COHOMOLOGIES ON SYMPLECTIC QUOTIENTS OF LOCALLY EUCLIDEAN FRÖLICHER SPACES

by

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Declaration

I declare that Cohomologies on Symplectic Quotients of Locally Euclidean Frölicher Spaces is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

______________________________
(Signature of candidate)

____________________day of _______________2015__
Dedication

I dedicate this thesis to all members of my family and friends for inestimable support during this time of sacrifices and various challenges.

Their lovely contribution has got us to this ultimate step.

Some have already left this life without living this memorable achievement.

Thanks to All for sharing with us the difficult moment we went through.

May The Almighty stream On Each One His Grace, Love and Blessing.
Abstract

This thesis deals with cohomologies on the symplectic quotient of a Frölicher space which is locally diffeomorphic to a Euclidean Frölicher subspace of \( \mathbb{R}^n \) of constant dimension equal to \( n \). The symplectic reduction under consideration in this thesis is an extension of the Marsden-Weinstein quotient (also called, the reduced space) well-known from the finite-dimensional smooth manifold case. That is, starting with a proper and free action of a Frölicher-Lie-group on a locally Euclidean Frölicher space of finite constant dimension, we study the smooth structure and the topology induced on a small subspace of the orbit space. It is on this topological space that we will construct selected cohomologies such as: sheaf cohomology, Alexander-Spanier cohomology, singular cohomology, Čech cohomology and de Rham cohomology. Some natural questions that will be investigated are for instance: the impact of the symplectic structure on these different cohomologies; the cohomology that will give a good description of the topology on the objects of category of Frölicher spaces; the extension of the de Rham cohomology theorem in order to establish an isomorphism between the five cohomologies.

Beside the algebraic, topological and geometric study of these new objects, the thesis contains a modern formalism of Hamiltonian mechanics on the reduced space under symplectic and Poisson structures.

Key terms:

- Differential geometry on Frölicher spaces, Constant dimension, Locally Euclidean Frölicher spaces, Symplectic structure, Symplectic geometry, Symplectic quotient or reduced space, Exterior algebra,

- Hausdorff paracompact Frölicher topologies, Ringed Frölicher space, Smooth Gelfand representation,

- Sheaf cohomology, Alexander-Spanier cohomology, Singular cohomology, Čech cohomology and de Rham cohomology, Isomorphism of cohomologies on the reduced space,

- Poisson and Hamiltonian geometries on the reduced space, Vector fields and mechanics.
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Chapter 1

Introduction

1.1 Review of literature

A Frölicher space $M$ is known as a smooth space with a smooth (differential) structure given by a set $C_M$ of paths $c : \mathbb{R} \to M$ and another set $\mathcal{F}_M$ of scalar functions $f : M \to \mathbb{R}$. In [43], Frölicher and Kriegl proved that such smooth spaces form a category FrI (now called Frölicher spaces) that is complete and co-complete, so that all limits and colimits exist. They showed that this category is Cartesian closed and topological over the category Set of sets, (see [27, 28]). Two topologies are defined on a Frölicher space. The initial topology, that is $\tau_{F_M}$, is generated by structure real-valued functions. Its base is the collection $B = \{ f^{-1}(0, +\infty) \}_{f \in \mathcal{F}_M}$ (see [36]) and its subbase is $S = \{ f^{-1}(0, 1) \}_{f \in \mathcal{F}_M}$ (see [43]). But it was merely noticed that a Frölicher space carries another natural topology, that is $\tau_{C_M}$, generated by curves. It can easily be proved that the topology induced by structure functions is a subset of the one induced by structure curves and also, structure curves and structure functions are smooth and continuous maps. Then, the category FrI is a subcategory of the category Top of topological spaces. Nevertheless, one can note that on some Frölicher spaces, $\tau_{F_M}$ and $\tau_{C_M}$ are equal, in which case A. Cap called them balanced spaces (see [23]).

In this work, we are dealing with the smoothness structure in the sense of Alfred Frölicher (see [29, 43]). The category of Frölicher spaces contains among its objects: $C^\infty$-manifolds (smooth manifolds), manifolds with boundaries and/or with corners (singularities). Among relevant references we have [8, 29, 43, 63]. In the light of many studies concerning smooth spaces, one concludes (see [106]) that the category of Frölicher spaces is a full subcategory of both the category of differential spaces in the sense of Sikorski (see [98, 99]) and the category of diffeological spaces (introduced by Souriau, see [45, 105]). On the one hand the differential spaces (Sikorski spaces) form a full subcategory of structured spaces (initiated by Mostow, see [81]) and on the other hand the diffeological spaces form a full subcategory
of Chen spaces (the differential structure induced by a set of plots as for Souriau spaces, see [24, 25]). There is also another category that of Smith spaces (with smooth structure defined by a set of functions, see [101]). Naturally, this one contains Frölicher spaces while it is itself contained in the category of Sikorski spaces. We have three kinds of smoothness or of smooth structures in the argument above. The smooth structure on Smith, Sikorski and Mostow spaces is a set of maps "going out", from the smooth space to a Euclidean space, say $\mathbb{R}$. Instead, the smooth structure on Souriau and Chen spaces is the set of maps "coming into", from a family of Euclidean spaces to the smooth space. Therefore, at the intersection of the previous approaches stands the category of Frölicher spaces, where the smooth structure is a pair of sets, one of "going out" maps and another of "coming into" maps (see [7, 29, 65, 82, 83, 84, 96, 106]). In [43], A. Frölicher and A. Kriegel have provided a general concept of smooth spaces where the family of Euclidean spaces is replaced by a family of general topological spaces. Finally, the reader may notice that the category of Sikorski spaces is a natural full subcategory of the category of ringed spaces, when we impose the separation point property the the differential structure and so is the category of Frölicher spaces ([56]). Since, the defining condition of smooth real functions Frölicher space is exactly the definition of ring structure for ringed spaces ([92]).

By the way, we wish to get into light the valuable contribution of Paul Cherenack in the study of Frölicher spaces. It can be well appreciated by the papers he published [27, 28, 29, 30, 31, 32] and by doctoral theses he supervised [8, 36]. A. Frölicher and A. Kriegel called the objects of their studies "Smooth spaces" (see [43]). Thereafter, P. Cherenack decided for the first time to name them "Frölicher spaces" as it appears in [27, 29]. The work of Cherenack gave the way to the background of topology, differential geometry and the symplectization of Frölicher spaces as it can be caught by the title of [8, 36]. Later on, A. Batubenge, P. Ntumba and B. Dugmore, both former PhD students of Cherenack, have continued researches in the field and have produced following papers [9, 10, 37, 85, 86]. These preliminary investigations were an attempt to lay down the foundation of what can be of need for further applications in geometry and dynamics.

The interest to the category of Frölicher spaces is growing among researchers as it can be seen from the amount of works aiming to compare it to others categories of smooth spaces. The comparisons and flourish recent works [7, 65, 106] point out the exciting fact that this is a convenient category able to host extensions of many theories from smooth manifolds: symplectic geometry [20, 22, 39, 40, 41, 51, 54, 58, 78, 85, 91, 109], symplectic reduction [68, 77, 89, 93, 112], Lie-Frölicher group and Lie-Frölicher-algebra of a Lie-Frölicher group [66, 85], sheaf theory and sheaf cohomology theory [7, 16, 26, 33, 46, 48, 96, 106], de Rham cohomology [17, 50, 53, 55, 60, 65, 73, 95, 100, 101, 110, 111, 115, 116], modern Hamiltonian formalism of mechanics [1, 8, 9, 21, 34, 47, 57, 70, 76, 86, 88, 93], Pois-
son geometry [21, 74, 90, 102, 120] and synthetic geometry [83, 84].

1.2 Thesis outline

Our research concerns a study entitled *Cohomologies on Symplectic Quotients of Locally Euclidean Frölicher Spaces*. We will deal with some specific selected cohomologies on symplectic quotients of these Frölicher spaces which are locally diffeomorphic to Euclidean Frölicher subspaces of $\mathbb{R}^n$ of constant dimension equal to $n$. That is a subcategory of the category of Frölicher spaces. The Frölicher spaces on which we perform the Marsden-Weinstein symplectic reduction in this introductory work are very close to $C^\infty$-manifolds. Such a space is a locally Euclidean Frölicher space whose open covering allows to map open sets diffeomorphically to open subsets of $\mathbb{R}^n$. The symplectic geometry on Frölicher spaces is developed in [8, 109, 112]. We assume that more information can be obtained using tools from smooth manifolds (differential geometry, symplectic geometry, cohomology and sheaf theories). In the category of smooth manifolds, it is known that the Marsden-Weinstein reduction performs a symplectic reduction in such a way as to obtain a quotient that is still a smooth manifold with a symplectic structure induced from the original one on the ambient space. We rather use objects of a more wider category that contains also smooth manifolds. The basic concepts will cover the differential geometry objects and topological properties that intervene in the symplectic reduction process and the construction of selected cohomologies. We must ensure that all definitions make sense in this new setting. In what follows we say "space" on the one hand, instead of "locally Euclidean space" or "locally diffeomorphic to Euclidean Frölicher subspaces of $\mathbb{R}^n". And also on the other hand, the term "reduced space" will mean "symplectic quotient" of a "symplectic locally Euclidean space".

We start with the usual assumptions in the Marsden-Weinstein symplectic reduction process as in the case of the category of smooth manifolds. For, on a space of the category under consideration, in the sight of given references [5, 8, 9, 65, 66, 77, 85, 89, 91, 112], we define a free, proper and Hamiltonian action provided with an equivariant moment map. We need to present simple and clearly understandable examples of concepts like: tangent vectors, vector fields, flows and their integral curves, exterior forms and exterior derivative, inner and exterior products, regular elements of the moment map. We can refer to [22, 36, 37, 39, 54, 78, 79, 80, 86, 88, 109]. Next we investigate some useful properties (as for example: existence, uniqueness and smoothness) of these concepts and also study the resulting equivalent concepts on the reduced space. And then, we raise new fundamental questions such as:

- What is the relationship between the differential geometry or topology on the ambient space and that induced on the reduced space? [40, 52]
1.2 Thesis outline

- Are the exterior derivative and the vector fields local concepts?
- Why is the compactness assumption for the operating Frölicher-Lie group so relevant?

Secondly, we construct the selected cohomologies: sheaf cohomology, Alexander-Spanier cohomology, singular cohomology, Čech cohomology and de Rham cohomology. The definition of the $p$th de Rham cohomology space depends on the existence, uniqueness and smoothness of a coboundary operator, this is in our case the exterior derivative. Some natural questions should be investigated (see [7, 16, 65, 73, 106, 114]) such as

- Which cohomology will give a good description of the topology of our spaces in the new category?

- Does the de Rham cohomology theorem still hold? (see [55, 95, 101])

We intend to use the approach in [114], where the last four cohomologies are both isomorphic to the sheaf cohomology under certain conditions. As a consequence, we establish an isomorphism between the five cohomologies. Recall that the Poincaré Lemma is a cornerstone in the proof of the de Rham theorem in the smooth manifolds case.

- What about the induced cohomologies on the reduced space? (see [96, 100, 110, 111])

- Recall that the cohomology space is the algebraic dual space of the homology space.

- Can we define in this new category the homotopy operator as it is a useful tool in the cohomology theory of cochain complexes for the category of smooth manifolds?

- Is the singular cohomology space a topological or differential invariant in the new category as it is for the category of smooth manifolds?

Recall that the study of several physical systems requires a good understanding of the topological background of the modeling space, which is usually a space provided with a smooth structure. In [10, 112], the preliminary investigations are given as an attempt to equip ourselves with information that can be of need for further applications in geometry and dynamics. Finally, as a possible application in the new category, we should investigate a modern formalism of mechanics on the reduced space by means of Poisson [102], Hamiltonian methods (see [1, 9, 21, 34, 47, 52, 57, 70, 76, 86, 88, 90, 93]).

The work will be presented as follows.

**Introduction and review of literature.**

**Frölicher spaces:** introduces basic concepts such as -Frölicher space, -Locally Euclidean Frölicher space, -Tangent structures on a $\mathbb{F}$-space, -Exterior algebra and -Examples.

**Symplectic reduction:** recalls -Symplectic structure and group action, -Flows, integral curves, -Moment map, -Reduced space.

**Ringed spaces:** presents -Structure ring, -Smooth representations and smooth Gelfand representation, -Natural Hausdorff paracompact property and others induced topological properties on the reduced space.

**Formalism of mechanics:** covers -Poisson geometry, -Hamiltonian systems.
Chapter 2

Frölicher spaces

2.1 Basic concepts on \(\mathbb{F}\)-spaces

2.1.1 \(\mathbb{F}\)-smooth structure

A nonempty set \(M\) whose smooth structure is determined by a pair of function sets \((\mathcal{C}_M, \mathcal{F}_M)\), where \(c \in \mathcal{C}_M\) and \(f \in \mathcal{F}_M\) are all those maps \(c : \mathbb{R} \to M\) and \(f : M \to \mathbb{R}\) satisfying the so called compatibility condition \(f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\). This is a smooth space in the sense of A. Frölicher [43], and P. Cherenack [29]. For convenience, we write \(C^\infty(\mathbb{R})\) for means of \(C^\infty(\mathbb{R}, \mathbb{R})\). Otherwise stated, given \(M\) a nonempty set, the pair \((\mathcal{C}_M, \mathcal{F}_M)\) is a \(\mathbb{F}\)-structure on \(M\) if it satisfies the compatibility (or duality) condition, that is,

\[
\mathcal{C}_M := \Gamma \mathcal{F}_M = \{ c : \mathbb{R} \to M | f \circ c \in C^\infty(\mathbb{R}) := C^\infty(\mathbb{R}, \mathbb{R}), \text{for all } f \in \mathcal{F}_M \}. \tag{2.1}
\]

\[
\mathcal{F}_M := \Phi \mathcal{C}_M = \{ f : M \to \mathbb{R} | f \circ c \in C^\infty(\mathbb{R}) := C^\infty(\mathbb{R}, \mathbb{R}), \text{for all } c \in \mathcal{C}_M \}. \tag{2.2}
\]

The triple \((M, \mathcal{C}_M, \mathcal{F}_M)\) is called a Frölicher space (or \(\mathbb{F}\)-space, smooth space). The elements of map sets \(\mathcal{C}_M\) and \(\mathcal{F}_M\) are structure curves and structure functions respectively. However, we shall often refer to \(M\) as a Frölicher space, or say \(\mathbb{F}\)-space instead of the triple above. The pair \((\mathcal{C}_M, \mathcal{F}_M)\) can be refered to as \(\mathbb{F}\)-smooth structure or Frölicher structure. The general setting for smooth spaces in the sense of Frölicher is given in [43, pp. 2 – 6, Section 1.1]. The generalization consists in taking any fixed nonempty set, say \(S\) as source of curves into \(M\) and any fixed nonempty set, say \(R\) as target of functions on \(M\) such that the compatibility condition invoked in Equations (2.1) and (2.2) are satisfied. However, in this thesis we are assuming \(S = R = \mathbb{R}\). In the literature on the concept of Frölicher spaces there is a tradition of using this simplified and practical way of thinking. Some mere examples of Frölicher spaces are the following.
Example 2.1.1. The triple \((\mathbb{R}^n, \mathcal{C}, \mathcal{F})\) is called the canonical \(\mathbb{F}\)-space. The smooth structure is given by Boman’s lemma (see [13, 14]). That is, \(\mathcal{C} = C^\infty(\mathbb{R}, \mathbb{R}^n)\) and \(\mathcal{F} = C^\infty(\mathbb{R}^n, \mathbb{R})\) so that the differentiability of elements of \(\mathcal{F}\) is tested by \(\mathcal{C}\) and the other way around. When \(n = 1\), the smooth curves and smooth functions coincide with the canonical smooth (differentiable) functions.

Example 2.1.2. Let \(M\) be a linear space and \(\hat{M}\) its algebraic dual. The linearly generated \(\mathbb{F}\)-structure is given by \((\Gamma \mathcal{F}_0, \Phi \mathcal{F}_0)\), where \(\mathcal{F}_0 \subseteq \hat{M}\) separates points in \(M\). That is, for each two distinct points \(p, q \in M\) there exists \(f \in \mathcal{F}_0 \subseteq \Phi \mathcal{F}_0 \subseteq \hat{M}\) such that \(f(p) \neq f(q)\).

2.1.2 Generating set of a \(\mathbb{F}\)-structure

A \(\mathbb{F}\)-structure on a set \(M\) can be generated by either a subset \(\mathcal{F}_0 \subset \mathbb{R}^M\) of functions or a subset \(\mathcal{C}_0 \subset \mathbb{R}^M\) of curves as from Equation (2.1) and Equation (2.2). The properties given below are a straightforward consequence of the \(\mathbb{F}\)-structure generating process on \(M\). Let \(\mathcal{F}_0, \mathcal{F}_1 \subset \mathbb{R}^M\) and \(\mathcal{C}_0, \mathcal{C}_1 \subset M^\mathbb{R}\), where \(M\) is a non-empty set. The following hold.

\[
\text{If } \mathcal{F}_0 \subset \mathcal{F}_1 \text{ then } \Gamma \mathcal{F}_0 \supseteq \Gamma \mathcal{F}_1, \mathcal{F}_0 \subseteq \Phi \Gamma \mathcal{F}_0 \text{ and } \Gamma \mathcal{F}_0 = \Gamma \Phi \Gamma \mathcal{F}_0. \quad (2.3)
\]

\[
\text{If } \mathcal{C}_0 \subset \mathcal{C}_1 \text{ then } \Phi \mathcal{C}_0 \supseteq \Phi \mathcal{C}_1, \mathcal{C}_0 \subseteq \Gamma \Phi \mathcal{C}_0 \text{ and } \Phi \mathcal{C}_0 = \Phi \Gamma \Phi \mathcal{C}_0. \quad (2.4)
\]

Notice that by setting \(\Gamma \mathcal{F}_0 = \mathcal{C}_M\) and \(\Phi \Gamma \mathcal{F}_0 = \mathcal{F}_M\), the \(\mathbb{F}\)-structure \((\Gamma \mathcal{F}_0, \Phi \Gamma \mathcal{F}_0)\) on \(M\) is said to be generated by \(\mathcal{F}_0\) or by functions in \(\mathcal{F}_0\). Also, given \(\Phi \mathcal{C}_0 = \mathcal{F}_M\) and \(\Gamma \Phi \mathcal{C}_0 = \mathcal{C}_M\), the \(\mathbb{F}\)-structure \((\Gamma \Phi \mathcal{C}_0, \Phi \mathcal{C}_0)\) on \(M\) is said to be generated by \(\mathcal{C}_0\) or by curves in \(\mathcal{C}_0\). Furthermore, if the power sets \(\mathcal{P}(\mathbb{R}^M)\) and \(\mathcal{P}(M^\mathbb{R})\) are considered ordered by inclusion, then both \(\mathcal{P}(\mathbb{R}^M)\) and \(\mathcal{P}(M^\mathbb{R})\) together with the identity \((I)\) and inclusion \((i)\) mappings are categories (see [65]) which we denote by \(\mathcal{C}_I\) and \(\mathcal{C}_c\) respectively in such a way that functions \(\Gamma\) and \(\Phi\) are order-reversing functors (see [11]) in the category \textbf{Set}. They imply the following important inclusions in the process of generating a Frölicher structure with regard to Equations (2.3) and (2.4) above. Using them, it is easy to show for instance that the Euclidean space \(\mathbb{R}\) is a Frölicher space generated by the identity function \(id\), in which case all \(C^\infty\) curves and real-valued functions form the canonical structure. A further natural example is that \(\Gamma\) and \(\Phi\) applied to \(\{a\}\), with \(a(x) = |x|\) on \(\mathbb{R}\) induce a Frölicher structure in which smooth curves are only those real functions whose graphs become flatter in crossing the origin. The \(\mathbb{F}\)-structure is finitely generated, countably generated or infinitely generated if the generating set is respectively a finite set, a countable set or an infinite set. The \(\mathbb{F}\)-structure is linearly generated if \(\mathcal{F}_0\) or \(\mathcal{C}_0\) is a set of linear functions or linear curves provided that \(M\) is a linear space (see [43]). Moreover, the diagrams below show that both \(\Phi\) and \(\Gamma\) are contravariant functors.
2.1 Basic concepts on $\mathbb{F}$-spaces

Their composition $\Phi \Gamma$ of gives:

The dual composition $\Gamma \Phi$ of functors can be dealt with in the same way.

The arrows $(1)$, $(2)$ and $(3)$ in the diagram above are compositions of morphisms in respective categories. Obviously, they read as follows:

$(1) = \iota_{\mathcal{C}_1} \circ \iota_{\mathcal{C}_0}$

$(2) = \Phi(\iota_{\mathcal{C}_1} \circ \iota_{\mathcal{C}_0}) = \Phi(\iota_{\mathcal{C}_0}) \circ \Phi(\iota_{\mathcal{C}_1}) = \iota_{\Phi \mathcal{C}_1} \circ \iota_{\Phi \mathcal{C}_2}$

$(3) = \Gamma \Phi(\iota_{\mathcal{C}_1} \circ \iota_{\mathcal{C}_0}) = \Gamma(\iota_{\Phi \mathcal{C}_1} \circ \iota_{\Phi \mathcal{C}_2}) = \Gamma(\iota_{\Phi \mathcal{C}_2}) \circ \Gamma(\iota_{\Phi \mathcal{C}_1}) = \iota_{\Gamma \Phi \mathcal{C}_1} \circ \iota_{\Gamma \Phi \mathcal{C}_0}$
2.1 Basic concepts on $\mathbb{F}$-spaces

2.1.3 $\mathbb{F}$-smooth maps and category of $\mathbb{F}$-spaces

**Definition 2.1.1.** Let $M$ and $N$ be two $\mathbb{F}$-spaces. A set map $\varphi : M \to N$ is said to be smooth if $\varphi$ is structure (curves and functions) preserving. That is, $\varphi \circ C_M \subseteq C_N$ and $F_N \circ \varphi \subseteq F_M$, which can be written as $F_N \circ \varphi \circ C_M \subseteq (C^\infty(\mathbb{R}, \mathbb{R}))$.

Naturally, any structure curve is a smooth map when one replaces $M$ by $\mathbb{R}$ in the Definition of a smooth map above and so is each structure function when one replaces $N$ by $\mathbb{R}$. We will denote by

$$C^\infty(M, N) := \{ \varphi : M \to N \mid \varphi \text{ is } \mathbb{F}\text{-smooth} \},$$

the set of all smooth maps between two $\mathbb{F}$-spaces $M$ and $N$. Foremost, we can now confirm the existence of $\text{FrI}$ the category of Frölicher spaces. Indeed, a $\text{FrI}$-morphism or a morphism will mean a smooth map in the sense of Frölicher as by the Definition above. It follows that

$$\theta \circ \varphi \in C^\infty(M, P), \text{ whenever } \varphi \in C^\infty(M, N) \text{ and } \theta \in C^\infty(N, P). \quad (2.6)$$

Consequently, given $M$ and $N$ two $\text{FrI}$-objects, and a set map $\varphi : M \to N$, we have the following

$$\varphi \in C^\infty(M, N) \text{ if, and only if } \theta \circ \varphi \in C^\infty(M, P), \text{ where } \theta \in C^\infty(N, P). \quad (2.7)$$

In [11] are summarized some interesting properties owned by the resulting category $\text{FrI}$. It is complete, and cocomplete (see [27, 29, 32, 43, 63]). So, in this category, subspaces, quotients, products and coproducts exist as limits or colimits lifted from the category $\text{Set}$ of sets.

Furthermore, it is also topological over the category $\text{Set}$ (see [19, 27, 29, 63]). We say that the forgetful functor $\text{FrI} \to \text{Set}$ is faithful and topological, this means that $\text{FrI}$ behaves as $\text{Top}$. It follows that one can naturally define induced $\mathbb{F}$-smooth structures and induced topologies on subsets, quotients, equalizers, coequalizers, products and coproducts in the category $\text{Set}$ of sets.

Another important feature of Frölicher spaces is of their category $\text{FrI}$ being Cartesian closed. This means that there are finite products and terminal objects. Furthermore, given any three Frölicher spaces $M, N$ and $P$, there is a natural diffeomorphism, known as the exponential law,

$$C^\infty(M, C^\infty(N, P)) \cong C^\infty(M \times N, P),$$

such that $C^\infty(N, P)$ can be endowed in canonical way with a $\mathbb{F}$-smooth structure. Otherwise, any set of $\mathbb{F}$-smooth maps between Frölicher spaces is also a Frölicher space (see [27, 29, 30, 43, 44, 63, 67]).
A $\mathbb{F}$-smooth map $\varphi \in C^\infty(M,N)$ is said to be a diffeomorphism of $M$ onto $N$ if $\varphi$ is a bijective map such that the inverse map $\varphi^{-1} : N \rightarrow M$ is $\mathbb{F}$-smooth (see [8, 85] for more on diffeomorphisms).

A linear space $M$ is a linear $\mathbb{F}$-space if the structure functions and structure curves are linear, provided that the addition map $+: M \times M \rightarrow M$, such that $(x, y) \mapsto + (x, y) := x + y$ and the scalar multiplication map $\bullet : \mathbb{R} \times M \rightarrow M$ such that $\bullet (t, x) := t \bullet x$ are $\mathbb{F}$-smooth maps, respectively.

### 2.1.4 $\mathbb{F}$-topologies

Two topologies are defined on a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$. The initial topology generated by structure functions $f \in \mathbb{R}^M$ is

$$
\tau_{\mathcal{F}_M} = \{ U \subset M | U = \bigcup_{f \in \mathcal{F}_M} f^{-1}(V) \}, \text{ (see [43])} \tag{2.9}
$$

where $V$ lies in $\tau_{\mathbb{R}}$, the standard topology of $\mathbb{R}$. The collection $B = \{ f^{-1}(0, +\infty) \}_{f \in \mathcal{F}_M}$ is its base (see [36]) and its subbase is $S = \{ f^{-1}(0, 1) \}_{f \in \mathcal{F}_M}$ (see [43]).

The second natural topology is generated by curves, that is,

$$
\tau_{\mathcal{C}_M} = \{ U \subset M | c^{-1}(U) \in \tau_{\mathbb{R}} \}, \tag{2.10}
$$

where $c \in \mathcal{C}_M$ (see [43]).

We call $\tau_{\mathcal{F}_M}$ the topology of the Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$.

**Lemma 2.1.1.** $\tau_{\mathcal{F}_M} \subset \tau_{\mathcal{C}_M}$.

**Proof.** Let $U \in \tau_{\mathcal{F}_M}$. That is, $U = \bigcup_{f \in \mathcal{F}_M} f^{-1}(V)$, where, $V$ is open in $\mathbb{R}$. For an arbitrary $c \in \mathcal{C}_M$, $c^{-1}(U) = c^{-1} \left( \bigcup_{f \in \mathcal{F}_M} f^{-1}(V) \right) = \bigcup_{f \in \mathcal{F}_M} (f \circ c)^{-1}(V) \in \tau_{\mathcal{C}_M}$. But $V \in \tau_{\mathbb{R}}$ and $f \circ c \in C^\infty(\mathbb{R}) := C^\infty(\mathbb{R}, \mathbb{R})$. Hence $c^{-1}(U)$ is an open set in $\mathbb{R}$ as arbitrary union of elements of $\tau_{\mathbb{R}}$. Thus $U \in \tau_{\mathcal{C}_M}$. \hfill $\square$

**Lemma 2.1.2.** If $\varphi$ is a Frölicher morphism then $\varphi$ is a Top-morphism for both $\tau_{\mathcal{C}_M}$ and $\tau_{\mathcal{F}_M}$.

Also, structure curves and functions are continuous in both $\mathbb{F}$-topologies.

**Proof.** Let $U \in \tau_{\mathcal{C}_M}$ ie for every $c \in \mathcal{C}_M$, $c^{-1}(U)$ is open in $\mathbb{R}$. $\varphi$ smooth means that $\varphi^{-1}(g^{-1}(0, 1))$ is a subbasis open for $\tau_{\mathcal{F}_M}$, iff for every $c \in \mathcal{C}_M$, $\varphi \circ c = d$ where $d \in \mathcal{C}_N$.

Assume $U \subset N$ such that $U \in \tau_{\mathcal{C}_N}$. It follows that $d^{-1}(U) = c^{-1}(\varphi^{-1}(U))$ is open in $\mathbb{R}$ where

$$
\tau_{\mathcal{C}_N} = \{ U | d^{-1}(U) \text{ open in } \mathbb{R}, d \in \mathcal{C}_N, U \subset N \}.
$$
Thus $\varphi^{-1}(U)$ is open in $M$ for
\[ \tau_{C_M} = \{ V \mid c^{-1}(V) \text{ open in } \mathbb{R}, d \in C_M, V \subset M \}. \]

Hence $\varphi$ is continuous for $\tau_{C_M}$.

Also, assume $U \subset N$ such that $U \in \tau_{F_N}$. It follows that
\[ U = \bigcup_{i \in I} \bigcap_{j=1}^{n} g_j^{-1}(0,1), \quad \text{where } g_j \in F_N. \tag{2.11} \]

Now we can show that $\varphi^{-1}(U)$ is open in $M$ for $\tau_{F_N}$.
\[ \varphi^{-1}(U) = \varphi^{-1}\left( \bigcup_{i \in I} \bigcap_{j=1}^{n} g_j^{-1}(0,1) \right) \]
\[ = \bigcup_{i \in I} \bigcap_{j=1}^{n} \varphi^{-1}g_j^{-1}(0,1) \]
\[ = \bigcup_{i \in I} \bigcap_{j=1}^{n} (g_j \circ \varphi)^{-1}(0,1) \]
\[ = \bigcup_{i \in I} f_i^{-1}(0,1), \]
which is open in $M$ for $\tau_{F_M}$. Here $\varphi$ is continuous, but for $\tau_{F_M}$. Also, since structure curves and structure functions are smooth then they are continuous in both topologies. \qed

Lemma 2.1.3. Let $\varphi : M \longrightarrow N$ be a map between underlying sets of Frl-objects. If $(U_i)_{i \in I}$ is a $\tau_{C_M}$-open covering of $M$ such that for any $i$, the restriction of $\varphi$ to $U_i$ is a Frl-morphism then $\varphi$ is a Frl-morphism.

Proof. (See [23]).

Example 2.1.3. In the F-structure $(C_Q, F_Q)$ generated on $Q$ by the inclusion map, only constant curves are structure (smooth) curves and all real-valued functions on $Q$ are structure (smooth) functions. That is, the F-structure $(C_Q, F_Q)$ is discrete since $F_Q = \mathbb{R}^Q$. One concludes that $\tau_{C_Q} = \mathcal{P}(Q)$, that is, a discrete topology. Hence, $Q$ is a balanced space since $\tau_{F_Q} = \mathcal{P}(Q) = \tau_{C_Q}$ (see [10]).

The topology and geometry on fiber bundle and associated objects are closely dependent of these preliminary results. In any case, when performing a symplectic reduction on the space under consideration, one must refer to the following objects: subspaces, quotients, products,
coproducts and tangent bundles. Therefore, the aforementioned induced (initial and final) objects are concurrently carrying natural three induced topologies as objects in both Top and Frl. We studied and compared the topologies on a subspace and on a quotient space in this category (see [10]), as well as the product and the coproduct of Frölicher spaces (See [11]). We show that the $F$-topologies on a $F$-subspace and the $F$-product space contain the trace topology and the product topology respectively. Both the $F$-subspace and the $F$-product space are initial objects in Frl. Whereas, for the $F$-quotient space and the $F$-coproduct space, the final objects, the $F$-quotient topology is equal to the identification topology and $F$-coproduct topology coincides with the coproduct topology.

### 2.1.5 $F$-subspace

Let $M$ be a Frl-object and $S$ any nonempty subset in its underlying set. The $F$-structure on $S$ is generated by $\iota_S : S \hookrightarrow M$, the canonical inclusion. For this purpose, let $F_{o,S} = \{ f|_S \mid f|_S = f \circ \iota_S, f \in F_{o,M} \} = F_{o,M} \circ \iota_S = \iota_S^* F_{o,M} = F_{o,M}|_S$ be a set generating the $F$-structure on $S$. The structure functions set is given below with regard to the compatibility condition. $C_S = \bigcap F_o = \{ c' : \mathbb{R} \rightarrow S \mid f|_S \circ c' \in C^\infty(\mathbb{R}) \text{ for all } f|_S \in F_{o,S} \}$ or equivalently $C_S = \{ c' : \mathbb{R} \rightarrow S \mid f \circ (\iota_S \circ c') \in C^\infty(\mathbb{R}) \text{ for all } f \in F_{o,M} \} = \{ c' : \mathbb{R} \rightarrow S \mid \iota_S \circ c' \in C_M \}$. Also the following holds: $C_S = \{ c' : \mathbb{R} \rightarrow S \mid c'(\mathbb{R}) \subseteq S \}$. The compatibility condition yields the structure curves set as follows: $F_S = \Phi C_S = \{ f' : S \rightarrow \mathbb{R} \mid f' \circ c' \in C^\infty(\mathbb{R}) \text{ for all } c' \in C_S \}$ or equivalently $F_S = \{ f' : S \rightarrow \mathbb{R} \mid f' \circ c' \in C^\infty(\mathbb{R}), c'(\mathbb{R}) \subseteq S \}$. Also the following holds: $F_S = \{ f' : S \rightarrow \mathbb{R} \mid f'(c'(\mathbb{R})) \subseteq f'(S) \}$. It follows that $F_M \circ \iota_S \subseteq F_S$. Therefore, $F_S$ is not the restriction of $F_M$ on $S$. That is the case when $S$ is open or closed set.

The $F$-space $(S, \Gamma F_{o,S}, \Phi \Gamma F_{o,S}) = (S, C_S, F_S)$ is the $F$-subspace of $(M, C_M, F_M)$. Also the pair $(C_S, F_S)$ is called the initial $F$-structure on $S$ induced by $(C_M, F_M)$, making $\iota_S$ a smooth map. That is, $\iota_S$ is smooth if, and only if $\iota_S \circ C_S \subseteq C_M$ if, and only if $F_M \circ \iota_S \subseteq F_S$.

Every subset $S$ of a $F$-space $M$ is canonically a $F$- subspace with regard to the construction of the $F$-structure on $S$ done above. Since the category Frl is topological over Set, complete and co-complete, $S$ can be made into a subobject so as to carry two $F$-topologies. That is $\tau_{F_S}$ and $\tau_{C_S}$. These are topologies, where, all smooth functions and smooth curves are continuous. The collection $\{ g^{-1}(0, \infty) \mid g \in F_S \}$ is a base for $\tau_{F_S}$. Moreover, $S$ has the relative topology as a Top-subobject, that is,

$$
\tau_{F_M}(S) = \{ S \cap U \mid U \in \tau_{F_M} \}.
$$

\(\text{Lemma 2.1.4.}\) Let $M$ be a Frl-object, $S$ a subset of its underlying set and $f \in F_M$. Then

1. $S \cap f^{-1}(0, +\infty)$ is $\tau_{F_M}(S)$-basic open in $S$.
2. $S \cap f^{-1}(0, 1)$ is $\tau_{F_M}(S)$-subbasic open in $S$.
3. $\iota_S$ is continuous in $\tau_{F_M}(S)$.
2.1 Basic concepts on $\mathbb{F}$-spaces

Proof.

1. In the topology $\tau_{F_M}$, a set $V$ is open if $V = \bigcup_{f \in F_M} [f^{-1}(0, \infty)]$. Thus

$$S \cap V = S \cap \left( \bigcup_{f \in F_M} [f^{-1}(0, \infty)] \right) = \bigcup_{f \in F_M} [S \cup f^{-1}(0, \infty)].$$

It remains to show that the family $\{S \cap f^{-1}(0, \infty) \mid f \in F_M\}$ is closed under finite intersection. Let $\{f_i(0, \infty) \mid 1 \leq i \leq n\}$ be a finite collection of $\tau_{F_M}$-basic open sets. Since $\{f_i^{-1}(0, \infty) \mid f \in F_M\}$ is closed under finite intersection, $\bigcap_{i=1}^{n} f_i^{-1}(0, \infty) = g^{-1}(0, \infty)$ with $g \in F_M$. Since $S \cap f^{-1}_i(0, \infty)$ lies in the collection $\{S \cap f^{-1}(0, \infty) \mid f \in F_M\}$, then $\bigcap_{i=1}^{n} (S \cap f_i^{-1}(0, \infty)) = S \cap (\bigcap_{i=1}^{n} f_i^{-1}(0, \infty)) = S \cap g^{-1}(0, \infty)$ also lies in $\{S \cap f^{-1}(0, \infty) \mid f \in F_M\}$. In the sequel $\{S \cap f^{-1}(0, \infty) \mid f \in F_M\}$ is closed under finite intersection. Hence it is a base for $\tau_{F_M}(S)$. That is, $S \cap f^{-1}(0, \infty)$ is a $\tau_{F_M}(S)$-basic open set in $S$.

2. For the second assertion, let $V \in \tau_{F_M}$. That is, $V = \bigcup_{j \in J} \bigcap_{i=1}^{n} f_{ij}^{-1}(0, 1)$, where $f_{ij} \in F_M$ and $S \cap V \in \tau_{F_M}(S)$. It follows that $S \cap V = S \cap \bigcup_{j \in J} \bigcap_{i=1}^{n} f_{ij}^{-1}(0, 1) = \bigcup_{j \in J} \bigcap_{i=1}^{n} (S \cap f_{ij}^{-1}(0, 1)).$

Therefore, $\bigcap_{i=1}^{n} (S \cap f_{i}^{-1}(0, 1))$ is a $\tau_{F_M}(S)$-basic open set. So $S \cap f_{i}^{-1}(0, 1)$ is a $\tau_{F_M}(S)$-subbasic open set.

3. For $U \in \tau_{F_M}$, we have $i_{S}^{-1}(U) = S \cap U$. So $i_{S}^{-1}(U) \in \tau_{F_M}(S)$. Thus, $i_{S}$ is continuous. Moreover, $i_{S}^{-1}(f^{-1}(0, \infty)) = S \cap f^{-1}(0, \infty)$ is a $\tau_{F_M}(S)$-basic open set. \qed

Proposition 2.1.5. Let $S$ be a $\mathbb{F}$-subspace of a Frl-object $M$. Then $\tau_{F_M}(S) \subset \tau_{F_S} \subset \tau_{C_S}$. That is, the trace topology $\tau_{F_M}(S)$ is the smallest topology on $S$ for which the inclusion map $i_{S}$ is continuous.

Proof.

1. Let $U \in \tau_{F_M}(S)$. That is, $U = S \cap V = \bigcup_{i \in I} (S \cap f_{i}^{-1}(0, \infty))$, with $V = \bigcup_{i} f_{i}^{-1}(0, \infty)$. It follows that

$$U = \bigcup_{i \in I} (i_{S}^{-1}(f_{i}^{-1}(0, \infty))) = \bigcup_{i \in I} ((f_{i} \circ i_{S})^{-1}(0, \infty)) = \bigcup_{i \in I} (g_{i}^{-1}(0, \infty)) \in \tau_{F_S},$$

where $g_{i} = f_{i} \mid S$. So the required inclusions hold.
2. Let \( V \in \tau_{F_M}(S) \) and \( \tau \) is any topology on \( S \), where \( \iota_S \) is continuous. Then \( V = S \cap U \), with \( U \in \tau_{F_M} \) and \( \iota_S^{-1}(U) \in \tau \) since \( \iota_S \) is continuous for \( \tau \). So, \( \iota_S^{-1}(U) = S \cap U = V \). Hence \( V \in \tau \) and \( \tau_{F_M}(S) \subseteq \tau \) is the smallest topology on \( S \) for which \( \iota_S \) is continuous. \( \square \)

**Proposition 2.1.6.** Let \( M \) be a Frölicher object and \( S \) a subset of its underlying set. The following hold: If \( S \in \tau_{F_M} \), then \( \tau_{F_S} = \tau_{F_M}(S) \). If \( S \in \tau_{C_M} \), then \( \tau_{C_S} = \tau_{C_M}(S) \).

**Proof.**

1. Assume \( U \in \tau_{F_S} \), that is \( U = \bigcup_{i \in I} (f_i|S)^{-1}(0, \infty) \), where \( f_i \in F_M \) and \( f_i|S \) is a generator of the structure \( (C_S, F_S) \). It follows that

\[
U = \iota_S^{-1}\iota_S(U) = \bigcup_{i \in I} \iota_S^{-1}\iota_S(f_i|S)^{-1}(0, \infty) = \bigcup_{i \in I} f_i^{-1}(0, \infty)
\]

using the fact that \( S \) is open and \( \iota_S \) is an open map. That is, \( \tau_{F_S} \subseteq \tau_{F_M} \). The reverse inclusion \( \tau_{F_M}(S) \subseteq \tau_{F_S} \) was proved in Proposition 2.1.5 above.

2. Assume \( U \in \tau_{C_S} \), that is \( d^{-1}(U) \in \tau_{C_R} \), with \( d \in C_S \). But \( S \in \tau_{C_M} \), hence for some \( c \in C_M \), \( c^{-1}(S) \in \tau_R = \tau_R \). (We used the fact that \( \tau_{C_R} = \tau_R \)). Let \( d \in C_S \). It follows that

\[
d^{-1}(U) = d^{-1}(\iota_S^{-1}(\iota_S(U))) = (\iota_S \circ d)^{-1}(\iota_S(U)) = c^{-1}(U),
\]

where \( c \in C_M \). Since \( \iota_S \) is smooth, \( d^{-1}(U) = c^{-1}(U) \in \tau_R \). Now \( U \in \tau_{C_M} \), that is \( U \subseteq S \subseteq M \). It follows that

\[
U = \iota_S^{-1}(\iota_S(U)) = S \cap \iota_S(U) = S \cap U \in \tau_{C_M}(S)
\]

since \( U \in \tau_{C_M} \). Therefore \( \tau_{C_M}(S) \supseteq \tau_{C_S} \). Hence, \( \tau_{C_S} = \tau_{C_M}(S) \). \( \square \)

### 2.1.6 Ñ-product space

Let \( M^* := \prod_{i \in I} M_i \) denote the product in \textbf{Sets}. The initial structure on \( M^* \) in \textbf{Frl} is the \( \mathbb{F} \)-structure generated by the family \( (f \circ p_i : M^* \to \mathbb{R}) \); where \( f : M_i \to \mathbb{R} \), and \( p_i : M^* \to M_i \) are projections in \textbf{Sets}. Now let \( F_{oM_i} \) generate the \( \mathbb{F} \)-structure \( (C_{M_i}, F_{M_i}) \) on each \( M_i \). The resulting Frölicher product structure \( (C_{M^*}, F_{M^*}) \) is generated by the collection \( F_o \) of arbitrary unions of functions of the form \( f_i \circ p_i \) for all \( f_i \in F_{oM_i} \). The \( \mathbb{F} \)-space \( (M^*, C_{M^*}, F_{M^*}) \) is called the \( \mathbb{F} \)-product of \( M_i \) or a product of \( \mathbb{F} \)-spaces \( (M_i, C_{M_i}, F_{M_i}) \).
The pair \((\mathcal{C}_{M^*}, \mathcal{F}_{M^*})\) is the initial \(\mathbb{F}\)-product structure such that all the projections maps \(p_i\) are smooth maps. The topologies \(\tau_{\mathcal{F}_{M^*}}\) and \(\tau_{\mathcal{C}_{M^*}}\) induced by the Frölicher structure are called \(\mathbb{F}\)-topologies on \(M^*\) or \(\mathbb{F}\)-product topologies.

**Lemma 2.1.7.** Let \(M^*\) be a topological product of \(\mathbb{F}\)-spaces \(M_i\). Let \(p_j : M^* \to M_j\) be the \(j\)th projection and \(f_{ji} \in \mathcal{F}_{M_j}\) an arbitrary collection of structure functions on \(M_j\) (for a fixed \(j\)). Let \(f_{ji}^{-1}(0,1)\) be a subbasic open set in \(\tau_{\mathcal{F}_{M_j}}\), for each \(i \in I\). Then \(p_j^{-1}(f_{ji}^{-1}(0,1))\) is also a subbasic open set in \(\tau_{\Pi}\).

This is a well-known result from point-set topology. Also it implies that the canonical projection \(p_j\) is a continuous, onto and an open map for \(\tau_{\Pi}\). We shall make use of the fact that smooth maps on a Frölicher space \(M\) are continuous in both \(\tau_{\mathcal{F}_{M}}\) and \(\tau_{\mathcal{C}_{M}}\), as proved in [10] and notice that \(\tau_{\mathcal{F}_{M^*}}\) and \(\tau_{\mathcal{C}_{M^*}}\) on \(M^*\) are respectively the smallest and the largest ones in which all smooth functions from \(M^*\) and all smooth curves into \(M^*\) are continuous. Now we can compare the usual topology \(\tau_{\Pi}\) of the product of \(\mathbb{F}\)-spaces with the two topologies arising from the \(\mathbb{F}\)-product structure. That is the topology induced by the topologies \(\tau_{\mathcal{F}_{M_i}}\) of factors, such that if \(U_{jk} \in \tau_{\mathcal{F}_{M_{jk}}}\) is open in the coordinate space \(M_{jk}\) and \(p_{jk} : M^* \to M_{jk}\) is the projection then \(p_{jk}^{-1}(U_{jk}) = \Pi\{M_i : i \neq j_k\} \times U_{jk}\) form a subbase in \(\tau_{\Pi}\), and the set \(B = \{\cap_{j=1}^n p_j^{-1}(U_j) \mid U_j \in \tau_{\mathcal{F}_{M_j}}\}\) is its base. Without loss of generality, we are concerned with a countable collection of Frölicher spaces.

**Theorem 2.1.8.** Let \(\tau^*\) and \(\tau_{\mathcal{C}}^*\) be the \(\mathbb{F}\)-topologies induced by the Frölicher structure on countable product \(M^* = \Pi M_i\) of Frölicher spaces. Let \(\tau_{\Pi}\) the Tychonoff topology on \(M^*\). Then \(\tau_{\Pi} = \tau^* \subseteq \tau_{\mathcal{C}}^*\).

**Proof.**

(a) We first prove the inclusion \(\tau^* \subseteq \tau_{\Pi}\). We may assume that \(V = f^{-1}(0,1)\) or an arbitrary union of them, where \(f \in \mathcal{F}_{M^*}\) is a \(\tau_{\mathcal{F}_{M^*}}\)-subbasic open set. Referring to the characterization of open sets we have what follows. For each \(x \in V = f^{-1}(0,1) \subset M^*\), \(x \in f^{-1}(t)\), for some \(t \in (0,1)\) such that \(f(x) = t\) and \(x = (x_i)_i\) with \(x_i \in M_i\) for \(i\) ranging in a countable set \(I\). Hence, there is an open set \(U_{i_0}\) such that \(x_{i_0} \in U_{i_0} \subset M_{i_0}\) with \(U_{i_0} \neq M_k\) for \(k = 1, \ldots\) and \(U_j = M_j\) for \(j \neq 1, \ldots\). That is, \((x_k) \in \prod_{k=1}^n U_k\) and \((x_{j\neq1,n}) \in \prod M_j\). Thus, \(f = f_i \circ p_i\) yields
\[
t = f(x_i)_{i \in I} = (f_i \circ p_i)((x_i)_{i \in I}) = f_i(x_i).
\]
It follows that \(x = (x_i)_{i \in I} \in f^{-1}(t) = p_k^{-1} \circ f_k^{-1}(t)\) and even better,
\[
x \in f^{-1}(0,1) = p_k^{-1} \circ f_k^{-1}(0,1)
\]
for \( k = 1, \ldots \). Hence, there exists \( U \in \tau \Pi \), where \( x \in U = \prod_{k=1}^{\infty} U_k \times \prod_{j \neq 1, \ldots} M_j \) from the definition of a \( \tau \Pi \)-basic open set. And since \( p_k \) is open and smooth, so continuous.

\[
U = \bigcap_{k=1}^{\infty} p_k^{-1}(U_k) \quad \text{and} \quad U_k = f_k^{-1}(0,1)
\]

Recalling that \( f \) is among the \( g_k \), one has

\[
U = \bigcap_{k=1}^{\infty} p_k^{-1} f_k^{-1}(0,1) = \bigcap_{k=1}^{\infty} (g_k^{-1}(0,1)) \subset f^{-1}(0,1) = V.
\]

Therefore, \( U \subset V \). That is, \( V \) contains a basic open set \( U \) of \( \tau \Pi \). Thus, \( V \in \tau \Pi \), which gives \( \tau^* \subset \tau \Pi \).

The reverse inclusion is a classical result in point-set topology, which we would verify its validity in the category \textsc{Frli} as follows.

(b) Let \( V \in \tau \Pi \). For some countable indexing set \( J \subset \mathbb{N} \) we know that \( \{p_j^{-1}(U_{ij})\} \) \((i,j) \in I \times J\), where \( U_{ij} \in \tau_{M_j} \) is a subbase for \( \tau \Pi \), and \( p_j \) are continuous in \( \tau \Pi \) since they are \( \mathbb{F} \)-smooth by definition. Hence \( n \bigcap_{j=1}^{n} \{p_j^{-1}(U_{ij})\} \) is a base for \( \tau^* \). Therefore,

\[
V = \bigcup_{k \in \Lambda} \bigcap_{j=1}^{n} \{p_j^{-1}(U_{ij})\} \in I.
\]

But for all \( f_j \in \mathcal{F}_{M_j} \), \( \{f_j^{-1}(0,1)\} \) is a subbase for \( \tau_{M_j} \) so that \( \bigcap_{j=1}^{n} \{f_j^{-1}(0,1)\} \) is a base and

\[
p_j^{-1} \bigcap_{j=1}^{n} \{f_j^{-1}(0,1)\} = \bigcap_{j=1}^{n} \{p_j^{-1}(f_j^{-1}(0,1))\} = \bigcap_{j=1}^{n} (f_j \circ p_j)^{-1}(0,1)
\]

is a basic open set in \( \tau^* \). Now let \( f_{ij} \circ p_j = g_i \), which is a generator for the Frölicher product structure. We have \( \bigcup_{k \in \Lambda} \bigcap_{j=1}^{n} (g_i^{-1}(0,1)) \in \tau^* \). Thus, \( V \in \tau^* \), which shows that \( \tau \Pi \subseteq \tau^* \). The equality \( \tau \Pi = \tau^* \) is proved.

(c) We finally show that \( \tau^* \) is weaker than \( \tau C \). Let \( V \in \tau^* \). That is, \( V = \bigcup_{\lambda \in \Lambda} \bigcap_{j=1}^{n} p_j^{-1}(U_{j\lambda}) \), with \( U_{j\lambda} \in \tau_{M_j} \). Notice that \( V \) will be in \( \tau_{CM} \), if \( c^{-1}(V) \in \tau_R \) for all \( c \in \mathcal{C}_{M^*} \). Recall that
\( c \in C_{M^*} \) is such that \( c = (c_i)_{i \in I} \) where \( c_i \in C_{M_i} \) for all \( i \). We have

\[
c^{-1}(V) = c^{-1}\left( \bigcup_{\lambda \in \Lambda} \bigcap_{j=1}^{n} p_j^{-1}(U_{jk}) \right) = \bigcup_{\lambda \in \Lambda} \bigcap_{j=1}^{n} (c^{-1}(p_j^{-1}(U_{jk}))) = \bigcup_{\lambda \in \Lambda} \bigcap_{j=1}^{n} (p_j \circ c)^{-1}(U_{jk}) = \bigcup_{\lambda \in \Lambda} \bigcap_{j=1}^{n} c_j^{-1}(U_{jk}),
\]

where \( U_{jk} \) is a \( \tau_{C_{M_j}} \)-open set, and \( c_j \in C_{M_j} \). Therefore, \( \bigcup_{\lambda \in \Lambda} \bigcap_{j=1}^{n} c_j^{-1}(U_{jk}) \in \tau_R \). Thus \( V \in C_{M^*} \) and the inclusion \( \tau^* \subseteq \tau_{C_{M^*}} \) holds true.

The theorem above read as the following inclusions \( \tau_{\Pi} \subseteq \tau_{F_{M^*}} \subseteq \tau_{C_{M^*}} \).

### 2.1.7 \( F \)-quotient space

We are given an equivalence relation \( \sim \) on the underlying set of a Fril-object \( M \) such that the quotient \( \tilde{M} := M/\sim \) in Sets is given the final Frölicher structure generated by the canonical map \( \pi_{\sim} : M \rightarrow \tilde{M} \). Recall the universality condition as follows. For an arbitrary object \( N \) in Sets and a map \( f : M \rightarrow N \), one obtains an equivalence relation \( \sim_f \) in the underlying set \( M \) by defining \( (x, y) \in \sim_f \) if and only if \( f(x) = f(y) \), for \( x, y \in M \), the equivalence classes of which are the fibers of \( f \). They are \( f^{-1}(s) \), where \( s \in \text{im}(f) \). Taking \( f = \pi_{\sim} \), it is clear that every equivalence relation \( \sim \) arises in this way. The map \( f \) is said to be consistent with \( \sim \) if \( x \sim y \) implies \( f(x) = f(y) \), ie \( f \) is constant on each equivalence class modulo \( \sim \) and there exists a unique one-to-one map \( \tilde{g} : \tilde{M} \rightarrow N \) such that \( \tilde{g} \circ \pi_{\sim} = f \). That is, \( f(x) = \tilde{g}(\pi_{\sim}(x)) = \tilde{g}([x]) \). This associated map \( \tilde{g} \) is one-to-one due to the fact that \( \sim_f \) is the kernel equivalence of \( f \), that is, the consistency of \( \sim \) with the smooth map \( f \in F_M \).

We define a \( F \)-structure on \( \tilde{M} \) as follows:

\( F_{\tilde{M}} = \Phi C_o = \{ \tilde{g} : M \rightarrow \mathbb{R}, \tilde{g} \circ \pi \in F_M \} \), where, \( C_o = \{ \pi \circ c \mid c \in C_oM \} \) and \( C_{\tilde{M}} = \Gamma \Phi C_o = \{ \tilde{g} : M \rightarrow \mathbb{R}, \tilde{g} \circ \pi \in F_M \} = \{ \pi \circ c, c \in \mathcal{C}_M \} \) (see [10]). The smoothness of \( \pi \) the canonical surjection reads \( F_{\tilde{M}} \circ \pi \subseteq F_M \) if, and only if \( \pi \circ C_M \subseteq C_{\tilde{M}} \). The \( F \)-space \( (\tilde{M}, C_{\tilde{M}}, F_{\tilde{M}}) \) is called a \( F \)-quotient space of the \( F \)-space \( M \) by the equivalence relation \( \sim \). The pair \( (C_{\tilde{M}}, F_{\tilde{M}}) \) is the final \( F \)-quotient structure (quotient structure for short) making \( \pi \) into a smooth map. Let \( M \) be a Fril-object and \( \sim_f \) a kernel equivalence on \( M \). The topology generated on the quotient space \( \tilde{M} = M/\sim_f \) by structure functions is \( \tau_{F_{\tilde{M}}} = \{ U \subseteq \tilde{M} \mid f^{-1}(V) = U, V \in \tau_{F_{\tilde{M}}}, f \in F_{\tilde{M}} \} \) with subbasis \( S = \{ f^{-1}(0, 1) \mid f \in F_{\tilde{M}} \} \) and base given by \( B = \{ f^{-1}(0, +\infty) \mid f \in F_{\tilde{M}} \} \).
The topology generated by structure curves on the quotient space is given by \( \tau_{\mathcal{S}} = \{ O \subseteq \tilde{M} | c^{-1}(O) \in \tau_{\mathcal{F}_M} \} \), where \( c \) is a structure curve on \( \tilde{M} \). Both \( \tau_{\mathcal{F}_M} \) as well as \( \tau_{\mathcal{S}} \) are called \( \mathcal{F} \)-topologies on \( \tilde{M} \) or \( \mathcal{F} \)-quotient topologies. Recall that the quotient topology (or standard quotient topology or identification topology) on \( \tilde{M} \) is the one which is generated by the canonical map \( \pi : M \rightarrow M = M/\sim \). It is defined by \( \tau_{\sim} = \{ V \subseteq \tilde{M} : \pi^{-1}(V) \in \tau_{\mathcal{F}_M} \} \) and known to be the strongest one in which \( \pi \) is continuous. The identification topology is Hausdorff. For, let \( \text{Proposition 2.1.11.} \) Given the three topologies dened on \( \tilde{M} \) for which \( \pi \) is continuous. The identification topology is the largest (nest) topology in \( \tilde{M} \) for which \( \pi \) is continuous. So \( \tau_{\mathcal{F}_M} \subseteq \tau_{\sim} \). For, let \( \tau \) be another topology making \( \pi \) a continuous map on \( \tilde{M} \). Let \( V \in \tau \). It follows from the continuity of \( \pi \), that \( \pi^{-1}(V) \in \tau_{\mathcal{F}_M} \) is surjective, so \( \tilde{F} \). and \( \tau_{\mathcal{F}_M} \subseteq \tau_{\mathcal{C}_M} \subseteq \tau_{\sim} \).

**Lemma 2.1.9.** Let \( \pi : M \rightarrow \tilde{M} \) be the canonical projection. Let \( \tilde{g} \in \mathcal{F}_{\tilde{M}} \) such that \( \tilde{g} \circ \pi = f \), \( f \in \mathcal{F}_M \). Then \( \tilde{g} \) is open (closed) map with regard to \( \tau_{\sim} \) and \( \tau_{\mathcal{R}} \) if, and only if \( f(U) \) is open (closed) set for each open (closed) set \( U = \pi^{-1}(\pi(U)) \). Let us say that \( U \) is \( \pi \)-saturated.

**Proof.** (see [10]).

**Lemma 2.1.10.** Let \( \tau_{\mathcal{F}_M} \) and \( \tau_{\mathcal{F}_{\tilde{M}}} \) be given on \( M \). Then \( \mathcal{B} = \{ \pi U | U \in \tau_{\mathcal{F}_M} \} \) is a base for \( \tau_{\sim} \) and \( \mathcal{B} = \{ \pi (f^{-1}(0, +\infty)) | f \in \mathcal{F}_M \} \) is a base for \( \tau_{\mathcal{F}_M} \).

**Proof.** Let \( V \in \tau_{\sim} \). That is, \( V = \pi U \) with \( U \in \tau_{\mathcal{F}_M} \) by definition of \( \tau_{\sim} \) and Lemma 2.1.9. Thus \( \mathcal{B} = \tau_{\sim} \) is the trivial base. From the universality condition, \( \pi(f^{-1}(0, +\infty)) = \tilde{g}^{-1}(0, +\infty) \). Thus \( \mathcal{B} \) is the standard base of the \( \mathcal{F} \)-space \( \tilde{M} \).

**Proposition 2.1.11.** Given the three topologies defined on \( \tilde{M} \). Then \( \tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M} = \tau_{\sim} \).

**Proof.** In the above section, we proved that \( \tau_{\mathcal{F}_M} \subseteq \tau_{\mathcal{C}_M} \subseteq \tau_{\sim} \). We need to show that one can reverse these inclusions. Let \( V \in \tau_{\sim} \). From assumption, \( \pi^{-1}(V) \) lies in \( \tau_{\mathcal{F}_M} \), the weakest topology on \( M \) in which \( \pi \) is continuous. Hence, \( \pi^{-1}(V) = \bigcup_{f \in \mathcal{F}_M} f^{-1}(0, \infty) \). But \( \pi \) is surjective, so \( \pi(\pi^{-1}(V)) = V = \bigcup_{f \in \mathcal{F}_M} \pi f^{-1}(0, \infty) \). From the universality condition on \( \mathcal{F} \)-quotient, there exists a unique map \( \tilde{g} \in \mathcal{F}_{\tilde{M}} \) such that \( f = \tilde{g} \circ \pi \). So,

\[
f^{-1}(0, \infty) = (\tilde{g} \pi)^{-1}(0, \infty) = \pi^{-1} \tilde{g}^{-1}(0, \infty)
\]

and

\[
\pi f^{-1}(0, \infty) = \pi (\pi^{-1}(\tilde{g}^{-1}(0, \infty))) = \tilde{g}^{-1}(0, \infty)
\]

again since \( \pi \) is surjective. This ends the proof.
### 2.1 Basic concepts on \( \mathbb{F} \)-spaces

#### 2.1.8 \( \mathbb{F} \)-coproduct space

The \( \mathbb{F} \)-coproduct space in the category \( \mathbb{F}_{\text{rl}} \) is the final object obtained by lifting the coproduct in the category \( \text{Sets} \) to \( \mathbb{F}_{\text{rl}} \). Let \( \bar{M} = \coprod_{i \in I} M_i \) denote the coproduct in \( \text{Sets} \). There are natural inclusion maps \( (s_i : M_i \longrightarrow \bar{M})_{i \in I} \) such that \( f \circ s_i = f_i \), where \( f : \bar{M} \longrightarrow \mathbb{R} \) is defined by \( f = (f_i)_{i \in I} \). The Frölicher structure \((\mathcal{C}_{\bar{M}}, \mathcal{F}_{\bar{M}})\) is the final one that is generated by the set \( \mathcal{C}_o \) of curves, given by

\[
\mathcal{C}_o = \bigcup_{i \in I} \{ s_i \circ c_{ij} \},
\]

\( c_{ij} \) running over \( \mathcal{C}_{oM_i} \) which is the generating set for the smooth structure \((\mathcal{C}_{M_i}, \mathcal{F}_{M_i})\) on each \( M_i \). It follows that

\[
\mathcal{F}_{\bar{M}} = \Phi \mathcal{C}_o
\]

where \( f = (f_i)_{i \in I} : \bar{M} \longrightarrow \mathbb{R} \) if \( f \vert_{M_i} = f_i \in \mathcal{F}_{M_i} \) and

\[
\mathcal{C}_{\bar{M}} = \Gamma \mathcal{F}_{\bar{M}} = \Gamma \Phi \mathcal{C}_o
\]

where \( c : \mathbb{R} \longrightarrow \bar{M} \colon c = s_i \circ c_i \), \( c_i \in \mathcal{C}_{M_i} \). The \( \mathbb{F} \)-space \((\bar{M}, \mathcal{C}_{\bar{M}}, \mathcal{F}_{\bar{M}})\) is called a \( \mathbb{F} \)-coproduct space of \( M_i \) or a \( \mathbb{F} \)-coproduct of \( \mathbb{F} \)-spaces \( M_i \). Also the pair \((\mathcal{C}_{\bar{M}}, \mathcal{F}_{\bar{M}})\) is the \( \mathbb{F} \)-coproduct structure (coproduct structure for short) such that all \( s_i \) are smooth maps. Note that

\[
\mathcal{F}_{\bar{M}} \circ s_i = \mathcal{F}_{M_i} \simeq \mathcal{F}_s(M_i) = \mathcal{F}_{\bar{M}} |_{s_i(M_i)}.
\]

In what follows we want to study and compare the topologies underlying a \( \mathbb{F} \)-coproduct space. The topologies \( \tau_{\mathcal{F}_{\bar{M}}} \) and \( \tau_{\mathcal{C}_{\bar{M}}} \) are called \( \mathbb{F} \)-topologies on \( \bar{M} \) or \( \mathbb{F} \)-coproduct topologies. These are respectively the smallest and the strongest topologies in which the canonical inclusions are continuous. The subbase for \( \tau_{\mathcal{F}_{\bar{M}}} \) is the collection \( \mathcal{S} = \{ f^{-1}(0,1) \}_{f \in \mathcal{F}_{\bar{M}}} \). The topological coproduct space \( \bar{M} \) is the coproduct of the family \((M_i)_{i \in I}\) in \( \text{Sets} \), endowed with the topology in which open sets are unions of \( s_i(U_i) \), \( i \in I \) and, where \( U_i \) is an arbitrary \( \tau_{\mathcal{F}_{M_i}} \)-open set in \( M_i \). We shall denote by \( \tau_{\Pi} \) the topology on \( \bar{M} \) and call it the coproduct topology for \( \mathbb{F} \)-topological spaces \( M_i \).

**Lemma 2.1.12.** Let \( \bar{M} \) be the coproduct of the family \((M_i)_{i \in I}\) and \( \tau_{\Pi} \) its coproduct topology. Then:

1. \( s_i \) is a continuous map for \( \tau_{\Pi} \),
2. The family \( \mathcal{B} = \{ s_i(U_i) \mid U_i \in \tau_{\mathcal{F}_{M_i}} \text{ for all } i \in I \} \) is a basis for \( \tau_{\Pi} \),
3. \( s_i(M_i) \) is a basic open set in \( \tau_{\Pi} \) for all \( i \in I \),
4. \( s_i \) is an open map for \( \tau_{\Pi} \).

**Proof.** (see [11]).
**Lemma 2.1.13.** Let $U \in \tau_I$. Then $U$ is a $\tau_I$-open (closed) set in $\bar{M}$ if, and only if $U \cap s_i(M_i)$ is a $\tau_I$-open (closed) set in $\bar{M}$ for all $i \in I$.

**Proof.** (see [11]).

**Lemma 2.1.14.** The topology $\tau_I$ is the finest one in which all canonical inclusions $s_i: M_i \rightarrow \bar{M}$ are continuous, and also we have $\tau_{\bar{M}} \subset \tau_{\bar{C}} \subset \tau_I$.

**Proof.** (see [11]).

**Lemma 2.1.15.** Let $\tau_{\bar{M}}$ be the coproduct topology and $\tau_{\bar{F}}$ the $F$-topology on $\bar{M}$. Then $\tau_{\bar{M}} = \tau_{\bar{F}}$.

**Proof.** First of all, let us draw the diagram below:

This diagram commutes in all its components. It is worth noticing that $M_i$ and $\bar{M}$ are endowed with their $F$-structures, whereas $s_i(M_i)$ is equipped with the $F$-subspace structure of $\bar{M}$. Moreover, the maps are related as follows. $f \circ t = g_i$, $g_i \circ \tilde{s}_i = f_i = f \circ s_i$ with $f$, $g_i$, and $f_i$ the structure functions, $t$, $s_i$, $\tilde{s}_i$ smooth and injective maps such that $\tilde{s}_i$ is a diffeomorphism (see [8, p. 20] and [63, p. 80]). Now let us assume that $U \in \tau_I$. So $U = s_i(U_i)$ with $U_i \in \tau_{F_i}$ for some $i \in I$. It follows that $U_i = \bigcup_{j \in J} f_{ji}^{-1}(0,1)$, with $f_{ji}$ running over $F_{M_i}$ and $j$ describing structures functions on $M_i$. Therefore,

$$U = \bigcup_{i \in I} s_i(\bigcup_{f_{ij} \in F_{M_i}} f_{ji}^{-1}(0,1))$$

$$= \bigcup_{i \in I} \bigcup_{f_{ij} \in F_{M_i}} (s_i f_{ji}^{-1}(0,1)).$$

Now,

$$U = \bigcup_{i \in I} \bigcup_{j \in J} (s_i(\tilde{s}_i^{-1} \circ g_{ji}^{-1}(0,1)))$$

$$= \bigcup_{(i,j) \in I \times J} s_i \tilde{s}_i^{-1}(g_{ji}^{-1}(0,1)).$$
Thus,

\[ U = \bigcup_{(i,j) \in I \times J} \left( g_{ji}^{-1}(0, +\infty) \cap s_i(M_i) \right) \]

\[ = \bigcup_{(i,j) \in I \times J} g_{ji}^{-1}(0, 1) \]

since \( g_{ji}^{-1}(0, 1) \subset s_i(M_i) \) for each \( i \in I \). If we fix \( i \), then \( \bigcup_{j} g_{ji}^{-1}(0, 1) \in \tau_{\mathcal{F}s_i(M_i)} \) and since \( f = (f_i)_i \), one has

\[ \bigcup_{j} g_{ji}^{-1}(0, 1) = \bigcup_{j} (\iota^{-1} \circ f_j^{-1}(0, 1)) \]

\[ = \bigcup_{j} [f_j^{-1}(0, 1) \cap s_i(M_i)] \].

Hence, \( \bigcup_{j} g_{ji}^{-1}(0, 1) \in \tau_{\mathcal{F}M}(s_i(M_i)) \), the trace topology on \( s_i(M_i) \). Thus, \( \tau_{\mathcal{F}s_i(M_i)} = \tau_{\mathcal{F}M}(s_i(M_i)) \) with \( f_j \in \mathcal{F}M \) such that \( f_j \circ \iota = g_{ji} \) and \( \bigcup_{j} g_{ji}^{-1}(0, +\infty) \subset s_i(M_i) \subset M \). That is, \( s_i(M_i) \in \tau_{\mathcal{F}M} \) and \( \bigcup_{j} g_{ji}^{-1}(0, 1) \in \tau_{\mathcal{F}M} \). Hence \( \tau_\Pi \subset \tau_{\mathcal{F}M} \). It follows from Lemma 2.1.14 that \( \tau_\Pi = \tau_{\mathcal{F}M} \). So, \( \tau_\Pi = \tau_{\mathcal{F}M} = \tau_{\mathcal{C}M} \).

\[ \square \]

2.2 Locally Euclidean Frölicher spaces

2.2.1 \( \mathbb{F} \)-spaces locally diffeomorphic to \( \mathbb{R}^n \)

The material in this subsection is drawn from [112], where the locally Euclidean Frölicher spaces were called "pseudomanifolds" and "locally Euclidean spaces". The concept of locally Euclidean spaces is also used in geometric topology, where, it defines other notions. In this work we chose to call them the locally Euclidean Frölicher spaces rather.

**Definition 2.2.1.** A locally Euclidean \( \mathbb{F} \)-space is a Hausdorff Frölicher space \( M \) which is locally diffeomorphic to a \( \mathbb{F} \)-subspace of \( \mathbb{R}^n \), where, \( \mathbb{R}^n \) is endowed with its canonical \( \mathbb{F} \)-structure.

That is, a locally Euclidean Frölicher space comes equipped with an open cover \( \{ \mathcal{U}_\alpha \}_{\alpha \in I} \) of \( M \) such that for every \( x \in M \), there exist a \( \mathcal{F}M \)-open neighborhood \( \mathcal{U} \) of \( x \) contained in a certain \( \mathcal{U}_\alpha \) and a \( \mathbb{F} \)-diffeomorphism \( \varphi \) of \( \mathcal{U} \) onto the \( \mathbb{F} \)-subspace \( V := \varphi(\mathcal{U}) \subset \mathbb{R}^n \), \( \varphi: \mathcal{U} \to \varphi(\mathcal{U}) \).

In the definition above the subspace \( \varphi(\mathcal{U}) \subset \mathbb{R}^n \) can be either open or closed, or neither open nor closed \( \mathbb{F} \)-subspace of \( \mathbb{R}^n \).
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**Definition 2.2.2.** A locally Euclidean $\mathbb{F}$-space $M$ is said to be of first kind if $M$ is locally diffeomorphic to open $\mathbb{F}$-subspaces $F_i$ of $\mathbb{R}^n$, of the second kind if $F_i$ are closed and not all of constant dimension $n$, of third kind if $F_i$ are closed and of constant maximal dimension with nonempty interior.

**Example 2.2.1.** A locally Euclidean $\mathbb{F}$-space of the first kind.

$(\mathbb{R}^n, C, F)$ is $\mathbb{F}$-space, where $(C, F)$ is the canonical $\mathbb{F}$-structure given by all $C^\infty$ real valued functions and curves, following Boman’s theorem (see [13]). So is the real line $\mathbb{R}$. Now, let $\mathbb{R}$ be also endowed with the canonical $\mathbb{F}$-structure. Let $\pi_i$ be the $i^{th}$ natural projection of $\mathbb{R}^n$ onto $\mathbb{R}$. Let $U \subset \mathbb{R}^n$ be an open set, with $i_U$ its canonical inclusion. Let $(C_U, F_U)$ be the $\mathbb{F}$-structure induced on $U$ by maps $f_i : U \to \mathbb{R}$, where $(i = 1, \ldots, n)$. Assume that the map $\varphi : U \to \mathbb{R}^n$ is a one-to-one map and defined by $\varphi(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Hence, $\varphi = (f_1, \ldots, f_n)$ is a diffeomorphism onto the $\mathbb{F}$-subspace $\varphi(U)$ of $\mathbb{R}^n$ such that $F_o \mathbb{R}^n \circ i_U = F_{oU}$ and $F_o \mathbb{R}^n$ contains a separating point function. Notice that, $F_o \mathbb{R}^n$ and $F_{oU}$ are generating sets on $\mathbb{R}^n$ and $U$, respectively. Thus, $f \circ i_U$ is a separating point function on $U$. From [85, p.80, Corollary 1.2] the set $\{f_1, \ldots, f_n\} = F_{oU}$ is a generating set for $(C_U, F_U)$ and $\varphi$ is a diffeomorphism onto the $\mathbb{F}$-subspace $\varphi(U)$ of $\mathbb{R}^n$, where $f_i = \pi_{i\varphi(U)} \circ \varphi$ and $\pi_{i\varphi(U)}$ is the restriction of the projection to $\varphi(U)$. Therefore, $\varphi : U \to \varphi(U)$ is a $\mathbb{F}$-diffeomorphism.

**Example 2.2.2.** A locally Euclidean $\mathbb{F}$-space of the third kind.

Let $f_1 = \sin : (0, \frac{\pi}{2}) \to (0, 1)$, $f_2 : (0, \frac{\pi}{2}) \to \{0\}$ and $\varphi = (f_1, f_2) : (0, \frac{\pi}{2}) \to (0, 1) \times \{0\} \subset \mathbb{R}^2$. It is known that $\{(x, 0) \mid 0 \leq x \leq 1\} = [0, 1] \times \{0\}$ is closed in $\mathbb{R}^2$ as a product of closed sets, whereas $\{(x, 0) \mid 0 < x < 1\} = (0, 1) \times \{0\}$ is neither closed nor open set. Therefore, its complement $A = \mathbb{C}((0, 1) \times \{0\})$ in $\mathbb{R}^2$ is computed by

\[
A = \mathbb{R}^2 - (0, 1) \times \{0\} \\
= \{(x, y) \in \mathbb{R}^2 \mid x \notin (0, 1), y \neq 0\} \\
= ( -\infty, 0] \cup [1, +\infty) ) \times \{0\} \cup (0, 1) \times \mathbb{R}^* ) \cup ( -\infty, 0] \cup [1, +\infty) ) \times \mathbb{R}^* ) \\
= ( -\infty, 0] \cup [1, +\infty) ) \times \{0\} \cup (\mathbb{R} \times \mathbb{R}^* )
\]

This is a union of an open set $\mathbb{R} \times \mathbb{R}^* = \mathbb{R}^2 - \{(x, y) \mid y = 0\}$ and a closed set $( -\infty, 0] \cup [1, +\infty) ) \times \{0\}$.

Smooth functions in the smooth $n$-manifold $\mathbb{R}^n$ coincide with $\mathbb{F}$-smooth functions. In the sequel, smooth curves and smooth functions is a smooth manifold coincide with smooth curves and smooth functions when it is viewed as a $\mathbb{F}$-space.

**Remark**

The rest of this thesis is devoted to locally Euclidean $\mathbb{F}$-spaces of constant dimension and of the first kind. From now on, we will restrict ourselves to locally Euclidean $\mathbb{F}$-spaces of the
2.2 Locally Euclidean Frölicher spaces

first kind. The study of locally Euclidean $\mathbb{F}$-spaces of the second and third kind falls beyond the scope of this thesis. We will say fort short, "locally Euclidean $\mathbb{F}$-space of dimension $n$" or indiscriminately "$n$-spaces", that is locally Euclidean $\mathbb{F}$-space of constant maximal dimension. The maximal dimension is related to subsets of $\mathbb{R}^n$ with non-empty interior. In the context of $n$-spaces, the dimension is one of all minimal submanifolds in $\mathbb{R}^n$, each of them containing an open neighborhood $V$ of $\varphi(p)$, for each $p \in U \subset M$, with $U$ an open neighborhood and $\varphi$ a local diffeomorphism from $U$ to $V$. So the object of interest in practice should be $\varphi(U)$. The first kind is new concept, while the third was first studied by Batubenge under the denomination of locally Euclidean spaces in [8]. There should exist a transfer of local features from $\mathbb{R}^n$ back to the $\mathbb{F}$-space $M$ by a local diffeomorphism. The examples we shall deal with in the next Section lie in either finitely generated structures or graphs of smooth maps.

**Lemma 2.2.1.** Each locally Euclidean $n$-$\mathbb{F}$-space is locally finitely generated by $n$ functions. 

**Proof.** The $\mathbb{F}$-substructure on an open set $U$ in a $\mathbb{F}$-space $M$ is the restriction of the $\mathbb{F}$-structure of $M$ to $U$. Then the lemma holds from [85, p.80, Corollary 1.2].

**Example 2.2.3.** $(\mathbb{R}^n, C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ is a natural model of locally Euclidean $\mathbb{F}$-spaces of the first kind.

**Example 2.2.4.** A $n$-dimensional smooth manifold is an example of $n$-space of the first kind.

**Example 2.2.5.** Let $M:= (0, 2\pi)$. With $U=(0, \pi)$, $V=(\frac{\pi}{2}, \frac{3\pi}{2})$ and $W=(\pi, 2\pi)$, it is clear that $U \cup V \cup W = (0, 2\pi)$. Let $f_1 := \cos : (0, 2\pi) \to \mathbb{R}$ and $f_2 := \exp : (0, 2\pi) \to \mathbb{R}$. Thus $\varphi=(f_1, f_2) : (0, 2\pi) \to \mathbb{R}^2$ is smooth and one-to-one since $\exp$ separates points in $(0, 2\pi)$. It follows that $\varphi(x)=\varphi(y)$ yields $(\cos x, e^y) = (\cos y, e^y)$. It follows from $x = \pm y + 2k\pi$ and $x = y$ that $k$ must be $0$ since $M = (0, 2\pi)$. Also $x, y > 0$ since $x, y \in M$. So, $\varphi(0, 2\pi) = (-1, 1) \times (1, e^{2\pi})$ is an open set as finite product of open sets. Therefore, $(0, 2\pi) \simeq (-1, 1) \times (1, e^{2\pi})$. Hence, $\varphi_1 = \varphi|_U$, $\varphi_2 = \varphi|_V$, and $\varphi_3 = \varphi|_W$ are local diffeomorphisms. Hence, for the $\mathbb{F}$-structure generated by $\{f_1, f_2\}$, one has $\dim (0, 2\pi) = \dim (-1, 1) \times (1, e^{2\pi}) = 2$. But, $\dim (0, 2\pi) = 1$ in the canonical $\mathbb{F}$-structure induced from $\mathbb{R}$.

**Example 2.2.6.** Let $N := (0, 2\pi)$ and $\varphi=(\cos, \sin)$ given by $\varphi(x)=(\cos x, \sin x)$ for all $x \in N$. It follows from the definition of $\varphi$ that

$$
\begin{align*}
\varphi(0, \frac{\pi}{2}) &= U_1 = [0, 1) \times (0, 1] = [0, 1] \times [0, 1] - \{(1, 0)\} \\
\varphi(\frac{\pi}{2}, \frac{\pi}{2}) &= U_2 = [-1, 0] \times [0, 1] \\
\varphi(\frac{\pi}{2}, \frac{3\pi}{2}) &= U_3 = [-1, 0] \times [-1, 0] \\
\varphi(\frac{3\pi}{2}, 2\pi) &= U_4 = [0, 1) \times [-1, 0) = [0, 1] \times [-1, 0) - \{(1, 0)\}.
\end{align*}
$$
Thus, \( U_1 \cup U_2 \cup U_3 \cup U_4 = S^1 - \{(1, 0)\} = \varphi(0, 2\pi) \) is \( \mathbb{R}^2 \)-open set since \( \{(0, 1)\} \) is \( \mathbb{R}^2 \)-closed set. We want to define \( \psi = \varphi \times id_\mathbb{R} \), that is, \( \psi(x, y) = (\varphi(x), id_\mathbb{R}(y)) \) such that

\[
\psi : (0, 2\pi) \times \mathbb{R} \longrightarrow \varphi(0, 2\pi) \times \mathbb{R} \hookrightarrow \mathbb{R}^3
\]

\[
(x, y) \mapsto (\varphi(x), y) \mapsto (\cos x, \sin x, y)
\]

or equivalently, \( \psi : (0, 2\pi) \times \mathbb{R} \longrightarrow (S^1 - \{(1, 0)\}) \times \mathbb{R} \hookrightarrow \mathbb{R}^3 \). Thus,

\[
\varphi(0, 2\pi) = (S^1 - \{(1, 0)\}) \times \mathbb{R}
\]

\[
= \{(\cos x, \sin x, y) \mid 0 \leq x \leq 2\pi, y \in \mathbb{R}\} - \{(1, 0, y) \mid y \in \mathbb{R}\}
\]

\[
= \{(\cos x, \sin x, y) \mid 0 < x < 2\pi, y \in \mathbb{R}\}.
\]

Hence, \( (S^1 - \{(1, 0)\}) \times \mathbb{R} = S^1 \times \mathbb{R} - \{(1, 0)\} \times \mathbb{R} \) is an open set.

We would like to retrace here the theoretical foundation of such a \( \psi \):

\[
(0, 2\pi) \times \mathbb{R} \xrightarrow{\pi_1|_{(0,2\pi) \times \mathbb{R}}} (0, 2\pi)
\]

\[
f_1 \circ \pi_1|_{(0,2\pi) \times \mathbb{R}} \quad \quad f_1 = \cos
\]

\[
[-1, 1) = f_1(0, 2\pi) = \{\cos x \mid x \in (0, 2\pi)\},
\]

that is \( \hat{\pi}_1 = \pi_1|_{(0,2\pi) \times \mathbb{R}} \)

\[
(0, 2\pi) \times \mathbb{R} \xrightarrow{\pi_1|_{(0,2\pi) \times \mathbb{R}}} (0, 2\pi)
\]

\[
f_2 \circ \pi_1|_{(0,2\pi) \times \mathbb{R}} \quad \quad f_2 = \sin
\]

\[
[-1, 1) = f_2(0, 2\pi) = \{\sin x \mid x \in (0, 2\pi)\},
\]

that is \( \hat{\pi}_1 = \pi_1|_{(0,2\pi) \times \mathbb{R}} \)

\[
(0, 2\pi) \times \mathbb{R} \xrightarrow{\pi_2|_{(0,2\pi) \times \mathbb{R}}} \mathbb{R}
\]

\[
id_\mathbb{R} \circ \pi_2|_{(0,2\pi) \times \mathbb{R}} \quad \quad id_\mathbb{R}
\]

\[
\mathbb{R}
\]
that is $\hat{\pi}_2 = \pi_2|_{(0,2\pi) \times \mathbb{R}}$.

Let $\psi := (f_1 \circ \hat{\pi}_1, f_2 \circ \hat{\pi}_1, id_{\mathbb{R}} \circ \hat{\pi}_2)$ and $\psi$ is one-to-one since one of its components separates points, that is, $id_{\mathbb{R}} \circ \hat{\pi}_1(x_1, y_1) = id_{\mathbb{R}} \circ \hat{\pi}_1(x_2, y_2) \Rightarrow id_{\mathbb{R}}(y_1) = id_{\mathbb{R}}(y_2) \Rightarrow y_1 = y_2$. Thus,

$$
\psi(x, y) = (f_1 \circ \hat{\pi}_1(x,y), f_2 \circ \hat{\pi}_1(x,y), id_{\mathbb{R}} \circ \hat{\pi}_2(x,y))
= (f_1(x), f_2(x), id_{\mathbb{R}}(y))
= (\cos x, \sin x, y).
$$

Making use of the parity and symmetries of $\cos$ and $\sin$ functions we would say: if $x_1 \neq x_2$ where $x_1$ and $x_2$ are symmetric arcs with regard to $x$-axis ($y$-axis) then $\cos x_1 = \cos x_2$ ($\sin x_1 = \sin x_2$) and $\sin x_1$ ($\cos x_1$) is opposed to $\sin x_2$ ($\cos x_2$). Hence $\psi(x_1, y_1) \neq \psi(x_2, y_2)$ whenever $x_1 \neq x_2$. Therefore $\psi$ is one-to-one and $(0, 2\pi) \times \mathbb{R} \simeq \psi((0, 2\pi) \times \mathbb{R})$:

$$
(0, 2\pi) \times \mathbb{R} \simeq \psi((0, 2\pi) \times \mathbb{R}) \\
\simeq f_1(0, 2\pi) \times f_2(0, 2\pi) \times \mathbb{R} \\
\simeq \{(\cos x, \sin x)| x \in (0, 2\pi)\} \times \mathbb{R} \\
\simeq \{(s,t) \in \mathbb{R}^2 | s^2 + t^2 = 1\} - \{(1,0)\} \times \mathbb{R} \\
\simeq S^1 \times \mathbb{R} - \{(1,0)\} \times \mathbb{R} \\
\simeq [S^1 - \{(1,0)\}] \times \mathbb{R} \\
$$

$\dim (0, 2\pi) \times \mathbb{R} = \dim (S^1 - \{(1,0)\}) + \dim \mathbb{R} = 2$.

**Lemma 2.2.2.** Let $f : M \rightarrow N$ and $g : M \rightarrow G(f)$ such that $x \mapsto g(x) = (x, f(x))$ be set maps, where $M, N$ are $F$-spaces, and $G(f) = \{(x, f(x))| x \in M\} \subset M \times N$ is a $F$-subspace of $M \times N$. Then $f$ is smooth if, and only if $g$ is diffeomorphism.

**Proof.**

"$\implies$" Let $f$ be a smooth map. That is $f$ smooth if, and only if $F_N \circ f \subset F_M$ that is to say $f_N \circ f \in F_M$. But, the following diagrams tell us more about the sequel of proof:

In fact $f = p_N \circ \iota_{G(f)} \circ g$ is smooth by assumption. And $f = p_{N|G(f)} \circ g$ with $p_{N|G(f)}$ smooth. Thus $g$ is smooth with regard to Equation (2.7). Now we have to show that $g$ is a diffeomorphism.
First, \( p_{M|G(f)} \) is obviously a smooth bijective map from \( p_{M|G(f)} \circ g = id_M \) and \( g \circ p_{M|G(f)} = id_G(f) \). Therefore \( g = p_{M|G(f)}^{-1} \) and \( g^{-1} = p_{M|G(f)} \) are smooth bijective maps. Hence \( g \) is a diffeomorphism and \( M \simeq G(f) \), such that the diagram below is commutative.

The series of examples below makes use of the diffeomorphism built on a graph of a smooth map.

**Example 2.2.7.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth map. Then \( g : \mathbb{R} \rightarrow G(f) \subset \mathbb{R} \times \mathbb{R} \) defined as in Lemma 2.2.2 is a diffeomorphism, that is \( \mathbb{R} \simeq G(f) \).

The series of examples below makes use of the diffeomorphism built on a graph of a smooth map.

**Example 2.2.8.** Let \( G(f) \) be the graph of the real function \( f = | \cdot | \), that is, \( G(f) = \{(x, |x|) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R} \). Let \( g : \mathbb{R} \rightarrow G(f) \subset \mathbb{R} \times \mathbb{R} \) defined as in Lemma 2.2.2 and in the Corollary above. Thus, \( G(f) = \{(x, f(x)), x \in \mathbb{R}, f(x) = |x|\} = g(\mathbb{R}) \). In \( \mathbb{F} \)-spaces setting, \( f = | \cdot | : \mathbb{R} \rightarrow \mathbb{R} \) should be a smooth map for the \( \mathbb{F} \)-structure generated by \( \{ | \cdot | \} \). It is known that \( f \) is not smooth in the canonical \( \mathbb{F} \)-structure. Now, assume \( G(f) \) endowed with the \( \mathbb{F} \)-structure generated by \( \pi_{1G(f)} \) and \( \pi_{2G(f)} \). With regard to Lemma 2.2.2, \( \mathbb{R} \simeq G(f) \) and \( p_{1|G(f)} \) are diffeomorphisms such that \( g : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \) and \( g = p_{1|G(f)}^{-1} \). The graph of \( f \) reveals two situations as shown on the figure below:
On the branches of $G(f)$ in (I) and (II), at each point $m_1$ or $m_2$ there exists an open neighborhood $G_1$ or $G_2$ such that $G_1$ is mapped into $U_1$ and $G_2$ into $U_2$; with $U_1$ and $U_2$ being open sets in $\mathbb{R}$. In the sequel $\dim G(f) = \dim \mathbb{R}$, that is, $\dim G(f) = 1$ on the two branches. At the origin, the open neighborhood $G_3$ in $G(f)$, is of dimension 2. Thus $G(f)$ is not of constant dimension at each point.

**Example 2.2.9.** Let $M = B^n = \{ x = (x_1, \ldots, x_n) \mid ||x|| \leq 1 \}$, that is, the $n$-closed unit ball. Let the $\mathbb{F}$-structure on $M$ be generated by functions $\hat{\pi}_i, f : M \to \mathbb{R}$, where $\hat{\pi}_i$ are restrictions of natural projections such that $\pi_i(x) = x_i$, $f(x) = \sqrt{1 - ||x||^2} = \sqrt{1 - \sum_{i=1}^{n} x_i^2}$ with $0 \leq f(x) \leq 1, 1 \leq i \leq n$. Now, we define $g : M \to M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$ by $g(x) = (x, f(x))$, that is, $g = (\hat{\pi}_1, \ldots, \hat{\pi}_n, f)$ and then $g(x) = (x_1, \ldots, x_n, f(x)) = (\hat{\pi}_1(x), \ldots, \hat{\pi}_n(x), f(x)) = (\pi_1, \ldots, \pi_n, f)(x)$. So, $g(M) = \{(x, f(x)) \mid x \in M\} = G(f)$. By definition, $G(f)$ is the closed hemisphere viewed as a closed $\mathbb{F}$-subspace of $\mathbb{R}^{n+1}$. We have $M \simeq G(f) \subset M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$, from Lemma 2.2.2. So $\dim M = \dim G(f) = n < \dim \mathbb{R}^{n+1}$. Furthermore, if we take $\text{Int}(B^n)$, with the same generating functions $\pi_i$, $f$, then $\text{Int}(B^n)$ is an open $\mathbb{F}$-subspace of $\mathbb{R}^{n+1}$ such that inclusion $\text{Int}(B^n) \hookrightarrow \mathbb{R}^{n+1}$ is a smooth map of $\mathbb{F}$-spaces. Therefore $\text{Int}(B^n)$ is diffeomorphic to the open top hemisphere $h(\text{Int}(B^n)) = \{(x, f(x)) \mid x \in \text{Int}(B^n)\}$, where $h = g|_{\text{Int}(B^n)}$. It follows that $h(\text{Int}(B^n)) = g|_{\text{Int}(B^n)}(\text{Int}(B^n)) = \{(x, f(x)) \mid x_1^2 + \cdots + x_n^2 + f(x)^2 < 1 \text{ and } 0 < f(x) < 1\}$ and $\text{Int}(B^n) \simeq h(\text{Int}(B^n))$, that is, $\text{Int}(B^n)$ is $n$-space of the first kind.

**Example 2.2.10.** Let $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (\cos x, \sin x)$, be a smooth map in the canonical $\mathbb{F}$-structures. Let $g : \mathbb{R} \to G(f)$ such that $g(x) = (x, f(x))$ and $G(f) \subset \mathbb{R} \times \mathbb{R}^2$. We have $\mathbb{R} \simeq G(f) = \{(x, \cos x, \sin x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^3 = \mathbb{R} \times S^1 \subset \mathbb{R}^3$ with regard to Lemma 2.2.2. The graph $G(f)$ is a helix drawn on a unit cylinder whose axis and basis are respectively the $x$-axis and the unit circle $S^1$ in $yz$-plane of $\mathbb{R}^3$. Now, any open neighborhood at any point $q$ in $G(f)$ is mapped on an open neighborhood at $p_1(q)$ in $\mathbb{R}$, where $q = (x, f(x)) = g(x)$ and $p_1 : G(f) \to \mathbb{R}$ the canonical projection that is $p_1(x, f(x)) = x$. Hence $G(f)$ is a 1-space of first kind.
2.2 Smooth maps between locally Euclidean $\mathbb{F}$-spaces

**Definition 2.2.3.** Let $M$ be a locally Euclidean $n$-$\mathbb{F}$-space. Then at each point $p \in M$, there exists a pair $(U, \varphi)$, where $U$ is an open neighborhood of $p$ and $\varphi: U \rightarrow \varphi(U) \subset \mathbb{R}^n$, a local diffeomorphism. The pair $(U, \varphi)$ is called a local chart (or a coordinate neighborhood) at $p \in M$, that is, at each point $p \in U$ correspond $n$ coordinates $x^1(p), \ldots, x^n(p)$ of $\varphi(p) \in \varphi(U) \subset \mathbb{R}^n$, where $\varphi(p) = x^1(p), \ldots, x^n(p)$ and $n$ is constant for every point in $U$. Each $x^i(p)$ is called the $i^{th}$ coordinate, where $x^i: U \rightarrow \mathbb{R}$ are smooth such that $\varphi = (x^1, \ldots, x^n)$. The open set $U$ is called the domain of the chart.

**Definition 2.2.4.** Let $(U, \varphi)$ be a chart at $p$ and $(V, \psi)$ be chart at $q$, where $p, q \in M$, $\varphi = (x^1, \ldots, x^n)$ and $\psi = (y^1, \ldots, y^n)$ with $x^i, y^i$ the $i^{th}$ smooth coordinate functions. Let $U \cap V \neq \emptyset$. The maps between open sets of $\mathbb{R}^n$, $\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V)$ and $\varphi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \varphi(U \cap V)$ are called transition functions. The charts $(U, \varphi)$ and $(V, \psi)$ are called $\mathbb{F}$-related (or $\mathbb{F}$-compatible) if $U \cap V \neq \emptyset$ and the transition function $\varphi \circ \psi^{-1}, \psi \circ \varphi^{-1}$ are diffeomorphisms of the open sets $\varphi(U \cap V)$ and $\psi(U \cap V)$ in $\mathbb{R}^n$.

That is an equivalence relation among charts.

**Definition 2.2.5.** Let $M$ be a $n$-space and $(U, \varphi)$ a chart in $M$. A collection $\mathcal{A}$ of $\mathbb{F}$-related charts is called a $\mathbb{F}$-atlas if the domain of charts in $\mathcal{A}$ form an open covering for $M$. The chart $(U, \varphi)$ is $\mathbb{F}$-compatible ($\mathbb{F}$-related) with an atlas $\mathcal{A}$ if $(U, \varphi)$ is $\mathbb{F}$-related to each chart of the atlas $\mathcal{A}$.

**Definition 2.2.6.** Let $\mathcal{A}_1, \mathcal{A}_2$ be two $\mathbb{F}$-atlases in a $n$-space $M$. $\mathcal{A}_1$ is equivalent to $\mathcal{A}_2$ and denoted by $\mathcal{A}_1 \sim \mathcal{A}_2$ if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again a $\mathbb{F}$-atlas, that is each chart of one is $\mathbb{F}$-related to the other $\mathbb{F}$-atlas. The union of all equivalent $\mathbb{F}$-atlases is the maximal $\mathbb{F}$-atlas that is the biggest $\mathbb{F}$-atlas equivalent to all members of an equivalence class of $\mathbb{F}$-atlases. Each chart $(U, \varphi)$ in the maximal $\mathbb{F}$-atlas is called an admissible local chart.

Actually $\sim$ is an equivalence relation.

**Definition 2.2.7.** Let $M, N$ be locally Euclidean $\mathbb{F}$-spaces of dimensions $m$ and $n$ respectively. A set map $\varphi: M \rightarrow N$ is said to be smooth map of locally Euclidean $\mathbb{F}$-spaces if for every $p \in M$, there is some chart $(U_\alpha, \varphi_\alpha)$ in $M$ with $p \in U_\alpha$ and $(V_\beta, \psi_\beta)$ in $N$ with $\varphi(p) \in V_\beta$, $\alpha, \beta$ belonging to some set of indices for a covering of $M$, such that $\psi_\beta \circ \varphi \circ \varphi^{-1}_\alpha: \varphi_\alpha(U_\alpha \cap \varphi^{-1}(V_\beta)) \rightarrow \psi_\beta[\varphi(U_\alpha \cap V_\beta)]$ (or equivalently, such that $\varphi(U_\alpha) \subset V_\beta$ and $\psi_\beta \circ \varphi \circ \varphi^{-1}_\alpha: \varphi_\alpha(U_\alpha) \rightarrow \psi_\beta(V_\beta)$ ) is a smooth map of $\mathbb{F}$-subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.
Note that $\varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta \neq \emptyset$ since $\varphi(p) \in \varphi(\mathcal{U}_\alpha)$ and $\varphi(p) \in \mathcal{V}_\beta$. But, $\varphi(p) \in \mathcal{V}_\beta$ implies $p \in \varphi^{-1}(\mathcal{V}_\beta)$, thus, $\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta) \neq \emptyset$. Also, $\varphi(\mathcal{U}_\alpha) \cap \varphi^{-1}(\mathcal{V}_\beta) = \varphi(\mathcal{U}_\alpha) \cap [\mathcal{V}_\beta \cap \varphi(M)] \subseteq \varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta$. It can be shown that $\varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta$ is an open set in $\varphi(\mathcal{U}_\alpha)$ and $\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)$ is an open set in $\varphi^{-1}(\mathcal{V}_\beta)$, both for $\mathbb{F}$-subspace topologies, since the given intersections are open sets for the respective trace topologies. But the trace topology is contained in the $\mathbb{F}$-subspace topology with regard to Lemma 2.1.5, so the claim holds. Now, the following diagram of restricted maps makes sense:

$$
\begin{array}{ccc}
\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta) & \xrightarrow{\varphi} & \varphi(\mathcal{U}_\alpha) \cap \varphi^{-1}(\mathcal{V}_\beta) \\
\varphi^{-1} & \xrightarrow{\varphi} & \varphi \\
\varphi[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] & \xrightarrow{\psi^{-1}} & \psi \beta(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta
\end{array}
$$

where $\varphi[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] = \varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta$, which is a set theoretical property and $\varphi\alpha[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] = \varphi\alpha(\mathcal{U}_\alpha) \cap \varphi\alpha[\varphi^{-1}(\mathcal{V}_\beta)]$ since $\varphi\alpha$ is an injective set map.

**Lemma 2.2.4.** Let $M$ be a locally Euclidean $\mathbb{F}$-space. Let $(\mathcal{U}_\alpha, \varphi_\alpha)$ and $(\mathcal{V}_\beta, \psi_\beta)$ be two charts at $p \in M$. The maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ in Definition 2.2.4 are smooth. They are inverse to each other. If $\mathcal{U}_\alpha \subset \mathcal{V}_\beta$ then the transition functions become $\psi_\beta \circ \varphi^{-1}_\alpha : \varphi_\alpha(\mathcal{U}_\alpha) \mapsto \psi_\beta(\mathcal{U}_\alpha)$ and $\varphi \circ \psi^{-1}_\beta : \psi_\beta(\mathcal{U}_\beta) \mapsto \varphi(\mathcal{U}_\alpha)$.

**Proof.** First, since $\varphi$ and $\phi$ are $\mathbb{F}$-diffeomorphisms, thus $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are $\mathbb{F}$-diffeomorphisms and $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$. Now, let $y^i(\rho) = h^i(x^1(\rho), \ldots, x^n(\rho))$ and $x^i(\rho) = g^i(y^1(\rho), \ldots, y^n(\rho))$. Thus, $h^i$ and $g^i$ are smooth functions as components of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$. These imply

$$
\psi \circ \varphi^{-1}(\varphi(p)) = \psi(p) = (y^1(p), \ldots, y^n(p)) = (h^1(x^1(p), \ldots, x^n(p))), \ldots, h^n(x^1(p), \ldots, x^n(p))) = (h^1[g^1(y^1(p), \ldots, y^n(p)), \ldots, g^n(y^1(p), \ldots, y^n(p))], \ldots, h^n[g^1(y^1(p), \ldots, y^n(p)), \ldots, g^n(y^1(p), \ldots, y^n(p))]),
$$

that is, $y^i(p) = h^i[g^1(y^1(p), \ldots, g^n(y^p))]$ with $y(p) = (y^1(p), \ldots, y^n(p))$ for $i = 1, \ldots, n$. Therefore, by analogy to the previous equalities, $\varphi \circ \psi^{-1}(\psi(p)) = \varphi(p) = (x^1(p), \ldots, x^n(p))$ yields $x^j(p) = g^j[h^1(x(p)), \ldots, h^n(x(p))]$ with $x(p) = (x^1(p), \ldots, x^n(p))$ for $j = 1, \ldots, n$. Hence, the transition maps are dually invertible. Secondly, let $\mathcal{U}_\alpha \subset \mathcal{V}_\beta$. Hence, $\mathcal{U}_\alpha \cap \mathcal{V}_\beta = \mathcal{U}_\alpha$ and the charts $(\mathcal{U}_\alpha, \varphi_\alpha)$, $(\mathcal{U}_\alpha, \psi_\beta|\mathcal{U}_\alpha)$ such as $\psi_\beta|\mathcal{U}_\alpha = \psi_\beta \circ \iota$ with $\iota : \mathcal{U}_\alpha \hookrightarrow \mathcal{V}_\beta$ the canonical inclusion. This ends the proof. \qed
**Proposition 2.2.5.** Let \( \varphi : M \rightarrow N \) be a smooth map of locally Euclidean \( \mathbb{F} \)-spaces, with \( \dim M = m \) and \( \dim N = n \). Then \( \varphi \) is \( \mathbb{F} \)-smooth map.

**Proof.** Assume that \( \varphi \) is a smooth map of locally Euclidean \( \mathbb{F} \)-spaces. It follows from Definition 2.2.7 that for every \( p \in M \), there exists some chart \( (U_\alpha, \varphi_\alpha) \) in \( M \) with \( p \in U_\alpha \) and some chart \( (V_\beta, \psi_\beta) \) in \( N \) with \( \varphi(p) \in V_\beta \) such that \( \psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha[U_\alpha \cap \varphi_\alpha^{-1}(V_\beta)] \rightarrow \psi_\beta(V_\beta) \) is smooth function of \( \mathbb{F} \)-subspaces of \( \mathbb{R}^m \) and \( \mathbb{R}^n \) respectively. That is, \( \psi_\beta \circ \varphi = (\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1}) \circ \varphi_\alpha \) is \( \mathbb{F} \)-smooth as the composite of smooth maps as shown in the diagram below:

\[
\begin{align*}
U_\alpha \cap \varphi_\alpha^{-1}(V_\beta) & \xrightarrow{\varphi} \varphi(U_\alpha) \cap V_\beta \\
\exists f_\alpha = f_\beta \circ \psi_\beta \circ \varphi & \quad \exists f_\beta \circ \psi_\beta \quad \forall f_\beta
\end{align*}
\]

In the light of Corollary 2.6, \( \varphi \) is smooth on \( U_\alpha \cap \varphi_\alpha^{-1}(V_\beta) \). Now, we have to show that the smoothness of \( \varphi \) does not depend on the choice of a chart. For, let \( (U'_\alpha, \varphi'_\alpha) \) be another chart at \( p \) in \( M \). It is clear that \( U_\alpha \cap U'_\alpha \) is nonempty and we can define the transition maps \( \varphi_\alpha \circ \varphi'_\alpha^{-1} : \varphi'_\alpha(U_\alpha \cap U'_\alpha) \rightarrow \varphi_\alpha(U_\alpha \cap U'_\alpha) \), and \( \varphi'_\alpha \circ \varphi^{-1}_\alpha : \varphi_\alpha(U_\alpha \cap U'_\alpha) \rightarrow \varphi'_\alpha(U_\alpha \cap U'_\alpha) \), which are \( \mathbb{F} \)-diffeomorphisms. In the sequel \( \psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} \circ (\varphi_\alpha \circ \varphi'_\alpha^{-1}) = \psi_\beta \circ \varphi \circ \varphi_{\alpha^{-1}} \) is a composition of \( \mathbb{F} \)-diffeomorphisms with \( \psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \varphi'_\alpha[U_\alpha \cap \varphi_\alpha^{-1}(V_\beta)] \rightarrow \psi_\beta[\varphi_\alpha(U_\alpha) \cap V_\beta] \). Therefore, for any chart \( (U_\alpha, \varphi_\alpha) \), \( \varphi \) is smooth on \( U_\alpha \cap \varphi_\alpha^{-1}(V_\beta) \). Without loss of generality, we may choose \( U_\alpha \) and \( V_\beta \) such that \( U_\alpha \subset \varphi^{-1}(V_\beta) \) and \( (U_\alpha, \varphi_\alpha)_{\alpha \in A} \) is open covering of \( M \) for \( \tau_{\mathcal{F}_M} \) and \( \tau_{\mathcal{C}_M} \) since \( \tau_{\mathcal{F}_M} \subset \tau_{\mathcal{C}_M} \). So \( \varphi \) is smooth on each \( U_\alpha \) member of a covering of \( M \) in \( \tau_{\mathcal{C}_M} \).

We conclude, by means of Lemma 2.1.3, that \( \varphi \) is smooth on the whole set \( M \).

**Proposition 2.2.6.** Let \( M, N \) be locally Euclidean \( \mathbb{F} \)-spaces with \( \dim M = m \) and \( \dim N = n \). Let \( \varphi : M \rightarrow N \) be a \( \mathbb{F} \)-smooth map. Then \( \varphi \) is smooth map of locally Euclidean \( \mathbb{F} \)-spaces.

**Proof.** Since \( \varphi \) is \( \mathbb{F} \)-smooth then it is continuous. That is, for every \( p \in M \) and each neighborhood \( W_{\varphi(p)} \) in \( N \), there exists a neighborhood \( V_p \) containing \( p \) such that \( \varphi(V_p) \subset W_{\varphi(p)} \). Assume \( U_\alpha = V_p \) and \( V_\beta = W_{\varphi(p)} \) with \( \varphi(U_\alpha) \subset V_\beta \). Thus, \( U_\alpha \subset \varphi^{-1}(\varphi(U_\alpha)) \subset \varphi^{-1}(V_\beta) \). It follows that \( U_\alpha \cap \varphi^{-1}(V_\beta) = U_\alpha \) and \( \varphi(U_\alpha) \cap V_\beta = \varphi(U_\alpha) \). Hence, \( \psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \varphi(U_\alpha) \rightarrow \psi_\beta(V_\beta) \) is smooth as the composite of smooth maps. Furthermore \( \varphi \) is a smooth map of locally Euclidean \( \mathbb{F} \)-spaces by Definition 2.2.7. The diagram below is related to the situation...
Corollary 2.2.7. Let $M$, $N$ be locally Euclidean $\mathbb{F}$-spaces with $\dim M = m$ and $\dim N = n$. A set map $\varphi : M \rightarrow N$ is $\mathbb{F}$-smooth if, and only if $\varphi$ is smooth map of locally Euclidean $\mathbb{F}$-spaces.

Proof. That is a straightforward consequence of Proposition 2.2.5 and Proposition 2.2.6.

Corollary 2.2.8. Let $M$ be a $n$-space and $f : M \rightarrow \mathbb{R}$ a function. Then $f \in \mathcal{F}_M$ if, and only if for every $p \in M$ there is some chart $(\mathcal{U}_a, \varphi_a)$ at $p$ in $M$ so that $f \circ \varphi_a^{-1} : \varphi_a(\mathcal{U}_a) \rightarrow \mathbb{R}$ is smooth, that is, $f \circ \varphi_a^{-1} \in \mathcal{F}_{\varphi_a(\mathcal{U}_a)}$, with $\varphi_a(\mathcal{U}_a) \subset \mathbb{R}^n$.

Proof. 

$\Longrightarrow$ Assume $f \in \mathcal{F}_M$. Then if $N = \mathbb{R}$ and $\varphi = f$ in Corollary 2.2.7, $f$ is smooth map of locally Euclidean $\mathbb{F}$-spaces. Now, refering to Definition 2.2.7 we have for every $p \in M$ there is some chart $(\mathcal{U}_a, \varphi_a)$ at $p$ and some chart $(I_\beta, \text{id}_{I_\beta})$ at $f(p)$ such that $f(\mathcal{U}_a) \subset I_\beta$ and $\text{id}_{I_\beta} \circ \varphi \circ \varphi_a^{-1} : \varphi_a(\mathcal{U}_a) \rightarrow I_\beta$ is a smooth map of $\mathbb{F}$-subspaces, where $\mathcal{V}_\beta = I_\beta \subset \mathbb{R}$ and $\psi_\beta = \text{id}_{I_\beta}$. Therefore, for every $p \in M$, there is some chart $(\mathcal{U}_a, \varphi_a)$ at $p$ so that $f \circ \varphi_a^{-1} : \varphi_a(\mathcal{U}_a) \rightarrow \mathbb{R}$ is smooth, that is $f \circ \varphi_a^{-1} \in \mathcal{F}_{\varphi_a(\mathcal{U}_a)}$.

$\Longleftarrow$ Proposition 2.2.5 yields $N = \mathbb{R}$, $\varphi = f$. Thus, $f$ is smooth map of $\mathbb{F}$-spaces, that is, $f \in \mathcal{F}_M$.

Corollary 2.2.9. Let $N$ be a $n$-locally Euclidean $\mathbb{F}$-space and $c : \mathbb{R} \rightarrow N$ be a curve. Then $c \in \mathcal{C}_N$ if, and only if for every $t \in \mathbb{R}$ there exists some chart $(\mathcal{V}_\beta, \psi_\beta)$ at $c(t)$ in $N$ such that $\psi_\beta \circ c : c^{-1}(\mathcal{V}_\beta) \rightarrow \psi_\beta(\mathcal{V}_\beta)$ is smooth, that is $\psi_\beta \circ c \in \mathcal{C}_{\psi_\beta(\mathcal{V}_\beta)}$ with $c^{-1}(\mathcal{V}_\beta) \subset \mathbb{R}$, $\psi_\beta(\mathcal{V}_\beta) \subset \mathbb{R}^n$.

Proof. 

$\Longrightarrow$ Assume $c \in \mathcal{C}_N$, $M = \mathbb{R}$, $\varphi = c$ in Proposition 2.2.6. Thus $c$ is a smooth map of locally Euclidean $\mathbb{F}$-spaces. By Definition 2.2.7, one has, for every $t \in \mathbb{R}$ there exists some chart $(c^{-1}(\mathcal{V}_\beta), \text{id}_{c^{-1}(\mathcal{V}_\beta)})$ at $t$ and some chart $(\mathcal{V}_\beta, \psi_\beta)$ such that $c(c^{-1}(\mathcal{V}_\beta)) = \mathcal{V}_\beta \cap c(\mathbb{R}) \subset \mathcal{V}_\beta$ and...
2.2 Locally Euclidean Frölicher spaces

\( \psi_\beta \circ c_\beta \circ id_{c^{-1}(V_\beta)} : c^{-1}(V_\beta) \rightarrow \psi_\beta(V_\beta) \) is smooth map of \( \mathbb{F} \)-subspaces, where \( U_\alpha = c^{-1}(V_\beta) \subset \mathbb{R} \) and \( \varphi_\alpha = id_{c^{-1}(V_\beta)} \). It follows a diagram of smooth maps:

\[
\begin{array}{c}
\mathbb{R} & \xrightarrow{c} & N \\
\downarrow{\psi_\beta} & & \downarrow{\psi_\beta} \\
\psi_\beta(V_\beta) & \xrightarrow{c^{-1}(V_\beta)} & V_\beta \\
\downarrow{id_{c^{-1}(V_\beta)}} & & \downarrow{id_{c^{-1}(V_\beta)}} \\
\mathbb{R} \supset c^{-1}(V_\beta) & \xrightarrow{\psi_\beta \circ c^{-1}(V_\beta) \circ id_{c^{-1}(V_\beta)}} & \psi_\beta(V_\beta) \subset \mathbb{R}^n \\
\end{array}
\]

Therefore, since \( id_{c^{-1}(V_\beta)} = id_{c^{-1}(V_\beta)} \), for every \( t \in \mathbb{R} \), there is some chart \((V_\beta, \psi_\beta)\) at \( c(t) \) such that \( \psi_\beta \circ c \circ c^{-1}(V_\beta) \rightarrow \psi_\beta(V_\beta) \) is smooth, that is, \( \psi_\beta \circ c \in C_{\psi_\beta(V_\beta)} \).

"\( \Longleftrightarrow \)" Proposition 2.2.5 yields \( M = \mathbb{R}, \ \varphi = c, \ p = t \). Thus, \( c \) is smooth map of \( \mathbb{F} \)-spaces, that is, \( c \in \mathcal{C}_N \). \( \square \)

**Corollary 2.2.10.** Let \( M, N \) be locally Euclidean \( \mathbb{F} \)-spaces and \( \varphi : M \rightarrow N \) a set map. Let \( c \) be any curve in \( \mathcal{C}_M \). The following conditions are equivalent.

1. \( \varphi \) is smooth.
2. \( \varphi \circ c \in \mathcal{C}_N \).
3. \( \psi_\beta \circ \varphi \circ c : c^{-1}(\varphi^{-1}(V_\beta)) \rightarrow \psi_\beta(V_\beta) \) is smooth, where \( t \in \mathbb{R} \) and \((V_\beta, \psi_\beta)\) a chart at \( c(t) \).
4. \( \varphi \circ c : c^{-1}(\varphi^{-1}(V_\beta)) \rightarrow V_\beta \) is smooth.

**Proof.**

(1) \( \Longrightarrow \) (2) Obvious from the definition of a smooth map.

(2) \( \Longrightarrow \) (3) From Corollary 2.2.9, \( \varphi \circ c \in \mathcal{C}_N \) if, and only if for every \( t \in \mathbb{R} \) there exists \((V_\beta, \psi_\beta)\) a chart at \((\varphi \circ c)(t)\) in \( N \) such that \( \psi_\beta \circ (\varphi \circ c) : (\varphi \circ c)^{-1}(V_\beta) \rightarrow \psi_\beta(V_\beta) \) is smooth.

(3) \( \Longrightarrow \) (4) Assume \( \psi_\beta \circ \varphi \circ c : c^{-1}(\varphi^{-1}(V_\beta)) \rightarrow \psi_\beta(V_\beta) \) smooth. Note that \( \varphi \circ c \) smooth implies \( \varphi \circ c \) continuous for both \( \tau_{\mathcal{C}_N} \) and \( \tau_{\mathcal{F}_N} \), that is, \((\varphi \circ c)^{-1}(V_\beta) \) is a \( \mathbb{R} \)-open set. Therefore \( c^{-1}(\varphi^{-1}(V_\beta)) \) is a \( \mathbb{R} \)-open set. This yields \( \varphi^{-1}(V_\beta) \in \tau_{\mathcal{C}_N} \). As composite of \( \psi_\beta \) smooth and \( \varphi \circ c \), it follows from Corollary 2.6 that \( \varphi \circ c : c^{-1}(\varphi^{-1}(V_\beta)) \rightarrow V_\beta \) is smooth.

(4) \( \Longrightarrow \) (1) Assume \( c(\mathbb{R}) \subset \varphi^{-1}(V_\beta) \). It follows that \( c^{-1}(c(\mathbb{R})) \subset c^{-1}\varphi^{-1}(V_\beta) \) and \( \mathbb{R} \subset (\varphi \circ c)^{-1}(V_\beta) \). Thus, \( \mathbb{R} = (\varphi \circ c)^{-1}(V_\beta) \). So \( \varphi \circ c : \mathbb{R} \rightarrow N \) is a smooth curve. Hence, \( \varphi \) is smooth as a straightforward consequence of the definition of a smooth map. \( \square \)
Corollary 2.2.11. Let $\gamma : \mathbb{R} \to N$ be a set map. Let $N$ be a locally Euclidean $\mathbb{F}$-space. If for every $t \in \mathbb{R}$, there exists $c \in C_N$ and $I_\alpha$ a $\mathbb{R}$-open set, with $t \in I_\alpha$ such that $\gamma_{|I_\alpha} = c_{|I_\alpha}$ then $\gamma \in C_N$.

Proof. We may make use of Corollary 2.2.9. That is, $c \in C_N$ if, and only if for any $t \in \mathbb{R}$ there exists some chart $(\mathcal{V}_\beta, \psi_\beta)$ at $c(t)$ in $N$ so that $\psi_\beta \circ c : c^{-1}(\mathcal{V}_\beta) \to \psi_\beta(\mathcal{V}_\beta)$ is smooth. One can assume $I_\alpha = c^{-1}(\mathcal{V}_\beta)$, and so $t \in I_\alpha$ and $(I_\alpha)_{\alpha \in A}$ is a $\tau_{\mathbb{C}_\mathbb{R}} = \tau_{\mathbb{F}_\mathbb{R}}$ open covering of $\mathbb{R}$ with $c : I_\alpha \to \mathcal{V}_\beta$ smooth. Now, one can make use of the assumption $\gamma_{|I_\alpha} = c_{|I_\alpha}$. It follows that the set map $\gamma : \mathbb{R} \to N$ has its restriction smooth on each $I_\alpha$, for any $\alpha$, in the covering. From Lemma 2.1.3 $\gamma$ is a smooth curve on the whole $\mathbb{R}$ considered as $\mathbb{F}$-space or locally Euclidean $\mathbb{F}$-space. \hfill $\square$

2.3 Locally Euclidean $\mathbb{F}$-subspace

Definition 2.3.1. Let $f : M \to N$ be a smooth mapping of locally Euclidean $\mathbb{F}$-spaces, with $\dim M = m$ and $\dim N = n$. The rank of $f$ at $p \in M$ is the rank at $\varphi(p) \in \varphi(\mathcal{U})$ of the map $\hat{f} = \psi \circ f \circ \varphi^{-1} : \varphi(\mathcal{U}) \to \psi(\mathcal{V})$, with $(\mathcal{U}, \varphi)$ a chart at $p$ in $M$ and $(\mathcal{V}, \psi)$ a chart at $f(p)$ in $N$, that is the rank at $\varphi(p)$ of the Jacobian matrix

$$
\begin{bmatrix}
\frac{\partial^1 f}{\partial x^1} & \ldots & \frac{\partial^1 f}{\partial x^m} \\
\vdots & \ddots & \vdots \\
\frac{\partial^n f}{\partial x^1} & \ldots & \frac{\partial^n f}{\partial x^m}
\end{bmatrix}
$$

of the map $\hat{f}(x^1, \ldots, x^m) = (\psi \circ f \circ \varphi^{-1})(x^1, \ldots, x^m) = (f^1(x^1, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m))$ expressing $f$ in the local coordinates.

The rank must be independent of the choice of coordinates since the smoothness of $f$ is independent of the choice of coordinates (charts). Let $M \xrightarrow{f} N$ be a smooth map. Thus, we get the following: $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$, $y = (f^1(x^1, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m)) \in \mathbb{R}^n$. The important case for our study will be in which the rank is constant at each point $p \in M$.

Example 2.3.1. [15, p.47] Let $f(x_1, x_2) = (x_1^2 + x_2^2, 2x_1x_2)$. Its Jacobian is given by

$$Df(x_1, x_2) = \begin{bmatrix}
\frac{\partial(x_1^2 + x_2^2)}{\partial x_1} & \frac{\partial(x_1^2 + x_2^2)}{\partial x_2} \\
\frac{\partial(2x_1x_2)}{\partial x_1} & \frac{\partial(2x_1x_2)}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
2x_1 & 2x_2 \\
2x_2 & 2x_1
\end{bmatrix}$$

and $\det Df(x_1, x_2) = 4x_1^2 + 4x_2^2$. First, $4x_1^2 + 4x_2^2 = 0 \iff x_1^2 + x_2^2 = 0 \iff x_1 = x_2 = 0$. Hence, $\Rightarrow Df(0, 0) = \begin{bmatrix}0 & 0 \\
0 & 0\end{bmatrix}$, that is, rank$f = 0$ at $(0,0)$. Secondly, $4x_1^2 + 4x_2^2 \neq 0$ as a sum of
squares \Rightarrow x_1 \neq 0 \text{ or } x_2 \neq 0. \text{ Thus, } rank f = 2 \text{ at } (x_1, x_2) \neq (0, 0).

Let \( f(x_1, x_2) = ((x_1)^2, 2x_1x_2) \). Then \( Df(x_1, x_2) = \begin{bmatrix} \frac{\partial x_1^2}{\partial x_1} & \frac{\partial x_1^2}{\partial x_2} \\ \frac{\partial 2x_1x_2}{\partial x_1} & \frac{\partial 2x_1x_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 0 \\ 2x_2 & 2x_1 \end{bmatrix} \). Thus, \( det Df(x_1, x_2) = 4x_1^2 \). Then \( rank f = 0 \) at \((0, 0)\) and \( rank f = 2 \) at \((x_1, x_2) \neq (0, 0)\).

The relationship between global diffeomorphisms and local diffeomorphisms on locally Euclidean \( \mathbb{F} \)-spaces setting is given by Definition 2.2.7 and Corollary 2.2.7.

**Definition 2.3.2.** Let \( f : M \rightarrow N \) be a smooth map of locally Euclidean \( \mathbb{F} \)-spaces with \( \dim M = m, \dim N = n \). The map \( f \) is said to be:

1. a submersion if \( rank f = n \) at every point \( p \in M \), with \( n \geq m \).
2. a immersion if \( rank f = m \) at every point \( p \in M \), with \( m \leq n \).
3. a diffeomorphism if \( f \) maps \( M \) one-to-one onto \( N \) and \( f^{-1} \) is smooth.
4. a local diffeomorphism if \( \dim M = \dim N \) and \( f \) a submersion (or equivalently, \( f \) is an immersion)

We will deal with the concepts of substructure in the locally Euclidean \( \mathbb{F} \)-space setting, that is, in \( \mathbb{F} \)-setting.

**Definition 2.3.3.** Let \( F : M \rightarrow N \) be a smooth map of locally Euclidean \( \mathbb{F} \)-spaces and \( \dim M = m, \dim N = n \). \( F(M) := (M, F) \) is an immersed locally Euclidean subspace of \( N \) if, and only if \( F \) is an injective immersion.

**Remark 2.3.1.**

1. Some authors, Frank W. Warner (see [114]) among them, denote \( F(M) := (M, F) \) to stress the fact that the structure lies on the nature of \( F \) as in Definition 2.3.3. And changing \( F \) to \( G \), another map, should yield \( (M, G) \neq (M, F) \). Also locally Euclidean \( \mathbb{F} \)-subspace will mean immersed locally Euclidean \( \mathbb{F} \)-subspace if no confusion is expected.

2. In Definition 2.3.3, \( F(M) := (M, F) \) is endowed with topology and \( \mathbb{F} \)-structure which makes \( F : M \rightarrow F(M) \) a \( \mathbb{F} \)-diffeomorphism.
3. We will be aware of this $\mathbb{F}$-diffeomorphism: even when $F$ is one-to-one immersion, it is not necessary a $\mathbb{F}$-diffeomorphism with $F(M)$ as a $\mathbb{F}$-subspace of $N$ since the structure on $F(M)$ is generated by $\mathcal{G}_{oF(M)} = \{ f \circ F^{-1} : f : F(M) \to \mathbb{R} \mid f \in \mathcal{F}_M \}$, that is, co-induced from that of $M$. The smoothness of $F$ yields $(f \circ F^{-1}) \circ F = f \in \mathcal{F}_M$. But $\mathcal{F}_{oF(M)} = \{ h \circ \iota_{F(M)} = h|_{F(M)} \mid h \in \mathcal{F}_N \} = \mathcal{F}_{N|_{F(M)}}$ generates the $\mathbb{F}$-substructure on $F(M)$, as shown in the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{F} & F(M) \\
\sim & & \iota_{F(M)} \\
\mathbb{R} & \xrightarrow{\sim} & \mathbb{R}
\end{array}
\]

\[
(F(M), \Gamma \mathcal{G}_{oF(M)}, \Phi \Gamma \mathcal{G}_{oF(M)}):= (F(M), \Gamma \mathcal{G}_{oF(M)}, \mathcal{G}(M))
\]
\[
(F(M), \Gamma \mathcal{F}_{oF(M)}, \Phi \Gamma \mathcal{F}_{oF(M)}):= (F(M), \Gamma \mathcal{F}_{oF(M)}, \mathcal{F}(M)).
\]

4. Since $F : M \to F(M) \subset N$ is one-to-one and onto, we do understand Definition 2.3.3 in the following way. By defining open set of $F(M)$ to be images of open sets of $M$ and coordinate neighborhood $(W, \eta)$ of $F(M)$ to be of the form $W = F(U)$, $\eta = \varphi \circ F^{-1}$, where $(U, \varphi)$ is a coordinate neighborhood of $M$, we will carry over the topology and $\mathbb{F}$-structure of $M$ to $F(M)$ as shown below:

\[
\begin{array}{ccc}
M & \xrightarrow{F} & F(M) \xrightarrow{\iota_{F(M)}} N \\
\sim & & \\
\mathbb{R} & \xrightarrow{\sim} & \mathbb{R}
\end{array}
\]

\[
\begin{array}{ccc}
U & \xrightarrow{F_{|U}} & F(U) = W \\
\sim & & \\
\varphi(M) & \xrightarrow{\sim} & \mathbb{R}^m
\end{array}
\]

\[
\begin{array}{ccc}
M & \xrightarrow{F^{-1}} & F(M) \\
\sim & & \\
\mathbb{R} & \xrightarrow{\sim} & \mathbb{R}
\end{array}
\]

The fact that $F : M \to N$ is continuous implies that, if $\mathcal{V} \in \tau_{FM}$ then $F^{-1}(\mathcal{V}) \in \tau_{FM}$ and also $F(F^{-1}(\mathcal{V})) = \mathcal{V} \cap F(M) = \iota_{F(M)}^{-1}(\mathcal{V})$ since $\iota_{F(M)}$ is the canonical inclusion. We recall that $\tau_{\mathcal{G}_{FM}} \supset \tau_{\mathcal{F}_N}(F(M))$, that is, $\mathbb{F}$-topology is finer than the relative topology. Thus, there may be open sets of $F(M)$ which are not of the form $\mathcal{V} \cap F(M)$. Nothing can allow us to state that all open sets in $M$ are of the form $F^{-1}(\mathcal{V})$. Thus, there exists $\mathcal{U} \subset M$ such that $\mathcal{U} \neq F^{-1}(\mathcal{V})$ for any $\mathcal{V}$ open in $N$. Notice that $\mathcal{V} \subset \mathcal{N}$ and not in $F(M)$.

**Definition 2.3.4.** Let $F : M \to N$ be a smooth map of locally Euclidean $\mathbb{F}$-spaces and
\[ \dim M = m, \dim N = n. \ F(M) := (M, F) \text{ is an embedded locally Euclidean } \mathbb{F}\text{-subspace of } N \text{ if, and only if } F \text{ is an injective immersion such that } F : M \rightarrow F(M) \text{ is a } \mathbb{F}\text{-diffeomorphism with } F(M) \text{ a } \mathbb{F}\text{-subspace}, \text{ but the topology is the trace topology of } \tau_{F_N} \text{ on } F(M). \text{ Such a smooth map } F \text{ is called an embedding of } M \text{ to } N. \]

**Remark 2.3.2.**

1. \( \tau_{F_N}(F(M)) \) is the smallest topology on \( F(M) \) for which \( \iota_{F(M)} \) (the canonical inclusion) is continuous.

2. \( F : M \rightarrow F(M) \) is an open map, that is \( F(\mathcal{U}) \) is open in \( F(M) \) and is of the form \( F(\mathcal{U}) = \mathcal{V} \cap F(M) \) with any \( \mathcal{V} \) open in \( N \) for \( \tau_{F_N} \) and \( \mathcal{U} \) open in \( M \).

3. An embedded locally Euclidean \( \mathbb{F}\text{-subspaces} \) is a particular type of immersed locally Euclidean \( \mathbb{F}\text{-subspace}. \)

4. We will denote \( \tau_o = \tau_{F_N}(F(M)), \tau_1 = \tau_{F(M)}, \text{ the } \mathbb{F}\text{-subspace topology on } F(M), \text{ and } \tau_2 \text{ the co-induced topology from } M \text{ to } F(M) \text{ as in Remark 2.3.1 (4). It is already known that } \tau_1 \supset \tau_o \text{ and } \tau_2 \supset \tau_o. \text{ We need to know what is it about } \tau_1 \text{ and } \tau_2. \text{ From their definitions } G_{F(M)} = \Phi \Gamma G_{o_F(M)} \text{ and } F_{F(M)} = \Phi \Gamma F_{o_F(M)}. \text{ That is, } h \circ \iota_{F(M)} = h|_{F(M)} \in F_{o_F(M)} \text{ with for all } h \in \mathcal{F}_N, h \circ F = f \in \mathcal{F}_M \text{ since } F \text{ is smooth also } F = \iota_{F(M)} \circ F \text{ by definition of } F. \text{ Since } F : M \rightarrow F(M). \text{ Thus } h|_{F(M)} = h \circ \iota_{F(M)} = h \circ \iota_{F(M)} \circ F \circ F^{-1} = h \circ F \circ F^{-1} = f \circ F^{-1} \in G_{o_F(M)}. \text{ It yields } F_{o_F(M)} \subset G_{o_F(M)}, \text{ that is, } F_{F(M)} = G_{F(M)}. \text{ Now } B = \{ h|_{F(M)}(0, +\infty) \mid h \in \mathcal{F}_N \} \text{ and } B' = \{ g^{-1}(0, +\infty), g \in G_{F(M)} \}. \text{ Thus, } B \subset B'. \text{ That implies } \tau_o \subset \tau_1 \subset \tau_2 \text{ since for all } \mathcal{U} \subset B, \text{ there exists } \mathcal{V} \subset B' \text{ such that } \mathcal{V} \subset \mathcal{U}. \]

**Definition 2.3.5.** Let \( N \) be an locally Euclidean \( n-\mathbb{F}\text{-space. Let } M \subset N \text{ be a } \mathbb{F}\text{-subspace of } N. \text{ } M \text{ is said to be a regular locally Euclidean } m-\mathbb{F}\text{-subspace of } N \text{ with } 0 \leq m \leq n \text{ if, and only if for every point } p \in M, \text{ there is a chart } (\mathcal{U}, \varphi) \in N \text{ with } p \in \mathcal{U} \text{ so that } \varphi(\mathcal{U} \cap M) = \varphi(\mathcal{U}) \cap (\mathbb{R}^m \times \{(0, \ldots, 0)\}) \text{ and the topology on } M \text{ is } \tau_{F_M} = \tau_{F_N}(M). \text{ The subset } \mathcal{U} \cap M \text{ is called a slice of } (\mathcal{U}, \varphi) \text{ and the chart } (\mathcal{U}, \varphi) \text{ is said to be adapted to } M. \]

**Definition 2.3.6.** A subset \( M \) of a \( n \)-locally Euclidean \( \mathbb{F}\text{-space } N \) is a regular \( m \)-locally Euclidean \( \mathbb{F}\text{-subspace of } N \text{ with } 0 \leq m \leq n \text{ if, and only if for each point } p \in M, \text{ there exists a coordinate neighborhood } (\mathcal{U}, \varphi) \text{ on } N \text{ with } p \in \mathcal{U}, \text{ with local coordinates } x^1, \ldots, x^n \text{ such that}

1. \( \varphi(p) = (0, \ldots, 0) \in \mathbb{R}^n. \text{ That is, the coordinate system is centered at } p. \)

2. \( \varphi(\mathcal{U}) = C^m_ε(0) = C^m_ε(0) = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid |x^i| < ε \text{ for all } 1 \leq i \leq n\}. \text{ That is, the open cube with sides of length } 2ε \text{ and centered at the origin } (0, \ldots, 0) \in \mathbb{R}^n. \)

3. \( \varphi(\mathcal{U} \cap M) = \{x \in C^m_ε(0) \mid x^{m+1} = \cdots = x^n = 0\} \)
Definition 2.3.5 and Definition 2.3.6 are equivalent. See [119, 1.4 Submanifolds] for details on regular point and regular locally Euclidean $\mathbb{F}$-subspace. A coordinate system $(U, \varphi)$ is called a cubic (cubical) coordinate system if $\varphi(U) = C^m_\epsilon(0)$.

**Example 2.3.2.** The sphere $S^2 = \{x \in \mathbb{R}^3 \mid ||x|| = 1\}$ is a regular locally Euclidean $\mathbb{F}$-subspace of $\mathbb{R}^3$. The examples given in [15, 18] are natural examples of regular locally Euclidean $\mathbb{F}$-subspaces.

**Definition 2.3.7.** Let $N$ be an locally Euclidean $n$-$\mathbb{F}$-space and $M \subset N$ a $\tau_{F_N}$-closed. $M$ is called a closed regular locally Euclidean $m$-$\mathbb{F}$-subspace if, and only if for each $p \in M$, there exists $(U, \varphi)$ a chart at $p$ in $N$ with $p \in U$ such that $\varphi(U \cap M) = \varphi(U) \cap (\mathbb{R}^m \times \{0, \ldots, 0\})$. $M$ is also called properly (or regularly) embedded locally Euclidean $\mathbb{F}$-subspace of $N$.

**Remark 2.3.3.**

1. If $M$ is a regular locally Euclidean $\mathbb{F}$-subspace of $N$ then $\iota_M \hookrightarrow N$ is a smooth map of locally Euclidean $\mathbb{F}$-spaces such that $\iota_M : M \hookrightarrow \iota_M(M)$ is the identity map, that is injective and immersion $(\psi \circ \iota_M \circ \varphi^{-1} = \psi \circ \varphi^{-1}$ a diffeomorphism, thus $\text{rank} \iota_M = m$), that is $M$ and $\iota_M(M)$ have the same $\mathbb{F}$-subspace structure and $\mathbb{F}$-subspace topology. Hence $\iota_M$ is naturally an embedding. Therefore $M$ is an embedded locally Euclidean subspace of $N$.

2. If $F : M \rightarrow N$ is an embedding, then $F(M)$ is an embedded locally Euclidean $\mathbb{F}$-subspace and $F : M \rightarrow F(M)$ is a diffeomorphism and the family of pairs $(U \cap M, \varphi|_{U\cap M})$, where $(U, \varphi)$ ranges over the charts over any atlas for $N$, is an atlas for $M$, where $M$ is given the topology $\tau_{F_M} = \tau_{F_N}(M)$.

3. Here are some Observations.

   (a) On $F(M)$, $\tau_o \subset \tau_1 \subset \tau_2$ as shown in Remark 2.3.2 (4).

   (b) $F(M)$ Embedded locally Euclidean $\mathbb{F}$-subspace $\Rightarrow$ Immersed (with $\tau_2$) and $\tau_o \Rightarrow \tau_2 = \tau_o \Rightarrow \tau_o \supset \tau_1 \cap \tau_2 \Rightarrow \tau_1 = \tau_o \Rightarrow \iota_{F(M)} : (F(M), \tau_o) \rightarrow (F(M), \tau_1)$ is a $\mathbb{F}$-diffeomorphism $\Rightarrow \iota_{F(M)}$ injective immersion such that such that $F(M)$ is a $\mathbb{F}$-subspace and $\tau_1 = \tau_o \Rightarrow F(M)$ is a regular locally Euclidean subspace of $N$.

   (c) $F(M)$ regular locally Euclidean $\mathbb{F}$-subspace of $N$ $\Rightarrow$

      i. $\iota_{F(M)}$ is an embedding $\Rightarrow F(M)$ is an embedded locally Euclidean $\mathbb{F}$-subspace of $N$. 

2.3 Locally Euclidean $\mathbb{F}$-subspace
ii. Since $\mathcal{F}_{F(M)} \subset \mathcal{G}_{F(M)}$ and $\tau_0 \subset \tau_1 \implies \tau_1 \supset B = \{ h_{F(M)}^{-1}(0, +\infty) \mid h \in \mathcal{F}_N \} \subset \{ g_{F(M)}^{-1}(0, +\infty) \mid g \in \mathcal{G}_{F(M)} \} = B' \subset \tau_2 \implies$ for all $\mathcal{U} \in B$ there exists $\mathcal{V} = \mathcal{U} \in B'$ such that $\mathcal{V} \subset \mathcal{U} \implies \tau_1 \subset \tau_2$. Also, since $F$ is a $\mathbb{F}$-diffeomorphism, any $\mathcal{V} = g_{F(M)}^{-1}(0, +\infty) = (f \circ F^{-1})^{-1}(0, +\infty) = F \circ f^{-1}(0, +\infty) = (f^{-1}(0, 0))$ is a $\tau_2$-basic open in $F(M)$. Thus $F(f^{-1}(0, +\infty)) = FF^{-1}(h_{F(M)}^{-1})(0, +\infty) = h_{F(M)}^{-1}(0, +\infty)$ $\tau_1$-basic open and $\tau_2$-basic open in $F(M)$. From closeness under finite intersections $f^{-1}(0, +\infty) \cap f_1^{-1}(0, +\infty) = f_2^{-1}(0, +\infty)$, $f_2 \in \mathcal{F}_N$ and $F(f_2^{-1}(0, +\infty)) = F\tau f^{-1}(0, +\infty) \cap F\tau f^{-1}(0, +\infty) = g^{-1}(0, +\infty) \cap h_{F(M)}^{-1}(0, +\infty)$ $\mathcal{V}$ is a $\tau_2$-basic open and $\tau_1$-basic open of $F(M)$. Hence for all $\mathcal{V} \in B'$ there exists $\mathcal{U} = g^{-1}(0, +\infty) \cap h_{F(M)}^{-1}(0, +\infty) \subset B$ such that $\mathcal{U} \subset \mathcal{V} \implies \tau_2 \subset \tau_1$. Therefore $\tau_2 = \tau_1$ and $\tau_1 = \tau_0$, that is $\tau_2 = \tau_0 \implies F(M)$ immersed (with $\tau_2$) and $\tau_0 \implies F$ embedding $\implies F(M)$ embedded locally Euclidean $\mathbb{F}$-subspace of $N$.

Example 2.3.3. Examples of open and closed locally Euclidean $\mathbb{F}$-subspaces

1. $\mathcal{U} = GL(n, \mathbb{R}) \subset M = \mu_n(\mathbb{R})$, $\times n$ matrices over $\mathbb{R}$, which consists of all non-singular $n \times n$ matrices $\mathcal{U} = \{ A \in \mu_n(\mathbb{R}) \mid det A \neq 0 \}$ since $det A$ is a polynomial function of its entries $a_{ij}$, it is a continuous (smooth) function of its entries and of $A$ in the topology of identification with $\mathbb{R}^{n^2}$. Thus $\mathcal{U} = GL(n, \mathbb{R})$ is an open set-the complement of the closed set of those $A$ such that $det A \neq 0$, and we see that $\mathcal{U} = GL(n, \mathbb{R})$ is an open locally Euclidean $\mathbb{F}$-subspace

2. $S^2$ is a closed, regular locally Euclidean $\mathbb{F}$-subspace

One can make the construction of a locally Euclidean $\mathbb{F}$-space from a given one by means of the following results borrowed from [15, 18].

Lemma 2.3.1. [15, Theorem 5.8] Let $M$ be a locally Euclidean $m$-$\mathbb{F}$-subspace, $N$ a locally Euclidean $n$-$\mathbb{F}$-space and $F : M \rightarrow N$ a smooth map. Suppose that $F$ has constant rank $k$ on $M$ and that $q \in F(M)$. Then $F^{-1}(q)$ is closed, regular $(m-k)$-locally Euclidean $\mathbb{F}$-subspace on $M$.

Corollary 2.3.2. [15, Corollary 5.9] If $F : M \rightarrow N$ is a smooth map of locally Euclidean $\mathbb{F}$-spaces with $dim N = n \leq dim M = m$ and if $rank F = n$ at every point of $F^{-1}(q)$, with $q \in M$ then $F^{-1}(q)$ is closed, regular locally Euclidean $\mathbb{F}$-subspace of $M$. 
2.4 Tangent spaces on $\mathbb{F}$-spaces

In what follows we assume that the concept of tangent spaces and bundles are known as described in the relevant literature (see [8, 29, 85, 86, 109]). Notice that the notions of tangent and cotangent $\mathbb{F}$-bundles on a $n$-$\mathbb{F}$-space are similar to those of tangent and cotangent bundles on a $n$-dimensional smooth manifold (see [47]). It follows that $(\mathbb{R}^n, \pi, \mathbb{R})$ is a trivial $\mathbb{F}$-bundle where $\mathbb{R}^n$ is the total space, $\mathbb{R}$ the base space, $\pi : \mathbb{R}^n \to \mathbb{R}$ the projection and $\pi^{-1}(x_i) := \{x = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n | \pi(x) = x_i\}$ is the fiber of the $\mathbb{F}$-bundle over $x_i \in \mathbb{R}$.

**Definition 2.4.1.** Let $(M, C_M, \mathcal{F}_M)$ be an $n$-$\mathbb{F}$-space. An operational tangent vector $v$ to $M$ at the point $p \in M$ is a smooth derivation (linear operator) of the algebra $\mathcal{F}_M$ at $p$. That is, $v := d_p = e v_p \circ d : \mathcal{F}_M \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}_M$, $\alpha \in \mathbb{R}$ we have $v(f + \alpha g) = v(f) + \alpha v(g)$ and $v(fg) = f(p)v(g) + g(p)v(g)$, the so-called Leibniz condition (or rule).

We denote by $\text{Der}(M) := \{d : \mathcal{F}_M \to \mathcal{F}_M | d$ is a smooth derivation of $\mathcal{F}_M$ on $M\}$ the $\mathcal{F}_M$-module containing all smooth derivations. The operational tangent vector $v$ is also called the contravariant tangent vector or the derivative.

**Lemma 2.4.1.** [8] Let $(M, C_M, \mathcal{F}_M)$ be a locally Euclidean $\mathbb{F}$-space and $p \in M$. Let $v : \mathcal{F}_M \to \mathbb{R}$ be a linear map. Then $v$ is an operational tangent vector to $M$ at $p$, if and only if $v$ satisfies the following conditions: $v(f) = 0$ if $f$ is constant, and $v|_{\alpha^2_p} = 0$, where $\alpha^2_p := \{(f - f(p))(g - g(p)) | f, g \in \mathcal{F}_M\}$.

**Definition 2.4.2.** Let $(M, C_M, \mathcal{F}_M)$ be a locally Euclidean $\mathbb{F}$-space and $p \in M$. The set $T_p M \subseteq C^\infty(\mathcal{F}_M, \mathbb{R})$ of all operational tangent vectors at $p$ is called the operational tangent space at $p$ on $M$.

Let $M$ be a $\mathbb{F}$-space and $p$ is running through $M$. The set denoted by

$$TM := \prod_{p \in M} \{p\} \times T_p M = M \times (\prod_{p \in M} T_p M) = \{(p, v_p) | p \in M, v_p \in T_p M\} \quad (2.13)$$

is called the operational tangent bundle on $M$, while

$$T^*M = \{(p, \theta_p) | p \in M, \theta_p \in T^*_p M\} = \prod_{p \in M} \{p\} \times T^*_p M = M \times (\prod_{p \in M} T^*_p M) \quad (2.14)$$
is called the operational cotangent bundle on $M$. We denote by,

$$\text{Der}(M) = \{ d : \mathcal{F}_M \rightarrow \mathcal{F}_M | d \text{ is a smooth derivation of } \mathcal{F}_M \text{ on } M \}, \quad (2.15)$$

the $\mathcal{F}_M$-module containing all smooth derivations. The operational tangent vector $v$ is also called the contravariant tangent vector or the derivative. It can be seen that

$$TM \subseteq M \times \text{Der}(M) \subseteq M \times C^\infty(\mathcal{F}_M, \mathbb{R}). \quad (2.16)$$

**Definition 2.4.3.** Let $M$ be a $\mathbb{F}$-space and $p \in M$. A map $\chi : M \rightarrow TM$ defined by $p \mapsto v_p \in T_p M$ such that $\pi \circ \chi = id_M$ is called a tangent vector field to $M$ (or a section of $\pi$).

That is, $\chi(p) : \mathcal{F}_M \rightarrow \mathbb{R}$ with $f \mapsto \chi(p)(f) = (\chi f)(p) = v_p(f)$ and $\chi f \in \mathbb{R}^M$, for any $f \in \mathcal{F}_M$. $\chi$ is a smooth tangent vector field if $\chi f = \chi(f) \in \mathcal{F}_M$. It follows that $\chi$ induces a map also denoted by $\chi : \mathcal{F}_M \rightarrow \mathcal{F}_M$ which is a smooth derivation. Definition 2.4.3 gives both local and global interpretation of the concept of tangent vector field. The evaluation of $\chi$ at $p$ can be understood as follows. $ev_p \circ \chi : \mathcal{F}_M \rightarrow \mathcal{F}_M \rightarrow \mathbb{R}$, $f \mapsto \chi(f) \mapsto (ev_p \circ \chi)(f) = \chi(p)(f) = \chi(f)(p)$. The set of all smooth tangent vector fields on $M$ is denoted by $\mathfrak{X}(M)$.

**Lemma 2.4.2.** [8] Let $M$ be a $\mathbb{F}$-space and $f \in \mathcal{F}_M$. Let $\chi$ be a tangent vector field on $M$. Then $\chi$ is smooth if, and only if $(f \circ \pi) \circ \chi \in \mathcal{F}_M$ and $df \circ \chi \in \mathcal{F}_M$. There exists $\chi^* : T^*M \rightarrow \mathbb{R}$ defined by $\chi^*(\theta) = \theta(\chi(\tau(\theta)))$, where $\theta \in T^*M$ and $\tau$ is the canonical projection $\tau : T^*M \rightarrow M$.

**Definition 2.4.4.** Let $M$ be a $\mathbb{F}$-space. $M$ is said to be of constant dimension $n$ if either

1. $\dim T_p M = \dim T_q M = n$ for any $p, q \in M$, with $p \neq q$ and for all $v \in T_p M$, there exists $\chi \in \mathfrak{X}(M)$ such that $\chi(p) = v$; or

2. for each $p \in M$, there exists an open neighborhood $U$ of $p$ in $M$ and a local basis of vector fields over $U$ making $\mathfrak{X}(U)$ a free module on $\mathcal{F}_M$.

We may have different dimensions at different points of a $\mathbb{F}$-space. The rest of this work is devoted to $\mathbb{F}$-spaces of constant dimension, "$\mathbb{F}$-spaces of dimension $n^n$" or indiscriminately "$n$-$\mathbb{F}$-spaces". A $n$-$\mathbb{F}$-spaces looks like $\mathbb{R}^n$ at both the $\mathbb{F}$-structure and the $\mathbb{F}$-topology points of view. Therefore, $(\mathbb{R}^n, C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ is a natural model of $\mathbb{F}$-spaces of constant dimension, as well as a $n$-dimensional smooth manifold. [8]
Proposition 2.4.3. Let $TM$ and $T^*M$ as defined in Equations (2.13), (2.14) and (2.16). If $M$ is a $\mathbb{F}$-space, then $TM$ and $T^*M$ are so. And if $M$ is locally Euclidean $n$-$\mathbb{F}$-space then $TM$ and $T^*M$ are locally $2n$-$\mathbb{F}$-spaces.

Proof. The pair $(x,v) \in TM$ is given in local coordinates by $(x_i, \frac{\partial}{\partial x_i})$, and $(x, \theta) \in T^*M$ is given by $(x_i, dx_i)$. Thus, $TM$ and $T^*M$ are both $2n$-$\mathbb{F}$-spaces. Now, referring to Equation (2.16), we have $TM \subset M \times C^\infty(F_M, \mathbb{R})$ as a $\mathbb{F}$-subspace of a $\mathbb{F}$-product of $\mathbb{F}$-spaces $M$ and $C^\infty(F_M, \mathbb{R})$. The $\mathbb{F}$-structure on $TM$ is generated by the set of functions $F_o = \{df \mid f \in F_M\}$ and $\{f \circ \pi \mid f \in F_M\}$. Thus, $(TM, \Gamma F_o, \Phi F_o) := (TM, T\mathcal{C}_M, T\mathcal{F}_M)$. Now, the cotangent bundle $T^*M$ on $M$ has the natural structure generated by the set of functions $G_o = \{\chi^* \mid \chi \in \mathcal{X}(M)\} \cup \{f \circ \tau \mid f \in \tau_M\}$, where $\mathcal{X}(M)$ is the set of all smooth vector fields on $M$. □

Let $\varphi : M \rightarrow N$ be a diffeomorphism of $\mathbb{F}$-spaces. Thus, $T^*\varphi := ((\varphi)^{-1})^* = (\varphi^*)^{-1}$ and $\varphi^* = (T^*\varphi)^{-1}$ such that $\varphi^*(\theta) := \theta \circ \varphi_* = \alpha$ if, and only if $(\varphi^*)^{-1}(\alpha) := \alpha \circ \varphi_*^{-1} = \theta$. Obviously, $(TM, \pi, M)$ and $(T^*M, \tau, M)$ are $\mathbb{F}$-bundles.

Lemma 2.4.4. [8, 109] Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a locally Euclidean $n$-$\mathbb{F}$-space and $p \in M$. The operational tangent and cotangent spaces $T_pM$ and $T^*_pM$ at the point $p$ of $M$ are linear Frölicher spaces, linear locally Euclidean $\mathbb{F}$-spaces.

Proof. Let $M$ be a $n$-$\mathbb{F}$-space and $p \in M$, and $(U, \varphi)$ a local chart at $p$. The sets $T_pM$ and $T^*_pM$ are linear $n$-$\mathbb{F}$-spaces diffeomorphic to $\mathbb{R}^n$ with respective bases $\{\frac{\partial}{\partial x_i}\}$ and $\{dx_i\}$, where $(x_i)$ are local coordinates of $p \in U \subset M$ such that $\varphi(p) = (x_1, \ldots, x_n)$. There exists natural projections defined as follows: $\pi : TM \rightarrow M$, $(p, v_p) \mapsto p$ and $\tau : T^*M \rightarrow M$, $(p, \theta_p) \mapsto p$. The structure, on the coproduct space given above, are here generated by the families $(t_p)_{p \in M}$ and $(\iota_p)_{p \in M}$ of canonical inclusion maps $v_p : T_pM \hookrightarrow TM$ and $\iota_p : T^*_pM \hookrightarrow T^*M$. At each point $p \in M$, $d : \mathcal{F}_M \rightarrow \mathcal{F}_M$ induces a map $d_p : \mathcal{F}_M \rightarrow \mathbb{R}$ such that, for all $f \in \mathcal{F}_M$, $d_p(f) = (df)_p = ev_p(df) = (ev_p \circ d)(f)$ with $ev_p$ the evaluation map at $p$. It follows that $d_p = ev_p \circ d$ is a smooth linear map and a derivation. As $(df)_p$ is defined for each $p \in M$, it determines globally a smooth map $df : TM \rightarrow \mathbb{R}$ such that $(df)|_{t_pM} = (df)_p = d_p(f)$. Also, $\pi^{-1}(p) = \{v \in TM \mid \pi(v) = p\} = T_pM$ is the fiber of $TM$ at $p$ and $\tau^{-1}(p) = T^*_pM$ is the fiber of $T^*M$ at $p$. □

Definition 2.4.5. Let $M$ be a Frölicher object. A set $S = \{f_1, \ldots, f_n\}$ of structure functions on $M$ is called functionally independent in the neighborhood of a point $p \in M$ if \{$(df_1)(p), \ldots, (df_n)(p)$\} is a linearly independent set in the cotangent space $T_p^*M$ to $M$ at $p$.

Lemma 2.4.5. Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. Let $f_1, \ldots, f_n$ be some smooth functions defined in an open neighborhood $U$ of $p \in M$ such that one of them is injective. Then the map $\psi := (f_1, \ldots, f_n)$ is a diffeomorphism of $(U, \mathcal{C}_U, \mathcal{F}_U)$ onto $(\psi(U), \mathcal{C}_{\psi(U)}, \mathcal{F}_{\psi(U)})$.
**Lemma 2.4.6.** Let \((M, C_M, F_M)\) be a \(n\)-\(F\)-space. Let \(\{f_1, \ldots, f_n\}\) be a generating set of \(F\)-structure on \(M\) such that the map given by \(\psi(p) = (f_1(p), \ldots, f_n(p))\) for all \(p \in M\) is one-to-one. Then the associated tangent map \(\psi_p : T_p M \longrightarrow T_{\psi(p)} \psi(M)\) is an isomorphism of linear spaces.

**Proof.** From linear algebra, it is known that the map \(\varphi : T_p M \longrightarrow \mathbb{R}^n\) defined by \(\varphi(v) := (v_1, \ldots, v_n)\), where the \(v_i\) are the coordinates of \(v \in T_p M\), is an isomorphism. Recall that the canonical \(F\)-structure on \(\mathbb{R}^n\) is generated by \(\{\pi_1, \ldots, \pi_n\}\). And \(\psi(M)\) is a \(n\)-\(F\)-subspace of \(\mathbb{R}^n\), generated by the restrictions \(\hat{\pi}_i = \pi_i |_{\psi(M)}\) while each \(f_i = \pi_i |_{\psi(M)} \circ \psi\) for \(i = 1, \ldots, n\). Thus, \(n = \dim \mathbb{R}^n = \dim \psi(M) = \dim T_p M = \dim T_{\psi(p)} \psi(M) = \dim \mathcal{X}(\mathcal{U})\), where \(\mathcal{U}\) is an open neighborhood of \(p\). Let \((M, C_M, F_M)\) be a \(n\)-\(F\)-space and \(p \in M\). Assume that \(F\)-structure on \(M\) is generated by the set \(\{f_1, \cdots, f_n\} \subset F_M\), such that \(\varphi(p) := (f_1(p), \cdots, f_n(p))\) is a \(F\)-diffeomorphism on a neighborhood of \(p\) onto a \(F\)-subspace of \(\mathbb{R}^n\) endowed with the canonical \(F\)-structure. \(\square\)

**Definition 2.4.6.** Let \(M\) be a \(n\)-\(F\)-space. A slit tangent bundle over \(M\) is the set denoted by \(T M_o = \{(p, y) \in TM \mid p \in M, y \in T_p M, y \neq 0\}\).

Since \(T_p M_o = T_p M - \{0\} \subset T_p M \subset TM\) and \(TM = \bigsqcup_{p \in M} \{p\} \times T_p M\), then \(TM\) is a balanced space and the coproduct topology is equal to the underlying \(F\)-topologies, since the coproduct of Frölicher spaces is a final object (see [23]). It follows that \(T_p M_o\) is an open set in \(T_p M\). Thus, \(\dim T_p M_o = n, TM_o = \bigsqcup_{p \in M} \{p\} \times T_p M_o\) is an open set in \(TM\), and so \(\dim TM_o = 2n\).

That is, \(T_p M_o\) and \(TM_o\) are respectively \(n\)-\(F\)-space and \(2n\)-\(F\)-space. Let \((x, y), (\bar{x}, \bar{y}) \in T M_o\). We define a relation on \(T M_o\) by \((x, y) \sim (\bar{x}, \bar{y})\) if, and only if there exists a real \(\lambda > 0\) such that \(x = \bar{x}\) and \(y = \lambda \bar{y}\). The relation \(\sim\) is an equivalence relation on \(T M_o\). The equivalence class of \((x, y) \in T M_o\) is of the form \((x, [y]) := \{(x, \lambda y) \mid \lambda > 0, (x, y) \in T M_o\}\), where \([y] = \{\bar{y} = \lambda y \mid \lambda > 0\}\). It is called a ray or a direction. The quotient \(F\)-space \(T M_o/\sim\) is called the projective sphere bundle denoted by \(T M_o/\sim := SM = \{(x, [y]) \mid (x, y) \in T M_o\}\). Note that each \(T_x M\) is partitioned by the equivalence classes. \(SM\) is a \((2n-1)\)-\(F\)-subspace of \(TM\). The fibers at \((x, [y]) \in SM\), denoted by \(S_x M := \tau^{-1}(x)\) and \(S_x^* M := \Gamma^{-1}(x)\), where \(\Gamma : S^* M \longrightarrow M\) is the canonical projection, are diffeomorphic to \((n-1)\)-\(F\)-subspaces in \(T_x M\) and \(T_x^* M\) respectively. Thus, \(S_x M\) and \(S_x^* M\) are diffeomorphic to \(S^{n-1}\), and are called projective spheres at \(x\).
2.5 Exterior algebra on ℱ-spaces

The reader is referred to [47, 62, 94, 107] for more details on the exterior algebra in smooth manifolds.

**Definition 2.5.1.** Let \( M \) be a \( n \)-space. A \( k \)-form on \( M \) or a form of degree \( k \) is a section of the \( ℱ \)-bundle \( \bigwedge^k T^*M = \bigcup_{x \in M} \bigwedge^k T_x^*M \) with base space \( M \) and fibers \( \bigwedge^k T_x^*M \). The set of all \( k \)-forms on \( M \) is denoted by \( \Omega^k(M) := (M, \bigwedge^k T^*M) \).

The set \( \Omega^k(M) := (M, \bigwedge^k T^*M) \) of all \( k \)-forms on \( M \) is a module on the algebra \( ℱ_M \). One can conclude that \( \bigwedge^k T^*M \) is a locally Euclidean space as a coproduct and with respect to Proposition 2.4.3 and Lemma 2.4.4 (see also [85, 112, 114]). Note that we are dealing with smooth vector fields as defined in Definition 2.4.3. Moreover, the sections (\( k \)-forms) of the \( ℱ \)-bundle \( \bigwedge^k T^*M \) are smooth maps with respect to Lemma 2.4.2.

For \( k = 0, 1, 2 \), we have: \( \bigwedge^0 T^*_x M = ℜ, \Omega^0(M) = ℱ_M, \bigwedge^1 T^*_x M = T^*_x M, \) and \( \bigwedge^2 T^*_x M = \) the set of all 2-linear alternating functions \( \omega : T^*_x M \times T^*_x M \rightarrow ℜ, \) with \( \Omega^2(M) := (M, \bigwedge^2 T^*M) \). If \( X_1, X_2, \ldots, X_k \) are \( k \) smooth vector fields on a Fr"olicher space \( M \) and \( \omega \) is the \( k \)-form above, then \( \omega(X_1, X_2, \ldots, X_k)(x) = \omega(x)(X_1(x), X_2(x), \ldots, X_k(x)) = \omega_x(X_1(x), X_2(x), \ldots, X_k(x)) \), where \( \omega(x) := \omega_x \) is a smooth function for all \( x \in M \). That is, the \( k \)-form \( \omega \) on \( M \) is a collection of smoothly varying \( k \)-linear alternating maps \( \omega_x \in \bigwedge^k T^*_x M \). In local coordinates any 1-form \( \omega \) and any vector field \( Z \) on \( M \) are given by, \( \omega = \sum_{i=1}^n h_i(x) dx^i \) and \( Z = \sum_{i=1}^n \xi_i(x) \frac{\partial}{\partial x^i} \), where \( x^1, x^2, \ldots, x^n, h_i, \xi_i \in ℱ_M \). Thus, \( \langle \omega, Z \rangle := \sum_{i=1}^n h_i(x)\xi_i(x) \) is a smooth function.

**Definition 2.5.2.** Let \( M \) be a \( n \)-dimensional locally Euclidean space. The operator denoted by \( \wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M) \), called the exterior product (also wedge or Grassmann product), is a \( ℱ \)-smooth multilinear and alternating map defined by: given \( \alpha \in \bigwedge^k T^*_x M \) and \( \beta \in \bigwedge^l T^*_x M \), the \((k + l)\)-form \( \alpha \wedge \beta : M \rightarrow \bigwedge^{k+l} T^*_x M \) is their exterior product.

The operator \( \wedge \) satisfies the following properties.
2.5 Exterior algebra on $\mathbb{F}$-spaces

1. Given $k$ $1$-forms $\omega_1, \omega_2, \ldots, \omega_k$ then $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k$ is a $k$-form defined, as a determinant of order $k$, by $< \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k; Z_1(x), Z_2(x), \ldots, Z_k(x) > = det(< \omega_i, Z_i(x) >)_{1 \leq i, j \leq k}$, where $Z_i(x)$ is any vector of $T_x M$. This is, a smooth real valued function on $T_x M \times T_x M \times \ldots \times T_x M$, with $k$ factors.

2. $\bigwedge^k T_x^* M$ is spanned by the basic $k$-forms $dx^i := dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$, with $I$ running over all strictly increasing multi-indices $1 \leq i_1 < i_2 < \ldots < i_k \leq dim M$. Thus, any $k$-form $\omega$ on $M$ has the local coordinates expression $\omega = \sum_{i=1}^n h_i(x) dx^I$, where $h_I$ is a smooth function, $dx^I$ a $k$-form as the exterior product of $k$ $1$-forms $dx^{i_1}, \ldots, dx^{i_k}$.

**Definition 2.5.3.** Let $M$ be a $n$-dimensional locally Euclidean Frölicher space. The operator $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, called the exterior derivative, is a linear map $k$-forms to $(k+1)$-forms, such that:

1. For $k = 0$ and each $x \in M$, the map $d: 0 \bigwedge T_x^* M \longrightarrow \bigwedge^k T_x^* M$ is defined by $df(Z) = Z(f)$, where $f \in \bigwedge^0 T_x^* M$, $df \in \bigwedge^1 T_x^* M$ (this is the natural differential) and $Z \in \mathfrak{X}(M)$;

2. For $k \geq 1$ and $\omega \in \Omega^k(M)$ and for any $Z_1, Z_2, \ldots, Z_k, Z_{k+1} \in \mathfrak{X}(M)$, the exterior derivative (differential) of a $k$-form $\omega$ defined by

$$d\omega(Z_1, Z_2, \ldots, Z_k, Z_{k+1}) = \sum_{i=1}^{k+1} (-1)^i Z_i(\omega(Z_1, Z_2, \ldots, Z_i, \ldots, Z_k, Z_{k+1}))$$

$$+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([Z_i, Z_j], Z_1, Z_2, \ldots, \widehat{Z_i}, \ldots, Z_k, Z_{k+1}),$$

where, $\widehat{Z_i}$ and $\widehat{Z_j}$ are omitted;

3. For a $k$-form $\alpha$, $d(d\alpha) = 0$.

For a $k$-form $\alpha$ and another $l$-form $\beta$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$. A $k$-form $\alpha \in \Omega(M)$ is called a closed form if $d\alpha = 0$. In local coordinates if $\omega = \sum_I h_I(x) dx^I$ for any $k$-form $\omega$ then $d\omega = \sum_I dh_I \wedge dx^I = \sum_{i_1 < \ldots < i_k} dh_{i_1 \ldots i_k} \wedge dx^1 \wedge dx^2 \wedge \ldots \wedge dx^k$.

**Example 2.5.1.** [47, 62, 94] The exterior derivative satisfies the following properties. If $\omega$ is a $1$-form then $d\omega$ is a $2$-form defined by $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$ for $X, Y$ vector fields on $M$. If $\omega$ is a $2$-form then $d\omega(X, Y, Z) = \bigwedge Z\omega(Y, Z) - \bigwedge \omega([X,Y], Z)$ is a $3$-form, where $\bigwedge$ means the summation over cyclic permutations of $X, Y, Z \in \mathfrak{X}(M)$. 

Definition 2.5.4. Let \( \varphi : M \to N \) be a \( \mathbb{F} \)-smooth map of finite dimensional locally Euclidean spaces. The pullback \( \varphi^* : \Omega(N) \to \Omega(M) \) is a smooth morphism of algebra which pulls back \( k \)-forms on \( N \) to \( k \)-form on \( M \), and satisfies three requirements as below:

1. \( \varphi^* : \bigwedge^k T^*_x N \to \bigwedge^k T^*_x M \) is the restriction of \( \varphi^* \) on \( \bigwedge^k T^*_x N \subset \Omega(N) \);
2. For each \( f \in \mathcal{F}_M \), that is, for each 0-form one has \( \varphi^* f = f \circ \varphi \);
3. For \( k > 0 \), each \( \mathbb{F} \)-smooth \( k \)-form \( \omega \) on \( N \) induces a \( \mathbb{F} \)-smooth \( k \)-form \( \varphi^* \omega = \omega \circ \varphi \) on \( M \), such that \( \varphi^* \omega(v_1, v_2, \ldots, v_k) = \omega_{\varphi(x)}(\varphi_{*x}(v_1), \varphi_{*x}(v_2), \ldots, \varphi_{*x}(v_k)) \) for \( v_1, v_2, \ldots, v_k \in T_x M \).

Proposition 2.5.1. Let \( f \in \bigwedge^0 T^*_x(M) \) and \( \alpha, \beta \in \bigwedge^1 T^*_x(N) \). Let \( \varphi : M \to N \) and \( \psi : P \to M \) be two \( \mathbb{F} \)-smooth maps. Then the pullbacks \( \varphi^* \) and \( \psi^* \) have the following properties.

1. \( \varphi^*(\alpha + \beta) = \varphi^*\alpha + \varphi^*\beta \).
2. \( \varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta \).
3. \( \varphi^*(d\alpha) = d(\varphi^*\alpha) \), that is, \( \varphi^* \) commutes with \( d \).
4. \( \varphi^*(df) = df \circ \varphi = (f \circ \varphi) \).
5. \( \varphi^*(f\alpha) = \varphi^*(f)\varphi^*\alpha = (f \circ \varphi)\varphi^*\alpha \).
6. \( (\varphi \circ \psi)^*\alpha = (\psi^* \circ \varphi^*)\alpha \).
7. If \( \varphi \) is a \( \mathbb{F} \)-diffeomorphism, then \( \varphi^* \) is a \( \mathbb{F} \)-isomorphism of \( \Omega(N) \) onto \( \Omega(M) \) and \( (\varphi^*)^{-1} = (\varphi^{-1})^* \).
8. If \( \varphi \) is the identity map of a finite dimensional locally Euclidean space \( M \) then \( \varphi^* \) is the identity of \( \Omega(M) \).

Definition 2.5.5. Let \( M \) be a \( n \)-dimensional locally Euclidean space. The \( \mathbb{F} \)-smooth map \( [\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \), denoted by \( (X, Y) \mapsto [X, Y] := XY - YX \in \mathfrak{X}(M) \), is called the commutator or the Lie-bracket and is defined by \( [X, Y](f) := X(Y f) - Y(X f) \in \mathfrak{X}(M) \), for any \( f \in \mathcal{F} \), for each \( x \in M \) and for all \( X, Y \in \mathfrak{X}(M) \).

For any \( f \in \mathcal{F} \) and for all \( X, Y, Z \in \mathfrak{X}(M) \). The Lie-bracket satisfies the properties below:

1. Closure: \( [X, Y] := XY - YX \in \mathfrak{X}(M) \).
2. Bilinearity: \( [X, Y + Z] = [X, Y] + [X, Z] \) and \( [X, fY] = (X \cdot f)Y + f[X, Y] \). That is, the linearity in both two components.
3. Antisymmetry: \([X, Y] = -[Y, X]\).


The \(\mathcal{F}_M\)-module \(\mathfrak{X}(M)\) together with the Lie-bracket is called a \(\mathcal{F}\)-Lie algebra of vector fields on \(M\).

**Definition 2.5.6.** Let \(\omega\) be a \(k\)-form and \(Z\) a vector field on a finite dimensional locally Euclidean space \(M\). The interior product (inner product or contraction) of \(Z\) and \(\omega\) is a \((k-1)\)-form denoted by \(\iota_Z \omega := Z \lrcorner \omega\), whose evaluation at every \(Z_1, Z_2, \ldots, Z_{k-1} \in \mathfrak{X}(M)\) is given by \(<Z \lrcorner \omega; Z_1, Z_2, \ldots, Z_{k-1} > = <\omega; Z, Z_1, Z_2, \ldots, Z_{k-1} >\).

The inner product \(Z \lrcorner \omega\) satisfies the following properties:

1. \(Z \lrcorner f = 0\), for any \(0\)-form \(f\).

2. \(<Z; \omega > = \iota_Z \omega = Z \lrcorner \omega\) is a \(0\)-form, for any \(1\)-form \(\omega\).

3. \(<\omega; Z_1, Z_2, \ldots, Z_{k-1}, Z_k > = \sum_{\sigma \in S_k} (-1)^{\sigma(k)} \omega \lrcorner Z_{\sigma(1)} \lrcorner \cdots \lrcorner Z_{\sigma(k-1)} \lrcorner Z_{\sigma(k)}\) for all \(Z_1, Z_2, \ldots, Z_{k-1}, Z_k\) vector fields and \(\omega\) any \(k\)-form on \(M\).

4. \(Z \lrcorner (Z \lrcorner \omega) = (Z \lrcorner Z \lrcorner \omega) = 0\) for any \(Z\) a vector field and \(\omega\) a \(k\)-form on \(M\).

5. \(Z \lrcorner (\alpha \wedge \beta) = (Z \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (Z \lrcorner \beta)\) for \(\alpha\) a \(k\)-form, \(\beta\) \(l\)-forms and \(Z\) a vector field on \(M\).

6. \(Z \lrcorner Y \lrcorner \omega = -Y \lrcorner Z \lrcorner \omega\) for \(Z, Y\) vector fields and \(\omega\) a \(k\)-form on \(M\).

**Proposition 2.5.2.** Let \(U\) be an open neighborhood in \(M\), \(Z, Y\) vector fields, \(\omega\) a \(k\)-form and \(f\) a \(0\)-form on \(M\). Then, the interior product satisfies the following properties.

1. There exists \(\iota: \mathfrak{X}(M) \times \Omega(M) \rightarrow \Omega(M)\) defined by \((Z, \omega) \mapsto \iota_Z \omega\), where, \(\iota_Z: \Omega^k \rightarrow \Omega^{k-1}\) is an operator that is, \(\mathbb{R}\)-linear map.

2. \(\iota_Z\) is a local operator, that is, \(\iota_Z \omega|_U = \iota_Z \omega|_U\).

3. \(\iota_{Z+Y} = \iota_Z + \iota_Y\) and \(\iota_{fZ} = f \iota_Z\).

4. \(\iota_Z^2 = \iota_Z \circ \iota_Z = 0\).
2.5 Exterior algebra on \( \mathbb{F} \)-spaces

5. In local coordinate, if \( \omega = \sum_{i_1 < \cdots < i_k} h_{i_1,\ldots,i_k}(x) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \), then

\[
\iota_Z \omega = \sum_{1 \leq i_1 < \cdots < i_k} Z^{i_1} h_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_l}} \wedge \ldots \wedge dx^{i_k}
\]
with \( Z = \sum_{i=1}^{\dim M} Z^i \frac{\partial}{\partial x^i} \).

6. \( \iota_Z (\varphi^* \omega) = \varphi^*(\iota_{\varphi_* Z} \omega) \), that is, \( \iota_Z (\varphi^*) = \varphi^*(\iota_{\varphi_* Z}) \). In particular, If \( \varphi_* Z = Z \), that is, \( Z \) is invariant then \( \iota_Z (\varphi^*) = \varphi^*(\iota_Z) \).

**Definition 2.5.7.** The Lie derivative of a \( k \)-form \( \omega \) with regard to a vector field \( Z \) is a \( \mathbb{F} \)-smooth \( \mathbb{R} \)-bilinear map \( \mathcal{L} : \mathfrak{X}(M) \times \Omega^k(M) \rightarrow \Omega^k(M) \) defined by the formula called the Cartan identity, that is, \( \mathcal{L}_Z \omega = \iota_Z(d\omega) + d(\iota_Z \omega) \), where, \( \iota_Z \), is the interior product and of \( d \), is the exterior derivative.

The following hold for the Lie derivative \( \mathcal{L} : \mathfrak{X}(M) \times \Omega^k(M) \rightarrow \Omega^k(M) \).

1. \( \mathcal{L}_Z = \iota_Z \circ d + d \circ \iota_Z \) is obviously an \( \mathbb{F} \)-smooth map, since \( \iota_Z \) and \( d \) are \( \mathbb{F} \)-smooth maps.
2. \( \mathcal{L}_Z f = Z(f) \) with \( \mathcal{L}_Z(c) = 0 \) for \( f \) a 1-form and \( f = c \) a constant.
3. \( \mathcal{L}_Z \), applying a \( k \)-form to a \( k \)-form, is a \( \mathbb{R} \)-linear map and a local operator.
4. \( \mathcal{L}_Z(Y) = [Z,Y] \).

**Proposition 2.5.3.** Let \( Z,Y \in \mathfrak{X}(M) \), \( f, g \) two 0-forms, \( \omega \) a \( k \)-form, and \( a \in \mathbb{R} \). The Lie derivative satisfies the following properties.

1. \( \mathcal{L}_{Z+Y} = \mathcal{L}_Z + \mathcal{L}_Y \).
2. \( \mathcal{L}_{af} = a \mathcal{L}_f \).
3. \( \mathcal{L}_{fZ} \neq f \mathcal{L}_Z \).
4. \( \mathcal{L}_Z (f \omega) = f \mathcal{L}_g + g \mathcal{L}_f \).

**Proof.**

1. The property holds with regard to the linearity of the interior product.
2. The same argument yields the property.
3. Since \( \mathcal{L}_Z \) is a derivation, it follows as a consequence of following computations. On the one hand side, \( \mathcal{L}_{fZ} \omega = df \wedge (\iota_Z \omega) + f(\mathcal{L}_Z \omega) \). On the other hand, \( \mathcal{L}_Z (f \omega) = \mathcal{L}_Z(f) \omega + f(\mathcal{L}_Z \omega) = (Z(f)) \omega + f(\mathcal{L}_Z \omega) \). Thus, the inequality holds.
Proposition 2.5.4. The Lie derivative has the following properties.

1. Let \( \alpha \) be a \( k \)-form, \( \beta \) a \( l \)-form and \( Z \in \mathfrak{X}(M) \). Then, the Lie derivative of \( \alpha \land \beta \) is given by
   \[ \mathcal{L}_Z (\alpha \land \beta) = (\mathcal{L}_Z \alpha) \land \beta + \alpha \land (\mathcal{L}_Z \beta). \]

2. Let \( \alpha \) be any exterior form and \( Z \) any vector field. Then the operators \( d \) and \( \mathcal{L}_Z \) commute, that is, \( \mathcal{L}_Z (d\alpha) = d(\mathcal{L}_Z \alpha) \).

3. \( (\mathcal{L}_Z) \)\( \varphi^* = \varphi^*(\mathcal{L}_Z) \). In particular, \( (\mathcal{L}_Z) \varphi^* = \varphi^*(\mathcal{L}_Z) \) if \( \varphi_* Z = Z \), that is, \( Z \) is invariant.

4. \( \mathcal{L}_{[Z,Y]} \alpha = [\mathcal{L}_Z, \mathcal{L}_Y] \alpha. \)

5. \( \mathcal{L}_Z (Y \cdot \alpha) = \mathcal{L}_Z Y \cdot \alpha + Y \cdot \mathcal{L}_Z \alpha. \)

6. \( \iota_{[Z,Y]} = \mathcal{L}_Z \circ \iota_Y - \iota_Y \circ \mathcal{L}_Z = [\mathcal{L}_Z, \iota_Y]. \)

7. \( \mathcal{L}_Z (\varphi^* \alpha) = \varphi^*(\iota_{\mathcal{L}_Z \varphi} \alpha). \)
Chapter 3

Symplectic reduction on locally Euclidean Frölicher spaces

In this chapter, we present an extension of the fundamentals of the "Marsden-Weinstein quotient" (also called reduced space) that is well known in the category of finite dimensional smooth manifolds. The process producing a reduced space is named the symplectic reduction (see [22, 77]). The modeling space for the extension of the symplectic reduction process is the finite dimensional locally Euclidean Frölicher Space. We say just "space" for "Locally Euclidean Frölicher Space". For a space of the category under consideration we define a free, proper and Hamiltonian action of a Frölicher Lie group provided with an equivariant moment map, as in the case of the category of smooth manifolds (see [8, 36, 66, 85, 112]). The result is a quotient that is still a finite dimensional space with a symplectic structure induced from the original one on the ambient space (see [8, 112]). And then, we raise some fundamental questions such as:

- Why the properness and the freeness of the action?
- Why is the compactness assumption for the operating Frölicher-Lie group so relevant?
- What happens when one drops one of the previous conditions comparatively to the category of smooth manifolds?

3.1 Symplectic locally Euclidean Frölicher spaces

3.1.1 Symplectic structure

The symplectic framework in the category FrI was introduced in [8], the F-bundles in [109] and the symplectic reduction process on locally Euclidean Frölicher spaces was investigated
in [112]. The main references on symplectic linear spaces and symplectic manifolds should be [1, 47, 70]. We introduce below the symplectic structure on linear locally Euclidean Frölicher spaces. Thereafter, we will present the general non-linear case.

**Definition 3.1.1.** Let $M$ be a finite dimensional linear $\mathbb{F}$-space and $\omega$ a $\mathbb{F}$-smooth 2-form on $M$. The form $\omega$ is called a symplectic form or a symplectic structure on $M$ if it is both skew-symmetric and non-degenerate. That is, $\omega(x, y) + \omega(y, x) = 0$ for all $x, y \in M$ and $\omega(x, y) = 0$ for all $y \in M$ implies that $x = 0$. The pair $(M, \omega)$ is called a symplectic linear $\mathbb{F}$-space.

For an exterior 2-form denoted by $\omega$, the induced linear map $\omega^\flat : M \to M^*$ is defined by $\omega^\flat(x) = \iota_x \omega = \omega(x, \cdot)$, for any $x \in M$. That is, for all $y \in M$,

$$\omega^\flat(x)(y) = (\iota_x \omega)(y) = \omega(x, y).$$

(3.1)

The map $\omega^\flat$ is an isomorphism if, and only if the 2-form $\omega$ is non-degenerate if, and only if $\text{rank}(\omega) = \dim M^* = n$.

**Remark 3.1.1.** The non-degeneracy of the symplectic form $\omega$ is equivalent to the following statements.

1. The linear map $\omega^\flat : M \to M^*$, as defined in Equation (3.1), is a smooth isomorphism of linear $\mathbb{F}$-spaces.

2. The transpose $\omega^t = -\omega$ is non-degenerate.

3. The dimension $\dim M = \dim \omega^\flat(M) = \text{rank}(\omega) = 2p$, that is maximal even integer, where $p$ is independent of the choice of a basis in $M$.

4. There is a basis $\{u_1, \ldots, u_p, v_1, \ldots, v_p\}$ in $M$, such that $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$ and $\omega(u_i, v_j) = \delta_{ij}$, where $i, j \in \{1, 2, \ldots, p\}$ and $\delta_{ij}$ is the Kronecker symbol. This basis is called the canonical or symplectic basis.

5. Let $(\omega_{ij})_{1 \leq i,j \leq p}$ be the matrix of $\omega$ in any basis and $(\omega^t_{ij})_{1 \leq i,j \leq p}$ its transpose. It follows that $\det(\omega_{ij})_{1 \leq i,j \leq p} \neq 0$ and also $\det(\omega^t_{ij})_{1 \leq i,j \leq p} \neq 0$. Moreover $\text{rank}(\omega_{ij})_{1 \leq i,j \leq p} = 2p$. It follows that any symplectic linear space is even dimensional.

6. $\omega \wedge \omega \wedge \cdots \wedge \omega = \omega^n (n \text{ copies of } \omega)$ is the volume form, that is, nowhere vanishing.
**Definition 3.1.2.** Let $(M, \omega)$ be a symplectic linear $\mathbb{F}$-space of dimension $n$. Let $W$ and $W'$ be two linear subspaces of $M$. Two vectors $x$ and $y$ in $M$ are called orthogonal with regard to $\omega$ (or $\omega$-orthogonal) if $\omega(x, y) = 0$. The linear subspaces $W$ and $W'$ are called $\omega$-orthogonal if every $x \in W$ is $\omega$-orthogonal to every $y \in W'$. The set $\{ x \in M \mid \omega(x, y) = 0 \text{ for every } y \in W \} := \text{orth}_\omega W := W^\perp := N^\omega$ is called the $\omega$-orthogonal of $W$ and it is the maximal element in the set of all linear subspaces of $M$ which are $\omega$-orthogonal to $W$.

Equation (3.1) and the first item in Remark 3.1.1 allow the following definition.

**Definition 3.1.3.** Let $(M, \omega)$ be a symplectic linear $\mathbb{F}$-space of dimension $2n$ and $N$ its linear $\mathbb{F}$-subspace of dimension $s$. Let $\omega_N$ and $\omega^b$ be the restrictions of $\omega$ and $\omega^b$ to $N$, respectively. The kernel of $\omega_N$ is given by $\ker \omega_N = \ker \omega^b_N = \{ x \in N \mid \omega^b(x) = \iota_x \omega = 0 \} = N \cap N^\perp$.

The kernel of $\omega_N$ is not necessarily equal to $\{0\}$. It raises the need of a characterization among linear $\mathbb{F}$-subspaces of $M$ with regard to $\omega_N$ as in the forthcoming Definition.

**Definition 3.1.4.** Let $(M, \omega)$ be a symplectic linear $\mathbb{F}$-space of dimension $2n$ and $N$ its linear $\mathbb{F}$-subspace of dimension $s$. Let $\text{orth}_\omega W$ as defined in Definition 3.1.2.

The linear $\mathbb{F}$-subspace $N$ is called symplectic if $\omega_N$ is a symplectic structure on $N$ defined by $\omega_N := \iota_N^* \omega$, where $\iota_N$ is the canonical inclusion of $N$ into $M$. That is, if $N \cap \text{orth}_\omega N = \{0\}$. The linear $\mathbb{F}$-subspace $N$ is called isotropic if $\omega_N = 0$. That is, if $N \subset \text{orth}_\omega N$. The linear $\mathbb{F}$-subspace $N$ is called co-isotropic if $\omega_{\text{orth}_\omega N} = 0$. That is, if $\text{orth}_\omega N \subset N$. The linear $\mathbb{F}$-subspace $N$ is called Lagrangian if $N$ is both isotropic and co-isotropic. That is, $N = \text{orth}_\omega N$.

**Proposition 3.1.1.** If $M$ is a linear $\mathbb{F}$-space of dimension $m$ and $\omega$ any 2-form (skew symmetric) on $M$ with $N = \ker \omega$, then there exists a symplectic form $\overline{\omega}$ on the linear $\mathbb{F}$-quotient space $M/N$, such that the form $\omega$ is the pullback $\pi^* \overline{\omega}$ of $\overline{\omega}$, where $\pi$ is the canonical projection of $M$ onto the quotient.

**Proposition 3.1.2.** Let $(M, \omega)$ be a symplectic linear $\mathbb{F}$-space of dimension $2n$ and $N$ its linear $\mathbb{F}$-subspace. The formula $\omega_N := \iota_N^* \omega = \pi_N^* \overline{\omega}_N$ defines a symplectic form, on the quotient linear $\mathbb{F}$-space $\overline{N} = N/(N \cap \text{orth}_\omega N)$, induced by $\omega$, where $\pi$ is the canonical projection of $M$ onto the quotient and $\iota_N$ the canonical inclusion of $N$ into $M$.

**Definition 3.1.5.** The symplectic linear $\mathbb{F}$-space $(\overline{N}, \overline{\omega}_N)$ is called the reduced symplectic linear $\mathbb{F}$-space associated to $N$. 
Propositions 3.1.1 and 3.1.2, are used in analytical mechanics to reduce the number of degrees of freedom of a Hamiltonian system by means of first integrals.

**Proposition 3.1.3.** Let \((M, \omega)\) be a symplectic linear \(\mathbb{F}\)-space of dimension \(2n\) and \(N\) its linear \(\mathbb{F}\)-subspace of dimension \(s\). Let \(\text{orth}_\omega W\) as defined in Definition 3.1.2. If \(N\) is isotropic, that is, \(\omega_N \equiv 0\). Then, \(\omega\) induces a canonical symplectic form \(\varpi_N\) on \(N\omega/N\).

**Definition 3.1.6.** Let \(\varphi\) be a linear smooth map of symplectic linear \(\mathbb{F}\)-spaces, from \((M, \omega)\) to \((N, \eta)\). The map \(\varphi\) is called a symplectic \(\mathbb{F}\)-smooth map if it preserves the symplectic structures in the sense that \(\varphi^* \eta = \omega\). That is, for all \(u, v \in M\), \(\varphi^* \eta(u, v) = \omega(\varphi(u), \varphi(v)) = (\omega \circ \varphi)(u, v)\). A symplectic linear transformation of \((M, \omega)\) is called a linear symplectomorphism.

The set of all symplectomorphisms on the symplectic linear \(\mathbb{F}\)-spaces \((M, \omega)\) is denoted by \(\text{Symp}(M)\). From some results in [8, 85, 112], we note that the set \(\text{Symp}(M)\) is a group for the composition of maps. Moreover, it is even a \(\mathbb{F}\)-Lie group.

Two symplectic linear \(\mathbb{F}\)-spaces \((M, \omega)\) and \((N, \eta)\) of the same dimension are symplectically isomorphic, that is, there exists a \(\mathbb{F}\)-smooth isomorphism \(\varphi : M \rightarrow N\) such that \(\varphi^* \eta = \omega\). All \(2n\)-dimensional symplectic linear \(\mathbb{F}\)-spaces \((M, \omega)\) are symplectically isomorphic to \((\mathbb{R}^{2n}, \omega_0)\), where \(\omega_0\) is the canonical symplectic form defined by \(\omega_0(x, y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)\) for \(x = (x_1, \ldots, x_{2n}), y = (y_1, \ldots, y_{2n}) \in \mathbb{R}^{2n}\). Let us choose the canonical basis \((u_1, \ldots, u_n; v_1, \ldots, v_n)\) on \((M, \omega)\) such that \(\omega(u_i, u_j) = \omega(v_i, v_j) = 0\) and \(\omega(u_i, v_j) = \delta_{ij}\), where \(i, j \in \{1, 2, \ldots, n\}\) and \(\delta_{ij}\) is the Kronecker symbol. It follows that there exists a symplectic \(\mathbb{F}\)-smooth isomorphism \(\varphi : \mathbb{R}^{2n} \rightarrow M\) such that \(\varphi^* \omega = \omega_0\), that is, \(\omega \circ \varphi = \omega_0\). In the sequel, all symplectic linear \(\mathbb{F}\)-spaces of the same dimension are symplectically isomorphic. That is, they all look alike.

Note that the non-degeneracy and the skew-symmetry of the symplectic form were both purely algebraic conditions on the linear spaces. Now, we restate these conditions in the general setting of locally Euclidean Frölicher spaces. The non-degeneracy will give rise to a symplectic form on each linear space \(T_x M\) where \(x \in M\). While the closedness of a symplectic form on a Frölicher space is a geometric condition, as it is related to the smooth structure on the locally Euclidean Frölicher space. From now on, Frölicher space means locally Euclidean Frölicher space of constant dimension.

**Definition 3.1.7.** Let \(M\) be Frölicher space. A 2-form \(\omega\) which is both closed and non-degenerate is called a symplectic form on \(M\). A symplectic Frölicher space is a pair \((M, \omega)\) where \(M\) is a Frölicher space and \(\omega\) a symplectic form on \(M\).
It follows from Definition 3.1.7 that a symplectic form $\omega$, also called symplectic structure, is such that

1. $d\omega = 0$, and

2. $\omega(X, Y) = 0$ for all $Y$ implies $X = 0$, where $X, Y \in \mathfrak{X}(M)$.

**Remark 3.1.2.** [8, 47, 70, 112] The non-degeneracy of the 2-form $\omega$ does equivalently say:

1. For all $x \in M$ and $Y_x \in T_x M$, if $\omega_x(X_x, Y_x) = 0$, then $X_x = 0$, where, $\omega_x := \omega|_{T_x M}$ is a skew-symmetric smooth bilinear form associated with the exterior form $\omega$ at the point $x$. That is, for all $x \in M$, the pair $(T_x M, \omega_x)$ is a symplectic linear $\mathbb{F}$-space, and its dimension is even.

2. The $\mathbb{F}$-smooth map $\omega^\flat : TM \to T^* M$ is a smooth isomorphism of vector bundles, that is, $\omega^\flat_x : T_x M \to T^*_x M$ is a smooth isomorphism of linear spaces such that $\omega^\flat_x(v) = \iota_v \omega_x$ for every $x \in M$ and every $v \in T_x M$. That is, $\omega^\flat_x(v)$ is the unique (1-form) element of $T^*_x M$ such that for every $u \in T_x M$ one has $<\omega^\flat_x(v), u> = \omega_x(v, u)$.

3. The $\mathbb{F}$-smooth map $\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is an isomorphism of $\mathcal{F}_M$-modules. In the latter case, using analogy in notations with the linear spaces setting, we denote by $\omega^\sharp$ the inverse of $\omega^\flat$. Hence, $\omega^\flat(X) = \omega(X, \cdot) = \iota_X \omega = \alpha \in \Omega^1(M)$ if, and only if $\omega^\sharp(\alpha) = X = X_{\alpha} \in \mathfrak{X}(M)$ if, and only if $\iota_{\omega^\sharp(\alpha)} \omega = \alpha$. That is, the vector field $X \in \mathfrak{X}(M)$ and the 1-form (Pfaffian form) $\alpha \in \Omega^1(M)$ are related in a bijective correspondence.

**Definition 3.1.8.** Let $(M, \omega)$ and $(N, \sigma)$ be two finite dimensional symplectic Frölicher spaces. A $\mathbb{F}$-map $\varphi : M \to N$ is called symplectic if $\varphi^* \sigma = \omega$. Moreover, if $\varphi$ is a symplectic $\mathbb{F}$-diffeomorphism, it is called a symplectomorphism.

**Proposition 3.1.4.** [8, 109, 112] Let $M$ be a finite dimensional space and $(N, \omega)$ be a finite dimensional symplectic space. Let $\varphi : M \to N$ be a $\mathbb{F}$-map. If $\varphi$ is a $\mathbb{F}$-diffeomorphism, then $\varphi^* \omega$ is a symplectic form on $M$.

**Lemma 3.1.5.** [8, 112] Let $(M, \omega)$ be a symplectic space of dimension $2n$ and $N$ a subspace of maximal constant dimension, that is, $\dim N = \dim M = 2n$. Then

1. There exists on $N$ a symplectic structure induced by $\omega$ such that $\iota_N^* \omega = \omega_N$, where $\iota_N$ is the canonical inclusion of $N$ into $M$. That is, $\iota_N$ is a symplectomorphism.
2. For every $x \in M$ there exists an open neighborhood $U$ of $x \in M$ and $2n$ smooth functions $q^1, \ldots, q^n; p_1, \ldots, p_n \in \mathcal{G}_x$, the germ of the $\mathbb{F}$-smooth functions at $x \in U$ such that $\omega|_U = \sum_{i=1}^{n} dq^i \wedge dp_i$. This is the Darboux’s theorem in $\mathbb{F}$-spaces setting.

3. Every local basis of smooth vector fields $\{W_1, \ldots, W_{2n}\} \subset \mathfrak{X}(M)$ induces a local basis of smooth vector fields $\{V_1, \ldots, V_{2n}\} \subset \mathfrak{X}(N)$.

4. Let $x \in M$ and $\varphi = (x^1, x^2, \ldots, x^{2n})$ a coordinate system of $M$ at $x$ with domain $U$. Then,

- every local basis over $U$ induces a basis on the open subspace $\varphi(U) \subseteq \mathbb{R}^{2n}$;
- moreover, the symplectic structure on $U$ induces a symplectic structure on $\varphi(U)$ with regard to the chart $(U, \varphi)$.

**Definition 3.1.9.** Let $(M, \omega)$ be a symplectic locally Euclidean Frölicher space. Let $\varphi: N \rightarrow M$ be a smooth map from a locally Euclidean Frölicher space $N$ into the symplectic locally Euclidean Frölicher space $M$. Let $x \in M$. Assume that the map $\varphi$ is an immersion at $x$, that is, the tangent map $T_x \varphi: T_x N \rightarrow T_{\varphi(x)} M$ is injective. Let $T_{\varphi(x)} \varphi(T_x N)$ be the linear subspace of the symplectic linear space $(T_{\varphi(x)} M, \omega_{\varphi(x)})$. The map $\varphi$ is isotropic, co-isotropic, Lagrangian or symplectic immersion at $x$, if $T_{\varphi(x)} \varphi(T_x N)$ is respectively isotropic, co-isotropic, Lagrangian or symplectic in $(T_{\varphi(x)} M, \omega_{\varphi(x)})$. Let $\varphi$ be the canonical inclusion, then $N$ is isotopic, co-isotropic, Lagrangian or symplectic at $x$, if $T_x N$ is respectively isotropic, co-isotropic, Lagrangian or symplectic in $(T_x M, \omega_x)$. The map $\varphi$ is isotropic, co-isotropic, Lagrangian or symplectic immersion on $N$, if $\varphi$ is isotropic, co-isotropic, Lagrangian or symplectic at every point $x \in N$. In particular, let $N \subset M$, $N$ is isotropic, co-isotropic, Lagrangian or symplectic locally Euclidean Frölicher subspace of $(M, \omega)$, if $N$ possesses the property at every point $x \in N$.

**Lemma 3.1.6.** Let $N$ be a locally Euclidean Frölicher subspace of dimension $n$ in the symplectic locally Euclidean Frölicher space $(M, \omega)$ of dimension $2m$. Let $\iota_N: N \rightarrow M$ be its canonical inclusion. Then,

1. The 2-form $\omega_N = \iota_N^* \omega$, induced by $\omega$ on $N$, has its kernel at a point $x$ of $N$ defined by $\text{Ker}_x \omega_N = T_x N \cap \text{orth}(T_x N)$, where $\text{orth}(T_x N)$ is the orthogonal of $T_x N$ in the symplectic linear space $(T_x M, \omega_x)$.

2. The rank of $\omega_N$ at the point $x \in N$ is an even integer $2p(x)$, equal to the co-dimension of $\text{Ker}_x \omega_N$ such that it satisfies the inequalities $\sup(0, 2(n - m)) \leq 2p(x) \leq n$. 
These inequalities come from the fact that \( \dim Ker_x\omega_N \) is positive and bounded by \( \dim T_xN \) and \( \dim orth(T_xN) \). The rank of \( \omega_N \) reaches its least possible value in the inequalities (that is, \( sup(0,2(n-m)) \)) if, and only if the locally Euclidean Fröhlicher subspace \( N \) is either co-isotropic (that is, \( n \geq m \)), isotropic (that is, \( n \leq m \)) or Lagrangian (that is, \( n = m \)) at \( x \in N \). The rank of \( \omega_N \) reaches its greatest possible value in the inequalities (that is, \( n \)) if, and only if the locally Euclidean Fröhlicher subspace \( N \) is even-dimensional and symplectic at \( x \in N \). Similar consequences can be drawn in the case of an immersion of a locally Euclidean Fröhlicher subspace \( N \) into a symplectic locally Euclidean Fröhlicher space \((M,\omega)\).

### 3.1.2 Basic concepts of group actions

The notion of a Fröhlicher-Lie-group (see [63, 85]) was introduced in [43] as smooth group by Fröhlicher and Kriegl.

**Definition 3.1.10.** Let \( G \) be a \( \mathbb{F} \)-space and a group with identity (unit) element \( e \) and let \( H \subset G \). The Triple \((G,C_G,F_G)\) is called a Fröhlicher-Lie group or \( \mathbb{F} \)-Lie group for short if the multiplication map \( \sigma : G \times G \rightarrow G \) given by \( \sigma(g,h) = gh \) is \( \mathbb{F} \)-smooth and, the map \( \theta : G \rightarrow G \) given by \( \theta(g) = g^{-1} \) is \( \mathbb{F} \)-smooth. Equivalently, the map \( \varsigma : G \times G \rightarrow G \) given by \( \varsigma(g,h) = gh^{-1} \) is \( \mathbb{F} \)-smooth. The subset \( H \) is called a \( \mathbb{F} \)-Lie subgroup of the group \( G \) if \( H \) is a subgroup of \( G \) which is a Fröhlicher subspace.

**Definition 3.1.11.** Let \( G, H \) be two \( \mathbb{F} \)-Lie groups, that is, \( G \) and \( H \) are finite dimensional Fröhlicher spaces and groups also. The map \( \varphi : G \rightarrow H \) is a \( \mathbb{F} \)-Lie group map if it is a smooth map of Fröhlicher spaces on the one hand and a homomorphism of groups on the other hand.

**Definition 3.1.12.** Let \( M \) be a \( n \)-\( \mathbb{F} \)-space and \( G \) a \( \mathbb{F} \)-Lie group. Assume that for each \( g \in G, x \in M \) the maps defined by \( \sigma : G \times M \rightarrow M, (g,x) \mapsto \sigma(g,x) := g.x \) and \( \delta : M \times G \rightarrow M, (x,g) \mapsto \delta(x,g) := x.g \) are smooth maps of \( \mathbb{F} \)-spaces such that the induced maps \( \sigma_g : M \rightarrow M, x \mapsto \sigma_g(x) := \sigma(g,x) \) and \( \delta_g : M \rightarrow M, x \mapsto \delta_g(x) := \delta(g,x) \) are diffeomorphisms of the \( \mathbb{F} \)-space \( M \). The map \( \sigma \) is called a left action of \( G \) on \( M \) if \( (\sigma_g \circ \sigma_h)(x) = (\sigma_{gh})(x) \) and \( \sigma_e = id_M \), for all \( g,h \in G, x \in M \). The map \( \delta \) is called a right action of \( G \) on \( M \) if \( (\delta_h \circ \delta_g)(x) = (\delta_{gh})(x) \) and \( \delta_e = id_M \), for all \( g,h \in G, x \in M \).

The equation \( \sigma_{gh}(x) = (\sigma_g \circ \sigma_h)(x) \) reads \( \sigma(gh,x) = \sigma_g((\sigma_h)(x)) = \sigma(g,\sigma(h,x)) \), that is, \( (gh).x = g.(h.x) \), for all \( g,h \in G, x \in M \). Likewise, \( (\delta_{gh})(x) = (\delta_h \circ \delta_g)(x) \) reads \( \delta(x,gh) = \delta_h((\delta_g)(x)) = \delta(h,\delta(g,x)) \), that is, \( x.(gh) = (x.g).h \), for all \( g,h \in G, x \in M \). It follows that
\[ \sigma_{gh} = \sigma_g \circ \sigma_h, \ \delta_{gh} = \delta_h \circ \delta_g \] and \( \sigma_e = \delta_e = \text{id}_M \). In what follows, we will say actions for means of left group actions. Whenever the right actions will be concerned the distinction will be stated. In what follows, \( \text{Diff}(M) \) denotes the group of diffeomorphisms of a Frölicher space \( M \) under the composition of maps. We recall that \( \text{Diff}(M) \) is a Frölicher-Lie group (see [43]). From now on, any reference to the word group alone will mean \( \mathbb{F} \)-Lie group.

**Lemma 3.1.7.** Let \( M \) and \( G \) be locally Euclidean Frölicher spaces, where \( G \) is a \( \mathbb{F} \)-Lie group. Let \( \sigma : G \times M \rightarrow M \) be a left action of \( G \) on \( M \).

1. The set \( GM := \{ \sigma_g \mid g \in G \} \) is a \( \mathbb{F} \)-Lie group of transformations of \( G \) on \( M \) and \( GM \subset \text{Diff}(M) \).

2. The map \( \rho : G \rightarrow \text{Diff}(M), g \mapsto \rho g := \sigma_g \) is a \( \mathbb{F} \)-smooth map, an injective homomorphism of abstract groups and \( \rho(G) = GM \).

**Definition 3.1.13.** Let \( M \) be a locally Euclidean Frölicher space. Let \( G \) be a \( \mathbb{F} \)-Lie group acting on \( M \) on the left and \( \sigma : G \times M \rightarrow M \) the action map. An element \( m_0 \in M \) is a fixed point (element) for \( \sigma \) if \( \sigma_g(m_0) = m_0 \) for each \( g \in G \), where \( \sigma_g \) is a transformation on \( M \).

The orbit map \( \sigma_x : G \rightarrow M, g \mapsto \sigma_x(g) := \sigma(g, x) = \sigma_g(x) \) is a \( \mathbb{F} \)-smooth map with respect to Lemma 2.2.2, since \( \sigma_{gh}(x) = (\sigma_g \circ \sigma_h)(x) = \sigma_h(\sigma_g(x)) = \sigma_g(\sigma_h(h)) = (\sigma_g \circ \sigma_x)(h) \).

**Definition 3.1.14.** Let \( M \) be a \( n \)-\( \mathbb{F} \)-space and \( G \) a \( \mathbb{F} \)-Lie group. Let \( x \in M \) be a fixed element and \( \sigma : G \times M \rightarrow M \) a left action of \( G \) on \( M \). The image of the orbit map \( \sigma_x : G \rightarrow M, g \mapsto \sigma_x(g) := \sigma(g, x) \), denoted by \( G.x := \sigma_x(G) \subset M \), is called the orbit (of) through \( x \) for the action \( \sigma \). That is, \( G.x = \{ \sigma_x(g) = g.x \mid g \in G \} \). The subset of \( G \) given by \( G_x := \{ g \in G \mid g.x = x \} \) is called the stabilizer (or the isotropy group) of \( x \in M \). At last the graph of \( \sigma_g \) will be denoted by \( G(\sigma_g) \).

**Definition 3.1.15.** Let \( M \) be a \( n \)-\( \mathbb{F} \)-space and \( \sigma : \mathbb{R} \times M \rightarrow M \) a \( \mathbb{F} \)-smooth map. The map \( \sigma \) is a one-parameter group of transformations of \( M \) if it has the following properties:

1. For each \( t \in \mathbb{R} \), \( \sigma_t : M \rightarrow M \), is a \( \mathbb{F} \)-diffeomorphism (a transformation) of \( M \) such that \( \sigma_t(x) = \sigma(t, x) \) and \( \sigma_0(x) = x \) or \( \sigma_0 = \text{id}_M \) for all \( x \in M \).

2. For each \( x \in M \), \( \sigma_x : \mathbb{R} \rightarrow M \) is a \( \mathbb{F} \)-smooth curve on \( M \) going through \( x \) such that \( \sigma_x(t) = \sigma(t, x) \), for all \( t \in \mathbb{R} \) and \( \sigma_x(0) = \sigma(0, x) = x \).

3. For all \( t, s \in \mathbb{R} \), \( \sigma_{t+s} = \sigma_t \circ \sigma_s \).
Lemma 3.1.8. Let $M$ be a $n$-$\mathbb{F}$-space and $\sigma : \mathbb{R} \times M \rightarrow M$ a one-parameter group of transformations of $M$. There exists $X \in \mathfrak{X}(M)$ such that $X = (x, X_x)_{x \in M}$ and $X_x(f) = \frac{d}{dt}(f \circ \sigma_x)(t)|_{t=0}$ for some $f \in \mathcal{F}_M$.

Definition 3.1.16. Let $G$ be a $\mathbb{F}$-Lie group. A one-parameter subgroup of $G$ is a smooth curve $\gamma : \mathbb{R} \rightarrow G$, $g \mapsto \gamma(t)$ satisfying the following conditions: $\gamma(t+s) = \gamma(t)\gamma(s)$, for all $t, s \in \mathbb{R}$ and $\gamma(0) = e$, where $e$ is the unit element of $G$.

The following transformations of $G$ onto itself play a central role in the theory of $\mathbb{F}$-Lie groups.

Definition 3.1.17. Let $G$ be a $\mathbb{F}$-Lie group and $g \in G$, a fixed element. Let $h \in G$, be any element. The transformation $L_g : G \rightarrow G$, defined by $L_g(h) := gh$ is called the left translation, that is, a left multiplication by $g$. The transformation $R_g : G \rightarrow G$, defined by $R_g(h) := hg$ is called the right translation, that is, a right multiplication by $g$. The transformation $R_g^{-1} : G \rightarrow G$, defined by $R_g^{-1}(h) := R_{g^{-1}}(h) = hg^{-1}$ is called the inverse right translation, that is, a right multiplication by $g^{-1}$. The transformation $L_gR^{-1}_g : G \rightarrow G$, called the inner automorphism or the conjugation, is defined by $L_gR^{-1}_g(h) := ghg^{-1}$, that is the composition map $(L_g \circ R^{-1}_g)(h) = L_g(R^{-1}_g(h)) = L_g(hg^{-1}) = L_g(gh^{-1}) = ghg^{-1}$.

3.2 Flows, Integral curve and Exponential map

In this Section we will derive concepts in the Frölicher setting by mimicking those from smooth manifolds (see [20, 21, 22, 51]).

Definition 3.2.1. Let $M$ be a $n$-$\mathbb{F}$-space. Let $X \in \mathfrak{X}(M)$ and $c : \mathbb{R} \rightarrow M$ a smooth curve. The curve $c$ is the integral curve of the vector field $X$ if $c_r(\frac{d}{dt}|_{t=r}) = dc(\frac{d}{dt}|_{t=r}) = X(c(r)) := X_{c(r)} = X_x$, where $t \in \mathbb{R}$, $c(r) = x \in M$ and $(\frac{d}{dt}|_{t=r}) = X(c(r))$. That is, the following diagram is commutative

\[ M \xrightarrow{\pi} X \xrightarrow{TM} \mathbb{R} \]

\[ c \]

\[ X \circ c = \frac{dc}{dt} := c' \]

Definition 3.2.2. Let $G$ be a $\mathbb{F}$-Lie group and $X \in \mathfrak{X}(G)$ a vector field on $G$. The vector field $X$ is called left invariant if $dL_g(X(h)) = X(L_g)(h) = X(gh)$, for all $g, h \in G$. 

Proposition 3.2.1. Let $G$ be a $\mathbb{F}$-Lie group and $e \in G$ its unit element. Let $T_eG$ be the set of tangent vectors to $G$ at $e$. Let $\mathcal{G}$ be the set of left invariant vector fields on $G$. Then, we have,

1. $X \in \mathcal{G}$ if and only if $dL_g(X(e)) = X(g)$.

2. $\mathcal{G}$ is a real linear space.

3. The map $\alpha : \mathcal{G} \to T_eG$, defined by $\alpha(X) = X(e)$, is a linear $\mathbb{F}$-diffeomorphism of $n$-spaces. Thus, $T_eG \cong \mathcal{G}$ and $\dim \mathcal{G} = \dim T_eG = \dim G$.

Definition 3.2.3. Let $G$ be a $n$-$\mathbb{F}$-Lie group. The $\mathbb{F}$-Lie algebra $\mathcal{G}$ of left invariant vector fields on $G$ is called the $n$-$\mathbb{F}$-Lie algebra of the $n$-$\mathbb{F}$-Lie group, such that every $X \in \mathcal{G}$ is characterized by $X = X_\xi$ with $\xi = X(e) \in T_eG$. That is, the vector fields $X_\xi \in \mathcal{G}$ are vector fields on $G$ and they are invariant under left translation by any element of $G$. The Lie bracket operation on vector fields of $\mathcal{G}$, defined by $[\xi, \eta] = [X_\xi, X_\eta]$ such that $[X_\xi, X_\eta](e) = X_{[\xi, \eta]}$, makes the map $\alpha : \mathcal{G} \to T_eG$ (see 3.2.1 Item 3.) into an isomorphism of $\mathbb{F}$-Lie algebras.

Definition 3.2.4. Let $G$ be a $n$-$\mathbb{F}$-Lie group, $\mathcal{G}$ the $\mathbb{F}$-Lie algebra of $G$ and $X \in \mathcal{G}$. Let the commutative diagram

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\gamma} & G \\
\downarrow t & & \downarrow X \\
\mathbb{R}^2 & \xrightarrow{\gamma_* = \gamma_* \circ t} & \mathcal{G} \\
\end{array}
\]

where $\gamma(t) = (t, 0)$, $\gamma(t + s) = \gamma(t)\gamma(s)$ and $\gamma_* (s) = X(\gamma(s))$ for all $t, s \in \mathbb{R}$, with $\gamma$ a smooth curve on $M$ and $\gamma_*$ its tangent. The curve $\gamma(\mathbb{R})$ is called the one-parameter subgroup of $G$ corresponding to $X$ or generated by $X$. This curve is the integral curve of $X$ which passes through $e$.

Definition 3.2.5. Let $G$ be a $\mathbb{F}$-Lie group, $\mathcal{G}$ the $\mathbb{F}$-Lie algebra of $G$ and $X \in \mathcal{G}$. Let $\gamma_X : \mathbb{R} \to G$ be the curve integral of $X$ starting at the identity, that is, $\frac{d}{dt}\gamma_X(t)|_{t=0} = X(e)$. The map $exp : \mathcal{G} \equiv T_eG \to G$ defined by $X \mapsto exp(X) = \gamma_X(1)$ is called the exponential map.

Remark 3.2.1.

1. The Lie-bracket $[\cdot, \cdot]$ on $\mathcal{G}$ is the first derivation of the Lie-group multiplication (see [20, p.21]).
2. As for right and left actions, there is a way to go from $G^{\text{opp}}$ to $G$. For details see [94, pp.148, 149]. We state and comment this reversible process.

- Let $X \in G$ or $X \in G^{\text{opp}}$ and $Y$ be the vector field defined by $Y : G \to TG$, with $g \mapsto Y(g) = d(\text{inv})(X(g^{-1}))$, where $\text{inv} : G \to G$; $g \mapsto g^{-1}$. Then, it follows that $dL_gX + dR_g^{-1}Y = 0$ and $dL_gY + dR_g^{-1}X = 0$, where $dL_g : TG \to TG$ and $dL_g(h) : T_gG \to T_{gh}G$.

- The invariance of $X$ and $Y$ may be read as follows. If $X \in G$, then $dL_gX(e) + dR_g^{-1}Y(e) = 0$ and $dL_gY(e) + dR_g^{-1}X(e) = 0$. Therefore, $X(g) = -dR_g^{-1}$ and $Y(g^{-1}) = d(\text{inv})(X(g)) = -X(g)$. Hence, we have $Y(g^{-1}) = X(e) + dR_g^{-1}Y(e) = 0$ and $dL_gY(e) + dR_g^{-1}X(e) = 0$. Thus, $X \in G$ implies $Y \in G^{\text{opp}}$. In particular, $Y(g^{-1}) = -X(g)$ becomes $Y(e) = -X(e)$ for $g = e$. This yields $Y(e) + X(e) = 0$, where $X(e) = \xi$. Thus, $Y(e) = -\xi$. The same arguments yield $Y \in G$.

3. Since $\text{inv} : G \to G$ is a bijective map, then $d(\text{inv})$ is an isomorphism. The Lie-bracket $[,]$ on $G$ induces a related Lie-bracket on $G^{\text{opp}}$, defined by $[X^{\text{opp}}, Y^{\text{opp}}]^{\text{opp}} = -[X, Y]$ for all $X^{\text{opp}}, Y^{\text{opp}} \in G^{\text{opp}}$ and $X, Y \in G$ with $X^{\text{opp}}(g) = d(\text{inv})(X(g^{-1}))$ and $Y^{\text{opp}}(g) = d(\text{inv})(Y(g^{-1}))$. Actually, there is an anti-homomorphism of $\mathbb{F}$-Lie algebras between $G$ and $G^{\text{opp}}$. [94]

**Definition 3.2.6.** Let $G$ be a $\mathbb{F}$-Lie group and $G$, its $\mathbb{F}$-Lie algebra. Let $\xi \in G$. The flow of the left invariant vector field $X_\xi$ on $G$ is denoted by $\Phi : \mathbb{R} \times G \to G$, and defined by $(t, g) \mapsto \Phi_\xi(t, g) = g \exp(t\xi)$. The flow of the right invariant vector field $X_\xi$ on $G$ is denoted by $\Psi : G \times \mathbb{R} \to G$, and defined by $(g, t) \mapsto \Psi_\xi(g, t) = \exp(t\xi)g$.

**Definition 3.2.7.** Let $G$ be a $\mathbb{F}$-Lie group and $M$ a $m$-$\mathbb{F}$-space. Suppose $G$ acts smoothly on $M$ by the action map $\sigma : G \times M \to M$ such that $(g, m) \mapsto \sigma(g, m) = \sigma_g(m) = g.m$. Let $G = T_eG$ be the set of all left invariant vector fields on $G$. Let $\mathcal{A} : G \times M \to TM$ be the map defined by $(X, m) \mapsto \mathcal{A}(X, m) = \mathcal{A}(X)(m) = A_X(m) = X_m \in T_mM$, where $X_m := (\frac{d}{dt}\exp(tX)|_{t=0}).m$ with $\exp(tX) \in G$, the one-parameter group generated by $X$, and $\frac{d}{dt}\exp(tX)|_{t=0} \in G$. The map $\mathcal{A}$ is called the infinitesimal action of $G$ on $M$ associated to the action $\sigma$ of $G$ on $M$.

**Remark 3.2.2.** As for $\sigma : G \times M \to M$, we can define the infinitesimal analogous of $\sigma_g$ and $\sigma_m$.

1. The map $\mathcal{A}_X : M \to TM$, $m \mapsto X_m \in T_mM$, for all $m \in M$, is defined in such a way that $\mathcal{A}_X(M) = \{X_m \in T_mM \mid m \in M\} = (m, X_m)_{m \in M}$. It follows that every $X \in G$ determines a vector field on $M$ denoted by $X_M$. Therefore, $X_M :=$
Proposition 3.2.2. Let \( M \) be a space and \( \mathcal{G} \) the \( \mathbb{F} \)-Lie algebra of the \( \mathbb{F} \)-Lie group \( G \) acting on \( M \). The map \( \alpha : \mathcal{G} \to \mathfrak{X}(M) \) defined by \( \alpha(X) = X_M = \mathcal{A}_X \) is an anti-homomorphism of \( \mathbb{F} \)-Lie algebras.

**Proof.** [20, p.44, Proposition 1] Note that \( Y_\xi^\sigma \circ \sigma \) is a section for the surjective map \( \tau \circ \sigma_s^{-1} \). Thereafter, for any fixed \( g \in G \), we have \( Y_\xi^\sigma \circ \sigma_g = \sigma_g \circ Y_\xi^\sigma \). Therefore, \( \sigma_g(\xi) = Y_\xi^\sigma = \sigma_g \circ Y_\xi \circ \sigma_g^{-1} \). Below is given a commutative diagram for the infinitesimal part of the proof.

\[
\begin{array}{c}
M \xrightarrow{\Psi_\xi} TM \xrightarrow{\Psi_{\xi^*}} TM \\
\sigma_g \downarrow \quad \sigma_g \downarrow \quad \sigma_g^{-1} \quad \sigma_g \downarrow \\
M \xrightarrow{\sigma_g^*} Y_\xi \xrightarrow{\sigma_g^{-1}} \sigma_g^* \xrightarrow{Y_\xi^*} TM \\
\end{array}
\]
We compute $\sigma_{g*}([\xi,\eta])$ as follows:

\[ [Y_\xi^\sigma, Y_\eta^\sigma] = [\sigma_{g*} \circ Y_\xi \circ \sigma_g^{-1}, \sigma_{g*} \circ Y_\eta \circ \sigma_g^{-1}] \]

\[ = \sigma_{g*} \circ [Y_\xi, Y_\eta] \circ \sigma_g^{-1} \]

\[ = \sigma_{g*} \circ (-Y_{[\xi,\eta]}) \circ \sigma_g^{-1} \]

\[ = -Y_{[\xi,\eta]}^\sigma \]

\[ = -\sigma_{g*}([\xi,\eta]) \]

Hence, $\sigma_{g*}([\xi,\eta]) = -[Y_\xi^\sigma, Y_\eta^\sigma]$. Since $\sigma_{g*} = \alpha$, we have: $\sigma_{g*}(\xi,\eta) = Y_{\xi+\eta}^\sigma = Y_\xi^\sigma + Y_\eta^\sigma$ and $\sigma_{g*}(a\xi) = Y_{a\xi}^\sigma = aY_\xi^\sigma$. Therefore, we have proved that $\sigma_{g*} : G \rightarrow \mathfrak{X}(M)$, \( \xi \mapsto Y_\xi^\sigma = \xi_M \)

is actually an anti-homomorphism of $\mathbb{F}$-Lie algebras. 

\[ \square \]

\textbf{Remark 3.2.3.}

1. Note that $Y_\xi^\sigma(m) = \xi_M(m) = (m, \xi_M(m)) = \xi_m$. This yields the following composition of smooth maps $m \mapsto \sigma_g^{-1}(m) = g^{-1}.m \mapsto Y_\xi(\sigma_g^{-1}(m)) = \sigma_g^{-1}(Y_\xi^\sigma(m))$ which is a smooth map.

2. From [51, pp.167,177], and [20, p.53], we can define an action of $G$ on $C^\infty(M)$ by pullback. Let $\sigma : G \times M \rightarrow M$ be a group action of a $\mathbb{F}$-Lie group $G$ on a space $M$. The map $\varrho$ given by $g \mapsto \varrho(g) = \sigma_g^*$ ensures the rules below:

$gh \mapsto (\sigma_{gh})^* = (\sigma_g \circ \sigma_h)^* = \sigma_h^* \circ \sigma_g^* \circ (f + h) = (f + h) \circ \sigma_g = \sigma_g^*(f) + \sigma_g^*(h)$,

$\sigma_g^*(af) = (af) \circ a \sigma_g^*(f)$ and $\varrho(e) = \sigma_e^*$, with $e \in G$ the unit element. We can conclude that $\varrho$ is an injective anti-representation of groups. Finally, $\varrho$ is an anti-representation since the map $\sigma_g^* \in Aut(C^\infty(M))$ is linear a map. From the diagram below we can get interesting conclusions:

\[ \begin{array}{cccc}
G & \xrightarrow{\varrho} & Aut(C^\infty(M)) & \\
X \uparrow exp & & Y \downarrow exp & \\
G & \xrightarrow{\varrho_{se} \sigma_\xi^*} & T_{\sigma_\xi^*}(Aut(C^\infty(M))) & \\
& & \xi = X_\xi & \\
\end{array} \]

where $X$ and $Y$ are smooth vector fields, with $X_g = X(g) = dL_g(X(e)) = dL_g(\xi) \in T_gG$ for $X \in G$. The linearity of $\varrho_{se}$ yields the following: $\varrho_{se}(sX_\xi) = s(X_\xi)_m = s\varrho_{se}(X_\xi)$ and $\varrho_{se}(X_\xi + X_\eta) = \varrho_{se}(X_{\xi+\eta}) = (X_{\xi+\eta})_m = X_{\xi|m} + X_{\eta|m} = \varrho_{se}(X_\xi) + \varrho_{se}(X_\eta)$.

Now, we are concerned with the action of $\varrho_{se}$ on the Lie Bracket, that is, for every $[X_g, X_h] = [X_\xi, X_\eta] \in G \mapsto \varrho_{se}([X_\xi, X_\eta]) \in T_{\sigma_\xi^*}(Aut(C^\infty(M)))$, for all $g$, $h \in G$ and $X \in G$. We have,

\[ \begin{array}{cccc}
G & \xrightarrow{\varrho} & Aut(C^\infty(M)) & \\
X \uparrow exp & & Y \downarrow exp & \\
G & \xrightarrow{\varrho_{se} \sigma_\xi^*} & T_{\sigma_\xi^*}(Aut(C^\infty(M))) & \\
& & \xi = X_\xi & \\
\end{array} \]
\[ \varrho_{se}([-X_{\xi}, X_{\eta}]) = \varrho_{se}(-X_{[\xi, \eta]}) = -\varrho_{se}(X_{[\xi, \eta]}) = -(X_{[\xi, \eta]}) = -([X_{\xi}, X_{\eta}]) = -([\varrho_{se}(X_{\xi}), \varrho_{se}(X_{\eta})]. \]

Hence, we have proved that \( \varrho_{se} : \mathcal{G} \to \mathfrak{X}(M) \) is an anti-homomorphism of \( \mathbb{R} \)-Lie algebras.

3. The arguments used above lie on the diagrams below:

\[ \begin{array}{ccc}
\mathbb{R} & \xrightarrow{A_{\xi}} & \mathcal{G} \\
\gamma_{\xi} & \downarrow & \sigma_m \\
\exp_{\xi} & \downarrow & \exp_{\xi_m} \\
G & \xrightarrow{\sigma_m} & G.m \subset M \\
\end{array} \]

\[ \begin{array}{ccc}
\mathbb{R} & \xrightarrow{A_{\xi}} & \mathcal{G} \\
\gamma_{\xi} & \downarrow & \sigma_m \\
\exp_{\xi} & \downarrow & \exp_{\xi_m} \\
T_m M \subset TM & \xrightarrow{\sigma_m} & \exp_{\xi_m}(t\xi).m \\
\end{array} \]

\[ \gamma_{\xi}(t) = \exp_{\xi}(t\xi) \]

4. Note that if \( \mathcal{A} : \mathcal{G} \times M \to TM \) is the infinitesimal action of \( \mathcal{G} \) on \( M \), then for all \( X, Y \in \mathcal{G} \), \( m \in M \) and \( X_m, Y_m \in T_m M \) one has the following:

\[ \mathcal{A}(X + Y, m) = \mathcal{A}_{X+Y}(m) = (X + Y)_m = X_m + Y_m = \mathcal{A}_X(m) + \mathcal{A}_Y(m). \]

Thus, \( \mathcal{A}(X + Y, m) = (\mathcal{A}_X + \mathcal{A}_Y)(m) \). Let \( \theta \in \mathcal{G} \) be defined by \( \theta : G \to TG \) such that \( \theta(g) = (g, \theta_g) = (g, 0_g) \), that is, \( \theta \) is the nil vector field, with \( 0_g \) the zero vector of \( T_0 G \).

It follows that \( \theta(G) \) is the zero section. Therefore, \( \mathcal{A}(\theta, m) = \mathcal{A}_\theta(m) = \theta_m = (m, 0) \in T_m M \). Hence, \( \mathcal{A}_\theta(m) = (e, 0_e)_m = e \), since \( \theta \) is determined by its value at \( e \).

5. If we set \( M = G \), then the infinitesimal action \( \mathcal{A} : \mathcal{G} \times G \to TG \) is determined by \( (X, g) \mapsto (g, X_g) = X(g) = \mathcal{A}_X(g) \). It follows the commutativity of the diagram below:
Now, let $c \in C_G$. Then $c(t) = X \in G$ with $X(g) = dL_g X(e)$. From Proposition 3.2.2, the isomorphism $\alpha : G \rightarrow T_e G \subset TG$ is clearly equal to $A_e$. This yields the following diagram:

As from the characterization of smooth maps, $\alpha$ is a smooth map if, and only if $\gamma = \alpha \circ c$ is a smooth curve. Also, $\gamma$ is smooth if, and only if $df \circ \gamma, f \circ \pi \circ \gamma \in C^\infty(\mathbb{R})$.

6. The map $A : \mathbb{R} \times G \rightarrow G$ defined by $A(t, X) = tX$ is an action of $\mathbb{R}$ on $G$.

7. From the infinitesimal action of $X \in G$ on $M$ with value in $\mathfrak{X}(M)$ yields the following $(\sigma_m \circ \exp_G \circ A)(X) = (\exp_{T_m M} \circ \sigma_{ms} \circ A)(X)$ with regard to Part (3) and the action above, that is, $\sigma_m(\exp_G(tX)) = \exp_{T_m M}(\sigma_{ms}(tX))$. Hence, $\exp_{T_m M}(tX_m) = \exp_G(tX).m$ at $m \in M$. Therefore, for all $m \in M$ we have $(\exp_{T_m M}(t(X_M(m))))_{m \in M} = (\exp_G(tX).m)_{m \in M}$. Finally, it follows that we can globally define $\exp_{TM}(tX_M) := \exp_G(tX).M$.

8. The rule transforming left invariant to right invariant vector fields is the following: a left action $\sigma$ relates a right invariant vector field $Y_\xi \in G^{opp}$ to a vector field $\sigma_*(\xi) \in \mathfrak{X}(M)$ with $\xi \in G$.

3.3 $G$-equivariance, Adjoint and Co-adjoint representations

The references on smooth manifolds used in this Section are [20, 22].
Definition 3.3.1. Let \((M, \omega)\) be a symplectic space, \(G\) a \(\mathbb{F}\)-Lie group acting on \(M\) on the left by an action \(\sigma\). Let \(\text{Sympl}(M)\) be the group of symplectomorphisms on \(M\). The symplectic form \(\omega\) is invariant under the action \(\sigma\) of \(G\) on \(M\) if \(G\) acts by symplectomorphisms. That is, the map \(\rho : G \rightarrow \text{Sympl}(M)\) such that for each \(g \in G\), \(\rho(g) := \sigma_g : M \rightarrow M\) is a symplectomorphism on \(M\). In other words, \(\sigma_g^*\omega = \omega\). Such an action \(\sigma\) of \(G\) on \(M\) is called a symplectic action.

Remark 3.3.1. As from the definition of \(\sigma_g^*\), for all \(X, Y \in \mathfrak{X}(M)\) we have the following defining equalities \(\sigma_g^*\omega(X, Y) = \omega(d\sigma_g(X), d\sigma_g(Y)) = \omega(X \circ \sigma_g, Y \circ \sigma_g) = \omega(X, Y)\), since \(X(\sigma_g(M)) = X(M)\). Note that \(d\sigma_g = \sigma_g^*\) is the tangent map associated to \(\sigma_g\). Also, the infinitesimal action of \(G = T_eG\) on \(M\) is \(A : G \times M \rightarrow TM\), \((\xi, m) \mapsto \xi_m = A(\xi)(m) = \left(\frac{d}{dt}\exp(t\xi)|_{t=0}\right).m \in T_mM\). If we set \(G = \mathbb{R}\) in Definition 3.3.1 then this yields a smooth map \(\rho : \mathbb{R} \rightarrow \text{Sympl}(M)\) such that for each \(t \in \mathbb{R}\), \(\rho(t) := \rho_t : M \rightarrow M\) is a symplectomorphism.

Let \((M, \omega)\) be a symplectic space, \(G\) a \(\mathbb{F}\)-Lie group acting on \(M\) by symplectomorphisms, \(G\) the \(\mathbb{F}\)-Lie algebra of \(G\) and \(G^*\) the dual of \(G\). The action of \(G\) on \(M\) induces a map \(\alpha : G \rightarrow \mathfrak{X}(M)\), such that \(\rho_t(m) = \exp(t\xi).m = \gamma_X(t).m\), where \(\rho_t\) is the flow of \(\alpha(X) = X_M\).

Definition 3.3.2. Let \((M, \omega)\) be a symplectic locally Euclidean Frölicher space. Let \(X\) be a vector field on \(M\) preserving \(\omega\), that is, \(\mathcal{L}_X \omega = 0\). Such a vector field is called symplectic vector field. The space of symplectic vector fields on \(M\) is denoted by \(\text{Sp}(\omega)\).

Lemma 3.3.1. Let \((M, \omega)\) be a symplectic space. Let \(G\) be a \(\mathbb{F}\)-Lie group acting by symplectomorphisms on \(M\) on the left by an action \(\sigma\). The symplectic form \(\omega\) on \(M\) is invariant under the action of \(G\) if, and only if the one-form \(\iota_X \omega = X \cdot \omega\) is closed for all \(X := X_M \in \mathfrak{X}(M)\) with regard to Remark 3.3.1.

Definition 3.3.3. Let \((M, \omega)\) be a symplectic space, and \(H : M \rightarrow \mathbb{R}\) any structure function. A vector field on \(M\) denoted by \(X_H\) and defined by \(\iota_{X_H} \omega = dH\) is called the Hamiltonian vector field associated to \(H\) and \(H\) is called the Hamiltonian function.

The set of all Hamiltonian vector fields on \(M\) is denoted by \(\mathfrak{h}(\omega)\). In other words the 1-form \(\iota_{X_H}\) is an exact 1-form and \(H\) is the primitive of \(\iota_{X_H}\) with regard to \(\omega\) defined from \(\mathfrak{X}(M) \times \mathfrak{X}(M)\) into \(C^\infty(M)\). The symplectic and Hamiltonian vector fields are both related to 1-forms.

Definition 3.3.4. Let \(\varphi : M \rightarrow N\) be a smooth map of locally Euclidean Frölicher spaces. Let \(G\) be a \(\mathbb{F}\)-Lie group acting, with respect to Definition 3.1.12, on \(M\) and \(N\) on the left by
actions \( \sigma \) and \( \delta \) respectively. That is, for \( g \in G \), \( m \in M \) and \( n \in N \), we have \( \sigma(g, m) = g.m \) and \( \delta(g, n) = g \cdot n \). The map \( \varphi \) is called \( G \)-equivariant if \( \varphi(\sigma(g, m)) = \varphi(g.m) = \delta(g, \varphi(m)) = g \cdot \varphi(m) \) for all \( m \in M \) and \( g \in G \). In other words, we say that \( \varphi \) preserves the actions \( \sigma \) and \( \delta \), or the diagram below is commutative:

\[
\begin{array}{ccc}
G \times M & \xrightarrow{\varphi} & M \\
| & & \\
\downarrow{id_G \times \varphi} & & \downarrow{\varphi} \\
G \times N & \xrightarrow{\delta} & N \\
\end{array}
\]

It will be worth noticing that \( \varphi \circ \sigma_g \) and \( \delta_g \circ \varphi \) are not equal in general.

**Lemma 3.3.2.** Let \( M \) be a locally Euclidean Frölicher space and \( G \) a \( \mathbb{F} \)-Lie group acting on \( M \) on the left by \( \sigma : G \times M \longrightarrow M \). Assume that \( m_0 \in M \) is a fixed point for \( \sigma \) as stated in Definition 3.1.13. Then the map \( \psi : G \longrightarrow \text{Aut}(T_{m_0}M) \) defined by \( \psi(g) := d\sigma_{g|T_{m_0}M} \) is a representation of \( G \) in the linear space \( \text{Aut}(T_{m_0}M) \).

**Definition 3.3.5.** Let \( G \) be a \( \mathbb{F} \)-Lie group acting on itself by inner automorphisms, that is, \( L_g R_{g^{-1}} : G \longrightarrow G \) such that \( a_g(h) := L_g R_{g^{-1}}(h) = ghg^{-1} \). This conjugation action on \( G \) induces a map \( \rho : G \longrightarrow \text{Aut}(G) \subset \text{Diff}(G) \) defined by \( \rho(g)(h) = a_g(h) \) for any \( h \in G \) and a fixed element \( g \in G \).

**Definition 3.3.6.** Let \( G \) be a \( \mathbb{F} \)-Lie group acting on itself by inner automorphisms, that is, by \( a_g \). Let \( \text{Lin}(G, \mathcal{G}) \) be the space of linear maps from \( G \) to \( \mathcal{G} \). Let \( \text{Ad} : G \longrightarrow \text{Aut}(\mathcal{G}) \subset \text{Lin}(G, \mathcal{G}) \) be the map \( g \mapsto \text{Ad}(g) = da_g|_{T_eG} = a_{g*e} \) with \( T_eG \simeq \mathcal{G} \). The equality \( \text{Ad}(g)(X) = gXg^{-1} \) defines the action of \( G \) on \( \mathcal{G} \) that is, \( \text{Ad} \) is called the adjoint representation of \( G \) into \( \text{Aut}(\mathcal{G}) \), or the adjoint representation of \( G \) on \( \mathcal{G} \) following the action of \( G \) on \( \mathcal{G} \).

It is easy to see that the unit element \( e \in G \) is a fixed point for the conjugation action. That is, \( a_e(h) = h \) for each \( h \in G \). The adjoint representation \( \text{Ad} : G \longrightarrow \text{Aut}(G) \) is actually \( \psi : G \longrightarrow \text{Aut}(T_{m_0}M) \), where \( \text{Aut}(T_{m_0}M) = \text{Aut}(T_eG) = \text{Aut}(\mathcal{G}) \) when \( M = G \) and \( m_0 = e \). Usually in the literature on has \( \text{Ad}(g) := \text{Ad}_g \). The differential \( d(\text{Ad}) \) of \( \text{Ad} \) is denoted by \( ad \), that is, \( ad := d(\text{Ad}) \). Now the map \( ad(X) := ad_X \) is the associated tangent map to \( \text{Ad}_g \) with \( X \in \mathcal{G} \) when \( g \in G \).

**Definition 3.3.7.** Let \( G \) be a \( \mathbb{F} \)-Lie group acting on itself by conjugation action \( a_g : G \longrightarrow G \). Let \( \mathcal{G} \) and \( \mathcal{G}^* \) be the \( \mathbb{F} \)-Lie algebra of invariant vector fields and its dual. Let \( Ad : G \longrightarrow \)}
3.4 Moment map

\(Aut(G)\) be the adjoint representation of \(G\). The action of \(G\) on \(G^*\) denoted by \(Ad^*\) and given by \(<Ad^*(g)\zeta,X> = <\zeta,Ad(g^{-1})(X)>\) for all \(g \in G, \zeta \in G^*\) and \(X \in G\) is called the co-adjoint action (co-adjoint representation). The corresponding infinitesimal action of \(G\) on \(G^*\) denoted by \(ad^*\) is given by \(<ad^*(X)\zeta,Y> = <\zeta,−[X,Y]>\) for all \(X,Y \in G\) and \(\zeta \in G^*\).

**Proposition 3.3.3.** Let \(X \in G\) and \(g \in G\). Let \(ad_X\) be the tangent map associated to \(Ad_g\). Then \(ad_X Y = −[X,Y]\).

**Proof.** The proof is a straightforward consequence of Definition 3.3.7.

3.4 Moment map

This is a pivotal notion in the process of symplectic reduction. In manifolds setting we have retained following references: [5, 20, 51, 54, 88, 91].

**Definition 3.4.1.** Let \((M,\omega)\) be a symplectic locally Euclidean Frölicher space, \(G\) a \(\mathbb{R}\)-Lie group acting on \(M\) by a symplectic action \(\sigma, G\) its \(\mathbb{R}\)-Lie algebra and \(G^*\) the dual of \(G\). The moment map for the action \(\sigma\) of \(G\) on \(M\) is a \(\mathbb{R}\)-smooth map \(\mu : M \rightarrow G^*\) defined by the following property: for all \(\xi \in G\) we have \(<\mu(m),\xi> = \mu_\xi(m)\) and \(d\mu_\xi = X_{\mu_\xi\omega}\), where \(<,>\) is the duality pairing (bracket) on \(G^* \times G\) and \(\mu_\xi := \mu(\xi) \in \mathcal{F}_M\). That is the evaluation of \(\mu(m)\) at \(\xi\): \(\mu(m)(\xi) = \mu_\xi(m) = \mu(\xi)(m)\), with \(\mu(m) \in G^*, \mu_\xi = \mu(\xi) \in \mathcal{F}_M\).

There is an abuse of notation in \(\mu_\xi = \mu(\xi) \in \mathcal{F}_M\), which we will explain in Subsection 6.2.2.

**Definition 3.4.2.** A symplectic \(\mathbb{R}\)-Lie group action on a symplectic Frölicher space is called Hamiltonian if a moment map \(\mu\) exists.

That is, each left invariant vector field \(\xi\) is associated to a Hamiltonian vector field \(X_{\xi}\) on \(M\) such that \(X_{\xi}(m) = X_M(m) = (\frac{d}{dt}|_{t=0})exp(t\xi).m\), where \(X \in G\) and \(X(e) = \xi\).

**Definition 3.4.3.** The moment map \(\mu : M \rightarrow G^*\) is \(G\)-equivariant if we have \((\mu \circ \sigma_g)(m) = (Ad^*(g) \circ \mu)(m)\) for every \((g,m) \in G \times M\). That is, with \(Ad^*(g) := (Ad(g^{-1}))^*\) and for \(X \in G\), \((\mu \circ \sigma_g)(m))(X) = ((Ad^*(g) \circ \mu)(m))(X) = (Ad(g^{-1})(\mu(m)))(X) = (\mu(m)(Ad(g^{-1}))(X).

**Lemma 3.4.1.** The moment map is \(G\)-equivariant if, and only if \(\mu_X \circ \sigma_g = \mu_{g^{-1}Xg}\) for every \(g \in G\) and \(X \in G\).
Proof. For all \( g \in G \), \( X \in \mathcal{G} \) and \( m \in M \), we have following equivalent statements.

\[
[(\mu \circ \sigma_g)(m)](X) = (\mu(m))(Ad(g^{-1}))(X) \iff \quad (\mu_X \circ \sigma_g)(m) = \mu_{(Ad(g^{-1}))X}(m) \iff \quad \mu_X \circ \sigma_g = \mu_{g^{-1}Xg} \quad \text{since} \quad (Ad_g)(X) = gXg^{-1}.
\]

\( \square \)

Definition 3.4.4. Let \( G \) be a \( \mathbb{F} \)-Lie group acting on a symplectic locally Euclidean Frolicher space \((M, \omega)\) and leaving \( \omega \) invariant, that is, acting by symplectomorphisms. The action is called Hamiltonian if there is a \( G \)-equivariant moment map of the action.

Remark 3.4.1.

1. The \( G \)-equivariance of the moment map \( \mu : M \longrightarrow \mathcal{G}^* \) means that it intertwines the action of \( G \) on \( M \) and the co-adjoint action of \( G \) on \( \mathcal{G}^* \).

2. At the infinitesimal level, the moment map intertwines the infinitesimal action of \( \mathcal{G} \) on \( M \) with the infinitesimal co-adjoint action of \( \mathcal{G} \) on \( \mathcal{G}^* \). That is, \( ad^*(X) : \mathcal{G}^* \longrightarrow \mathcal{G}^* \), \( \langle ad^*X\zeta, Y \rangle = \langle \zeta, -[X, Y] \rangle \), for \( \zeta \in \mathcal{G}^* \), \( X, Y \in \mathcal{G} \), such that \( Y \mapsto ad_X(Y) = -[X, Y] \). It follows \( ad^*X\mu = -\xi_X\mu \).

3. We need to know the nature of \( d\mu_m \). We recall that the moment map is defined by \( \mu : M \longrightarrow \mathcal{G}^* \), \( m \mapsto \mu(m) = \mu_m \) with \( \mu(m) : G \longrightarrow \mathbb{R} \), and \( X \mapsto \mu(m)(X) = \mu_m(X) \). Thus, for all \( m \in M \), the differential of \( \mu \) at \( m \) is \( (d\mu)_m = \mu_m : T_mM \longrightarrow T_{\mu_m}\mathcal{G}^* \). While for all \( X \in \mathcal{G} \), the differential of \( \mu_m \) at \( X \) is \( (d\mu_m)_X : T_X\mathcal{G} \longrightarrow \mathbb{R} \).

4. The diagram below is commutative,

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\alpha} & T\mathcal{G} \\
\downarrow \exp & & \downarrow \text{dexp} \\
G & \xrightarrow{\beta} & TG \\
\end{array}
\begin{array}{ccc}
X & \xrightarrow{\alpha} & \alpha(X) \\
\downarrow \exp & & \downarrow \text{dexp} \\
\exp(tX) & \xrightarrow{\beta} & \beta(\exp(tX)) = (\text{dexp})(\alpha(X)) \\
\end{array}
\]

Where \( \alpha(X) \in T_X\mathcal{G} \) and \( \beta(\exp(tX)) = (\text{dexp})(\alpha(X)) \in T_{\exp(tX)}G \), with \( \alpha \) and \( \beta \) being vector fields on \( \mathcal{G} \) and \( G \) respectively.

5. We will consider the case where the existence of a \( G \)-equivariant moment map is granted. There exist in the literature some results concerning this purpose. But there are using cohomological arguments (see for example [79, 80] for details), which are beyond those considered in this work.
Lemma 3.4.2. Let \((M, \omega)\) be a symplectic locally Euclidean Frölicher space, \(G\) a \(\mathbb{R}\)-Lie group acting on \(M\) by symplectomorphisms. The flow of a symplectic vector field is a one-parameter subgroup of \(\text{Sympl}(M)\) in \(G\). The flow of a Hamiltonian vector field is a one-parameter subgroup of \(\text{Ham}(M)\) in \(\text{Sympl}(M)\).

Definition 3.4.5. Let \(G\) be the \(\mathbb{R}\)-Lie algebra of a \(\mathbb{R}\)-Lie group \(G\) acting on a locally Euclidean Frölicher space \(M\). Let \(m \in M, \xi \in G\) and \(A : G \times M \to TM\) be the infinitesimal action of \(G\) on \(M\) such that \(A(\xi) = \frac{d}{dt}\exp(t\xi)|_{t=0} : M \to TM, \quad A(\xi)(m) = (\frac{d}{dt}\exp(t\xi)|_{t=0}).m = (\frac{d}{dt}\exp(t\xi).m)|_{t=0}\). The \(\mathbb{R}\)-Lie subalgebra of \(G\) defined by \(G_m := \{\xi \in G \mid A(\xi)(m) = A(\xi, m) = 0\} = \{\xi \in G \mid X_M^\xi(m) = 0\}\) is called the isotropy (symmetry) subalgebra of \(m\).

Definition 3.4.6. Let \(\sigma\) be the \(\mathbb{R}\)-Lie group action of \(G\) on a locally Euclidean Frölicher space \(M\). The action \(\sigma\) is called free if \(g \neq e\) implies that \(g.m \neq m\) for all \(m \in M\). That is, \(G_m = \{e\}\), for every \(m \in M\). Equivalently, all stabilizers are equal to the trivial subgroup \(\{e\}\). The action \(\sigma\) is called locally free if \(G_m = \{0\}\), for every \(m \in M\). That is, all stabilizers are discrete.

Definition 3.4.7. Let \(\sigma\) be the \(\mathbb{R}\)-Lie group action of \(G\) on a locally Euclidean Frölicher space \(M\). The action \(\sigma\) is called proper action if the map \(\Phi : G \times M \to M \times M\) defined by \((g, m) \mapsto (m, g.m)\), where \(g.m = \sigma(g, m) = \sigma_g(m) = \sigma_m(g)\), is a proper map. That is, the pre-image of any compact set is a compact set with regard to \(\tau_{\mathcal{F}M \times M}\) and \(\tau_{\mathcal{F}G \times M}\), where \(M \times M\) and \(G \times M\) are finite products of locally Euclidean Frölicher spaces, thus they are also locally Euclidean Frölicher spaces.

Lemma 3.4.3. The properness of an action implies the compactness of \(G_m\) and the closeness of \(\Phi\) defined above.

Proof. First of all, we claim that \(\Phi\) is a smooth map of Frölicher spaces since it is so in all its \(G\)-components \(\Phi_g\) and \(M\)-components \(\Phi_m\), with \(g \in G\) and \(m \in M\). From Lemma 2.2.2 we have that, \(\sigma_g\) is smooth if, and only if \(\Phi_g : M \to G(\sigma_g) = M \times M\) is a diffeomorphism into the graph \(G(\sigma_g)\) of \(\sigma_g\). Now, the component \(\Phi_m : G \to G(\sigma_g) = M \times M\) is defined by \(\Phi_m(g) = (m, \sigma_m(g)) = (m, \sigma_g(m)) = \Phi(g, m)\). We consider two smooth maps: the projection map on the second factor denoted by \(\text{proj}_2 : M \times M \to M\) and the orbit map \(\sigma_m\) such that \(\text{proj}_2 \circ \Phi_m = \sigma_m\). Thus, \(\Phi_m\) is smooth with respect to Lemma 2.2.2. Hence, \(\Phi\) is smooth since it is so in all its components. In other words \(\Phi = \iota \circ (\sigma_g \times id_M)\), where \(\iota : M \times M \to M \times M\) is given by \(\iota(\sigma_g(m), m) = (m, \sigma_g(m))\). Thus, \(\Phi\) is smooth as the composite of smooth maps and in the sequel a continuous map. Now, we need to derive
\[ \Phi^{-1}\{(m, m)\} = \Phi^{-1}(\{m\} \times \{m\}) \]. For, the following equalities hold:

\[
\begin{align*}
\Phi^{-1}(\{(m, m)\}) &= \{(g, m) \in G \times \{m\} \mid \Phi(g, m) = (m, m)\} \\
&= \{(g, m) \in G \times \{m\} \mid (m, g.m) = (m, m)\} \\
&= \{(g, m) \in G \times \{m\} \mid g.m = m\} \\
&= \{(g, m) \in G \times \{m\} \mid g \in G_m\} \\
&= G_m \times \{m\}.
\end{align*}
\]

From the diffeomorphism \( \Phi_g : M \rightarrow G(\sigma_g) \subset M \times M \), it follows that for each \( g \in G_m \), we have \( \Phi_g(m) = (m, g.m) = (m, m) \). Hence, \( \Phi(G_m \times \{m\}) = \{(m, m)\} \). We note that \( \{(m, m)\} = \{(m) \times \{m\}\} \) is compact. In any topological space, all finite subsets and \( \emptyset \) are compact subsets. The product of a family of topological spaces is compact if, and only if each factor of the product is a compact set. It follows from the properness of \( \Phi \) that \( G_m \times \{m\} \subset G \times M \) is compact as the preimage of a compact set. Therefore, \( G_m \) is a compact set as a factor of a product of compact sets. It will be noted that the continuous image of a compact set is compact. This confirms the compactness of \( G_m \) shown above.

Since the canonical projection \( \pi : G \times M \rightarrow G \) is smooth, thus continuous, it follows that \( \pi(G_m \times \{m\}) = G_m \) is a compact set in \( G \). Finally, a compact set in a Hausdorff space is closed and the Cartesian product of closed sets is always a closed set. It follows that \( \{m\} \subset M, \{(m, m)\} \subset M \times M, G_m \subset G \) are closed sets, with \( m \in M \). From the definition of a proper action, one can see that \( \Phi \) is a closed map. \( \square \)

**Remark 3.4.2.** Let \( \varphi : M \rightarrow N \) be a smooth map of locally Euclidean Frölicher spaces with \( \text{dim} M = m \) and \( \text{dim} N = n \). If \( \varphi \) has constant rank \( k \) on \( N \) and \( y = \varphi(x) \), where \( y \in N \) and \( x \in M \). Then,

1. The set \( \varphi^{-1}(y) \) is a closed, regular locally Euclidean Frölicher subspace of \( N \) of dimension \( m - k \), that is, of codimension \( k \). (see Definition 2.3.7, Lemma 2.3.1 and Corollary 2.3.2)

2. For any point \( x \in M \), we have \( T_x(\varphi^{-1}(y)) = \text{Ker}(d_x\varphi) \).

3. For any point \( x \in M \), for any \( U_x \), a sufficiently small neighborhood of \( x \), the image \( \varphi(U_x) \) is a \( k \)-dimensional locally Euclidean subspace in \( N \).

4. For any point \( x \in M \), we have \( T_{\varphi(x)}\varphi(U_x) = \text{im}(d_x\varphi) \).

5. If \( \varphi(M) \) is a locally Euclidean Frölicher subspace of \( N \) then \( \text{dim} \varphi(M) = k \).

**Lemma 3.4.4.** Let \( \sigma \) be an action of a \( \mathbb{F} \)-Lie group \( G \) on a locally Euclidean Frölicher space \( M \). Then,
1. The orbit map $\sigma_m : G \rightarrow M$, where $\sigma_m(g) = g.m$, is a smooth map of locally Euclidean Frölicher spaces. Its rank is constant, that is, for any $g \in G$, $\text{rank}(\sigma_m) = \text{rank}(d_g \sigma_m) = k$.

2. $T_e(G_m) = \text{Ker}(d_e \sigma_m)$.

3. The stabilizer $G_m$ is a closed $F$-Lie subgroup of $G$ such that $\text{dim} G_m = \text{dim} G - \text{rank}(\sigma_m) = \text{dim} G - k$.

4. For any sufficiently small neighborhood $U_e$ of the unit element of $G$, the subset $\sigma_m(U_e) = U_e.m$ is a locally Euclidean Frölicher subspace of dimension $k$ in $M$.

5. $T_m(U_e.m) = \text{im} d_e \sigma_m$.

6. If the orbit $G.m$ is a locally Euclidean Frölicher subspace in $M$ then $\text{dim} G.m = k$.

**Proof.** The proof is based on the literature (see [20, p.41], [87, pp.6, 17]).

1. If one gives a glance to the following commutative diagrams, the constant rank of $\sigma_m$ follows:

   $\begin{array}{ccc}
   G & \xrightarrow{L_g} & G \\
   \downarrow \sigma_m & & \downarrow \sigma_m \\
   M & \xrightarrow{\sim} & M \\
   \end{array}
   \quad \begin{array}{ccc}
   h & \xrightarrow{L_g} & L_g(h) = gh \\
   \downarrow \sigma_m & & \downarrow \sigma_m \\
   \sigma_m(h) = h.m & \xrightarrow{\sim} & \sigma_g(\sigma_m(h)) = \sigma_m(L_g(h)) \\
   \end{array}
   \quad \begin{array}{ccc}
   T_hG & \xrightarrow{d_h L_g} & T_{gh}G \\
   \downarrow d_h \sigma_m & & \downarrow d_{gh} \sigma_m \\
   T_{\sigma_m(h)}M & \xrightarrow{d_{\sigma_m(h)} \sigma_g} & T_{\sigma_g(h.m)}M = T_{\sigma_m(gh)}M \\
   \end{array}$

   Indeed, $\sigma_g(\sigma_m(h)) = \sigma_g(h.m) = \sigma_{gh}(m) = \sigma_m(gh) = \sigma_m(L_g(h))$. This yields:

   Thus, for $h = e$: 
3.4 Moment map

\[ T_eG = G \xrightarrow{d_eL_g} T_gG \]

\[ T_mM \xrightarrow{d_m\sigma_g} T_gM \]

Note that both \( d_eL_g \) and \( d_m\sigma_g \) are isomorphisms of linear spaces, and \( d_g\sigma_m \circ d_eL_g = d_m\sigma_g \circ d_e\sigma_m \). It follows that \( d_g\sigma_m = d_m\sigma_g \circ d_e\sigma_m \circ (d_eL_g)^{-1} \). Hence,

\[
Ker d_g\sigma_m = (d_g\sigma_m)^{-1}(0) \\
= (d_m\sigma_g \circ (d_eL_g)^{-1})^{-1}(0) \\
= (d_eL_g \circ (d_m\sigma_g)^{-1} \circ (d_m\sigma_g)^{-1})(0) \\
= d_eL_g((d_m\sigma_g)^{-1})(0), \text{since } d_m\sigma_g \text{ is a linear isomorphism.}
\]

We need to investigate the nature of \((d_e\sigma_m)^{-1}(0)\) for a fixed \( m \in M \):

\[
(d_e\sigma_m)^{-1}(0) = Ker d_e\sigma_m \\
= \{ X \in G \mid d_e\sigma_m(X) = 0 \}, \text{but } A(m) = d_e\sigma_m \\
= \{ X \in G \mid A(m)(X) = A(X)(m) = X_m = 0 \} \\
= G_m.
\]

It follows that \( Ker d_g\sigma_m = d_eL_g(Ker d_e\sigma_m) = d_eL_g(G_m) \). Therefore, \( X \in G_m \) corresponds to \( A(m)(X) = d_e\sigma_m(X) = 0 \). That is, \( \sigma_m \) is a constant map for some \( h \in G \).

This means that \( h.m = \sigma_m(h) = \sigma_h(m) = m \). That is equivalent to saying that \( h \in G_m \) and \( T_eG_m = G_m \). Now, considering \( G_m \) and \( G \) as linear spaces, we have:

\[
dim G = dim Ker d_e\sigma_m + dim im d_e\sigma_m \\
= dim G_m + rank(d_e\sigma_m) \\
= dim d_eL_g G \\
= dim T_g G
\]

Hence, for all \( g \in G \), we have \( rank(d_g\sigma_m) = rank(d_e\sigma_m) \). That is, the rank of \( (d_g\sigma_m) \) is constant for all \( g \in G \). Therefore, \( \sigma_m \) has a constant rank, \( rank(\sigma_m) = k \), say.

2. From Part 1. in the proof, \( T_eG_m = G_m = Ker d_e\sigma_m \).

3. Since \( G_m = \{ g \in G \mid \sigma_m(g) = m \} = \sigma_m^{-1}(m) \) and \( \{ m \} \) is a closed set in the Hausdorff space \( M \). Thus, \( G_m \) is closed set in \( G \) by the smoothness of \( \sigma_m \). Let \( g,h \in G_m \). Then \( (gh).m = g.(h.m) = g.m = m \), that is, \( gh \in G_m \). Also, \( g.m = m \) implies \( m = g^{-1}.m \),
that is, \( g^{-1} \in G_m \). Hence, \( G_m \) is a closed subgroup of \( G \). From Remark 3.4.2, (1), \( G_m \) is a closed, regular locally Euclidean Frölicher subspace of \( G \). Therefore, \( G_m \) is a \( \mathbb{F} \)-Lie subgroup of \( G \). From Part (2) above in the proof we have,

\[
\dim G_m = \dim T_e G_m \\
= \dim \ker d_e \sigma_m \\
= \dim G - \operatorname{rank}(\sigma_m) \\
= \dim G - k.
\]

4. Since \( U_e \) is an open set in \( G \) then \( T_g U_e = T_g G \) for all \( g \in U_e \) with regard to [8]. It follows that \( \sigma_m(U_e) = U_e, m \) is a locally Euclidean Frölicher subspace of dimension \( k \) with regard to Lemma 3.4.4, (4) and the computation of \( \dim G \) in Part (1) of the proof above.

5. We have \( T_m \sigma_m(U_e) = T_m(U_e, m) = \text{im} d_e \sigma_m \) from Part (1) above in the proof.

6. Recall that the orbit \( G.m = \sigma_m(G) \) and \( T_m \sigma_m(G) = \text{im} d_e \sigma_m \). Therefore, \( \dim G.m = \dim \text{im} d_e \sigma_m = \operatorname{rank}(\sigma_m) = k \). \( \square \)

**Remark 3.4.3.**

1. The quotient \( M/G = \{ G.m \mid m \in M \} \) is the set of orbits of the action of \( G \) on \( M \).

2. Proper maps:

   (a) It is obvious that the action of a compact group on a locally Euclidean Frölicher space is naturally proper.

   (b) If the action is proper the orbit map \( \sigma_m : G \to M \) is proper for each \( m \in M \) (see [51, p.174, LemmaB.3]).

   (c) Note that if \( \sigma : G \times M \to M, \quad (g, m) \mapsto g.m \) is a proper map then \( \Phi : G \times M \to M \times M \) is a proper map. That is, the action \( \sigma \) is proper.

   (d) The reciprocal statement statement to the above (see [51, p.174, Remark B.4]), is not true in general.

3. From Part (6) in the proof of Lemma 3.4.4, one concludes that the tangent space of \( G.m \) at \( m \) is \( T_m(G.m) = \text{span}(\text{im} A_m) = \text{span}(\{ A_m(X) \mid X \in G \}) = \text{span}(\{ X_m \mid X \in G \}) \), with \( T_m(G.m) \subset T_m M \).

**Lemma 3.4.5.** Let \( G \) be a \( \mathbb{F} \)-Lie group acting from the left on a space \( M \) by an action \( \sigma \). For each \( f \in \mathcal{F}_M \) and for each \( g \in G \), there exists a unique \( h \in \mathcal{F}_M \), such that:
1. \( f = h \circ \sigma_g \), where \( \sigma_g : M \rightarrow M \) is defined by \( m \rightarrow \sigma_g(m) = \sigma(g, m) = g.m \).

2. \( \pi \circ \sigma_g = \pi \), where \( \pi \) is the canonical surjection.

3. The \( \mathbb{F} \)-structure on the quotient space is isomorphic to the sub-algebra of \( G \)-invariant functions in \( \mathcal{F}_M \).

**Proof.** The diagram below shows the position of the problem.

\[
\begin{array}{c}
M \xrightarrow{\sigma_g} M \xrightarrow{\pi} \bar{M} = M/G \\
\downarrow f \quad \downarrow h \\
\mathbb{R} \xrightarrow{\bar{h}} \end{array}
\]

1. Since the pullback \( \sigma_g^* : \mathcal{F}_M \rightarrow \mathcal{F}_M \) is one-to-one and onto, then \( h \) exists and is unique. And \( h = f \circ \sigma_g^{-1} \) is a smooth function as the composite of smooth maps.

2. The equivalence relation induced by the action \( \sigma \) of \( G \) on \( M \) is defined by: Given \( x, y \in M \), \( x \sim y \) if, and only if \( y \in G.x \) or \( G.x = G.y \) or there exists \( g \in G \), such that \( y = g.x = \sigma_g(x) = \sigma(x) = \sigma(g, x) \). It follows that \( G.y = G.(g.x) = (Gg).x = G.x \) if, and only if \( \pi(y) = \pi(g.x) = \pi(x) \) if, and only if \( \pi \circ \sigma_g(x) = \pi(x) \) if, and only if \( \sigma_g^* \pi = \pi \circ \sigma_g = \pi \) if, and only if \( \pi \) is invariant under the action of \( \sigma \) of \( G \) on \( M \). Note that \( Gg = R_g(G) = \{ hg \mid h \in G \} = G \) since \( R_g : G \rightarrow G \) is a diffeomorphism.

3. Now, we want to show the compatibility of the relation above with the structure functions. From the definition, one has \( x \sim y \) if, and only if \( y = g.x \) for some \( g \in G \). We get \( \bar{h}(G.y) = \bar{h}(G.x) \) if and only if \( \bar{h}(\pi(y)) = \bar{h}(\pi(x)) \) if, and only if \( h(y) = h(x) \) since \( h = \bar{h} \circ \pi \), which is equivalently the formula \( \bar{h} = h \circ \pi^{-1} \), where \( h \) one-to-one. Let us denote by \( C^\infty(M, \mathbb{R})^G := \mathcal{F}_M^G \) the algebra of \( G \)-invariant smooth functions on \( M \) and by \( C^\infty(\bar{M}, \mathbb{R}) := \mathcal{F}_{M/G} \) the smooth structure on the orbit space. From the diagram above we have \( \mathcal{F}_M \xrightarrow{\pi^*} \mathcal{F}_M^G \), and \( \bar{h} \rightarrow h \), where the pullback \( \pi^* \) is an isomorphism of \( \mathbb{R} \)-algebras. It follows that \( h \) is constant on each orbit (the equivalence class). That is, \( h \circ \sigma_g = h \) if, and only if \( h \) is invariant under the action of \( \sigma \) of \( G \) on \( M \) if, and only if \( h \in \mathcal{F}_M^G \), the space of smooth invariant functions on \( M \). Linking this to Part 1. above, we draw the following consequence, \( f = h \circ \sigma_g = h \), that is, \( \mathcal{F}_M^G \subseteq \mathcal{F}_M \). Therefore, the \( \mathbb{F} \)-structure on the quotient space is compatible with \( G \)-invariant smooth functions on \( M \).

**Remark 3.4.4.** We present some invariance properties:
1. It follows from the construction of the quotient space of $M$ by the relation $\sim$ induced from the action $\sigma$ of $G$ on $M$, that the orbit set $M/G = \{g.x \mid x \in M\}$ is a locally Euclidean space. This was proved in [10].

2. It seems that in locally Euclidean space setting, there is no need for $\sigma$ to be proper for $M/G$ being a locally Euclidean space since we do not use this assumption in the construction of the quotient locally Euclidean space. The detailed proof is given in [10].

3. We need the properness of $\sigma$ to get the compactness of $G_m$ and consequently its closeness. Thus, the restriction of the action to $G_m$ is proper.

4. If $y \in G.x$ then $G_y = gG_xg^{-1}$, for all $g \in G$ (see [20, p.39]). Now, $G_x = \{h \in G \mid h.x = x\}$ and $G_y = \{k \in G \mid k.x = x\}$, with $y = g.x$, if $g \in G$. Thus, $G_{g.x} = \{k \in G \mid k.(g.x) = g.x\} = \{k \in G \mid (kg).x = g.(h.x)\}$ for all $h \in G_x$, $g$ fixed in $G$. It follows that $kg = gh$ for all $k \in G_{g.x}$ and for all $h \in G_x$. Thus, $k = ghg^{-1}$. Hence, $G_{g.x} = gG_xg^{-1}$. That is, if $y$ is in the orbit of $x$, then $G_y$ and $G_x$ are conjugate subgroups of $G$. Otherwise stated: A conjugate $gG_xg^{-1}$ of the stabilizer $G_x$ is also a stabilizer.

5. Recall that the canonical projection $\pi : M \rightarrow M/G$ is actually a smooth map. Thus, it is a continuous map. Therefore, the continuous image of a compact is compact and closed too, since $M/G$ is Hausdorff topological space.

6. If $h \in F_M^G$ then $X_h$ is $G$-invariant, that is, $T_m\sigma_gX_h(m) = X_h(\sigma_g(m))$ (see [34]).

**Lemma 3.4.6.** If the $G$-action $\sigma$ is proper, its restriction to any closed subgroup $H \subseteq G$ is a proper $H$-action on $M$, and its restriction to any invariant subset $S$ of $M$ is a proper $G$-action on $S$.

**Lemma 3.4.7.** If the $G$-action $\sigma$ is proper, every orbit is closed subset of $M$ and a locally Euclidean Frölicher subspace with $\text{dim } G.m = \text{rank}(\sigma_m)$.

**Proof.** Let $\sigma_m : G \rightarrow M$ be the orbit map as in Definition 3.1.14. From Lemma 3.4.4 and Remark 3.4.3 (2) (b), the orbit map is proper. Hence a closed map. Therefore, $\sigma_m(G) = G.m \subset M$ is a closed subset of $M$ since $G$ is a closed set. Now, let $\sigma_m$ be considered into its image $\sigma_m(G) = G.m$. That is, $\sigma_m^{-1}(m) = \{g \in G \mid \sigma_m(g) = g.m = m\}$, where $\sigma_m : G \rightarrow G.m$ is onto. It follows that $g$ and $h$ belong to $\sigma_m^{-1}(m)$ if, and only if $\sigma_m(g) = \sigma_m(h) = m$. Equivalently, it says, $g.m = h.m$ if, and only if $h^{-1}gm = m$. It follows that $h^{-1}g \in G_m$ if, and only if there exists $k \in G_m$ such that $h^{-1}g = k$. Now, we have, $g = hk$ if, and only if $gk^{-1} = h$. $h \in gG_m$. Thus, $g \sim h$, (where $\sim$ is the orbit relation) if, and only
if the fibers of $\sigma_m$ are left $G_m$-cosets in $G$. Therefore, there exists a bijection denoted by $\bar{\sigma}_m: G/G_m \sim G.m$, with $gG_m \mapsto g.m$. If $\iota$ is the canonical inclusion of $G.m$ into $M$ then the following diagram is commutative:

\[
\begin{array}{ccc}
G & \xrightarrow{\sigma_m} & G.m \\
\downarrow{\pi} & & \downarrow{\iota} \\
G/G_m & \xrightarrow{\bar{\sigma}_m} & M \\
\downarrow{f|_{G.m}} & & \downarrow{f} \\
\mathbb{R} & \xrightarrow{\iota} & M
\end{array}
\]

That is, $\sigma_m = \bar{\sigma}_m \circ \pi$, $l = f|_{G.m} \circ \sigma_m$ and $l = f|_{G.m} \circ \bar{\sigma}_m \circ \pi$, with $f \in C^\infty(M)$, $l \in C^\infty(G)$, and $f|_{G.m} \in C^\infty(G.m) = C^\infty(M)|_{G.m}$, by the closeness of $G.m$ in $M$. But, $l\pi^{-1} = f|_{G.m} \circ \sigma_m$ is smooth from the construction of the quotient locally Euclidean Frölicher space $G/G_m$. From Equation (2.7), it follows that $\bar{\sigma}_m$ is a diffeomorphism of locally Euclidean Frölicher spaces. Thus, $G.m$ is a closed regular locally Euclidean Frölicher subspace of $M$. From Lemma 3.4.4 (6), one can conclude that $\dim G.m = \text{rank}(\sigma_m)$. \qed

**Remark 3.4.5.** The map $\iota: G.m \rightarrow M$ is a one-to-one smooth immersion such that the composite map $\iota \circ \bar{\sigma}_m: G/G_m \rightarrow M$ is a one-to-one smooth immersion. The orbit map $\sigma_m: G \rightarrow G.m$ is a surjective smooth submersion. The orbit $G.m$ is open and closed since the orbits form a partition of $M$. The map $\iota \circ \bar{\sigma}_m: G/G_m \rightarrow M$ is an open and closed map.

**Lemma 3.4.8.** Let $\mathcal{G}$ be the $\mathbb{F}$-Lie algebra of a $\mathbb{F}$-Lie group acting on a locally Euclidean Frölicher space $M$. If the $\mathbb{F}$-Lie algebra action is given by the infinitesimal generators, then

1. The algebra $\mathcal{G}_m$ is a closed $\mathbb{F}$-Lie subalgebra of $\mathcal{G}$ as the $\mathbb{F}$-Lie algebra of $G_m$. Thus, $\dim \mathcal{G}_m = \dim G.m$.
2. If $G_m = \{e\}$, that is, the action is free, then the orbit map $\sigma_m: G \rightarrow M$ is a one-to-one immersion.

**Proof.**

1. From Lemma 3.4.4, (2), the set $\mathcal{G}_m$ is a linear subspace of $\mathcal{G}$ as the kernel of the linear map $d_e \sigma_m$ and $\dim \mathcal{G}_m = \dim T_e G_m = \dim G.m$. The fact that $\mathcal{G}_m = (d_e \sigma_m)^{-1}(0)$ implies that $\mathcal{G}_m$ is a closed locally Euclidean Frölicher subspace in $\mathcal{G}$. Now, let $X,Y \in \mathcal{G}_m$. 

Thus, $[X,Y]_m = -[X_m,Y_m] = -[0,0] = 0$. One concludes that $[X,Y] \in G_m$ with regard to Remark 3.2.1, (1),(2). It follows that $G_m$ is a closed $\mathbb{F}$-Lie subalgebra of $G$ with its dimension given by $\dim G_m = \dim G_m$.

2. Let us assume that $G_m = \{e\}$. It follows from the diagram drawn in the proof of Lemma 3.4.7 that the diffeomorphism $\bar{\sigma}_m: G/G_m \rightarrow G.m$ changes to $\bar{\sigma}_m = \sigma_m: G/\{e\} \rightarrow G.m$. Therefore, $\sigma_m: G \rightarrow M$ is a one-to-one immersion. □

Part 2 in Lemma 3.4.8 is a characterization of a free action.

**Definition 3.4.8.** Let $M$ be a locally Euclidean Frölicher space, $\mu: M \rightarrow G^*$ a moment map associated to a Hamiltonian action of $G$ on $M$. An element $m \in M$ is called a regular point (element) of the moment map $\mu$ if the tangent map of $\mu$, that is, $\mu_*: T_m M \rightarrow T_{\mu(m)}G^* = G^*$ is onto. An element $\theta \in G^*$ is called a regular value of the moment map $\mu$ if all elements in the inverse image $\mu^{-1}(\theta)$ are regular elements of $\mu$.

### 3.5 Symplectic reduction on locally Euclidean Frölicher spaces

This Section is mainly drawn from [112] for the Frölicher setting. While the symplectic reduction in manifolds setting can be found in [20, 22, 34, 51, 70, 77, 91].

**Theorem 3.5.1.** Let $(M,\omega)$ be a symplectic locally Euclidean Frölicher space and $\mu: M \rightarrow G^*$ a moment map associated to a Hamiltonian $G$-action on $M$, with $\dim G = n$, $\dim M = q$ and $q \geq n$. Let $\theta \in G^*$ such that $\mu^{-1}(\theta)$ is nonempty and $\theta$ a regular value of $\mu$. Let $\iota_\theta: \mu^{-1}(\theta) \hookrightarrow M$ be the canonical inclusion. Then the subset $\mu^{-1}(\theta)$ is a closed embedded locally Euclidean Frölicher subspace of $M$ and $\dim \mu^{-1}(\theta) = \dim M - \dim G = q - n$. There is an induced 2-form on $\mu^{-1}(\theta)$, that is $\omega_\theta$ and defined by $\omega_\theta := \iota_\theta^* \omega$.

**Proof.** From Remark 3.4.2, $\mu^{-1}(\theta)$ is a closed embedded (regular) locally Euclidean Frölicher subspace with $\dim \mu^{-1}(\theta) = \dim M - \text{rank}_m \mu$, where $m \in M$ such that $\mu(m) = \theta$, since the inclusion $\iota_\theta$ is an injective immersion because of the closeness of $\mu^{-1}(\theta)$. As a consequence of the regularity of $\theta$, it follows that $\mu_*: T_m M \rightarrow T_{\mu(m)}G^* = G^*$ is surjective (or $\mu$ is a submersion). Thus, $\text{rank} \mu = \dim G^* = n$. Therefore, $\dim \mu^{-1}(\theta) = q - n$.

Now, we need to endow $\mu^{-1}(\theta)$ with a 2-form $\omega$ in a natural way. Therefore, one can draw the following diagram for both the tangent map and the pullback of $\iota_\theta$:

3.5 Symplectic reduction on locally Euclidean Frölicher spaces

\[
\begin{array}{ccc}
T_m\mu^{-1}(\theta) \times T_m\mu^{-1}(\theta) & \overset{d_mt_\theta \times d_mt_\theta}{\longrightarrow} & T_mM \times T_mM \\
(\omega_\theta)_m & \downarrow & \omega_m \\
\mathbb{R} \\
\end{array}
\]

where \( \theta \in G^* \), \( m \in \mu^{-1}(\theta) \) and \((\omega_\theta)_m = (t_\theta^* \omega)_m = \omega_m \circ (d_m t_\theta \times d_m t_\theta) \). Thus, for all \( u, v \in T_m\mu^{-1}(\theta) \), one has \((\omega_\theta)_m(u, v) = \omega_m(d_m t_\theta u, d_m t_\theta v) = \omega_m(u, v) \), since \( d_m t_\theta \) is the canonical inclusion \( T_m\mu^{-1}(\theta) \subset T_mM \). Therefore, \((\omega_\theta)_m = \omega_m|_{T_m\mu^{-1}(\theta)} \) if, and only if \( \omega_\theta = \omega_{|_{\mu^{-1}(\theta)}} \). □

**Definition 3.5.1.** Let \( G \) be the \( \mathbb{F} \)-Lie algebra of a \( \mathbb{F} \)-Lie group \( G \) and \( \theta \in G^* \). The \( \mathbb{F} \)-Lie subgroup of \( G \) denoted by \( G_\theta = \{ g \in G \mid Ad^* g \theta = \theta \} \) is called the isotropy \( \mathbb{F} \)-Lie subgroup of \( \theta \) with regard to the co-adjoint action of \( G \) on \( G^* \). The set \( G.\theta = \{ Ad^* g \theta \mid g \in G \} \subset G^* \) is the orbit of the co-adjoint action of \( G \) on \( G^* \). The \( \mathbb{F} \)-Lie algebra of \( G_\theta \), denoted by \( G_\theta \), is the isotropy \( \mathbb{F} \)-Lie subalgebra of \( \theta \).

**Lemma 3.5.2.** Let \( G \) be the \( \mathbb{F} \)-Lie algebra of a \( \mathbb{F} \)-Lie group \( G \) and \( \theta \in G^* \), a regular value of a moment map \( \mu : M \longrightarrow G^* \) associated to a Hamiltonian \( G \)-action on a symplectic locally Euclidean Frölicher space \((M, \omega)\). Assume the \( G \)-action free and proper. Then

1. The subgroup \( G_\theta \) is a compact (thus, closed) set in \( G \), acting smoothly on \( \mu^{-1}(\theta) \).

2. \( G_m \subset G_\theta \) for all \( m \in \mu^{-1}(\theta) \).

3. \( \mu^{-1}(\theta) \) is invariant under the restricted action of \( G_\theta \).

4. Every \( \alpha \in G.\theta = \{ Ad^* g \theta \mid g \in G \} \subset G^* \) is a regular value of the moment map \( \mu \).

5. \( G_\theta \) acts freely and properly on the locally Euclidean Frölicher subspace \( \mu^{-1}(\theta) \).

**Lemma 3.5.3.** Let \((M, \omega)\) be a symplectic locally Euclidean Frölicher space and \( G \) a \( \mathbb{F} \)-Lie group. Assume that the \( G \)-action on \( M \) is Hamiltonian. Let \( \mu : M \longrightarrow G^* \) be its moment map. For every \( \xi \in G \), every \( g \in G \) and every \( m \in M \) the following equality holds:

\[
T_m\mu(\xi_M(m)) = \xi_{G^*}(\mu(m)).
\]

**Proof.** Since the \( G \)-action is Hamiltonian, then the moment map \( \mu \) is co-adjoint equivariant. That is, \( \mu \circ \sigma_g = Ad^*_g \mu \) or the diagrams below are commutative:
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\[ M \xrightarrow{\sim} M \xrightarrow{\sigma_g} M \]

\[ \mu \]

\[ G^* \xrightarrow{Ad_g^*} G^* \]

\[ T_{m \mu} \]

\[ G^* \xrightarrow{\sim} G^* \]

\[ T_{m \sigma_g} \]

\[ T_{m \mu} \]

The arguments above are true if we restrict the action to the one-parameter group \( \sigma_{\exp(t \xi)} \), so, one gets \( \mu(\sigma_{\exp(t \xi)}(m)) = Ad^*(\sigma_{\exp(t \xi)})(\mu(m)) \). The infinitesimal version reads \( T_{m \mu}(\xi_M(m)) = \frac{d}{dt}\exp(t \xi)|_{t=0} = \frac{d}{dt}Ad^*(\sigma_{\exp(t \xi)})(\mu(m))|_{t=0} = \xi_G^*(\mu(m)) \), as required.

\[ \square \]

**Lemma 3.5.4.** Let \( G \) be a \( \mathbb{F} \)-Lie group. Let \( \mu : M \rightarrow G^* \) be the moment map associated to a Hamiltonian \( G \)-action on a symplectic locally Euclidean Frölicher space \((M, \omega)\). If the \( G \)-action is free then \( \text{Ker} T_{m \mu} = T_{m \mu}^{-1}(\theta) \) and \( \text{im} T_{m \mu} = G_m^0 = G_m^{\omega_m} \), where \( m \in \mu^{-1}(\theta) \), \( \theta \in G^* \), is a regular value of the moment map \( \mu \) and \( G_m^0 = G_m^{\omega_m} \) is the annihilator of the \( \mathbb{F} \)-Lie algebra \( G_m \) of the stabilizer of \( m \), with regard to \( \omega \).

**Lemma 3.5.5.** Let \( G \) be the \( \mathbb{F} \)-Lie algebra of a \( \mathbb{F} \)-Lie group acting on a locally Euclidean Frölicher space \( M \) by infinitesimal generators. Then the range \( \text{im} A_m = im d_e \sigma_m \) is spanned by \( \{ \xi_M(m) \mid \xi \in G \} \). Furthermore, if the \( G \)-action is free then \( G.m = T_m(G.m) = \{ \xi_M(m) \mid \xi \in G \} \simeq G \).

**Proof.** We know from linear Algebra that \( \text{im} A_m = \text{span}\{ \xi_M(m) \mid \xi \in G \} \). So, the first statement is a straightforward consequence of the linearity of \( A_m \). The second one follows from the characterization of a free group action stated in Part 2. of Lemma 3.4.8.

**Corollary 3.5.6.** We have with regard to the assumptions of Lemma 3.5.4 and Lemma 3.5.5 both

\[ G.m = T_m(G.m) = \text{Ker} T_{m \mu}^{\omega_m} = T_{m \mu}^{-1}(\theta)^{\omega_m} \text{ and } \]

\[ G.m^{\omega_m} = T_m(G.m)^{\omega_m} = \text{Ker} T_{m \mu} = T_{m \mu}^{-1}(\theta). \]

**Remark 3.5.1.**

1. The set \( G.m = T_m(G.m) \) is the tangent space at \( m \) to the orbit \( G.m \), while \( G.m^{\omega_m} \) is the symplectic orthogonal complement space to \( G.m \) in the symplectic linear space \((T_m M, \omega_m)\). The relation \( \text{Ker} T_{m \mu} = G.m^{\omega_m} = T_m(G.m)^{\omega_m} \) is called the bifurcation Lemma since it establishes a link between the symmetry of a point and the rank of the moment map at that point (see [91]).
2. Therefore, since $T_m \mu$ and $A_m$ are linear maps, it follows that the kernel of $T_m \mu \circ A_m$ is defined by: 

$$Ker(T_m \mu \circ A_m) = \{ \xi \in G \mid (T_m \mu \circ A_m)(\xi) = 0 \} = \{ \xi \in G \mid T_m \mu(A_m(\xi)) = 0 \} = \{ \xi \in G \mid A_m(\xi) \in KerT_m \mu \}$$

since $T_m \mu$ and $A_m$ are linear maps. This implies that $v \in im A_m$ if, and only if $A_m(\xi) = v$ for some $\xi \in G$ if, and only if $\xi_M(m) = v$ for some $\xi \in G$ if, and only if $T_m \mu v \xi = 0$ for some $\xi \in G$.

3. $im T_m \mu = T_m \mu(T_m M) = \{ T_m \mu(v) = \alpha \in G^* \mid v \in T_m M \}$. Equivalently, $< T_m \mu v, \xi > = \iota_v \omega_m(\xi_M(m)) = \alpha(\xi)$, for all $\xi \in G$.

4. $G_m = T_m G_m$ and $G_m^0 = T_m G_m^\perp$

5. $Ker(T_m \mu) = Ker\omega_m = KerT_m \mu = T_m \mu^{-1}(\theta)$

Lemma 3.5.7. Let $G$ be a $\mathbb{F}$-Lie group acting on a symplectic locally Euclidean Frölicher space $(M, \omega)$ by a Hamiltonian action $\sigma$. Let $\theta \in G^*$ be a regular value of $\mu : M \to G^*$, the moment map of the action. For all $m \in \mu^{-1}(\theta)$, we have:

1. $T_m(G_\theta.m) = T_m(G.m) \cap T_m \mu^{-1}(\theta)$.

2. $T_m(G_\theta.m) = KerT_m \mu^\omega m \cap KerT_m \mu$.

3. $G_\theta.m = G.m \cap G.m^\omega m$.

Lemma 3.5.8. Let $(M, \omega)$ be a symplectic locally Euclidean $\mathbb{F}$-space and $\mu : M \to G^*$, the moment map of a Hamiltonian $G$-action on $M$, with $dim G = n$ and $dim M = q$. Let $\theta \in G^*$ be a regular value of $\mu$ such that $\mu^{-1}(\theta)$ is nonempty. If $\iota_\theta : \mu^{-1}(\theta) \hookrightarrow M$ is the canonical inclusion, then the induced 2-form $\omega_{|\mu^{-1}(\theta)} := (\iota_\theta^* \omega)$ has constant rank.

Proof. The 2-form $\omega_\theta := \iota_\theta^* \omega = \omega_{|\mu^{-1}(\theta)}$ was constructed in Theorem 3.5.1, (2). Now, we have $Ker_{m} \omega_{|\mu^{-1}(\theta)} = T_m \mu^{\omega m}(\theta) \cap T_m \mu^{-1}(\theta) \cap T_m(G.m) = T_m(G_\theta.m)$ from Lemma 3.1.5, where $N := \mu^{-1}(\theta)$, $x := m \in \mu^{-1}(\theta)$. As in [70, chapterIII, Remark 2.3], we can state: the rank of $\omega_{|\mu^{-1}(\theta)}$ at the point $m$ is an even integer $k = 2p(m)$ that is equal to the co-dimension of $Ker_{m} \omega_{|\mu^{-1}(\theta)}$ such that the inequalities $sup(0, 2(n - q)) \leq 2p(m) \leq n$ hold. Recall that $dim Ker_{m} \omega_{|\mu^{-1}(\theta)}$ is both non negative and bounded by the dimensions of $T_m \mu^{-1}(\theta)$, where $T_m \mu^{-1}(\theta) = KerT_m \mu$ and $im T_m \mu = T_m(G.m) = T_m \mu^{-1}(\theta) \cap T_m(G.m) = KerT_m \mu^\omega m$. It follows that $Ker_{m} \omega_{|\mu^{-1}(\theta)}$ is of (maximal) constant dimension, since $G.m$, $T_m(G.m)$ and $KerT_m \mu$ are of constant dimensions. Hence, $\omega_{|\mu^{-1}(\theta)}$ has constant rank on $\mu^{-1}(\theta)$.  

Corollary 3.5.9. Under the assumptions of Lemma 3.5.8 and let \( m \in \mu^{-1}(\theta) \); we have: \( T_m \mu^{-1}(\theta) \) and \( T_m(G.m) \) are orthogonal complement in the symplectic linear space \((T_m M, \omega_m)\). It follows that the set \( T_m(G.m) \) is an isotropic linear subspace of the symplectic linear space \((T_m M, \omega_m)\). That is, \( T_m(G.m) \subset T_m(G.m)^\omega = \text{Ker}d\mu_m = T_m\mu^{-1}(\theta) \). Also, \( \text{Ker}d\omega_{\mu^{-1}(\theta)} = T_m(G_\theta.m) \) is an isotropic linear subspace of \( T_m\mu^{-1}(\theta) \).

Proof.

1. Since they are symplectic orthogonal to each other, then the conclusion follows.

2. From Definition of isotropic linear subspace, it is enough to show that \( \omega|_{T_m(G.m)} = 0 \). For, let \( m \in \mu^{-1}(\theta) \) and \( \xi, \eta \in G \) be any left invariant vector fields. It follows that \( T_m\mu^{-1}(\theta) = \text{ker}T_m\mu \simeq \text{im}A_{\omega_m} \) and \( T_m(G.m) = \text{im}A_{m} \simeq \text{im}T_m\mu \). from the proof of Lemma 3.5.10. Therefore, \( T_m\mu(\xi_m(m)) = T_m\mu(\eta_m(m)) = 0 \) and \( \xi_m(m) \perp \eta_m(m) \). Equivalently, \( \omega_m(\xi_m(m), \eta_m(m)) = 0 \). Hence, \( \omega|_{T_m(G.m)} = 0 \).

3. We have \( \text{Ker}d\omega_{\mu^{-1}(\theta)} = T_m(G_\theta.m) \), and \( \omega|_{\mu^{-1}(\theta)m} := (\iota^*_\theta \omega)_m \), with regard to Lemma 3.5.8. Thus, we have that \( (\iota^*_\theta \omega)_m^{-1}(0) = T_m(G_\theta.m) \). Hence, \( (\iota^*_\theta \omega)_m(T_m(G_\theta.m)) = \{0\} \), that is, \( (\iota^*_\theta \omega)_m(u, v) = 0 \), for all \( u, v \in T_m(G_\theta.m) \). But, the former tangent space is given by \( T_m(G_\theta.m) = T_m\mu^{-1}(\theta) \cap T_m\mu^{-1}(\theta)^\omega \). Hence, it is an isotropic linear subspace of \( T_m\mu^{-1}(\theta) \). □

Lemma 3.5.10. Let \( G \) be a \( \mathbb{F} \)-Lie group and \( \mu : M \rightarrow G^* \), the moment map associated to a Hamiltonian \( G \)-action on a symplectic locally Euclidean Frölicher space \((M, \omega)\). Then, there exists an induced \( \mathbb{F} \)-smooth map \( \overline{\mu} : M/G \rightarrow G^*/G \).

Proof. [20, pp.121 – 123] The existence of \( \overline{\mu} \) is a consequence of the \( G \)-equivariance of the moment map \( \mu \). For, we have \( \pi G \circ \text{Ad}_g \circ \mu = \pi G \circ \mu \circ \sigma_g \). We set \( \overline{\mu}(\lfloor m \rfloor) := \pi G^{-1}(\lfloor \mu(m) \rfloor) \) by definition. Thus, \( \overline{\mu} \circ \pi_M = \pi G \circ \mu \). Let \( n \in \lfloor m \rfloor \), that is, \( n = \sigma_g(m) \). It follows that \( \overline{\mu}(n) = (\overline{\mu} \circ \pi_M)(n) = (\overline{\mu} \circ \pi_M \circ \sigma_g)(m) = \overline{\mu} \circ \pi_M(m) = \overline{\mu}(\lfloor m \rfloor) \). Therefore, \( \overline{\mu} \) is well-defined and smooth map. □

Lemma 3.5.11. If the Hamiltonian \( G \)-action on a symplectic locally Euclidean Frölicher space \((M, \omega)\) is free and proper, then, every \( \theta \in G^* \) is a regular value of the moment map \( \mu : M \rightarrow G^* \) associated to the \( G \)-action.

Remark 3.5.2. \( T_m(\mu^{-1}(\theta)/G_\theta.m) = T_m\mu^{-1}(\theta)/T_mG_\theta.m \)

Theorem 3.5.12. Let \( \mu : M \rightarrow G^* \) the moment map associated to a Hamiltonian, free and proper \( G \)-action on a symplectic locally Euclidean Frölicher space \((M, \omega)\). Let \( \theta \) be a regular value of \( \mu \). Let \( \pi : M \rightarrow M/G \) be the canonical projection and \( \pi_\theta = \pi|_{\mu^{-1}(\theta)} \) the restriction
of the canonical projection to $\mu^{-1}(\theta)$. Let $\iota_\theta = \iota|_{\mu^{-1}(\theta)} : \mu^{-1}(\theta) \to M$ be the canonical inclusion of $\mu^{-1}(\theta)$ to $M$ and $\omega_\theta = \omega|_{\mu^{-1}(\theta)}$ the restriction of $\omega$ to $\mu^{-1}(\theta)$. The reduced space $M_\theta = \pi(\mu^{-1}(\theta)) = \mu^{-1}(\theta)/G_\theta$ is a symplectic locally Euclidean Frölicher subspace of $\overline{M} = M/G$ with the symplectic form $\omega_\theta$ defined by $\pi^*\omega_\theta = \iota_\theta^*\omega_\theta$.

**Proof.** Theorem 3.5.1, Lemma 3.5.7 together with Propositions 3.1.1, 3.1.2 and 3.1.3 allow the construction of $\omega_\theta$ defined by $\pi_\theta^*\omega_\theta = \iota_\theta^*\omega$. It is well-defined, non-degenerate. Also, it can be shown to be a closed 2-form. For, let $\text{Ker} \pi^* = \{\alpha \in \Omega(M_\theta) \mid \pi^*(\alpha) = 0_{\Omega(\mu^{-1}(\theta))}\}$, where $\pi^* : \Omega(M_\theta) \to \Omega(\mu^{-1}(\theta))$ and $\alpha \circ \pi_\theta = 0_{\Omega(\mu^{-1}(\theta))}$. Since the tangent map of $\pi$ at $m \in \mu^{-1}(\theta)$ is surjective, one has $\alpha \circ \pi_\theta \circ \pi_\theta^{-1} = 0_{\Omega(\mu^{-1}(\theta))}$ at each $m \in \mu^{-1}(\theta)$. It follows that $\alpha = 0$. Hence, the pullback of $\pi$ is a one-to-one map. Therefore, $\pi^* d\omega_\theta = d\pi^*\omega_\theta = d\iota_\theta^*\omega = \iota_\theta^*d\omega = 0$. So, $d\omega_\theta = 0$. That is, $\omega_\theta$ is closed. □
Chapter 4

Topological inheritance on symplectic quotients of ringed Frölicher spaces

In this chapter we introduce the notion of a ringed space à la Richard Palais [92] in the category $\mathsf{FrI}$ of Frölicher spaces. It is shown that the topology on the ringed $\mathsf{FrI}$-objects is Hausdorff and paracompact, and that these properties are hereditary in the passage to the Marsden-Weinstein symplectic quotient. Moreover, it is proved that the Zariski, the Witney and the Frölicher topologies coincide on a ringed Frölicher space and, using the Cartesian closedness property of this category, it is proved that the Gelfand representation is a smooth map. Recall that a Frölicher structure is defined on a set by two sets of functions which determine one another as described below. This structure induces a category, denoted by $\mathsf{FrI}$. From the differential geometry point of view the category $\mathsf{Man}$ of smooth manifolds is known to be the most important subcategory of $\mathsf{FrI}$. We refer the reader to Chapter 2 and to the literature in [23], [29], [43], and [63] for the foundations of the theory of Frölicher spaces. A framework of differential geometry on these spaces is laid in [8], [36], and [65]. For a generalized notion of a Frölicher Lie group, the basic ingredients can be found in [85]. The existence of the Marsden-Weinstein quotient in this category was investigated in [112], and the topologies on objects are studied in [10], [11], and [23]. The main object of this work being the topological properties of a ringed Frölicher space as well as its symplectic quotient, we lean heavily on the unifying presentations which appeared in [75] and [77]. The details on the reduction process will not be given in this work. The most general properties of topological spaces in the main also hold true for Frölicher spaces and will be either simply recalled or omitted. The books [17] and [61] contain the essential information on general topological notions. The concept of a ringed space which was introduced in [92] induces a subcategory of the category of Frölicher spaces under consideration in the present work, and extends to the smoothness of the Gelfand representation. They will be covered in Sections 1
4.1 Ringed \( \mathbb{F} \)-space

and 2. Most importantly, it is shown in Section 4 that a ringed Frölicher space is Hausdorff and paracompact. These two topological properties are induced on the Marsden-Weinstein quotient. This preliminary result helps to construct the sheaf cohomology and lead to proving the de Rham theorem in Chapter 5. Let \((M, \mathcal{C}_M, \mathcal{F}_M)\) be a Frölicher space, and assume that \(\mathbb{R}\) is provided with its standard topology. We describe the two natural topologies on the underlying set \(M\) as follows. On the one hand, \(\tau_{\mathcal{C}_M}\) is the topology induced by structure curves, that is, the collection of subsets of \(M\) whose preimages by structure curves are open sets in \(\mathbb{R}\). On the other hand, \(\tau_{\mathcal{F}_M}\) is induced by structure functions. Its members are those subsets of \(M\) which are the union of preimages of open sets of \(\mathbb{R}\) by the structure functions. More often, \(\tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M}\). Such spaces were called balanced spaces by Cap (see [23]). It is clear that the topology on \(\mathbb{R}^n\) coincides with the topology in the sense of Frölicher, and because of the compatibility condition defining the Frölicher structure, the subset \(\mathbb{Q}\) of rational numbers has a discrete topology. The structure on this set is trivial, with smooth curves being only constant functions and any function being smooth.

4.1 Ringed \( \mathbb{F} \)-space

Now, let \(M\) and \(N\) be two \( \mathbb{F} \)-spaces. Recall that a map \(\varphi : M \rightarrow N\) in \(\text{Set}\) is said to be smooth if \(\varphi\) is structure (curve and function) preserving, that is, \(\varphi \circ \mathcal{C}_M \subseteq \mathcal{C}_N\) and \(\mathcal{F}_N \circ \varphi \subseteq \mathcal{F}_M\), which can be written as \(\mathcal{F}_N \circ \varphi \circ \mathcal{C}_M \subseteq C^\infty(\mathbb{R}, \mathbb{R})\).

Clearly, any structure curve is a smooth map and so is each structure function. Also, any smooth map is continuous in both \(\tau_{\mathcal{C}_M}\) and \(\tau_{\mathcal{F}_M}\). Next, \(\varphi : M \rightarrow N\) is a map in \(\text{Set}\), \(\theta \circ \varphi \in C^\infty(M, P)\) and \(\theta \in C^\infty(N, P)\) then \(\varphi \in C^\infty(M, N)\). Most importantly, \(\theta \circ \varphi \in C^\infty(M, P)\) whenever \(\varphi \in C^\infty(M, N)\) and \(\theta \in C^\infty(N, P)\). The resulting category is \(\text{Frl}\) is complete, co-complete, and Cartesian closed. The Frölicher structure on objects extends naturally to the set \(C^\infty(M, N)\) since the exponential law \(C^\infty(M \times N, P) \cong C^\infty(M, C^\infty(N, P))\) holds true.

**Definition 4.1.1.** Let \(M, N\) be nonempty sets and \(\mathbb{R}^M, \mathbb{R}^N\) be \(\mathbb{R}\)-algebras of \(\mathbb{R}\)-valued functions on \(M\) and \(N\) respectively, endowed with pointwise operations. A subalgebra \(A\) of \(\mathbb{R}^M\) which separates points is a *structure ring* over \(\mathbb{R}\) for \(M\), that is, for any two different elements \(x, y \in M\) there is a function \(f \in A\) such that \(f(x) \neq f(y)\). The pair \((M, A)\) is called a ringed space (see [92]).

**Definition 4.1.2.** A map \(\varphi : M \rightarrow N\) is a morphism of ringed spaces \((M, A)\), \((N, B)\) if its pullback \(\varphi^* : \mathbb{R}^N \rightarrow \mathbb{R}^M\) is a \(\mathbb{R}\)-algebras homomorphism of \(B\) into \(A\) such that \(f \mapsto \varphi^*(f) = f \circ \varphi\) lies in \(A\) whenever \(f \in B\).
4.2 Smooth Gelfand representation

It can be shown that if $M, N$ and $P$ are ringed spaces, and $\varphi : M \to N$ and $\psi : N \to P$ are morphisms of ringed spaces then $\psi \circ \varphi : M \to P$ is a morphism of ringed spaces so that ringed spaces and morphisms on them form a category which we denote by $\text{R}$. 

Definition 4.1.3. A Frölicher space $(M, C_M, F_M)$ where $F_M$ separates points is called a ringed Frölicher space.

### 4.2 Smooth Gelfand representation

In what follows we will make substantial use of the inspiring work of Richard S. Palais (see [92]). We will assume that $F_M$ separates points as in Definition 4.1.1. This will allow an extension of Palais’ results to ringed Frölicher spaces. The category of ringed Frölicher spaces is denoted by $\text{RFrl}$. We will use the following smooth maps in $\text{Fr}$ (see [43, 63, 85]).

**Definition 4.2.1.**

(i) The evaluation map, $ev : C^\infty(M, N) \times M \to N$ is defined by $ev(\varphi, p) := \varphi(p) \in N$, for $p \in M$ and $\varphi \in C^\infty(M, N)$.

(ii) The evaluation at a point, $ev_p : C^\infty(M, N) \to N$ is defined by $ev_p(\varphi) := \varphi(p) \in N$, for $\varphi \in C^\infty(M, N)$ and $p \in M$.

(iii) The insertion map, $ins : M \to C^\infty(N, M \times N)$ is defined by $ins(p)(q) := (p, q) \in M \times N$, for $p \in M, q \in N$ and $ins(p) \in C^\infty(N, M \times N)$.

(iv) The composition map, $comp : C^\infty(N, P) \times C^\infty(M, N) \to C^\infty(M, P)$ is defined by $comp(\theta, \varphi) := \theta \circ \varphi$, for $\varphi \in C^\infty(M, N)$, $\theta \in C^\infty(N, P)$ and $comp(\theta, \varphi) \in C^\infty(M, P)$.

Let $(M, C_M, F_M)$ be a ringed Frölicher space with $F_M \subset \mathbb{R}^M$. Consider the dual real vector spaces $F_M^*$ and $F_M$ as well as the dual real algebras $\hat{F}_M$ and $F_M$. Then, clearly, $\hat{F}_M \subset F_M^* \subset C^\infty(F_M, \mathbb{R})$.

Now, we need to express the exponential law $C^\infty(M \times N, P) \cong C^\infty(M, C^\infty(N, P))$ in the category $\text{Fr}$ through the following diagrams of smooth maps. Let $\psi \in C^\infty(M \times N, P)$ and
\[ \tilde{\psi} \in C^\infty(M, C^\infty(N, P)) \]. That is

\[
\psi : M \times N \longrightarrow P \iff \tilde{\psi} : M \longrightarrow C^\infty(N, P)
\]

\[
(p, q) \mapsto \psi(p, q) \iff p \mapsto \tilde{\psi}(p)
\]

\[
\psi_p := \psi(p, \ldots) : N \longrightarrow P \iff \tilde{\psi}(p) : N \longrightarrow P
\]

\[
\psi_p(q) := \psi(p, \ldots)(q) := \psi(p, q) \iff \tilde{\psi}(p)(q) := (\psi \circ \text{ins}(p))(q) = \psi(p, q)
\]

\[
\psi_p \iff \tilde{\psi}(p) = \psi \circ \text{ins}(p)
\]

\[
\psi : M \times N \longrightarrow P \iff \tilde{\psi} : M \longrightarrow C^\infty(N, P) \quad \text{and} \quad \psi_p(q) = \psi(p, q).
\] (4.1)

The map \( \tilde{\psi} \) is called the associated map of \( \psi \). By Cartesian closedness of the category \( \mathbf{FrI} \), all these maps are \( \mathbb{F} \)-smooth and uniquely determined. So, to ease notations we will use the same symbol for both maps in Equation (4.1), such as \( ev : M \longrightarrow C^\infty(C^\infty(M, N), N) \) for the evaluation map, and \( ev : C^\infty(M, \mathbb{R}) \times M \longrightarrow \mathbb{R} \) and \( ev : M \longrightarrow C^\infty(\mathcal{F}_M, \mathbb{R}) \). The point-separating condition on the structure ring implies that \( ev : M \longrightarrow \hat{\mathcal{F}}_M \) is one-to-one, with \( p \mapsto ev_p \) and \( ev_p : \mathcal{F}_M \longrightarrow \mathbb{R} \) a \( \mathbb{R} \)-algebra homomorphism defined by \( ev_p(\varphi) := \varphi(p) \in \mathbb{R} \), when \( \varphi \in \mathcal{F}_M \).

Generally speaking, and without a smoothness assumption, a representation of \( \mathcal{F}_M = C^\infty(M, \mathbb{R}) \) is defined to be a \( \mathbb{R} \)-algebra homomorphism \( \rho : \mathcal{F}_M \longrightarrow \mathbb{R}^M \) which separates points in \( M \). That is, given \( p, q \in M \) with \( p \neq q \) there exists \( f \in \mathcal{F}_M \) such that \( \rho(f)(p) \neq \rho(f)(q) \), and \( \rho(f) : M \longrightarrow \mathbb{R} \).

For any representation \( \rho \), one may consider that there exists a canonical associated map \( \psi : M \longrightarrow \hat{\mathcal{F}}_M \) defined by \( \psi_p(f) = \rho(f)(p) \), where \( \psi(p) := \psi_p \in \hat{\mathcal{F}}_M \). So, the identities \( \psi = ev \circ \rho \) and \( \psi(p) = \psi_p = ev_p \circ \rho \) are canonical. It can be easily seen that for any representation \( \rho \), the function \( \rho(f) \) is one-to-one by definition. Hence, so is \( \psi \). It follows that \( M \) can be embedded in \( \hat{\mathcal{F}}_M \). That is, it can be identified with a subset of \( \hat{\mathcal{F}}_M \). Thus, \( \mathbb{R}^M \subset \mathbb{R}^{\hat{\mathcal{F}}_M} \).

Therefore, any algebra can be represented as an algebra of functions.

Now, let us deal with a Frölicher smooth representation. Let \( \rho : \mathcal{F}_M \longrightarrow C^\infty(\hat{\mathcal{F}}_M, \mathbb{R}) \subset \mathbb{R}^{\hat{\mathcal{F}}_M} \) be a map that separates points in \( \hat{\mathcal{F}}_M \). Notice that the source and the range of \( \rho \) are ringed \( \mathbb{F} \)-spaces.

Let \( f \in \mathcal{F}_M, \varphi \in \hat{\mathcal{F}}_M \). Equation (4.1) becomes

\[
\psi : \mathcal{F}_M \times \mathcal{F}_M \longrightarrow \mathbb{R} \iff \psi : \mathcal{F}_M \longrightarrow C^\infty(\hat{\mathcal{F}}_M, \mathbb{R});
\]

\[
\psi_f(\varphi) = \psi(f)(\varphi) = \psi(f, \varphi);
\] (4.2)

\[
\theta : \mathcal{F}_M \times \mathcal{F}_M \longrightarrow \mathbb{R} \iff \theta : \mathcal{F}_M \longrightarrow C^\infty(\mathcal{F}_M, \mathbb{R});
\]

\[
\theta_{\varphi}(f) = \theta(\varphi)(f) = \theta(\varphi, f).
\] (4.3)

We can define two canonical maps that are inverse of each other from Equations (4.2) and (4.3) as follows.

\[
\eta : \mathcal{F}_M \times \hat{\mathcal{F}}_M \longrightarrow \hat{\mathcal{F}}_M \times \mathcal{F}_M \iff \zeta : \hat{\mathcal{F}}_M \times \mathcal{F}_M \longrightarrow \mathcal{F}_M \times \hat{\mathcal{F}}_M.
\] (4.4)
4.2 Smooth Gelfand representation

We set \( \eta(f, \varphi) := (\varphi, f) \leftrightarrow \zeta(\varphi, f) := (f, \varphi) \), where \( f \in \mathcal{F}_M \) and \( \varphi \in \hat{\mathcal{F}}_M \). From the diagrams below \( \eta \) and \( \zeta \) are \( \mathbb{F} \)-smooth maps since the composition map is smooth.

\[
\begin{align*}
\mathcal{F}_M \times \hat{\mathcal{F}}_M & \quad \psi \quad \mathcal{F}_M \\
\hat{\mathcal{F}}_M \times \mathcal{F}_M & \quad \theta \quad \hat{\mathcal{F}}_M
\end{align*}
\]

From now on we use an abuse of notation as follows.

\[
\psi(f, \varphi) = \theta(\varphi, f) \quad \text{and then} \quad \psi_f(\varphi) = \psi(f)(\varphi) = \psi(\varphi)(f) \in \mathbb{R}. \tag{4.6}
\]

This will help to link the associated map in representation theory and the associated map used in the exponential law in the category \( \text{Fril} \) of Frölicher spaces as follows. Firstly, notice that as a representation, Equations (4.2) and (4.3) translate into \( \psi : \mathcal{F}_M \times \hat{\mathcal{F}}_M \to \mathbb{R} \leftrightarrow \psi : \mathcal{F}_M \to C^\infty(\hat{\mathcal{F}}_M, \mathbb{R}); f \mapsto \psi(f) \) with \( \psi(f) = \psi_f : \hat{\mathcal{F}}_M \to \mathbb{R} \), which is a one-to-one map. Hence the associated map is \( \psi : \hat{\mathcal{F}}_M \to C^\infty(\mathcal{F}_M, \mathbb{R}) \), where \( \varphi \mapsto \psi(\varphi) \), with \( \psi(\varphi) = \psi_\varphi : \mathcal{F}_M \to \mathbb{R} \). It is clear that \( f \in \mathcal{F}_M \), \( \psi_f := \hat{\psi}_f : \mathcal{F}_M \to \mathbb{R}, \varphi \in \hat{\mathcal{F}}_M, \psi_\varphi : \mathcal{F}_M \to \mathbb{R} \) are \( \text{Fril} \)-smooth maps. Now we are able to introduce the following definition:

**Definition 4.2.2.** A representation \( \rho^G : \mathcal{F}_M \to C^\infty(\hat{\mathcal{F}}_M, \mathbb{R}) \subset \mathbb{R}^{\hat{\mathcal{F}}_M} \) such that \( \rho^G(f)(\varphi) = \psi_f(\varphi) = \hat{\psi}_f(\varphi) = \varphi(f) \in \mathbb{R} \), where \( \rho^G \in C^\infty(\mathcal{F}_M, C^\infty(\hat{\mathcal{F}}_M, \mathbb{R})), \rho^G(f) := \hat{\psi}_f \in C^\infty(\hat{\mathcal{F}}_M, \mathbb{R}), \varphi \in \hat{\mathcal{F}}_M \) and \( f \in \mathcal{F}_M \). This is the so-called Gelfand representation of \( \mathcal{F}_M \).

For ringed spaces, a Gelfand representation is defined as an algebra homomorphism \( \rho^G : \mathcal{F}_M \to \mathbb{R}^{\hat{\mathcal{F}}_M} \). Then \( \rho^G(f)(\varphi) := \psi_f(\varphi) = \hat{\psi}_f(\varphi) = \varphi(f) \in \mathbb{R} \), where \( \rho^G(f) := \hat{\psi}_f \in \mathbb{R}^{\hat{\mathcal{F}}_M}, \varphi \in \hat{\mathcal{F}}_M \) and \( f \in \mathcal{F}_M \).

**Theorem 4.2.1.** The Gelfand representation defined in Definition 4.2.2 is a smooth map.

**Proof.** The smooth maps involved in the construction of the Gelfand representation are clearly smooth \( \mathbb{R} \)-algebra homomorphisms, except for \( \rho^G(f) := \hat{\psi}_f \in \mathbb{R}^{\hat{\mathcal{F}}_M} \) of course. Since \( C^\infty(\mathcal{F}_M, C^\infty(\hat{\mathcal{F}}_M, \mathbb{R})) \) contains the set of all \( \mathbb{F} \)-smooth representations, then a Gelfand representation is a \( \mathbb{F} \)-smooth map by definition. We recall here that \( \psi \in C^\infty(\mathcal{F}_M \times \hat{\mathcal{F}}_M, \mathbb{R}) \), which proves that \( \psi \) is a smooth function on a product. Then the exponential law induces the associated map \( \psi \in C^\infty(\mathcal{F}_M, C^\infty(\hat{\mathcal{F}}_M, \mathbb{R})) \), which is a representation in this context. In other words, \( \psi_f : \hat{\mathcal{F}}_M \to \mathbb{R} \) is a one-to-one map. Thus \( \psi \in C^\infty(\hat{\mathcal{F}}_M, C^\infty(\mathcal{F}_M, \mathbb{R})) \) is the one-to-one associated map to the representation. \( \square \)
However, replacing $\hat{F}_M$ by $M$ in Theorem 4.2.1, specially in the three last statements involving $\psi$, yields a general smooth representation. It follows that all smooth maps $\psi_p : F_M \to \mathbb{R}$ are $\mathbb{R}$-algebra homomorphisms. We have just built a smooth embedding of $M$ into $\hat{F}_M \subset C^\infty(F_M, \mathbb{R})$. Hence, in the category Frél, $M$ is diffeomorphic to a subset of $\hat{F}_M$ by any smooth representation. The facts that $\psi_f(\varphi) = \hat{f} : \hat{F}_M \to \mathbb{R}$ is a one-to-one map and also $f \mapsto \hat{f}$ is the Gelfand representation of $F_M$ induce $f = \hat{f}_{|M}$, that is, each $f$ is a restriction to $M$ of a certain $\hat{f}$.

Now we need to elaborate briefly on initial and final objects in the category of ringed Frölicher spaces. We refer mostly to our previous studies (see [10] and [11]) and assume that the reader will find it easy to verify the following statements. In the sequel, each initial object in the category of ringed Frölicher spaces has two natural structure rings. In the first case, each object is endowed with the structure ring consisting of the set of generating functions on the one hand (for ringed subspace and ringed product space), and the induced structure on the other. Each final object (for ringed quotient and ringed coproduct) has a unique structure ring, that is, the final $\mathbb{F}$-structure simply because the set of structure functions is generated by curves.

### 4.3 Topology on a ringed Frölicher space

We refer to the topologies defined above, and state what follows. Firstly, $\tau_{F_M}$ and $\tau_{C_M}$ on $M$ are Hausdorff by the point separation condition. Next, the Whitney topology $\tau_W$ on $M$ is the weakest Hausdorff topology such that each $f : M \to \mathbb{R}$ is continuous. Also, the Zariski topology is the weakest topology on $\hat{F}_M$ such that all maps $\hat{f}$ are continuous. The basic $\tau_Z$-open sets on $\hat{F}_M$ are the subsets $\hat{F}_{M_f} := \{ \varphi \in \hat{F}_M | \hat{f}(\varphi) = \varphi(f) \neq 0, f \in F_M \}$. It follows that the induced $Z$-topology on $M$ is Hausdorff and its basic open sets are given by $M_f := \{ x \in M | f(x) \neq 0, f \in F_M \}$.

Generally, $\tau_W \supseteq \tau_Z$. The equivalence of these three topologies is a powerful tool which allows all important results in Palais’ book [92] to be restated, mainly in Section 1.5. We focus on complete and regular spaces that help investigate dense subspaces.

Let $(M, C_M, F_M)$ and $(N, C_N, F_N)$ be ringed $\mathbb{F}$-spaces. By definition, the smooth map $\varphi : M \to N$ of ringed Frölicher spaces is structure (curve and function) preserving. So, each structure function and also any smooth map is continuous with regard to $\tau_{F_M}$ and $\tau_{F_N}$. Moreover, the pullback map $\varphi^* : F_N \to F_M$ is one-to-one if and only if the image $\varphi(M) \subset N$ is $\tau_{F_N}$-dense.

**Definition 4.3.1.** The ringed $\mathbb{F}$-space $(M, C_M, F_M)$ is said to be complete if $M \cong \hat{F}_M$. 
4.3 Topology on a ringed Frölicher space

Equivalently, if the evaluation map \( ev : M \rightarrow \hat{\mathcal{F}}_M \) with \( p \mapsto ev_p \), is a diffeomorphism, and \( ev_p : \mathcal{F}_M \rightarrow \mathbb{R} \) a \( \mathbb{R} \)-algebra homomorphism defined by \( ev_p(\varphi) := \varphi(p) \in \mathbb{R} \), when \( \varphi \in \mathcal{F}_M \). However, since the evaluation map is one-to-one, it will only need to prove that it is onto. Thus, \((\hat{\mathcal{F}}_M)\) is a canonical example of a complete ringed Frölicher space when \( M \subset \hat{\mathcal{F}}_M \) strictly.

**Definition 4.3.2.** A completion of a ringed Frölicher space \( M \) is a complete ringed Frölicher space \( N \) such that \( M \) is a dense subset of \( N \) irrespective of topologies on \( M \).

In this definition, the closure with regard of \( \tau_{\mathcal{F}_N} \) is considered. However, the Frölicher topology and the Frölicher structure are discrete on a dense subspace. Furthermore, we say that \( M \) is both a general ringed subspace and a Frölicher subspace of \( N \) simply because on the general ringed subspace the structure ring comes from the restriction of \( \mathcal{F}_N \) to \( M \), while for the ringed Frölicher subspace the structure ring is generated by the functor \( \Phi \circ \Gamma \) applied to the former set of restrictions. Notice that in the case of completion we do not say that \( M \) is a complete ringed Frölicher subspace of \( N \), the latter being complete as a completion. In general, a completion is unique up to an algebra isomorphism, that is, the identity map is an algebra smooth isomorphism. The completion process consists of constructing a complete space containing the given one. Let us assume that \( M \) is a complete ringed Frölicher space and \( N \) its completion. It follows that \( \hat{\mathcal{F}}_M \cong M \) and \( \hat{\mathcal{F}}_N \cong N \). But, according to Palais (see [92]), the assumption that \( M \) is dense is equivalent to saying that every element of \( \mathcal{F}_M \) is uniquely extended to an element of \( \mathcal{F}_N \). It follows that \( M \cong \hat{\mathcal{F}}_M \cong \hat{\mathcal{F}}_N \cong N \), that is, \( N \) is diffeomorphic to its subspace \( M \).

In contrast the inverse problem will be that of asking at which conditions a subspace of a complete space is also complete. We have to consider both ringed subspaces \((M, \mathcal{F}_N|_M)\) and ringed Frölicher subspace \((M, \mathcal{C}_M, \mathcal{F}_M)\), that is, the general case versus the smooth one respectively. It is known that if \((N, \mathcal{F}_N)\) is a complete space, then so is any ringed subspace \((M, \mathcal{F}_N|_M)\) provided it is \( \tau_{\mathcal{F}_N} \)-closed.

We recall that any Frölicher space induces a ringed space as by Definition 4.1.3. Consequently, any nonempty subset is a ringed Frölicher subspace. Naturally, \( \mathcal{F}_M = \Phi \circ \Gamma(\mathcal{F}_N|_M) \), where \( \mathcal{F}_M \supset \mathcal{F}_N|_M \). So, we call \((M, \mathcal{C}_M, \mathcal{F}_M)\) a complete ringed Frölicher subspace if \( \hat{\mathcal{F}}_M \cong M \). Therefore, \((N, \mathcal{C}_N, \mathcal{F}_N)\) is a complete ringed Frölicher space if and only if \( \hat{\mathcal{F}}_N \cong N \). Hence, \( \hat{\mathcal{F}}_M \supset \hat{\mathcal{F}}_N|_M \) and also \( \hat{\mathcal{F}}_M \cong M \). We can define a diffeomorphism \( \hat{\mathcal{F}}_{N|_M} \cong \hat{\mathcal{F}}_M \).

Now and firstly, if \( \hat{\mathcal{F}}_{N|_M} \neq \hat{\mathcal{F}}_M \) then \( \hat{\mathcal{F}}_{N|_M} \) is dense in \( \hat{\mathcal{F}}_M \). Equivalently, every smooth algebra homomorphism \( \varphi : \mathcal{F}_N|_M \rightarrow \mathbb{R} \) extends uniquely to a smooth algebra homomorphism \( \psi : \mathcal{F}_M \rightarrow \mathbb{R} \).
Secondly, if $\hat{\mathcal{F}}_{N|M} = \hat{\mathcal{F}}_M$ and as $M$ is already $\tau_{\mathcal{F}_N}$-closed in $N$, irrespective of the relative topology on $M$ then we get $\psi = \varphi$, which in turn induces $\mathcal{F}_M = \mathcal{F}_{N|M}$ as sources of equal maps. Hence, $M$ is $\tau_{\mathcal{F}_N}$-closed in $N$ if and only if $\mathcal{F}_M = \mathcal{F}_{N|M}$. In fact, we have there proof of a well-known result in the category $\text{Fr}l$. Furthermore, $\hat{\mathcal{F}}_M$ is in fact the Stone-Čech compactification of $M \subset \hat{\mathcal{F}}_M$. Thus, a ringed Frölicher space is complete if and only if $M$ is $\tau_{\mathcal{F}_M}$-compact (see [92]).

We will now focus on some aspects of a regular ringed space and the regularization of a ringed space. It is a fact that if $(M, \mathcal{C}_M, \mathcal{F}_M)$ is a ringed Frölicher space, then a function $h : M \to \mathbb{R}$ is regular when $f, g \in \mathcal{F}_M$ exist such that $h := \frac{f}{g}$ and $g(x) \neq 0$ for every $x \in M$. It can be proved easily that $h^{-1}(0) = f^{-1}(0)$. The ring of regular functions on $M$, induced by $\mathcal{F}_M$ will be denoted by $\mathcal{F}_{M\text{reg}}$ and $(M, \mathcal{F}_{M\text{reg}})$ will be the regularization of the ringed Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$. We will then call $(M, \mathcal{C}_M, \mathcal{F}_M)$ a regular ringed Frölicher space if $\mathcal{F}_M = \mathcal{F}_{M\text{reg}}$. That is, for every $f \in \mathcal{F}_M$ such that $f(x) \neq 0$ at each $x \in M$ we have $\frac{1}{f} \in \mathcal{F}_M$. In general, $\mathcal{F}_M \subset \mathcal{F}_{M\text{reg}}$ and with properties of functors $\Phi$ and $\Gamma$, we have the following:

\[
\Gamma(\mathcal{F}_M) = \mathcal{C}_M \supset \Gamma(\mathcal{F}_{M\text{reg}}) := \mathcal{C}_{M\text{reg}} \quad \text{and} \quad (4.7)
\]

\[
\Phi \Gamma(\mathcal{F}_M) = \Phi \mathcal{C}_M = \mathcal{F}_M \subset \Phi \Gamma(\mathcal{F}_{M\text{reg}}) := \Phi \mathcal{C}_{M\text{reg}} = \mathcal{F}_{M\text{reg}} \quad \text{and} \quad (4.8)
\]

\[
\mathcal{F}_M \subset \mathcal{F}_{M\text{reg}} \subset \mathcal{F}_{M\text{reg}} \quad \text{and} \quad (4.9)
\]

It follows that the Frölicher structure $(\mathcal{C}_{M\text{reg}}, \mathcal{F}_{M\text{reg}})$ is finer than the Frölicher structure $(\mathcal{C}_M, \mathcal{F}_M)$, and it follows that the latter is coarser than the former. Also, we now call $(M, \mathcal{C}_{M\text{reg}}, \mathcal{F}_{M\text{reg}})$ a Frölicher regularization of $(M, \mathcal{C}_M, \mathcal{F}_M)$. If $(M, \mathcal{C}_M, \mathcal{F}_M)$ is regular ringed Frölicher space then $\tau_{\mathcal{F}_M} = \tau_{\mathcal{F}_{M\text{reg}}}$. That is, they are equivalent topologies. But, we have now four topologies on $M$ with regard to the two Frölicher structures as shown in the diagram below, where arrows indicate canonical inclusions:

\[
\begin{array}{ccc}
\tau_{\mathcal{F}_M} & \quad & \tau_{\mathcal{C}_M} \\
\downarrow & & \downarrow \\
\tau_{\mathcal{F}_{M\text{reg}}} & \quad & \tau_{\mathcal{C}_{M\text{reg}}}
\end{array}
\]

Furthermore, a compact regular ringed Frölicher space is a complete ringed Frölicher space. It follows that $(M, \mathcal{C}_{M\text{reg}}, \mathcal{F}_{M\text{reg}})$, the Frölicher regularization of a compact ringed Frölicher space, is a complete ringed Frölicher space (see [92]). The Frölicher regularization replaces the Frölicher topology $\tau_{\mathcal{F}_M}$ by a finer Frölicher topology $\tau_{\mathcal{F}_{M\text{reg}}}$. But, it can be proved as in
that the standard regularization \((M, C^\text{reg}_M, F^\text{reg}_M)\) does not change the topology, that is, \(\tau_{F_M} = \tau_{F^\text{reg}_M}\). This is again an immediate consequence of the equality \(\tau_{F_M} = \tau_W = \tau_Z\) on \(M\).

4.4 Hausdorff and paracompactness inheritance

The following three lemmas refer to Hausdorff or paracompactness properties for topological spaces in general (see [17], [38], and [61]). They hold true for a ringed \(\mathbb{F}\)-space. Therefore, we state them without proof.

**Lemma 4.4.1.** Let \(M\) and \(N\) be topological spaces. Let \(S \subset M\) be a subspace. Let \(I\) be (a finite or infinite) set of indices and \((M_i)_{i \in I}\) be a family of topological spaces. Let \(\psi : M \rightarrow N\) be an arbitrary map. Then:

1. If \(\psi\) is a closed bijection and if \(M\) is Hausdorff, then \(N\) is Hausdorff.
2. If \(\psi\) is a continuous injection and \(N\) is Hausdorff, then \(M\) is Hausdorff.
3. If \(M\) is Hausdorff, then \(S\) is Hausdorff.
4. The product space \(\prod_{i \in I} M_i\) is Hausdorff if and only if \(M_i\) is Hausdorff for each \(i \in I\).

**Lemma 4.4.2.** Let \(I\) be a countable set (finite set or \(I = \mathbb{N}\)). Let \(M, N\) and \((M_i, \tau_{F_M})\) be Hausdorff spaces, for each \(i \in I\). Let \(S \subset M\) be a subspace. Let \(\psi : M \rightarrow N\) be a set map. Then:

1. If \(\psi\) is continuous open surjection and \(M\) is second countable, so is \(N\).
2. If \(M\) is second countable, then \(S\) is second countable.
3. If for all \(i \in I\) the space \((M_i, \tau_{F_M})\) is first countable or second countable, so is the product space \(M^* := \prod_{i \in I} M_i\) for \(\tau_H = \tau_{F_M^*}\).
4. Let \(I\) be an uncountable set. Then the product space \(M^*\) is not first countable for \(\tau_H = \tau_{F_M^*}\), that is, \(M^*\) is not second countable.

**Lemma 4.4.3.** The following hold:

1. If \(\psi : M \rightarrow N\) is a continuous closed surjection and if \(M\) is paracompact, then \(N\) is paracompact.
2. If \(S\) is closed, then \(S\) is paracompact.
3. If the product space \(M^* := \prod_{i \in I} M_i\) is paracompact, then \(M_i\) is paracompact for each \(i \in I\).
4. If \(M\) is second countable, then \(M\) is paracompact.

**Proposition 4.4.4.** Let \(M\) be a \(\mathbb{F}\)-space with a \(\mathbb{F}\)-structure generated by a set \(F\) that separates points of \(M\). Then:
4.4 Hausdorff and paracompactness inheritance

(1) The inclusion map \( \varphi : M \hookrightarrow \mathbb{R}^F \), such that \( x \mapsto \varphi(x) = (f(x))_{f \in F} \) is a \( \mathbb{F} \)-smooth map.
(2) The map \( \varphi \) identifies \( M \) with a \( \mathbb{F} \)-subspace \( \varphi(M) \subset \mathbb{R}^F \) (see [85]).

Corollary 4.4.5. The following holds:

(1) If \( F \) the generating set in Proposition 4.4.4 is a finite set with \( n \) elements, then the map \( \varphi : M \hookrightarrow \mathbb{R}^n \) defined by \( \varphi(x) = (f_i(x)) \) is one-to-one, with \( f_i = \pi_i \circ \varphi \) for each \( i \in \{1, 2, 3, \ldots, n\} \). Furthermore, notice that in \( \mathbb{F}rl \), \( \varphi \) is a diffeomorphism onto \( \varphi(M) \subset \mathbb{R}^n \) and the \( \mathbb{F} \)-structure on \( M \) is finitely generated.

(2) If \( F \) in Proposition 4.4.4 is an infinite countable set, then the map \( \varphi : M \hookrightarrow \mathbb{R}^N \) defined by \( \varphi(x) = (f_i(x))_{i \in \mathbb{N}} \) is one-to-one, with \( f_i = \pi_i \circ \varphi \) for each \( i \in \mathbb{N} \). Moreover, \( \varphi \) is a diffeomorphism onto \( \varphi(M) \subset \mathbb{R}^N \) and the \( \mathbb{F} \)-structure on \( M \) is countably generated.

Remark 4.4.1. Let \( D \) be a dense subset in \( \mathbb{R} \). Then:

(1) There are discrete \( \mathbb{F} \)-structure and topology on \( \mathbb{Q} \) or on any other general dense subset of \( \mathbb{R} \).

(2) If \( (D_i)_{i \in I} \) are dense subsets of \( \mathbb{R} \), then the Cartesian product \( \prod_{i \in I} D_i \) is dense in the Cartesian product \( \prod_{i \in I} \mathbb{R} \) of the ambient spaces \( \mathbb{R} \), that is,
\[
\text{cl}(\prod_{i \in I} D_i) = \prod_{i \in I} \text{cl}(D_i) = \prod_{i \in I} \mathbb{R} = \mathbb{R}^I.
\]

(3) The diffeomorphism constructed in Proposition 4.4.4 is not necessarily a homeomorphism in the category of Frölicher spaces. It is a homeomorphism irrespective of the topologies \( \tau_F \), but may fail to enjoy this property on subspaces. For, it was proved in [10, 23, 36] that the topology on a Frölicher space is generally finer than the relative topology on its subspaces. We then assume that \( \varphi(M) \) is not a dense subspace of \( \mathbb{R}^n \), in which case the relative topology agrees with the induced Frölicher topology. In order to understand this argument, which is particular to these spaces and different from the usual topological spaces, let us give a few examples.

Example 4.4.1. Let \( M = (0, \pi) \) and \( \varphi(x) = (\cos(x), h(x)) \) such that \( h(x) = -1 \) for all \( x \in M \).
\[
\varphi(M) = \varphi((0, \pi)) = (-1, 1) \times \{-1\} = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1 \text{ and } y = -1\}
\]

Hence, \( \varphi(M) \subset \mathbb{R}^2 \) is neither an open, nor closed set.

Example 4.4.2. Let \( \varphi = (f_1, f_2): (0, \frac{\pi}{2}) \rightarrow (0, 1) \times \{0\} \subset \mathbb{R}^2 \), where \( f_1 = \sin : (0, \frac{\pi}{2}) \rightarrow (0, 1) \), \( f_2 : (0, \frac{\pi}{2}) \rightarrow \{0\} \).
It is known that \( \{(x, 0) \mid 0 \leq x \leq 1\} \) is closed in \( \mathbb{R}^2 \) as a product of closed sets, whereas \( \{(x, 0) \mid 0 < x < 1\} \) is neither a closed nor open set. Therefore,
its complement in \( \mathbb{R}^2 \) is given by

\[
\mathcal{C}[(0, 1) \times \{0\}] = \mathbb{R}^2 - (0, 1) \times \{0\} = \{(x, y) \in \mathbb{R}^2 \mid x \notin (0, 1), y \neq 0\} = \left(\left(-\infty, 0\right] \cup [1, +\infty)\right) \times \{0\} \cup \left(\left(0, 1\right) \times \mathbb{R}^*\right) \cup \left(\left(-\infty, 0\right] \cup [1, +\infty\right) \times \mathbb{R}^*\right)
\]

This is a union of an open set \( \mathbb{R} \times \mathbb{R}^* = \mathbb{R}^2 - \{(x, y) \mid y = 0\} \) and a closed set \( [(-\infty, 0] \cup [1, +\infty) \times \{0\} \). Thus, \( \varphi(U) \subset \mathbb{R}^2 \) is neither an open, nor a closed set.

**Theorem 4.4.6.** The topology of the countably generated ringed \( \mathbb{F} \)-space \( M \) is Hausdorff paracompact.

**Proof.** The spaces \( \mathbb{R}^I \), (with \( I \) finite or \( I = \mathbb{N} \)) and \( \varphi(M) \) are Hausdorff as product space and subspace by Lemma 4.4.1. They are second countable by Lemma 4.4.2 as countable product and as subspace. Also, they are both paracompact by Lemma 4.4.3. Thus, by Corollary 4.4.5, the topological properties of \( \varphi(M) \subseteq \mathbb{R}^n \) are induced on \( M \) by means of the diffeomorphism \( \varphi : M \to \varphi(M) \). Therefore, \( M \) is Hausdorff, second countable and paracompact for the trace topology on \( \varphi(M) \). Moreover, if the \( \mathbb{F} \)-topology and the Cartesian topology coincide on \( \mathbb{R}^I \), then they coincide on \( \varphi(M) \). \( \square \)

**Proposition 4.4.7.** The quotient ringed \( \mathbb{F} \)-space \( M/G \) is Hausdorff paracompact.

**Proof.** The topology of the quotient ringed \( \mathbb{F} \)-space is Hausdorff by definition. The canonical quotient map is continuous closed surjection and does preserve paracompactness. So, the quotient ringed \( \mathbb{F} \)-space is paracompact by Lemma 4.4.3. \( \square \)

This result extends to a symplectic quotient of a Frölicher space \( M \), which was investigated in [112]. A detailed treatment of the subject will be given in a further work. So, we shall assume that the reader is familiar with the quotient of a Hamiltonian action of a Lie group \( G \) on a symplectic space, to which is associated a momentum mapping \( \mu \) with regular value \( \mu \), say. The process can be summarised as follows. Let \( (M, \omega) \) be a symplectic ringed Frölicher space of constant dimension, hence, a Hausdorff topological space. Let \( G \times M \to \mathbb{R} \) be a Hamiltonian action of a connected Frölicher-Lie group \( G \) on \( M \). Assume that the action is free and proper, with a \( Ad^\mu \)-equivariant momentum map \( \mu : M \to G^* \), where \( G \) is the Lie algebra of the Lie group \( G \). Let \( G_\mu \) denote the isotropy subgroup of the regular value \( \theta \mu \) of \( \mu \) such that the action of \( G_\theta \) on the level set \( \mu^{-1}(\mu) \) is free and proper. Then under these conditions it turns out that the Frölicher subspace \( M_\mu = \mu^{-1}(\mu)/G_\mu \) is a symplectic ringed space provided with the symplectic form

\[
\pi^*_\mu \omega_\theta = i^*_\theta \omega,
\]
where \( \pi_\mu : \mu^{-1}(\theta) \longrightarrow M_\theta \) is the projection to the quotient space and \( \iota_\theta : \mu^{-1}(\theta) \longrightarrow M \) is the inclusion. By Sard’s lemma it is known that the pre-image of such a \( \mu \) is a closed subspace of \( M \). Our main references for symplectic reduction process are the work by Marsden and Weinstein [77] in manifolds setting. While [85, 112] used in Frölicher setting.

**Proposition 4.4.8.** The subspace \( K = \mu^{-1}(\theta) \subset M \) is Hausdorff paracompact.

**Proof.** The topology of \( K = \mu^{-1}(\theta) \) is Hausdorff since this property descends to any subset by Lemma 4.4.1. Moreover, the topology of \( K = \mu^{-1}(\theta) \) is paracompact since this property descends to any closed subset by Lemma 4.4.3. \( \square \)

**Proposition 4.4.9.** The symplectic quotient is Hausdorff paracompact.

**Proof.** The topology on the symplectic quotient is Hausdorff, as it is a subspace of \( M/G \) by Lemma 4.4.1. The canonical quotient map is continuous closed surjection (see [10]). Thus, the image of the closed subset \( K = \mu^{-1}(\theta) \subset M \) is the symplectic quotient, which is closed subset in the quotient space \( M/G \). Thus, the symplectic quotient is paracompact by Lemma 4.4.3. Note: The restriction of the quotient map on \( K = \mu^{-1}(\theta) \subset M \) is still a continuous and closed surjection. Hence, the symplectic quotient is paracompact by Lemma 4.4.3. \( \square \)
Chapter 5

Sheaf and cohomology on reduced spaces of locally Euclidean $\mathbb{F}$-spaces

In the previous chapter we introduced the notion of ringed spaces à la Richard Palais (see [92]) in the category of Frölicher spaces. The new objects are called ringed $\mathbb{F}$-spaces (for ringed Frölicher spaces). We showed that the topology of these spaces is Hausdorff paracompact, property that is inherited by the symplectic quotient of a Frölicher space. These are conditions that allow the construction of sheaf cohomology theory which is a generalization of classical cohomology theories such as De Rham and singular cohomologies. Next, we introduce a preliminary result for the fundamental de Rham theorem of differential topology, following Frank Warner’s approach. We then extend the de Rham Theorem to the symplectic quotient of a Frölicher space.

In algebraic and differential topologies for instance, the constant sheaf $\mathcal{G} = M \times G$ is given the product topology, and $G$ is a $\mathbb{K}$-module with discrete topology. Instead, in this work, we will consider $\mathbb{F}$-topology since all objects are $\mathbb{F}$-spaces and the morphisms between them are $\mathbb{F}$-smooth maps and we investigate the relationship between Alexander-Spanier cohomology, de Rham cohomology and Sheaf cohomology, that is, $k^{th}$ cohomology groups on $M$ with coefficients in the constant sheaf $\mathcal{R} = M \times \mathbb{R}$. It has to be pointed out that Hausdorff paracompact topology is the minimum requirement prior to define a sheaf theory without referring to the differential structure in the category of Frölicher spaces. An extension of this theorem based on the sheaf cohomology module $\mathcal{H}^k(M, \mathcal{R})$ with coefficients in the constant sheaf $\mathcal{R} = M \times \mathbb{R}$ was made by Frank W. Warner. He has established canonical isomorphisms between: de Rham, Alexander-Spanier, Continuous Singular, differentiable Singular and Čech, Cohomology modules of a differentiable manifold with coefficients in a $\mathbb{R}$-module $\mathbb{R}$.

Therefore, the isomorphism of cohomology theories on the reduced symplectic Frölicher space
The aim of this research is to prove that the homomorphism from the de Rham cohomology ring to the differentiable singular cohomology ring given by integration of closed forms over differentiable singular cocycles is a ring isomorphism.

As a first step towards this result, we need construct a second countable (then paracompact) and Hausdorff topological space. To this end, we shall consider a frölicher space $M$ whose functions form a structure ring in the sense of R. Palais. Then we define sheaves and presheaves of $K$-modules over $M$ and establish the relationship between them. From this later, we define Alexander-Spanier cohomology theory as well de Rham cohomology and deduce the canonical isomorphisms of the $k^{th}$ cohomology groups on $M$ with coefficients in the constant sheaf $\mathcal{R} = M \times \mathbb{R}$ onto the $k^{th}$ de Rham cohomology group.

In the second part, again on the ambient space, we should define singular cohomology and establish its isomorphism with a sheaf cohomology. In the addition to the requirements in the previous part, we will need to involve the differential structure. So, we have to conclude by the isomorphism of all five cohomology theories.

The last part concerns cohomologies on the symplectic quotient. We will concentrate on de Rham cohomology on symplectic quotient and make use of the previous isomorphisms to extend the concept to the others four cohomologies.

The natural question is to understand how the de Rham Theorem can descend to the Marsden-Weinstein symplectic reduced space. For, we are currently investigating the characterisation of exterior forms on the symplectic quotient. So far, we are trying to extend two defining conditions for the symplectic form to others forms, that is, the invariance under the action and the restricted form on the fiber.

### 5.1 Sheaves, Presheaves and Properties

This section deals with the construction of sheaf theories and cohomologies. More of concepts we will use in this construction, are based on that of general topology. The topological space $M$ needs to be Hausdorff paracompact. The field $\mathbb{R}$ is viewed as a topological space with its canonical topology so that the $\mathbb{R}$-module is a real linear space.

Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a $\mathbb{F}$-space and $(M, \mathcal{F}_M)$ be simultaneously a differential space in the sense of Sikorski (see [98, 99] and a $\mathbb{F}$-ringed space. Let $(U_i)_{i \in I}$ be an open covering of $M$ for $\tau_{\mathcal{F}_M}$. It is known that for each $i \in I$, the $\mathbb{F}$-space $(U_i, \mathcal{C}_{U_i}, \mathcal{F}_{U_i})$ is a $\mathbb{F}$-subspace of $(M, \mathcal{C}_M, \mathcal{F}_M)$ with $\mathcal{C}_{U_i} = \mathcal{C}_{M|U_i}$ and $\mathcal{F}_{U_i} = \mathcal{F}_{M|U_i}$. These restrictions yield the presheaf of rings on $M$.

Recall that the category of $\mathbb{F}$-spaces is a full subcategory of differential spaces in the sense of Sikorski. It follows that the closure with regard to localization is satisfied by $\mathbb{F}$-spaces and transforms the local structures (presheaf) into a global one (associated sheaf). Moreover, the
structure functions set $\mathcal{F}_M$ is a $\mathbb{R}$-algebra (thus, a ring). This section is devoted to presheaves, sheaves and their essential properties for the construction of cohomology theories on spaces, required to be Hausdorff paracompact and also on $\mathbb{K}$-modules where the ring $\mathbb{K}$ is assumed to be a principal ideal domain. We shall restrict ourselves to the field of real numbers $\mathbb{R}$. The main references for this section are F.W. Warner (see [114]) and Glen E. Bredon (see [16]). All objects and morphisms referred to in this chapter are assumed smooth with regard to the category $\mathfrak{FrI}$ of Frölicher spaces.

5.1.1 Sheaves

Definition 5.1.1. Let $\Theta$ and $(M, \tau_{\mathcal{F}_M})$ be $\mathbb{F}$-topological spaces. The triple $(\Theta, \pi, M)$ is a sheaf of $\mathbb{R}$-modules ($\mathbb{R}$-vector spaces) over $M$ if it satisfies the following:

1. The projection $\pi : \Theta \rightarrow M$ is a local diffeomorphism ($\tau_{\mathcal{F}_M}$-homeomorphism).

2. The fiber $\pi^{-1}(m) = \Theta_m$ is a $\mathbb{R}$-module for each $m \in M$ under pointwise operations. It is called the stalk over $m \in M$.

3. The compositions laws are $\mathbb{F}$-smooth maps (continuous in the $\mathbb{F}$-topology of $\Theta$).

This work is concerned with sheaves on Hausdorff spaces, though this property is not needed in general. Moreover, the notation $\Theta$ and $M$ represent arbitrary objects of $\mathfrak{FrI}$.

Definition 5.1.2. Let $\Theta$ and $(M, \tau_{\mathcal{F}_M})$ be $\mathbb{F}$-topological spaces. Let $(\Theta, \pi, M)$ be a sheaf of $\mathbb{R}$-modules ($\mathbb{R}$-vector spaces) over $M$. Let $U \subset M$ be an open subset. A $\mathbb{F}$-smooth (continuous) map $s : U \rightarrow \Theta$ such that $\pi \circ s = id_U$ is called a section of $\Theta$ over $U$.

For example, the 0-section sends $m \in U$ into $0 \in \Theta_m$. We denote by $\Gamma(\Theta, U)$ the $\mathbb{R}$-module of all sections of $\Theta$ over $U$. When $U = M$, one speaks of global sections of $\Theta$ over $M$ and denotes $\Gamma(\Theta, M) := \Gamma(\Theta)$. Since, $\pi s(U) = id_M(U) = U$ then $\pi^{-1} \pi s(U) = s(U) = \pi^{-1}(U)$, we have that sections are open injective maps.

Example 5.1.1. A natural example of a sheaf over $M$ is the constant sheaf defined by $(\mathfrak{R} = M \times \mathbb{R}, \pi, M)$, where $\pi(m, r) = m$, the $\mathbb{R}$-module $\mathbb{R}$ and the sheaf $\mathfrak{R}$ are endowed with $\mathbb{F}$-topologies (instead of discrete topology for $\mathbb{R}$ and product topology for $\mathfrak{R} = M \times \mathbb{R}$, respectively as in general case.
Recall that $\tau_{\Pi} \subseteq \tau_{F_{M^*}} \subseteq \tau_{C_{M^*}}$ by Theorem 2.1.8 and $\tau_{F_{M}}(S) \subseteq \tau_{F_S} \subseteq \tau_{C_S}$ by Proposition 2.1.5.

**Definition 5.1.3.** Let $(\Theta, \pi, M)$ and $(\Upsilon, \Pi, M)$ be sheaves over $M$.

1. A sheaf map is a smooth (continuous) map, $\varphi : \Theta \rightarrow \Upsilon$ such that $\Pi \circ \varphi = \pi$, which is a local diffeomorphism (homeomorphism) mapping stalks into stalks.

2. A sheaf homomorphism (isomorphism) is a $\mathbb{R}$-module homomorphism (isomorphism) on each stalk. Notice that $\text{Ker} \varphi$ and $\text{im} \varphi$ are defined as in linear case.

3. A subsheaf of a sheaf $(\Theta, \pi, M)$ is an open subset $\mathcal{R} \subset \Theta$ such that $\mathcal{R}_m = \mathcal{R} \cap \Theta_m$ for each $m \in M$. Notice that $(\mathcal{R}, \pi_{\mathcal{R}}, \pi(\mathcal{R}))$ is a sheaf with the induced subspace topologies and the restricted projection.

4. Let $\mathcal{R}$ be a subsheaf of $\Theta$, let $\mathcal{G}_m = \Theta_m/\mathcal{R}_m$ be the quotient $\mathbb{R}$-module for each $m \in M$. The set $\Theta/\mathcal{R} := \mathcal{G} = \bigcup_{m \in M} \mathcal{G}_m$ is the quotient sheaf of $\Theta$ modulo $\mathcal{R}$, with $\mathcal{G}$ a sheaf over $M$ and the projection map $\tau : \Theta \rightarrow \mathcal{G}$ a sheaf homomorphism.

**Remark 5.1.1.** In the smooth case, we have considered the following for the definition above:

1. replace "continuous map" by "smooth map", indeed the latter is continuous,

2. the trace topology coincides with the $F$-subspace topology on an open set,

3. a homomorphism is to be seen as a smooth map, so an isomorphism is a diffeomorphism,

4. the quotient topology coincides with the $F$-quotient topology,

5. for more details on the above remarks (see [10, 11]).

### 5.1.2 Presheaves

Notice that $\text{TopM}$ will represent the category of open subsets of $(M, \tau_{F_M})$ with inclusions maps between objects while $\mathcal{M}$ will represent the category of $\mathbb{R}$-modules (real linear spaces).

**Definition 5.1.4.** A presheaf of $\mathbb{R}$-modules ($\mathbb{R}$-vector spaces) over $M$ is a system of pairs $S = \{S(U), \rho_{U,V}\}$, where $U, V \in \text{TopM}$, $U \subset V$, $S(U) \in \mathcal{M}$, and $\rho_{U,V} \in \text{Hom}_{\mathcal{M}}(S(V), S(U))$, such that $S(\emptyset) = \{0\}$ the trivial $\mathbb{R}$-module, $\rho_{U,U}$ is the identity $id_{S(U)}$ and whenever $U \subset V \subset W$ then $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$.

Categorically speaking, a presheaf is a contravariant functor $S : \text{TopM} \rightarrow \mathcal{M}$, such that $S(\iota_{U,V}) = \rho_{U,V}$ where $\iota_{U,V} : U \hookrightarrow V$ is the inclusion map.
Definition 5.1.5. Let $S = \{S(U), \rho_{U,V}\}$, $S' = \{S'(U), \rho_{U,V}\}$ be presheaves on $M$. A presheaf homomorphism of $S$ to $S'$ is a family $(\varphi_U : S(U) \rightarrow S'(U))_{U \in \text{TopM}}$ of $\mathbb{R}$-module homomorphisms satisfying $\rho'_{U,V} \circ \varphi_V = \varphi_U \circ \rho_{U,V}$ for $U \subset V$.

5.1.3 Relationship between sheaves and presheaves

Definition 5.1.6. Let $S = \{S(U), \rho_{U,V}\}$ be a presheaf on $M$ and let $U = \bigcup_{k \in I} U_k$. The presheaf $S$ is complete if it satisfies the following conditions:

1. Suppose that $a, b \in S(U)$ and $\rho_{U_k,U}(a) = \rho_{U_k,U}(b)$ for all $k \in I$, then $a = b$.

2. Suppose that $a_k \in S(U_k)$ for each $k \in I$ and $\rho_{U_k\cap U_j,U_k}(a_k) = \rho_{U_k\cap U_j,U_j}(a_j)$ for all $k, j \in I$ then there exists $a \in S(U)$ such that $a_k = \rho_{U_k,U}(a)$ for each $k \in I$.

From the definition of a presheaf it can be shown that each sheaf $(\Theta, \pi, M)$ induces a natural presheaf $\{\Gamma(\Theta, U), \rho_{U,V}\}$ such that $\Gamma(\Theta, U)$ is a $\mathbb{R}$-module of sections of $\Theta$ over $U$, where for $U, V \in \text{TopM}$ with $\iota_{U,V} : U \rightarrow V$ one maps a section $s_V$ of $\Theta$ over $V$ to its restriction over $U$, that is, $s_U = s_V \circ \iota_{U,V} = s_{V|U}$. So, a map $\alpha : \Theta \rightarrow \Gamma(\Theta, U)$ from sheaves to presheaves is defined.

The converse $\beta$ maps each presheaf to its associated sheaf. This can be seen from the example of the presheaf $(\mathcal{F}_U = C^{\infty}(U, \mathbb{R}), \rho_{U,V})$ to $\mathcal{F}_M = C^{\infty}(M, \mathbb{R})$ its sheaf of germs of smooth functions on $M$. So, $\beta(\mathcal{F}_U) = \mathcal{F}_M$. There is a functorial behavior for $\alpha$ and $\beta$ as show in the lemma below.

Lemma 5.1.1. 1. If $\varphi : \Theta \rightarrow \Upsilon$ is a sheaf homomorphism such that $\Pi \circ \varphi = \pi$, then its induced presheaf homomorphism $\alpha(\Theta) = \Gamma(\Theta, U) \rightarrow \beta(\Upsilon) = \Gamma(\Upsilon, U)$ is defined by $s_U \rightarrow \varphi \circ s_V$.

2. If $(\varphi_U : S(U) \rightarrow S'(U))_{U \in \text{TopM}}$ is a presheaf homomorphism of $S$ to $S'$ then the induced sheaf homomorphism $\varphi : \beta(S) = \Theta \rightarrow \beta(S') = \Upsilon$ of the associated sheaves is defined by $\rho'_{m,U} \circ \varphi_U = \varphi \circ \rho_{m,U}$ for $U \in \text{TopM}$ and $m \in U$ such that $\rho_{m,U} : S(U) \rightarrow \Theta_m$ and $\rho'_{m,U} : S' \rightarrow \Upsilon_m$ are quotient maps (natural projections).

3. If $S = \{S(U), \rho_{U,V}\}$ is a complete presheaf on $M$, then $\alpha \beta(S)$ is canonically isomorphic to $S$. That is, a complete presheaf is a sheaf.

4. For any sheaf $P$, one has $\beta \alpha(P) = P$. 
5.1 Sheaves, Presheaves and Properties

Proof. From the first item, we have $\Pi \circ \varphi = \pi$, which is a local homeomorphism mapping stalks into stalks. It follows that $\Pi \circ \varphi \circ s_U = \pi \circ s_U$. Thus, $\Pi \circ \varphi \circ s_U = id_U$, since $s_U$ is a section of $\Theta$ over $U$. Finally, $\varphi \circ s_U$ is section of $\Upsilon$ over $U$.

The second item comes from the composition of well-defined maps that is $\rho_{U,V} \circ \varphi_U = \varphi \circ \rho_{U,V}$ where $\varphi : \Theta_m \to \Upsilon_m$. □

We consider the notions on tensor products known from the reader. Since, we are working with Hausdorff paracompact space, we should assume the existence of a partition of unity.

Definition 5.1.7. Let $\Theta$ be a sheaf over $M$, $\{U_k\}$ a finite open cover of $M$, and consider a family $\{\kappa_k\}$ of endomorphisms of the sheaf $\Theta$.

The support of $\kappa_k$, denoted $\text{supp}(\kappa_k)$, is the closure of $\{m|\kappa_k \Theta_m \neq 0\} \subset M$.

The family $\{\kappa_k\}$ is the partition of unity for $\Theta$ subordinate to the cover $\{U_k\}$ if:

1. $\text{supp}(\kappa_k) \subset \{U_k\}$.
2. $\sum_k \kappa_k = id_{\Theta_m}$
3. The sheaf $\Theta$ is fine if it admits a partition of unity $\{\kappa_k\}$ subordinate to the cover $\{U_k\}$.

The sheaf $F_M = C^\infty(M,\mathbb{R})$ of germs of smooth functions is fine. Since the endomorphisms $\tilde{\kappa}_k$ of presheaves $\{C^\infty(U,\mathbb{R}), \rho_{U,V}\}$ are given by $\tilde{\kappa}_k(f) = (\varphi_k)|_U f$ for $f \in C^\infty(U,\mathbb{R})$. The associated sheaf endomorphisms $\{\kappa_k\}$ of $C^\infty(M,\mathbb{R})$ form a partition of unity subordinate to the cover $U_k$ of $M$. We should point out the fact that a sheaf of $\mathbb{R}$-modules is torsionless because each stalk is torsionless as a real vector space.

Another type of sheaves is the one so called flabby (flasque) sheaf.

Definition 5.1.8. A sheaf $\mathcal{F}$ over $M$ is called flabby if for any inclusion of open sets $U \subset V \subset M$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. Equivalently, for all open sets $U \subset M$ the restriction map $\mathcal{F}(M) \to \mathcal{F}(U)$ is surjective.

Subsequently, each local section of $\mathcal{F}$ is extendable to a global section.

Definition 5.1.9.

1. A sequence of sheaves and their homomorphisms

$$\ldots \to \Theta_{k-1} \to \Theta_k \to \Theta_{k+1} \to \Theta_{k+2} \to \ldots$$

is said exact if the kernel of an arrow is the image of the previous. That is, for each $m \in M$, the sequence of $\mathbb{R}$-modules and homomorphisms

$$\ldots \to (\Theta_{k-1})_m \to (\Theta_k)_m \to (\Theta_{k+1})_m \to (\Theta_{k+2})_m \to \ldots$$
given by stalks over $m$ is exact.

2. Given the sequence of sheaves

$$0 \to \mathcal{R} \to \Theta \to \Upsilon \to 0,$$

where $0$ is the constant sheaf with stalk $0_m$ over $m$ is the trivial $\mathbb{R}$-module, $\mathcal{R}$ is a subsheaf of $\Theta$ and $\Upsilon = \Theta / \mathcal{R}$. The sequence above is exact.

3. A short exact sequence is an exact sequence with only five terms such that the source and the tail of the sequence are Trivial (0 sheaves).

**Definition 5.1.10.** Let $\Theta$ and $\Upsilon$ be sheaves over $M$. Let $\alpha$ and $\beta$ as in the text below Definition 5.1.6. The tensor product of $\Upsilon$ and $\Theta$ is the sheaf given by the formula $\Upsilon \otimes \Theta = \beta(\alpha(\Upsilon)) \otimes \alpha(\Theta)$, that is, the associated sheaf of the tensor product of presheaves of section of $\Upsilon$ and $\Theta$.

Below we are giving some properties of fine torsionless sheaf and the tensor product.

**Lemma 5.1.2.** Let $\Theta$ and $\Upsilon$ be sheaves over $M$. Let $\varphi : \Theta \to \Upsilon$ be a sheaf homomorphism. Let $\Gamma(\Theta)$ and $\Gamma(\Upsilon)$ be $\mathbb{R}$-modules of global sections.

1. If $\Theta$ is fine then $\Theta \otimes \Upsilon$ is indeed a fine sheaf.
2. The sheaf homomorphism $\varphi$ induces a $\mathbb{R}$-modules homomorphism $\Gamma(\Theta) \to \Gamma(\Upsilon)$ by composition of sections of $\Theta$ with $\varphi$.
3. If the sheaf homomorphism $\varphi$ is surjective such that $\ker(\varphi) = \mathcal{R}$ and $\mathcal{R}$ is a fine sheaf then the induced $\mathbb{R}$-modules homomorphism $\Gamma(\Theta) \to \Gamma(\Upsilon)$ is also surjective.
4. If $0 \to \Theta' \to \Theta \to \Theta'' \to 0$ is a short exact sequence then $0 \to \Gamma(\Theta') \to \Gamma(\Theta) \to \Gamma(\Theta'') \to 0$ is an exact sequence.

**Proposition 5.1.3.** Let $\mathbb{R}$ be taken as a principal ideal domain, $S$ be a $\mathbb{R}$-module and $\mathbb{R} = M \times \mathbb{R}$ be the constant sheaf. Let $\Theta$ and $\Upsilon$ be sheaves over $M$.

1. If $0 \to \Theta' \to \Theta \to \Theta'' \to 0$ and if either $\Theta''$ or $\Upsilon$ is torsionless then $0 \to \Theta' \otimes \Upsilon \to \Theta \otimes \Upsilon \to \Theta'' \otimes \Upsilon \to 0$ is an exact sequence.
2. Moreover, if either $\Theta'$ or $\Upsilon$ is fine then the sequence $0 \to \Gamma(\Theta' \otimes \Upsilon) \to \Gamma(\Theta \otimes \Upsilon) \to \Gamma(\Theta'' \otimes \Upsilon) \to 0$ is exact.
3. The tensor product of $\mathbb{R}$-modules $S$ and $\mathbb{R}$ is $S \otimes \mathbb{R} \cong S$.
4. The tensor product of $\Theta$ and $\mathbb{R}$ is $\Theta \otimes \mathbb{R} \cong \Theta$.

### 5.1.4  Cochain complex and sheaf cohomology

A cochain complex $C^*$ is a sequence of $\mathbb{K}$-modules and homomorphisms $d = d_k$ given by

$$\cdots \to C^{k-1} \to C^k \to C^{k+1} \cdots$$

such that (for all $k \geq 0$) at each stage the image of
a given homomorphism is contained in the kernel of the next. The homomorphism \( d : C^k \rightarrow C^{k+1} \) is the \( k \)th coboundary operator. The kernel \( Z^k(C^*) \) of \( d \) is the module of \( k \)th degree cocycles for the cochain complex \( C^* \) and the image \( B^k(C^*) \) is the module of \( k \)th degree coboundaries. It follows that the \( k \)th sheaf cohomology module is the quotient module \( H^k(C^*) = Z^k(C^*)/B^k(C^*) \), since \( B^k(C^*) \subset Z^k(C^*) \) is an inclusion of submodules of \( C^k \). We shall notice this principle in classical cohomology theories.

A cochain map \( C^* \rightarrow D^* \) is a family of homomorphisms \( C^k \rightarrow D^k \) defined such that \( C^k \rightarrow C^{k+1} \rightarrow D^{k+1} = C^k \rightarrow D^k \rightarrow D^{k+1} \). That is, the equality of compositions or commutativity of diagram. It follows that \( k \)-cocycles of \( C^* \) are sent into \( k \)-cocycles of \( D^* \) and \( k \)-coboundaries of \( C^* \) are sent into \( k \)-coboundaries of \( D^* \).

A cochain map \( C^* \rightarrow D^* \) induces a homomorphism of cohomology modules \( H^k(C^*) \rightarrow H^k(D^*) \). It follows that the composition of cochain maps induces the composition of the homomorphisms of cohomology modules.

A sequence of cochain maps \( 0 \rightarrow C^* \rightarrow D^* \rightarrow E^* \rightarrow 0 \) forms a short exact sequence if for each \( k \) the sequence of \( \mathbb{K} \)-modules \( 0 \rightarrow C^k \rightarrow D^k \rightarrow E^k \rightarrow 0 \) is short exact.

A homomorphism of short exact sequences \( 0 \rightarrow C^* \rightarrow D^* \rightarrow E^* \rightarrow 0 \) and \( 0 \rightarrow \tilde{C}^* \rightarrow \tilde{D}^* \rightarrow \tilde{E}^* \rightarrow 0 \) of cochain complexes is a family of cochain maps \( C^* \rightarrow \tilde{C}^* \), \( D^* \rightarrow \tilde{D}^* \) and \( E^* \rightarrow \tilde{E}^* \) such that all the maps above form a commutative diagram.

**Proposition 5.1.4.** Let \( 0 \rightarrow C^* \rightarrow D^* \rightarrow E^* \rightarrow 0 \) and \( 0 \rightarrow \tilde{C}^* \rightarrow \tilde{D}^* \rightarrow \tilde{E}^* \rightarrow 0 \) be two short exact sequences of cochains maps. Then

1. there exist homomorphisms \( H^k(E^*) \xrightarrow{\partial} H^{k+1}(C^*) \) for each \( k \), such that

2. there exist a long exact sequence

\[
\cdots \rightarrow H^{k-1}(E^*) \xrightarrow{\partial} H^k(C^*) \xrightarrow{\partial} H^k(D^*) \xrightarrow{\partial} H^k(E^*) \xrightarrow{\partial} H^{k+1}(C^*) \xrightarrow{\partial} \cdots,
\]

3. and for any homomorphism between two given short exact sequences of cochain maps, the following hold:

\[
H^k(E^*) \xrightarrow{\partial} H^{k+1}(C^*) \xrightarrow{\partial} H^{k+1}(\tilde{C}^*) = H^k(E^*) \xrightarrow{\partial} H^k(\tilde{C}^*) \xrightarrow{\partial} H^{k+1}(\tilde{C}^*)
\]

**Lemma 5.1.5.** Let \( \Theta, \Lambda \) and \( \Upsilon \) be \( \mathbb{K} \)-modules on a principal ideal domain. The following holds.

1. The sequence, \( 0 \rightarrow \Theta \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} \Upsilon \rightarrow 0 \) of \( \mathbb{K} \)-modules and homomorphisms, is short exact if and only if \( \alpha \) is injective, \( \beta \) is surjective, \( \text{Ker}(\beta) = \text{Im}(\alpha) \) and \( \Upsilon \cong \Lambda/\text{Im}(\alpha) \),

2. If the sequence of \( \mathbb{R} \)-vector spaces and homomorphisms, \( 0 \rightarrow \Theta \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} \Upsilon \rightarrow 0 \), is short exact, both \( \text{dim}(\Theta) < \infty \) and \( \text{dim}(\Upsilon) < \infty \) then \( \text{dim}(\Lambda) < \infty \) and \( \Lambda \cong \Theta \oplus \Upsilon \),
3. The sequence, \(0 \to \Theta \xrightarrow{\alpha} \Lambda \to 0\) of \(\mathbb{K}\)-modules and homomorphisms, is exact if and only if \(\alpha\) is an isomorphism,

4. The sequence, \(0 \xrightarrow{\alpha} \Theta \xrightarrow{\beta} \Lambda\) of \(\mathbb{K}\)-modules and homomorphisms, is exact if and only if \(\text{Ker}(\alpha) = \{0\}\), that is \(\alpha\) is injective,

5. The sequence, \(\Theta \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} 0\) of \(\mathbb{K}\)-modules and homomorphisms, is short exact if and only if \(\text{Im}(\alpha) = \Lambda\).

**Definition 5.1.11.** [53] A short exact sequence \(0 \to \Theta \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} \Upsilon \to 0\) of \(\mathbb{K}\)-modules and homomorphisms, with \(\mathbb{K}\) a commutative ring, is called split when there is \(\mathbb{K}\)-isomorphism \(\theta : \Lambda \to \Theta \oplus \Upsilon\) of \(\mathbb{K}\)-modules such that the following diagram commutes:

\[
\begin{array}{ccc}
0 & \to & \Theta \\
\downarrow{id_{\Theta}} & & \downarrow{\theta} \\
0 & \to & \Theta \oplus \Upsilon
\end{array}
\]

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{\beta} & \Upsilon \\
\downarrow{\theta} & & \downarrow{id_{\Upsilon}} \\
\Theta \oplus \Upsilon & \xrightarrow{id_{\Lambda}} & \Upsilon \\
\end{array}
\]

**Theorem 5.1.6.** Let \(0 \to \Theta \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} \Upsilon \to 0\) be a short exact sequence of \(\mathbb{K}\)-modules. Then, the following statement are equivalent

1. The short exact sequence splits,
2. There exists \(\bar{\alpha} \in \text{Hom}_{\mathbb{K}}(\Lambda, \Theta)\) such that \(\bar{\alpha} \circ \alpha = id_{\Theta}\),
3. There exists \(\bar{\beta} \in \text{Hom}_{\mathbb{K}}(\Upsilon, \Lambda)\) such that \(\beta \circ \bar{\beta} = id_{\Upsilon}\)

### 5.1.5 Fine torsionless resolution and cohomology

**Definition 5.1.12.** 1. The exact sheaf sequence \(0 \to \Theta \to \Upsilon_0 \to \Upsilon_1 \to \Upsilon_2 \to \cdots\) is said to be a resolution of the sheaf \(\Theta\). A resolution is fine if each \(\Upsilon_i\) is a fine sheaf. A resolution is torsionless if each \(\Upsilon_i\) is a torsionless sheaf.

2. Given the resolution above and any sheaf \(\Lambda\) the associated cochain complex \(0 \to \Gamma(\Upsilon_0 \otimes \Lambda) \to \Gamma(\Upsilon_1 \otimes \Lambda) \to \Gamma(\Upsilon_2 \otimes \Lambda) \to \cdots\) is an exact sheaf sequence and shall be denoted by \(\Gamma(\Upsilon^* \otimes \Lambda)\). Where, for \(k \geq 0\), the module of \(k\)-cochains is \(\Gamma(\Upsilon_k \otimes \Lambda)\), whereas for \(k < 0\) the module of \(k\)-cochain is the zero module.

Notice that a sheaf homomorphism \(\Lambda \to \Lambda'\) can be tensored with the identity homomorphisms of the sheaves \(\Upsilon_i\) to induce two new homomorphisms \(\Upsilon_i \otimes \Lambda \to \Upsilon_i \otimes \Lambda'\) and \(\Gamma(\Upsilon_i \otimes \Lambda) \to \Gamma(\Upsilon_i \otimes \Lambda')\) which commute with the coboundary homomorphisms of the respective cochain complexes. Therefore, they give rise to a cochain map \(\Gamma(\Upsilon^* \otimes \Lambda) \to \Gamma(\Upsilon^* \otimes \Lambda')\).
An axiomatic sheaf cohomology theory $\mathcal{K}$ is defined in the literature in a way that each $k^{th}$ cohomology module of $M$ with coefficients in the sheaf $\Lambda$ is built with regard to the theory $\mathcal{K}$.

**Theorem 5.1.7.** Given the constant sheaf $\mathcal{G} = M \times G$, where $G$ is an arbitrary $\mathbb{R}$-module, ($\mathcal{R} = M \times \mathbb{R}$ in particular). Each fine torsionless resolution of the constant sheaf $\mathcal{G} = M \times G$ ($\mathcal{R}$ in particular) induces naturally a cohomology theory for $M$ with coefficients in sheaves of $\mathbb{R}$-modules over $M$.

From now on, we should consider fine torsionless resolution of the constant sheaf $0 \to \mathcal{R} \to \mathcal{Y}_0 \to \mathcal{Y}_1 \to \mathcal{Y}_2 \to \cdots$ with regard to the previous theorem. The context should determine the use of $\mathcal{G} = M \times G$ or $\mathcal{R}$.

**Remark 5.1.2.** The sheaf cohomology theory $\mathcal{K}$ is obtained in the following four steps:

1. For each integer $k$, one sets $\mathcal{H}^k(M, \Lambda) = \mathcal{H}^k(\Gamma(\mathcal{Y}^* \otimes \Lambda))$, that is, one associates the $k^{th}$ cohomology module of $M$ with coefficients in the sheaf $\Lambda$ to the $k^{th}$ cohomology module of the cochain complex $\Gamma(\mathcal{Y}^* \otimes \Lambda)$.

2. For each integer $k$ and for each sheaf homomorphism $\Lambda \to \Lambda'$, there exists the cochain map $\Gamma(\mathcal{Y}^* \otimes \Lambda) \to \Gamma(\mathcal{Y}^* \otimes \Lambda')$ which induces $\mathcal{H}^k(M, \Lambda) \to \mathcal{H}^k(M, \Lambda')$.

3. Given a short exact sheaf sequence $0 \to \Lambda' \to \Lambda \to \Lambda'' \to 0$ and fine torsionless sheaves $\mathcal{Y}_i$, there exist a short exact sequence of cochain maps $0 \to \Gamma(\mathcal{Y}^* \otimes \Lambda') \to \Gamma(\mathcal{Y}^* \otimes \Lambda) \to \Gamma(\mathcal{Y}^* \otimes \Lambda'') \to 0$ and its associated homomorphism $\mathcal{H}^k(\Gamma(\mathcal{Y}^* \otimes \Lambda'')) \to \mathcal{H}^{k+1}(\Gamma(\mathcal{Y}^* \otimes \Lambda'))$, that is, $\mathcal{H}^k(M, \Lambda'') \to \mathcal{H}^{k+1}(M, \Lambda')$, for each integer $k$ as by step 1. above.

4. Any sheaf on a Hausdorff paracompact space is fine.

**Lemma 5.1.8.** Let $\mathcal{F}$ be an arbitrary sheaf over $M$. Then

1. If $\mathcal{F}$ is a fine sheaf then the $k^{th}$ sheaf cohomology $\mathcal{H}^k(M, \mathcal{F}) = \{0\}$.

2. If $\mathcal{F}$ is a sheaf of $\mathbb{K}$-modules then $\mathcal{F}$ is a fine sheaf.

3. If $\mathcal{F}$ is a flabby sheaf then the $k^{th}$ sheaf cohomology $\mathcal{H}^k(M, \mathcal{F}) = \{0\}$ for all $k > 0$.

4. There exist flabby resolutions for any sheaf $\mathcal{F}$

**Definition 5.1.13.** Let $\mathcal{K}$ and $\tilde{\mathcal{K}}$ be two sheaf cohomology theories on $M$ with coefficients in sheaves of $\mathbb{R}$-modules over $M$.

A homomorphism $\mathcal{K} \to \tilde{\mathcal{K}}$ of the sheaf cohomology theories is defined as follows:

1. An homomorphism $\mathcal{H}^k(M, \Lambda) \to \tilde{\mathcal{H}}^k(M, \Lambda)$ for each integer $k$ and for each sheaf $\Lambda$ such that;
2. For $k = 0$, $\mathcal{H}^k(M, \Lambda) \cong \Gamma(\Lambda) \rightarrow \Gamma(\Lambda) = \mathcal{H}^k(M, \Lambda) \rightarrow \widetilde{\mathcal{H}}^k(M, \Lambda) \cong \Gamma(\Lambda)$, that is, the equality of composition maps, say, the commutativity of diagram;

3. For each homomorphism $\Lambda \rightarrow \Lambda''$ and each integer $k$ the following holds:

$\mathcal{H}^k(M, \Lambda) \rightarrow \mathcal{H}^k(M, \Lambda'') \rightarrow \widetilde{\mathcal{H}}^k(M, \Lambda) \rightarrow \widetilde{\mathcal{H}}^k(M, \Lambda'')$.

4. For each short exact sequence of sheaves $0 \rightarrow \Lambda' \rightarrow \Lambda \rightarrow \Lambda'' \rightarrow 0$ and for each integer $k$, the following holds:

$\mathcal{H}^k(M, \Lambda'') \rightarrow \mathcal{H}^{k+1}(M, \Lambda') \rightarrow \widetilde{\mathcal{H}}^{k+1}(M, \Lambda') = \mathcal{H}^k(M, \Lambda'') \rightarrow \widetilde{\mathcal{H}}^k(M, \Lambda'') \rightarrow \widetilde{\mathcal{H}}^{k+1}(M, \Lambda')$.

Naturally, a homomorphism of sheaf cohomology theories is an isomorphism if for each integer $k$ and for each sheaf $\Lambda$ the homomorphisms $\mathcal{H}^k(M, \Lambda) \rightarrow \widetilde{\mathcal{H}}^k(M, \Lambda)$ are isomorphisms.

Therefore, if the homomorphisms $\mathcal{H}^k(M, \Lambda) \rightarrow \mathcal{H}^k(M, \Lambda)$ are identity homomorphisms for all integers $k$ then $\mathcal{K} \rightarrow \widetilde{\mathcal{K}}$ is the identity homomorphism.

**Theorem 5.1.9.** Let $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ be two sheaf cohomology theories on $M$ with coefficients in sheaves of $\mathbb{R}$-modules over $M$. Then the homomorphism $\mathcal{K} \rightarrow \widetilde{\mathcal{K}}$ defined above is unique.

**Corollary 5.1.10.** Let $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ be two sheaf cohomology theories on $M$ with coefficients in sheaves of $\mathbb{R}$-modules over $M$. Then the homomorphism $\mathcal{K} \rightarrow \widetilde{\mathcal{K}}$ defined above is an isomorphism. Therefore, any two sheaf cohomology theories on $M$ with coefficients in sheaves of $\mathbb{R}$-modules over $M$ are uniquely isomorphic.

**Theorem 5.1.11.** Let $\mathcal{K}$ be a sheaf cohomology theory for $M$ with coefficients in sheaves of $\mathbb{R}$-modules over $M$. Let a fine resolution of the sheaf $\Theta$ given by the exact sheaf sequence $0 \rightarrow \Theta \rightarrow \Upsilon_0 \rightarrow \Upsilon_1 \rightarrow \Upsilon_2 \rightarrow \cdots$.

Then, for all $k$ there are canonical isomorphisms $\mathcal{H}^k(M, \Theta) \cong \mathcal{H}^k(\Gamma(\Upsilon^*))$.

After the presentation of a summary of the axiomatic sheaf cohomology theory for sheaves of $\mathbb{R}$-modules, we should now study four classical cohomology theories: (Alexander-Spanier, de Rham, Čech and Singular cohomologies) versus the corresponding sheaf cohomology theories on $\mathbb{F}$-spaces.

Recall that $\mathbb{F}$-spaces are Hausdorff paracompact, the minimum requirement for building sheaf theory. Also recall that all sheaf cohomology theories are uniquely isomorphic. We are referring the reader to constructions used in Frank W. Warner ([114]). The main technique lies on the construction of a fine resolution of the constant sheaf $\mathcal{R} = M \times \mathbb{R}$ of $\mathbb{R}$-module which is actually a real linear space. Naturally, this resolution is torsionless and then this should canonically induces a cohomology theory for the $\mathbb{F}$-space $M$ with coefficients in sheaves of $\mathbb{R}$-linear spaces over $M$. 

5.2 Alexander-Spanier cohomology

5.2.1 Coboundary operator and cochain complex

We are now considering the Alexander-Spanier sheaf cohomology. Let $U \in \tau_{\mathcal{F}_M}$, and $\mathbb{R}$ taken as a principal ideal domain. Let $U^{k+1}$ be the fold Cartesian product with $k+1$ factors $U$. Let $A^k(U, \mathbb{R})$ be the $\mathbb{R}$-module of functions $f : U^{k+1} \rightarrow \mathbb{R}$, that is, a real vector space for pointwise operations.

The coboundary map $d : U^k \rightarrow U^{k+1}$ is defined by

$$df(x_0, x_1, \ldots, x_i, \ldots, x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i f(x_0, x_1, \ldots, \tilde{x}_i, \ldots, x_{k+1}),$$

for each $k \geq 0$, $f \in A^k(U, \mathbb{R})$ and $(x_0, x_1, \ldots, x_i, \ldots, x_{k+1}) \in U^{k+2}$, where $\tilde{x}_i$ indicates the omission of the entry.

$d$ increases by 1 the number of factors in the fold product and so $d^2 := d \circ d = 0$. Furthermore, this induces a cochain complex denoted by $A^*(U, \mathbb{R})$, and defined by the sequence of $\mathbb{R}$-module homomorphisms:

$$\cdots \rightarrow 0 \hookrightarrow A^0(U, \mathbb{R}) \xrightarrow{d} A^1(U, \mathbb{R}) \xrightarrow{d} A^2(U, \mathbb{R}) \xrightarrow{d} \cdots,$$

where the modules of $k$-cochains are all zero modules for $k < 0$. We are entitled now to discuss the construction of a presheaf.

5.2.2 Presheaf and associated sheaf

Let $U, V \in \text{TopM}$, $U \subset V$, where $\text{TopM}$ is the category of $\tau_{\mathcal{F}_M}$-open sets in $M$ with the inclusions maps as homomorphisms. It follows that the fold Cartesian products satisfy $U^{k+1} \subset V^{k+1}$ and we can define $\rho_{U,V} \in \text{Hom}(A^k(V, \mathbb{R}), A^k(U, \mathbb{R}))$ as the corresponding restriction homomorphisms of $\mathbb{R}$-modules. So the family $\{A^k(U, \mathbb{R}); \rho_{U,V}\}$ forms a presheaf of $\mathbb{R}$-modules on $M$ and satisfies Item 2, for $k \geq 0$, but does not satisfy Item 1 for $k \geq 1$ with regard to Definition 5.1.6. This is the presheaf of Alexander-Spanier $k$-cochains. It forms an exact sequence of presheaves. So, its associated sheaf of germs of Alexander-Spanier $k$-cochains is denoted by $\mathcal{A}^k(M, \mathbb{R})$, and it is endowed with the induced coboundary operator:

$$d : \mathcal{A}^k(M, \mathbb{R}) \rightarrow \mathcal{A}^{k+1}(M, \mathbb{R})$$

for each $k \geq 0$.

5.2.3 Fine torsionless resolution and sheaf Alexander-Spanier cohomology

It follows a fine torsionless resolution of the constant sheaf $\mathcal{R} = M \times \mathbb{R}$:

$$0 \rightarrow \mathcal{R} \hookrightarrow \mathcal{A}^0(M, \mathbb{R}) \xrightarrow{d} \mathcal{A}^1(M, \mathbb{R}) \xrightarrow{d} \mathcal{A}^2(M, \mathbb{R}) \xrightarrow{d} \mathcal{A}^3(M, \mathbb{R}) \xrightarrow{d} \cdots$$

(5.2)
Indeed, the above sheaf sequence is exact as a consequence of exactness of the former presheaf sequence. Hence, it is a resolution of the constant sheaf $\mathcal{R}$. The resolution is torsionless since the field $\mathbb{R}$ is an integral domain and each element of $\mathcal{A}^k(M, \mathbb{R})$ is an equivalence class of functions with values in $\mathbb{R}$. Finally, $M$ is Hausdorff paracompact, thus, all sheaves $\mathcal{A}^k(M, \mathbb{R})$ admit a partition of unity. It follows that, sheaves $\mathcal{A}^k(M, \mathbb{R})$ are fine, that is, the resolution above is fine. The fine torsionless resolution induces an Alexander-Spanier sheaf cohomology theory with regard to Theorem 5.1.7 in which we set

$$\mathcal{H}^k(M, \Lambda) = \mathcal{H}^k(\Gamma(\mathcal{A}^*(M, \mathbb{R}) \otimes \Lambda))$$

(5.3)

for each integer $k$ as in Item 1. of Remark 5.1.2. Finally, under the power of Corollary 5.1.10, we claim that any other sheaf cohomology theory for $M$ with coefficients in sheaves of $\mathbb{R}$-modules is uniquely isomorphic. So, when we set $\Lambda = \mathcal{G}$ or $\Lambda = \mathcal{R}$, it follows that

$$\mathcal{H}^k(M, \mathcal{G}) = \mathcal{H}^k(\Gamma(\mathcal{A}^*(M, \mathbb{R}))), \quad \mathcal{H}^k(M, \mathcal{R}) = \mathcal{H}^k(\Gamma(\mathcal{A}^*(M, \mathbb{R}))).$$

(5.4)

We have to show that the classical Alexander-Spanier cohomology modules of $M$ with coefficients in a $\mathbb{R}$-module $G$ are canonically isomorphic with the sheaf cohomology modules $\mathcal{H}^k(M, \mathcal{G})$ with coefficients in the constant sheaf $\mathcal{G} = M \times G$.

### 5.2.4 Classical Alexander-Spanier cohomology

Let us replace $\mathbb{R}$ by a general $\mathbb{R}$-module $G$ in all previous subsections in the current section. Therefore, let $\mathcal{A}^k(U, G)$ be the $\mathbb{R}$-module of functions $U^{k+1} \rightarrow G$. Consequently, replace $\mathcal{R}$ by $\mathcal{G}$. Now, we should define the coboundary operator $d$ by means of an equivalence relation with regard to the sub-module $\mathcal{A}_0^k(U, G) = \{f \in \mathcal{A}^k(U, G) : \rho_{m,M}(f) = 0, \text{ for all } m \in M\}$ of $\mathcal{A}^k(U, G)$. Hence, the homomorphism $\rho_{m,M} : \mathcal{A}^k(U, G) \rightarrow \mathcal{A}_m^k(M, G)$ assigns each $f$ to its equivalence class in the stalk over $m$ of the sheaf $\mathcal{A}^k(M, G)$, as in Lemma 5.1.1. This sheaf is associated to the presheaf $\{\mathcal{A}^k(U, G), \rho_{U,V}\}$. It follows that the coboundary operator $d : \mathcal{A}^k(M, \mathcal{G}) \rightarrow \mathcal{A}^{k+1}(M, \mathcal{G})$ restricted to $\mathcal{A}_0^k(U, G)$ sends $\mathcal{A}_0^k(U, G)$ into $\mathcal{A}_0^{k+1}(U, G)$. Thus, the module homomorphisms on quotients, that is, $\mathcal{A}^k(U, G)/\mathcal{A}_0^k(U, G) \rightarrow \mathcal{A}^{k+1}(U, G)/\mathcal{A}_0^{k+1}(U, G)$ can be defined. They yield in turn a sequence of modules and homomorphisms for $k \geq 0$ which forms a cochain complex $\mathcal{A}^*(U, G)/\mathcal{A}_0^*(U, G)$. As usual for $k < 0$ the modules of $k$-cochains are all the zero module. By definition, $\mathbb{H}^k_{A-S}(M, G)$, the classical Alexander-Spanier cohomology modules for $M$ with coefficients in the $\mathbb{R}$-module $G$ are given by:

$$\mathbb{H}^k_{A-S}(M, G) = \mathcal{H}^k(\mathcal{A}^*(M, G)/\mathcal{A}_0^*(M, G)).$$

(5.5)

As by the convention of replacement made at the beginning of the subsection, the following exact sequence:

$$0 \rightarrow \mathcal{G} \hookrightarrow \mathcal{A}^0(M, G) \xrightarrow{d} \mathcal{A}^1(M, G) \xrightarrow{d} \mathcal{A}^2(M, G) \xrightarrow{d} \mathcal{A}^3(M, G) \xrightarrow{d} \cdots$$

(5.6)
5.3 de Rham cohomology

of the constant sheaf $\mathcal{G} = M \times G$ is a fine resolution of the constant sheaf $\mathcal{G}$. If follows from Theorem 5.1.11 that there exist canonical isomorphisms

$$\mathcal{H}^k(M, \mathcal{G}) \cong \mathcal{H}^k(\Gamma(A^*(M, G))).$$  \hfill (5.7)

**Lemma 5.2.1.** Let \( \{A^k(U, G), \rho_U\} \) be a presheaf on \( M \) satisfying Item 2. of Definition 5.1.6. Let \( \mathcal{A}^k(M, G) \) be the associated sheaf. Let \( A^k_0(U, G) = \{ f \in A^k(U, G) : \rho_{m,M}(f) = 0 \text{ for all } m \in M \} \) as defined above. Then:

1. the sequence \( 0 \to A^k_0(M, G) \to A^k(M, G) \to \Gamma(\mathcal{A}^k(M, G)) \to 0 \) is exact,
2. the natural cochain map \( A^*(M, G)/A^*_0(M, G) \to \Gamma(\mathcal{A}^*(M, G)) \) is an isomorphism.

The Lemma above induces canonical isomorphisms

$$\mathbb{H}^k_{A-S}(M, G) \cong \mathcal{H}^k(M, \mathcal{G}).$$  \hfill (5.8)

Finally, with regard to Equations (5.4) and (5.8), we have got the isomorphism of sheaves we needed:

$$\mathbb{H}^k_{A-S}(M, \mathbb{R}) \cong \mathcal{H}^k(\Gamma(A^*(M, \mathbb{R}))) = \mathcal{H}^k(M, \mathbb{R}).$$  \hfill (5.9)

This ends the construction of an isomorphism between the sheaf Alexander-Spanier cohomology and the classical Alexander-Spanier cohomology.

### 5.3 de Rham cohomology

In this section we rely mainly on the material presented in Section 2.4 for the tangent structure and Section 2.5 for differential operators. In this Section, the reader is referred to the following references [47, 65].

#### 5.3.1 Differential forms and coboundary operator

**Definition 5.3.1.** Let \( M \) be a \( n \)-space. A \( k \)-form on \( M \) or a form of degree \( k \) is a section of the \( \mathbb{F} \)-bundle \( \bigwedge^k T^* M = \bigsqcup_{x \in M} \bigwedge^k T^*_x M \) with base space \( M \) and fibers \( \bigwedge^k T^*_x M \). The set \( \Omega^k(M) := (M, \bigwedge^k T^* M) = \{ k \text{-forms on } M \} \), is a module on the algebra \( \mathcal{F}_M \) and a linear space on \( \mathbb{R} \) (see Definition 2.5.1 for more details).
For $k = 0$, we have: $\bigwedge^0 T_x^* M = \mathbb{R}$, with $\Omega^0(M) = F_M$.

For $k = 1$, we have: $\bigwedge^1 T_x^* M = T_x^* M$, $\omega : T_x M \to \mathbb{R}$, with $\Omega^1(M) = (M, \bigwedge^1 T^* M)$.

For $k = 2$, we have: $\bigwedge^2 T_x^* M$, the set of all 2-linear alternating functions $\omega : T_x M \times T_x M \to \mathbb{R}$, with $\Omega^2(M) = (M, \bigwedge^2 T^* M)$. If $X_1, X_2, \ldots, X_k \in \mathfrak{X}(M)$, then $\omega(X_1, X_2, \ldots, X_k)(x) = \omega(x)(X_1(x), X_2(x), \ldots, X_k(x))$, where $\omega(x) := \omega_x$ is a smooth function for all $x \in M$.

That is, the $k$-form $\omega$ on $M$ is a collection of smoothly varying $k$-linear alternating maps $\omega_x \in \bigwedge^k T^* M$ (see [47]). As consequence of Cartesian closedness, completeness, one can conclude that $\bigwedge T^* M$ is a $\mathbb{F}$-space. Moreover, the sections, $k$-forms, of the $\mathbb{F}$-bundle $\bigwedge^k T^* M$ are smooth.

**Definition 5.3.2.** Let $M$ be a $n$-dimensional $\mathbb{F}$-space. Let $\alpha \in \bigwedge^k T_x^* M$ and $\beta \in \bigwedge^l T_x^* M$. The operator $\wedge$, called the exterior product (also wedge or Grassmann product), is a $\mathbb{F}$-smooth multilinear and alternating map. The exterior product of $\alpha$ and $\beta$ is the $(k+l)$-form $\alpha \wedge \beta : M \to \bigwedge^{k+l} T_x^* M$ (see Definition 2.5.2 for more details).

**Definition 5.3.3.** Let $M$ be a $n$-dimensional $\mathbb{F}$-space. The operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$, called the exterior derivative, is defined by:

1. $d : \bigwedge^k T_x^* M \to \bigwedge^{k+1} T_x^* M$ is a linear map that takes each $k$-form to a $(k+1)$-form, such that:

2. $df(Z) = Z(f)$ for $f \in \bigwedge^0 T_x^* M$, $df \in \bigwedge^1 T_x^* M$ and $Z \in \mathfrak{X}(M)$. That is, the usual differential. And for $\alpha$ a $k$-form we have,

3.

$$d(d\alpha) = 0, \text{ i.e. } d_{k+1} \circ d_k. \quad (5.10)$$

(see Definition 2.5.3 for more details)

For each $k$-form $\omega$ on $N$, with $(k > 0)$ the $\mathbb{F}$-smooth $\varphi^* \omega = \omega \circ \varphi_*$ induces a $k$-form on $M$, such that

$$\varphi^* \omega(v_1, v_2, \ldots, v_k) = \omega_{\varphi(x)}(\varphi_*(v_1), \varphi_*(v_2), \ldots, \varphi_*(v_k))$$
for \( v_1, v_2, \ldots, v_k \in T_x M \).

Note that

\[
\text{If } \alpha \in \Omega(M) \text{ and } (d\alpha) = 0 \text{ then } \alpha \text{ is called a closed form.} \quad (5.11)
\]

\[
\text{If } \alpha, \beta \in \Omega(M) \text{ and } \beta = d\alpha \text{ then } \beta \text{ is called exact form.} \quad (5.12)
\]

Each exact form is a closed form since \((d\beta) = d(d\alpha) = 0\).

\[
\text{We also have the following property mixing operators } d \text{ and } \wedge \text{ on a } k\text{-form and a } l\text{-form:}
\]

\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \text{ for } \alpha \text{ a } k\text{-form and } \beta \text{ a } l\text{-form.}
\]

**Theorem 5.3.1.** [53, 114] **Poincaré lemma:**

Let \( U \) be an open unit ball in Euclidean space \( \mathbb{R}^n \), let \( \Omega^k(M) \) be the space of differential \( k \)-forms on \( U \), and \( d \) the exterior derivative. Then, for each \( k \geq 1 \) there is a linear transformation \( h_k : \Omega^k(M) \rightarrow \Omega^{k-1}(M) \) such that \( h_{k+1} \circ d + d \circ h_k = id \).

**Corollary 5.3.2.** [53, 114] If \( \alpha \) is a \( k \)-form on the open ball in \( \mathbb{R}^n \), where \( k \geq 1 \), and \( d\alpha = 0 \), then there exists a \((k-1)\)-form \( \beta = h_k(\alpha) \) such that \( d\beta = \alpha \).

### 5.3.2 Presheaf and associated sheaf

Let \( M \) be a \( n \)-dimensional \( \mathbb{F} \)-space and \( U, V \in \text{TopM} \), with \( U \subset V \). It follows that \( U, V \) are open \( \mathbb{F} \)-subspaces of \( M \). Therefore, one may restrict \( k \)-forms to \( U \) by setting \( \Omega^k(U) := \Omega^k(M)|_U \) the \( \mathbb{R} \)-linear space of \( k \)-forms on \( U \). One can endow \( \Omega^k(U) \) with natural restriction homomorphisms \( \rho_{U,V} \) to define a complete presheaf

\[
\{\Omega^k(U), \rho_{U,V}\}. \quad (5.14)
\]

Recall that by Definition 5.3.3 we have \( d \circ d = 0 \). So does the restriction of \( d \) to \( U \). We should now define the presheaf homomorphisms

\[
\{\Omega^k(U), \rho_{U,V}\} \xrightarrow{d} \{\Omega^{k+1}(U), \rho_{U,V}\} \text{ for } k \geq 0. \quad (5.15)
\]

Its associated sheaf, with the induced sheaf homomorphism \( d \), is the sheaf of germs of \( k \)-forms, which we shall denote by \( \Omega^k(M) \). It is a sheaf of \( \mathbb{R} \)-linear spaces, thus they are torsionless sheaves for all \( k \geq 0 \). Since the \( \mathbb{F} \)-space \( M \) is paracompact by assumption then there exist partitions of unity. Therefore, they are all fine sheaves. We have to define a fine torsionless resolution of the constant sheaf \( \mathcal{R} \).
5.3 Fine torsionless resolution and sheaf de Rham cohomology

Let us consider the following cochain complex of fine and torsionless sheaves:

\[ 0 \to \mathcal{R} \hookrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \cdots \quad (5.16) \]

This is actually an exact sequence, since \( d \circ d = 0 \) and \( 0 \to \mathcal{R} \hookrightarrow \Omega^0(M) \) is a composition of one-to-one sheaf homomorphisms. In conclusion, one has a fine torsionless resolution of the constant sheaf \( \mathcal{R} \). Therefore, there exists a cohomology theory over \( M \) with coefficients in sheaves of \( \mathbb{R} \)-linear spaces, that is,

\[ H^k(M, \mathcal{R}) = H^k(\Gamma(\Omega^* \otimes \mathcal{R})) \cong H^k(\Gamma(\Omega^* (M))) \quad (5.17) \]

For each integer \( k \) as in Item 1. of Remark 5.1.2. Finally, under the power of Corollary 5.1.10, we claim that any other sheaf cohomology theory for \( M \) with coefficients in sheaves of \( \mathbb{R} \)-linear spaces is uniquely isomorphic.

We have to show that the classical de Rham cohomology for \( M \) is canonically isomorphic to the sheaf cohomology over \( M \) with coefficients in the constant sheaf \( \mathcal{R} = M \times \mathbb{R} \) of \( \mathbb{R} \)-linear spaces.

5.3.4 Classical de Rham cohomology

Let \( \Omega(M) = \{ \alpha \mid \alpha \text{ is a } k \text{-form, } k \geq 0 \} \). A cochain complex \( \Omega^*(M) \) consists of a sequence of \( \Omega^k(M) \) and homomorphisms \( d = d_k \) such that (for all \( k \geq 0 \)) at each stage the image of a given homomorphism is contained in the kernel of the next homomorphism.

The diagram below describes a cochain complex.

\[ \cdots \xrightarrow{d=d_{k-2}} \Omega^{k-1}(M) \xrightarrow{d=d_{k-1}} \Omega^k(M) \xrightarrow{d=d_k} \Omega^{k+1}(M) \xrightarrow{d=d_{k+1}} \cdots \]

\[ \cdots \xrightarrow{\alpha_{(k-1)}} \xrightarrow{\alpha_{k}} \xrightarrow{\alpha_{(k+1)}} \cdots \]

\( d = d_k \) is called the \( k \text{th} \) coboundary operator.

We have below a list of some objects used in the process of building de Rahm cohomology:

- \( \text{Im}(d_{k-1}) := B^k(M) \) is called the module of \( k \text{th} \) cocycles of the cochain, containing exact \( k \)-forms on \( M \).
5.3 de Rham cohomology

- $Ker(d_k) := Z^k(M)$ is called the module of $k^{th}$ coboundaries of the cochain, containing closed forms.

- $B^1(M) \subset Z^1(M) \subset \Omega^1(M)$.

- $B^k(M) \subset Z^k(M) \subset \Omega^k(M)$, for $k \geq 0$.

- If $\alpha$ and $\beta$ are two closed forms then $\alpha \wedge \beta$ is also a closed form with regard to $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for $\alpha$ a $k$-form and $\beta$ a $l$-form.

The above inclusions on $\mathbb{R}$-modules ($\mathbb{R}$-linear spaces) induce the quotient linear space called the $k^{th}$ de Rham Cohomology:

\[
\mathbb{H}^k_{dR}(M) := \frac{Z^k(M)}{B^k(M)} = \frac{Ker(d_k)}{Im(d_{k-1})} = \frac{(d_k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M))}{(d_{k-1} : \Omega^{k-1}(M) \longrightarrow \Omega^k(M))}, \quad \text{where} \quad (5.18) \\
\mathbb{H}^0_{dR}(M) = \{0\} \quad \text{if} \quad k \leq 0 \\
\mathbb{H}^k_{dR}(M) = Ker(d_0), \quad \text{if} \quad k = 0, \quad \text{where} \quad d_0 : \mathcal{F}_M \longrightarrow \Omega^1(M).
\]

We set $[\alpha]$ to be the cohomology class (equivalence class) of $\alpha$, so

\[
[\alpha \wedge \beta] := [\alpha] \wedge [\beta], \quad (5.19)
\]

and this does not depend on the choice of the representative. Therefore, we may extend the exterior product to the cohomology classes and define by this means the cohomology ring, as in the Subsection 5.6.1 on multiplicative structures. From now on and for seek of simplicity in notations we should denote a cohomology class without brackets, and the contest will prevail. Let us recall the cochain complex $\Omega^*(M)$:

\[
\ldots \longrightarrow \Omega^{k-1}(M) \xrightarrow{d=d_{k-2}} \Omega^k(M) \xrightarrow{d=d_{k-1}} \Omega^{k+1}(M) \xrightarrow{d=d_k} \Omega^{k+2}(M) \xrightarrow{d=d_{k+1}} \ldots \quad (5.20)
\]

whose presheaf $P = \{\Omega^k(U), \rho_{U,V}\}$, given in Equation (5.14), is complete. That is, $\alpha(\beta(P)) = P$, where $\alpha(\beta(P))$ is the presheaf of sections of $\beta(P)$, whereas the later is the associated sheaf of $P$. The aforementioned sequence canonically induces another cochain complex:

\[
\ldots \longrightarrow \Gamma(\Omega^{k-1}(M)) \xrightarrow{d=d_{k-2}} \Gamma(\Omega^k(M)) \xrightarrow{d=d_{k-1}} \Gamma(\Omega^{k+1}(M)) \xrightarrow{d=d_k} \Gamma(\Omega^{k+2}(M)) \xrightarrow{d=d_{k+1}} \ldots \quad (5.21)
\]

Accordingly, there exist, as in Subsection sec : sec14, natural homomorphisms $\Omega^k(M) \longrightarrow \Gamma(\Omega^k(M))$ that are bijections and commute with coboundaries operators in Equations (5.20) and (5.21). Therefore, we end with a cochain map $\Omega^*(M) \longrightarrow \Gamma(\Omega^*(M))$. Moreover, this is
an isomorphism. After all, we have obvious isomorphisms of following cohomology \( \mathbb{R} \)-linear spaces:

\[
H^k(\Omega^*(M)) \cong \mathcal{H}^k(\Gamma(\Omega^*(M)))
\]  

(5.22)

The left hand side in Equation (5.22) is the de Rham cohomology in Equation (5.18). Hence, Equations (5.17) and (5.22) give rise to the wanted isomorphisms

\[
\mathbb{H}^k_{dRH}(M) \cong H^k(M, \mathbb{R}).
\]  

(5.23)

5.4 Singular cohomology

This section is built with regard to following references: [53, 114].

5.4.1 Singular \( k \)-(co)chains and coboundary operator

Bearing in mind the same approach as in two previous sections, we have to exhibit a fine torsionless resolution of the constant sheaf \( \mathfrak{R} \). For, differential singular \( k \)-simplex concept is to be defined. Let \( k \) be a non negative integer. A standard \( k \)-simplex in \( \mathbb{R}^k \), is the set \( \Delta^k = \{(a_1, a_2, a_3, \cdots, a_k) \mid \sum_{i=1}^{k} a_i \leq 1, \text{ for all } a_i \geq 0 \} \subset \mathbb{R}^k \), with the convention that \( \Delta^0 = \{0\} \) is the standard 0-simplex. Let \( M \) be a locally Euclidean \( \mathbb{F} \)-space and \( U \subset M \) an open set. A differentiable singular \( k \)-simplex, denoted by \( \sigma : \Delta^k \to U \) is a smooth map from an open neighborhood of \( \Delta^k \) into \( U \). Let us consider \( \nabla_k(U) \) to be the set of all possible differentiable singular \( k \)-simplexes on \( U \), \( \sigma \), that is, \( \sigma \in \nabla_k(U) \). We then consider the set \( S_k(U) \) as the free abelian group with basis \( \nabla_k(U) \). Each element of the former group is a finite linear combination (sum) of differentiable \( k \)-simplexes and called differentiable singular \( k \)-chains with integers coefficients. In fact, \( \nabla_k(U) \) is a \( \mathbb{Z} \)-module, where \( \mathbb{Z} \) is the set of integers.

The boundary of a differentiable singular \( k \)-simplex \( \sigma \) in \( U \) is the differentiable singular \( (k-1) \)-chain, that is, a finite linear combination of differentiable singular \( (k-1) \)-simplexes. A differentiable singular \( (k-1) \)-simplex in \( U \), called the \( i \)th face of \( \sigma \) and also denoted by \( \sigma^i \in \nabla_{k-1}(U) \), is obtained by deleting the \( i \)th vertex in the differentiable singular \( k \)-simplex. Formally, we shall denote and define the boundary of the differentiable singular \( k \)-simplex \( \sigma \) by:

\[
\delta \sigma := \sum_{i=0}^{k} (-1)^i \sigma^i \quad \text{such that } \delta \circ \delta = 0. \quad \text{(see [114]).}
\]  

(5.24)

Since \( M \) is a locally Euclidean \( \mathbb{F} \)-space and the standard \( k \)-simplex is diffeomorphic to the \( k \)-closed unit ball in \( \mathbb{R}^{n+1} \), it follows the embedding of differentiable singular \( k \)-simplices and
differentiable singular $k$-chains, while the boundary operation and the summation commute. Therefore, the boundary $\delta$ extends linearly to a homomorphism of abelian groups, inducing a new boundary operator:

$$\partial : S_k(U) \to S_{k-1}(U) \text{ for each } k \geq 1, \text{ such that } \partial \circ \partial = 0. \quad (5.25)$$

The later boundary operator induces a chain complex of abelian groups called singular complex. We need to build dual concepts of differential singular $k$-cochains and cochain complex of $\mathbb{R}$-linear spaces. For, we let $S^k(U, \mathbb{R})$ be the $\mathbb{R}$-linear space of homomorphisms $f : S_k(U, \mathbb{R}) \to \mathbb{R}$, say the set of differentiable singular $k$-cochains.

### 5.4.2 Presheaf and associated sheaf

The restriction homomorphism $\rho_{U,V} : S^k(V, \mathbb{R}) \to S^k(U, \mathbb{R})$, with $U \subset V$, is mapping $f$ to its restriction on differentiable $k$-simplexes such that $\sigma(\Delta^k) \subset U$. It follows that

$$\{S^k(U, \mathbb{R}); \rho_{U,V}\}, \quad (5.26)$$

is the presheaf of differentiable singular $k$-cochains on $M$. As we saw in Alexander-Spanier case, we are dealing with presheaf of $\mathbb{R}$-modules (real linear spaces) on $M$ and it satisfies Item 2. for $k \geq 0$, but does not satisfy Item 1 for $k \geq 1$ with regard to Definition 5.1.6. The dual coboundary homomorphism (operator) with regard to Equations (5.24) and (5.25), may be defined as follows. For each $k \geq 1$,

$$d : S^k(U, \mathbb{R}) \to S^{k+1}(U, \mathbb{R}), \text{ such that } df(\sigma) := f(\partial \sigma), \text{ and } d \circ d = 0. \quad (5.27)$$

Moreover, $d$ commutes with restriction homomorphisms $\rho_{U,V}$, and so induces a presheaf homomorphism $\{S^k(U, \mathbb{R}); \rho_{U,V}\} \to \{S^{k+1}(U, \mathbb{R}); \rho_{U,V}\}$, which turns into the following cochain complex $S^*(U, \mathbb{R})$, with the $\mathbb{R}$-linear spaces $S^k(U, \mathbb{R}) = \{0\}$ for $k \leq 0$:

$$\cdots \to 0 \hookrightarrow S^0(U, \mathbb{R}) \xrightarrow{d} s^1(U, \mathbb{R}) \xrightarrow{d} s^2(U, \mathbb{R}) \xrightarrow{d} \cdots, \quad (5.28)$$

So, its associated sheaf of germs of differentiable singular $k$-cochains is denoted by $S^k(M, \mathbb{R})$, and it is endowed with the induced coboundary operator (a sheaf homomorphism): $d : S^k(M, \mathbb{R}) \to S^{k+1}(M, \mathbb{R})$ for each $k \geq 0$. Particularly, $S^0(M, \mathbb{R})$ is the sheaf of germs of functions on $M$ with values in $\mathbb{R}$. Hence, the constant sheaf $\mathfrak{R} = M \times \mathbb{R}$ of $\mathbb{R}$-module which is actually a real linear space is naturally embedded into $S^0(M, \mathbb{R})$, if we map $k \in \mathfrak{R}_p$ to the germ at $p$ of the function on $M$ and taking the constant value $k$. 
5.4.3 Fine torsionless resolution and sheaf singular cohomology

It follows a fine torsionless resolution of the constant sheaf \( \mathcal{R} = M \times \mathbb{R} \):

\[
0 \to \mathcal{R} \to S^0(M, \mathbb{R}) \xrightarrow{d} S^1(M, \mathbb{R}) \xrightarrow{d} S^2(M, \mathbb{R}) \xrightarrow{d} S^3(M, \mathbb{R}) \xrightarrow{d} \cdots
\]  

(5.29)

Indeed, this sequence is exact. Since \( d \circ d = 0 \) implies that the kernel contains the image of \( d \). It remains to show that the image contains the kernel of \( d \). Recall that \( M \) is a locally Euclidean \( \mathbb{F} \)-space and the standard \( k \)-simplex is diffeomorphic to the \( k \)-closed unit ball. Let \( U \) be an open unit ball diffeomorphic to an open unit ball in \( \mathbb{R}^{\dim M} \). Then, by Poincaré Lemma it can be proven that if \( f \) is a differentiable singular \( k \)-cochain on the open unit ball \( U \) such that for \( k \geq 1 \), \( df = 0 \), that is, \( f \in \text{Ker}(d) \) or \( f \) is a cocycle, then there exists a differentiable singular \( (k-1) \)-cochain \( g \) such that \( f = dg \), that is, \( f \in \text{Im}(d) \) or \( f \) is a coboundary (see [114]). Naturally, this is a torsionless resolution since we deal here with \( \mathbb{R} \)-linear spaces. Finally, this is a fine resolution since \( M \) is paracompact Hausdorff space.

The fine torsionless resolution of \( \mathcal{R} \) should canonically induce a singular sheaf cohomology theory for the \( \mathbb{F} \)-space \( M \) with coefficients in sheaves of \( \mathbb{R} \)-linear spaces with regard to Theorem 5.1.7 in which we set

\[
\mathcal{H}^k(M, \Lambda) = \mathcal{H}^k(\Gamma(S^*(M, \mathbb{R}) \otimes \Lambda))
\]  

(5.30)

for each integer \( k \) as in Item 1. of Remark 5.1.2. Finally, under the power of Corollary 5.1.10, we claim that any other sheaf cohomology theory for \( M \) with coefficients in sheaves of \( \mathbb{R} \)-modules is uniquely isomorphic. So, when we set \( \Lambda = \mathcal{R} \), it follows that

\[
\mathcal{H}^k(M, \mathcal{R}) = \mathcal{H}^k(\Gamma(S^*(M, \mathbb{R}))).
\]  

(5.31)

We have to show that the classical singular cohomology modules of \( M \) with coefficients in a \( \mathbb{R} \)-module \( G \) are canonically isomorphic with the sheaf cohomology modules \( \mathcal{H}^k(M, G) \) with coefficients in the constant sheaf \( G = M \times G \). After what, we should set \( G = \mathbb{R} \) and \( \mathcal{G} = \mathcal{R} \).

5.4.4 Classical singular cohomology

Let us replace \( \mathbb{R} \) by a general \( \mathbb{R} \)-module \( G \) in all previous subsections in the current section, mainly in all constructs from Equation (5.24) to Equation (5.31). Therefore, let \( S^k(U, G) \) be the \( \mathbb{R} \)-module of functions which map each differentiable singular \( k \)-simplex in \( U \) to an element in \( G \). That is, we let \( S^k(U, G) \) be the \( \mathbb{R} \)-linear space of homomorphisms \( f : S_k(U, G) \to G \), say the set of differentiable singular \( k \)-cochains. Consequently, one should read \( \mathcal{G} = M \times G \) where \( \mathcal{R} \) appears in the aforementioned Equations. Recall that the cochain complex of presheaves \( \{S^k(U, G); \rho_{UV}\} \) should be denoted by \( S^*(U, G) \). By definition,

\[
\mathbb{H}^k_{\Delta_\infty}(M, G) = \mathcal{H}^k(S^*(M, G)),
\]  

(5.32)
5.5 Čech cohomology

will represent the classical singular cohomology groups of $M$ with coefficients in the $\mathbb{R}$-module $G$, where $G = M \times G$, $S^\ast(M, G)$ is the cochain complex of sheaves associated to the presheaves above. Similarly to Alexander-Spanier constructions, the Lemma 5.2.1 above induces the exactness of the short sequence of cochains complexes:

$$0 \to S^\ast_0(M, G) \to S^\ast(M, G) \to \Gamma(S^\ast(M, G)) \to 0.$$ (5.33)

From Item 2. of Proposition 5.1.4 there exists a long exact sequence

$$\cdots \to \mathcal{H}^k(S^\ast_0(M, G)) \to \mathcal{H}^k(S^\ast(M, G)) \to \mathcal{H}^k(\Gamma(S^\ast(M, G))) \to \cdots$$ (5.34)

By using some homotopy techniques (see [114]), it can be proven that $\mathcal{H}^k(S^\ast_0(M, G)) = \{0\}$, for all $k$. Thus, the exactness of Equation (5.34), induces canonical isomorphisms

$$\mathcal{H}^k(S^\ast(M, G)) \cong \mathcal{H}^k(\Gamma(S^\ast(M, G))).$$ (5.35)

It follows, with regard to Equations (5.32) and (5.35) that

$$\mathbb{H}^k_{\Delta\infty}(M, G) \cong \mathcal{H}^k(\Gamma(S^\ast(M, G))).$$ (5.36)

Now, in Equation (5.31), if we replace $\mathbb{R}$ by $G$ and $\mathfrak{R}$ by $\mathcal{G}$, and combining with Equation (5.36), it follows that

$$\mathbb{H}^k_{\Delta\infty}(M, G) \cong \mathcal{H}^k(\Gamma(S^\ast(M, G))) \cong \mathcal{H}^k(M, \mathcal{G}).$$ (5.37)

We have got the isomorphism of sheaves we needed and this ends the construction of an isomorphism between the sheaf singular cohomology and the classical singular cohomology. Moreover, if we set $G = \mathbb{R}$ and $\mathcal{G} = \mathfrak{R} = M \times \mathbb{R}$, we get the isomorphism

$$\mathbb{H}^k_{\Delta\infty}(M, \mathbb{R}) \cong \mathcal{H}^k(M, \mathfrak{R}).$$ (5.38)

5.5 Čech cohomology

We recall that the topological space $(M, \tau_{\mathcal{F}M})$ considered in this chapter is a Hausdorff paracompact space. We know that each smooth map is $\tau_{\mathcal{F}M}$-continuous. Now, we start building Čech Cohomology with coefficients in a sheaf $\mathcal{F}$ in the continuous context. The following references [16, 26, 46, 114] are relevant for this section.
5.5.1 Coboundary operator and cochain complex

Let $J$ be an ordered index set and the collection $\mathcal{U} = \{U_j | j \in J\}$ be an open cover of the topological space $(M, \tau_M)$. Let $K$ be a finite subset of $J$, with its cardinal $|K| = k + 1$. Let us consider $\sigma := (U_0, U_1, U_2, U_3, \cdots, U_k)$ a finite subcollection of the cover $\mathcal{U}$, such that $U_K = U_0 \cap U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_k$ is not an empty set. We should call the subcollection above the $k$-simplex with support $U_K$. It follows that the $i$th face of $\sigma$ is the $(k-1)$-simplex denoted by $\sigma^i = (U_0, U_1, \cdots, U_{i-1}, U_{i+1}, \cdots, U_k) = (U_0, U_1, \cdots, U_{i-1}, \widehat{U}_i, U_{i+1}, \cdots, U_k)$, where $\widehat{U}_i$ means that $U_i$ is deleted from the finite subcollection of open sets. Now, one defines a $k$-cochain as a function which assigns to each $k$-simplex $\sigma = (U_0, U_1, U_2, U_3, \cdots, U_k)$ a section of the sheaf $\mathcal{F}$ on its support $U_K = U_0 \cap U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_k$. We denote the $\mathbb{K}$-module of $k$-cochains by

$$C^k(\mathcal{U}, \mathcal{F}) := \prod_{|K| = k+1} \mathcal{F}(U_K). \quad (5.39)$$

Note that the product is running over all sets $K \subset J$ such that $|K| = k + 1$. As usual, the forthcoming step for defining a cohomology theory should be the coboundary operator which assigns each $k$-cochain to a $(k + 1)$-cochain. First of all, the $k$-cochain $\alpha \in C^k(\mathcal{U}, \mathcal{F})$, is defined by

$$\alpha = ((\alpha_K) \in \mathcal{F}(U_K)), \text{ for all } K \subset J \text{ and } |K| = k + 1, \quad (5.40)$$

such that the $k^{th}$ component of $\alpha$ is

$$\alpha_K = \alpha(K) = \alpha(\sigma) = \alpha(U_0, U_1, U_2, U_3, \cdots, U_k) = \alpha|_{U_0 \cap U_1 \cap U_2 \cap \cdots \cap U_k}. \quad (5.41)$$

It is obvious that by definition of $k$-cochains the following hold, where $K \subset J$:

- for $k = 0$, one gets $\alpha = ((\alpha_{U_0}) \in \mathcal{F}(U_0)) \in C^0(\mathcal{U})$, with $|K| = o + 1 = 1$,
- for $k = 1$, one gets $\alpha = ((\alpha_{U_0, U_1}) \in \mathcal{F}(U_0 \cap U_1)) \in C^1(\mathcal{U})$, with $|K| = 1 + 1 = 2$,
- for $k = 2$, one gets $\alpha = ((\alpha_{U_0, U_1, U_2}) \in \mathcal{F}(U_0 \cap U_1 \cap U_2)) \in C^2(\mathcal{U})$, with $|K| = 2 + 1 = 3$,

and so on for all higher values $k \geq 0$. We let

$$C^k(\mathcal{U}, \mathcal{F}) = \{0\} \text{ for all } k < 0. \quad (5.42)$$

It is known that the coboundary operator increases the order of the cochain by one. Let us define any well-ordering relation $\leq$ on the index set $J$. Thus, the finite subset $K \subset J$ inherits this well-ordering on its elements, that is, $0 < 1 < 2 < \cdots < k$. We set $K_i := K \setminus \{i\}$ for $i$ running from $0$ to $k$ in $K$. Let $d = d^k : C^k(\mathcal{U}, \mathcal{F}) \rightarrow C^{k+1}(\mathcal{U}, \mathcal{F})$ be the coboundary
5.5 Čech cohomology

operator. We should explain how the action of a coboundary operator must work on cochains from small values of $k$ and then after give the general definition. Indeed, $d(\alpha_{U_0}) = \alpha_{U_0, U_1}$, a 1-cochain defined over $U_0 \cap U_1$, but $d(\alpha_{U_0, U_1}) = \alpha_{U_0, U_1, U_2}$. It appears that $d$ measures how far $\alpha_{U_0}$ and $\alpha_{U_1}$ fail to be respectively restrictions of a section on $U_0 \cup U_1$. Therefore, the difference $\alpha_{U_0} - \alpha_{U_1}$ over the intersection may be the relevant tool to check the failure. We set $d(\alpha_{U_0, U_1}) = \alpha_{U_0} - \alpha_{U_1}$. For a 2-cochain, $d(\alpha_{U_0, U_1, U_2}) = \alpha_{U_0, U_1, U_2} + \alpha_{U_0, U_2} + \alpha_{U_1, U_2}$. A symmetry property is readable in the process. By taking a permutation $\epsilon$ on $\{0, 1, 2, \cdots, k\}$ we get,

$$\alpha_K = \alpha_{U_0 \cap U_1 \cap U_2 \cap \cdots \cap U_k} = \text{sign}(\epsilon)\alpha_{U_{\epsilon(0) \cap U_{\epsilon(1)} \cap U_{\epsilon(2)} \cap \cdots \cap U_{\epsilon(k)}}}. \quad (5.43)$$

Naturally, we define the coboundary operator $d : C^k(U, F) \to C^{k+1}(U, F)$ by

$$(d\alpha)(K) = (d\alpha)(U_0, U_1, U_2, \cdots, U_k)$$

$$= \sum_{i=0}^{k+1} (-1)^i \alpha(\sigma^i)|_{U_K} \text{ such that } d \circ d = d^{k+1} \circ d^k = 0 \quad (5.44)$$

Note that $\text{Ker}(d) = Z^k(U, F) \subset C^k(U, F)$ and $\text{Im}(d) = d(C^{k+1}(U, F)) \subset C^k(U, F)$. It follows that $\text{Im}(d) \subset \text{Ker}(d)$ with regard to Equation (5.44). Hence, a cochain complex, (called the Čech complex) $C^*(U, F)$ can be considered for each sheaf $F$ and each open cover $U$ on $M$, so that it allows the definition of the $k^{th}$ Čech cohomology group of (with coefficients in) $F$ relative to the cover $U$:

$$\check{H}^k(U, F) := \mathcal{H}^k(C^*(U, F)) = \mathcal{H}^k(U, F). \quad (5.45)$$

By the functorial behavior of sheaves on $M$, we have that at any sheaf homomorphism $F \to \Lambda$ corresponds a homomorphism of Čech complexes $C^*(U, F) \to C^*(U, \Lambda)$ for each $k$, which commutes with the coboundary operator. Therefore, we get a homomorphism of cohomologies for each $k$

$$\check{H}^*(U, F) \to \check{H}^*(U, \Lambda). \quad (5.46)$$

5.5.2 Partial ordering of refinements of a cover

Let $\mathcal{V} = (V_s)_{s \in S}$ be a refinement of an open cover $U = (U_j)_{j \in J}$ of $M$, that is, every open set $V_s$ of $\mathcal{V} = (V_s)_{s \in S}$ is contained in some open set $U_j$ of $U = (U_j)_{j \in J}$. That is, $\mathcal{V} = (V_s)_{s \in S}$ is finer then $U = (U_j)_{j \in J}$, denoted by $\mathcal{V} < U$. The refinement induces a partial ordering relation on the set of all open covers of $M$, so that, there exists a map $\diamond : S \to J$, defined by $s \to \diamond(s) = j$, such that $V_s \subset U_{\diamond(s)}$, for all $s \in S$. Therefore, an induced map $\check{\diamond} : \mathcal{V} \to U$ is defined by $V_s \subset \check{\diamond}(V_s) = U_{\diamond(s)}$ for each $V_s \in \mathcal{V}$, such that, the $q$-simplex
\[ \sigma = (V_0, V_1, V_2, V_3, \ldots, V_k) \] over \( \mathcal{V} \) is mapped on the \( q \)-simplex over \( \mathcal{U} \) as below:

\[
\tilde{\delta}(\sigma) = (\tilde{\delta}(V_0), \tilde{\delta}(V_1), \tilde{\delta}(V_2), \tilde{\delta}(V_3), \ldots, \tilde{\delta}(V_k))
\]
\[ = (U_{\sigma(0)}, U_{\sigma(1)}, U_{\sigma(2)}, \ldots, U_{\sigma(k)}), \text{ with } U_{\sigma(0)} \supset V_s. \] (5.47)

Let \( \sigma = (V_0, V_1, V_2, V_3, \ldots, V_k) \), and \( (\alpha_K) \) as in Equations (5.40) and (5.41). Recall that \( V_s \subset \tilde{\delta}(V_s) = U_{\sigma(0)} \). Then, it follows that the map \( \tilde{\delta} \) induces a cochain map \( \tilde{\delta} : \mathcal{C}^k(\mathcal{U}, \mathcal{F}) \rightarrow \mathcal{C}^k(\mathcal{V}, \mathcal{F}) \) of Čech complexes, that is, \( \tilde{\delta}^k := \tilde{\delta} \) for simplification of notation. The later is defined by

\[ \tilde{\delta}(\{\alpha_K\}) = \tilde{\delta}(\{\alpha|_{U_0 \cap U_1 \cap U_2 \cap \cdots \cap U_k}\}) := \{\alpha_K\}(\tilde{\delta}), \text{ such that (5.48)} \]

\[ \tilde{\delta}(\{\alpha_K\})(\sigma) = \{\alpha_K\}(\tilde{\delta})(\sigma) \]
\[ = \{\alpha_K\}(\tilde{\delta})(V_0, V_1, V_2, V_3, \ldots, V_k) \]
\[ = \{\alpha_K\}(\tilde{\delta}(V_0), \tilde{\delta}(V_1), \tilde{\delta}(V_2), \tilde{\delta}(V_3), \ldots, \tilde{\delta}(V_k)) \] (5.49)
\[ = \{\alpha_K\}(U_{\sigma(0)}, U_{\sigma(1)}, U_{\sigma(2)}, U_{\sigma(3)}, \ldots, U_{\sigma(k)}) \]
\[ = \{(\alpha|_{U(\sigma(0)) \cap U_{\sigma(1)} \cap U_{\sigma(2)} \cap U_{\sigma(3)} \cap \cdots \cap U_{\sigma(k)})}\}(\sigma) \]

### 5.5.3 Čech sheaf cohomology

The cochain map \( \tilde{\delta} \) above commutes with the coboundary operator. So, there exists a map \( \tilde{\delta}^* : \tilde{\mathcal{H}}^*(\mathcal{U}, \mathcal{F}) \rightarrow \tilde{\mathcal{H}}^*(\mathcal{V}, \mathcal{F}) \) of Čech cohomologies with regard to the two covers. If two refining maps \( \tilde{\delta} \) and \( \tilde{\delta}^* \) are given from \( \mathcal{V} \) into \( \mathcal{U} \), we have that \( \tilde{\delta}^* = \tilde{\delta}^* \) up to homotopy as proved in [114]. Thus, \( \tilde{\delta}^* \) are natural homomorphisms of modules such that by the composition of homomorphisms we have \( \tilde{\mathcal{H}}^*(\mathcal{U}, \mathcal{F}) \rightarrow \tilde{\mathcal{H}}^*(\mathcal{V}, \mathcal{F}) \rightarrow \tilde{\mathcal{H}}^*(\mathcal{W}, \mathcal{F}) \) when \( \mathcal{W} < \mathcal{V} < \mathcal{U} \) is a chain of refinements of \( \mathcal{U} \) on \( M \). Now, we have a direct system yielded by the collection of modules \( \tilde{\mathcal{H}}^*(\mathcal{U}, \mathcal{F}) \) and refinement homomorphisms \( \tilde{\delta}^* \). We should define the \( k \)th Čech cohomology module of \( M \) with coefficients in the sheaf of \( \mathbb{K} \)-modules \( \mathcal{F} \) by setting

\[ \tilde{\mathcal{H}}^k(M, \mathcal{F}) := \lim_{\mathcal{U}} \tilde{\mathcal{H}}^k(\mathcal{U}, \mathcal{F}). \] (5.50)

By the universality of the direct limit we end up with a natural homomorphism

\[ \tilde{\mathcal{H}}^*(M, \mathcal{F}) \rightarrow \mathcal{H}^*(M, \mathcal{F}). \] (5.51)

As it is not easy to manipulate the direct limit, we are stating below some properties of the cover \( \mathcal{U} \) under which we reach \( \tilde{\mathcal{H}}(M, \mathcal{F}) = \mathcal{H}(M, \mathcal{F}) \).

**Lemma 5.5.1.** [26] Let \( \mathcal{F} \) be an arbitrary sheaf on \( M \) and \( \mathcal{U} = (U_j)_{j \in J} \) be an open cover of \( M \). Then, there exists a canonical isomorphism

\[ \tilde{\mathcal{H}}^0(\mathcal{U}, \mathcal{F}) \cong \Gamma(M, \mathcal{F}) = \Gamma(\mathcal{F}) = \mathcal{F}(M) \]
Lemma 5.5.2. [26] As by Definition 5.1.8, let \( F \) be a flasque sheaf on \( M \). Then,

\[
\mathcal{H}^k(M, F) = \Gamma(F) = F(M) = \{0\} \text{ for all } k > 0.
\]

Lemma 5.5.3. [33, 26] Let \( F \) be an arbitrary sheaf on \( M \), \( \mathcal{U} = (U_j)_{j \in J} \) be an open cover of \( M \), and let \( K \) be a finite set, with its cardinal \( |K| = k + 1 \). We set \( U_K := U_0 \cap U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_k \).

If \( \mathcal{H}^k(U_K, F) = \{0\} \) for each \( k \geq 1 \) (that is, \( \mathcal{U} \) is acyclic for \( F \)), then

\[
\check{H}^k(\mathcal{U}, F) = \mathcal{H}^k(M, F) \text{ for all } k \geq 0.
\]

Lemma 5.5.4. [33] If a sheaf \( F \) is fine then \( \mathcal{H}^k(M, F) = \{0\} \), that is, \( F \) is acyclic.

Theorem 5.5.5. [33, 26] Leray Theorem

Let \( M \) be a topological space, \( \mathcal{U} = (U_j)_{j \in J} \) an open cover of \( M \), and \( F \) a sheaf of abelian groups on \( M \). Let \( U_K := U_0 \cap U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_k \), with \( |K| = k + 1 \). Then, if \( \mathcal{H}^k(U_K, F|_{U_K}) = \{0\} \) then \( \check{H}^k(\mathcal{U}, F) \to \mathcal{H}^k(M, F) \) are isomorphisms for all \( k \).

Theorem 5.5.6. [26] Cartan Theorem

Let \( M \) be a topological space, \( F \) a sheaf of abelian groups on \( M \). Let \( \mathcal{U} = (U_j)_{j \in J} \) be an open cover of \( M \) such that \( U_K := U_0 \cap U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_k \in \mathcal{U} \), with \( |K| = k + 1 \). That is, \( \mathcal{U} = (U_j)_{j \in J} \) is closed under finite intersections and it contains arbitrary small open sets. If \( \check{H}^k(U, F) = \{0\} \) for all \( U \in \mathcal{U} \) and \( k > 0 \), then the homomorphisms \( \check{H}^k(M, F) \to \mathcal{H}^k(M, F) \) defined in Equation (5.51) are natural isomorphisms for all \( k \).

5.5.4 Classical Čech cohomology

The classical \( k \)th Čech cohomology module of \( M \) with coefficients in a \( K \)-modules \( G \) is given by \( \check{H}^k(M, G) \cong \mathcal{H}^k(M, G) \). Recall that \( G = M \times G \) is the constant sheaf. The previous isomorphism becomes

\[
\check{H}^k(M, \mathbb{R}) \cong \mathcal{H}^k(M, \mathbb{R}), \text{ with } \mathbb{R} = M \times \mathbb{R}.
\] (5.52)

5.6 Multiplicative structure and de Rham theorem

5.6.1 Multiplicative structure and Kunneth formulas

A multiplicative structure in cohomology theory evokes the definition of a correspondence between the cohomology of the Cartesian product of spaces and the Cartesian product of
cohomology modules. That is, we would like to express the cohomology of the Cartesian product of spaces in terms of the cohomologies of its factors. The multiplicative structure is an important concept in the way it makes the $\mathbb{K}$-module cohomology

$$\mathcal{H}^*(M, \mathcal{F}) := \bigoplus_{k \geq 0} \mathcal{H}^k(M, \mathcal{F}),$$

the direct sum of $\mathcal{H}^k(M, \mathcal{R})$, into a ring and more better into a $\mathbb{K}$-algebra. So, for any $a \in \mathcal{H}^k(M, \mathcal{F})$ and $b \in \mathcal{H}^l(M, \mathcal{F})$ there exists $a \cup b \in \mathcal{H}^{k+l}(M, \mathcal{F})$ a kind of product called cohomology cup product, that is, an internal product. The detailed machinery to build the cup product can be found in [50, 53, 114]. Nevertheless, we expose here a summary.

Recall that $\mathcal{H}^k(M, \mathcal{R})$ is the $k^{th}$ cohomology $\mathbb{R}$-vector space of $M$ with coefficients in the constant sheaf $\mathcal{R} = M \times \mathbb{R}$, while $\mathcal{H}^*(M, \mathcal{R})$ is the cohomology cochain complex. We consider $\sigma = (V_0, \cdots, V_k, V_{k+1}, \cdots, v_{k+l})$ as $(k+l)$-simplex. Thus, for cochains $\alpha$ and $\beta$, we define the cup product by $\alpha \cup \beta$ as a $(k+l)$-cochain, such that

$$(\alpha \cup \beta)(\sigma) := \alpha(\sigma_K)\beta(\sigma_L),$$

where $\sigma_K := (V_0, \cdots, V_k)$ and $\sigma_L := (V_k, V_{k+1}, \cdots, v_{k+l})$. Apparently, $\cup$ is compatible with the coboundary operator in the sense that

$$d((\alpha \cup \beta)) := (d\alpha) \cup \beta + (-1)^k \alpha \cup (d\beta).$$

We can now build $\mathcal{H}^k(M, \mathcal{R}) \times \mathcal{H}^l(M, \mathcal{R}) \xrightarrow{\cup} \mathcal{H}^{k+l}(M, \mathcal{R})$ the induced map of cohomologies. This new cup product on cohomology $\mathbb{R}$-vector spaces inherits the associative and distributive properties from the cup product of cochains. It follows from this construction that any map $f : M \rightarrow N$, induces a linear map $f^* : \mathcal{H}^k(N, \mathcal{R}) \rightarrow \mathcal{H}^k(M, \mathcal{R})$ satisfying $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$. Let $\pi_1$ and $\pi_2$ be the canonical projections of the Cartesian Product $M \times M$ onto $M$. Let $a$ be a cohomology class of $k$-cochains $\alpha$, $b$ a cohomology class of $l$-cochains $\beta$.

We call cross product or external cup product, the map

$$\mathcal{H}^k(M, \mathcal{R}) \times \mathcal{H}^l(M, \mathcal{R}) \xrightarrow{\times} \mathcal{H}^{k+l}(M, \mathcal{R})$$

defined by $a \times b := \pi_1^*(a) \cup \pi_2^*(b)$.

In taking one step forward we shall try to make $\mathcal{H}^*(M, \mathcal{R}) := \bigoplus_{k \geq 0} \mathcal{H}^k(M, \mathcal{R})$ in a graded ring, where elements are finite sums $\sum_{k \geq 0} a_k$ with $a_k \in \mathcal{H}^k(M, \mathcal{R})$. The multiplication on finite sums $\sum_{k \geq 0} a_k$ and $\sum_{l \geq 0} b_l$ gives as product the sum $\sum_{k,l} a_k b_l$. The later multiplication endows $\mathcal{H}^*(M, \mathcal{R})$ with a ring structure and unity. Now, we should link cup product to product spaces by redefining the cross product as a bilinear map

$$\mathcal{H}^*(M, \mathcal{R}) \times \mathcal{H}^*(M, \mathcal{R}) \xrightarrow{\times} \mathcal{H}^*(M \times M, \mathcal{R})$$

such that $a \times b := \pi_1^*(a) \cup \pi_2^*(b)$. 5.6 Multiplicative structure and de Rham theorem

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However, the bilinear map above induces a linear map:

\[ H^*(M, \mathcal{R}) \otimes H^*(M, \mathcal{R}) \rightarrow H^*(M \times M, \mathcal{R}) \]
defined by \( a \otimes b \rightarrow a \times b \)
and called also cross product on the tensor product. Obviously, this is an isomorphism and read

\[ H^k(M, \mathcal{R}) \otimes H^l(M, \mathcal{R}) \rightarrow H^{k+l}(M \times M, \mathcal{R}) \]
or

\[ \bigoplus_{k+l=r} H^k(M, \mathcal{F}) \otimes H^l(M, \mathcal{R}) \rightarrow H^r(M \times M, \mathcal{R}). \]

If we introduce the multiplication on the tensor product by setting

\[(a \otimes b)(c \otimes d) := (a \times b)(c \times d),\]

we get ring structure, with following properties:

**Lemma 5.6.1.** [50] Let \( a \in H^k(M, \mathcal{R}) \), \( b \in H^l(M, \mathcal{R}) \) and \( c \in H^m(M, \mathcal{R}) \). Then the following statements are true:

1. \[(a \times b) \times c = a \times (b \times c) \in H^{k+l+m}(M \times M \times M, \mathcal{R}),\]
2. \[a \times b = (-1)^{kl} t^*(b \times a) \in H^{k+l}(M \times M, \mathcal{R}), \]
   with \( M \times M \xrightarrow{t:=\pi} M \times M, (x, y) \rightarrow (y, x)\),
3. \[a \times 1 = \pi_1^*(a) \in H^k(M \times M, \mathcal{R}) \text{ and } 1 \times b = \pi_2^*(b) \in H^l(M \times M, \mathcal{R}).\]

Let \( \Delta : M \rightarrow M \times M \) be the diagonal map. By the composition map \( \Delta^* \circ \times \) we will define a new internal cup product for \( k, l \geq 0 \) by

\[ \cup : H^k(M, \mathcal{R}) \otimes H^l(M, \mathcal{R}) \rightarrow H^{k+l}(M \times M, \mathcal{R}) \xrightarrow{\Delta^*} H^{k+l}(M, \mathcal{R}). \]

We should notice that the cohomology class 1 \( \in H^0(M, \mathcal{R}) \) is the class of the evaluation map on \( ev : \mathcal{F}_M \rightarrow \mathbb{R} \)

**Lemma 5.6.2.** [50] Let \( a \in H^k(M, \mathcal{R}) \), \( b \in H^l(M, \mathcal{R}) \) and \( c \in H^m(M, \mathcal{R}) \). Then the following statements are true:

1. \[(a \cup b) \cup c = a \cup (b \cup c) \in H^{k+l+m}(M, \mathcal{R}),\]
2. \[a \cup b = (-1)^{kl}(b \cup a) \in H^{k+l}(M, \mathcal{R}),\]
3. \[a \cup 1 = (a) = 1 \cup a \in H^k(M, \mathcal{R}),\]
5.6 Multiplicative structure and de Rham theorem

4. $\mathcal{H}^k(M, \mathcal{R}) \otimes \mathcal{H}^l(M, \mathcal{R}) \cong \mathcal{H}^l(M, \mathcal{R}) \otimes \mathcal{H}^k(M, \mathcal{R})$, $a \otimes b \mapsto (-1)^{kl} b \otimes a$.

We have got a graded ring called the cohomology ring of the space $M$ with regard to the constant sheaf $\mathcal{R}$. The cohomology ring $\mathcal{H}^*(M, \mathcal{R})$ and $\mathbb{R}$ are $\mathbb{F}$-spaces. Some curves into the cohomology ring are homomorphisms of rings. Thus, the cohomology ring is actually a $\mathbb{R}$-algebra. The Kunneth formula allows the construction of the cohomology of a product as the tensor product of the cohomology of the factors. We now have a graded cohomology ring (say, algebra), such that any map $f : M \to N$ induces a homomorphism of graded rings (algebra) $f^* : \mathcal{H}^k(N, \mathcal{R}) \to \mathcal{H}^k(M, \mathcal{R})$ satisfying $f^*(a \cup b) = f^*(a) \cup f^*(b)$.

5.6.2 de Rham cohomology algebra

A natural correspondence can be drawn between the extension of exterior product to cohomology classes of exterior forms and the cup product in Subsection 5.6.1. Indeed, from Definitions 5.3.1, 5.3.2, 5.3.3, and with regard to Equations (5.10) thru (5.13) and (5.19) we may rewrite the whole subsection of multiplicative structure by replacing the cross product by the exterior product. Hence, $\mathbb{H}^*_{dRH}(M) = \bigoplus_{k \geq 0} \mathbb{H}^k_{dRH}(M)$ is naturally endowed with the ring structure, it is an algebra on $\mathbb{R}$.

5.6.3 Singular cohomology algebra

The definition of a cup product of singular cochains and its extension to singular cohomology classes is provided in [53, 114].

5.6.4 de Rham theorem for $\mathbb{R}$-modules cohomology

**Theorem 5.6.3.** Let $M$ be a locally Euclidean space, $\mathbb{H}^k_{A-S}(M, \mathbb{R})$, $\mathbb{H}^k_{dRH}(M)$, $\mathbb{H}^k_{\Delta}(M, \mathbb{R})$, $\mathbb{H}^k(M, \mathbb{R})$ and $\mathcal{H}^k(M, \mathcal{R})$ respective cohomologies under consideration. Then, the five cohomology $\mathbb{R}$-vector spaces are isomorphic.

**Proof.** We recall all the previous isomorphisms of $\mathbb{R}$-vector spaces encountered in this work with regard to the constant sheaf cohomology $\mathcal{R}$:

In Equation (5.9) we had $\mathbb{H}^k_{A-S}(M, \mathbb{R}) \cong \mathcal{H}^k(M, \mathcal{R})$ for Alexander-Spanier cohomology. For the de Rham cohomology we had in Equation (5.23) the isomorphism $\mathbb{H}^k_{dRH}(M) \cong \mathcal{H}^k(M, \mathcal{R})$. The singular cohomology isomorphism $\mathbb{H}^k_{\Delta}(M, \mathbb{R}) \cong \mathcal{H}^k(M, \mathcal{R})$ is given in Equation (5.38).
5.7 Isomorphism on reduced space

Finally, in Equation (5.52) the isomorphism $\tilde{H}^k(M, \mathbb{R}) \cong \check{H}^k(M, \mathcal{R})$ is stated for Čech Cohomology. The composition of all the above isomorphisms gives

$$\mathbb{H}^*_{A-S}(M, \mathbb{R}) \cong \mathbb{H}^*_{dRH}(M) \cong \mathbb{H}^*_{\Delta_{\infty}}(M, \mathbb{R}) \cong \tilde{H}^*(M, \mathbb{R}) \cong H^*(M, \mathcal{R}). \Box$$

5.6.5 de Rham theorem for ring cohomologies

Theorem 5.6.4. Let $M$ be a locally Euclidean space, $\mathbb{H}^k_{A-S}(M, \mathbb{R})$, $\mathbb{H}^k_{dRH}(M)$, $\mathbb{H}^k_{\Delta_{\infty}}(M, \mathbb{R})$, $\tilde{H}^k(M, \mathbb{R})$ and $H^k(M, \mathcal{R})$ respective cohomologies under consideration. Then, the five cohomologies are $\mathbb{R}$-algebras and isomorphic.

Proof. The $\mathbb{R}$-linear isomorphisms are bijective maps. So, given $a, b \in H^*(M, \mathcal{R})$, the bijective map yields $a \cup b \rightarrow \alpha \wedge \beta$, with $\alpha, \beta \in \mathbb{H}^k_{dRH}(M)$ and $a \rightarrow \alpha, b \rightarrow \beta$. It follows that the multiplicative structure on $H^*(M, \mathcal{R})$ is transferable on all others cohomologies to make them into $\mathbb{R}$-algebras. Thus, the isomorphisms of linear spaces in Theorem 5.6.3 become isomorphisms of $\mathbb{R}$-algebras. \Box

5.7 Isomorphism on reduced space

5.7.1 Cohomologies on a symplectic $\mathbb{F}$-space

In this section the word space would refer to a locally Euclidean symplectic $\mathbb{F}$-space of constant dimension, if not otherwise stated. The five isomorphisms established in Theorem 5.6.4 allow us to work with at least one of them on the symplectic quotient space. We have opted for $H^k(M, \mathcal{R})$ and $\mathbb{H}^k_{dRH}(M)$, respectively the cohomology of constant sheaf and the de Rham cohomology. So, all others remaining isomorphisms shall be deduced naturally. It is worth recalling that we are dealing here with a Frölicher (compact) Lie group $G$ acting freely and properly, in a Hamiltonian fashion on a symplectic space $(M, \omega)$. Given the equivariant moment map $\mu : M \rightarrow G^*$ of this action. We set $Z := \mu^{-1}(\theta) \subset M$. Let $G$ and $G^*$ be the $\mathbb{F}$-Lie algebra of the group $G$ and its algebraic dual. Hence, for any regular value $\theta \in G^*$ of the moment map $\mu$, the quotient space $M_\theta := \mu^{-1}(\theta)/G_\theta$ inherits a reduced symplectic form $\overline{\omega}_\theta$ defined by $\pi^*_\theta \overline{\omega}_\theta = \iota^*_\theta \omega$, where $\pi_\theta$ is the restriction of the canonical quotient map, $\omega_\theta := \omega|_Z$ and $\iota_\theta$ is the canonical inclusion map $Z \subset M$, as closed subset. Thus, on $Z$, the relative topology coincides with the $\mathbb{F}$-subspace topology, and its set of Frölicher structure functions is $\mathcal{F}_Z = \mathcal{F}_{M|Z}$. From induced Hausdorff paracompact topology shown in Section 4.2 the
symplectic quotient is second countable as by Proposition 4.4.2, paracompact Hausdorff as by Proposition 4.4.9. Therefore, with regard to [112] Sections 3.5, 3.8, the existence of the local diffeomorphisms extends naturally to the orbits space and the symplectic quotient, so that we are dealing with Subcartesian spaces (see [6, 71, 72, 102, 103, 104]). It follows that we may make preferably use of constructions and properties provided in [58, 59, 65, 115, 116] with regard to forms on a symplectic quotient, de Rham complexes and cohomologies for differential (mainly subcartesian) and diffeological spaces. Moreover, a Frölicher space is both differential and diffeological space. So, we will state definitions and state properties without proofs, and we recommend the aforementioned references to the reader.

The cohomologies defined on a general locally Euclidean space in preceding Sections 5.1 through 5.5, stand unchanged on \((M, \omega)\) since the symplectic structure does not impact on these different constructions. The objects of interest should now be the quotient space (the orbits space of the action described above), the symplectic quotient and their de Rham complexes of differential forms.

### 5.7.2 de Rham complexes for the spaces \(M\) and \(M/G\)

The starting point should be the cochain complexes in Equations (5.20) and (5.21) with regard to the cochain complex \(\Omega^*(M)\):

\[
\begin{array}{ccccccc}
\cdots & \Omega^{k-1}(M) & \overset{d}{\longrightarrow} & \Omega^k(M) & \overset{d}{\longrightarrow} & \Omega^{k+1}(M) & \overset{d}{\longrightarrow} & \cdots\\
\cdots & \Gamma(\Omega^{k-1}(M)) & \overset{d}{\longrightarrow} & \Gamma(\Omega^k(M)) & \overset{d}{\longrightarrow} & \Gamma(\Omega^{k+1}(M)) & \overset{d}{\longrightarrow} & \cdots
\end{array}
\]

where, \(\Omega^k(M) \rightarrow \Gamma(\Omega^k(M))\) are natural homomorphisms that are bijections and commute with coboundaries operators. We should from now be interested in the smooth structure on the quotient space and the following lemma gives its relation with a special set of functions on the ambient set \(M\), that is, \(\mathcal{F}_M \overset{\pi}{\rightarrow} \mathcal{F}_M^G\) as in the proof of Lemma 3.4.5, and Remark 3.4.4, with, \(\mathcal{F}_M^G\) the algebra of \(G\)-invariant smooth functions on \(M\), \(C^\infty(M, \mathbb{R}) := \mathcal{F}_M\) the smooth structure on the orbit space, and \(\mathcal{F}_M \overset{\pi}{\rightarrow} \mathcal{F}_M^G\), the pullback \(\pi^*\) which is an isomorphism of \(\mathbb{R}\)-algebras. So, \(h\) is constant on each orbit (the equivalence class) \(i\), and only if \(h \circ \sigma_g = h\) if, and only if \(h\) is invariant under the action of \(\sigma\) of \(G\) on \(M\) if, and only if \(h \in \mathcal{F}_M^G\). Thus, \(f = h \circ \sigma_g = h\) implies \(\mathcal{F}_M^G \subseteq \mathcal{F}_M\). We will now look to the construction of de Rham complexes homomorphisms on the quotient space. For, we define respectively the infinitesimal action of \(G\) and a basic form on \(M\) by:

**Definition 5.7.1.** Let \(G\) be the Lie algebra of a Lie group \(G\) acting on a locally Euclidean space \(M\). \(L\) and \(m \in M\), \(\xi \in \mathcal{G}\) and \(A : \mathcal{G} \times M \rightarrow TM\) be the infinitesimal action of \(\mathcal{G}\) on
$M$ such that $A(\xi) = \frac{d}{dt} \exp(t\xi)|_{t=0} : M \to TM$,
with $A(\xi, m) := A(\xi)(m) = (\frac{d}{dt} \exp(t\xi)|_{t=0}).m = (\frac{d}{dt} \exp(t\xi).m)|_{t=0} := A(m)(\xi)$.

The map $A_m := A(m) : G \to T_m M$, $\xi \mapsto A_m(\xi)$ is one-to-one, since the action is free. That is, $A_m(G) := G.m \subset T_m M$ and the following holds: $G \simeq im A_m = A_m(G) = T_m(G.m) = G.m$. The Lie subalgebra of $G$ defined by $G_m := \{\xi \in G | A(\xi)(m) = A(m)(\xi) = 0\} = \{\xi \in G | \xi_M(m) = 0\} = Ker A_m$ is called the isotropy (symmetry) subalgebra of $m \in M$.

**Definition 5.7.2.** Let $\alpha \in \Omega^k(M)$ be a $k$-differential form on $M$, $G$ be a group acting freely and properly, in a Hamiltonian fashion on a symplectic space $(M, \omega)$, and $G$ be the $\mathbb{F}$-Lie algebra of the group $G$. Then $\alpha$ is a basic form if it satisfies both following conditions:

(1) For each $g \in G$, one has $\sigma^*_g \alpha = \alpha$, that is, $\alpha$ is $G$-invariant, and

(2) For each $\xi \in G$, one has $\xi_M \cdot \alpha = 0$, that is, $\alpha$ is horizontal with regard to the vector field $\xi_M : M \to TM$ defined in Definition 5.7.1.

Equivalently, we define a basic form as a $G$-invariant form $\alpha \in \Omega^k(M)^G$ such that at each $m \in M$ and for each $v \in T_m(G.m) = G.m$, one has $v \cdot \alpha = 0$.

We will write $\Omega^k_{basic}(M)$ for the $\mathbb{R}$-algebra of all basic $k$-forms on $M$. Furthermore, the Koszul Theorem [60] asserts that the de Rham complex $\Omega^*_{basic}(M)$ of basic forms on $M$ is a subcomplex of the cochain complex $\Omega^*(M)$ of all differential forms on $M$ and there exists an isomorphism of cohomologies with real coefficients between the singular cohomology and the de Rham cohomology as by Theorems 5.6.3 and 5.6.4, that is, $H^*_basic\text{dR}(M) \cong H^*_\Delta_{\infty}(\overline{M}, \mathbb{R})$. Therefore, the pullback $\pi^* : \Omega^k(\overline{M}) \to \Omega^k_{basic}(M)$ is actually an isomorphism of exterior algebras (complexes), it follows that $\pi^*(\Omega^k(\overline{M})) = \Omega^k_{basic}(M)$ and a detailed proof is provided in [115]. Thus, $\beta \in \Omega^k(\overline{M})$ if and only if there exists a unique $\alpha \in \Omega^k_{basic}(M)$ such that $\pi^* \beta = \alpha$. In the same reference, it is also shown that there is a $\mathbb{R}$-algebra homomorphism $\iota : \Omega^k(M)^G \to \Omega^k(\overline{M})$ such that $\iota \circ \pi^* = \text{id}_{\Omega^k(\overline{M})}$. In addition, if $\alpha \in \Omega^k(M)^G$ is basic at $m \in M$ then $\iota(\alpha)|_{T_m(G.m)} = \alpha|_{T_m(G.m)}$. Recall that the pullback of $\pi$ is defined by $\pi^*(\alpha) = \alpha(\pi_*(v_1), \pi_*(v_2), \pi_*(v_3), \ldots, \pi_*(v_k))$ for any $\alpha \in \Omega^k(\overline{M})$ and for any $v_1, v_2, v_3, \ldots, v_k \in T_m(G.m) = G.m$, where the tangent map $\pi_*$ is surjective onto $T\overline{M}$.

Finally, for any $k \geq 0$ the short exact sequence of complexes is split:

$$
0 \to \Omega^k(\overline{M}) \to \Omega^k(M)^G \to \Omega^k(M)^G/\pi^*(\Omega^k(\overline{M})) \to 0.
$$

That is, with respect to Definition 5.1.11 and Theorem 5.1.6.

### 5.7.3 Effects of the symplectic reduction process

We assume that the reader is aware of concepts and notations used hereafter and can refer to the appropriate part of the thesis. From there we can notice that the compatibility of the
The dualitiy between the kernel and the range of a linear map, we have:

We claim that we can write:

\[ G \cong \text{following holds:} \]

\[ \forall \]

\[ \text{im} \mathcal{T}_m \mathcal{\mu} = \mathcal{T}_m \mathcal{\mu}^{-1}(\theta) \text{ and } \mathcal{im} \mathcal{T}_m \mathcal{\mu} = \mathcal{G}_m = \mathcal{G}_m^\omega, \text{ where } m \in \mu^{-1}(\theta), \theta \in \mathcal{G}^*, \text{ is a regular value of the moment map } \mu \text{ and } \mathcal{G}_m = \mathcal{G}_m^\omega \text{ is the annihilator of the Lie algebra } \mathcal{G}_m \text{ of the stabilizer of } m, \text{ with regard to } \omega. \]

**Proof.** One has, \( \dim M = q, \dim \mathcal{G}^* = n, \) and the moment map \( \mu : M \longrightarrow \mathcal{G}^* \) is smooth. Also, since \( \theta \) is a regular value of \( \mu \) then \( m \in \mu^{-1}(\theta) \) is a regular element of \( \mu \). That is, \( T_m \mu \) is surjective at each \( m \in \mu^{-1}(\theta) \). That is, \( \text{rank} (T_m \mu) = \text{rank} (\mu) = k \). Thus, \( T_m \mu^{-1}(\theta) = \text{Ker} T_m \mu \). But, we know from Linear Algebra that \( \dim T_m M = \dim \text{Ker} T_m \mu + \dim \text{im} T_m \mu \) (the rank theorem for a linear map). We have \( \dim V = \dim W + \dim W^\perp \) and \( (V/W)^* = W^\omega = W^\perp \). Now, we can set \( W = \text{Ker} T_m \mu \). Then, \( K\text{er} T_m \mu^\omega = (T_m M/\text{Ker} T_m \mu)^* \cong (\text{im} T_m \mu)^* \cong \text{im} T_m \mu \). Recall that the tangent map \( T_m \mu : T_m M \longrightarrow \mathcal{G}^*, \; v \longmapsto T_m \mu(v) = \alpha, \) for \( v \in T_m M \) and \( \alpha \in \mathcal{G}^* \) since \( T_m(\mathcal{G})^* \cong \mathcal{G}^* \) from Linear Algebra. The map \( \mathcal{A}_m : \mathcal{G} \longrightarrow T_m M, \; \xi \longmapsto \xi_M(m) \) is one-to-one, since the action is free. That is, \( \mathcal{G} \cdot m \subset T_m M \) and the following holds: \( \mathcal{G} \cong \text{im} \mathcal{A}_m = \mathcal{A}_m(\mathcal{G}) = T_m(\mathcal{G} \cdot m) = \mathcal{G} \cdot m. \) Since \( \mathcal{A}_m \) is a linear map, it follows that \( \text{Ker} \mathcal{A}_m = \{ \xi \in \mathcal{G} \mid \xi_M(m) = 0 \} = \mathcal{G} \subset \mathcal{G}. \) By the same arguments as above we can write:

\[ \text{im} \mathcal{A}_m^\omega = (T_m M/\text{im} \mathcal{A}_m)^* \cong (\text{Ker} \mathcal{A}_m)^* \cong \text{Ker} \mathcal{A}_m = \mathcal{G}_m. \]

We claim that \( \text{im} \mathcal{A}_m^\omega = \text{Ker} T_m \mu \subset T_m M. \) For,

\[ \text{Ker} T_m \mu = \{ v \in T_m M \mid T_m \mu(v) = \alpha = 0, \; \alpha = \iota_{\xi_M(m)} \omega_m, \text{ for all } \xi \in \mathcal{G} \} \]

\[ = \{ v \in T_m M \mid T_m \mu(v)(\xi) = \alpha(\xi) = 0, \alpha = \iota_{\xi_M(m)} \omega_m, \text{ for all } \xi \in \mathcal{G} \} \]

\[ = \{ v \in T_m M \mid < T_m \mu(v), \xi > = d_m \mu(\xi)(v) = 0, \text{ for all } \xi \in \mathcal{G} \} \]

\[ = \{ v \in T_m M \mid \omega_m(\xi_M(m), v) = 0, \text{ for all } \xi \in \mathcal{G} \} \]

\[ = \{ v \in T_m M \mid v \perp \xi_M(m), \text{ for all } \xi \in \mathcal{G} \} \]

\[ = \{ v \in T_m M \mid v \perp \text{im} \mathcal{A}_m \} \]

\[ = \{ v \in T_m M \mid v \in \text{im} \mathcal{A}_m^\omega \} \]

\[ = \text{im} \mathcal{A}_m^\omega. \]  

(5.53)

The duality between the kernel and the range of a linear map, we have:

\[ \text{Ker} T_m \mu = \text{im} \mathcal{A}_m^\omega \cong \text{Ker} \mathcal{A}_m = \mathcal{G}_m \text{ and } \text{im} \mathcal{A}_m = \text{Ker} T_m \mu^\omega \cong \text{im} T_m \mu. \]
It follows that $Ker T_m \mu^\omega \simeq Ker A_m \omega = G_m^\circ$. Therefore, $im T_m \mu = G_m^\circ$. □

The notations in the formula $\xi_M \perp \alpha = 0$ to define horizontal forms in Definition 5.7.2, is embodying a complex process of symplectic reduction as stated in Lemma 5.7.1, but particularly as show in Equation (5.53) in the proof of the latter lemma. This gives the full meaning and interconnection between objects of interest in the symplectic reduction. It is the cornerstone of the whole arguments in the construction of complexes of the quotient space and its symplectic quotient.

Recall Lemma 3.5.5 for the statement: if the $G$-action is free then $G.m = T_m(G.m) = \{\xi_M(m) \mid \xi \in G\} \simeq G$. And also the Corollary 3.5.6 for the equalities $G.m = T_m(G.m) = Ker T_m \mu^\omega = T_m \mu_1^{-1}(\theta) \omega_m$ and $G.m^\omega = T_m(G.m)^\omega = Ker T_m \mu = T_m \mu_1^{-1}(\theta)$. We should recall here the Remark 3.5.1 particularly its two first items. That is, let $m \in Z = \mu^{-1}(\theta)$. We have: $T_m \mu_1^{-1}(\theta)$ and $T_m(G.m)$ are orthogonal complement in the symplectic linear space $(T_m M, \omega_m)$. And then $T_m(G.m)$ is an isotropic linear subspace of the symplectic linear space $(T_m M, \omega_m)$. That is, $T_m(G.m) \subset T_m(G.m)^\omega = Ker \mu_m = T_m \mu_1^{-1}(\theta)$. The kernel $Ker \mu_1^{-1}(\theta) = T_m(G_\theta.m)$, of the restriction of the symplectic structure to $Z = \mu^{-1}(\theta)$, is an isotropic linear subspace of $T_m \mu_1^{-1}(\theta)$. From Lemma 3.5.7, we retain the following: $T_m(G_\theta.m) = T_m(G.m) \cap T_m \mu_1^{-1}(\theta)$, $T_m(G_\theta.m) = Ker T_m \mu^\omega \cap Ker T_m \mu$, and $G_\theta.m = G.m \cap G.m^\omega$.

**Definition 5.7.3.** Let $G$ be the Lie algebra of a Lie group $G$ and $\theta \in G^*$. The subgroup of $G$ denoted by $G_\theta = \{g \in G \mid Ad_g^*\theta = \theta\}$ is called the isotropy subgroup of $\theta$ with regard to the co-adjoint action of $G$ on $G^*$. The set $G.\theta = \{Ad_g^*\theta \mid g \in G\} \subset G^*$ is the orbit of the co-adjoint action of $G$ on $G^*$. The Lie algebra of $G_\theta$, denoted by $G_\theta = \{\xi \in G \mid ad^*(\xi)\theta = \theta\}$, is the isotropy subalgebra of $\theta$.

**Lemma 5.7.2.** Let $G$ be the Lie algebra of a $\mathbb{F}$-Lie group $G$ and $\theta \in G^*$, a regular value of a moment map $\mu : M \longrightarrow G^*$ associated to a Hamiltonian $G$-action on a symplectic locally Euclidean space $(M, \omega)$. Assume the $G$-action free and proper. Then

1. The subgroup $G_\theta$ is a compact (thus, closed) set in $G$, acting smoothly on $\mu^{-1}(\theta)$.
2. $G_m \subset G_\theta$ for all $m \in \mu^{-1}(\theta)$.
3. $\mu^{-1}(\theta)$ is invariant under the restricted action of $G_\theta$.
4. Every $\alpha \in G.\theta = \{Ad_g^*\theta \mid g \in G\} \subset G^*$ is a regular value of the moment map $\mu$.
5. $G_\theta$ acts freely and properly on the locally Euclidean subspace $\mu^{-1}(\theta)$. 

Notice that if $i_\theta : \mu^{-1}(\theta) \hookrightarrow M$ is the canonical inclusion, then the induced 2-form $\omega_{|\mu^{-1}(\theta)} := (i_\theta^* \omega)$ has a constant rank, under the assumptions of Lemma 3.5.8. It comes from Corollary 3.5.9 that $T_m(G.m)$ is an isotropic linear subspace of the symplectic linear space $(T_m M, \omega_m)$. That is, $T_m(G.m) \subset T_m(G.m)^* = \text{Ker}d\mu_m = T_m\mu^{-1}(\theta)$ and also $\text{Ker}_m\omega_m|\mu^{-1}(\theta) = T_m(G_\theta.m)$ is an isotropic linear subspace of $T_m\mu^{-1}(\theta)$. The existence of the induced smooth moment map $\bar{\mu} : M/G \longrightarrow G^*/G$ is ensured by Lemma 3.5.10, whereas, Lemma 3.5.10 asserts that all $\theta \in G^*$ are regular value for the moment map $\mu : M \longrightarrow G^*$ associated to the Hamiltonian, free and proper $G$-action. Now, before we come back to the cochain complexes on $Z = \mu^{-1}(\theta)$ (the fiber of the moment map) and $M_\theta = \mu^{-1}(\theta)/G_\theta = Z/G_\theta$ (the reduced space), we give a important remark below.

**Remark 5.7.1.**

1. The action of $G$ on $Z := \mu^{-1}(\theta) \subset M$ is reduced to the one from $G_\theta$,

2. $Z$ is invariant for the action of $G_\theta$, and if $m \in Z$, then $T_{[m]}(\mu^{-1}(\theta)/G_\theta.m) = T_m\mu^{-1}(\theta)/T_m(G_\theta.m) = T_m(G_\theta.m)\omega_m/T_m(G_\theta.m)$.

### 5.7.4 de Rham complexes for $Z = \mu^{-1}(\theta)$ and $M_\theta = Z/G_\theta$

First of all, let us consider the commutative diagram of smooth maps below and it will play a central role in the definition of forms on $M_\theta$ and the construction of complex isomorphisms:

$$
\begin{array}{ccc}
Z & \xrightarrow{\iota_Z} & M \\
\downarrow{\pi_Z} & & \downarrow{\pi} \\
Z/G & \xrightarrow{\mathcal{I}} & M/G
\end{array}
$$

This reads, $\pi \circ \iota_Z = \mathcal{I} \circ \pi_Z$ or equivalently $\iota_Z^* \pi = \pi_Z^* \mathcal{I}$, with $\iota_Z = \iota_\theta$.

The smooth structure on $Z/G$ is induced by the inclusion map $\mathcal{I}$. It will be denoted as usual by $C^\infty(Z/G)$. So, the map $\pi_Z^* : C^\infty(Z/G) \longrightarrow C^\infty(Z)^G$ is naturally an isomorphism of $\mathbb{R}$-algebras because it is the restriction of an isomorphism $\pi^* : C^\infty(M/G) \longrightarrow \mathcal{F}_M^G$ encountered
a while before in the preceding Subsection. We have that the space $Z/G$ is a closed subset in $M/G$ and therefore $C^\infty(Z/G) = C^\infty(M/G)|_{Z/G}$ is the set of 0-forms on $Z$. We are now interested in the characterization of exterior differential forms on the symplectic quotient. We are once again making use of the reference [115]. Therefore, we define a basic form on $Z$ as a form $\zeta$ which is $G$-invariant (that is, for each $g \in G$, one has $\sigma_g^* \zeta = \zeta$) and horizontal (that is, for each $\xi \in \mathfrak{g}$, one has $\xi_Z \cdot \zeta = 0$, with $\xi_Z : Z \rightarrow T_Z$). Equivalently and with regard to Definitions 5.7.1 and 5.7.2, we define a basic form on $Z$ as a $G$-invariant form $\zeta \in \Omega^k(Z)^G$ such that at each $m \in Z$ and for each $v \in T_m(G.m) = G.m = A_{m|Z}(G) = im A_{m|Z} \subset T_mZ$, one has $v \cdot \zeta = 0$. We will write $\Omega^k_{\text{basic}}(Z)$ for the $\mathbb{R}$-algebra of all basic $k$-forms on $Z$. We can characterize basic forms on $Z$ by:

$$\zeta \in \Omega^k_{\text{basic}}(Z) \iff \text{there exists } \alpha \in \Omega^k_{\text{basic}}(M) \text{ such that } \iota_Z^* \alpha = \alpha|_Z = \zeta. \quad (5.54)$$

Now, for $k$-forms where $k \geq 0$ we can build cochain complexes on $Z$ and $Z/G$. Since the coboundary operator commutes with the pullback we have the following: $d\zeta = d\iota_\theta^* \alpha = \iota_\theta^* d\alpha$, with $d\zeta \in \Omega^{k+1}_{\text{basic}}(Z)$ and $d\alpha \in \Omega^{k+1}_{\text{basic}}(M)$. Hence, $d\zeta$ and $d\alpha$ are basic forms. It follows that $\Omega^*_{\text{basic}}(Z)$ is a subcomplex of $\Omega^*_{\text{basic}}(M)$ and if we set $\pi_\theta = \pi|_Z$ then $\pi_Z^* : \Omega^*(M_\theta) \rightarrow \pi_Z^*(\Omega^*(M_\theta)) \subset \Omega^*_{\text{basic}}(Z)$ is a one-to-one $\mathbb{R}$-algebra homomorphism. We must characterize the basic forms on $Z$ that are in $\pi_Z^*(\Omega^*(M_\theta))$. For, we recall defining properties of a stratification induced by the action of a Lie group $G$ acting properly, freely on a space $M$ and in a Hamiltonian fashion with a moment map $\mu$.

**Definition 5.7.4.** Let $H$ be a closed subgroup of $G$, with $H$ running over closed subgroups of $G$.

Let $M(H) := \{m \in M|G_m \text{ is conjugate of } H\} = \{m \in M|gG_mg^{-1} = H\} = \{m \in M|gG_m = Hg\}$. Then, the disjoint unions below are true

1. $\bigcup_{H \in G} M(H) = M$,
2. $\bigcup_{H \in G} Z(H) = Z$, where $Z(H) := Z \cap M(H)$,
3. $\bigcup_{H \in G} \pi(M(H)) = M/G$, where $\pi(M(H)) = M(H)/G = (M/G)(K)$,
4. $\bigcup_{H \in G} \pi_Z(Z(H)) = Z/G = M_\theta$, where $\pi_Z(Z(H)) = Z(H)/G = (Z/G)(K) = M_\theta(H)$,
5. and the respective restrictions of $\pi_Z$ and $\iota_Z$ with regard to each $H$ are $\pi_{Z|Z(H)} := \pi(H)$ and $\iota_{Z|Z(H)} := \iota(H)$.

**Theorem 5.7.3.** Principal Orbit Theorem [68]

1. The partitions defined above induce stratification on $M$, $Z$, $M/G$, $Z/G$ respectively, and strata are connected components of the sets in the disjoint unions.
2. All strata $M$ and $Z$ are $G$-invariant.

3. There is a closed subgroup $K$ of $G$ such that $M_{(K)}$, $\pi(M_{(K)}) = M_{(K)}/G = (M/G)_{(K)}$, and $Z_{(K)}$ are open dense strata of $M$, $M/G$ and $Z$.

4. Since the moment map $\mu$ is a proper map then $\pi(Z_{(K)}) = Z_{(K)}/G = (Z/G)_{(K)}$ is an open dense connected stratum in $Z/G$.

5. Also, the map $\iota_Z$, $\pi$, $\pi_Z$ and $\Theta$, descend to strata with regard to the given stratification.

The map $\iota_Z$, $\pi$, $\pi_Z$ and $\Theta$ are given in the first diagram in Subsection 5.7.4. Recall that any $F$-smooth map is continuous with regard to $F$-topologies, and the open dense property stated in the previous theorem induces uniquely extended continuous maps, that is smooth maps, for following respective inclusions: $M_{(K)} \subset M$, $(M/G)_{(K)} \subset M/G$, $Z_{(K)} \subset Z$ and $(Z/G)_{(K)} \subset Z/G$. We have all the ingredients that allow us to define the differential forms on the symplectic quotient $M_\theta = Z/G_\theta$. For,

$$\beta \in \Omega^k(Z/G_\theta)$$

$$\Downarrow$$

there exists $\zeta \in \Omega^k_{\text{basic}}(Z)$ such that $\pi^* \beta = \zeta$

$$\Downarrow$$

there exists $\alpha \in \Omega^k_{\text{basic}}(M)$ such that $\iota_Z^* \alpha = \zeta$,

$$\Downarrow$$

$$\pi_Z^* \beta = \zeta = \iota_Z^* \alpha.$$  \hspace{1cm} (5.55)

Moreover, the restrictions of later maps on $Z_{(K)}$ give the following equation:

$$\pi_{(K)}^* \beta = \zeta|_{Z_{(K)}} = \iota_{(K)}^* \alpha.$$  \hspace{1cm} (5.56)

When we evoke the open dense property invoked in the Principal Orbit Theorem above, the forms $\beta$, $\zeta$ and $\alpha$ become the unique extensions of maps in Equation (5.55). It follows that Equation (5.55) defines the $\pi_Z^* : \Omega^*(M_\theta) \rightarrow \pi_Z^*(\Omega^*(M_\theta))$ and it characterizes the set of forms $\beta \in \Omega^k(Z/G_\theta)$ if and only if there exists unique $\zeta \in \Omega^k_{\text{basic}}(Z)$ such that $\pi^* \beta = \zeta$ if and only if there exists a unique $\alpha \in \Omega^k_{\text{basic}}(M)$ such that $\iota_Z^* \alpha = \zeta$. Equivalently,

$$\beta \in \Omega^k(Z/G_\theta) \text{ if } \beta \in \Omega^k((Z/G_\theta)_{(K)}) \text{ and there exists a unique }$$

$$\alpha \in \Omega^k_{\text{basic}}(M) \text{ such that Equation (5.55) is satisfied.}$$  \hspace{1cm} (5.56)
Naturally, the isomorphism \( \pi_{Z}^{*} : \Omega^{*}(M_{\theta}) \rightarrow \pi_{Z}^{*}(\Omega^{*}(M_{\theta})) \) is well-defined by \( \pi_{Z}^{*}(\beta) = \zeta \) such that \( \pi_{(K)}^{*}\beta = \zeta_{Z_{(K)}} = \iota_{(K)}^{*}\alpha \). Otherwise, a form \( \beta \in \Omega^{k}(Z/G_{\theta}) \) is a unique extension of a form \( \beta \in \Omega^{*(Z_{(K)})} \), the latter form \( \beta \) is sent on \( \zeta \in (\Omega^{*(Z)})_{(K)} \) and finally is extended uniquely to a form \( \zeta \in (\Omega^{*}_{basic}(Z)) \). Let us denote by \( (\Omega^{*}_{basic}(Z))_{(K)} := \pi_{Z}^{*}(\Omega^{*}(M_{\theta})) \subset \Omega^{*}_{basic}(Z) \). The situation we are in is visualized by the diagrams of complexes below:

\[
\begin{array}{ccc}
\Omega^{*}(Z/G) & & \Omega^{*}((Z/G)_{(K)}) \\
\downarrow \pi_{Z}^{*} & & \downarrow \pi_{(K)}^{*} \\
\Omega^{*}_{basic}(M) & & \Omega^{*}_{basic}(M_{(K)}) \\
\end{array}
\]

One can conclude that the isomorphism \( \pi_{Z}^{*} : \Omega^{*}(M_{\theta}) \rightarrow (\Omega^{*}_{basic}(Z))_{(K)} \) is an isomorphism of cochain complexes since any pullback commutes with the coboundary operator \( d \) which descends canonically to all the sets of forms in the diagrams above. So, Equation (5.55) yields:

\[
d\pi_{(K)}^{*}\beta = d\zeta_{Z_{(K)}} = d\iota_{(K)}^{*}\alpha \iff \pi_{(K)}^{*}d\beta = d\zeta_{Z_{(K)}} = \iota_{(K)}^{*}d\alpha. \quad (5.57)
\]

It follows that \( d\beta \in \Omega^{*}(M_{\theta}) \) since it satisfies the defining condition with regard to the equation above.

### 5.7.5 de Rham cohomology on a reduced \( \mathbb{F} \)-space

In [115], it is shown that the pullback of any diffeomorphism between two orbit spaces descends to a diffeomorphism of their respective open dense connected subsets. This is actually an isomorphism of complexes of orbit spaces. In particular, this property holds for two diffeomorphic symplectic quotients, too. The functorial behavior of the quotient guarantees the existence of a de Rham algebra cohomology on the symplectic quotient. The natural process of the construction of the cohomology on the symplectic quotient is starting from the cohomology on \( M \), then a cohomology on \( Z \), which in turn induces a cohomology of the complex \( \Omega^{*}_{basic}(Z) \). And finally, a cohomology of the subcomplex \( (\Omega^{*}_{basic}(Z))_{(K)} \) of the complex of \( \Omega^{*}_{basic}(Z) \). The last cohomology is isomorphic to the algebra cohomology of the symplectic quotient, provided that the exterior product passes to the quotient as by Definition 5.3.3 in Subsection 5.3.4.
5.7 Isomorphism on reduced space

5.7.6 Isomorphisms of classical cohomologies and sheaf cohomology on the reduced space

All the constructs made in this Section 5.7, and back to Section 5.3 confirm the existence of a de Rham cohomology algebra on the symplectic quotient $M_\theta$ as any Frl-object, with respect to Theorems 5.6.3 and 5.6.4. Nevertheless, our aim in this thesis is to endow the symplectic quotient $M_\theta$ with a de Rham cohomology canonically induced from the de Rham cohomology on the ambient space $M$. Now, we can, with regard to Theorems 5.6.3 and 5.6.4, Equations (5.54) to (5.57)) in Subsection 5.7.4, give rise to a theorem which concludes the main question of this thesis in the following terms.

**Theorem 5.7.4.** Let $(M, \omega)$ be a symplectic locally Euclidean space endowed with a free, proper and Hamiltonian action with an equivariant moment map. Let $\mathbb{H}^k_{\Delta-S}(M, \mathbb{R})$, $\mathbb{H}^k_{dRH}(M)$, $\mathbb{H}^k_{\Delta\infty}(M, \mathbb{R})$, $\mathcal{H}^k(M, \mathbb{R})$ and $\mathcal{H}^k(M, \mathcal{R})$ be respective cohomologies on $M$. Then, the five cohomologies $\mathbb{H}^k_{\Delta-S}(M_\theta, \mathbb{R})$, $\mathbb{H}^k_{dRH}(M_\theta)$, $\mathbb{H}^k_{\Delta\infty}(M_\theta, \mathbb{R})$, $\mathcal{H}^k(M_\theta, \mathbb{R})$ and $\mathcal{H}^k(M_\theta, \mathcal{R})$ on the symplectic quotient are $\mathbb{R}$-algebras and isomorphic.

5.7.7 Multiplicative structure and Kunneth formulas

The idea is to introduce a sort of multiplicative operation on the cohomology module $\mathcal{H}^*(M, \mathcal{F})$. So, given $a \in \mathcal{H}^k(M, \mathcal{F})$ and $b \in \mathcal{H}^l(M, \mathcal{F})$ we are willing to define a product $ab \in \mathcal{H}^{k+l}(M, \mathcal{F})$ such that $\mathcal{H}^*(M, \mathcal{F})$ becomes a ring and more better an algebra. In this order we introduce the cohomology cross product which turns into the cup product by using the diagonal map arguments. So, $\mathcal{H}^*(M, \mathcal{F})$ becomes a graded ring by this operation (see [50, 53]).
Chapter 6

Modern Formalism of mechanics

In third chapter, we have studied symplectic Frölicher spaces, more particularly those locally Euclidean. Thereafter, we introduced the notion of Hamiltonian vector fields in the category of Frölicher spaces. Also we showed the link between 1-forms and some specific vector fields. In the present chapter we will introduce a more general structure than the symplectic structure, the so called Poisson structure (see [74, 90, 102, 117, 120]). We shall show that the two structures are closely related in the sense that every symplectic Frölicher space is a Poisson space. While the converse is not true since the zero bracket makes any Frölicher space into a Poisson space. The purpose is of setting down the formulation of the Poisson structure, also called Poisson bracket, in the language of Frölicher structure rather than in the setting of manifolds.

Whereas, the phase spaces arising in classical mechanics have often an additional geometric structure due to the symplectic structure, although it may be considered that in practice the operation given by the Poisson structure on the set of structure functions is more important than those given on the phase space by the symplectic structure. We will then point out that we need both structures on the configuration space (see [1, 108].)

6.1 Poisson geometry

Let \( (M, \mathcal{C}_M, \mathcal{F}_M) \) be a \( \mathbb{F} \)-space and the topology \( \tau_{\mathcal{F}_M} \) on \( M \) be Hausdorff paracompact. Recall that \( \mathcal{F}_M \) is a real algebra.

6.1.1 Almost-Poisson structure

**Definition 6.1.1.** An almost-Poisson structure (or bracket) on \( (M, \mathcal{C}_M, \mathcal{F}_M) \) is a Frölicher smooth map, defined by \( \{,\} : \mathcal{F}_M \times \mathcal{F}_M \to \mathcal{F}_M \) with the assignment \( (f, g) \to \{f, g\} \) of
The pair $((M, \mathcal{C}_M, \mathcal{F}_M), \{,\})$ is called an almost Poisson Frölicher space. We give an interpretation of the definition above in the light of Cartesian closedness property and derivation in the category $\text{Frl}$. Indeed, one may notice that for a fixed $h \in \mathcal{F}_M = C^\infty(M, \mathbb{R})$ the almost Poisson structure (bracket) $\{,\}$ induces a smooth map $\{h,\} : \mathcal{F}_M \rightarrow \mathcal{F}_M$ defined by $\{h,\}(f) : M \rightarrow \mathbb{R}$ for all $f \in \mathcal{F}_M$. As a matter of easing notation, we denote $p := \{h,\}$. And then by the Cartesian closedness of the category $\text{Frl}$ of Frölicher spaces with regard to the exponential law: $C^\infty(\mathcal{F}_M \times M, \mathbb{R}) \cong C^\infty(\mathcal{F}_M, C^\infty(M, \mathbb{R})) \cong C^\infty(\mathcal{F}_M, \mathcal{F}_M)$, it follows that $p$ is smooth if and only if the associated map $\hat{p} : \mathcal{F}_M \times M \rightarrow \mathbb{R}$ is smooth. We have $p \in C^\infty(\mathcal{F}_M, \mathcal{F}_M) \iff \hat{p} \in C^\infty(\mathcal{F}_M \times M, \mathbb{R}) \iff \hat{p} = p \circ \text{rev}^{-1} \in C^\infty(M \times \mathcal{F}_M, \mathbb{R})$ where $\text{rev} : \mathcal{F}_M \times M \rightarrow M \times \mathcal{F}_M$ is a swap of components for a given pair and such that for $x \in M, f \in \mathcal{F}_M, p(f) \in \mathcal{F}_M$ we have $p(f)(x) = \hat{p}(f, x) = \hat{p}(x)(f)$. This comes from a judicious adaptation of Equations (4.3), (4.4), (4.5) and (4.6) as in the diagram below:

![Diagram](image)

But, smooth maps in $C^\infty(\mathcal{F}_M \times M, \mathbb{R})$ are generated by $\{(df)_x | f \in \mathcal{F}_M, x \in M\}$. We put $\hat{p}(f, x) := (df)_x$. Therefore, it turns out that $p$ is smooth globally. Moreover, $p$ is $\mathbb{R}$-linear and has the Leibniz property. Hence, by a conjunction of Definition 2.4.1 and Definition 2.5.3, the map $p = \{h,\}$ defines a smooth derivation induced by $h$ on $\mathcal{F}_M$. So, we shall denote $p := X_h$ as a smooth vector field induced by $h$. Furthermore, observe that from the skew-symmetry property of $\{,\}$ follows the identity:

$$X_h(f) = -X_f(h) \quad (6.1)$$

**Proposition 6.1.1.** The set of all almost Poisson structures on $(M, \mathcal{C}_M, \mathcal{F}_M)$ is a module over $\mathcal{F}_M$. 
Proof. The proof is based on Definition 6.1.1. For a fixed \( h \in \mathcal{F}_M \), the almost Poisson structure induces a smooth derivation \( \{ h, \} : \mathcal{F}_M \to \mathcal{F}_M \) such that the following properties are immediate from Definition 6.1.1 of \( \{ , \} \):

1. \( k\{ h, f \} = \{ h, kf \} \) is an almost Poisson structure for all \( f \in \mathcal{F}_M \), that is, \( \{ h, k \} = 0 \) for all \( k \in \mathbb{R} \).
2. \( \{ h, f \}_1 + \{ h, g \}_2 = \{ h, f + g \} \) is an almost Poisson structure for any two almost Poisson structures \( \{ , \}_1, \{ , \}_2 \).

First of all, the derivation \( X_h \) induced by \( h \) is smooth by definition. Secondly, the product and sum of two smooth functions are smooth too. Hence one can conclude that the two operations are smooth. It turns out that the set of all almost Poisson structures is an module over \( \mathcal{F}_M \). That is, the aforementioned set is a submodule of \( \text{Der}(M) \), which is the set of smooth derivations \( d : \mathcal{F}_M \to \mathcal{F}_M \). \( \square \)

6.1.2 Poisson structure, Poisson maps and Poisson structure functions

Definition 6.1.2. One calls Poisson structure on a Frölicher space \((M, C_M, \mathcal{F}_M)\) a smooth map satisfying the two following properties on \( \mathcal{F}_M \):

1. \( \{ , \} \) is an almost Poisson structure,
2. Jacobi identity: \( \{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0 \), for all \( f, g, h \in \mathcal{F}_M \)

In this case the pair \(( (M, C_M, \mathcal{F}_M), \{ , \} ) \) is called Poisson Frölicher space. Equivalently, \(( (M, C_M, \mathcal{F}_M), \{ , \} ) \) is Poisson Frölicher space if and only if the bracket is a derivation in each component and \( \mathcal{F}_M \) is a Lie algebra when the bracket is an internal operator taken as a multiplication.

Example 6.1.1. Let \((M, C_M, \mathcal{F}_M)\) be a Frölicher space and consider the Abelian Lie algebra of \( 2n \) smooth vectors fields \( X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n \) such that \( [X_i, X_j] = [X_i, Y_j] = [Y_i, Y_j] = 0 \) for \( i \neq j \), where \([ , ]\) is the usual Lie commutator (bracket) on \( \mathfrak{X}(M) \). A map \( \{ , \} : \mathcal{F}_M \times \mathcal{F}_M \to \mathcal{F}_M \) is defined by \( \{ f, g \} = \sum_{i=1}^{n} (X_i(f)Y_i(g) - X_i(g)Y_i(f)) \) for all \( f, g \in \mathcal{F}_M \), is by some algebraic computations clearly smooth and a Poisson structure.
In line with Definition 6.1.2, a case worth of consideration is given by special smooth function $h \in \mathcal{F}_M$ whose Poisson bracket $\{h, f\} = X_h(f) = 0$ for all $f \in \mathcal{F}_M$. Such a function $h$ is said to be vanishing identically and it is called distinguished or Casimir function. That is, $h$ is a Casimir function if and only if its associated Hamiltonian vector field $X_h$ vanishes everywhere.

Another situation that deserves attention is the vector field $X_h$ on $M$ associated to (or generated by) a fixed function $h$ with regard to Equation (6.1). From now on, the vector field $X_h$ will be called the Hamiltonian vector field associated to $h$ with regard to the Poisson structure.

**Proposition 6.1.2.** Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space and $\{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2$ be two Poisson structures on $(M, \mathcal{C}_M, \mathcal{F}_M)$.

Then, on $(M, \mathcal{C}_M, \mathcal{F}_M)$ the difference $\{\cdot, \cdot\}_1 - \{\cdot, \cdot\}_2$ is an almost Poisson structure.

**Proof.** Obviously, $\{\cdot, \cdot\}_1 - \{\cdot, \cdot\}_2$ is bilinear and skew-symmetric by definition of almost Poisson structures $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$. We can now check the Leibniz property. In fact for all $f, g \in \mathcal{F}_M$

$$k(\{f, g\}_1 - \{f, g\}_2) = \{f, kg\}_1 - \{f, kg\}_2$$

$$= \{f, k\}_1 g + k\{f, g\}_1 - \{f, k\}_2 g - k\{f, g\}_2$$

$$= (\{f, k\}_1 - \{f, k\}_2)g + (\{f, g\}_1 - \{f, g\}_2)k$$

(6.2)

**Definition 6.1.3.** A smooth map $\phi : M \rightarrow N$ of Poisson Frölicher spaces is called a Poisson map if $\phi^* \{f, g\}_N = \{\phi^* f, \phi^* g\}_M$, for all $f, g \in \mathcal{F}_N$ and $\phi^* f, \phi^* g \in \mathcal{F}_M$. That is, $\{f, g\}_N \circ \phi = \{f \circ \phi, g \circ \phi\}_M$. A Poisson map is also called a canonical map as it preserves the Poisson bracket.

Notice that from the literature on Poisson maps (see [21] and references therein) the following statements summarize some of their properties.

**Lemma 6.1.3.** Let $(M, \{\cdot, \cdot\})$, $(N, \{\cdot, \cdot\})$ be two Frölicher spaces and $X_H$ be a Hamiltonian field associated to $H : N \rightarrow \mathbb{R}$. Then, The flow $H_t = exp(tX_H)$ of the Hamiltonian vector field $X_H$ is a Poisson map. That is, for $F, G : N \rightarrow \mathbb{R}$ two Hamiltonian, we have $H_t^* \{F, G\} = \{F \circ H_t, G \circ H_t\}$.

**Corollary 6.1.4.** Let $X_H$ be a Hamiltonian vector field on a Poisson Frölicher space $M$ and $H_t = exp(tX_H)$ its flow. Then, for any $t \in \mathbb{R}$ and $x \in M$, $M$ has the same rank at $x$ and at $H_t(x) = exp(tX_H)(x)$.
Lemma 6.1.5. Let \((M, \{, \}), (N, \{, \})\) be two Frölicher spaces and \(X_H\) be a Hamiltonian field associated to \(H : N \to \mathbb{R}\), with \(H_t = \exp(tX_H)\), the flow of \(X_H\). Let \(\phi : M \to N\) a smooth Poisson map of Frölicher spaces, such that at each \(x \in M\), the tangent map \(\phi_x : T_x M \to T_{\phi(x)} N\). If \(\psi_t\) is the flow of \(X_{H \circ \phi}\). Then, \(H_t \circ \phi = \phi \circ \psi_t\) and \(\phi_x \circ X_{H \circ \phi} = X_H \circ \phi\).

Definition 6.1.4. Let \(((M, \mathcal{C}_M, \mathcal{F}_M), \{, \})\) be a locally Euclidean \(n\)-\(\mathbb{R}\)-space, \(H : M \to \mathbb{R}\) a Hamiltonian function associated to its Hamiltonian vector fields \(X_H\), and \(U\) an open neighborhood of \(x \in M\). We can express \(X_H\) in local coordinates \(x^1, \ldots, x^n\) at \(x \in U \subset M\) by

\[
X_H = \sum_{i=1}^{n} h^i(x) \frac{\partial}{\partial x^i},
\]

where \(h^i : U \to \mathbb{R}\) are smooth functions and with respect to notations in Lemma 2.2.4, Definition 2.2.3 and Definition 2.4.4. The structure functions of the Poisson structure (also called Poisson structure functions) on \(M\) are the basic brackets \(\{x^i, x^j\}\), with \(1 < i < n\) and \(1 < j < n\).

In general and without assuming any Poisson related conditions, when given any smooth vector fields \(X_H, Y\), and any smooth function \(F\), we have \(X_H(F) = dF(X_H), \mathcal{L}_{X_H}(F) = X_H(F)\) with \(\mathcal{L}_{X_H}(c) = 0\) for \(F\) a 0-form and \(F = c\) a constant, respectively. But, \(dF\) is a 1-form, \(\mathcal{L}_{X_H}\), applying a \(k\)-form to a \(k\)-form, is a \(\mathbb{R}\)-linear map and a local operator satisfying \(\mathcal{L}_{X_H}(Y) = [X_H, Y]\) such that \(\mathcal{L}_{[X_H,Y]} = [\mathcal{L}_{X_H}, \mathcal{L}_Y]\) as by Proposition 2.5.4 in Section 2.5 (also see [1, Definition 2.2.12, p85]). Now, with regard to Poisson structure we have

\[
\mathcal{L}_{X_H}(F) = X_H(F) = dF(X_H) = \{F, H\}
\]

Equation (6.3) and Definition 6.1.4 yield the following equations.

\[
\{F, H\} = X_H(F) = \sum_{i=1}^{n} h^i(x) \frac{\partial F}{\partial x^i}
\]

(6.4)

\[
\{x^i, H\} = X_H(x^i) = \sum_{i=1}^{n} h^i(x) \frac{\partial x^i}{\partial x^j} = h^i(x)
\]

(6.5)

\[
\{F, H\} = X_H(F) = \sum_{i=1}^{n} \{x^i, H\} \frac{\partial F}{\partial x^i}
\]

(6.6)

\[
\{H, x^i\} = X_{x^i}(H) = \sum_{i=1}^{n} \{x^i, x^i\} \frac{\partial H}{\partial x^j}
\]

(6.7)

\[
\{F, H\} = X_H(F) = \sum_{i=1}^{n} \sum_{j=1}^{n} \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial x^j}
\]

(6.8)
6.1.3 Poisson structure induced by a symplectic structure

In Section 2.5 we presented a review of different operators like $d$, $\wedge$, $[,]$, $\iota$ and $\mathcal{L}$ and in Chapter 3 they were presented with regard to the symplectic structure $\omega$, the flow of a vector field and integral curve for a vector field.

$$\mathcal{L}_{X_H}(F) = X_H(F) = dF(X_H) = \omega(X_F, X_H) \quad (6.9)$$

The combination of Equation (6.3) and Equation (6.9) yields a new equation as follows:

$$\{F, H\} = \mathcal{L}_{X_H}(F) = X_H(F) = dF(X_H) = \omega(X_F, X_H) \quad (6.10)$$

We know that the symplectic structure $\omega$ induces a linear map of modules:

$$\omega^\sharp : \mathfrak{X}(M) \rightarrow \Omega^1(M) \text{ defined by } \omega^\sharp(X) = \iota_X \omega = \omega(X, \cdot) \quad (6.11)$$

which is injective by construction since $\omega$ is nondegenerate. The inverse map $\omega^\flat$ associates to every 1-form $\alpha = \iota_X \omega$ a unique vector field denoted $X_\alpha$ or $X_f$ if $\alpha = df$ for $f \in \mathcal{F}_M$. The vector field surely exists if $\omega^\flat$ is an isomorphism. This correspondence fails in the general case where $\omega^\flat$ is not an isomorphism. In this case the solution restricts to the domain of definition of the vector field. We need to show that $\omega^\flat$ induces Poisson structure on $\mathcal{F}_M$ and on $\Omega^1(M)$ respectively. Then, we can present some properties of the Poisson structure that should serve, from now on, as tools in defining ongoing concepts in this work. We define two brackets (induced by the symplectic form $\omega$) on $\mathcal{F}_M$ and $\Omega^1(M)$ by

$$\{f, g\} = -\iota_{X_f} \circ \iota_{X_g} \omega \quad \text{and} \quad \{\alpha, \beta\} = -\iota_{[X_\alpha, X_\beta]} \omega, \text{ respectively.} \quad (6.12)$$

We shall prove the following lemmas prior we prove that the two brackets are Poisson structures:

Lemma 6.1.6. Let $\alpha, \beta \in \Omega^1(M)$ and $\{,\}$ as defined above. Then,

$$\{\alpha, \beta\} = -\mathcal{L}_{X_\alpha}(\beta) + \mathcal{L}_{X_\beta}(\alpha) + d(\iota_{X_\alpha} \circ \iota_{X_\beta} \omega)$$

\textbf{Proof.} From the definition of the bracket we have:

$$\{\alpha, \beta\} = -\iota_{[X_\alpha, X_\beta]} \omega$$

$$= -[\mathcal{L}_{X_\alpha}, \mathcal{L}_{X_\beta}](\omega)$$

$$= -\mathcal{L}_{X_\alpha} \iota_{X_\beta} \omega - \iota_{X_\beta} \mathcal{L}_{X_\alpha} \omega)$$

$$= -\mathcal{L}_{X_\alpha} \mathcal{L}_{X_\beta} \omega + \iota_{X_\beta} \mathcal{L}_{X_\alpha} \omega$$

$$= -\mathcal{L}_{X_\alpha} \beta + \iota_{X_\beta} (d\iota_{X_\alpha} \omega)$$

$$= -\mathcal{L}_{X_\alpha} \beta + \mathcal{L}_{X_\beta} (d\iota_{X_\alpha} \omega)$$

$$= -\mathcal{L}_{X_\alpha} \beta + \mathcal{L}_{X_\beta} \alpha + d\iota_{X_\beta} (\iota_{X_\alpha} \omega)$$

$$= -\mathcal{L}_{X_\alpha} \beta + \mathcal{L}_{X_\beta} \alpha + d\iota_{X_\alpha} \iota_{X_\beta} \omega$$

$$\square$$
Lemma 6.1.7. Let $\alpha, \beta \in \Omega^1(M)$ be closed 1-forms that is $d\alpha = d\beta = 0$ and let $\{\, , \}$ be as defined above. Then, $\{\alpha, \beta\}$ is an exact 1-form.

Proof. With reference to Lemma 6.1.6 and the Cartan magic formula above, we have:

$$\{\alpha, \beta\} = -\mathcal{L}_{X_\alpha} \beta + \mathcal{L}_{X_\beta} \alpha + dt_{X_\alpha} t_{X_\beta} \omega$$

$$= -(dt_{X_\alpha} \beta + t_{X_\alpha} d\beta) + (dt_{X_\beta} \alpha + t_{X_\beta} d\alpha) + dt_{X_\alpha} t_{X_\beta} \omega$$

$$= -dt_{X_\alpha} \beta + dt_{X_\beta} \alpha + dt_{X_\alpha} t_{X_\beta} \omega$$

$$= d(-t_{X_\alpha} \beta + t_{X_\beta} \alpha + t_{X_\alpha} t_{X_\beta} \omega)$$

since $\alpha$ and $\beta$ are closed forms.

which is an exact 1-form. □

We can now derive some properties of the bracket on 0-forms in the following lemma.

Lemma 6.1.8. Let $f, g \in F_M$. Then,

1. $\{f, g\} = -\mathcal{L}_{X_f}(g) = \mathcal{L}_{X_g}(f)$

2. $d\{f, g\} = \{df, dg\}$

3. $\{f, gh\} = \{f, g\} h + g\{f, h\}$ Poisson identity.

4. $X_{\{f, g\}} = -[X_f, X_g] = [X_g, X_f]$

Proof.

(1) $\mathcal{L}_{X_g}(f) = -\mathcal{L}_{X_f}(g)$

$$= -(t_{X_f} d(g) + dt_{X_f}(g))$$

Cartan identity,

$$= -(t_{X_f} d(g) - dt_{X_f}(g))$$

$$= -(t_{X_f} d(g))$$

since $dt_{X_f}(g) = 0$ for $g \in \Omega^0(M) = F_M$

$$= -t_{X_f} t_{X_g} \omega$$

since $dg = t_{X_g} \omega$

$$= \{f, g\}$

(2) $\{df, dg\} = -\mathcal{L}_{X_f} dg + \mathcal{L}_{X_g} df + dt_{X_f} t_{X_g} \omega$

by Lemma 6.1.6

$$= -d\mathcal{L}_{X_f} g + d\mathcal{L}_{X_g} f + dt_{X_f} t_{X_g} \omega$$

since $\mathcal{L}_{X_f} d = d\mathcal{L}_{X_f}$

$$= d(-\mathcal{L}_{X_f} g + \mathcal{L}_{X_g} f + dt_{X_f} t_{X_g} \omega)$$

$$= d\{f, g\} + \{f, g\} - \{f, g\}$$

$$= d\{f, g\}$

(3) $\{f, gh\} = \{f, g\} h + g\{f, h\}$ this is obvious since the bracket is a derivation from (1) above.
\[ (4) \, \iota_{X_{\{f,g\}}} \omega = d\{f, g\} \]
\[ = \{df, dg\} \quad \text{by definition} \]
\[ = -\iota_{[X_f, X_g]} \omega \quad \text{from (2) above} \]
\[ = \iota_{-\{X_f, X_g\}} \omega \quad \text{However, the form } \omega \text{ is nondegenerate.} \]

It follows that \( X_{\{f,g\}} = -[X_f, X_g] \).

We are now able to prove that the brackets induced by the symplectic form \( \omega \) in Equation (6.12) are actually Poisson structures.

**Proposition 6.1.9.** Let \( ((M, C_M, F_M), \omega) \) be a symplectic Frölicher space. For all \( f, g \in F_M \) and \( \alpha, \beta \in \Omega_1^1(M) = (M, \bigwedge T^*M) \), the following brackets define Poisson structures on \( F_M \) and on \( \Omega_1^1(M) \):

\[
\{f, g\} = -\iota_{X_f} \circ \iota_{X_g} \omega \quad \text{and} \quad \{\alpha, \beta\} = -\iota_{[X_\alpha, X_\beta]} \omega = -\{\beta, \alpha\}
\]

turning \( F_M \) and \( \Omega_1^1(M) \) into infinite dimensional Lie algebras over \( \mathbb{R} \).

**Proof.** The bracket \( \{\alpha, \beta\} = -\iota_{[X_\alpha, X_\beta]} \omega = \iota_{[X_\beta, X_\alpha]} \omega \) is an internal operator in \( \Omega_1^1(M) \) which is \( \mathbb{R} \)-bilinear, skew-symmetric and satisfies the Jacobi identity. In fact the bilinearity is obvious. So, is skew-symmetry from that of the Lie bracket \([,]\). By a straightforward calculation one proves the Jacobi identity.

Furthermore, we will prove that the bracket defined on \( \Omega_0^1(M) = F_M \) is also a Poisson structure. In fact the bracket is bilinear and skew-symmetric together with \( \omega \). To show that it has the Jacobi property we shall use previous Lemmas 6.1.6, 6.1.7 and 6.1.8. Indeed, we are evaluating the three brackets in the Jacobi identity separately. Therefore, for \( f, g, h \in F_M \) we have:

\[
(1) \, \{f, \{g, h\}\} = -\mathcal{L}_{X_f}(\{g, h\})
\]
\[ = -\mathcal{L}_{X_f}(\mathcal{L}_{X_g}(h)) \]
\[ = (\mathcal{L}_{X_f} \circ \mathcal{L}_{X_g})(h) \]
\[
(2) \, \{g, \{h, f\}\} = -\mathcal{L}_{X_g}(\{h, f\})
\]
\[ = -\mathcal{L}_{X_g}(-\mathcal{L}_{X_h}f) \]
\[ = (\mathcal{L}_{X_g} \circ \mathcal{L}_{X_h})(f) \]
\[ = -\mathcal{L}_{X_g} \circ \mathcal{L}_{X_f}(h) \quad \text{since } \mathcal{L}_{X_g} \circ \mathcal{L}_{X_f} = -\mathcal{L}_{X_f} \circ \mathcal{L}_{X_g} \]
\[
(3) \, \{h, \{f, g\}\} = -\mathcal{L}_{X_{\{f,g\}}}(h)
\]
\[ = \mathcal{L}_{-\{X_f, X_g\}}(h) \]
\[ = -\mathcal{L}_{[X_f, X_g]}(h) \]
Finally, then we have the Jacobi identity satisfied as by the following:

\[
\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = (\mathcal{L}_{X_f} \circ \mathcal{L}_{X_g})(h) - (\mathcal{L}_{X_g} \circ \mathcal{L}_{X_f})(h) - \mathcal{L}_{[X_f, X_g]}(h) = 0 \text{ since } [\mathcal{L}_{X_f}, \mathcal{L}_{X_g}] = \mathcal{L}_{[X_f, X_g]}.
\]

Now, with regard to Definition 3.3.3 and Proposition 6.1.9, Equation (6.10) becomes

\[
\{ F, H \} = \mathcal{L}_{X_H}(F) = X_H(F) = dF(X_H) = \iota_{X_H}dF = \iota_{X_H}(\iota_{X_F}\omega) = \omega(X_F, X_H)
\]

6.1.4 Poisson subspace, Foliation and Darboux’s Theorems

In Section 2.3 we have discussed immersed, embedded and regular locally Euclidean \( \mathbb{F} \)-subspaces with respect to injective immersion. Now, we discuss Poisson immersion and immersed Poisson locally Euclidean \( \mathbb{F} \)-subspaces (see [21] for details and proofs).

**Definition 6.1.5.** Let \( M \) be a Poisson locally Euclidean \( \mathbb{F} \)-space and \( N \) a nonempty set. Let \( \phi : N \rightarrow M \) be an injective immersion. A injective immersion is a Poisson injective immersion if any Hamiltonian vector field defined on an open subset \( U \) of \( M \) containing \( \phi(N) \) is in the range of \( \phi_{x,2} \), at all point \( \phi(x) \) for \( x \in N \). So, \( N \) is an immersed locally Euclidean \( \mathbb{F} \)-space.

**Lemma 6.1.10.** Characterization of Poisson injective immersion

Let \( M \) be a Poisson locally Euclidean \( \mathbb{F} \)-space and \( N \) an immersed locally Euclidean \( \mathbb{F} \)-space. The following statements are equivalent.

1. An injective immersion \( \phi : N \rightarrow M \), is a Poisson injective immersion.
2. If for \( \mathcal{V} \subset N \) is an open set and \( F, G : \mathcal{V} \rightarrow \mathbb{R} \) are smooth functions and if \( \bar{F}, \bar{G} : U \rightarrow \mathbb{R} \) are extensions of \( F \circ \phi^{-1}, G \circ \phi^{-1} : \phi(\mathcal{V}) \rightarrow \mathbb{R} \) to an open neighborhood \( U \) of \( \phi(\mathcal{V}) \subset M \), then the Poisson bracket \( \{ \bar{F}, \bar{G} \}_{\phi(\mathcal{V})} \) is well-defined and independent of the choice of the extensions.

From the proof provided in [21] we can define a Poisson structure on \( N \) by:

\[
\{ F, G \}_N(x) := \{ \bar{F}, \bar{G} \}_M(\phi(x)), \text{ for any } x \in \mathcal{V}, \text{ where } \mathcal{V} \subset N \text{ is an open set}.
\]
It follows from Equation (6.14) and Lemma 6.1.10 above that the injective immersion \( \phi : N \rightarrow M \) is a Poisson map as shown in the following equation.

\[
[\phi^*\{\bar{F}, \bar{G}\}_M](x) \equiv \{\bar{F}, \bar{G}\}_M \circ \phi(x) = \{F, G\}_N(x) \equiv \{\phi^*\bar{F}, \phi^*\bar{G}\}_M(x)
\] (6.15)

Now, with regard to Lemma 6.1.5 and the fact that \( \phi : N \rightarrow M \) is a Poisson map we have:

\[
(\phi_*x)(X_{H \circ \phi}(x)) = X_H(\phi(x)) \quad \text{for} \quad x \in N \quad \text{such that} \quad \phi(x) \in U.
\] (6.16)

That is, \( \phi \) pushes \( X_{H \circ \phi} \) to \( X_H \) and \( X_H(\phi(x)) \) is in the range of \( \phi_*x \).

**Definition 6.1.6.** Let the Poisson injective immersion \( \phi : N \rightarrow M \) be the canonical inclusion \( \iota_N : N \hookrightarrow M \). We call \( N \) a Poisson Locally Euclidean \( \mathbb{F} \)-subspace of \( M \).

The foliation is induced by an equivalence relation defined below and which we name foliation equivalence.

**Definition 6.1.7.** Let \( M \) be a Poisson \( \mathbb{F} \)-space and \( x, y \in M \). The two elements are in relation, that is, \( x \sim y \) if there exists a piecewise \( \mathbb{F} \)-smooth curve joining them on \( M \), such that each piece of which is an integral curve of locally defined Hamiltonian vector fields. An orbit of the foliation equivalence at \( x \in M \) is called a leaf through \( x \) and denoted by \( N_x \). We say that the Poisson \( \mathbb{F} \)-space \( M \) is foliated in a disjoint union of leaves.

**Theorem 6.1.11.** Let \( M \) be a Poisson locally Euclidean \( \mathbb{F} \)-space endowed with the foliation equivalence relation. Each leaf is a symplectic immersed locally Euclidean \( \mathbb{F} \)-subspace of \( M \). The induced Poisson bracket on a leaf is symplectic. The dimension of each leaf \( N_x \) is equal to the rank of the Poisson bracket at \( x \). The tangent space to \( N_x \) is given by \( TN_x := \{X \in T_x M \mid X = \omega^2(\alpha), \text{ for some } \alpha \in T^*_x M\} = \{X_{H}(x) \mid H \in \mathcal{F}_U, \ U \in \tau_F\} \)

**Proof.** (see [21]).

The theorem have to be understood as meaning that the evaluation of the Poisson bracket of two functions \( F, G \in \mathcal{F}_M \) at a point \( x \in M \) is restricted to the one done for the Poisson bracket induced by the symplectic form defined on the symplectic leaf \( N_x \) through the point \( x \). Also a distinguished function \( H \in \mathcal{F}_M \), whose Poisson bracket \( \{F, H\} = \{H, F\} = X_H(F) = 0 \) for all \( F \in \mathcal{F}_M \), is constant on the symplectic leaf \( N_x \) of \( M \) for each \( x \in M \). □

**Remark 6.1.1.** In Lemma 3.1.5 and in [8] a version of Darboux theorem is given for \( \mathbb{F} \)-space setting, that is to say, a symplectic locally Euclidean \( 2n \)-\( \mathbb{F} \)-space looks locally like \( \mathbb{R}^{2n} \). Whereas, the Poisson locally Euclidean \( m \)-\( \mathbb{F} \)-space looks locally like \( \mathbb{R}^m = \mathbb{R}^{2n+s} \), where \( 2n \leq m \) and \( s = m - 2n \). This reference [21] on manifolds appeared to be useful.
Theorem 6.1.12. Darboux theorem
Let \( M \) be a Poisson locally Euclidean \( m \)-\( \mathbb{F} \)-space, let \( U \) be an open neighborhood of \( x \in M \). If \( F, G \in \mathcal{F}_M \) and \( x \) is a point with constant rank equal to \( 2n \leq m \) with \( s = m - 2n \) then there exists a local coordinate \((q, p, z) = (q^1, \ldots, q^n; p^1, \ldots, p^n; z^1, \ldots, z^s)\) on a smaller open neighborhood \( \mathcal{V} \subset U \) of \( x \), in which the Poisson bracket is defined by
\[
\{F, H\} = X_H(F) = \sum_i^n \left( \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right).
\]

As for the symplectic case the sum above induces the Poisson brackets for coordinate functions by \( \{q^i, q^j\} = \{p^i, p^j\} = \{q^i, z^j\} = \{p^i, z^j\} = \{z^i, z^j\} = 0 \) and \( \{q^i, p^j\} = \delta_{ij} \), the Kronecker symbols. When a function \( F(q, p, z) = F(z) \) then \( \{F, H\} = 0 \) for all \( H \in \mathcal{F}_M \), that is, \( F \) is a distinguished function.

6.1.5 Quotient of a Poisson locally Euclidean \( \mathbb{F} \)-space

For notions in this section the following references [20, 21, 22, 90], lead to manifolds and [102], to differential spaces. Notice that in Subsection 3.4 we have presented important properties on the quotient space of a group acting from the left on \( M \). More of these properties are not related to the symplectic structure. We call to the attention of the reader the following Lemma 3.4.5, Remarks 3.4.3, 3.4.4 and 3.4.5. Thus, we can extend them to the Poisson context. Since the action of \( G \) on \( M \) is proper and free, then the canonical surjection \( \pi : M \rightarrow M/G \) is a \( \mathbb{F} \)-smooth submersion. Hence, its pullback is an isomorphism. Moreover, if we assume that the Frölicher Lie group \( G \) acts on Poisson locally Euclidean \( \mathbb{F} \)-space \((M, \{\cdot, \cdot\}_M)\) such that for each \( g \in G \) the diffeomorphism \( \sigma_g : M \rightarrow M \) is a Poisson map, that is,
\[
\sigma^*\{\bar{f}, \bar{h}\}_M = \{\sigma^*\bar{f}, \sigma^*\bar{h}\}_M
\]

Therefore, the pullback \( \pi^* \) induces uniquely a Poisson structure \( \{\cdot, \cdot\}_{M/G} \) on the quotient space \( M/G \) by:
\[
\pi^*\{\bar{f}, \bar{h}\}_{M/G} = \{f, h\}_M = \{\bar{f} \circ \pi, \bar{h} \circ \pi\}_M = \{\pi^*\bar{f}, \pi^*\bar{h}\}_M \quad \text{and} \quad (6.18)
\]

From the proof of Lemma 3.4.5 and the diagram therein reproduced below:

\[
\begin{array}{ccc}
M & \xrightarrow{\sigma_g} & M \\
\downarrow f & & \downarrow \pi \\
\mathbb{R} & \xrightarrow{h} & \hat{M} = M/G \\
\downarrow \bar{h} & & \\
\end{array}
\]
we get $\bar{h}(G.y) = \bar{h}(G.x)$ if and only if, $\bar{h}(\pi(y)) = \bar{h}(\pi(x))$ if, and only if $h(y) = h(x)$ since $h = \bar{h} \circ \pi$, which is equivalently the formula $\bar{h} = h \circ \pi^{-1}$, where $h$ one-to-one. Let $\mathcal{F}_M^G$ be the algebra of $G$-invariant smooth functions on $M$ and $\mathcal{F}_{\bar{M}}$ be the smooth structure on the orbit space. From the diagram above we have $\mathcal{F}_{\bar{M}} \xrightarrow{\pi^*} \mathcal{F}_M^G$, where the pullback $\pi^*$ is an isomorphism of $\mathbb{R}$-algebras. It follows that $h$ is constant on each orbit (the equivalence class). That is, $h \circ \sigma_g = h$ if, and only if $h$ is invariant under the action of $\sigma$ of $G$ on $M$ if, and only if $h \in \mathcal{F}_M^G$, the spaces of smooth invariant functions on $M$. Therefore, $\mathcal{F}_M^G \subseteq \mathcal{F}_M$.

When we consider $f, g \in \mathcal{F}_M^G$, that is $f$ and $g$ are constant on orbits, Equation (6.17) implies that $\{f, h\}_{M} \in \mathcal{F}_M^G$ is also constant on orbits as started in the following equation.

$$\sigma^*\{\bar{f}, \bar{h}\}_M(x) = (\{\bar{f}, \bar{h}\}_M \circ \sigma_g)(x) = \{\sigma^*\bar{f}, \sigma^*\bar{h}\}_M(x) = \{\bar{f} \circ \sigma, \bar{h} \circ \sigma\}_M(x) = \{\bar{f}, \bar{h}\}_M(x) \quad (6.19)$$

**Definition 6.1.8.** The Poisson bracket defined on $M/G$ in Equation (6.18) is called the reduced Poisson bracket.

The reduced bracket in Definition 6.1.8 is compatible for the reduction of Hamiltonian vectors and their flows.

**Lemma 6.1.13.** If $h \in \mathcal{F}_M^G$ is a $G$-invariant smooth Hamiltonian function then the Hamiltonian flow $H_t$ of $X_h$ commutes with the $G$-action and so induces a Hamiltonian flow $H_t^\bar{M}$ on $\bar{M} = M/G, \{\}. \}$ for the reduced smooth Hamiltonian function $\bar{h} \in \mathcal{F}_{\bar{M}}$ defined by $\bar{h} \circ \pi = h$ or $\bar{h} = h \circ \pi^{-1}$ as in Lemma 3.4.5.

From the Lemma above and with respect to Lemma 6.1.5, Lemma 6.1.3 and Corollary 6.1.4, we have that the canonical surjection $\pi$ transform $X_h$ on $M$ into $X_{\bar{h}}$ on $\bar{M}$ such that

$$\bar{H}_t \circ \pi = \pi \circ H_t \quad \text{and} \quad \pi_* \circ X_h = X_{\bar{h}} \circ \pi \quad (6.20)$$

This is meaning that $X_h$ and $X_{\bar{h}}$ are $\pi$-related. Otherwise said, the Hamiltonian system $X_h$ on $M$ is reduced to the Hamiltonian $X_{\bar{h}}$ on $\bar{M}$.

### 6.2 Hamiltonian systems

#### 6.2.1 Hamiltonian systems

Let $(\mathcal{M}, \mathcal{C}_M, \mathcal{F}_M, \omega)$ be a symplectic Frölicher space and $\mathfrak{X}(M)$ be the set of smooth vector fields. Let $\sigma : (\mathcal{F}_M, \{\}) \longrightarrow (\mathfrak{X}(M), [\cdot])$ be a smooth morphism of Algebras such that it
6.2 Hamiltonian systems

By the characterization of Hamiltonian vector fields, the definition and properties of an equivariant moment map as by Definition 3.4.1, Definition 3.4.3, Lemma 3.4.1 and Remark 3.4.1 in Section 3.4, it follows that $X_{\xi} = X_{\mu_{\xi}} = X_{\hat{\mu}(\xi)}$ such that $X_{\hat{\mu}(\xi)} \cdot \omega = X_{\mu_{\xi}} \cdot \omega = d\mu_{\xi} = d\hat{\mu}(\xi)$, where $\mu_{\xi} = \hat{\mu}(\xi) \in \mathcal{F}_M$. Hence, we can define a map $\hat{\mu} : \mathcal{G} \rightarrow \mathcal{F}_M$ such that $\hat{\mu}(X) := \mu_X$. The map $\hat{\mu}$ is called the comoment map of the action $\sigma$ of $G$ on $M$. This is an homomorphism of Lie-algebras $(\mathcal{G}, [,])$ and $(\mathcal{F}_M, \{ , \})$. Let us take $\xi \in \mathcal{G}$. By differentiating $\hat{\mu}(X)Ad(g^{-1}) = \mu_{\xi}Ad(g^{-1}) = \sigma_{\hat{\mu}_g}^{\mathcal{G}}\mu_{\xi} = \sigma_{\hat{\mu}}^{\mathcal{G}}\mu(X)$, the co-adjoint equivariance will be equivalent to $\hat{\mu}(X)[X,Y] = \mu_{\xi}[X,Y] = -\{\mu_{\xi}(X),\mu_{\xi}(Y)\} = \{\hat{\mu}(X)(X),\hat{\mu}(X)(Y)\}$, for any $X,Y \in \mathcal{G}$. That is, a Lie algebra anti-homomorphism of $\mathcal{G}$ into $\mathcal{F}_M$ (see [80, p.195], by sign convention. In comoment setting, we have $\hat{\mu}[X,Y] = -\{\hat{\mu}(X),\hat{\mu}(Y)\}$ and the proof is similar to the one provided in [54, p.4].

In [51, p.15, Definition 1.1], $\hat{\mu}(X) = \mu_X$ is defined by the following property: for every tangent vector $\eta$ we have $\eta\hat{\mu}(X) = \eta\mu_X = \omega(X_M,\eta)$, where $\eta$ is taken as a derivation on the smooth function $\mu(X) = \mu_X \in \mathcal{F}_M$.

From now on, we will use $\hat{\mu}(X)$ in the place of $\mu_X$ for seek of a clear understanding. In Mechanics, $X_H$ is the Hamiltonian System and $H$ carries the total energy of the system. In this case the triple $((\mathcal{M},\mathcal{C}_M,\mathcal{F}_M),\omega,H)$ is said to be the Hamiltonian Dynamical System.

Definition 6.2.2. Let $((\mathcal{M},\mathcal{C}_M,\mathcal{F}_M),\omega)$ be a symplectic Frölicher space. A vector field $X \in \mathfrak{X}(M)$ is said to be the locally Hamiltonian vector field if at every point $p \in M$ there exists a neighborhood $U$ such that the restriction of $X$ to $U$ is an Hamiltonian vector field. Hence, on $U$ we have $X = X_H$ and $H$ is called local Hamiltonian of $X_H$, that is,

$$\forall \ p \in \ M, \exists \ U \ni p \text{ such that } \iota_{X_{\hat{\mu}}}, \omega = dH_{\hat{\mu}}.$$
**Definition 6.2.3.** Let \((M, C_M, F_M)\) be a Frölicher space. A smooth function \(f \in F_M\) is called a first integral of a vector field \(X_h = \{h, \cdot\}\) if \(\{h, f\} = 0\).

**Proposition 6.2.1.** Let \((M, C_M, F_M)\) be a symplectic Frölicher space and \(H \in F_M\) be the locally Hamiltonian function of the vector field \(X = X_H\). Then, \(X_H(H) = 0\). That is, \(H\) is constant on the trajectories of the flow of \(X_H\) and the energy is conserved in the system. In other words \(H\) is a first integral of \(X_H\).

**Proof.** The statement above is true since \(X_H(H) = \mathcal{L}_{X_H}(H) = \{H, H\} = 0\).

**Proposition 6.2.2.** Let \(((M, C_M, F_M), \omega)\) be a symplectic Frölicher space. Let \(X = X_H\) a locally Hamiltonian vector field with local Hamiltonian function \(H\). Then, \(H \circ c\) is constant if and only if \(c\) is the integral curve of \(X_H\). That is, the integral trajectories of Hamiltonian system lie on energy surfaces \(H = \text{constant}\).

**Proof.** First of all, \(c\) is the integral curve for \(X\) if and only if \(\dot{c}(t) = X(c(t))\) for all \(t \in \mathbb{R}\). And then by using the chain rule we have:

\[
\frac{d}{dt}(H \circ c)(t) = dH(c(t))\dot{c}(t) \\
= dH(c(t))(X_H(c(t))) \\
= \omega(X_H(c(t)), X_H(c(t))) \\
= 0 \quad \text{since } \omega \text{ is skew-symmetric.}
\]

**Proposition 6.2.3.** Let \(((M, C_M, F_M), \omega)\) be a symplectic Frölicher space. The set \(L_s = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0\}\) of locally Hamiltonian vector fields is a real Lie subalgebra of \(\mathfrak{X}(M)\).

**Proof.** Obviously, \(0 \in L_s\). We show that \(L_s\) is stable for any linear combination of its elements. That is, for all \(a, b \in \mathbb{R}\) and for all \(X, Y \in L_s\) we have:

\[
\mathcal{L}_{aX + bY} \omega = a \mathcal{L}_X \omega + b \mathcal{L}_Y \omega \\
= a0 + b0 \\
= 0.
\]

It follows that \(aX + bY \in L_s\).

**Proposition 6.2.4.** Let \(((M, C_M, F_M), \omega)\) be a symplectic Frölicher space. Every globally Hamiltonian vector field is locally Hamiltonian vector field.
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Proof. Let \( L^0_s = \{ X \in \mathfrak{X}(M) \mid \iota_X \omega = dh, \text{ for } h \in \mathcal{F}_M \} \) be the set of globally Hamiltonian vector fields on a symplectic Frölicher space \( (\mathcal{M}, \mathcal{C}_M, \mathcal{F}_M, \omega) \). Then from \( X \in L^0_s \) one has \( d_t X \omega = dh = d^2 h = 0 \). Thus, from Cartan magic formula, \( 0 = \mathcal{L}_X \omega - \iota_X d \omega = \mathcal{L}_X \omega \), which proves that \( X \in L_s \).

Corollary 6.2.5. \( L^0_s \) is an ideal of \( L_s \) and \( L_s / L^0_s \) is a commutative Lie algebra.

Definition 6.2.4. Let \( (\mathcal{M}, \mathcal{C}_M, \mathcal{F}_M, \omega) \) be a symplectic Frölicher space. A smooth function \( h \in \mathcal{F}_M \) is called Casimir function if \( \{ h, f \} = 0 \) for any \( f \in \mathcal{F}_M \).

Note that a Casimir function is naturally associated with the zero vector field \( X_h = \{ h, \} = 0 \).

Proposition 6.2.6. Let \( (\mathcal{M}, \mathcal{C}_M, \mathcal{F}_M, \omega) \) be a symplectic Frölicher space. The set \( \mathcal{C}^{FM} \subset \mathcal{F}_M \) of all Casimir functions is a Lie subalgebra of \( (\mathcal{F}_M, \{ , \}) \) and a module over \( \mathcal{F}_M \).

Proof. As a nonempty subset, \( \mathcal{C}^{FM} \subset \mathcal{F}_M \) inherits the induced Frölicher structure making it into a Frölicher subspace of \( \mathcal{F}_M \). Moreover, \( \mathcal{C}^{FM} \subset \mathcal{F}_M \) contains the zero function and is closed under linear combinations. Hence, \( \mathcal{C}^{FM} \subset \mathcal{F}_M \) is also obviously a module.

Proposition 6.2.7. Let \( (\mathcal{M}, \mathcal{C}_M, \mathcal{F}_M, \omega) \) be a symplectic Frölicher space. The set of all first integrals of the vector field \( X_h \) is a Lie subalgebra of the Lie algebra \( (\mathcal{F}_M, \{ , \}) \).

Proof. Recall the definition of a symplectomorphism: \( \sigma^g_\omega \in \text{Sympl}(\mathcal{M}) \) if \( \sigma^g_\omega \omega = \omega \), with the map \( \omega : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathbb{R} \) such that \( X \circ \sigma^g_\omega = X \circ \omega \) and \( d(X \circ \sigma^g_\omega) = d(X \circ \omega) \). 

\( \implies \) Let \( \sigma^g_\omega \omega = \omega \) for all \( g \in G \) by assumption. Then the flow associated to the vector field \( X = X_M \in \mathfrak{X}(\mathcal{M}) \) is \( \rho_t : M \to M, m \mapsto \rho_t(m) = e^{\text{exp}(t \xi)} m = e^{\text{exp}(tX_M)} \), with regard to Remark 3.2.3 (7). Thus, \( \rho_t \in G \) are symplectomorphisms for all \( t \in \mathbb{R} \). That is, \( \rho^*_t \omega = \omega \). [22, Section 18.1.1]. As from the definition of the flow one has \( \rho_t(m) = \text{exp}(t \xi) \text{exp}(tX_M) \). Hence, \( \rho^*_t \omega = \text{exp}(t \xi) \text{exp}^*(tX) = \text{exp}(tX) \omega \), with regard to [20, 22, 39, 94]. This yields, \( \mathcal{L}_X \omega = \frac{d}{dt} \rho^*_t \omega \big|_{s=0} = \frac{d}{dt} (\text{exp}^*(sX)) \big|_{s=0} \). From the assumption and Remark 3.3.1, for all \( t \in \mathbb{R} \), the curve \( t \mapsto \sigma^*_{\text{exp}(tx)} \omega = \omega \) is constant. That is, it takes the same value for all \( t \in \mathbb{R} \) as for \( t = 0 \). It follows that \( \sigma^*_{\text{exp}(tx)} \mathcal{L}_X \omega = \frac{d}{dt} \omega = 0 \). Therefore, \( \mathcal{L}_X \omega = 0 \) since \( \sigma^*_{\text{exp}(tx)} \) is a linear isomorphism. From Cartan identity we draw \( d(\iota_X \omega) = 0 \), that is, \( \iota_X \omega \) is closed.

\( \impliedby \) Assume \( \iota_X \omega \) is closed. Thus, from Cartan identity, we draw \( \mathcal{L}_X \omega = 0 \) which means the vector field \( X \) preserves \( \omega \). That is, \( \rho^*_t \omega = \omega \). This equality can be extended to a general \( g \in G \) with regard to Remark 3.2.2. Therefore, \( \omega \) is invariant under the action of \( G \).
**Definition 6.2.5.** Let $(M, \omega)$ be a symplectic locally Euclidean Frölicher space, and $H : M \rightarrow \mathbb{R}$ any structure function. A vector field on $M$ denoted by $X_H$ and defined by $\iota_{X_H} \omega = dH$ is called the Hamiltonian vector field associated to $H$ and $H$ is called the Hamiltonian function.

The set of all Hamiltonian vector fields on $M$ is denoted by $\mathfrak{h}(\omega)$. In other words the 1-form $\iota_{X_H} \omega$ is an exact 1-form and $H$ is the primitive of $\iota_{X_H} \omega$ with regard to $\omega$ defined from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ into $\mathcal{F}_M$. The symplectic and Hamiltonian vector fields are both related to 1-forms. We want to explain below these relationships.

**Lemma 6.2.8.** Let $(M, \omega)$ be a symplectic locally Euclidean Frölicher space and $\omega^b : \mathfrak{X}(M) \rightarrow \Omega^1(M)$ a map defined such that $X \mapsto \omega^b(X) := \iota_X \omega = \alpha = \omega(X,.).$ The map $\omega^b$ is an isomorphism of $C^\infty(M)$-modules.

**Proof.** The map $\omega^b$ is linear. Indeed, for all $X,Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ the following holds: $\omega^b(X + Y) = \iota_{X+Y} \omega = (X+Y) \lrcorner \omega = X \lrcorner \omega + Y \lrcorner \omega = \omega^b(X) + \omega^b(Y)$, $\omega^b(fX) = \iota_{fX} \omega = (fX) \lrcorner \omega = f(X \lrcorner \omega) = f(\omega^b(X))$. After the linearity has been proven, we need now to show that $\omega^b$ one-to-one and onto. Recall that $\omega$ is non-degenerate. Now, let $X,Y \in \mathfrak{X}(M)$. If $\omega(X,Y) = 0$ for all $Y$, then we have $\omega(X,Y) = \omega^b(X)(Y) = (\iota_X \omega)(Y) = 0$. That is, $X = 0$ and $\text{Ker} \omega^b = \{0\}$. It follows that the linear map $\omega^b$ is one-to-one since. But, $\omega^b$ can be equivalently considered as a map of $TM$ into $T^*M$, which have the same dimension. Then from the rank theorem, if $\text{Ker} \omega^b = \{0\}$, then $\omega^b$ is an isomorphism. \hfill \Box

It follows that $\omega^b : TM \rightarrow T^*M$ is a bijection and also a smooth map since it is smooth into all its components $\omega^b_{\alpha \beta}$. Since to each 1-form $\alpha$ on $M$ corresponds a unique vector field $X$ on $M$ such that $\omega^b(X) = \alpha$, we can define the inverse map $(\omega^b)^{-1} := \omega^b : \Omega^1(M) \rightarrow \mathfrak{X}(M)$, such that $\omega^b(\alpha) = X$ with $\iota_X \omega = \alpha$.

**Corollary 6.2.9.** Let $\omega \in \Omega^2(M)$. The map $\omega^b$ is an isomorphism of $\mathcal{F}_M$-modules if, and only if $\omega$ is non-degenerate.

**Definition 6.2.6.** Let $(M, \omega)$ be a symplectic locally Euclidean Frölicher space. Let $\alpha, \beta \in \Omega^1$ and $\rho_t$ be the flow of $X_\alpha$ such that $X_\alpha$ is uniquely associated to $\alpha$ by $X_\alpha \mapsto \omega^b$, where, $\omega^b := \iota_{X_\alpha} \omega = \alpha = \omega(X_\alpha, \ldots) = dH_{X_\alpha}$. One calls Poisson bracket of $\alpha$ and $\beta$ the one-form $\{\alpha, \beta\} := -\iota_{[X_\alpha, X_\beta]} \omega$ on $M$ with $[X_\alpha, X_\beta] = \lim_{t \rightarrow 0} \frac{1}{t} (X_\beta - d \rho_t \circ H_\beta)$.

**Corollary 6.2.10.** Let $\mathcal{Z}^1(M)$ be the $\mathcal{F}_M$- submodule of $\Omega^1(M)$ containing closed 1-forms on $M$ and $\mathcal{B}^1(M)$ the $C^\infty(M)$-submodule of $\mathcal{Z}^1(M)$ containing exact 1-forms on $M$, that is, $\mathcal{B}^1(M) = d(C^\infty(M))$. Then,
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1. $\mathfrak{sp}(\omega) = \omega^\flat(Z^1(M))$, that is, $\omega^\flat(\mathfrak{sp}(\omega)) = Z^1(M)$.
2. $\mathfrak{h}(\omega) = \omega^\flat(B^1(M))$, that is, $\omega^\flat(\mathfrak{h}(\omega)) = B^1(M)$.
3. $T_{id}\text{Symp}(M) \simeq \{\alpha \in \Omega^1(M) \mid d\alpha = 0\} = Z^1(M)$.
4. $T_{id}\text{Ham}(M) \simeq \{\alpha = h \mid h \in C^\infty(M)\} = B^1(M)$.
5. $[\mathfrak{sp}(\omega), \mathfrak{sp}(\omega)] \subset \mathfrak{h}(\omega)$.

**Proof.** The proof of the first four identities is straightforward from definitions. For the last identity, we refer the reader to [20, p.90, Proposition 4]. For $X, Y \in \mathfrak{sp}(\omega)$, we have $[X, Y] = X_{\omega(X,Y)}$, where $\omega(X,Y) \in C^\infty(M)$. Thus, $[X, Y] \in \mathfrak{sp}(\omega)$. In particular, since the Poisson bracket of $f, g \in C^\infty(M)$ is defined by $\{f, g\} = \omega(X_f, X_g)$, it follows that $\{X_f, X_g\} = X_{\{f,g\}}$, with regard to Definition 6.2.6. □

A symplectic vector field is of the form $X = \omega^\flat(\alpha)$, with $\alpha \in \Omega^1(M)$ and $d\alpha = 0$, that is, $\omega^\flat(X) = \alpha$. A Hamiltonian vector field is of the form $X_H = X = \omega^\flat(dH)$ with $H \in C^\infty(M)$, $dH \in \Omega^1(M)$, that is, $\omega^\flat(X_H) = dH$. It follows that $\mathfrak{h}(\omega)$ is an ideal of the Lie subalgebra $\mathfrak{sp}(\omega)$ of $\mathfrak{X}(M)$ with regard to the inclusion $[\mathfrak{sp}(\omega), \mathfrak{sp}(\omega)] \subset \mathfrak{h}(\omega)$. Thus, we have the following chain of inclusions of Lie algebras: $\mathfrak{h}(\omega) \subset \mathfrak{sp}(\omega) \subset \mathfrak{X}(M)$. The 1-forms counterpart of the inclusions above is: $B^1(M) \subset Z^1(M) \subset \Omega^1(M)$. The set diffeomorphism counter part of these inclusions is: $\text{Ham}(M) \subset \text{Symp}(M) \subset \text{Diff}(M)$.

**Proposition 6.2.11.** Let $(M, \omega)$ be a symplectic locally Euclidean Frölicher space and $\{\alpha, \beta\}$ the Poisson bracket of one-forms as given in Definition 6.2.6.

1. $(\Omega^1, \{,\})$ is a Lie algebra on $\mathbb{R}$.
2. $\{\alpha, \beta\} = -\mathcal{L}_{X_\alpha}(\beta) + \mathcal{L}_{X_\beta}(\alpha) + d(\iota_{X_\alpha} \circ \iota_{X_\beta} \omega)$.
3. If $\alpha$ and $\beta$ are closed one-forms, then $\{\alpha, \beta\}$ is an exact form.
4. If $\alpha$ and $\beta$ are exact one-forms, then $\{\alpha, \beta\}$ is an exact form.
5. The $\omega^\flat$ is a smooth bijective antimorphism of Lie algebras.

**Proof.**

1. The definition links Lie bracket and interior product to Poisson bracket. Thus, the Poisson bracket satisfies the same properties than the Lie bracket.
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2. Another mixed property involving the Lie derivative, the interior product and the Poisson bracket is given by the formula $\iota_{[X_\alpha,X_\beta]} = [\mathcal{L}_{X_\alpha},\iota_{X_\beta}]$. Combining the formula above with the Cartan magic identity, we break out with the proof by using the closedness of the symplectic form.

3. That is the straightforward consequence of previous item.

4. We are done, if we use the definition of an exact form for $\alpha$ and $\beta$ and the previous item.

5. Let $\omega^\flat(X) = \alpha$ and $\omega^\flat(Y) = \beta$. By the definition of the Poisson bracket we have 
\[ \{\omega^\flat(X),\omega^\flat(Y)\} = \{\alpha,\beta\} = -\iota_{[X,Y]} = -\omega^\flat([X,Y]). \]

6.3 Equivariance of moment map

We have widely presented flow, integral curve, equivariance, and moment map in symplectic context through Sections 3.2, 3.3, 3.4, and 3.5. Now, we want succinctly to go through these concepts in the Poisson setting.

6.3.1 Conservation of moment maps

In contrast with symplectic case where the infinitesimal generator of the action was locally Hamiltonian, we are now considering in Poisson setting, for all $\xi \in \mathcal{G}$, we have $\xi_M(x) = \left.\frac{d}{dt}\exp(t\xi).x\right|_{t=0}$ being globally Hamiltonian, so that, when $\xi_M$ acts on a Poisson bracket of two smooth functions $F, H \in \mathcal{F}_M$, we have $\xi_M(\{F,H\}) = \{\xi_M(F),H\} + \{F,\xi_M(H)\}$. The vector field $\xi_M$ is called infinitesimal Poisson automorphism. One can notice the fact that Remark 3.2.2 the infinitesimal Poisson automorphisms are closed under the Lie bracket. Moreover, it follows the existence of a smooth map $\hat{\mu} : \mathcal{G} \rightarrow \mathcal{F}_M$, where $\hat{\mu}(\xi) : M \rightarrow \mathbb{R}$, such that
\[ X_{\hat{\mu}(\xi)} = \xi_M. \quad (6.22) \]

When we introduce the Poisson bracket in the Equation (6.22), we have,
\[ \xi_M(F) = \{F,\hat{\mu}(\xi)\} \text{ for all } F \in \mathcal{F}_M, \text{ for all } \xi \in \mathcal{G}. \quad (6.23) \]

Let $\xi, \eta \in \mathcal{G}$, and $\xi_M, \eta_M \in \mathcal{X}(M)$. Using Remark 3.2.2 in its item 1. and Lemma 6.1.8 we get the following equation:
\[ X_{\hat{\mu}([\xi,\eta])} = [\xi,\eta]_M = -[\xi_M,\eta_M] = -[X_{\hat{\mu}(\xi)},X_{\hat{\mu}(\eta)}] = [X_{\hat{\mu}(\xi),\hat{\mu}(\eta)}] \quad (6.24) \]
6.3 Equivariance of moment map

Recall in Poisson setting we are dealing with globally Hamiltonian infinitesimal generator of the group action. We assume that the map \( \hat{\mu} : \mathcal{G} \rightarrow \mathcal{F}_M \) is linear such that the map \( \xi \in \mathcal{G} \mapsto \xi_M \in \mathfrak{X}(M) \) is the composition of \( \hat{\mu} \) and \( F \in \mathcal{F}_M \mapsto X_F \in \mathfrak{X}(M) \) as shown below

\[
\mathcal{G} \rightarrow \mathcal{F}_M \rightarrow \mathfrak{X}(M)
\]

Now, in Poisson setting, the moment map \( \mu : M \rightarrow \mathcal{G}^* \) and the associated comoment map \( \hat{\mu} : \mathcal{G} \rightarrow \mathcal{F}_M \) are in bijective correspondence and also are related by the following equation.

\[
\mu(m)(\xi) = \langle \mu(m); \xi \rangle =: \hat{\mu}(\xi)(m) \quad \text{for all } \xi \in \mathcal{G} \text{ and } m \in M \quad (6.25)
\]

This equation may be interpreted as follows: for \( m \in M \) and \( \xi \in \mathcal{G} \), we have

\[
\mu(m) : \mathcal{G} \rightarrow \mathbb{R}, \text{ with } \xi \mapsto \mu(m)(\xi) \quad (6.26)
\]

Making use of both definitions of \( \omega^\# \) and \( \omega^\flat \), we can set a Hamiltonian equation as a defining condition for the moment map on a Poisson space, on the one hand by,

\[
\omega^\flat(d\hat{\mu}(\xi)) = \xi_M, \quad (6.27)
\]

and also by applying \( \omega^\flat \) on Equation (6.26) we obtain the moment map on a symplectic space on the other hand as given below,

\[
(d\hat{\mu}(\xi))(.) = \omega(\xi_M,.). \quad (6.28)
\]

Under the assumption of all infinitesimal generators being globally Hamiltonian in Poisson setting we are giving important statements on the conservation of the moment map. Nevertheless, the Cartan magic identity was pivotal in defining a vector field that preserves the symplectic as a locally Hamiltonian vector field with regard to the following statement:

\[
X_F(H) = \{H,F\} = 0 \quad \text{if, and only if} \quad X_H(F) = F, H = 0 \quad (6.29)
\]

That is, the Hamiltonian function \( H \) is constant under the flow induced by \( F \) if, and only if \( F \) is a constant of motion under the dynamical flow \( X_H \). But, this equivalence collapses when it comes to Poisson setting in the sense that an infinitesimal Poisson automorphism need not to be locally Hamiltonian (see [21]). The equivalence that holds in Poisson setting is given in the following lemma.

**Lemma 6.3.1.** Let \( M \) be Poisson locally Euclidean \( \mathbb{F} \)-space and \( F, H \in \mathcal{F}_M \). We have: \( H \) is constant along the integral curves of \( X_F \) if, and only if \( \{F,H\} = 0 \) if, and only if \( F \) is constant along the integral curves of \( X_H \)
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**Theorem 6.3.2.** Noether’s theorem

Let $M$ be Poisson locally Euclidean $\mathbb{F}$-space. Let $g \in G$ and $G$ be a Frölicher Lie group acting on $M$ by canonical Poisson maps $\sigma_g$ from the left. If the Poisson action has a moment map $\mu : M \rightarrow \mathfrak{g}^*$ and if $H \in \mathcal{F}_M$ is invariant under $\xi_M$ for all $\xi \in \mathcal{G}$, that is, $\{H, \hat{\mu}(\xi)\} = \xi_M(H) = 0$ for all $\xi \in \mathcal{G}$ as by Equation (6.23) and if $H_t$ is the flow of $X_H$. Then the moment map $\mu$ is a constant of motion under the dynamical flow of $X_H$. That is, $H_t^*\mu = \mu \circ H_t = \mu$.

**Proof.** The invariance of $H$ under $\xi_M$, that is, $\{H, \hat{\mu}(\xi)\} = \xi_M(H) = 0$ for all $\xi \in \mathcal{G}$ if, and only if $\hat{\mu}(\xi)$ is constant along the integral curves of $X_H$ if, and only if $\hat{\mu}(\xi)$ is constant along the flow of $X_H$ if, and only if a constant of motion under the dynamical flow of $X_H$ with regards to Lemma 6.3.1. By Equation (6.25) we have $<\mu(m); \xi > := \hat{\mu}(\xi)m$ for all $\xi \in \mathcal{G}$ and $m \in M$. In fact, this statement encodes the bijective correspondence between comoment map $\hat{\mu} : \mathcal{G} \rightarrow \mathcal{F}_M$ and the moment map $\mu : M \rightarrow \mathcal{G}^*$. In turn, $\mu : M \rightarrow \mathcal{G}^*$ is also a constant of motion under the dynamical flow of $X_H$. $\square$

### 6.3.2 Poisson equivariant moment maps

We have presented the general concept of the $G$-equivariance of a moment map with respect to the symplectic action in Section 3.3, while in Section 3.4 we have elaborated on the infinitesimal version of the equivariance of the moment map. Notice that Equation (6.25), stating that $\mu(m)(\xi) = <\mu(m); \xi > := \hat{\mu}(\xi)m$ for all $\xi \in \mathcal{G}$ and $m \in M$ should be taken as the defining condition of the moment map. However, the defining condition of the comoment map in Poisson setting is provided by Equation (6.23) that reads $\xi_M(F) = \{F, \hat{\mu}(\xi)\}$ for all $F \in \mathcal{F}_M$, for all $\xi \in \mathcal{G}$. Notice that the condition for equivariance was given in Definition 3.4.3 in symplectic setting, it will be the same in Poisson setting. Let $M$ be a Poisson locally Euclidean $\mathbb{F}$-space and $G$ a $\mathbb{F}$-Lie group. Let $\sigma$ be a Poisson action of $G$ on $M$ and acting from the left. Let $\mu : M \rightarrow \mathcal{G}^*$ be a moment map for the action. The equivariance of the moment map $\mu$ is given by the following equation.

$$\mu \circ \sigma_g = \text{Ad}(g^{-1})^* \circ \mu \quad \text{for all } g \in G$$

(6.30)

If we add appropriate evaluation maps into $\mathbb{R}$ to Equation (6.30) we get the following pairings:

$<\text{Ad}(g^{-1})^*(\eta); \text{Ad}(g)(\xi) > = <\eta; \text{Ad}(g^{-1})\text{Ad}(g)(\xi) > = <\xi; \eta >$ for all $\eta \in \mathcal{G}^*$ and $\xi \in \mathcal{G}$.

That is, the following diagram is commutative and all its sub-diagrams as well.
It follows that the equivariance of the moment map can be thought of by an equivalent characterization. That is,

\[ \mu(\sigma_g(m))(Ad(g)(\xi)) = \mu(g.m)(Ad(g)(\xi)) = \mu(Ad(g)(\xi))(g.m) = \hat{\mu}(\xi)(m) = \mu(m)(\xi) \]  \hspace{1cm} (6.31)

We have extended to a Poisson action the infinitesimal version of the equivariance of the moment map on a Poisson locally Euclidean \( \mathbb{F} \)-space by Equation (6.24) which states that \( X_{\tilde{\mu}(\xi,\eta)} = [\xi,\eta]_M = -[\xi_M,\eta_M] = -[X_{\mu(\xi)},X_{\mu(\eta)}] = [X_{\tilde{\mu}(\xi)},X_{\tilde{\mu}(\eta)}] \). First of all, we make clear the definition of the Poisson bracket on \( \mathcal{G}^* \). So, given \( F,H \in \mathcal{F}_{\mathcal{G}^*} \), it is known that for all \( \alpha \in \mathcal{G}^* \), we have \( X_F(\alpha),X_H(\alpha) \in \mathcal{G}^{**} := \mathcal{G}^{**} \cong \mathcal{G} \). We have \( X_F,X_H : \mathcal{G}^* \to \mathcal{G}^{**} \) with respect to Definition 3.2.3 and the canonical Lie bracket \([,]_\mathcal{G}\) on \( \mathcal{G} \). It follows that the canonical Poisson bracket on \( \mathcal{G}^* \) is defined by:

\[ \{ F,H \}_{\mathcal{G}^*}(\alpha) = \langle \alpha, [X_F(\alpha),X_H(\alpha)]_{\mathcal{G}} \rangle \quad \text{for all } F,H \in \mathcal{F}_{\mathcal{G}^*}, \text{ and } \alpha \in \mathcal{G}^* \]  \hspace{1cm} (6.32)

The existence and uniqueness of such \( X_F \) and \( X_H \) are guaranteed in the Cartesian closed category of \( \mathbb{F} \)-spaces when one uses the Gelfand representation as presented in Section 4.2 through Equations (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), Definition 4.2.2 and Theorem 4.2.1 in the following way: the fact that \( X_F(\alpha) \in \mathcal{G}^{**} = C^\infty(\mathcal{G}^*,\mathbb{R}) \) means that \( X_F(\alpha) : \mathcal{G}^* \to \mathbb{R}, \alpha \in \mathcal{G}^* \) and \( X_F : \mathcal{G}^* \to C^\infty(\mathcal{G}^*,\mathbb{R}) = \mathcal{G}^{**} \cong \mathcal{G} \). The Cartesian closedness property ensures that we are dealing with \( \text{FrI} \)-objects and \( \text{FrI} \)-morphisms and also we can define the associated map to \( X_F \) which we denote by the same symbol. That is, \( X_F : \mathcal{G}^* \times \mathcal{G}^* \to \mathbb{R} \), such that \( (\alpha,\beta) \mapsto X_F(\alpha,\beta) \). The maps \( X_F(\alpha),X_F(\beta) : \mathcal{G}^* \to \mathbb{R} \) read \( X_F(\alpha),X_F(\beta) \in \mathcal{G}^{**} \cong \mathcal{G} \). Now, the forthcoming theorem is a very important result as it makes provision of the fact that the equivariance of a map in Poisson setting implies that such a moment map is a Poisson map.

**Theorem 6.3.3.** Let \( M \) be a Poisson locally Euclidean \( \mathbb{F} \)-space and \( G \) a \( \mathbb{F} \)-Lie group. Let \( \sigma \) be a Poisson action of \( G \) on \( M \) and acting from the left. Let \( \mu : M \to \mathcal{G}^* \) be a moment map for the Poisson action of \( G \) on \( M \). If the moment map \( \mu : M \to \mathcal{G}^* \) is equivariant then
μ is a Poisson map. That is,
$$
\mu^* \{ F, H \}_G = \{ \mu^* F, \mu^* H \}_M \quad \iff \quad \{ F, H \}_G \circ \mu = \{ F \circ \mu, H \circ \mu \}_M, \quad \text{for all } F, H \in F_G.
$$

**Proof.** ([21]) We will work on LHS and RHS separately to prove that they are both equal to the same quantity. Indeed, for \( m \in M \), \( \mu(m) \in G^* \), and \( \xi = X_F(\mu(x)), \eta = X_H(\mu(m)) \in G^{**} = G \), we have on one side

$$
LHS = \{ F, H \}_G(\mu(m)) = \langle \mu(m); [X_F(\mu(m)), X_H(\mu(m))] \rangle > \quad \text{by Equation (6.32) and pullback}
$$

$$
= \langle \mu(m); [\xi, \eta] \rangle > \\
= \hat{\mu}(\langle [\xi, \eta] \rangle(m)) \quad \text{by Equation (6.25)}
$$

$$
= \{ \hat{\mu}(\eta), \hat{\mu}(\xi) \}(m) \quad \text{by infinitesimal equivariance in Equation (6.24).}
$$

On the second side, for \( m \in M \), \( F, H \in F_G \), \( v_m \in T_m M \), and \( \xi = X_F(\mu(m)), \eta = X_H(\mu(m)) \in G^{**} = G \), we will pairwise compute \( d(F \circ \mu)(m) \) and \( d(H \circ \mu)(m) \), respectively. Thereafter, we will deal with the RHS of the equality stated in the theorem.

$$
d(F \circ \mu)(m).v_m = dF(\mu(m)).T_m \mu(v_m) \quad \text{by the chain rule}
$$

$$
= \langle T_m \mu(v_m); X_F(\mu(m)) \rangle > \\
= d\hat{\mu}(\xi)(m).v_m \quad \text{by the defining condition of } \mu \text{ and } \hat{\mu} \text{ in Equation (6.25)}.
$$

By using the similar arguments we show that

$$
d(H \circ \mu)(m).v_m = d\hat{\mu}(\eta)(m).v_m.
$$

Notice that \( \mu(m) \in F_G \) and \( \hat{\mu}(\xi) \in F_M \), so we can apply Lemma 6.1.8 as follows:

$$
\{ d(F \circ \mu); d(H \circ \mu) \} = \{ d\hat{\mu}(\xi); d\hat{\mu}(\eta) \} \quad \iff \quad d\{ (F \circ \mu); (H \circ \mu) \} = d\{ \hat{\mu}(\xi); \hat{\mu}(\eta) \}
$$

$$
\iff \quad \{ (F \circ \mu); (H \circ \mu) \} = \{ \hat{\mu}(\xi); \hat{\mu}(\eta) \}.
$$

Hence, combining all results collected so far in first and second part of the proof, we have

$$
LHS = \{ F, H \}_G(\mu = \{ \hat{\mu}(\eta), \hat{\mu}(\xi) \} = \{ (F \circ \mu); (H \circ \mu) \} = RHS. \quad \square
$$

**Remark 6.3.1.** From Equation (2.8) on the Cartesian closedness of the category Frl we have \( C^\infty(M, G^*) = C^\infty(M, C^\infty(G, \mathbb{R})) \cong C^\infty(M \times G, \mathbb{R}) \). This means uniqueness and smoothness of the moment map \( \mu \) which is determined by its unique comoment \( \hat{\mu} \). The latter is uniquely defined by a smooth function on the product \( M \times G \). The equivariance is a selective criteria among moment maps.
6.4 Symplectic reduction of dynamical systems

We have treated the Poisson reduction process through Definition 6.1.8 which gives the Poisson bracket on the quotient space, Lemma 6.1.13 which states that $\mathcal{F}_M \xrightarrow{\pi^*} \mathcal{F}_M^G$, the pullback of $\pi$ (the canonical surjection) is an isomorphism of $\mathbb{R}$-algebras, and Equation (6.20) which reads $H_t \circ \pi = \pi \circ H_t$ and $\pi_x \circ X_h = X_h \circ \pi$. We summarize this process as follows. The canonical surjection is a Poisson map. So, it transforms a Hamiltonian vector field $X_H$ on $M$ to a Hamiltonian vector field $X_h$ on $M = M/G$. This is what we read in the symplectic reduction where the Poisson bracket is nondegenerate and the fiber of the moment map being closed makes the canonical inclusion in a Poisson injective immersion and a Hamiltonian function on this fiber obeys the Equation (6.20).

**Theorem 6.4.1.** Marsden-Westen reduction of dynamic

Let $\mu : M \rightarrow G^*$ the moment map associated to a Hamiltonian, free and proper $G$-action on a symplectic locally Euclidean Frölicher space $(M, \omega)$. Let $\theta$ be a regular value of $\mu$.

Let $\pi : M \rightarrow M/G$ be the canonical projection and $\pi_{\theta} = \pi|_{\mu^{-1}(\theta)}$ the restriction of the canonical projection to $\mu^{-1}(\theta)$. Let $\iota_{\theta} = \iota|_{\mu^{-1}(\theta)} : \mu^{-1}(\theta) \rightarrow M$ be the canonical inclusion of $\mu^{-1}(\theta)$ to $M$ and $\omega_{\theta} = \omega|_{\mu^{-1}(\theta)}$ the restriction of $\omega$ to $\mu^{-1}(\theta)$. Let $H \in \mathcal{F}_M^G$ be a $G$-invariant Hamiltonian function under the action of $G$ on $M$ from the left. Then,

1. the reduced space $M_\theta = \pi(\mu^{-1}(\theta)) = \mu^{-1}(\theta)/G_\theta$ is a symplectic locally Euclidean Frölicher subspace of $M/M/G$ with the symplectic form $\omega_{\theta}$ defined by $\pi_{\theta}^* \omega_{\theta} = \iota_{\theta}^* \omega$;
2. the moment map $\mu$ is conserved by the Noether’s Theorem that is, the flow $H_t$ of $X_H$ leaves $\mu^{-1}(\theta)$ invariant;
3. the flow $H_t$ commutes with the action of $G_\theta$ on $\mu^{-1}(\theta)$, that is, $H_t \circ \sigma_g = \sigma_g \circ H_t$ for $g \in G_{\theta}$;
4. the flow $H_t$ induces a flow $H_t^0$ on $M_\theta$ satisfying $\pi_{\theta} \circ H_t = H_t^0 \circ \pi_{\theta}$;
5. the vector field generated by the flow $H_t^0$ on $(M_\theta, \omega_\theta)$ is Hamiltonian with its associated Hamiltonian function $H_\theta \in C^\infty(M_\theta)$, called reduced Hamiltonian and defined by $H_\theta \circ \pi_\theta = H \circ \iota_{\theta}$;
6. moreover, the vector field $X_H$ and $X_{H_\theta}$ are $\pi_{\theta}$-related, as by Equation (6.20): that is, $\pi_{\theta} \circ X_H = X_{H_\theta} \circ \pi_{\theta}$.
7. Let $K \in \mathcal{F}_M^G$ be another $G$-invariant function. Then $\{H, K\}$ is also $G$-invariant and $\{H, K\}_\theta = \{H_\theta, K_\theta\}_{M_\theta}$, where $\{., .\}_{M_\theta}$ denotes the Poisson bracket associated to the symplectic form $\omega_\theta$ on $M_\theta$. 

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Conclusion

Our contribution: The differential geometry has reached out to some new categories. These categories are an attempt to extending notions from smooth manifolds to a large concept of smooth spaces which integrate smooth manifolds as a full subcategory. New categories such as that of differential spaces and diffeological spaces serve as ground for the extension of important results in differential geometry which are known on smooth manifolds.

However, we point out that there is no good notion of differential forms in the category of differential spaces (in the sense of Sikorski) that can lead to a cohomology isomorphic to Čech cohomology. In the contrary, they have some nice generalizations of notions from manifolds, such as that of vector fields.

Whereas, in the category of diffeological spaces, there is a good definition for differential forms. But, there are problems when it comes to defining the notion of vector fields.

Both the differential form and the vector field notions are naturally extended to Frölicher spaces, which form a full subcategory of both the category of differential spaces and the category of diffeological spaces. The category of $\mathbb{F}$-spaces inherits the good properties of its two super-categories. Therefore, it is a convenient ground for the extension of the de Rham Theorem well-known on topological spaces and smooth manifolds.

We have selected five cohomologies which are: de Rham cohomology, Alexander-Spanier cohomology, Continuous Singular and differentiable Singular cohomologies, Čech cohomology and sheaf cohomology (cohomology real algebra of a differentiable manifold with coefficients in a $\mathbb{R}$-algebra $\mathbb{R}$). We have proved that the five selected cohomologies on symplectic locally Euclidean spaces are canonically isomorphic (Theorem 5.6.3 and Theorem 5.6.4). Therefore, the isomorphism of cohomology theories on the reduced symplectic Frölicher space follows naturally. The main result of the thesis is Theorem 5.7.4 which confirms that the two isomorphisms established by the preceding theorems descend to the symplectic quotient of a locally Euclidean Frölicher space. The result was elaborated through Subsection 5.7.3 in order for us to reach the purpose of this thesis. We have presented the work over six chapters ending
6.4 Symplectic reduction of dynamical systems

by the conclusion, where we explain what we have done in the thesis and what may be part of further researches.

A review of the literature is presented in the Introduction. In Chapter 2 we recall basic concepts in the category of Frölicher spaces. This amounts to a summary of our two papers [10, 11], definitions and examples, tangent structures and exterior algebra on locally Euclidean spaces. We have also proved that the power set of smooth functions with identity and canonical inclusions as morphisms forms a category. So is the power set of smooth curves. It comes out that the compatibility conditions for \( \mathbb{F} \)-smooth structures are actually defining two functors. In Chapter 3 we presented a summary of steps and concepts referred to when proceeding with a symplectic reduction. In Chapter 4, we defined a new concept of ringed Frölicher spaces, on which we defined the Gelfand representation and proved its smoothness. Thereafter, in this chapter we proved that objects of interest in the the process of symplectic reduction are Hausdorff paracompact spaces by a \( \mathbb{F} \)-diffeomorphism built onto \( \mathbb{R}^n \). It follows that the symplectic quotient is a Hausdorff paracompact topological space. The Chapter 5 is devoted to the proof of the de Rham theorem on a symplectic quotient. We have established an isomorphism between each selected cohomology and the sheaf cohomology. Hence, we deducted the isomorphism between all five selected cohomologies. We have proved that this isomorphism of cohomologies holds when we consider a symplectic quotient. In Chapter 6, we aimed to present a modern formalism of mechanics in the category of Frölicher spaces. So, we defined a Poisson structure and extented it to a Poisson quotient. We have proved some properties of the Poisson structure with respect to exterior algebra operators and the symplectic structure. We have presented the canonical Poisson maps such as the equivariant moment map and the canonical projection map. Furthermore, we have dealt with Poisson reduction and compared it to symplectic reduction. Finally, we presented the symplectic reduction of dynamical systems.

Further researches: At first stage we will be interested in the proof of the existence, uniqueness and smoothness of a moment map in the category of Frölicher spaces. But, also we will check the moment maps for cotangent lifted action. Secondly, we will look at applications of cohomologies and the consequences of the isomorphism of cohomologies, such as topological and cohomological invariance. Thirdly, projects of research can arise from chapters 4, 5 and 6. We must explain the three kinds of locally Euclidean spaces as stated in Definition 2.2.2 and seek avenues on how to name the first kind, the second kind and the third kind in such a way as to avoid confusion with the existing notions. Finally, we will explore some questions such the ones below.

Can a theory of smooth orbifolds and gluing process be introduced in the new setting? What about Lagrangian mechanics and Legendre transforms? How to work out the diffeomorphism \( T^*G/G \longrightarrow \mathcal{G}^* \) and its consequences on the reduction of dynamics from \( T^*G \) to \( \mathcal{G}^* \)?
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