GEOMETRIC STEINER MINIMAL TREES

by

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Summary

In 1992 Du and Hwang published a paper confirming the correctness of a well known 1968 conjecture of Gilbert and Pollak suggesting that the Euclidean Steiner ratio for the plane is $2/\sqrt{3}$. The original objective of this thesis was to adapt the technique used in this proof to obtain results for other Minkowski spaces. In an attempt to create a rigorous and complete version of the proof, some known results were given new proofs (results for hexagonal trees and for the rectilinear Steiner ratio) and some new results were obtained (on approximation of Steiner ratios and on transforming Steiner trees).

The most surprising result, however, was the discovery of a fundamental gap in the proof of Du and Hwang. We give counterexamples demonstrating that a statement made about inner spanning trees, which plays an important role in the proof, is not correct. There seems to be no simple way out of this dilemma, and whether the Gilbert-Pollak conjecture is true or not for any number of points seems once again to be an open question. Finally we consider the question of whether Du and Hwang's strategy can be used for cases where the number of points is restricted. After introducing some extra lemmas, we are able to show that the Gilbert-Pollak conjecture is true for 7 or fewer points. This is an improvement on the 1991 proof for 6 points of Rubinstein and Thomas.

Key terms: Minkowski space, spanning tree, minimum spanning tree, Steiner tree, Steiner minimal tree, Steiner ratio, rectilinear tree, hexagonal tree, surface, inner spanning tree, inner Steiner tree.

Contents

Preface

A geometric Steiner tree is a tree, embedded in some defined metric space, which interconnects a given set of vertices. Finding such a tree of minimal length, called a Steiner minimal tree, is not a trivial task. (The problem is generally NP-complete [22].) An interesting problem is to compare the lengths of Steiner minimal trees with those of minimum spanning trees (which many algorithms can find in polynomial time) by using the Steiner ratio.

In 1992 Du and Hwang [8] published a paper confirming the correctness of a well known 1968 conjecture of Gilbert and Pollak suggesting that the Euclidean Steiner ratio for the plane is $2/\sqrt{3}$. The original objective of this thesis was to adapt the technique used in this proof to obtain results for other Minkowski spaces. In an attempt to create a rigorous and complete version of the proof, some known results were given new proofs (results for hexagonal trees in Chapter 2; the rectilinear Steiner ratio in Chapter 3) and some new results were obtained (approximation of Steiner ratios in Chapter 4; transforming Steiner trees in Chapter 5). The results of Chapter 4 were obtained through the development of an interesting technique which can also be employed in other Minkowski spaces.

The most surprising result, however, was the discovery of a fundamental gap in the proof of Du and Hwang. In Chapter 6 we give counterexamples demonstrating that a statement made about inner spanning trees, which plays an important role in the proof, is not correct. There seems to be no simple way out of this dilemma, and whether the Gilbert-Pollak conjecture is true or not for any number of points seems once again to be an open question.

In the last chapter we consider the question of whether Du and Hwang's strategy can be used for cases where the number of points is restricted. Here, after introducing some extra lemmas, we are able to show that the Gilbert-Pollak conjecture is true for 7 or fewer points. This is an improvement on the proof for 6 points, published by Rubinstein and Thomas [25] in 1991.

I would like to sincerely thank my promoter, Konrad Swanepoel, for his support throughout this project, as well as for cultivating in me a genuine interest for discrete geometry. I would furthermore like to thank the Department of Decision Sciences at Unisa for the very supportive atmosphere in which I could work on this thesis.

Oloff de Wet
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Chapter 1

Preliminaries

This chapter introduces the basic concepts used in the rest of the thesis. We do not define vertices and arcs simply as subsets of the space in which they occur, as it is usually done, but rather as elements that can be embedded in space. This definition allows for these elements to coincide, which is a useful property.

1.1 Background

Fermat (1601–1665) raised the question as to where a point in the plane should be so that the sum of the distances to three given points is minimal. In 1934 Jarník and Kössler [17] considered the more general problem of finding a shortest network which interconnects n points in the plane. Courant and Robbins, noticing the relationship between these problems, referred to it as "the Steiner problem" in their 1941 book "What is mathematics?"[2]. Although Steiner himself did some work on a related problem, this name remained, and when Gilbert and Pollak wrote their important paper on the subject [17], they furthermore introduced the terms Steiner minimal tree and Steiner point, which are in common use today. They also introduced the idea of Steiner trees in higher dimensions, while others, like Hanan, began to explore Steiner trees in other (non-Euclidean) Minkowski spaces.

1.2 Basic concepts

A *norm* on a finite dimensional real vector space V is a real valued function $\|\cdot\|$ on V such that

$$\|x\| \geq 0 \text{ with equality if and only if } x = 0,$$

$\|\alpha x\| = |\alpha| \ \|x\|$ where $\alpha \in \mathbf{R}$ and

$\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The norm defines a metric where the *distance* between x and y is $\|xy\| = \|x - y\|$. It follows from the triangle inequality that

$$\|x - y\| = \|(x - z) + (z - y)\| \leq \|x - z\| + \|z - y\|.$$

A *Minkowski space* $\mathcal{M}^d$ (or simply $\mathcal{M}$) is a d-dimensional real vector space on which a norm has been defined. We say that $A \subset \mathcal{M}$ is a *centrally symmetric* set if $x \in A$ implies $-x \in A$. The unit ball $B = \{x \in \mathcal{M} : \ \|x\| \leq 1\}$ is centrally symmetric, convex, compact, and has non-empty interior. Conversely, given a convex, compact, centrally symmetric set with non-empty interior in a finite-dimensional real vector space, then a norm can be defined which has this set as its unit ball [29].

A finite set V is called a set of *vertices* in $\mathcal{M}$ if it is equipped with a mapping $\beta : V \to \mathcal{M}$. We define an *arc* as any set of the form $\gamma([0, \lambda]) = \{\gamma(x) : 0 \leq x \leq \lambda\}$, where $\lambda \geq 0$ and $\gamma : [0, \lambda] \to \mathcal{M}$ is a continuous injective mapping. Note that a single point in $\mathcal{M}$ is considered to be an arc. Now let $\mathcal{A}$ be the set of all arcs in $\mathcal{M}$. A finite set E is called a set of *edges* in $\mathcal{M}$ if it is equipped with a mapping $\alpha : E \to \mathcal{A}$. The pair (V, E) is called a *topological graph* in $\mathcal{M}$ if we have for every $e \in E$ an $x \in V$ and a $y \in V$ such that the endpoints of $\alpha(e)$ are $\beta(x)$ and $\beta(y)$. (This is not the usual definition of a topological graph.) A topological graph defines a graph in a natural way, making it possible for us to use terms from graph theory when discussing topological graphs. In this regard we call a topological graph a *tree* if its graph is a tree. A tree is a *spanning tree* of some finite set V in $\mathcal{M}$ if it has V as its vertex set. A *Steiner tree* of V is a spanning tree of some finite vertex set S in $\mathcal{M}$ such that $S \supseteq V$ and each vertex in $S \backslash V$ has degree at least 3. We call the vertices in V the *terminals* of the Steiner tree, and the vertices in $S \backslash V$ the *Steiner points* of the Steiner tree. A Steiner tree is a *full Steiner tree* if all terminals have degree 1. A Steiner tree which is not full, is the union of edge disjoint full Steiner trees. We call these smaller trees *full subtrees* and the original tree is called the *main tree*.

The *length of an arc* $\gamma([0, \lambda])$ is

$$\|\gamma([0, \lambda])\| = \sup\{\sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| : \ n \in \mathbf{N}, \ 0 = t_0 < t_1 < \cdots < t_n = \lambda\}.$$

The *length of an edge* is the length of its associated arc and the *length of a tree* T is the sum of the lengths of its edges, denoted as $\|T\|$. A *minimum*

spanning tree (MST) of V is a spanning tree of V of smallest length. A *Steiner minimal tree* (SMT) of V is a Steiner tree of V of smallest length.

For a fixed set V with n elements in $\mathcal{M}$ a MST always exists. To show that a SMT exists, we argue as follows: A Steiner tree for V with m Steiner points has $n+m-1$ edges. Since terminals have degree at least 1 and Steiner points have degree at least 3, there are at least $n/2+3m/2$ edges, and thus

$$n+m-1 \geq n/2+3m/2.$$

It follows that there are at most $n-2$ Steiner points, and thus a finite number of possible graph structures for the Steiner trees of V. A shortest Steiner tree with a specific graph structure will have edges which are straight line segments and Steiner points which are all within a closed ball containing V. For a specific graph structure we can now consider the length of a Steiner tree for which the Steiner points are variable within such a ball. This length is a continuous function of the positions of the Steiner points and is defined on a compact subset of $\mathbf{R}^{2m}$. It follows that a SMT always exists, because a continuous function always achieves a minimum on a compact set.

We define the *Steiner ratio* ρ for $\mathcal{M}$ as

$$\rho = \sup_{\text{any } V \text{ in } \mathcal{M}} \frac{\|\mathrm{MST}(V)\|}{\|\mathrm{SMT}(V)\|}$$

where $\mathrm{MST}(V)$ is a MST of V and $\mathrm{SMT}(V)$ is a SMT of V. (Note that ρ is sometimes inversely defined as $\inf \frac{\|\mathrm{SMT}(V)\|}{\|\mathrm{MST}(V)\|}$, but that our definition provides for the fact that we will often have a SMT of length 1.)

A lower bound for the Steiner ratio is usually found by considering a specific vertex set. For an upper bound we need to consider only full Steiner trees:

Lemma 1.1 *Consider a Steiner minimal tree which is not full. If for every subtree SMT $'$ we have a MST $'$ for its terminals V' such that*

$$\|MST\,'(V')\| \leq c\ \|SMT\,'(V')\|,$$

then we also have

$$\|MST(V)\| \leq c\ \|SMT(V)\|$$

for the main Steiner tree.

Proof: We have

$$\|\mathrm{MST}(V)\| \leq \sum \|\mathrm{MST}'(V')\| \leq c \sum \|\mathrm{SMT}'(V')\| = c\ \|\mathrm{SMT}(V)\|$$

for the main Steiner tree, because the union of the smaller spanning trees is a spanning tree of V. QED

Figure 1 shows a main Steiner tree consisting of two full subtrees. Terminals are shown as solid dots and the edges of MSTs as dotted lines. Note that the union of the two MSTs is a spanning tree, but not necessarily a MST, of V.

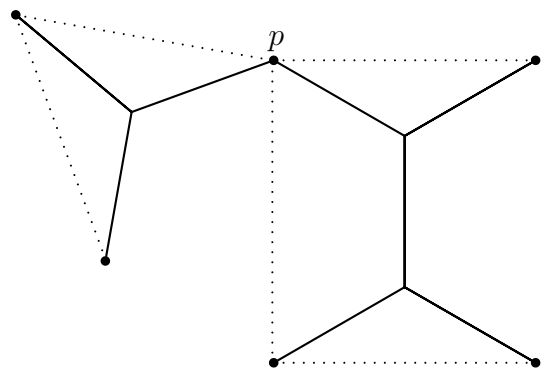


Figure 1

Figure 1 can also be interpreted as a full Steiner tree which has a terminal and a Steiner point coinciding at p with the length of the edge between them 0. An upper bound for the length of this MST can be found by considering the same smaller full Steiner trees as in the previous paragraph.

Chapter 2

Rectilinear trees and hexagonal trees

In Section 1 we establish that a rectilinear Steiner minimal tree can be assumed to lie on some form of grid. The result for two dimensions is used in a proof of the rectilinear Steiner ratio in Chapter 3. The result for three dimensions is combined with results from Section 2 (dealing with the slightly artificial concepts of diagonals, square grids and grids) to culminate in Section 3 with a novel proof for Lemma 2.6 which is much shorter than the original. This lemma is essential for proving Lemma 4.2, which plays an important part in the last two chapters.

2.1 Rectilinear trees

Given n vertices in the plane, a grid can be created by constructing a horizontal line and a vertical line through each vertex. This network is commonly called the grid graph [22] of the vertices. We can generalize this idea to more dimensions: Given n vertices in $\mathbf{R}^k$ we can construct a hyperplane perpendicular to each of the axes through each vertex. The intersection of any $n-1$ hyperplanes, with distinct normals, forms a line. The collection of all of these lines is known as the *grid graph* of the vertices.

The norm we will first investigate is the L_1 or *taxicab norm* where the distance between points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $\mathbf{R}^n$ is defined as $|y_1 - x_1| + \ldots + |y_n - x_n|$. We can assume that a SMT in a Minkowski space with the L_1 norm consists of line segments, each of which is parallel to one of the axes. We call such a tree a *rectilinear tree*. The following is a result of Hanan [14]. (We provide a new proof.)

Lemma 2.1 *Given n terminals in the plane with the L_1 norm, then there*

exists a SMT with all segments on the grid graph of the terminals. Furthermore, each such segment that is maximal (consisting of a maximal sequence of adjacent collinear segments) contains at least one of the terminals.

Proof: First consider a SMT such that the number of horizontal maximal segments which do not contain a terminal, is a minimum and assume it to be greater than zero. Consider the topmost of these maximal segments. Since we have a SMT, this maximal segment can be moved up or down by a sufficiently small Δx without decreasing (or increasing) the length of the tree (Figure 1).

Figure 1

We move the maximal segment upwards until a terminal or horizontal segment is reached, thus decreasing the number of horizontal maximal segments not containing terminals and providing a contradiction. It follows that there is a SMT where each horizontal maximal segment contains a terminal. Among all such SMTs we may consider one in which the number of vertical maximal segments not containing terminals is a minimum. As above, it follows that this number is 0. The SMT obtained lies on the grid graph. QED

We can extend the previous lemma to three dimensions if we replace the concept of a maximal segment by that of a maximal planar tree. (See [28] for more on Steiner points in higher dimensions.) By a *maximal planar tree* we mean a maximal tree which lies in a plane perpendicular to one of the axes.

Lemma 2.2 *Given n terminals in $\boldsymbol{R}^3$ with the L_1 norm, then there exists a SMT with all segments on the grid graph. Furthermore, each maximal planar tree of the SMT contains at least one of the terminals.*

Proof: First consider a SMT such that the number of maximal planar trees in planes perpendicular to the z-axis which do not contain a terminal, is a

minimum and assume it to be more than zero. Consider the topmost (i.e. with largest z-coordinate) of these maximal planar trees. Since we have a SMT, this maximal planar tree can be moved up or down by a sufficiently small Δz without decreasing (or increasing) the length of the tree.

We move the maximal planar tree upwards until a terminal or horizontal maximal planar tree is reached, thus decreasing the number of horizontal maximal planar trees not containing terminals and providing a contradiction. It follows that there is a SMT where each maximal planar tree which is perpendicular to the z-axis contains a terminal. Among all such SMTs we may consider one in which the number of maximal planar trees perpendicular to the y-axis not containing terminals is a minimum. As above, it follows that this number is 0. Finally the same is done for the x-axis. QED

It is possible to extend the previous lemma to k dimensions if we introduce the concept of a *maximal hyperplanar tree* – a maximal tree which lies in a hyperplane perpendicular to one of the axes. We will not use this fact, but state it for completeness. The proof is similar to that of the previous lemma.

Lemma 2.3 *Given n terminals in $\boldsymbol{R}^k$ with the L_1 norm, then there exists a SMT with all segments on the grid graph. Furthermore, each maximal hyperplanar tree of the SMT contains at least one of the terminals.*

2.2 Diagonals, square grids and cube grids

If we construct a horizontal and a vertical line through each integer coordinate pair in the plane, we obtain an infinite grid graph. We call any translate of this grid graph a *square grid*. For three dimensions we define the *cube grid* similarly, by constructing three lines through all integer points and considering translates.

In the plane we collectively call all lines parallel to $y = x$ (i.e. parallel to the vector $(1, 1)$) through all integer points, *diagonals*. A vertex on one of these diagonals naturally defines a square grid in the plane if we let an intersection of the square grid coincide with this vertex (Figure 2). Similarly diagonals in three dimensions pass through all integer points and are parallel to the vector $(1, 1, 1)$. A vertex on a diagonal now naturally defines a cube grid.

We note that two points on diagonals in the plane (three dimensional space) define the same square grid (cube grid) if they have the same x or y (x or y or z) coordinate.

We keep to the L_1 norm and explore the following problem: Given n different diagonals, each with a terminal on it, where should the terminals

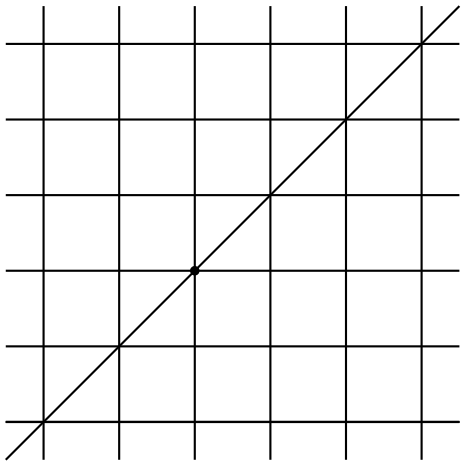


Figure 2

be for the SMT to have minimal length? For two dimensions it is not difficult to see that the following lemma is true.

Lemma 2.4 *Given n terminals on diagonals in the plane, then we can slide the terminals along the diagonals to new positions so that they all have the same y-coordinate and so that the new SMT is not longer than the initial one.*

For three dimensions the problem is more complex. We notice that the SMT of $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ has minimal length and that this is not possible if the points are moved along diagonals to lie in the same plane. To see why this is so, let us start with the three diagonals which go through $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Fix one terminal at $(0,0,1)$ while allowing the other two terminals to be moved on their respective diagonals. Next we construct around each terminal a ball of radius 1, as shown in Figure 3. (We indicate the diagonals, as projected onto a plane parallel to vector $(1,1,1)$, by dots.)

We see that the only positions for the two terminals not yet fixed which will assure that the union of the three balls is connected, are $(1,0,0)$ and $(0,1,0)$. Now since any SMT for the three terminals will for each terminal contain a path which connects the point to the surface of the ball, the SMT can not be shorter than 3, and this is only achieved when $(0,0,0)$ is a Steiner point. Finally we note that the SMT will at best remain the same if we introduce another terminal, so that the SMT remains the same after we introduce the terminal $(0,0,0)$.

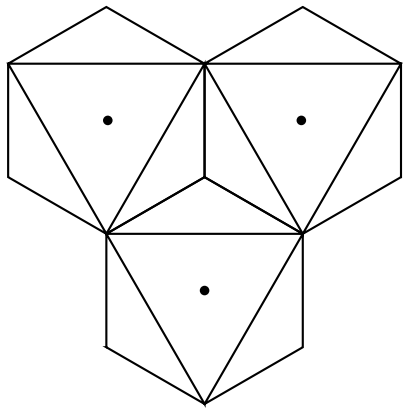


Figure 3

Lemma 2.5 *Given n terminals on diagonals in three dimensional space, then we can slide the terminals along the diagonals to new positions so that they all define the same cube grid and so that the new SMT is not longer than the initial one.*

Proof: Assume that the lemma is false. This implies that if the positions of the terminals on the diagonals are such that the length of the SMT is minimal, then the terminals define more than one cube grid. Assume that the number of cube grids defined is as small as possible and that the SMT is in the form described by Lemma 2.2.

Consider the set A of all terminals which define a particular cube grid. Each horizontal maximal planar tree which contains a terminal or terminals from A may be moved up and down (together with the terminals) by sufficiently small Δz so that the change in the length of the SMT is linear. If we move all horizontal maximal planar trees with terminals from A upwards simultaneously, then the change in the length of the SMT remains linear until some terminal in A has the same z-coordinate as some terminal which is not in A. Let Z be the length of the upward movement for this to happen.

In a similar way we can begin by moving the maximal planar trees which are perpendicular to the x-axis and which contain terminals in A in a positive direction until some terminal in A has the same x-coordinate as some terminal which is not in A. Let us call the length of this movement X. Let Y be the length of the corresponding movement in the positive y-direction. The three movements can be combined to achieve movement of the elements of A along the direction of vector $(1, 1, 1)$ with linear change in the length of the SMT if this Δd is sufficiently small. Since the SMT has minimal length

the length of the SMT has to stay constant. Let $D = \min(X, Y, X)$. If we move the elements of A by distance D in the positive x, y and z-directions, then we obtain a SMT with the same length, with all terminals on diagonals, and for which the number of cube grids defined by the terminals is one less, providing a contradiction and showing that it is possible for all terminals to define the same cube grid. QED

2.3 Hexagonal trees

Given three directions, each two of which form an angle of 120°, then a Steiner tree on n points in the plane for which all edges are line segments parallel to these directions, is called a *hexagonal tree*. A shortest such tree is called a *minimum hexagonal tree* (MHT). The length of the hexagonal tree is defined to be the sum of the lengths of the segments. We collectively call Steiner points and points (non-terminal) where two segments join at different angles, *junctions*. The example in Figure 4 has 4 junctions. For terminals on an equilateral triangular lattice, we have the following result. (The result is known [9], but our approach to proving it is novel.)

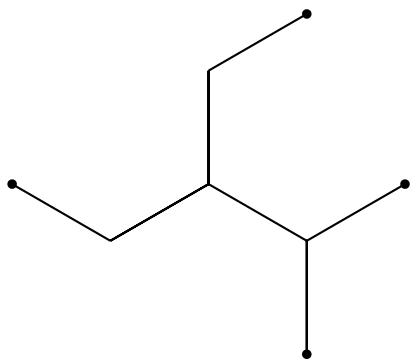


Figure 4

Lemma 2.6 *Consider any set of n terminals on an equilateral triangular lattice. Let the three directions for hexagonal trees be parallel to the edges of the equilateral triangles in the lattice. Then there is a MHT for which all junctions are lattice points.*

Proof: We will use Lemma 2.5 of the previous section. We begin by noticing that the projection of all diagonals in $\mathbf{R}^3$ onto a plane perpendicular to the vector $(1,1,1)$ forms an equilateral triangular lattice. Furthermore, a hexagonal tree for n lattice points can be lifted to a rectilinear tree in $\mathbf{R}^3$ with the terminals on diagonals, such that the projection of this rectilinear tree onto the plane parallel to $(1,1,1)$ returns the hexagonal tree. We now obtain the desired minimum hexagonal tree as follows: Begin with any minimum hexagonal tree, lift this tree into $\mathbf{R}^3$, replace it by a tree of equal length according to Lemma 2.5 (terminals now all lie on vertices of the same cube grid), modify this tree by using Lemma 2.2 (all segments now also lie on the cube grid), finally project the tree back to the plane (perpendicular to $(1,1,1)$) and note that this is still a minimum hexagonal tree and that it satisfies the theorem. QED

Chapter 3

The rectilinear Steiner ratio for the plane

3.1 Introduction

Our main aim in this chapter is to prove that the rectilinear Steiner ratio for the plane is 3/2. It was originally proven in 1976 by Hwang [15] by first characterizing Steiner trees and then obtaining the Steiner ratio. Richards and Salowe [23] discuss only the characterization and Salowe [26] proceeds from there to give a proof of Hwang's result. Our proof uses the same two main steps, but we use continuity so that we only have to characterize a dense subset of all possible Steiner trees. Our characterization can then be done in a more global way and we finally apply Salow's strategy to obtain a short and self-contained proof of the rectilinear Steiner ratio. Most of this chapter, together with some sections of the previous chapter, is already published in [5].

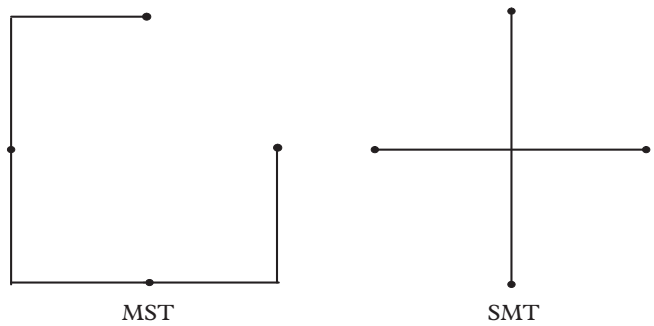


Figure 1

In Figure 1 we have an example of a MST and a SMT for terminals $\{(-1,0),(1,0),(0,-1),(0,1)\}$. The length of the MST is 6 and that of the SMT is 4. The SMT has one Steiner point $(0,0)$. It follows that ρ is at least $3/2$, and it remains to be shown that it will never exceed this value.

3.2 Restricted point sets

Given $n > 1$ vertices $(x_1, y_1), \ldots, (x_n, y_n)$ in the plane we assume, without loss, that the first vertex is at the origin, and use the rest of the vertices to define a point $z = (x_2, y_2, \ldots, x_n, y_n) \in \mathbf{R}^{2(n-1)} \backslash \{0\}$. We define

$$f : \mathbf{R}^{2(n-1)} \backslash \{0\} \to \mathbf{R}$$

as the length of a MST of the n vertices and

$$g : \mathbf{R}^{2(n-1)} \backslash \{0\} \to \mathbf{R}$$

as the length of a SMT of the n vertices.

Let us also equip $\mathbf{R}^{2(n-1)}$ with the L_1 norm. Since each terminal has less than n edges connected to it, we note that we have $|f(z_2) - f(z_1)| \leq n|z_2 - z_1|$ and $|g(z_2) - g(z_1)| \leq n|z_2 - z_1|$ and thus that f and g are continuous. Since g is never 0 we also have that f/g is continuous. Thus, to show

$$\rho = \max_{z \in \mathbf{R}^{2(n-1)}} \frac{f(z)}{g(z)} \leq \frac{3}{2},$$

we only have to consider a dense subset of $\mathbf{R}^{2(n-1)}$ and will consequently assume that no two vertices have the same x or the same y coordinates. (For any z in an Euclidean space one can find a point arbitrarily close to it such that none of the coordinates of the point are the same.)

We say that a set of terminals is a *restricted set* if no two terminals have the same x or the same y coordinates and if all SMTs on the terminals are full. To prove the Steiner ratio we only need to consider restricted sets.

3.3 Simple SMTs

We introduce some definitions which we will use in this chapter. We collectively refer to terminals, Steiner points and 90° angles as *points*. A *segment* connects two points without containing any other points. A *line* is a sequence of adjacent collinear segments. In Figure 1 we see that the SMT has 5 points, 4 segments and 2 lines.

We call a SMT on a restricted set *simple* if each line has a terminal at one of its endpoints and if the SMT contains no crosses (Steiner points of degree 4).

Lemma 3.1 *For every restricted point set there exists a simple SMT.*

Proof: First consider a SMT such that the number of horizontal lines that do not end in a terminal, is a minimum and assume it to be more than 0. Consider the topmost of these lines. Since we have a SMT, this line can be moved up or down by a sufficiently small Δx without decreasing (or increasing) the length of the tree (Figure 2(a)).

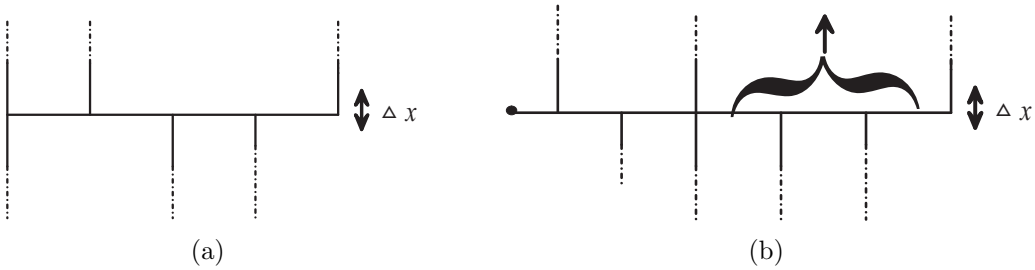


Figure 2

We move the line upwards until a terminal or horizontal line is reached, thus decreasing the number of horizontal lines not ending in terminals and providing a contradiction. (A terminal thus reached will have to be at the end of the line not to contradict fullness.) It follows that there is a SMT where each horizontal line has a terminal as one of its endpoints. Among all such SMTs we may consider one in which the number of vertical lines not ending in terminals is a minimum. As above, it follows that this number is 0.

Next we consider those SMTs with lines ending in terminals and the smallest number of crosses and choose one for which the topmost cross is as high as possible (Figure 2(b)).

We move the left or right horizontal arm of the cross (depending on which side does not contain the terminal) upwards, until we reach another horizontal line or terminal. (Again the terminal will have to be at the end of the line.) We still have that all lines end in terminals. We now either have one fewer cross, or the same number of crosses of which one is higher than before – providing a contradiction. QED

3.4 Characterisation of a longest simple SMT

We consider a simple SMT (on a given restricted point set) which has a maximum number of points on a line. We call this tree a *longest* simple SMT, the line is called the *trunk* and the subgraphs attached to the trunk are called *branches*. We want to characterise these trees for $n > 4$. The basic structure is as in Figure 3, where we put the terminal of the trunk at the top.

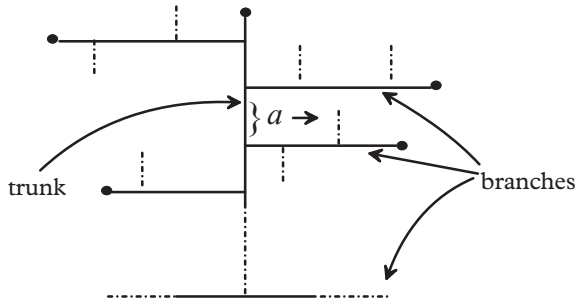


Figure 3

Lemma 3.2 *Adjacent branches of a longest simple SMT are on opposite sides of the trunk, i.e. branches alternate.*

Proof: If two adjacent branches (neither the lowest) of a longest SMT are on the same side of the trunk, as in Figure 3, a segment a of the trunk can be made to slide between the branches until a terminal is reached – contradicting fullness. At the lowest branch we may have the situation of Figure 4 (as indicated by the solid lines).

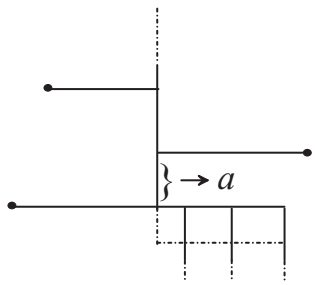


Figure 4

If segment a is slid to reach the terminal or a vertical segment on top of the lowest branch, we again contradict our assumptions. (Fullness is contradicted if the terminal is reached first, and the fact that it is a SMT is contradicted if segment a overlaps another vertical segment.) If this does not happen, we can replace the lowest branch with a new lowest branch (the dotted lower branch in the sketch) to obtain a simple SMT with a longer trunk – contradicting the fact that we had a longest simple SMT. QED

Lemma 3.3 *The branches of a longest simple SMT, except possibly for the lowest branch, each consists of only one segment; i.e. branches, except possibly the lowest, have no subbranches.*

Proof: Assume there are subbranches. As before, a segment a (the first segment on the branch) can be slid up or down until a terminal is reached or another branch is overlapped as in Figure 5(a).

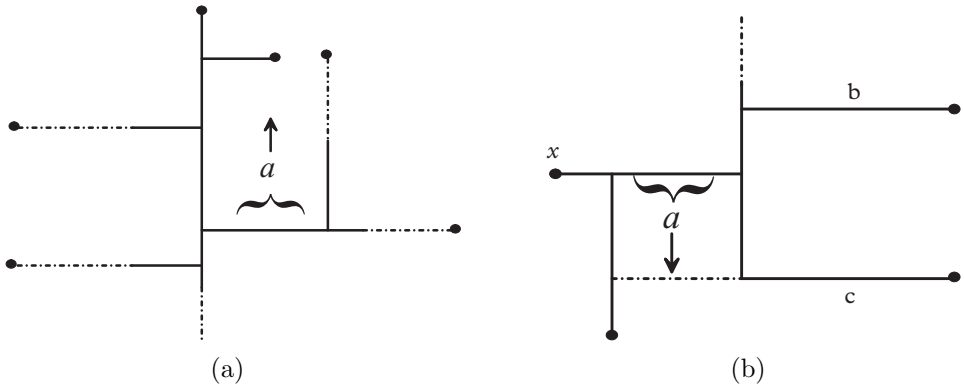


Figure 5

At the second lowest branch we may, however, reach the end of the trunk. In this case we see (Figure 5(b)) that we have created non-alternating branches b and c not allowed by Lemma 3.2.

Note: Branch b must exist, otherwise the line with the most points would have passed through x. QED

Lemma 3.4 *The lowest branch of a longest simple SMT, can have at most one subbranch, which consists of a single segment and faces downwards.*

Proof: Assume there are subbranches facing upwards. Consider the first and note that there has to be a single downward going subbranch between it and the trunk to ensure that it is a minimal tree and full (Figure 6).

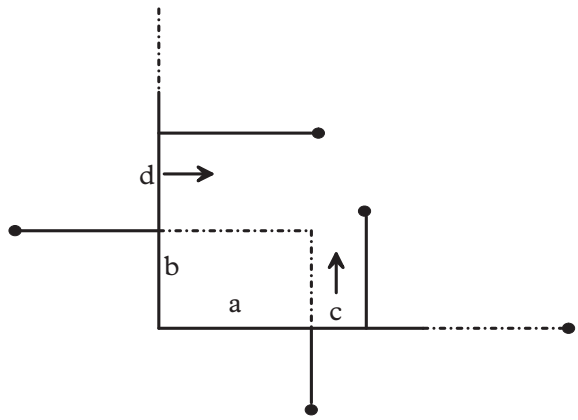


Figure 6

We replace segments a and b with the parallel dotted segments and let segments c and d slide to obtain a contradiction of fullness. The fact that there is only one downward subbranch and that it consists of a single segment is easily verified by sliding of segments. QED

3.5 The Steiner ratio

According to the previous three lemmas, a longest simple SMT with $n > 4$ will have the general structure as shown in Figure 7, where the subbranch at the bottom may or may not exist. We can now prove our main theorem.

Theorem 3.5 *The rectilinear Steiner ratio for the plane is* $3/2$.

Proof: We use induction and assume that the theorem is correct for all sets with less than $n > 4$ terminals. We only have to consider a longest simple SMT. We first assume that there is a subbranch and number the terminals and segments as in Figure 7, taking h_n to be 0.

If $h_4 < h_2$, as in Figure 8(a), we consider the rectangle with perimeter $2(h_2 + h_3 + v_1 + v_2 + v_3)$ and note that a path W lies on it which connects the terminals such that

$$\text{length } (W) \leq \frac{3}{4} \, (2(h_2 + h_3 + v_1 + v_2 + v_3)).$$

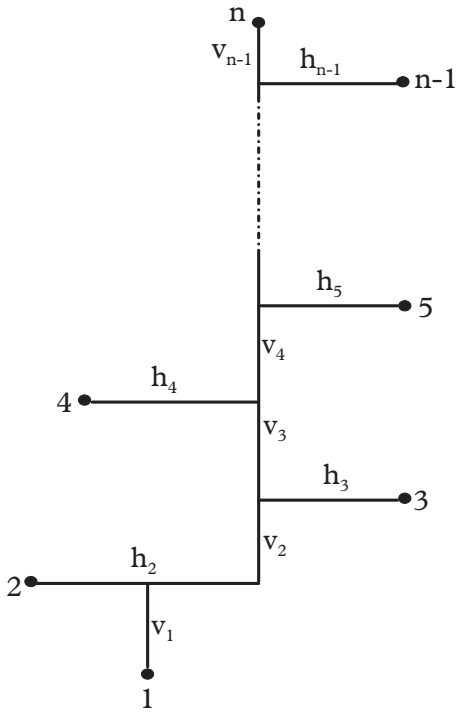


Figure 7

We can now combine W and MST$(4, \ldots, n)$ to obtain the MST we want:

$$\begin{aligned}
\text{length (MST}(1, \ldots, n)) &\leq \text{length } (W) + \text{length (MST}(4, \ldots, n)) \\
&\leq \frac{3}{2}\,(h_2 + h_3 + v_1 + v_2 + v_3) + \frac{3}{2}\,(\sum_{i=4}^{n} h_i + \sum_{i=4}^{n-1} v_i) \\
&= \frac{3}{2} \text{ length (SMT}(1 \ldots n)).
\end{aligned}$$

If $h_4 > h_2$ we find the first p such that $h_{p+3} < h_{p+1}$, as in Figure 8(b). (This must eventually happen, because $h_n = 0$.)

Again we find a path W such that

$$\text{length } (W) \leq \frac{3}{2}\,(h_{p+1} + h_{p+2} + v_p + v_{p+1} + v_{p+2})$$

and combine MST$(1, \ldots, p)$, W and MST$(p+3, \ldots, n)$ to obtain the MST we want.

We have now shown that the theorem is true for a longest simple SMT on n points with a subbranch, under the hypothesis that the theorem is generally

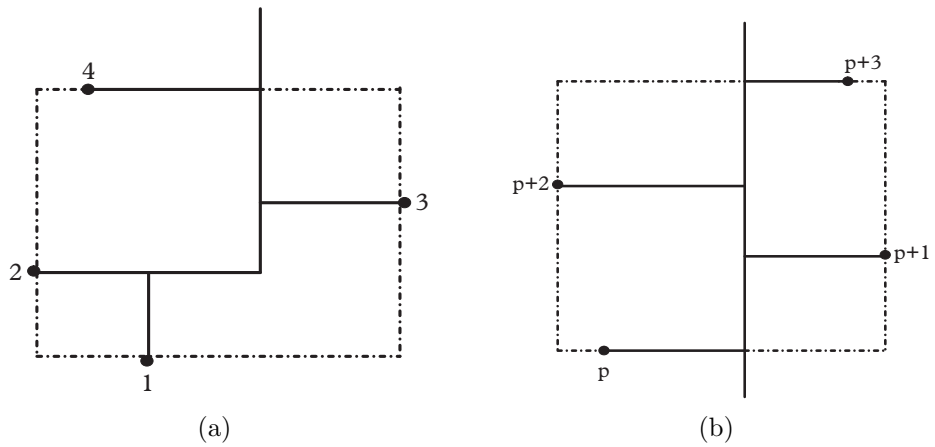


Figure 8

true for less than n points. We can repeat the proof for $n+1$ points under the same hypothesis, because we remove 4 points at a time in the proof. By noting that a longest simple SMT without a subbranch on n points can be seen as a longest simple SMT with a subbranch on $n+1$ points, where points 1 and 2 overlap, we see that the theorem is also true for a longest simple SMT on n points without a subbranch. To show that the theorem is true for $n \leq 4$ is straightforward, again using a path W. We omit the detail. QED

Chapter 4

The Euclidean Steiner ratio for special cases

In this chapter we consider the Euclidean Steiner ratio for two special cases. In the first case we examine the situation where all the terminals lie on an equilateral triangular lattice in such a way that all lines of a MST connect adjacent points of the lattice. Lemma 4.2 plays an important role in Du and Hwang's proof of the Euclidean Steiner ratio. The second case to be examined is a generalization of the first and requires for all lines of a MST to have the same length. If Du and Hwang's proof for the Euclidean Steiner ratio is incorrect, then Theorem 4.5 is an important result, because this upper bound is lower than the general upper bound obtained by Chung and Graham [3]. It might also be noted that the method developed in this section is general enough to be of use in studies of higher dimensions and non-Euclidean Minkowski spaces.

4.1 Vertices on an equilateral triangular lattice

The following lemma is due to Weng [30]. (Minimum hexagonal trees were introduced in Chapter 2.)

Lemma 4.1 *Given a set P of vertices in the plane together with the directions for hexagonal trees, we have*

$$\frac{\|MHT(P)\|}{\|SMT(P)\|} \leq 2/\sqrt{3}.$$

Proof: For a triangle ABC with a $120°$ angle at B, we note that $\|AB\| + \|BC\| \leq 2/\sqrt{3}\|AC\|$. Now each line of a SMT can be replaced by two lines along the given directions, thus forming a sufficiently short hexagonal tree for the lemma to hold. QED

For terminals on an equilateral triangular lattice we can now prove the following lemma, which plays an important part in Du and Hwang's proof of the general Euclidean Steiner ratio.

Lemma 4.2 *For n vertices on an equilateral triangular lattice such that all edges of a MST connect adjacent points of the lattice, we have the Steiner ratio $\rho \leq 2/\sqrt{3}$.*

Proof: We choose the directions for hexagonal trees parallel to the sides of a smallest triangle in the lattice. From Lemma 2.6 it follows that all lines of a MHT will connect adjacent points on the lattice. It follows that the length of a MHT is equal to that of a MST. We now use Lemma 4.1 to complete the proof. QED

4.2 When all lines of a MST have the same length

We extend the notion of vector summation to allow for the sum of a tree T and a centrally symmetric set A:

$$T + A = \{z : z = x + y, x \in T, y \in A\}$$

Figure 1 illustrates a tree and the sum of the tree and a ball.

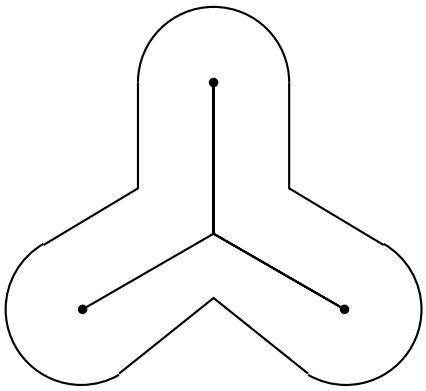


Figure 1

We begin by giving a simple illustration of how summation can be used to approximate the Steiner ratio. Consider n vertices in the plane with a

MST for which all edges are of length 2. Clearly the distance between any two vertices is at least 2. It follows that a ball of radius 1 can be constructed around each vertex such that the area of intersection of any two balls is 0.

For the area of the sum of a SMT S and a ball B of radius 1 we have

$$\text{Area}(S + B) \leq \pi + 2\|S\|.$$

From the fact that this area is larger than the area of n balls of radius 1, we also have

$$\text{Area}(S + B) \geq n\pi.$$

It follows that

$$\|S\| \geq (n - 1)\pi/2.$$

Since the MST has length $2(n - 1)$, we have for the Steiner ratio

$$\rho \leq 4/\pi.$$

We devote the rest of the section to finding a better approximation than the above. We do this by summing a SMT and a regular hexagon. Properties of the construction shown in Figure 2 are often used in what follows. It consists of two circles and two regular hexagons with parallel sides.

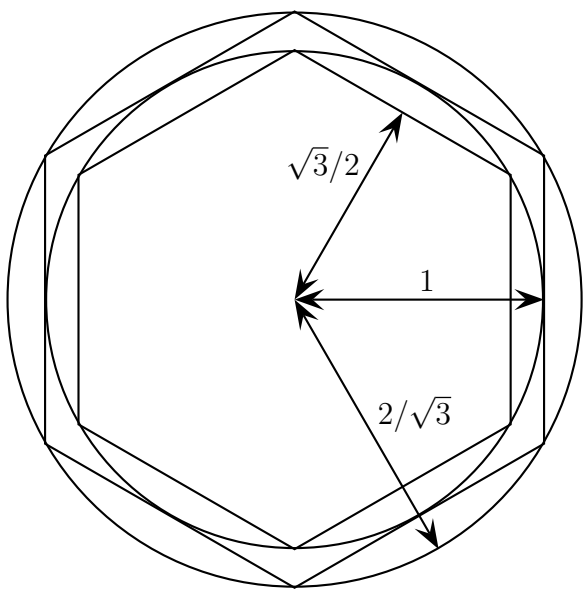


Figure 2

Lemma 4.3 *Consider a circle centered at the origin with radius* $2/\sqrt{3}$ *and chords, with non-intersecting interiors, of maximal length* $2/\sqrt{3}$*. Then the area enclosed by* $x = -\sqrt{3}/2$ *to the left, by* $x = \sqrt{3}/2$ *to the right, by the* x*-axis at the top and by chords and the arcs between the chords at the bottom, is at least* $2 - \sqrt{3}/4$*. (See Figure 3(a).)*

Proof: We can assume that the curve consists exclusively of chords, of length at most $2/\sqrt{3}$, which intersects at their endpoints, as shown in Figure 3(b). (The arc in Figure 3(a) has been replaced by a chord.) We can furthermore replace any chord shorter than $2/\sqrt{3}$ which intersects with one of the lines $x = \pm\sqrt{3}/2$, by a chord of length $2/\sqrt{3}$ without increasing the enclosed area (Figure 3(c)).

We make the following general observation regarding lengths of chords: Given chords ab and bc on a circle such that $\|ab\| \geq \|bc\|$, and then replacing b by b' so that we have $\|ab'\| > \|ab\|$ and $\|b'c\| < \|bc\|$, then we have $\text{Area}(\triangle abc) > \text{Area}(\triangle ab'c)$.

The above observation is used to replace any two neighboring chords both shorter than $2/\sqrt{3}$ by one chord, or two chords of which one has length $2/\sqrt{3}$ (Figure 3(d)). Clearly the enclosed area is smaller after this step. This process is repeated until there are no two chords, both with lengths less than $2/\sqrt{3}$, next to each other.

Now, if a chord is shorter than $2/\sqrt{3}$, then it has a neighboring chord of length $2/\sqrt{3}$ which crosses one of the lines $x = \pm\sqrt{3}/2$, because two chords of length $2/\sqrt{3}$ cannot fit between the vertical lines. We illustrate these chords by ab and bc in Figure 3(e) and Figure 3(f). We now construct $ab' = bc$ and $b'c = ab$ with e the intersection of ab and $b'c$. We note that e can lie to the left (Figure 3(e)) or right (Figure 3(f)) of the vertical line. In the first case it is immediately clear that the enclosed area is reduced by the replacement of b with b'. In the second case we note that the areas of $\triangle ab'e$ and $\triangle cbe$ are the same and also find that the enclosed area is reduced by the replacement of b with b'. If the situation is such that ab' crosses the vertical line, then we can again replace it with a line of length $2/\sqrt{3}$. We have now shown that it is possible to systematically eliminate or replace all chords which have length less than $2/\sqrt{3}$ without increasing the enclosed area. It remains only to show that the lemma is true for the case where all chords at the lower boundary have length $2/\sqrt{3}$.

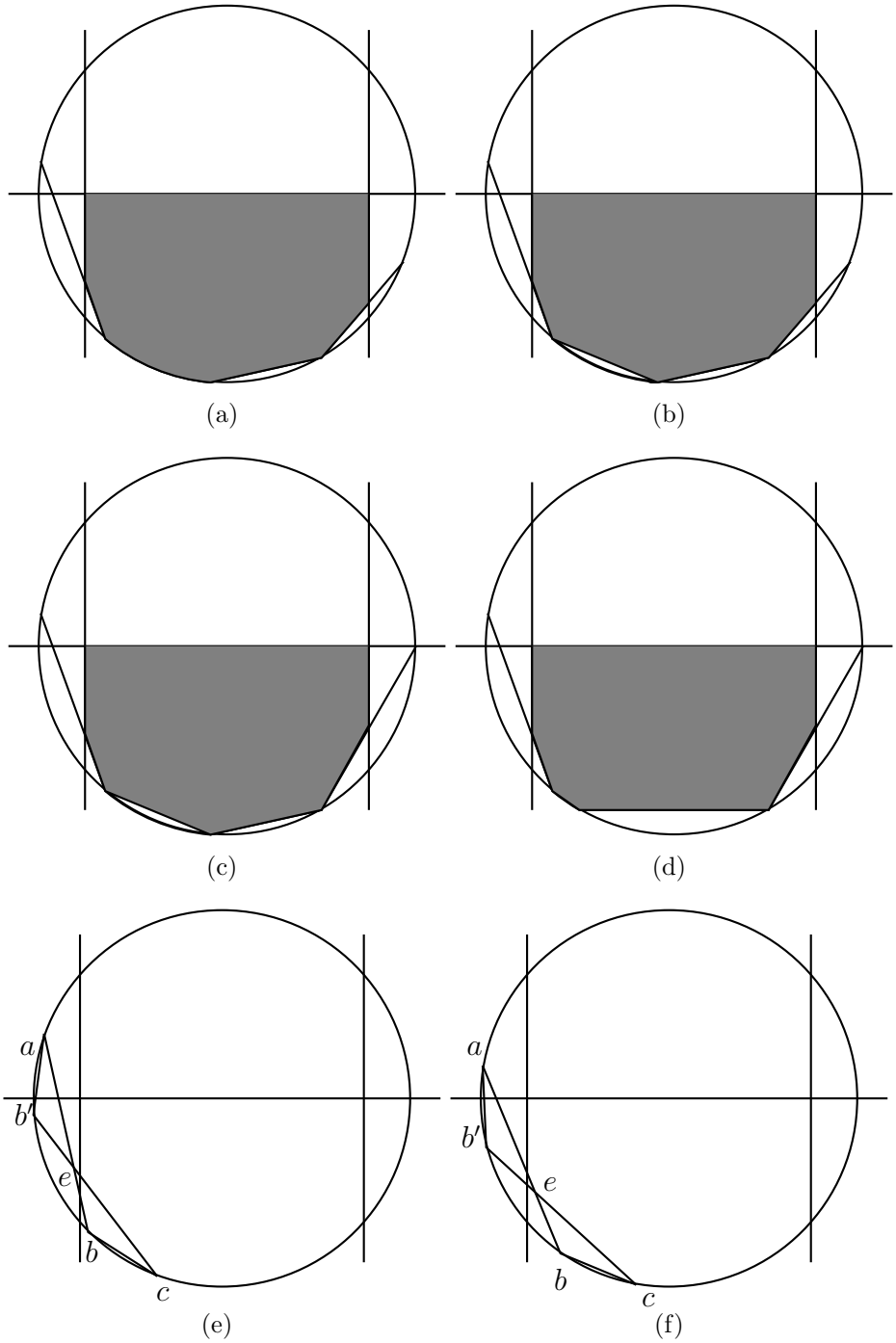
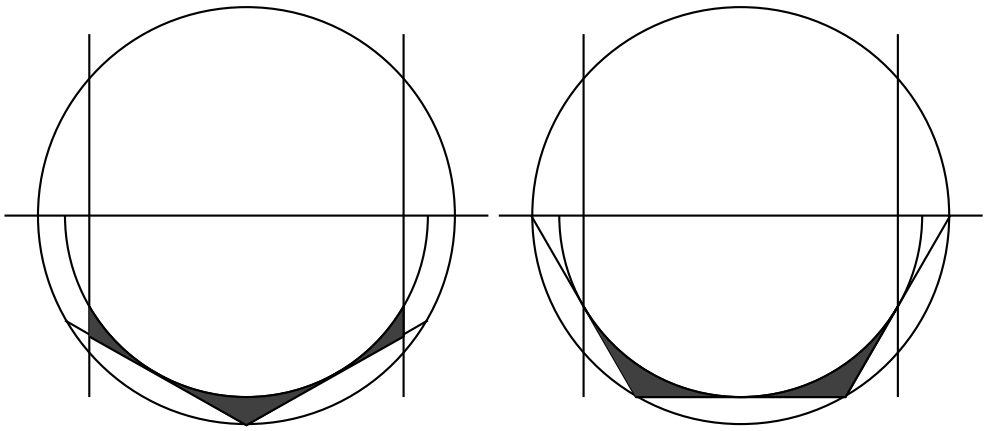


Figure 3

The best and worst cases for chords all of length $2/\sqrt{3}$ are seen to be as in Figure 4(a) and Figure 4(b) respectively. (The darkly shaded area should make this fact apparent.) Using Figure 5 we calculate the smallest possible enclosed area as

$$3 \cdot \frac{\sqrt{3}}{4} + (\sqrt{3}(\frac{2}{\sqrt{3}} - 1)),$$

where the first term is the combined area of the triangles and the second that of the parallelograms. We simplify and obtain $2 - \sqrt{3}/4$. QED



(a) (b)

Figure 4

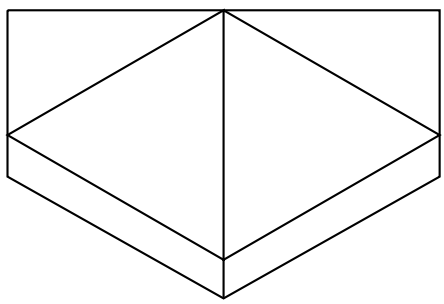


Figure 5

Lemma 4.4 *Consider a regular hexagon H contained in a convex set K. The hexagon is moved out of K along a path P which consists of a finite number of line segments which are all parallel to some side of H. Then the area of the intersection of $P + H$ and K is a minimum for some path which consists of a single segment.*

Proof: As H is moved out of K, it will at some stage happen that two opposite sides of H simultaneously intersect the boundary of K, say at p and q. From this stage onwards H will move parallel to these sides out of K, along the last segment of P. We use induction and show that the last two segments of P can be replaced by a single segment. Let the movement associated with the last two segments be from A to B and then from B to C as in Figure 6. We refer to this part of the path as ABC. We need to show that the area of the intersection of K and $ABC + H$ is larger than either the area of the vertically shaded region or the area of the diagonally shaded region. (These areas are both associated with movements along a single segment.) We assume that the boundary of the convex set is a straight line. (Replacing the convex boundary by a straight line will increase the area of $(ABC + H) \cap K$ more than those of X and Y.)

If θ is larger than $30°$ then the area of the diagonally shaded region is smaller than the area of $(ABC + H) \cap K$, otherwise the area of the vertically shaded region is smaller than or equal to the area of $(ABC + H) \cap K$. QED

We can now approximate the Steiner ratio.

Theorem 4.5 *Given n vertices in the plane, such that there is a MST for which all edges have the same length, then we have for the Euclidean Steiner ratio*

$$\rho \leq 4\sqrt{3}/(4 + \sqrt{3}) \approx 1.209.$$

Proof: We assume, without loss, that the edges of a MST are all of length 2. Around each terminal we construct a circle of radius $2/\sqrt{3}$ and where two circles intersect, we connect the two points of intersection with a line segment. Since all terminals are at least length 2 apart, we find that all segments have maximum length $2/\sqrt{3}$. We have now constructed a convex set around each terminal in such a way that the interiors of these convex sets do not intersect.

Now consider a full subtree T of a SMT S of the n vertices. Let T have $m \leq n$ terminals. We will see in Chapter 7 that the edges of a full subtree are all parallel to some side of a regular hexagon H. We let the sides of H

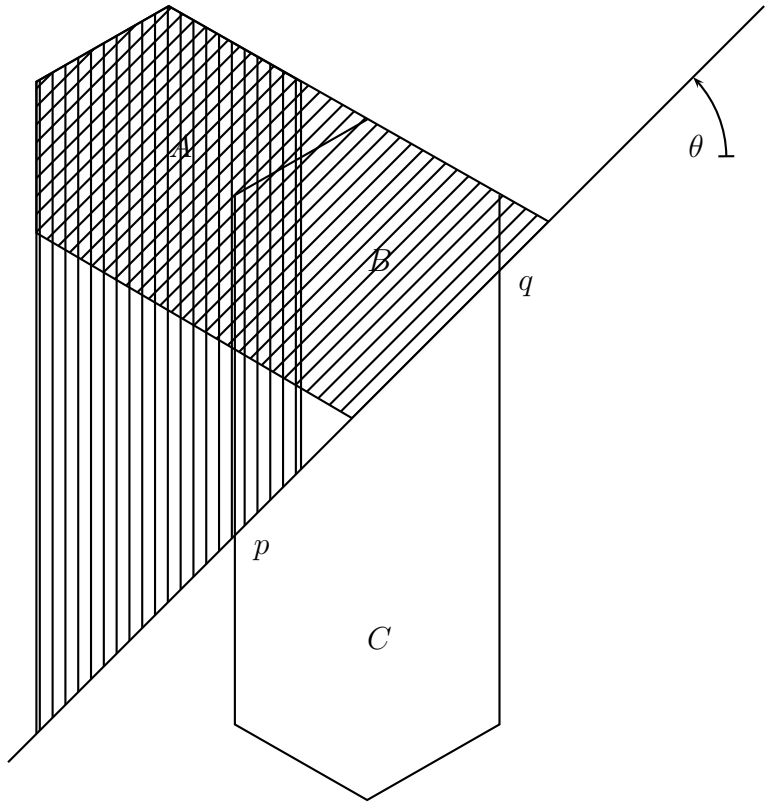


Figure 6

have length 1 and use it as the convex set in our summation method. For the area of $T + H$ we have

$$\text{Area}(T + H) \leq 3/2 \cdot \sqrt{3} + \|T\|\sqrt{3}.$$

Furthermore, the area of $T + H$ contained inside the convex set which surrounds every terminal is, according to Lemmas 4.3 and Lemma 4.4, at least

$$(2 - \sqrt{3}/4) + 3\sqrt{3}/4 = 2 + \sqrt{3}/2.$$

We now have

$$\text{Area}(T + H) \geq m(2 + \sqrt{3}/2).$$

It follows that

$$3/2 \cdot \sqrt{3} + \|T\|\sqrt{3} \geq m(2 + \sqrt{3}/2)$$

and thus that

$$\|T\| \geq m(2/\sqrt{3} + 1/2) - 3/2.$$

Since the length of the main Steiner tree is the sum of the lengths of the full trees, and since the number of terminals of the main tree is at least that of the sum of the terminals of the full trees, we also have

$$\|S\| \geq n(2/\sqrt{3} + 1/2) - 3/2.$$

Finally, from the fact that a MST has length $2(n-1)$, we see that

$$\rho \leq \sup_n \frac{2(n-1)}{n(2/\sqrt{3} + 1/2) - 3/2} = 4\sqrt{3}/(4 + \sqrt{3}) \approx 1.209.$$

QED

Chapter 5

Transformation of Steiner trees

5.1 Introduction

In this chapter we will only consider full Euclidean Steiner trees for which all edges are segments with positive length and for which all Steiner points are of degree 3 with the edges intersecting at 120°. We will say that we *transform* such a tree if we change the lengths of the edges, rotate the tree, and/or translate the tree. We want to prove the following theorem.

Theorem 5.1 *Consider a full Euclidean Steiner tree. Then for each terminal there exists a neighborhood such that the position of the terminal can be adjusted to any point in this neighborhood. Any subset of the terminals can be adjusted in this way by transforming the Steiner tree.*

One approach to proving the theorem might be by saying that the function from the $2n$ dimensional configurational space, defined by the lengths of the $2n-3$ edges plus rotation plus the x and y-coordinates of the first terminal ($2n$ variables in total), mapping to the $2n$ variables in $\mathbf{R}^{2n}$, consisting of the x and y-coordinates of all the terminals, has to have a local inverse. Such a function is known as a local diffeomorphism and is characterized by the well known inverse function theorem.

Theorem 5.2 (The inverse function theorem) *Let $f : U \to \boldsymbol{R}^n$ be a differentiable function and let $x \in U$ where $U \subset \boldsymbol{R}^n$ is open and let $f(x) = y$. Then there exists a neighborhood of x where f has an inverse f^{-1} which is defined on a neighborhood of y if and only if the Jacobian matrix $Df(x)$ is invertible (non-singular).*

The Jacobian matrices turn out to be quite complex with no obvious way for simplifying the determinants. We prove the theorem by using two alternative approaches: The first is more constructive and geometric, while the second uses a strong theorem from topology.

5.2 Geometric approach

We call a ray a *variable ray* if it has the following property: the angle of the ray θ to the positive x-axis is a variable ($\alpha \leq \theta \leq \beta$, α and β fixed), and the position of the vertex of the ray is a continuous function of θ. We denote the ray by $r(\theta)$. Furthermore, if for any distinct θ_1 and θ_2 (both angles in the domain of r) we have that $r(\theta_1)$ and $r(\theta_2)$ do not intersect, except possibly the two vertices, then we call the ray a *simple variable ray*. (See Figure 1.)

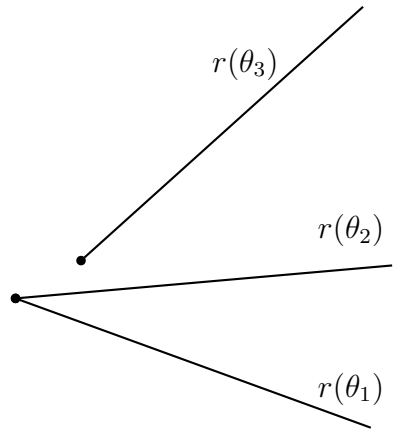


Figure 1

The following property of a simple variable ray follows directly from the definition:

Lemma 5.3 *If p is a point in the plane such that it lies on the interior of ray $r(\theta)$ for some θ_1, then p has a neighborhood such that for any x in this neighborhood there exists a θ_2 for which x lies on the interior of $r(\theta_2)$.*

Proof: For $r(\theta_1)$ construct line l between the vertex and p, perpendicular to $r(\theta_1)$ as in Figure 2. Now choose $\theta_3 > \theta_1$ such that the vertex of $r(\theta_1)$ remains on the same side of l. We now note that as θ increases from θ_1 to θ_3, so the point of intersection of the ray with l continuously moves further from the intersection of $r(\theta_1)$ and l. (This has to be the case for the variable ray to be simple.) It follows that, as θ increases from θ_1 to θ_3, the ray goes through all points in the shaded area for some θ. Lastly we note that the same argument can be used for the case with $\theta_3 < \theta_1$, proving the existence of the required neighborhood. QED

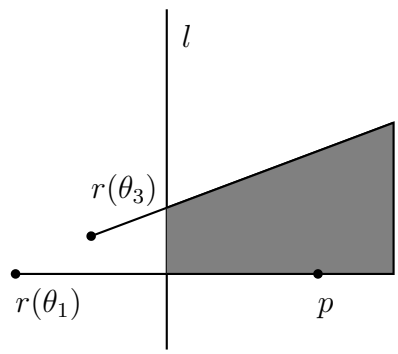


Figure 2

Lemma 5.4 *Consider two simple variable rays which intersect on their interiors at an angle of* $120°$*. We denote them by* $r_1(\theta + 60°)$ *and* $r_2(\theta - 60°)$*, as shown in Figure 3. Let* $r_3(\theta)$ *be the variable ray which has the point of intersection as vertex and* θ *as angle. Then* $r_3(\theta)$ *is also a simple variable ray.*

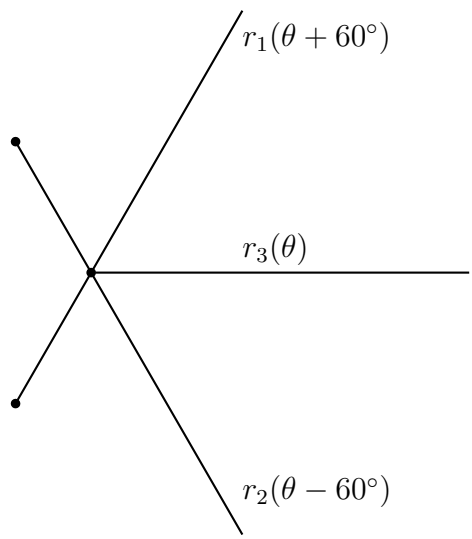


Figure 3

Proof: We assume that the lemma is false and that θ_1 and θ_2 exist such that $r_3(\theta_1)$ and $r_3(\theta_2)$ intersect at their interiors. To simplify the illustrations we omit the initial parts of rays r_1 and r_2 (from vertex to point of intersection). Without loss of generality we are now left with two cases as in Figure 4(a) and Figure 4(b). In the first case $r_1(\theta + 60°)$ is not a simple variable ray and in the second case $r_2(\theta - 60°)$ is not, providing the required contradictions. QED

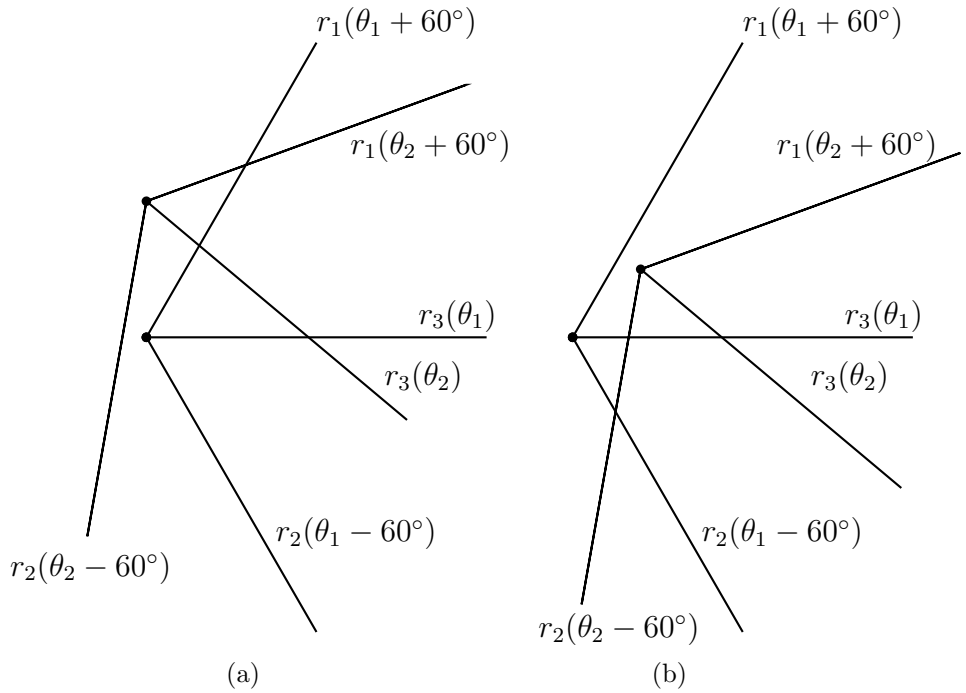


Figure 4

From the proof of the lemma it should be clear that the domain of r_3 is smaller than or equal to that of both r_1 and r_2. In fact, if r_3 is at an edge of its domain, then we have that either r_1 or r_2 is at the edge of its domain or that the vertex of r_3 coincides with that of either r_1 or r_2.

Lemma 5.5 *Consider a full Euclidean Steiner tree and let all terminals, except one, be fixed. Then there exists a neighborhood within which the position of this one vertex can be adjusted to any point by transforming the Steiner tree.*

Proof: We begin by converting all edges of the tree into rays. First extend the line through the adjustable point p as shown in Figure 5.

The Steiner point adjacent to p is the vertex of this generation 0 of rays. The previous generation of rays, generation -1, is created by extending the other two edges through this Steiner point as shown. In this way we work progressively backwards, changing edges into rays. When a ray has a terminal as vertex, it has no parents and the process stops, so that we eventually reach a point where all edges have been converted into rays.

We now view all rays with terminals for vertices as simple variable rays, parameterized by the same variable θ. From Lemma 5.4 it follows that a

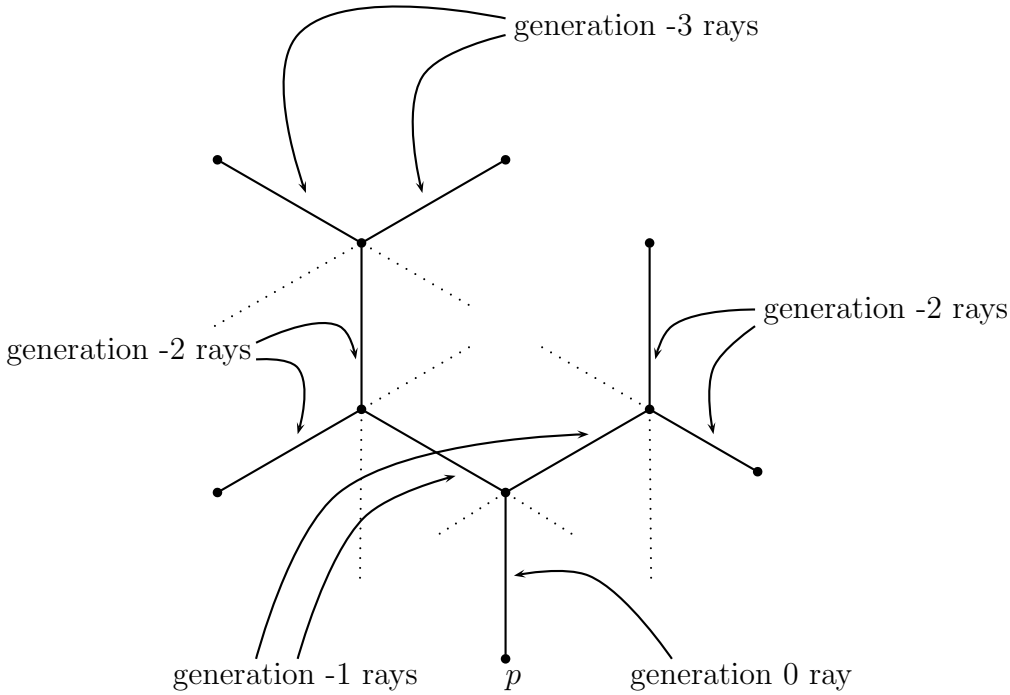


Figure 5

ray is a simple variable ray if its parents are simple variable rays. Since all rays have simple variable rays as ancestors, we have that all rays are simple variable rays. It follows that p also lies on a simple variable ray, and the theorem follows from Lemma 5.3. QED

We can also prove a stronger version of the above.

Lemma 5.6 *For any given a, b and λ such that $0 < b < a$ and $0 < \lambda < (a-b)/2$, together with an integer $n \geq 2$, there exists an $\epsilon > 0$ such that for any given full Euclidean Steiner tree with n terminals and all edge lengths between $a-\lambda$ and $b+\lambda$, we have the following: If all terminals except one are kept fixed, then this one terminal can be adjusted to any position within its neighborhood of radius ϵ by transforming the Steiner tree without increasing or decreasing any of the edges of the Steiner tree by more than λ.*

Proof: We see from the previous proof that movement of the adjustable terminal is associated with continuous changes in the lengths of the edges. It is sufficient to show that $\Delta l/\Delta p$ is smaller than some constant k, where

Δl is the change in length of any edge and Δp is the size of the change in position of the adjustable point p. We use three steps.

1. **We show that $\Delta s/\Delta\theta$ is smaller than some constant k_1, where Δs is the size of the change in position of any Steiner point s and $\Delta\theta$ is the change in the angles of the edges:**

   We replace the tree by a system of simple variable rays, as in the previous proof. If we look at the oldest generation of points, we get the situation of Figure 6(a). Since all edges are shorter than a, we have that $\Delta s/\Delta\theta$ for the Steiner point cannot be larger than some constant which we call I.

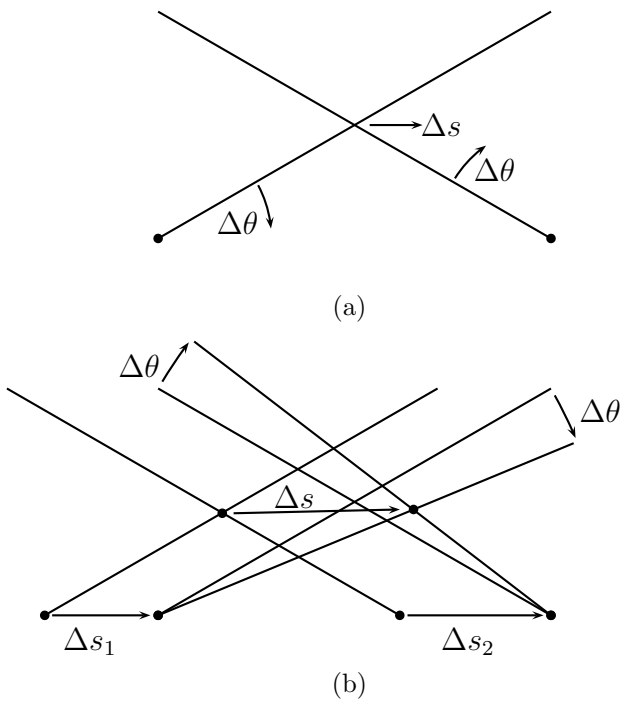


Figure 6

   Next we look at two simple variable rays which have the Steiner points s_1 and s_2 as their terminals and s as their offspring. Let us assume that we have $\Delta s_1/\Delta\theta < nI$ and $\Delta s_2/\Delta\theta < mI$, where n and m are positive integers with $n \leq m$, as shown in Figure 6(b). The movement

of s can be seen to be the sum of the effects of the movement of s_1 and s_2 as well as the effect of the change of θ. It follows that

$$\Delta s/\Delta\theta < 2/\sqrt{3}nI + 2/\sqrt{3}mI + I.$$

By induction and from the fact that n is finite, it follows that $\Delta s/\Delta\theta$ is smaller than some constant k_1 for any Steiner point s.

2. **We show that $\Delta\theta/\Delta p$ is smaller than some constant k_2, where Δp is the size of the change in position of the adjustable terminal.**

   In Lemma 5.3 we saw that a point p on a simple variable ray can be adjusted in any direction with an associated change in the angle θ of the ray. We see that $\Delta\theta/\Delta p$ is smaller than $1/b$ (where we use radials to measure $\Delta\theta$), because all edges are longer than b. If this is not the case, then the variable ray is not simple, as demonstrated in Figure 7.

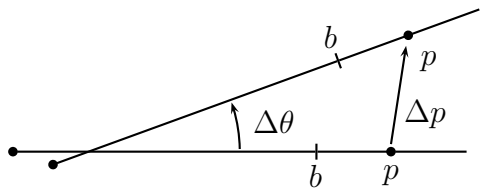


Figure 7

3. **We show that $\Delta l/\Delta p$ is smaller than some constant k, where Δl is the change in length of any edge.**

   From steps 1 and 2 we have $\Delta s/\Delta p < k_1k_2$ for any Steiner point s. It follows that $\Delta l/\Delta p < k_1k_2 + 1$ if l is the length of the edge at p, and that $\Delta l/\Delta p < 2k_1k_2$ for any other edge, and thus $\Delta l/\Delta p$ is smaller than some constant k.

   QED

Proof of Theorem 5.1: Choose b to be half the length of the shortest edge of the Steiner tree, a to be at least b larger than the length of the longest edge of the Steiner tree, and λ to be b/n, where n is the number of terminals. We can now apply the previous lemma consecutively for every terminal. QED

5.3 Topological approach

Proof of Theorem 5.1: For a given full Euclidean Steiner tree we can associate with it a point $z = (z_1, \ldots, z_{2n}) \in \mathbf{R}^{2n}$ if we use z_1 and z_2 to represent the components of the position of the first terminal, z_3 to represent the rotation of the edge at this point, and $z_4, \ldots, z_{2n}$ to represent the lengths of the $2n-3$ edges. We let A be a closed ball with z as its center which is small enough for all points in A to represent full Euclidean Steiner trees in a similar way with no two points in A having third components differing by 2π or more. Let $f : A \to \mathbf{R}^{2n}$ be the function which associates with each point in A a point in $\mathbf{R}^{2n}$ which has the x and y-coordinates of the n terminals as its components. Theorem 5.1 will be proved if $f(z)$ has a neighborhood which is contained in $f(A)$.

The function $f : A \to f(A)$ is known to be injective, and the well known Melzak algorithm [16] can be used to calculate the positions of the Steiner points, given the positions of the terminals. Since $f : A \to f(A)$ is also continuous and surjective, we can use the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism [27], to determine that $f : A \to f(A)$ is in fact a homeomorphism.

We can now use a theorem of Brouwer (see invariance of domain [24], p. 129) which claims the following: Let U and V be subsets of $\mathbf{R}^m$ having a homeomorphism $h : U \to V$. If U is open, then V is open.

We let U be an open ball with z as its center which is contained in A. Now V is a neighborhood of $f(z)$ which is contained in $f(A)$. QED

Chapter 6

Du and Hwang's proof for the Euclidean Steiner ratio

In this chapter we take a critical look at Du and Hwang's proof for the Euclidean Steiner ratio. We give background in Section 1, make some general comments in Section 2 (where we compare the different versions of the proof) and demonstrate the method of proof with 4 points in Section 3. Section 4 is the heart of the chapter. We mention there the fact that inner spanning trees are not properly defined (not an insurmountable problem, as we will see in the next chapter) and, more importantly, a serious problem concerning the decomposition of Steiner trees into subtrees. Here we have to add that Yue [31] also criticized Du and Hwang's proof, but that this critique relies on a strange interpretation of what constitutes an edge of an inner spanning tree and that (as we show in the next chapter) it cannot be seen as a fundamental problem. Yue proposes an alternative proof, said to be based on the methods used by Chung and Hwang [4] and Chung and Graham [3]. An analysis of [31] falls outside the scope of this thesis, but the author would like to say that as far as he is concerned, the question of whether the Gilbert-Pollak conjecture is true or not is still very much an open one.

6.1 Background

The Steiner ratio for the Euclidean plane was conjectured to be $2/\sqrt{3}$ by Gilbert and Pollak [12] in 1968. This was shown to be true for 4 points by Pollak [20], for 5 points by Du, Hwang and Yao [10], and in 1991 for 6 points by Rubinstein and Thomas [25].

Du and Hwang's proof concerns the Gilbert-Pollak conjecture for the general case of n points. In 1990 they published a paper entitled *The Steiner*

ratio conjecture of Gilbert and Pollak is true [7], which they described as an abridged proof for the conjecture. This was followed by a paper, *A Proof of the Gilbert-Pollak conjecture on the Steiner ratio* [8], in 1992 in Algorithmica and a chapter, *The state of art on Steiner ratio problems* [9], in the book Computing in Euclidean geometry, also in 1992. Discussions on their proof and considerations on generalizing the method also appeared in [16] and [11]. The most detailed proofs appear in [8] and [9] and we will focus on them.

The fact that $2/\sqrt{3}$ is a lower bound for the Steiner ratio follows from Figure 1, which shows 3 vertices which are equal distances from one another, a SMT with edges meeting at 120°, as well as three dotted lines, any two of which form a MST.

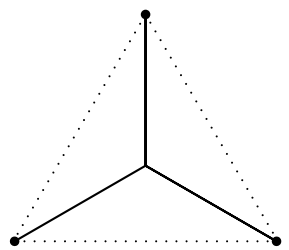


Figure 1

Clearly every MST and every SMT have straight line segments as edges. For any SMT it is also true that no two edges intersect at an angle of less than 120°. To see why, we consider two lines α and β meeting at less than 120°, and construct oa, ob and oc as shown in Figure 2, such that angles at o are 120° and such that $\|ab\| = \|ac\|$. It is sufficient to show that $\|oa\| + \|ob\| + \|oc\|$ is shorter than $\|ab\| + \|ac\|$. We note that

$$\|oa\| + \|ob\| + \|oc\| = \|oa\| + 4\|od\| = \sqrt{\|oa\|^2 + 8\|oa\|\|od\| + 16\|od\|^2},$$

but from

$$\|ab\| = \|ac\| = \sqrt{(\|oa\| + \|od\|)^2 + \|bd\|^2} = \sqrt{(\|oa\| + \|od\|)^2 + 3\|od\|^2}$$

we have that

$$\begin{aligned} \|ab\| + \|ac\| &= 2\sqrt{\|oa\|^2 + 2\|oa\|\|od\| + 4\|od\|^2} \\ &= \sqrt{4\|oa\|^2 + 8\|oa\|\|od\| + 16\|od\|^2}. \end{aligned}$$

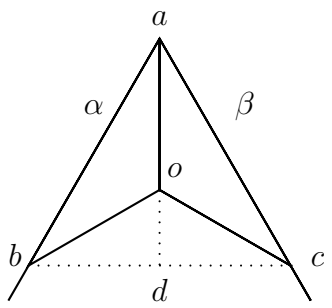


Figure 2

It follows that for a SMT Steiner points will always have degree 3 with edges meeting at 120°. Furthermore angles between edges at terminals are at least 120°. Any tree satisfying these criteria is referred to as a *Steiner tree* (ST) by Du and Hwang, and we do the same in the rest of the chapter. We also note that an edge between two Steiner points cannot have length 0.

6.2 Some general comments

Du and Hwang's approach contains some interesting ideas, in particular the use of their minimax theorem and the introduction of inner spanning trees. A precise analysis is however made difficult by a general lack of rigour. In what follows we will try to present the ideas behind the proof in a simple and straightforward way. This will enable us to identify problems and to determine to what an extent the proof can be remedied. We try to keep our notation and expressions as simple as possible and do not always use Du and Hwang's original notation. (Different notation is used in [8], [9] and [16].) In [8] and [9] the minimax theorem is stated in terms of concave functions and convex functions are then algebraically manipulated in order to apply it. In [16] convex functions are used throughout and a maximin theorem is used instead of the minimax theorem. This is simpler and we do the same.

We describe the general outline of the proof in this section by breaking it down into steps. We then take a detailed look at certain aspects of the proof in subsequent sections.

The following steps are used repeatedly (albeit in different orders) in all versions of the proof:

1. A maximin (or minimax) theorem is introduced, which, for a class of convex functions g_i, characterizes points where the function $\min_i g_i$ attains its maximum.

2. A class of spanning trees is introduced, the lengths of which are shown to be convex functions h_i on a domain where lengths of Steiner trees are 1.

3. The Gilbert-Pollak conjecture (or some stronger statement which implies it) is assumed to be false.

4. From this assumption it is shown to follow that $\min_i h_i$ reaches its maxima only on the interior of its domain.

5. The maximin theorem is applied to show that all terminals lie on a triangular lattice.

6. It is shown that for points on a triangular lattice the Gilbert-Pollak conjecture is true – a contradiction.

In [9] the steps above are first used to prove the Gilbert-Pollak conjecture for 4 points in order to demonstrate the method. After this the steps are repeated for the general case. The class of spanning trees considered here is called inner spanning trees. Unfortunately inner spanning trees are not precisely defined in [9], although what is meant seems clear for simple cases. A more complex minimax theorem is introduced and the rest of the steps also become more complex, but much detail is omitted as it is simply mentioned to be similar to the case with 4 points. Lastly, under the heading of "Final touches for a rigorous proof" in [9] it is noted that inner spanning trees were indeed not defined rigorously for all situations. It is then proposed how this can be remedied by considering a smaller domain, and how each of the previous steps might be altered accordingly. Little detail of these final steps is given.

In [8], which is an older version of the proof, less detail is given than in [9]. It does not start with a demonstration for 4 points, and immediately introduces inner spanning trees. Again the uncertainty regarding inner spanning trees make the steps difficult to follow. The proof also ends with an attempted rework of the steps for situations where inner spanning trees are not clearly defined.

We end this section with some notation and lemmas which we will need in the next two sections.

Any full ST with n terminals can be described by its *topology* s and a vector $x = (x_1, \ldots, x_{2n-3}) \in \mathbf{R}^{2n-3}$ which specifies the lengths of the edges. ([8] differs here from [9] in that fullness is not assumed from the beginning. It uses variables for the angles at terminals, but shows later that only full topologies need to be considered.) For 6 points there are 3 full topologies, as illustrated in Figure 3. In each case the vector $x \in \mathbf{R}^9$ represents the lengths of the 9 edges. Note that to simply say, as Du and Hwang do, that the topology of the ST is the graph structure of the network, is not quite correct, for the first and last examples differ, but have the same graph structure. It would be more accurate to say that topology is an equivalence relation, where two Steiner trees have the same topology if they are congruent when all edges of both trees are altered to have length 1.

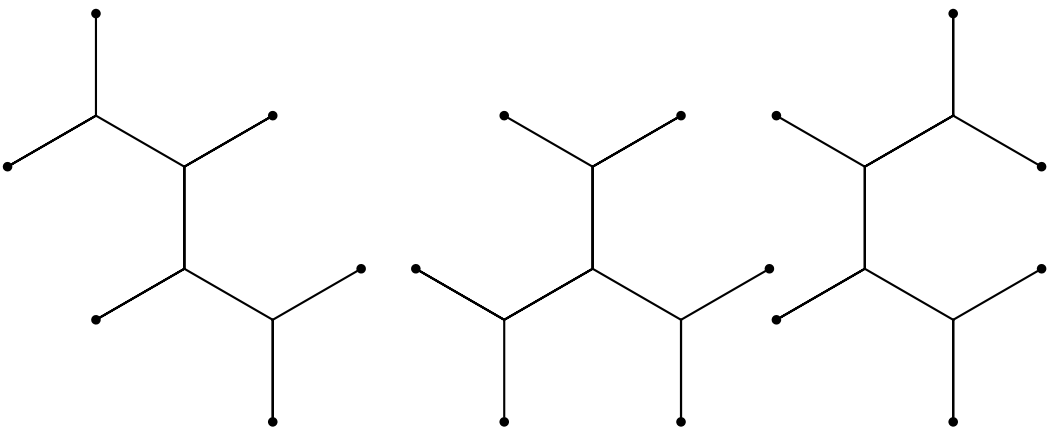


Figure 3

We can now use (s, x) to denote a Steiner tree with topology s and edge lengths x, and MST(s, x) to denote a MST on the terminals of (s, x). We will assume that (s, x) is normalized and thus that we have $x_1 + \ldots + x_{2n-3} = 1$. This will not influence our results, because the Steiner ratio does not depend on scale. Now let I be the set of spanning trees on the variable n points associated with x and denote the spanning tree $i \in I$ associated with x by $T_i(x)$ and its length by $g_i(x)$. We now have the following important lemma:

Lemma 6.1 *For each $i \in I$, $g_i(x)$ is a convex function of x.*

Proof: We only need to show that the distance $\|AB\|$ between any 2 terminal points A and B is a convex function, because each $g_i(x)$ is the sum

of such distances. Find the path on the Steiner tree that joins A and B and suppose it consists of k edges with lengths $x_1, \ldots, x_k$ and directions (unit vectors) $e_1, \ldots, e_k$. Now we have

$$\|AB\| = \|x_1 e_1 + \ldots + x_k e_k\|$$

where $\|\cdot\|$ is the Euclidean norm. We note that a norm is a convex function and that the part inside the norm is linear with respect to x. QED

The following lemma is due to Rubinstein and Thomas [25].

Lemma 6.2 *Edges of different MSTs of the same set of terminals intersect only at terminal points.*

Proof: We begin by showing that a MST cannot cross itself. Assume that edges AD and BC of the same MST do cross (Figure 4). Remove AD and BC. We can assume without loss that A and B are in the same component. (Removing two edges from a tree leaves three components.) We now add edges AC and BD to obtain a shorter tree (by the triangle inequality), contradicting the fact that we started with a MST. It follows that a MST cannot cross itself.

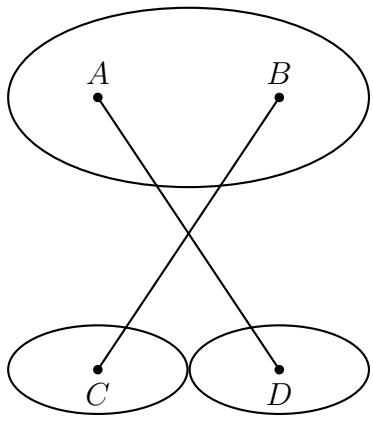


Figure 4

Let us now assume that two MSTs (tree 1 and tree 2) cross. Let an intersecting edge of tree 1 be incident to points A and B. Removal of this edge partitions tree 1 into two connected components α (containing A) and β (containing B). Since there is also a path in tree 2 which connects points A and B, there has to be an edge of this path which connects a point in α with a point in β. Let CD be such an edge, as in Figure 5(a). We notice that AB and CD must have the same length for trees 1 and 2 both to be

MSTs. (If the length of AB is larger than that of CD then tree 1 can be made shorter by replacing AB with CD; if the length of CD is larger than that of AB then tree 2 can be made shorter by replacing CD with AB.) We now alter tree 2 by removing CD and adding AB to obtain tree $2'$. But since AB crosses some edge of tree 2 this results in tree $2'$ crossing itself (which we have shown not to be possible), unless CD is the edge that crosses AB. The situation is thus that of Figure 5(b). Since AB and CD have the same length, we have from the triangle inequality that either AD or CB will be shorter than AB. This edge can be used to replace AB in tree 1, forming a shorter tree and providing the required contradiction. QED

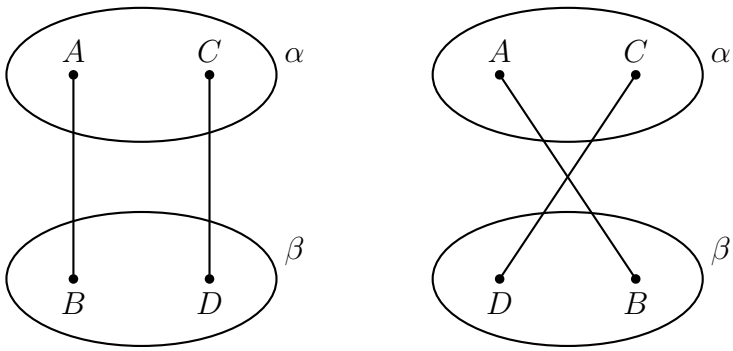


Figure 5

6.3 Four points

In [9] and [11] the method of proof is first demonstrated by proving the Gilbert-Pollak conjecture for 4 points. We will do the same here because this will help us to understand the problems arising in the general case.

Step 1: For a real valued function f defined by $f(x) = \min_i g_i(x)$, where the g_i are a finite number of functions, we use $I(x)$ to denote the set of those i for which $g_i(x) = f(x)$.

Lemma 6.3 *Let $f(x) = \min_i g_i(x)$, where the g_i are a finite number of continuous functions defined on an open set $A \subset \boldsymbol{R}^k$. Then for every $x^* \in A$ there is a neighborhood of x^* such that for any y in this neighborhood $I(y) \subseteq I(x^*)$.*

Proof: From continuity we have that for every g_i such that $g_i(x^*) > f(x^*)$ there is a neighborhood of x^* which has $g_i(x) > f(x)$ for every x in this neighborhood. The intersection of all such neighborhoods provides the neighborhood which the theorem requires. QED

Theorem 6.4 (Maximin) *Let $f(x) = \min_i g_i(x)$, where the g_i are a finite number of continuous convex functions defined on a k-dimensional polytope $X \subset \boldsymbol{R}^k$. Suppose that every maximum point of f is an interior point of X. Then there exists a maximum point x^* such that there does not exist a point y, different from x^*, with $I(y) \supseteq I(x^*)$.*

Proof: Fix any point p of X and let x^* be a maximum point such that the distance from p is a maximum. (Such a point exists because X is closed and bounded.)

Assume there exists a $y \neq x^*$ in X such that $\|y - p\| \leq \|x^* - p\|$ and for which $I(y) \supseteq I(x^*)$. Construct a straight line from y through x^* to any point z such that $I(z) \subseteq I(x^*)$. (This is possible by the previous lemma.)

For $i \in I(x^*)$ we now have $g_i(y) = f(y) \leq f(x^*)$ and thus $g_i(z) \geq f(x^*)$ and $f(z) \geq f(x^*)$, since all g_i are convex. This is a contradiction, because z is further from p than x^* and f cannot have a maximum here.

Now assume that there exists a y in X such that $\|y - p\| > \|x^* - p\|$ with $I(y) \supseteq I(x^*)$. Construct a straight line from y through x^* to any point z such that $I(z) \subseteq I(x^*)$. Since $f(y) < f(x^*)$ we have $f(z) > f(x^*)$; a contradiction. QED

Step 2: We take the class of spanning trees we consider to be all spanning trees. We assume that the Gilbert-Pollak conjecture is true for less than 4 points. Thus, as we saw in Lemma 1.1, we only need to consider full Steiner trees. There is only one full topology on 4 points, shown in Figure 6.

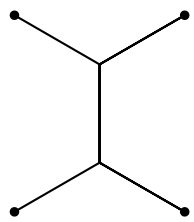


Figure 6

We denote a tree with this topology by $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5$ and assume that $|x| = 1$, i.e. $x_1 + \ldots + x_5 = 1$. This is a simplex on which the length of any spanning tree can be described as a convex function $g_i(x)$, as we saw in the previous section.

Step 3: The Gilbert-Pollak conjecture for 4 points can now be stated as follows:

$$f(x) = \min_i g_i(x) \leq 2/\sqrt{3}$$

for all $|x| = 1$. Let us assume that this is false.

Step 4: It now follows that:

Lemma 6.5 *The function $f(x)$ with $|x| = 1$ assumes its maximum values only when none of $x_1, \ldots, x_5$ are zero.*

Proof: If the edge at a terminal has length 0 then, as we saw in Chapter 1, we only need to consider smaller full Steiner trees. But we have assumed that the conjecture is true for less than 4 terminals, so the edge at a terminal cannot have 0 length.

If the edge between the 2 Steiner points has length 0, then there exists a shorter Steiner tree on the terminal points. (See end of Section 6.1.) This shorter Steiner tree has the same MSTs on its terminals. When this shorter Steiner tree is normalized, the MSTs become longer and thus f is greater at some other x. QED

Step 5: We can now apply the maximin theorem. According to it there is a Steiner tree x^* such that the terminals of no $y \neq x^*$ has the same set of MSTs, thus the set of MSTs is maximal at x^*. The topological graph consisting of the 4 terminals, together with the set of all edges which forms part of some MST of the 4 terminals, is now denoted by Γ. Du and Hwang make the following claim: Γ must consist of two equilateral triangles for the set of MSTs to be maximal, as shown by the dotted lines in Figure 7. From Lemma 6.2 we know that edges of Γ cannot cross. The rest of Du and Hwang's argument (Lemma 4.4 in [8] and Lemma 10 in [9]) seems incomplete, with at least Theorem 5.1 being taken for granted. In what follows we arrive at the same conclusion using rigidity and Kruskal's algorithm. (See [13] for more on rigidity and [18] for more on Kruskal's algorithm.)

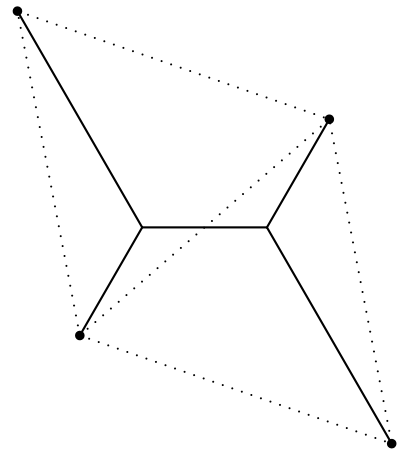


Figure 7

A topological graph for which all edges are straight line segments, is called *rigid* if it is true that if we move all vertices by an arbitrary distance in such a way that edge lengths remain the same, then the result is congruent to the original topological graph.

Lemma 6.6 *The Gilbert-Pollak conjecture is true for* 4 *points if it is true for any* 4 *points for which* Γ *is rigid.*

Proof: Assume that Γ is not rigid for x^*. Figure 8 illustrates an example. We can alter it slightly to obtain a new topological graph with the same edge lengths. This perturbation is made by changing the edge lengths of the underlying Steiner tree to y (Theorem 5.1) and is made small enough for Lemma 6.3 to apply. It follows that the length of a MST for y is the same as one for x^* and thus that any MST for x^* is also a MST for y. By normalizing y we obtain a point in our simplex with the same MSTs as x^*, contradicting the maximin theorem. It follows that the Γ framework is indeed rigid for x^*. QED

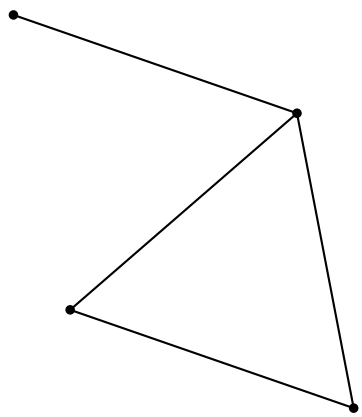


Figure 8

Lemma 6.7 *For the terminals of* x^* *we have that* Γ *consists of two equilateral triangles.*

Proof: We have seen that Γ has no crossing edges and is rigid. Thus it has to look as in Fig 7.

We want to show that all edges of Γ have the same length. Let us assume that this is not the case and let A be the set consisting of the shortest edge or edges. We now increase the lengths of the edges in A slightly by the same $\epsilon > 0$. (This is possible because the lengths of the edges can be changed

independently.) This perturbation is made by changing the edge lengths of the underlying Steiner tree to y (Theorem 5.1) and is made small enough for Lemma 6.3 to apply. It follows that an edge which was not in Γ cannot be part of a MST of y.

We now consider the 5 edges in Γ and order the edges according to increasing length: $e_1 \leq \ldots \leq e_5$. The order of these edges remains the same for the terminals of y if we make ϵ small enough.

The following version of Kruskal's algorithm constructs all MSTs on the terminals of x^*:

```
Let T={}
While T is not a spanning tree do:
Add to T a shortest edge e_i from Γ such that T contains no circuits
Stop.
```

The decision of which e_i to add when there are edges of equal length, is not stipulated by the algorithm. By making these choices in different ways the above algorithm can be used to create all possible MSTs. Clearly a MST for the terminals of x^* is also a MST for the terminals of y. By normalizing y we obtain a point in our simplex with the same set of MSTs as for x^*, contradicting the maximin theorem. It follows that all edges of Γ must have the same length. QED

Step 6: It remains to be shown that the Gilbert-Pollak conjecture is true for vertices on a triangular lattice. We have shown this to be true in previous chapters – providing a contradiction and thus showing that the Gilbert-Pollak conjecture is true for 4 points.

6.4 The general case of n points

In this section we demonstrate what we believe to be the main gap in the proof of Du and Hwang.

Step 1: Another version of the minimax theorem is needed for the general case, because the convex functions concerned are not everywhere defined, as we will shortly see. We will not concern ourselves with the detail here.

At this stage it is tempting to continue in the same manner as in the previous section: We assume that the Gilbert-Pollak conjecture is false and we fix n to be the smallest number of terminals for which a counterexample can be constructed. Furthermore we fix s to be the topology of a best counterexample on these n terminals so that $\max_{|x|=1} \|\mathrm{MST}(s,x)\| \geq \max_{|x|=1} \|\mathrm{MST}(t,x)\|$ for any topology t on n terminals. We can now prove the following lemma:

Lemma 6.8 *The function* $f(x) = \|MST(s,x)\|$ *with* $|x| = 1$ *assumes its maximum values only when none of* $x_1, \ldots, x_{2n-3}$ *are zero.*

Proof: If the edge at a terminal point has length 0, then we only need to consider smaller Steiner trees (Chapter 1), but by our assumption the Gilbert-Pollak conjecture is true for smaller Steiner trees, so this is not possible.

If the edge between two Steiner points has length 0, then there exists a shorter Steiner tree on the terminal points. This shorter Steiner tree has the same MSTs on its terminals. When this shorter Steiner tree is normalized we get a better counterexample, but we assumed that s provided the best counterexample – a contradiction. QED

The following lemma can now be derived in the same manner as before:

Lemma 6.9 *The Gilbert-Pollak conjecture is true if it is true for any set of* n *vertices for which* Γ *is rigid.*

Figure 9 shows a rigid topological graph for which the vertices are not on the same triangular lattice. This topological graph is in fact "over rigid" and edge lengths cannot be altered independently. The result is that the remainder of the method of proof for 4 points cannot be used. For this reason Du and Hwang introduce the idea of inner spanning trees.

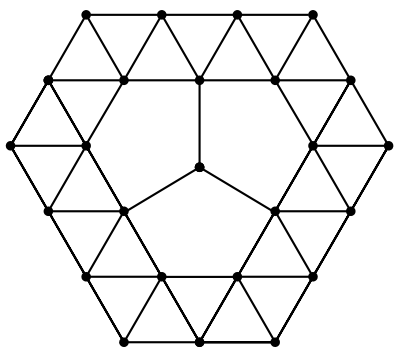


Figure 9

Step 2: The descriptions of *inner spanning trees* are slightly different in [8] and [9], but the basic idea is essentially the same. (In [8] it is furthermore stated, without detail being given, that the description of inner spanning trees can be generalized for situations where the Steiner tree is not full; we will not need to do that.) We have noted in Section 2 that inner spanning trees are not precisely defined. It is however quite clear what is meant for

simple cases, and consequently this fact will not hinder our discussion now. Furthermore we may note already that it is possible to solve this "definition problem", as we will show in the next chapter.

Du and Hwang's basic idea is to first consider a full topology and to define the concept of adjacent terminals. Next they note that the adjacent terminals can be connected to form a polygon, called the characteristic area of the Steiner tree. This is clear for simple examples, as shown in Figure 10.

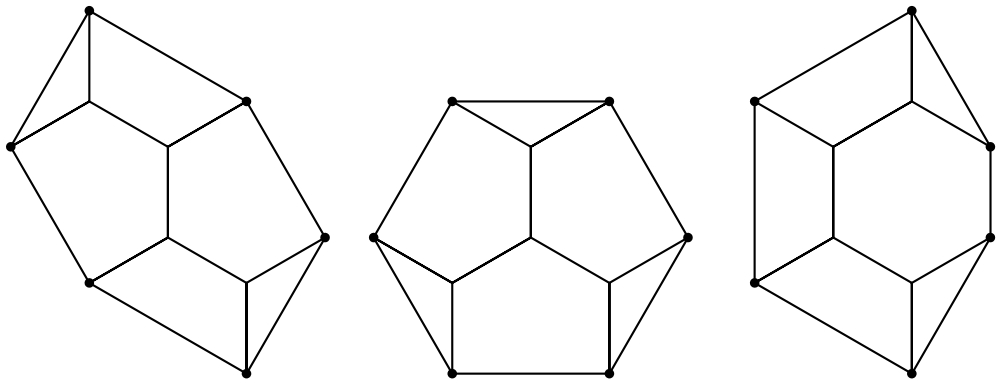


Figure 10

For more complex situations (as in Figure 11) Du and Hwang suggests that a "spiral surface" (they also refer to it as a Riemann surface) be used. We will look at surfaces in more detail in the next chapter.

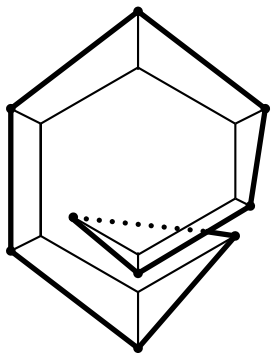


Figure 11

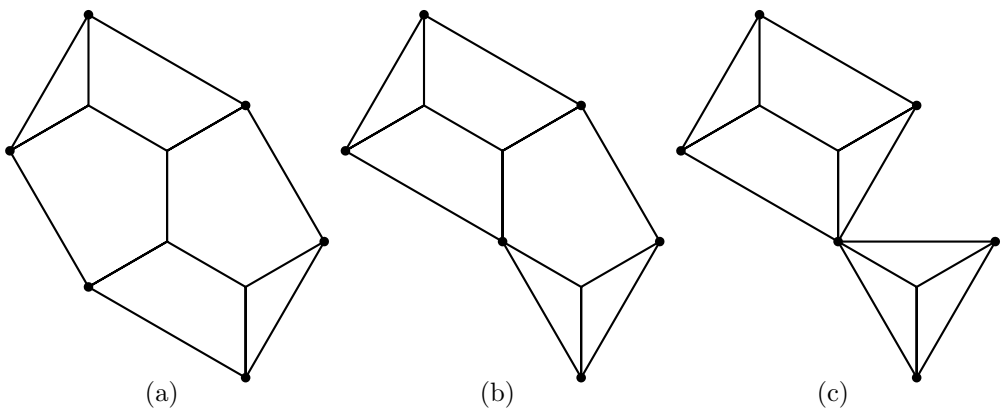
A spanning tree for the terminals of (s, x) is now called an inner spanning tree for s at x if it lies in the characteristic area. The lengths of the inner spanning trees are clearly also convex functions, although not everywhere defined.

Step 3: The following statement implies the Gilbert-Pollak conjecture: For any Steiner tree (s, x) there exists an inner spanning tree T such that $\|T\| \leq 2/\sqrt{3}\ \|(s, x)\|$.

Assume that this is false.

Step 4: Here Du and Hwang uses the same approach as in Lemma 6.8. They start by saying that if an edge incident to a terminal vanishes, then the ST can be decomposed into edge disjoint smaller STs. They then claim, without proof, that a union of inner spanning trees for the smaller STs is an inner spanning tree for the whole Steiner tree. We examine this claim.

In Figure 12(a) we show a ST and its characteristic area. We let one of the edges vanish in Figure 12(b). We now see in Figure 12(c) that it is possible to decompose the ST into smaller STs, and that the characteristic areas of the smaller STs lie within the characteristic area of the main ST. It follows that the union of inner spanning trees for the smaller STs is indeed an inner spanning tree for the main Steiner tree.



(a) (b) (c)

Figure 12

We look at another example in Figure 13(a). We let one of the edges vanish in Figure 13(b). One of the smaller STs into which the main ST can be decomposed is now shown Figure 13(c). (The other is simply a single edge.) It is important to note that the characteristic area of this smaller ST is in fact larger than that of the main ST. **Consequently we cannot assume that the union of the inner spanning trees for the smaller STs is an inner spanning tree for the main Steiner tree.** We consider this to be the main gap in the proof of Du and Hwang. A solution to this problem, like defining inner spanning trees differently, does not seem apparent. We will show in the next chapter that there is a way to define inner spanning trees rigorously, and that the ideas of Du and Hwang can be used to prove the Gilbert-Pollak conjecture for 7 points.

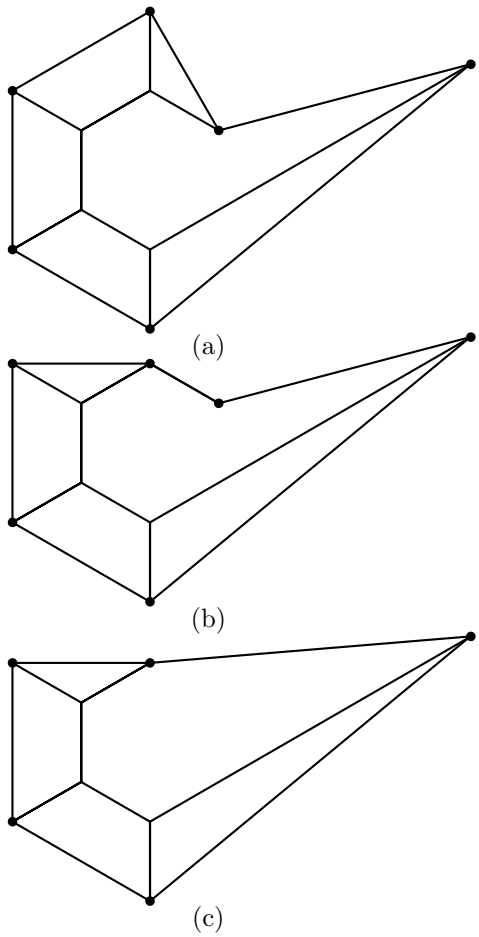


Figure 13

We end this chapter with another example for which the characteristic area of a subtree is larger than the characteristic area of the main tree, illustrated in Figure 14.

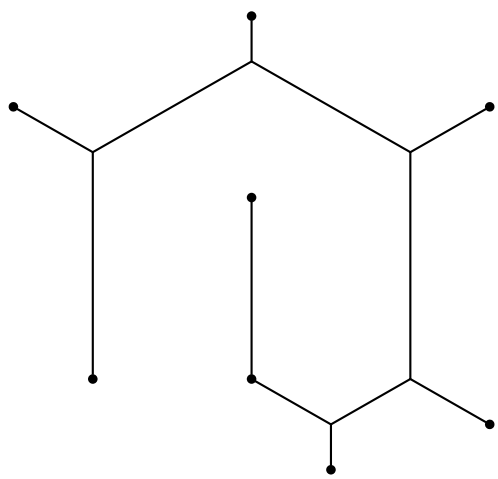


Figure 14

Chapter 7

The Gilbert-Pollak conjecture for 7 points

In this chapter we demonstrate that Du and Hwang's strategy can be altered to obtain a rigorous method for which inner spanning trees are well defined, and that this method can be used to prove the Gilbert-Pollak conjecture for 7 points.

In Section 1 we introduce a new version of the maximin theorem. In Section 2 we first define surfaces and then, for each surface, we define inner spanning trees as well as inner Steiner trees. This differs from Du and Hwang's proof, where the idea of inner Steiner trees was not used, and where the definition of a characteristic area (surface) in terms of a given Steiner tree was problematic. Sections 3 and 4 discuss aspects of a general proof and Section 6 proves the Gilbert-Pollak conjecture for 6 points. In Section 7 we first introduce some extra lemmas and then show that the Gilbert-Pollak conjecture is true for 7 points. This is an improvement on the proof for 6 points, published by Rubinstein and Thomas [25] in 1991.

7.1 A maximin theorem

The following theorem generalizes Theorem 6.4.

Theorem 7.1 *Let $f = \min_i g_i$, where the g_i are a finite number of continuous functions defined on a closed and bounded set $A \subset \boldsymbol{R}^k$. Suppose that every maximum point x of f is the center of a ball B_x in A where all g_i with $i \in I(x)$ are convex. Then there exists a maximum point x^* such that no $y \neq x^*$ in B_{x^*} has $I(y) \supseteq I(x^*)$.*

Proof: Fix any point p of A and let x^* be a maximum point such that the distance from p is a maximum. (Such a point exists because A is closed and bounded.)

Assume there exists a $y \neq x^*$ in B_{x^*} such that $\|y - p\| \leq \|x^* - p\|$ and for which $I(y) \supseteq I(x^*)$. Construct a straight line from y through x^* to any point z in B_{x^*} such that $I(z) \subseteq I(x^*)$. (This is possible by Lemma 6.3 of the previous chapter.) For $i \in I(x^*)$ we now have $g_i(y) = f(y) \leq f(x^*)$ and thus $g_i(z) \geq f(x^*)$ and $f(z) \geq f(x^*)$, since the g_i with $i \in I(x^*)$ are convex. This is a contradiction, because z is further from p than x^* and f cannot have a maximum here.

Now assume that there exists a y in B_{x^*} such that $\|y - p\| > \|x^* - p\|$ and for which $I(y) \supseteq I(x^*)$. Construct a straight line from y through x^* to any point z in B_{x^*} such that $I(z) \subseteq I(x^*)$. Since $f(y) < f(x^*)$ we have $f(z) > f(x^*)$; a contradiction. QED

7.2 Surfaces, inner spanning trees and inner Steiner trees

The word surface usually refers to a connected 2-dimensional manifold. The reader is referred to [1] and [21] for more on surfaces, surfaces with boundary and the "gluing together" of triangles to create surfaces. For our purposes it will suffice to define a *surface* inductively by the way in which it can be constructed:

- A triangle (we also allow degenerate triangles – see the notes below) is a surface, the edges of the triangle are the edges of the surface and the vertices of the triangle are the vertices of the surface.

- An edge of a triangle can be glued to an edge of a surface (given that these edges are of the same length and at the same angle) to create a surface which has one more edge and one more vertex, as shown in Figure 1. The edges and vertices together are called the *boundary* of the surface.

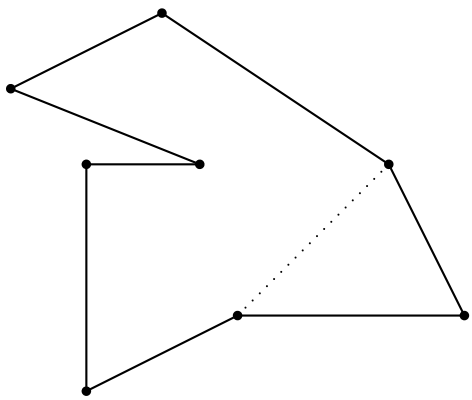
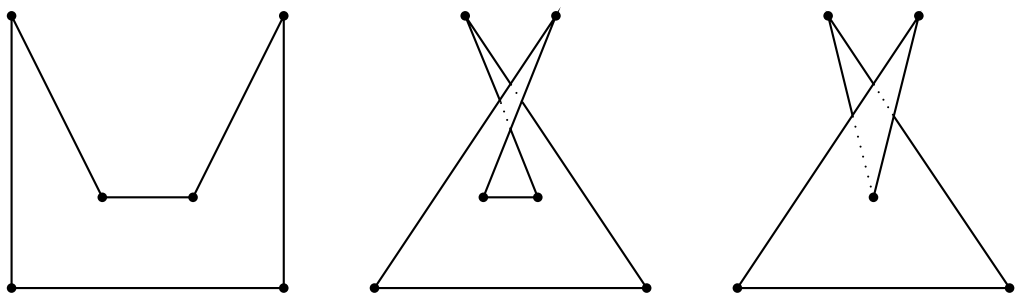


Figure 1

Notes: A surface is not necessarily a subset of $\mathbf{R}^2$, as illustrated in Figure 2(a). In Figure 2(b) the same surface is illustrated twice, showing (dotted lines) that a surface can be constructed out of triangles in different ways. We say that the surface has different *triangulations*. The third surface has only one triangulation. Since we allow degenerate triangles (triangles with edges or angles that are zero) we have surfaces with boundaries that intersect themselves so that the surface is not homeomorphic to a closed ball in $\mathbf{R}^2$. This "limit case" is shown in Figure 2(c). A surface with a triangulation of non-degenerate triangles is homeomorphic to a closed ball in $\mathbf{R}^2$ and its boundary does not intersect itself.



(a)

Figure 2

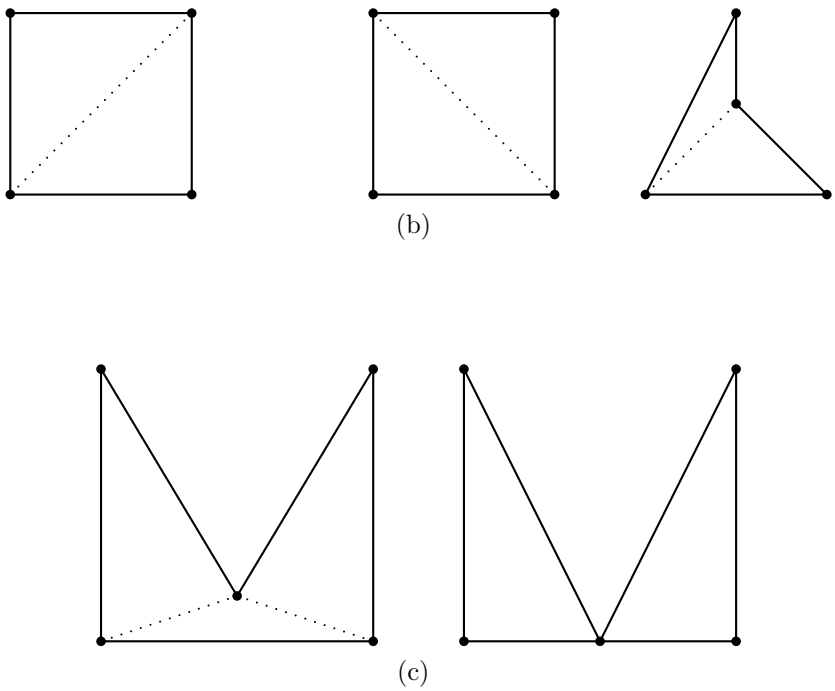


Figure 2

Lemma 7.2 *Any surface for which the boundary does not intersect itself and which has no edges of length zero, can be subdivided into non-degenerate triangles such that the vertices of all triangles are vertices of the surface.*

Proof: We will show that such a surface (which is not already a triangle) can be divided into two surfaces, each also with non-intersecting boundary and no edges of length zero, and with some of the original vertices as its vertices. If the surface is a convex set in $\mathbf{R}^2$ and its vertices a strictly convex set, then this is trivial, because any two non-adjacent vertices can be connected to divide the surface into two smaller surfaces. So we assume that there is a vertex o with an internal angle of at least 180° and neighboring vertices a' and b'. From o we construct a line into the surface to intersect the boundary of the surface at x on edge ab. (See Figure 3.) It is not possible for oa' to lie on oa and ob' to lie on ob, because the angle at o is at least 180°. Let us assume that oa' does not lie on oa. The position of x can now be adjusted on ab in the direction of vertex a until x either reaches a or until ox intersects some other vertex or vertices, the closest to o we name y. Either oa or oy now divides the surface in a suitable way. QED

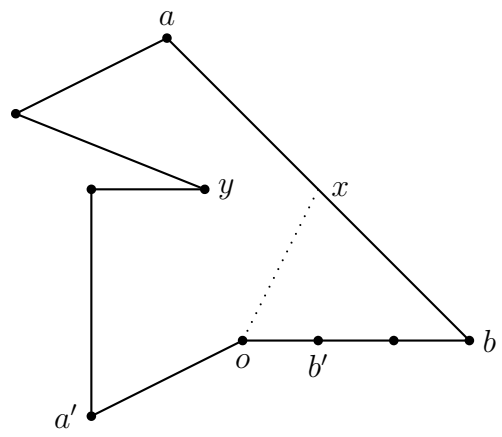


Figure 3

A triangle can be parametrized by the lengths of its sides, and thus the set of all surfaces $\mathcal{C}_\alpha$ with n vertices and the same triangulation α can be parametrized and given a topology in a natural way. Now let α and σ be different triangulations for surfaces with n vertices and $\mathcal{C}_\alpha$ and $\mathcal{C}_\sigma$ the sets of associated surfaces. Let Y be $\mathcal{C}_\alpha \cap \mathcal{C}_\sigma$ regarded as subspace (with subspace topology) of $\mathcal{C}_\alpha$ and let Z be $\mathcal{C}_\alpha \cap \mathcal{C}_\sigma$ regarded as subspace (with subspace topology) of $\mathcal{C}_\sigma$. Both subspaces are closed and the identity map $Y \leftrightarrow Z$ is continuous, with the result that $\mathcal{C}_\alpha$ and $\mathcal{C}_\sigma$ induce the same subspace topology on $\mathcal{C}_\alpha \cap \mathcal{C}_\sigma$. Let $\mathcal{C}$ be the set of all surfaces with n vertices, then we can (see [27] for detail) topologize $\mathcal{C}$ in a natural way by giving it the weak topology relative to the subspaces $\{\mathcal{C}_\alpha : \ \alpha$ is a triangulation for a surface with n vertices$\}$. In a similar way we can denote by $\mathcal{C}^*_\alpha \subset \mathcal{C}_\alpha$ the compact space consisting of surfaces with n vertices, boundary of length 1 and triangulation α. We now have that $\mathcal{C}^* \subset \mathcal{C}$, the set of all surfaces with boundary length 1, is also a compact space, as it is a union of a finite set of compact spaces.

Given a surface A and a spanning tree for the vertices of A, then we call the tree an *inner spanning tree* if every edge ab of the tree is the shortest path in A between a and b. We call a shortest inner spanning tree an *inner minimum spanning tree* (IMST).

Lemma 7.3 *Let A be a surface with a boundary which does not intersect itself and which has no edges of length zero. Then for every edge of any IMST which does not have adjacent vertices of the boundary as endpoints, we have that the edge intersects the boundary only at its endpoints.*

Proof: An edge ab of an IMST, as well as the boundary of A, consists of straight line segments. It follows that, if we assume the theorem to be false, then at least one of the following has to be true:

- There is a point on edge ab where two of its segments intersect, which intersects the boundary. (See Figure 4(a).)
- There is a vertex of the boundary (not a or b) which intersects edges ab. (See Figure 4(b).)

In the first case we have that the edge (and thus the IMST) can be made shorter. In the second case we can consider the IMST to contain a circuit, because the vertex which is intersected must also be a terminal of the IMST. Both are contradictions. QED

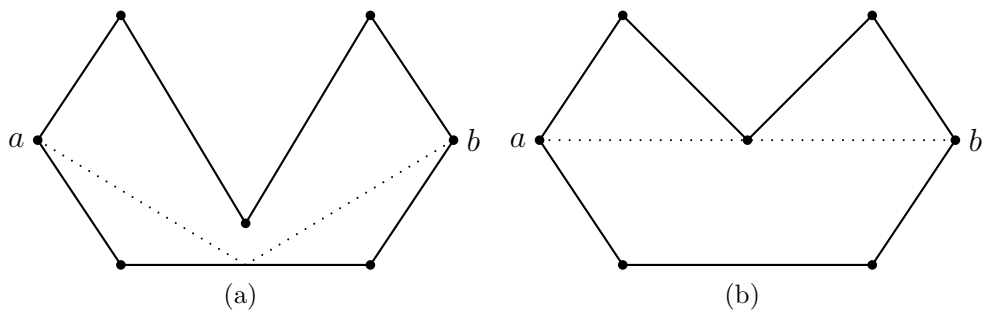


Figure 4

We define an *inner Steiner tree* as a Steiner tree for the vertices of A with all edges in A, and an *inner Steiner minimal tree* (ISMT) as a shortest such tree.

Lemma 7.4 *Let A be a surface with a boundary which does not intersect itself and which has no edges of length zero. Then for any ISMT we have that Steiner points are of degree 3 with edges meeting at* $120°$ *and that edges between Steiner points cannot have length 0. Furthermore, for every edge which does not have adjacent vertices of the boundary as endpoints, we have that the edge intersects the boundary only where it ends at a terminal.*

Proof: The first part of the lemma follows as for Euclidean Steiner points (see previous chapter). The proof for the second part of the lemma is similar to that of the previous lemma. QED

Lemma 7.5 *For any full Steiner tree which has no edges of length 0 and for which none of the edges intersect, we can construct a surface such that the Steiner tree is an inner Steiner tree of the surface.*

Proof: We begin by giving a Steiner tree infinitesimal thickness, as in Figure 5(a), and will then glue triangles onto this structure until there is

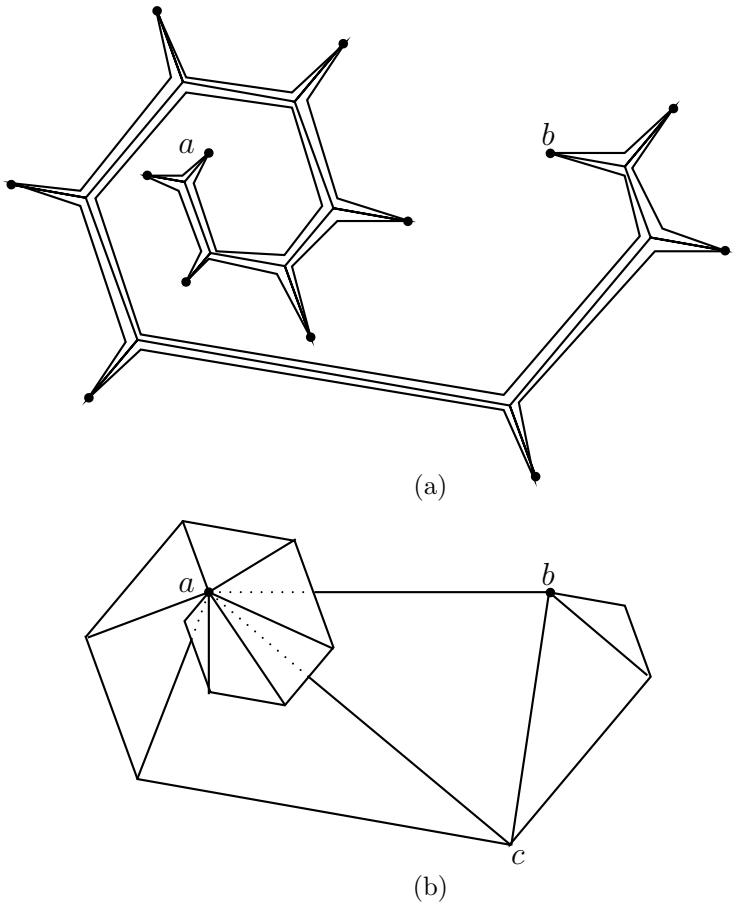


Figure 5

one edge connecting every two adjacent terminals. Finally we will show that the 2-dimensional manifold with boundary which we have constructed is a surface according to our definition.

It will be sufficient to look at two adjacent terminals and to show how they become adjacent vertices of a surface. Figure 5(b) shows two adjacent points, a and b, taken from Figure 5(a). We assume without loss that a and b lie on the same horizontal level and that the path between them is orientated as in the figure. Note that the path between any two adjacent terminals has edges intersecting at 120°. A lowest point on this path is called c. We now begin at point a, adding triangles as indicated until line ac has been used. The same is then done from point b, until line bc has been used. Finally triangle abc is added.

If this construction is carried out for all pairs of adjacent terminals we obtain a 2-dimensional manifold with boundary which is homeomorphic to

a closed ball in $\mathbf{R}^2$, which has a boundary that does not intersect itself and which has no edges of length zero. To show that this is a surface follows as in the proof of Lemma 7.2. QED

7.3 The Gilbert-Pollak conjecture in general

For a given n, the Gilbert-Pollak conjecture for n points follows from the following conjecture.

Conjecture 7.6 *For any surface A with n vertices we have $\|IMST(A)\| \leq (2/\sqrt{3})\ \|ISMT(A)\|$.*

We will now attempt to prove this conjecture in general. For n fixed we have that $\|\mathrm{IMST}(A)\|$ and $\|\mathrm{ISMT}(A)\|$ are continuous functions of $A \in \mathcal{C}$. This follows from the fact that $\|\mathrm{IMST}(A)\|$ and $\|\mathrm{ISMT}(A)\|$ are continuous functions on any particular $\mathcal{C}_\alpha$ [27]. Since $\mathcal{C}^*$ is closed and bounded, we have that $\frac{\|\mathrm{IMST}(A)\|}{\|\mathrm{ISMT}(A)\|}$ attains a maximum on $\mathcal{C}^*$. We will attempt to arrive at a contradiction, so let us fix n to be the smallest number of vertices for which the conjecture is not true.

We will use the following:

Assumption 7.7 *If n is the smallest number of vertices for which the conjecture is not true, then $\frac{\|IMST(A)\|}{\|ISMT(A)\|}$ attains a maximum on $\mathcal{C}^*$ only where all ISMTs have a full topology.*

Proof of Conjecture 7.6 (Subject to Assumption 7.7): The basic idea of the proof is the same as that which was used for 4 points in the previous chapter.

Let n be the smallest number of vertices for which the conjecture is not true. Consider an element of $\mathcal{C}^*$ for which $\frac{\|\mathrm{IMST}(A)\|}{\|\mathrm{ISMT}(A)\|}$ is maximum. We see that this surface has no edges of length 0, because then the conjecture would be false for fewer than n vertices. We also have that the boundary of this surface does not intersect itself, because if it does then we can view the surface as the union of two smaller surfaces (see Figure 2(c)), each with fewer than n vertices, and the conjecture would have to be false for one of these. (We used the same argument in Chapter 1 when we considered the Steiner ratio for full subtrees of a main tree.)

By Assumption 7.7 any ISMT has a full topology. We fix s to be the topology of such an ISMT. We now rescale (the criterion that $A \in \mathcal{C}^*$ is relaxed) so that we can use a vector $x' \in S = \{x : |x| = 1\}$, together with s, to describe this inner Steiner tree by (s, x'). We name the rescaled surface

A'. From Lemma 7.4 we have that edges of (s, x') (except for the terminals) lie within A', as illustrated in Figure 6. As in the previous chapter we regard x as our independent variable. If x changes, then the surface changes. We describe this with the function J and have $J(x') = A'$. For x' the following three statements are true:

- No component of x' is 0.
- $J(x')$ is a surface with a boundary which does not intersect itself and which has no edges of length 0.
- Edges of (s, x') (except for the terminals) lie within $J(x')$.

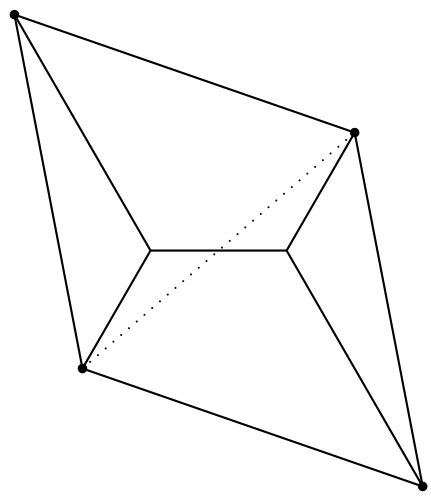


Figure 6

We see that x' has a neighborhood such that the three statements are true for all the points in the neighborhood. Let us consider the set $P \subset S$ for which the three statements are true. We find that P is an open set, because every point in P has a neighborhood such that the three statements are true for all the points in the neighborhood.

Next we consider the closure of P, which we denote by $\overline{P}$. For a point $y \in \overline{P}$ which is not in P we must have that:

- y has a component which is 0 or
- $J(y)$ is a surface with a boundary which intersects itself or $J(y)$ is a surface with an edge of length 0 or
- an edge of (s, y) intersects $J(y)$ at a point which is not a terminal.

We can now consider the function $f(x) = \|\mathrm{IMST}(s, x)\|$ on domain $\overline{P}$. We see that $f(x)$ cannot be maximal for $x \in \overline{P},\ x \notin P$. We also see that $f(x)$ is of the form $f = \min_i g_i$, where g_k is the length of a particular inner spanning tree. If $k \in I(x)$ then the inner spanning tree $T_k(x)$ is an IMST, and from Lemma 7.3 we have that all edges of $T_k(x)$ are straight line segments. Since edges which are not on the boundary are within the surface, we have that the edges remain straight line segments on some neighborhood of x. We can consequently show that g_k is a convex function on this neighborhood, as we did in the previous chapter. The maximin theorem can now be applied.

According to the maximin theorem there is an inner Steiner tree (s, x^*) such that x^* is a maximum point for $f(x)$ and such that the set of IMSTs is maximal. The topological graph in the surface which consists of the terminals as well as all edges which form part of some IMST is denoted by Λ. Edges belonging to Λ cannot cross. (The proof is the same as in the previous chapter.) As before, we can show that Λ has to be rigid. This is so because a non-rigid Λ can be altered slightly (by changing the edge lengths of the underlying ISMT) to obtain a y which has the same set of IMSTs as x^*, thus contradicting the maximin theorem.

Next we look at the graph structure obtained if we consider faces of Λ to be vertices, connecting two vertices if they share an edge. If we use a homeomorphism to represent the surface as a ball, as in Figure 7, we see that the graph can contain no circuits. It follows that the faces of Λ form a tree structure and, since rigidity is required, that all faces are triangles.

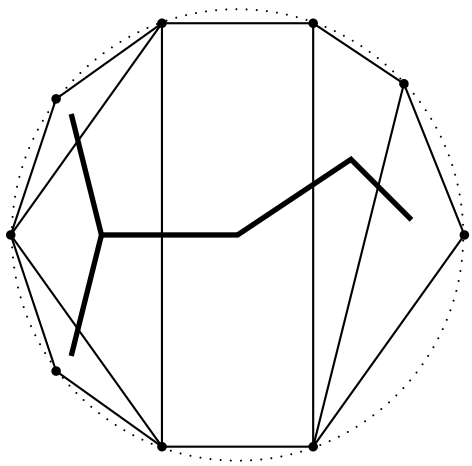


Figure 7

We end by showing that all edges of Λ have the same length. Let us assume that this is not the case. Because of the tree structure of Λ, the lengths of the edges of Λ can be changed independently. Let A be the set consisting of the shortest edge or edges. We now increase the length of the edges in A slightly by the same $\epsilon > 0$ (by changing the edge lengths of the underlying ISMT) and rescale so that the sum of the edges of the underlying inner Steiner tree is once again 1. As in the previous chapter we find that if $\epsilon > 0$ is small enough, then a MST before these perturbations remains a MST afterwards. But by the maximin theorem this is not allowed, so all edges must have the same length. The truth of the conjecture follows from the fact that the Gilbert-Pollak conjecture is true for points on a triangular lattice. QED

In the remaining sections we consider the validity of Assumption 7.7.

7.4 Assumption 7.7 in general

We consider a surface which has an ISMT which is not full and use a homeomorphism to represent it as a ball as in Figure 8. We introduce some definitions.

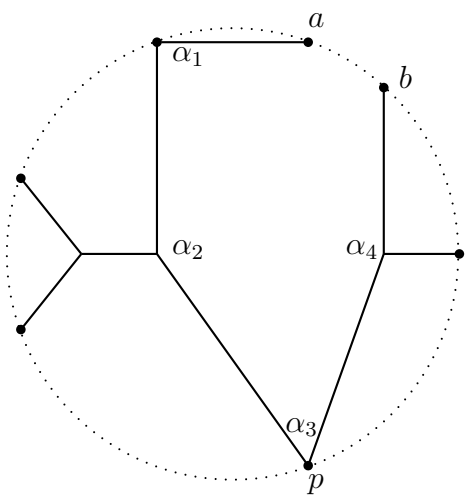


Figure 8

Consider a boundary point p with degree more than 1. Then there exists a path P between two boundary points a and b which are adjacent on the boundary of the surface, such that p is on P. We refer to P as a *path associated with* p. We indicate the internal angles of P in the figure, where we adopt the convention that the symbols for the angles refer to the angles

of the actual ISMT and not those of the figure. (The homeomorphism does not preserve angles.) For the inner Steiner tree to be minimal we have for all these internal angles that $\alpha_i \geq 120°$. We now connect all boundary points which are adjacent on P with edges of minimal length, as shown by the thick lines in Figure 9. We call these edges *quasi edges*. We will see shortly that a quasi edge can consist of more than one straight line segment. Between a quasi edge and the boundary of the surface a smaller surface is formed which we call a *quasi surface*. Each quasi surface contains a part of the ISMT and the union of the quasi surfaces contains the whole ISMT. If the quasi edge of a quasi surface is a single straight line segment, then the quasi surface is itself a surface and the part of the ISMT in it is an inner Steiner tree for this surface.

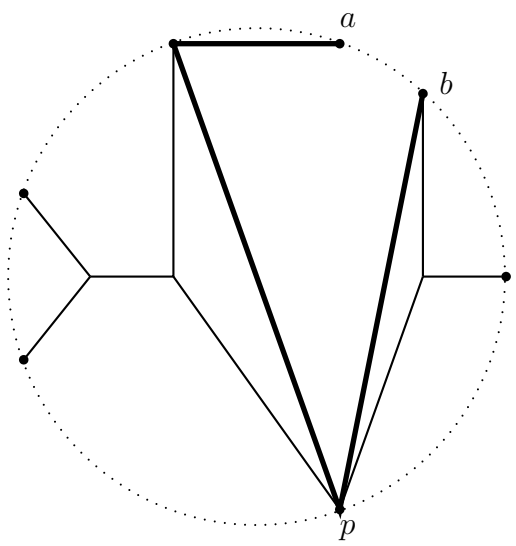


Figure 9

In Figure 10(a) we consider an actual (not a homeomorphism) point p with an associated path P which consists of 4 edges. We let p be at one of the positions p_1, p_2 or p_3, and the remaining positions can be either Steiner points or terminals. It follows, from the fact that internal angles on the path are at least 120°, that any quasi edge is a straight line segment. We illustrate one possible quasi edge by a thick line. In Figure 10(b) we have an example where one of the internal angles is larger than 120°, in which case the point concerned has to be a terminal, with two quasi edges at it.

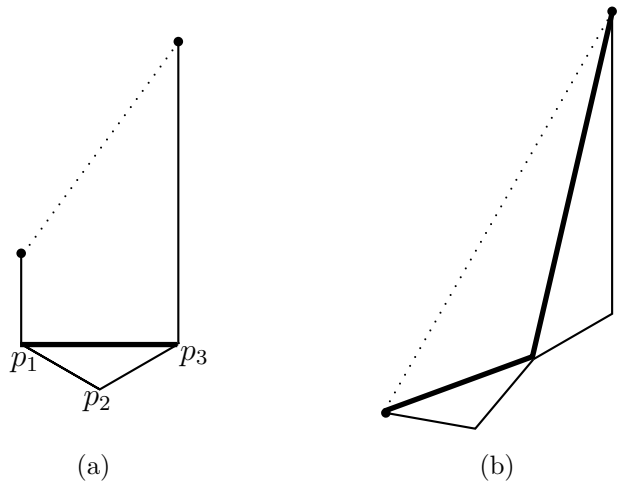


Figure 10

If the associated path of p has 5 edges then a quasi edge can consist of one or two segments, as illustrated in Figure 11(a) and Figure 11(b) respectively. We note that in the latter case the segments of the quasi edge cannot form an internal angle of more than 240°. The point where the two segments intersect is part of two distinct quasi surfaces: the one has these segments as quasi edge and the other contains the terminal point which is present at this point.

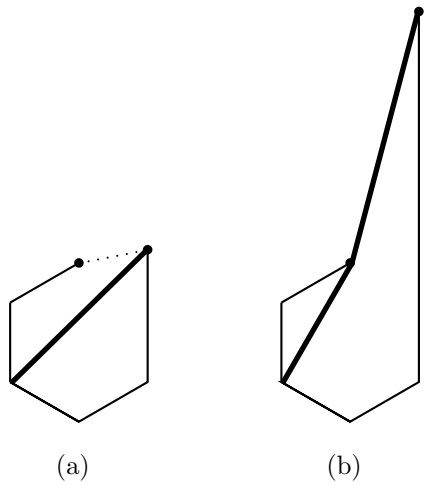
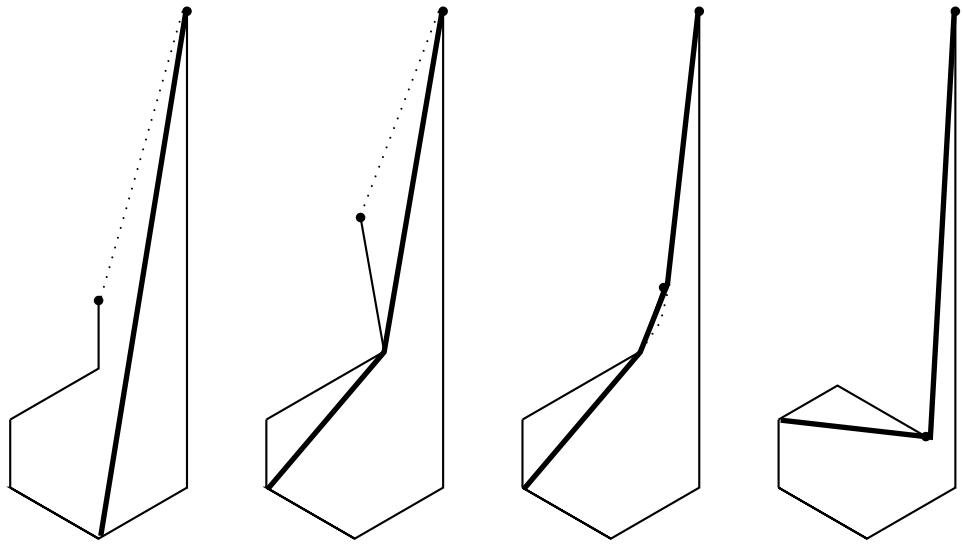


Figure 11

If the associated path of p has 6 edges then a quasi edge can consist of one, two or three segments, as illustrated in Figure 12. If a quasi edge consists of two segments, then we note that these segments cannot form an internal angle of more than 300°. If a quasi edge consists of three segments, then when we extend the outer segments to meet, the internal angle formed cannot be more than 240°. (See dotted line in Figure 12(c).)



(a) (b) (c) (d)

Figure 12

We summarize our discussion by the following lemma.

Lemma 7.8 *For a surface with an ISMT and a terminal p with degree more than one and associated path P, the following is true:*

- *If P has fewer than 5 edges, then all quasi edges are straight line segments.*
- *If P has 5 edges, then any quasi edge consists of either one or two edges. Two segments of a quasi edge cannot form an internal angle of more than* 240°.
- *If P has 6 edges, then any quasi edge consists of either one, two or three edges. Two segments of a quasi edge cannot form an internal angle of more than* 300°. *If a quasi edge consists of three segments, and we extend the outer segments to meet, then the internal angle formed cannot be more than* 240°.

Let us assume that Conjecture 7.6 is false and that $n \leq 5$ is the smallest number of boundary points for which this is so. This, by our previous discussion, implies that Assumption 7.7 has to be false. We consider A' for which $\frac{\|\mathrm{IMST}(A)\|}{\|\mathrm{ISMT}(A)\|}$ is maximum with an ISMT which is not full. We have seen that there exists a terminal p and associated path P as well as quasi surfaces and that the ISMT can be decomposed into smaller parts, each contained in a quasi surface. Since P has fewer than 5 edges, the quasi edge of each quasi surface consists of only one straight line segment, and any particular quasi surface is itself a surface such that the part of the ISMT in it is an internal Steiner tree. Let us refer to this part of the ISMT as ISMT$'$. The quasi surface contains IMST$'$ such that $\|\mathrm{IMST}'\| \leq (2/\sqrt{3})\|\mathrm{ISMT}'\|$. The union of all these smaller spanning trees is a spanning tree for A' and its length is such that Conjecture 7.6 is true. This is a contradiction and it follows that the Gilbert-Pollak conjecture is true for $n \leq 5$. We saw, however, that it is not always the case that quasi edges consist of single line segments, and thus this simple inductive method cannot be used in general. A simple proof for Assumption 7.7 seems unlikely in general, and in what follows we consider surfaces with fewer than 8 boundary points.

7.5 The Gilbert-Pollak conjecture for 6 points

We assume that Conjecture 7.6 is false for $n = 6$, and thus we have A' for which $\frac{\|\mathrm{IMST}(A)\|}{\|\mathrm{ISMT}(A)\|}$ is maximum with an ISMT which is not full, such that there is a quasi surface with quasi edge which consists of more than one straight line segment. A path cannot have more than 5 edges and it follows from Lemma 7.8 that the quasi edge consists of two segments with an internal angle which is at most 240°. Let us call the point where the segments intersect q. Figure 13(a) illustrates the quasi surface.

If we replace the quasi edge by a straight edge (Figure 13(b)) then the quasi surface (call it B) becomes a surface, C, which has terminals as boundary points. Since there are fewer than 6 boundary points, we know that C has IMST$'$ such that Conjecture 7.6 is true. The union of IMST$'$ and the IMSTs of the other quasi surfaces (which are all surfaces contained in A') is a tree, T with length less than $(2/\sqrt{3})\|\mathrm{ISMT}(A')\|$. For the conjecture to be false for $n = 6$, some edge (or edges) of IMST$'$ is in C, but not in B. Let ab be such an edge (see Figure 13(c) and Figure 13(d)). We delete this edge of T and are left with two subtrees T_1 (which has a as a vertex) and T_2 (which has b as a vertex). Both aq and bq are shorter than ab, because the internal angle of the segments is at most 240°. (This is true whenever the internal angle is less than 270°.) Now, since there is some vertex at q, we can assume

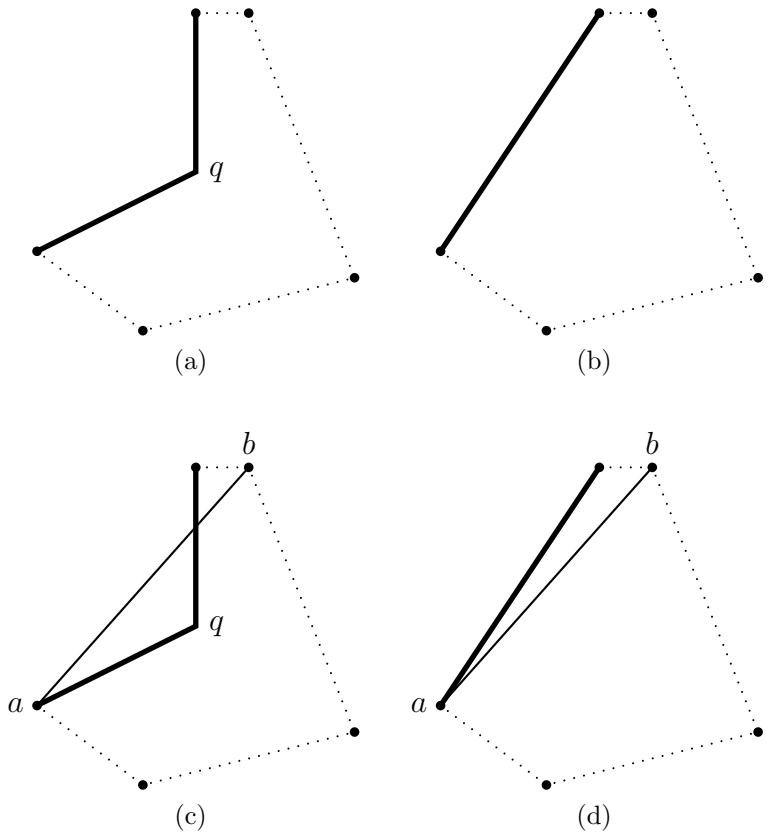


Figure 13

without loss that it is a vertex of T_1. We now add edge bq to obtain a new spanning tree which is shorter than T and which has edge bq (which is in B) in stead of edge ab (which was not in B). It follows that there exists an inner spanning tree with length less than $(2/\sqrt{3})\|\mathrm{ISMT}(A')\|$.

7.6 The Gilbert-Pollak conjecture for 7 points

If we assume that Conjecture 7.6 is false for $n = 7$ then Assumption 7.7 has to be false and there has to exist A' for which $\frac{\|\mathrm{IMST}(A)\|}{\|\mathrm{ISMT}(A)\|}$ is maximum with an ISMT which is not full. Furthermore there has to be a quasi surface with quasi edge which consists of two straight line segments with an internal angle of more than 270°, otherwise we can use the same scheme as in the previous section to obtain a contradiction. By Lemma 7.8 this is possible for 7 points, as illustrated in Figure 14(a) for ISMT. In this illustration c is a terminal

and gac is the quasi edge, but either b or d could have been a terminal and gab or gad respectively a quasi edge. For the internal angle of the segments of the quasi edge to be more than 270°, we must however have that e and f are Steiner points.

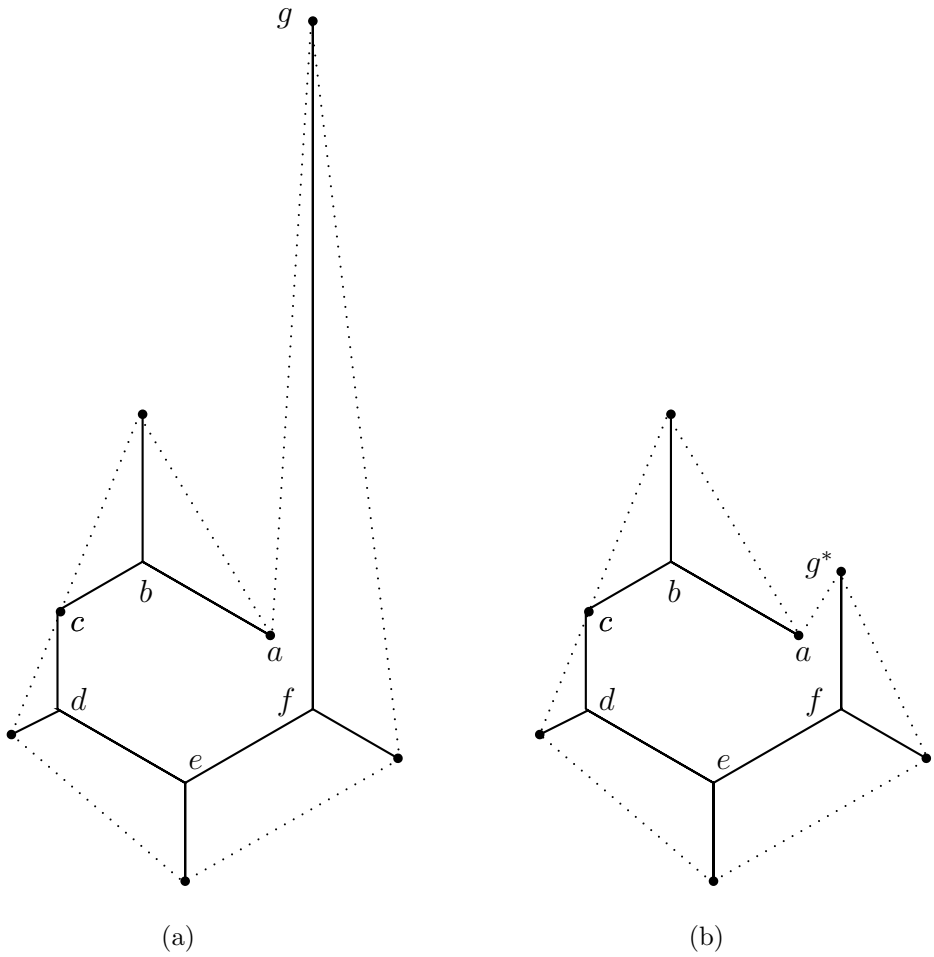


Figure 14

Consider the surface $abcdefg$. For the internal angle of the quasi edge to be more than 270°, we must have that the internal angle at a is more than 270°. Internal angles at b, c, d, e and f all have to be at least 120°, and it follows, from the fact that the sum of the internal angles of surface $abcdefg$ is 900°, that the internal angle at g is less than 30°. We now reduce the length of the edge at g by length y so that the angle becomes 30°, and we obtain A^* with ISMT* as in Figure 14(b). For any terminal it is true that the distance in A^* is less to a than to g^*. (We consider the perpendicular

bisector of a and g^* to see this.) From this it follows that IMST* has degree 1 at g^*. To see why, assume that g^* has degree more than 1. Then there is a terminal x adjacent to g^* such that the path in IMST* between x and a includes an edge xg^*. But the tree would become shorter if we replace xg^* with xa, showing that g^* cannot have degree more than 1. It follows as before, from the fact that the segments of the quasi edge have an internal angle which is not larger than 270°, that $\|\text{IMST}^*\| \leq (2/\sqrt{3})\|\text{ISMT}^*\|$. But now we have that $\|\text{ISMT}\| = \|\text{ISMT}^*\| + y$ and $\|\text{IMST}\| \leq \|\text{IMST}^*\| + y$, and thus that $\|\text{IMST}\| \leq (2/\sqrt{3})\|\text{ISMT}\|$.

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