in Taber, 2009, p. 153). It allows experimental group students to progress as they justify a new concept based on prior knowledge. In the first place, it requires them to reflect on prior learning while, secondly, it allows them to explain and apply prior learning to a new scenario. It is considered to be positive for all students, not only for low achievers but also for high ability students who are engaged in the solution of the problem of the concept of cone and its SAC’s proof while it enables a model of eliciting activity skills to emerge concerning the concept of the tallest cone and prevents dramatic misconceptions (observed in the control group) concerning the construction/deconstruction and the creation of the SAC. Of great importance is the use of mathematical language and the sense of perception due to the Lakatosian method, in the experimental group compared to the control group. In chapter 7 the study is summarised and recommendations are made.
CHAPTER SEVEN

CONCLUSION AND RECOMMENDATIONS

7.1 INTRODUCTION
This chapter reviews and summarises the study, draws conclusions and makes recommendations for educational practice. The limitations of the study and suggestions for future research study are also highlighted.

7.2 SUMMARY OF THE STUDY
In an attempt to contribute to the reform in the Cypriot educational system, this study was mainly aimed at exploring the heuristic Lakatos method compared to the Euclidean method of teaching. The objective was to examine the effect of using Lakatos’ heuristic method to teach the SAC on students’ learning. To realize the objective, a pre-test and post-test quasi-experimental research design was employed. This research design entailed the use of two intact groups, i.e. control and experimental. The sample of the study consisted of 98 students in the experimental group and 100 students in the control group from four intact classes of grade 11 students. Data were collected using a cognitive test, lesson observations, a questionnaire and an interview.

The study found that the Lakatosian heuristic method enhanced students’ learning of the SAC. This is evident in the results of the experimental group of students who had higher achievement post-test scores in the test than the control group did, and who also attained higher order thinking in better way than the students taught using the traditional Euclidean method. The experimental group of students also showed a better conceptual understanding of the SAC and were able to retain their learning over a longer period more than the control group students could.

The role of teachers is to be able to analyze both mathematical and cognitive thinking when they place their students in situations that required problem solving (Duval, 1996 as cited in Duval, 2002, p. 313). Therefore, teachers should seek teaching methods that provide effective interventions during tutoring sessions (Matsuda et al., 2013) that will help students to the next level of learning (Diezmann, & Watters,
Students of superior ability could pass through the first three levels of Bloom’s taxonomy quickly if they are exposed to the more challenging tasks that might keep them motivated to learn (Harris, 2010). This could be achieved by promoting the heuristic methods such as the Lakatosian one. The Lakatosian heuristic method not only promoted catering for the needs of students at different levels of readiness (Table 5.23), it also made the students participate in knowledge construction through the discourse in which they engaged in the process of collectively finding solutions to a given problem concerning the cone and its SAC, e.g. to find the tallest cone hat. In addition, this method refined students’ perceptions on how to construct/deconstruct a cone, as well as how to create its SAC. The finding of this study is in consonance with the view of Kulik (2003), that as students (low achievers as well as gifted students) work together in the same class, in small groups, they “produce positive results and even dramatic improvements” (p. 274) in their learning. Finally, Hughes (1974) proposes that “the primary purpose of the heuristic teaching is to teach mathematical thinking, not mathematical thought” (p. 298). On the other hand Lakatosian as a “heuristic, clarifies the critical distinction between logical (convinces doubters) and psychological (brings understanding) approach to mathematics” (Hughes, 1974, p. 298).

7.3 CONCLUSION

The findings of this study imply that students in the experimental group achieved higher order thinking skills, compared to the control group, by changing their level of readiness. They also fostered the students’ model of eliciting activities, thinking and skills by emerging a model of how the smaller sector produced the tallest cone (section 4). The heuristic method inspired all students, high and low achievers.

7.4 RECOMMENDATIONS

The results of this study are promising. We recommend that the Lakatosian heuristic method be made part of the geometry curriculum in Cypriot schools as it could be a remedy for the problems faced by students in learning this subject. Pre-service teachers should be trained on how to implement effectively the Lakatosian heuristic method in their teaching. This method should be applied to the teaching of core students who are not high achievers in mathematics with three teaching hours per week, and also for optional students who have nine teaching hours per week.
(Appendix A). With this strategy it would become clear whether the method has comparatively better results when used to teach mathematics to low achievers or low profile students.

We would also recommend the use of lesson study to improve teachers’ geometry teaching in particular and mathematics in general. In lesson study a group of teachers comes together to develop, teach, observe, analyze and revise lessons. Through lesson study teachers can be equipped on how to teach geometry using the Lakatosian heuristic method. This is a necessary step, as teachers are not familiar with this method and clearly cannot apply teaching methods unknown to them. Novice teachers should be trained in order for them to learn how to implement this method at school. Studies (e.g. Lee, 2008) have shown the effectiveness of lesson study in preparing teachers to teach challenging topics effectively. Hence, this strategy may also be found as an effective way of showing teachers how to teach geometry using the Lakatosian heuristic method. In conclusion, this approach is relevant to all teachers because “the best teachers are always trying to improve their practice” (Van de Walle, Karp & Bay-Williams, 2007, p. 10).

The use of heuristic tools in geometric concepts has contributed greatly to the effective application of the study. Therefore, it is recommended that students and teachers be exposed to maths software in order to improve their knowledge of how to make effective math applets, as well as to understand how to use them, with respect to not only the concept of the cone but also to be able to competently present other concepts in geometry. The employment of the applets, created with the help of software (e.g. GeoGebra), and used in dynamic geometry teaching, had a positive effect on the understanding and knowledge of the students. Thus, “the role of teaching is not to lecture, explain, or otherwise ‘transfer’ mathematical knowledge, but to create situations for students that will foster their making the necessary mental constructions” (Ljubica, 2009, p. 192).

7.5 SUGGESTIONS FOR FUTURE RESEARCH
In future studies, a researcher could expand the use of the Lakatosian method to cover other concepts/topics, e.g. solid geometry, and also to cater for students at different levels of education, such as those in grades 9 or 10.
In addition, larger samples from different locations could be included in future studies offering the ability to compare the results of students in urban areas to those in rural, more isolated areas; or students from public schools and private institutions could be compared with those in the state system. The present study concerns the SAC of solid geometry. However, it would be interesting to investigate its efficiency in other areas of mathematics, such as Algebra.

Furthermore, the time given between the tests (post- and delayed) in this study was two weeks. This could be extended to a longer period in future studies to give the opportunity to observe further differences—if any—at longer time intervals between post-tests and delayed tests. In order to achieve this, robust time management should be implemented so that school holidays do not interrupt the experiment.

7.6 LIMITATIONS OF THE STUDY

In this study there are certain limitations which should be pointed out, concerning the sample population, the educators and the environment in which the experiment was conducted.

The sample employed for the study was solely from ‘option students’ who had selected enriched mathematics, thus, they were students with a special interest in the subject; students from the core course of mathematics (Appendix A) could also be included. The ages of the enrolled samples were 16 to 17 years so the results cannot be generalized to other age groups.

Even though the study was double blinded in the control group and involved more than one educator, differences in teaching skills and transmissibility may have influenced the results. Another matter of interest was the consent of the educators to have their lessons monitored and the impact of videotaping the process on the students’ psychology, which may have caused stress to the students leading to dropouts from the sample population because they were unwilling to be videotaped and evaluated by a different teaching approach.

There were only two participating schools. They were not randomly selected and belonged to the same district and socioeconomic background, out of the vast number of local schools, implying that the results may have been influenced by the
background of the students at these particular institutions. The schools selected were also public education institutions of the city and the sample size was limited.

In addition, data collection was very difficult as a single researcher captured and recorded the multiple, simultaneous discussions and activities of the groups, trying to be careful not to miss interesting notations by quieter students, in an attempt to pay attention to the more interactive students who were working independently.

Regarding the processing of the data collected from the questionnaires, it was hard to categorize open-ended questions statistically as they were unequivocal and as a result, difficult to group. Furthermore, due to time constraints because of the impending holidays, the post-test and delayed test were administrated precisely at two and four weeks after the interventions respectively, which was the minimum waiting time (as proposed by Niaz (1998). Therefore, it was not possible to explore the results of the study after a longer waiting period.

It was indeed hard for the researcher to pay attention to the multiple discussions and activities occurring simultaneously during the intervention in the groups (experimental and control), especially in the small teams. However, the researcher had multiple recorders, one in each group, and several ways (mobiles, lists) of recording the discussions, which helped her to fully decipher students’ discussions while working to prove the SAC.

7.7 EPILOGUE
The heuristic method is an innovation to Cypriot students. To enter university, 12th grade students have to take examinations. These examination requirements are defined within the context of textbooks and the curriculum; hence, it is extremely difficult to introduce any innovation or improvement in one’s teaching practice.

The examinations influence the content and methods of teaching in the classroom. According to the report of the Evaluation Committee in May 2009, for the Cyprus Pre-service program for Secondary school mathematics teachers, it concluded that “school practice in Cyprus schools is still very traditional and many mentors do not encourage student-teachers [pre-service teachers] to put into practice what they were taught in the course” (p. 3). Therefore, it is important to promote the development of such methods in younger students (even as early as in primary school) in order to nourish teacher
and students with thought–experiments; aiming later in the secondary school to teach students the “know how” (Pólya, 1973) to explore mathematical ideas, by finding counter-examples and justifying them while engaging in the process of the Lakatosian method of *improving by proving* (Lakatos, 1976). The Lakatosian method is based on an activity of reasoning-and–proving which “is at the heart of mathematical sense making and is important for all students’ learning as early as the elementary grades” (Stylianides et al., 2013).
REFERENCES


Duval, R. (1999). Representation, vision and visualizations: Cognitive functions in


Tikva, J. B. (2010). Socratic teaching is not teaching, but direct transmission is: Notes from 13 to 15-years olds’ conceptions of teaching. *Teaching and Teacher Education, 26*, 656-664.


Appendices

Appendix A: Program of Studies-Courses in Lyceum Schools

A student has the flexibility to form his own programme in accordance with his interests and aptitudes. Therefore, (s)he is assisted by the teachers of the Advisory and Vocational Education, as well as by the Committee of Evaluation of Students’ Choices. Form A is a common type for all students. All subjects are common core ones, which means that they are compulsory. This form gives students a chance to acquire a general rich core of knowledge and a rich social and emotional background. This constitutes a form of observation, guidance and orientation for the student.

Table 1: Lyceum-FORM A (15-16 years old)

<table>
<thead>
<tr>
<th>Common Core (Lyceum)</th>
<th>Mathematical periods per week/total periods of all common core subjects</th>
</tr>
</thead>
<tbody>
<tr>
<td>FORM A</td>
<td>444/35</td>
</tr>
</tbody>
</table>

In Forms B and Form C, a student (S) attends common core subjects that are considered essential for all students. Simultaneously, however, (s)he has an opportunity to select stream subjects that will help her/him in her/his preparation for her/his future career, as well as enrichment or special interest subjects that will satisfy or enrich her/his special interests or inclinations. The mathematical common core subject consists of 3 teaching periods (t.p.) in the Form B (from the total 19t.p. of all common subjects) and 2 t.p. (from the total 17t.p. of all common subjects) in the Form C. More specifically: Mathematics is an optional stream subject four teaching periods per week in the Form B or C. The Form B Mathematical subject is in one of the three or four stream subjects of totally 12 or 16 teaching periods (45min) per week respectively. As an optional stream of 4-periods per week which is continued in Form C (Mathematical subject) is a stream subject of totally 7 or 6 periods per week in the Form B or C receptively (table 3). The selection of the 4-periods as required by Form C is an equivalent requirement of subject in Form B. Also, a student can take the Mathematics Enrichment Course of two more teaching periods per week provided that the student had already taken the 4 periods in classes B or C as shown in Table 3.
Table 3: Lyceum: Form B (16-17 years old) & Form C (17-18 years old)

<table>
<thead>
<tr>
<th></th>
<th>Form B</th>
<th>Form C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical common core</td>
<td>3 periods per week</td>
<td>2 periods per week</td>
</tr>
<tr>
<td>Mathematics subject (optional)</td>
<td>7 periods per week</td>
<td>6 periods per week</td>
</tr>
<tr>
<td>Enrichment mathematics</td>
<td>9 periods per week</td>
<td>8 periods per week</td>
</tr>
</tbody>
</table>
Appendix B: Test on the Surface Area of a Cone (SAC)

The Surface Area of a Cone

Name:…………………………………………………………………………………. Class:………..
School:……….. Date:…/…/…

1. It is given a right angle triangle of sides l, h, r. Find the relationship between its sides if side l is the hypotenuse of a triangle.

Answer:…………………………..

2. Match the correct answers of column A with those of column B.
A circle (O, r) is given and θ° or μ° is the in-centre angle of a sector of the same circle with radius r and centre O.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of a circle</td>
<td>2πr</td>
</tr>
<tr>
<td>Area of a circle</td>
<td>2r</td>
</tr>
<tr>
<td>Area of a sector</td>
<td>(\frac{r^2}{2} \mu^\circ)</td>
</tr>
<tr>
<td>60°° corresponds to</td>
<td>(\pi r^2)</td>
</tr>
<tr>
<td>Perimeter of a circle</td>
<td>(\frac{\pi}{3}) radians</td>
</tr>
<tr>
<td>Length of arc</td>
<td>(\frac{1}{2} ab \sin C)</td>
</tr>
<tr>
<td>Surface area of a cone</td>
<td>(\frac{\pi r^2 \theta^\circ}{360})</td>
</tr>
<tr>
<td>Area of a triangle</td>
<td>(\frac{\pi r \theta^\circ}{180})</td>
</tr>
</tbody>
</table>
3. Complete the following sentences:

If angle $\theta = 30^\circ$, then it corresponds to $\ldots\ldots\ldots\ldots$ radians.

If angle $\phi = \pi$, then it corresponds to $\ldots\ldots\ldots\ldots$ degrees

If an angle is $\theta^\circ$ degrees, then it corresponds to $\ldots\ldots\ldots\ldots \mu^\circ$ radians

4. If a square turns $360^\circ$ over one of its sides then the shape/solid formed, will be a

(a) cylinder
(b) rectangle
(c) square
(d) rhombus
(e) $\ldots\ldots\ldots\ldots$

5. If a right angle triangle turns $360^\circ$ over one of its vertical sides, then the shape/solid formed, will be a $\ldots\ldots\ldots\ldots$

6. If a line segment AB turns $360^\circ$ over a line ($\epsilon$)//AB, then the shape formed, will be:

(a) infinite lines
(b) a Surface Area of a Cylinder
(c) a symmetrical segment of AB about the line ($\epsilon$)
(d) a Surface Area of a Cone
(e) $\ldots\ldots\ldots\ldots$
7. When a right-angle triangle is turned 360° about the vertical line, then a 3-dimensional solid is formed. Draw this solid and name it.

(a) The name of the solid is……………………………..
(b) What kind of triangle turns 180° about the above vertical line in order to form the same solid?
Answer:…………………………..

(c) If the hypotenuse on the above shape is turned 360° over the vertical side of a triangle (that is an axis of symmetry), what is the difference between the new and the previous solid shape?

8. What is the shape of a cross-section of a cone and a plane that is passing through the vertex of the cone and the centre of its base? (Thomaides, et.al., 2000, p.349).

The shape is………………………………………………

9. What is the shape of a cross-section of a cone and a plane that is passing through the vertex of the cone and the centre of its base? (Thomaides, et.al., 2000, p.349).
A 3-dim cone is given in the figure below.

If we cut it by one side from the top to the base and open the shape in a 2-dimensional shape, which one is TRUE from the following to be the Surface Area of the Cone?

A right-angle triangle
An isosceles triangle
A circle
A sector of a circle

.....................
11 An equilateral cone called the solid which is formed when an equilateral triangle side’s $a$ is turned of $180^\circ$ about its height. Find: 1) the surface area of this cone 2) the side of the middle cross-section of an equilateral cone having surface area $S=2\pi cm^2$.

12 A cone-hat having surface area the sector of a circle with in-centre angle of $60^\circ$ and radius $r=12$cm is to be made using a material. Find the height of this hat.

(Papanikolaou, 1975, p.368).
Appendix C: Questionnaires (A&B)

Questionnaire A (Pilot study)

<table>
<thead>
<tr>
<th>Name</th>
<th>Class:</th>
</tr>
</thead>
<tbody>
<tr>
<td>School:</td>
<td>Date:</td>
</tr>
</tbody>
</table>

**Information:** This questionnaire is based on what you have learned from the current lesson. You are kindly requested to supplement it in one teaching period of 45min giving clarified and precised answers. Please answer all the following questions. I want to thank you for your time which contributed positively to the achievement of the aim of this study.

I. The lesson today was interesting

Yes/No

Explain why. Give an example.

II. Please write down the most interesting/or not point(s) of the lesson that you want to repeat or avoid in the following lessons.

III. The current lesson has helped you to comprehend certain misunderstandings comparing to the previous lessons.

Yes/No

Explain why. What makes you solve your misunderstanding? Give an example.

IV. Write down briefly what you have learnt today.
Questionnaire B

Name Class:

School: Date:

Information: This questionnaire is based on what you have learned from the current lesson. You are kindly requested to supplement it in 25-30min giving clarified and precised answers in part B. Please answer all the following questions in part B and follow the instructions in Part A.

Part A

Instructions:

Please put x in the box next to each question which you consider to be right. If you don’t know the answer, leave it blank.

<table>
<thead>
<tr>
<th>Questions</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>When a right angle triangle turns over one of its vertical sides, then the solid formed, will be a cone.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>When a line segment AB turns 360° over a line (e)//AB, then the shape formed, will be a cylinder.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The Surface Area of the Cone in 2-dim is a right angle triangle.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The Surface Area of a Cone equals to the sector having radius the lateral high of a cone</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The Surface Area of a Cone equals to $\pi rl$, where r is the radius of a base circle and l the lateral height.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A vertical cross section of a Cone, passing through its vertex, is an equilateral triangle.

The arc of a circle (O, l) centre O and radius l equals to \( l\theta = 2\pi \rho \), \( \rho \) = base radius of a Cone formed by this sector.

and \( \theta \) rad is the incentre angle of a sector of the circle radius l.

The use of the experiment (cut and paste method) in this lesson helps you to prove the formula of the Surface Area of a Cone.

The teacher centered lesson helps you in comprehend the lesson

The use of the mathematical applet/videos in teaching method contribute in better understand it.
Part B

The lesson today has helped you to resolve your queries that concern the properties and the definition of the surface area of a cone. Yes/No

A) Please mark which one of the following definitions you consider more suitable to define the surface area of a cone. Put in a circle only one answer.

1. The surface area is formed by the rotation of the right angle triangle around one of its vertical sides.

2. The surface area is formed by the rotation of the plane of a right angle triangle around one of its vertical sides.

3. The surface area is formed by the rotation of the hypotenuse of a right angle triangle around one of its vertical sides.

B) 1. Write down briefly what you have learned in the current lesson about the rotation of a segment about an axis of symmetry?

2. The lesson today was interesting Yes/No

Explain why. Give an example.
3. The current lesson has helped you to comprehend certain misunderstandings for the construction/deconstruction of the surface area of cone between 2-dim and 3-dim. 

Yes/No

Explain why. What makes you solve your misunderstanding? Give an example.

4. Write down briefly what you have learnt today about the proving of the surface area of a cone, \( S = \pi rl \).
Appendix D: Interviews Schedule based on the questionnaire (Experimental group)

<table>
<thead>
<tr>
<th>Student’s names in a group</th>
<th>Group:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A, B, C, D</td>
</tr>
</tbody>
</table>

**Introduction:** The researcher will cover the following in a congenial manner:

- The researcher will first warmly thank students for their participation in the intervention lasting two teaching periods under video recording situations. Also the researcher will thank them for their good behavior trying to give the best effort during the intervention annoying any disturbing from the videotaping that may be doing them feel uncomfortable as their first participation in such an experience.
- The researcher will explain to them that she is using the voice/video recorder to capture the interview.
- The researcher will go through the whole information letter, drawing particular attention to the following:
  - The student may withdraw his/her name will be kept confidential (i.e. known only to the researcher) but the researcher may anonymously quote the things she/he says.
  - The researcher will destroy the video/audio tapes after transcribing.
- The researcher will ask if they have any questions
- The researcher will ask them to discuss the questionnaire filling after the intervention just to justify some points deeply and by explain her their way of thinking in their groups especially those who write few lines.
- The researcher will start to ask them one by one all the students in each of the groups spending about 10min in each one beginning from the first question by comparing what they have written in their questionnaire (the researcher will have their questionnaire in front of her have a look of their answers).
- Finally the researcher will thank each group.
Follow up the questions: (start with positive feedback)

The researcher wait from students to explain her what they meant exactly comparing the two methods i.e. what they mean by interesting lesson, not monotonous etc. and what made the lesson interesting for them.

If they want to repeat such a lesson and why?

(give points they like more /less).

What make them understand the lesson than previous lessons and trying to explain her what they have learn from a lesson.

Finally the researcher will ask them to explain what they write down, what they have learnt and to show her all the steps of the SAC if they remember the proves of the formula just to realize the points that remain unclear or not.
Appendix E: Student’s Consent Form

Institute for Science and Technology Education
University of South Africa (Unisa)

Students’ participation in research study consent form

Title: The effect of using Lakatosian heuristic method to teach surface area of cone (SAC) on students’ learning

Dear Respondent,

I am a secondary school teacher of mathematics in the position of a deputy manager in a Cyprus Lyceum school. I am carrying out a doctoral study in the Unisa University of S.A. I believe that this research will help to improve the teaching methods of future teachers of mathematics, especially with the Educational reconstruction already begun in the Cyprus Educational System. I have obtained the permission of the headmaster of this school to do this research in the enrichment classes of Mathematics during the timetable lesson of the curriculum. This method will apply only in one subject the (SAC) depending on the current curriculum, lasting two teaching periods including the intervention and the questionnaire that will be given immediately after the intervention in both the experimental and the control classes. Also, three teaching periods will be spent on the three tests in all classes (control and experimental) for the needs of the pre-test (one week before the intervention), for the post (two weeks after the intervention) and the delayed tests (two weeks after the post-test).

I intend to audio record and transcribe the teaching lessons in all classes and the some interviews following the intervention in the experimental group only. I will take field notes during the lessons and will also videotape the lessons to allow me to discuss the lesson afterwards in our groups. However, for the most part, the lessons will not be transcribed. Transcripts of the interviews and lessons will not contain the participants’ names and participants will be allocated pseudonyms for the analysis, thereby ensuring anonymity of the participants. The audiotapes will be kept in my portable in school and
they will be destroyed after they have been transcribed two months after the intervention.

I would like to make it clear that participation in this study is entirely voluntary, and no harm is envisaged. You may choose to accept or decline to answer any question, and you may withdraw from the study at any time. You may also freely choose after the study to decline video segments being used as described above. I will provide you with a summary of my research results on completion if you would like me to do so.

Thank you in advance for the so kind participation in my study that I ensure you that I will do the best for the improvement of the teaching practices and methods I have been studying for the lesson of mathematics in Geometry and our educational reconstruction.

Please note that:

(1) Your participation in the study is voluntary. You may choose not to participate and may withdraw your participation at any time without any negative consequences.
(2) Your information will be treated as confidential and your identity will by no means be revealed in any publication.
(3) The result of this study will be used for academic purposes only and may be published in the academic journal. I will provide you with a summary of the results of my findings on request.
(4) Should you have any queries, please do not hesitate to contact me on 0035799370890 or by email at chrysoh@cytanet.com.cy

Please sign this form to indicate that:

- You have read and understood the information above.
- You give your consent to participate in the study on voluntary basis.

__________________________________________  ________________________
Respondent’s signature                        Date
Appendix F: Class list based on the intervention of the experimental group

<table>
<thead>
<tr>
<th>School:………………………………………………..Date:../../...</th>
<th>The researcher will circle the appropriate group: A, B, C, D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instructions to the observer:</td>
<td></td>
</tr>
<tr>
<td>Please complete Student’s names and surname of a group.......</td>
<td></td>
</tr>
<tr>
<td>✓ ……………………………….(S1)</td>
<td></td>
</tr>
<tr>
<td>✓ ……………………………….(S2)</td>
<td></td>
</tr>
<tr>
<td>✓ ……………………………….(S3)</td>
<td></td>
</tr>
<tr>
<td>✓ ……………………………….(S4)</td>
<td></td>
</tr>
<tr>
<td>✓ ……………………………….(S5)</td>
<td></td>
</tr>
<tr>
<td>✓ ……………………………….(S6)</td>
<td></td>
</tr>
</tbody>
</table>

**Instructions:** The observer is part of each team of experimental group and follows the instructions below:

1. Introduces himself/herself to the team giving his/her name.
2. Takes his/her seat so as his/her presence will not disturb the members of the team but he will be able to watch all the participants by taking down their dialogues, as soon as the experiment starts, without interfering even if something is not heard well.
3. Informs them that their conversations will be recorded by using a mobile phone which is put in the middle of the team’s desk.
4. The students are represented by the symbols S1, S2,… and every conversation is taken down in a separate line such as:

   (S1)…………………………………………………………………………………..
    …………

   (S4)…………………………………………………………………………………..
    …………

5. As soon as the process is completed the Lists and the mobile phone are given to the researcher.
## Appendix G: Bloom’s Taxonomy and Norman Webb’s analysis of the Test

<table>
<thead>
<tr>
<th>Pre-existing knowledge about the Cone</th>
<th>Bloom taxonomy/ Webb Norman Levels/ Aims of the test</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1.</strong></td>
<td><strong>Knowledge (K)</strong></td>
</tr>
<tr>
<td>It is given a right angle triangle of side’s l, h, r.</td>
<td>Level 1: Recall</td>
</tr>
<tr>
<td>Find the relationship between its sides if side l is the hypotenuse of a triangle.</td>
<td>Aims: If the student knows the basic knowledge of Pythagoras theorem</td>
</tr>
<tr>
<td>Answer:………………………...</td>
<td></td>
</tr>
</tbody>
</table>

202
2. Match the correct answers of column A with those of column B.

A circle \((O, r)\) is given and \(\theta^\circ\) or \(\mu^c\) is the in-centre angle of a sector of the same circle with radius \(r\) and centre \(O\).

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of a circle</td>
<td>(2\pi r)</td>
</tr>
<tr>
<td>Area of a circle</td>
<td>(2r)</td>
</tr>
<tr>
<td>Area of a sector</td>
<td>(\frac{r^2}{2} \mu^c)</td>
</tr>
<tr>
<td>(60^\circ) corresponds to</td>
<td>(\pi r^2)</td>
</tr>
<tr>
<td>Perimeter of a circle</td>
<td>(\frac{\pi}{3}) radians</td>
</tr>
<tr>
<td>Length of arc</td>
<td>(\frac{1}{2} ab \sin C)</td>
</tr>
<tr>
<td>Surface area of a cone</td>
<td>(\frac{\pi r^2 9^\circ}{360})</td>
</tr>
<tr>
<td>Area of a triangle</td>
<td>(\frac{\pi r 9^\circ}{180})</td>
</tr>
<tr>
<td></td>
<td>(r \mu^c)</td>
</tr>
<tr>
<td></td>
<td>(\pi rl)</td>
</tr>
</tbody>
</table>

3. Complete the following sentences:

If angle \(\theta = 30^\circ\), then it corresponds to ………………. radians.

If angle \(\phi = \pi\), then it corresponds to ………………. degrees

If an angle is \(\theta^\circ\) degrees, then it corresponds to ………………. \(\mu^c\) radians.
<table>
<thead>
<tr>
<th><strong>Notion of the Surface Area of a Cone constructively</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4. If a square turns 360° over one of its sides then the shape/solid formed, will be a</td>
<td>(K)</td>
</tr>
<tr>
<td>(a) cylinder</td>
<td>L.2: Recognize &amp; Classify</td>
</tr>
<tr>
<td>(b) rectangle</td>
<td></td>
</tr>
<tr>
<td>(c) square</td>
<td></td>
</tr>
<tr>
<td>(d) rhombus</td>
<td></td>
</tr>
<tr>
<td>(e) ...............</td>
<td></td>
</tr>
<tr>
<td>5. If a right angle triangle turns 360° over one of its vertical sides, then the shape/solid formed, will be a</td>
<td>(K)</td>
</tr>
<tr>
<td>.....................</td>
<td>L.2: Recognize</td>
</tr>
<tr>
<td>6. If a line segment AB turns 360° over a line ( \varepsilon )//AB, then the shape formed, will be:</td>
<td>(K)</td>
</tr>
<tr>
<td>(a) infinite lines</td>
<td>L.3: Interpret abstract &amp; complex cognitive</td>
</tr>
<tr>
<td>(b) a Surface Area of a Cylinder</td>
<td></td>
</tr>
<tr>
<td>(c) a symmetrical segment of AB about the line ( \varepsilon )</td>
<td></td>
</tr>
<tr>
<td>(d) a Surface Area of a Cone</td>
<td></td>
</tr>
<tr>
<td>(e) ..................</td>
<td></td>
</tr>
</tbody>
</table>
7. When a right-angle triangle is turned $360^\circ$ about the vertical line, then a 3-dimensional solid is formed. Draw this solid and name it.

(d) The name of the solid is ……..

(e) What kind of triangle turns $180^\circ$ about the above vertical line in order to form the same solid?
Answer: ……………………….

<table>
<thead>
<tr>
<th>Acquisition /Understand (U)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aims:</strong> If the student knows the geometrical meaning of the SAC and the volume of a cone.</td>
</tr>
<tr>
<td>(i.e. if he/she knows that volumes formed by turning areas and areas formed by turning lines)</td>
</tr>
<tr>
<td>L.3:Describe</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Application (A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L.3:Describe</td>
</tr>
</tbody>
</table>

8. What is the shape of a cross-section of a cone and a plane that is passing through the vertex of the cone and the centre of its base?

(Thomaides, et. al., 2000, p.349)

The shape is…………………………………….
9. A cone-shaped tall hat is asked to be made for the junior school carnival show. Circle only one of the following shapes that is the proper one to be used for the model of the hat.

10. A 3-dim cone is given in the figure below.

If we cut it by one side from the top to the base and open the shape in a 2-dimensional shape, which one is TRUE from the following to be the Surface area of the cone?

a) A right-angle triangle
b) An isosceles triangle
c) A circle
d) A sector of a circle
e) .........................
<table>
<thead>
<tr>
<th>Problem solving</th>
<th>Analysis-Synthesis (A-S)</th>
</tr>
</thead>
</table>
| 11. An **equilateral cone** called the solid which is formed, when an equilateral triangle, side’s $a$, is turned $180^0$ about its height. Find: i) the Surface Area of this Cone  
ii) Find the **side** of the middle cross-section of an equilateral cone having surface area $S=2\pi \text{ cm}^2$ | **Aims:** If the student knows how to solve problems deductively.  
L.4: Complex reasoning |
| 12. A cone hat having surface area the sector of a circle with in-centre angle of $60^0$ and radius $r=12\text{cm}$ is to be made using a material. Find the height of this hat. (Papanikolaou, 1975, p.368). | **Aims:** If the student knows how to apply their knowledge of an area of a cone in problem solving  
L.4: Complex reasoning |
## Appendix H: Bloom’s taxonomy percentage results of the Pilot’s study within time (pre-to post-test)

<table>
<thead>
<tr>
<th>Bloom’s Taxonomy</th>
<th>Students’ notion of the cone/SAC</th>
<th>Students’ perceptions about the construction/deconstruction of the cone</th>
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</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td>K      K      K      U      U      A      A      8</td>
<td>A      A      A      A      A      U</td>
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<tr>
<td>Experimental group (N=21)</td>
<td>13(62)</td>
<td>17(81)</td>
</tr>
<tr>
<td>Control group</td>
<td>27(71)</td>
<td>32(84)</td>
</tr>
<tr>
<td>Post-test</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental group (N=21)</td>
<td>17(81)</td>
<td>13(62)</td>
</tr>
<tr>
<td>Control group</td>
<td>31(82)</td>
<td>32(84)</td>
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### Appendix I: Bloom’s taxonomy percentage analysis of the Main study within time (pre-to post and delayed)

Section C: Students’ perceptions about the construction/deconstruction (tasks:9-10)

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<tr>
<th>Bloom’s Taxonomy</th>
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<th>K</th>
<th>K</th>
<th>U</th>
<th>U</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
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<th>A-S</th>
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<th>A-S</th>
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<td><strong>Pre-test</strong></td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7a</td>
<td>7b</td>
<td>7c</td>
<td>8</td>
<td>9a</td>
<td>9b</td>
<td>9c</td>
<td>9d</td>
<td>10d</td>
<td>11a</td>
<td>11b</td>
<td>12</td>
</tr>
<tr>
<td><strong>Experimental (n=98)</strong></td>
<td>53.06</td>
<td>58.16</td>
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<td>42.85</td>
<td>10.2</td>
<td>21.43</td>
<td>22.45</td>
<td>32.65</td>
<td>19.39</td>
<td>23.47</td>
<td>32.65</td>
<td>10.2</td>
<td>6.12</td>
<td>1.02</td>
</tr>
<tr>
<td><strong>Control group(n=100)</strong></td>
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<td>69</td>
<td>65</td>
<td>63</td>
<td>52</td>
<td>26</td>
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<td>11</td>
<td>33</td>
<td>9</td>
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<td>1</td>
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<tr>
<td><strong>Post-test</strong></td>
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<td>5</td>
<td>6</td>
<td>7a</td>
<td>7b</td>
<td>7c</td>
<td>8</td>
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<td><strong>Experimental (n=98)</strong></td>
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<td>82.65</td>
<td>73.47</td>
<td>43.88</td>
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<td>18.36</td>
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<td>61.22</td>
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<td>44.9</td>
<td>30.6</td>
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<td>53</td>
<td>68</td>
<td>50</td>
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<tr>
<td><strong>Delayed test</strong></td>
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<td>6</td>
<td>7a</td>
<td>7b</td>
<td>7c</td>
<td>8</td>
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<td>9b</td>
<td>9c</td>
<td>9d</td>
<td>10d</td>
<td>11a</td>
<td>11b</td>
<td>12</td>
</tr>
<tr>
<td><strong>Experimental (n=98)</strong></td>
<td>95</td>
<td>69.38</td>
<td>73.46</td>
<td>98.98</td>
<td>89.8</td>
<td>82.65</td>
<td>28.57</td>
<td>65.3</td>
<td>17.34</td>
<td>3.06</td>
<td>1.02</td>
<td>72</td>
<td>65.31</td>
<td>55.1</td>
<td>40.81</td>
</tr>
<tr>
<td><strong>Control group(n=100)</strong></td>
<td>90</td>
<td>88</td>
<td>71</td>
<td>87</td>
<td>44</td>
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<td>19</td>
<td>43</td>
<td>33</td>
<td>15</td>
<td>0</td>
<td>64.29</td>
<td>29</td>
<td>26</td>
<td>19</td>
</tr>
</tbody>
</table>
Appendix J: Analysis of the Bloom’s Taxonomy tasks test

The test on the SAC (Appendix B) consists of 4 sections (A, B, C and D) of a total of twelve questions aligned to the curriculum. The tasks are based on Bloom’s taxonomy cognitive levels. However, there are no clear limits/borders between levels, each level is characterized by descriptive process verbs. The knowledge (K) level can be described by the verbs of the lower level of cognitive skills such as: define, label, listen, list, name, read, recall, record, relate and repeat, where this characterizes section A; The process verbs in the understanding (U) level are such as: solve, tell, describe, explain, locate, report and recognize, where this characterized especially the tasks of the section B; The process verbs in the application (A) level are such as: apply, demonstrate, illustrate and use, where this characterized especially the tasks of section C and finally in section D the process verbs are mainly calculate and solve for the analysis level and construct, create, design, compose for the synthesis level.

For example, task 9 which referred to the construction of the SAC was ranking in this test, in the highest level of the application (demonstrate), as well as it could be in the lowest level of the analysis-synthesis (construct). Also task 10 in this test, was ranking in the understanding level as a process of development from 3-dim in 2-dim, where the process verbs explained that this deconstruction must be describe and explain and finally recognize the shape in 2-dim, that is a sector of a circle.

The first section A: Pre-existing knowledge of the cone consists of the first three questions representing the knowledge (K) level of the Bloom’s taxonomy. The first question is open and aims to identify whether the student knows the basic knowledge of Pythagoras theorem. The second question is about matching A to B and its aim is to identify whether the student has the basic knowledge of the elements of a circle (radius, area and perimeter of a circle, area and perimeter of a sector, the relationship between radians and degrees). The third question is about completing sentences referring to the relationship between radius and degrees.
The second section B: *Notion about the construction/deconstruction of the surface area of a cone*, consists of five questions (tasks: 4-8). Three of them (tasks 4-6) belong in the knowledge (K) level, about the cone, of the Bloom’s taxonomy. Task 7 consists of 3 parts; part (a) and (b) belong to the comprehensive/understanding (U) level of the Bloom’s taxonomy, whereas part (c) belongs to the application level (A). Its aim is to ascertain that the student has knowledge of the geometrical meaning of the Surface Area and the volume of a Cone, (i.e. that the student is aware that volumes are formed by rotating areas and that areas are formed by rotating lines). Task 8 belongs to the application level (A) of the Bloom’s taxonomy. Task 7c and 8 are open-ended questions, whereas 7c refers to the shape that is formed in the previous tasks (7a and 7b) and task 8 examines the cross section of a cone.

The third section C: *Perceptions of the students about the construction of a cone*, consists of two multiple-choice questions, one question (task 9) in the application level (A) of the Bloom’s taxonomy with the aim to explore if the student knows how to construct a cone from 2-dim to 3-dim and the other (task 10) in the (U) level of the Bloom’s taxonomy aiming to identify whether the student knows how to deconstruct a cone from 3-dim to 2-dim.

The fourth section D: *Problem solving* consists of two open-ended (tasks 11 and 12 (Papanikolaou, 1975, p.368) in the level of analysis-synthesis (A-S) of the Bloom’s taxonomy regarding the Surface Area of a Cone.
Appendix K: Norman Webb’s cognitive analysis

Norman Webb’s cognitive analysis explains the conceptual learning skills of the tasks

The aim of task 1 was: “If the student knows the basics of the Pythagorean theorem”. The Webb’s Norman lower level (level 1: recall information) explained that students need to “recall” the concept of the Pythagorean theorem and find the relationship of the sides of a right angle triangle. Task 2 aimed: “If the student knows the basics of the elements of a circle (radius, area and perimeter of a circle, area and perimeter of a sector, the relationship between radians and degrees)”, the Webb’s Norman level 1 also explained that students ‘recognize’ the concepts of the elements of the circle to be able to use them in the subsequent tasks. Also in task 3: “Complete the following sentences…” referred to the relations between the radians and the degrees as well as in task 4: “If a square turns 360° over one of its sides then the shape/solid formed, will be a …” Students’ have to “recall” information (level 1). In task 3 students have to “recall” first the concept of the relations between the degrees and the radians and then to translate them between each other. In task 4 students have to “Recognize & Classify” (level 2: basic reasoning/skill & concepts) the solid formed. In task 5: “If a right angle triangle turns 360° over one of its vertical sides, then the shape/solid formed, will be a…….” Students’ have to “recognize” (level 2) the shape/solid formed while first have to known the definition of the solid formed when the proper area is rotated about its axis of rotation. This task 5 was a transformation from verbally to graphically, stimulating students’ abstract thinking. In task 6: “If a line segment AB turns 360° over a line (e)//AB, then the shape formed, will be……” similar to task 5, students have to know the definition of the curved surface area however, the Webb’s Norman level 3 was to “Interpret abstract & complex cognitive” while what is required first, is to interpret the translation of the line about the axis and then to ‘Recognize & Classify’ (level 2) the solid formed. Task 7: “When a right-angle triangle is turned 360° about the vertical line, then a 3-dimensional solid is formed. Draw this solid and name it”. Three different questions are asked: first tasks 7a asked to “recognize” the shape/solid formed, similar to task 5 but students have to translate from graphically to verbally, and then to name it (task 7a). The task 7b: “What kind of
triangle turns $180^\circ$ about the top vertical line in order to form the same solid?” is a similar question to task 5, however, more difficult, while students have to ‘recognize’ first the definition and then to ‘describe’, so it was in level 3 (strategy thinking) of the Norman Webb’s taxonomy. Task 7c: “If the hypotenuse on the above shape is turned $360^\circ$ over the vertical line, what is the difference between the new shape and the previous solid shape?” was the target of the tests while it was examined whether students know the definition of the SAC, as well as the difference between the concepts of the solid cone formed and the SAC. Tasks 7c and 8: “What is the shape of a cross-section of a cone and a plane that is passing through the vertex of the cone and the centre of its base?” They are both in the level 3 (complex reasoning) of Norman Webb’s taxonomy while in task 8 students must also ‘recognize’ the cross section of the cone and then ‘describe’ it.

The following tasks 9 & 10 are referred to in the students’ perceptions about the construction/deconstruction of a cone. Task 9: “A cone-shaped tall hat is required to be made for the junior school carnival show. Circle only one of the following shapes that is the proper one to be used for the model of the hat”,

![Diagram of shapes A, B, C, and D]
Aims: “If the student knows how to construct a cone from 2-dim to 3-dim” and task 10: “A 3-dim cone is given in the figure below.

If we cut it by one side from the top to the base and open the shape in a 2-dimensional shape, which one is true from the following to be the Surface area of the cone?” Aims: “If the student knows how to deconstruct a cone from 3-dim to 2-dim”. Both tasks 9 &10 of the test are in level 3 (strategic thinking/complex reasoning). It requires “reasoning, planning, using evidence and a higher level thinking than the previous two levels” (Webb, 2002, p.4). In most instances, students must explain their thinking as well as “Visualizations skills and activities such as observation” (Webb, 2002, p.4). Cognitive demand in level 3 is ‘complex and abstract’.

For example, in task 9, students must first be able to abstractly observe and analyze the problem’s situations to give the proper ‘reason’ of the shape used for the construction of the SAC and then to probe and guess the mathematical problem about the tall hat by answering the ‘whys’ via proof and refutations stages (stage 3) of the Lakatosian method.

The following tasks 11 & 12 of the test are referred to in the problem solving of a Cone. Task 11: “An equilateral Cone called the solid which is formed when an equilateral triangle, side \(a\), is turned 180° about its height: i) Find the Surface Area of this Cone. ii) Find the side of the middle cross-section of an equilateral cone having surface area \(S=2\pi \text{ cm}^2\) requires the prior knowledge of task 8 which recalls the cross section of a cone. The aim of task 8 is to examine: “If the student knows how to apply his/ her knowledge in problems about a cone deductively”. This task 11 is in level 4 (extended reasoning). It requires Complex reasoning of Webb’s Norman as well as in
task 12: A cone hat having as a surface area the sector of a circle with in-centre angle of 60° and radius r=12cm is to be made using a material. Find the height of this hat. In this level 4, in order to solve task 12, students’ have to know all the previous levels of cognitive thinking about the concepts of the SAC. Students must have an “extended period of time to apply significant conceptual understanding and higher order thinking” (Webb, 2002, p.4) in order to acquire the problem solving skills of the target task of the test in Webb’s level 4. In order for students to solve this task 12, they have to realize how the SAC related to both dimensions, construct /deconstruct it (cone) and know how they must be able to use their knowledge deductively to prove the SAC, by observing first, the relations that are connected to the SAC in 3-dim by its development in 2-dim.
### Appendix L: Analysis of Bloom’s taxonomy Levels within time (pre-to post test)

<table>
<thead>
<tr>
<th>Bloom’s Taxonomy levels</th>
<th>Source</th>
<th>TIME</th>
<th>Type III Sum of Squares</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
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<td>2,88</td>
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<td></td>
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# Appendix M: Analysis of Bloom’s taxonomy Levels within time (post-to delayed test)

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